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On the Core of Second Smarandache Bol Loops

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Abstract: Let $(G,\cdot)$ be a loop. A loop $(G_H,\cdot)$ is called a special loop of $(G,\cdot)$ if the pair $(H,\cdot)$ is an arbitrary a non-empty subloop of $(G,\cdot)$. In general, $(G_H,\cdot)$ is called second Smarandache Bol loop (S_{2nd}BL) if it obey the identity $(xs\cdot z)s = x(sz\cdot s)$ for all $s\in H$ and $x,z\in G$. This paper presents some algebraic characterizations of the core of a second Smarandache Bol loop (S_{2nd}BL). Some results in this paper extend or generalize the results of the classical studies of the core of a Bol loop. The conditions for the core of $S_{2nd}BL$ to be left symmetric, left(right) idempotents, left self-distributive, and flexible was shown. A necessary and sufficient condition for a core of (S_{2nd}BL) to be right(left) alternative property was revealed. The characterization of $S-$isotopic and $S-$isomorphic invariance was also presented in this paper.

Key Words: Core, special loop, Smarandache Bol loops.

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§1. Introduction

Let Q be a non -empty set. Define a binary operation " $\cdot$ " on Q. If $x\cdot y\in Q$ for all $x,y\in Q$, then the pair $(Q,\cdot)$ is called a groupoid or magma. If the equations: $a\cdot x=b$ and $y\cdot a=b$ have unique solutions $x,y\in Q$ for all $a,b\in Q$, then $(Q,\cdot)$ is called a quasigroup. Let $(Q,\cdot)$ be a quasigroup and there exist a unique element $e\in Q$ called the identity element such that for all $x\in Q, x\cdot e=e\cdot x=x$, then $(Q,\cdot)$ is called a loop. At times, we shall write xy instead of $x\cdot y$ and stipulate that $\cdot$ has lower priority than juxtaposition among factors to be multiplied. Let $(Q,\cdot)$ be a groupoid and a be a fixed element in Q, then the left and right translations L_a and R_a of a are respectively defined by $xL_a=a\cdot x$ and $xR_a=x\cdot a$ for all $x\in Q$. It can now be seen that a groupoid $(Q,\cdot)$ is a quasigroup if its left and right translation mappings are permutations. Since the left and right translation mappings of a quasigroup are bijective, then the inverse mappings L_x^{-1} and R_x^{-1} exist.

Let

$$x\backslash y = yL_x^{-1} = xP_y \qquad \text{and} \qquad x/y = xR_y^{-1} = yP_x^{-1}$$

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and note that

$$x\backslash y = z \Longleftrightarrow x \cdot z = y \qquad \text{and} \qquad x/y = z \Longleftrightarrow z \cdot y = x.$$

Thus, for any quasigroup $(Q, \cdot)$, we have two new binary operations; right division $(/)$ and left division $(\backslash)$ and middle translation P_a for any fixed $a \in Q$. Consequently, $(Q, \backslash)$ and $(Q, /)$ are also quasigroups. Using the operations $(\backslash)$ and $(/)$, the definition of a loop can be restated as follows.

Definition 1.1 *A loop $(Q, \cdot, /, \backslash, e)$ is a set G together with three binary operations $(\cdot)$, $(/)$, $(\backslash)$ and one nullary operation e such that*

(i) $x \cdot (x\backslash y) = y$, $(y/x) \cdot x = y$ for all $x, y \in Q$;
(ii) $x\backslash(x \cdot y) = y$, $(y \cdot x)/x = y$ for all $x, y \in Q$;
(iii) $x\backslash x = y/y$ or $e \cdot x = x$ for all $x, y \in Q$.

We also stipulate that $(/)$ and $(\backslash)$ have higher priority than $(\cdot)$ among factors to be multiplied. For instance, $x \cdot y/z$ and $x \cdot y\backslash z$ stand for $x(y/z)$ and $x(y\backslash z)$ respectively.

In a loop $(Q, \cdot)$ with identity element e, the *left inverse element* of $x \in Q$ is the element $xJ_\lambda = x^\lambda \in Q$ such that

$$x^\lambda \cdot x = e$$

while the *right inverse element* of $x \in G$ is the element $xJ_\rho = x^\rho \in G$ such that

$$x \cdot x^\rho = e.$$

For more on quasigroups and loops, the reader can check Jaiyéọlá [13], Pflugfelder [5] and Shcherbacov [3] for details.

The study of Smarandache concept in groupoid was first introduced by (W. B Vasantha Kandasamy [18], 2002). The paper [20] and her book on Smarandache concept in the study of loops [19], where she initially defined Smarandache loop (S-loop) as a loop with at least a subloop which forms a subgroup under the binary operations of the loop have started receiving an attention of researchers.

Smarandache quasigroup was defined by (Muktibodh, [21, 22]), as a non-trivial subset H of a quasigroup $(G, \cdot)$ such that $(H, \cdot)$ is a associative subquasigroup of the quasigroup $(G, \cdot)$.

Immediately after the work of Muktibodh, (Jaiyéọlá [6], 2006) introduced the study of holomorphic structures of a loop under Smarandache quasigroup. It was revealed that a loop is a Smarandache loop if and only if its holomorph is a Smarandache loop and further shown that the statement is also true for some weak Smarandache loops such as inverse property, weak inverse property but false for others(conjugacy closed, Bol, central, extra, Burn, A- homogeneous except if their holomorphs are nuclear or central.

In (Jaiyéọlá [10, 11, 12, 14, 15], 2008), more characterizations of a Smarandache concept in quasigroups and loops are presented. In particular, a Smarandache isotopic quasigroup and holomorphic study of Smarandache automorphism and cross inverse property loops were investigated in the same manner the isotopy theory was carried out for groupoids, quasigroups, and loops. The same author [15], introduced and studied double cryptography using the concept of Smarandache Keedwell Cross inverse quasigroup.

In [16, 17], the author furthered his exploration of Smarandache quasigroups (loops) theory by classifying the algebraic structures into first Smarandache quasigroup (loop) and second Smarandache quasigroup (loop). The author announced that the most comprehensive study in Bol-Moufang type identities called Bol loop falls into the second class of Smarandache loops. Hence, the second Smarandache loop is a particular case of the first Smarandache loops and the second Smarandache Bol loop is a generalization of Bol loops.

In (Jaiyéọlá [8, 9], 2006), the authors studied parastrophic invariants of Smarandache quasigroups, and presented a ground view of the studies of the universality of some Smarandache loops of Bol-Moufang type. His results showed that Smarandache quasigroup (loop) is universal if all its $f, g-$principal isotopes are Smarandache $f, g-$ principal isotopes.

In (Osoba et al. [28, 29], 2018), the authors studied the relationship of multiplication groups and isostrophic quasigroups and some algebraic characterizations of middle Bol loops.

In 2022, Osoba and Jaiyéọlá [24] presented algebraic connections between the middle Bol loop and right Bol loop and their cores. A necessary and sufficient condition for the core of a right Bol loop to be elastic property and right idempotent law was established. It was further revealed that If a middle Bol loop is right (left) symmetric then, the core of its corresponding (RBL) is a medial (semimedial). The results in [16, 17] were extended by the first author of this paper in [27].

In 2023, Jaiyéọlá et al. [23] presented a study on the Bryant-Schneider group of a middle Bol loop. The authors used the concept of the Bryant-Schneider group to link some of the isostrophy-group invariance results of Grecu and Syrbu. In particular, it was established that some subgroups of the Bryant-Schneider group of a middle Bol loop are isomorphic to the automorphism and pseudo-automorphism groups of its corresponding right (left) Bol loop. Some elements of the Bryant-Schneider group of a middle Bol loop were shown to induce automorphisms and middle pseudo-automorphisms. It was discovered that if a middle Bol loop is of exponent two then, its corresponding right (left) Bol loop is a left (right) G-loop while more results on the algebraic properties of a middle Bol loop using its parastrophes were unveiled by Osoba and Oyebo [26] in 2022.

Recently, the characterization of the cry-automorphism group of some quasigroups was studied in [25].

§2. Preliminaries

Definition 2.1 *A groupoid (quasigroup)* $(G, \cdot)$ *is said to have*

(1) *the left inverse property* (LIP) *if there exists a mapping* $J_\lambda : x \mapsto x^\lambda$ *such that* $x^\lambda \cdot xy = y$ *for all* $x, y \in G$*;*

(2) *the right inverse property* (RIP) *if there exists a mapping* $J_\rho : x \mapsto x^\rho$ *such that* $yx \cdot x^\rho = y$ *for all* $x, y \in G$*;*

(3) *the inverse property (IP) if it has both the LIP and RIP;*

(4) *the right alternative property (RAP) if* $y \cdot xx = yx \cdot x$ *for all* $x, y \in G$*;*

(5) *the left alternative property (LAP) if* $y \cdot xx = yx \cdot x$ *for all* $x, y \in G$*;*

(6) *the flexibility or elasticity if* $xy \cdot x = x \cdot yx$ *holds for all* $x, y \in G$*;*

(7) *the cross inverse property* (CIP) *if there exist mapping* $J_\lambda : x \mapsto x^\lambda$ *or* $J_\rho : x \mapsto x^\rho$ *such that* $xy \cdot x^\rho = y$ *or* $x \cdot yx^\rho = y$ *or* $x^\lambda \cdot yx = y$ *or* $x^\lambda y \cdot x = y$ *for all* $x, y \in G$*.*

Definition 2.2 *A loop* $(G, \cdot)$ *is said to be right power alternative property loop (RPAPL) if its obeys the identity* $xy^n = ((((xy)y)y)y)y...y$ *that is* $R_{y^n} = R_y^n$ *for all* $x, y \in G$*.*

Definition 2.3 *A special quasigroup(loop)* $(G_H, \cdot)$ *is called:*

(1) *a second Smarandache left inverse property quasigroup(loop)* $S_{2nd}LIPQ(S_{2nd}LIPL)$ *if it obeys the second Smarandache left inverse property* $(S_{2nd}LIP)$ $s^\lambda \cdot sx = x$ *for all* $x \in G$ *and* $s \in H$*;*

(2) *a second Smarandache right inverse property quasigroup(loop)* $S_{2nd}RIPQ(S_{2nd}RIPL)$ *if it obeys the second Smarandache right inverse property* $(S_{2nd}LIP)$ $xs \cdot s^\rho = x$ *for all* $x \in G$ *and* $s \in H$*;*

(3) *a second Smarandache inverse property quasigroup(loop)* $S_{2nd}IP$ *if it has both the* $S_{2nd}RIP$ *and* $S_{2nd}LIP$*;*

(4) *a second Smarandache right alternative property quasigroup(loop)* $S_{2nd}RAPQ(S_{2nd}RAPL)$ *if* $x \cdot ss = xs \cdot s$ *for all* $x \in G$ *and* $s \in H$*;*

(5) *a second Smarandache left alternative property quasigroup(loop)* $S_{2nd}LAPQ(S_{2nd}LAPL$ *if* $ss \cdot x = s \cdot sx$ *for all* $x \in G$ *and* $s \in H$ *for all* $x, y \in G$*;*

(6) *a second Smarandache flexible or elastic quasigroup(loop) if* $sx \cdot s = s \cdot xs$ *holds for all* $x \in G$ *and* $s \in H$*;*

(7) *a second Smarandache right power alternative property loop* $S_{2nd}RPAPL$ *if its obeys the identity* $sx^n = ((((xs)s)s)s)s...s$ *that is* $R_{s^n} = R_s^n$ *for all* $x \in G$ *and* $s \in H$*.*

Definition 2.4 *A Smarandache groupoid (quasigroup)* $(Q, \cdot)$ *is called:*

(1) *the second Smarandache right symmetric (*$S_{2nd}RS$*) if* $xs \cdot s = x$ *for all* $x \in Q$ *and* $s \in H$*;*

(2) *the second Smarandache left symmetric (*$S_{2nd}LS$*) if* $s \cdot sx = x$ *for all* $x \in Q$ *and* $s \in H$*;*

(3) *the second Smarandache middle symmetric (*$S_{2nd}MS$*) if* $s \cdot xs = x$ *or* $xs \cdot x = x$ *for all* $x \in Q$ *and* $s \in H$*;*

(4) *the third Smarandache middle symmetric (*$S_{3nd}MS$*) if* $xs \cdot x = s$ *or* $x \cdot sx = s$ *for all* $x \in Q$ *and* $s \in H$*;*

(5) *the second Smarandache idempotent (*$S_{2nd}I$*) if* $s \cdot s = s$ *for all* $x \in Q$ *and* $s \in H$*;*

(6) *the second Smarandache left idempotent (*S_2ndLI*) if* $ss \cdot x = sx$ *for all* $x \in Q$ *and* $s \in H$*;*

(7) *the second Smarandache right idempotent (*$S_{2nd}RI$*) if* $x \cdot ss = sx$ *for all* $x \in Q$ *and* $s \in H$*;*

(8) *the second Smarandache commutative (*$S_{2nd}CP$*) if* $x \cdot s = s \cdot x$ *for all* $x \in Q$ *and* $s \in H$*;*

(9) *the second Smarandache anti-automorphic inverse property (*$S_{2nd}AAIP$*) if* $(x \cdot s)^\rho = (s \cdot x)^\rho$ *or* $(x \cdot s)^\lambda = (s \cdot x)^\lambda$ *for all* $x \in Q$ *and* $s \in H$*;*

(10) *the second Smarandache totally quasigroup ($S_{2nd}TQ$) if and only if (1) or (2) and (8) hold.*

Definition 2.5 *Let $(Q,\cdot)$ be a $S_{2nd}TQ$. If $(Q,\cdot)$ is a special loop, then it is called second Smarandache Steiner loop ($S_{2nd}SL$).*

Theorem 2.6 (Jaiyeola [16]) *Let the special loop $(G_H,\cdot)$ be a $S_{2nd}BL$. Then, $S_{2nd}BL$ is satisfies $S_{2nd}RIPL$ and $S_{2nd}RAPL$.*

Theorem 2.7 (Jaiyeola [16]) *If the special loop $(G_H,\cdot)$ is a $S_{2nd}BL$. Then,*

$$xs^n = xs^{n-1}\cdot s = xs\cdot s^{n-1}$$

for all $n\in\mathbb{Z}, s\in H$ and $x\in G$.

Theorem 2.8 (Jaiyeola [16]) *If the special loop $(G_H,\cdot)$ is a $S_{2nd}BL$. Then, $xs^m\cdot s^n = xs^{m+n}$ for all $m,n\in\mathbb{Z}, s\in H$ and $x\in G$.*

Theorem 2.9 (Jaiyeola [16]) *If the special loop $(G_H,\cdot)$ is a $S_{2nd}BL$. Then, G_H is a $S_{2nd}SAIPL$ if and only if G_H is a $S_{3rd}RIPL$.*

Corollary 2.10 (Jaiyeola [16]) *Every $S_{2nd}BL$ is a Smarandache right power associative property loop.*

Definition 2.11 (Jaiyeola [17]) *Let $(G_H,\cdot)$ and $(Q_N,\circ)$ be spacial groupiods and let G_H, Q_N be Smarandache isotopes (S-isotopes). Then, $(Q_N,\circ)$ is a Smarandache isotopic of $(G_H,\cdot)$ if and only if there is a bijective $(A,B,C): H\mapsto N$ such that the triple $(A,B,C):(G_H,\cdot)\mapsto(Q_N,\cdot)$ is isotopism. Suppose that the triple $A=B=C$, then $(G_H,\cdot)$ and $(Q_N,\circ)$ are said to be Smarandache isomorphic (S-isomorphic).*

Definition 2.12 *Let the spacial loop $(G_H,\cdot)$ be a $S_{2nd}BL$. The groupoid $(G_H,+)$ called the core of $(G_H,\cdot)$ is define as $x+y=xy^\lambda\cdot x$ for all $x\in H$ and $y\in G$.*

Definition 2.13 *A special groupoid $(Q,+)$ is called:*

(1) *Smarandache left self distributive (SLSD) if $s+(y+z)=(s+y)+(s+z)$ for all $y,z\in Q$ and $s\in H$;*

(2) *Smarandache left distributive(SLD) if $s(y+z)=(sy)+(sz)$ for all $y,z\in Q$ and $s\in H$;*

(3) *Smarandache right distributive(SRD) if $(y+z)s=(ys)+(zs)$ for all $y,z\in Q$ and $s\in H$.*

Definition 2.14 (Jaiyeola [17], 2011) *Let $(G_H,\cdot)$ is called a special loop with special subloop $(H,\cdot)$. If $(H,\cdot)$ is of exponent 2, then $(G_H,\cdot)$ is called a special loop of Smarandache exponent two.*

Definition 2.15 *Let $(G_H,\cdot)$ be a special loop. H is called an ideal of $(G,\cdot)$ if $sx\in H$ for all $s\in H$, and $x\in G$*

Furtherance to the past research, this paper is posted to extend the results in [16, 24]. Some new definitions were established and were used to characterize the core of the second Smarandache Bol loop.

§3. Main Results

Lemma 3.1 *Let $(G_H,\cdot)$ be a special quasigroup.*

(1) *if $(G_H,\cdot)$ is a $S_{3nd}RIP$ and H is a right ideal of $(G,\cdot)$, then $x^{\rho^2}=x$ and $x^\rho=x^\lambda$ for all $x\in G$;*

(2) *if $(G_H,\cdot)$ is a $S_{3nd}LIP$ and H is a left ideal of $(G,\cdot)$, then $x^{\lambda^2}=x$ and $x^\rho=x^\lambda$ for all $x\in G$;*

(3) *if $(G_H,\cdot)$ is a $S_{2nd}LIP$, then $sx=b\Rightarrow x=s^\lambda b$ for all $s\in H$ and $x\in G$;*

(4) *if $(G_H,\cdot)$ is a $S_{3nd}RIP$, then $xs=b\Rightarrow x=bs^\rho$ for all $s\in H$ and $x\in G$;*

(5) *if $(G_H,\cdot)$ is a $S_{2nd}RIP$, then $ys=b\Rightarrow y=bs^\rho$ for all $s\in H$ and $x\in G$;*

(6) *if $(G_H,\cdot)$ is a $S_{3nd}LIP$, then $ys=b\Rightarrow s=y^\lambda b$ for all $s\in H$ and $x\in G$;*

(7) *if $(G_H,\cdot)$ is a $S_{2nd}RIP$ and $S_{3nd}LIP$, then $s^\lambda=(as)^\lambda a$ $-S_{3rd}LWIP$ for all $s\in H$ and $a\in G$;*

(8) *if $(G_H,\cdot)$ is a $S_{2nd}RIP$, $S_{3rd}LIP$, $S_{3nd}RIP$ and H is $\lambda-$ideal, then $s^{-1}a^{-1}=(as)^{-1}$ for all $s\in H$ and $a\in G$;*

(9) *if $(G_H,\cdot)$ is a $S_{2nd}LIP$ and $S_{3nd}RIP$, then $s^\rho=b(sb)^\rho-$ $S_{3rd}RWIP$ for all $s\in H$ and $b\in G$;*

(10) *if $(G_H,\cdot)$ is a $S_{2nd}LIP$, $S_{3nd}RIP$, $S_{3rd}LIP$ and H is $\rho-$ideal, then $b^{-1}s^{-1}=(sb)^{-1}$ for all $s\in H$ and $b\in G$;*

(11) *$(G_H,\cdot)$ has $S_{2nd}RIP\Leftrightarrow R_{s^{-1}}=R_s^{-1}$;*

(12) *$(G_H,\cdot)$ has $S_{2nd}LIP\Leftrightarrow L_{s^{-1}}=L_s^{-1}$;*

(13) *if $(G_H,\cdot)$ is a $S_{2nd}RIP$, $S_{3nd}IP$ and $\lambda-$ideal, $J_\lambda R_sJ_\rho=L_{s^{-1}}$ for all $s\in H$;*

(14) *if $(G_H,\cdot)$ is a $S_{2nd}LIP$, $S_{3nd}IP$ and $\rho-$ideal, $J_\lambda L_sJ_\rho=R_{s^{-1}}$ for all $s\in H$.*

Proof (1) Consider the expression $(sx\cdot x^\rho)(x^\rho)^\rho$, then

$$(sx\cdot x^\rho)(x^\rho)^\rho \underbrace{=}_{3_{nd}RIP} s(x^\rho)^\rho=sx\Rightarrow x^{\rho^2}=x\Rightarrow J_\rho^2=I\Rightarrow J_\rho^{-1}=J_\rho\Rightarrow J_\lambda=J_\rho.$$

(2) Consider the expression $(x^\lambda\cdot xs)(x^\lambda)^\lambda$, then

$$(x^\lambda\cdot xs)(x^\lambda)^\lambda \underbrace{=}_{3_{nd}LIP} (x^\lambda)^\lambda s=xs\Rightarrow x^{\lambda^2}=x\Rightarrow J_\lambda^2=I\Rightarrow J_\lambda{-}1=J_\rho\Rightarrow J_\lambda=J_\rho.$$

(3) Let $sx=b$. Multiplying both sides by s^λ on the left, we have $x=\underbrace{=}_{2_{nd}LIP} s^\lambda b$.

(4) Let $xs=b$. Multiplying both sides by s^ρ on the right, we have $x=\underbrace{=}_{2_{nd}RIP} bs^\rho$.

(5) Let $ys=b$. Multiplying both sides by s^ρ on the right, we have $y=\underbrace{=}_{2_{nd}RIP} bs^\rho$.

(6) Let $ys = b$. Multiplying both sides by y^λ on the left, we have $s = \underbrace{=}_{3_{nd}LIP} y^\lambda b$.

(7) Let $as = c$, then $a \underbrace{=}_{2_{nd}RIP} cs^\rho \underbrace{\Rightarrow}_{S_{3nd}LIP} s^\rho = (as)^\lambda a \Rightarrow s^\lambda = (as)^\lambda a$.

(8) So, $s^\rho = (as)^\lambda a \Rightarrow s^\lambda = (as)^\lambda a \underbrace{\overset{\lambda-ideal}{\Rightarrow}}_{S_{3rd}RIP} s^\lambda a^\rho = (as)^\lambda \Rightarrow s^{-1}a^{-1} = (as)^{-1}$.

(9) Let $sb = c \underbrace{\Rightarrow}_{S_{2nd}LIP} b = s^\lambda c \underbrace{\Rightarrow}_{S_{3nd}RIP} bc^\rho s^\lambda \Rightarrow b(sb)^\rho = s^\rho$.

(10) So, $b(sb)^\rho = s^\lambda \Rightarrow b(sb)^\rho = s^\rho \underbrace{\overset{\rho-ideal}{\Rightarrow}}_{S_{3rd}LIP} b^\lambda s^\rho = (bs)^\rho \Rightarrow b^{-1}s^{-1} = (sb)^{-1}$.

(11) $ys \cdot s^{-1} = y \Leftrightarrow yR_sR_{s^{-1}} = y \Leftrightarrow R_sR_{s^{-1}} = I \Leftrightarrow R_s^{-1} = R_{s^{-1}}$.

(12) $s^\lambda \cdot sx = x \Leftrightarrow xL_sL_{s^{-1}} = x \Leftrightarrow L_sL_{s^{-1}} = I \Leftrightarrow L_s^{-1} = L_{s^{-1}}$.

(13) $xJ_\lambda R_sJ_\rho = (x^\lambda s)^\rho \underbrace{\overset{S_{3rd}IP}{\Rightarrow}}_{S_{2rd}RIP} s^{-1}(x^{-1})^{-1} = s^{-1}x = xL_{s^{-1}}$.

(14) $xJ_\lambda L_sJ_\rho = (sx^\lambda)^\rho \underbrace{\overset{S_{2nd}LIP}{\Rightarrow}}_{S_{3rd}IP} (x^{-1})^{-1}s^{-1} = xs^{-1} = xR_{s^{-1}}$.

This completes the proof. □

Theorem 3.2 *Let the spacial loop $(G_H, \cdot)$ be a $S_{2nd}BL$. Then,*

(1) $(G_H, +)$ is (S_{2nd} LS) if $(G_H, \cdot)$ is a (S_{2nd} RIPL) and H is $\rho-$ideal of $(G, \cdot)$;

(2) $(G_H, +)$ is (S_{2nd} LI);

(3) $(G_H, +)$ is (S_{2nd} RI);

(4) if $(G_H, \cdot)$ is (S_{2nd} RIP), S_{2nd}elastic, S_{3rd}RIP and H is $\rho-$ideal, then $(G_H, \cdot)$ satisfies commutative if and only if $(G_H, +)$ is S_{2nd} middle symmetric;

(5) if $(G_H, \cdot)$ is S_{3rd}RIP and H is $\rho-$ideal, then $(G_H, +)$ is S_{2rd} LD if and only if $(sy^\rho \cdot z)y^\rho = s(yz^\rho \cdot y)^\rho$ for all $s \in H$ and $y, z \in G$;

(6) if $(G_H, \cdot)$ is S_{3rd}SAIP then $(G_H, +)$ is S_{2nd} flexible if and only if $(sy^\rho \cdot s)y^\rho = s(y^\rho s \cdot y^\rho)$ for all $y \in G$, and $s \in H$.

Proof (1) By symmetric property, we have $s + (s + y) = s(s + y)^\lambda \cdot s = [s(sy^\lambda \cdot s)^\lambda]s = [s(s^\lambda y^{\lambda^2} \cdot s^\lambda)]s$. Apply Theorem 2.9, we have $((ss^\lambda \cdot y^{\lambda^2})s^\lambda)s = y^{\lambda^2}s^\lambda \cdot s = y$.

(2) By left idempotent: $(s+s)+y = (s+s)y^\lambda \cdot (s+s) = [(ss^\lambda \cdot s)y^\lambda]ss^\lambda \cdot s = sy^\lambda \cdot s = s+y$.

(3) By right idempotent: $y + (s + s) = y(s + s)^\lambda \cdot y = y(ss^\lambda \cdot s)^\lambda \cdot y = ys^\lambda \cdot y = y + s$.

(4) By middle symmetric:

$$\begin{aligned} s + (x + s) = x \quad &\Leftrightarrow \quad [s(x + s)^\lambda]s = x \\ &\Leftrightarrow \quad [s(xs^{-1} \cdot x)^\lambda]s = x \underbrace{\Leftrightarrow}_{S_{2nd}IP} (xs^\lambda \cdot x)^\lambda = s^\lambda \cdot xs^\lambda \\ &\underbrace{\Leftrightarrow} \quad _{S_{3rd}\text{ (RIP) and H is }\rho-\text{ideal}} (xs^\lambda \cdot x)^\lambda = (s \cdot x^\lambda s)^\lambda \\ &\underbrace{\Leftrightarrow} \quad _{S_{2nd}\text{(elasticity)}} (xs^\lambda \cdot x)^\lambda = (sx^\lambda \cdot s)^\lambda \\ &\Leftrightarrow \quad (x + s)^\lambda = (s + x)^\lambda \Leftrightarrow x + s = s + x \end{aligned}$$

for all $x \in G$ and $s \in H$.

(5) By left self distributive: $s + (y + z) = (s + y) + (s + z)$. For all $s \in H$ and $y, z \in G$, we have

$$\begin{aligned}
LHS = (s+y)+(s+z) &= [(sy^{\lambda} \cdot s)(sz^{\lambda} \cdot s)^{\lambda}](sy^{\lambda} \cdot s) \\
&\underbrace{=}_{S_{3rd}\text{ RIP and H is }\rho-\text{ideal}} [(sy^{\lambda} \cdot s)(s^{-1}z \cdot s^{-1})](sy^{\lambda} \cdot s) \\
&= \left[\left(((sy^{\lambda} \cdot s)s^{\lambda})z\right)s^{\lambda}\right](sy^{\lambda} \cdot s) \\
&= \left[((sy^{\lambda} \cdot z))s^{\lambda}\right](sy^{\lambda} \cdot s) \\
&= \left[\left(((sy^{\lambda} \cdot z)s^{-1})s\right)y^{\lambda}\right]s \\
&= (sy^{\lambda} \cdot z)y^{\lambda} \cdot s
\end{aligned}$$

and $RHS = s+(y+z) = s(y+z)^{\rho} \cdot s = s(yz^{\lambda} \cdot y)^{\rho} \cdot s$. So, $(G_H, +)$ has S_{2rd}LSD $\Leftrightarrow (sy^{\rho} \cdot z)y^{\rho} = s(yz^{\rho} \cdot y)^{\rho}$ for all $s \in H$ and $y, z \in G$.

(6) $(G_H, +)$ is S_{2nd} flexible $\Leftrightarrow (s + y) + s = s + (y + s)$ for all $y \in G$, and $s \in H$.

$$\begin{aligned}
RHS &= LSH \\
&\Leftrightarrow (s+y)s^{\lambda} \cdot (s+y) = s(y+s)^{\lambda} \cdot s \\
&\Leftrightarrow [(sy^{\lambda} \cdot s)s^{\lambda}](sy^{\lambda} \cdot s) = (s(ys^{\lambda} \cdot y)^{\lambda})s \\
&\Leftrightarrow [(sy^{\lambda} \cdot s)s^{\lambda}](sy^{\lambda} \cdot s) \underbrace{=}_{S_{3rd}SAIP} (s(y^{\lambda}s \cdot y^{\lambda}))s \\
&\Leftrightarrow sy^{\lambda}(sy^{\lambda} \cdot s) = s(y^{\lambda}s \cdot y^{\lambda}) \cdot s \\
&\Leftrightarrow (sy^{\lambda} \cdot s)y^{\lambda} \cdot s = s(y^{\lambda}s \cdot y^{\lambda}) \cdot s \\
&\Leftrightarrow (sy^{\lambda} \cdot s)y^{\lambda} = s(y^{\lambda}s \cdot y^{\lambda}).
\end{aligned}$$

This completes the proof. □

Theorem 3.3 *Let the spacial loop $(G_H, \cdot)$ be a $S_{2nd}BL$. Then,*

(1) *$(G_H, +)$ is $S_{2nd}RAP$ if and only if $y + s = s$ for all $s \in H$;*

(2) *$(G_H, +)$ is $S_{2nd}LAP$ if and only if $s + y = y$ for all $s \in H$;*

(3) *if $(s + y)x = sx + yx$ for all $s \in H$ and $x, y \in G$, then $(G_H, \cdot)$ is satisfies $S_{2nd}LAP$ if and only if it satisfies $S_{3nd}RIP$;*

(4) *if $x(s + y) = xs + xy$, then $(G_H, \cdot)$ is satisfies $SAAIP$ for all $s \in H$ and $x, y \in G$.*

Proof (1) Notice that

$$\begin{aligned}(G_H,+) \text{ is } S_{2nd}RAP &\Leftrightarrow (y+s)+s = y+(s+s)\\ &\Leftrightarrow (y+s)s^{\lambda}(y+s) = y(s+s)^{\lambda}\cdot y\\ &\Leftrightarrow [(ys^{\lambda}\cdot y)s^{\lambda}](ys^{-1}\cdot y) = [y(ss^{\lambda}\cdot s)^{\lambda}]y\\ &\Leftrightarrow [(ys^{\lambda}\cdot y)s^{\lambda}](ys^{\lambda}\cdot y) = ys^{\lambda}\cdot y\\ &\Leftrightarrow (ys^{\lambda}\cdot y)s^{\lambda} = e\\ &\Leftrightarrow ys^{\lambda}\cdot y = s^{(\lambda)^{\rho}}\\ &\Leftrightarrow y+s = s.\end{aligned}$$

(2) Notice that

$$\begin{aligned}(G_H,+) \text{ is } S_{2nd}LAP &\Leftrightarrow (s+s)+y = s+(s+y)\\ &\Leftrightarrow [(s+s)y^{\lambda}](s+s) = [s(s+y)^{\lambda}]s\\ &\Leftrightarrow [(ss^{-1}\cdot s)y^{\lambda}](ss^{-1}\cdot s) \underbrace{=}_{S_{3rd}\text{RIP and H is } \rho-\text{ideal}} [s(sy^{\lambda}\cdot s)^{\lambda}]s\\ &\Leftrightarrow sy^{\lambda}\cdot s = [s(s^{\lambda}y\cdot s^{\lambda})]s\\ &\Leftrightarrow sy^{\lambda}\cdot s = ((ss^{\lambda}\cdot y)s^{\lambda})s\\ &\Leftrightarrow sy^{\lambda}\cdot s = ys^{\lambda}\cdot s\\ &\Leftrightarrow sy^{\lambda}\cdot s = y\\ &\Leftrightarrow s+y = y.\end{aligned}$$

(3) if $(s+y)x = sx+yx$ then,

$$(sy^{\lambda}\cdot s)x = [(sx)(yx)^{\lambda}](sx)$$

Put $y = e$, the identity element in G, we have $ss\cdot x = (sx\cdot x^{\lambda})(sx) \underbrace{\Rightarrow}_{S_{3nd}\text{RIP}} ss\cdot x = s\cdot sx$ for all $x\in G$ and $s\in H$.

(4) if $x(s+y) = xs+yx$ then,

$$\begin{aligned}x(sy^{\lambda}\cdot s) &= [(xs)(xy)\lambda](xs)\\ &\Rightarrow (xs\cdot y^{\lambda})s = [(xs)(xy)^{\lambda}](xs).\end{aligned}$$

Let $x = s^{\lambda}$ for all $s\in H$, get $y^{\rho}s = (s^{\lambda}y)^{\rho}$. □

Corollary 3.4 *Let $(G_H,+)$ be a $S_{2nd}RAP(LAP)$ of $S_{2nd}BL$. Then, $(G_H,\cdot)$ is $S_{3nd}MS(S_{2nd}MS)$ respectively if and only if it is Smarandache exponent two.*

Proof The proof follows from Theorem 3.3. □

Corollary 3.5 *Let $(G_H,+)$ be an alternative property of $S_{2nd}BL$. Then, $(G_H,\cdot)$ is S_2ndC if* *and only if it $S_{2nd}RIP$ and Smarandache exponent two.*

Proof By Theorem 3.3, we have $sy^\rho \cdot s = y \Leftrightarrow s \cdot y^\rho = y \cdot s^\rho \Leftrightarrow s \cdot y = y \cdot s$. □

Corollary 3.6 *Let $(G_H, +)$ be an alternative property of $S_{2nd}BL$. Then, $(G_H, \cdot)$ is a second Smarandache Steiner loop if and only if it $S_{2nd}RIP$ and Smarandache exponent two.*

Proof Following from Theorem 3.3, this is true base on Corollaries 3.4 and 3.5. □

Theorem 3.7 *Let the spacial loop $(G_H, \cdot)$ be a $S_{2nd}BL$ with Smarandache AIPL and $(G_H, +)$ be its core.*

(1) *If $(s + y)x = sx + yx$, then $(G_H, \cdot)$ is Smarandache loop. For all $x, z \in G$ and $s \in H$;*

(2) *If $x(s + y) = xs + yx$, then $(G_H, \cdot)$ is S_{2nd} flexible. For all $x, z \in G$ and $s \in H$.*

Proof (1) By using Theorems 3.3 and 2.6, we have $(sy^\rho \cdot s)x = [sx(yx)^\rho](sx) \Leftrightarrow (sy^\rho \cdot s)x = [sx(x^{-1}y^\rho)](sx) \Leftrightarrow (sy^\rho \cdot s)x = (sy^\rho)(sx)$. Let $y^\rho = t$, for all $s \in H$ and $x, t \in G$, we have $tL_sR_s \cdot x = tL_s \cdot xL_s$. Let $t = tL_s^{-1}$, then $tR_s \cdot x = t \cdot xL_s \Leftrightarrow ts \cdot x = t \cdot sx$. For all $s \in H$.

(2) By using Theorems 3.3 and 2.6, we have $x(sy^\rho \cdot s) = [xs(xy)^\rho](xs) \Leftrightarrow x(sy^\rho \cdot s) = [xs(y^\rho x^{-1}](xs) \Leftrightarrow x(sy^\rho \cdot s) = (xs)[(y^\rho x^{-1} \cdot (xs)] \Leftrightarrow x(sy^\rho \cdot s) = (xs)(y^\rho s)$. Let $t = y^\rho$, then $x(st \cdot s) = (xs)(ts) \Leftrightarrow x \cdot tL_sR_s = xL_s \cdot tR_s$. Put $x = xL_s^{-1}$, get $xL_s^{-1} \cdot tL_sR_s = x \cdot tR_s \Leftrightarrow sy \cdot s = s \cdot ys$. By letting $x = s$ for all $s \in H$ and $y \in G$. □

Theorem 3.8 *Let the spacial loop $(G_H, \cdot)$ be a $S_{2nd}BL$. Let $(G_H, \circ)$ be a $S-$principal isotope of $(G_H, \cdot)$, where $x \circ y = xR_g \cdot yL_g^{-1}$ for all $x, y \in G$ and some $g \in H$. Let $(G_H, +)$ and $(G_H, \oplus)$ be the cores of $(G_H, \cdot)$ and $(G_H, \circ)$ respectively. Then $s\phi \oplus y\phi = (s + y)\phi$ if and only if*

$$((gs \cdot t^{-1})s)\phi^{-1} = [(gs)\phi^{-1} \cdot (gt)\phi^{-1}J] \cdot (gs)\phi^{-1}$$

where ϕ is $S-$permutation in G_H.

Proof Let $(G_H, \cdot)$ be a $S_{2nd}BL$. By Theorem 2.6, $(G_H, \cdot)$ is a $S_{2nd}RIPL$. Let $(G_H, \circ)$ be a S-principal isotope of $(G_H, \cdot)$ defined as $x \circ y = xR_g \cdot yL_g^{-1}$ for all $g \in H$ is the identity element in G_H. Then $y \circ y^\rho = yR_g \cdot yJ_\rho = e \Rightarrow yR_fJ = yJ_\rho L_g^{-1} \Rightarrow R_gJL_g = J_\rho$, where $J : y \mapsto y^{-1}$ and $y^\rho = yJ_\rho$ the right inverse element in G_H.

$$\begin{aligned} s \oplus y &= (s \circ yJ_\rho) \circ s \\ &= (sR_g \cdot yJ_\rho L_g^{-1})R_g \cdot sL_g^{-1} \\ &= (sR_g \cdot yR_gJL_fL_g^{-1})R_g \cdot sL_g^{-1} \\ &= (sR_g \cdot yR_fJ)R_g \cdot sL_g^{-1} \end{aligned}$$

So, $s\phi \oplus y\phi = (s + y)\phi \Leftrightarrow (sR_g\phi \cdot y\phi R_fJ)R_g \cdot s\phi L_g^{-1} = (sy^{-1} \cdot s)\phi$ for all $s \in H$ and $y \in G$.

Doing the following steps: Replace s by $sL_g\phi^{-1}$ and y by

$$\begin{aligned} yR_g^{-1}\phi^{-1} &\Leftrightarrow (sLgR_g \cdot yJ)R_g \cdot s = [(sL_g\phi^{-1} \cdot yR_g^{-1}\phi^{-1}J)sL_g\phi^{-1}]\phi \\ &\Leftrightarrow ((sLgR_g \cdot yJ)R_g \cdot s)\phi^{-1} = [(sL_g\phi^{-1} \cdot yR_g^{-1}\phi^{-1}J)sL_g\phi^{-1}] \\ &\Leftrightarrow [((gs)(gy^{-1} \cdot g)s]\phi^{-1} = [(gs)\phi^{-1} \cdot (gg^{-1} \cdot yg^{-1})\phi^{-1}J] \cdot (gs)\phi^{-1} \\ &\Leftrightarrow [((gs)(gy^{-1} \cdot g)s]\phi^{-1} = [(gs)\phi^{-1} \cdot g(g^{-1}y \cdot g^{-1})\phi^{-1}J] \cdot (gs)\phi^{-1}. \end{aligned}$$

Now, set $t = g^{-1}y \cdot g^{-1}$, we have

$$\Leftrightarrow [(gs \cdot t^{-1})s]\phi^{-1} = [(gs)\phi^{-1} \cdot (gt)\phi^{-1}J] \cdot (gs)\phi^{-1}$$

for any $t \in G_H$. □

Theorem 3.9 *The core $(G_H, +)$ is $S-$isotopic invariant for $S_{2nd}BL$ $(G_H, \cdot)$. That is $S-$isotopic $(G_H, \cdot)$ have $S-$isomorphic $(G_H, +)$.*

Proof Use Theorem 3.8, we consider the those S-isotopes $(G_H, \circ)$, where $x \circ y = xR_g \cdot yL_g^{-1}$ for all x, y, G_H and some $g \in H$. Let $(G_H, +)$ and $(G_H, \oplus)$ be the $S_{2nd}cores$ of $(G_H, \cdot)$ and $(G_H, \circ)$ respectively. Since $(G_H, \cdot)$ is a $S_{2nd}BL$, we have $(gs \cdot t^{-1})s = g(st^{-1} \cdot s)$ for all $t \in G_H$, and $s \in H$. Using Theorem 3.8 by replacing ϕ^{-1} by L_g, we have

$$\begin{aligned} [gs \cdot t^{-1})s]L_g^{-1} &= st^{-1} \cdot s \\ &\Rightarrow [(gs \cdot t^{-1})s]L_g^{-1} = [(gs)L_g^{-1} \cdot (gt)L_g^{-1}J] \cdot (gs)L_g^{-1} \end{aligned}$$

for some $g \in H$. So, by Theorem 3.8, $(G_H, +)$ and $(G_H, \oplus)$ are S-isomorphic. □

Corollary 3.10 *A $S_{2nd}BL$ is SAAIPL if and only if for each S-principal isotope of $(G_H, \circ)$, where $x \circ y = xR_g \cdot yL_g^{-1}$ for all $x, y \in G$ and some $g \in H$, $s \oplus y = (s + y)$ for all $s \in H$, and $y \in G$ where $(G_H, +)$ and $(G_H, \oplus)$ are the cores of $(G_H, \cdot)$ and $(G_H, \circ)$ respectively.*

Proof Put $\phi = e$ in Theorem 3.9, we have $((gs \cdot t^{-1})s) = [(gs) \cdot (gt)^{-1}] \cdot (gs)$ for all $s \in H$ and $t \in G_H$. Put $s = g^{-1}$, then $t^{-1}g^{-1} = (gt)^{-1}$ for all $t \in G_H$ and $g \in H$. □

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