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Pair Difference Cordial Labeling of Subdivision of Wheel and Comb Graphs

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Abstract: In this paper, we discuss about the pair difference cordial labeling behavior of subdivision of wheel and comb graphs.

Key Words: Pair difference cordial labeling, pair difference cordial graph, Smarandachely pair difference cordial labeling, Smarandachely pair difference cordial graph, cycle, path, wheel, subdivision.

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§1. Introduction

We consider only finite, undirected and simple graphs. The notion of pair difference cordial labeling of graphs was introduced in [4]. Pair difference cordial labeling behaviour of several graphs like path, cycle, star, Mirror graph, Shadow graph,double fan, mangolian tent, grid etc have been investigated in [4-10]. In this we investigate the pair difference cordial labeling behaviour of subdivision of wheel and comb graphs. Terms not defined here follow from Harary [2,3].

§2. Preliminaries

Definition 2.1([7]) *A subdivision graph $S(G)$ of a graph G is obtained by replacing each edge uv by a path uvw.*

Definition 2.2([4]) *Let $G = (V, E)$ be a (p, q) graph. Define*

$$\rho = \begin{cases} \frac{p}{2}, & \text{if } p \text{ is even} \\ \frac{p-1}{2}, & \text{if } p \text{ is odd} \end{cases}$$

and $L = \{\pm 1, \pm 2, \pm 3, \cdots, \pm \rho\}$ called the set of labels. Consider a mapping $f : V \longrightarrow L$

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by assigning different labels in L to the different elements of V when p is even and different labels in L to p-1 elements of V and repeating a label for the remaining one vertex when p is odd.The labeling as defined above is said to be a pair difference cordial labeling if for each edge uv of G there exists a labeling $|f(u) - f(v)|$ such that $\left|\Delta_{f_1} - \Delta_{f_1^c}\right| \leq 1$. Otherwise, it is called a Smarandachely pair difference cordial labeling if $\left|\Delta_{f_1} - \Delta_{f_1^c}\right| \geq 2$, where Δ_{f_1} and $\Delta_{f_1^c}$ respectively denote the number of edges labeled with 1 and number of edges not labeled with 1.

A graph G for which there exists a pair difference cordial labeling or Smarandachely pair difference cordial labeling is called a pair difference cordial graph or Smarandachely pair difference cordial graph.

Theorem 2.3([7]) *A wheel W_n is pair difference cordial if and only if n is even.*

§3. Main Results

Theorem 3.1 *A subdivision of the wheel $W_n, S(W_n)$ is pair difference cordial for all values of $n \geq 3$.*

Proof Let us take the vertex set and edge set of $S(W_n)$ as follows: $V(S(W_n)) = \{a, a_i, b_i, u_i : 1 \leq i \leq n\}$ and $E(S(W_n)) = \{aa_i, a_ib_i, b_iu_i : 1 \leq i \leq n\} \cup \{u_ib_{i+1} : 1 \leq i \leq n-1\} \cup \{b_nu_1\}$. This graph has $3n + 1$ vertices and $4n$ edges.

Case 1. n is even.

Assign the labels $1, 4, 7, \cdots, \frac{3n-4}{2}$ respectively to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n}{2}}$ and assign the labels $2, 5, 8, \cdots, \frac{3n-2}{2}$ to the vertices $b_1, b_2, b_3, \cdots, b_{\frac{n}{2}}$ respectively. Next assign the labels $-1, -4, -7, \cdots, -(\frac{3n-4}{2})$ to the vertices $a_{\frac{n+2}{2}}, a_{\frac{n+4}{2}}, a_{\frac{n+6}{2}}, \cdots, a_n$ respectively and assign the labels $-2, -5, -8, \cdots, -(\frac{3n-2}{2})$ to the vertices $b_{\frac{n+2}{2}}, b_{\frac{n+4}{2}}, b_{\frac{n+6}{2}}, \cdots, b_n$. Now we assign the labels $3, 6, 9, \cdots, \frac{3n}{2}$ respectively to the vertices $u_1, u_2, u_3, \cdots, u_{\frac{n}{2}}$ and assign the labels $-3, -6, -9, \cdots, -(\frac{3n}{2})$ to the vertices $u_{\frac{n+2}{2}}, u_{\frac{n+4}{2}}, u_{\frac{n+6}{2}}, \cdots, u_n$ respectively. Finally assign the label 1 to the vertex a. Clearly in this case $\Delta_{f_1^c} = \Delta_{f_1} = 2n$.

Case 2. n is odd.

Assign the labels $1, 4, 7, \cdots, \frac{3n-7}{2}$ respectively to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n-1}{2}}$ and assign the labels $2, 5, 8, \cdots, \frac{3n-5}{2}$ to the vertices $b_1, b_2, b_3, \cdots, b_{\frac{n-1}{2}}$ respectively. Next assign the labels $-1, -4, -7, \cdots, -(\frac{3n-7}{2})$ to the vertices $a_{\frac{n+1}{2}}, a_{\frac{n+3}{2}}, a_{\frac{n+5}{2}}, \cdots, a_{n-1}$ respectively and assign the labels $-2, -5, -8, \cdots, -(\frac{3n-5}{2})$ to the vertices $b_{\frac{n+1}{2}}, b_{\frac{n+3}{2}}, b_{\frac{n+5}{2}}, \cdots, b_{n-1}$. Now we assign the labels $3, 6, 9, \cdots, \frac{3n-3}{2}$ respectively to the vertices $u_1, u_2, u_3, \cdots, u_{\frac{n-1}{2}}$ and assign the labels $-3, -6, -9, \cdots, -(\frac{3n-3}{2})$ to the vertices $u_{\frac{n+1}{2}}, u_{\frac{n+3}{2}}, u_{\frac{n+5}{2}}, \cdots, u_{n-1}$ respectively. Finally assign the label $\frac{3n-1}{2}, -(\frac{3n+1}{2}), -(\frac{3n-1}{2}), \frac{3n+1}{2}$ to the vertices a, a_n, b_n, u_n . Clearly in this case $\Delta_{f_1^c} = \Delta_{f_1} = 2n$. □

Theorem 3.2 *A subdivision of the spokes of wheel W_n is pair difference cordial for all values of $n \geq 3$.*

Proof Let G_s be the subdivision of the spokes of the wheel W_n with the vertex set $V(G_s) =$

$\{a, a_i, b_i : 1 \leq i \leq n\}$ and edge set $E(G_s) = \{aa_i, a_ib_i : 1 \leq i \leq n\} \cup \{b_ib_{i+1} : 1 \leq i \leq n-1\} \cup \{b_na_1\}$. Here the graph G_s has $2n+1$ vertices and $3n$ edges.

Case 1. $n \equiv 0(mod4)$.

Assign the labels $2, 6, 10, \cdots, n-2$ respectively to the vertices $a_1, a_3, a_5, \cdots, a_{\frac{n-2}{2}}$ and assign the labels $5, 9, 13, \cdots, n-3$ to the vertices $a_2, a_4, a_6, \cdots, a_{\frac{n-4}{2}}$ respectively. Next assign the labels $-1, -5, -9, \cdots, -(n-3)$ to the vertices $a_{\frac{n+2}{2}}, a_{\frac{n+6}{2}}, a_{\frac{n+10}{2}}, \cdots, a_{n-1}$ respectively and assign the labels $-4, -8, -12, \cdots, -n$ to the vertices $a_{\frac{n+4}{2}}, a_{\frac{n+8}{2}}, a_{\frac{n+12}{2}}, \cdots, a_n$. Now we assign the labels $3, 7, 11, \cdots, n-1$ respectively to the vertices $b_1, b_3, b_5, \cdots, b_{\frac{n-2}{2}}$ and assign the labels $4, 8, 12, \cdots, n$ to the vertices $b_2, b_4, b_6, \cdots, b_{\frac{n-4}{2}}$ respectively. Next assign the labels $-2, -6, -10, \cdots, -(n-2)$ to the vertices $b_{\frac{n+2}{2}}, b_{\frac{n+6}{2}}, b_{\frac{n+10}{2}}, \cdots, b_{n-1}$ respectively and assign the labels $-3, -7, -11, \cdots, -(n-1)$ to the vertices $b_{\frac{n+4}{2}}, b_{\frac{n+8}{2}}, b_{\frac{n+12}{2}}, \cdots, b_n$. Finally assign the labels $1, n-1, n$ to the vertices $a, a_{\frac{n}{2}}, b_{\frac{n}{2}}$.

Case 2. $n \equiv 1(mod4)$.

Assign the labels $2, 6, 10, \cdots, n-3$ respectively to the vertices $a_1, a_3, a_5, \cdots, a_{\frac{n-3}{2}}$ and assign the labels $5, 9, 13, \cdots, n$ to the vertices $a_2, a_4, a_6, \cdots, a_{\frac{n-1}{2}}$ respectively. Next assign the labels $-1, -5, -9, \cdots, -(n-4)$ to the vertices $a_{\frac{n+1}{2}}, a_{\frac{n+5}{2}}, a_{\frac{n+9}{2}}, \cdots, a_{n-2}$ respectively and assign the labels $-4, -8, -12, \cdots, -(n-1)$ to the vertices $a_{\frac{n+3}{2}}, a_{\frac{n+7}{2}}, a_{\frac{n+11}{2}}, \cdots, a_{n-1}$. Now we assign the labels $3, 7, 11, \cdots, n-2$ respectively to the vertices $b_1, b_3, b_5, \cdots, b_{\frac{n-3}{2}}$ and assign the labels $4, 8, 12, \cdots, n-1$ to the vertices $b_2, b_4, b_6, \cdots, b_{\frac{n-1}{2}}$ respectively. Next assign the labels $-2, -6, -10, \cdots, -(n-3)$ to the vertices $b_{\frac{n+1}{2}}, b_{\frac{n+5}{2}}, b_{\frac{n+9}{2}}, \cdots, b_{n-2}$ respectively and assign the labels $-3, -7, -11, \cdots, -(n-2)$ to the vertices $b_{\frac{n+3}{2}}, b_{\frac{n+7}{2}}, b_{\frac{n+11}{2}}, \cdots, b_{n-1}$. Finally assign the labels $1, -(n-1), -n$ to the vertices a, a_n, b_n.

Case 3. $n \equiv 2(mod4)$.

Assign the labels $2, 6, 10, \cdots, n-4$ respectively to the vertices $a_1, a_3, a_5, \cdots, a_{\frac{n-4}{2}}$ and assign the labels $5, 9, 13, \cdots, n-1$ to the vertices $a_2, a_4, a_6, \cdots, a_{\frac{n-2}{2}}$ respectively. Next assign the labels $-1, -5, -9, \cdots, -(n-1)$ to the vertices $a_{\frac{n+2}{2}}, a_{\frac{n+6}{2}}, a_{\frac{n+10}{2}}, \cdots, a_n$ respectively and assign the labels $-4, -8, -12, \cdots, -(n-2)$ to the vertices $a_{\frac{n+4}{2}}, a_{\frac{n+8}{2}}, a_{\frac{n+12}{2}}, \cdots, a_{n-1}$. Now we assign the labels $3, 7, 11, \cdots, n-3$ respectively to the vertices $b_1, b_3, b_5, \cdots, b_{\frac{n-4}{2}}$ and assign the labels $4, 8, 12, \cdots, n-2$ to the vertices $b_2, b_4, b_6, \cdots, b_{\frac{n-2}{2}}$ respectively. Next assign the labels $-2, -6, -10, \cdots, -n$ to the vertices $b_{\frac{n+2}{2}}, b_{\frac{n+6}{2}}, b_{\frac{n+10}{2}}, \cdots, b_n$ respectively and assign the labels $-3, -7, -11, \cdots, -(n-3)$ to the vertices $b_{\frac{n+4}{2}}, b_{\frac{n+8}{2}}, b_{\frac{n+12}{2}}, \cdots, b_{n-1}$. Finally assign the labels $1, n-1, n$ to the vertices $a, a_{\frac{n}{2}}, b_{\frac{n}{2}}$.

Case 4. $n \equiv 3(mod4)$.

Assign the labels $2, 6, 10, \cdots, n-5$ respectively to the vertices $a_1, a_3, a_5, \cdots, a_{\frac{n-5}{2}}$ and assign the labels $5, 9, 13, \cdots, n-2$ to the vertices $a_2, a_4, a_6, \cdots, a_{\frac{n-3}{2}}$ respectively. Next assign the labels $-1, -5, -9, \cdots, -(n-2)$ to the vertices $a_{\frac{n+1}{2}}, a_{\frac{n+5}{2}}, a_{\frac{n+9}{2}}, \cdots, a_{n-1}$ respectively and assign the labels $-4, -8, -12, \cdots, -(n-3)$ to the vertices $a_{\frac{n+3}{2}}, a_{\frac{n+7}{2}}, a_{\frac{n+11}{2}}, \cdots, a_{n-2}$. Now we assign the labels $3, 7, 11, \cdots, n-4$ respectively to the vertices $b_1, b_3, b_5, \cdots, b_{\frac{n-5}{2}}$ and assign the labels $4, 8, 12, \cdots, n-3$ to the vertices $b_2, b_4, b_6, \cdots, b_{\frac{n-3}{2}}$ respectively. Next assign the labels

$-2, -6, -10, \cdots, -(n-1)$ to the vertices $b_{\frac{n+1}{2}}, b_{\frac{n+5}{2}}, b_{\frac{n+9}{2}}, \cdots, b_{n-1}$ respectively and assign the labels $-3, -7, -11, \cdots, -(n-4)$ to the vertices $b_{\frac{n+3}{2}}, b_{\frac{n+7}{2}}, b_{\frac{n+11}{2}}, \cdots, b_{n-2}$. Finally assign the labels $1, -n, , -(n-1)$ respectively to the vertices a, a_n, b_n and assign the labels $n, n-1$ to the vertices $a_{\frac{n-1}{2}}, b_{\frac{n-1}{2}}$ respectively.

Table 1 given below establishes that this vertex labeling gives subdivision of spoke of the wheel is pair difference cordial.

Nature of n	$\Delta_{f_1^c}$	Δ_{f_1}
$n \equiv 0 \pmod 4$	$\frac{3n}{2}$	$\frac{3n}{2}$
$n \equiv 1 \pmod 4$	$\frac{3n-1}{2}$	$\frac{3n+1}{2}$
$n \equiv 2 \pmod 4$	$\frac{3n}{2}$	$\frac{3n}{2}$
$n \equiv 3 \pmod 4$	$\frac{3n-1}{2}$	$\frac{3n+1}{2}$

Table 1

This completes the proof. □

Theorem 3.3 *A subdivision of the rim edges of the wheel W_n is pair difference cordial for all values of $n \geq 3$.*

Proof Let G_r be the subdivision of rim edges of the wheel graph with the vertex set $V(G_r) = \{a, a_i, b_i : 1 \leq i \leq n\}$ and edge set

$$E(G_r) = \{aa_i : 1 \leq i \leq n\} \cup \{a_ib_i, a_{i+1}b_i : 1 \leq i \leq n-1\} \cup \{b_na_1\}.$$

Certainly, the graph G_r has $2n+1$ vertices and $3n$ edges.

Case 1. $3 \leq n \leq 11$.

Tables 2 and 3 shows that subdivision of rim edges of the wheel is pair difference cordial for all values of $3 \leq n \leq 11$. Assign the label 1 to the vertex a.

n	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a
3	2	4	-2									1
4	2	4	-1	-3								1
5	2	4	-1	-3	-5							1
6	2	4	6	-2	-4	-5						1
7	2	4	6	-1	-3	-5	-6					1
8	2	4	6	8	-2	-4	-7	-8				1
9	2	4	6	8	-1	-3	-5	-8	-9			1
10	2	4	6	8	10	-2	-4	-6	-10	-7		1
11	2	4	6	8	10	-1	-3	-5	-7	-11	-8	1

Table 2

n	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}
3	3	-1	-3								
4	3	4	-2	-4							
5	3	5	-2	-4	4						
6	3	5	-1	-3	-6						
7	3	5	7	-2	-4	-7	-8				
8	3	5	7	-1	-3	-5	-6	8			
9	3	5	7	9	-2	-4	-6	-7	9		
10	3	5	7	9	-1	-3	-5	-8	-9	10	
11	3	5	7	9	11	-2	-4	-6	-9	-10	11

Table 3

Case 2. $n \equiv 0(mod4), n \geq 12$.

Assign the labels $2, 4, 6, \cdots, n$ respectively to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n}{2}}$ and assign the labels $-2, -4, -6, \cdots, -(\frac{n+4}{2})$ to the vertices $a_{n+22}, a_{\frac{n+4}{2}}, a_{\frac{n+6}{2}}, \cdots, a_{\frac{3n+4}{4}}$ respectively and assign the labels $-(\frac{n+6}{2})$,$-(\frac{n+8}{2})$,$-(\frac{n+10}{2}), \cdots, -(\frac{3n+4}{4})$ respectively to the vertices $a_{\frac{3n+8}{4}}$, $a_{\frac{3n+12}{4}}$, $a_{\frac{3n+16}{4}}, \cdots, a_n$. Now we assign the labels $3, 5, 7, \cdots, n-1$ respectively to the vertices b_1, b_2, b_3, $\cdots, b_{\frac{n-2}{2}}$. Next assign the labels $-1, -3, -5, \cdots, -(\frac{n+2}{2})$ to the vertices $b_{n2}, b_{\frac{n+2}{2}}, b_{\frac{n+4}{2}}, \cdots, b_{\frac{3n}{4}}$ respectively and assign the labels $-(\frac{3n+8}{4})$,$-(\frac{3n+12}{4})$,$-(\frac{3n+16}{4}), \cdots, -n$ respectively to the vertices $b_{\frac{3n+4}{4}}$, $b_{\frac{3n+8}{4}}$, $b_{\frac{3n+12}{4}}, \cdots, b_{n-1}$. Next assign the labels $1, n$ respectively to the vertices a, b_n.

Case 3. $n \equiv 1(mod4), n \geq 13$.

Assign the labels $2, 4, 6, \cdots, n-1$ respectively to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n-1}{2}}$ and assign the labels $-1, -3, -5, \cdots, -(\frac{n+5}{2})$ to the vertices $a_{n+12}, a_{\frac{n+3}{2}}, a_{\frac{n+5}{2}}, \cdots, a_{\frac{3n+5}{4}}$ respectively and assign the labels $-(\frac{n+7}{2})$,$-(\frac{n+9}{2})$,$-(\frac{n+11}{2}), \cdots, -(\frac{3n+5}{4})$ respectively to the vertices $a_{\frac{3n+9}{4}}$, $a_{\frac{3n+13}{4}}$, $a_{\frac{3n+17}{4}}, \cdots, a_n$. Now we assign the labels $3, 5, 7, \cdots, n$ respectively to the vertices $b_1, b_2, b_3, \cdots, b_{\frac{n-1}{2}}$. Next assign the labels $-2, -4, -6, \cdots, -(\frac{n+3}{2})$ to the vertices $b_{n+12}, b_{\frac{n+3}{2}}, b_{\frac{n+5}{2}}, \cdots, b_{\frac{3n+1}{4}}$ respectively and assign the labels $-(\frac{3n+9}{4})$,$-(\frac{3n+13}{4})$,$-(\frac{3n+17}{4}), \cdots$, $-n$ respectively to the vertices $b_{\frac{3n+5}{4}}$, $b_{\frac{3n+9}{4}}$, $b_{\frac{3n+13}{4}}, \cdots, b_{n-1}$. Next assign the labels $1, n$ respectively to the vertices a, b_n.

Case 4. $n \equiv 2(mod4), n \geq 14$.

Assign the labels $2, 4, 6, \cdots, n$ respectively to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n}{2}}$ and assign the labels $-2, -4, -6, \cdots, -(\frac{n+2}{2})$ to the vertices $a_{n+22}, a_{\frac{n+4}{2}}, a_{\frac{n+6}{2}}, \cdots, a_{\frac{3n+2}{4}}$ respectively and assign the labels $-(\frac{n+4}{2})$,$-(\frac{n+6}{2})$,$-(\frac{n+8}{2}), \cdots, -(\frac{3n+2}{4})$ respectively to the vertices $a_{\frac{3n+6}{4}}$, $a_{\frac{3n+10}{4}}$, $a_{\frac{3n+14}{4}}, \cdots, a_n$. Now we assign the labels $3, 5, 7, \cdots, n-1$ respectively to the vertices b_1, b_2, b_3, $\cdots, b_{\frac{n-2}{2}}$. Next assign the labels $-1, -3, -5, \cdots, -(\frac{n+2}{2})$ to the vertices $b_{n2}, b_{\frac{n+2}{2}}, b_{\frac{n+4}{2}}, \cdots, b_{\frac{3n}{4}}$ respectively and assign the labels $-(\frac{3n+2}{4})$,$-(\frac{3n+6}{4})$,$-(\frac{3n+10}{4}), \cdots, -n$ respectively to the vertices $b_{\frac{3n+2}{4}}$, $b_{\frac{3n+6}{4}}$, $b_{\frac{3n+10}{4}}, \cdots, b_{n-1}$. Next assign the labels $1, 1$ respectively to the vertices a, b_n.

Case 5. $n \equiv 3(mod4), n \geq 15$.

Assign the labels $2, 4, 6, \cdots, n-1$ respectively to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n-1}{2}}$ and assign the labels $-1, -3, -5, \cdots, -(\frac{n+3}{2})$ to the vertices $a_{n+12}, a_{\frac{n+3}{2}}, a_{\frac{n+5}{2}}, \cdots, a_{\frac{3n+3}{4}}$ respectively and assign the labels $-(\frac{n+5}{2}), -(\frac{n+7}{2}), -(\frac{n+9}{2}), \cdots, -(\frac{3n+3}{4})$ respectively to the vertices $a_{\frac{3n+7}{4}}$, $a_{\frac{3n+11}{4}}$, $a_{\frac{3n+15}{4}}, \cdots, a_n$. Now we assign the labels $3, 5, 7, \cdots, n$ respectively to the vertices $b_1, b_2, b_3, \cdots, b_{\frac{n-1}{2}}$. Next assign the labels $-2, -4, -6, \cdots, -(\frac{n+1}{2})$ to the vertices $b_{n+12}, b_{\frac{n+3}{2}}, b_{\frac{n+5}{2}}, \cdots, b_{\frac{3n-1}{4}}$ respectively and assign the labels $-(\frac{3n+7}{4}), -(\frac{3n+11}{4}), -(\frac{3n+15}{4}), \cdots,$ $-n$ respectively to the vertices $b_{\frac{3n+3}{4}}$, $b_{\frac{3n+7}{4}}$, $b_{\frac{3n+11}{4}}, \cdots, b_{n-1}$. Next assign the labels $1, n$ respectively to the vertices a, b_n.

Table 4 given below establishes that this vertex labeling gives subdivision of rim edges of the wheel is pair difference cordial.

Nature of n	$\Delta_{f_1^c}$	Δ_{f_1}
$n \equiv 0 \pmod 4$	$\frac{3n}{2}$	$\frac{3n}{2}$
$n \equiv 1 \pmod 4$	$\frac{3n-1}{2}$	$\frac{3n+1}{2}$
$n \equiv 2 \pmod 4$	$\frac{3n}{2}$	$\frac{3n}{2}$
$n \equiv 3 \pmod 4$	$\frac{3n-1}{2}$	$\frac{3n+1}{2}$

Table 4

A pair difference cordial labeling on subdivision of rim edges of the wheel W_5 is shown in Figure 1.

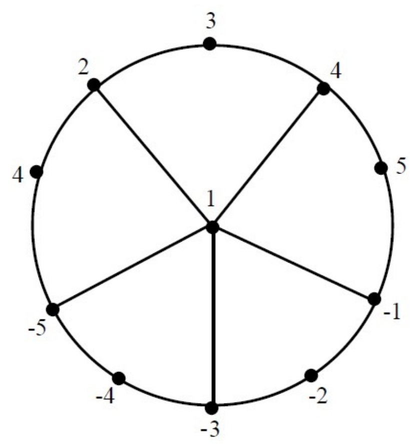

Figure 1

This completes the proof. □

Theorem 3.4 *A subdivision of comb $P_n \odot K_1$ is pair difference cordial for all values of $n \geq 2$.*

Proof Let the vertex set and edge set be $V(P_n \odot K_1) = \{a_i, b_i, c_i : 1 \leq i \leq n\} \cup \{d_i : 1 \leq i \leq n-1\}$ and $E(P_n \odot K_1) = \{a_ib_i, b_ic_i : 1 \leq i \leq n\} \cup \{a_id_i, d_ia_{i+1} : 1 \leq i \leq n-1\}$. There are $4n-1$ vertices and $4n-2$ edges.
There are four cases arises.

Case 1. n is even.

Assign the labels $3, 6, 9, \cdots, \frac{3n}{2}$ to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n}{2}}$ respectively and assign the labels $2, 5, 8, \cdots, \frac{3n-2}{2}$ respectively to the vertices $b_1, b_2, b_3, \cdots, b_{\frac{n}{2}}$. Next assign the labels $1, 4, 7, \cdots, \frac{3n-4}{2}$ to the vertices $c_1, c_2, c_3, \cdots, c_{\frac{n}{2}}$ respectively and assign the labels $-3, -6, -9, \cdots, -(\frac{3n-6}{2})$ respectively to the vertices $a_{\frac{n+2}{2}}, a_{\frac{n+4}{2}}, a_{\frac{n+6}{2}}, \cdots, a_{n-1}$. Now we assign the labels $-2, -5, -8, \cdots, -(\frac{3n-8}{2})$ to the vertices $b_{\frac{n+2}{2}}, b_{\frac{n+4}{2}}, b_{\frac{n+6}{2}}, \cdots, b_{n-1}$ and assign the labels $-1, -4, -7, \cdots, -(\frac{3n-4}{2})$ to the vertices $c_{\frac{n+2}{2}}, c_{\frac{n+4}{2}}, c_{\frac{n+6}{2}}, \cdots, c_n$. Next assign the labels $\frac{3n+2}{2}, \frac{3n+4}{2}, \frac{3n+6}{2}, \cdots, 2n-1$ respectively to the vertices $d_1, d_2, d_3, \cdots, d_{\frac{n-2}{2}}$ and assign the labels $-(\frac{3n+2}{2}), -(\frac{3n+4}{2}), -(\frac{3n+6}{2}), \cdots, -(2n-1)$ to the vertices $d_{\frac{n}{2}}, d_{\frac{n+2}{2}}, d_{\frac{n+4}{2}}, \cdots, d_{n-2}$ respectively. Finally assign the labels $-(\frac{3n}{2}), -(\frac{3n-2}{2}), 1$ respectively to the vertices b_n, a_n, u_{n-1}.

Case 2. $n = 3$.

Assign the labels $1, 2, 3, -1, -2, -3, 4, -4, -4$ respectively to the vertices $c_1, b_1, a_1, c_2, b_2, a_2, c_3, b_3, a_3$ and assign the labels $5, -5$ to the vertices d_1, d_2 respectively.

Case 3. $n = 5$.

Assign the labels $1, 2, 3, 4, 5, 6$ respectively to the vertices $c_1, b_1, a_1, c_2, b_2, a_2$ and assign the labels $-1, -2, -3, -4, -5, -6$ to the vertices $c_3, b_3, a_3, c_4, b_4, a_4$ respectively. Next assign the labels $7, -7, 8, -8$ respectively to the vertices d_1, d_2, d_3, d_4 and assign the labels $-9, -9, 9$ to the vertices a_5, b_5, c_5 respectively.

Case 4. n is odd $n \geq 7$.

Assign the labels $3, 6, 9, \cdots, \frac{3n-3}{2}$ to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n-1}{2}}$ respectively and assign the labels $2, 5, 8, \cdots, \frac{3n-9}{2}$ respectively to the vertices $b_1, b_2, b_3, \cdots, b_{\frac{n-1}{2}}$. Next assign the labels $1, 4, 7, \cdots, \frac{3n-4}{2}$ to the vertices $c_1, c_2, c_3, \cdots, c_{\frac{n}{2}}$ respectively and assign the labels $-3, -6, -9, \cdots, -(\frac{3n-6}{2})$ respectively to the vertices $a_{\frac{n+2}{2}}, a_{\frac{n+4}{2}}, a_{\frac{n+6}{2}}, \cdots, a_{n-1}$. Now we assign the labels $-2, -5, -8, \cdots, -(\frac{3n-8}{2})$ to the vertices $b_{\frac{n+2}{2}}, b_{\frac{n+4}{2}}, b_{\frac{n+6}{2}}, \cdots, b_{n-1}$ and assign the labels $-1, -4, -7, \cdots, -(\frac{3n-4}{2})$ to the vertices $c_{\frac{n+2}{2}}, c_{\frac{n+4}{2}}, c_{\frac{n+6}{2}}, \cdots, c_n$. Next assign the labels $\frac{3n+2}{2}, \frac{3n+4}{2}, \frac{3n+6}{2}, \cdots, 2n-1$ respectively to the vertices $d_1, d_2, d_3, \cdots, d_{\frac{n-2}{2}}$ and assign the labels $-(\frac{3n+2}{2}), -(\frac{3n+4}{2}), -(\frac{3n+6}{2}), \cdots, -(2n-1)$ to the vertices $d_{\frac{n}{2}}, d_{\frac{n+2}{2}}, d_{\frac{n+4}{2}}, \cdots, d_{n-2}$ respectively. Finally assign the labels $-(\frac{3n}{2}), -(\frac{3n-2}{2}), 1$ respectively to the vertices b_n, a_n, u_{n-1}.

In all the cases, we have $\Delta_{f_1} = \Delta_{f_1^c} = 2n - 1$. □

Theorem 3.5 *A subdivision of the edges of the path P_n in the comb $P_n \odot K_1$ is pair difference cordial for all values of $n \geq 2$.*

Proof Let G be subdivision of the edges of the path P_n in the comb graph $P_n \odot K_1$. Let the vertex set and edge set be $V(G) = \{a_i, b_i : 1 \leq i \leq n\} \cup \{c_i : 1 \leq i \leq n-1\}$ and $E(G) = \{a_ib_i : 1 \leq i \leq n\} \cup \{a_ic_i, c_ia_{i+1} : 1 \leq i \leq n-1\}$, which has $3n - 1$ vertices and $3n - 2$ edges. There are two cases arises.

Case 1. n is even.

Subcase 1.1 $n = 2$.

Assign the labels $1, 2, 1, -2, -1$ respectively to the vertices b_1, a_1, c_1, a_2, b_2 respectively.

Subcase 1.2 $n \geq 4$.

Assign the labels $3, 6, 9, \cdots, \frac{3n-6}{2}$ to the vertices $b_2, b_3, b_4, \cdots, b_{\frac{n}{2}}$ respectively and assign the labels $1, 4, 7, \cdots, \frac{3n-4}{2}$ respectively to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n}{2}}$. Next assign the labels $5, 8, 11, \cdots, \frac{3n-2}{2}$ to the vertices $c_1, c_2, c_3, \cdots, c_{\frac{n-2}{2}}$ respectively and assign the labels $-1, -3, -5, \cdots, -n+1$ respectively to the vertices $a_{\frac{n+2}{2}}, a_{\frac{n+4}{2}}, a_{\frac{n+6}{2}}, \cdots, a_n$. Now we assign the labels $-2, -4, -6, \cdots, -n$ to the vertices $b_{\frac{n+2}{2}}, b_{\frac{n+4}{2}}, b_{\frac{n+6}{2}}, \cdots, b_n$ and assign the labels $-(n+1), -(n+2), -(n+3) \cdots, -(\frac{3n-2}{2})$ to the vertices $c_{\frac{n}{2}}, c_{\frac{n+2}{2}}, c_{\frac{n+4}{2}}, \cdots, c_{n-2}$. Finally assign the labels $2, -(\frac{3n-2}{2})$ respectively to the vertices b_1, c_{n-1}.

Case 2. n is odd.

Subcase 2.1 $n = 3$.

Assign the labels $1, 2, -1, -2, 3, -3, 4, -4$ respectively to the vertices $b_1, a_1, b_2, a_2, b_3, a_3, c_1, c_2$.

Subcase 2.2 $n \geq 5$.

Assign the labels $3, 6, 9, \cdots, \frac{3n-3}{2}$ to the vertices $c_1, c_2, c_3, \cdots, c_{\frac{n-3}{2}}$ respectively and assign the labels $1, 4, 7, \cdots, \frac{3n-7}{2}$ respectively to the vertices $b_1, b_2, b_3, \cdots, b_{\frac{n-1}{2}}$. Next assign the labels $2, 5, 8, \cdots, \frac{3n-5}{2}$ to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n-1}{2}}$ respectively and assign the labels $-1, -3, -5, \cdots, -n$ respectively to the vertices $a_{\frac{n+1}{2}}, a_{\frac{n+3}{2}}, a_{\frac{n+5}{2}}, \cdots, a_n$. Now we assign the labels $-2, -4, -6, \cdots, -n-1$ to the vertices $b_{\frac{n+1}{2}}, b_{\frac{n+3}{2}}, b_{\frac{n+5}{2}}, \cdots, b_n$ and assign the labels $-(n+2), -(n+3), -(n+4), \cdots, -(\frac{3n-1}{2})$ to the vertices $c_{\frac{n+1}{2}}, c_{\frac{n+3}{2}}, c_{\frac{n+5}{2}}, \cdots, c_{n-1}$. Finally assign the labels $\frac{3n-11}{2}$ to the vertex $c_{\frac{n-1}{2}}$.

Table 5 given below establishes that this vertex labeling gives subdivision of the edges of the path P_n in the comb graph is pair difference cordial.

Nature of n	$\Delta_{f_1^c}$	Δ_{f_1}
n is odd	$\frac{3n-4}{2}$	$\frac{3n}{2}$
n is even	$\frac{3n-2}{2}$	$\frac{3n-2}{2}$

Table 5

This completes the proof. □

Theorem 3.6 *A subdivision of the pendant edges of the comb $P_n \odot K_1$ in the comb is pair difference cordial for all values of $n \geq 2$.*

Proof Let G be subdivision of the pendant edges of the path P_n in the comb $P_n \odot K_1$. Let the vertex set and edge set be $V(G) = \{a_i, b_i, c_i : 1 \leq i \leq n\}$ and $E(G) = \{a_ib_i, b_ic_i : 1 \leq i \leq n\}$, which has $3n-1$ vertices and $3n-2$ edges.

Case 1. n is even.

Subcase 1.1 $n = 2$.

Assign the labels $1, 2, 1, -2, -1$ respectively to the vertices b_1, a_1, c_1, a_2, b_2 respectively.

Subcase 1.2 $n \geq 4$.

Assign the labels $3, 6, 9, \cdots, \frac{3n-6}{2}$ to the vertices $b_2, b_3, b_4, \cdots, b_{\frac{n}{2}}$ respectively and assign the labels $1, 4, 7, \cdots, \frac{3n-4}{2}$ respectively to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n}{2}}$. Next assign the labels $5, 8, 11, \cdots, \frac{3n-2}{2}$ to the vertices $c_1, c_2, c_3, \cdots, c_{\frac{n-2}{2}}$ respectively and assign the labels $-1, -3, -5, \cdots, -n+1$ respectively to the vertices $a_{\frac{n+2}{2}}, a_{\frac{n+4}{2}}, a_{\frac{n+6}{2}}, \cdots, a_n$. Now we assign the labels $-2, -4, -6, \cdots, -n$ to the vertices $b_{\frac{n+2}{2}}, b_{\frac{n+4}{2}}, b_{\frac{n+6}{2}}, \cdots, b_n$ and assign the labels $-(n+1), -(n+2), -(n+3) \cdots, -(\frac{3n-2}{2})$ to the vertices $c_{\frac{n}{2}}, c_{\frac{n+2}{2}}, c_{\frac{n+4}{2}}, \cdots, c_{n-2}$. Finally assign the labels $2, -(\frac{3n-2}{2})$ respectively to the vertices b_1, c_{n-1}.

Case 2. n is odd.

Subcase 2.1 $n = 3$.

Assign the labels $1, 2, -1, -2, 3, -3, 4, -4$ respectively to the vertices $b_1, a_1, b_2, a_2, b_3, a_3, c_1, c_2$.

Subcase 2.2 $n \geq 5$.

Assign the labels $3, 6, 9, \cdots, \frac{3n-3}{2}$ to the vertices $c_1, c_2, c_3, \cdots, c_{\frac{n-3}{2}}$ respectively and assign the labels $1, 4, 7, \cdots, \frac{3n-7}{2}$ respectively to the vertices $b_1, b_2, b_3, \cdots, b_{\frac{n-1}{2}}$. Next assign the labels $2, 5, 8, \cdots, \frac{3n-5}{2}$ to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n-1}{2}}$ respectively and assign the labels $-1, -3, -5, \cdots, -n$ respectively to the vertices $a_{\frac{n+1}{2}}, a_{\frac{n+3}{2}}, a_{\frac{n+5}{2}}, \cdots, a_n$. Now we assign the labels $-2, -4, -6, \cdots, -n-1$ to the vertices $b_{\frac{n+1}{2}}, b_{\frac{n+3}{2}}, b_{\frac{n+5}{2}}, \cdots, b_n$ and assign the labels $-(n+2), -(n+3), -(n+4), \cdots, -(\frac{3n-1}{2})$ to the vertices $c_{\frac{n+1}{2}}, c_{\frac{n+3}{2}}, c_{\frac{n+5}{2}}, \cdots, c_{n-1}$. Finally assign the labels $\frac{3n-11}{2}$ to the vertex $c_{\frac{n-1}{2}}$.

Table 6 given below establishes that this vertex labeling gives subdivision of the edges of the path P_n in the comb graph is pair difference cordial.

Nature of n	$\Delta_{f_1^c}$	Δ_{f_1}
n is odd	$\frac{3n-4}{2}$	$\frac{3n}{2}$
n is even	$\frac{3n-2}{2}$	$\frac{3n-2}{2}$

Table 6

A subdivision of the pendant edges of the comb $P_7 \odot K_1$ is pair difference cordial is shown in Figure 2.

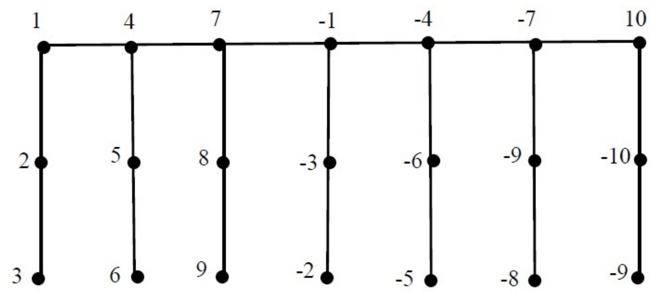

Figure 2

This completes the proof. □

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