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On the M-Polynomials and Degree-Based Topological Indices of an Important Class of Graphs

Shilpa H. C[1], Gayathri K[1], Nagesh H. M[2] and Narahari N[3]

1. Department of Mathematics, School of Applied Sciences, REVA University, Kattigenahalli, Bangalore, India

2. Department of Science & Humanities, PES University - Electronic City Campus, Hosur Road, Bangalore, India

3. Department of Mathematics, University College of Science, Tumkuru University, Tumakuru, India

E-mail: shilpahc539@gmail.com, gayathri.k@reva.edu.in, nageshhm@pes.edu, narahari@tumkuruniversity.in

Abstract: For a graph G, the M-polynomial is defined as

$$M(G;x,y)=\sum_{\delta\leq i\leq j\leq\Delta} m_{ij}(G)x^iy^j,$$

where m_{ij}, $(i,j\geq 1)$, is the number of edges uv of G such that $deg_G(u)=i$ and $deg_G(v)=j$; and δ and Δ are the minimum and maximum degree of G, respectively. The topological indices play an important role in determining physio-chemical properties of chemical graphs, among them the degree-based topological indices can be driven from an algebraic formula corresponding to the chemical graphs called M-polynomial. In this paper, we compute the closed forms of M-polynomial for cycle-star graph and the line graph of cycle-star graph. Further, we give the graphical representation of M-polynomial and derive some degree-based topological indices from M-polynomial.

Key Words: M-polynomial, degree-based topological indices, cycle-star graph, line graph.

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§1. Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [5]. Throughout this paper, by a graph $G=(V,E)$, we mean a simple, undirected, finite graph of order n and size m. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. A *chemical graph (or, molecular graph)* is a labeled graph whose vertices and edges correspond to the atoms and chemical bonds of the compound, respectively. The numerical parameters of a graph which describe its topology are said to be the *topological indices* or *graph invariants.* The topological indices of a chemical or molecular graph helps us to investigate the physio-chemical properties and boiling activities.

The study of topological indices was first initiated by H. Wiener [13] in the year 1947. He introduced Weiner index in order to understand the correlation of the measured properties of

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molecules in a compound with their structural properties. In the year 1972, the Weiner index was interpreted by Hosoya [6] using distances between vertices in a graph. Over the last decade, various topological indices were introduced and studied by different authors [1,3, 4].

There are many algebraic polynomials such as Hosoya polynomial (or, Weiner polynomial), which plays an important role in determining distance-based topological indices. Among many other algebraic polynomials, M-polynomial [2] introduced in 2015 plays an important role in determining the closed form of many degree-based topological indices. Related papers on finding topological indices using M-polynomials can be found in [7,8,9,10,11].

Sedlar [12] introduced the concept of cycle-star graph while studying additively weighted Harary index for extremal unicyclic graphs.

Definition 1.1 *A cycle-star graph, written $CS_{k,n-k}$, is a graph with n vertices consisting of the cycle graph of length k and $n-k$ leafs appended to the same vertex of the cycle.*

Clearly, cycle-star graphs are the unicyclic graphs (i.e., connected graphs containing exactly one cycle). Recently, the topological indices of unicyclic graphs attracted much attention. Studies along this line include general multiplicative Zagreb indices of unicyclic graphs, Zagreb eccentricity indices of unicyclic graphs, Maximal hyper-Zagreb index of unicyclic graphs with a given order, and matching number. However, the studies on the topological indices of the intersection graph on the vertex set of cycle-star graph was not attempted. In this paper we have made an attempt to fill this gap and study the topological indices of the cycle-star graph and line graph of the cycle-star graph through a polynomial approach.

The cycle-star graphs $CS_{3,4}$ and $CS_{4,3}$ are shown in Figure 1.

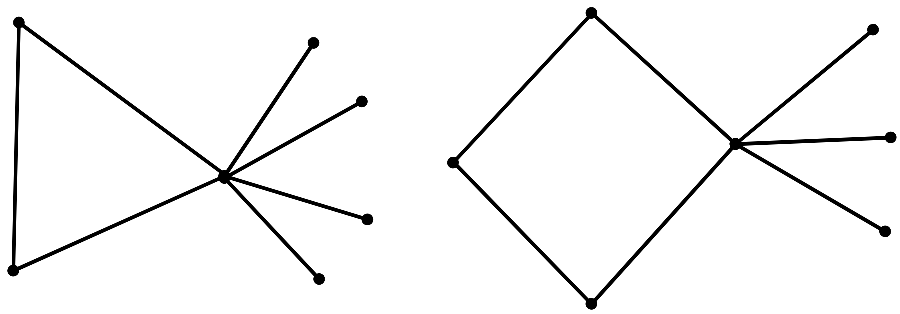

Figure 1

§2. Methodology

We first divide the edge set of cycle-star graph and the line graph of cycle-star graph into different classes based on the degree of end vertices. With the help of this edge division, we compute the M-polynomial of cycle-star graph and the line graph of cycle-star graph. Further, by using M-polynomial, we compute the degree-based topological indices as listed in Table 1. The 3-D graph of M-polynomials are sketched by using MATLAB.

§3. Preliminaries

Definition 3.1 *For a graph G, the* M-polynomial *is defined as*

$$M(G;x,y)=\sum_{\delta\leq i\leq j\leq\Delta} m_{ij}(G)x^i y^j,$$

where $m_{ij},(i,j\geq 1)$, is the number of edges uv of G such that $deg_G(u)=i$ and $deg_G(v)=j$; and δ and Δ are the minimum and maximum degree of G, respectively.

Table 1. Operations to derive degree-based topological indices from M-polynomial

Notation	Topological Index	Derivation from $M(G;x,y)$
$M_1(G)$	First Zagreb index	$(D_x+D_y)(M(G;x,y))\|_{x=y=1}$
$M_2(G)$	Second Zagreb index	$(D_xD_y)(M(G;x,y))\|_{x=y=1}$
${}^mM_2(G)$	Second modified Zagreb index	$(S_xS_y)(M(G;x,y))\|_{x=y=1}$
$R_\alpha(G)$	Randić index	$(D_x^\alpha D_y^\alpha)(M(G;x,y))\|_{x=y=1}$
$RR_\alpha(G)$	Inverse Randić index	$(S_x^\alpha S_y^\alpha)(M(G;x,y))\|_{x=y=1}$
$SSD(G)$	Symmetric division index	$(D_xS_y+D_yS_x)(M(G;x,y))\|_{x=y=1}$
$H(G)$	Harmonic index	$2S_xJ(M(G;x,y))\|_{x=1}$
$I(G)$	Inverse sum index	$S_xJD_xD_y(M(G;x,y))\|_{x=1}$
$A(G)$	Augmented Zagreb index	$S_x^3Q_{-2}JD_x^3D_y^3(M(G;x,y))\|_{x=1}$

Here,

$$\begin{aligned} M(G;x,y) &= f(x,y),\ D_x(f(x,y))=x\frac{\partial f(x,y)}{\partial x},\ D_y(f(x,y))=y\frac{\partial f(x,y)}{\partial y}, \\ S_x(f(x,y)) &= \int_o^x \frac{f(t,y)}{t}dt,\ S_y(f(x,y))=\int_o^y \frac{f(x,t)}{t}dt, \\ J(f(x,y)) &= f(x,x) \text{ and } Q_\alpha f(x,y)=x^\alpha f(x,y). \end{aligned}$$

are the operators.

As discussed in [2], each of these topological indices can be found using M-polynomials as given in Table 1.

§4. M-Polynomial of Cycle-Star Graph $CS_{k,n-k}$

In this section, we find the M-polynomial of cycle-star graph $CS_{k,n-k}$.

Theorem 4.1 *Let $G=CS_{k,n-k}$ be the cycle-star graph. Then the M-polynomial of G is*

$$M(G;x,y)=(k-2)x^2y^2+2x^2y^{n-k+2}+(n-k)xy^{n-k+2}.$$

Proof Let $G = CS_{k,n-k}$ be the cycle-star graph. It is easy to see from Figure 1 that $|V(G)| = n$ and $|E(G)| = k + n - k = n$. Since each of the vertices of G is of degree either 1 or 2 or $n - k + 2$, the vertex set of G has three partitions with respect to degree:

$V_1(G) = \{u \in V(G) : deg_G(u) = 1\}$;
$V_2(G) = \{u \in V(G) : deg_G(u) = 2\}$;
$V_3(G) = \{u \in V(G) : deg_G(u) = n - k + 2\}$.

Clearly, $|V_1(G)| = n - k; |V_2(G)| = k - 1; |V_3(G)| = 1$.

Further, the edge set of G has three partitions based on the degree of end vertices.

$E_1(G) = \{e = uv \in E(G) : deg_G(u) = 2, deg_G(v) = 2\}$;
$E_2(G) = \{e = uv \in E(G) : deg_G(u) = 2, deg_G(v) = n - k + 2\}$;
$E_3(G) = \{e = uv \in E(G) : deg_G(u) = 1, deg_G(v) = n - k + 2\}$.

Clearly, $|E_1(G)| = k - 2; |E_2(G)| = 2; |E_3(G)| = n - k$. Now, from the definition of M-polynomial,

$$\begin{aligned} M(G; x, y) &= \sum_{\delta \le i \le j \le \Delta} m_{ij}(G) x^i y^j \\ &= m_{22}(G)x^2y^2 + m_{2(n-k+2)}(G)x^2y^{n-k+2} + m_{1(n-k+2)}(G)xy^{n-k+2} \\ &= (k-2)x^2y^2 + 2x^2y^{n-k+2} + (n-k)xy^{n-k+2}. \end{aligned}$$

This completes the proof. □

We now compute some degree-based topological indices of the cycle-star graph using this M-polynomial.

Theorem 4.2 *Let $G = CS_{k,n-k}$ be the cycle-star graph. Then,*

$$\begin{aligned} M_1(G) &= n^2 + (5-2k)n + k^2 - k, \\ M_2(G) &= n^2 + (6-2k)n + k^2 - 2k, \\ {}^mM_2(G) &= \frac{(k+2)n - k^2}{4n - 4k + 8}, \\ R_\alpha(G) &= 4^\alpha(n-k+2) + (n-k)(n-k+2) + 4^\alpha(k-2), \\ RR_\alpha(G) &= \frac{1}{n-k+2} + 4^\alpha(n-4) + (k-2)n - k^2 + 4k - 4, \\ SSD(G) &= \frac{n^3 + (5-3k)n^2 + (3k^2 - 8k + 5)n - k^3 + 3k^2 - k}{n-k+2}, \\ H(G) &= \frac{2\left((k+2)n^2 + (-2k^2 + 11k - 14)n + (16-8k)n + 8n + k^3 - 5k^2 + 2k\right)}{4n^2 + (28-8k)n + 4k^2 - 28k + 48}, \\ I(G) &= \frac{n^3 + (8-2k)n^2 + (k^2 - 9k + 14)n + k^2 - 2k}{n^2 + (7-2k)n + k^2 - 7k + 12}, \\ A(G) &= \frac{n^4 + (4k+6)n^3 + (-18k^2 + 6k + 12)n^2 + (20k^3 - 30k^2 + 8)n - 7k^4 + 18k^3 - 12k^2}{n^3 + (3-3k)n^2 + (3k^2 - 6k + 3)n - k^3 + 3k^2 - 3k + 1}. \end{aligned}$$

Proof From Theorem 4.1, we have

$$M(G;x,y) = f(x,y) = (k-2)x^2y^2 + 2x^2y^{n-k+2} + (n-k)xy^{n-k+2}.$$

Then, we have the following:

$$D_x(f(x,y)) = 4x^2y^{n-k+2} + (n-k)xy^{n-k+2} + 2(k-2)x^2y^2,$$
$$D_y((f(x,y)) = 2(n-k+2)x^2y^{n-k+2} + (n-k)(n-k+2)xy^{n-k+2} + 2(k-2)x^2y^2,$$
$$(D_yD_x)(f(x,y)) = 4(n-k+2)x^2y^{n-k+2} + (n-k)(n-k+2)xy^{n-k+2} + 4(k-2)x^2y^2,$$
$$S_x(f(x,y)) = x^2y^{n-k+2} + (n-k)xy^{n-k+2} + \tfrac{1}{2}(k-2)x^2y^2,$$
$$S_y(f(x,y)) = \tfrac{1}{2n-2k+4}\left(4x^2y^{n-k+2} + (2n-2k)xy^{n-k+2} + ((k-2)n-k^2+4k-4)x^2y^2\right),$$
$$S_xS_y(f(x,y)) = \tfrac{1}{4n-4k+8}\left(4x^2y^{n-k+2} + (4n-4k)xy^{n-k+2} + ((k-2)n-k^2+4k-4)x^2y^2\right),$$
$$S_yD_x(f(x,y)) = \tfrac{1}{n-k+2}\left((4x+n-k)xy^{n-k+2} + ((k-2)n-k^2+4k-4)x^2y^2\right),$$
$$S_xD_y(f(x,y)) = (n-k+2)x^2y^{n-k+2} + \left(n^2+(2-2k)n+k^2-2k\right)xy^{n-k+2} + (k-2)x^2y^2,$$

$$\begin{aligned} 2S_xJ(f(x,y)) &= \frac{2((k-2)n^2+(-2k^2+11k-14)n+k^3-9k^2+26k-24)x^4}{4n^2+(28-8k)n+4k^2-28k+48} \\ &\quad + \frac{(8n-8k+24)x^{n+4-k}+(4n^2+(16-8k)n+4k^2-16k)x^{n+3-k}}{4n^2+(28-8k)n+4k^2-28k+48}, \end{aligned}$$

$$\begin{aligned} S_xJD_xD_y(f(x,y)) &= \frac{\left((k-2)n^2+(-2k^2+11k-14)n+k^3-9k^2+26k-24\right)x^4}{n^2+(7-2k)n+k^2-7k+12} \\ &\quad + \frac{\left(4n^2+(20-8k)n+4k^2-20k+24\right)x^{n+4-k}}{n^2+(7-2k)n+k^2-7k+12} \\ &\quad + \frac{\left(n^3+(6-3k)n^2+(3k^2-12k+8)n-k^3+6k^2-8k\right)x^{n+3-k}}{n^2+(7-2k)n+k^2-7k+12}, \end{aligned}$$

$$\begin{aligned} S_x^3Q_{-2}JD_x^3D_y^3(f(x,y)) &= \frac{\left((8k-16)n^3+(-24k^2+72k-48)n^2+(24k^3-96k^2+120k-48)n\right)x^2}{n^3+(3-3k)n^2+(3k^2-6k+3)n-k^3+3k^2-3k+1} \\ &\quad + \frac{\left(-8k^4+40k^3-72k^2+56k-16\right)x^2}{n^3+(3-3k)n^2+(3k^2-6k+3)n-k^3+3k^2-3k+1} \\ &\quad + \frac{\left(16n^3+(48-48k)n^2+(48k^2-96k+48)n\right)x^{n-k+2}}{n^3+(3-3k)n^2+(3k^2-6k+3)n-k^3+3k^2-3k+1} \\ &\quad + \frac{\left(-16k^3+48k^2-48k+16\right)x^{n-k+2}}{n^3+(3-3k)n^2+(3k^2-6k+3)n-k^3+3k^2-3k+1} \\ &\quad + \frac{\left(n^4+(6-4k)n^3+(6k^2-18k+12)n^2+(-4k^3+18k^2-24k+8)n\right)x^{n-k+1}}{n^3+(3-3k)n^2+(3k^2-6k+3)n-k^3+3k^2-3k+1} \\ &\quad + \frac{\left(k^4-6k^3+12k^2-8k\right)x^{n-k+1}}{n^3+(3-3k)n^2+(3k^2-6k+3)n-k^3+3k^2-3k+1}. \end{aligned}$$

Now, we have the following from Table 1:

1. First Zagreb index

$$M_1(G) = (D_x + D_y)(f(x,y))|_{x=y=1} = n^2 + (5-2k)n + k^2 - k.$$

2. Second Zagreb index

$$M_2(G) = (D_x D_y)(f(x,y))|_{x=y=1} = n^2 + (6-2k)n + k^2 - 2k.$$

3. Second modified Zagreb index

$$^{m}M_2(G) = (S_x S_y)(f(x,y))|_{x=y=1} = \frac{(k+2)n - k^2}{4n - 4k + 8}.$$

4. Randić index

$$R_\alpha(G) = (D_x^\alpha D_y^\alpha)(f(x,y))|_{x=y=1} = 4^\alpha(n-k+2) + (n-k)(n-k+2) + 4^\alpha(k-2).$$

5. Inverse Randić index

$$\begin{aligned} RR_\alpha(G) &= (S_x^\alpha S_y^\alpha)(f(x,y))|_{x=y=1} \\ &= \frac{1}{n-k+2} + 4^\alpha(n-4) + (k-2)n - k^2 + 4k - 4. \end{aligned}$$

6. Symmetric division index

$$\begin{aligned} SSD(G) &= (D_x S_y + D_y S_x)(f(x,y))|_{x=y=1} \\ &= \frac{n^3 + (5-3k)n^2 + (3k^2 - 8k + 5)n - k^3 + 3k^2 - k}{n-k+2}. \end{aligned}$$

7. Harmonic index

$$\begin{aligned} H(G) &= 2S_x J(f(x,y))|_{x=1} \\ &= \frac{2\left((k+2)n^2 + (-2k^2 + 11k - 14)n + (16-8k)n + 8n + k^3 - 5k^2 + 2k\right)}{4n^2 + (28-8k)n + 4k^2 - 28k + 48}. \end{aligned}$$

8. Inverse sum index

$$I(G) = S_x J D_x D_y(f(x,y))|_{x=1} = \frac{n^3 + (8-2k)n^2 + (k^2 - 9k + 14)n + k^2 - 2k}{n^2 + (7-2k)n + k^2 - 7k + 12}.$$

9. Augmented Zagreb index

$$\begin{aligned} A(G) &= S_x^3 Q_{-2} J D_x^3 D_y^3(f(x,y))|_{x=1} \\ &= \frac{n^4 + (4k+6)n^3 + (-18k^2 + 6k + 12)n^2 + (20k^3 - 30k^2 + 8)n - 7k^4 + 18k^3 - 12k^2}{n^3 + (3-3k)n^2 + (3k^2 - 6k + 3)n - k^3 + 3k^2 - 3k + 1}. \end{aligned}$$

This completes the proof. □

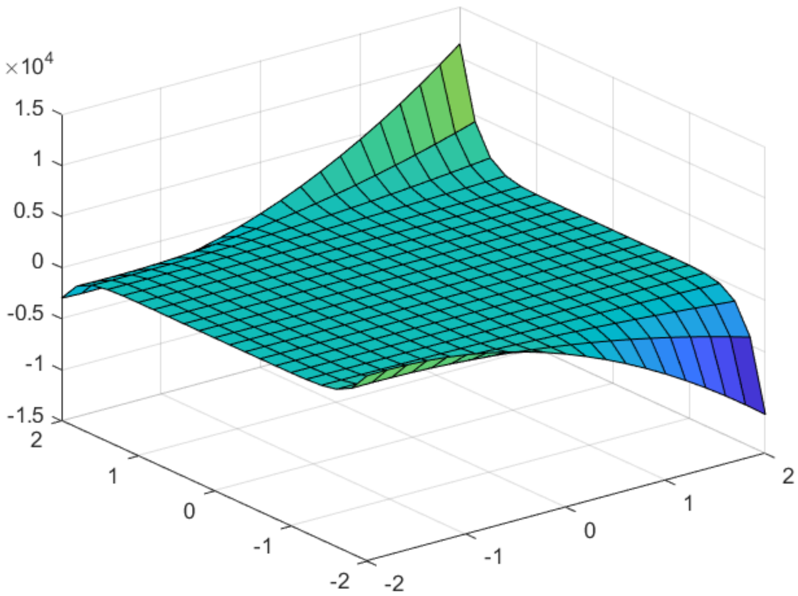

Figure 2. Plot of M-polynomial of the cycle-star graph $CS_{7,7}$

§5. M-Polynomial of Line Graph of Cycle-Star Graph $CS_{k,n-k}$

There are many graph operators with which one can construct a new graph from a given graph, such as the line graphs, total graphs, middle graphs, and their generalizations.

Definition 5.1 *A line graph of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G have a vertex in common.*

In the next Theorem, we find the M-polynomial of the line graph of cycle-star graph.

Theorem 5.1 *Let $G = CS_{k,n-k}$ be the cycle-star graph. Then the M-polynomial of $L(G)$ is*

$$\begin{aligned} M(L(G);x,y) &= (k-3)x^2y^2 + 2x^2y^{n-k+2} + x^{n-k+2}y^{n-k+2} + 2(n-k)x^{n-k+1}y^{n-k+2} \\ &\quad + \binom{n-k}{2} x^{n-k+1}y^{n-k+1}. \end{aligned}$$

Proof Let $G = CS_{k,n-k}$ be the cycle-star graph. Then, $|V(L(G))| = n$ and $|E(L(G))| = \frac{1}{2}\left(n^2 + k^2 - 2nk + 3n - k\right)$. Since each of the vertices of $L(G)$ is of degree either 2 or $n-k+1$

or $n-k+2$, the vertex set of $L(G)$ has three partitions with respect to degree:

$$\begin{aligned}
V_1(L(G)) &= \{u \in V(L(G)) : deg_{L(G)}(u) = 2\};\\
V_2(L(G)) &= \{u \in V(L(G)) : deg_{L(G)}(u) = n-k+1\};\\
V_3(L(G)) &= \{u \in V(L(G)) : deg_{L(G)}(u) = n-k+2\}.
\end{aligned}$$

Clearly, $|V_1(L(G))| = k-2; |V_2(L(G))| = n-k; |V_3(L(G))| = 2$.

Furthermore, the edge set of $L(G)$ has five partitions based on the degree of the end vertices.

$$\begin{aligned}
E_1(L(G)) &= \{e = uv \in E(L(G)) : deg_{L(G)}(u) = 2, deg_{L(G)}(v) = 2\};\\
E_2(L(G)) &= \{e = uv \in E(L(G)) : deg_{L(G)}(u) = 2, deg_{L(G)}(v) = n-k+2\};\\
E_3(L(G)) &= \{e = uv \in E(L(G)) : deg_{L(G)}(u) = n-k+2, deg_{L(G)}(v) = n-k+2\};\\
E_4(L(G)) &= \{e = uv \in E(L(G)) : deg_{L(G)}(u) = n-k+1, deg_{L(G)}(v) = n-k+2\};\\
E_5(L(G)) &= \{e = uv \in E(L(G)) : deg_{L(G)}(u) = n-k+1, deg_{L(G)}(v) = n-k+1\}.
\end{aligned}$$

Clearly,

$$|E_1(L(G))| = k-3,\ |E_2(L(G))| = 2,\ |E_3(L(G))| = 1,$$

$$|E_4(L(G))| = 2(n-k) \text{ and } |E_5(L(G))| = \binom{n-k}{2}.$$

Now, from the definition of M-polynomial,

$$\begin{aligned}
M(L(G); x, y) &= \sum_{\delta \le i \le j \le \Delta} m_{ij}(G) x^i y^j = m_{22}(L(G)) x^2 y^2\\
&\quad + m_{2(n-k+2)}(L(G)) x^2 y^{n-k+2} + m_{(n-k+2)(n-k+2)}(L(G)) x^{n-k+2} y^{n-k+2}\\
&\quad + m_{(n-k+1)(n-k+2)}(G) x^{n-k+1} y^{n-k+2}\\
&\quad + m_{(n-k+1)(n-k+1)}(L(G)) x^{n-k+1} y^{n-k+1}\\
&= (k-3) x^2 y^2 + 2x^2 y^{n-k+2} + x^{n-k+2} y^{n-k+2}\\
&\quad + 2(n-k) x^{n-k+1} y^{n-k+2} + \binom{n-k}{2} x^{n-k+1} y^{n-k+1}
\end{aligned}$$

This completes the proof. □

We now compute some degree-based topological indices of the line graph of cycle-star graph using this M-polynomial.

Theorem 5.2 *Let $G = CS_{k,n-k}$ be the cycle-star graph. Then,*

$$\begin{aligned} M_1(L(G)) &= 4n^2 + (-8k + 2a + 10)n + 4k^2 + (-2a - 6)k + 2a, \\ M_2(L(G)) &= 2n^3 + (-6k + a + 7)n^2 + (6k^2 + (-2a - 14)k + 2a + 12)n - 2k^3 \\ &\quad +(a + 7)k^2 + (-2a - 8)k + a, \\ {}^{m}M_2(G) &= \frac{A_1}{B_1} + \frac{A_2}{B_2}, \\ R_\alpha(G) &= 4^\alpha(n - k + 2) + (n - k)(n - k + 2) + 4^\alpha(k - 2), \\ RR_\alpha(G) &= \frac{1}{n - k + 2} + 4^\alpha(n - 4) + (k - 2)n - k^2 + 4k - 4, \\ SSD(G) &= \frac{C_1}{D_1}, \\ H(G) &= \frac{E_1}{F_1} + \frac{E_2}{F_2}, \\ I(G) &= \frac{G_1}{H_1} + \frac{G_2}{H_2}, \end{aligned}$$

where, $A_1 = (k-3)n^4 + (-4k^2 + 18k - 6)n^3 + (6k^3 - 36k^2 + 31k + 4a + 5)n^2$, $B_1 = 4n^4 + (24 - 16k)n^3 + (24k^2 - 72k + 52)n^2 + (-16k^3 + 72k^2 - 104k + 48)n + 4k^4 - 24k^3 + 52k^2 - 48k + 16$, $A_2 = (-4k^4 + 30k^3 - 44k^2 + (2 - 8a)k + 16a + 8)n + k^5 - 9k^4 + 19k^3 + (4a - 7)k^2 + (-16a - 4)k + 16a$, $B_2 = 4n^4 + (24 - 16k)n^3 + (24k^2 - 72k + 52)n^2 + (-16k^3 + 72k^2 - 104k + 48)n + 4k^4 - 24k^3 + 52k^2 - 48k + 16$, $C_1 = 5n^3 + (-13k + 2a + 13)n^2 + (11k^2 + (-4a - 20)k + 6a + 10)n - 3k^3 + (2a + 7)k^2 + (-6a - 6)k + 4a$, $D_1 = n^2 + (3 - 2k)n + k^2 - 3k + 2$, $E_1 = (2k + 2)n^4 + (-8k^2 + 9k + 4a + 25)n^3 + (12k^3 - 39k^2 + (-12a - 26)k + 30a + 63)n^2$, $F_1 = 4n^4 + (34 - 16k)n^3 + (24k^2 - 102k + 98)n^2 + (-16k^3 + 102k^2 - 196k + 116)n + 4k^4 - 34k^3 + 98k^2 - 116k + 48$, $E_2 = (-8k^4 + 43k^3 + (12a - 23)k^2 + (-60a - 68)k + 68a + 40)n + 2k^5 - 15k^4 + (24 - 4a)k^3 + (30a + 5)k^2 + (-68a - 16)k + 48a$, $F_2 = 4n^4 + (34 - 16k)n^3 + (24k^2 - 102k + 98)n^2 + (-16k^3 + 102k^2 - 196k + 116)n + 4k^4 - 34k^3 + 98k^2 - 116k + 48$, $G_1 = 4n^4 + (-16k + 2a + 30)n^3 + (24k^2 + (-6a - 86)k + 13a + 75)n^2$, $H_1 = 4n^2 + (22 - 8k)n + 4k^2 - 22k + 24$, $G_2 = (-16k^3 + (6a + 82)k^2 + (-26a - 128)k + 23a + 56)n + 4k^4 + (-2a - 26)k^3 + (13a + 53)k^2 + (-23a - 32)k + 12a$, $H_2 = 4n^2 + (22 - 8k)n + 4k^2 - 22k + 24$.

Proof From Theorem 5.1, we have

$$\begin{aligned} M(L(G); x, y) &= (k - 3)x^2y^2 + 2x^2y^{n-k+2} + x^{n-k+2}y^{n-k+2} \\ &\quad +2(n - k)x^{n-k+1}y^{n-k+2} + ax^{n-k+1}y^{n-k+1}, \end{aligned}$$

where $a = \binom{n-k}{2}$. Then, we have the following:

$$\begin{aligned} D_x(f(x, y)) &= (n - k + 2)x^{n-k+2}y^{n-k+2} + 2(n - k)(n - k + 1)x^{n-k+1}y^{n-k+2} \\ &\quad +4x^2y^{n-k+2} + a(n - k + 1)x^{n-k+1}y^{n-k+1} + 2(k - 3)x^2y^2; \\ D_y((f(x, y)) &= (n - k + 2)x^{n-k+2}y^{n-k+2} + 2(n - k)(n - k + 2)x^{n-k+1}y^{n-k+2} \\ &\quad +2(n - k + 2)x^2y^{n-k+2} + a(n - k + 1)x^{n-k+1}y^{n-k+1} + 2(k - 3)x^2y^2; \end{aligned}$$

$$
\begin{aligned}
(D_yD_x)(f(x,y)) &= (n-k+2)^2x^{n-k+2}y^{n-k+2} \\
&+2(n-k)(n-k+1)(n-k+2)x^{n-k+1}y^{n-k+2} \\
&+4(n-k+2)x^2y^{n-k+2}+a(n-k+1)^2x^{n-k+1}y^{n-k+1}+4(k-3)x^2y^2; \\
S_x(f(x,y)) &= \frac{(k-3)x^2y^2}{2}+x^2y^{n-k+2}+\frac{x^{n-k+2}y^{n-k+2}}{n-k+2} \\
&+\frac{2(n-k)x^{n-k+1}y^{n-k+2}}{n-k+1}+a\frac{x^{n-k+1}y^{n-k+1}}{n-k+1};
\end{aligned}
$$

$$
\begin{aligned}
S_y(f(x,y)) &= \frac{(k-3)x^2y^2}{2}+\frac{2x^2y^{n-k+2}}{n-k+2}+\frac{x^{n-k+2}y^{n-k+2}}{n-k+2} \\
&+\frac{2(n-k)x^{n-k+1}y^{n-k+2}}{n-k+2}+a\frac{x^{n-k+1}y^{n-k+1}}{n-k+1}; \\
S_xS_y(f(x,y)) &= \frac{(k-3)x^2y^2}{4}+\frac{x^2y^{n-k+2}}{n-k+2}+\frac{x^{n-k+2}y^{n-k+2}}{(n-k+2)^2} \\
&+\frac{2(n-k)x^{n-k+1}y^{n-k+2}}{(n-k+1)(n-k+2)}+a\frac{x^{n-k+1}y^{n-k+1}}{(n-k+1)^2}; \\
S_yD_x(f(x,y)) &= \frac{I}{(n-k+2)y^k}; \\
S_xD_y(f(x,y)) &= \frac{((k-3)n-k^2+4k-3)x^{k+2}y^2}{(n-k+1)x^2}+\frac{J}{(n-k+1)x^2};
\end{aligned}
$$

where, $I = x^{1-k}((((k-3)n-k^2+5k-6)x^{k+1}y^{k+2}+y^n((4x^{k+1}+x^n((n-k+2)x+2n^2+(2-4k)n+2k^2-2k))y^2+(an-ak+2a)x^ny))$, $J = y^n((((n^2+(3-2k)n+k^2-3k+2)x^{k+2}+x^n((n-k+1)x^2+(2n^2+(4-4k)n+2k^2-4k)x))y+(an-ak+a)x^{n+1})$,

$$
2S_xJ(f(x,y)) = \frac{K_1}{L_1}+\frac{K_2}{L_2}+\frac{K_3}{L_3}+\frac{K_4}{L_4}+\frac{K_5}{L_5}+\frac{K_6}{L_6}+\frac{K_7}{L_7},
$$

where, $K_1 = 2((16n^3+(72-48k)n^2+(48k^2-144k+104)n-16k^3+72k^2-104k+48)x^{n+k+4}$, $L_1 = (8n^4+\alpha n^3+\beta n^2+\gamma n+8k^4-68k^3+196k^2-232k+96)x^{2k}$, $K_2 = ((2k-6)n^4+(-8k^2+41k-51)n^3+(12k^3-87k^2+202k-147)n^2)x^{2k+4}$, $L_2 = (8n^4+\alpha n^3+\beta n^2+\gamma n+8k^4-68k^3+196k^2-232k+96)x^{2k}$, $K_3 = ((-8k^4+75k^3-251k^2+352k-174)n+2k^5-23k^4+100k^3-205k^2+198k-72)x^{2k+4})$, $L_3 = (8n^4+\alpha n^3+\beta n^2+\gamma n+8k^4-68k^3+196k^2-232k+96)x^{2k}$, $K_4 = x^{2n}((4n^3+(26-12k)n^2+(12k^2-52k+46)n-4k^3+26k^2-46k+24)x^4$, $L_4 = (8n^4+\alpha n^3+\beta n^2+\gamma n+8k^4-68k^3+196k^2-232k+96)x^{2k}$, $K_5 = (8n^4+(56-32k)n^3+(48k^2-168k+112)n^2+(-32k^3+168k^2-224k+64)n)x^3$, $L_5 = (8n^4+\alpha n^3+\beta n^2+\gamma n+8k^4-68k^3+196k^2-232k+96)x^{2k}$, $K_6 = (8k^4-56k^3+112k^2-64k)x^3$, $L_6 = (\alpha n^3+\beta n^2+\gamma n+8k^4-68k^3+196k^2-232k+96)x^{2k}$, $K_7 = (4an^3+(30a-12ak)n^2+(12ak^2-60ak+68a)n-4ak^3+30ak^2-68ak+48a)x^2)$, $L_7 = (8n^4+\alpha n^3+\beta n^2+\gamma n+8k^4-68k^3+196k^2-232k+96)x^{2k}$, $\alpha = 68-32k, \beta =$

$48k^2 - 204k + 196, \gamma = -32k^3 + 204k^2 - 392k + 232.$

$$\begin{aligned} S_xJD_xD_y(f(x,y)) &= \frac{(16n^2 + (56-32k)n + 16k^2 - 56k + 48)x^{n+k+4}}{(4n^2 + (22-8k)n + 4k^2 - 22k + 24)x^{2k}} \\ &+ \frac{M_1}{N_1} + \frac{M_2}{N_2} + \frac{M_3}{N_3} + \frac{M_4}{N_4} + \frac{M_5}{N_5}, \end{aligned}$$

where, $M_1 = ((4k-12)n^2 + (-8k^2 + 46k - 66)n + 4k^3 - 34k^2 + 90k - 72)x^{2k+4}$, $N_1 = (4n^2 + (22-8k)n + 4k^2 - 22k + 24)x^{2k}$, $M_2 = x^{2n}((2n^3 + (15-6k)n^2 + (6k^2 - 30k + 34)n - 2k^3 + 15k^2 - 34k + 24)x^4$, $N_2 = (4n^2 + (22-8k)n + 4k^2 - 22k + 24)x^{2k}$, $M_3 = 4n^4 + (28-16k)n^3 + (24k^2 - 84k + 56)n^2 + (-16k^3 + 84k^2 - 112k + 32)n$, $N_3 = (4n^2 + (22-8k)n + 4k^2 - 22k + 24)x^{2k}$, $M_4 = 4k^4 - 28k^3 + 56k^2 - 32k$, $N_4 = (4n^2 + (22-8k)n + 4k^2 - 22k + 24)x^{2k}$, $M_5 = (2an^3 + (13a - 6ak)n^2 + (6ak^2 - 26ak + 23a)n - 2ak^3 + 13ak^2 - 23ak + 12a)x^2)$, $N_5 = (4n^2 + (22-8k)n + 4k^2 - 22k + 24)x^{2k}$.

Now, we have the following from Table 1:

1. First Zagreb index

$$M_1(L(G)) = (D_x + D_y)(f(x,y))|_{x=y=1} = 4n^2 + (-8k + 2a + 10)n + 4k^2 + (-2a - 6)k + 2a.$$

2. Second Zagreb index

$$\begin{aligned} M_2(L(G)) &= (D_xD_y)(f(x,y))|_{x=y=1} = 2n^3 + (-6k + a + 7)n^2 \\ &+ (6k^2 + (-2a - 14)k + 2a + 12)n - 2k^3 + (a+7)k^2 + (-2a - 8)k + a. \end{aligned}$$

3. Second modified Zagreb index

$$^{m}M_2(L(G)) = (S_xS_y)(f(x,y))|_{x=y=1} = \frac{O_1}{P_1} + \frac{O_2}{P_2},$$

where, $O_1 = (k-3)n^4 + (-4k^2 + 18k - 6)n^3 + (6k^3 - 36k^2 + 31k + 4a + 5)n^2$, $P_1 = 4n^4 + (24 - 16k)n^3 + (24k^2 - 72k + 52)n^2 + (-16k^3 + 72k^2 - 104k + 48)n + 4k^4 - 24k^3 + 52k^2 - 48k + 16$, $O_2 = (-4k^4 + 30k^3 - 44k^2 + (2-8a)k + 16a + 8)n + k^5 - 9k^4 + 19k^3 + (4a-7)k^2 + (-16a-4)k + 16a$, $P_2 = 4n^4 + (24-16k)n^3 + (24k^2 - 72k + 52)n^2 + (-16k^3 + 72k^2 - 104k + 48)n + 4k^4 - 24k^3 + 52k^2 - 48k + 16$.

4. Randić index

$$\begin{aligned} R_\alpha(L(G)) &= (D_x^\alpha D_y^\alpha)(f(x,y))|_{x=y=1} = (n-k+2)^2 + 2^\alpha(n-k)(n-k+1)(n-k+2) \\ &+ 4^\alpha(n-k+2) + a(n-k+1)^2 + 4^\alpha(k-3). \end{aligned}$$

5. Inverse Randić index

$$\begin{aligned} RR_\alpha(L(G)) &= (S_x^\alpha S_y^\alpha)(f(x,y))|_{x=y=1} = \frac{k-3}{4^\alpha} + \frac{1}{n-k+2} + \frac{1}{(n-k+2)^2} \\ &+ \frac{2^\alpha(n-k)}{(n-k+1)(n-k+2)} + a\frac{1}{(n-k+1)^2}. \end{aligned}$$

6. Symmetric division index

$$\begin{aligned} SSD(L(G)) &= (D_xS_y + D_yS_x)(f(x,y))|_{x=y=1} \\ &= \frac{Q}{n^2+(3-2k)n+k^2-3k+2}, \end{aligned}$$

where $Q = 5n^3 + (-13k+2a+13)n^2 + (11k^2 + (-4a-20)k + 6a + 10)n - 3k^3 + (2a+7)k^2 + (-6a-6)k + 4a$.

7. Harmonic index

$$H(L(G)) = 2S_xJ(f(x,y))|_{x=1} = \frac{R_1}{S_1} + \frac{R_2}{S_2},$$

where, $R_1 = (2k+2)n^4 + (-8k^2+9k+4a+25)n^3 + (12k^3-39k^2+(-12a-26)k+30a+63)n^2$, $S_1 = 4n^4 + (34-16k)n^3 + (24k^2-102k+98)n^2 + (-16k^3+102k^2-196k+116)n + 4k^4 - 34k^3 + 98k^2 - 116k + 48$, $R_2 = (-8k^4 + 43k^3 + (12a-23)k^2 + (-60a-68)k + 68a + 40)n + 2k^5 - 15k^4 + (24-4a)k^3 + (30a+5)k^2 + (-68a-16)k + 48a$, $S_2 = 4n^4 + (34-16k)n^3 + (24k^2 - 102k + 98)n^2 + (-16k^3 + 102k^2 - 196k + 116)n + 4k^4 - 34k^3 + 98k^2 - 116k + 48$.

8. Inverse sum index

$$I(L(G)) = S_xJD_xD_y(f(x,y))|_{x=1} = \frac{U_1}{V_1} + \frac{U_2}{V_2},$$

where, $U_1 = 4n^4 + (-16k+2a+30)n^3 + (24k^2 + (-6a-86)k + 13a + 75)n^2$, $V_1 = 4n^2 + (22-8k)n + 4k^2 - 22k + 24$, $U_2 = (-16k^3 + (6a+82)k^2 + (-26a-128)k + 23a + 56)n + 4k^4 + (-2a-26)k^3 + (13a+53)k^2 + (-23a-32)k + 12a$, $V_2 = 4n^2 + (22-8k)n + 4k^2 - 22k + 24$.

This completes the proof. □

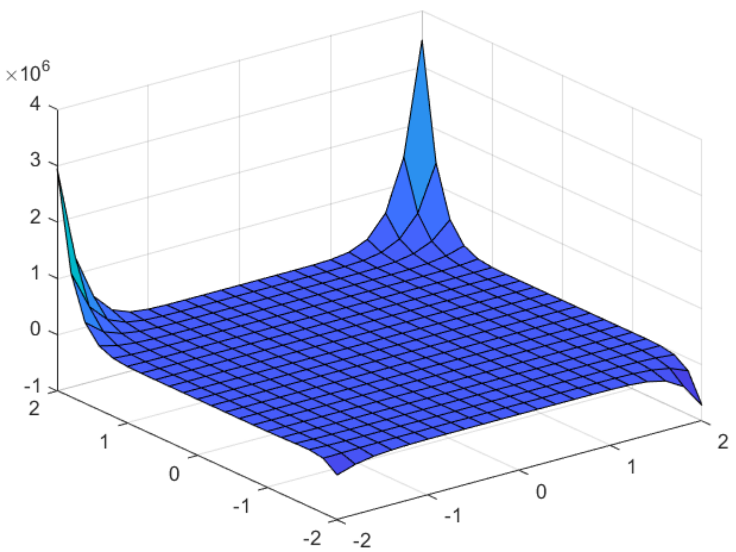

Figure 3. Plot of M-polynomial of the line graph of cycle-star graph $CS_{7,7}$

§6. Conclusion

Topological indices play an important role in understanding many physical and chemical properties of a chemical compound. Some of the degree-based topological indices can be found by means of the M-polynomial of the corresponding chemical graph. In this paper, we have determined some of these topological indices using the closed form of the M-polynomial of cycle-star graph and the line graph of cycle-star graph. The study on M-polynomials with respect to different types of graph operators seem to be much promising.

References

[1] Bruckler, F, M., Došlić, T., Graovac. A., & Gutman, I. (2011), On a class of distance - based molecular structure descriptors, *Chemical Physics Letters*, 503, (4-6), 336-338.

[2] Deutsch, E., & Klavźar, S. (2015), M-polynomial and degree-based topological indices, *Iranian Journal of Mathematical Chemistry*, 6(2), 93-102.

[3] Das, K. C., & Trinajstić, N. (2010), Comparison between first geometric - arithmetic index and atom-bond connectivity index, *Chem. Phys. Lett*, 497, 149-151.

[4] Fath-Tabar, G., Furtula, B., & Gutman, I. (2010), A new geometric-arithmetic index, *J. Math. Chem*, 47, 477-486.

[5] Harary, F. (1969), *Graph Theory*, Addison Wesley, Reading.

[6] Hosoya, H. (1971), Topological index, a newly proposed quantity characterizing the topological nature of structure isomers of saturated hydrocarbons, *Bull. Chem. Soc. Jpn*, 44, 2332-2339.

[7] Khalaf, A, J, M., Hussain, S., Afzal, D., Afzal, F., & Maqbool, A. (2020), M-polynomial and topological indices of book graph, *Journal of Discrete Mathematical Sciences and Cryptography*, 23(6), 1217-1237.

[8] Kwun, Y. C., Munir, M., Nazeer, W., Rafique, S., & Min Kang, S. (2017), M-Polynomials and topological indices of V-Phenylenic nanotubes and nanotori, *Scientific reports*, 7(1), 1-9.

[9] Munir, M., Nazeer, W., Rafique, S., & Kang, S. M. (2016), M-polynomial and degree-based topological indices of polyhex nanotubes, *Symmetry*, 8(12), 149.

[10] Munir, M., Nazeer, W., Nizami, A. R., Rafique, S., & Kang, S. M. (2016), M-polynomials and topological indices of titania nanotubes. *Symmetry*, 8(11), 117.

[11] Swamy, N. N., Gangappa, C. K., Poojary, P., Sooryanarayana, B., & Nagesh, H. M. (2022), Topological indices of the subdivision graphs of the nanostructure $TUC_4C_8(R)$ using M-polynomials, *Journal of Discrete Mathematical Sciences and Cryptography*, 25(1), 265-282.

[12] Sedlar, J. (2013), Extremal unicyclic graphs with respect to additively weighted Harary index, *Miskolic mathematical Notes*, 16(2), 1-16.

[13] Wiener, H. (1947), Structural determination of the paraffin boiling points, *J. Am. Chem. Soc*, 69, 17-20.