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Some New Inequalities for N-Times Differentiable Strongly Godunova-Levin Functions

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Abstract: In this manuscript, by using an integral identity together with the Hölder integral inequality and Hölder-İşcan integral inequality we establish several new inequalities for n-times differentiable strongly Godunova-Levin functions. In addition, the results obtained in this article coincide with those obtained previously in special cases.

Key Words: Convex function, Godunova-Levin function, Hölder integral inequality.

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§1. Introduction

For some inequalities, generalizations and applications about convexity theory and inequalities (see [6, 7, 8, 10]). Recently, in the literature there are so many studies about n-times differentiable functions on several kinds of convexities. In references [3, 4, 5, 15, 19], readers can find some results about this study. Many papers have been written by a number of mathematicians concerning inequalities for different classes of convex and Godunova-Levin functions see for instance the recent papers [13, 14, 16, 17, 20] and the references within these papers. Strongly convex functions play an important role in optimization theory, mathematical economics and some other branches of science. Since strongly convexity is a strengthening of the notion of convexity, some properties of strongly convex functions are just "stronger versions" of known properties of convex functions.

Definition 1.1 *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality*

$$f\left(tx + (1-t)y\right) \leq tf\left(x\right) + (1-t)\,f\left(y\right)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$. This definition is well known in the literature.

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Definition 1.2 *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be Godunova-Levin function, if*

$$f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}$$

where $\forall x, y \in I,\ t \in (0,1)$.

Definition 1.3([18]) *Let $I \subset \mathbb{R}$ be an interval and c be a positive number. A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called strongly convex with modulus c if*

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) - ct(1-t)(b-a)^2$$

for all $a, b \in I$ and $t \in [0,1]$.

If a function $f : I \to \mathbb{R}$ is strongly convex with modulus c, then

$$f\left(\frac{a+b}{2}\right) + \frac{c}{12}(b-a)^2 \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} - \frac{c}{6}(b-a)^2$$

for all $a, b \in I,\ a < b$. In this definition, if we take $c = 0$, we get the definition of convexity in the classical sense.

Definition 1.4 *Let $h : J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : I \to \mathbb{R}$ is an h-convex function, or that f belongs to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I$, $\alpha \in (0,1)$ we have*

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha)f(x) + h(1-\alpha)f(y).$$

If this inequality is reversed, then f is said to be h-concave, i.e. $f \in SV(h, I)$. It is clear that, if we choose $h(\alpha) = \alpha$ and $h(\alpha) = 1$, then the h-convexity reduces to convexity and definition of P-function, respectively.

Readers can look at [2, 11] for studies on h-convexity.

Definition 1.5([1]) *Let $(X, \|.\|)$ be a real normed space, D stands for a convex subset of X, $h : (0,1) \to (0,\infty)$ is a given function and c is a positive constant. Then we say that a function $f : D \to \mathbb{R}$ is strongly h-convex with module c if*

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y) - ct(1-t)\|x-y\|^2 \tag{1.1}$$

for all $x, y \in D$ and $t \in (0,1)$. In particular, if f satisfies (1.1) with $h(t) = t$, $h(t) = t^s$ $(s \in (0,1))$, $h(t) = \frac{1}{t}$, and $h(t) = 1$, then f is said to be strongly convex, strongly s-convex, strongly Godunova-Levin functions and strongly P-function, respectively. The notion of h-convex function corresponds to the case $c = 0$.

Theorem 1.1(Hölder-İşcan integral inequality, [9]) *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are* *real functions defined on interval $[a,b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a,b]$, then*

$$\begin{aligned}\int_a^b |f(x)g(x)|\,dx \quad &\leq \quad \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (b-x)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\int_a^b (x-a)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}. \qquad (1.2)\end{aligned}$$

Theorem 1.2 *Let $h:(0,1)\to(0,\infty)$ be a given function. If a function $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is Lebesgue integrable and strongly h-convex with module $c>0$, then*

$$\begin{aligned}\frac{1}{2h\left(\frac{1}{2}\right)}\left[f\left(\frac{a+b}{2}\right)+\frac{c}{12}(b-a)^2\right] \quad &\leq \quad \frac{1}{b-a}\int_a^b f(x)dx \\ &\leq \quad (f(a)+f(b))\int_0^1 h(t)dt-\frac{c}{6}(b-a)^2\end{aligned}$$

for all $a,b\in I$, $a<b$.

In [1], the authors gave the following definition.

Throughout this paper we will use the following notations and conventions. Let $J=[0,\infty)\subset\mathbb{R}=(-\infty,+\infty)$, and $a,b\in J$ with $0<a<b$ and

$$\begin{aligned}A(a,b) \quad &= \quad \frac{a+b}{2}, \\ L_p(a,b) \quad &= \quad \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}},\ a\neq b,\ p\in\mathbb{R},\ \ p\neq -1,0\end{aligned}$$

be the arithmetic, geometric, identic, harmonic, logarithmic, generalized logarithmic mean for $a,b>0$ respectively.

For we obtain the main results we will use the following Lemma [15].

Lemma 1.1 *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be n-times differentiable mapping on I° for $n\in\mathbb{N}$ and $f^{(n)}\in L[a,b]$, where $a,b\in I^{\circ}$ with $a<b$, we have the identity*

$$\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)\,dx=\frac{(-1)^{n+1}}{n!}\int_a^b x^n f^{(n)}(x)\,dx.$$

In [13], the authors proved the following theorems.

Theorem 1.3 ([13]) *For $\forall n\in\mathbb{N}$, let $f:I\subset(0,\infty)\to\mathbb{R}$ be n-times differentiable function on I° and $a,b\in I^{\circ}$ with $a<b$. If $\left|f^{(n)}\right|^q$ for $q>1$ is Godunova-Levin function on $[a,b]$, then the* *following inequality holds:*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|$$
$$\leq \frac{1}{n!}(b-a)^{\frac{3}{q}}C^{\frac{1}{p}}(a,b,n,p)A^{\frac{1}{q}}\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right),$$

where $\frac{1}{p}+\frac{1}{q}=1,\ 1<p<2$ *and* $C(a,b,n,p)=\int_a^b\frac{x^{np}}{(x-a)^{p-1}(b-x)^{p-1}}dx.$

Theorem 1.4([13]) *For* $n\in\mathbb{N}$*; let* $f:(0,\infty)\subset\mathbb{R}\to\mathbb{R}$ *be n-times differentiable function and* $0\leq a<b$. *If* $\left|f^{(n)}\right|^q\in L[a,b]$ *and* $\left|f^{(n)}\right|^q$ *for* $q>1$ *is Godunova-Levin function on* $[a,b]$, *then the following inequality*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|\leq\frac{1}{n!}(b-a)^{\frac{2}{q}}D^{\frac{1}{p}}(a,b,n,p)$$
$$\times\left[\left|f^{(n)}(b)\right|^q\left\{bL_n^n(a,b)-L_{n+1}^{n+1}(a,b)\right\}+\left|f^{(n)}(a)\right|^q\left\{L_{n+1}^{n+1}(a,b)-aL_n^n(a,b)\right\}\right]^{\frac{1}{q}},$$

holds, where $\frac{1}{p}+\frac{1}{q}=1,\ 1<p<2$ *and* $D(a,b,n,p)=\int_a^b\frac{x^{n}}{(x-a)^{p-1}(b-x)^{p-1}}dx.$

§2. Main Results

Theorem 2.1 *For* $\forall n\in\mathbb{N}$, *let* $f:I\subset(0,\infty)\to\mathbb{R}$ *be n-times differentiable function on* I° *and* $a,b\in I^\circ$ *with* $a<b$. *If* $\left|f^{(n)}\right|^q$ *for* $q>1$ *is a strongly Godunova-Levin function with modulus c on* $[a,b]$, *then the following inequalities*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|$$
$$\leq \frac{1}{n!}(b-a)^{\frac{3}{q}}C^{\frac{1}{p}}(a,b,n,p)\left[A\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right)-c\frac{(b-a)^2}{30}\right]^{\frac{1}{q}},$$

holds, where $\frac{1}{p}+\frac{1}{q}=1,\ 1<p<2$ *and* $C(a,b,n,p)=\int_a^b\frac{x^{np}}{(x-a)^{p-1}(b-x)^{p-1}}dx.$

Proof Firstly, let $x\in(a,b)$. Then, we can write the following inequalities

$$\left|f^{(n)}(x)\right|^q=\left|f^{(n)}\left(\frac{x-a}{b-a}b+\frac{b-x}{b-a}a\right)\right|^q\leq\frac{\left|f^{(n)}(b)\right|^q}{\frac{x-a}{b-a}}+\frac{\left|f^{(n)}(a)\right|^q}{\frac{b-x}{b-a}}-c\frac{x-a}{b-a}\frac{b-x}{b-a}(b-a)^2,$$

$$\left|f^{(n)}(x)\right|^q\leq\frac{b-a}{x-a}\left|f^{(n)}(b)\right|^q+\frac{b-a}{b-x}\left|f^{(n)}(a)\right|^q-c\,(x-a)\,(b-x)$$

$$(x-a)(b-x)\left|f^{(n)}(x)\right|^q \leq (b-a)(b-x)\left|f^{(n)}(b)\right|^q+(b-a)(x-a)\left|f^{(n)}(a)\right|^q -c\,(x-a)^2\,(b-x)^2. \quad (2.1)$$

The last inequality is also valid in case of $x \in [a,b]$. If $\left|f^{(n)}\right|^q$ for $q>1$ is a strongly Godunova-Levin function on the interval $[a,b]$, using Lemma 1.1, the Hölder integral inequality and the inequality (2.1) we have

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right| \leq \frac{1}{n!}\int_a^b x^n\left|f^{(n)}(x)\right|dx$$

$$\begin{aligned}
&\leq \frac{1}{n!}\left(\int_a^b \frac{x^{np}}{(x-a)^{p-1}(b-x)^{p-1}}dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)(b-x)\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}\\
&\leq \frac{1}{n!}C^{\frac{1}{p}}(a,b,n,p)\left(\int_a^b\left\{\begin{array}{c}(b-a)(b-x)\left|f^{(n)}(b)\right|^q+(b-a)(x-a)\left|f^{(n)}(a)\right|^q\\ -c\,(x-a)^2(b-x)^2\end{array}\right\}dx\right)^{\frac{1}{q}}\\
&= \frac{1}{n!}C^{\frac{1}{p}}(a,b,n,p)\left(\begin{array}{c}(b-a)\left|f^{(n)}(b)\right|^q\int_a^b(b-x)dx+(b-a)\left|f^{(n)}(a)\right|^q\int_a^b(x-a)dx\\ -c\int_a^b(x-a)^2(b-x)^2dx\end{array}\right)^{\frac{1}{q}}\\
&= \frac{1}{n!}C^{\frac{1}{p}}(a,b,n,p)\left[(b-a)\left|f^{(n)}(b)\right|^q\frac{(b-a)^2}{2}+(b-a)\left|f^{(n)}(a)\right|^q\frac{(b-a)^2}{2}-c\frac{(b-a)^5}{30}\right]^{\frac{1}{q}}\\
&= \frac{1}{n!}C^{\frac{1}{p}}(a,b,n,p)\left[\left|f^{(n)}(b)\right|^q\frac{(b-a)^3}{2}+\left|f^{(n)}(a)\right|^q\frac{(b-a)^3}{2}-c\frac{(b-a)^5}{30}\right]^{\frac{1}{q}}\\
&= \frac{1}{n!}(b-a)^{\frac{3}{q}}C^{\frac{1}{p}}(a,b,n,p)\left[A\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right)-c\frac{(b-a)^2}{30}\right]^{\frac{1}{q}}
\end{aligned}$$

It is stated that the improper integral $C(a,b,n,p)$ is convergent for $1<p<2$. □

Corollary 2.1 *Under the conditions in Theorem* 1.3 *for* $c=0$, *we have the following inequalities*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|$$
$$\leq \frac{1}{n!}(b-a)^{\frac{3}{q}}C^{\frac{1}{p}}(a,b,n,p)A^{\frac{1}{q}}\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right).$$

The last inequality coincides with the inequality in Theorem 2.1 in [13].

Corollary 2.2 *Under the conditions in Theorem* 2.1 *for* $n=1$, *we have the following inequality:*

$$\left|\frac{f(b)\,b-f(a)\,a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq (b-a)^{\frac{3}{q}-1}C^{\frac{1}{p}}(a,b,1,p)\left[A\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)-c\frac{(b-a)^2}{30}\right]^{\frac{1}{q}}.$$

Corollary 2.3 *Under the conditions in Theorem* 2.1 *for* $n=1$ *and* $c=0$*, we have the following inequality*

$$\left|\frac{f(b)\,b-f(a)\,a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|\le (b-a)^{\frac{3}{q}-1}C^{\frac{1}{p}}(a,b,1,p)A^{\frac{1}{q}}\left(|f'(a)|^q,|f'(b)|^q\right).$$

This inequality coincides with the inequality in [13].

Theorem 2.2 *For* $n\in\mathbb{N}$*; let* $f:(0,\infty)\subset\mathbb{R}\to\mathbb{R}$ *be n-times differentiable function and* $0\le a<b$*. If* $\left|f^{(n)}\right|^q\in L[a,b]$ *and* $\left|f^{(n)}\right|^q$ *for* $q>1$ *is a strongly Godunova-Levin function with modulus c on the interval* $[a,b]$*, then the following inequality*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|$$
$$\le\frac{1}{n!}(b-a)^{\frac{2}{q}}D^{\frac{1}{p}}(a,b,n,p)\times\left[\left|f^{(n)}(b)\right|^q\left\{bL_n^n(a,b)-L_{n+1}^{n+1}(a,b)\right\}\right.$$
$$\left.+\left|f^{(n)(a)}\right|^q\left\{L_{n+1}^{n+1}(a,b)-aL_n^n(a,b)\right\}-\frac{cE(a,b,n)}{(b-a)^2}\right]^{\frac{1}{q}}$$

holds, where $\frac{1}{p}+\frac{1}{q}=1$*,* $1<p<2$*,* $D(a,b,n,p)=\int_a^b\frac{x^n}{(x-a)^{p-1}(b-x)^{p-1}}dx$ *and* $E(a,b,n)=\int_a^b x^n(x-a)^2(b-x)^2dx$*.*

Proof Firstly, let $x\in(a,b)$ (Notices that this proof is also valid in case of $x\in[a,b]$). From Lemma 1.1, Hölder integral inequality and the inequality (2.1), we obtain

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|$$
$$\le\frac{1}{n!}\int_a^b x^n\left|f^{(n)}(x)\right|dx$$
$$\le\frac{1}{n!}\left(\int_a^b\frac{x^n}{(x-a)^{p-1}(b-x)^{p-1}}dx\right)^{\frac{1}{p}}\left(\int_a^b x^n(b-x)(x-a)\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}$$
$$\le\frac{1}{n!}D^{\frac{1}{p}}(a,b,n,p)\left[\int_a^b x^n\left[\begin{array}{c}(b-a)(b-x)\left|f^{(n)}(b)\right|^q+(b-a)(x-a)\left|f^{(n)}(a)\right|^q\\-c(x-a)^2(b-x)^2\end{array}\right]dx\right]^{\frac{1}{q}}$$
$$\le\frac{1}{n!}D^{\frac{1}{p}}(a,b,n,p)$$

$$\times\left[\begin{array}{c}(b-a)\left|f^{(n)}(b)\right|^{q}\left\{b\left(\frac{b^{n+1}-a^{n+1}}{n+1}\right)-\left(\frac{b^{n+2}-a^{n+2}}{n+2}\right)\right\}\\+(b-a)\left|f^{(n)}(a)\right|^{q}\left\{\left(\frac{b^{n+2}-a^{n+2}}{n+2}\right)-a\left(\frac{b^{n+1}-a^{n+1}}{n+1}\right)\right\}-c\int_a^b x^n(x-a)^2(b-x)^2dx\end{array}\right]^{\frac{1}{q}}$$

$$\leq\frac{1}{n!}D^{\frac{1}{p}}(a,b,n,p)\left[\begin{array}{c}(b-a)^2\left|f^{(n)}(b)\right|^{q}\left\{b\left(\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}\right)-\left(\frac{b^{n+2}-a^{n+2}}{(b-a)(n+2)}\right)\right\}\\+(b-a)^2\left|f^{(n)}(a)\right|^{q}\left\{\left(\frac{b^{n+2}-a^{n+2}}{(b-a)(n+2)}\right)-a\left(\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}\right)\right\}-\frac{cE(a,b,n)}{(b-a)^2}\end{array}\right]^{\frac{1}{q}}$$

$$\leq\frac{1}{n!}(b-a)^{\frac{2}{q}}D^{\frac{1}{p}}(a,b,n,p)\left[\begin{array}{c}\left|f^{(n)}(b)\right|^{q}\left\{bL_n^n(a,b)-L_{n+1}^{n+1}(a,b)\right\}\\+\left|f^{(n)}(a)\right|^{q}\left\{L_{n+1}^{n+1}(a,b)-aL_n^n(a,b)\right\}-\frac{cE(a,b,n)}{(b-a)^2}\end{array}\right]^{\frac{1}{q}}$$

It is stated that the improper integral $D(a,b,n,p)$ is convergent for $1<p<2$. □

Corollary 2.4 *Under the conditions in Theorem* 2.2 *for* $c=0$, *we have the following inequality*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|\leq\frac{1}{n!}(b-a)^{\frac{2}{q}}D^{\frac{1}{p}}(a,b,n,p)$$

$$\times\left[\left|f^{(n)}(b)\right|^{q}\left\{bL_n^n(a,b)-L_{n+1}^{n+1}(a,b)\right\}+\left|f^{(n)}(a)\right|^{q}\left\{L_{n+1}^{n+1}(a,b)-aL_n^n(a,b)\right\}\right]^{\frac{1}{q}}.$$

This inequality coincides with the inequality in Theorem 2.2 in [13].

Corollary 2.5 *Under the conditions in Theorem* 2.2 *for* $n=1$ *we have the following inequality*

$$\left|\frac{f(b)\,b-f(a)\,a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq(b-a)^{\frac{3}{q}-1}D^{\frac{1}{p}}(a,b,1,p)\left[(b+2a)|f'(b)|^q+(2b+a)|f'(a)|^q-c\frac{(a+b)(b-a)^2}{10}\right]^{\frac{1}{q}}.$$

Corollary 2.6 *Under the conditions in Theorem* 2.2 *for* $n=1$ *and* $c=0$, *we have the following inequality*

$$\left|\frac{f(b)\,b-f(a)\,a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq(b-a)^{\frac{3}{q}-1}D^{\frac{1}{p}}(a,b,1,p)\left[(b+2a)|f'(b)|^q+(2b+a)|f'(a)|^q\right]^{\frac{1}{q}}.$$

This inequality coincides with the inequality in [13].

Theorem 2.3 *For* $\forall n\in\mathbb{N}$; *let* $f:I\subset(0,\infty)\to\mathbb{R}$ *be* n*-times differentiable function on* I° *and* $a,b\in I^{\circ}$ *with* $a<b$. *If* $\left|f^{(n)}\right|^q$ *for* $q>1$ *is a strongly Godunova-Levin function with modulu* c

on $[a,b]$, *then the following inequality*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|$$

$$\leq \frac{(b-a)^{\frac{4}{q}}}{n!}\left(bC(a,b,n,p)-D(a,b,n,p)\right)^{\frac{1}{p}}\left(\frac{\left|f^{(n)}(b)\right|^q}{3}+\frac{\left|f^{(n)}(a)\right|^q}{6}-c\frac{(b-a)^2}{60}\right)^{\frac{1}{q}}$$

$$+\frac{(b-a)^{\frac{4}{q}}}{n!}\left(D(a,b,n,p)-aC(a,b,n,p)\right)^{\frac{1}{p}}\left(\frac{\left|f^{(n)}(b)\right|^q}{6}+\frac{\left|f^{(n)}(a)\right|^q}{3}-c\frac{(b-a)^6}{60}\right)^{\frac{1}{q}},$$

holds, where $\frac{1}{p}+\frac{1}{q}=1,\ 1<p<2$ *and* $C(a,b,n,p)=\int_a^b\frac{x^{np}}{(x-a)^{p-1}(b-x)^{p-1}}dx$ *and* $D(a,b,n,p)=\int_a^b\frac{x^{np+1}}{(x-a)^{p-1}(b-x)^{p-1}}dx$.

Proof Firstly, let $x\in(a,b)$. If $\left|f^{(n)}\right|^q$ for $q>1$ is a strongly Godunova-Levin function with modulus c on the interval $[a,b]$, by using the Lemma 1.1, the Hölder-İşcan integral inequality and the inequality (2.1) we obtain

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|\leq\frac{1}{n!}\int_a^b x^n\left|f^{(n)}(x)\right|dx$$

$$\leq\frac{1}{n!}\left(\int_a^b\frac{(b-x)x^{np}}{(x-a)^{p-1}(b-x)^{p-1}}dx\right)^{\frac{1}{p}}\left(\int_a^b(x-a)(b-x)^2\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}$$

$$+\frac{1}{n!}\left(\int_a^b\frac{(x-a)x^{np}}{(x-a)^{p-1}(b-x)^{p-1}}dx\right)^{\frac{1}{p}}\left(\int_a^b(x-a)^2(b-x)\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}$$

$$\leq\frac{1}{n!}\left(b\int_a^b\frac{x^{np}}{(x-a)^{p-1}(b-x)^{p-1}}dx-\int_a^b\frac{x^{np+1}}{(x-a)^{p-1}(b-x)^{p-1}}dx\right)^{\frac{1}{p}}$$

$$\times\left(\int_a^b(b-x)\left[(b-a)(b-x)\left|f^{(n)}(b)\right|^q+(b-a)(x-a)\left|f^{(n)}(a)\right|^q-c(x-a)^2(b-x)^2\right]dx\right)^{\frac{1}{q}}$$

$$+\frac{1}{n!}\left(\int_a^b\frac{x^{np+1}}{(x-a)^{p-1}(b-x)^{p-1}}dx-a\int_a^b\frac{x^{np}}{(x-a)^{p-1}(b-x)^{p-1}}dx\right)^{\frac{1}{p}}$$

$$\times\left(\int_a^b(x-a)\left[(b-a)(b-x)\left|f^{(n)}(b)\right|^q+(b-a)(x-a)\left|f^{(n)}(a)\right|^q-c(x-a)^2(b-x)^2\right]dx\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}\left(bC(a,b,n,p)-D(a,b,n,p)\right)^{\frac{1}{p}}$$

$$\times\left(\begin{array}{c}(b-a)\left|f^{(n)}(b)\right|^q\int_a^b(b-x)^2dx\\+(b-a)\left|f^{(n)}(a)\right|^q\int_a^b(x-a)(b-x)dx-c\int_a^b(x-a)^2(b-x)^3dx\end{array}\right)^{\frac{1}{q}}$$

$$+\frac{1}{n!}\left(D(a,b,n,p)-aC(a,b,n,p)\right)^{\frac{1}{p}}$$

$$\times\left(\begin{array}{c}(b-a)\left|f^{(n)}(b)\right|^{q}\int_{a}^{b}(x-a)(b-x)dx\\+(b-a)\left|f^{(n)}(a)\right|^{q}\int_{a}^{b}(x-a)^{2}dx-c\int_{a}^{b}(x-a)^{3}(b-x)^{2}dx\end{array}\right)^{\frac{1}{q}}$$

$$=\frac{1}{n!}(bC(a,b,n,p)-D(a,b,n,p))^{\frac{1}{p}}\left(\frac{(b-a)^{4}}{3}\left|f^{(n)}(b)\right|^{q}+\frac{(b-a)^{4}}{6}\left|f^{(n)}(a)\right|^{q}-c\frac{(b-a)^{6}}{60}\right)^{\frac{1}{q}}$$

$$+\frac{1}{n!}(D(a,b,n,p)-aC(a,b,n,p))^{\frac{1}{p}}\left(\frac{(b-a)^{4}}{6}\left|f^{(n)}(b)\right|^{q}+\frac{(b-a)^{4}}{3}\left|f^{(n)}(a)\right|^{q}-c\frac{(b-a)^{6}}{60}\right)^{\frac{1}{q}}$$

$$=\frac{(b-a)^{\frac{4}{q}}}{n!}(bC(a,b,n,p)-D(a,b,n,p))^{\frac{1}{p}}\left(\frac{\left|f^{(n)}(b)\right|^{q}}{3}+\frac{\left|f^{(n)}(a)\right|^{q}}{6}-c\frac{(b-a)^{2}}{60}\right)^{\frac{1}{q}}$$

$$+\frac{(b-a)^{\frac{4}{q}}}{n!}(D(a,b,n,p)-aC(a,b,n,p))^{\frac{1}{p}}\left(\frac{\left|f^{(n)}(b)\right|^{q}}{6}+\frac{\left|f^{(n)}(a)\right|^{q}}{3}-c\frac{(b-a)^{6}}{60}\right)^{\frac{1}{q}}.$$

It is stated that the improper integral $C(a,b,n,p)$ and $D(a,b,n,p)$ are convergent for $1<p<2$. □

Corollary 2.7 *Under the conditions in Theorem* 2.3 *for* $c=0$, *we have the following inequality*

$$\left|\sum_{k=0}^{n-1}(-1)^{k}\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_{a}^{b}f(x)dx\right|$$

$$\leq\frac{(b-a)^{\frac{4}{q}}}{n!}(bC(a,b,n,p)-D(a,b,n,p))^{\frac{1}{p}}\left(\frac{\left|f^{(n)}(b)\right|^{q}}{3}+\frac{\left|f^{(n)}(a)\right|^{q}}{6}\right)^{\frac{1}{q}}$$

$$+\frac{(b-a)^{\frac{4}{q}}}{n!}(D(a,b,n,p)-aC(a,b,n,p))^{\frac{1}{p}}\left(\frac{\left|f^{(n)}(b)\right|^{q}}{6}+\frac{\left|f^{(n)}(a)\right|^{q}}{3}\right)^{\frac{1}{q}}.$$

Corollary 2.8 *Under the conditions in Theorem* 2.3 *for* $n=1$, *we have the following inequality*

$$\left|\frac{f(b)\,b-f(a)\,a}{b-a}-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|$$

$$\leq\frac{(b-a)^{\frac{4}{q}}}{n!}(bC(a,b,1,p)-D(a,b,1,p))^{\frac{1}{p}}\left(\frac{|f'(b)|^{q}}{3}+\frac{|f'(a)|^{q}}{6}-c\frac{(b-a)^{2}}{60}\right)^{\frac{1}{q}}$$

$$+\frac{(b-a)^{\frac{4}{q}}}{n!}(D(a,b,1,p)-aC(a,b,1,p))^{\frac{1}{p}}\left(\frac{|f'(b)|^{q}}{6}+\frac{|f'(a)|^{q}}{3}-c\frac{(b-a)^{6}}{60}\right)^{\frac{1}{q}},$$

Corollary 2.9 *Under the conditions in Theorem* 2.3 *for* $n=1$ *and* $c=0$, *we have the following* *inequality*

$$\left|\frac{f(b)\,b-f(a)\,a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{(b-a)^{\frac{4}{q}}}{n!}\left(bC\left(a,b,1,p\right)-D\left(a,b,1,p\right)\right)^{\frac{1}{p}}\left(\frac{|f'(b)|^q}{3}+\frac{|f'(a)|^q}{6}\right)^{\frac{1}{q}}$$

$$+\frac{(b-a)^{\frac{4}{q}}}{n!}\left(D\left(a,b,1,p\right)-aC\left(a,b,1,p\right)\right)^{\frac{1}{p}}\left(\frac{|f'(b)|^q}{6}+\frac{|f'(a)|^q}{3}\right)^{\frac{1}{q}}.$$

Theorem 2.4 *For $n\in\mathbb{N}$; let $f:(0,\infty)\subset\mathbb{R}\to\mathbb{R}$ be n-times differentiable function and $0\leq a<b$. If $\left|f^{(n)}\right|^q\in L[a,b]$ and $\left|f^{(n)}\right|^q$ for $q>1$ is strongly Godunova-Levin function with modulus c on the interval $[a,b]$, then the following inequality*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|$$

$$\leq\frac{1}{n!}\left[aD\left(a,b,n,p\right)-E\left(a,b,n,p\right)\right]^{\frac{1}{p}}$$

$$\times\left[\begin{array}{c}(b-a)^2\left|f^{(n)}(b)\right|^q\left(b^2L_n^n(a,b)-2bL_{n+1}^{n+1}(a,b)+L_{n+2}^{n+2}(a,b)\right)\\ +(b-a)^2\left|f^{(n)}(a)\right|^q\left(-abL_n^n(a,b)+(a+b)L_{n+1}^{n+1}(a,b)-L_{n+2}^{n+2}(a,b)\right)\\ -c\frac{G(a,b,n)}{(b-a)^2}\end{array}\right]^{\frac{1}{q}}$$

$$+\frac{1}{n!}\left[E\left(a,b,n,p\right)-aD\left(a,b,n,p\right)\right]^{\frac{1}{p}}$$

$$\times\left[\begin{array}{c}(b-a)^2\left|f^{(n)}(b)\right|^q\left(-abL_n^n(a,b)+(a+b)L_{n+1}^{n+1}(a,b)-(b-a)L_{n+2}^{n+2}(a,b)\right)\\ +(b-a)^2\left|f^{(n)}(a)\right|^q\left(a^2L_n^n(a,b)-2aL_{n+1}^{n+1}(a,b)+L_{n+2}^{n+2}(a,b)\right)\\ -c\frac{H(a,b,n)}{(b-a)^2}\end{array}\right]^{\frac{1}{q}},$$

holds, where $\frac{1}{p}+\frac{1}{q}=1$, $1<p<2$ and

$$D\left(a,b,n,p\right)=\int_a^b\frac{x^n}{(x-a)^{p-1}(b-x)^{p-1}}dx,\qquad E\left(a,b,n,p\right)=\int_a^b\frac{x^{n+1}}{(x-a)^{p-1}(b-x)^{p-1}}dx,$$

$$G\left(a,b,n\right)=\int_a^b x^n\left(x-a\right)^2\left(b-x\right)^3dx,\qquad H\left(a,b,n\right)=\int_a^b x^n\left(x-a\right)^3\left(b-x\right)^2dx.$$

Proof Firstly, let $x\in(a,b)$ (The proof is also valid in case of $x\in[a,b]$). By using the Lemma 1.1, well known Hölder integral inequality and the inequality

$$(x-a)(b-x)\left|f^{(n)}(x)\right|^q\leq(b-a)(b-x)\left|f^{(n)}(b)\right|^q+(b-a)(x-a)\left|f^{(n)}(a)\right|^q-c\left(x-a\right)^2\left(b-x\right)^2,$$

we get that

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right|\le\frac{1}{n!}\int_a^b x^n\left|f^{(n)}(x)\right|dx$$
$$\le\frac{1}{n!}\left(\int_a^b\frac{(b-x)x^n}{(x-a)^{p-1}(b-x)^{p-1}}dx\right)^{\frac{1}{p}}\left(\int_a^b x^n\,(b-x)^2(x-a)\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}$$
$$+\frac{1}{n!}\left(\int_a^b\frac{(x-a)x^n}{(x-a)^{p-1}(b-x)^{p-1}}dx\right)^{\frac{1}{p}}\left(\int_a^b x^n\,(b-x)\,(x-a)^2\left|f^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}$$
$$\le\frac{1}{n!}\left(b\int_a^b\frac{x^n}{(x-a)^{p-1}(b-x)^{p-1}}dx-\int_a^b\frac{x^{n+1}}{(x-a)^{p-1}(b-x)^{p-1}}dx\right)^{\frac{1}{p}}$$
$$\times\left(\int_a^b x^n\,(b-x)\left[(b-a)(b-x)\left|f^{(n)}(b)\right|^q+(b-a)(x-a)\left|f^{(n)}(a)\right|^q-c\,(x-a)^2\,(b-x)^2\right]dx\right)^{\frac{1}{q}}$$
$$+\frac{1}{n!}\left(\int_a^b\frac{x^{n+1}}{(x-a)^{p-1}(b-x)^{p-1}}dx-a\int_a^b\frac{x^n}{(x-a)^{p-1}(b-x)^{p-1}}dx\right)^{\frac{1}{p}}$$
$$\times\left(\int_a^b x^n\,(x-a)\left[(b-a)(b-x)\left|f^{(n)}(b)\right|^q+(b-a)(x-a)\left|f^{(n)}(a)\right|^q-c\,(x-a)^2\,(b-x)^2\right]dx\right)^{\frac{1}{q}}$$
$$=\frac{1}{n!}\left[bD\,(a,b,n,p)-E\,(a,b,n,p)\right]^{\frac{1}{p}}$$
$$\times\left[\begin{array}{c}(b-a)\left|f^{(n)}(b)\right|^q\int_a^b x^n(b-x)^2dx\\+(b-a)\left|f^{(n)}(a)\right|^q\int_a^b x^n(b-x)(x-a)dx-c\int_a^b x^n\,(x-a)^2\,(b-x)^3\,dx\end{array}\right]^{\frac{1}{q}}$$
$$+\frac{1}{n!}\left[E\,(a,b,n,p)-aD\,(a,b,n,p)\right]^{\frac{1}{p}}$$
$$\times\left[\begin{array}{c}(b-a)\left|f^{(n)}(b)\right|^q\int_a^b x^n(x-a)(b-x)dx\\+(b-a)\left|f^{(n)}(a)\right|^q\int_a^b x^n\,(x-a)^2dx-c\int_a^b x^n\,(x-a)^3\,(b-x)^2\,dx\end{array}\right]^{\frac{1}{q}}$$
$$=\frac{1}{n!}\left[aD\,(a,b,n,p)-E\,(a,b,n,p)\right]^{\frac{1}{p}}$$
$$\times\left[\begin{array}{c}(b-a)^2\left|f^{(n)}(b)\right|^q\left(b^2L_n^n(a,b)-2bL_{n+1}^{n+1}(a,b)+L_{n+2}^{n+2}(a,b)\right)\\+(b-a)^2\left|f^{(n)}(a)\right|^q\left(-abL_n^n(a,b)+(a+b)L_{n+1}^{n+1}(a,b)-L_{n+2}^{n+2}(a,b)\right)\\-c\frac{G(a,b,n)}{(b-a)^2}\end{array}\right]^{\frac{1}{q}}$$
$$+\frac{1}{n!}\left[E\,(a,b,n,p)-aD\,(a,b,n,p)\right]^{\frac{1}{p}}$$
$$\times\left[\begin{array}{c}(b-a)^2\left|f^{(n)}(b)\right|^q\left(-abL_n^n(a,b)+(a+b)L_{n+1}^{n+1}(a,b)-(b-a)L_{n+2}^{n+2}(a,b)\right)\\+(b-a)^2\left|f^{(n)}(a)\right|^q\left(a^2L_n^n(a,b)-2aL_{n+1}^{n+1}(a,b)+L_{n+2}^{n+2}(a,b)\right)\\-c\frac{H(a,b,n)}{(b-a)^2}\end{array}\right]^{\frac{1}{q}}$$
$$\times\left[\begin{array}{c}(b-a)\left|f^{(n)}(b)\right|^q\left\{b\left(\frac{b^{n+1}-a^{n+1}}{n+1}\right)-\left(\frac{b^{n+2}-a^{n+2}}{n+2}\right)\right\}\\+(b-a)\left|f^{(n)}(a)\right|^q\left\{\left(\frac{b^{n+2}-a^{n+2}}{n+2}\right)-a\left(\frac{b^{n+1}-a^{n+1}}{n+1}\right)\right\}-c\int_a^b x^n\,(x-a)^2\,(b-x)^2\,dx\end{array}\right]^{\frac{1}{q}}$$

It is stated that the improper integral $D(a,b,n,p)$, $E(a,b,n,p)$ are convergent for $1<p<2$. □

Corollary 2.10 *Under the conditions in Theorem* 2.4 *for* $c=0$, *we have the following inequality*

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)\,b^{k+1}-f^{(k)}(a)\,a^{k+1}}{(k+1)!}\right)-\int_a^b f(x)dx\right| \\
&\leq \frac{1}{n!}\left[aD(a,b,n,p)-E(a,b,n,p)\right]^{\frac{1}{p}} \\
&\quad\times\left[\begin{array}{c}(b-a)^2\left|f^{(n)}(b)\right|^q\left(b^2L_n^n(a,b)-2bL_{n+1}^{n+1}(a,b)+L_{n+2}^{n+2}(a,b)\right)\\+(b-a)^2\left|f^{(n)}(a)\right|^q\left(-abL_n^n(a,b)+(a+b)L_{n+1}^{n+1}(a,b)-L_{n+2}^{n+2}(a,b)\right)\end{array}\right]^{\frac{1}{q}} \\
&\quad+\frac{1}{n!}\left[E(a,b,n,p)-aD(a,b,n,p)\right]^{\frac{1}{p}} \\
&\quad\times\left[\begin{array}{c}(b-a)^2\left|f^{(n)}(b)\right|^q\left(-abL_n^n(a,b)+(a+b)L_{n+1}^{n+1}(a,b)-(b-a)L_{n+2}^{n+2}(a,b)\right)\\+(b-a)^2\left|f^{(n)}(a)\right|^q\left(a^2L_n^n(a,b)-2aL_{n+1}^{n+1}(a,b)+L_{n+2}^{n+2}(a,b)\right)\end{array}\right]^{\frac{1}{q}}.
\end{aligned}$$

Corollary 2.11 *Under the conditions in Theorem* 2.4 *for* $n=1$ *we have the following inequality*

$$\begin{aligned}
&\left|\frac{f(b)\,b-f(a)\,a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{1}{n!}\left[aD(a,b,1,p)-E(a,b,1,p)\right]^{\frac{1}{p}} \\
&\quad\times\left[\begin{array}{c}(b-a)^2|f'(b)|^q\left(b^2L_1^1(a,b)-2bL_2^2(a,b)+L_3^3(a,b)\right)\\+(b-a)^2|f'(a)|^q\left(-abL_1^1(a,b)+(a+b)L_2^2(a,b)-L_3^3(a,b)\right)\\-c\frac{G(a,b,1)}{(b-a)^2}\end{array}\right]^{\frac{1}{q}} \\
&\quad+\frac{1}{n!}\left[E(a,b,1,p)-aD(a,b,1,p)\right]^{\frac{1}{p}} \\
&\quad\times\left[\begin{array}{c}(b-a)^2|f'(b)|^q\left(-abL_1^1(a,b)+(a+b)L_2^2(a,b)-(b-a)L_3^3(a,b)\right)\\+(b-a)^2|f'(a)|^q\left(a^2L_1^1(a,b)-2aL_2^2(a,b)+L_3^3(a,b)\right)\\-c\frac{H(a,b,1)}{(b-a)^2}\end{array}\right]^{\frac{1}{q}}.
\end{aligned}$$

Corollary 2.12 *Under the conditions in Theorem* 2.4 *for* $n=1$ *and* $c=0$, *we have the following inequality*

$$\begin{aligned}
&\left|\frac{f(b)\,b-f(a)\,a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \left[aD(a,b,1,p)-E(a,b,1,p)\right]^{\frac{1}{p}} \\
&\quad\times\left[\begin{array}{c}(b-a)^2|f'(b)|^q\left(b^2L_1^1(a,b)-2bL_2^2(a,b)+L_3^3(a,b)\right)\\+(b-a)^2|f'(a)|^q\left(-abL_1^1(a,b)+(a+b)L_2^2(a,b)-L_3^3(a,b)\right)\end{array}\right]^{\frac{1}{q}} \\
&\quad+\left[E(a,b,1,p)-aD(a,b,1,p)\right]^{\frac{1}{p}} \\
&\quad\times\left[\begin{array}{c}(b-a)^2|f'(b)|^q\left(-abL_1^1(a,b)+(a+b)L_2^2(a,b)-(b-a)L_3^3(a,b)\right)\\+(b-a)^2|f'(a)|^q\left(a^2L_1^1(a,b)-2aL_2^2(a,b)+L_3^3(a,b)\right)\end{array}\right]^{\frac{1}{q}}.
\end{aligned}$$

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