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Some Common Fixed Point Theorems for Contractive Type Conditions in Complex Valued S-Metric Spaces

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Abstract: The aim of this paper is to establish some common fixed point theorems for four self-mappings satisfying $(E.A)$ and weak compatible properties and contractive type condition involving rational expressions in the setting of complex valued $\mathcal{S}$-metric spaces. The results presented in this paper extend and generalize several results from the existing literature.

Key Words: Common fixed point, $(E.A)$-property, weak compatibility, contractive type condition, complex valued $\mathcal{S}$-metric space.

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§1. Introduction

In the development of nonlinear analysis, fixed point theory plays a very important role. Banach contraction principle [6] was the starting point for many researchers during the last decades in the field of nonlinear analysis. The Banach contraction principle with rational expressions have been expanded and some fixed point and common fixed point theorems have been obtained in [12], [13], [14], [15].

In the existing literature there are a great number of generalizations of the Banach contraction principle (see [3, 4] and others). Some generalization of the notion of a metric space have been proposed by some authors, such as, partial metric spaces, probabilistic metric spaces, fuzzy metric spaces, D-metric spaces, cone metric spaces, b-metric spaces and cone b-metric spaces (see, [7, 9, 10, 11, 18, 19, 20, 21, 22, 23, 24, 26, 27, 32, 46]).

Also, as an extension of the fixed point problem there are many results in finding a common fixed point for two self mappings on different types of metric spaces; see, for example, [2], [41], [34], [35], [38], [44] and the references therein. But all of these results were found in real valued metric spaces.

In 2011, Azam *et al.* [5] introduced the notion of complex valued metric space and established sufficient conditions for the existence of common fixed points of a pair of mappings satisfying a contractive condition. The results proved by Azam *et al.* [5] and Bhatt *et al.* [8] via rational inequality in a complex valued metric space as a contractive condition. Complex valued

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metric space is very useful in many branches of mathematics, including algebraic geometry, number theory, applied mathematics, applied physics, mechanical engineering, thermodynamics and electrical engineering. After the establishment of complex valued metric spaces, Rouzkardand *et al.* [33] established some common fixed point theorems satisfying certain rational expressions in these spaces which generalize the result of [5]. In 2012, Sintanuvarat and Kumam [42] extend and improve the results of [5] by replacing the constant of contractive conditions to some control functions. Verma and Pathak in [44] introduced the notion of $(E.A)$-property in complex valued metric space and proved some common fixed point results for two pairs of weakly compatible mappings satisfying a "max" type contractive condition. After that many authors have contributed different concepts in this space (see, for example, [29], [36], [37], [42], [39] and many others).

Recently, Mlaiki [28] *(Adv. Fixed Point Theory 4(4) (2014), 509-524)* introduced the concept of complex valued $\mathcal{S}$-metric spaces and investigate the existence and uniqueness of a common fixed point of two self-mappings in such space via various contractive conditions. After Mlaiki's results many authors have established a lot of results in complex valued S-metric space under various contractive conditions (see, for example, [31], [45] and many others).

In this paper, we prove some common fixed point theorems for contractive type conditions involving rational expressions in the framework of complex valued S-metric spaces. Our results extend, generalize and enrich several results from the existing literature.

§2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$, $Im(z_1) \leq Im(z_2)$. It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2)$, $Im(z_1) < Im(z_2)$;
(ii) $Re(z_1) < Re(z_2)$, $Im(z_1) = Im(z_2)$;
(iii) $Re(z_1) < Re(z_2)$, $Im(z_1) < Im(z_2)$;
(iv) $Re(z_1) = Re(z_2)$, $Im(z_1) = Im(z_2)$.

In particular, we will write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of $(i), (ii)$ and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied. Note that

$$0 \lesssim z_1 \lnsim z_2 \Rightarrow |z_1| < |z_2|,$$

$$z_1 \precsim z_2,\ z_2 \prec z_3 \Rightarrow z_1 \prec z_3.$$

In 2014, the following definition was introduced by Mlaiki in [28].

Definition 2.1([28]) *Let X be a nonempty set and $\mathbb{C}$ be the set of all complex numbers. A complex valued $\mathcal{S}$-metric space on X is a function $\mathcal{S}\colon X^3 \to \mathbb{C}$ that satisfies the following conditions, for all $x, y, z, t \in X$:*

($\mathcal{CS}1$) $0 \precsim \mathcal{S}(x, y, z)$;
($\mathcal{CS}2$) $\mathcal{S}(x, y, z) = 0$ *if and only if* $x = y = z$*;*
($\mathcal{CS}3$) $\mathcal{S}(x, y, z) \precsim \mathcal{S}(x, x, t) + \mathcal{S}(y, y, t) + \mathcal{S}(z, z, t)$.

Then, $\mathcal{S}$ *is called a complex valued* $\mathcal{S}$*-metric on* X *and the pair* $(X, \mathcal{S})$ *is called a complex valued* $\mathcal{S}$*-metric space.*

Example 2.2([28]) Let $X = \mathbb{C}$ be the set of complex numbers. Define a mapping $\mathcal{S}\colon \mathbb{C}^3 \to \mathbb{C}$ by $\mathcal{S}(z_1, z_2, z_3) = |\max\{Re(z_1), Re(z_2)\} - Re(z_2)| + i|\max\{Im(z_1), Im(z_2)\} - Im(z_2)|$. Then it is not difficult to verify that $(\mathbb{C}, \mathcal{S})$ is a complex valued $\mathcal{S}$-metric space.

Definition 2.3([28]) *If* $(X, \mathcal{S})$ *is called a complex valued* $\mathcal{S}$*-metric space, then,*

($\mathbf{\Gamma_1}$) *A sequence* $\{u_n\}$ *in* X *converges to* u *if and only if for every* $\varepsilon \in \mathbb{C}$ *with* $0 \prec \varepsilon$*, there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$*, we have* $\mathcal{S}(u_n, u_n, u) \prec \varepsilon$ *and we denote this by* $u_n \to u$ *or* $\lim_{n\to\infty} u_n = u$*;*

($\mathbf{\Gamma_2}$) *A sequence* $\{u_n\}$ *in* X *is called a Cauchy sequence if for every* $\varepsilon \in \mathbb{C}$ *with* $0 \prec \varepsilon$*, there exists* $n_0 \in \mathbb{N}$ *such that for all* $n, m \geq n_0$*, we have* $\mathcal{S}(u_n, u_n, u_m) \prec \varepsilon$*;*

($\mathbf{\Gamma_3}$) *An* $\mathcal{S}$*-metric space* $(X, \mathcal{S})$ *is said to be complete if every Cauchy sequence is convergent.*

Definition 2.4 *Let* X *be a non-empty set and let* $\mathcal{R}, h\colon X \to X$ *be two self mappings of* X*. Then a point* $v \in X$ *is called a*

($\mathbf{\Lambda_1}$) *fixed point of operator* $\mathcal{R}$ *if* $\mathcal{R}(v) = v$*;*

($\mathbf{\Lambda_2}$) *common fixed point of* $\mathcal{R}$ *and* h *if* $\mathcal{R}(v) = h(v) = v$*.*

Definition 2.5([1]) *Let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be single valued self-mappings on a set* X*. If* $u = \mathcal{P}z = \mathcal{Q}z$ *for some* $z \in X$*, then* z *is called a coincidence point point of* $\mathcal{P}$ *and* $\mathcal{Q}$*, and* u *is called a point of coincidence of* $\mathcal{P}$ *and* $\mathcal{Q}$*.*

Definition 2.6([16]) *Let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be single valued self-mappings on a set* X*. Mappings* $\mathcal{P}$ *and* $\mathcal{Q}$ *are said to be commuting if* $\mathcal{P}\mathcal{Q}v = \mathcal{Q}\mathcal{P}v$ *for all* $v \in X$*.*

Definition 2.7([17]) *Let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be single valued self-mappings on a set* X*. Mappings* $\mathcal{P}$ *and* $\mathcal{Q}$ *are said to be weakly compatible if they commute at their coincidence points, i.e., if* $\mathcal{P}u = \mathcal{Q}u$ *for some* $u \in X$ *implies* $\mathcal{P}\mathcal{Q}u = \mathcal{Q}\mathcal{P}u$*.*

Definition 2.8([44]) *Let* (X, d) *be a complex valued metric space and let* $\mathcal{R}, \mathcal{Q}\colon X \to X$ *be two self mappings of* X*. The pair* $(\mathcal{R}, \mathcal{Q})$ *is said to satisfy* $(E.A)$*-property if there exists a sequence* $\{r_n\}$ *in* X *such that* $\lim_{n\to\infty} \mathcal{R}r_n = \lim_{n\to\infty} \mathcal{Q}r_n = d$ *for some* $d \in X$*.*

Note that weakly compatibility and $(E.A)$-property are independent of each other (see [30] for details).

Example 2.9 Let $X = \mathbb{C}$ and let $\mathcal{R}, \mathcal{Q}\colon X \to X$ be defined by $\mathcal{R}(z) = 4z - 2i$ and $\mathcal{Q}(z) = z + i$ for all $z \in X$. Let $\{z_n\} = \{i + \frac{1}{n}\}_{n\geq 1}$ be the sequence in X. Then

$$\lim_{n\to\infty} \mathcal{R}z_n = \lim_{n\to\infty} \left(4i + \frac{4}{n} - 2i\right) = 2i,$$

and

$$\lim_{n\to\infty} \mathcal{Q}z_n = \lim_{n\to\infty} \Big(i + \frac{1}{n} + i\Big) = 2i.$$

Thus there exists a sequence $\{z_n\}$ in X such that $\lim_{n\to\infty} \mathcal{R}z_n = \lim_{n\to\infty} \mathcal{Q}z_n = 2i \in X$. Hence $\mathcal{R}$ and $\mathcal{Q}$ satisfy $(E.A)$-property.

Liu *et al.* [25] introduced common $(E.A)$-property which is an extension of $(E.A)$-property were define common $(E.A)$-property in the complex valued metric space as follows.

Definition 2.10([25]) *Let (X, d) be a complex valued metric space and let $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{T} \colon X \to X$ be four self mappings of X. The pairs $(\mathcal{P}, \mathcal{R})$ and $(\mathcal{Q}, \mathcal{T})$ satisfy the common $(E.A)$-property if there exist two sequences $\{u_n\}$ and $\{v_n\}$ in X such that*

$$\lim_{n\to\infty} \mathcal{P}u_n = \lim_{n\to\infty} \mathcal{R}u_n = \lim_{n\to\infty} \mathcal{Q}v_n = \lim_{n\to\infty} \mathcal{T}v_n = z \in X.$$

Example 2.11 Let $X = \mathbb{C}$ and let d be a complex valued metric and let $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{T} \colon X \to X$ be four self-maps defined by $\mathcal{P}(z) = 3+iz$, $\mathcal{Q}(z) = -i-3z^2$, $\mathcal{R}(z) = -i-3z$ and $\mathcal{T}(z) = 3+(z-2i)$ for all $z \in X$. Let $\{x_n\} = \{-1 + \frac{1}{n}\}_{n\geq 1}$ and $\{y_n\} = \{\frac{1}{n} + i\}_{n\geq 1}$ be two sequences in X. Then

$$\lim_{n\to\infty} \mathcal{P}x_n = \lim_{n\to\infty} \Big(3 - i + \frac{i}{n}\Big) = 3 - i,$$

$$\lim_{n\to\infty} \mathcal{R}x_n = \lim_{n\to\infty} \Big(-i + 3 - \frac{3}{n}\Big) = 3 - i,$$

$$\lim_{n\to\infty} \mathcal{Q}y_n = \lim_{n\to\infty} \Big(-i - 3(\frac{1}{n} + i)^2\Big) = 3 - i,$$

and

$$\lim_{n\to\infty} \mathcal{T}y_n = \lim_{n\to\infty} \Big(3 + (\frac{1}{n} + i - 2i)\Big) = 3 - i.$$

Thus there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} \mathcal{P}x_n = \lim_{n\to\infty} \mathcal{R}x_n = \lim_{n\to\infty} \mathcal{Q}y_n = \lim_{n\to\infty} \mathcal{T}y_n = 3 - i \in X.$$

Hence the pairs $(\mathcal{P}, \mathcal{R})$ and $(\mathcal{Q}, \mathcal{T})$ satisfy common $(E.A)$-property.

Now, we redefine the common $(E.A)$-property in the setting of complex valued S-metric space as follows.

Definition 2.12 *Let $(X, \mathcal{S})$ be a complex valued $\mathcal{S}$-metric space and let $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{T} \colon X \to X$ be four self mappings of X. The pairs $(\mathcal{P}, \mathcal{R})$ and $(\mathcal{Q}, \mathcal{T})$ are said to satisfy the common $(E.A)$-property if there exist two sequences $\{p_n\}$ and $\{q_n\}$ in X such that*

$$\lim_{n\to\infty} \mathcal{P}p_n = \lim_{n\to\infty} \mathcal{R}p_n = \lim_{n\to\infty} \mathcal{Q}q_n = \lim_{n\to\infty} \mathcal{T}q_n = t \in X.$$

Example 2.13 Let $X = \mathbb{C}$ and let $\mathcal{S} \colon \mathbb{C}^3 \to \mathbb{C}$ be defined by $\mathcal{S}(z_1, z_2, z_3) = |\max\{Re(z_1), Re(z_2)\} - Re(z_2)| + i|\max\{Im(z_1), Im(z_2)\} - Im(z_2)|$. Then $(\mathbb{C}, \mathcal{S})$ is a complex valued $\mathcal{S}$-metric space.

Let $\mathcal{P},\mathcal{Q},\mathcal{R},\mathcal{T}\colon X\to X$ be four self-maps defined by $\mathcal{P}(z)=z+i$, $\mathcal{Q}(z)=z+(1+2i)$, $\mathcal{R}(z)=3i-z$ and $\mathcal{T}(z)=-z+(2i-1)$ for all $z\in X$. Let $\{p_n\}=\{i+\frac{1}{n}\}_{n\geq 1}$ and $\{q_n\}=\{-1+\frac{i}{n}\}_{n\geq 1}$ be two sequences in X and that $\mathcal{P}p_n=p_n+i=2i+\frac{1}{n}$ and $\mathcal{R}p_n=3i-p_n=2i-\frac{1}{n}$ for all $n\in\mathbb{N}$. This implies that

$$\mathcal{S}(\mathcal{P}p_n,\mathcal{P}p_n,0)=\mathcal{S}\Big(2i+\frac{1}{n},2i+\frac{1}{n},0\Big)\to 0 \text{ as } n\to\infty.$$

This shows that $\mathcal{P}p_n\to 0$ as $n\to\infty$ and by similar way, we have

$$\mathcal{S}(\mathcal{R}p_n,\mathcal{R}p_n,0)=\mathcal{S}\Big(2i-\frac{1}{n},2i-\frac{1}{n},0\Big)\to 0 \text{ as } n\to\infty.$$

This shows that $\mathcal{R}p_n\to 0$ as $n\to\infty$. Thus the pair $(\mathcal{P},\mathcal{R})$ satisfies $(E.A)$-property.

Similarly, note that $\mathcal{Q}q_n=q_n+(1+2i)=2i+\frac{i}{n}$ and $\mathcal{T}q_n=-q_n+(2i-1)=2i-\frac{i}{n}$ for all $n\in\mathbb{N}$. This implies that

$$\mathcal{S}(\mathcal{Q}q_n,\mathcal{Q}q_n,0)=\mathcal{S}\Big(2i+\frac{i}{n},2i+\frac{i}{n},0\Big)\to 0 \text{ as } n\to\infty.$$

This shows that $\mathcal{Q}q_n\to 0$ as $n\to\infty$ and by similar way, we have

$$\mathcal{S}(\mathcal{T}q_n,\mathcal{T}q_n,0)=\mathcal{S}\Big(-2i-\frac{i}{n},-2i-\frac{i}{n},0\Big)\to 0 \text{ as } n\to\infty.$$

This shows that $\mathcal{T}q_n\to 0$ as $n\to\infty$. Thus the pair $(\mathcal{Q},\mathcal{T})$ satisfies $(E.A)$-property. Thus there exist two sequences $\{p_n\}$ and $\{q_n\}$ in X such that

$$\lim_{n\to\infty}\mathcal{P}p_n=\lim_{n\to\infty}\mathcal{R}p_n=\lim_{n\to\infty}\mathcal{Q}q_n=\lim_{n\to\infty}\mathcal{T}q_n=0\in X.$$

Hence the pairs $(\mathcal{P},\mathcal{R})$ and $(\mathcal{Q},\mathcal{T})$ satisfy common $(E.A)$-property.

Lemma 2.14([28]) *Let $(X,\mathcal{S})$ be a complex valued $\mathcal{S}$-metric space and let $\{u_n\}$ be a sequence in X. Then $\{u_n\}$ converges to u if and only if $\lim_{n\to\infty}|\mathcal{S}(u_n,u_n,u)|=0$ or $|\mathcal{S}(u_n,u_n,u)|\to 0$ as $n\to\infty$.*

Lemma 2.15([28]) *Let $(X,\mathcal{S})$ be a complex valued $\mathcal{S}$-metric space and let $\{u_n\}$ be a sequence in X. Then $\{u_n\}$ is a Cauchy sequence if and only if $\lim_{n,m\to\infty}|\mathcal{S}(u_n,u_n,u_{n+m})|=0$ or $|\mathcal{S}(u_n,u_n,u_{n+m})|\to 0$ as $n,m\to\infty$.*

Lemma 2.16([28]) *Let $(X,\mathcal{S})$ be a complex valued $\mathcal{S}$-metric space, then $\mathcal{S}(x,x,y)=\mathcal{S}(y,y,x)$ for all $x,y\in X$.*

§3. **Common Fixed Point Theorems**

In this section, we shall prove some common fixed point theorems under contractive type conditions involving rational expression and satisfies $(E.A)$ property in the framework of complex valued $\mathcal{S}$-metric spaces.

Theorem 3.1 *Let $(X, \mathcal{S})$ be a complex valued $\mathcal{S}$-metric space and let $\mathcal{A}, \mathcal{B}, \mathcal{Q}, \mathcal{T}\colon X \to X$ be four self-mappings of X satisfying the following conditions:*

(i) For all $u, v \in X$,

$$\begin{aligned}\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v) \;\precsim\; r \max\Big\{&\mathcal{S}(\mathcal{Q}u, \mathcal{Q}u, \mathcal{T}v), \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{A}u), \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v),\\ &\frac{1}{2}[\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{T}v) + \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{Q}u)],\\ &\frac{\mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{A}u)[1 + \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v)]}{[1 + \mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v)]}\Big\}, \qquad (3.1)\end{aligned}$$

where $r \in [0, 1)$ is a constant;

(ii) The pairs $(\mathcal{A}, \mathcal{Q})$ and $(\mathcal{B}, \mathcal{T})$ are weakly compatible;

(iii) One of the pairs $(\mathcal{A}, \mathcal{Q})$ and $(\mathcal{B}, \mathcal{T})$ satisfy $(E.A)$ property;

(iv) $\mathcal{A}(X) \subseteq \mathcal{T}(X)$ and $\mathcal{B}(X) \subseteq \mathcal{Q}(X)$.

If the range of one of the mappings $\mathcal{Q}(X)$ or $\mathcal{T}(X)$ is a complete subspace of $(X, \mathcal{S})$, then $\mathcal{A}$, $\mathcal{B}$, $\mathcal{Q}$ and $\mathcal{T}$ have a unique common fixed point in X.

Proof First, we suppose that the pair $(\mathcal{A}, \mathcal{Q})$ satisfies $(E.A)$ property. Then by Definition 2.8, there exists a sequence $\{u_n\}$ in X such that $\lim_{n\to\infty} \mathcal{A}u_n = \lim_{n\to\infty} \mathcal{Q}u_n = t$ for some $t \in X$. Further, since $\mathcal{A}(X) \subseteq \mathcal{T}(X)$, there exists a sequence $\{v_n\}$ in X such that $\lim_{n\to\infty} \mathcal{A}u_n = \lim_{n\to\infty} \mathcal{T}v_n$. Hence $\lim_{n\to\infty} \mathcal{T}v_n = t$. We claim that $\lim_{n\to\infty} \mathcal{B}v_n = t$. If not, then putting $u = u_n$, $v = v_n$ in inequality (3.1), using Lemma 2.16 and $(\mathcal{CS}2)$, we have

$$\begin{aligned}\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n) \;&\precsim\; r \max\Big\{\mathcal{S}(\mathcal{Q}u_n, \mathcal{Q}u_n, \mathcal{T}v_n), \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n), \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{T}v_n),\\ &\qquad\frac{1}{2}[\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{T}v_n) + \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{Q}u_n)],\\ &\qquad\frac{\mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n)[1 + \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{T}v_n)]}{[1 + \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)]}\Big\}\\ &= r \max\Big\{\mathcal{S}(\mathcal{Q}u_n, \mathcal{Q}u_n, \mathcal{A}u_n), \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n), \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n),\\ &\qquad\frac{1}{2}[\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{A}u_n) + \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n)],\\ &\qquad\frac{\mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n)[1 + \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n)]}{[1 + \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)]}\Big\}\\ &= r \max\Big\{0, \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n), \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n),\\ &\qquad\frac{1}{2}[\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)], \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)\Big\}\\ &\precsim\; r\, \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n). \qquad (3.2)\end{aligned}$$

Thus

$$|\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)| \leq r\, |\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)|,$$

which is a contradiction since $r \in [0,1)$. Letting $n \to \infty$, we have

$$\lim_{n\to\infty} |\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)| \le r\,.0 = 0,$$

which is a contradiction by condition $(\mathcal{CS}1)$. Thus, we get $\lim_{n\to\infty} \mathcal{A}u_n = \lim_{n\to\infty} \mathcal{B}v_n = t$.

Now, first we assume that $\mathcal{T}(X)$ is a complete subspace of $(X, \mathcal{S})$, then $t = \mathcal{T}p$ for some $p \in X$. Subsequently, we have

$$\lim_{n\to\infty} \mathcal{B}v_n = \lim_{n\to\infty} \mathcal{A}u_n = \lim_{n\to\infty} \mathcal{Q}u_n = \lim_{n\to\infty} \mathcal{T}v_n = \mathcal{T}p = t.$$

We claim that $\mathcal{B}p = \mathcal{T}p$. For this, putting $u = u_n$ and $v = p$ in inequality (3.1), using Lemma 2.16 and $(\mathcal{CS}2)$, we have

$$\begin{aligned} \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}p) \quad &\precsim \quad r \max\Big\{\mathcal{S}(\mathcal{Q}u_n, \mathcal{Q}u_n, \mathcal{T}p), \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{A}u_n), \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{T}p), \\ &\qquad \frac{1}{2}[\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{T}p) + \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{Q}u_n)], \\ &\qquad \frac{\mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{A}u_n)[1 + \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{T}p)]}{[1 + \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}p)]}\Big\}. \end{aligned} \tag{3.3}$$

Letting $n \to \infty$ in (3.3), using Lemma 2.16 and $(\mathcal{CS}2)$, we get

$$\begin{aligned} \mathcal{S}(\mathcal{T}p, \mathcal{T}p, \mathcal{B}p) \quad &\precsim \quad r \max\Big\{\mathcal{S}(\mathcal{T}p, \mathcal{T}p, \mathcal{T}p), \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{T}p), \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{T}p), \\ &\qquad \frac{1}{2}[\mathcal{S}(\mathcal{T}p, \mathcal{T}p, \mathcal{T}p) + \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{T}p)], \\ &\qquad \frac{\mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{T}p)[1 + \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{T}p)]}{[1 + \mathcal{S}(\mathcal{T}p, \mathcal{T}p, \mathcal{B}p)]}\Big\} \\ &= \quad r \max\Big\{0, \mathcal{S}(\mathcal{T}p, \mathcal{T}p, \mathcal{B}p), \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{T}p), \\ &\qquad \frac{1}{2}[\mathcal{S}(\mathcal{T}p, \mathcal{T}p, \mathcal{B}p)], \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{T}p)\Big\} \precsim r\,\mathcal{S}(\mathcal{T}p, \mathcal{T}p, \mathcal{B}p). \end{aligned} \tag{3.4}$$

Thus, $|\mathcal{S}(\mathcal{T}p, \mathcal{T}p, \mathcal{B}p)| \le r\,|\mathcal{S}(\mathcal{T}p, \mathcal{T}p, \mathcal{B}p)|$, which is a contradiction since $r \in [0,1)$. Hence, we have $\mathcal{S}(\mathcal{T}p, \mathcal{T}p, \mathcal{B}p) = 0$, that is, $\mathcal{T}p = \mathcal{B}p = t$. Hence p is a coincidence point of the mappings $\mathcal{B}$ and $\mathcal{T}$, that is, the pair $(\mathcal{B}, \mathcal{T})$. Now, the weak compatibility of the pair $(\mathcal{B}, \mathcal{T})$ implies that $\mathcal{B}\mathcal{T}p = \mathcal{T}\mathcal{B}p$ or $\mathcal{B}t = \mathcal{T}t$.

On the other hand, since $\mathcal{B}(X) \subseteq \mathcal{Q}(X)$, there exists $\nu \in X$ such that $\mathcal{B}p = \mathcal{Q}\nu$. Thus $\mathcal{T}p = \mathcal{B}p = \mathcal{Q}\nu = t$. Let us show that ν is a coincidence point of the pair $(\mathcal{A}, \mathcal{Q})$, that is, $\mathcal{A}\nu = \mathcal{Q}\nu = t$. If not, then putting $u = \nu$ and $v = p$ in inequality (3.1), using Lemma 2.16 and $(\mathcal{CS}2)$, we get

$$\begin{aligned} \mathcal{S}(\mathcal{A}\nu, \mathcal{A}\nu, \mathcal{B}p) \quad &\precsim \quad r \max\Big\{\mathcal{S}(\mathcal{Q}\nu, \mathcal{Q}\nu, \mathcal{T}p), \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{A}\nu), \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{T}p), \\ &\qquad \frac{1}{2}[\mathcal{S}(\mathcal{A}\nu, \mathcal{A}\nu, \mathcal{T}p) + \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{Q}\nu)], \\ &\qquad \frac{\mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{A}\nu)[1 + \mathcal{S}(\mathcal{B}p, \mathcal{B}p, \mathcal{T}p)]}{[1 + \mathcal{S}(\mathcal{A}\nu, \mathcal{A}\nu, \mathcal{B}p)]}\Big\} \end{aligned}$$

$$\begin{aligned} &= r\max\Big\{0, \mathcal{S}(\mathcal{A}\nu,\mathcal{A}\nu,\mathcal{B}p), 0, \frac{\mathcal{S}(\mathcal{A}\nu,\mathcal{A}\nu,\mathcal{B}p)}{2}, \\ &\qquad \mathcal{S}(\mathcal{A}\nu,\mathcal{A}\nu,\mathcal{B}p)\Big\} \\ &\precsim r\,\mathcal{S}(\mathcal{A}\nu,\mathcal{A}\nu,\mathcal{B}p). \end{aligned} \tag{3.5}$$

Thus, $|\mathcal{S}(\mathcal{A}\nu,\mathcal{A}\nu,\mathcal{B}p)| \leq r\,|\mathcal{S}(\mathcal{A}\nu,\mathcal{A}\nu,\mathcal{B}p)|$, which is a contradiction since $r \in [0,1)$. Hence, we have $\mathcal{S}(\mathcal{A}\nu,\mathcal{A}\nu,\mathcal{B}p) = 0$, that is, $\mathcal{S}(\mathcal{A}\nu,\mathcal{A}\nu,\mathcal{Q}\nu) = 0$ and hence $\mathcal{A}\nu = \mathcal{Q}\nu = t$. Thus ν is a coincidence point of the mappings $\mathcal{A}$ and $\mathcal{Q}$, that is, the pair $(\mathcal{A},\mathcal{Q})$. Further, the weak compatibility of the pair $(\mathcal{A},\mathcal{Q})$ implies that $\mathcal{A}\mathcal{Q}\nu = \mathcal{Q}\mathcal{A}\nu$ or $\mathcal{A}t = \mathcal{Q}t$. Hence t is a common coincidence point of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{Q}$ and $\mathcal{T}$.

Now to show that t is a common fixed point of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{Q}$ and $\mathcal{T}$. For this, we put $u = \nu$ and $v = t$ in (3.1), using Lemma 2.16 and $(\mathcal{CS}2)$, we get

$$\begin{aligned} \mathcal{S}(t,t,\mathcal{B}t) &= \mathcal{S}(\mathcal{A}\nu,\mathcal{A}\nu,\mathcal{B}t) \\ &\precsim r\max\Big\{\mathcal{S}(\mathcal{Q}\nu,\mathcal{Q}\nu,\mathcal{T}t), \mathcal{S}(\mathcal{B}t,\mathcal{B}t,\mathcal{A}\nu), \mathcal{S}(\mathcal{B}t,\mathcal{B}t,\mathcal{T}t), \\ &\qquad \frac{1}{2}[\mathcal{S}(\mathcal{A}\nu,\mathcal{A}\nu,\mathcal{T}t) + \mathcal{S}(\mathcal{B}t,\mathcal{B}t,\mathcal{Q}\nu)], \\ &\qquad \frac{\mathcal{S}(\mathcal{B}t,\mathcal{B}t,\mathcal{A}\nu)[1+\mathcal{S}(\mathcal{B}t,\mathcal{B}t,\mathcal{T}t)]}{[1+\mathcal{S}(\mathcal{A}\nu,\mathcal{A}\nu,\mathcal{B}t)]}\Big\} \\ &= r\max\Big\{\mathcal{S}(t,t,\mathcal{B}t), \mathcal{S}(\mathcal{B}t,\mathcal{B}t,t), \mathcal{S}(\mathcal{B}t,\mathcal{B}t,\mathcal{B}t), \\ &\qquad \frac{1}{2}[\mathcal{S}(t,t,\mathcal{B}t) + \mathcal{S}(\mathcal{B}t,\mathcal{B}t,t)], \\ &\qquad \frac{\mathcal{S}(\mathcal{B}t,\mathcal{B}t,t)[1+\mathcal{S}(\mathcal{B}t,\mathcal{B}t,\mathcal{B}t)]}{[1+\mathcal{S}(t,t,\mathcal{B}t)]}\Big\} \\ &= r\max\Big\{\mathcal{S}(t,t,\mathcal{B}t), \mathcal{S}(t,t,\mathcal{B}t), 0, \mathcal{S}(t,t,\mathcal{B}t), \\ &\qquad \frac{\mathcal{S}(t,t,\mathcal{B}t)}{[1+\mathcal{S}(t,t,\mathcal{B}t)]}\Big\} \\ &\precsim r\max\Big\{\mathcal{S}(t,t,\mathcal{B}t), \mathcal{S}(t,t,\mathcal{B}t), 0, \mathcal{S}(t,t,\mathcal{B}t), \mathcal{S}(t,t,\mathcal{B}t)\Big\} \\ &\precsim r\,\mathcal{S}(t,t,\mathcal{B}t). \end{aligned} \tag{3.6}$$

Thus, $|\mathcal{S}(t,t,\mathcal{B}t)| \leq r\,|\mathcal{S}(t,t,\mathcal{B}t)|$, which is a contradiction since $r \in [0,1)$. Hence, we have $\mathcal{S}(t,t,\mathcal{B}t) = 0$, that is, $\mathcal{B}t = t$. Consequently, $\mathcal{A}t = \mathcal{B}t = \mathcal{Q}t = \mathcal{T}t = t$. This shows that t is a common fixed point of the mappings $\mathcal{A}$, $\mathcal{B}$, $\mathcal{Q}$ and $\mathcal{T}$.

Similar argument arises if we assume that $\mathcal{Q}(X)$ is a complete subspace of $(X,\mathcal{S})$.

Similarly, the property $(E.A)$ of the pair $(\mathcal{B},\mathcal{T})$ will give the similar result.

Now, we show the uniqueness of the common fixed point. For this, let us assume that t' be another common fixed point of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{Q}$ and $\mathcal{T}$ with $t' \neq t$. From inequality (3.1), using Lemma 2.16 and $(\mathcal{CS}2)$ for $u = t'$ and $v = t$, we have

$$\mathcal{S}(t',t',t) = \mathcal{S}(\mathcal{A}t',\mathcal{A}t',\mathcal{B}t)$$

$$\begin{aligned}
&\precsim\ r\ \max\Big\{\mathcal{S}(\mathcal{Q}t',\mathcal{Q}t',\mathcal{T}t),\mathcal{S}(\mathcal{B}t,\mathcal{B}t,\mathcal{A}t'),\mathcal{S}(\mathcal{B}t,\mathcal{B}t,\mathcal{T}t),\\
&\qquad\qquad \frac{1}{2}[\mathcal{S}(\mathcal{A}t',\mathcal{A}t',\mathcal{T}t)+\mathcal{S}(\mathcal{B}t,\mathcal{B}t,\mathcal{Q}t')],\\
&\qquad\qquad \frac{\mathcal{S}(\mathcal{B}t,\mathcal{B}t,\mathcal{A}t')[1+\mathcal{S}(\mathcal{B}t,\mathcal{B}t,\mathcal{T}t)]}{[1+\mathcal{S}(\mathcal{A}t',\mathcal{A}t',\mathcal{B}t)]}\Big\}\\
&=\ r\ \max\Big\{\mathcal{S}(t',t',t),\mathcal{S}(t,t,t'),\mathcal{S}(t,t,t),\\
&\qquad\qquad \frac{1}{2}[\mathcal{S}(t',t',t)+\mathcal{S}(t,t,t')],\\
&\qquad\qquad \frac{\mathcal{S}(t,t,t')[1+\mathcal{S}(t,t,t)]}{[1+\mathcal{S}(t',t',t)]}\Big\}\\
&=\ r\ \max\Big\{\mathcal{S}(t',t',t),\mathcal{S}(t',t',t),0,\mathcal{S}(t',t',t),\\
&\qquad\qquad \frac{\mathcal{S}(t',t',t)}{[1+\mathcal{S}(t',t',t)]}\Big\}\\
&\precsim\ r\ \max\Big\{\mathcal{S}(t',t',t),\mathcal{S}(t',t',t),0,\mathcal{S}(t',t',t),\\
&\qquad\qquad \mathcal{S}(t',t',t)\Big\}\\
&\precsim\ r\,\mathcal{S}(t',t',t).
\end{aligned}\tag{3.7}$$

Thus

$$|\mathcal{S}(t',t',t)|\leq r\,|\mathcal{S}(t',t',t)|,$$

which is a contradiction since $r\in[0,1)$. Hence, we have

$$\mathcal{S}(t',t',t)=0,$$

that is, $t'=t$. Hence $\mathcal{A}t=\mathcal{B}t=\mathcal{Q}t=\mathcal{T}t=t$ and t is the unique common fixed point of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{Q}$ and $\mathcal{T}$. This completes the proof. □

If we take $\mathcal{A}=\mathcal{B}$ and $\mathcal{Q}=\mathcal{T}$ in Theorem 3.1, then we have the following result.

Corollary 3.2 *Let $(X,\mathcal{S})$ be a complex valued $\mathcal{S}$-metric space and let $\mathcal{A},\mathcal{Q}\colon X\to X$ be two self-mappings of X satisfying the following conditions:*

(i) For all $u,v\in X$,

$$\begin{aligned}
\mathcal{S}(\mathcal{A}u,\mathcal{A}u,\mathcal{A}v)\ &\precsim\ r\ \max\Big\{\mathcal{S}(\mathcal{Q}u,\mathcal{Q}u,\mathcal{Q}v),\mathcal{S}(\mathcal{A}v,\mathcal{A}v,\mathcal{A}u),\mathcal{S}(\mathcal{A}v,\mathcal{A}v,\mathcal{Q}v),\\
&\qquad\qquad \frac{1}{2}[\mathcal{S}(\mathcal{A}u,\mathcal{A}u,\mathcal{Q}v)+\mathcal{S}(\mathcal{A}v,\mathcal{A}v,\mathcal{Q}u)],\\
&\qquad\qquad \frac{\mathcal{S}(\mathcal{A}v,\mathcal{A}v,\mathcal{A}u)[1+\mathcal{S}(\mathcal{A}v,\mathcal{A}v,\mathcal{Q}v)]}{[1+\mathcal{S}(\mathcal{A}u,\mathcal{A}u,\mathcal{A}v)]}\Big\},
\end{aligned}\tag{3.8}$$

where $r\in[0,1)$ is a constant;

(ii) The pairs $(\mathcal{A},\mathcal{Q})$ is weakly compatible;

(iii) The pair $(\mathcal{A},\mathcal{Q})$ satisfies $(E.A)$ property;

(iv) $\mathcal{A}(X)\subseteq\mathcal{Q}(X)$.

If the range of the mapping $\mathcal{Q}(X)$ is a complete subspace of $(X, \mathcal{S})$, then $\mathcal{A}$ and $\mathcal{Q}$ have a unique common fixed point in X.

Theorem 3.3 *Let $(X, \mathcal{S})$ be a complex valued $\mathcal{S}$-metric space and let $\mathcal{A}, \mathcal{B}, \mathcal{Q}, \mathcal{T}$: $X \to X$ be four self-mappings of X satisfying the following conditions:*

(*i*) *For all $u, v \in X$,*

$$\begin{aligned} \mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v) &\precsim n_1\, \mathcal{S}(\mathcal{Q}u, \mathcal{Q}u, \mathcal{T}v) + n_2\, \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{A}u) + n_3\, \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v) \\ &\quad + n_4\, \mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{T}v) \frac{[1 + \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v)]}{[1 + \mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v)]} \\ &\quad + n_5\, \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{A}u) \frac{[1 + \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v)]}{[1 + \mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v)]}, \end{aligned} \tag{3.9}$$

where $n_1, n_2, n_3, n_4, n_5 > 0$ are nonnegative reals with $n_1 + n_2 + n_3 + n_4 + n_5 < 1$;

(*ii*) *The pairs $(\mathcal{A}, \mathcal{Q})$ and $(\mathcal{B}, \mathcal{T})$ are weakly compatible;*

(*iii*) *One of the pairs $(\mathcal{A}, \mathcal{Q})$ and $(\mathcal{B}, \mathcal{T})$ satisfy $(E.A)$ property;*

(*iv*) *$\mathcal{A}(X) \subseteq \mathcal{T}(X)$ and $\mathcal{B}(X) \subseteq \mathcal{Q}(X)$.*

If the range of one of the mappings $\mathcal{Q}(X)$ or $\mathcal{T}(X)$ is a complete subspace of $(X, \mathcal{S})$, then $\mathcal{A}$, $\mathcal{B}$, $\mathcal{Q}$ and $\mathcal{T}$ have a unique common fixed point in X.

Proof First, we suppose that the pair $(\mathcal{A}, \mathcal{Q})$ satisfies $(E.A)$ property. Then by Definition 2.8, there exists a sequence $\{u_n\}$ in X such that $\lim_{n\to\infty} \mathcal{A}u_n = \lim_{n\to\infty} \mathcal{Q}u_n = t$ for some $t \in X$. Further, since $\mathcal{A}(X) \subseteq \mathcal{T}(X)$, there exists a sequence $\{v_n\}$ in X such that $\lim_{n\to\infty} \mathcal{A}u_n = \lim_{n\to\infty} \mathcal{T}v_n$. Hence $\lim_{n\to\infty} \mathcal{T}v_n = t$. We claim that $\lim_{n\to\infty} \mathcal{B}v_n = t$. If not, then putting $u = u_n$ and $v = v_n$ in inequality (3.9), using Lemma 2.16 and $(\mathcal{CS}2)$, we have

$$\begin{aligned} \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n) &\precsim n_1\, \mathcal{S}(\mathcal{Q}u_n, \mathcal{Q}u_n, \mathcal{T}v_n) + n_2\, \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n) \\ &\quad + n_3\, \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{T}v_n) \\ &\quad + n_4\, \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{T}v_n) \frac{[1 + \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{T}v_n)]}{[1 + \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)]} \\ &\quad + n_5\, \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n) \frac{[1 + \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{T}v_n)]}{[1 + \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)]} \\ &= n_1\, \mathcal{S}(\mathcal{Q}u_n, \mathcal{Q}u_n, \mathcal{Q}u_n) + n_2\, \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n) \\ &\quad + n_3\, \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n) \\ &\quad + n_4\, \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{A}u_n) \frac{[1 + \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n)]}{[1 + \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)]} \\ &\quad + n_5\, \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n) \frac{[1 + \mathcal{S}(\mathcal{B}v_n, \mathcal{B}v_n, \mathcal{A}u_n)]}{[1 + \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)]} \\ &= n_1\, .0 + n_2\, \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n) + n_3\, \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n) \\ &\quad + n_4\, .0 + n_5\, \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n) \\ &= (n_2 + n_3 + n_5)\, \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n) \\ &\precsim (n_1 + n_2 + n_3 + n_4 + n_5)\, \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n) \\ &= m\, \mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n) \end{aligned}$$

where $m = n_1 + n_2 + n_3 + n_4 + n_5 < 1$. Thus

$$|\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)| \leq m\,|\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)|,$$

which is a contradiction since $m \in [0, 1)$. Letting $n \to \infty$, we have

$$\lim_{n\to\infty} |\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n)| \leq m\,.0 = 0,$$

which is a contradiction by condition $(\mathcal{CS}1)$. Thus, we get $\lim_{n\to\infty} \mathcal{A}u_n = \lim_{n\to\infty} \mathcal{B}v_n = t$.

Now, first we assume that $\mathcal{T}(X)$ is a complete subspace of $(X, \mathcal{S})$, then $t = \mathcal{T}p$ for some $p \in X$. Subsequently, we have

$$\lim_{n\to\infty} \mathcal{B}v_n = \lim_{n\to\infty} \mathcal{A}u_n = \lim_{n\to\infty} \mathcal{Q}u_n = \lim_{n\to\infty} \mathcal{T}v_n = \mathcal{T}p = t.$$

Rest of the proof follows from Theorem 3.1. This completes the proof. □

Theorem 3.4 *Let $(X, \mathcal{S})$ be a complex valued $\mathcal{S}$-metric space and let $\mathcal{A}, \mathcal{B}, \mathcal{Q}, \mathcal{T}$: $X \to X$ be four self-mappings of X satisfying the following conditions:*

(i) *For all $u, v \in X$,*

$$\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v) \precsim \mathcal{R}_1\, \mathcal{D}^{\mathcal{S}}_{\mathcal{C}_1}(u, u, v) + \mathcal{R}_2\, \mathcal{D}^{\mathcal{S}}_{\mathcal{C}_2}(u, u, v), \tag{3.10}$$

where $\mathcal{R}_1, \mathcal{R}_2 > 0$ are nonnegative reals with $\mathcal{R}_1 + \mathcal{R}_2 < 1$ and

$$\mathcal{D}^{\mathcal{S}}_{\mathcal{C}_1}(u, u, v) = \max\Big\{\mathcal{S}(\mathcal{Q}u, \mathcal{Q}u, \mathcal{T}v), \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{A}u), \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v)\Big\},$$

$$\begin{aligned}\mathcal{D}^{\mathcal{S}}_{\mathcal{C}_2}(u, u, v) = \max\Big\{&\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{T}v)\frac{[1 + \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v)]}{[1 + \mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v)]},\\ &\mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{A}u)\frac{[1 + \mathcal{S}(\mathcal{B}v, \mathcal{B}v, \mathcal{T}v)]}{[1 + \mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{B}v)]}\Big\},\end{aligned}$$

(ii) *The pairs $(\mathcal{A}, \mathcal{Q})$ and $(\mathcal{B}, \mathcal{T})$ are weakly compatible;*
(iii) *One of the pairs $(\mathcal{A}, \mathcal{Q})$ and $(\mathcal{B}, \mathcal{T})$ satisfy $(E.A)$ property;*
(iv) *$\mathcal{A}(X) \subseteq \mathcal{T}(X)$ and $\mathcal{B}(X) \subseteq \mathcal{Q}(X)$.*

If the range of one of the mappings $\mathcal{Q}(X)$ or $\mathcal{T}(X)$ is a complete subspace of $(X, \mathcal{S})$, then $\mathcal{A}$, $\mathcal{B}$, $\mathcal{Q}$ and $\mathcal{T}$ have a unique common fixed point in X.

Proof First, we suppose that the pair $(\mathcal{A}, \mathcal{Q})$ satisfies $(E.A)$ property. Then by Definition 2.8, there exists a sequence $\{u_n\}$ in X such that $\lim_{n\to\infty} \mathcal{A}u_n = \lim_{n\to\infty} \mathcal{Q}u_n = t$ for some $t \in X$. Further, since $\mathcal{A}(X) \subseteq \mathcal{T}(X)$, there exists a sequence $\{v_n\}$ in X such that $\lim_{n\to\infty} \mathcal{A}u_n = \lim_{n\to\infty} \mathcal{T}v_n$. Hence $\lim_{n\to\infty} \mathcal{T}v_n = t$. We claim that $\lim_{n\to\infty} \mathcal{B}v_n = t$. If not, then putting $u = u_n$ and $v = v_n$ in inequality (3.10), using Lemma 2.16 and $(\mathcal{CS}2)$, we have

$$\mathcal{S}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{B}v_n) \precsim \mathcal{R}_1\, \mathcal{D}^{\mathcal{S}}_{\mathcal{C}_1}(u_n, u_n, v_n) + \mathcal{R}_2\, \mathcal{D}^{\mathcal{S}}_{\mathcal{C}_2}(u_n, u_n, v_n), \tag{3.11}$$

where

$$\begin{aligned}
\mathcal{D}^{\mathcal{S}}_{\mathcal{C}_1}(u_n,u_n,v_n) &= \max\Big\{\mathcal{S}(\mathcal{Q}u_n,\mathcal{Q}u_n,\mathcal{T}v_n),\mathcal{S}(\mathcal{B}v_n,\mathcal{B}v_n,\mathcal{A}u_n),\mathcal{S}(\mathcal{B}v_n,\mathcal{B}v_n,\mathcal{T}v_n)\Big\}\\
&= \max\Big\{\mathcal{S}(\mathcal{Q}u_n,\mathcal{Q}u_n,\mathcal{Q}u_n),\mathcal{S}(\mathcal{B}v_n,\mathcal{B}v_n,\mathcal{A}u_n),\mathcal{S}(\mathcal{B}v_n,\mathcal{B}v_n,\mathcal{A}u_n)\Big\}\\
&= \max\Big\{0,\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n),\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n)\Big\}\\
&= \mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n), \qquad (3.12)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{D}^{\mathcal{S}}_{\mathcal{C}_2}(u_n,u_n,v_n) &= \max\Big\{\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{T}v_n)\frac{[1+\mathcal{S}(\mathcal{B}v_n,\mathcal{B}v_n,\mathcal{T}v_n)]}{[1+\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n)]},\\
&\qquad \mathcal{S}(\mathcal{B}v_n,\mathcal{B}v_n,\mathcal{A}u_n)\frac{[1+\mathcal{S}(\mathcal{B}v_n,\mathcal{B}v_n,\mathcal{T}v_n)]}{[1+\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n)]}\Big\}\\
&= \max\Big\{\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{A}u_n)\frac{[1+\mathcal{S}(\mathcal{B}v_n,\mathcal{B}v_n,\mathcal{A}u_n)]}{[1+\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n)]},\\
&\qquad \mathcal{S}(\mathcal{B}v_n,\mathcal{B}v_n,\mathcal{A}u_n)\frac{[1+\mathcal{S}(\mathcal{B}v_n,\mathcal{B}v_n,\mathcal{A}u_n)]}{[1+\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n)]}\Big\}\\
&= \max\Big\{0,\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n)\Big\}\\
&= \mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n). \qquad (3.13)
\end{aligned}$$

Using equations (3.12) and (3.13) in equation (3.11), we get

$$\begin{aligned}
\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n) &\precsim \mathcal{R}_1\,\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n)+\mathcal{R}_2\,\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n)\\
&= \Big(\mathcal{R}_1+\mathcal{R}_2\Big)\,\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n)\\
&= \mathcal{W}\,\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n), \qquad (3.14)
\end{aligned}$$

where $\mathcal{W}=\mathcal{R}_1+\mathcal{R}_2<1$.

Thus

$$|\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n)|\le \mathcal{W}\,|\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n)|,$$

which is a contradiction since $\mathcal{W}\in[0,1)$. Letting $n\to\infty$, we have

$$\lim_{n\to\infty}|\mathcal{S}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{B}v_n)|\le \mathcal{W}\,.0=0,$$

which is a contradiction by condition $(\mathcal{CS}1)$. Thus, we get $\lim_{n\to\infty}\mathcal{A}u_n=\lim_{n\to\infty}\mathcal{B}v_n=t$.

Now, first we assume that $\mathcal{T}(X)$ is a complete subspace of $(X,\mathcal{S})$, then $t=\mathcal{T}p$ for some $p\in X$. Subsequently, we have

$$\lim_{n\to\infty}\mathcal{B}v_n=\lim_{n\to\infty}\mathcal{A}u_n=\lim_{n\to\infty}\mathcal{Q}u_n=\lim_{n\to\infty}\mathcal{T}v_n=\mathcal{T}p=t.$$

Rest of the proof follows from Theorem 3.1. This completes the proof. □

From Corollary 3.2 we obtain the following special case.

Corollary 3.5 *Let $(X,\mathcal{S})$ be a complete complex valued $\mathcal{S}$-metric space and let $\mathcal{A}\colon X\to X$ be* *a self-mapping of X satisfies the contractive condition:*

$$\mathcal{S}(\mathcal{A}u, \mathcal{A}u, \mathcal{A}v) \precsim q\,\mathcal{S}(u, u, v),$$

for all $u, v \in X$, where $q \in [0, 1)$ is a constant. Then $\mathcal{A}$ has a unique fixed point in X.

Remark 3.6 Corollary 3.5 extends Theorem 3.1 of Sedghi et al. [40] from complete S-metric space to the setting of complete complex valued S-metric space.

Remark 3.7 Corollary 3.5 also extends the well-known Banach fixed theorem [6] from complete metric space to the setting of complete complex valued S-metric space.

Corollary 3.8([28], Corollary 2.5) *Let $(X, \mathcal{S})$ be a complete complex valued $\mathcal{S}$-metric space and let $\mathcal{A}\colon X \to X$ be a self-mapping of X satisfies the contractive condition:*

$$\mathcal{S}(\mathcal{A}^n u, \mathcal{A}^n u, \mathcal{A}^n v) \precsim q\,\mathcal{S}(u, u, v),$$

for all $u, v \in X$, where n is some positive integer and $q \in [0, 1)$ is a constant. Then $\mathcal{A}$ has a unique fixed point in X.

Proof By Corollary 3.5, there exists $p \in X$ such that $\mathcal{A}^n p = p$. Then

$$\begin{aligned}\mathcal{S}(\mathcal{A}p, \mathcal{A}p, p) &= \mathcal{S}(\mathcal{A}\mathcal{A}^n p, \mathcal{A}\mathcal{A}^n p, \mathcal{A}^n p)\\ &= \mathcal{S}(\mathcal{A}^n\mathcal{A}p, \mathcal{A}^n\mathcal{A}p, \mathcal{A}^n p)\\ &\precsim q\,\mathcal{S}(\mathcal{A}p, \mathcal{A}p, p).\end{aligned}$$

Thus

$$|\mathcal{S}(\mathcal{A}p, \mathcal{A}p, p)| \leq q\,|\mathcal{S}(\mathcal{A}p, \mathcal{A}p, p)|,$$

which is a contradiction since $0 \leq q < 1$ and so $\mathcal{S}(\mathcal{A}p, \mathcal{A}p, p) = 0$, that is, $\mathcal{A}p = p$. This shows that $\mathcal{A}$ has a unique fixed point in X. This completes the proof. □

Remark 3.9 (i) Completeness of the space X is relaxed in Theorems 3.1, 3.3 and 3.4.

(ii) Continuity of the mappings $\mathcal{A}$, $\mathcal{B}$, $\mathcal{Q}$ and $\mathcal{T}$ is relaxed in Theorems 3.1, 3.3 and 3.4.

Finally, we give the following example which is an application of Corollary 3.5.

Example 3.10 Let $X_1 = \{z \in \mathbb{C} : Re(z) \geq 0, Im(z) = 0\}$ and $X_2 = \{z \in \mathbb{C} : Im(z) \geq 0, Re(z) = 0\}$. Now, let $X = X_1 \cup X_2$ and define a mapping $\mathcal{S}\colon X^3 \to \mathbb{C}$ By:

$$\mathcal{S}(z_1, z_2, z_3) = \begin{cases} \max\{x_1, x_2, x_3\} + i\max\{x_1, x_2, x_3\}, & \text{if } z_1, z_2, z_3 \in X_1,\\ \max\{y_1, y_2, y_3\} + i\max\{y_1, y_2, y_3\}, & \text{if } z_1, z_2, z_3 \in X_2,\\ (\max\{x_1, x_2\} + y_3) + i(\max\{x_1, x_2\} + y_3), & \text{if } z_1, z_2 \in X_1, z_3 \in X_2,\\ (\max\{y_1, y_2\} + x_3) + i(\max\{y_1, y_2\} + x_3), & \text{if } z_1, z_2 \in X_2, z_3 \in X_1,\end{cases}$$

where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ and $z_3 = x_3 + iy_3$. It is very easy to verify that $(X, \mathcal{S})$ is a complete complex valued $\mathcal{S}$-metric space.

Now, we define a self-mapping $\mathcal{A}$ on X (with $z = (x, y)$) as

$$\mathcal{A}(z) = \begin{cases} (\frac{x}{2}, 0), & \text{if } z \in X_1, \\ (0, \frac{y}{2}), & \text{if } z \in X_2. \end{cases}$$

Now, we show that $\mathcal{A}$ satisfies the conditions of Corollary 3.5. Here, we note that

$$0 \precsim \mathcal{S}(z_1, z_2, z_3), \mathcal{S}(\mathcal{A}z_1, \mathcal{A}z_2, \mathcal{A}z_3).$$

. Now, let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Hence, we have the following four cases.

Case 1. If $z_1, z_2 \in X_1$, then we have

$$\begin{aligned} \mathcal{S}(\mathcal{A}z_1, \mathcal{A}z_1, \mathcal{A}z_2) &= \mathcal{S}\Big((\frac{x_1}{2}, 0), (\frac{x_1}{2}, 0), (\frac{x_2}{2}, 0)\Big) \\ &= \max\Big\{\frac{x_1}{2}, \frac{x_2}{2}\Big\} + i \max\Big\{\frac{x_1}{2}, \frac{x_2}{2}\Big\} = \max\Big\{\frac{x_1}{2}, \frac{x_2}{2}\Big\}(1+i) \\ &= \frac{1}{2}\max\{x_1, x_2\}(1+i) \precsim \frac{1}{2}\mathcal{S}(z_1, z_1, z_2) = q\,\mathcal{S}(z_1, z_1, z_2), \end{aligned}$$

Case 2. If $z_1, z_2 \in X_2$, then we have

$$\begin{aligned} \mathcal{S}(\mathcal{A}z_1, \mathcal{A}z_1, \mathcal{A}z_2) &= \mathcal{S}\Big((0, \frac{y_1}{2}), (0, \frac{y_1}{2}), (0, \frac{y_2}{2})\Big) \\ &= \max\Big\{\frac{y_1}{2}, \frac{y_2}{2}\Big\} + i \max\Big\{\frac{y_1}{2}, \frac{y_2}{2}\Big\} = \max\Big\{\frac{y_1}{2}, \frac{y_2}{2}\Big\}(1+i) \\ &= \frac{1}{2}\max\{y_1, y_2\}(1+i) \precsim \frac{1}{2}\mathcal{S}(z_1, z_1, z_2) = q\,\mathcal{S}(z_1, z_1, z_2), \end{aligned}$$

Case 3. If $z_1 \in X_1$, $z_2 \in X_2$, then we have

$$\begin{aligned} \mathcal{S}(\mathcal{A}z_1, \mathcal{A}z_1, \mathcal{A}z_2) &= \mathcal{S}\Big((\frac{x_1}{2}, 0), (\frac{x_1}{2}, 0), (0, \frac{y_2}{2})\Big) = \Big(\frac{x_1}{2} + \frac{y_2}{2}\Big)(1+i) \\ &= \frac{1}{2}(x_1 + y_2)(1+i) \precsim \frac{1}{2}\mathcal{S}(z_1, z_1, z_2) = q\,\mathcal{S}(z_1, z_1, z_2), \end{aligned}$$

Case 4. If $z_2 \in X_1$, $z_1 \in X_2$, then we have

$$\begin{aligned} \mathcal{S}(\mathcal{A}z_1, \mathcal{A}z_1, \mathcal{A}z_2) &= \mathcal{S}\Big((0, \frac{y_1}{2}), (0, \frac{y_1}{2}), (\frac{x_2}{2}, 0)\Big) \\ &= \Big(\frac{x_2}{2} + \frac{y_1}{2}\Big)(1+i) = \frac{1}{2}(x_2 + y_1)(1+i) \\ &\precsim \frac{1}{2}\mathcal{S}(z_1, z_1, z_2) = q\,\mathcal{S}(z_1, z_1, z_2), \end{aligned}$$

where $q = \frac{1}{2}$. If we take $0 \le q < 1$, then all the conditions of Corollary 3.5 are satisfied. Hence by applying Corollary 3.5, $\mathcal{A}$ has a unique fixed point in X. Indeed, in this case $0 \in X$ is the unique fixed point.

§4. Conclusion

In this paper, we prove some common fixed point theorems for contractive type conditions involving rational expressions and using common $(E.A)$ property in the framework of complex-valued S-metric spaces. Also, we give an example in support of the result. The results presented in this paper extend, generalize and enrich several results from the current existing literature.

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