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Quarter Symmetric Metric Connection on Generalized Semi Pseudo Ricci Symmetric Manifold

Kalyan Halder

(Faculty of Science, Department of Mathematics, Sidho Kanho Birsha University, Purulia 723104, India)

Arindam Bhattacharyya

(Faculty of Science, Department of Mathematics, Jadavpur University, Kolkata 700032, India)

E-mail: halderk@rediffmail.com, bhattachar1968@yahoo.co.in

Abstract: Object of this paper is to find some properties of generalized semi pseudo Ricci symmetric manifold (denoted by $G(SPRS)_n$) admitting quarter symmetric metric connection. At last we have given an example of this manifold.

Key Words: Generalized semi pseudo Ricci symmetric manifold, semi pseudo Ricci symmetric manifold, pseudo Ricci symmetric manifold,quarter symmetric metric connection.

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§1. Introduction

The notion of locally symmetric and Ricci symmetric Riemannian manifold began with work of Cartan [10] and Eisenhert[8] respectively. A Riemannian manifold is said to be locally symmetric if its curvature tensor R satisfies the relation

$$\nabla R = 0, \tag{1}$$

where ∇ is the operator of covariant differentiation w.r.t. the metric tensor g. Again a Ricci symmetric manifold is a Riemannian manifold with the Ricci tensor S of type (0,2) satisfying

$$\nabla S = 0. \tag{2}$$

After them these notions have flowed in several branches such as recurrent manifold, Ricci-recurrent manifold, semi-symmetric manifold, pseudo-symmetric manifold[4], pseudo Ricci-symmetric manifold[6] and so on.

A non flat Riemannian manifold $(M^n, g), (n > 2)$ is said to be pseudo Ricci symmetric manifold $((PRS)_n)$[5] if Ricci tensor S is not identically zero and satisfies

$$(\nabla_X S)(Y, Z) = 2A(X)S(Y, Z) + A(Y)S(X, Z) + A(Z)S(X, Y), \tag{3}$$

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where A is nonzero 1-form satisfying

$$g(X,U) = A(X) \tag{4}$$

for a particular vector field U.

A non flat Riemannian manifold $(M^n, g), (n > 2)$ is said to be semi pseudo Ricci symmetric manifold $((SPRS)_n)$[2] if Ricci tensor S is not identically zero and satisfies

$$(\nabla_X S)(Y,Z) = A(Y)S(X,Z) + A(Z)S(X,Y), \tag{5}$$

where A is nonzero 1-form satisfying

$$g(X,U) = A(X) \tag{6}$$

for a particular vector field U.

A non flat Riemannian manifold $(M^n, g), (n > 2)$ is said to be generalised semi pseudo Ricci symmetric manifold $(G(SPRS)_n)$[1] if Ricci tensor S is not identically zero and satisfies

$$(\nabla_X S)(Y,Z) = A(Y)S(X,Z) + B(Z)S(X,Y) \tag{7}$$

where A, B are nonzero 1-forms satisfying

$$g(X,V) = A(X) \tag{8}$$

$$g(X,W) = B(X) \tag{9}$$

for particular vector fields V, W respectively.

From the above definition we observe that when $\delta = A - B$ is identically zero, $G(SPRS)_n$ reduces $(SPRS)_n$.

Consider a non flat Riemannian manifold $(M^n, g), (n > 2)$with a Riemannian connection ∇. We define a linear connection D on M by

$$D_X Y = \nabla_X Y + H(X,Y), \tag{10}$$

where H is a tensor field of type $(0,2)$.

Now D is said to be quarter symmetric connection on M if the torsion tensor $\bar{T}$ with respect to D satisfies

$$\bar{T}(X,Y) = \eta(Y)QX - \eta(X)QY, \tag{11}$$

where Q is the symmetric endomorphism of the tangent space at each point of a $G(SPRS)_n$ corresponding to the Ricci tensor S.

D is said to be metric connection if

$$(D_X g)(Y,Z) = 0. \tag{12}$$

The above relations will be used in the followings.

§2. Preliminaries

Let a non flat Riemannian manifold $(M^n, g), (n > 2)$ which is $G(SPRS)_n$ and Q be the symmetric endomorphism of the tangent space at each point of M corresponding to the Ricci tensor S. Then M satisfies (7) and

$$g(QX, Y) = S(X, Y). \tag{13}$$

Again from (7) we can get

$$(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = A(Y)S(X, Z) - A(X)S(Y, Z). \tag{14}$$

Contracting above with respect to Y and Z, we have

$$dr(X) = 2\bar{A}(X) - 2A(X)r, \tag{15}$$

where $\bar{A}(X) = A(QX)$.

Now let ∇ be the Riemannian connection and D be a quarter symmetric metric connection on M and $\bar{T}$ is the torsion tensor with respect to D. Then from (10) and (12), we have,

$$g(H(X, Y), Z) + g(H(X, Z), Y) = 0. \tag{16}$$

From (10) and (11) we can obtain,

$$H(X, Y) - H(Y, X) = A(Y)QX - A(X)QY. \tag{17}$$

Then (16) and (17) gives us the following,

$$H(X, Y) = A(Y)QX - S(X, Y)V, \tag{18}$$

where V is a particular vector field such that $g(X, V) = A(X)$. Then (10) can be written as

$$D_X Y = \nabla_X Y + A(Y)QX - S(X, Y)V. \tag{19}$$

§3. Curvature Tensor with Respect to a Quarter Symmetric Metric Connection on $G(SPRS)_n$

Let R, S, r be the curvature tensor, Ricci tensor, and scalar curvature respectively with respect to Reimannian connection ∇ on M. Again let $\bar{R}, \bar{S}, \bar{r}$ be the curvature tensor, Ricci tensor, and scalar curvature respectively with respect to a quarter symmetric metric connection D on M. Then

$$\bar{R}(X, Y, Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z. \tag{20}$$

Then (14), (19) and (20) gives us,

$$\begin{aligned}\bar{R}(X,Y,Z) =& R(X,Y,Z) + [(\nabla_X A)(Z) - A(X)A(Z) + \frac{1}{2}S(X,Z)A(V)]QY \\ & - [(\nabla_Y A)(Z) - A(Y)A(Z) + \frac{1}{2}S(Y,Z)A(V)]QX \\ & + S(X,Z)[\nabla_Y V - A(Y)V + \frac{1}{2}A(V)QY] \\ & - S(Y,Z)[\nabla_X V - A(X)V + \frac{1}{2}A(V)QX].\end{aligned} \tag{21}$$

Now let

$$\lambda(X,Z) = (\nabla_X A)(Z) - A(X)A(Z) + \frac{1}{2}S(X,Z)A(V) = g(LX,Z), \tag{22}$$

where L is the symmetric endomorphism corresponding to the Ricci tensor with respect to the quarter symmetric metric connection D on M.

Hence (21) can be reduced to

$$\bar{R}(X,Y,Z) = R(X,Y,Z) + \lambda(X,Z)QY - \lambda(Y,Z)QX + S(X,Z)LY - S(Y,Z)LX. \tag{23}$$

Thus we can state the followings.

Theorem 1 *If an $G(SPRS)_n$ admits a quarter symmetric metric connection D, then the curvature tensor with respect to D is of the form* (23).

Corollary 1 *On a $G(SPRS)_n$ admitting a quarter symmetric metric connection D, λ defined by* (22) *is symmetric iff A is closed.*

Corollary 2 *On a $G(SPRS)_n$ admitting a quarter symmetric metric connection D,the necessary and sufficient condition for $\bar{R} = R$ is that*

$$\lambda(X,Z)QY - \lambda(Y,Z)QX + S(X,Z)LY - S(Y,Z)LX = 0, \tag{24}$$

where λ is defined by (22).

§4. Ricci Tensor and Scalar Curvature with Respect to a Quarter Symmetric Metric Connection on $G(SPRS)_n$

Now contracting (23) with respect to X we get,

$$\bar{S}(Y,Z) = S(Y,Z) + \lambda(QY,Z) - r\lambda(Y,Z) + \lambda(Y,QZ) - aS(Y,Z), \tag{25}$$

where

$$a = tr.L = divA + \frac{r-2}{2}A(\rho). \tag{26}$$

Again contracting (25) with respect to Y, Z and using (22) and (15) we can get,

$$\bar{r} = r + (\nabla_{QX}A)(X) + (\nabla_X A)(QX) - dr(X)A(X) + 2A(X)A(X)r. \tag{27}$$

These give us the following theorem.

Theorem 2 *If an $G(SPRS)_n$ admits a quarter symmetric metric connection D, then the Ricci tensor and scalar curvature with respect to D is of the form* (25) *and* (27) *respectively.*

§5. Examples

Let us consider M^3 be an open subsets of R^3 with the basis $\{e_1, e_2, e_3\}$ where

$$e_1 = \frac{1}{x^1x^3}\frac{\partial}{\partial x^1},\ e_2 = \frac{\partial}{\partial x^2},\ e_3 = \frac{\partial}{\partial x^3}. \tag{28}$$

Let us define the metric g as

$$g(e_i, e_j) = \begin{cases} 1, & \text{i = j} \\ 0, & \text{otherwise.} \end{cases}$$

Then form of the metric is

$$g = g_{ij}dx^i dx^j = (x^1x^3)^2(dx^1)^2 + (dx^2)^2 + (dx^3)^2, \quad i, j = 1, 2, 3. \tag{29}$$

Obviously it is a Riemannian metric. Then the Ricci tensor is

$$S_{11} = (x^1)^2, S_{11,1} = 2x^1 \tag{30}$$

and all others vanish identically, where (,) denotes the covariant differentiation with respect to metric g.

Now we define

$$A_i(x) = \begin{cases} -\frac{1}{x^1}, & \text{i = 1} \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad B_i(x) = \begin{cases} \frac{3}{x^1}, & \text{i = 1} \\ 0, & \text{otherwise,} \end{cases}$$

for any point $x \in M$, Then

$$S_{11} = A_1 S_{11} + B_1 S_{11} \tag{31}$$

and all other forms vanish identically. The relation (31) implies that the above Riemannian manifold(M^3, g) is a $G(SPRS)_3$.

Now let the symmetric endomorphism Q defined by

$$Q(e_1) = (x^1)^2 e_1, \quad Q(e_2) = e_3, \quad Q(e_3) = e_2 \tag{32}$$

and let the vector field $V = -\frac{1}{x^1}e_1$ so that $A(X) = g(X,V)$. Then we get,

$$[e_i, e_j] = 0, \ \ \forall \ \ i,j = 1,2,3. \tag{33}$$

Then using Koszul's formula we have

$$\nabla_{e_i} e_j = 0, \ \ \forall \ \ i,j = 1,2,3, \tag{34}$$

where ∇ is Levi-Civita connection with respect to g.

Again using (19) we can define a connection D on M as follows:

$$D_{e_2}e_1 = -\frac{1}{x^1}e_3, D_{e_3}e_1 = -\frac{1}{x^1}e_2 \tag{35}$$

and all others vanish identically.

Using (11) we can find the torsion tensor with respect to D as follows:

$$T(e_1,e_2) = \frac{1}{x^1}e_3, T(e_1,e_3) = \frac{1}{x^1}e_2 \tag{36}$$

and all others vanish identically.

Using (12) we have

$$(D_{e_1}g)(e_2,e_3) = (D_{e_2}g)(e_1,e_3) = (D_{e_3}g)(e_2,e_1) = 0. \tag{37}$$

The above approves that D is a quarter symmetric metric connection on M. Thus (M^3, g) is a $G(SPRS)_3$ with a quarter symmetric metric connection D.

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