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The Order of the Sandpile Group of Infinite Complete Expansion Regular Graphs

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Abstract: The sandpile group or critical group of a graph is an Abelian group whose order is the number of spanning trees of the graph. In the paper, the order of the sandpile group of infinite complete expansion regular graphs is obtained.

Key Words: Sandpile group, expansion graph, infinite complete expansion graph.

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§1. Introduction

The sandpile group or critical group $K(G)$ of a graph G is an isomorphism invariant that comes in the form of a finite Abelian group; its order the *complexity* $\kappa(G)$, that is, the number of spanning trees in G. The interested reader can find standard results on the subject in [2,Chapter 13] and in [3,4].

We explore here the order of sandpile group of infinite expansion transformation graph. The concept of expansion transformation graph of a graph was given by [5](Fig.1 is a simple example), and complete expansion graph of a graph G is the special expansion graphs of G(see Fig.1), that is the line of subdivision of G. *Subdivision graph* sd$G$, obtained from placing a new vertex in the center of every edge of G, and the line of sd$G$ is complete expansion graph, we denote it as $EXP(G)$.

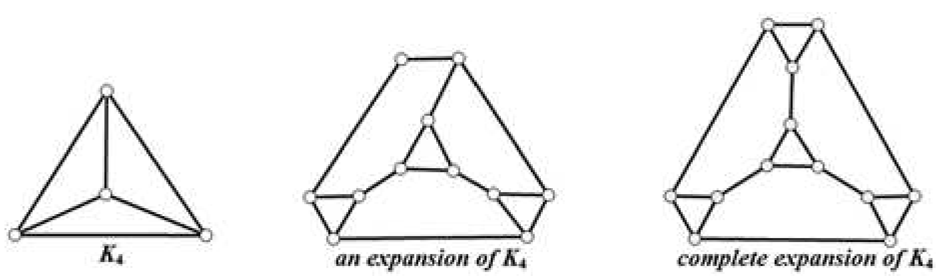

Fig.1 K_4**, an expansion and complete expansion graph of** K_4

Hence graph G and $EXP(G)$ are both special graphs of the expansions graph of G. We said G be an ordinary expansion of G. The complete expansion of $EXP(G)$, we denote as $EXP^2(G)$,

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that is, $EXP^2(G) = EXP(EXP(G))$, similarly, $m \in \mathbb{N}, EXP^m(G) = EXP(EXP^{m-1}(G))$. We call $\{EXP^m(G)\},\ m = 1, 2, \cdots$, be an infinite complete expansion of graph G, simply denoted by ICEG.

Let $\beta(G)$ denote the number of linearly independent elements in the cycle space of G. The present paper refer to following results [6,7].

lemma 1.1(Sachs, Cvetković) *Let G be a connected with $lineG$ regular.*

If G is d-regular, then

$$\kappa(lineG) = d^{\beta(G)-2} 2^{\beta(G)}. \tag{1.1}$$

If G is bipartite and (d_1, d_2)-semiregular with bipartition V_1, V_2, then

$$\kappa(lineG) = \frac{(d_1 + d_2)^{\beta(G)}}{d_1 d_2} (\frac{}{d_1} d_2)^{|V_1|-|V_2|} \kappa(G). \tag{1.2}$$

Lemma 1.2 *Let G be a connected graph. Then*

$$\kappa(sdG) = 2^{\beta(G)} \kappa(G). \tag{1.3}$$

These results suggests some close relationship between the sandpile group $K(G)$ and K(lineG) in either of these situations.

§2. Main Results and Proofs

Theorem 2.1 *Let G be a k- regular graph with n vertices, ε edges, then $\forall m \in \mathbb{N}, m \geq 1$,*

(1) *$EXP^m(G)$ have $2k^{m-1}\varepsilon$ vertices and εk^m edges;*

(2) *$SdEXP^m(G)$ have $2k^{m-1}\varepsilon + k^m\varepsilon$ vertices and $2k^m\varepsilon$ edges.*

Proof We use n, ε to denote $n(G), \varepsilon(G)$ in the following, and show that the results by induction for m. Since subdivision graph sdG obtained from G placing a new vertex in the center of every edge of G, hence $n(sdG) = \varepsilon + n, \varepsilon(sdG) = nk = 2\varepsilon$, and so the $linesdG$ have 2ε vertices and $k\varepsilon$ edges, that is, $n(EXP(G)) = 2\varepsilon, \varepsilon(EXP(G)) = \varepsilon k$, the result is true for $m = 1$.

Assume that result is true for $m-1$, that is, $n(EXP^{m-1}(G)) = 2k^{m-2}\varepsilon, \varepsilon(EXP^{m-1}(G)) = \varepsilon k^{m-1}$. Then subdivision graph $sdEXP^{m-1}(G)$ obtained from $EXP^{m-1}(G)$ placing a new vertex in the center of every edge of $EXP^{m-1}(G)$, hence $n(sdEXP^{m-1}(G)) = \varepsilon k^{m-1} + 2k^{m-2}\varepsilon, \varepsilon(sdG) = \varepsilon k^{m-1}$. The details of the direct proof refer to the table below.

ICEG of k regular graph G_0				
	graphs	*vertex-number*	*edge-number*	*Remarks*
0	G_0	n	ε	
0	sdG_0	$n+\varepsilon$	2ε	subd G_0
1	$G_1 = EXP(G_0)$	2ε	εk	linesubd G_0
1	sd$G_1 = sdEXP(G_0)$	$2\varepsilon + \varepsilon k$	$2\varepsilon k$	subd G_1
2	$G_2 = EXP^2(G_0)$	$2\varepsilon k$	εk^2	linesubd G_1
2	sd$G_2 = sdEXP^2(G_0)$	$2\varepsilon k + \varepsilon k^2$	$2\varepsilon k^2$	subdG_2
3	$G_3 = EXP^3(G_0)$	$2\varepsilon k^2$	εk^3	linesubd G_2
3	sd$G_3 = sdEXP^3(G_0)$	$2\varepsilon k^2 + \varepsilon k^3$	$2\varepsilon k^3$	subd G_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m-1$	sd$G_{m-1} = sdEXP^{m-1}(G_0)$	$2\varepsilon k^{m-2} + \varepsilon k^{m-1}$	$2\varepsilon k^{m-1}$	subd G_{m-1}
m	$G_m = EXP^m(G_0)$	$2\varepsilon k^{m-1}$	εk^m	linesubd G_{m-1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Remark *subdG* denotes subdivision of graph G; *linesubdG* denotes the line graph of the subdivision of the graph G. □

Theorem 2.2 *Let G be a k- regular graph with n vertices, ε edges, then $\forall m \in \mathbb{N}, m \geq 1$, the order of sandpile group of $EXP^m(G)$ equals to*

$$2^{m(\omega-1)} \cdot (2+k)^{\frac{k^{m-1}(k-2)+1}{k-1}\varepsilon - n + m\omega} \cdot k^{\frac{k^{m-1}(k-2)+1}{k-1}\varepsilon - n - m}\kappa(G). \tag{2.1}$$

Proof The proof is by mathematical induction on m. If $m = 1$, by (1.3) $\kappa(sd(G)) = 2^{\varepsilon-n+\omega} \cdot \kappa(G)$, and by (1.2)

$$\kappa(linesd(G)) = \frac{(2+k)^{\beta(sdG)}}{2k} \cdot (\frac{k}{2})^{\varepsilon-n} \cdot \kappa(sd(G)) \tag{2.2}$$

Put $\kappa(linesd(G)) = \kappa(EXP(G))$ and $\beta(sdG) = 2\varepsilon - (n+\varepsilon) + \omega = \varepsilon - n + \omega$ and $\kappa(sd(G)) = 2^{\varepsilon-n+\omega}$ into the (2.2),

$$\kappa(EXP(G)) = \frac{(2+k)^{\varepsilon-n+\omega}}{2k} \cdot (\frac{k}{2})^{\varepsilon-n} \cdot 2^{\varepsilon-n+\omega}\kappa(G),$$

that is

$$\kappa(EXP(G)) = 2^{\omega-1}(2+k)^{\varepsilon-n+\omega} \cdot k^{\varepsilon-n-1}\kappa(G), \tag{2.3}$$

hence (2.1) is true for $m = 1$.

Now assume (2.1) be true for $m-1$. Since $\kappa(EXP^m(G)) = \kappa(linesdG_{m-1})$, and

$$\kappa(linesdG_{m-1}) = \frac{(2+k)^{\beta(sdG_{m-1})}}{2k} \cdot (\frac{k}{2})^{\varepsilon(G_{m-1})-n(G_{m-1})} \cdot \kappa(sdG_{m-1}), \tag{2.4}$$

we have

$$\beta(sdG_{m-1}) = \varepsilon(G_{m-1}) - n(G_{m-1}) + \omega,$$
$$\varepsilon(G_{m-1}) = 2\varepsilon k^{m-1}, n(G_{m-1}) = 2\varepsilon k^{m-2} + \varepsilon k^{m-1},$$

and by inductive hypothesis $\kappa(sdG_{m-1}) = 2^{\beta(G_{m-1})} \cdot \kappa(G_{m-1})$,

$$\kappa(G_{m-1}) = 2^{(m-1)(\omega-1)} \cdot (2+k)^{\frac{k^{m-2}(k-2)+1}{k-1}\varepsilon - n + (m-1)\omega} \cdot k^{\frac{k^{m-2}(k-2)+1}{k-1}\varepsilon - n - (m-1)}\kappa(G).$$

Substitute all of above into the (2.4), we get that

$$\kappa(EXP^m(G)) = \frac{(2+k)^{\varepsilon k^{m-1} - 2\varepsilon k^{m-2} + \omega}}{2k} \cdot (\frac{k}{2})^{\varepsilon k^{m-1} - 2\varepsilon k^{m-2}} \cdot 2^{\beta(G_{m-1})} \cdot \kappa(G_{m-1}),$$

that is,

$$\kappa(EXP^m(G)) = 2^{m(\omega-1)} \cdot (2+k)^{\frac{k^{m-1}(k-2)+1}{k-1}\varepsilon - n + m\omega} \cdot k^{\frac{k^{m-1}(k-2)+1}{k-1}\varepsilon - n - m}\kappa(G). \qquad \square$$

Corollary 2.1 *Let G be a k- regular graph with n vertices, ε edges, if G is connected graph, then $\forall m \in \mathbb{N}, m \geq 1$*

$$\kappa(EXP^m(G)) = (2+k)^{\frac{k^{m-1}(k-2)+1}{k-1}\varepsilon - n + m} \cdot k^{\frac{k^{m-1}(k-2)+1}{k-1}\varepsilon - n - m}\kappa(G). \tag{2.5}$$

Specially, if $k = 2$, then

$$\kappa(EXP^m(G)) = 2^m \cdot \kappa(G) \tag{2.6}$$

and if $k = 3$, then

$$\kappa(EXP^m(G)) = 5^{\frac{3^{m-1}+1}{2} \cdot \varepsilon - n + m} \cdot 3^{\frac{3^{m-1}+1}{2} \cdot \varepsilon - n - m}\kappa(G), \tag{2.7}$$

if $m = 1$, then

$$\kappa(EXP(G)) = (2+k)^{\varepsilon - n + 1} \cdot k^{\varepsilon - n - 1} \cdot \kappa(G), \tag{2.8}$$

if $m = 2$, then

$$\kappa(EXP^2(G)) = (2+k)^{(k-1)\varepsilon - n + 2} \cdot k^{(k-1)\varepsilon - n - 2} \cdot \kappa(G). \tag{2.9}$$

Proof Let $\omega = 1$ in (2.1), we have (2.5) at once; and $k = 2, 3$ in (2.5) obtained (2.6) and (2.7); $m = 1, 2$ in (2.5) obtained (2.8) and (2.9). $\square$

§3. Examples

Example 3.1 Let G be a loop, then $\kappa(G) = 1$, and $EXP(G) = C_2$.[C_t denotes t-cycle] By (2.6), the order of sandpile group of $EXP(G)$, that is,

$$\kappa(C_2) = \kappa(EXP(G)) = 2.$$

Similarly

$$EXP^2(G) = C_{2^2}, \kappa(C_4) = \kappa(EXP^2(G)) = 2^2;$$

$$\cdots\cdots;$$

$$EXP^m(G) = C_{2^m}, \kappa(C_{2^m}) = \kappa(EXP^m(G)) = 2^m.$$

Example 3.2 Let θ be a θ-graph, then $\kappa(\theta) = 3$, and $EXP(\theta)$ is a Prism(see Fig.2). Then by (2.7), we have

$$\kappa(EXP(\theta)) = 5^{3-2+1} \cdot 3^{3-2-1} \cdot 3 = 75; \tag{3.1}$$

and

$$\kappa(EXP^2(\theta)) = 5^{\frac{3^{2-1}+1}{2}\cdot 3-2+2} \cdot 3^{\frac{3^{2-1}+1}{2}\cdot 3-2-2} \cdot 3 = 421875, \tag{3.2}$$

or by (2.8) , (2.1) and graph $EXP(\theta)$, we also have

$$\begin{aligned} &\kappa(EXP^2(\theta)) \\ &= (2+k)^{\varepsilon(EXP(\theta))-n(EXP(\theta))+1} \cdot k^{\varepsilon(EXP(\theta))-n(EXP(\theta))-1} \cdot \kappa(EXP(\theta)) \\ &= 5^{9-6+1} \cdot 3^{9-6-1} \cdot 75 = 421875. \end{aligned}$$

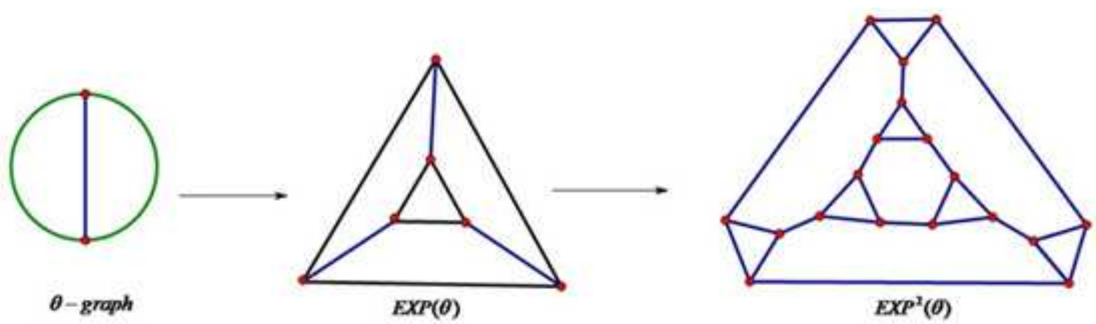

Fig.2 **θ-graph , 1th and 2th complete expansion graphs of the θ-graph**

Generally, if G is a multiplicity k's edges graph, that is, G have 2 vertices and k edges no loop connected graph, then by (2.5),

$$\kappa(EXP^m(G)) = (2+k)^{\frac{k^{m-1}(k-2)+1}{k-1}\cdot k-2+m} \cdot k^{\frac{k^{m+1}-2k^m+1}{k-1}-m}. \tag{3.3}$$

By (2.1),

$$m = 1 \Longrightarrow \kappa(EXP(G)) = (2+k)^{k-2} \cdot k^{k-1}; \tag{3.4}$$

and

$$m = 2 \Longrightarrow \kappa(EXP^2(G)) = (2+k)^{(k-1)k} \cdot k^{(k-1)k-3}. \tag{3.5}$$

Example 3.3 Let G be K_4, then $\kappa(K_4) = 4^2$, and $EXP(K_4)$ as Figure 3. By (2.5), we have

$$\kappa(EXP(K_4)) = 5^{6-4+1} \cdot 3^{6-4-1}\kappa(K_4) = 5^3 \cdot 3 \cdot 4^2 = 6000 \tag{3.6}$$

and by (2.9)

$$\kappa(EXP^2(K_4)) = (2+3)^{6(3-1)-4+2} \cdot 3^{6(3-1)-4-2} \cdot 4^2 = 11390625 \times 10^4,$$

or by (2.8) and (3.6) we have

$$
\begin{aligned}
&\kappa(EXP^2(K_4))\\
&= (2+k)^{\varepsilon(EXP(K_4))-n(EXP(K_4))+1}\cdot k^{\varepsilon(EXP(K_4))-n(EXP(K_4))-1}\cdot\kappa(EXP(K_4))\\
&= 5^{6+1}\cdot 3^{6-1}\cdot 6000 = 11390625\times 10^4.
\end{aligned}
$$

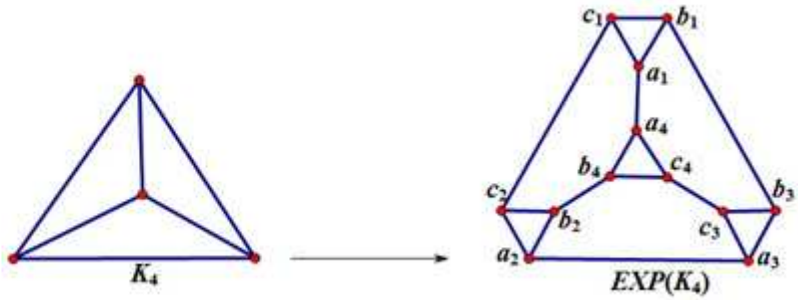

Fig.3 K_4 **and its expansion**

By Example 3.3 we have the following conclusion.

Proposition 3.1 *The order of sandpile group of Cayley graph* $Cay(A_4, \{(12),(123),(132)\})$ *is* 6000.

Proof Since $EXP(K_4) = Cay(A_4, \{(12),(123),(132)\})$, by[6], the proof is finished. □

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, North-Holland, New York, 1976.

[2] Chris Godsil and Gordon Royle, *Algebraic Graph Theory*, Springer-Verlag, 2004.

[3] N.Biggs, *Algebraic Graph Theory*, Cambridge University Press, 1993.

[4] N.Biggs, Algebraic potential theory on graphs, *Bull. London Math. Soc.*, 29(1997),641-682.

[5] Ayongga and Siqin, The expansion graph and the properties of its spectrum, *Journal of Baoji University of Arts and Sciences* (Natural Science), 29(2009),1-3.

[6] D.Cvetkoić, M.Doob and Sachs, Spectra of graphs, Theory and Application, *Pure and Applied Mathematics* 87, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980.

[7] Andrew Berget,Andrew Manion, Molly Maxwell et., The critical group a of regular line graph, *www.math.ucdavis.edu/ berget/research/*