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Some New Ramanujan Type Series for $1/\pi$

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Abstract: In this paper, we derive some new Ramanujan-type series for $1/\pi$ as well as proved existing series, using Eisenstein series representations of the form $-P(q) + nP(q^n)$ and $P(q) + nP(q^n)$, along with Clausen's formulas.

Key Words: Eisenstein series, theta-functions, modular equations.

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§1. Introduction

Ramanujan [43] recorded 17 hypergeometric series like representations for $1/\pi$ in which he gave the brief proof of first three series which are belong to the classical theory of elliptic functions. J. M. Borwein and P. B. Borwein were first proved all the 17 identities in 1987 [17]. Further they derived more series for $1/\pi$ [18], [19], [22]. Also many authors derived several new Ramanujan type series for $1/\pi$ as well as proved the existing identities in the subsequent years.

B. C. Berndt and H. H. Chan used Eisenstein series identities to prove Ramanujan type series for $1/\pi$ in their papers [12] and [13, where the latter one is coauthored with Wen-Chin Liaw. On the basis of the idea of above two papers and with the guidance of Chan, Baruah and Berndt used Eisenstein series identities of the form

$$-P(q^2) + nP(q^{2n}) \text{ and } P(q^2) + nP(q^{2n})$$

for $n = 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 9,\ 10,\ 13,\ 14,\ 15,\ 17,\ 18,\ 22$ and 25, to prove series of Ramanujan type series for $1/\pi$ in [3] and [4], by invoking the hints of Ramanujan. Further, Baruah and N. Nayak worked on Ramanujan type series for $1/\pi$ using Eisenstein series identities of the form $-P(-q)+nP(-q^n)$ and $P(-q) + nP(-q^n)$ for $n = 3,\ 5,\ 7,\ 9,$ and 25. Motivated by this, using Clausen's formulas and Eisenstein series representations of the form $-P(q) + nP(q^n)$ and $P(q) + nP(q^n)$ for $n = 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9$ and 10, we proved 9 series out of 17 series that are recorded by Ramanujan in his famous paper [43] and some other existing series. Besides, we have recorded some new Ramanujan type series for $1/\pi$. A brief details of the existing identities which are

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proved in the Sections 3-11 is given in the below table.

Sl. No.	Authors	Equations
1.	S. Ramanujan [43], [41]	(3.2), (4.3), (5.4), (6.1), (6.5), (8.4), (9.1), (9.6),(10.4)
2.	G. Bauer [7]	(3.2)
3.	J. Guillera [36]	(7.2)
4.	G. H. Hardy [39], [45]	(3.2)
5.	W. N. Bailey [2]	(3.2)
6.	J. M. Borwein and P. B. Borwein [17], [18]	(3.4), (7.5)
7.	B. C. Berndt, H. H. Chan and W. -C. Liaw [13]	(7.4), (9.5)
8.	N.D.Baruah and B.C.Berndt [3]	(3.1), (3.2), (3.3), (3.4), (4.1), (4.2), (4.4), (5.3), (5.4), (6.1), (6.2), (6.3), (6.4), (6.5), (7.1), (7.2), (7.4), (7.5), (8.3), (8.4), (9.1), (9.2), (9.3), (9.4), (9.5), (9.6), (10.2), (10.4)

The Section 2 contains preliminary definitions and results, in which (2.10) and (2.18) plays an important role in proving our results in the Sections 3-11, where (2.18) seems to be new.

§2. **Preliminaries**

Throughout the sequel, we use the following notation:

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n),$$

where a and q are complex numbers with $|q|<1$. For $|ab|<1$, Ramanujan's general theta function is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

Further, Ramanujan [9, p36] considers following three special cases of $f(a,b)$:

$$\varphi(q) := f(q,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = \frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty},$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty},$$

and

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}}.$$

After Ramanujan, we define

$$\chi(q) := (-q; q^2)_\infty.$$

The generalized hypergeometric functions ${}_pF_{p-1}, \ \ p \geq 1$, are defined by

$${}_pF_{p-1}[a_1, \ a_2, \cdots, \ a_p; \ b_1, \ b_2, \cdots, \ b_{p-1}; \ x] := \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_{p-1})_n} \ \frac{x^n}{n!} \ ,$$

where $|x| < 1, \ (a)_n := a(a+1)\cdots(a+n-1)$ and $(a)_0 := 1$. Ramanujan recorded the following identities in his Second Notebook [44] which give the relationship between hypergeometric series and theta functions. Moreover these identities are frequently used to derive our results. A proof of the below identities can be seen in [9, pp 120-124].

Lemma 2.1 *If*

$$q = e^{-y}, \ y = -\pi \ \frac{{}_2F_1[\frac{1}{2}, \ \frac{1}{2}; \ 1 \ ; \ 1-x \ ]}{{}_2F_1[\frac{1}{2}, \ \frac{1}{2}; \ 1 \ ; \ x \ ]} \ \ \textit{and} \ \ z = \ {}_2F_1\left[\frac{1}{2}, \ \frac{1}{2}; \ 1 \ ; \ x \ \right], \tag{2.1}$$

then

$$\varphi(q) = \sqrt{z}, \tag{2.2}$$

$$\varphi(-q) = \sqrt{z}(1-x)^{1/4}, \tag{2.3}$$

$$\psi(q) = \sqrt{\frac{z}{2}} \left(\frac{x}{q}\right)^{1/8}, \tag{2.4}$$

$$\psi(q^2) = \frac{\sqrt{z}}{2} \left(\frac{x}{q}\right)^{1/4}, \tag{2.5}$$

$$\psi(-q) = \sqrt{\frac{z}{2}} \left(\frac{x(1-x)}{q}\right)^{1/8}, \tag{2.6}$$

$$f(-q) = \frac{\sqrt{z}}{2^{1/6}}(1-x)^{1/6}\left(\frac{x}{q}\right)^{1/24}, \tag{2.7}$$

$$f(-q^2) = \frac{\sqrt{z}}{2^{1/3}}\left(\frac{x(1-x)}{q}\right)^{1/12}, \tag{2.8}$$

$$\chi(-q) = 2^{1/6}(1-x)^{1/12}\left(\frac{q}{x}\right)^{1/24}, \tag{2.9}$$

and

$$\frac{dy}{dx} = -\frac{1}{x(1-x)z^2}. \tag{2.10}$$

Let $P(q)$ denote Ramanujan's Eisenstein series defined by

$$P(q) := 1 - 24\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}, \quad |q| < 1. \tag{2.11}$$

Further, Ramanujan [9, p.129] gave the representation for $P(q)$ in terms of x, y and z:

$$P(q) := P(e^{-y}) = (1-5x)z^2 + 12x(1-x)z\frac{dz}{dx}. \tag{2.12}$$

In the sequel, set

$$q := e^{-\pi/\sqrt{n}}, \quad x_n := x(e^{-\pi\sqrt{n}}) \text{ and } z_n := z(e^{-\pi\sqrt{n}}). \tag{2.13}$$

From (2.2), (2.3), (2.5), (2.13) and [44, Entry 27, Chapter 16], we obtain that

$$x_{1/n} := x(e^{-\pi/\sqrt{n}}) = 1 - x_n \text{ and } z_{1/n} := z(e^{-\pi/\sqrt{n}}) = \sqrt{n}z_n. \tag{2.14}$$

The number x_n is called classical singular modulus. We often used the values of these numbers recorded by Ramanujan in [44]. For sometimes we borrow from [11] and [42]. Now employing (2.13) and (2.14) in (2.12) to obtain the following identities:

$$P(q) := P(e^{-\pi/\sqrt{n}}) = (1-5x_{1/n})z_{1/n}^2 + 12x_{1/n}(1-x_{1/n})z_{1/n}\frac{dz_{1/n}}{dx_{1/n}}. \tag{2.15}$$

and

$$P(q^n) := P(e^{-\pi\sqrt{n}}) = (1-5x_n)z_n^2 + 12x_n(1-x_n)z_n\frac{dz_n}{dx_n}. \tag{2.16}$$

The following theorem seems to be new and it produces the representations of the form $P(q) + nP(q^n)$, and with the help of Eisenstein series identities of the form $-P(q) + nP(q^n)$ [44, 47], we are able to derive some new Ramanujan-type series for $1/\pi$ as well as an alternate proof for the existing identities.

Theorem 2.2 *we have*

$$z_{1/n}\,\frac{dz_{1/n}}{dx_{1/n}}=-nz_n\,\frac{dz_n}{dx_n}+\frac{\sqrt{n}}{\pi x_n(1-x_n)} \tag{2.17}$$

and

$$P(e^{-\pi/\sqrt{n}})+nP(e^{-\pi\sqrt{n}})=\frac{12\sqrt{n}}{\pi}-3nz_n^2. \tag{2.18}$$

Proof of (2.17) From (2.14), we have

$$z_{1/n}^2=nz_n^2. \tag{2.19}$$

Differentiating (2.19) with respect to $x_{1/n}$ and using chain rule, we deduce that

$$2z_{1/n}\frac{dz_{1/n}}{dx_{1/n}}=2nz_n\frac{dz_n}{dx_n}\frac{dx_n}{dx_{1/n}}+z_n^2\frac{dn}{dy}\frac{dy}{dx_{1/n}}. \tag{2.20}$$

From (2.14), we obtain that

$$\frac{dx_n}{dx_{1/n}}=-1. \tag{2.21}$$

From (2.1) and (2.13), we easily seen that

$$y=\frac{\pi}{\sqrt{n}}. \tag{2.22}$$

Differentiating (2.22) with respect to n, we find that

$$\frac{dn}{dy}=\frac{-2n\sqrt{n}}{\pi}. \tag{2.23}$$

Employing (2.14) in (2.10) to obtain

$$\frac{dy}{dx_{1/n}}=-\frac{1}{x_n(1-x_n)nz_n^2}. \tag{2.24}$$

Substituting (2.21), (2.23) and (2.24) into (2.20), we arrive at (2.17). □

Proof of (2.18) By employing (2.14) and (2.17) in (2.15), we find that

$$P(e^{-\pi/\sqrt{n}})=n(-4+5x_n)z_n^2-12x_n(1-x_n)z_n\frac{dz_n}{dx_n}+\frac{12\sqrt{n}}{\pi}. \tag{2.25}$$

Then (2.18) follows from (2.16) and (2.25). □

Now our task is to obtain the relationship between Eisenstein series and ${}_3F_2$ hypergeometric series. To achieve this let us recall Clausen's formulas and Borwein's proofs [17, pp. 180-181]. Throughout the sequel, set

$$A_k := \frac{(\frac{1}{2})_k^3}{k!^3}, \quad B_k := \frac{(\frac{1}{4})_k(\frac{1}{2})_k(\frac{3}{4})_k}{k!^3} \text{ and } C_k := \frac{(\frac{1}{6})_k(\frac{1}{2})_k(\frac{5}{6})_k}{k!^3}. \tag{2.26}$$

If

$$X := 4x(1-x), \quad Y := \frac{4x}{(1-x)^2}, \quad U := \frac{x^2}{4(1-x)}, \quad V := \frac{4\sqrt{x}(1-x)}{(1+x)^2},$$
$$W := \frac{2\sqrt{X}}{1-X}, \quad L := \frac{27X^2}{(4-X)^3}, \quad \text{and} \quad M := \frac{27X}{(1-4X)^3},$$

then

$$z^2 = {}_3F_2\left[\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;X\right] = \sum_{k=0}^{\infty} A_k X^k\ , \quad 0 \le x \le \frac{1}{2}, \tag{2.27}$$

$$= \frac{1}{1-x}\ {}_3F_2\left[\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;-Y\right] = \frac{1}{1-x}\sum_{k=0}^{\infty}(-1)^k A_k Y^k\ , \ 0 \le x \le 3-2\sqrt{2}, \tag{2.28}$$

$$= \frac{1}{\sqrt{1-x}}\,{}_3F_2\left[\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;-U\right] = \frac{1}{\sqrt{1-x}}\sum_{k=0}^{\infty}(-1)^k A_k U^k, \ 0 \le x \le 2\sqrt{2}-2, \tag{2.29}$$

$$= \frac{1}{1+x}\ {}_3F_2\left[\frac{1}{4},\frac{1}{2},\frac{3}{4};1,1;V^2\right] = \frac{1}{1+x}\sum_{k=0}^{\infty} B_k V^{2k}, \quad 0 \le x \le 3-2\sqrt{2}, \tag{2.30}$$

$$= \frac{1}{1-2x}\ {}_3F_2\left[\frac{1}{4},\frac{1}{2},\frac{3}{4};1,1;-W^2\right] = \frac{1}{1-2x}\sum_{k=0}^{\infty}(-1)^k B_k W^{2k},$$
$$0 \le x \le \frac{1}{2}\left(1-2^{1/4}\sqrt{2-\sqrt{2}}\right), \tag{2.31}$$

$$= \frac{2}{\sqrt{4-X}}\ {}_3F_2\left[\frac{1}{6},\frac{1}{2},\frac{5}{6};1,1;L\right] = \frac{2}{\sqrt{4-X}}\sum_{k=0}^{\infty} C_k L^k\ , \ 0 \le x \le \frac{1}{2}, \tag{2.32}$$

$$= \frac{1}{\sqrt{1-4X}}\ {}_3F_2\left[\frac{1}{6},\frac{1}{2},\frac{5}{6};1,1;-M\right] = \frac{1}{\sqrt{1-4X}}\sum_{k=0}^{\infty}(-1)^k C_k M^k\ , \ 0 \le x \le \frac{1}{2}. \tag{2.33}$$

Differentiating (2.27) with respect to x, we find that

$$2z\frac{dz}{dx}=\sum_{k=0}^{\infty}A_k k X^{k-1}\cdot 4(1-2x). \tag{2.34}$$

Substituting (2.34) into (2.12) and using (2.27), we deduce that

$$P(q)=\sum_{k=0}^{\infty}\{6k(1-2x)+(1-5x)\}\,A_kX^k. \tag{2.35}$$

Setting $q=e^{-\pi\sqrt{n}}$ in (2.35), we obtain that

$$P(e^{-\pi\sqrt{n}})=\sum_{k=0}^{\infty}\left\{\frac{6k(1+x_n)+x_n}{1-x_n}+(1-5x_n)\right\}A_kX_n^k, \tag{2.36}$$

where $X_n=4x_n(1-x_n)$. Similarly, differentiating each of (2.28)-(2.33) with respect to x, and proceeding as above, we deduce that

$$P(e^{-\pi\sqrt{n}})=\frac{1+x_n}{1-x_n}\sum_{k=0}^{\infty}(6k+1)\,(-1)^kA_kY_n^k, \tag{2.37}$$

$$=\frac{1}{\sqrt{1-x_n}}\sum_{k=0}^{\infty}\{6k(2-x_n)+1-2x_n\}\,(-1)^kA_kU_n^k, \tag{2.38}$$

$$=\frac{1}{(1+x_n)^2}\sum_{k=0}^{\infty}\left\{6k(x_n^2-6x_n+1)+x^2-10x_n+1\right\}(-1)^kB_kV_n^{2k}, \tag{2.39}$$

$$=\frac{-1}{(1-2x_n)^2}\sum_{k=0}^{\infty}\left\{6k(4x_n^2-4x_n-1)+2x_n^2-5x_n-1\right\}(-1)^kB_kW_n^{2k}, \tag{2.40}$$

$$=\sum_{k=0}^{\infty}\left\{\frac{2(1-5x_n)}{\sqrt{4-X_n}}+\frac{3k(4x_n^3-6x_n^2-6x_n+4)+6x_n^3-9x_n^2+3x_n}{(1-x_n+x_n^2)^{\frac{3}{2}}}\right\}C_kL_n^k, \tag{2.41}$$

$$=\sum_{k=0}^{\infty}\left\{\frac{1-5x_n}{\sqrt{1-4X_n}}+\frac{6k(64x_n^3-9x_n^2+30x_n+1)+96x_n^3-144x_n^2+48x_n}{(1-16x_n+16x_n^2)^{\frac{3}{2}}}\right\}$$
$$\times(-1)^kC_kM_n^k, \tag{2.42}$$

where $X_n:=4x_n(1-x_n)$, $Y_n:=\dfrac{4x_n}{(1-x_n)^2}$, $U_n:=\dfrac{x_n^2}{4(1-x_n)}$, $V:=\dfrac{4\sqrt{x_n}(1-x_n)}{(1+x_n)^2}$, $W_n:=$

$\dfrac{2\sqrt{X_n}}{1-X_n}$, $L_n := \dfrac{27X_n^2}{(4-X_n)^3}$ and $M_n := \dfrac{27X_n}{(1-4X_n)^3}$. Put $n = 1$ in (2.18), we obtain that

$$P(e^{-\pi}) = \frac{6}{\pi} - \frac{3}{2}z_1^2,$$

Employing (2.27), we find that

$$\frac{6}{\pi} = P(e^{-\pi}) + \frac{3}{2}\sum_{k=0}^{\infty} A_k, \tag{2.43}$$

where $x_1 = \frac{1}{2}$ and $X_1 = 1$. The series (2.43) seems to be new and this is similar to the series recorded by Ramanujan in [8, p. 256].

§3. Example: $n = 2$

Theorem 3.1 *We have*

$$\frac{1}{\pi} = \sum_{k=0}^{\infty} \left\{(8-5\sqrt{2})k + 3 - 2\sqrt{2}\right\} A_k(2\sqrt{2}-2)^{3k}, \tag{3.1}$$

$$\frac{2}{\pi} = \sum_{k=0}^{\infty} (-1)^k(4k+1)A_k, \tag{3.2}$$

$$\frac{2\sqrt{\sqrt{2}-1}}{\pi} = \sum_{k=0}^{\infty} \left\{(4\sqrt{2}-2)k + \sqrt{2}-1\right\}(-1)^k A_k \left(\frac{\sqrt{2}-1}{2}\right)^{3k}, \tag{3.3}$$

$$\frac{5\sqrt{5}}{\pi} = \sum_{k=0}^{\infty} (28k+3)C_k \left(\frac{3}{5}\right)^{3k}. \tag{3.4}$$

Proof From Entry 13(viii) in Chapter 17 of Ramanujan's second notebook [44] (Also [9, p.127]), we see that

$$-P(q) + 2P(q^2) = (1+x)z^2. \tag{3.5}$$

Setting $q = e^{-\pi/\sqrt{2}}$ in (3.5), then using (2.14) and the value of the singular modulus $x_2 = (\sqrt{2}-1)^2$ [11, p. 281], we find that

$$-P(e^{-\pi/\sqrt{2}}) + 2P(e^{-\pi\sqrt{2}}) = 2(-1+2\sqrt{2})z_2^2. \tag{3.6}$$

Setting $n = 2$ in (2.18), we obtain that

$$P(e^{-\pi/\sqrt{2}}) + 2P(e^{-\pi\sqrt{2}}) = \frac{12\sqrt{2}}{\pi} - 6z_2^2. \tag{3.7}$$

Adding (3.6) and (3.7), we immediately deduce that

$$P(e^{-\pi\sqrt{2}}) = \frac{3\sqrt{2}}{\pi} - (2-\sqrt{2})z_2^2. \tag{3.8}$$

By employing (2.27) in (3.8), one can rewrite (3.8) as

$$P(e^{-\pi\sqrt{2}}) = \frac{3\sqrt{2}}{\pi} - (2-\sqrt{2})\sum_{k=0}^{\infty} A_k X_2^k, \tag{3.9}$$

where $X_2 = (2\sqrt{2} - 2)$. Now, setting $n = 2$ in (2.36), then with the aid of (2.27) and the value of the singular modulus $x_2 = (\sqrt{2} - 1)^2$ [44, p. 214], [11, p. 281], we easily obtain that

$$P(e^{-\pi\sqrt{2}}) = \sum_{k=0}^{\infty}(-14 + 10\sqrt{(2)} + (-30 + 24\sqrt{(2)})k)A_k X_2^k, \tag{3.10}$$

From (3.9) and (3.10), we arrive at (3.1). Similarly the proofs of (3.2), (3.3) and (3.4) are follows, by employing (2.28), (2.29) and (2.32) in (3.8) and setting $n = 2$ in (2.37), (2.38) and (2.41), respectively. □

§4. **Example:** $n = 6$

Theorem 4.1 *We have*

$$\frac{\sqrt{6}+\sqrt{2}+1}{\pi} = \sum_{k=0}^{\infty}\left\{\left(6\sqrt{3}+3\sqrt{6}-6\right)k + 2\sqrt{3}+\sqrt{6}-3-\sqrt{2}\right\}(-1)^k A_k \times \left(8(\sqrt{2}+1)^2(\sqrt{3}-\sqrt{2})^3(2-\sqrt{3})^3\right)^k, \tag{4.1}$$

$$\frac{2}{\pi} = \sum_{k=0}^{\infty}(-1)^k\left\{(12\sqrt{2}-12)k + 4\sqrt{4} - 5\right\}A_k(\sqrt{2}-1)^{4k}, \tag{4.2}$$

$$\frac{2\sqrt{3}}{\pi} = \sum_{k=0}^{\infty}(8k+1)B_k\frac{1}{9^k}. \tag{4.3}$$

Proof From [47], we have

$$-P(q)+6P(q^6)=f_1f_2f_3f_6\left(\frac{5\chi^6(-q^3)}{\chi^6(-q)}-\frac{q\chi^6(-q)}{\chi^6(-q^3)}\right). \tag{4.4}$$

This is

$$-P(q)+6P(q^6)=\frac{\chi^9(-q)\chi(-q^3)}{\chi^8(-q^2)}\frac{f^2(q^6)}{f^2(q^2)}\left(\frac{5\chi^6(-q^3)}{\chi^6(-q)}-\frac{q\chi^6(-q)}{\chi^6(-q^3)}\right)\varphi^4(q). \tag{4.5}$$

If $q=e^{-\pi/\sqrt{6}}$, then we obtain from [6, Theorem 4.1] that

$$\frac{f^2(e^{-\pi\sqrt{6}})}{f^2(e^{-2\pi/\sqrt{6}})}=\frac{(\sqrt{2}+1)^{1/3}}{\sqrt{3}}. \tag{4.6}$$

Setting $q=e^{-\pi/\sqrt{6}}$ in (4.5), using (2.9), (4.6), (2.14) and the values of the singular moduli $x_{3/2}=2\left(-3-2\sqrt{2}+2\sqrt{3}+\sqrt{6}\right)\left(\sqrt{2}-1\right)^2\left(\sqrt{3}+\sqrt{2}\right)^2$ [42] and $x_6=\left(2-\sqrt{3}\right)^2\left(\sqrt{3}-\sqrt{2}\right)^2$ [11, p. 282], we deduce that

$$-P(e^{-\pi/\sqrt{6}})+6P(e^{-\pi\sqrt{6}})=6(15+12\sqrt{2}-8\sqrt{3}-6\sqrt{6})z_6^2. \tag{4.7}$$

Setting $n=2$ in (2.18), we see that

$$P(e^{-\pi/\sqrt{6}})+6P(e^{-\pi\sqrt{6}})=\frac{12\sqrt{6}}{\pi}-18z_6^2. \tag{4.8}$$

It follows from (4.7) and (4.8) that

$$P(e^{-\pi\sqrt{6}})=(6+6\sqrt{2}-4\sqrt{3}-6\sqrt{6})z_6^2. \tag{4.9}$$

As in the previous Section, by employing (2.27), (2.28) and (2.30) in (4.9) and setting $n=6$ in (2.36), (2.37) and (2.39), we easily deduce the identities (4.1), (4.2) and (4.3), respectively.□

§5. **Example:** $n=9$

Theorem 5.1 *We have*

$$\frac{2}{(3\sqrt{3}-5)a\pi}=\sum_{k=0}^{\infty}\left(12k+3-\sqrt{3}\right)A_k\left(2-\sqrt{3}\right)^{4k}, \tag{5.1}$$

$$\frac{1+(6\sqrt{3}-10)a}{\pi}=\sum_{k=0}^{\infty}(-1)^k\left[\left\{9+(30-18\sqrt{3})a\right\}k+3+(6-4\sqrt{3})a\right]$$

$$\times A_k \frac{2^k(\sqrt{3}-1)^{4k}(a-1-\sqrt{3})^{2k}}{\{1+(6\sqrt{3}-10)a\}^k}, \tag{5.2}$$

$$\frac{2\sqrt{2+(12\sqrt{3}-20)a}}{\pi} = \sum_{k=0}^{\infty}(-1)^k\left[\left\{18+(36\sqrt{3}-60)a\right\}k+3+(10\sqrt{3}-18)a\right] \times A_k\left(\frac{(\sqrt{3}+1)(\sqrt{2}-3^{1/4})^3}{4}\right)^{2k}, \tag{5.3}$$

$$\frac{16}{\pi\sqrt{3}} = \sum_{k=0}^{\infty}(-1)^k(28k+3)B_k\frac{1}{48^k}, \tag{5.4}$$

where $a=\sqrt{3+2\sqrt{3}}$.

Proof On page 475 and page 345 of [9], we have

$$-P(q)+9P(q^9) = \frac{8f_3^6}{f_1^2f_9^2}\left\{f_1^6+9qf_1^3f_9^3+27q^2f_9^6\right\}^{1/3} \tag{5.5}$$

and

$$1+9q\frac{f_9^3}{f_1^3} = \left\{1+27q\frac{f_3^{12}}{f_1^{12}}\right\}^{1/3}. \tag{5.6}$$

The above identity can be written as

$$f_3^6 = \frac{f_1^6}{\sqrt{27q}}\left\{\left(1+9q\frac{f_9^3}{f_1^3}\right)^3-1\right\}^{1/2}. \tag{5.7}$$

Setting $q=e^{-\pi/3}$ in (5.5), then using (2.7), (5.7), (2.14) and the value of the singular modulus $x_9=\frac{1}{2}\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^4\left(\sqrt{4+2\sqrt{3}}-\sqrt{3+2\sqrt{3}}\right)^2$ [11, p. 290], we find that

$$-P(e^{-\pi/3})+9P(e^{-3\pi}) = 18\sqrt{3+2\sqrt{3}}(\sqrt{3}-1)Z_9^2. \tag{5.8}$$

Setting $n=9$ in (2.18), we see that

$$P(e^{-\pi/3})+9P(e^{-3\pi}) = \frac{36}{\pi}-27z_9^2. \tag{5.9}$$

From (5.8) and (5.9), we obtain that

$$P(e^{-3\pi}) = \frac{2}{\pi}\left\{\sqrt{3+2\sqrt{3}}(\sqrt{3}-1)-\frac{3}{2}\right\}z_9^2. \tag{5.10}$$

Now, employing (2.27), (2.28), (2.29) and (2.31) in (5.10) and setting $n=9$ in (2.36), (2.37), (2.38) and (2.40), we arrive at (5.1), (5.2), (5.3) and (5.4), respectively. □

The proofs of the Sections 6-11 follow along the similar lines as those in previous sections, so we do not record the proofs.

§6. **Example:** $n = 3$

Theorem 6.1 *We have*

$$\frac{4}{\pi} = \sum_{k=0}^{\infty} (6k+1)\, A_k \frac{1}{4^k}, \tag{6.1}$$

$$\frac{1}{\pi} = \sum_{k=0}^{\infty} (-1)^k \left\{(15\sqrt{3}-24)k + 6\sqrt{3} - 10\right\} A_k 2^k (\sqrt{3}-1)^{6k}, \tag{6.2}$$

$$\frac{4\sqrt{2}}{\pi} = \sum_{k=0}^{\infty} (-1)^k \left\{(30-6\sqrt{3})k + 7 - 3\sqrt{3}\right\} A_k \frac{(2-\sqrt{3})^{3k}}{2^{4k}}, \tag{6.3}$$

$$\frac{8\sqrt{2}}{\pi} = \sum_{k=0}^{\infty} \left\{(85\sqrt{3}-135)k + 8\sqrt{3} - 12\right\} B_k \left(\frac{8\sqrt{2}}{51\sqrt{3}-75}\right)^{2k+1}, \tag{6.4}$$

$$\frac{5\sqrt{5}}{2\sqrt{3}\pi} = \sum_{k=0}^{\infty} (11k+1) C_k \left(\frac{4}{125}\right)^k. \tag{6.5}$$

We note that $-P(e^{-\pi/\sqrt{3}}) + 3P(e^{-\pi\sqrt{3}}) = \frac{9\sqrt{3}}{2} z_3^2$ and $x_3 = \frac{2-\sqrt{3}}{4}$.

§7. **Example:** $n = 4$

Theorem 7.1 *We have*

$$\frac{1}{\pi} = \sum_{k=0}^{\infty} \left\{(48\sqrt{2}-66)k + 20\sqrt{2} - 28\right\} A_k (1584\sqrt{2} - 2240)^k, \tag{7.1}$$

$$\frac{2\sqrt{2}}{\pi} = \sum_{k=0}^{\infty} (-1)^k (6k+1) A_k \frac{1}{8^k}, \tag{7.2}$$

$$\frac{2\sqrt{3\sqrt{2}-4}}{\pi} = \sum_{k=0}^{\infty}(-1)^k \left\{(24\sqrt{2}-30)k + 8\sqrt{2}-11\right\} A_k \left(\frac{(\sqrt{2}-1)^6}{16\sqrt{2}}\right)^k, \tag{7.3}$$

$$\frac{9}{2\pi} = \sum_{k=0}^{\infty}(7k+1)B_k \left(\frac{32}{81}\right)^k, \tag{7.4}$$

$$\frac{\sqrt{33}}{\pi} = \sum_{k=0}^{\infty}(126k+10)C_k \left(\frac{2}{11}\right)^{3k+1}. \tag{7.5}$$

We note that $-P(e^{-\pi/2}) + 3P(e^{-2\pi}) = 12z_4^2$ and $x_4 = (\sqrt{2}-1)^4$.

§8. **Example:** $n = 5$

Theorem 8.1 *We have*

$$\frac{\sqrt{2}}{b\pi} = \sum_{k=0}^{\infty}\left\{(5+\sqrt{5})k+1\right\} A_k(\sqrt{5}-2)^{2k}, \tag{8.1}$$

$$\frac{1}{\pi} = \sum_{k=0}^{\infty}\left[\left\{80+35\sqrt{5}-(30\sqrt{2}+14\sqrt{10})b\right\} k + 34 + 15\sqrt{5} - (13\sqrt{2}+6\sqrt{10})b\right] \times (-1)^k A_k 8^k \left\{617+276\sqrt{5}-(485+217\sqrt{5})\frac{b}{\sqrt{2}}\right\}^k, \tag{8.2}$$

$$\frac{8}{\pi} = \sum_{k=0}^{\infty}\left[2\left\{(15+5\sqrt{5})b - 7\sqrt{10}-5\sqrt{2}\right\} k + (9+3\sqrt{5})b - 7\sqrt{2} - 5\sqrt{10}\right](-1)^k \times A_k \left(\frac{\sqrt{5}-1}{4}\right)^{3k}\left(\frac{b^2}{2}-\frac{b}{\sqrt{2}}\right)^{6k}, \tag{8.3}$$

$$\frac{8}{\pi} = \sum_{k=0}^{\infty}(-1)^k(20k+3)B_k\frac{1}{4^k}, \tag{8.4}$$

$$\frac{2(-5+4\sqrt{5})^{3/2}}{b\sqrt{10}\pi} = \sum_{k=0}^{\infty}\left\{(142-58\sqrt{5})k + 21 - 9\sqrt{5}\right\} C_k \left(\frac{27(-9875+4420\sqrt{5})}{55^3}\right)^k, \tag{8.5}$$

where $b = \sqrt{\sqrt{5}+1}$.

We note that $-P(e^{-\pi/\sqrt{5}}) + 5P(e^{-\pi\sqrt{5}}) = \frac{b(15-\sqrt{5})}{\sqrt{2}} z_5^2$ and the singular modulus for $n = 5$ is $x_5 = \frac{1}{2}\left(\frac{\sqrt{5}-1}{2}\right)^3 \left(\frac{b^2}{2} - \frac{b}{\sqrt{2}}\right)^2$.

§9. **Example:** $n = 7$

Theorem 9.1 *We have*

$$\frac{16}{\pi} = \sum_{k=0}^{\infty} (42k+5) A_k \frac{1}{2^{6k}}, \tag{9.1}$$

$$\frac{1}{\pi} = \sum_{k=0}^{\infty} (-1)^k \left\{(255\sqrt{7} - 672)k + 112\sqrt{7} - 296\right\} A_k (32 - 12\sqrt{7})^{3k}, \tag{9.2}$$

$$\frac{8\sqrt{2}}{\pi} = \sum_{k=0}^{\infty} (-1)^k \left\{(255\sqrt{7} - 672)k + 112\sqrt{7} - 296\right\} A_k \left(\frac{8-3\sqrt{7}}{4}\right)^{3k}, \tag{9.3}$$

$$\frac{29241}{\pi} = \sum_{k=0}^{\infty} \left\{(76160 - 455\sqrt{7})k + 6728 - 784\sqrt{7}\right\} B_k \left(\frac{8\sqrt{2}(325+119\sqrt{7})}{29241}\right)^{2k}, \tag{9.4}$$

$$\frac{9\sqrt{7}}{\pi} = \sum_{k=0}^{\infty} (65k+8)(-1)^k B_k \left(\frac{16}{63}\right)^{2k}, \tag{9.5}$$

$$\frac{85\sqrt{85}}{18\pi\sqrt{3}} = \sum_{k=0}^{\infty} (133k+8) C_k \left(\frac{4}{85}\right)^{3k}. \tag{9.6}$$

We note that $-P(e^{-\pi/\sqrt{7}}) + 7P(e^{-\pi\sqrt{7}}) = \frac{75\sqrt{7}}{8} z_7^2$ and $x_7 = \frac{8-3\sqrt{7}}{16}$.

§10. **Example:** $n = 10$

Theorem 10.1 *We have*

$$\frac{310}{(680 - 480\sqrt{2} + 304\sqrt{5} - 215\sqrt{10})\pi} = \sum_{k=0}^{\infty} \left(930 + 220k - 50\sqrt{2} + 16\sqrt{5} - 29\sqrt{10}\right)$$

$$\times A_k \left\{(3+\sqrt{5})(2+\sqrt{5})(3\sqrt{2}-\sqrt{5}-2)\right\}^{3k}, \qquad (10.1)$$

$$\frac{2}{\pi} = \sum_{k=0}^{\infty}(-1)^k \left\{(60-24\sqrt{5})k + 23 - 10\sqrt{5}\right\} A_k(\sqrt{5}-2)^{4k}, \qquad (10.2)$$

$$\frac{\sqrt{10}\sqrt{102\sqrt{10}-144\sqrt{5}+228\sqrt{2}-322}}{\pi}$$
$$= \sum_{k=0}^{\infty}\left\{\left(1020\sqrt{10}-1440\sqrt{5}+2280\sqrt{2}-3210\right)k + 407\sqrt{10}-576\sqrt{5}+910\sqrt{2}-1285\right\}$$
$$\times (-1)^k A_k \left(\frac{207\sqrt{10}-288\sqrt{5}+450\sqrt{2}-647}{8}\right)^k, \qquad (10.3)$$

$$\frac{9}{2\sqrt{2}\pi} = \sum_{k=0}^{\infty}(10k+1)B_k\frac{1}{9^{2k}}. \qquad (10.4)$$

We note that $x_{10} = 323 - 228\sqrt{2} + 144\sqrt{5} - 102\sqrt{10}$ and $-P(e^{-\pi/\sqrt{10}}) + 10P(e^{-\pi\sqrt{10}}) = \left(2550 - 1800\sqrt{2} + 1152\sqrt{5} - 804\sqrt{10}\right) z_{10}^2$.

§11. **Example:** $n = 8$

Theorem 11.1 *We have*

$$\frac{7}{2\sqrt{2}\pi} = \sum_{k=0}^{\infty}\left[\left\{(560+392\sqrt{2})c - 1575 - 1120\sqrt{2}\right\}k + (248+174\sqrt{2})c - 700 - 497\sqrt{2}\right]$$
$$\times A_k 16^k \left\{(4490+3175\sqrt{2})c - 12756 - 9020\sqrt{2}\right\}^k, \qquad (11.1)$$

$$\frac{14}{c\pi} = \sum_{k=0}^{\infty}(-1)^k \left(14k + 3 - \sqrt{2}\right) A_k \left(\frac{5\sqrt{2}-7}{8}\right)^k, \qquad (11.2)$$

$$\frac{7\sqrt{2}\sqrt{(10+7\sqrt{2})c - 28 - 20\sqrt{2}}}{\pi}$$
$$= \sum_{k=0}^{\infty}\left[\left\{(560+392\sqrt{2})c - 1554 - 1120\sqrt{2}\right\}k + (216+152\sqrt{2})c - 609 - 434\sqrt{2}\right]$$
$$\times (-1)^k A_k \left(\frac{(320+225\sqrt{2})c - 908 - 640\sqrt{2}}{32}\right)^k, \qquad (11.3)$$

$$\frac{343}{\pi(32-13\sqrt{2})}=\sum_{k=0}^{\infty}(70k+12-3\sqrt{2})B_k\frac{2^{5k}(325\sqrt{2}-457)^k}{7^{4k}}, \tag{11.4}$$

where $c=\sqrt{1+5\sqrt{2}}$.

We note that $x_8=113+80\sqrt{2}-4c(7\sqrt{2}-10)$ and

$$-P(e^{-\pi/\sqrt{8}})+10P(e^{-\pi\sqrt{8}})=\left\{600-416\sqrt{2}-\left(\frac{1408+1024\sqrt{2}}{7}\right)c\right\}z_8^2.$$

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Data Availability

Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

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