International J.Math. Combin. Vol.1(2024), 89-99

Several Fundamental Findings on Intuitionistic Fuzzy Strong ∅-b-Normed Linear Spaces

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Abstract: Following the concept of fuzzy normed linear space that Bag and Samanta provided in general t-norm settings, a definition of fuzzy strong-b-normed linear space is provided in this study. In this case, a general function $\varnothing(c)$ that satisfies certain requirements is used in place of the scalar function $|c|$. We study some fundamental results on finite dimensional fuzzy strong b-normed linear space.

Key Words: Intuitionistic fuzzy norm, t-norm, intuitionistic fuzzy normed linear space, neutrosophic set, intuitionistic fuzzy strong ϕ-b-normed linear space.

AMS(2010): 54A40, o3E72.

§1. Introduction

Zadeh [19] was the first to develop the idea of a fuzzy set in 1965. The theory of fuzzy sets has since been extensively expanded by other authors. Fuzzy metric spaces were first introduced by Osmo Kaleva [10], Kramosil and Michalek [14], Georage and Veeramani [9], et al. in various ways. On the other hand, the concept of fuzzy normed linear spaces has been provided in several ways by Katsaras [11], Felbin [6], Cheng and Mordeson [4] and Bag and Samanta [1].

Different generalised metric and norm types, such as the 2-metric [7], b-metric [5], strong-b-metric [13], G-metric [15], 2-norm [13], G-norm [12], etc., as well as generalised fuzzy metric and fuzzy norm types, such as the fuzzy b-metric [16], strong-fuzzy b-metric [18], fuzzy cone metric [17], fuzzy cone norm [2], G-fuzzy norm [3], etc.

Oner proposed fuzzy strong b-metric spaces and produced some topological findings on these spaces in [18].

§2. Preliminaries

In this section, some definitions and results are collected which are used in this paper.

Definition 2.1 *A binary operation* $*: [0,1]\times[0,1]\to[0,1]$ *is called a t-norm if it satisfies the following conditions:*

[1]Received October 6, 2023, Accepted March 10, 2024.

$(a) *$ *is commutative and associative;*
$(b) *$ *is continuous;*
(c) $a * 1 = a$ *for all* $a \in [0, 1]$*;*
(d) $a * \ b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$ *for each* $a, b, c, d \in [0, 1]$.

If $*$ *is continuous, then it is called continuous t-norm.*

The following are examples of some t-norms.

(i) Standard intersection: $a * b = \min\{a, b\}$.
(ii) Algebraic product: $a * b = ab$.
(iii) Bounded difference: $a * b = \max\{0, a + b - 1\}$.

Definition 2.2 *A binary operation* $\diamond : [0, 1] \times [0, 1] \rightarrow [0, 1]$ *is a continuous tconorm if it satisfies the following conditions:*

(a) *is commutative and associative;*
(b) *is continuous;*
(c) $\diamond =$ a *for all* $a \in [0, 1]$*;*
(d) $a \diamond b \leq c \diamond d$ *whenever* $a \leq c$ *and* $b \leq d$ *for each of* $a, b, c, d \in [0, 1]$.
If $*$ *is continuous, then it is called continuous t-norm.*

The following are examples of some t-norms.

(i) Standard intersection: $abb = \max\{a, b\}$.
(ii) Algebraic product: $a \downarrow b = ab$.
(iii) Bounded difference: $a \diamond b = \min\{0, a + b - 1\}$.

Definition 2.3 *A three tuple* $(X, M, *)$ *is said to be a fuzzy metric space, a case of neutrosophic set if* X *is an arbitrary set,* $*$ *a continuous t-norm and* M *a fuzzy set on* $X^2 \times [0, \infty)$ *satisfying the following condition, for all* $\mathrm{x, y, z \in X}$ *and* $\mathrm{t, s} > 0$ *:*

(a) $M(x, y, 0) = 0$*;*
(b) $M(x, y, t) = 1$ *for all* $t > 0$ *iff* $x = y$*;*
(c) $M(x, y, t) = M(y, x, t)$*;*
(d) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$*;*
(e) $M(x, y,) : .[0, \infty) \rightarrow [0, 1]$ *is left continuous;*
(f) $\lim_{\mathrm{n} \rightarrow \infty} M(x, y, t) = 1$.

Definition 2.4 *A 5-tuple* $(X, M, N, *, \diamond)$*issaidtobeanintuitionisticfuzzymetric space (shortly IFM-Space) if* X *is an arbitrary set,* $*$ *is a continuous t-norm,* $\forall$ *is a continuous t-conorm and* M, N *are fuzzy sets on* $X^2 \times [0, \infty)$ *satisfying the following conditions:*

(a) $M(x, y, t) + N(x, y, t) \leq 1$ *for all* $x, y \in X$ *and* $t > 0$*;*
(b) $M(x, y, 0) = 0$ *for all* $x, y \in X$*;*
(c) $M(x, y, t) = 1$ *for all* $x, y \in X$ *and* $t > 0$ *if and only if* $x = y$*;*
(d) $M(x, y, t) = M(y, x, t)$ *for all* $x, y \in X$ *and* $t > 0$*;*
(e) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ *for all* $x, y, z \in X$ *and* $s, t > 0$*;*
(f) $M(x, y,) : .[0, \infty) \rightarrow [0, 1]$ *is left continuous for all* $x, y \in X$*;*
(g) $\lim_{\mathrm{n} \rightarrow \infty} M(x, y, t) = 1$*;*

(h) $N(x,y,0)=1$ *for all* $x,y\in X$;

(i) $N(x,y,t)=0$ *for all* $x,y\in X$ *and* $t>0$ *if and only if* $x=y$;

(j) $N(x,y,t)=N(y,x,t)$ *for all* $x,y\in X$ *and* $t>0$;

(k) $N(x,y,t)\diamond N(y,z,s)\geq N(x,z,t+s)$ *for all* $x,y,z\in X$ *and* $s,t>0$;

(l) $M(x,y,):.[0,\infty)\rightarrow[0,1]$ *is right continuous for all* $x,y\in X$;

(m) $\lim_{\mathrm{n}\rightarrow\infty} N(x,y,t)=0$ *for all* $x,y\in X$,

then, (M,N) *is called an intuitionistic fuzzy metric on* X. *The functions* $M(x,y,t)$ *and* $N(x,y,t)$ *denote the degree of nearness and degree of non nearness between x and y with respect to* t, *respectively.*

Definition 2.5 *Let* $(X,M,N,*,\diamond)$ *be an intuitionistic fuzzy metric space. Then,*

(a) *A sequence* $\{x_n\}$ *is said to be convergent* x *in* X *if for each* $\epsilon>0$ *and* $t>0$, *there exist* $\mathrm{n}_0\in N$ *such that* $M\left(x_n,x,t\right)>1-\epsilon$ *and* $N\left(x_n,x,t\right)<0-\epsilon$ *for all* $n\geq n_0$;

(b) *A sequence* $\{\mathrm{x_n}\}$ *is said to be Cauchy if for each* $\epsilon>0$ *and* $\mathrm{t}>0$, *there exist* $\mathrm{n}_0\in\mathrm{N}$ *such that* $M\left(\mathrm{x_n},\mathrm{x_m},\mathrm{t}\right)>1-\epsilon$ *and* $N\left(\mathrm{x_n},\mathrm{x_m},\mathrm{t}\right)<0-\epsilon$ *for all* $\mathrm{n},\mathrm{m}\geq\mathrm{n}_0$;

(c) *An intuitionistic fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.*

Definition 2.6 *A sequence* $\{\mathrm{Si}\}$ *of self maps on a complete intuitionistic fuzzy metric space* $(X,M,N,*,\diamond)$ *is said to be intuitionistic mutually contractive if for* $t>0$ *and* $\mathrm{i}\epsilon\mathrm{N}$

$$M\left(S_{\mathrm{i}}x,S_jy,t\right)\geq M\left(x,y,\frac{t}{p}\right)\quad and\quad N\left(S_ix,S_jy,t\right)\leq N\left(x,y,\frac{t}{p}\right),$$

where $x,y\in X,p\in(0,1),i\neq j$ *and* $x\neq y$.

Definition 2.7 *Let* $(X,M,N,*,\diamond)$ *be an intuitionistic fuzzy normed linear space.*

(i) *A sequence* $\{x_n\}$ *is said to be convergent if there exists* $x\in X$ *such that*

$$\lim_{n\rightarrow\infty}\mathrm{M}\left(x_n-x,t\right)=1\quad and\quad \lim_{n\rightarrow\infty}N\left(x_n-x,t\right)=0$$

for all $t>0$. *Then* x *is called the limit of the sequence* $\{x_n\}$ *and denoted by* $\lim_{n\rightarrow\infty}x_n$;

(ii) *A sequence* $\{x_n\}$ *in an intuitionistic fuzzy normed linear space* (X,N) *is said to be Cauchy if*

$$\lim_{n\rightarrow\infty}\mathrm{M}\left(x_{n+p}-x_n,t\right)=1\quad and\quad \lim_{n\rightarrow\infty}N\left(x_{n+p}-x_n,t\right)=0$$

for all $t>0$ *and* $p=1,2,\cdots$;

(iii) $A\subseteq X$ *is said to be closed if for any sequence* $\{x_n\}$ *in* A *converges to* $x\in A$;

(iv) $A\subseteq X$ *is said to be the closure of* A, *denoted by* $\bar{A}$ *if for any* $x\in\bar{A}$, *if there is a sequence* $\{x_n\}\subseteq A$ *such that* $\{x_n\}$ *converges to* x. *(v)* $A\subseteq X$ *is said to be compact if any sequence* $\{x_n\}\subseteq A$ *has a subsequence converging to an element of* A.

Lemma 2.6 *Let* $(X,M,N,*,\diamond)$ *be an intuitionistic fuzzy normed linear space and let* $M(x,.)$, $N(x,\cdot)$ *be with* $x\neq 0)$. *If the set* $A=\{x:M(x,1)>0$ *and* $\mathrm{N(x,1)}<0\}$ *is compact, then* X *is finite dimensional.*

§3. Intuitionistic Fuzzy Strong ϕ-b-Normed Linear Space

In this section, we give the definition of intuitionistic fuzzy normed linear space in a new approach.

Definition 3.1 *Let ϕ be a function defined on $\mathbb{R}$ to $\mathbb{R}^+$ with the following properties*

(ϕ1) $\phi(-t) = \phi(t)$ *for all* $t \in \mathbb{R}$;

(ϕ2) $\phi(1) = 1$;

(ϕ3) ϕ *is strictly increasing and continuous on* $(0, \infty)$;

(ϕ4) $\lim\limits_{\alpha \to 0} \phi(\beta) = 0$ *and* $\lim_{\alpha \to \infty} \phi(\beta) = \infty$.

The followings are examples of such functions.

(i) $\phi(\beta) = |\beta|$ for all $\beta \in \mathbb{R}$.

(ii) $\phi(\beta) = |\beta|^p$ for all $\beta \in \mathbb{R}, p \in \mathbb{R}^+$.

(iii) $\phi(\beta) = \dfrac{2\beta^{2n}}{|\beta| + 1}$ for all $\beta \in \mathbb{R}, n \in \mathbb{N}$.

Definition 3.2 *Let X be a linear space over the field $\mathbb{R}$ and $b \geq 1$ be a given real number. A fuzzy subset N of $X \times \mathbb{R}$ is called intuitionistic fuzzy strong ϕ-b-norm on X if for all $x, y \in X$ the following conditions hold:*

(i) $\forall t \in \mathbb{R}$ *with* $t \leq 0, M(x, t) = 0$;

(ii) $(\forall t \in \mathbb{R}, t > 0, M(x, t) = 1)$ *iff* $x = \theta$;

(iii) $\forall t \in \mathbb{R}, t > 0, M(cx, t) = M\left(x, \frac{t}{\phi(c)}\right)$ *if* $\phi(c) \neq 0$;

(iv) $\forall s, t \in \mathbb{R}, M(x + y, s + bt) \geq M(x, s) * N(y, t)$;

(v) $M(x, \cdot)$ *is a non-decreasing function of t and* $\lim_{t \to \infty} M(x, t) = 1$;

(vi) $\forall t \in \mathbb{R}$ *with* $t \geq 0, N(x, t) = 0$;

(vii) $(\forall t \in \mathbb{R}, t = 0, N(x, t) = 0)$ *iff* $x = \theta$;

($viii$) $\forall t \in \mathbb{R}, t < 0, N(cx, t) = N\left(x, \frac{t}{\phi(c)}\right)$ *if* $\phi(c) \neq 0$;

(ix) $\forall s, t \in \mathbb{R}, N(x + y, s + bt) \leq N(x, s) \diamond N(y, t)$;

(x) $N(x, \cdot)$ *is a non-increasing function of t and* $\lim_{t \to \infty} N(x, t) = 0$.

Then $(X, \mathrm{M}, N, \phi, b, *)$ is called intuitionistic fuzzy strong ϕ-b-normed linear space.

§4. Finite Dimensional Intuitionistic Fuzzy Strong ϕ-b-Normed Linear Spaces

In this section, some basic results on finite dimensional intuitionistic fuzzy strong ϕ-b-normed linear spaces are established.

Lemma 4.1 *Let $(X, \mathrm{M}, N, \phi, b, *, \diamond)$ be a Intuitionistic fuzzy strong ϕ-b-normed linear space with the underlying t-norm $*$ continuous and t-co norm at $(1, 1)$ and $\{x_1, x_2, \cdots x_n\}$ be a linearly independent set of vectors in X. Then there exists $c > 0$ and $\delta \in (0, 1)$ such that for any set of* *scalars* $\{\beta_1, \beta_2, \cdots, \beta_n\}$ *with* $\sum_{i=1}^{n} |\beta_i| \neq 0$

$$M\left(\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n, \frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{n}|\beta_i|}\right)}\right) < 1 - \delta. \tag{4.1}$$

and

$$N\left(\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n, \frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{n}|\beta_i|}\right)}\right) > 0 - \delta. \tag{4.2}$$

Proof Notice that the equations

$$M\left(\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n, \frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{n}|\beta_i|}\right)}\right) < 1 - \delta.$$

and

$$N\left(\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n, \frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{n}|\beta_i|}\right)}\right) > 0 - \delta.$$

are equivalent to the relations

$$M\left(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n, bc\right) < 1 - \delta$$

and

$$N\left(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n, bc\right) > 0 - \delta$$

for some $c > 0$, $\delta \in (0,1)$ and for all set of scalars $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ with $\sum_{i=1}^{n} |\alpha_i| = 1$ If possible, suppose that (4.1) does not hold. Thus, for each $c > 0$ and $\delta \in (0,1)$, there exists a set of scalars $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ with $\sum_{i=1}^{n} |\alpha_i| = 1$ for which

$$M\left(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n, bc\right) \geq 1 - \delta$$

and

$$N\left(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n, bc\right) \leq 0 - \delta.$$

Then, for $c = \delta = \frac{1}{m}, m = 1, 2, \ldots$, there exists a set of scalars $\left\{\alpha_1^{(m)}, \alpha_2^{(m)}, \ldots, \alpha_n^{(m)}\right\}$ with $\sum_{i=1}^{n} \left|\alpha_i^{(m)}\right| = 1$ such that

$$M\left(y_m, \frac{b}{m}\right) \geq 1 - \frac{1}{m}$$

and

$$N\left(y_m, \frac{b}{m}\right) \leq 0 - \frac{1}{m},$$

where $y_m = \alpha\beta_1^{(m)} x_1 + \beta_2^{(m)} x_2 + \cdots + \beta_n^{(m)} x_n$. Since $\sum_{i=1}^{n} \left|\alpha_i^{(m)}\right| = 1$, we have $0 \leq \left|\alpha_i^{(m)}\right| \leq 1$ for $i = 1, 2, \ldots, n$. So for each fixed i, the sequence $\left\{\alpha_i^{(m)}\right\}$ is bounded and hence $\left\{\alpha_i^{(m)}\right\}$ has a convergent subsequence. Let α_1 denotes the limit of that subsequence and let $\{y_{1,m}\}$ denotes the corresponding subsequence of $\{y_m\}$. By the same argument $\{y_{1,m}\}$ has a subsequence $\{y_{2,m}\}$ for which the corresponding subsequence of scalars $\left\{\alpha_2^{(m)}\right\}$ converges to α_2. Continuing in this way, after n steps we obtain a subsequence $\{y_{n,m}\}$ where

$$y_{n,m} = \sum_{i=1}^{n} \gamma_i^{(m)} x_i \quad \text{with} \quad \sum_{i=1}^{n} \left|\gamma_i^{(m)}\right| = 1$$

and $\gamma_i^{(m)} \to \alpha_i$ as $m \to \infty$ for each $i = 1, 2, \cdots, n$.

Let $y = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$. Now,

$$\begin{aligned}
M(y_{n,m} - y, t) &= M\left(\sum_{j=1}^{n}\left(\gamma_j^{(m)} - \alpha_j\right) x_j, t\right) \\
&= M\left(\left(\gamma_1^{(m)} - \alpha_1\right) x_1 + \sum_{j=2}^{n}\left(\gamma_j^{(m)} - \alpha_j\right) x_j, \frac{t}{n} + b(n-1)\frac{t}{nb}\right) \\
&\geq M\left(\left(\gamma_1^{(m)} - \alpha_1\right) x_1, \frac{t}{n}\right) * M\left(\sum_{j=2}^{n}\left(\gamma_j^{(m)} - \alpha_j\right) x_j, (n-1)\frac{t}{nb}\right) \\
&= M\left(\left(\gamma_1^{(m)} - \alpha_1\right) x_1, \frac{t}{n}\right) \\
&\quad * M\left(\left(\gamma_2^{(m)} - \alpha_2\right) x_2 + \sum_{j=3}^{n}\left(\gamma_j^{(m)} - \alpha_j\right) x_j, \frac{t}{nb} + b\left(1 - \frac{2}{n}\right)\frac{t}{b^2}\right) \\
&\geq M\left(\left(\gamma_1^{(m)} - \alpha_1\right) x_1, \frac{t}{n}\right) * M\left(\left(\gamma_2^{(m)} - \alpha_2\right) x_2, \frac{t}{nb}\right) \\
&\quad * M\left(\sum_{j=3}^{n}\left(\gamma_j^{(m)} - \alpha_j\right) x_j, \left(1 - \frac{2}{n}\right)\frac{t}{b^2}\right) \\
&\geq M\left(\left(\gamma_1^{(m)} - \alpha_1\right) x_1, \frac{t}{n}\right) * M\left(\left(\gamma_2^{(m)} - \alpha_2\right) x_2, \frac{t}{nb}\right) \\
&\quad * \cdots * M\left(\left(\gamma_n^{(m)} - \alpha_n\right) x_n, \frac{t}{nb^{n-1}}\right) \\
&= M\left(x_1, \frac{t}{n\phi\left(\left(\gamma_1^{(m)} - \alpha_1\right)\right)}\right) * \cdots * M\left(x_n, \frac{t}{nb^{n-1}\phi\left(\left(\gamma_n^{(m)} - \alpha_n\right)\right)}\right)
\end{aligned}$$

and

$$\begin{aligned}
N(y_{n,m} - y, t) &= N\left(\sum_{j=1}^{n}\left(\gamma_j^{(m)} - \alpha_j\right) x_j, t\right) \\
&= N\left(\left(\gamma_1^{(m)} - \alpha_1\right) x_1 + \sum_{j=2}^{n}\left(\gamma_j^{(m)} - \alpha_j\right) x_j, \frac{t}{n} + b(n-1)\frac{t}{nb}\right)
\end{aligned}$$

$$
\begin{aligned}
&= N\left(\left(\gamma_1^{(m)}-\alpha_1\right)x_1,\frac{t}{n}\right)\\
&\le N\left(\gamma_1^{(m)}-\alpha_1\right)x_1,\frac{t}{n}\right)\diamond N\left(\sum_{j=2}^{n}\left(\gamma_j^{(m)}-\alpha_j\right)x_j,(n-1)\frac{t}{nb}\right)\\
&\quad \diamond N\left(\left(\gamma_2^{(m)}-\alpha_2\right)x_2+\sum_{j=3}^{n}\left(\gamma_j^{(m)}-\alpha_j\right)x_j,\frac{t}{nb}+b\left(1-\frac{2}{n}\right)\frac{t}{b^2}\right)\\
&\le N\left(\left(\gamma_1^{(m)}-\alpha_1\right)x_1,\frac{t}{n}\right)\diamond N\left(\left(\gamma_2^{(m)}-\alpha_2\right)x_2,\frac{t}{nb}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&N\left(\sum_{j=3}^{n}\left(\gamma_j^{(m)}-\alpha_j\right)x_j,\left(1-\frac{2}{n}\right)\frac{t}{b^2}\right)\\
&\ge N\left(\left(\gamma_1^{(m)}-\alpha_1\right)x_1,\frac{t}{n}\right)\diamond N\left(\left(\gamma_2^{(m)}-\alpha_2\right)x_2,\frac{t}{nb}\right)\diamond\cdots\nabla N\left(\left(\gamma_n^{(m)}-\alpha_n\right)x_n,\frac{t}{nb^{n-1}}\right)\\
&= N\left(x_1,\frac{t}{n\phi\left(\left(\gamma_1^{(m)}-\alpha_1\right)\right)}\right)\diamond\cdots\nabla N\left(x_n,\frac{t}{nb^{n-1}\phi\left(\left(\gamma_n^{(m)}-\alpha_n\right)\right)}\right).
\end{aligned}
$$

Now taking limit as $m\to\infty$ on both sides, we have

$$\lim_{m\to\infty} M\left(y_{n,m}-y,t\right)\ge 1*1*\cdots*1,\quad \forall t>0$$

and

$$\lim_{m\to\infty} N\left(y_{n,m}-y,t\right)\le 0>0\diamond\cdots\vee 0,\quad \forall t>0$$

i.e

$$\lim_{m\to\infty} M\left(y_{n,m}-y,t\right)=1,\quad \forall t>0$$

and

$$\lim_{m\to\infty} N\left(y_{n,m}-y,t\right)=0,\quad \forall t>0$$

Now, for $r>0$, choose m such that $\frac{1}{m}<\frac{r}{b^2}$. We have

$$
\begin{aligned}
M\left(y_{n,m},\frac{r}{b}\right)&=M\left(y_{n,m}+\theta,\frac{b}{m}+b\left(\frac{r}{b^2}-\frac{1}{m}\right)\right)\\
&\ge\left(y_{n,m},\frac{b}{m}\right)*M\left(\theta,\frac{r}{b^2}-\frac{1}{m}\right)\ge\left(1-\frac{b}{m}\right)*1
\end{aligned}
$$

and

$$
\begin{aligned}
N\left(y_{n,m},\frac{r}{b}\right)&=N\left(y_{n,m}+\theta,\frac{b}{m}+b\left(\frac{r}{b^2}-\frac{1}{m}\right)\right)\\
&\le N\left(y_{n,m},\frac{b}{m}\right)\diamond N\left(\theta,\frac{r}{b^2}-\frac{1}{m}\right)\le\left(1-\frac{b}{m}\right)\diamond 0
\end{aligned}
$$

which implies

$$\lim_{m\to\infty} M\left(y_{n,m}, \frac{r}{b}\right) \geq 1 \quad \text{i.e.,} \quad \lim_{m\to\infty} M\left(y_{n,m}, \frac{r}{b}\right) = 1$$

and

$$\lim_{m\to\infty} N\left(y_{n,m}, \frac{r}{b}\right) \leq 0 \quad \text{i.e.,} \lim_{m\to\infty} N\left(y_{n,m}, \frac{r}{b}\right) = 0.$$

Again,

$$\begin{aligned} M(y, 2r) &= M\left(y - y_{n,m} + y_{n,m}, r + b \cdot \frac{r}{b}\right) \\ &\geq M\left(y - y_{n,m}, r\right) \cdot N\left(y_{n,m}, \frac{r}{b}\right) \end{aligned}$$

and

$$\begin{aligned} N(y, 2r) &= N\left(y - y_{n,m} + y_{n,m}, r + b \cdot \frac{r}{b}\right) \\ &\leq N\left(y - y_{n,m}, r\right) \diamond N\left(y_{n,m}, \frac{r}{b}\right). \end{aligned}$$

Thus,

$$\begin{aligned} M(y, 2r) &\geq \lim_{m\to\infty} M\left(y - y_{n,m}, r\right) * \lim_{m\to\infty} M\left(y_{n,m}, \frac{r}{b}\right) \\ &\Rightarrow M(y, 2r) \geq 1 \cdot 1 = 1 \Rightarrow M(y, 2r) = 1 \end{aligned}$$

and

$$\begin{aligned} N(y, 2r) &\leq \lim_{m\to\infty} N\left(y - y_{n,m}, r\right) \diamond \lim_{m\to\infty} N\left(y_{n,m}, \frac{r}{b}\right) \\ &\Rightarrow N(y, 2r) \leq 0 \diamond 0 = 0 \Rightarrow N(y, 2r) = 0. \end{aligned}$$

Since $r > 0$ is arbitrary, so $y = \theta$. Again since $\sum_{i=1}^{n} \left|\alpha_i^{(m)}\right| = 1$ and $\{x_1, x_2, \ldots, x_n\}$ is a linearly independent set of vectors so $y = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \neq \theta$, thus we arrive at a contradiction and Lemma is proved. □

Theorem 4.2 *Every finite dimensional Intuitionistic fuzzy strong ϕ-b-normed linear space with the underlying t-norm * continuous and t-co norm $\diamond$ Continuous at $(1,1)$ is complete.*

Proof Let $(X, \mathrm{M}, N, \phi, b, *, \diamond)$ be a Intuitionistic fuzzy strong ϕ-b-normed linear space where $b(> 1)$ is a real constant. Let $\dim X = r$ and $\{e_1, e_2, \cdots, e_r\}$ be a basis for X. Let $\{x_p\}$ be a Cauchy sequence in X. Then, $x_n = \sum_{k=1}^{r} \alpha_k^{(n)} e_k$ for suitable scalars $\alpha_1^{(n)}, \alpha_2^{(n)}, \cdots, \alpha_r^{(n)}$ · So

$$\lim_{m,n\to\infty} \mathrm{M}\left(x_m - x_n, t\right) = 1, \quad \forall t > 0$$

and

$$\lim_{m,n\to\infty} N\left(x_m - x_n, t\right) = 0, \quad \forall t > 0.$$

Now, by Lemma 4.1 it follows that $\exists c > 0$ and $\delta \in (0,1)$ such that

$$M\left(\sum_{i=1}^{r} \left(a_i^{(m)} - \alpha_i^{(n)}\right) e_i, \frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{r}\left|\alpha_i^{(m)} - \alpha_i^{(n)}\right|}\right)}\right) < 1 - \delta$$

and

$$N\left(\sum_{i=1}^{r}\left(\alpha_i^{(m)}-\alpha_i^{(n)}\right)e_i,\frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{r}\left|\alpha_i^{(m)}-\alpha_i^{(n)}\right|}\right)}\right)>0-\delta. \quad (4.3)$$

If

$$\sum_{i=1}^{r}\left|\alpha_i^{(m)}-\alpha_i^{(n)}\right|=0$$

then $\alpha_i^{(m)}=\alpha_i^{(n)}$ for any integer i implies that $\{x_n\}$ is a constant sequence and hence follows the theorem. So we may assume

$$\sum_{i=1}^{r}\left|\alpha_i^{(m)}-\alpha_i^{(n)}\right|\neq 0.$$

Again, for $0<\delta<1$ from (4-3) it follows that there exists a positive integer $n_0(\delta,t)$ such that

$$M\left(\sum_{i=1}^{r}\left(\alpha_i^{(m)}-\alpha_i^{(n)}\right)e_i,t\right)>1-\delta,\quad \forall m,n\geq n_0(\delta,t) \quad (4.4)$$

and

$$N\left(\sum_{i=1}^{r}\left(\alpha_i^{(m)}-\alpha_i^{(n)}\right)e_i,t\right)<0-\delta,\quad \forall m,n\geq n_0(\delta,t). \quad (4.5)$$

Now, from (4.4) and (4.5), $\forall m,n\geq n_0(\delta,t)$ we have

$$M\left(\sum_{i=1}^{r}\left(\alpha_i^{(m)}-\alpha_i^{(n)}\right)e_i\frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{r}\left|\alpha_i^{(m)}-\alpha_i^{(n)}\right|}\right)}\right)<M\left(\sum_{i=1}^{r}\left(\alpha_i^{(m)}-\alpha_i^{(n)}\right)e_{i,t}\right)$$

and

$$N\left(\sum_{i=1}^{r}\left(\alpha_i^{(m)}-\alpha_i^{(n)}\right)e_i\frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{r}\left|\alpha_i^{(m)}-\alpha_i^{(n)}\right|}\right)}\right)>N\left(\sum_{i=1}^{r}\left(\alpha_i^{(m)}-\alpha_i^{(n)}\right)e_i,t\right).$$

Thus,

$$\frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{r}\left|\alpha_t^{(m)}-\alpha_i^{(m)}\right|}\right)}<t$$

since $M(x,t)$ is non-decreasing with respect to t and

$$\frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{r}\left|\alpha_t^{(m)}-\alpha_i^{(m)}\right|}\right)}>t$$

since $N(x,t)$ is non-increasing with respect to t. Hence, since $t>0$ is arbitrary, namely

$$\lim_{m,n\to\infty} \frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{r}\left|\alpha_i^{(m)}-\alpha_i^{(n)}\right|}\right)} = 0$$

then

$$\lim_{m,n\to\infty} \phi\left(\frac{1}{\sum_{i=1}^{r}\left|\alpha_i^{(m)}-\alpha_i^{(n)}\right|}\right) = \infty.$$

Thus,

$$\phi\left(\frac{1}{\lim_{m\to\infty}\sum_{i=1}^{r}\left|\alpha_i^{(m)}-\alpha_i^{(n)}\right|}\right) = \infty$$

since ϕ is continuous. Then

$$\lim_{m,n\to\infty} \sum_{i=1}^{r}\left|\alpha_i^{(m)}-\alpha_i^{(n)}\right| = 0$$

since $\lim_{\alpha\to\infty}\phi(\beta)=\infty$. Therefore, $\left\{\alpha_i^{(m)}\right\}$ is a Cauchy sequence of scalars for each $i=1,2,\cdots,r$. So each sequence $\left\{\alpha_i^{(m)}\right\}$ converges. Let $\lim_{n\to\infty}\alpha_i^{(n)}=\alpha_i$ for $i=1,2,\ldots,r$. Define $x=\sum_{i=1}\alpha e_i$. Then clearly $x\in X$. By similar calculation as in Lemma 4.1 , it can be shown that $\lim_{n\to\infty} M(x_n-x,t)=1, \lim_{n\to\infty} N(x_n-x,t)=0, \forall t>0$. Hence X is complete.

§5. Conclusion

Recently, various writers have constructed various kinds of generalised fuzzy metric spaces as well as generalised fuzzy normed linear spaces. The concept of fuzzy strong b-normed linear spaces was presented after the introduction of fuzzy strong b-metric spaces, and various findings in finite finite dimensions fuzzy strong b-normed linear spaces were examined. We believe there is a vast area of research to be done in order to create fuzzy strong b-normed linear spaces. Open issues in such spaces include results on completeness and compactness, operator standards, etc.

References

[1] T. Bag and S.K. Samanta, Finite dimensional fuzzy normed linear spaces, *The J. Fuzzy Math.*, 11 (2003), pp. 687-705.

[2] T. Bag, Finite dimensional fuzzy cone normed linear spaces, *Int. J. Math. Scientific Computing*, 3 (2013), pp. 9-14.

[3] S. Chatterjee, T. Bag and S.K. Samanta, Some results on G-fuzzy normed linear space, *Int. J. Pure Appl. Math.*, 5 (2018), pp. 1295-1320.

[4] S.C. Cheng and J.N. Mordeson, Fuzzy linear operators and fuzzy normed linear spaces, *Bull. Cal. Math. Soc.*, 86 (1994), pp. 429-436.

[5] S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math Inf Univ Ostraviensis*, 1 (1993), pp. 5-11.

[6] C. Felbin, Finite dimensional fuzzy normed linear spaces, *Fuzzy Sets Syst.*, 48 (1992), pp. 239-248.

[7] S. Gahler, 2-metrische Raume und thre topologische Struktur, *Mathematische Nachrichten*, 26 (1963), pp. 115-118.

[8] S. Gahler, Lineare 2-normierte mume, *Math. Nachr.*, 28 (1964), pp. 1 - 43.

[9] A. George and P. Veeramani, On some results in fuzzy metric spaces, *Fuzzy Sets Syst.*, 64 (1994), pp. 395-399.

[10] O. Kaleva and S. Seikkala, On fuzzy metric spaces, *Fuzzy Sets Syst.*, 12 (1984), pp. 215-229.

[11] A.K. Katsaras, Fuzzy topological vector spaces I, *Fuzzy Sets Syst.*, 12 (1984), pp. 143-154.

[12] K.A. Khan, Generalized normed spaces and fixed point theorems, *J. Math. Computer Sci.*, 13 (2014), pp. 157-167.

[13] W. Kirk and N. Shahzad, *Fired Point Theory in Distance Spaces*, Springer, Cham, 2014.

[14] I. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, *Kybernetica*, 11 (1975), pp. 326-334.

[15] Z. Mustafa, H. Obiedat and F. Awawdeh, Some fixed point theorem for mapping on complete G-metric spaces, *Fixed Point Theory Appl.*, 2008, 12 pages.

[16] S. Nădăban, Fuzzy b-metric spaces, *Int. J. Computers Communications and Control*, 11 (2016), pp. 273-281.

[17] T. Oner, M.B. Kandemir and B. Tanay, Fuzzy cone metric spaces, *J. Nonlinear Sci. Appl.*, 8(2015), pp. 610-616.

[18] Oner, On topology of fuzzy strong b-metric spaces, *J. New Theory*, 21 (2018), pp. 59-67.

[19] L.A. Zadeh, Fuzzy sets, *Inf. Control*, 8 (1965), pp. 338-353.