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On Line-Block Signed Graphs

Swamy[1], K. V. Madhusudhan[2] and Khaled A. A. Alloush[3]

1. Department of Mathematics, Maharani's Science College for Women, Mysore-570 005, India

2. Department of Mathematics, ATME College of Engineering, Mysore-570 028, India

3. Department of Mathematics, Dar Al-Uloom University, Riyadh-13314, Saudi Arabia

E-mail: sgautham48@gmail.com, kvmadhu13@gmail.com, khaledindia@gmail.com

Abstract: In this paper we introduced the new notion called line-block signed graph of a signed graph and its properties are studied. Also, we obtained the structural characterization of this new notion and presented some switching equivalent characterizations.

Key Words: Signed graphs, neutrosophic signed graph, balance, switching, line-block signed graph.

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§1. Introduction

For standard terminology and notion in graph theory, we refer the reader to the text-book of Harary [1]. The non-standard will be given in this paper as and when required.

Given a graph $G = (V, E)$, the line-block graph of $G = (V, E)$, denoted $\mathcal{LBG}(G)$, is defined to be that graph with $V(\mathcal{LBG}(G)) = E(G) \cup B$, where B is set of blocks of G and any two vertices in $V(\mathcal{LBG}(G))$ are joined by an edge if, and only if, the corresponding blocks are adjacent or one corresponds to a block of G and other to a line incident with it (see [4]).

To model individuals' preferences towards each other in a group, Harary [2] introduced the concept of signed graphs in 1953. A signed graph $S = (G, \sigma)$ is a graph $G = (V, E)$ whose edges are labeled with positive and negative signs (i.e., $\sigma : E(G) \to \{+, -\}$). The vertices of a graph represent people and an edge connecting two nodes signifies a relationship between individuals. The signed graph captures the attitudes between people, where a positive (negative edge) represents liking (disliking). A neutrosophic signed graph $S^N = (G, \sigma, H)$ for a subgraph $H \subset G$ with property $\mathscr{P}$ is such a graph that $G\backslash H$ is a signed graph but H is indefinite for those of uncertainties in reality. Certainly, if there are no indefinite subgraph in G, it must be a signed graph. An unsigned graph is a signed graph with the signs removed. Similar to an unsigned graph, there are many active areas of research for signed graphs.

The sign of a cycle (this is the edge set of a simple cycle) is defined to be the product of the signs of its edges; in other words, a cycle is positive if it contains an even number of negative edges and negative if it contains an odd number of negative edges. A signed graph S is said to be balanced if every cycle in it is positive. A signed graph S is called totally unbalanced if

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every cycle in S is negative. A chord is an edge joining two non adjacent vertices in a cycle.

A *marking* of S is a function $\zeta : V(G) \to \{+,-\}$. Given a signed graph S one can easily define a marking ζ of S as follows: For any vertex $v \in V(S)$,

$$\zeta(v) = \prod_{uv \in E(S)} \sigma(uv),$$

the marking ζ of S is called *canonical marking* of S. For more new notions on signed graphs refer the papers (see [5, 9-13, 13-22]).

The following are the fundamental results about balance, the second being a more advanced form of the first. Note that in a bipartition of a set, $V = V_1 \cup V_2$, the disjoint subsets may be empty.

Theorem 1.1 *A signed graph S is balanced if and only if either of the following equivalent conditions is satisfied:*

(i) *Its vertex set has a bipartition $V = V_1 \cup V_2$ such that every positive edge joins vertices in V_1 or in V_2, and every negative edge joins a vertex in V_1 and a vertex in V_2 (Harary [2]).*

(ii) *There exists a marking μ of its vertices such that each edge uv in Γ satisfies $\sigma(uv) = \zeta(u)\zeta(v)$ (Sampathkumar [6]).*

Switching S with respect to a marking ζ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs.

Two signed graphs $S_1 = (G_1, \sigma_1)$ and $S_2 = (G_2, \sigma_2)$ are said to be *weakly isomorphic* (see []23) or *cycle isomorphic* (see [24]) if there exists an isomorphism $\phi : G_1 \to G_2$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known (see [24]).

Theorem 1.2(T. Zaslavsky [24]) *Given a graph G, any two signed graphs in $\psi(G)$, where $\psi(G)$ denotes the set of all the signed graphs possible for a graph G, are switching equivalent if and only if they are cycle isomorphic.*

§2. Line-Block Signed Graph of a Signed Graph

Motivated by the existing definition of complement of a signed graph, we now extend the notion of line-block graphs to signed graphs as follows: The *line-block signed graph* $\mathcal{LBS}(S) = (\mathcal{LBG}(G), \sigma')$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $\mathcal{LBG}(G)$ and sign of any edge uv is $\mathcal{LBS}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S. Further, a signed graph $S = (G, \sigma)$ is called a line-block signed graph, if $S \cong \mathcal{LBS}(S')$ for some signed graph S'. The following result restricts the class of line-block signed graphs.

Theorem 2.1 *For any signed graph $S = (G, \sigma)$, its line-block signed graph $\mathcal{LBS}(S)$ is balanced.*

Proof Since sign of any edge $e = uv$ in $\mathcal{LBS}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S, by Theorem 1.1, $\mathcal{LBS}(S)$ is balanced. $\square$

For any positive integer k, the k^{th} iterated line-block signed graph, $\mathcal{LBS}^k(S)$ of S is defined as follows:

$$\mathcal{LBS}^0(S) = S,\ \mathcal{LBS}^k(S) = \mathcal{LBS}(\mathcal{LBS}^{k-1}(S)).$$

Corollary 2.2 *For any signed graph $S = (G, \sigma)$ and for any positive integer k, $\mathcal{LBS}^k(S)$ is balanced.*

In [4], the authors remarked that $\mathcal{LBG}(G)$ and G are isomorphic if and only if G is K_2. We now characterize the signed graphs and its line block signed graphs are cycle isomorphic.

Theorem 2.3 *For any signed graph $S = (G, \sigma)$, the line-block signed graph $\mathcal{LBS}(S)$ and S are cycle isomorphic if and only if the underlying of S is is isomorphic to K_2 and S is balanced.*

Proof Suppose $\mathcal{LBS}(S) \sim S$. This implies, $\mathcal{LBG}(G) \cong G$ and hence G is isomorphic to K_2. Then $\mathcal{LBS}(S)$ is balanced and hence if S is unbalanced and its line-block signed graph $\mathcal{LBS}(S)$ being balanced can not be switching equivalent to S in accordance with Theorem 1.2. Therefore, S must be balanced.

Conversely, suppose that S balanced signed graph with the underlying graph G is isomorphic to K_2. Then, since $\mathcal{LBS}(S)$ is balanced as per Theorem 2.1 and since $\mathcal{LBG}(G) \cong G$, the result follows from Theorem 1.2 again. $\square$

Corollary 2.4 *Let $S = (G, \sigma)$ be a connected signed graph. Then the n^{th}-iterated line-block signed graph $\mathcal{LBS}^n(S)$, $n \geq 1$ and S are cycle isomorphic if and only if the underlying of S is isomorphic to K_2 and S is balanced.*

Corollary 2.5 *Let $S = (G, \sigma)$ be any signed graph with no isolated vertices, the n^{th}-iterated line-block signed graph $\mathcal{LBS}^n(S)$, $n \geq 1$ and S are cycle isomorphic if and only if the underlying of S is isomorphic to mK_2, $m \geq 1$ and S is balanced.*

The following result characterize signed graphs which are line-block signed graphs.

Theorem 2.6 *A signed graph $S = (G, \sigma)$ is a line-block signed graph if, and only if, S is balanced signed graph and its underlying graph G is a line-block graph.*

Proof Suppose that S is balanced and G is a line-block graph. Then there exists a graph G' such that $\mathcal{LBG}(G') \cong G$. Since S is balanced, by Theorem 1.1, there exists a marking ζ of G such that each edge uv in S satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. Now consider the signed graph $S' = (G', \sigma')$, where for any edge e in G', $\sigma'(e)$ is the marking of the corresponding vertex in G. Then clearly, $\mathcal{LBS}(S') \cong S$. Hence S is a line-block signed graph.

Conversely, suppose that $S = (G, \sigma)$ is a line-block signed graph. Then there exists a signed graph $S' = (G', \sigma')$ such that $\mathcal{LBS}(S') \cong S$. Hence, G is the line-block graph of G' and by Theorem 2.1, S is balanced. $\square$

The notion of *negation* $\eta(S)$ of a given signed graph S defined in [3] as follows: $\eta(S)$ has the same underlying graph as that of S with the sign of each edge opposite to that given to it in S. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in S while applying the unary operator $\eta(.)$ of taking the negation of S.

For a signed graph $S = (G, \sigma)$, the $\mathcal{LBS}(S)$ is balanced (Theorem 2.1). We now examine, the conditions under which negation $\eta(S)$ of $\mathcal{LBS}(S)$ is balanced.

Proposition 2.7 *Let $S = (G, \sigma)$ be a signed graph. If $\mathcal{LBG}(G)$ is bipartite then $\eta(\mathcal{LBS}(S))$ is balanced.*

Proof Since, by Theorem 2.1, $\mathcal{LBS}(S)$ is balanced, it follows that each cycle C in $\mathcal{LBS}(S)$ contains even number of negative edges. Also, since $\mathcal{LBG}(G)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $\mathcal{LBS}(S)$ is also even. Hence $\eta(\mathcal{LBS}(S))$ is balanced. □

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