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Second Order Connectivity Indices of Some Chemical Trees

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Abstract: Based on the higher order Randić index, we propose three other high order connectivity indices, and obtain the calculation formulas for the new connectivity indices of some chemical trees.

Key Words: Connectivity indices, higher order Randić index, chemical trees.

AMS(2010): 05C05, 05C07, 05C09, 05C92.

§1. Introduction

In 1975, the connectivity index (now also called the Randić index or the branching index) of a graph G, denoted by $R(G)$, introduced by the chemist Milan Randić [6], is the degree-based topological index that is most frequently applied in quantitative structure-property and structure-activity studies. For a simple undirected graph $G = (V, E)$ with vertex set $V(G)$ and edge set $E(G)$, its Randić index is defined as

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}},$$

where d_u denotes the degree of the vertex u in G.

The sum-connectivity index [14], the atom-bond connectivity index [3] and the atom-bond sum-connectivity index [1] are the class of successful variants of the connectivity index, and defined as

$$\begin{aligned} SCI(G) &= \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u + d_v}}, \\ ABC(G) &= \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}}, \\ ABS(G) &= \sum_{uv \in E(G)} \sqrt{1 - \frac{2}{d_u + d_v}}. \end{aligned}$$

In 1976, Kier et al. [4] modified the Randić index, and proposed the higher order Randić

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index ${}^hR(G)$ of a graph G, that is,

$$ {}^hR(G) = \sum_{v_1,v_2,\ldots,v_{h+1}\in E_h(G)} \frac{1}{\sqrt{d_{v_1}d_{v_2}\cdots d_{v_{h+1}}}}, $$

where $E_h(G)$ is all paths of length h in G. Clearly, $E_1(G)$ is the edge of G. Thus the higher order Randić index is a natural extension of the Randić index. The higher order Randić index is of great interest in mathematics [2,7,9,11] and the theory of molecular topology [10]. In particular, the lower order Randić index has attracted widespread attention from scholars, and extensively studied [5,8,12,13]. However, these investigations all focused on benzenoid systems. The results of the chemical trees (alkanes) have not been reported.

Naturally, we need to study other high order connectivity indices, especially the second order connectivity indices. By definition, the second sum-connectivity index, second atom-bond connectivity index and the second atom-bond sum-connectivity index are respectively

$$ \begin{aligned} {}^2SCI(G) &= \sum_{uvw\in E_2(G)} \frac{1}{\sqrt{d_u+d_v+d_w}}, \\ {}^2ABC(G) &= \sum_{uvw\in E_2(G)} \sqrt{\frac{d_u+d_v+d_w-3}{d_ud_vd_w}}, \\ {}^2ABS(G) &= \sum_{uvw\in E_2(G)} \sqrt{1-\frac{3}{d_u+d_v+d_w}}. \end{aligned} $$

In this paper, the expression of the second order connectivity indices of some chemical trees is found.

§2. Chemical Trees of Module 2

The chemical trees of T_2^0 and T_2^1 are shown in Figure 1.

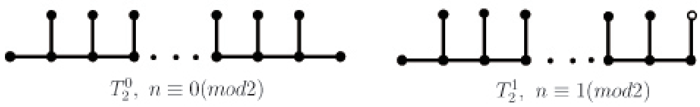

Figure 1. T_2^0 and T_2^1

Theorem 2.1 *The second order connectivity indices of T_2^0 with n vertices are given by*

$$ \begin{aligned} {}^2R(T_2^0) &= \frac{(6+\sqrt{3})n}{18} + \frac{\sqrt{3}-2}{3}, \\ {}^2SCI(T_2^0) &= \frac{(7+6\sqrt{7})n}{42} + \frac{14\sqrt{5}-10\sqrt{7}-35}{35}, \\ {}^2ABC(T_2^0) &= \frac{(4+\sqrt{2})n}{6} + \frac{2\sqrt{6}-3\sqrt{2}-4}{3}, \\ {}^2ABS(T_2^0) &= \frac{(12\sqrt{7}+7\sqrt{6})n}{42} + \frac{14\sqrt{10}-20\sqrt{7}-35\sqrt{6}}{35}. \end{aligned} $$

Proof Let $m(i,j,k)$ denote the number of paths with degree sequence (i,j,k). Then we can obtain the basic information on T_2^0 in the following table.

$m(1,3,1)$	$m(1,3,3)$	$m(3,3,3)$
2	$n-2$	$\frac{n-6}{2}$

Thus, we have

$$\begin{aligned}
{}^2R(T_2^0) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_ud_vd_w}}\\
&= 2\times\frac{1}{\sqrt{3}}+(n-2)\times\frac{1}{\sqrt{9}}+\frac{n-6}{2}\times\frac{1}{\sqrt{27}}\\
&= \frac{(6+\sqrt{3})n}{18}+\frac{\sqrt{3}-2}{3},\\
{}^2SCI(T_2^0) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_u+d_v+d_w}}\\
&= 2\times\frac{1}{\sqrt{5}}+(n-2)\times\frac{1}{\sqrt{7}}+\frac{n-6}{2}\times\frac{1}{\sqrt{9}}\\
&= \frac{(7+6\sqrt{7})n}{42}+\frac{14\sqrt{5}-10\sqrt{7}-35}{35},\\
{}^2ABC(T_2^0) &= \sum_{uvw\in E_2(T)} \frac{\sqrt{d_u+d_v+d_w-3}}{\sqrt{d_ud_vd_w}}\\
&= 2\times\frac{\sqrt{6}}{3}+\frac{2n-4}{3}+\frac{n-6}{2}\times\frac{\sqrt{2}}{3}\\
&= \frac{(4+\sqrt{2})n}{6}+\frac{2\sqrt{6}-3\sqrt{2}-4}{3},\\
{}^2ABS(T_2^0) &= \sum_{uvw\in E_2(T)} \sqrt{1-\frac{3}{d_u+d_v+d_w}}\\
&= 2\times\frac{\sqrt{2}}{\sqrt{5}}+(n-2)\times\frac{2}{\sqrt{7}}+\frac{n-6}{2}\times\frac{\sqrt{6}}{3}\\
&= \frac{(12\sqrt{7}+7\sqrt{6})n}{42}+\frac{14\sqrt{10}-20\sqrt{7}-35\sqrt{6}}{35}.
\end{aligned}$$

This completes the proof. □

Theorem 2.2 *The second order connectivity indices of T_2^1 with n vertices are given by*

$$\begin{aligned}
{}^2R(T_2^1) &= \frac{(6+\sqrt{3})n}{18}+\frac{6\sqrt{6}-\sqrt{3}+3\sqrt{2}-24}{18},\\
{}^2SCI(T_2^1) &= \frac{(7+6\sqrt{7})n}{42}+\frac{84\sqrt{5}-240\sqrt{7}+140\sqrt{6}-490+105\sqrt{2}}{420},\\
{}^2ABC(T_2^1) &= \frac{(4+\sqrt{2})n}{6}+\frac{2\sqrt{6}-16-\sqrt{2}+\sqrt{10}}{6},\\
{}^2ABS(T_2^1) &= \frac{(7\sqrt{6}+12\sqrt{7})n}{42}+\frac{189\sqrt{10}-490\sqrt{6}-480\sqrt{7}+420\sqrt{2}}{420}.
\end{aligned}$$

Proof Let $m(i,j,k)$ denote the number of paths with degree sequence (i,j,k). Then we can obtain the basic information on T_2^1 in the following table.

$m(1,3,1)$	$m(1,3,3)$	$m(3,3,3)$	$m(1,2,3)$	$m(1,3,2)$	$m(3,3,2)$
1	$n-4$	$\frac{n-7}{2}$	1	1	1

Thus we have

$$\begin{aligned}
{}^2R(T_2^1) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_u d_v d_w}}\\
&= 1\times\frac{1}{\sqrt{3}}+(n-4)\times\frac{1}{\sqrt{9}}+\frac{n-7}{2}\times\frac{1}{\sqrt{27}}+1\times\frac{1}{\sqrt{6}}+1\times\frac{1}{\sqrt{6}}+1\times\frac{\sqrt{2}}{6}\\
&= \frac{1}{\sqrt{3}}+\frac{n-4}{3}+\frac{n-7}{6}\times\frac{1}{\sqrt{3}}+\frac{\sqrt{6}}{6}+\frac{\sqrt{6}}{6}+\frac{\sqrt{2}}{6}\\
&= \frac{(6+\sqrt{3})n}{18}+\frac{6\sqrt{6}-\sqrt{3}+3\sqrt{2}-24}{18},\\
{}^2SCI(T_2^1) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_u+d_v+d_w}}\\
&= 1\times\frac{1}{\sqrt{5}}+(n-4)\times\frac{1}{\sqrt{7}}+\frac{n-7}{2}\times\frac{1}{\sqrt{9}}+1\times\frac{1}{\sqrt{6}}+1\times\frac{1}{\sqrt{6}}+1\times\frac{1}{\sqrt{8}}\\
&= \frac{\sqrt{5}}{5}+\frac{\sqrt{7}n-4\sqrt{7}}{7}+\frac{n-7}{6}+\frac{2\sqrt{6}}{6}+\frac{\sqrt{2}}{4}\\
&= \frac{(7+6\sqrt{7})n}{42}+\frac{84\sqrt{5}-240\sqrt{7}+140\sqrt{6}-490+105\sqrt{2}}{420},\\
{}^2ABC(T_2^1) &= \sum_{uvw\in E_2(T)} \frac{\sqrt{d_u+d_v+d_w-3}}{\sqrt{d_u d_v d_w}}\\
&= 1\times\frac{\sqrt{6}}{3}+(n-4)\times\frac{2}{3}+\frac{n-7}{2}\times\frac{\sqrt{2}}{3}+1\times\frac{\sqrt{2}}{2}+1\times\frac{\sqrt{2}}{2}+1\times\frac{\sqrt{10}}{6}\\
&= \frac{\sqrt{6}}{3}+\frac{2n-8}{3}+\frac{n-7}{2}\times\frac{\sqrt{2}}{3}+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}+\frac{\sqrt{10}}{6}\\
&= \frac{(4+\sqrt{2})n}{6}+\frac{2\sqrt{6}-16-\sqrt{2}+\sqrt{10}}{6},\\
{}^2ABS(T_2^1) &= \sum_{uvw\in E_2(T)} \sqrt{1-\frac{3}{d_u+d_v+d_w}}\\
&= \frac{\sqrt{10}}{5}+(n-4)\times\frac{2}{\sqrt{7}}+\frac{n-7}{2}\times\frac{\sqrt{6}}{3}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}+\frac{\sqrt{10}}{4}\\
&= \frac{(7\sqrt{6}+12\sqrt{7})n}{42}+\frac{189\sqrt{10}-490\sqrt{6}-480\sqrt{7}+420\sqrt{2}}{420}.
\end{aligned}$$

This completes the proof. □

§3. Chemical Trees of Module 3

The chemical trees of T_3^0, T_3^1 and T_3^2 are shown in Figure 2.

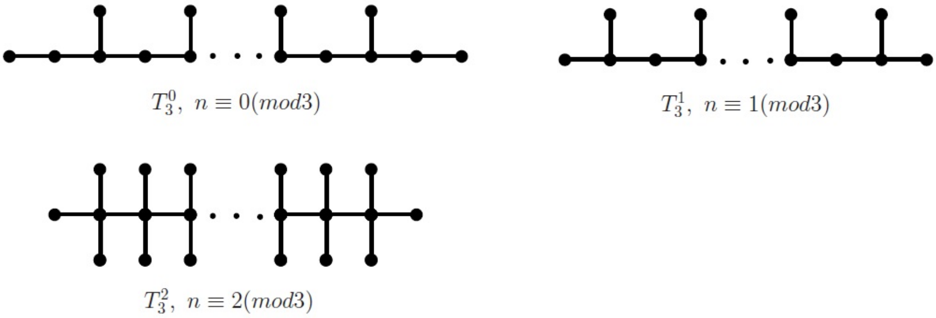

Figure 2. T_3^0, T_3^1 and T_3^2

Theorem 3.1 *The second order connectivity indices of T_3^0 with n vertices are given by*

$$\begin{aligned}
{}^2R(T_3^0) &= \frac{(2\sqrt{6}+\sqrt{3}+\sqrt{2})n}{18} - \frac{2\sqrt{2}+\sqrt{3}}{6},\\
{}^2SCI(T_3^0) &= \frac{(28\sqrt{6}+12\sqrt{7}+21\sqrt{2})n}{252} - \frac{2\sqrt{7}+7\sqrt{2}}{14},\\
{}^2ABC(T_3^0) &= \frac{(6\sqrt{2}+2\sqrt{3}+\sqrt{10})n}{18} - \frac{\sqrt{3}+\sqrt{10}}{3},\\
{}^2ABS(T_3^0) &= \frac{(28\sqrt{2}+8\sqrt{7}+7\sqrt{10})n}{84} - \frac{4\sqrt{7}+7\sqrt{10}}{14}.
\end{aligned}$$

Proof Let $m(i,j,k)$ denote the number of paths with degree sequence (i,j,k). Then we can obtain the basic information on T_3^0 in the following table.

$m(1,2,3)$	$m(2,3,1)$	$m(2,3,2)$	$m(3,2,3)$
2	$\frac{2n-6}{3}$	$\frac{n-3}{3}$	$\frac{n-6}{3}$

Thus, we have

$$\begin{aligned}
{}^2R(T_3^0) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_u d_v d_w}}\\
&= 2\times\frac{\sqrt{6}}{6} + \frac{\sqrt{6}}{6}\times\frac{2n-6}{3} + \frac{\sqrt{3}}{6}\times\frac{n-3}{3} + \frac{\sqrt{2}}{6}\times\frac{n-6}{3}\\
&= \frac{\sqrt{6}}{3} + \frac{\sqrt{6}n-3\sqrt{6}}{9} + \frac{\sqrt{3}n-3\sqrt{3}+\sqrt{2}n-6\sqrt{2}}{18}\\
&= \frac{(2\sqrt{6}+\sqrt{3}+\sqrt{2})n}{18} - \frac{2\sqrt{2}+\sqrt{3}}{6},
\end{aligned}$$

$$
\begin{aligned}
{}^2SCI(T_3^0) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_u+d_v+d_w}}\\
&= 2\times\frac{\sqrt{6}}{6}+\frac{\sqrt{6}}{6}\times\frac{2n-6}{3}+\frac{\sqrt{7}}{7}\times\frac{n-3}{3}+\frac{\sqrt{2}}{4}\times\frac{n-6}{3}\\
&= \frac{(28\sqrt{6}+12\sqrt{7}+21\sqrt{2})n}{252}-\frac{2\sqrt{7}+7\sqrt{2}}{14},\\
{}^2ABC(T_3^0) &= \sum_{uvw\in E_2(T)} \frac{\sqrt{d_u+d_v+d_w-3}}{\sqrt{d_ud_vd_w}}\\
&= 2\times\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\times\frac{2n-6}{3}+\frac{\sqrt{3}}{3}\times\frac{n-3}{3}+\frac{\sqrt{10}}{6}\times\frac{n-6}{3}\\
&= \frac{(6\sqrt{2}+2\sqrt{3}+\sqrt{10})n-6\sqrt{3}-6\sqrt{10}}{18}\\
&= \frac{(6\sqrt{2}+2\sqrt{3}+\sqrt{10})n}{18}-\frac{\sqrt{3}+\sqrt{10}}{3},\\
{}^2ABS(T_3^0) &= \sum_{uvw\in E_2(T)} \sqrt{1-\frac{3}{d_u+d_v+d_w}}\\
&= 2\times\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\times\frac{2n-6}{3}+\frac{2}{\sqrt{7}}\times\frac{n-3}{3}+\frac{\sqrt{5}}{2\sqrt{2}}\times\frac{n-6}{3}\\
&= \frac{(28\sqrt{2}+8\sqrt{7}+7\sqrt{10})n}{84}-\frac{4\sqrt{7}+7\sqrt{10}}{14}.
\end{aligned}
$$

This completes the proof. □

Theorem 3.2 *The second order connectivity indices of T_3^1 with n vertices are given by*

$$
\begin{aligned}
{}^2R(T_3^1) &= \frac{(2\sqrt{6}+\sqrt{3}+\sqrt{2})n}{18}+\frac{5\sqrt{3}-2\sqrt{6}-4\sqrt{2}}{18},\\
{}^2SCI(T_3^1) &= \frac{(28\sqrt{6}+12\sqrt{7}+21\sqrt{2})n}{252}-\frac{15\sqrt{2}+5\sqrt{6}+15\sqrt{7}-18\sqrt{5}}{45},\\
{}^2ABC(T_3^1) &= \frac{(6\sqrt{2}+2\sqrt{3}+\sqrt{10})n}{18}+\frac{6\sqrt{6}-3\sqrt{2}-7\sqrt{3}-2\sqrt{10}}{9},\\
{}^2ABS(T_3^1) &= \frac{(28\sqrt{2}+8\sqrt{7}+7\sqrt{10})n}{84}+\frac{\sqrt{10}-5\sqrt{2}-10\sqrt{7}}{15}.
\end{aligned}
$$

Proof Let $m(i,j,k)$ denote the number of paths with degree sequence (i,j,k). Then we can obtain the basic information on T_3^1 in the following table.

$m(1,3,1)$	$m(1,3,2)$	$m(3,2,3)$	$m(2,3,2)$
2	$\frac{2n-2}{3}$	$\frac{n-4}{3}$	$\frac{n-7}{3}$

Thus, we have

$$\begin{aligned}
{}^2R(T_3^1) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_ud_vd_w}} \\
&= 2\times\frac{\sqrt{3}}{3}+\frac{\sqrt{6}}{6}\times\frac{2n-2}{3}+\frac{\sqrt{2}}{6}\times\frac{n-4}{3}+\frac{\sqrt{3}}{6}\times\frac{n-7}{3} \\
&= \frac{(2\sqrt{6}+\sqrt{3}+\sqrt{2})n}{18}+\frac{5\sqrt{3}-2\sqrt{6}-4\sqrt{2}}{18}, \\
{}^2SCI(T_3^1) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_u+d_v+d_w}} \\
&= 2\times\frac{\sqrt{5}}{5}+\frac{\sqrt{6}}{6}\times\frac{2n-2}{3}+\frac{\sqrt{2}}{4}\times\frac{n-4}{3}+\frac{\sqrt{7}}{7}\times\frac{n-7}{3} \\
&= \frac{(28\sqrt{6}+12\sqrt{7}+21\sqrt{2})n}{252}-\frac{15\sqrt{2}+5\sqrt{6}+15\sqrt{7}-18\sqrt{5}}{45}, \\
{}^2ABC(T_3^1) &= \sum_{uvw\in E_2(T)} \frac{\sqrt{d_u+d_v+d_w-3}}{\sqrt{d_ud_vd_w}} \\
&= 2\times\frac{\sqrt{6}}{3}+\frac{\sqrt{2}}{2}\times\frac{2n-2}{3}+\frac{\sqrt{10}}{6}\times\frac{n-4}{3}+\frac{\sqrt{3}}{3}\times\frac{n-7}{3} \\
&= \frac{(6\sqrt{2}+2\sqrt{3}+\sqrt{10})n}{18}+\frac{6\sqrt{6}-3\sqrt{2}-7\sqrt{3}-2\sqrt{10}}{9}, \\
{}^2ABS(T_3^1) &= \sum_{uvw\in E_2(T)} \sqrt{1-\frac{3}{d_u+d_v+d_w}} \\
&= 2\times\frac{\sqrt{10}}{5}+\frac{2n-2}{3}\times\frac{\sqrt{2}}{2}+\frac{n-4}{3}\times\frac{\sqrt{10}}{4}+\frac{n-7}{3}\times\frac{2\sqrt{7}}{7} \\
&= \frac{(28\sqrt{2}+8\sqrt{7}+7\sqrt{10})n}{84}+\frac{\sqrt{10}-5\sqrt{2}-10\sqrt{7}}{15}.
\end{aligned}$$

This completes the proof. □

Theorem 3.3 *The second order connectivity indices of T_3^2 with n vertices are given by*

$$\begin{aligned}
{}^2R(T_3^2) &= \frac{4+13n}{24}, \\
{}^2SCI(T_3^2) &= \frac{(\sqrt{6}+8+\sqrt{3})n}{18}+\frac{5\sqrt{6}-14-4\sqrt{3}}{9}, \\
{}^2ABC(T_3^2) &= \frac{(3+4\sqrt{3}+8\sqrt{6})n}{24}+\frac{10\sqrt{3}-6-7\sqrt{6}}{6}, \\
{}^2ABS(T_3^2) &= \frac{(3\sqrt{2}+3\sqrt{3}+8\sqrt{6})n}{18}+\frac{15\sqrt{2}-12\sqrt{3}-14\sqrt{6}}{9}.
\end{aligned}$$

Proof Let $m(i,j,k)$ denote the number of paths with degree sequence (i,j,k). Then we can obtain the basic information on T_3^2 in the following table.

$m(1,4,1)$	$m(1,4,4)$	$m(4,4,4)$
$\frac{n+10}{3}$	$\frac{4n-14}{3}$	$\frac{n-8}{3}$

Thus, we have

$$\begin{aligned}
{}^2R(T_3^2) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_u d_v d_w}} \\
&= \frac{n+10}{3}\times\frac{1}{2}+\frac{4n-14}{3}\times\frac{1}{4}+\frac{n-8}{3}\times\frac{1}{8} \\
&= \frac{4+13n}{24},
\end{aligned}$$

$$\begin{aligned}
{}^2SCI(T_3^2) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_u+d_v+d_w}} \\
&= \frac{n+10}{3}\times\frac{1}{\sqrt{6}}+\frac{4n-14}{3}\times\frac{1}{\sqrt{9}}+\frac{n-8}{3}\times\frac{1}{\sqrt{12}} \\
&= \frac{4n-14}{9}+\frac{10\sqrt{6}+\sqrt{6}n+(n-8)\sqrt{3}}{18} \\
&= \frac{(\sqrt{6}+8+\sqrt{3})n}{18}+\frac{5\sqrt{6}-14-4\sqrt{3}}{9},
\end{aligned}$$

$$\begin{aligned}
{}^2ABC(T_3^2) &= \sum_{uvw\in E_2(T)} \frac{\sqrt{d_u+d_v+d_w-3}}{\sqrt{d_u d_v d_w}} \\
&= \frac{n+10}{3}\times\frac{\sqrt{3}}{2}+\frac{4n-14}{3}\times\frac{\sqrt{6}}{4}+\frac{n-8}{3}\times\frac{3}{8} \\
&= \frac{(3+4\sqrt{3}+8\sqrt{6})n}{24}+\frac{10\sqrt{3}-6-7\sqrt{6}}{6},
\end{aligned}$$

$$\begin{aligned}
{}^2ABS(T_3^2) &= \sum_{uvw\in E_2(T)} \sqrt{1-\frac{3}{d_u+d_v+d_w}} \\
&= \frac{n+10}{3}\times\frac{\sqrt{2}}{2}+\frac{4n-14}{3}\times\frac{\sqrt{6}}{3}+\frac{n-8}{3}\times\frac{\sqrt{3}}{2} \\
&= \frac{(3\sqrt{2}+3\sqrt{3}+8\sqrt{6})n}{18}+\frac{15\sqrt{2}-12\sqrt{3}-14\sqrt{6}}{9}.
\end{aligned}$$

This completes the proof. □

§4. Chemical Trees of Module 4

The chemical trees of T_4^0, T_4^1, T_4^2 and T_4^3 are shown in Figure 2.

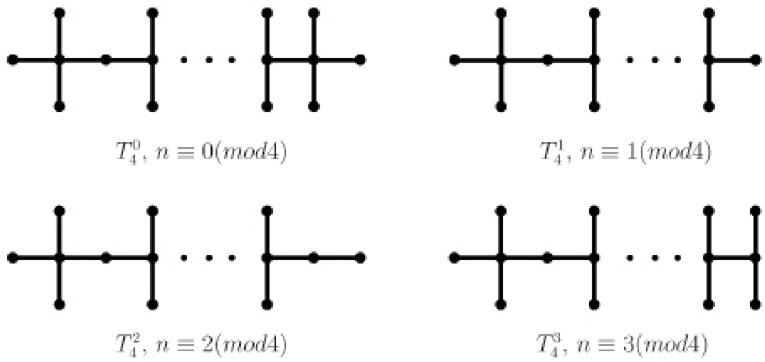

Figure 3. T_4^0, T_4^1, T_4^2 and T_4^3

Theorem 4.1 *The second order connectivity indices of T_4^0 with n vertices are given by*

$$\begin{aligned}
{}^2R(T_4^0) &= \frac{(6+9\sqrt{2})n}{32}+\frac{20-15\sqrt{2}}{8},\\
{}^2SCI(T_4^0) &= \frac{(70\sqrt{6}+42\sqrt{10}+105\sqrt{2}+240\sqrt{7})n}{1680}\\
&\quad +\frac{40\sqrt{6}-6\sqrt{10}-45\sqrt{2}-60\sqrt{7}+100}{60},\\
{}^2ABC(T_4^0) &= \frac{(4\sqrt{3}+2\sqrt{5}+\sqrt{14}+16\sqrt{2})n}{32}+\frac{10\sqrt{6}+16\sqrt{3}-28\sqrt{2}+\sqrt{14}-6\sqrt{5}}{8},\\
{}^2ABS(T_4^0) &= \frac{(35\sqrt{10}+14\sqrt{70}+70\sqrt{2}+160\sqrt{7})n}{560}\\
&\quad +\frac{120\sqrt{2}-6\sqrt{70}-45\sqrt{10}-120\sqrt{7}+100\sqrt{6}}{60}.
\end{aligned}$$

Proof Let $m(i,j,k)$ denote the number of paths with degree sequence (i,j,k). Then, we can obtain the basic information on T_4^0 in the following table.

$m(1,4,1)$	$m(1,4,2)$	$m(1,4,4)$	$m(2,4,4)$	$m(4,2,4)$	$m(2,4,2)$
$\frac{n+16}{4}$	$n-7$	5	1	$\frac{n-8}{4}$	$\frac{n-12}{4}$

Thus, we have

$$\begin{aligned}
{}^2R(T_4^0) &= \sum_{uvw\in E_2(T)}\frac{1}{\sqrt{d_ud_vd_w}}\\
&= \frac{n+16}{4}\times\frac{1}{2}+\frac{n-7}{2\sqrt{2}}+\frac{5}{4}+\frac{1}{4\sqrt{2}}+\frac{1}{4\sqrt{2}}\times\frac{n-8}{4}+\frac{1}{4}\times\frac{n-12}{4}\\
&= \frac{(6+9\sqrt{2})n}{32}+\frac{20-15\sqrt{2}}{8}.
\end{aligned}$$

$$\begin{aligned}
{}^2SCI(T_4^0) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_u+d_v+d_w}}\\
&= \frac{n+16}{4}\times\frac{1}{\sqrt{6}}+\frac{n-7}{\sqrt{7}}+\frac{5}{3}+\frac{1}{\sqrt{10}}+\frac{1}{\sqrt{10}}\times\frac{n-8}{4}+\frac{1}{2\sqrt{2}}\times\frac{n-12}{4}\\
&= \frac{\sqrt{6}n-18\sqrt{2}}{24}+\frac{\sqrt{7}n}{7}+\frac{\sqrt{10}n-4\sqrt{10}}{40}+\frac{\sqrt{2}n}{16}+\frac{2\sqrt{6}+5}{3}-\sqrt{7}\\
&= \frac{(70\sqrt{6}+42\sqrt{10}+105\sqrt{2}+240\sqrt{7})n}{1680}+\frac{40\sqrt{6}-6\sqrt{10}-45\sqrt{2}-60\sqrt{7}+100}{60},\\
{}^2ABC(T_4^0) &= \sum_{uvw\in E_2(T)} \frac{\sqrt{d_u+d_v+d_w-3}}{\sqrt{d_ud_vd_w}}\\
&= \frac{n+16}{4}\times\frac{\sqrt{3}}{2}+(n-7)\frac{4\sqrt{2}}{8}+\frac{5\sqrt{6}}{4}+\frac{\sqrt{14}}{8}+\frac{\sqrt{14}}{8}\times\frac{n-8}{4}+\frac{\sqrt{5}}{4}\times\frac{n-12}{4}\\
&= \frac{\sqrt{3}n-\sqrt{14}}{8}+\frac{\sqrt{2}n-7\sqrt{2}}{2}+\frac{5\sqrt{6}-3\sqrt{5}-\sqrt{14}}{4}+\frac{\sqrt{14}n+2\sqrt{5}n}{32}\\
&= \frac{(4\sqrt{3}+2\sqrt{5}+\sqrt{14}+16\sqrt{2})n}{32}+\frac{10\sqrt{6}+16\sqrt{3}-28\sqrt{2}-\sqrt{14}-6\sqrt{5}}{8},\\
{}^2ABS(T_4^0) &= \sum_{uvw\in E_2(T)} \sqrt{1-\frac{3}{d_u+d_v+d_w}}\\
&= \frac{n+16}{4}\times\frac{\sqrt{2}}{2}+(n-7)\frac{2\sqrt{7}}{7}+\frac{5\sqrt{6}}{3}+\frac{\sqrt{70}}{10}\times\frac{n-4}{4}+\frac{\sqrt{10}}{4}\times\frac{n-12}{4}\\
&= \frac{5\sqrt{2}n-4\sqrt{70}+\sqrt{70}n}{40}+\frac{6\sqrt{7}n+35\sqrt{6}}{21}+\frac{\sqrt{10}n-12\sqrt{10}}{16}+2\sqrt{2}-2\sqrt{7}\\
&= \frac{(35\sqrt{10}+14\sqrt{70}+70\sqrt{2}+160\sqrt{7})n}{560}+\frac{120\sqrt{2}-6\sqrt{70}-45\sqrt{10}-120\sqrt{7}+100\sqrt{6}}{60}.
\end{aligned}$$

This completes the proof. □

Theorem 4.2 *The second order connectivity indices of* T_4^1 *with* n *vertices are given by*

$$\begin{aligned}
{}^2R(T_4^1) &= \frac{(6+9\sqrt{2})n}{32}+\frac{42-29\sqrt{2}}{32},\\
{}^2SCI(T_4^1) &= \frac{(70\sqrt{6}+42\sqrt{10}+105\sqrt{2}+240\sqrt{7})n}{1680}+\frac{210\sqrt{6}-189\sqrt{2}-144\sqrt{7}-42\sqrt{10}}{336},\\
{}^2ABC(T_4^1) &= \frac{(4\sqrt{3}+2\sqrt{5}+\sqrt{14}+16\sqrt{2})n}{32}+\frac{60\sqrt{3}-48\sqrt{2}-5\sqrt{14}-18\sqrt{5}}{32},\\
{}^2ABS(T_4^1) &= \frac{(35\sqrt{10}+14\sqrt{70}+70\sqrt{2}+160\sqrt{7})n}{560}+\frac{210\sqrt{2}-96\sqrt{7}-63\sqrt{10}-14\sqrt{70}}{112}.
\end{aligned}$$

Proof Let $m(i,j,k)$ denote the number of paths with degree sequence (i,j,k). Then, we can obtain the basic information on T_4^1 in the following table.

$m(1,4,1)$	$m(1,4,2)$	$m(4,2,4)$	$m(2,4,2)$
$\frac{n+15}{4}$	$n-3$	$\frac{n-5}{4}$	$\frac{n-9}{4}$

Thus, we have

$$\begin{aligned}
{}^2R(T_4^1) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_u d_v d_w}} \\
&= \frac{n+15}{4}\times\frac{1}{2}+\frac{n-3}{2\sqrt{2}}+\frac{n-5}{4}\times\frac{1}{4\sqrt{2}}+\frac{n-9}{16} \\
&= \frac{(6+9\sqrt{2})n}{32}+\frac{42-29\sqrt{2}}{32}, \\
{}^2SCI(T_4^1) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_u+d_v+d_w}} \\
&= \frac{n+15}{4}\times\frac{\sqrt{6}}{6}+\frac{n-3}{\sqrt{7}}+\frac{1}{\sqrt{10}}\times\frac{n-5}{4}+\frac{n-9}{8\sqrt{2}} \\
&= \frac{(70\sqrt{6}+42\sqrt{10}+105\sqrt{2}+240\sqrt{7})n}{1680}+\frac{210\sqrt{6}-189\sqrt{2}-144\sqrt{7}-42\sqrt{10}}{336}, \\
{}^2ABC(T_4^1) &= \sum_{uvw\in E_2(T)} \frac{\sqrt{d_u+d_v+d_w-3}}{\sqrt{d_u d_v d_w}} \\
&= \frac{n+15}{4}\times\frac{\sqrt{3}}{2}+\frac{n-3}{\sqrt{2}}+\frac{\sqrt{14}}{8}\times\frac{n-5}{4}+\frac{n-9}{4}\times\frac{\sqrt{5}}{4} \\
&= \frac{(4\sqrt{3}+2\sqrt{5}+\sqrt{14}+16\sqrt{2})n}{32}+\frac{60\sqrt{3}-48\sqrt{2}-5\sqrt{14}-18\sqrt{5}}{32}, \\
{}^2ABS(T_4^1) &= \sum_{uvw\in E_2(T)} \sqrt{1-\frac{3}{d_u+d_v+d_w}} \\
&= \frac{n+15}{4}\times\frac{\sqrt{2}}{2}+\frac{2n-6}{\sqrt{7}}+\frac{7}{\sqrt{70}}\times\frac{n-5}{4}+\frac{\sqrt{10}n-9\sqrt{10}}{16} \\
&= \frac{(35\sqrt{10}+14\sqrt{70}+70\sqrt{2}+160\sqrt{7})n}{560}+\frac{210\sqrt{2}-96\sqrt{7}-63\sqrt{10}-14\sqrt{70}}{112}.
\end{aligned}$$

This completes the proof. □

Theorem 4.3 *The second order connectivity indices of T_4^2 with n vertices are given by*

$$\begin{aligned}
{}^2R(T_4^2) &= \frac{(6+9\sqrt{2})n}{32}+\frac{6-11\sqrt{2}}{16}, \\
{}^2SCI(T_4^2) &= \frac{(70\sqrt{6}+42\sqrt{10}+105\sqrt{2}+240\sqrt{7})n}{1680}+\frac{140\sqrt{6}-240\sqrt{7}-126\sqrt{10}-315\sqrt{2}}{560}, \\
{}^2ABC(T_4^2) &= \frac{(4\sqrt{3}+2\sqrt{5}+\sqrt{14}+16\sqrt{2})n}{32}+\frac{12\sqrt{3}-16\sqrt{2}-6\sqrt{5}-3\sqrt{14}}{16}, \\
{}^2ABS(T_4^2) &= \frac{(35\sqrt{10}+14\sqrt{70}+70\sqrt{2}+160\sqrt{7})n}{560}+\frac{210\sqrt{2}-42\sqrt{70}-105\sqrt{10}-160\sqrt{7}}{280}.
\end{aligned}$$

Proof Let $m(i,j,k)$ denote the number of paths with degree sequence (i,j,k). Then we can obtain the basic information on T_4^2 in the following table.

$m(1,4,1)$	$m(1,4,2)$	$m(4,2,4)$	$m(4,2,1)$	$m(2,4,2)$
$\frac{n+6}{4}$	$n-3$	$\frac{n-6}{4}$	1	$\frac{n-6}{4}$

Thus, we have

$$\begin{aligned}
{}^2R(T_4^2) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_u d_v d_w}}\\
&= \frac{n+6}{4}\times\frac{1}{2}+\frac{n-3}{2\sqrt{2}}+\frac{1}{4\sqrt{2}}\times\frac{n-6}{4}+1\times\frac{1}{2\sqrt{2}}+\frac{1}{4}\times\frac{n-6}{4}\\
&= \frac{n+6}{4}\times\frac{1}{2}+\frac{n-6}{4}\times(\frac{1}{4\sqrt{2}}+\frac{1}{4})+\frac{n-2}{2\sqrt{2}}\\
&= \frac{(6+9\sqrt{2})n}{32}+\frac{6-11\sqrt{2}}{16},\\
{}^2SCI(T_4^2) &= \sum_{uvw\in E_2(T)} \frac{1}{\sqrt{d_u+d_v+d_w}}\\
&= \frac{n+6}{4}\times\frac{1}{\sqrt{6}}+\frac{n-3}{\sqrt{7}}+\frac{1}{\sqrt{10}}\times\frac{n-6}{4}+\frac{1}{\sqrt{7}}+\frac{1}{2\sqrt{2}}\times\frac{n-6}{4}\\
&= \frac{n+6}{4}\times\frac{1}{\sqrt{6}}+\frac{n-6}{4}\times(\frac{1}{\sqrt{10}}+\frac{1}{2\sqrt{2}})+\frac{n-2}{\sqrt{7}}\\
&= \frac{(70\sqrt{6}+42\sqrt{10}+105\sqrt{2}+240\sqrt{7})n}{1680}+\frac{140\sqrt{6}-160\sqrt{7}-84\sqrt{10}-210\sqrt{2}}{560},\\
{}^2ABC(T_4^2) &= \sum_{uvw\in E_2(T)} \frac{\sqrt{d_u+d_v+d_w-3}}{\sqrt{d_u d_v d_w}}\\
&= \frac{n+6}{4}\times\frac{\sqrt{3}}{2}+\frac{n-3}{\sqrt{2}}+\frac{\sqrt{5}}{4}\times\frac{n-6}{4}+\frac{1}{\sqrt{2}}+\frac{\sqrt{7}}{\sqrt{32}}\times\frac{n-6}{4}\\
&= \frac{n+6}{4}\times\frac{\sqrt{3}}{2}+\frac{n-6}{4}\times(\frac{\sqrt{5}}{4}+\frac{\sqrt{7}}{4\sqrt{2}})+\frac{n-2}{\sqrt{2}}\\
&= \frac{(4\sqrt{3}+2\sqrt{5}+\sqrt{14}+16\sqrt{2})n}{32}+\frac{12\sqrt{3}-16\sqrt{2}-6\sqrt{5}-3\sqrt{14}}{16},\\
{}^2ABS(T_4^2) &= \sum_{uvw\in E_2(T)} \sqrt{1-\frac{3}{d_u+d_v+d_w}}\\
&= \frac{n+6}{4}\times\frac{\sqrt{3}}{\sqrt{6}}+\frac{2n-6}{\sqrt{7}}+\frac{\sqrt{7}}{\sqrt{10}}\times\frac{n-6}{4}+\frac{2}{\sqrt{7}}+\frac{\sqrt{5}}{2\sqrt{2}}\times\frac{n-6}{4}\\
&= \frac{(35\sqrt{10}+14\sqrt{70}+70\sqrt{2}+160\sqrt{7})n}{560}+\frac{210\sqrt{2}-42\sqrt{70}-105\sqrt{10}-160\sqrt{7}}{280}.
\end{aligned}$$

This completes the proof. □

Theorem 4.4 *The second order connectivity indices of* T_4^3 *with* n *vertices are given by*

$$\begin{aligned}
{}^2R(T_4^3) &= \frac{(6+9\sqrt{2})n}{32}+\frac{8\sqrt{6}+96\sqrt{3}-165\sqrt{2}-6}{96},\\
{}^2SCI(T_4^3) &= \frac{(70\sqrt{6}+42\sqrt{10}+105\sqrt{2}+240\sqrt{7})n}{1680}+\frac{35\sqrt{2}}{112}+\frac{8\sqrt{5}-7\sqrt{10}}{40}\\
&\quad -\frac{6\sqrt{7}}{7}+\frac{35\sqrt{6}+56}{168},\\
{}^2ABC(T_4^3) &= \frac{(4\sqrt{3}+2\sqrt{5}+\sqrt{14}+16\sqrt{2})n}{32}+\frac{8\sqrt{6}-10\sqrt{3}-48\sqrt{2}-11\sqrt{5}+8}{16}\\
&\quad -\frac{7\sqrt{14}}{32}+\frac{35\sqrt{15}}{96},\\
{}^2ABS(T_4^3) &= \frac{(35\sqrt{10}+14\sqrt{70}+70\sqrt{2}+160\sqrt{7})n}{560}\\
&\quad +\frac{350\sqrt{2}-980\sqrt{70}+287\sqrt{10}-960\sqrt{7}}{560}+\frac{\sqrt{6}}{3}.
\end{aligned}$$

Proof Let $m(i,j,k)$ denote the number of paths with degree sequence (i,j,k). Then we can obtain the basic information on T_4^3 in the following table.

$m(1,4,1)$	$m(1,4,2)$	$m(4,2,4)$	$m(2,4,2)$	$m(1,3,1)$	$m(1,4,3)$	$m(1,3,4)$	$m(2,4,3)$
$\frac{n+5}{4}$	$n-6$	$\frac{n-7}{4}$	$\frac{n-11}{4}$	1	2	2	1

Thus, we have

$$\begin{aligned}
{}^2R(T_4^3) &= \sum_{uvw\in E_2(T)}\frac{1}{\sqrt{d_ud_vd_w}}\\
&= \frac{n+5}{4}\times\frac{1}{2}+\frac{n-6}{2\sqrt{2}}+\frac{n-7}{4}\times\frac{1}{4\sqrt{2}}+\frac{n-11}{4}\times\frac{1}{4}+\frac{1}{2\sqrt{6}}+\frac{1}{\sqrt{3}}+4\times\frac{1}{2\sqrt{3}}\\
&= \frac{(6+9\sqrt{2})n}{32}+\frac{8\sqrt{6}+96\sqrt{3}-165\sqrt{2}-6}{96},\\
{}^2SCI(T_4^3) &= \sum_{uvw\in E_2(T)}\frac{1}{\sqrt{d_u+d_v+d_w}}\\
&= \frac{n+5}{4}\times\frac{\sqrt{6}}{6}+\frac{n-6}{\sqrt{7}}+\frac{1}{\sqrt{10}}\times\frac{n-7}{4}+\frac{1}{\sqrt{8}}\times\frac{n-11}{4}\\
&\quad +\frac{1}{3}+\frac{1}{\sqrt{5}}+2\times\frac{1}{\sqrt{8}}\\
&= \frac{(70\sqrt{6}+42\sqrt{10}+105\sqrt{2}+240\sqrt{7})n}{1680}+\frac{35\sqrt{2}}{112}+\frac{8\sqrt{5}-7\sqrt{10}}{40}\\
&\quad -\frac{6\sqrt{7}}{7}+\frac{35\sqrt{6}+56}{168}.
\end{aligned}$$

$$
\begin{aligned}
{}^2ABC(T_4^3) &= \sum_{uvw\in E_2(T)} \frac{\sqrt{d_u+d_v+d_w-3}}{\sqrt{d_ud_vd_w}} \\
&= \frac{n+5}{4}\times\frac{\sqrt{3}}{2}+(n-6)\times\frac{\sqrt{2}}{2}+\frac{n-7}{4}\times\frac{\sqrt{14}}{8}+\frac{\sqrt{5}}{4}\times\frac{n-11}{4} \\
&\quad +\frac{1}{2}+\frac{\sqrt{6}}{3}+2\times\frac{\sqrt{5}}{\sqrt{3}} \\
&= \frac{(4\sqrt{3}+2\sqrt{5}+\sqrt{14}+16\sqrt{2})n}{32}+\frac{8\sqrt{6}-10\sqrt{3}-48\sqrt{2}-11\sqrt{5}+8}{16}-\frac{7\sqrt{14}}{32}+\frac{35\sqrt{15}}{96}, \\
{}^2ABS(T_4^3) &= \sum_{uvw\in E_2(T)} \sqrt{1-\frac{3}{d_u+d_v+dw}} \\
&= \frac{n+5}{4}\times\frac{\sqrt{2}}{2}+(n-6)\times\frac{2\sqrt{7}}{7}+\frac{\sqrt{7}}{\sqrt{10}}\times\frac{n-7}{4}+\frac{\sqrt{5}}{\sqrt{8}}\times\frac{n-11}{4}+\frac{\sqrt{8}}{\sqrt{5}} \\
&\quad +\frac{\sqrt{2}}{\sqrt{5}}+4\times\frac{\sqrt{5}}{\sqrt{8}} \\
&= \frac{(35\sqrt{10}+14\sqrt{70}+70\sqrt{2}+160\sqrt{7})n}{560}+\frac{350\sqrt{2}-980\sqrt{70}+287\sqrt{10}-960\sqrt{7}}{560}+\frac{\sqrt{6}}{3}.
\end{aligned}
$$

This completes the proof. □

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