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# Perfect Roman Domination of Some Cycle Related Graphs

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**Abstract**: In this paper, we continue the study of perfect Roman dominating functions in graphs. A perfect Roman dominating PRD function on a graph $G = (V, E)$ is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex $v$ with $f(v) = 0$ is adjacent to exactly one vertex neighbor $u$ with $f(u) = 2$. The weight of PRD function is the sum of its function values over all the vertices. The perfect Roman domination number of $G$ denoted by $\gamma_R^P(G)$ is the minimum weight of a PRD function in $G$. We present the perfect Roman domination number of some cycle related graphs such as helm graphs, sunlet graphs and flower snark graphs. If $G$ is a spider web graph, we show that $\gamma_R^P(G) \leq \frac{2}{3}|G|$.

**Key Words**: Perfect Roman domination, Smarandachely perfect Roman domination, domination number, web graph.

**AMS(2010)**: 05C69, 05C76.

## §1. Introduction and Preliminaries

Let $G = (V, E)$ be an undirected simple graph with vertex set $V$ and edge set $E$. The graph $G$ has order $n = |V|$. For every vertex $v \in V$, the open neighborhood of $v$ is the set $\{u|uv \in E\}$ denoted by $N(v)$ and the closed neighborhood of $v$ is the set $N(v) \cup \{v\}$ denoted as $N[v]$. The cardinality of $N(v)$ is the degree of vertex $v$ denoted by $d_G(v) = |N(v)|$. A vertex of degree one is called a pendant vertex. We denote the cycle graph with $n$ vertices by $C_n$. A *wheel* graph $W_n$ is the join of $K_1 + C_n$. The vertex corresponding to $K_1$ in $W_n$ is known as apex vertex and vertices of the cycle $C_n$ in $W_n$ are rim vertices. And a *helm* graph $H_n$ is the graph obtained from a wheel graph $W_n$ by attaching a pendant vertex to each rim vertex, a sunlet graph denoted by $L_n$ is a graph that contains a cycle $C_{\frac{n}{2}}$ with pendant vertex attached to each vertex of the cycle $C_{\frac{n}{2}}$.

A *dominating set* of $G$ is a set $V \setminus D$ such that every vertex not in $D$ has a neighbor in $D$. The minimum cardinality of all dominating sets of $G$ is the *domination number*, denoted as $\gamma(G)$. More on dominating sets can be found in [9].

A *Roman dominating function* of a graph $G$, denoted as a *RD-function* is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u with $f(u) = 0$ is adjacent to at least one vertex $v$ with $f(v) = 2$. The weight of a vertex $v$ is its value, $f(v)$, assigned to it

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under $f$. The weight, $wt_f$, of $f$ is the sum, $\sum_{u\in V(G)} = f(u)$, of the weights of the vertices. The *Roman domination number*, denoted $\gamma_R(G)$ is the minimum weight of an RD-function in $G$; i.e. $\gamma_R(G) = min\{wt_f | f \text{ is an } RD - \text{function in } G\}$. In [7], Roman domination was first studied, see the papers [2], [4], [5], [8], [12],[13], [14], [15] for more results on Roman domination number. In [6], upper bounds for Roman domination were considered.

A *perfect Roman dominating function* (or *PRD-function*) on $G$ is a Roman dominating function $f(V) = \{0, 1, 2\}$ on $G$ such that for each vertex $u$ with $f(u) = 0$ there exists exactly one vertex $v$ with $f(v) = 2$, for which $uv \in E(G)$ and contrarily, a *Smarandachely perfect Roman domination* is such a Roman dominating function $f(V) = \{0, 1, 2\}$ on $G$ that each vertex $u$ with $f(u) = 0$ has at least two vertices $v$ with $f(v) = 2$ in its neighborhood $N_G(u)$. Generally, the *perfect Roman domination number* of $G$, denoted by $\gamma_R^P(G)$ is the minimum weight of a $PRD$-function on $G$. A $PRD$-function $f$ with weight $wt_f(G) = \gamma_R^P(G)$ is called $\gamma_R^P(G)$-function of $G$. The perfect Roman domination is a variation of the Roman domination. It was introduced and first studied in 2018 by Henning et al. [11] and further studied in [10] for regular graphs. More recent studies on perfect Roman domination number can be found in [1], [16], [17], [3].

In this paper, we continue the study of perfect Roman domination number for some irregular graphs which are cycle related graphs, mainly Helm graphs, Sunlet graphs and spider web graphs. Also, we gave perfect Roman domination number of flower snark graph.

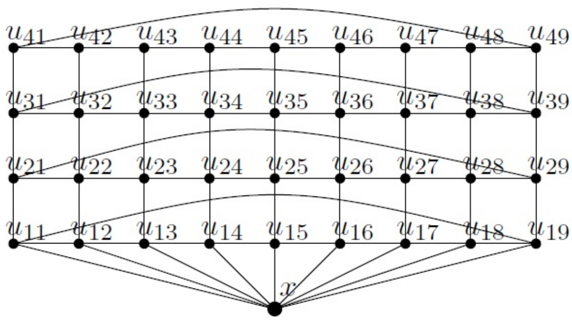


**Figure 1.** Web graph $W(4, 9)$

## §2. Perfect Roman Domination Number of Cycle Related Graphs

In this section, we shall consider the perfect Roman dominating functions of Helm graph, sunlet graph and flower snark graph. We begin with the following definition.

**Definition** 2.1 (Flower Snark Graph) *For $n \geq 3$, take the union of $n$ copies of $K_{1,3}$. Denote the vertex with degree 3 in the i-th copy of $K_{1,3}$ as $x^i$, $1 \leq i \leq n$ and the other three vertices in the i-th copy as $w^i, y^i, z^i$. Next, construct cycle $C_n$ through vertices $w^1, w^2, \cdots, w^n$ and cycle $C_{2n}$ through vertices $y^1, y^2, \cdots, y^n, z^1, z^2, \cdots, z^n$. Let the $i-th$ copy of $K_{1,3}$ be denoted by $J^i$ and its vertices are $x^i, w^i, y^i, z^i$. Denote flower snark graph by $J_n$ and note that $J_n$ contains $n$ copies of $K_{1,3}$.*

Next, we have the following results on the perfect Roman domination number of Helm graph, sunlet graph and flower snark graph.

**Theorem** 2.1 *Let $H_n$ be a Helm graph forn $\geq 9$. Then,* $\gamma_R^P(H_n) = \frac{n-1}{2} + 2$.

*Proof* Let $x, z_i, y_i, 1 \leq i \leq \frac{n-1}{2}$ denotes the center vertex, vertices of the cycle contained in $H_n$ and pendant vertices respectively. Define a function $f : V(H_n) \to \{0, 1, 2\}$ as follow: $f(x) = 2$, $f(z_i) = 0$, $f(y_i) = 1$, where $1 \leq i \leq \frac{n-1}{2}$. It+ is clear to see from the labeling that $f$ is a perfect Roman dominating function of $H_n$ since each vertex with label 0 is adjacent to exactly one vertex with label 2. Hence, from the above labeling

$$\gamma_R^P(H_n) = \frac{n-1}{2} + 2.$$

Assume that $\gamma_R^P(H_n) < \frac{n-1}{2} + 2$. Then we split the problem into the following two cases.

**Case** 1. $f(x) < 2$.

In this case, $\sum\limits_{i=1}^{\frac{n-1}{2}} f(z_i) > 2$ which implies that

$$wt_f(H_n) > \frac{n-1}{2} + 2,$$

which is a contradiction.

**Case** 2. $\sum\limits_{i=1}^{\frac{n-1}{2}} f(y_i) < \frac{n-1}{2}$.

In this case, $f(y_t) = 0$ for some $1 \leq t \leq \frac{n-1}{2}$. If $f(y_t) = 0$, this implies that $1 \leq f(z_t) \leq 2$, which will give the condition in Case 1, that is

$$wt_f(H_n) > \frac{n-1}{2} + 2$$

a contradiction. Hence $\gamma_R^P(H_n) = \frac{n-1}{2} + 2$. $\square$

**Theorem** 2.2 *Let $L_n$ be a sunlet graph for $n \geq 6$. Then,*

$$\gamma_R^P(L_n) = \begin{cases} \frac{2}{3}n, & \textit{if } n \equiv 0 \text{mod } 3 \\ \frac{2}{3}(n+2), & \textit{if } n \equiv 1 \text{mod } 3 \\ \frac{2}{3}(n+1), & \textit{if } n \equiv 2 \text{mod } 3. \end{cases}$$

*Proof* Notice that a sunlet graph $L_n$ consist of cycle $C_t, t = \frac{n}{2}$ with pendant vertex attached to each vertex of the cycle $C_t$. Let $x_1, x_2, \cdots, x_t$ be the vertices of the cycle in $L_n$ and $y_1, y_2, \cdots, y_t$ be the pendant vertices in $L_n$ such that $x_i y_i \in E(L_n)$ for $1 \leq i \leq t$. Next, we describe the construction of perfect Roman domination of $L_n$. For any $t = 3q + r$, where $q \geq 1$ and $0 \leq r \leq 2$, partition $V(C_t) = \{x_1, x_2, \cdots, x_t\}$ into $q$ sets of cardinality 3. Define a function

$f : V(L_n) \to \{0,1,2\}$ as follows: Assign labeling 2 to a single vertex and 0 to the remaining vertices in each $q$ set. such that vertex with label 2 in $q_i$ is not adjacent to any vertices in $q_{i+1}$. Furthermore, assign 1 to $r$ vertices. Assign 0 to $q$ pendant vertices $y_i$ such that for every $x_iy_i \in E(L_n)$, $f(x_i) = 2$. Lastly, assign label 1 to the remaining pendant vertices. From the above labeling, the function $f$ is a perfect Roman dominating function of $L_n$ since each vertex with label 0 is adjacent to exactly one vertex with label 2. Thus we have

$$\begin{aligned} wt_f(L_n) &= \sum_{i=1}^{t} f(x_i) + \sum_{i=1}^{t} f(y_i) \\ &= 2q + r + t - q = 2q + r + 3q + r - q = 4q + 2r. \end{aligned} \tag{1}$$

Now, we consider the problem in the following three cases.

**Case** 1. $n \equiv 0 \mod 3$ i.e. $r = 0$ and $t = 3q$.

By equation (1),

$$wt_f(L_n) = 4q = \frac{4}{3}.t = \frac{2}{3}n \quad \text{because of} \quad t = \frac{n}{2}.$$

**Case** 2. $n \equiv 1 \mod 3$ i.e. $r = 2$ and $t = 3q + 2$.

By equation (1),

$$\begin{aligned} wt_f(L_n) &= 4q + 4 = \frac{4}{3} \cdot t + 4 \\ &= \frac{2}{3}n + \frac{4}{3} \quad (\text{since } t = \frac{n}{2}) \\ &= \frac{2}{3}(n+2). \end{aligned}$$

**Case** 3. $n \equiv 2 \mod 3$, i.e. $r = 1$ and $t = 3q + 1$.

By equation (1),

$$\begin{aligned} wt_f(L_n) &= 4q + 2 = \frac{4}{3} \cdot t + 2 \\ &= \frac{2}{3}n + \frac{2}{3} \quad (\text{since } t = \frac{n}{2}) \\ &= \frac{2}{3}(n+1). \end{aligned}$$

Hence, we get the result. □

**Theorem** 2.3 *Let $J_n$ be a flower snark graph for $n \geq 3$. Then, $\gamma_R^P(J_n) = 2n$.*

*Proof* Define a function $f : V(J_n) \to \{0,1,2\}$ as follows:

$$f(x^i) = 2, f(w^i) = f(y^i) = f(z^i) = 0, \ 1 \leq i \leq n.$$

The function $f$ gives perfect Roman dominating function since each vertex with label 0 is adjacent to only one vertex with label 2. Thus we have $wt_f(J_n) = 2n$.

Assume that $wt_f(J_n) < 2n$, then we have that $f(x^t) < 2$ for some $t < n$. If $f(x^t) < 2$, then $f(w^t \neq 0$, $f(y^t) \neq 0$, $f(z^t) \neq 0$. This implies that the copy $J^t$ will have weight greater than 2. Assume that the statement is true for $J_{n-1}$, that is, $\gamma_R^P(J_{n-1}) = 2(n-1)$. Then we have

$$\begin{aligned} wt_f(J_n) &= wt_f(J^i) + wt_f(J^t), \quad i = 1, 2, \cdots, n-1 \\ &= 2(n-1) + wt_f(J^t) > 2n. \end{aligned}$$

since $wt_f(J^t) > 2$, a contradiction. Hence, $\gamma_R^P(J_n) = 2n$. □

## §3. Perfect Roman Domination Number of Spider Web Graph

The following result present the upper bound for the perfect Roman domination number of a spider web graph. We begin with the following definition.

**Definition** 3.1 (Web graph) *Let $p, q \geq 5$, the spider web graph $W(p, q)$ is constructed from $p$ cycles of length $q$ and a vertex $x$, as shown in Figure* 1. *Let $\{u_1, u_2, \cdots, u_q\}$ be the vertices in each cycle and let $x$ denote the center vertex. The spider web graph has $p$ rows and $q$ columns, where all the vertices on the row $p-th$ row are adjacent to a common vertex $x$. Denote the vertex in row $i$ and column $j$ by $u_{ij}, 1 \leq i \leq p$ and $1 \leq j \leq q$. The vertex set of*

$$V(W(p,q)) = \{x, u_{11}, u_{12}, \cdots, u_{1q}, u_{21}, u_{22}, \cdots, u_{2q}, u_{31}, u_{32}, \cdots, u_{3q}, \cdots, u_{p1}, u_{p2}, \cdots, u_{pq}\}.$$

*To define the edge set of this graph, let*

$$\begin{aligned} A_i &= \{u_{i1} - u_{i2}, u_{i2} - u_{i3}, \cdots, u_{i(q-1)} - u_{iq}, u_{iq} - u_{i1}\} \\ B_j &= \{x - u_{pj}, u_{pj} - u_{p-1j}, \cdots, u_{(2)j} - u_{1j}\}. \end{aligned}$$

*Then, $E(W(p,q)) = \bigcup_{i,j}(A_i \cup B_j)$. A web graph $W(p,q)$ has $pq+1$ vertices.*

The next result gives the upper bound of the perfect Roman domination number of spider web graph.

**Theorem** 3.1 *Let $p, q \geq 5$. If $G = W(p,q)$ then, $\gamma_R^P(G) \leq \frac{2}{3}|G|$.*

*Proof* Note that the web graph $W(p,q)$ contains $p$ rows, $q$ columns and a common vertex, i.e. $W(p,q)$ contains $pq+1$ vertices. The problem will be split into the following 3 cases.

**Case** 1. $q \equiv 0$ *mod* 3 or $p \equiv 0$ *mod* 3.

If $q = 3k$ for some $k$, for $i = 1, 2, 3, \cdots, p-1$, label each vertex in the column $2+3t$ for $t \in \{0, 1, \cdots, k\}$ and the common vertex $x$ with 2. When $i = p$, label the vertices on the column $2+3t$ for $t \in \{0, 1, \cdots, k\}$ with 1 and the remaining vertices with 0, such as those shown in Figure 2.

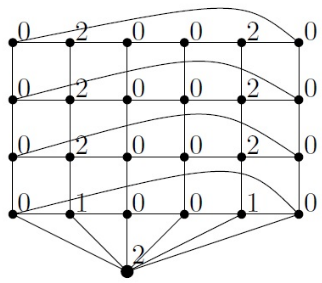


**Figure 2.** Web graph $W(4, 6)$

The labelling holds for all $p$. It is easy to see that the labeling in Figure 2 gives a perfect Roman domination since each vertex label with 0 is adjacent to exactly one vertex with the label 2.

Let assume that $q = 3k$ and let $f$ be the perfect Roman domination functing on $G$. If $j = 2 + 3t$, $t \in \{0, 1, \cdots, k-1\}$,

$$\begin{aligned} \sum_{i=p} &= 2(p-1)(\frac{q}{3}) \\ f(x) &= 2. \end{aligned}$$

Thus,

$$\begin{aligned} wt_f(G) &= 2(p-1)(\frac{q}{3}) + \frac{q}{3} + 2 = \frac{2pq}{3} + 2 - \frac{q}{3} \\ &\leq \frac{2pq}{3} + \frac{2}{3} \quad \text{(since } q > 5) \\ &= \frac{2}{3}(pq+1) = \frac{2}{3}|G| \end{aligned}$$

We need the following functions for the remaining cases.

Define a function $f : V(G) \to \{0, 1, 2\}$ as follows

$$f(u_{i,j}) = \begin{cases} 2, & \text{if } i \equiv 0 \text{mod } 3, i \neq p \text{ and } j \equiv 3 \text{mod } 6 \\ 2, & \text{if } i \equiv 1 \text{mod } 3, i \neq p \text{ and } j \equiv 1 \text{mod } 6 \\ 2, & \text{if } i \equiv 2 \text{mod } 3, i \neq p \text{ and } j \equiv 5 \text{mod } 6 \\ 1, & \text{if } i \equiv 0 \text{mod } 3, i \neq p \text{ and } j \equiv 0 \text{mod } 6 \\ 1, & \text{if } i \equiv 1 \text{mod } 3, i \neq p \text{ and } j \equiv 4 \text{mod } 6 \\ 1, & \text{if } i \equiv 2 \text{mod } 3, i \neq p \text{ and } j \equiv 2 \text{mod } 6 \\ 0, & \text{otherwise.} \end{cases}$$

and $f(x) = 2$. The function $f$ reoccur at every six columns and at every three rows. The labeling above gives a perfect Roman dominating function $f$ since each vertex $u_{ij}$ with label zero is adjacent to only one vertex with label 2.

**Case** 2. $q \equiv 1 \ mod \ 3$.

Next, we divide the proof into the following three subcases.

**Subcase 2.1** $p \equiv 0 \quad mod \quad 3$ and $q \equiv 1\text{mod } 6$.

In this subcase, define the function $f^* : V(G) \to \{0, 1, 2\}$ as follows:

$$f^*(u_{i,j}) = \begin{cases} 1, & \text{if } i = 1, \text{ and } j \equiv 3\text{mod } 6 \\ 1, & \text{if } i = p-1, \text{ and } j \not\equiv 1\text{mod } 6 \\ f(u_{ij}), & \text{otherwise.} \end{cases}$$

. Then, the function $f^*$ is a PRD on $G$, see Figure 3.

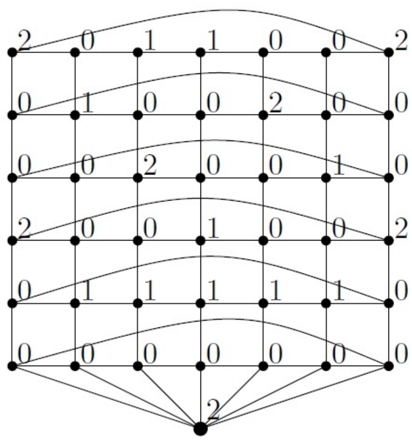


**Figure 3.** Web graph $W(p,q)$, $p \equiv 0\text{mod } 6$ and $q \equiv 1\text{mod } 6$

Notice that

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-3)}{3} + 2 \text{ if } j \equiv 1\text{mod } 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-3)}{3} + 1 \text{ if } j \equiv 2\text{mod } 6,$$
$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-3)}{3} + 2 \text{ if } j \equiv 3\text{mod } 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-3)}{3} + 2\text{if } j \equiv 4\text{mod } 6,$$
$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-3)}{3} + 1 \text{ if } j \equiv 5\text{mod } 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-3)}{3} + 1 \text{ if} j \equiv 6\text{mod } 6.$$

Then, we have

$$\begin{aligned} wt_f(G) &= \left(9\left(\frac{p-3}{3}\right) + 9\right)\left(\frac{q-1}{6}\right) + 2\left(\frac{p-3}{3}\right) + 2 + 2 \\ &= \frac{1}{2}pq + \frac{1}{6}p + 2 \le \frac{1}{2}pq + \frac{1}{6}pq + \frac{2}{3} \\ &= \frac{2}{3}pq + \frac{2}{3} = \frac{2}{3}(pq+1) = \frac{2}{3}|G| \end{aligned}$$

**Subcase 2.2** $p = 3y + 1$, for some integer $y$.

In this subcase, if $q \equiv 1 \bmod 6$ define a function $f^* : V(G) \to \{0, 1, 2\}$ as follows:

$$f^*(u_{i,j}) = \begin{cases} 1, & \text{if } i = 1, \text{ and } j \equiv 3 \bmod 6 \\ 1, & \text{if } i = p-2 \text{ and } j \equiv 3 \bmod 6 \\ 1, & \text{if } i = p-1, \text{ and } j \not\equiv 5 \bmod 6 \\ 1, & \text{if } i = p-2, \text{ and } j = 3 \bmod 6 \\ f(u_{ij}), & \text{otherwise.} \end{cases}$$

The function $f^*$ is a PRD on $G$, see Figure 4. Notice that

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-4)}{3} + 3 \text{ if } j \equiv 1 \bmod 6, \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-4)}{3} + 2 \text{ if } j \equiv 2 \bmod 6,$$
$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-4)}{3} + 3 \text{ if } j \equiv 3 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-4)}{3} + 2 \text{ if } j \equiv 4 \bmod 6,$$
$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-4)}{3} + 2 \text{ if } j \equiv 5 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-4)}{3} + 1 \text{ if } j \equiv 6 \bmod 6.$$

Then, we have

$$\begin{aligned} wt_f(G) &= \left(9\left(\frac{p-4}{3}\right) + 13\right)\left(\frac{q-1}{6}\right) + 2\left(\frac{p-4}{3}\right) + 3 + 2 \\ &= \frac{1}{2}pq + \frac{1}{6}p + \frac{1}{6}q + \frac{13}{6} \le \frac{1}{2}pq + \frac{1}{6}pq + \frac{2}{3} \\ &= \frac{2}{3}pq + \frac{2}{3} = \frac{2}{3}(pq+1) = \frac{2}{3}|G| \end{aligned}$$

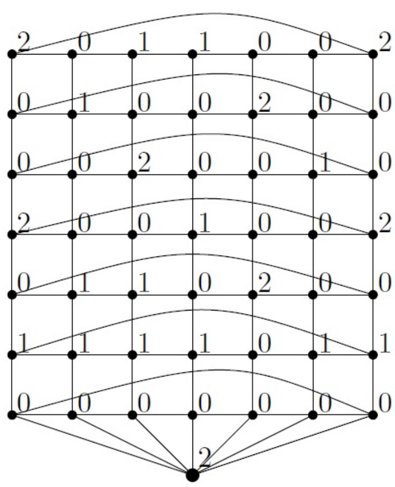


**Figure 4.** Web graph $W(p, q)$, $p \equiv 1 \bmod 6$ and $q \equiv 1 \bmod 6$

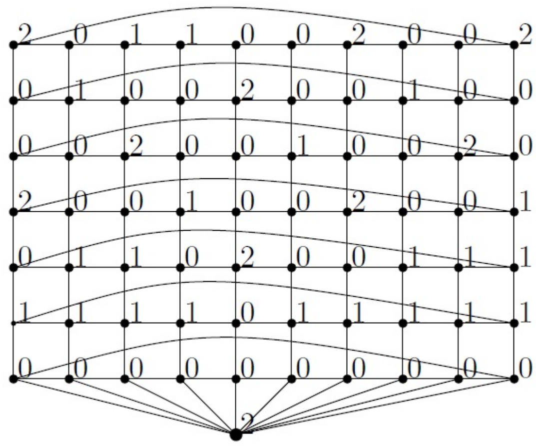

**Figure 5.** Web graph $W(p,q)$, $p \equiv 1 \bmod 6$ and $q \equiv 4 \bmod 6$

If $q \equiv 4 \bmod 6$, define a function $f^*: V(G) \to \{0,1,2\}$ such that

$$f^*(u_{i,j}) = \begin{cases} 1, & \text{if } i = p-1, \text{ and } j \not\cong 5 \bmod 6 \\ 2, & \text{if } i = 1 \text{ and } j = q \\ 1, & \text{if } i = 1, \text{ and } j \equiv 3 \bmod 6 \text{ and } j \neq q-1 \\ 1, & \text{if } i = p-2, \text{ and } j \equiv 3 \bmod 6 \\ 1, & \text{if } i \equiv 2 \bmod 3, i \neq 2, j = q \\ f(u_{ij}), & \text{otherwise.} \end{cases}$$

Therefore, $f^*$ is a PRD function on $G$, see Figure 5. Notice that

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-4)}{3} + 3 \text{ if } j \equiv 1 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-4)}{3} + 2 \text{ if } j \equiv 2 \bmod 6,$$

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-4)}{3} + 3 \text{ if } j \equiv 3 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-4)}{3} + 2 \text{ if } j \equiv 4 \bmod 6,$$

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-4)}{3} + 2 \text{ if } j \equiv 5 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-4)}{3} + 1 \text{ if } j \equiv 6 \bmod 6.$$

Now, we have

$$\begin{aligned} wt_f(G) &= \left(9\left(\frac{p-4}{3}\right) + 13\right)\left(\frac{q-4}{6}\right) + 7\left(\frac{p-4}{3}\right) + 10 + 2 \\ &= \frac{1}{2}pq + \frac{1}{3}p + \frac{1}{6}q + 2 \leq \frac{1}{2}pq + \frac{1}{6}pq + \frac{2}{3}, \quad (p,q \geq 7) \\ &= \frac{2}{3}pq + \frac{2}{3} = \frac{2}{3}(pq+1) = \frac{2}{3}|G|. \end{aligned}$$

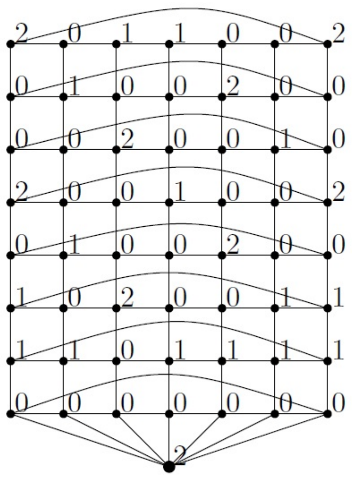


**Figure 6.** Web graph $W(p,q)$ with function $f^*$, $q = 1\text{mod } 6$ and $p = 3y+2$

**Subcase 2.3** $p = 3y + 2$ for some integer $y$.

Let $q \equiv 1\text{mod } 6$. Define the function $f^* : V(G) \to \{0, 1, 2\}$ as follows:

$$f^*(u_{i,j}) = \begin{cases} 1, & \text{if } i = 1, \text{ and } j \equiv 3\text{mod } 6 \\ 1, & \text{if } i = p-2 \text{ and } j \equiv 1\text{mod } 6 \\ 1, & \text{if } i = p-1, \text{ and } j \not\equiv 3\text{mod } 6 \\ f(u_{ij}), & \text{otherwise.} \end{cases}$$

Therefore, $f^*$ is a PRD function on $G$, see Figure 6. Notice that

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-5)}{3} + 4 \text{ if } j \equiv 1\text{mod } 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-5)}{3} + 2 \text{ if } j \equiv 2\text{mod } 6,$$

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-5)}{3} + 3 \text{ if } j \equiv 3\text{mod } 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-5)}{3} + 2 \text{ if } j \equiv 4\text{mod } 6,$$

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-5)}{3} + 3 \text{ if } j \equiv 5\text{mod } 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-5)}{3} + 2 \text{ if } j \equiv 6\text{mod } 6.$$

Now, we have

$$\begin{aligned} wt_f(G) &= \left(9\left(\frac{p-5}{3}\right) + 16\right)\left(\frac{q-1}{6}\right) + 2\left(\frac{p-5}{3}\right) + 4 + 2 \\ &= \frac{1}{2}pq + \frac{1}{6}p + \frac{1}{6}q + \frac{15}{6} \le \frac{1}{2}pq + \frac{1}{6}pq + \frac{2}{3}, \quad (p \ge 8, q \ge 7) \\ &= \frac{2}{3}pq + \frac{2}{3} = \frac{2}{3}(pq+1) = \frac{2}{3}|G| \end{aligned}$$

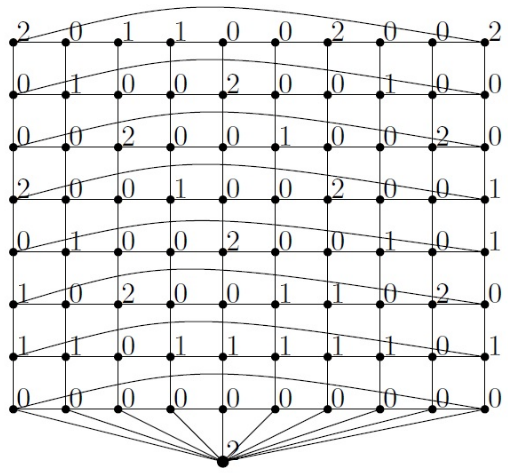

**Figure 7.** Web graph $W(p,q)$ with function $f^*$, $q = 4\text{mod } 6$ and $p = 3y + 2$

If $q \equiv 4\text{mod } 6$, define a function $f^* : V(G) \to \{0,1,2\}$ as follows:

$$f^*(u_{i,j}) = \begin{cases} 1, & \text{if } i = 1, \text{ and } j \equiv 3\text{mod } 6, j \neq q-1 \\ 1, & \text{if } i = p-2 \text{ and } j \equiv 1\text{mod } 6 \\ 1, & \text{if } i = p-1, \text{ and } j \not\equiv 3\text{mod } 6 \\ 0, & \text{if } i = p-1, \text{ and } j \equiv 3\text{mod } 6 \\ 2, & \text{if } i = 1, \text{ and } j = q \\ 1, & \text{if } i \equiv 2\text{mod } 3, j = q, \\ f(u_{ij}), & \text{otherwise.} \end{cases}$$

Then, $f^*$ is a PRD function on $G$, see Figure 7. Notice that

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-5)}{3} + 3 \text{ if } j \equiv 1\text{mod } 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-5)}{3} + 2 \text{ if } j \equiv 2\text{mod } 6,$$
$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-5)}{3} + 3 \text{ if } j \equiv 3\text{mod } 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-5)}{3} + 2 \text{ if } j \equiv 4\text{mod } 6,$$
$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-5)}{3} + 3 \text{ if } j \equiv 5\text{mod } 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-5)}{3} + 2 \text{ if } j \equiv 6\text{mod } 6.$$

Now, we have

$$\begin{aligned} wt_f(G) &= \left(9\left(\frac{p-5}{3}\right) + 16\right)\left(\frac{q-4}{6}\right) + 7\left(\frac{p-5}{3}\right) + 11 + 2 \\ &= \frac{1}{2}pq + \frac{1}{3}p + \frac{1}{6}q + \frac{2}{3} \leq \frac{1}{2}pq + \frac{1}{6}pq + \frac{2}{3}, \quad (p,q \geq 8) \\ &= \frac{2}{3}pq + \frac{2}{3} = \frac{2}{3}(pq+1) = \frac{2}{3}|G| \end{aligned}$$

**Case** 3. $q \equiv 2 \ mod \ 3$.

**Subcase** 3.1 If $p = 3y$, assume that $q \equiv 2 \text{mod } 6$. Then define a function $f^* : V(G) \to \{0, 1, 2\}$ as follows:

$$f^*(u_{i,j}) = \begin{cases} 1, & \text{if } i = 1, \text{ and } j \equiv 3 \text{mod } 6 \\ 1, & \text{if } i \equiv 1 \text{mod } 3, \text{ and } j = r \\ 1, & \text{if } i \equiv 0 \text{mod } 3, \text{ and } j = r \\ 1, & \text{if } i = p-1, \text{ and } j \not\equiv 1 \text{mod } 6 \\ f(u_{ij}), & \text{otherwise.} \end{cases}$$

Then, $f^*$ is a PRD function on $G$, see Figure 8.

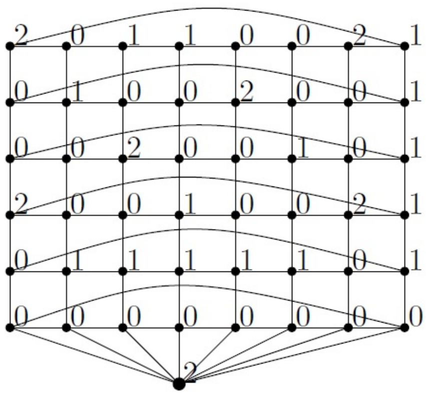

**Figure 8.** Web graph $W(p, q)$ with function $f^*$, $q \equiv 2 \text{mod } 6$ and $p = 3y$

Notice that

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-3)}{3} + 2 \text{ if } j \equiv 1 \text{mod } 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-3)}{3} + 1 \text{ if } j \equiv 2 \text{mod } 6,$$

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-3)}{3} + 2 \text{ if } j \equiv 3 \text{mod } 6; \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-3)}{3} + 2 \text{ if } j \equiv 4 \text{mod } 6,$$

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-3)}{3} + 1 \text{ if } j \equiv 5 \text{mod } 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-3)}{3} + 2 \text{ if } j \equiv 6 \text{mod } 6.$$

Thus,

$$\begin{aligned} wt_f(G) &= \left(9\left(\frac{p-3}{3}\right) + 9\right)\left(\frac{q-2}{6}\right) + 3\left(\frac{p-3}{3}\right) + 4 + 2 \\ &= \frac{1}{2}pq + 5 \leq \frac{1}{2}pq + \frac{1}{6}pq + \frac{2}{3}, \quad (p \geq 6, q \geq 8) \\ &= \frac{2}{3}pq + \frac{2}{3} = \frac{2}{3}(pq+1) = \frac{2}{3}|G| \end{aligned}$$

**Subcase 3.2** If $p = 3y + 1$, assume that $q \equiv 2 \bmod 6$. Then define a function $f^* : V(G) \to \{0, 1, 2\}$ as follows:

$$f^*(u_{i,j}) = \begin{cases} 1, & \text{if } i = 1, \text{ and } j \equiv 3 \bmod 6 \\ 1, & \text{if } i \equiv 1 \bmod 3, \text{ and } j = r \\ 1, & \text{if } i \equiv 0 \bmod 3, \text{ and } j = r \\ 1, & \text{if } i = p-2, \text{ and } j \equiv 3 \bmod 6 \\ 1, & \text{if } i = p-1, \text{ and } j \not\equiv 5 \bmod 6 \\ f(u_{ij}), & \text{otherwise.} \end{cases}$$

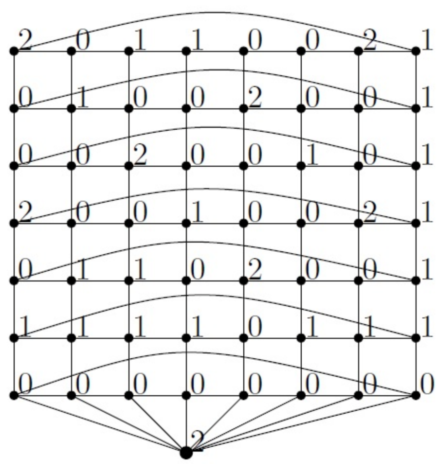

**Figure 9.** Web graph $W(p, q)$ with function $f^*$, $q \equiv 2 \bmod 6$ and $p = 3y + 1$

Then, $f^*$ is a PRD function on $G$, see Figure 9. Notice that

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-4)}{3} + 3 \text{ if } j \equiv 1 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-4)}{3} + 2 \text{ if } j \equiv 2 \bmod 6,$$
$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-4)}{3} + 3 \text{ if } j \equiv 3 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-4)}{3} + 2 \text{ if } j \equiv 4 \bmod 6,$$
$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-4)}{3} + 2 \text{ if } j \equiv 5 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-4)}{3} + 1 \text{ if } j \equiv 6 \bmod 6.$$

Thus,

$$\begin{aligned} wt_f(G) &= \left(9\left(\frac{p-4}{3}\right) + 13\right)\left(\frac{q-2}{6}\right) + 4\left(\frac{p-4}{3}\right) + 7 + 2 \\ &= \frac{1}{2}pq + \frac{1}{6}p + \frac{1}{6}q + \frac{10}{3} \leq \frac{1}{2}pq + \frac{1}{6}pq + \frac{2}{3}, \quad (p \geq 7, q \geq 8) \\ &= \frac{2}{3}pq + \frac{2}{3} = \frac{2}{3}(pq + 1) = \frac{2}{3}|G| \end{aligned}$$

Assume that $q \equiv 5 \bmod 6$ and $p = 3y + 1$. Define a function $f^* : V(G) \to \{0, 1, 2\}$ as follows:

$$f^*(u_{i,j}) = \begin{cases} 1, & \text{if } i = 1, \text{ and } j \equiv 3 \bmod 6 \\ 1, & \text{if } i \equiv 2 \bmod 3, \text{ and } j = q-1, q \\ 1, & \text{if } i \equiv 0 \bmod 3 \text{ and } j = q \\ 1, & \text{if } i = p-1, \text{ and } j \not\equiv 5 \bmod 6 \text{ and } j \neq q \\ 1, & \text{if } i = p-2, \text{ and } j \equiv 3 \bmod 6 \\ 1, & \text{if } i = p-1, j = q \\ f(u_{ij}), & \text{otherwise.} \end{cases}$$

Now, $f^*$ is a PRD function on $G$, see Figure 10.

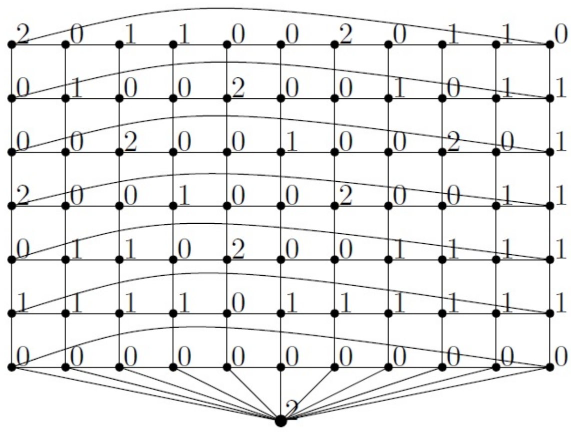

**Figure 10.** Web graph $W(p, q)$ with function $f^*$, $q \equiv 5 \bmod 6$ and $p = 3y + 1$

Notice that

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-4)}{3} + 3 \text{ if } j \equiv 1 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-4)}{3} + 2 \text{ if } j \equiv 2 \bmod 6,$$

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-4)}{3} + 3 \text{ if } j \equiv 3 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-4)}{3} + 2 \text{ if } j \equiv 4 \bmod 6,$$

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-4)}{3} + 2 \text{ if } j \equiv 5 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-4)}{3} + 1 \text{ if } j \equiv 6 \bmod 6.$$

Thus

$$\begin{aligned} wt_f(G) &= \left(9\left(\frac{p-4}{3}\right) + 16\right)\left(\frac{q-5}{6}\right) + 9\left(\frac{p-4}{3}\right) + 13 + 2 \\ &= \frac{1}{2}pq + \frac{1}{2}p + \frac{2}{3}q - \frac{1}{3} \leq \frac{1}{2}pq + \frac{1}{6}pq + \frac{2}{3}, \quad (p \geq 7, q \geq 11) \\ &= \frac{2}{3}pq + \frac{2}{3} = \frac{2}{3}(pq+1) = \frac{2}{3}|G| \end{aligned}$$

**Subcase 3.3** $q = 3k + 2$ and $p = 3y + 2$ for some positive integer $y$.

In this case, assume that $q \equiv 2 \bmod 6$. Define a function $f^* : V(G) \to \{0, 1, 2\}$ as follows:

$$f^*(u_{i,j}) = \begin{cases} 1, & \text{if } i = 1, \text{ and } j \equiv 3 \bmod 6 \\ 1, & \text{if } i \equiv 1 \bmod 3, \text{ and } j = q \\ 1, & \text{if } i \equiv 0 \bmod 3 \text{ and } j = q \\ 1, & \text{if } i = p - 2, \text{ and } j \equiv 1 \bmod 6 \\ 1, & \text{if } i = p - 1, j \not\equiv 3 \bmod 6 \\ f(u_{ij}), & \text{otherwise.} \end{cases}$$

Now, $f^*$ is a PRD function on $G$, see Figure 11.

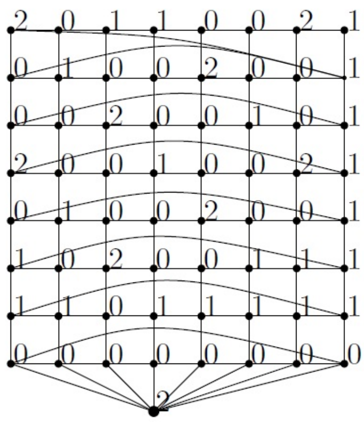

**Figure 11.** Web graph $W(p, q)$ with function $f^*$, $q \equiv 2 \bmod 6$ and $p = 3y + 2$

Notice that

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-5)}{3} + 4 \text{ if } j \equiv 1 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-5)}{3} + 2 \text{ if } j \equiv 2 \bmod 6,$$

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-5)}{3} + 3 \text{ if } j \equiv 3 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-5)}{3} + 2 \text{ if } j \equiv 4 \bmod 6,$$

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-5)}{3} + 2 \text{ if } j \equiv 5 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-5)}{3} + 2 \text{ if } j \equiv 6 \bmod 6.$$

Thus,

$$\begin{aligned} wt_f(G) &= \left(9\left(\frac{p-5}{3}\right) + 16\right)\left(\frac{q-2}{6}\right) + 5\left(\frac{p-5}{3}\right) + 8 + 2 \\ &= \frac{1}{2}pq + \frac{2}{3}p + \frac{1}{6}q + \frac{4}{3} \le \frac{1}{2}pq + \frac{1}{6}pq + \frac{2}{3}, \quad (p, q \ge 8) \\ &= \frac{2}{3}pq + \frac{2}{3} = \frac{2}{3}(pq + 1) = \frac{2}{3}|G| \end{aligned}$$

Assume that $q \equiv 5 \bmod 6$ and $p = 3y + 2$. We define a function $f^* : V(G) \to \{0, 1, 2\}$ as follows:

$$f^*(u_{i,j}) = \begin{cases} 1, & \text{if } i = 1, \text{ and } j \equiv 3 \bmod 6 \\ 1, & \text{if } i \equiv 2 \bmod 3, \text{ and } j = q-1, q \\ 1, & \text{if } i \equiv 0 \bmod 3 \text{ and } j = q \\ 1, & \text{if } i = p-2, \text{ and } j \equiv 1 \bmod 6 \\ 1, & \text{if } i = p-1, j \not\equiv 3 \bmod 6 \\ f(u_{ij}), & \text{otherwise.} \end{cases}$$

Therefore, $f^*$ is a PRD function on $G$, see Figure 12.

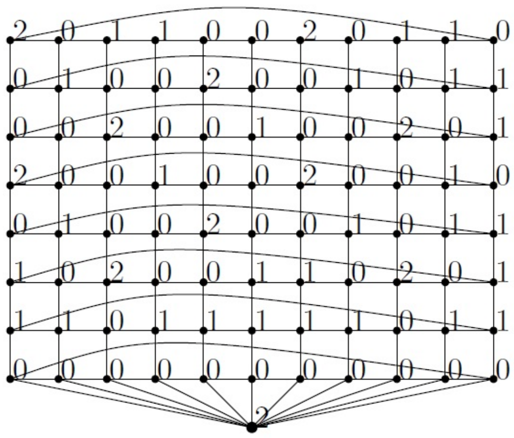

**Figure 12.** Web graph $W(p, q)$ with function $f^*$, $q \equiv 5 \bmod 6$ and $p = 3y + 2$

Notice that

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-5)}{3} + 4 \text{ if } j \equiv 1 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-5)}{3} + 2 \text{ if } j \equiv 2 \bmod 6,$$

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-5)}{3} + 3 \text{ if } j \equiv 3 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-5)}{3} + 2 \text{ if } j \equiv 4 \bmod 6,$$

$$\sum_{i=1}^{p} f^*(u_{i,j}) = \frac{2(p-5)}{3} + 3 \text{ if } j \equiv 5 \bmod 6; \quad \sum_{i=1}^{p} f^*(u_{i,j}) = \frac{(p-5)}{3} + 2 \text{ if } j \equiv 6 \bmod 6.$$

Thus,

$$\begin{aligned} wt_f(G) &= \left(9\left(\frac{p-5}{3}\right) + 16\right)\left(\frac{q-5}{6}\right) + 9\left(\frac{p-5}{3}\right) + 15 + 2 \\ &= \frac{1}{2}pq + \frac{1}{3}p + \frac{1}{6}q + \frac{7}{6} \leq \frac{1}{2}pq + \frac{1}{6}pq + \frac{2}{3}, \quad (p \geq 8, q \geq 11) \\ &= \frac{2}{3}pq + \frac{2}{3} = \frac{2}{3}(pq+1) = \frac{2}{3}|G| \end{aligned}$$

This completes the proof. $\square$

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