Smarandachely Bondage Number of a Graph

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Abstract: A dominating set \( D \) of a graph \( G \) is called a Smarandachely dominating \( s \)-set if for an integer \( s \), each vertex \( v \) in \( V - D \) is adjacent to a vertex \( u \in D \) such that \( \deg u + s = \deg v \). The minimum cardinality of Smarandachely dominating \( s \)-set in a graph \( G \) is called the Smarandachely dominating \( s \)-number of \( G \), denoted by \( \gamma^s_s(G) \). Such a set with minimum cardinality is called a Smarandachely dominating \( s \)-set. The Smarandachely bondage \( s \)-number \( b^s_s(G) \) of a graph \( G \) is defined to be the minimum cardinality among all sets of edges \( E' \subseteq E \) such that \( \gamma^s_s(G - E') > \gamma^s_s(G) \). Particularly, the set with minimum Smarandachely bondage \( s \)-number for all integers \( s \geq 0 \) or \( s \leq 0 \) is called the strong or weak dominating number of \( G \), denoted by \( \gamma_s(G) \) or \( \gamma_w(G) \), respectively. In this paper, we present some bounds on \( b_s(G) \) and \( b_w(G) \) and give exact values for \( b_s(G) \) and \( b_w(G) \) for complete graphs, paths, wheels and bipartite complete graphs. Some general bounds are also given.

Key Words: Smarandachely dominating \( s \)-set, Smarandachely dominating \( s \)-number, Smarandachely bondage \( s \)-number, strong or weak bondage numbers.


§1. Introduction

In this paper, we follow the notation of [6,7]. Specifically, let \( G = (V,E) \) be a graph with vertex set \( V \) and edge set \( E \). A set \( D \subseteq V \) is a dominating set of \( G \) if every vertex \( v \) in \( V - D \) there exists a vertex \( u \) in \( D \) such that \( u \) and \( v \) are adjacent in \( G \). The domination number of \( G \), denoted \( \gamma(G) \), is the minimum cardinality of a dominating set of \( G \). The concept of domination in graphs, with its many variations, is well studied in graph theory. A thorough study of domination appears in [6,7]. Let \( uv \in E \). Then, \( u \) and \( v \) dominate each other. A dominating set \( D \) of a graph \( G \) is called a Smarandachely dominating \( s \)-set if for an integer \( s \), each vertex

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v in \( V - D \) is adjacent to a vertex \( u \in D \) such that \( \deg u + s = \deg v \). The minimum cardinality of Smarandachely dominating \( s \)-set in a graph \( G \) is called the Smarandachely dominating \( s \)-number of \( G \), denoted by \( \gamma_s^\ast(G) \). Such a set with minimum cardinality is called a Smarandachely dominating \( s \)-set. The Smarandachely bondage \( s \)-number \( b_s^\ast(G) \) of a graph \( G \) is defined to be the minimum cardinality among all sets of edges \( E' \subseteq E \) such that \( \gamma_s^\ast(G - E') > \gamma_s^\ast(G) \). Particularly, the set with minimum Smarandachely bondage \( s \)-number for all integers \( s \geq 0 \) or \( s \leq 0 \) is called the strong or weak dominating number of \( G \), denoted by \( \gamma_s(G) \) or \( \gamma_w(G) \), respectively.

As a special case of Smarandachely bondage number, the strong (weak) domination was introduced by E. Sampathkumar and L. Pushpa Latha in [8]. For any undefined term, we refer Harary [4]. By definition, the bondage number \( b(G) \) of a nonempty graph \( G \) is the minimum cardinality among all sets of edges \( E' \subseteq E \) for which \( \gamma(G - E') > \gamma(G) \). Thus, the bondage number of \( G \) is the smallest number of edges whose removal renders every minimum dominating set of \( G \) a nondominating set in the resulting spanning subgraph. Since the domination number of every spanning subgraph of a nonempty graph \( G \) is at least as great as \( \gamma(G) \), the bondage number of a nonempty graph is well defined. This concept was introduced by Bauer, Harary, Nieminen and Suffel [1] and has been further studied by Fink, Jacobson, Kinch and Roberts [2], Hartnell and Rall [5], etc. The strong bondage number of \( G \), denoted \( b_s(G) \), as the minimum cardinality among all sets of edges \( E' \subseteq E \) such that \( \gamma_s(G - E') > \gamma_s(G) \). This concept was introduced by J. Ghoshal, R. Laskar, D. Pillone and C. Wallis [3].

We define the weak bondage number of \( G \), denoted \( b_w(G) \), as the minimum cardinality among all sets of edges \( E' \subseteq E \) such that \( \gamma_w(G - E') > \gamma_w(G) \), and we deal with the strong bondage number of a nonempty graph \( G \).

§2. Exact Values for \( b_s(G) \) and \( b_w(G) \)

We begin our investigation of the strong and weak bondage numbers by computing its value for several well known classes of graphs. In several instances we shall have cause to use the ceiling function of a number \( x \). This is denoted \([x]\) and takes the value of the least integer greater than or equal to \( x \). We begin with a rather straightforward evaluation of the strong and weak bondage numbers of the complete graph of order \( n \).

**Proposition 2.1** The strong bondage number of the complete graph \( K_n \) (\( n \geq 2 \)) is

\[
b_s(K_n) = \lceil n/2 \rceil.
\]

*Proof.* Let \( u_1, u_2, \ldots, u_n \) be the \( n \) vertices of degree \( n - 1 \). Then clearly removal of fewer than \( n/2 \) edges results in a graph \( H \) having maximum degree \( n - 1 \). Hence \( b_s(K_n) \geq \lceil n/2 \rceil \). Now we consider the following cases.

**Case 1.** If \( n \) is even, then the removal of \( n/2 \) independent edges \( u_1u_2, u_3u_4, \ldots, u_{n-1}u_n \) results in a graph \( H' \) regular of degree \( n - 2 \). Hence \( b_s(K_n) = n/2 \).

**Case 2.** If \( n \) is odd, then the removal of \((n-1)/2\) independent edges \( u_1u_2, u_3u_4, \ldots, u_{n-2}u_{n-1} \)
yields a graph $H''$ containing exactly one vertex $u_n$ of degree $n - 1$. Thus by removing an edge incident with $u_n$ we obtain a graph $H'''$ with maximum degree $n - 2$. Hence $b_s(K_n) = (n - 1)/2 + 1$.

Combining cases (1) and (2) it follows that $b_s(K_n) = \lceil n/2 \rceil$.

**Proposition 2.2** The weak bondage number of the complete graph $K_n$ ($n \geq 2$) is

$$b_s(K_n) = 1.$$ 

**Proof** If $H$ is a spanning subgraph of $K_n$ that is obtained by removing any edge from $K_n$, then $H$ contains two vertices of degree $n - 2$. Whence $\gamma_w(H) = 2 > 1 = \gamma_w(K_n)$. Hence $b_w(K_n) = 1$. □

If $G$ is a regular graph, then $\gamma(G) = \gamma_s(G)$ because in a regular graph, the degrees of all the vertices are equal. We next consider paths $P_n$ and cycles $C_n$ on $n$ vertices and find that $\gamma(C_n) = \gamma_s(C_n)$ because $C_n$ is a regular graph. Also $\gamma(P_n) = \gamma_s(P_n)$ since we can choose from all the $\gamma$ sets of $P_n$, one which dose not include either end vertex. Such a $\gamma$ set is also a $\gamma_s$ set and hence we get $\gamma(P_n) \geq \gamma_s(P_n)$ but since $\gamma(G) \geq \gamma_s(G)$ for all graphs $G$, which follows

**Lemma 2.3** The strong domination number of the $n$-cycle and the path of order $n$ are respectively

(i) $\gamma_s(C_n) = \lceil n/3 \rceil$ for $n \geq 3$ and

(ii) $\gamma_s(P_n) = \lceil n/3 \rceil$ for $n \geq 2$.

**Lemma 2.4** The weak domination number of the $n$-cycle and the path of order $n$ are respectively

(i) $\gamma_w(C_n) = \lceil n/3 \rceil$ for $n \geq 3$ and

(ii) $\gamma_w(P_n) = \begin{cases} \lceil n/3 \rceil & \text{if } n \equiv 1 \mod 3, \\ \lceil n/3 \rceil + 1 & \text{otherwise.} \end{cases}$

**Proof**

(i) Since $C_n$ is a regular graph, so $\gamma_w(C_n) = \gamma(C_n)$ and proof techniques in [2].

(ii) $\gamma_w(P_n) = \lceil (n - 4)/3 \rceil + 2 = \gamma(P_{n-4}) + 2$, the proof is the same as in [2]. □

**Theorem 2.5** The strong bondage number of the $n$-cycle (with $n \geq 3$) is

$$b_s(C_n) = \begin{cases} 3 & \text{if } n \equiv 1 \mod 3, \\ 2 & \text{otherwise.} \end{cases}$$

**Proof** Since $\gamma_s(C_n) = \gamma_s(P_n)$ for $n \geq 3$, we see that $b_s(C_n) \geq 2$. If $n \equiv 1 \mod 3$ the removal of two edges from $C_n$ leaves a graph $H$ consisting of two paths $P$ and $Q$. If $P$ has order $n_1$ and $Q$ has order $n_2$, then either $n_1 \equiv n_2 \equiv 2 \mod 3$, or, without loss of generality,
n_1 \equiv 0 \pmod{3} and n_2 \equiv 1 \pmod{3}. In the former case,
\[
\gamma_s(H) = \gamma_s(P) + \gamma_s(Q) = \lfloor n_1/3 \rfloor + \lfloor n_2/3 \rfloor
\]
\[
= (n_1 + 1)/3 + (n_2 + 1)/3 = (n_1 + n_2 + 2)/3 = (n + 2)/3 = \lfloor n/3 \rfloor = \gamma_s(C_n).
\]
In the latter case,
\[
\gamma_s(H) = \gamma_s(P) + \gamma_s(Q) = n_1/3 + (n_2 + 2)/3 = (n + 2)/3 = \lfloor n/3 \rfloor = \gamma_s(C_n).
\]
In either case, when \( n \equiv 1 \pmod{3} \) we have \( b_s(C_n) \geq 3 \). Now we consider two cases.

**Case 1** Suppose that \( n \equiv 0, 2 \pmod{3} \). The graph \( H \) obtained removing two adjacent edges from \( C_n \) consist of an isolated vertex and a path of order \( n - 1 \). Thus
\[
\gamma_s(H) = \gamma_s(P_1) + \gamma_s(P_{n-1}) = 1 + \lfloor (n - 1)/3 \rfloor = 1 + \lfloor n/3 \rfloor = 1 + \gamma_s(C_n),
\]
whence \( b_s(C_n) \leq 2 \) in this case. Combining this with the upper strong bondage obtained earlier, we have \( b_s(C_n) = 2 \) if \( n \equiv 0, 2 \pmod{3} \).

**Case 2** Suppose now that \( n \equiv 1 \pmod{3} \). The graph \( H \) resulting from the deletion of three consecutive edges of \( C_n \) consists of two isolated vertices and a path of order \( n - 2 \). Thus,
\[
\gamma_s(H) = 2 + \lfloor (n - 2)/3 \rfloor = 2 + (n - 1)/3 = 2 + \lfloor n/3 \rfloor - 1 = 1 + \gamma_s(C_n),
\]
so that \( b_s(C_n) \leq 3 \). With the earlier inequality we conclude that \( b_s(C_n) = 3 \) when \( n \equiv 1 \pmod{3} \).  

**Theorem 2.6** The weak bondage number of the \( n \)-cycle (with \( n \geq 3 \)) is
\[
b_w(C_n) = \begin{cases} 
2 & \text{if } n \equiv 1 \pmod{3}, \\
1 & \text{otherwise.}
\end{cases}
\]

**Proof** Assume \( n \not\equiv 1 \pmod{3} \) since \( \gamma_w(P_n) = \lfloor n/3 \rfloor + 1 = \gamma_w(C_n) + 1 > \gamma_w(C_n) \). Hence \( b_w(C_n) = 1 \). Now assume \( n \equiv 1 \pmod{3} \) since \( \gamma_w(C_n) = \gamma_w(P_n) \) it follows that \( b_w(C_n) \geq 2 \).

Let \( H \) be the graph obtained by the removal of two edges from \( C_n \) such that \( P_3 \) and \( P_{n-3} \) are formed. Then
\[
\gamma_w(H) = \gamma_w(P_3) + \gamma_w(P_{n-3}) = 2 + \lfloor (n - 3)/3 \rfloor = 2 + \lfloor n/3 \rfloor - 1 = \lfloor n/3 \rfloor + 1 > \gamma_w(C_n).
\]
Hence \( b_w(C_n) \leq 2 \) thus \( b_w(C_n) = 2 \).  

As an immediate Corollary to Theorem 2.5 we have the following.

**Corollary 2.7** The strong bondage number of the path (with \( n \geq 3 \)) is given by
\[
b_s(P_n) = \begin{cases} 
2 & \text{if } n \equiv 1 \pmod{3}, \\
1 & \text{otherwise.}
\end{cases}
\]

**Theorem 2.8** The weak bondage number of the path (with \( n \geq 3 \)) is
Proposition 2.10
The strong bondage number of the wheel

\[ b_w(P_n) = \begin{cases} 
  2 & \text{if } n = 3, 5, \\
  1 & \text{otherwise.} 
\end{cases} \]

Proof
It is easy to verify that \( b_w(P_n) = 2 \) for \( n = 3, 5 \).

Let \( H \) be the graph obtained by the removal of one edge from \( P_n \) such that \( P_3 \) and \( P_{n-3} \) are formed. Then \( \gamma_w(H) = \gamma_w(P_3) + \gamma_w(P_{n-3}) \). Now we consider the following cases.

Case 1 If \( n \equiv 1 \mod 3 \) then \( \gamma_w(H) = \gamma_w(P_3) + \gamma_w(P_{n-3}) = 2 + \lfloor (n-3)/3 \rfloor = 2 + \lfloor n/3 \rfloor - 1 = \lfloor n/3 \rfloor + 1 \) then \( \gamma_w(H) > \gamma_w(P_n) \). Hence \( b_w(P_n) = 1 \).

Case 2 If \( n \not\equiv 1 \mod 3 \) we have \( \gamma_w(H) = 2 + \lfloor (n-3)/3 \rfloor + 1 = 2 + \lfloor n/3 \rfloor - 1 + 1 = 2 + \lfloor n/3 \rfloor > \gamma_w(P_n) \) then \( \gamma_w(H) > \gamma_w(P_n) \). Hence \( b_w(P_n) = 1 \). \qed

Lemma 2.9 The strong and weak domination numbers of the wheel \( W_n \) (with \( n \geq 4 \)) are

(i) \( \gamma_s(W_n) = 1 \);
(ii) \( \gamma_w(W_n) = \lfloor (n-1)/3 \rfloor \).

Proof
(i) Since \( \gamma(W_n) = \gamma_s(W_n) \) so proof techniques same in [2].
(ii) Since \( \gamma_w(W_n) = \gamma(C_{n-1}) = \lfloor (n-1)/3 \rfloor \) so proof techniques same in [2]. \qed

Proposition 2.10 The strong bondage number of the wheel \( W_n \) (with \( n \geq 4 \)) is \( b_s(W_n) = 1 \).

Proof
Let \( x \) be the vertex of maximum degree of \( W_n \). Let \( v \) be a vertex of \( W_n \) such that \( \deg v < \deg x \). Let \( H \) be the graph obtained from \( W_n \) by removing edge \( xv \). Then no one vertex strongly dominates \( H \). So \( \gamma_s(W_n - xv) > \gamma_s(W_n) \). Hence \( b_s(W_n) = 1 \). \qed

Proposition 2.11 The weak bondage number of \( W_n \) (with \( n \geq 4 \)) is given by

\[ b_w(W_n) = \begin{cases} 
  2 & \text{if } n \equiv 2 \mod 3, \\
  1 & \text{otherwise.} 
\end{cases} \]

Proof
Assume \( n \equiv 0, 1 \mod 3 \), let \( e \) be an edge on the \((n-1)\)-cycle. Then \( \gamma_w(W_n - e) = \lfloor (n-1)/3 \rfloor + 2 = \lfloor (n-2)/3 \rfloor + 1 = \lfloor (n-1)/3 \rfloor + 1 + \lfloor (n-1)/3 \rfloor = \gamma_w(W_n) \), whence \( b_w(W_n) = 1 \).

Now assume \( n \equiv 2 \mod 3 \), the removal of any one edge from \( W_n \) will not alter \( \gamma_w(W_n) \). So when \( n \equiv 2 \mod 3 \) we have \( b_w(W_n) \geq 2 \).

Let \( H \) be the graph obtained by the removal of two adjacent edges from \( W_n \) such that these edges are not incident with the vertex of maximum degree. Then \( \gamma_w(H) = \lfloor (n-6)/3 \rfloor + 3 = \lfloor n/3 \rfloor + 1 = \lfloor (n-1)/3 \rfloor + 1 + \lfloor (n-1)/3 \rfloor = \gamma_w(W_n) \), whence \( b_w(W_n) = 2 \). \qed

Lemma 2.12 The strong and weak domination numbers of the \( K_{r,t} \) are

(i)

\[ \gamma_s(K_{r,t}) = \begin{cases} 
  2 & \text{if } 2 \leq r = t, \\
  r & \text{if } 1 \leq r < t. 
\end{cases} \]
\[(ii) \]
\[
\gamma_w(K_{r,t}) = \begin{cases}  
  t & \text{if } 1 \leq r < t, \\
  2 & \text{if } 2 \leq r = t.
\end{cases}
\]

**Proof** (i) see [3].

(ii) Note that the vertices in the second partite set have the smallest degree. If \(1 \leq r < t\), then to weakly dominate these vertices, we need include all of them in any \(wd\)-set and these suffice to weakly dominate the rest. If \(r = t \geq 2\), we claim \(\gamma_w = 2\). Since \(t \geq 2\), none of the vertices in the graph are of full degree hence \(\gamma_w\) in this case is greater than 1. Now to demonstrate a \(wd\)-set of cardinality 2, we can take one vertex from the first partite set which weakly dominate the rest of the vertices in the first partite set, we use a vertex from the second partite set. Note that a vertex from the second partite set has equal degree as the vertices in the first set since \(r = t\). □

The next theorem establishes the strong and weak bondage numbers of the complete bipartite graph \(K_{r,t}\).

**Theorem 2.13** Let \(K_{r,t}\) be a complete bipartite graph, where \(4 \leq r \leq t\), then

\[
b_s(K_{r,t}) = \begin{cases}  
  2r & \text{if } t = r + 1, \\
  r & \text{otherwise}.
\end{cases}
\]

**Proof** Let \(V = V_1 \cup V_2\) be the vertex set of \(K_{r,t}\) such that \(|V_1| = r\) and \(|V_2| = t\). We consider the following cases.

**Case 1** Suppose \(t = r + 1\) and \(v \in V_2\), then by removing all edges incident with \(v\), we obtain a graph \(H\) containing two components \(K_1\) and \(K_{r,t-1}\). Hence

\[
\gamma_s(H) = \gamma_s(K_1) + \gamma_s(K_{r,t-1}) = 1 + 2 < r = \gamma_s(K_{r,t}).
\]

Now let \(v \in V_2\) and \(u \in V_1\) be a vertex of \(K_{r,t}\), then by removing all edges incident to both \(u\) and \(v\), we obtain a graph \(H\) containing two components \(2K_1\) and \(K_{r-1,t-1}\), thus

\[
\gamma_s(H) = 2\gamma_s(K_1) + \gamma_s(K_{r-1,t-1}) = 2 + r - 1 = r + 1 > r = \gamma_s(K_{r,t}).
\]

Hence

\[
b_s(K_{r,t}) = \deg u + \deg v - 1 = |V_2| + |V_1| - 1 = t + r - 1 = 2r
\]

for \(t = r + 1\).

**Case 2** Suppose \(r = t\), then by Lemma 2.12, \(\gamma_s(K_{r,t}) = 2\). Let \(v \in V_2\), then by removing all edges incident with \(v\), we obtain a graph \(H\) containing two components \(K_1\) and \(K_{r,t-1}\), thus

\[
\gamma_s(H) = \gamma_s(K_1) + \gamma_s(K_{r,t-1}) = 1 + t - 1 = t = r > 2 = \gamma_s(K_{r,t}).
\]

Hence \(b_s(K_{r,t}) = \deg v = |V_1| = r\) for \(r = t\).

**Case 3** Suppose \(r + 1 < t\), then by Lemma 2.12, \(\gamma_s(K_{r,t}) = r\). Let \(v \in V_2\), then by removing all edges incident with \(v\), we obtain a graph \(H\) containing two components \(K_1\) and \(K_{r,t-1}\). Hence
\( \gamma_s(H) = \gamma_s(K_1) + \gamma_s(K_{r,t-1}) = 1 + r > r = \gamma_s(K_{r,t}). \) Thus \( b_s(K_{r,t}) = \deg v = |V_1| = r \) for \( r + 1 < t. \)

**Theorem 2.14** Let \( K_{r,t} \) be a complete bipartite graph, where \( 1 \leq r \leq t \), then \( b_w(K_{r,t}) = t. \)

**Proof** Let \( V = V_1 \cup V_2 \) be the vertex set of \( K_{r,t} \) where \( |V_1| = r \) and \( |V_2| = t \). Let \( v \in V_1 \) and \( r = t \geq 2 \), then by removing all edges incident with \( v \), we obtain a graph \( H \) containing two components \( K_1 \) and \( K_{r-1,t} \). Hence
\[
\gamma_w(H) = \gamma_w(K_1) + \gamma_w(K_{r-1,t}) = 1 + t > 2 = \gamma_w(K_{r,t}).
\]
Thus
\[
b_w(K_{r,t}) = \deg v = |V_2| = t.
\]

Now suppose \( r < t \), then by removing all edges incident with \( v \), we obtain a graph \( H \) containing two components \( K_1 \) and \( K_{r-1,t} \). Hence
\[
\gamma_w(H) = \gamma_w(K_1) + \gamma_w(K_{r-1,t}) = 1 + t > t = \gamma_w(K_{r,t}).
\]
Thus
\[
b_w(K_{r,t}) = \deg v = |V_2| = t.
\]

§3. **The Strong and Weak Bondage Numbers of a Tree**

We now consider the strong and weak bondage numbers for a tree \( T \). Define a support to be a vertex in a tree which is adjacent to an end-vertex (see [3]).

**Proposition 3.1** Every tree \( T \) with \( (n \geq 4) \) has at least one of the following characteristics.

1. A support adjacent to at least 2 end-vertex.
2. A support is adjacent to a support of degree 2.
3. A vertex is adjacent to 2 support of degree 2.
4. The support of a leaf and the vertex adjacent to the support are both of degree 2.

**Proof** See [3] for the proof.

**Theorem 3.2** If \( T \) is a nontrivial tree then \( b_s(T) \leq 3. \)

**Proof** See [3] for the proof.

**Proposition 3.3** If any vertex of tree \( T \) is adjacent with two or more end-vertices, then \( b_s(T) = 1. \)

**Proof** Let \( u \) be a cut vertex adjacent two or more end-vertices. Then \( u \) belongs to every minimum strong dominating set of \( T \). Let \( v \) be an end-vertex adjacent to \( u \). Then \( T - uv \) contains an isolated vertex and a tree \( T' \) of order \( n - 1 \). Therefore \( \gamma_s(T - uv) = \gamma_s(T') + 1 > \gamma_s(T) \). Hence \( b_s(T) = 1. \)
Theorem 3.4 If \( T \) is a nontrivial tree, then \( b_w(T) \leq \Delta(T) \).

Proof The statement is obviously true for trees order 2 or 3, so we shall suppose that \( T \) has at least 4 vertices. Now we consider the following cases.

Case 1 Suppose \( T \) has a support vertex \( s \) that is adjacent to two (and possibly more) end-vertex, that dose not belong to a weak dominating set. Let \( E_s \) denote the set of edges incident with \( s \). And let \( D \) be a minimum weak dominating set for \( T - E_s \). Then \( s \) is in \( D \) and \( D \setminus \{s\} \) is a weak dominating set for \( T \). Hence \( \gamma_w(T - E_s) > \gamma_w(T) \) thus \( b_w(T) \leq |E_s| = \deg s \leq \Delta(T) \).

Case 2 Suppose a support vertex is adjacent to a support vertex of degree 2. Delete the edge \((s, l)\). The vertex \( x \) then has two end-vertices an adjacent to \( s \) and \( m \). Let \( D \) be wd-set of \( T - \{(s, l)\} \). Then \( s \) is in \( D \) and \( D \setminus \{s\} \) is a weak dominating set for \( T \). Hence \( b_w(T) \) in this case equals 1.

Case 3 In this case delete the edge \((s, l)\). If \( \gamma_w(T - \{(s, l)\}) < \gamma_w(T) \), then it will contradict the assumption that the \( \gamma_w \)-set was the smallest wd-set for \( T \). If \( \gamma_w(T - \{(s, l)\}) \) is greater that \( \gamma_w(T) \) then we have done. If \( \gamma_w(T - \{(s, l)\}) = \gamma_w(T) \), then the vertex \( x \) has a one support vertex \( s \) in \( T - \{(s, l)\} \), that adjacent to it. Then by Case 2, deleting on more edge \((\{m, k\})\) will increase the weak domination number of the resulting graph. So in this case \( b_w(T) = 2 \).

Case 4 In the last case, either \( s \) or \( l \) is any weak dominating set of \( T \). By removing edges \((k, x)\) and \((x, s)\), we make the necessary for any \( \gamma_w \)-set for the resulting graph to contain \( x \) and so \( b_w(T) = 2 \) this completes the proof. \( \square \)

Theorem 3.5 Let \( T \) be a tree. Then \( b_w(T) = \Delta(T) \) if and only if \( T = K_{1,r} \).

Proof This follows from Theorem 3.4. \( \square \)

§4. General Bounds on Strong and Weak Bondage Numbers

Proposition 4.1([2]) If \( G \) is a nonempty graph, then

\[
b(G) \leq \min\{\deg u + \deg v - 1 : \ u \ and \ v \ are \ adjacent\}.
\]
Theorem 4.2 If $\gamma(G) = \gamma_s(G)$ and $\gamma(G) = \gamma_w(G)$ then,

(i) $b_s(G) \leq b(G)$;
(ii) $b_w(G) \leq b(G)$.

Proof Let $E$ be a $b$-set of $G$. Then $\gamma_s(G) = \gamma(G) < \gamma(G - E) \leq \gamma_s(G - E)$. Thus $b_s(G) \leq b(G)$ and for (ii) proof is same.

Theorem 4.3 If $G$ is a nonempty graph and $\gamma(G) = \gamma_s(G)$ then

$$b_s(G) \leq \min\{\deg u + \deg v - 1 \mid u \text{ and } v \text{ are adjacent}\}.$$

Proof This follows from Proposition 4.1 and Theorem 4.2.

Theorem 4.4 For any graph $G$, $b_s(G) \leq q - p + \gamma_s(G) + 1$

Proof Let $D$ be a $\gamma_s$-set of a graph $G$. For each vertex $v \in V \setminus D$ choose exactly one edge which is incident to $v$ and to a vertex in $D$. Let $E_0$ be the set of all such edges. Then clearly $\gamma_s(G - (E - E_0)) = \gamma_s(G)$ and $|E - E_0| = q - p + \gamma_s(G)$. So for any edge $e \in G - (E - E_0) = E_0$ we see that $\{E - E_0 \cup \{e\}$ is a strong bondage set of $G$. Thus

$$b_s(G) \leq q - p + \gamma_s(G) + 1$$

Corollary 4.5 For any graph $G$, $b_s(G) \leq q - \Delta(G) + 1$

Proof In [8], We have known that $\gamma_s(G) \leq p - \Delta(G)$. By applying Theorem 4.4, we get that $b_s(G) \leq q - \Delta(G) + 1$.

Theorem 4.6 If $G$ is a nonempty graph with strong domination number $\gamma_s(G) \geq 2$, Then

$$b_s(G) \leq (\gamma_s(G) - 1)\Delta(G) + 1.$$

Proof We proceed by induction on the strong domination number $\gamma_s(G)$. Let $G$ be a nonempty graph with $\gamma_s(G) = 2$, and assume that $b_s(G) \geq \Delta(G) + 2$, then, if $u$ is a vertex of maximum degree in $G$, we have $\gamma_s(G - u) = \gamma_s(G) - 1 = 1$, and $b_s(G - u) \geq 2$. Since $\gamma_s(G) = 2$ and $\gamma_s(G - u) = 1$, there is a vertex $v$ that is adjacent with every vertex of $G$ but $u$, that $\deg_G v = \Delta(G)$ also, and $u$ is adjacent with every vertex of $G$ except $v$. Since $b_s(G - u) \geq 2$, the removal from $G - u$ of any one edge incident with $v$ again leaves a graph with strong domination number 1. Thus there is a vertex $w \neq v$ that is adjacent with every vertex of $G - u$. But, since $v$ is the only vertex of $G$ that is not adjacent with $u$, vertex $w$ must be adjacent in $G$ with $u$. This however implies that $\gamma_s(G) = 1$, a contradiction. Thus $b_s(G) \leq \Delta(G) + 1$ if $\gamma_s(G) = 2$.

Now, let $(k \geq 2)$ be any integer for which the following statement is true: If $H$ is nonempty graph with $\gamma_s(G) = k$, then $\gamma_s(H) \leq (k - 1)\cdot\Delta(H) + 1$. Let $G$ be a graph nonempty graph with
\(\gamma_s(G) = k+1\), and assume that \(b_s(G) > k \cdot \Delta(G)+1\). Then. But then, \(b_s(G) \leq b_s(G-u) + \deg u\), and by the inductive hypothesis we have
\[
b_s(G) \leq |(k - 1) \cdot \Delta(G-u) + 1| + \deg u \leq (k - 1) \cdot \Delta(G) + 1 + \Delta(G),
\]
or
\[
b_s(G) \leq k \cdot \Delta(G) + 1,
\]
a contradiction to our assumption that \(b_s(G) > k \cdot \Delta(G)+1\). Thus, \(b_s(G) \leq k \cdot \Delta(G) + 1\), and, by the principle of mathematical induction, the proof is complete. \(\square\)

**Theorem 4.7** If \(G\) is a planar graph, then
\[
b_w(G) \leq \Delta(G).
\]

*Proof* Suppose \(G\) has a vertex \(u\) with maximum degree that does not belong to a weak dominating set. Let \(E_u\) denote the set of edges incident with \(u\). And let \(D\) be a minimum weak dominating set for \(G - E_u\). Then \(u\) is in \(D\) and \(D \setminus u\) is a weak dominating set for \(G\). Hence \(\gamma_w(G - E_u) > \gamma_w(G)\) thus \(b_w(G) \leq |E_u| = \deg u \leq \Delta(G)\). \(\square\)

§5. **Open Problems**

We strongly believe the following to be true.

**Theorem 5.1** If \(G\) is a nonempty graph of order \((n \geq 2)\) then \(b_w(G) \leq n - 1\).

**Theorem 5.2** If \(G\) is a nonempty graph of order \((n \geq 2)\) then \(b_w(G) \leq n - \delta(G)\).

**Theorem 5.3** If \(G\) is a nonempty graph of order \((n \geq 2)\) then \(b_s(G) \leq n - 1\).

Other bounds for the strong and weak bondage of a graph exist. For several classes of graphs, \(b_s(G) \leq \Delta(G)\) and \(b_w(G) \leq \Delta(G)\). Let \(F\) be the set of edges incident with a vertex of maximum degree. Then it can be shown that \(\gamma_s(G - F) \geq \gamma_s(G)\) and similarly \(\gamma_w(G - F) \geq \gamma_w(G)\). But it is not necessary that this action would result in an increase in the strong and weak domination numbers. See Fig.2. The calculation for the strong and weak bondage for multipartite graphs remains open. Unions, joins and product of graphs could be investigated for their strong and weak bondage in terms of the constituent graphs. This implies that we need to calculate the strong and weak domination of these graphs. The problem of strong and weak domination is virtually unexplored and so there are several classes of graphs for which the strong and weak domination numbers could be calculated.

![Fig. 2](image-url)
References


