Some Properties of a h-Randers Finsler Space

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Abstract: The purpose of the present paper is to obtain the relation between the imbedding class numbers of tangent Riemannian spaces to \((M^n, L)\) and \((M^n, L^*)\) where \(L^*(x, y)\) is obtained from the transformation of \(L(x, y)\) is given by

\[
L^*(x, y) \rightarrow L(x, y) + b_i(x, y)y^i
\]

Key Words: Riemannian metric, h-vector, imbedding class.


§1. Introduction

In 1971 Matsumoto [5] introduced the transformation of Finsler metric

\[
\bar{L}(x, y) \rightarrow L(x, y) + b_i y^i
\]  

(1.1)

and obtain the relation between the imbedding class numbers of a tangent Riemannian spaces to \((M^n, L)\) and a Finsler space \((M^n, \bar{L})\) which is obtained by the transformation of the Finsler metric \(L\) by the relation given by in the equation (1.1). Since a concurrent vector field is a function of \((x)\) i.e., position only, assuming \(b_i(x)\) as a concurrent vector field, Matsumoto [6] studied the R3-likeness of Finsler spaces \((M^n, L)\) and \((M^n, \bar{L})\). Singh and Prasad [14,11] generalized the concept of concurrent vector field and introduced the semi-parallel and concircular vector fields which are functions of \((x)\) only. Assuming \(b_i(x)\) as a concircular vector field, Prasad, Singh and Singh [11] studied the R3-likeness of \((M^n, L)\) and \((M^n, \bar{L})\).

If \(L(x, y)\) is a metric function of Riemannian space then \(\bar{L}(x, y)\) reduces to the metric function of Randers’s space. Such a Finsler metric was first introduced by G. Randers [13] from the standpoint of general theory of relativity and applied to the theory of the electron microscope by R. S. Ingarden [3] who first named it as Randers space. The geometrical properties of this space have been studied by various workers [2, 7, 9, 12, 15]. In 1970 Numata [10] has studied the properties of \((M^n, \bar{L})\) which is obtained from Minkowski space \((M^n, L)\) by transformation.
(1.1) In all those works the function \( b_i(x) \) are assumed to be functions of \( x \) only.

In 1980, Izumi [4] while studying the conformal transformation of Finsler spaces, introduced the h-vector \( b_i \) which is v-covariantly constant with respect to Cartan’s connection \( \Gamma \) and satisfies the relation

\[
LC_{ij}^h b_h = \rho h_{ij}
\]

Thus the h-vector \( b_i \) is not only a function of \( x \) but it is also a function of directional arguments satisfying \( L\dot{\partial}_j b_i = \rho h_{ij} \). The purpose of the present paper is to obtain the relation between imbedding class numbers of tangent Riemannian spaces to \((M^n, L)\) and \((M^n, L^*)\) where \( L^*(x, y) \) is obtained from the transformation of \( L(x, y) \) is given by

\[
L^*(x, y) \rightarrow L(x, y) + \beta(x, y),
\]

where \( \beta(x, y) = b_i(x, y)y^i \), i.e. \( b_i(x, y) \) is the function of position and direction both.

§2. An h-Vector in \((M^n, L)\)

Let \( b_i \) be a vector field in the Finsler space \((M^n, L)\). If \( b_i(x, y) \) satisfies the conditions

\[
b_i|_j = 0,
\]

\[
LC_{ij}^h b_h = \rho h_{ij},
\]

then the vector field \( b_i \) is called an h-vector [4]. Here \( |_j \) denotes the v-covariant derivative with respect to \( y^j \) in the case of Cartan’s connection \( \Gamma \), \( C_{ij}^h \) is the cartan’s C-tensor, \( h_{ij} \) is the angular metric tensor and \( \rho \) is given by

\[
\rho = \frac{LC_{ij}^h b_i}{(n-1)},
\]

where \( C^i \) is the torsion tensor given by \( C^i_{jk}g^{jk} \).

Lemma 2.1([4]) If \( b_i \) is an h-vector then the function \( \rho \) and are independent of \( y \).

Since The v-covariant derivation of \( b^2 = g^{ij}b_ib_j \) and the fact that \( g^{ij} \) is v-covariantly constant yield

\[
\dot{b}_k b = g^{ij}b_ib_j|_k.
\]

In the view of (2.1) we have

\[
\dot{b}_k b = 0.
\]

Thus we have

Lemma 2.2 The magnitude \( b \) of an h-vector is independent of \( y \).

From (2.1), Ricci identity [8] and the fact that \( S_{hjk} = g_{hr}S^r_{ijk} \) is skew-symmetric in \( h \) and
Thus we have

**Lemma 2.3** For an h-vector $b_i$ we have $S_{hi}b^h = 0$, where $S_{hi}b^h$ are components of v-curvature tensor of Cartan’s connection $C^\Gamma$.

The concept of concurrent vector field in $(M^n, L)$ has been introduced by Tachibana [16] and its properties have been studied by Matsumoto [6]. A vector field $b_i$ in $(M^n, L)$ is said to be concurrent if it satisfies the condition (2.1) and

$$b_{ij} = -g_{ij},$$  \tag{2.4}$$

where $|j$ denotes h-covariant differentiation with respect to $x^i$ in the sense of Cartan’s connection $C^\Gamma$.

Applying Ricci Identity [8]

$$b_{ij}k - b_{i}k_{j} = -b_{ik}P_{jk}^h - b_{ij}C_{jk}^h - b_{i}h_{jk}P^h_{jk}$$

and using (2.1) and (2.4) we have

$$P_{ijk}^h b^h + C_{ijk} = 0.$$

Since $P_{imjk} = g_{mh}P_{ijk}^h$ is skew-symmetric in $i$ and $m$, contraction of above equation with $b^i = g^{ij}b_j$ gives $C_{ijk}b^i = 0$. Hence we have the following

**Lemma 2.4** An h-vector $b_i$ with $\rho \neq 0$ is not a concurrent vector field.

§3. **Properties of the h-Randers Finsler Space**

Let $b_i$ be an h-vector in the Finsler space $(M^n, L)$ and $(M^n, L^*)$ be another Finsler space whose fundamental function $L^*(x, y)$ is given by (1.2).

Since $b_i$ is an h-vector, from (2.1) and (2.2), we get

$$\dot{\mathcal{h}}_{ij}b_i = L^{-1}\rho_{ij},$$  \tag{3.1}$$

which after using the indicatory property of $h_{ij}$ yields $\dot{\mathcal{h}}_{ij} = b_j$.

**Definition 3.1** Let $M^n$ be an n-dimensional differentiable manifold and $F^n$ be a Finsler space equipped with a fundamental function $L(x, y), (y^i = \dot{x}^i)$ of $M^n$. A change in the fundamental function $L$ by the equation (1.2) on the same manifold $M^n$ is called h-Randers change. A space equipped with fundamental metric $L^*$ is called h-Randers changed Finsler space $F^{*n}$.
Now differentiating (1.2) with respect to $y^i$ we have

$$l^*_i = l_i + b_i,$$

(3.2)

where $l_i = \dot{\partial}_i L$ is the normalized supporting element in $(M^n, L)$ and $l^*_i = \dot{\partial}_i L^*$ is the normalized element of support in $(M^n, L^*)$. The quantities of $(M^n, L^*)$ will be denoted by starred letter.

Now differentiating (3.2) with respect to $y^j$ then the angular metric tensor $h^*_{ij} = \dot{\partial}_j l^*_i$ is given by

$$h^*_{ij} = \sigma h_{ij},$$

(3.3)

where $\sigma = LL^{-1}(1 + \rho)$. Hence we have

$$g^*_{ij} = \sigma g_{ij} + (1 - \sigma)l_i l_j + (l_i b_j + l_j b_i) + b_i b_j.$$

(3.4)

From (3.4) the relation between the contravariant components of the fundamental tensors can be derived as follows

$$g^*_{ij} = \sigma^{-1} g_{ij} - (1 + \rho^2)\sigma^{-3}(1 - b^2 - \sigma)l^i l^j - (1 + \rho)\sigma^{-2}(l^i b^j + l^j b^i),$$

(3.5)

where $b$ is the magnitude of the vector $b_i$.

From the lemma (2.1) and (3.2) we have

$$\dot{\partial}_i \sigma = (1 + \rho) - m_i,$$

(3.6)

$$m_i = b_i - \beta L l_i.$$

(3.7)

Now differentiating (3.3) with respect to $y^k$ (3.2), (3.6), (3.3) and the fact

$$\dot{\partial}_k h_{ij} = 2C_{ijk} - L^{-1}(h_{ik} l_j + h_{jk} l_i),$$

we have

$$C^*_{ijk} = \sigma C_{ijk} + (1 + \rho) h_{ij} m_k + h_{jk} m_i + h_{ki} m_j.\]

(3.8)

From the definition of $m_i$, it is evident that

(a) $m_i l^i$,  
(b) $m_i b^i = b^2 - \frac{\beta^2}{L^2} = m^i m_i$,  
(c) $h_{ij} m^i = h_{ij} b^i = m_j$.  
(d) $C^*_{ih} m_h = L^{-1} \rho h_{ij}$.  

(3.9)

From (2.1), (3.5), (3.8) and (3.9) we have

$$C^*_{ijr} = \frac{C^*_{ij} + \frac{(h_{ij} m^r + h^r_i m_i + h^r_j m_j)}{2L^*} - \frac{1}{L^*}[\{\rho}$$

$$+ \frac{L}{2L^*}(b^2 - \frac{\beta^2}{L^2})h_{ij} + \frac{L}{L^*} m_i m_j]v^r.\]

(3.10)
Proposition 3.1 Let $F^* = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the h-Randers change of the Finsler space $F^n = (M^n, L)$, then the normalized supporting element $l_i^*$, angular metric tensor $h_{ij}^*$, fundamental metric tensor $g_{ij}^*$ and (h)hv-torsion tensor $C_{ijk}^*$ of $F^*$ are given by (3.2), (3.3), (3.4) and (3.8) respectively.

Proposition 3.2 Let $F^* = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the h-Randers change of the Finsler space $F^n = (M^n, L)$, then the reciprocal of the fundamental metric tensor $g_{ij}^*$ is given by (3.5).

The curvature tensor $S_{hijk}$ of $(M^n, L^*)$ is given by

$$S_{hijk}^* = C_{hkm}^* C_{ij}^* - C_{hjm}^* C_{ik}^*.$$  \hspace{1cm} (3.11)

From the equation (3.8) and (3.10), we have

$$C_{hkm}^* C_{ij}^* = \sigma C_{hkm} C_{ij} + \alpha h_{hk} h_{ik} + \frac{(1 + \rho)}{2L} \{ C_{ijk} m_h + C_{hjk} m_i 
+ C_{hik} m_j + C_{hij} m_k \} + \frac{(1 + \rho)}{4LL^*} \{ 2h_{ij} m_k m_h 
+ 2h_{ik} m_j m_h + h_{ik} m_j m_k + h_{hk} m_i m_h 
+ h_{hk} m_i m_k \},$$ \hspace{1cm} (3.12)

where $\alpha = \frac{(1+\rho)^2}{4L^2} + \frac{1+\rho}{4LL^*}(b^2 - \frac{a^2}{L^*})$. Thus from (3.11) we have

$$S_{hijk}^* = \sigma S_{hijk} + h_{ij} d_{hk} + h_{ik} d_{ij} - h_{ik} d_{jh} - h_{hj} d_{ik};$$ \hspace{1cm} (3.13)

where $d_{ij} = \frac{\sigma}{2} h_{ij} + \frac{1+\rho}{4LL^*} m_i m_j$.

If we define the tensor $A_{ij}$ and $B_{ij}$ as

$$A_{ij} = \frac{h_{ij} + d_{ij}}{\sqrt{2}}, \quad B_{ij} = \frac{h_{ij} - d_{ij}}{\sqrt{2}},$$ \hspace{1cm} (3.14)

then $S_{hijk}^*$ is written as

$$S_{hijk}^* = \sigma S_{hijk} - (A_{ij} A_{ik} - A_{hj} A_{hk}) + (B_{hj} B_{ik} - B_{hk} B_{ij}).$$ \hspace{1cm} (3.15)

Thus we have

Proposition 3.3 Let $F^* = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the h-Randers change of the Finsler space $F^n = (M^n, L)$, then the curvature tensor $S_{hijk}^*$ is given by (3.15).

If $\mid_j$ denotes v-covariant differentiation with respect to $y^j$ in $(M^n, L^*)$ then we have

$$h_{ij} \mid_k - h_{ik} \mid_j = \frac{(h_{ij} l_k - h_{ik} l_j)}{L},$$ \hspace{1cm} (3.16)
\[ m_i | j - m_j | i = \frac{(m_i | l_j - m_j | l_i)}{L}, \]  
(3.17)

\[ d_{ij} | k - d_{ik} | j = \frac{(d_{ij} l_k - d_{ik} l_j)}{L}. \]  
(3.18)

Hence from (3.14), (3.16) and (3.18), we get

\[ A_{ij} | k - A_{ik} | j = \frac{(B_{ij} l_k - B_{ik} l_j)}{L}, \]  
(3.19)

\[ B_{ij} | k - B_{ik} | j = \frac{(A_{ij} l_k - A_{ik} l_j)}{L}. \]  
(3.20)

§4. Imbedding Class Numbers of Tangent Riemannian Space to \((M^n, L)\) and \((M^n, L^*)\)

The tangent vector space \(M^n_x\) to \(M^n\) at every point \(x\) is regarded as \(n\)-dimensional Riemannian space \((M^n_x, g_x)\) with Riemannian metric \(g_x = g_{ij}(x,y)dy^idy^j\). Thus the component \(C^i_{jk}\) of Cartan’s C-tensor are the Christoffel symbols associated with \(g_x\), i.e.

\[ C^i_{jk} = \frac{1}{2}g^{ih}(\partial_k g_{jh} + \partial_j g_{hk} + \partial_h g_{jk}). \]

Hence \(C^i_{jk}\) defines the Riemannian connection on \(M^n_x\). It is observed from the definition if \(S_{hijk}\) that the curvature tensor of the Riemannian space \((M^n_x, g_x)\) at a point \(x\). The space \((M^n_x, g_x)\) equipped with such a Riemannian connection will be called the tangent Riemannian space.

In the theory of Riemannian space, we know that any \(n\)-dimensional Riemannian space \(V^n\), can be imbedded isometrically in a Euclidean space of dimension \(\frac{n(n-1)}{2}\). If \(n + r\) is the lowest dimension of the Euclidean space in which \(V^n\) is imbedded isometrically then the integer \(r\) is called imbedding class number of \(V^n\). The fundamental theorem of isometric imbedding [1] states that the tangent Riemannian n-space \((M^n_x, g_x)\) is locally imbedded isometrically in an Euclidean \(n + r\) space if and only if there exist \(r\) numbers, and \(\lambda = \pm 1, r\) symmetric tensor \(H(P_{ij})\) and \(\frac{r(r-1)}{2}\) covariant vector fields \(H(Q_{ij}) = H(Q_{ij})\), \(Q = 1, 2, 3, \cdots, r\) satisfying the Gauss equations,

\[ S_{hijk} = \Sigma \lambda(P) \{H(P)_{hk}H(P)_{ij} - H(P)_{ik}H(P)_{ij}\}, \]

where summation is given over \(P\).

The Codazzi equations

\[ H(P)_{ij}|k - H(P)_{ik}|j = \Sigma \lambda(Q) \{H(Q)_{ij}H(Q)_{k} - H(Q)_{ik}H(Q)_{ij}\}, \]

where summation is given over \(Q\) and Ricci-Kuhne equations

\[ H(P_{Qij})|j - H(P_{Qij})|i + \Sigma \lambda(R) \{H(RP)_{ij}H(RQ)_{ij} - H(RP)_{ij}H(RQ)_{ij}\} + g^{hk}\{H(P)_{hi}H(Q)_{kj} - H(P)_{kj}H(Q)_{hi}\} = 0. \]
For a special case when \((M^n_x, g_x)\) is of imbedding class 1, the above equations reduce to

\[
S_{hijk} = \lambda(H_{hj}H_{ik} - H_{hk}H_{ij}), \quad (4.1)
\]

\[
H_{ij}|k - H_{ik}|j = 0. \quad (4.2)
\]

Since \(S_{hijk} y^k = 0\), from (3.21), we have

\[
H_{hj}H_{i0} - H_{h0}H_{ij} = 0
\]

contracting above equation by \(y^i\), we have

\[
H_{hj}H_{00} - H_{h0}H_{0j} = 0,
\]

which implies that \(H_{0j} = 0\) or \(H_{ij} = H_{00}^{-1}H_{h0}H_{0j}\). In the latter case we get \(S_{hijk} = 0\). In the theory of spaces of imbedding class 1, [17] introduced the concept of type number \(t\), which is the rank of matrix \(\|H_{ij}\|\) provided to the rank is more than 1. If the rank is 0 or 1, then \(S\) vanishes. Therefore if \((M^n_x, g_x)\) is of imbedding class 1, the second fundamental tensor \(H_{ij}\) satisfies \(H_{ij}y^j = 0\) and thus the type number \(t\) is less than \(n\).

Again by virtue of Lemma 2.3 and equation (4.1), we get

\[
H_{hj}H_{ik} - H_{hk}H_{ij}b^h = 0.
\]

From this equation we have

\[
H_{hj}b^h b^j H_{ik} - H_{hk}b^h H_{ij}b^j = 0.
\]

This gives

\[
H_{hk}b^h = 0, \quad \text{or} \quad H_{ik} = \frac{H_{hk}b^h H_{ij}b^j}{H_{hj}b^h b^j}.
\]

In the latter case \(S_{hijk} = 0\). Thus for an imbedding class 1, \(H_{hk}b^h = 0\). Now we shall put

\[
H^*_{(1)ij} = \sqrt{\sigma} H_{ij}, \quad \varepsilon^*_1 = \varepsilon, \quad (4.3)
\]

\[
H^*_{(2)ij} = A_{ij}, \quad \varepsilon^*_2 = -1, \quad 4.4
\]

\[
H^*_{(3)ij} = B_{ij}, \quad \varepsilon^*_3 = 1, \quad 4.5
\]

then from (3.15) and (4.1), we get

\[
S^*_{hijk} = \Sigma \lambda P \{H^*_{(P)hj} H^*_{(P)ik} - H^*_{(P)hk} H^*_{(P)ij}\},
\]

where summation is varies from \(P = 1, 2, 3\). Thus the above equation is noting but Gauss equation of \((M^n_x, g^*_x)\).

Now we put

\[
H^*_{(21)ij} = -H^*_{(12)ij} = 0, \quad (4.6)
\]
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$$H^*_{(31)i} = -H^*_{(13)i} = 0, \quad (4.7)$$

$$H^*_{(32)i} = -H^*_{(23)i} = \frac{1}{L} l_i, \quad (4.8)$$

and using (4.2), (4.3), (3.3), Lemma 2.1 and the fact that $H_{i0} = 0$, we get

$$H^*_{(1)ij|k} - H^*_{(1)ik|j} = 0. \quad (4.9)$$

Again in view of (4.4), (4.5), (4.6), (4.7) and (4.8), equations (3.19) and (3.20) reduce to

$$H^*_{(2)ij|k} - H^*_{(2)ik|j} = \Sigma Q \{ H^*_{(Q)ij} H^*_{(Q2)k} - H^*_{(Q)ik} H^*_{(Q2)j} \}, \quad (4.10)$$

$$H^*_{(3)ij|k} - H^*_{(3)ik|j} = \Sigma Q \{ H^*_{(Q)ij} H^*_{(Q3)k} - H^*_{(Q)ik} H^*_{(Q3)j} \}, \quad (4.11)$$

where summation is varies from $Q = 1, 2, 3$.

The equations (4.9), (4.10) and (4.11) are the Codazzi equations of $(M^n_x, g_x^*)$. Now we have to verify Ricci-Kuhne equations, we have from (3.10),

$$l_i|j = L^{-1} h_{ij} + L^{-1} [\{ \rho + (2L^*)^{-1} (v^2 - \frac{\beta^2}{L^2}) \} h_{ij} + L^{*-1} m_i m_j]$$

from which we get $l_i|j - l_j|i = 0$. Hence from (4.10), we get

$$H^*_{(32)i|j} - H^*_{(23)i|j} = 0,$$

which are the Ricci-Kuhne equations of $(M^n_x, g_x^*)$ as

$$M^*_i(12) - M^*_i(21) = 0, \quad and \quad M^*_i(13) - M^*_i(31) = 0.$$

Thus from above we have

**Theorem 4.1** Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the h-Randers change of the Finsler space $F^n = (M^n, L)$, then if the tangent Riemannian n-space $(M^n_x, g_x^*)$ to $(M^n, L)$ is of imbedding class 1, then the tangent Riemannian n-space $(M^n_x, g_x^*)$ to $(M^n, L^*)$ is at most of imbedding class 3.

**References**


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