The Forcing Vertex Monophonic Number of a Graph

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Abstract: For any vertex $x$ in a connected graph $G$ of order $p \geq 2$, a set $S_x \subseteq V(G)$ is an $x$-monophonic set of $G$ if each vertex $v \in V(G)$ lies on an $x - y$ monophonic path for some element $y$ in $S_x$. The minimum cardinality of an $x$-monophonic set of $G$ is the $x$-monophonic number of $G$ and is denoted by $m_x(G)$. A subset $T_x$ of a minimum $x$-monophonic set $S_x$ of $G$ is an $x$-forcing subset for $S_x$ if $S_x$ is the unique minimum $x$-monophonic set containing $T_x$. An $x$-forcing subset for $S_x$ of minimum cardinality is a minimum $x$-forcing subset of $S_x$. The forcing $x$-monophonic number of $S_x$, denoted by $f_{m_x}(S_x)$, is the cardinality of a minimum $x$-forcing subset for $S_x$. The forcing $x$-monophonic number of $G$ is $f_{m_x}(G) = \min\{f_{m_x}(S_x)\}$, where the minimum is taken over all minimum $x$-monophonic sets $S_x$ in $G$. We determine bounds for it and find the forcing vertex monophonic number for some special classes of graphs. It is shown that for any three positive integers $a$, $b$ and $c$ with $2 \leq a \leq b < c$, there exists a connected graph $G$ such that $f_{m_x}(G) = a$, $m_x(G) = b$ and $cm_x(G) = c$ for some vertex $x$ in $G$, where $cm_x(G)$ is the connected $x$-monophonic number of $G$.

Key Words: monophonic path, vertex monophonic number, forcing vertex monophonic number, connected vertex monophonic number, Smarandachely geodetic $k$-set, Smarandachely hull $k$-set.

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§1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of $G$ are denoted by $p$ and $q$ respectively. For basic graph theoretic terminology we refer to Harary [6]. For vertices $x$ and $y$ in a connected graph $G$, the distance $d(x, y)$ is the length of a shortest $x - y$ path in $G$. An $x - y$ path of length $d(x, y)$ is called an $x - y$ geodesic. The neighbourhood of a vertex $v$ is the set $N(v)$ consisting of all vertices $u$ which are adjacent with $v$. The closed neighbourhood of a vertex $v$ is the set $N[v] = N(v) \cup \{v\}$. A vertex $v$ is a simplicial vertex if the subgraph induced by its neighbors is complete.
The closed interval \( I[x, y] \) consists of all vertices lying on some \( x - y \) geodesic of \( G \), while for \( S \subseteq V, I[S] = \bigcup_{x,y \in S} I[x, y] \). A set \( S \) of vertices is a geodetic set if \( I[S] = V \), and the minimum cardinality of a geodetic set is the geodetic number \( g(G) \). The geodetic number of a graph was introduced in [1,8] and further studied in [2,5]. A geodetic set of cardinality \( g(G) \) is called a \( g - \text{set of } G \). Generally, for an integer \( k \geq 0 \), a subset \( S \subseteq V \) is called a Smarandache geodetic \( k \)-set if \( I[S \cup S^+] = V \) and a Smarandache hull \( k \)-set if \( I_h(S \cup S^+) = V \) for a subset \( S^+ \subseteq V \) with \( |S^+| \leq k \). Let \( k = 0 \). Then a Smarandache geodetic 0-set and Smarandache hull 0-set are nothing else but the geodetic set and hull set, respectively.

The concept of vertex geodomination number was introduced in [9] and further studied in [10]. For any vertex \( x \) in a connected graph \( G \), a set \( S \) of vertices of \( G \) is an \( x \)-geodominating set of \( G \) if each vertex \( v \) of \( G \) lies on an \( x - y \) geodesic in \( G \) for some element \( y \) in \( S \). The minimum cardinality of an \( x \)-geodominating set of \( G \) is defined as the \( x \)-geodomination number of \( G \) and is denoted by \( g_x(G) \). An \( x \)-geodominating set of cardinality \( g_x(G) \) is called a \( g_x - \text{set} \).

A chord of a path \( P \) is an edge joining any two non-adjacent vertices of \( P \). A path \( P \) is called a monophonic path if it is a chordless path. A set \( S \) of vertices of a graph \( G \) is a monophonic set of \( G \) if each vertex \( v \) of \( G \) lies on an \( x - y \) monophonic path in \( G \) for some \( x, y \in S \). The minimum cardinality of a monophonic set of \( G \) is the monophonic number of \( G \) and is denoted by \( m(G) \).

The concept of vertex monophonic number was introduced in [11]. For a connected graph \( G \) of order \( p \geq 2 \) and a vertex \( x \) of \( G \), a set \( S_x \subseteq V(G) \) is an \( x \)-monophonic set of \( G \) if each vertex \( v \) of \( G \) lies on an \( x - y \) monophonic path for some element \( y \) in \( S_x \). The minimum cardinality of an \( x \)-monophonic set of \( G \) is defined as the \( x \)-monophonic number of \( G \), denoted by \( m_x(G) \). An \( x \)-monophonic set of cardinality \( m_x(G) \) is called a \( m_x - \text{set} \) of \( G \). The concept of upper vertex monophonic number was introduced in [13]. An \( x \)-monophonic set \( S_x \) is called a minimal \( x \)-monophonic set if no proper subset of \( S_x \) is an \( x \)-monophonic set. The upper \( x \)-monophonic number, denoted by \( m_x^+(G) \), is defined as the minimum cardinality of a minimal \( x \)-monophonic set of \( G \). The connected \( x \)-monophonic number was introduced and studied in [12]. A connected \( x \)-monophonic set of \( G \) is an \( x \)-monophonic set \( S_x \) such that the subgraph \( G[S_x] \) induced by \( S_x \) is connected. The minimum cardinality of a connected \( x \)-monophonic set of \( G \) is the connected \( x \)-monophonic number of \( G \) and is denoted by \( cm_x(G) \). A connected \( x \)-monophonic set of cardinality \( cm_x(G) \) is called a \( cm_x - \text{set} \) of \( G \).

The following theorems will be used in the sequel.

**Theorem 1.1** ([11]) Let \( x \) be a vertex of a connected graph \( G \).

1. Every simplicial vertex of \( G \) other than the vertex \( x \) (whether \( x \) is simplicial vertex or not) belongs to every \( m_x - \text{set} \);
2. No cut vertex of \( G \) belongs to any \( m_x - \text{set} \).

**Theorem 1.2** ([11]) (1) For any vertex \( x \) in a cycle \( C_p (p \geq 4), m_x(C_p) = 1 \);

2. For the wheel \( W_p = K_1 + C_{p-1} (p \geq 5), m_x(W_p) = p - 1 \) or 1 according as \( x \) is \( K_1 \) or \( x \) is in \( C_{p-1} \).
Theorem 1.3([11]) For \( n \geq 2 \), \( m_x(Q_n) = 1 \) for every vertex \( x \) in \( Q_n \).

Throughout this paper \( G \) denotes a connected graph with at least two vertices.

\[ \text{§2. Vertex Forcing Subsets in Vertex Monophonic Sets of a Graph} \]

Let \( x \) be any vertex of a connected graph \( G \). Although \( G \) contains a minimum \( x \)-monophonic set there are connected graphs which may contain more than one minimum \( x \)-monophonic set. For example, the graph \( G \) given in Figure 2.1 contains more than one minimum \( x \)-monophonic set. For each minimum \( x \)-monophonic set \( S_x \) in a connected graph \( G \) there is always some subset \( T \) of \( S_x \) that uniquely determines \( S_x \) as the minimum \( x \)-monophonic set containing \( T \). Such sets are called "vertex forcing subsets" and we discuss these sets in this section. Also, forcing concepts have been studied for such diverse parameters in graphs as the geodetic number [3], the domination number [4] and the graph reconstruction number [7].

Definition 2.1 Let \( x \) be any vertex of a connected graph \( G \) and let \( S_x \) be a minimum \( x \)-monophonic set of \( G \). A subset \( T \) of \( S_x \) is called an \( x \)-forcing subset for \( S_x \) if \( S_x \) is the unique minimum \( x \)-monophonic set containing \( T \). An \( x \)-forcing subset for \( S_x \) of minimum cardinality is a minimum \( x \)-forcing subset of \( S_x \). The forcing \( x \)-monophonic number of \( S_x \), denoted by \( f_{m_x}(S_x) \), is the cardinality of a minimum \( x \)-forcing subset for \( S_x \). The forcing \( x \)-monophonic number of \( G \) is \( f_{m_x}(G) = \min \{f_{m_x}(S_x)\} \), where the minimum is taken over all minimum \( x \)-monophonic sets \( S_x \) in \( G \).

Example 2.2 For the graph \( G \) given in Figure 2.1, the minimum vertex monophonic sets, the vertex monophonic numbers, the minimum forcing vertex monophonic sets and the forcing vertex monophonic numbers are given in Table 2.1.

<table>
<thead>
<tr>
<th>Vertex ( x )</th>
<th>Minimum ( x )-monophonic sets</th>
<th>( m_x(G) )</th>
<th>Minimum forcing ( x )-monophonic sets</th>
<th>( f_{m_x}(G) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u )</td>
<td>{u, r}, {u, y}, {r, z}, {r, s}</td>
<td>2</td>
<td>{y}, {z}, {s}</td>
<td>1</td>
</tr>
<tr>
<td>( v )</td>
<td>{u, r, y}, {u, r, z}, {u, r, s}</td>
<td>3</td>
<td>{y}, {z}, {s}</td>
<td>1</td>
</tr>
<tr>
<td>( w )</td>
<td>{u, r}</td>
<td>2</td>
<td>{}</td>
<td>0</td>
</tr>
<tr>
<td>( y )</td>
<td>{u, r}</td>
<td>2</td>
<td>{}</td>
<td>0</td>
</tr>
<tr>
<td>( z )</td>
<td>{u, r}</td>
<td>2</td>
<td>{}</td>
<td>0</td>
</tr>
<tr>
<td>( s )</td>
<td>{u, r}</td>
<td>2</td>
<td>{}</td>
<td>0</td>
</tr>
<tr>
<td>( t )</td>
<td>{u, r, w}, {u, r, y}, {u, r, z}</td>
<td>3</td>
<td>{w}, {y}, {z}</td>
<td>1</td>
</tr>
<tr>
<td>( r )</td>
<td>{u, w}, {u, y}, {u, z}</td>
<td>2</td>
<td>{w}, {y}, {z}</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2.1
The Forcing Vertex Monophonic Number of a Graph

Figure 2.1

**Theorem 2.3** For any vertex $x$ in a connected graph $G$, $0 \leq f_{m_x}(G) \leq m_x(G)$.

*Proof* Let $x$ be any vertex of $G$. It is clear from the definition of $f_{m_x}(G)$ that $f_{m_x}(G) \geq 0$. Let $S_x$ be a minimum $x$-monophonic set of $G$. Since $f_{m_x}(S_x) \leq m_x(G)$ and since $f_{m_x}(G) = \min \{f_{m_x}(S_x) : S_x \text{ is a minimum } x\text{-monophonic set in } G\}$, it follows that $f_{m_x}(G) \leq m_x(G)$. Thus $0 \leq f_{m_x}(G) \leq m_x(G)$. $\square$

Figure 2.2

**Remark 2.4** The bounds in Theorem 2.3 are sharp. For the graph $G$ given in Figure 2.2, $S = \{u, z, t\}$ is the unique minimum $w$-monophonic set of $G$ and the empty set $\phi$ is the unique minimum $w$-forcing subset for $S$. Hence $f_{m_w}(G) = 0$. Also, for the graph $G$ given in Figure 2.2, $S_1 = \{y\}$ and $S_2 = \{z\}$ are the minimum $u$-monophonic sets of $G$ and so $m_u(G) = 1$. It is clear that no minimum $u$-monophonic set is the unique minimum $u$-monophonic set containing any of its proper subsets. It follows that $f_{m_u}(G) = 1$ and hence $f_{m_u}(G) = m_u(G) = 1$. The inequalities in Theorem 2.3 can be strict. For the graph $G$ given in Figure 2.1, $m_u(G) = 2$ and $f_{m_u}(G) = 1$. Thus $0 < f_{m_u}(G) < m_u(G)$.

In the following theorem we characterize graphs $G$ for which the bounds in Theorem 2.3 are attained and also graphs for which $f_{m_x}(G) = 1$.

**Theorem 2.5** Let $x$ be any vertex of a connected graph $G$. Then
(1) \( f_{m_x}(G) = 0 \) if and only if \( G \) has a unique minimum \( x \)-monophonic set;
(2) \( f_{m_x}(G) = 1 \) if and only if \( G \) has at least two minimum \( x \)-monophonic sets, one of which is a unique minimum \( x \)-monophonic set containing one of its elements, and
(3) \( f_{m_x}(G) = m_x(G) \) if and only if no minimum \( x \)-monophonic set of \( G \) is the unique minimum \( x \)-monophonic set containing any of its proper subsets.

**Definition 2.6** A vertex \( u \) in a connected graph \( G \) is said to be an \( x \)-monophonic vertex if \( u \) belongs to every minimum \( x \)-monophonic set of \( G \).

For the graph \( G \) in Figure 2.1, \( S_1 = \{u, r, y\} \), \( S_2 = \{u, r, z\} \) and \( S_3 = \{u, r, s\} \) are the minimum \( v \)-monophonic sets and so \( u \) and \( r \) are the \( v \)-monophonic vertices of \( G \). In particular, every simplicial vertex of \( G \) other than \( x \) is an \( x \)-monophonic vertex of \( G \).

Next theorem follows immediately from the definitions of an \( x \)-monophonic vertex and forcing \( x \)-monophonic subset of \( G \).

**Theorem 2.7** Let \( x \) be any vertex of a connected graph \( G \) and let \( F_{m_x} \) be the set of relative complements of the minimum \( x \)-forcing subsets in their respective minimum \( x \)-monophonic sets in \( G \). Then \( \bigcap_{F \in F_{m_x}} F \) is the set of \( x \)-monophonic vertices of \( G \).

**Theorem 2.8** Let \( x \) be any vertex of a connected graph \( G \) and let \( M_x \) be the set of all \( x \)-monophonic vertices of \( G \). Then \( 0 \leq f_{m_x}(G) \leq m_x(G) - |M_x| \).

**Proof** Let \( S_x \) be any minimum \( x \)-monophonic set of \( G \). Then \( m_x(G) = |S_x| \), \( M_x \subseteq S_x \) and \( S_x \) is the unique minimum \( x \)-monophonic set containing \( S_x - M_x \) and so \( f_{m_x}(G) \leq |S_x - M_x| = m_x(G) - |M_x| \). \( \square \)

**Theorem 2.9** Let \( x \) be any vertex of a connected graph \( G \) and let \( S_x \) be any minimum \( x \)-monophonic set of \( G \). Then

1. no cut vertex of \( G \) belongs to any minimum \( x \)-forcing subset of \( S_x \);
2. no \( x \)-monophonic vertex of \( G \) belongs to any minimum \( x \)-forcing subset of \( S_x \).

**Proof**
(1) Since any minimum \( x \)-forcing subset of \( S_x \) is a subset of \( S_x \), the result follows from Theorem 1.1(2).

(2) Let \( v \) be an \( x \)-monophonic vertex of \( G \). Then \( v \) belongs to every minimum \( x \)-monophonic set of \( G \). Let \( T \subseteq S_x \) be any minimum \( x \)-forcing subset for any minimum \( x \)-monophonic set \( S_x \) of \( G \). If \( v \in T \), then \( T' = T - \{v\} \) is a proper subset of \( T \) such that \( S_x \) is the unique minimum \( x \)-monophonic set containing \( T' \) so that \( T' \) is an \( x \)-forcing subset for \( S_x \) with \( |T'| < |T| \), which is a contradiction to \( T \) a minimum \( x \)-forcing subset for \( S_x \). Hence \( v \notin T \). \( \square \)

**Corollary 2.10** Let \( x \) be any vertex of a connected graph \( G \). If \( G \) contains \( k \) simplicial vertices, then \( f_{m_x}(G) \leq m_x(G) - k + 1 \).

**Proof** This follows from Theorem 1.1(1) and Theorem 2.9(2). \( \square \)

**Remark 2.11** The bound for \( f_{m_x}(G) \) in Corollary 2.10 is sharp. For a non-trivial tree \( T \) with
k end-vertices, \( f_{m_x}(T) = 0 = m_x(T) - k + 1 \) for any end-vertex \( x \) in \( T \).

**Theorem 2.12**  
(1) If \( T \) is a non-trivial tree, then \( f_{m_x}(T) = 0 \) for every vertex \( x \) in \( T \);
(2) If \( G \) is the complete graph, then \( f_{m_x}(G) = 0 \) for every vertex \( x \) in \( G \).

**Proof** This follows from Theorem 2.9. \( \square \)

**Theorem 2.13** For every vertex \( x \) in the cycle \( C_p(p \geq 3) \), \( f_{m_x}(C_p) = \begin{cases} 0 & \text{if } p = 3, 4 \\ 1 & \text{if } p \geq 5 \end{cases} \).

**Proof** Let \( C_p : u_1, u_2, \ldots, u_p, u_1 \) be a cycle of order \( p \geq 3 \). Let \( x \) be any vertex in \( C_p \), say \( x = u_1 \). If \( p = 3 \) or 4, then \( C_p \) has unique minimum \( x \)-monophonic set. Then by Theorem 2.5(1), \( f_{m_x}(C_p) = 0 \). Now, assume that \( p \geq 5 \). Let \( y \) be a non-adjacent vertex of \( x \) in \( C_p \). Then \( S_x = \{ y \} \) is a minimum \( x \)-monophonic set of \( C_p \). Hence \( C_p \) has more than one minimum \( x \)-monophonic set and it follows from Theorem 2.5(1) that \( f_{m_x}(C_p) \neq 0 \). Now it follows from Theorems 1.2(1) and 2.3 that \( f_{m_x}(G) = m_x(G) = 1 \). \( \square \)

**Theorem 2.14** For any vertex \( x \) in a complete bipartite graph \( K_{m,n}(m, n \geq 2) \), \( f_{m_x}(K_{m,n}) = 0 \).

**Proof** Let \( (V_1, V_2) \) be the bipartition of \( K_{m,n} \). If \( x \in V_1 \), then \( S_x = V_1 - \{ x \} \) is the unique minimum \( x \)-monophonic set of \( G \) and so by Theorem 2.5(1), \( f_{m_x}(G) = 0 \). If \( x \in V_2 \), then \( S_x = V_2 - \{ x \} \) is the unique minimum \( x \)-monophonic set of \( G \) and so by Theorem 2.5(1), \( f_{m_x}(G) = 0 \).

**Theorem 2.15**  
(1) If \( G \) is the wheel \( W_p = K_1 + C_{p-1}(p = 4, 5) \), then \( f_{m_x}(G) = 0 \) for any vertex \( x \) in \( W_p \);
(2) If \( G \) is the wheel \( W_p = K_1 + C_{p-1}(p \geq 6) \), then \( f_{m_x}(G) = 0 \) or 1 according as \( x \) is \( K_1 \) or \( x \) is in \( C_{p-1} \).

**Proof** Let \( C_{p-1} : u_1, u_2, \ldots, u_{p-1}, u_1 \) be a cycle of order \( p - 1 \) and let \( u \) be the vertex of \( K_1 \).

(1) If \( p = 4 \) or 5, then \( G \) has unique minimum \( x \)-monophonic set for any vertex \( x \) in \( G \) and so by Theorem 2.5(1), \( f_{m_x}(G) = 0 \).

(2) Let \( p \geq 6 \). If \( x = u \), then \( S_x = \{ u_1, u_2, \ldots, u_{p-1} \} \) is the unique minimum \( x \)-monophonic set and so by Theorem 2.5(1), \( f_{m_x}(G) = 0 \). If \( x \in V(C_{p-1}) \), say \( x = u_1 \), then \( S_x = \{ u_i \} (3 \leq i \leq p - 2) \) is a minimum \( x \)-monophonic set of \( G \). Since \( p \geq 6 \), there is more than one minimum \( x \)-monophonic set of \( G \). Hence it follows from Theorem 2.5(1) that \( f_{m_x}(G) \neq 0 \). Now it follows from Theorems 1.2(2) and 2.3 that \( f_{m_x}(G) = m_x(G) = 1 \). \( \square \)

**Theorem 2.16** For any vertex \( x \) in the \( n \)-cube \( Q_n (n \geq 2) \), \( f_{m_x}(Q_n) = \begin{cases} 0 & \text{if } n = 2 \\ 1 & \text{if } n \geq 3 \end{cases} \).

**Proof** If \( n = 2 \), then \( Q_n \) has unique minimum \( x \)-monophonic set for any vertex \( x \) in \( Q_n \) and so by Theorem 2.5(1), \( f_{m_x}(Q_n) = 0 \). If \( n \geq 3 \), then it is easily seen that there is more than one minimum \( x \)-monophonic set for any vertex \( x \) in \( Q_n \). Hence it follows from Theorem 2.5(1)
that \( f_{m_x}(Q_n) \neq 0 \). Now it follows from Theorems 1.3 and 2.3 that \( f_{m_x}(Q_n) = m_x(Q_n) = 1 \). □

The following theorem gives a realization result for the parameters \( f_{m_x}(G) \), \( m_x(G) \) and \( m_x^+(G) \).

**Theorem 2.17** For any three positive integers \( a, b \) and \( c \) with \( 2 \leq a \leq b \leq c \), there exists a connected graph \( G \) with \( f_{m_x}(G) = a \), \( m_x(G) = b \) and \( m_x^+(G) = c \) for some vertex \( x \) in \( G \).

**Proof** For each integer \( i \) with \( 1 \leq i \leq a - 1 \), let \( F_i : u_{0,i}, u_{1,i}, u_{2,i}, u_{3,i} \) be a path of order 4. Let \( C_6 : t, u, v, x, y, t \) be a cycle of order 6. Let \( H \) be a graph obtained from \( F_i \) and \( C_6 \) by joining the vertex \( x \) of \( C_6 \) to the vertices \( u_{0,i} \) and \( u_{3,i} \) of \( F_i (1 \leq i \leq a - 1) \). Let \( G \) be the graph obtained from \( H \) by adding \( c - a \) new vertices \( y_1, y_2, \ldots, y_{c-b}, v_1, v_2, \ldots, v_{b-a} \) and joining each \( y_i (1 \leq i \leq c - b) \) to both \( u \) and \( y \), and joining each \( v_j (1 \leq j \leq b - a) \) with \( x \). The graph \( G \) is shown in Figure 2.3.

![Figure 2.3](image)

Let \( S = \{v_1, v_2, \ldots, v_{b-a}\} \) be the set of all simplicial vertices of \( G \). For \( 1 \leq j \leq a - 1 \), let \( S_j = \{u_{1,j}, u_{2,j}\} \). If \( b = c \), then let \( S_a = \{u, v, t\} \). Otherwise, let \( S_a = \{u, v\} \). Now, we observe that a set \( S_x \) of vertices of \( G \) is a \( m_x \)-set if \( S_x \) contains \( S \) and exactly one vertex from each set \( S_j (1 \leq j \leq a) \) so that \( m_x(G) \geq b \). Since \( S_x' = S \cup \{u, u_{1,1}, u_{1,2}, \ldots, u_{1,a-1}\} \) is an \( x \)-monophonic set of \( G \), we have \( m_x(G) = b \).

Now, we show that \( f_{m_x}(G) = a \). Let \( S_x = S \cup \{u, u_{1,1}, u_{1,2}, \ldots, u_{1,a-1}\} \) be a \( m_x \)-set of \( G \) and let \( T_x \) be a minimum \( x \)-forcing subset of \( S_x \). Since \( S \) is the set of all \( x \)-monophonic vertices of \( G \) and by Theorem 2.8, \( f_{m_x}(G) \leq m_x(G) - |S| = a \).
If \(|T_x| < a\), then there exists a vertex \(y \in S_x\) such that \(y \notin T_x\). It is clear that \(y \in S_j\) for some \(j = 1, 2, \ldots, a\), say \(y = u_{1,1}\). Let \(S'_x = (S_x - \{u_{1,1}\}) \cup \{u_{2,1}\}\). Then \(S'_x \neq S_x\) and \(S'_x\) is also a minimum \(x\)-monophonic set of \(G\) such that it contains \(T_x\), which is a contradiction to \(T_x\) a minimum \(x\)-forcing subset of \(S_x\). Thus \(|T_x| = a\) and so \(f_m^+(G) = a\).

Next, we show that \(m^+_x(G) = c\). Let \(U_x = S \cup \{u_{1,1}, u_{1,2}, \ldots, u_{1,a-1}, t, y_1, y_2, \ldots, y_{c-b}\}\). Clearly \(U_x\) is a minimal \(x\)-monophonic set of \(G\) and so \(m^+_x(G) \geq c\). Also, it is clear that every minimal \(x\)-monophonic set of \(G\) contains at most \(c\) elements and hence \(m^+_x(G) \leq c\). Therefore, \(m^+_x(G) = c\).

The following theorem gives a realization for the parameters \(f_m(G), m_x(G)\) and \(cm_x(G)\).

**Theorem 2.18** For any three positive integers \(a, b, c\) with \(2 \leq a \leq b < c\), there exists a connected graph \(G\) with \(f_m(G) = a\), \(m_x(G) = b\) and \(cm_x(G) = c\) for some vertex \(x\) in \(G\).

**Proof** We prove this theorem by considering three cases.

**Case 1.** \(2 \leq a < b < c\).

For each integer \(i\) with \(1 \leq i \leq a - 1\), let \(F_i : y_1, u_{i,1}, u_{i,2}, y_3\) be a path of order 4. Let \(P_{c-b+2} : y_1, y_2, y_3, \ldots, y_{c-b+2}\) be a path of order \(c - b + 2\) and let \(P : v_1, v_2, v_3\) be a path of order 3. Let \(H_1\) be a graph obtained from \(F_i(1 \leq i \leq a - 1)\) and \(P_{c-b+2}\) by identifying the vertices \(y_1\) and \(y_3\) of all \(F_i(1 \leq i \leq a - 1)\) and \(P_{c-b+2}\). Let \(H_2\) be the graph obtained from \(H_1\) and \(P\) by joining the vertex \(v_1\) of \(P\) to the vertex \(y_2\) of \(H_1\) and joining the vertex \(v_3\) of \(P\) to the vertex \(y_3\) of \(H_1\). Let \(G\) be the graph obtained from \(H_2\) by adding \(b - a\) new vertices \(z_1, z_2, \ldots, z_{b-a}\) and joining each \(z_i(1 \leq i \leq b - a)\) with the vertex \(y_{c-b+2}\). The graph \(G\) is shown in Figure 2.4.

![Figure 2.4](image-url)
vertices of $G$ is a $m_x$-set if $S_x$ contains $S$ and exactly one vertex from each set $S_j (1 \leq j \leq a)$. Hence $m_x(G) \geq b$. Since $S'_x = S \bigcup \{v_2, u_{1,1}, u_{1,2}, \ldots, u_{1,a-1}\}$ is an $x$-monophonic set of $G$ with $|S'_x| = b$, it follows that $m_x(G) = b$.

Now, we show that $f_{m_x}(G) = a$. Let $S_x = S \bigcup \{v_2, u_{1,1}, u_{1,2}, \ldots, u_{1,a-1}\}$ be a $m_x$-set of $G$ and let $T_x$ be a minimum $x$-forcing subset of $S_x$. Since $S$ is the set of all $x$-monophonic vertices of $G$ and by Theorem 2.8, $f_{m_x}(G) \leq m_x(G) - |S| = a$.

If $|T_x| < a$, then there exists a vertex $y \in S_x$ such that $y \notin T_x$. It is clear that $y \in S_j$ for some $j = 1, 2, \ldots, a$, say $y = u_{1,1}$. Let $S'_x = (S_x - \{u_{1,1}\}) \bigcup \{u_{2,1}\}$. Then $S'_x \neq S_x$ and $S'_x$ is also a minimum $x$-monophonic set of $G$ such that it contains $T_x$, which is a contradiction to $T_x$ an $x$-forcing subset of $S_x$. Thus $|T_x| = a$ and so $f_{m_x}(G) = a$.

Clearly, $S \bigcup \{v_3, u_{2,1}, u_{2,2}, \ldots, u_{2,a-1}, y_3, y_4, \ldots, y_{c-b+2}\}$ is the unique minimum connected $x$-monophonic set of $G$, we have $cm_x(G) = c$.

**Case 2.** $2 \leq a = b < c$ and $c = b + 1$.

Construct the graph $H_2$ in Case 1. Then $G = H_2$ has the desired properties ($S$ is the empty set).

**Case 3.** $2 \leq a = b < c$ and $c \geq b + 2$. For each $i$ with $1 \leq i \leq a - 1$, let $F_i : y_1, u_{i,1}, u_{i,2}, y_3$ be a path of order 4. Let $P_{c-a+1} : y_1, y_2, y_3, \ldots, y_{c-a+1}$ be a path of order $c - a + 1$ and let $C_5 : v_1, v_2, v_3, v_4, v_5, v_1$ be a cycle of order 5. Let $H$ be a graph obtained from $F_i$ and $P_{c-a+1}$ by identifying the vertices $y_1$ and $y_1$ of all $F_i (1 \leq i \leq a - 1)$ and $P_{c-a+1}$. Let $G$ be the graph obtained from $H$ by identifying the vertex $y_{c-a+1}$ of $P_{c-a+1}$ and $v_1$ of $C_5$. The graph $G$ is shown in Figure 2.5. Let $x = y_2$.

![Figure 2.5](image-url)

For $1 \leq j \leq a - 1$, let $S_j = \{u_{1,j}, u_{2,j}\}$ and let $S_a = \{v_3, v_4\}$. Now, we observe that a set $S_x$ of vertices of $G$ is a $m_x$-set if $S_x$ contains exactly one vertex from each set $S_j (1 \leq j \leq a)$ so that $m_x(G) \geq a$. Since $S'_x = \{v_3, u_{1,1}, u_{1,2}, \ldots, u_{1,a-1}\}$ is an $x$-monophonic set of $G$ with $|S'_x| = a$, we have $m_x(G) = a$. 
Now, we show that $f_{m_x}(G) = a$. Let $S_x = \{v_3, u_{1,1}, u_{1,2}, \cdots, u_{1,a-1}\}$ be a $m_x$-set of $G$ and let $T_x$ be a minimum $x$-forcing subset of $S_x$. Then by Theorem 2.3, $f_{m_x}(G) \leq m_x(G) = a$.

If $|T_x| < a$, then there exists a vertex $y \in S_x$ such that $y \notin T_x$. It is clear that $y \in S_j$ for some $j = 1, 2, \cdots, a$, say $y = u_{1,1}$. Let $S'_x = (S_x - \{u_{1,1}\}) \cup \{u_{1,2}\}$. Then $S'_x \neq S_x$ and $S'_x$ is also a minimum $x$-monophonic set of $G$ such that it contains $T_x$, which is a contradiction to $T_x$ an $x$-forcing subset of $S_x$. Thus $|T_x| = a$ and so $f_{m_x}(G) = a$.

Let $S = \{v_2, v_3, u_{2,1}, u_{2,2}, \cdots, u_{2,a-1}, y_3, y_4, \cdots, y_{c-a+1}\}$. It is easily verified that $S$ is a minimum connected $x$-monophonic set of $G$ and so $c_{m_x}(G) = c$.

\begin{problem}
For any three positive integers $a$, $b$ and $c$ with $2 \leq a \leq b = c$, does there exist a connected graph $G$ with $f_{m_x}(G) = a$, $m_x(G) = b$ and $c_{m_x}(G) = c$ for some vertex $x$ in $G$?
\end{problem}

References