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# Some Generalized Inequalities for Functions of Bounded Variation Involving Weighted Area Balance Functions

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**Abstract**: In this study, we first prove an identity for the integrable functions involving weighted area balance function. Then, using this equality, some generalized inequalities for mappings of bounded variation are obtained. Moreover, some generalized inequalities for Lipschitzian functions are given. The results in this paper generalize the inequalities obtained in [10] and [19].

**Key Words**: Function of bounded variation, Riemann-Stieltjes integrals, area balance functions.

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## §1. Introduction

Let $P : \kappa_1 = \varkappa_0 < \varkappa_1 < \cdots < \varkappa_n = \kappa_2$ be any partition of $[\kappa_1, \kappa_2]$ and let $\Delta\omega(\varkappa_i) = \omega(\varkappa_{i+1}) - \omega(\varkappa_i)$. Then $\omega$ is said to be of bounded variation if the sum

$$\sum_{i=1}^{m} |\Delta\omega(\varkappa_i)|$$

is bounded for all such partitions [4].

Let $\omega$ be of bounded variation on $[\kappa_1, \kappa_2]$, and $\sum \Delta\omega\,(P)$ denote the sum $\sum_{i=1}^{n} |\Delta\omega(\varkappa_i)|$ corresponding to the partition $P$ of $[\kappa_1, \kappa_2]$. The number

$$\bigvee_{\kappa_1}^{\kappa_2}(\omega) := \sup\left\{\sum \Delta F\,(P) : P \in P([\kappa_1, \kappa_2])\right\},$$

is called the total variation of $\omega$ on $[\kappa_1, \kappa_2]$. Here $P([\kappa_1, \kappa_2])$ denotes the family of partitions of $[\kappa_1, \kappa_2]$ [4]. For a function of bounded variation $\omega : [\kappa_1, \kappa_2] \to \mathbb{C}$ we define the *cumulative variation function* $CVF : [\kappa_1, \kappa_2] \to [0, \infty)$ by

$$CVF(\xi) := \bigvee_{\kappa_1}^{\xi}(\omega)$$

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the total variation $\omega$ on the interval $[\kappa_1, \xi]$. It is known that the $CVF$ is monotonic nondecreasing on $[\kappa_1, \kappa_2]$ and is continuous in a point $c \in [\kappa_1, \kappa_2]$ if and only if the generating function $\omega$ is continuing in that point.

Integral inequalities for functions of bounded variation have potential applications in mathematical sciences. They have applications in numerical integration, probability and optimization theory, stochastic, statistics, information and integral operator theory. In the past, many authors have worked on some inequalities for functions of bounded variation. see for example ([1]-[12], [14]-[21]).

In [18], Dragomir give the following lemma which will be used frequently in our paper.

**Lemma** 1.1 *Let $F, \omega : [\kappa_1, \kappa_2] \to \mathbb{C}$. If $F$ is a continuous on $[\kappa_1, \kappa_2]$ and $\omega$ is of bounded variation on $[\kappa_1, \kappa_2]$, then*

$$\left| \int_{\kappa_1}^{\kappa_2} F(\xi) d\omega(\xi) \right| \leq \int_{\kappa_1}^{\kappa_2} |F(\xi)| \, d\left( \bigvee_{\kappa_1}^{\xi} (\omega) \right) \leq \max_{\xi \in [\kappa_1, \kappa_2]} |F(\xi)| \bigvee_{\kappa_1}^{\kappa_2} (\omega). \tag{1.1}$$

In [19], Dragomir introduce the following area balance function:

Let $F : [\kappa_1, \kappa_2] \to \mathbb{C}$ be Lebesgue integrable function. Then we define the function $AB_F(\kappa_1, \kappa_2, .) : [\kappa_1, \kappa_2] \to \mathbb{C}$ by

$$AB_F(\kappa_1, \kappa_2, \varkappa) = \frac{1}{2} \left[ \int_{\varkappa}^{\kappa_2} F(\xi) d\xi - \int_{\kappa_1}^{\varkappa} F(\xi) d\xi \right]. \tag{1.2}$$

Moreover, Dragomir proved the following inequalities involving area balance function.

**Theorem** 1.1 *Let $F : [\kappa_1, \kappa_2] \to \mathbb{C}$ be a function of bounded variation on $[\kappa_1, \kappa_2]$. Then*

$$\begin{aligned}
&\left| AB_F(\kappa_1, \kappa_2, \varkappa) - \left( \frac{\kappa_1 + \kappa_2}{2} - \varkappa \right) F(\varkappa) \right| \\
&\leq AB_{\bigvee_{\kappa_1}^{\cdot}(F)}(\kappa_1, \kappa_2, \varkappa) - \left( \frac{\kappa_1 + \kappa_2}{2} - \varkappa \right) \bigvee_{\kappa_1}^{\varkappa}(F) \\
&= \frac{1}{2} \left[ \int_{\kappa_1}^{\varkappa} \left( \bigvee_{\xi}^{\varkappa}(F) \right) d\xi + \int_{\varkappa}^{\kappa_2} \left( \bigvee_{\varkappa}^{\xi}(F) \right) d\xi \right] \\
&\leq \frac{1}{2} \left[ (\varkappa - \kappa_1) \bigvee_{\kappa_1}^{\varkappa}(F) + (\kappa_2 - \varkappa) \bigvee_{\varkappa}^{\kappa_2}(F) \right] \\
&\leq \frac{1}{2} \times \begin{cases} \left[ \frac{1}{2}(\kappa_2 - \kappa_1) + \frac{1}{2} \left| \varkappa - \frac{\kappa_1 + \kappa_2}{2} \right| \right] \bigvee_{\kappa_1}^{\kappa_2}(F) \\ \left[ \frac{1}{2} \bigvee_{\kappa_1}^{\kappa_2}(F) \xi + \frac{1}{2} \left| \bigvee_{\kappa_1}^{\varkappa}(F) - \bigvee_{\varkappa}^{\kappa_2}(F) \right| \right] (\kappa_2 - \kappa_1) \end{cases}
\end{aligned} \tag{1.3}$$

*for any $\varkappa \in [\kappa_1, \kappa_2]$.*

Moreover, Dragomir prove some inequalities for the area balance of absolutely continuous functions in [20]. Delevar and Dragomir give some weighted trapezoidal inequalities related to the area balance of a function in [13]. On the other hand, in [10] Budak and Pehlivan establish some generalizations of the results in [19]. In this paper we obtain some new generalized weighted inequalities by the weighted area balance functions.

## §2. Main Results

First we recall the following weighted area functions given in [10]:

Let $\varpi : [\kappa_1, \kappa_2] \to [0, \infty)$ be Lebesgue integrable and let $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a function of bounded variation on $[\kappa_1, \kappa_2]$. Then we define the weighted area balance function $WAB_F(\kappa_1, \kappa_2, .; \varpi) : [\kappa_1, \kappa_2] \to \mathbb{R}$ by

$$WAB_F(\kappa_1, \kappa_2, \varkappa; \varpi) := \frac{1}{2}\left[\int_{\varkappa}^{\kappa_2} F(\xi)\varpi(\xi)d\xi - \int_{\kappa_1}^{\varkappa} F(\xi)\varpi(\xi)d\xi\right].$$

Throughout the paper, we denote the weighted area balance function $AB_F(\kappa_1, \kappa_2, \varkappa; \varpi)$ by $WAB_F(\kappa_1, \kappa_2, \varkappa)$. First we prove the following generalized identity involving the weighted area balance function.

**Lemma** 2.1 *Let $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a function of bounded variation on $[\kappa_1, \kappa_2]$. Then we have the following identity*

$$\begin{aligned} & WAB_F(\kappa_1, \kappa_2, \varkappa) - \lambda AB_{\varpi}(\kappa_1, \kappa_2, \varkappa)F(\varkappa) \\ & -\frac{(1-\lambda)}{2}\left[\left(\int_{\varkappa}^{\kappa_2} \varpi(\eta)d\eta\right)F(\kappa_2) - \left(\int_{\kappa_1}^{\varkappa} \varpi(\eta)d\eta\right)F(\kappa_1)\right] \\ = & \frac{1}{2}\int_{\kappa_1}^{\kappa_2} K_{\varpi}^{\lambda}(\xi, \varkappa)dF(\xi) \end{aligned} \tag{2.1}$$

*for all $\varkappa \in [\kappa_1, \kappa_2]$ and $\lambda \in [0, 1]$ where $K_{\varpi}^{\lambda}(\xi, \varkappa) : [\kappa_1, \kappa_2] \times [\kappa_1, \kappa_2] \to \mathbb{R}$ is defined by*

$$K_{\varpi}^{\lambda}(\xi, \varkappa) = \begin{cases} \lambda \int_{\kappa_1}^{\xi} \varpi(\eta)d\eta + (1-\lambda)\int_{\xi}^{\varkappa} \varpi(\eta)d\eta, & \kappa_1 \le \xi < \varkappa \\ \lambda \int_{\xi}^{\kappa_2} \varpi(\eta)d\eta + (1-\lambda)\int_{\varkappa}^{\xi} \varpi(\eta)d\eta, & \varkappa \le \xi \le \kappa_2 \end{cases}$$

*and the integrals in the right hand side are taken in the Riemann-Stieltjes sense.*

*Proof* From the definition of $K_{\varpi}^{\lambda}(\xi,\varkappa)$, we have

$$\begin{aligned}
&\int_{\kappa_1}^{\kappa_2} K_{\varpi}^{\lambda}(\xi,\varkappa)dF(\xi) \\
&= \int_{\kappa_1}^{\varkappa}\left[\lambda\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta+(1-\lambda)\int_{\xi}^{\varkappa}\varpi(\eta)d\eta\right]dF(\xi) \\
&\quad+\int_{\varkappa}^{\kappa_2}\left[\lambda\int_{\xi}^{\kappa_2}\varpi(\eta)d\eta+(1-\lambda)\int_{\varkappa}^{\xi}\varpi(\eta)d\eta\right]dF(\xi) \\
&= \lambda\int_{\kappa_1}^{\varkappa}\left(\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta\right)dF(\xi)+(1-\lambda)\int_{\kappa_1}^{\varkappa}\left(\int_{\xi}^{\varkappa}\varpi(\eta)d\eta\right)dF(\xi) \\
&\quad+\lambda\int_{\varkappa}^{\kappa_2}\left(\int_{\xi}^{\kappa_2}\varpi(\eta)d\eta\right)dF(\xi)+(1-\lambda)\int_{\varkappa}^{\kappa_2}\left(\int_{\varkappa}^{\xi}\varpi(\eta)d\eta\right)dF(\xi). \qquad (2.2)
\end{aligned}$$

Using the integration by parts for Riemann-Stieltjes integrals, we get

$$\begin{aligned}
&\int_{\kappa_1}^{\varkappa}\left(\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta\right)dF(\xi) \\
&= \left(\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta\right)F(\xi)\Bigg|_{\kappa_1}^{\varkappa}-\int_{\kappa_1}^{\varkappa}F(\xi)d\left(\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta\right) \\
&= \left(\int_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right)F(\varkappa)-\int_{\kappa_1}^{\varkappa}F(\xi)\varpi(\xi)d\xi. \qquad (2.3)
\end{aligned}$$

Similarly, we have

$$\int_{\kappa_1}^{\varkappa}\left(\int_{\xi}^{\varkappa}\varpi(\eta)d\eta\right)dF(\xi)=\left(\int_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right)F(\kappa_1)-\int_{\kappa_1}^{\varkappa}F(\xi)\varpi(\xi)d\xi, \qquad (2.4)$$

$$\int_{\varkappa}^{\kappa_2}\left(\int_{\xi}^{\kappa_2}\varpi(\eta)d\eta\right)dF(\xi)=\left(\int_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta\right)F(\varkappa)+\int_{\varkappa}^{\kappa_2}F(\xi)\varpi(\xi)d\xi \qquad (2.5)$$

and

$$\int_{\varkappa}^{\kappa_2}\left(\int_{\varkappa}^{\xi}\varpi(\eta)d\eta\right)dF(\xi)=\left(\int_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta\right)F(\kappa_2)+\int_{\varkappa}^{\kappa_2}F(\xi)\varpi(\xi)d\xi. \qquad (2.6)$$

If we substitute the equalities (2.3) – (2.6) in (2.2), then we obtain

$$
\begin{aligned}
&\int_{\kappa_1}^{\kappa_2} K_{\varpi}^{\lambda}(\xi,\varkappa)dF(\xi) \\
&= \lambda\left(\int_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right)F(\varkappa) - \lambda\int_{\kappa_1}^{\varkappa}F(\xi)\varpi(\xi)d\xi \\
&\quad +(1-\lambda)\left(\int_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right)F(\kappa_1) - (1-\lambda)\int_{\kappa_1}^{\varkappa}F(\xi)\varpi(\xi)d\xi \\
&\quad -\lambda\left(\int_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta\right)F(\varkappa) + \lambda\int_{\varkappa}^{\kappa_2}F(\xi)\varpi(\xi)d\xi \\
&\quad -(1-\lambda)\left(\int_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta\right)F(\kappa_2) + (1-\lambda)\int_{\varkappa}^{\kappa_2}F(\xi)\varpi(\xi)d\xi \\
&= -\lambda f(\varkappa)\left(\int_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta - \int_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right) + \lambda\left(\int_{\varkappa}^{\kappa_2}F(\xi)\varpi(\xi)d\xi - \int_{\kappa_1}^{\varkappa}F(\xi)\varpi(\xi)d\xi\right) \\
&\quad +(1-\lambda)\left(\int_{\varkappa}^{\kappa_2}F(\xi)\varpi(\xi)d\xi - \int_{\kappa_1}^{\varkappa}F(\xi)\varpi(\xi)d\xi\right) \\
&\quad -(1-\lambda)\left[\left(\int_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta\right)F(\kappa_2) - \left(\int_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right)F(\kappa_1)\right] \\
&= 2WAB_F(\kappa_1,\kappa_2,\varkappa) - 2\lambda AB_{\varpi}(\kappa_1,\kappa_2,\varkappa)F(\varkappa) \\
&\quad -(1-\lambda)\left[\left(\int_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta\right)F(\kappa_2) - \left(\int_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right)F(\kappa_1)\right]
\end{aligned}
$$

which completes the proof. □

**Remark** 2.1 If we take $\lambda = 1$ and $\lambda = 0$ in Lemma 2.1, then Lemma 2.1 reduces to Lemma 2 and Lemma 3 in [10], respectively.

**Corollary** 2.1 *If we choose* $\varpi(\varkappa) = 1$ *for all* $\varkappa \in [\kappa_1, \kappa_2]$ *in Lemma* 2.1*, then we have the following identity*

$$
\begin{aligned}
&WAB_F(\kappa_1,\kappa_2,\varkappa) - \lambda\left(\frac{\kappa_1+\kappa_2}{2} - \varkappa\right)F(\varkappa) \\
&\quad -\frac{(1-\lambda)}{2}\left[\frac{\kappa_2 f(\kappa_2)+\kappa_1 f(\kappa_1)}{2} - \frac{F(\kappa_1)+F(\kappa_2)}{2}\varkappa\right] \\
&= \frac{1}{2}\int_{\kappa_1}^{\kappa_2}\left[\lambda p(\varkappa,\xi) + (1-\lambda)\left|\varkappa-\xi\right|\right]dF(\xi) \qquad (2.7)
\end{aligned}
$$

*for all* $\varkappa \in [\kappa_1, \kappa_2]$ *and* $\lambda \in [0,1]$ *where* $p(\xi,\varkappa) : [\kappa_1,\kappa_2] \times [\kappa_1,\kappa_2] \to \mathbb{R}$ *is defined by*

$$p(\xi,\varkappa) = \begin{cases} \xi - \kappa_1 & \kappa_1 \le \xi < \varkappa \\ \\ \kappa_2 - \xi & \varkappa \le \xi \le \kappa_2. \end{cases}$$

**Remark** 2.2 If we take $\lambda = 1$ and $\lambda = 0$ in Corollary 2.1, then the equality (2.7) reduces to the equalities (2.1) and (2.2) of Theorem 1 in [19], respectively.

**Theorem** 2.2 *If* $F : [\kappa_1,\kappa_2] \to \mathbb{R}$ *is a function of bounded variation on* $[\kappa_1,\kappa_2]$, *then we have*

$$\begin{aligned}
&\Bigg| WAB_F(\kappa_1,\kappa_2,\varkappa) - \lambda AB_{\varpi}(\kappa_1,\kappa_2,\varkappa)F(\varkappa) \\
&\quad - \frac{(1-\lambda)}{2}\left[\left(\int_{\varkappa}^{\kappa_2} \varpi(\eta)d\eta\right)F(\kappa_2) - \left(\int_{\kappa_1}^{\varkappa} \varpi(\eta)d\eta\right)F(\kappa_1)\right]\Bigg| \\
&\le \frac{1}{2}\lambda \int_{\kappa_1}^{\varkappa}\left(\bigvee_{\xi}^{\varkappa}(F)\right)\varpi(\xi)d\xi + \frac{1}{2}\lambda\int_{\varkappa}^{\kappa_2}\left(\bigvee_{\varkappa}^{\xi}(F)\right)\varpi(\xi)d\xi \\
&\quad + \frac{1}{2}(1-\lambda)\int_{\kappa_1}^{\varkappa}\varpi(\xi)\left(\bigvee_{\kappa_1}^{\xi}(F)\right)d\xi + \frac{1}{2}(1-\lambda)\int_{\varkappa}^{\kappa_2}\varpi(\xi)\left(\bigvee_{\xi}^{\kappa_2}(F)\right)d\xi \\
&\le \frac{1}{2}\left[\bigvee_{\kappa_1}^{\varkappa}(F)\int_{\kappa_1}^{\varkappa}\varpi(\xi)d\xi + \bigvee_{\varkappa}^{\kappa_2}(F)\int_{\varkappa}^{\kappa_2}\varpi(\xi)d\xi\right] \\
&\le \frac{1}{2}\begin{cases} \left[\frac{1}{2}\int_{\kappa_1}^{\kappa_2}\varpi(\xi)d\xi + \frac{1}{2}\left|\int_{\kappa_1}^{\varkappa}\varpi(\xi)d\xi - \int_{\varkappa}^{\kappa_2}\varpi(\xi)d\xi\right|\right]\bigvee_{\kappa_1}^{\kappa_2}(F), \\ \left[\frac{1}{2}\bigvee_{\kappa_1}^{\kappa_2}(F)\xi + \left|\bigvee_{\kappa_1}^{\varkappa}(F) - \bigvee_{\varkappa}^{\kappa_2}(F)\right|\right]\int_{\kappa_1}^{\kappa_2}\varpi(\xi)d\xi \end{cases}
\end{aligned} \tag{2.8}$$

*for all* $\varkappa \in [\kappa_1,\kappa_2]$ *and* $\lambda \in [0,1]$.

*Proof* Taking the modulus identity (2.1), we have

$$\begin{aligned}
&\Bigg| WAB_F(\kappa_1,\kappa_2,\varkappa) - \lambda AB_{\varpi}(\kappa_1,\kappa_2,\varkappa)F(\varkappa) \\
&\quad - \frac{(1-\lambda)}{2}\left[\left(\int_{\varkappa}^{\kappa_2} \varpi(\eta)d\eta\right)F(\kappa_2) - \left(\int_{\kappa_1}^{\varkappa} \varpi(\eta)d\eta\right)F(\kappa_1)\right]\Bigg| \\
&= \frac{1}{2}\left|\int_{\kappa_1}^{\kappa_2} K_{\varpi}^{\lambda}(\xi,\varkappa)dF(\xi)\right| \\
&= \frac{1}{2}\left|\int_{\kappa_1}^{\varkappa}\left[\lambda\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta + (1-\lambda)\int_{\xi}^{\varkappa}\varpi(\eta)d\eta\right]dF(\xi)\right.
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\varkappa}^{\kappa_2}\left[\lambda\int_{\xi}^{\kappa_2}\varpi(\eta)d\eta+(1-\lambda)\int_{\varkappa}^{\xi}\varpi(\eta)d\eta\right]dF(\xi)\Bigg|\\
&\le\frac{1}{2}\left|\int_{\kappa_1}^{\varkappa}\left[\lambda\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta+(1-\lambda)\int_{\xi}^{\varkappa}\varpi(\eta)d\eta\right]dF(\xi)\right|\\
&+\frac{1}{2}\left|\int_{\varkappa}^{\kappa_2}\left[\lambda\int_{\xi}^{\kappa_2}\varpi(\eta)d\eta+(1-\lambda)\int_{\varkappa}^{\xi}\varpi(\eta)d\eta\right]dF(\xi)\right|.
\end{aligned}
$$

Using Lemma 1.1, we get

$$
\begin{aligned}
&\Bigg|WAB_F(\kappa_1,\kappa_2,\varkappa)-\lambda AB_\varpi(\kappa_1,\kappa_2,\varkappa)F(\varkappa)\\
&\quad-\frac{(1-\lambda)}{2}\left[\left(\int_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta\right)F(\kappa_2)-\left(\int_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right)F(\kappa_1)\right]\Bigg|\\
&\le\frac{1}{2}\left[\int_{\kappa_1}^{\varkappa}\left|\lambda\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta+(1-\lambda)\int_{\xi}^{\varkappa}\varpi(\eta)d\eta\right|d\left(\bigvee_{\kappa_1}^{\xi}(F)\right)\right]\\
&\quad+\frac{1}{2}\left[\int_{\varkappa}^{\kappa_2}\left|\lambda\int_{\xi}^{\kappa_2}\varpi(\eta)d\eta+(1-\lambda)\int_{\varkappa}^{\xi}\varpi(\eta)d\eta\right|d\left(\bigvee_{\xi}^{\kappa_2}(F)\right)\right]\\
&=\frac{1}{2}\left[\int_{\kappa_1}^{\varkappa}\left(\lambda\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta+(1-\lambda)\int_{\xi}^{\varkappa}\varpi(\eta)d\eta\right)d\left(\bigvee_{\kappa_1}^{\xi}(F)\right)\right]\\
&\quad+\frac{1}{2}\left[\int_{\varkappa}^{\kappa_2}\left(\lambda\int_{\xi}^{\kappa_2}\varpi(\eta)d\eta+(1-\lambda)\int_{\varkappa}^{\xi}\varpi(\eta)d\eta\right)d\left(\bigvee_{\xi}^{\kappa_2}(F)\right)\right]\\
&=\frac{1}{2}\left[\lambda\int_{\kappa_1}^{\varkappa}\left(\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta\right)d\left(\bigvee_{\kappa_1}^{\xi}(F)\right)+(1-\lambda)\int_{\kappa_1}^{\varkappa}\left(\int_{\xi}^{\varkappa}\varpi(\eta)d\eta\right)d\left(\bigvee_{\kappa_1}^{\xi}(F)\right)\right]\\
&\quad+\frac{1}{2}\left[\lambda\int_{\varkappa}^{\kappa_2}\left(\int_{\xi}^{\kappa_2}\varpi(\eta)d\eta\right)d\left(\bigvee_{\varkappa}^{\kappa_2}(F)\right)+(1-\lambda)\int_{\varkappa}^{\kappa_2}\left(\int_{\varkappa}^{\xi}\varpi(\eta)d\eta\right)d\left(\bigvee_{\varkappa}^{\kappa_2}(F)\right)\right].
\end{aligned}
\tag{2.9}
$$

By utilizing the integration by parts for Riemann-Stieltjes integrals, we obtain

$$
\int_{\kappa_1}^{\varkappa}\left(\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta\right)d\left(\bigvee_{\kappa_1}^{\xi}(F)\right)=\left(\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta\right)\bigvee_{\kappa_1}^{\xi}(F)\Bigg|_{\kappa_1}^{\varkappa}-\int_{\kappa_1}^{\varkappa}\left(\bigvee_{\kappa_1}^{\xi}(F)\right)d\left(\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta\right)
$$

$$
\begin{aligned}
&= \left(\int_{\kappa_1}^{\varkappa} \varpi(\eta)d\eta\right) \bigvee_{\kappa_1}^{\varkappa}(F) - \int_{\kappa_1}^{\varkappa} \left(\bigvee_{\kappa_1}^{\xi}(F)\right) \varpi(\xi)d\xi \\
&= \int_{\kappa_1}^{\varkappa} \left[\bigvee_{\kappa_1}^{\varkappa}(F) - \bigvee_{\kappa_1}^{\xi}(F)\right] \varpi(\xi)d\xi = \int_{\kappa_1}^{\varkappa} \left(\bigvee_{\xi}^{\varkappa}(F)\right) \varpi(\xi)d\xi, \qquad (2.10)
\end{aligned}
$$

$$
\begin{aligned}
\int_{\kappa_1}^{\varkappa} \left(\int_{\xi}^{\varkappa} \varpi(\eta)d\eta\right) d\left(\bigvee_{\kappa_1}^{\xi}(F)\right) &= \left(\int_{\xi}^{\varkappa} (\varpi(\eta)d\eta\right) \bigvee_{\kappa_1}^{\xi}(F)\Bigg|_{\kappa_1}^{\varkappa} - \int_{\kappa_1}^{\varkappa} \left(\bigvee_{\kappa_1}^{\xi}(F)\right) d\left(\int_{\xi}^{\varkappa} \varpi(\eta)d\eta\right) \\
&= \left(\int_{\kappa_1}^{\varkappa} \varpi(\eta)d\eta\right) \bigvee_{\kappa_1}^{\varkappa}(F) + \int_{\kappa_1}^{\varkappa} \left(\bigvee_{\kappa_1}^{\xi}(F)\right) d\left(\int_{\kappa_1}^{\varkappa} \varpi(\eta)d\eta\right) \\
&= \left(\int_{\kappa_1}^{\varpi} \varpi(\eta)d\eta\right) \bigvee_{\kappa_1}^{\varkappa}(F) + \int_{\kappa_1}^{\varkappa} \left(\bigvee_{\kappa_1}^{\xi}(F)\right) \varpi(\xi)d\xi \\
&= \int_{\kappa_1}^{\varkappa} \varpi(\xi) \bigvee_{\kappa_1}^{\xi}(F)d\xi, \qquad (2.11)
\end{aligned}
$$

$$
\begin{aligned}
\int_{\varkappa}^{\kappa_2} \left(\int_{\xi}^{\kappa_2} \varpi(\eta)d\eta\right) d\left(\bigvee_{\xi}^{\kappa_2}(F)\right) &= \left(\int_{\xi}^{\kappa_2} \varpi(\eta)d\eta\right) \bigvee_{\xi}^{\kappa_2}(F)\Bigg|_{\varkappa}^{\kappa_2} - \int_{\varkappa}^{\kappa_2} \left(\bigvee_{\xi}^{\kappa_2}(F)\right) d\left(\int_{\xi}^{\kappa_2} \varpi(\eta)d\eta\right) \\
&= -\left(\int_{\varkappa}^{\kappa_2} \varpi(\eta)d\eta\right) \bigvee_{\varkappa}^{\kappa_2}(F) + \int_{\varkappa}^{\kappa_2} \left(\bigvee_{\xi}^{\kappa_2}(F)\right) \varpi(\xi)d\xi \\
&= \int_{\varkappa}^{\kappa_2} \left(\bigvee_{\xi}^{\kappa_2}(F) - \bigvee_{\varkappa}^{\kappa_2}(F)\right) \varpi(\xi)d\xi = \int_{\varkappa}^{\kappa_2} \left(\bigvee_{\varkappa}^{\xi}(F)\right) \varpi(\xi)d\xi \qquad (2.12)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{\varkappa}^{\kappa_2} \left(\int_{\varkappa}^{\xi} \varpi(\eta)d\eta\right) d\left(\bigvee_{\xi}^{\kappa_2}(F)\right) &= \left(\int_{\varkappa}^{\xi} \varpi(\eta)d\eta\right) \bigvee_{\xi}^{\kappa_2}(F)\Bigg|_{\varkappa}^{\kappa_2} - \int_{\varkappa}^{\kappa_2} \left(\bigvee_{\xi}^{\kappa_2}(F)\right) d\left(\int_{\varkappa}^{\xi} \varpi(\eta)d\eta\right) \\
&= \left(\int_{\varkappa}^{\kappa_2} \varpi(\eta)d\eta\right) \bigvee_{\varkappa}^{\kappa_2}(F) - \int_{\varkappa}^{\kappa_2} \varpi(\xi) \left(\bigvee_{\xi}^{\kappa_2}(F)\right) d\xi \\
&= \int_{\varkappa}^{\kappa_2} \varpi(\xi) \left(\bigvee_{\varkappa}^{\kappa_2}(F) - \bigvee_{\xi}^{\kappa_2}(F)\right) d\xi = \int_{\varkappa}^{\kappa_2} \varpi(\xi) \bigvee_{\xi}^{\kappa_2}(F)d\xi. \qquad (2.13)
\end{aligned}
$$

By substituting the equalities (2.10)-(2.13) in (2.9), we establish

$$\left|WAB_F(\kappa_1,\kappa_2,\varkappa)-\lambda AB_F(\kappa_1,\kappa_2,\varkappa)-\frac{(1-\lambda)}{2}\left[\left(\int_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta\right)F(\kappa_2)-\left(\int_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right)F(\kappa_1)\right]\right|$$
$$\leq\frac{1}{2}\left[\lambda\int_{\kappa_1}^{\varkappa}\left(\bigvee_{\xi}^{\varkappa}(F)\right)\varpi(\xi)d\xi+(1-\lambda)\int_{\kappa_1}^{\varkappa}\varpi(\xi)\left(\bigvee_{\kappa_1}^{\xi}(F)\right)d\xi\right]$$
$$+\frac{1}{2}\left[\lambda\int_{\varkappa}^{\kappa_2}\left(\bigvee_{\varkappa}^{\xi}(F)\right)\varpi(\xi)d\xi+(1-\lambda)\int_{\varkappa}^{\kappa_2}\varpi(\xi)\left(\bigvee_{\xi}^{\kappa_2}(F)\right)d\xi\right]$$

which gives first inequality in (2.8). By using the facts that

$$\bigvee_{\xi}^{\varkappa}(F)\leq\bigvee_{\kappa_1}^{\varkappa}(F),\quad\bigvee_{\kappa_1}^{\xi}(F)\leq\bigvee_{\kappa_1}^{\varkappa}(F),\quad\xi\in[\kappa_1,\varkappa],$$
$$\bigvee_{\varkappa}^{\xi}(F)\leq\bigvee_{\varkappa}^{\kappa_2}(F),\quad\bigvee_{\xi}^{\kappa_2}(F)\leq\bigvee_{\varkappa}^{\kappa_2}(F),\quad\xi\in[\varkappa,\kappa_2]$$

we obtain

$$\frac{1}{2}\left[\lambda\int_{\kappa_1}^{\varkappa}\left(\bigvee_{\xi}^{\varkappa}(F)\right)\varpi(\xi)d\xi+(1-\lambda)\int_{\kappa_1}^{\varkappa}\varpi(\xi)\left(\bigvee_{\kappa_1}^{\xi}(F)\right)d\xi\right]$$
$$+\frac{1}{2}\left[\lambda\int_{\varkappa}^{\kappa_2}\left(\bigvee_{\varkappa}^{\xi}(F)\right)\varpi(\xi)d\xi+(1-\lambda)\int_{\varkappa}^{\kappa_2}\varpi(\xi)\left(\bigvee_{\varkappa}^{\kappa_2}(F)\right)d\xi\right]$$
$$\leq\frac{1}{2}\left[\lambda\int_{\kappa_1}^{\varkappa}\left(\bigvee_{\kappa_1}^{\varkappa}(F)\right)\varpi(\xi)d\xi+(1-\lambda)\int_{\kappa_1}^{\varkappa}\varpi(\xi)\bigvee_{\kappa_1}^{\varkappa}(F)d\xi\right]$$
$$+\frac{1}{2}\left[\lambda\int_{\varkappa}^{\kappa_2}\left(\bigvee_{\varkappa}^{\kappa_2}(F)\right)\varpi(\xi)d\xi+(1-\lambda)\int_{\varkappa}^{\kappa_2}\varpi(\xi)\bigvee_{\varkappa}^{\kappa_2}(F)d\xi\right]$$
$$=\frac{1}{2}\left[\lambda\bigvee_{\kappa_1}^{\varkappa}(F)\int_{\kappa_1}^{\varkappa}\varpi(\xi)d\xi+(1-\lambda)\bigvee_{\kappa_1}^{\varkappa}(F)\int_{\varkappa}^{\kappa_2}\varpi(\xi)d\xi\right]$$
$$+\frac{1}{2}\left[\lambda\bigvee_{\varkappa}^{\kappa_2}(F)\int_{\varkappa}^{\kappa_2}\varpi(\xi)d\xi+(1-\lambda)\bigvee_{\varkappa}^{\kappa_2}(F)\int_{\varkappa}^{\kappa_2}\varpi(\xi)d\xi\right]$$
$$=\bigvee_{\kappa_1}^{\varkappa}(F)\int_{\kappa_1}^{\varkappa}\varpi(\xi)d\xi+\bigvee_{\varkappa}^{\kappa_2}(F)\int_{\varkappa}^{\kappa_2}\varpi(\xi)d\xi.$$

This proves the second inequality in (2.8).

Notice that the last inequality in (2.8) is obvious from the fact that

$$\max\{p,q\}=\frac{p+q}{2}+\frac{1}{2}|p-q|$$

for $p,q\in\mathbb{R}$. □

**Remark** 2.3 If we take $\lambda=1$ and $\lambda=0$ in Theorem 2.2, then we have the following inequalities, respectively

$$\begin{aligned}
&|AB_F(\kappa_1,\kappa_2,\varkappa)-\lambda AB_\varpi(\kappa_1,\kappa_2,\varkappa)F(\varkappa)|\\
&\le\frac{1}{2}\int_{\kappa_1}^{\varkappa}\left(\bigvee_{\xi}^{\varkappa}(F)\right)\varpi(\xi)d\xi+\frac{1}{2}\int_{\varkappa}^{\kappa_2}\left(\bigvee_{\varkappa}^{\xi}(F)\right)\varpi(\xi)d\xi\\
&\le\frac{1}{2}\left[\bigvee_{\kappa_1}^{\varkappa}(F)\int_{\kappa_1}^{\varkappa}\varpi(\xi)d\xi+\bigvee_{\varkappa}^{\kappa_2}(F)\int_{\varkappa}^{\kappa_2}\varpi(\xi)d\xi\right]\\
&\le\frac{1}{2}\begin{cases}\left[\frac{1}{2}\int_{\kappa_1}^{\kappa_2}\varpi(\xi)d\xi+\frac{1}{2}\left|\int_{\kappa_1}^{\varkappa}\varpi(\xi)d\xi-\int_{\varkappa}^{\kappa_2}\varpi(\xi)d\xi\right|\right]\bigvee_{\kappa_1}^{\kappa_2}(F),\\[2ex]
\left[\frac{1}{2}\bigvee_{\kappa_1}^{\kappa_2}(F)\xi+\left|\bigvee_{\kappa_1}^{\varkappa}(F)-\bigvee_{\varkappa}^{\kappa_2}(F)\right|\right]\int_{\kappa_1}^{\kappa_2}\varpi(\xi)d\xi\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\left|AB_F(\kappa_1,\kappa_2,\varkappa)-\frac{1}{2}\left[\left(\int_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta\right)F(\kappa_2)-\left(\int_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right)F(\kappa_1)\right]\right|\\
&\le\frac{1}{2}\int_{\kappa_1}^{\varkappa}\varpi(\xi)\left(\bigvee_{\kappa_1}^{\xi}(F)\right)d\xi+\frac{1}{2}\int_{\varkappa}^{\kappa_2}\varpi(\xi)\left(\bigvee_{\xi}^{\kappa_2}(F)\right)d\xi\\
&\le\frac{1}{2}\left[\bigvee_{\kappa_1}^{\varkappa}(F)\int_{\kappa_1}^{\varkappa}\varpi(\xi)d\xi+\bigvee_{\varkappa}^{\kappa_2}(F)\int_{\varkappa}^{\kappa_2}\varpi(\xi)d\xi\right]\\
&\le\frac{1}{2}\begin{cases}\left[\frac{1}{2}\int_{\kappa_1}^{\kappa_2}\varpi(\xi)d\xi+\frac{1}{2}\left|\int_{\kappa_1}^{\varkappa}\varpi(\xi)d\xi-\int_{\varkappa}^{\kappa_2}\varpi(\xi)d\xi\right|\right]\bigvee_{\kappa_1}^{\kappa_2}(F),\\[2ex]
\left[\frac{1}{2}\bigvee_{\kappa_1}^{\kappa_2}(F)\xi+\left|\bigvee_{\kappa_1}^{\varkappa}(F)-\bigvee_{\varkappa}^{\kappa_2}(F)\right|\right]\int_{\kappa_1}^{\kappa_2}\varpi(\xi)d\xi\end{cases}
\end{aligned}$$

which are proved by Budak and Pehlivan in [10].

**Corollary** 2.2 *If we choose $\varpi(\varkappa)=1$ for all $\varkappa\in[\kappa_1,\kappa_2]$ in Theorem 2.2, then we have the* *following inequality*

$$\begin{aligned}
&\Bigg|WAB_F(\kappa_1,\kappa_2,\varkappa)-\lambda\left(\frac{\kappa_1+\kappa_2}{2}-\varkappa\right)F(\varkappa)\\
&\quad-\frac{(1-\lambda)}{2}\left[\frac{\kappa_2 f(\kappa_2)+\kappa_1 f(\kappa_1)}{2}-\frac{F(\kappa_1)+F(\kappa_2)}{2}\varkappa\right]\Bigg|\\
&\le\frac{\lambda}{2}\left[\int_{\kappa_1}^{\varkappa}\left(\bigvee_{\xi}^{\varkappa}(F)\right)d\xi+\int_{\varkappa}^{\kappa_2}\left(\bigvee_{\varkappa}^{\xi}(F)\right)d\xi\right]\\
&\quad+\frac{(1-\lambda)}{2}\left[\int_{\kappa_1}^{\varkappa}\left(\bigvee_{\kappa_1}^{\xi}(F)\right)d\xi+\int_{\varkappa}^{\kappa_2}\left(\bigvee_{\xi}^{\kappa_2}(F)\right)d\xi\right]\\
&\le\frac{1}{2}\left[\bigvee_{\kappa_1}^{\varkappa}(F)\int_{\kappa_1}^{\varkappa}\varpi(\xi)d\xi+\bigvee_{\varkappa}^{\kappa_2}(F)\int_{\varkappa}^{\kappa_2}\varpi(\xi)d\xi\right]\\
&\le\frac{1}{2}\begin{cases}\left[\frac{1}{2}(\kappa_2-\kappa_1)+\left|\varkappa-\frac{\kappa_1+\kappa_2}{2}\right|\right]\bigvee_{\kappa_1}^{\kappa_2}(F),\\[2ex]
\left[\frac{1}{2}\bigvee_{\kappa_1}^{\kappa_2}(F)\xi+\left|\bigvee_{\kappa_1}^{\varkappa}(F)-\bigvee_{\varkappa}^{\kappa_2}(F)\right|\right](\kappa_2-\kappa_1)\end{cases}
\end{aligned}$$

*for all* $\varkappa\in[\kappa_1,\kappa_2]$ *and* $\lambda\in[0,1]$.

**Remark** 2.4 If $\lambda=1$ in Corollary 2.2, the inequalities (2.7) reduce to the inequalities (1.3).

**Remark** 2.5 If we take $\lambda=0$ in Corollary 2.2, then the inequality (2.7) reduces to the equalities (3.9) of Theorem 3.4 in [19].

## §3. Inequalities for Lipschitzian Functions

In this section, we obtain some inequalities for Lipschitzian functions. First we give the following important fact:

If $\omega$ is Lipschitzian with the constant $L>0$; i.e.

$$|\omega(\xi)-\omega(\eta)|\le L|\xi-\eta| \text{ for any } \xi,\eta\in(\kappa_1,\kappa_2)$$

then, it is well known that for any Riemann integrable function $g:[\kappa_1,\kappa_2]\to\mathbb{R}$ the Riemann-Stieltjes integral $\int_{\kappa_1}^{\kappa_2} g(\xi)d\omega(\xi)$ exist and

$$\left|\int_{\kappa_1}^{\kappa_2} g(\xi)d\omega(\xi)\right|\le L\int_{\kappa_1}^{\kappa_2}|g(\xi)|\,d\xi. \tag{3.1}$$

**Theorem** 3.1 *If $\varpi$ is bounded on $[\kappa_1,\kappa_2]$, i.e.*

$$\|\varpi\|_\infty = \sup_{\xi\in[\kappa_1,\kappa_2]} |\varpi(\xi)| < \infty,$$

*and if* $: [\kappa_1,\kappa_2] \to \mathbb{R}$ *is a Lipschitzian with the constant $L>0$ on $[\kappa_1,\kappa_2]$, then we have*

$$\begin{aligned}
&\Bigg| WAB_F(\kappa_1,\kappa_2,\varkappa) - \lambda AB_\varpi(\kappa_1,\kappa_2,\varkappa)F(\varkappa) \\
&\quad - \frac{(1-\lambda)}{2}\left[\left(\int_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta\right)F(\kappa_2) - \left(\int_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right)F(\kappa_1)\right]\Bigg| \\
&\leq \frac{L\|\varpi\|_\infty}{2}\left[\frac{(\kappa_2-\kappa_1)^2}{4} + \left(\varkappa - \frac{\kappa_1+\kappa_2}{2}\right)^2\right] \qquad (3.2)
\end{aligned}$$

*for all $\varkappa \in [\kappa_1,\kappa_2]$.*

*Proof* By taking modulus in the inequality (2.1) and by using the inequality (3.1), we obtain

$$\begin{aligned}
&\Bigg| WAB_F(\kappa_1,\kappa_2,\varkappa) - \lambda AB_\varpi(\kappa_1,\kappa_2,\varkappa)F(\varkappa) \\
&\quad - \frac{(1-\lambda)}{2}\left[\left(\int_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta\right)F(\kappa_2) - \left(\int_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right)F(\kappa_1)\right]\Bigg| \\
&\leq \frac{1}{2}\left[\lambda\left|\int_{\kappa_1}^{\varkappa}\left(\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta\right)dF(\xi)\right| + (1-\lambda)\left|\int_{\kappa_1}^{\varkappa}\left(\int_{\xi}^{\varkappa}\varpi(\eta)d\eta\right)dF(\xi)\right|\right] \\
&\quad + \frac{1}{2}\left[\lambda\left|\int_{\varkappa}^{\kappa_2}\left(\int_{\xi}^{\kappa_2}\varpi(\eta)d\eta\right)dF(\xi)\right| + (1-\lambda)\left|\int_{\varkappa}^{\kappa_2}\left(\int_{\varkappa}^{\xi}\varpi(\eta)d\eta\right)dF(\xi)\right|\right] \\
&\leq \frac{L}{2}\left[\lambda\int_{\kappa_1}^{\varkappa}\left|\int_{\kappa_1}^{\xi}\varpi(\eta)d\eta\right|d\xi + (1-\lambda)\int_{\kappa_1}^{\varkappa}\left|\int_{\xi}^{\varkappa}\varpi(\eta)d\eta\right|d\xi\right] \\
&\quad + \frac{L}{2}\left[\lambda\int_{\varkappa}^{\kappa_2}\left|\int_{\xi}^{\kappa_2}\varpi(\eta)d\eta\right|d\xi + (1-\lambda)\int_{\varkappa}^{\kappa_2}\left|\int_{\varkappa}^{\xi}\varpi(\eta)d\eta\right|d\xi\right].
\end{aligned}$$

Since $\varpi$ is bounded on $[\kappa_1,\kappa_2]$, we have

$$\begin{aligned}
&\Bigg| WAB_F(\kappa_1,\kappa_2,\varkappa) - \lambda AB_\varpi(\kappa_1,\kappa_2,\varkappa)F(\varkappa) \\
&\quad - \frac{(1-\lambda)}{2}\left[\left(\int_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta\right)F(\kappa_2) - \left(\int_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right)F(\kappa_1)\right]\Bigg|
\end{aligned}$$

$$\leq \frac{L}{2}\left[\lambda\left\|\varpi\right\|_{\infty,[\kappa_1,\varkappa]}\int\limits_{\kappa_1}^{\varkappa}\left(\xi-\kappa_1\right)d\xi+(1-\lambda)\left\|\varpi\right\|_{\infty,[\kappa_1,\varkappa]}\int\limits_{\kappa_1}^{\varkappa}\left(\varkappa-\xi\right)d\xi\right]$$
$$+\frac{L}{2}\left[\lambda\left\|\varpi\right\|_{\infty,[\varkappa,\kappa_2]}\int\limits_{\varkappa}^{\kappa_2}\left(\kappa_2-\xi\right)d\xi+(1-\lambda)\left\|\varpi\right\|_{\infty,[\varkappa,\kappa_2]}\int\limits_{\varkappa}^{\kappa_2}\left(\xi-\varkappa\right)d\xi\right]$$
$$=\frac{L}{4}\left[\left\|\varpi\right\|_{\infty,[\kappa_1,\varkappa]}\left(\varkappa-\kappa_1\right)^2+\left\|\varpi\right\|_{\infty,[\varkappa,\kappa_2]}\left(\kappa_2-\varkappa\right)^2\right].$$

Using the facts that

$$\left\|\varpi\right\|_{\infty,[\kappa_1,\varkappa]}\leq\left\|\varpi\right\|_{\infty}\quad\text{and}\quad\left\|\varpi\right\|_{\infty,[\varkappa,\kappa_2]}\leq\left\|\varpi\right\|_{\infty}$$

for all $\varkappa\in[\kappa_1,\kappa_2]$, we obtain

$$\Bigg|WAB_F(\kappa_1,\kappa_2,\varkappa)-\lambda AB_\varpi(\kappa_1,\kappa_2,\varkappa)F(\varkappa)-\frac{(1-\lambda)}{2}\left[\left(\int\limits_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta\right)F(\kappa_2)-\left(\int\limits_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right)F(\kappa_1)\right]\Bigg|$$
$$=\frac{L\left\|\varpi\right\|_{\infty}}{4}\left[\left(\varkappa-\kappa_1\right)^2+\left(\kappa_2-\varkappa\right)^2\right]$$
$$=\frac{L\left\|\varpi\right\|_{\infty}}{2}\left[\frac{(\kappa_2-\kappa_1)^2}{4}+\left(\varkappa-\frac{\kappa_1+\kappa_2}{2}\right)^2\right]$$

which completes the proof. □

**Remark** 3.1 If $\lambda=1$ or $\lambda=0$ in Theorem 3.1, we have respectively the following inequalities,

$$\left|AB_F(\kappa_1,\kappa_2,\varkappa)-\lambda AB_\varpi(\kappa_1,\kappa_2,\varkappa)F(\varkappa)\right|$$
$$\leq\frac{L\left\|\varpi\right\|_{\infty}}{2}\left[\frac{(\kappa_2-\kappa_1)^2}{4}+\left(\varkappa-\frac{\kappa_1+\kappa_2}{2}\right)^2\right]$$

and

$$\left|AB_F(\kappa_1,\kappa_2,\varkappa)-\frac{1}{2}\left[\left(\int\limits_{\varkappa}^{\kappa_2}\varpi(\eta)d\eta\right)F(\kappa_2)-\left(\int\limits_{\kappa_1}^{\varkappa}\varpi(\eta)d\eta\right)F(\kappa_1)\right]\right|$$
$$\leq\frac{L\left\|\varpi\right\|_{\infty}}{2}\left[\frac{(\kappa_2-\kappa_1)^2}{4}+\left(\varkappa-\frac{\kappa_1+\kappa_2}{2}\right)^2\right]$$

which are proved by Budak and Pehlivan in [10].

**Corollary** 3.1 *Under assumptions of Theorem* 3.1 *with* $\varpi(\varkappa)=1$ *for all* $\varkappa\in[\kappa_1,\kappa_2]$, *we have* *the following inequality*

$$\begin{aligned}&\left|WAB_F(\kappa_1,\kappa_2,\varkappa)-\lambda\left(\frac{\kappa_1+\kappa_2}{2}-\varkappa\right)F(\varkappa)\right.\\&\left.-\frac{(1-\lambda)}{2}\left[\frac{\kappa_2F(\kappa_2)+\kappa_1F(\kappa_1)}{2}-\frac{F(\kappa_1)+F(\kappa_2)}{2}\varkappa\right]\right|\\&\leq\frac{L}{2}\left[\frac{(\kappa_2-\kappa_1)^2}{4}+\left(\varkappa-\frac{\kappa_1+\kappa_2}{2}\right)^2\right].\end{aligned}\tag{3.3}$$

**Remark** 3.2 Ifc$\lambda = 1$ or $\lambda = 0$ in Corollary 3.3, then the inequality (3.3) reduces to the equalities (4.2) of Theorem 4.1 and 4.5 of Theorem 4.2 in [19], respectively.

**Corollary** 3.2 *Let* $\varkappa = \frac{\kappa_1+\kappa_2}{2}$ *and let* $\varpi : [\kappa_1, \kappa_2] \to [0,\infty)$ *be symmetric about* $\varkappa = \frac{\kappa_1+\kappa_2}{2}$ *(i.e.* $\varpi(\varkappa) = \varpi(\kappa_1+\kappa_2-\varkappa)$*) in Theorem* 3.1*. Then we have*

$$\left|WAB_F\left(\kappa_1,\kappa_2,\frac{\kappa_1+\kappa_2}{2}\right)-(1-\lambda)\frac{F(\kappa_2)-F(\kappa_1)}{4}\int_{\kappa_1}^{\kappa_2}\varpi(\eta)d\eta\right|\leq\frac{L}{8}(\kappa_2-\kappa_1)^2\|\varpi\|_\infty.$$

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