International J.Math. Combin. Vol.3(2023), 30-47

On Lorentzian Sasakian Space Form with Respect to Generalized Tanaka Connection and Some Solitons

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Abstract: The object of the present paper is to study several type of symmetricness (semi-symmetric, Ricci semi-symmetric) of Lorentzian Sasakian space forms with respect to generalized Tanaka connection and nature of $*$-Ricci soliton, $*$-conformal Ricci soliton, $*-$conformal η-Ricci soliton, generalized Ricci soliton, generalized conformal Ricci soliton of this type of space forms with respect to generalized Tanaka connection.

Key Words: Semi-symmetry, Ricci semi-symmetry, pseudo-symmetry, Ricci-pseudo-symmetry, $*$-Ricci soliton, $*$-conformal Ricci soliton, generalized Ricci soliton.

AMS(2010): 53C25, 53C15.

§1. Introduction

In 1996, the authors [28] first time studied Sasakian manifold with Lorentzian metric i.e., a metric compatible Sasakiann manifold (M, η, ξ, φ) with Lorentzian metric g (a symmetric non-degenerated (0,2) tensor field of index 1) which we called Lorentzian Sasakiann manifold. A Lorentzian Sasakian manifold with φ-holomorphic sectional curvature, we called Lorentzian Sasakian space form. The curvature tensor and Ricci tensor of a Lorentzian Sasakiann space form with constant φ-holomorphic sectional curvature c proved by the authors [23] as follows:

$$\begin{aligned} R(X,Y)Z = {} & \frac{c-3}{4}[g(Y,Z)X - g(X,Z)Y] \\ & + \frac{c+1}{4}\{\eta(Z)[\eta(Y)X - \eta(X)Y] + [\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]\xi \\ & + g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z\}, \end{aligned} \tag{1}$$

and

$$S(X,Y) = \frac{n(c-3)+4}{2}g(X,Y) + \frac{n(c+1)}{2}\eta(X)\eta(Y), \tag{2}$$

Tanaka [16] and, independently Webster [29] defined the canonical affine connection on a

[1]Received June 27, 2023, Accepted August 13, 2023.

nondegenerate, integrable CR manifold. Tanno [26] generalized this connection extending its definition to the general contact metric manifold which called generalized Tanaka欽揥ebster connection or generalized Tanaka connection.

A manifold M is said to be locally symmetric (see [25]) if we have $\nabla_X R = 0$ for all $X \in \mathfrak{X}(M)$, where R is curvature tensor. A locally symmetric Riemannian manifold satisfies $R(X,Y) \cdot R = 0$ for all tangent vectors X and Y, where the linear endomorphism $R(X,Y)$ acts on R as a derivation. The spaces with $R(X,Y) \cdot R = 0$ have been investigated first by E. Cartan [5] which directly generalizes the notion of symmetric spaces. Conversely, does the condition $R(X,Y) \cdot R = 0$ imply the manifold M is locally symmetric? K. Nomizu [15] conjectured that an irreducible, complete Riemannian space with dim≥ 3 and with the above symmetric property of the curvature tensor is always a locally symmetric space. But this conjecture was refuted by H. Takagi [7] who constructed 3-dimensional complete irreducible nonlocally-symmetric hypersurfaces with $R(X,Y) \cdot R = 0$. According to Szabó [30], we call a space satisfying $R(X,Y) \cdot R = 0$ is a semi-symmetric space. Okumura [14] proved that a Sasakian manifold which is at the same time a locally symmetric space is a space of constant curvature. This fact means that a symmetric space condition is too strong for a Sasakian manifold. In a semi-symmetric manifold, the condition $R(X,Y) \cdot S = 0$ satisfies for all $X, Y \in \mathfrak{X}(M)$, where S is the Ricci tensor. But the converse statement is however not true. These two conditions $R(X,Y) \cdot R = 0$ and $R(X,Y) \cdot S = 0$ are equivalent for hypersurfaces of Euclidean spaces proved by P.J. Ryan [18]. The spaces which satisfies $R(X,Y) \cdot S = 0$ we called Ricci-semisymmetric spaces. Thus, every semisymmetric space is Ricci-semisymmetric. The generalized condition $R \cdot R = LQ(g,R)$, where L is a non zero function and $Q(g,R)$ is defined in [1] of the conditions $\nabla_X R = 0$ and $R(X,Y) \cdot R = 0$ (symmetric and semi-symmetric) was introduced by R. Deszcz [8] and if a manifold satisfies this condition then it called pseudo-symmetric. On the other hand M.C. Chaki [11] introduced a different definition of pseudo-symmetric manifold. In this paper we approach the Deszcz's definition. Deszcz also defined Ricci-pseudo-symmetric manifold [19] by the condition $R \cdot S = LQ(g,S)$.

In 1982 Hamilton [21] introduced the concept of Ricci flow and proved its existence. The Ricci flow equation is given by

$$\frac{\partial g}{\partial t} = -2S \tag{3}$$

on a compact Riemannian manifold M with Riemannian metric g, where S is the Ricci tensor. A self-similar solution to the Ricci flow (3) is called a Ricci soliton which moves under the Ricci flow simply by diffeomorphisms of the initial metric, that is, they are stationary points of the Ricci flow in space of metrics on M. A Ricci soliton is a generalization of an Einstein metric. The Ricci soliton equation is given by

$$\mathcal{L}_X g + 2S = 2\lambda g \tag{4}$$

where $\mathcal{L}$ is the Lie derivative, S is the Ricci tensor, g is Riemannian metric, X is a vector field and λ is a scalar. The Ricci soliton is said to be shrinking, steady, and expanding according as λ is positive, zero and negetive respectively.

Fischer during 2003-2004 developed the concept of conformal Ricci flow [3] which is a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. The conformal Ricci flow on M is defined by [24]

$$\frac{\partial g}{\partial t} + 2\left(S + \frac{g}{n}\right) = -pg \tag{5}$$

where $R(g) = -1$ and p is a non-dynamical scalar field(time dependent scalar field), $R(g)$ is the scalar curvature of the n-dimessional manifold M.

In 2015, N. Basu and A. Bhattacharyya [2] introduced the notion of conformal Ricci soliton and the equation is as follows

$$\mathcal{L}_X g + 2S = \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g \tag{6}$$

where λ is a scalar.

Cho and Kimura [9] introduced the notion of η-Ricci soliton in 2009, as follows

$$\mathcal{L}_\xi g + 2S = 2\lambda g + 2\mu\eta \otimes \eta \tag{7}$$

for some constants λ and μ, where ξ is a soliton vector field and η is an 1-form on M.

In 2018, Siddiqi [13] introduced the notion of conformal η-Ricci soliton, given by

$$\mathcal{L}_\xi g + 2S + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g + 2\mu\eta \otimes \eta = 0 \tag{8}$$

for some constants λ and μ, where ξ is a soliton vector field and η is an 1-form on M. where $\mathcal{L}_\xi$ is the Lie derivative along the vector field ξ , S is the Ricci tensor, λ, μ are constants, p is a scalar non-dynamical field (time dependent scalar field) and n is the dimension of manifold.

Tachibana [22] and Hamada [27] introduced the notion of $*$-Ricci tensor on almost Hermitian manifolds and on real hypersurfaces in non-flat complex space respectively and then in 2014, Kaimakamis and Panagiotidou [6] introduced the notion of $*$-Ricci soliton on non-flat complex space forms and the equation as

$$\mathcal{L}_V g + 2S^* + 2\lambda g = 0, \tag{9}$$

where $S^*(X,Y) = \frac{1}{2}\left[trace\{\varphi \circ R(X, \varphi Y)\}\right]$ for all vector fields X, Y on M and φ is a (1,1)-tensor field.

In 2022, the authors [24] have defined the $*$-conformal η-Ricci soliton on a Riemannian manifold as

$$\mathcal{L}_\xi g + 2S^* + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g + 2\mu\eta \otimes \eta = 0, \tag{10}$$

where $\mathcal{L}_\xi$ is the Lie derivative along the vector field ξ , S^* is the $*$-Ricci tensor, λ, μ are constants, p is a scalar non-dynamical field (time dependent scalar field) and n is the dimension of manifold.

In 2016, Nurowski and Randall [17] introduced the concept of generalized Ricci soliton as a class of over determined system of equations

$$\mathcal{L}_V g = -2aV^{\#} \odot V^{\#} + 2bS + 2\lambda g, \tag{11}$$

where $\mathcal{L}_V g$ and $V^{\#}$ denote, respectively, the Lie derivative of the metric g in the directions of vector field V and the canonical one-form associated to V, and some real constants a, b, and λ. Levy [10] acquired the necessary and sufficient conditions for the existence of such tensors. In 2018 M.D. Siddiqi [12] have studied generalized Ricci solitons on trans-Sasakian manifolds.

In this paper, we consider generalized Tanaka connection on Lorentzian Sasakian Space form and studied various symmetric properties of Lorentzian Sasakian space form with generalized Tanaka connection and solitons. After preliminaries in section-3, we consider consider generalized Tanaka connection on Lorentzian Sasakian space form, state and proved some results, finding curvature tensor and Ricci curvature tensor. In section-4, we study the semi-symmetric, Ricci semi-symmetric properties of Lorentzian Sasakian Space form. From section-5 to 7 $*$-Ricci soliton, $*$-conformal Ricci soliton, $*$-conformal η-Ricci soliton, generalized Ricci soliton, generalized conformal Ricci soliton have been studied on Lorentzian Sasakian space form with generalized Tanaka connection and obtained the values of the scalar λ of these solitons on which nature of solitons depend, whether it is shrinking, steady or expanding.

§2. **Preliminaries**

Let M be a $(2n+1)$ dimensional (denoted by M^{2n+1}) having almost contact structure (φ, ξ, η, g) i.e.,

$$\eta(\xi) = 1,\ \varphi^2 = -I + \eta \otimes \xi,\ \varphi(\xi) = 0,\ \eta \circ \varphi = 0, \tag{12}$$

where φ is a (1,1)-tensor field, ξ a contravariant vector field, η a covariant vector field.

A Lorentzian metric g is said to be compatible with the structure (φ, ξ, η, g) if

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y). \tag{13}$$

If the manifold M^{2n+1} equipped with an almost contact structure (φ, ξ, η, g) and a compatible Lorentzian metric g, is called an almost contact Lorentzian manifold.

Note that equations (12) and (13) imply

$$g(X, \xi) = -\eta(X) \quad \text{and} \quad g(\xi, \xi) = -1. \tag{14}$$

Also, equations (13) implies

$$g(X, \varphi Y) = -g(\varphi X, Y). \tag{15}$$

In almost contact Lorentzian manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$, the fundamental 2-form Φ is defined as

$$\Phi(X, Y) = g(X, \varphi Y) \quad \text{for all} \quad X, Y \in \mathfrak{X}(M).$$

An almost contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is Sasakian [4] if and only if it is normal and

$$d\eta = 0. \tag{16}$$

In Lorentzian Sasakian manifold, the following properties [23] hold good:

$$(\nabla_X\varphi)Y = \eta(Y)X + g(X,Y)\xi, \tag{17}$$

$$\nabla_X\xi = \varphi X, \tag{18}$$

$$(\nabla_X\eta)Y = g(X,\varphi Y). \tag{19}$$

Let (M,g) be an n-dimensional Riemannian manifold $n > 2$, its curvature tensor defined by

$$R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}.$$

and let T be $(0,k)$-tensor, define a $(0,2+k)$-tensor field $R\cdot T$ by

$$(R\cdot T)(X_1,X_2,\cdots,X_k,X,Y) = R(X,Y)(T(X_1,X_2,\cdots,X_k)) - T(R(X,Y)X_1,X_2,\cdots,X_k)$$
$$-T(X_1,R(X,Y)X_2,\cdots,X_k) - \cdots - T(X_1,X_2,\cdots,R(X,Y)X_k).$$

One has

$$R(X,Y)\cdot T = \nabla_X(\nabla_Y T) - \nabla_Y(\nabla_X T) - \nabla_{[X,Y]}T.$$

When $T = R$, then we have a $(0,6)$-tensor $R\cdot R$.

Also, we can determine a $(0,k+2)$-tensor field $Q(A,T)$, associated with any $(0,k)$-tensor field T and any symmetric $(0,2)$-tensor field A by

$$\begin{aligned} Q(A,T)(X_1,X_2,\cdots,X_k,X,Y) &= ((X\wedge_A Y)\cdot T)(X_1,X_2,\cdots,X_k) \\ &= -T((X\wedge_A Y)X_1,X_2,\cdots,X_k) - T(X_1,(X\wedge_A Y)X_2,\cdots,X_k) \\ &\quad -\cdots - T(X_1,X_2,\cdots,(X\wedge_A Y)X_k), \end{aligned}$$

where $(X\wedge_A Y)$ is the endomorphism given by

$$(X\wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y. \tag{20}$$

Particularly, if we put $A = g$ we get

$$(X\wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{21}$$

and we will write $(X\wedge_g Y)$ as $(X\wedge Y)$ in General.

§3. Generalized Tanaka Connection on Lorentzian Sasakian Space Form

For an $(2n+1)$-dimensional Lorentzian Sasakian manifold M with almost contact structure (φ,ξ,η,g), the relation between generalized Tanaka connection $\mathring{\nabla}$ and Levi-Civita connection

∇ is given by

$$\mathring{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi. \tag{22}$$

By (18) and (19),

$$\mathring{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + g(X, \varphi Y)\xi - \eta(Y)\varphi X. \tag{23}$$

Putting $Y = \xi$ in (22),

$$\mathring{\nabla}_X \xi = \nabla_X \xi + \eta(X)\varphi\xi + g(X, \varphi\xi)\xi - \eta(\xi)\nabla_X \xi$$

By (12),

$$\mathring{\nabla}_X \xi = 0. \tag{24}$$

$$(\mathring{\nabla}_X \eta)Y = \mathring{\nabla}_X \eta(Y) - \eta(\mathring{\nabla}_X Y)$$

From (23),

$$(\mathring{\nabla}_X \eta)Y = 0, \tag{25}$$

$$(\mathring{\nabla}_X g)(Y, Z) = 0. \tag{26}$$

Thus, we can state

Theorem 3.1 *In a Lorentzian Sasakian manifold ξ, η, g are parallel with respect to the generalized Tanaka connection.*

Now,

$$(\mathring{\nabla}_X \varphi)Y = \mathring{\nabla}_X \varphi Y - \varphi(\mathring{\nabla}_X Y).$$

Using (22),

$$(\mathring{\nabla}_X \varphi)Y = 0. \tag{27}$$

The curvature tensor of Lorentzian Sasakian manifold with respect to the generalized Tanaka connection is given by

$$\begin{aligned}
\mathring{R}(X,Y)Z &= \mathring{\nabla}_X \mathring{\nabla}_Y Z - \mathring{\nabla}_Y \mathring{\nabla}_X Z - \mathring{\nabla}_{[X,Y]} Z \\
&= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z + [\eta(X)\varphi(\nabla_Y Z) - \eta(Y)\varphi(\nabla_X Z)] \\
&\quad + [g(X, \varphi(\nabla_Y Z))\xi - g(Y, \varphi(\nabla_X Z))\xi] - [\eta((\nabla_Y Z))\varphi X - \eta((\nabla_X Z))\varphi Y] \\
&\quad + [\eta(\nabla_X Y)\varphi Z - \eta(\nabla_Y X)\varphi Z] + [g(X, \varphi Y)\varphi Z - g(Y, \varphi X)\varphi Z] \\
&\quad + [\eta(Y)\varphi(\nabla_X Z) - \eta(X)\varphi(\nabla_Y Z)] - [\eta(X)\eta(Y)Z - \eta(Y)\eta(X)Z] \\
&\quad + [g(\nabla_X Y, \varphi Z) - g(\nabla_Y X, \varphi Z)]\,\xi - [\eta(Y)g(\varphi X, \varphi Z)\xi - \eta(X)g(\varphi Y, \varphi Z)\xi] \\
&\quad + [g(Y, \varphi(\nabla_X Z))\xi - g(X, \varphi(\nabla_Y Z))\xi] + [\eta(Z)g(\varphi Y, \varphi X)\xi - \eta(Z)g(\varphi X, \varphi Y)\xi] \\
&\quad - [\eta(\nabla_X Z)\varphi Y - \eta(\nabla_Y Z)\varphi X] - [g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X] \\
&\quad - [\eta(Z)\varphi(\nabla_X Y) - \eta(Z)\varphi(\nabla_Y X)] + [\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X] \\
&\quad - \eta([X,Y])\varphi Z - g([X,Y], \varphi Z)\xi + \eta(Z)\varphi[X,Y].
\end{aligned}$$

So,

$$\mathring{R}(X,Y)Z = R(X,Y)Z + 2g(X,\varphi Y)\varphi Z - [\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]\,\xi$$
$$- [g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X] + \eta(Z)\,[\eta(X)Y - \eta(Y)X] \tag{28}$$

and

$$\mathring{S}(X,Y) = S(X,Y) + 2g(X,Y) - 2(n-1)\eta(X)\eta(Y)$$

If M has constant φ-holomorphic sectional curvature c, then by (1) and (28) we get

$$\mathring{R}(X,Y)Z = \frac{c-3}{4}\{[g(Y,Z)X - g(X,Z)Y] + \eta(Z)[\eta(Y)X - \eta(X)Y]$$
$$+[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]\xi + [g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X]\}$$
$$+\frac{c+5}{2}g(X,\varphi Y)\varphi Z. \tag{29}$$

and

$$\mathring{S}(X,Y) = \frac{n(c-3)+4}{2}g(X,Y)\frac{n(c+1)}{2}\eta(X)\eta(Y) + 2g(X,Y) - 2(n-1)\eta(X)\eta(Y)$$

Or,

$$\mathring{S}(X,Y) = \frac{n(c-3)+8}{2}g(\varphi X,\varphi Y). \tag{30}$$

§4. Semi-symmetry and Ricci-semisymmetry on Lorentzian Sasakian Space Form with Respect to Generalized Tanaka Connection

Applying (21) in (29), we get

$$\mathring{R}(X,Y)Z = \frac{c-3}{4}\{(\varphi X \wedge \varphi Y)Z + (\varphi^2 X \wedge \varphi^2 Y)Z\} + \frac{c+5}{2}g(X,\varphi Y)\varphi Z. \tag{31}$$

Lemma 4.1 *Let $M^{2n+1}(c)$ be a Lorentzian Sasakian space form with generalized Tanaka connection and $X, Y \in \mathfrak{X}(M)$, then the following properties hold:*

(*a*) $\varphi \cdot \mathring{S} = 0$;
(*b*) $(X \wedge Y) \cdot \mathring{S} = 0$ *if and only if* $c = \frac{3n-8}{n}$;
(*c*) $(\varphi X \wedge \varphi Y) \cdot \mathring{S} = 0$;
(*d*) $(\varphi^2 X \wedge \varphi^2 Y) \cdot \mathring{S} = 0$.

Proof (*a*) Since φ is a tensor field, we have

$$(\varphi \cdot \mathring{S})(U,V) = -\mathring{S}(\varphi U, V) - \mathring{S}(U, \varphi V) = \frac{n(c-3)}{2}\,[g(\varphi U, V) + g(U, \varphi V)]$$
$$+\frac{n(c+5)-4}{2}\,[\eta(\varphi U)\eta(V) + \eta(U)\eta(\varphi V)]$$
$$= \frac{n(c-3)}{2}\,[g(\varphi U, V) - g(\varphi U, V)] = 0.$$

Thus $(\varphi\cdot\mathring{S})(U,V)=0$ for any $U,V\in\mathfrak{X}(M)$.

(b) For any $U,V\in\mathfrak{X}(M)$, we have

$$\begin{aligned}((X\wedge Y).\mathring{S})(U,V) &= -\mathring{S}((X\wedge Y)U,V)-\mathring{S}(U,(X\wedge Y)V)\\ &= -g(Y,U)\mathring{S}(X,V)+g(X,U)\mathring{S}(Y,V)\\ &\quad -g(Y,V)\mathring{S}(U,X)+g(X,V)S(U,Y)\\ &= -\frac{n(c-3)+8}{2}[g(Y,U)\eta(X)\eta(V)-g(X,U)\eta(Y)\eta(V)\\ &\quad +g(Y,V)\eta(X)\eta(U)-g(X,V)\eta(Y)\eta(U)].\end{aligned}$$

Since, $-g(Y,U)\eta(X)\eta(V)+g(X,U)\eta(Y)\eta(V)-g(Y,V)\eta(U)\eta(X)$ $+\,g(X,V)\eta(U)\eta(Y)\neq 0$ always. Therefore

$$((X\wedge Y).\mathring{S})(U,V)=0 \text{ if and only if } n(c-3)+8=0, \quad i.e.,\ c=\frac{3n-8}{n}.$$

(c) For any $U,V\in\mathfrak{X}(M)$, we have

$$\begin{aligned}((\varphi X\wedge\varphi Y).\mathring{S})(U,V) &= -\mathring{S}((\varphi X\wedge\varphi Y)U,V)-\mathring{S}(U,(\varphi X\wedge\varphi Y)V)\\ &= -g(\varphi Y,U)\mathring{S}(\varphi X,V)+g(\varphi X,U)\mathring{S}(\varphi Y,V)\\ &\quad -g(\varphi Y,V)\mathring{S}(U,\varphi X)+g(\varphi X,V)\mathring{S}(U,\varphi Y).\end{aligned}$$

Using (12) and (21), we get the result.

(d) The proof is similar to (c). □

Theorem 4.2 *A Lorentzian Sasakian space form $M^{2n+1}(c)$ is Ricci-semi-symmetric with respect to generalized Tanaka connection.*

Proof In the equation (31), we see that the curvature tensor is of the form

$$\mathring{R}(X,Y)=\frac{c-3}{4}(\varphi X\wedge\varphi Y)+\frac{c-3}{4}(\varphi^2X\wedge\varphi^2Y)+\frac{c+5}{2}g(X,\varphi Y)\varphi.$$

So,

$$\mathring{R}(X,Y)\cdot\mathring{S}=\frac{c-3}{4}(\varphi X\wedge\varphi Y)\cdot\mathring{S}+\frac{c-3}{4}(\varphi^2X\wedge\varphi^2Y)\cdot\mathring{S}+\frac{c+5}{2}g(X,\varphi Y)\varphi\cdot\mathring{S}.$$

By the Lemma 4.1, we have $\mathring{R}\cdot\mathring{S}=0$. □

Lemma 4.3 *In a Lorentzian Sasakian space form $M^{2n+1}(c)$ with generalized Tanaka connection the following properties hold for all $X,Y\in\mathfrak{X}(M)$:*

(a) $\varphi\cdot\mathring{R}=0$;

(b) $(\varphi^2X\wedge\varphi^2Y)\cdot\mathring{R}=-(\varphi X\wedge\varphi Y)\cdot\mathring{R}$;

(c) $(X\wedge_S Y).\mathring{R}=0$.

Proof (*a*) For any $X, Y, U, V \in \mathfrak{X}(M)$

$$
\begin{aligned}
(\varphi \cdot \mathring{R})(X, Y, U, V) &= -\mathring{R}(\varphi X, Y, U, V) - \mathring{R}(X, \varphi Y, U, V) \\
&\quad -\mathring{R}(X, Y, \varphi U, V) - \mathring{R}(X, Y, U, \varphi V) \\
&= -2[g((\varphi X \wedge Y)U, V) + g((X \wedge \varphi Y)U, V) + g((X \wedge Y)\varphi U, V) + g((X \wedge Y)U, \varphi V)] \\
&\quad + \frac{c+5}{4}[g((\varphi^2 X \wedge \varphi Y)U, V) + g((\varphi X \wedge \varphi^2 Y)U, V) + g((\varphi X \wedge \varphi Y)\varphi U, V) \\
&\quad + g((\varphi X \wedge \varphi Y)U, \varphi V)] + \frac{c+5}{4}[g((\varphi^3 X \wedge \varphi^2 Y)U, V) + g((\varphi^2 X \wedge \varphi^3 Y)U, V) \\
&\quad + g((\varphi^2 X \wedge \varphi^2 Y)\varphi U, V) + g((\varphi^2 X \wedge \varphi^2 Y)U, \varphi V)] - \frac{c-3}{2}[g(\varphi^2 X, Y)g(\varphi U, V) \\
&\quad + g(\varphi X, \varphi Y)g(\varphi U, V) + g(\varphi X, Y)g(\varphi^2 U, V) + g(\varphi X, Y)g(\varphi U, \varphi V)].
\end{aligned}
$$

Using (12) and (21) we get result.

(*b*) For any $X, Y, Z, U, V, W \in \mathfrak{X}(M)$,

$$
\begin{aligned}
((\varphi X \wedge \varphi Y) \cdot \mathring{R})(Z, U, V, W) &= -g(\varphi Y, Z)\mathring{R}(\varphi X, U, V, W) + g(\varphi X, Z)\mathring{R}(\varphi Y, U, V, W) \\
&\quad -g(\varphi Y, U)\mathring{R}(Z, \varphi X, V, W) + g(\varphi X, U)\mathring{R}(Z, \varphi Y, V, W) \\
&\quad -g(\varphi Y, V)\mathring{R}(Z, U, \varphi X, W) + g(\varphi X, V)\mathring{R}(Z, U, \varphi Y, W) \\
&\quad -g(\varphi Y, W)\mathring{R}(Z, U, V, \varphi X) + g(\varphi X, W)\mathring{R}(Z, U, V, \varphi Y) \\
&= \frac{c-3}{4}\{-g(\varphi Y, Z)g(\varphi U, W)g(\varphi X, \varphi V) - g(\varphi Y, Z)g(U, \varphi V)g(\varphi X, \varphi W) \\
&\quad +g(\varphi U, W)g(\varphi X, Z)g(\varphi Y, \varphi V) + g(U, \varphi V)g(\varphi X, Z)g(\varphi Y, \varphi W) \\
&\quad +g(\varphi Y, U)g(Z, \varphi V)g(\varphi X, \varphi W) + g(\varphi Y, U)g(\varphi Z, W)g(\varphi X, \varphi V) \\
&\quad -g(\varphi X, U)g(Z, \varphi V)g(\varphi Y, \varphi W) - g(\varphi X, U)g(\varphi Z, W)g(\varphi Y, \varphi V) \\
&\quad +g(\varphi Y, V)g(\varphi U, W)g(\varphi Z, \varphi X) - g(\varphi Y, V)g(\varphi Z, W)g(\varphi U, \varphi X) \\
&\quad -g(\varphi X, V)g(\varphi U, W)g(\varphi Z, \varphi Y) + g(\varphi X, V)g(\varphi Z, W)g(\varphi U, \varphi Y) \\
&\quad -g(\varphi Y, W)g(Z, \varphi V)g(\varphi U, \varphi X) + g(\varphi Y, W)g(U, \varphi V)g(\varphi Z, \varphi X) \\
&\quad +g(\varphi X, W)g(Z, \varphi V)g(\varphi U, \varphi Y) - g(\varphi X, W)g(U, \varphi V)g(\varphi Z, \varphi Y) \\
&\quad +\frac{c+5}{2}\{-g(\varphi Y, Z)g(\varphi X, \varphi U)g(\varphi V, W) + g(\varphi X, Z)g(\varphi Y, \varphi U)g(\varphi V, W) \\
&\quad +g(\varphi Y, U)g(\varphi Z, \varphi X)g(\varphi V, W) - g(\varphi X, U)g(\varphi Z, \varphi Y)g(\varphi V, W) \\
&\quad +g(\varphi Y, V)g(Z, \varphi U)g(\varphi X, \varphi W) - g(\varphi X, V)g(Z, \varphi U)g(\varphi Y, \varphi W) \\
&\quad -g(\varphi Y, W)g(Z, \varphi U)g(\varphi V, \varphi X) + g(\varphi X, W)g(Z, \varphi U)g(\varphi V, \varphi Y)\}. \qquad (32)
\end{aligned}
$$

Now,

$$\begin{aligned}
&((\varphi^2X\wedge\varphi^2Y)\cdot\mathring{R})(Z,U,V,W)\\
&=-g(\varphi^2Y,Z)\mathring{R}(\varphi^2X,U,V,W)+g(\varphi^2X,Z)\mathring{R}(\varphi^2Y,U,V,W)\\
&\quad-g(\varphi^2Y,U)\mathring{R}(Z,\varphi^2X,V,W)+g(\varphi^2X,U)\mathring{R}(Z,\varphi^2Y,V,W)\\
&\quad-g(\varphi^2Y,V)\mathring{R}(Z,U,\varphi^2X,W)+g(\varphi^2X,V)\mathring{R}(Z,U,\varphi^2Y,W)\\
&\quad-g(\varphi^2Y,W)\mathring{R}(Z,U,V,\varphi^2X)+g(\varphi^2X,W)\mathring{R}(Z,U,V,\varphi^2Y)\\
&=\frac{c-3}{4}\{g(\varphi Y,\varphi Z)g(\varphi X,V)g(\varphi U,W)+g(\varphi Y,\varphi Z)g(U,\varphi V)g(\varphi X,W)\\
&\quad-g(\varphi Y,V)g(\varphi U,W)g(\varphi X,\varphi Z)-g(U,\varphi V)g(\varphi Y,W)g(\varphi X,\varphi Z)\\
&\quad-g(\varphi Y,\varphi U)g(Z,\varphi V)g(\varphi X,W)-g(\varphi Y,\varphi U)g(\varphi X,V)g(\varphi Z,W)\\
&\quad+g(\varphi X,\varphi U)g(Z,\varphi V)g(\varphi Y,W)+g(\varphi X,\varphi U)g(\varphi Y,V)g(\varphi Z,W)\\
&\quad-g(\varphi Y,\varphi V)g(Z,\varphi X)g(\varphi U,W)+g(\varphi Y,\varphi V)g(U,\varphi X)g(\varphi Z,W)\\
&\quad+g(\varphi X,\varphi V)g(Z,\varphi Y)g(\varphi U,W)-g(\varphi X,\varphi V)g(U,\varphi Y)g(\varphi Z,W)\\
&\quad+g(\varphi Y,\varphi W)g(Z,\varphi V)g(U,\varphi X)-g(\varphi Y,\varphi W)g(U,\varphi V)g(Z,\varphi X)\\
&\quad-g(\varphi X,\varphi W)g(Z,\varphi V)g(U,\varphi Y)+g(\varphi X,\varphi W)g(U,\varphi V)g(Z,\varphi Y)\}\\
&\quad+\frac{c+5}{2}\{g(\varphi Y,\varphi Z)g(\varphi X,U)g(\varphi V,W)-g(\varphi X,\varphi Z)g(\varphi Y,U)g(\varphi V,W)\\
&\quad-g(\varphi Y,\varphi U)g(Z,\varphi X)g(\varphi V,W)+g(\varphi X,\varphi U)g(Z,\varphi Y)g(\varphi V,W)\\
&\quad-g(\varphi Y,\varphi V)g(Z,\varphi U)g(\varphi X,W)+g(\varphi X,\varphi V)g(Z,\varphi U)g(\varphi Y,W)\\
&\quad+g(\varphi Y,\varphi W)g(Z,\varphi U)g(V,\varphi X)-g(\varphi X,\varphi W)g(Z,\varphi U)g(V,\varphi Y)\}.
\end{aligned}\tag{33}$$

From (32) and (33), we see that

$$((\varphi^2X\wedge\varphi^2Y)\cdot\mathring{R})(Z,U,V,W)=-((\varphi X\wedge\varphi Y)\cdot\mathring{R})(Z,U,V,W).$$

(c) The Ricci curvature tensor can be written as

$$\mathring{S}(X,Y)=\frac{n(c-3)+8}{2}g(\varphi X,\varphi Y).$$

So, we have

$$\begin{aligned}
(X\wedge_{\mathring{S}}Y)Z&=\mathring{S}(Y,Z)X-\mathring{S}(X,Z)Y\\
&=\frac{n(c-3)+8}{2}\{g(\varphi Y,\varphi Z)X-g(\varphi X,\varphi Z)Y\}.
\end{aligned}$$

Replacing Z by $\mathring{R}$, we obtain

$$(X\wedge_{\mathring{S}}Y)\cdot\mathring{R}=\frac{n(c-3)+8}{2}\{g(\varphi Y,\varphi\cdot\mathring{R})X-g(\varphi X,\varphi\cdot\mathring{R})Y\}.$$

Using (a), $\varphi\cdot\mathring{R}=0$, we get the result

$$(X\wedge_{\mathring{S}}Y)\cdot\mathring{R}=0.$$

□

Theorem 4.4 *A Lorentzian Sasakian space form $M^{2n+1}(c)$ is semi-symmetric with respect to generalized Tanaka connection.*

Proof From (31), the curvature tensor is of the form

$$\mathring{R}(X,Y)=\frac{c-3}{4}(\varphi X\wedge\varphi Y)+\frac{c-3}{4}(\varphi^2X\wedge\varphi^2Y)+\frac{c+5}{2}g(X,\varphi Y)\varphi.$$

So,

$$\mathring{R}(X,Y)\cdot\mathring{R}=\frac{c-3}{4}(\varphi X\wedge\varphi Y)\cdot\mathring{R}+\frac{c-3}{4}(\varphi^2X\wedge\varphi^2Y)\cdot\mathring{R}+\frac{c+5}{2}g(X,\varphi Y)\varphi\cdot\mathring{R}.$$

Using (a) and (b) of Lemma 4.3, we have

$$\mathring{R}(X,Y)\cdot\mathring{R}=\frac{c-3}{4}(\varphi X\wedge\varphi Y)\cdot\mathring{R}-\frac{c-3}{4}(\varphi X\wedge\varphi Y)\cdot\mathring{R}=0.$$ □

§5. *-Ricci Soliton on Lorentzian Sasakian Space Form with Respect to Generalized Tanaka Connection

In this section we first derived the $*$-Ricci tensor in Lorentzion Sasakian space form. The $*$-Ricci tensor first introduced by Kaimakamis and Panagiotidou [6] and given by

$$\mathring{S}^*(X,Y)=\frac{1}{2}\left[trace\{\varphi\circ R(X,\varphi Y)\}\right] \tag{34}$$

for all vector fields X,Y on M and φ is a (1,1)-tensor field.

Theorem 5.1 *In a Lorentzion Sasakian space form with generalized Tanaka connection, the $*$-Ricci tensor*

$$\mathring{S}^*(X,Y)=-\frac{n(c-3)+8}{4}g(\varphi X,\varphi Y). \tag{35}$$

Proof Replacing Z by φZ in (28), we get

$$\begin{aligned}\mathring{R}(X,Y)\varphi Z \;=\;& \frac{c-3}{4}\{[g(Y,\varphi Z)X-g(X,\varphi Z)Y]+\eta(\varphi Z)[\eta(Y)X-\eta(X)Y]\\ &+[\eta(Y)g(X,\varphi Z)-\eta(X)g(Y,\varphi Z)]\xi+[g(X,\varphi^2Z)\varphi Y-g(Y,\varphi^2Z)\varphi X]\}\\ &+\frac{c+5}{2}g(X,\varphi Y)\varphi^2Z.\end{aligned}$$

Or,

$$\begin{aligned}\mathring{R}(X,Y)\varphi Z \;=\;& \frac{c-3}{4}\{[g(Y,\varphi Z)X-g(X,\varphi Z)Y]+[\eta(Y)g(X,\varphi Z)-\eta(X)g(Y,\varphi Z)]\xi\\ &+[g(X,\varphi^2Z)\varphi Y-g(Y,\varphi^2Z)\varphi X]\}+\frac{c+5}{2}g(X,\varphi Y)\varphi^2Z.\end{aligned}$$

Taking inner product of the preceding equation with φW, we get

$$\begin{aligned} g(\mathring{R}(X,Y)\varphi Z,\varphi W) &= \frac{c-3}{4}\{[g(Y,\varphi Z)g(X,\varphi W)-g(X,\varphi Z)g(Y,\varphi W)] \\ &+[\eta(Y)g(X,\varphi Z)-\eta(X)g(Y,\varphi Z)]g(\xi,\varphi W) \\ &+[g(X,\varphi^2 Z)g(\varphi Y,\varphi W)-g(Y,\varphi^Z)g(\varphi X,\varphi W)]\} \\ &+\frac{c+5}{2}g(X,\varphi Y)g(\varphi^2 Z,\varphi W). \end{aligned}$$

Using (12), we get

$$\begin{aligned} -g(\varphi\mathring{R}(X,Y)\varphi Z,W) = \frac{c-3}{4}&\{[g(Y,\varphi Z)g(X,\varphi W)+g(X,\varphi Z)g(\varphi Y,W)] \\ &-[g(X,\varphi^2 Z)g(\varphi^2 Y,W)-g(\varphi Y,\varphi Z)g(\varphi X,\varphi W)]\} \\ &+\frac{c+5}{2}g(X,\varphi Y)g(\varphi Z,W). \end{aligned}$$

Contracting X and W and using definition, we get

$$-2\mathring{S}^*(Y,Z) = \frac{c-3}{4}\{g(\varphi Y,\varphi Z)-[g(\varphi Y,\varphi Z)-g(\varphi Y,\varphi Z)2(n-1)]\}+\frac{c+5}{2}g(\varphi Y,\varphi Z).$$

Or,

$$-2\mathring{S}^*(Y,Z) = \frac{(c-3)(n-1)}{2}g(\varphi Y,\varphi Z)+\frac{c+5}{2}g(\varphi Y,\varphi Z).$$

Or,

$$\mathring{S}^*(Y,Z) = -\frac{n(c-3)+8}{4}g(\varphi Y,\varphi Z).$$

Replacing Y,Z by X,Y respectively we get the result. □

Corollary 5.2 *In a Lorentzion Sasakian space form,*

$$\mathring{S}^*(X,\xi)=0. \tag{36}$$

Theorem 5.3 *If $M(c)$ is a Lorentzian Sasakian space form with generalized Tanaka connection and (M,V,g) a $*$-Ricci soliton, where V is a pointwise collinear vector field with ξ. Then V is a constant multiple of ξ and the soliton is steady.*

Proof Let V be pointwise collinear vector field with ξ i.e. $V=f\xi$, where f is a function on the Lorentzian Sasakian manifold M. Then $(\mathcal{L}_V g+2\mathring{S}^*+2\lambda g)(X,Y)=0$, implies

$$g(\mathring{\nabla}_X f\xi,Y)+g(\mathring{\nabla}_Y f\xi,X)+2S^*(X,Y)+2\lambda g(X,Y)=0.$$

By (24),

$$-(Xf)\eta(Y)-(Yf)\eta(X)+2\mathring{S}^*(X,Y)+2\lambda g(X,Y)=0. \tag{37}$$

Replacing Y by ξ in (37) it follows that

$$-(Xf) - (\xi f)\eta(X) + 2\mathring{S}^*(X,\xi) + 2\lambda\eta(X) = 0.$$

Using (36),

$$Xf + (\xi f)\eta(X) - 2\lambda\eta(X) = 0. \tag{38}$$

Put $X = \xi$,

$$\xi f = \lambda.$$

From (38),

$$Xf = \lambda\eta(X).$$

Or,

$$df = \lambda\eta. \tag{39}$$

Applying (d) in (39),

$$\lambda d\eta = 0.$$

Since $d\eta \neq 0$ for Lorentzian Sasakian manifold, we have $\lambda = 0$. So by (39), V is constant multiple of ξ and as $\lambda = 0$, the soliton is steady. □

§6. *-Conformal Ricci Soliton on Lorentzian Sasakian Space Form with Respect to Generalized Tanaka Connection

Theorem 6.1 *If $M(c)$ is a Lorentzian Sasakian space form with generalized Tanaka connection and (M, V, g) a $*$-conformal Ricci soliton, where V is a pointwise collinear vector field with ξ. Then V is a constant multiple of ξ and the soliton is expanding or steady or shrinking according as $p < -\frac{2}{2n+1}$, $p = -\frac{2}{2n+1}$ or $p > -\frac{2}{2n+1}$.*

Proof Let V be pointwise collinear vector field with ξ i.e. $V = h\xi$, where h is a function on the Lorentzian Sasakian manifold M. Then $\left(\mathcal{L}_V g + 2\mathring{S}^* + \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right]\right)(X,Y) = 0$, implies

$$g(\mathring{\nabla}_X h\xi, Y) + g(\mathring{\nabla}_Y h\xi, X) + 2S^*(X,Y) + \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y) = 0.$$

By (24),

$$-(Xh)\eta(Y) - (Yh)\eta(X) + 2\mathring{S}^*(X,Y) + \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y) = 0. \tag{40}$$

Replacing Y by ξ in (40) it follows that

$$-(Xh) - (\xi h)\eta(X) + 2\mathring{S}^*(X,\xi) + \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] \eta(X) = 0.$$

Using (36),

$$Xb + (\xi h)\eta(X) - \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right]\eta(X) = 0. \quad (41)$$

Puting $X = \xi$,

$$\xi h = \lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right).$$

From (41),

$$Xh = \left(\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right)\eta(X).$$

Or,

$$dh = \left(\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right)\eta. \quad (42)$$

Applying (d) in (42),

$$\left(\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right)d\eta = 0.$$

Since $d\eta \neq 0$, we have $\lambda = \frac{1}{2}\left(p + \frac{2}{2n+1}\right)$. So by (42), V is constant multiple of ξ. Also we see that the soliton is expanding or steady or shrinking according as $p < -\frac{2}{2n+1}$, $p = -\frac{2}{2n+1}$ or $p > -\frac{2}{2n+1}$. □

Theorem 6.2 *If $M(c)$ is a Lorentzian Sasakian space form with generalized Tanaka connection and (M, V, g) a $*$-conformal η−Ricci soliton, where V is a pointwise collinear vector field with ξ. Then V is a constant multiple of ξ and the soliton is expanding or steady or shrinking according as $p < 2\mu - \frac{2}{2n+1}$ or $p = 2\mu - \frac{2}{2n+1}$ or $p > 2\mu - \frac{2}{2n+1}$.*

Proof Let V be pointwise co-linear vector field with ξ i.e. $V = \rho\xi$, where ρ is a function on the Lorentzian Sasakian manifold M. Then

$$\left(\mathcal{L}_V g + 2\mathring{S}^* + \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] + +2\mu\eta\otimes\eta\right)(X, Y) = 0,$$

which implies

$$g(\mathring{\nabla}_X \rho\xi, Y) + g(\mathring{\nabla}_Y \rho\xi, X) + 2S^*(X, Y) + \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right]g(X, Y) + 2\mu\eta(X)\eta(Y) = 0.$$

By (24),

$$-(X\rho)\eta(Y) - (Y\rho)\eta(X) + 2\mathring{S}^*(X, Y) + \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right]g(X, Y) + 2\mu\eta(X)\eta(Y) = 0. \quad (43)$$

Replacing Y by ξ in (43) it follows that

$$-(X\rho) - (\xi\rho)\eta(X) + 2\mathring{S}^*(X, \xi) + \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right]\eta(X) + 2\mu\eta(X) = 0.$$

Using (36),

$$X\rho + (\xi\rho)\eta(X) - \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right]\eta(X) - 2\mu\eta(X) = 0. \tag{44}$$

Put $X = \xi$,

$$\xi\rho = \lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right) + \mu.$$

From (44),

$$X\rho = \left(\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right) + \mu\right)\eta(X).$$

Or,

$$d\rho = \left(\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right) + \mu\right)\eta. \tag{45}$$

Applying (d) in (45),

$$\left(\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right) + \mu\right)d\eta = 0.$$

Since $d\eta \neq 0$, we have $\lambda = \frac{1}{2}\left(p + \frac{2}{2n+1}\right) - \mu$. So by (45), V is constant multiple of ξ. Also we see that the soliton is expanding or steady or shrinking according as $p < 2\mu - \frac{2}{2n+1}$ or $p = 2\mu - \frac{2}{2n+1}$ or $p > 2\mu - \frac{2}{2n+1}$. □

§7. Generalized Ricci Soliton on Lorentzian Sasakian Space Form with Respect to Generalized Tanaka Connection

We defined $V^{\#}$ in the equation (11) by

$$V^{\#}(X) = g(V, X).$$

Replaced S by $\mathring{S}$, then (11) becomes

$$\mathcal{L}_V g = -2aV^{\#} \odot V^{\#} + 2b\mathring{S} + 2\lambda g. \tag{46}$$

Theorem 7.1 *If a Lorentzian Sasakian space form $M(c)$ with generalized Tanaka connection is a generalized Ricci soliton. Then*

$$\lambda = \frac{b[n(c-3)+8](n-1) - a}{2n-1}.$$

Proof The equation $\mathcal{L}_V g = -2aV^{\#} \odot V^{\#} + 2b\mathring{S} + 2\lambda g$, implies

$$g(\mathring{\nabla}_X\xi, Y) + g(X, \mathring{\nabla}_Y\xi) = -2a\eta(X)\eta(Y) + 2b\mathring{S}(X, Y) + 2\lambda g(X, Y).$$

Using (24), we get

$$a\eta(X)\eta(Y) - b\mathring{S}(X,Y) - \lambda g(X,Y) = 0. \tag{47}$$

Using (30), we have

$$a\eta(X)\eta(Y) - b\frac{n(c-3)+8}{2}g(\varphi X, \varphi Y) - \lambda g(X,Y) = 0.$$

Contracting X and Y, we get

$$-a + b[n(c-3)+8](n-1) - \lambda(2n-1) = 0.$$

Therefore, this implies

$$\lambda = \frac{b[n(c-3)+8](n-1) - a}{2n-1}. \tag{48}$$

This completes the proof. □

We introduce the generalized conformal Ricci soliton equation on a manifold of dimension n as

$$\mathcal{L}_V g = \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g - 2aV^{\#} \odot V^{\#} + 2bS. \tag{49}$$

where $V \in \Gamma(TM)$ and $\mathcal{L}_V g$ is the Lie-derivative of g along V and $V^{\#}$ the canonical one-form associated to V and a, b, λ some constants. Taking $V^{\#}(X) = g(V, X)$, and replace S by $\mathring{S}$. Then, (49) becomes

$$\mathcal{L}_V g(X,Y) = \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X,Y) - 2aV^{\#}(X) \odot V^{\#}(Y) + 2b\mathring{S}(X,Y). \tag{50}$$

Theorem 7.2 *If a Lorentzian Sasakian space form $M(c)$ with generalized Tanaka connection is a generalized conformal Ricci soliton. Then the soliton is expanding or steady or shrinking according as $p < 2a\mu^2 - \frac{2}{2n+1}$ or $p = 2a - \frac{2}{2n+1}$ or $p > 2a\mu^2 - \frac{2}{2n+1}$.*

Proof The equation (50) implies

$$g(\mathring{\nabla}_X \xi, Y) + g(X, \mathring{\nabla}_Y \xi) = \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y) - 2a\mu^2\eta(X)\eta(Y) + 2b\mathring{S}(X,Y).$$

By (24),

$$\left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y) - 2a\eta(X)\eta(Y) + 2b\mathring{S}(X,Y) = 0.$$

Replacing Y by ξ it follows that

$$-\left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right]\eta(X) - a\eta(X) + b\mathring{S}(X,\xi) = 0.$$

By equation (30), we have

$$-\left[\lambda-\frac{1}{2}\left(p+\frac{2}{2n+1}\right)\right]\eta(X)-a\eta(X)=0.$$

Or,

$$-\left[\lambda-\frac{1}{2}\left(p+\frac{2}{2n+1}\right)+a\right]\eta(X)=0.$$

This implies

$$\lambda=\frac{1}{2}\left(p+\frac{2}{2n+1}\right)-a.$$

Thus, the soliton is expanding or steady or shrinking according as $p<2a\mu^2-\frac{2}{2n+1}$ or $p=2a-\frac{2}{2n+1}$ or $p>2a\mu^2-\frac{2}{2n+1}$. □

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