International J.Math. Combin. Vol.3(2023), 57-65

Pairwise Balanced Designs Arising from Minimum Covering and Maximum Independent Sets of Circulant Graphs

Veena Bankapur

(Department of Mathematics, Jnana Bharathi, Campus, Bangalore University, Bengaluru-560 056, India)

Shaikh Ameer Basha

(Department of Mathematics, Bearys Institute of Technology, Mangaluru-574 199, India)

B. Chaluvaraju

(Department of Mathematics, Jnana Bharathi, Campus, Bangalore University, Bengaluru-560 056, India)

E-mail: veenabankapur88@gmail.com, shaikhameerbasha@gmail.com, bchaluvaraju@gmail.com

Abstract: The pairwise balanced designs (PBD's) is a pair (P, B), where P is a finite set of ν points and B is a family of subsets of P, called blocks such that every two distinct points in P appear in exactly one block. Let $\alpha(G) = \alpha$ and $\beta(G) = \beta$ be the vertex covering and independence number of a graph $G = (V, E)$ with the minimum and maximum cardinality of such sets are denoted by α-sets and β-sets of G, respectively (or, simply (α, β)-sets). In this paper, we obtain the total number of (α, β)-sets in different jump sizes of some circulant graphs apart from strongly regular graphs which are the blocks of PBD.

Key Words: Covering number, independence number, circulant graph, pairwise balanced designs, Smarandachely pairwise balanced designs.

AMS(2010): 05C51, 05E30, 05C69.

§1. Introduction

In this paper, we are focusing on nonempty, finite, simple, undirected graphs with notations $p = |V|$ and $q = |E|$ for the number of vertices and edges of a graph $G = (V, E)$, respectively. In general, we use to $\langle X \rangle$ denote the sub graph induced by the set of vertices X. We refer to [12] for unspecified terms of the paper.

1.1. (α, β)-Sets of a Graph

A vertex of graph G is said to cover the edges incident with it, and a vertex cover of G is a set of vertices covering all the edge of G. The smallest cardinality of a vertex cover is called the vertex covering number $\alpha(G)$ or α of G. Further, a subset S of the vertex set $V(G)$ is said to be an independent set if the induced sub graph $\langle S \rangle$ is a trivial graph. The largest number of vertices in such a set is called the vertex independence number $\beta(G)$ or β of a graph G.

[1]Received June 6, 2023, Accepted August 16, 2023.

The minimum and maximum cardinality of vertex covering and independence sets are denoted by α-sets and β-sets of a graph G, respectively. The maximum number of α-sets and β-sets of G is denoted by $\tau_\alpha(G)$ and $\tau_\beta(G)$, respectively. For more details on similar concepts, we refer to [6, 11, 12, 13, 16].

1.2. Strongly Regular Graph

A strongly regular graph with parameters (p, l, ω, μ) is a finite graph on p vertices, without loops or multiple edges, regular of degree l (with $0 < l < p - 1$, so that there are both edges and no edges), and such that any two distinct vertices have ω common neighbors when they are adjacent, and μ common neighbors when they are nonadjacent. For the related concepts of the strongly regular graph, we refer to [2, 9].

1.3. Pairwise Balanced Designs (PBD's)

The combinatorial design theory is a study of collection of subsets with certain intersection properties. Based on the group and the type of underlying association scheme, there are other subgroups that can be made.

The following conditions satisfy m classes of association scheme on ν vertices (elements or objects) are

(i) If the associates are symmetric, then any two vertices are m^{th} associates, where $1 \leq k \leq m$;

(ii) Each vertex x contains n_k k^{th} associates, the number n_k being independent of vertex x;

(iii) If two vertices x and y are k^{th} associates, then the number of vertices which are a^{th} associates of x and b^{th} associates of y is p^k_{ab} and is independent of the k^{th} associates x and y. Hence $p^k_{ab} = p^k_{ba}$.

The pairwise balanced designs (PBD's) are a specific type of experimental design that offer several advantages over other block designs in certain situations and is defined as follows:

The PBD is a pair $(P, \mathcal{B})$ such that $\mathcal{B}$ is a set of subsets (called blocks) of P, each of cardinality at least two such that every unordered pair of points (elements of P) is contained in a unique block in $\mathcal{B}$. If ν is a positive integer and K is a set of positive integers, each of which is greater than or equal to 2, then we say that $(P, \mathcal{B})$ is a (ν, K)-PBD if $|X| = \nu$, and $|\mathcal{B}| \in K$ for every $B \in \mathcal{B}$. Furthermore, a Smarandachely PBD is contrary to the pairwise balanced designs, which asks for every unordered pair of points in P containing in 2 blocks in $\mathcal{B}$ at least.

Generally, these PBD's are highly efficient in terms of the number of treatments they can accommodate with a limited number of experimental units. They allow for a large number of treatments to be compared while minimizing the number of experimental units required. This efficiency is particularly useful when resources, such as time, money, or subjects, are limited. These designs provide increased precision in estimating treatment effects compared to other block designs. By carefully selecting which treatments are paired together in each block, these designs allow for a more precise estimation of treatment effects by reducing the variation caused by extraneous factors or confounding variables. PBD's ensure that each treatment appears with every other treatment in a block a balanced number of times. This balance helps to control the influence of confounding factors or extraneous variables that may affect the response variable. By balancing the treatment combinations, PBD's can help to separate the true treatment effects from the effects of other variables. For more details on combinatorial design theory with some graph parameters, we refer to [5, 7, 8, 9, 14, 17].

§2. Main Results

For a given positive integer p, let s_1, s_2, $\cdots$, s_t be a sequence of integers where $0 < s_1 < s_2 < \cdots < s_t < \frac{p+1}{2}$. The circulant graph $C_p(S)$ where $S = s_1, s_2, \cdots, s_t$ is the graph on p vertices labeled as v_1, v_2, $\cdots$, v_p with vertex v_i adjacent to each vertex $v_{i\pm s_j (mod\ p)}$ and the values s_t are called jump sizes.

The circulant graphs have been investigated in the fields outside of graph theory. For example, in geometers, circulant graphs are known as star polygons [1]. They have been used to solve problems in group theory (particularly the families of Cayley graphs), as shown in [3] as well as number theory and analysis. They are also, used as models for interconnection networks in telecommunication, VLSI designs, parallel, and distributed computing. For applications and mathematical properties of circulant graphs, see [4], [10] and [15].

2.1 Circulant Graph $C_p(1)$

The jump size of circulant graph is one, known as cycle C_p with $p \geq 3$ vertices. That is, $C_p(1) \cong C_p$, $p \geq 3$. The circulant graph $C_4(1)$ is the only strongly regular graphs.

Proposition 2.1 ([11]) *For any Circulant graph $C_p(1)$; $p \geq 3$ vertices,*

$$\alpha(C_p(1)) = \left\lceil \frac{p}{2} \right\rceil \text{ and } \beta(C_p(1)) = \left\lfloor \frac{p}{2} \right\rfloor.$$

Theorem 2.1 *The collection of all (α, β)-sets of a circulant graph $C_p(1)$, $p = 2n$, $n \geq 2$ vertices form a PBD with parameters $\nu = p$, $b = 4$, $g_1 = \lceil \frac{p}{2} \rceil$, $g_2 = \lfloor \frac{p}{2} \rfloor$, $r = 2$ and*

$$\lambda_m = \begin{cases} 2, & m \equiv 0 (mod\, 2), \\ 0, & otherwise. \end{cases}$$

Proof Let $C_p(1)$ be a circulant graph with $p = 2n$, $n \geq 2$ vertices given by $v_1, v_2, \cdots, v_p$. By Proposition 2.1, we have $\alpha(C_p(1)) = \left\lceil \frac{p}{2} \right\rceil$ and $\beta(C_p(1)) = \left\lfloor \frac{p}{2} \right\rfloor$. Further, $C_p(1)$ with $p = 2n$, $n \geq 2$ have two blocks of (α, β)-set, it implies $b = \tau_\alpha(C_p(1)) + \tau_\beta(C_p(1)) = 4$. Also, we have $g_1 = \alpha(C_p(1)) = \lceil \frac{p}{2} \rceil$ and $g_2 = \beta(C_p(1)) = \lfloor \frac{p}{2} \rfloor$, where g_1 and g_1 are the number of elements contained exactly in their respective blocks. By virtue of the above facts, we have $r = 2$. To obtain the m-associates for the elements, where $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$. The two distinct elements are first associates, if they have jump size 1 and they are k^{th}-associates ($2 \leq k \leq \lfloor \frac{p}{2} \rfloor$), if they have k jump sizes. Hence the parameters of first kind are given by $\nu = p$, $b = 4$, $g_1 = \lceil \frac{p}{2} \rceil$, $g_2 = \left\lfloor \frac{p}{2} \right\rfloor$, $r = 2$ and

$$\lambda_m = \begin{cases} 2, & m \equiv 0 (mod\, 2), \\ 0, & \text{otherwise}. \end{cases}$$

Thus, the result follows. □

Theorem 2.2 *The collection of all (α, β)-sets of a circulant graph $C_p(1)$; $p = 2n+1$, $n \geq 1$ vertices form a PBD with parameters $\nu = p$, $b = 2p$, $g_1 = \left\lceil \frac{p}{2} \right\rceil$, $g_2 = \left\lfloor \frac{p}{2} \right\rfloor$, $r = p$ and*

$$\lambda_m = \begin{cases} m, & m \equiv 0 (mod\, 2), \\ p - \lambda_{m-1} - \lambda_{m+1}, & \textit{otherwise}. \end{cases}$$

Proof For a given circulant graph $C_p(1)$; $p = 2n+1$, $n \geq 2$ vertices labeled as $v_1, v_2, \cdots, v_p$. By Proposition 2.1, we have $\alpha(C_p(1)) = \left\lceil \frac{p}{2} \right\rceil$ and $\beta(C_p(1)) = \left\lfloor \frac{p}{2} \right\rfloor$. Further, $C_p(1)$; $p = 2n+1$, $n \geq 1$ have p blocks of α-set, it implies $\tau_\alpha(C_p(1)) = \tau_\beta(C_p(1)) = p$. Therefore $b = 2p$. Also, we have $g_1 = \alpha(C_p(1)) = \left\lceil \frac{p}{2} \right\rceil$ and $g_2 = \beta(C_p(1)) = \left\lfloor \frac{p}{2} \right\rfloor$, where g_1 and g_2 are the number of elements contained exactly in their respective blocks. From the above facts, we have $r_\alpha = \left\lceil \frac{p}{2} \right\rceil$ and $r_\beta = \left\lfloor \frac{p}{2} \right\rfloor$, therefore $r = p$. To obtain the m-associates for the elements, where $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$. The two distinct elements are first associates, if they have jump size 1 and otherwise they are k^{th}-associates ($2 \leq k \leq \lfloor \frac{p}{2} \rfloor$). Hence the parameters of first kind are given by $\nu = p$, $b = 2p$, $g_1 = \lceil \frac{p}{2} \rceil$, $g_2 = \lfloor \frac{p}{2} \rfloor$, $r = p$, and

$$\lambda_m = \begin{cases} m, & m \equiv 0 (mod\, 2), \\ p - \lambda_{m-1} - \lambda_{m+1}, & \text{otherwise}. \end{cases}$$

Thus, the result follows. □

2.2 Circulant Graph with Odd Jump Sizes

The circulant graph of odd jump size $(1, 3, \cdots, \lfloor \frac{p}{2} \rfloor)$ with $p \geq 2$ is known as a complete bipartite graph K_{p_1, p_2} for $p_1 = p_2$, that is $C_p\left(1, 3, \ldots, \left\lfloor \frac{p}{2} \right\rfloor\right) \cong K_{p_1, p_2}$. Further, all the sequence of an odd jump size from 1 to $\lfloor \frac{p}{2} \rfloor$ are strongly regular graphs. Apart from this, the circulant graphs are not strongly regular graph if the sequence of odd jump sizes were taken randomly excluding jump size 1.

Proposition 2.2 ([11]) *For any circulant graph $C_p(1, 3, \cdots, \lfloor \frac{p}{2} \rfloor)$; $p = 4n-2$ or $4n$, $n \geq 2$ vertices,*

$$\alpha(C_p(1, 3, \cdots, \lfloor \frac{p}{2} \rfloor)) = \beta(C_p(1, 3, \cdots, \lfloor \frac{p}{2} \rfloor)) = \frac{p}{2}.$$

Theorem 2.3 *The collection of all (α,β)-sets of a Circulant graph $C_p(1,3,\cdots,\lfloor\frac{p}{2}\rfloor)$; $p=4n-2$ or $4n$, $n\geq 2$ vertices form a PBD with parameters $\nu=p$, $b=4$, $g_1=g_2=\frac{p}{2}$, $r=2$ and*

$$\lambda_m=\begin{cases}2, & m\equiv 0(mod\,2),\\ 0, & otherwise.\end{cases}$$

Proof For a given circulant graph $C_p(1,3,\cdots,\lfloor\frac{p}{2}\rfloor)$; $\nu_\alpha=\nu_\beta=p=4n-2$ or $4n$, $n\geq 2$ vertices labeled as $v_1,v_2,\cdots,v_p$. By Proposition 2.2, we have $\alpha(C_p(1,3,\cdots,\lfloor\frac{p}{2}\rfloor)=\beta(C_p(1,3,\cdots,\lfloor\frac{p}{2}\rfloor)=\dfrac{p}{2}$. Further, $C_p(1,3,\cdots,\lfloor\frac{p}{2}\rfloor)$; $p=4n-2$ or $4n$, $n\geq 2$ have two blocks of (α,β)-sets, it implies $b=4$. Also, we have $g_1=g_2=\dfrac{p}{2}$, where g_1 and g_2 are the number of elements contained exactly in their respective blocks. From the above facts, we have $r=2$. To obtain the m-associates for the elements, where $1\leq m\leq\lfloor\frac{p}{2}\rfloor$. Two distinct elements are odd associates if they have odd jump size and they are even associates $(2\leq k\leq\lfloor\frac{p}{2}\rfloor)$. Hence the parameters of first kind are given by $\nu=p$, $b=4$, $g_1=g_2=\dfrac{p}{2}$, $r=2$ and

$$\lambda_m=\begin{cases}2, & m\equiv 0(mod\,2),\\ 0, & \text{otherwise}.\end{cases}$$

This completes the proof. □

2.3 Circulant Graph with Even Jump Sizes

The jump size of circulant graph is $2,4,\cdots,\lfloor\frac{p}{2}\rfloor$ is a $C_p(2,4,\cdots,\lfloor\frac{p}{2}\rfloor)$ with $p\geq 4$ vertices. The circulant graphs $C_5(2)$, $C_6(2)$, $C_8(2,4)$, $C_{10}(2,4)$, $C_{12}(2,4,6)$ are some examples of strongly regular graphs.

Proposition 2.3 ([11]) *For any circulant graph $C_p(2,4,\cdots,\lfloor\frac{p}{2}\rfloor)$ with $p\geq 4$ vertices,*

$$\alpha(C_p(2,4,\cdots,\lfloor\frac{p}{2}\rfloor))=\begin{cases}p-3, & p=4n+3,\\ p-2, & otherwise.\end{cases}$$

and

$$\beta(C_p(2,4,\cdots,\lfloor\frac{p}{2}\rfloor))=\begin{cases}3, & p=4n+3,\\ 2, & otherwise.\end{cases}$$

Proof Since the circulant graph $C_p(2,4,\cdots,\lfloor\frac{p}{2}\rfloor)$ is a $(2n-1)$-regular for $p=4n$ or $2n$-regular for $p=4n+1$ or $p=4n+2$ or $p=4n+3$, $n\geq 1$ vertices, the result follows. □

Theorem 2.4 *The collection of all (α,β)-sets of a Circulant graph $C_p(2,4,\cdots,\lfloor\frac{p}{2}\rfloor)$; $p\geq 4$ vertices form a PBD with parameters:*

(*i*) *For $\nu=p=4n$ or $\nu=p=4n+1, n\geq 1$, $b=p(n+1)$, $g_1=p-2$, $g_2=2$, $r=\frac{3p-4}{2}$,*

$\lambda_m(1) = p-4$, *and*

$$\lambda_m(2) = \begin{cases} 0, & m \equiv 0(mod\, 2), \\ 1, & otherwise. \end{cases}$$

(ii) For $\nu = p = 4n+2; n \geq 1$, $b = \frac{p^2}{2}$, $g_1 = p-2$, $g_2 = 2$, $r = \frac{3p-4}{2}$, $\lambda_m(1) = p-4$, *and*

$$\lambda_m(2) = \begin{cases} 0, & m \equiv 0(mod\, 2), \\ 1, & otherwise, \end{cases}$$

where $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$.

(iii) For $\nu = p = 4n+3; n \geq 1$, $b = 2p$, $g_1 = p-2$, $g_2 = \left\lceil \frac{p}{2n+1} \right\rceil$, $r = \frac{2(n+1)(p-1)}{2n+1}$ *and due to the variations in the values of* $\lambda_m(1)$ *there is no good relation between* $\lambda_m(2)$.

Proof For a given circulant graph $C_p(2, 4, \cdots, \lfloor \frac{p}{2} \rfloor)$, $\nu = p = 4n$ or $4n+1$ or $4n+2$ or $4n+3; n \geq 1$, vertices labeled as $v_1, v_2, \cdots, v_p$. By Proposition 2.3, we have the following cases.

Case 1. The circulant graph $C_p(2, 4, \cdots, \lfloor \frac{p}{2} \rfloor)$ with $\nu = p = 4n$ or $4n+1$, $n \geq 1$ have p blocks of α-set and np blocks of β-sets. This implies that $\tau_\alpha(C_p(2, 4, \cdots, \lfloor \frac{p}{2} \rfloor)) = p$ and $\tau_\beta(C_p(2, 4, \cdots, \lfloor \frac{p}{2} \rfloor)) = np$. Therefore $b = p(n+1)$. By Proposition 2.3, we have $g_1 = \alpha(C_p(2, 4, \ldots, \lfloor \frac{p}{2} \rfloor)) = p-2$ and $g_2 = \beta(C_p(2, 4, \ldots, \lfloor \frac{p}{2} \rfloor)) = 2$, where g_1 and g_2 are the number of elements contained exactly in their respective blocks. From the above facts, we have $r = \frac{3p-4}{2}$. To obtain the m-associates for the elements, where $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$. The two distinct elements odd associates, if they have odd jump size and they are even associates $(2 \leq k \leq \lfloor \frac{p}{2} \rfloor)$. Hence the parameters of first kind are given by $\nu = p$, $b = p(n+1)$, $g_1 = p-2$, $g_2 = 2$, $r = \dfrac{3p-4}{2}$, $\lambda_m(1) = p-4$ and

$$\lambda_m(2) = \begin{cases} 0, & m \equiv 0(mod\, 2), \\ 1, & \text{otherwise}, \end{cases}$$

where $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$.

Case 2. The circulant graph $C_p(2, 4, \cdots, \lfloor \frac{p}{2} \rfloor)$ with $\nu = p = 4n+2$; $n \geq 1$ have p blocks of α-set and $\frac{p^2}{4}$ blocks of β-sets, this implies $\tau_\alpha(C_p(2, 4, \cdots, \lfloor \frac{p}{2} \rfloor)) = \frac{p^2}{4}$ and $\tau_\beta(C_p(2, 4, \cdots, \lfloor \frac{p}{2} \rfloor)) = \frac{p^2}{4}$. Therefore $b = \frac{p^2}{2}$. By Proposition 2.3, we have $g_1 = \alpha(C_p(2, 4, \ldots, \lfloor \frac{p}{2} \rfloor)) = p-2$ and $g_2 = \beta(C_p(2, 4, \ldots, \lfloor \frac{p}{2} \rfloor)) = 2$, where g_1 and g_2 are the number of elements contained exactly in their respective blocks. From the above facts, we have $r = \frac{3p-4}{2}$. To obtain the m-associates for the elements, where $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$.

The two distinct elements are odd associates, if they have odd jump size and they are even associates $(2 \leq k \leq \lfloor \frac{p}{2} \rfloor)$. Hence, the parameters of first kind are given by $\nu = p$, $b = \frac{p^2}{2}$,

$g_1 = p-2$, $g_2 = 2$, $r = \dfrac{3p-4}{2}$, $\lambda_m(1) = p-4$ and

$$\lambda_m(2) = \begin{cases} 0, & m \equiv 0(mod\, 2), \\ 1, & \text{otherwise.} \end{cases}$$

where $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$.

Case 3. The circulant graph $C_p(2,4,\cdots,\lfloor \frac{p}{2} \rfloor)$ with $p = 4n+3$, $n \geq 1$ have p blocks of α-set and p blocks of β-sets. This implies that $\tau_\alpha(C_p(2,4,\cdots,\lfloor \frac{p}{2} \rfloor)) = \tau_\beta(C_p(2,4,\cdots,\lfloor \frac{p}{2} \rfloor)) = p$. Therefore $b = 2p$. By Proposition 2.3, we have $g_1 = \alpha(C_p(2,4,\ldots,\lfloor \frac{p}{2} \rfloor)) = p-2$ and $g_2 = \beta(C_p(2,4,\cdots,\lfloor \frac{p}{2} \rfloor)) = \left\lceil \dfrac{p}{2n+1} \right\rceil$, where g_1 and g_2 are the number of elements contained exactly in a block. From the above facts, we have $r = \dfrac{2(n+1)(p-1)}{2n+1}$. To obtain the m-associates for the elements, where $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$. The two distinct elements odd associates, if they have odd jump size and they are even associates $(2 \leq k \leq \lfloor \frac{p}{2} \rfloor)$.

Hence, the parameters of first kind are given by $\nu = p$, $b = 2p$, $g_1 = p-2$, $g_2 = \left\lceil \dfrac{p}{2n+1} \right\rceil$, $r = \dfrac{2(n+1)(p-1)}{2n+1}$ and

$$\lambda_m(1) = \begin{cases} 1, & \text{for } m = 1, \\ 0, & \text{for } 2 \leq m \leq \lfloor \frac{p-1}{2} \rfloor, \\ 2, & \text{for } m = \lfloor \frac{p}{2} \rfloor. \end{cases}$$

Thus the result follows. □

2.4 Circulant Graph $C_p(1,2,\cdots,\lfloor \frac{p}{2} \rfloor)$

The jump size of circulant graph is $(1,2,\cdots,\lfloor \frac{p}{2} \rfloor)$, known as complete graph K_p with $p \geq 3$ that is, $C_p\left(1,2,\cdots,\lfloor \frac{p}{2} \rfloor\right) \cong K_p$. Further, the complete graph K_p is strongly regular for all $p \geq 3$. The status of the trivial singleton graph K_1 is unclear. Since the parameter μ is not well defined on K_2. It is difficult to analyse and conclude whether K_2 is a strongly regular graph or not.

Proposition 2.4 ([11]) *For any circulant graph $C_p(1,2,\cdots,\lfloor \frac{p}{2} \rfloor)$ with $p \geq 3$ vertices,*

$$\alpha\left(C_p\left(1,2,\ldots,\left\lfloor \frac{p}{2} \right\rfloor\right)\right) = p-1 \text{ and } \beta\left(C_p\left(1,2,\ldots,\left\lfloor \frac{p}{2} \right\rfloor\right)\right) = 1.$$

Theorem 2.5 *The collection of all (α,β)-sets of a circulant graph $C_p(1,2,\cdots,\lfloor \frac{p}{2} \rfloor)$, $p \geq 3$ vertices form a PBD with parameters $\nu = p$, $b = 2p$, $g_1 = p-1$, g_2, $r = p$ and $\lambda_m = p-1$.*

Proof For a given circulant graph $C_p(1,2,\cdots,\lfloor \frac{p}{2} \rfloor)$; $\nu = p$ vertices labeled as $v_1, v_2, \cdots, v_p$. By Proposition 2.4, we have $\alpha(C_p(1,2,\cdots,\lfloor \frac{p}{2} \rfloor) = p-1$, and $\beta(C_p(1,2,\cdots,\lfloor \frac{p}{2} \rfloor) = \frac{p}{2}$. Further, $C_p(1,2,\cdots,\lfloor \frac{p}{2} \rfloor)$; have p blocks of (α,β)-sets, it implies $b = 2p$. Also, we have $g_1 = p-1$,

$g_2 = 1$, where g_1 and g_2 is the number of elements contained exactly in a block. From the above facts, we have $r = p$. To obtain the m-associates for the elements, where $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$. Two distinct elements are odd associates, if they have odd jump size and they are even associates $(2 \leq k \leq \lfloor \frac{p}{2} \rfloor)$. Hence the parameters of first kind are given by $\nu = p$, $b = 2p$, $g_1 = p - 1$, $g_2 = 1$, $r = p$, $\lambda_m = p - 1$, $1 \leq m \leq \lfloor \frac{p}{2} \rfloor)$. Thus, the result follows. □

§3. Conclusion

The construction and analysis of PBD's involve the combinatorial mathematics and statistical techniques, which have applications in various fields, including agriculture, biology, medicine, and social sciences. They are used in situations where it is important to compare treatments or conditions in a systematic and balanced way, while minimizing the number of required comparisons. This allows researchers to obtain reliable and statistically valid results with a reduced number of experimental runs or observations. Generally, the PBD's obtained from the families of strongly regular graphs. Interestingly, we determine the total number of (α, β)-sets with its association schemes in different jump sizes of some circulant graphs.

Finally, we pose the following open problem.

Problem 3.1 *Obtain the PBD's associated with (α, β)-sets of an integral circulant graph, a regular circulant graph or a Cayley graph.*

Acknowledgements

The authors wish to thank Prof.N.D.Soner for his help and valuable suggestions in the preparation of this article.

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