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The Crossing Number of the Cartesian Product of Star S_n with a 6-Vertex Graph

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Abstract: Calculating the crossing number of a given graph is in general an elusive problem and only the crossing numbers of few families of graphs are known. Most of them are the Cartesian product of special graphs. This paper determines the crossing number of the Cartesian product of star S_n with a 6-vertex graph.

Keywords: Smarandache $\mathscr{P}$-drawing, crossing number, Cartesian product, Star.

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§1. Introduction

For definitions not explained here, readers are referred to [1]. Let G be a simple graph with vertex set V and edge set E. By a *drawing* of G on the plane Π, we mean a collection of points P in Π and open arcs A in $\Pi - P$ for which there are correspondences between V and P and between E and A such that the vertices of an edge correspond to the endpoints of the open arcs. A drawing is called *good,* if for all arcs in A, no two with a common endpoint meet, no two meet in more than one point, and no three have a common point. A *crossing* in a good drawing is a point of intersection of two arcs in A. A *Smarandache $\mathscr{P}$-drawing* of a graph G for a graphical property $\mathscr{P}$ is such a good drawing of G on the plane with minimal intersections for its each subgraph $H \in \mathscr{P}$. A Smarandache $\mathscr{P}$-drawing is said to be *optimal* if $\mathscr{P} = G$ and it minimizes the number of crossings. The *crossing number* $cr(G)$ of a graph G is the number of crossings in any optimal drawing of G in the plane. Let D be a good drawing of the graph G, we denote by $cr(D)$ the number of crossings in D.

Let P_n and C_n be the path and cycle of length n, respectively, and the *star* S_n be the complete bipartite graph $K_{1,n}$.

Given two vertex disjoint graphs G_1 and G_2, the *Cartesian product* $G_1 \times G_2$ of G_1 and G_2

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is defined by

$$\begin{cases} V(G_1 \times G_2) = & V(G_1) \times V(G_2) \\ \\ E(G_1 \times G_2) = & \{(u_1,\, u_2)\,(v_1, v_2)\,|\, u_1 = v_1 \text{ and } u_2\, v_2 \in E(G_2), \\ & \qquad\qquad\quad \text{or } \ u_2 = v_2 \text{ and } u_1\, v_1 \in E(G_1)\} \end{cases}$$

Let G_1 be a graph homeomorphic to G_2, then $cr(G_1) = cr(G_2)$. And if G_1 is a subgraph of G_2, it is easy to see that $cr(G_1) \leq cr(G_2)$.

Calculating the crossing number of a given graph is in general an elusive problem [2] and only the crossing numbers of few families of graphs are known. Most of them are Cartesian products of special graphs, partly because of the richness of their repetitive patterns. The already known results on the crossing number of $G \times H$ fit into three categories:

(i) *G and H are two small graphs.* Harary, et al. obtained the crossing number of $C_3 \times C_3$ in 1973 [3]; Dean and Richter [4] investigated the crossing number of $C_4 \times C_4$; Richter and Thomassen [5] determined the crossing number of $C_5 \times C_5$; in [6] Anderson, et al. obtained the crossing number of $C_6 \times C_6$; Klešč [7]studied the crossing number of $K_{2,3} \times C_3$. These results are usually used as the induction basis for establishing the results of type (ii):

(ii) *G is a small graph and H is a graph from some infinite family.* In [8], the crossing numbers of $G \times C_n$ for any graph G of order four except S_3 were studied by Beineke and Ringeisen, this gap was bridged by Jendrol' et al. in [9]. The crossing numbers of Cartesian products of 4-vertex graphs with P_n and S_n are determined by Klešč in [10], he also determined the crossing numbers of $G \times P_n$ for any graph G of order five [11-13]. For several special graphs of order five, the crossing numbers of their products with C_n or S_n are also known, most of which are due to Klešč [14-17]. For special graphs G of order six, Peng et al. determined the crossing number of the Cartesian product of the Petersen graph $P(3, 1)$ with P_n in [18], Zheng et al. gave the bound for the crossing number of $K_m \times P_n$ for $m \geqslant 3, n \geqslant 1$, and they determined the exact value for $cr(K_6 \times P_n)$, see [19], and the authors [20] established the crossing number of the Cartesian product of P_n with the complete bipartite graph $K_{2,4}$.

(iii) *Both G and H belong to some infinite family.* One very long attention-getting problem of this type is to determine the crossing number of the Cartesian product of two cycles, C_m and C_n, which was put forward by Harary et al. [3], and they conjectured that $cr(C_m \times C_n) = (m-2)n$ for $n \geq m$. In the next three decades, many authors were devoted to this problem and the conjecture has been proved true for $m = 3, 4, 5, 6, 7$, see [8,21-24]. In 2004, the problem was progressed by Glebsky and Salazar, who proved that the crossing number of $C_m \times C_n$ equals its long-conjectured value for $n \geq m(m + 1)$ [25]. Besides the Cartesian product of two cycles, there are several other results. D.Bokal [26] determined the crossing number of the Cartesian product $S_m \times P_n$ for any $m \geq 3$ and $n \geq 1$ used a quite newly introduced operation: the zip product. Tang, et al. [27] and Zheng, et al. [28] independently proved that the crossing number of $K_{2,m} \times P_n$ is $2n\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$ for arbitrary $m \geq 2$ and $n \geq 1$.

Stimulated by these results, we begin to investigate the crossing number of the Cartesian product of star S_n with a 6-vertex graph G_2 shown in Figure 1, and get its crossing number is $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n + 2\lfloor \frac{n}{2} \rfloor$, for $n \geq 1$.

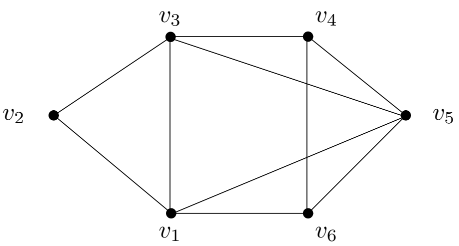


Figure1: The graph G_2

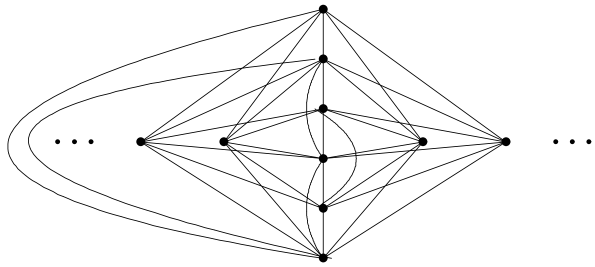


Figure 2: The graph H_n

§2. Some Basic Lemmas and the Main Result

Let A and B be two disjoint subsets of E. In a drawing D, the number of crossings made by an edge in A and another edge in B is denoted by $cr_D(A, B)$. The number of crossings made by two edges in A is denoted by $cr_D(A)$. So $cr(D) = cr_D(E)$. By counting the number of crossings in D, we have Lemma 1.

Lemma 1 *Let A, B, C be mutually disjoint subsets of E. Then*

$$\begin{aligned} cr_D(A \cup B, C) &= cr_D(A, C) + cr_D(B, C); \\ cr_D(A \cup B) &= cr_D(A) + cr_D(B) + cr_D(A, B). \end{aligned} \tag{1}$$

The crossing numbers of the complete bipartite graph $K_{m,n}$ were determined by Kleitman [29] for the case $m \leq 6$. More precisely, he proved that

$$cr(K_{m,n}) = \lfloor \tfrac{m}{2} \rfloor \lfloor \tfrac{m-1}{2} \rfloor \lfloor \tfrac{n}{2} \rfloor \lfloor \tfrac{n-1}{2} \rfloor, \quad \text{if } m \leq 6 \tag{2}$$

For convenience, $\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ is often denoted by $Z(m, n)$ in our paper. To obtain the main result of the paper, first we construct a graph H_n which is shown in Figure 2. Let $V(H_n) = \{v_1, v_2, v_3, v_4, v_5, v_6; t_1, t_2, \cdots, t_n\}$, $E(H_n) = \{v_i t_j | \ 1 \leqslant i \leqslant 6; 1 \leqslant j \leqslant n\} \cup \{v_1v_2, v_1v_3, v_1v_5, v_1v_6, v_2v_3, v_3v_4, v_3v_5, v_4v_5, v_4v_6, v_5v_6\}$. Let T^i be the subgraph of H_n induced by the edge set $\{v_i t_j | 1 \leqslant j \leqslant n\}$, and let t_i be the vertex of T^i of degree six. Clearly, the induced subgraph $[v_1, v_2, \cdots, v_6] \cong G_2$. Thus, we have

$$H_n = G_2 \cup K_{6,n} = G_2 \cup (\bigcup_{i=1}^{n} T^i) \tag{3}$$

For a graph G, the *removal number* $r(G)$ of G is the smallest nonnegative integer r such that the removal of some r edges from G results in a planar subgraph of G. By removing an edge from each crossing of a drawing of G in the plane we get a set of edges whose removal leaves a planar graph. Thus we have the following.

Lemma 2 *For any drawing D of G, $cr(D) \geqslant r(G)$.*

Lemma 3 $cr(H_1) = 1$, $cr(H_2) = 4$.

Proof A good drawing of H_1 in Figure 3 shows that $cr(H_1) \leqslant 1$, and a good drawing of H_2 in Figure 4 shows that $cr(H_2) \leqslant 4$. By Lemma 2, we only need to prove that $r(H_1) \geqslant 1$ and $r(H_2) \geqslant 4$.

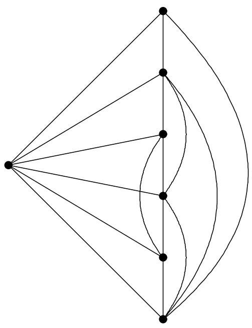


Figure 3: A good drawing of H_1

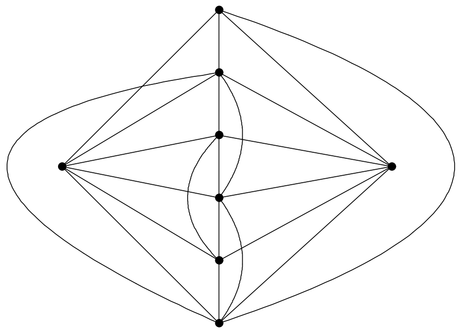


Figure 4: A good drawing of H_2

Let $r = r(H_1)$ and let H_1' be a planar subgraph of H_1 having $16 - r$ edges. It is easy to see that H_1' is a connected spanning subgraph of H_1. By Euler's formula, in any planar drawing of H_1', there are $11 - r$ faces. Since H_1' has girth at least 3, $2(16 - r) \geqslant 3(11 - r)$, so $r \geqslant 1$, that is $r(H_1) \geqslant 1$. Similarly, we can have $r(H_2) \geqslant 4$. □

In a drawing D, if an edge is not crossed by any other edge, we say that it is *clean* in D; if it is crossed by at least one edge, we say that it is *crossed* in D.

Lemma 4 *Let D be a good drawing of H_n. If there are two different subgraphs T^i and T^j such that $cr_D(T^i, T^j) = 0$, then $cr_D(G_2, T^i \cup T^j) \geqslant 4$.*

Proof We label the vertices of G_2, see Figure 1. Since the two subgraphs T^i and T^j do not cross each other in D, the induced drawing $D|_{T^i \cup T^j}$ of $T^i \cup T^j$ divides the plane into six regions that there are exactly two vertices of G_2 on the boundary of each region.

Assume to the contrary that $cr_D(G_2, T^i \cup T^j) \leqslant 3$. The degrees of vertices v_1, v_3 and v_5 in G_2 are all 4, so there are at least two crossings on the edges incident to v_1, v_3 and v_5, respectively. We can assert that edges v_1v_3, v_3v_5 and v_1v_5 must be crossed. Otherwise, without loss of generality, we may assume that the edge v_1v_3 is clean, then the vertices v_1 and v_3 must lie on the boundary of the same region, and there are at least two crossings on the edges (except the edge v_1v_3) incident to vertices v_1 and v_3, respectively, a contradiction. Since the degree of vertex v_4 in G_2 is 3, one can easily see that there is at least one more crossing on the edges incident to v_4, contradicts to our assumption and completes the proof. □

To obtain our main result, the following theorem is introduced.

Theorem 1 $cr(H_n) = Z(6, n) + n + 2\lfloor \frac{n}{2} \rfloor$, *for* $n \geqslant 1$.

Proof A good drawing in Figure 2 shows that $cr(H_n) \leq Z(6, n) + n + 2\lfloor \frac{n}{2} \rfloor$. Now we prove the reverse inequality by induction on n. By Lemma 3, the cases hold for $n = 1$ and $n = 2$. Now suppose that $n \geqslant 3$, and for all $l < n$, there is

$$cr(H_l) \geq Z(6, l) + l + 2\lfloor \tfrac{l}{2} \rfloor \tag{4}$$

and for a certain good drawing D of H_n, assume that

$$cr_D(H_n) < Z(6,n) + n + 2\lfloor \tfrac{n}{2} \rfloor \tag{5}$$

The following two cases are discussed:

Case 1. Suppose that there are at least two different subgraphs T^i and T^j that do not cross each other in D. Without loss of generality, assume that $cr_D(T^{n-1}, T^n) = 0$. By Lemma 4, $cr_D(G_2, T^{n-1} \cup T^n) \geq 4$. As $cr(K_{3,6}) = 6$, for all i, $i = 1, 2, \cdots, n-2$, $cr_D(T^i, T^{n-1} \cup T^n) \geqslant 6$. Using (1), (2), (3) and (4), we have

$$\begin{aligned}
cr_D(H_n) &= cr_D(G_2 \cup \bigcup_{i=1}^{n-2} T^i \cup T^{n-1} \cup T^n) \\
&= cr_D(G_2 \cup \bigcup_{i=1}^{n-2} T^i) + cr_D(T^{n-1} \cup T^n) + cr_D(G_2, T^{n-1} \cup T^n) \\
&\quad + \sum_{i=1}^{n-2} cr_D(T^i, T^{n-1} \cup T^n) \\
&\geqslant Z(6, n-2) + (n-2) + 2\lfloor \frac{n-2}{2} \rfloor + 4 + 6(n-2) \\
&= Z(6,n) + n + 2\lfloor \frac{n}{2} \rfloor
\end{aligned}$$

This contradicts (5).

Case 2. Suppose that $cr_D(T^i, T^j) \geqslant 1$ for any two different subgraphs T^i and T^j, $1 \leqslant i \neq j \leqslant n$. Using (1), (2) and (3), we have

$$\begin{aligned}
cr_D(H_n) &= cr_D(G_2) + cr_D(\bigcup_{i=1}^{n} T^i) + cr_D(G_2, \bigcup_{i=1}^{n} T^i) \\
&\geqslant cr_D(G_2) + Z(6,n) + \sum_{i=1}^{n} cr_D(G_2, T^i)
\end{aligned} \tag{6}$$

This, together with (5) implies that

$$cr_D(G_2) + \sum_{i=1}^{n} cr_D(G_2, T^i) < n + 2\lfloor \frac{n}{2} \rfloor$$

So, there is at least one subgraph T^i that $cr_D(G_2, T^i) \leqslant 1$.

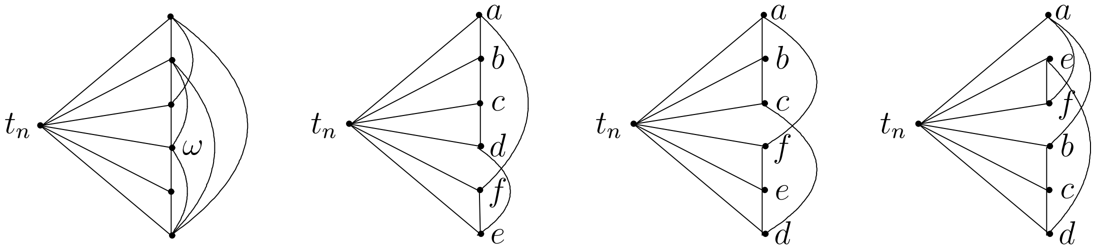
Subcase 2.1 Suppose that there is at least one subgraph T^i that do not cross the edges of G_2. Without loss of generality, we may assume that $cr_D(G_2, T^n) = 0$. Let us consider the 6-cycle C_6 of the graph G_2. Hence G_2 consists of C_6 and four additional edges.

Subcase 2.1.1 Suppose that the edges of C_6 do not cross each other in D. Since $cr_D(G_2, T^n) = 0$, then the possibility of $C_6 \cup T^n$ must be as shown in Figure 5(1). Consider the four additional edges of G_2, they cannot cross the edges of T^n and the edges of C_6 either, so the unique possibility is $cr_D(G_2 \cup T^n) = 2$, see Figure 5(1). Consider now a subdrawing of $G_2 \cup T^n \cup T^i$ of the drawing D for some $i \in \{1, 2, \ldots, n-1\}$. If t_i locates in the region labeled ω, then we have

$cr_D(G_2, T^i) \geqslant 4$, using $cr_D(T^n, T^i) \geqslant 1$, we get $cr_D(G_2 \cup T^n, T^i) \geqslant 5$. If t_i locates in the other regions, one can see that on the boundary of these regions there are at most three vertices of G_2, and there are at least two vertices of G_2 are in a region having no common edge with it, in this case we have $cr_D(G_2 \cup T^n, T^i) \geqslant 5$. Using (1), (2) and (3), we can get

$$\begin{aligned} cr_D(H_n) &= cr_D(G_2 \cup T^n \cup \bigcup_{i=1}^{n-1} T^i) \\ &= cr_D(G_2 \cup T^n) + cr_D(\bigcup_{i=1}^{n-1} T^i) + \sum_{i=1}^{n-1} cr_D(G_2 \cup T^n, T^i) \\ &\geqslant 1 + Z(6, n-1) + 5(n-1) \\ &\geqslant Z(6,n) + n + 2\lfloor \frac{n}{2} \rfloor \end{aligned}$$

which contradicts (5).



(1) (2) (3) (4)

Figure 5

Subcase 2.1.2 Suppose that the edges of C_6 cross each other in D. By the above arguments in Subcase 2.1.1, we can assert that in D there must exist a subgraph T^i, $i \in \{1, 2, \cdots, n-1\}$, such that $cr_D(G_2 \cup T^n, T^i) \leqslant 4$. The condition $cr_D(G_2, T^n) = 0$ implies that $cr_D(C_6, T^n) = 0$. In this case the vertex t_n of T^n lies in the region with all six vertices of C_6 on its boundary, and the condition $cr_D(G_2 \cup T^n, T^i) \leqslant 4$ enforces that in the subdrawing of $C_6 \cup T^n$ there is a region with at least three vertices of C_6 on its boundary. In this case C_6 cannot have more than two internal crossings. If C_6 has only one internal crossing, then the possibilities of $C_6 \cup T^n$ are shown in Figure 5(2) and Figure 5(3). If C_6 has two internal crossings, then the possibility of $C_6 \cup T^n$ is shown in Figure 5(4). The vertices of G_2 are labeled by a, b, c, d, e, f, respectively. Since $cr_D(G_2, T^n) = 0$, the four edges of G_2 not in C_6 do not cross the edges of T^n.

Consider the case shown in Figure 5(2). The three possible edges ac, ce, ae and the fourth possible edge bd or bf or df separate the subdrawing of $G_2 \cup T^n$ into several regions with at most three vertices of G_2 on each boundary. The three possible edges bd, bf, df and the fourth possible edge ac or ce or ae separate the subdrawing of $G_2 \cup T^n$ into several regions with at most three vertices of G_2 on each boundary. If the vertex t_i of T^i locates in the region with three vertices of G_2 on its boundary, one can note that there are at least 2 vertices of G_2 do not on the boundary of its neighborhood region, then $cr_D(G_2 \cup T^n, T^i) \geqslant 5$; if the vertex t_i of T^i locates in the region with at most two vertices of G_2 on its boundary, one can see that there is at least one vertex of G_2 is in a region having no common edge with it, then $cr_D(G_2 \cup T^n, T^i) \geqslant 5$, a contradiction. If the possibility of $C_6 \cup T^n$ is as shown in Figure 5(3) or Figure 5(4), then a similar contradiction can be made by the analogous arguments.

Subcase 2.2 Suppose that $cr_D(G_2, T^i) \geqslant 1$ for $1 \leqslant i \leqslant n$. Together with our former assumption, there is at least one subgraph T^i that $cr_D(G_2, T^i) = 1$. Without loss of generality, assume that $cr_D(G_2, T^n) = 1$.

Subcase 2.2.1 Suppose that $cr_D(C_6, T^n) = 0$. Then the possibilities of $C_6 \cup T^n$ are shown in Figure 5. It is clear that, in each region whose boundary composed of segments of edges that incident with t_n, there are at most two vertices of G_2. Adding the four additional possible edges of G_2 that have one crossing with the edges of T^n, then there are at most three vertices of G_2 on the boundary of each region. Consider now a subdrawing of $G_2 \cup T^n \cup T^i$ of the drawing D for some $i \in \{1, 2, \ldots, n-1\}$. If t_i locates in one of the regions with three vertices of G_2 on its boundary, then then we have $cr_D(G_2, T^i) \geqslant 3$, using $cr_D(T^n, T^i) \geqslant 1$, we have $cr_D(G_2 \cup T^n, T^i) \geqslant 4$. If t_i locates in one of the regions with at most two vertices of G_2 on its boundary, then one can see that there are at least two vertices of G_2 are in a region having no common edge with it, in this case we have $cr_D(G_2 \cup T^n, T^i) \geqslant 6$. Let

$$M = \{T^i | t_i \text{ lies in the region with three vertices of } \mathrm{G}_2 \text{ on its boundary}\}$$

Using (1), (2) and (3), we have

$$\begin{aligned} cr_D(H_n) &= cr_D(G_2 \cup T^n \cup \bigcup_{i=1}^{n-1} T^i) \\ &= cr_D(G_2 \cup T^n) + cr_D(\bigcup_{i=1}^{n-1} T^i) + \sum_{T^i \in M} cr_D(G_2 \cup T^n, T^i) \\ &\quad + \sum_{T^i \notin M} cr_D(G_2 \cup T^n, T^i) \\ &\geqslant 1 + Z(6, n-1) + 4|M| + 6(n-1-|M|) \end{aligned}$$

Together with (5), we can get

$$2|M| \geqslant 5n - 5 - 2\lfloor \frac{n}{2} \rfloor - 6\lfloor \frac{n-1}{2} \rfloor \geqslant 2\lfloor \frac{n}{2} \rfloor \tag{7}$$

Combined with (6) and (7), we can get

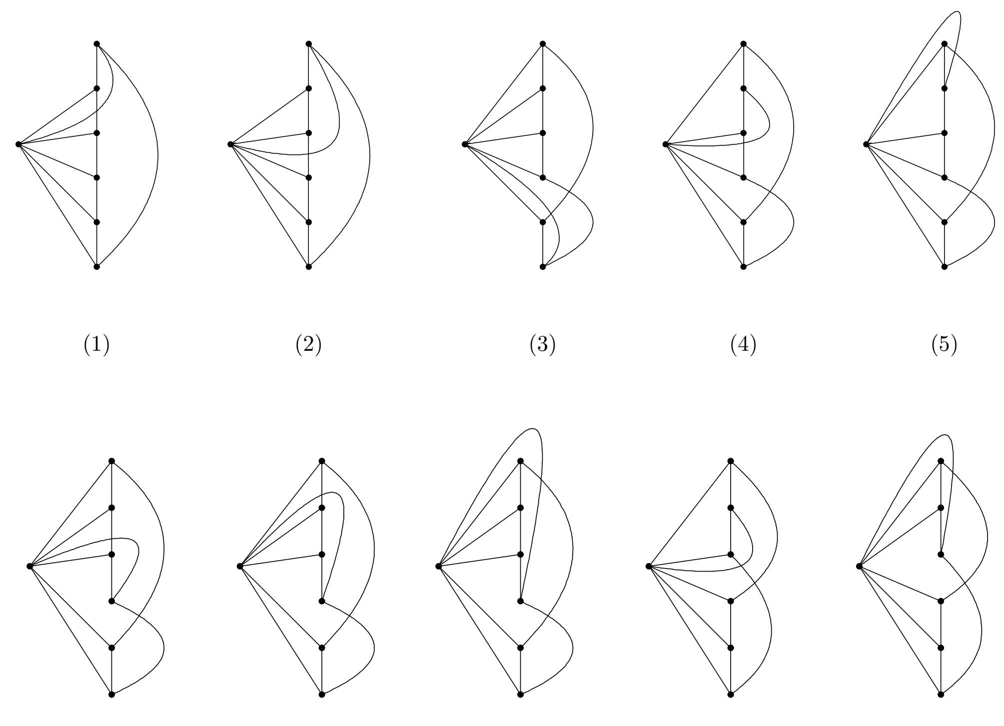
$$\begin{aligned} cr_D(H_n) &= cr_D(G_2) + cr_D(\bigcup_{i=1}^{n} T^i) + cr_D(G_2, \bigcup_{i=1}^{n} T^i) \\ &= cr_D(G_2) + cr_D(\bigcup_{i=1}^{n} T^i) + \sum_{T^i \in M} cr_D(G_2, T^i) + \sum_{T^i \notin M} cr_D(G_2, T^i) \\ &\geqslant Z(6, n) + 3|M| + (n - |M|) \\ &\geqslant Z(6, n) + n + 2\lfloor \frac{n}{2} \rfloor \end{aligned}$$

which contradicts (5).

Subcase 2.2.2 Suppose that $cr_D(C_6, T^n) = 1$, then the subdrawing of $C_6 \cup T^n$ must be one of the ten possibilities shown in Figure 6. Adding the four additional possible edges of G_2 that do not cross T^n, it is not difficult to see that there are at most three vertices of G_2 on the boundary of every region. Consider now a subdrawing of $G_2 \cup T^n \cup T^i$ of the drawing D for some $i \in \{1, 2, \ldots, n-1\}$. One can see that the number of crossings between the edges of $G_2 \cup T^n$ and the edges of T^i are divided into two classes:

(1) In the subdrawing of $G_2 \cup T^n$, we have $cr_D(G_2 \cup T^n, T^i) \geqslant 5$ no matter which region does t_i locate in, then a contradiction can be made by the similarly arguments in Subcase 2.1.1.

(2) In the subdrawing of $G_2 \cup T^n$, $cr_D(G_2 \cup T^n, T^i) = 4$ when t_i locates in the region with three vertices of G_2 on its boundary (and $cr_D(G_2 \cup T^n, T^i) = 4$ if and only if $cr_D(G_2, T^i) = 3$ and $cr_D(T^n, T^i) = 1$), and $cr_D(G_2 \cup T^n, T^i) \geqslant 6$ when t_i locates in the other regions, then a contradiction can be made by the similarly arguments in Subcase 2.2.1. That completes the proof of the theorem. □


(1) (2) (3) (4) (5)

(6) (7) (8) (9) (10)

Figure 6: Ten possibilities of $C_6 \cup T^n$

Lemmas 5 and 6 are trivial observations.

Lemma 5 *If there exists a crossed edge e in a drawing D and deleting it results in a new drawing D^*, then $cr(D) \geqslant cr(D^*) + 1$.*

Lemma 6 *If there exists a clean edge $e = uv$ in a drawing D and contracting it into a vertex*

$u = v$ results in a new drawing D^*, then $cr(D) \geqslant cr(D^*)$.

Let H be a graph isomorphic to G_2. Consider a graph G_H obtained by joining all vertices of H to six vertices of a connected graph G such that every vertex of H will only be adjacent to exactly one vertex of G. Let G_H^* be the graph obtained from G_H by contracting the edges of H.

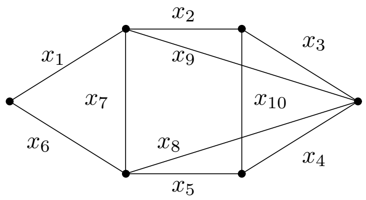


Figure 7

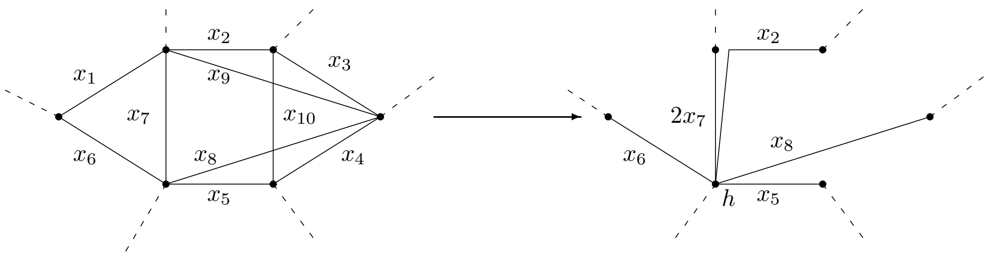


Figure 8

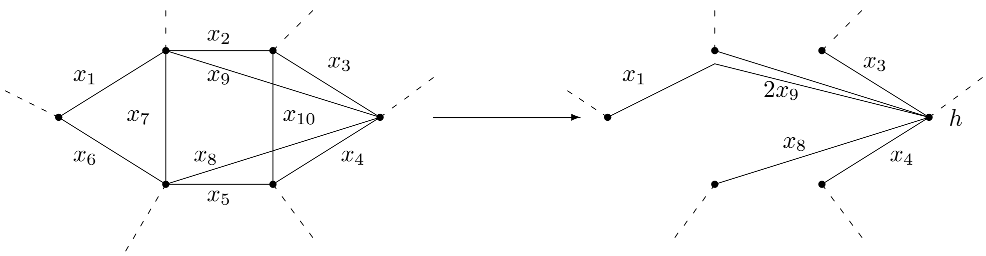


Figure 9

Lemma 7 $cr(G_H^*) \leqslant cr(G_H) - 1$.

Proof Let D be an optimal drawing of G_H. The subgraph H has ten edges and let $x_1, x_2, \ldots, x_9, x_{10}$ denote the numbers of crossings on the edges of H, see Figure 7. The following two cases are distinguished.

Case 1. Suppose that at least one of $x_1, x_2, \ldots, x_6, x_{10}$ is greater than 0, then either $x_7 < x_1 + x_3 + x_4 + x_9 + x_{10}$ or $x_9 < x_2 + x_5 + x_6 + x_7 + x_{10}$ holds. Figure 8 shows that H can be contracted to the vertex h with at least one crossing decreased if $x_7 < x_1 + x_3 + x_4 + x_9 + x_{10}$. Figure 9 shows that H can be contracted to the vertex h with at least one crossing decreased if $x_9 < x_2 + x_5 + x_6 + x_7 + x_{10}$. That means $cr(G_H^*) \leqslant cr_D(G_H) - 1 = cr(G_H) - 1$.

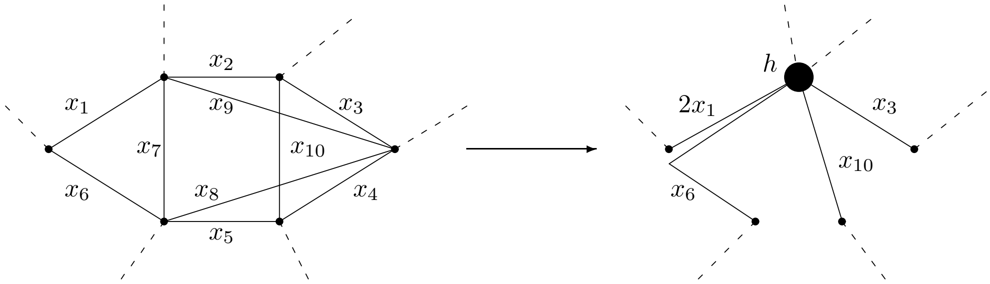


Figure 10

Case 2. Suppose that $x_1 = x_2 = \cdots = x_6 = x_{10} = 0$, then we have $x_7 + x_8 + x_9 \geqslant 1$ since $cr(H_1) = 1$. Figure 10 shows that H can be contracted to the vertex h in the following way: first, delete the edges x_7, x_8 and x_9, (for convenience, here we use x_i to denote the respective edge with x_i crossings), then redraw the former edge x_7 closely enough to edges x_1 and x_6, at last, contract the edge x_2 into a vertex h. By Lemma 5, the first step decreases at least one crossing. And by Lemma 6, the second and last steps do not increase the number of crossings. That means $cr(G_H^*) \leqslant cr_D(G_H) - 1 = cr(G_H) - 1$. This completes the proof. $\square$

Consider now the graph $G_2 \times S_n$. For $n \geqslant 1$ it has $6(n+1)$ vertices and edges that are the edges in $n+1$ copies G_2^i, $i = 0, 1, \cdots, n$, and in the six stars S_n, see Figure 11.

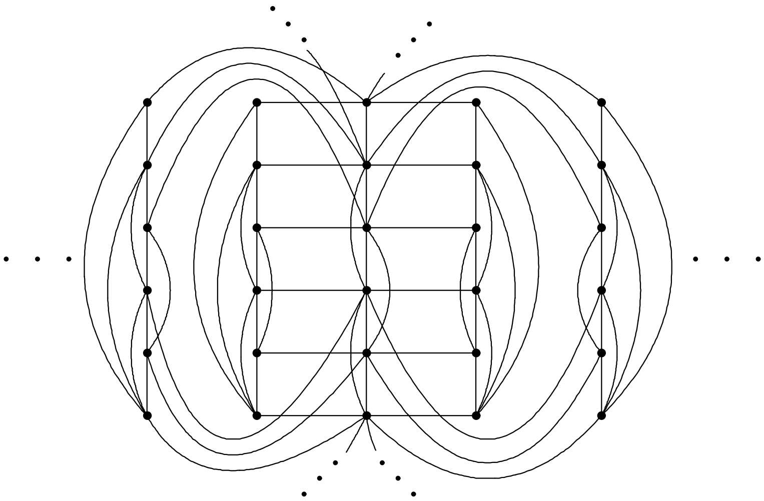


Figure 11: A good drawing of $G_2 \times S_n$

Now, we can get the main theorem.

Theorem 2 $cr(G_2 \times S_n) = Z(6, n) + 2n + 2\lfloor \frac{n}{2} \rfloor$, *for* $n \geqslant 1$.

Proof A drawing in Figure 11 shows that $cr(G_2 \times S_n) \leqslant Z(6, n) + 2n + 2\lfloor \frac{n}{2} \rfloor$. Assume that there is an optimal drawing D of $G_2 \times S_n$ with fewer than $Z(6, n) + 2n + 2\lfloor \frac{n}{2} \rfloor$ crossings. Contracting the edges of each G_2^i to a vertex t_i for all $i = 1, 2, \cdots, n$ in D results in a graph homeomorphic to H_n, and using Lemma 7 repeatedly, we have $cr(H_n) \leq cr(G_2 \times S_n) - n =$

$cr_D(G_2 \times S_n) - n < Z(6,n) + n + 2\lfloor \frac{n}{2} \rfloor$, a contradiction with Theorem 1. Therefore, $cr(G_2 \times S_n) = Z(6,n) + 2n + 2\lfloor \frac{n}{2} \rfloor$. $\square$

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