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The Natural Lift Curve of the Spherical Indicatrix of a Curve According to Bishop Frame in Euclidean 3-Space

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Abstract: In this study, we dealt with the natural lift curves of the spherical indicatrices of a curve according to Bishop frame.Furthermore, some interesting results about the original curve were obtained depending on the assumption that the natural lift curves should be the integral curve of the geodesic spray on the tangent bundle $T\left(S^2\right)$.

Key Words: Natural lift, bishop frame, geodesic spray.

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§1. Introduction

Classical differential geometry began with the emergence of derivative and integral calculus, which helped solve geometric problems such as determining the tangents and curvatures of a curve.

Richard L. Bishop [1] brought forward the best answer to this as "there are 3 more than one way to crack a curve". Bishop observed that parallel vector fields on a C^2 regular curve form a 3-dimensional vector space. He clarified the equations of the Bishop roof, which is named after him; hence it is sometimes referred to as the Relatively Parallel Adapted Frame, Bishop, [1]. Fenchel W. [7] stated that a point $\gamma(t)$ on a curve, when plotting the curve, the Frenet vectors $\{T, N, B\}$ change and thus spherical signs are formed.

Thorpe J.A. [4], together with the geodesic spray concepts, gave the theorem that "for a curve γ to be an integral curve for the geodesic spray X of the natural lift γ, and only if γ is a geodesic over "M.Çalışkan, Sivridağ and Hacısalihoğlu [5], using these concepts and theorem given by Thorpe [4] in E^3.

§2. Preliminaries

Let $\gamma : I \longrightarrow \mathbb{R}^3$ be a parameterized curve. We denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve γ, where T, N and B are the tangent, the principal normal and the binormal vector fields of the curve γ, respectively.

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Let γ be a reguler curve in $\mathbb{R}^3$. Then ([6])

$$T = \frac{\gamma^{'}}{\|\gamma'\|},\ N = B \times T,\ B = \frac{\gamma^{'} \times \gamma^{''}}{\|\gamma' \times \gamma''\|}$$

and if γ is a unit speed curve, then

$$T = \gamma^{'},\ N = \frac{\gamma^{''}}{\|\gamma''\|},\ B = T \times N.$$

Let γ be a unit speed space curve with curvature κ and torsion τ. Let Frenet vector fields of γ be $\{T, N, B\}$. Then, Frenet formulas are given ([6]) by

$$T^{'} = \kappa N,\ N^{'} = -\kappa T + \tau B,\ B^{'} = -\tau N,$$

where $\kappa = \left\langle T^{'}, N \right\rangle$ and $\tau = \left\langle N^{'}, B \right\rangle$.

Definition 2.1 *Let $\gamma : I \to \mathbb{R}^3$ be a unit speed spacelike or timelik space curve. Let $T = \dot{\gamma}$ be the tangent vector defined at each point of the curve. In this case, M_1 and M_2 vectors are perpendicular to the tangent vector T at each point and any two vector fields in the normal plane, on the curve γ, $\{T, N, B\}$, there is always a frame $\{T, M_1, M_2\}$,as an alternative to the moving frame. $\{T, M_1, M_2\}$ is Bishop frame to this alternative frame*

Then, Frenet formulas are given by [1]

$$\begin{aligned}
\dot{T} &= k_1 M_1 + k_2 M_2, \\
\dot{M}_1 &= k_1 T, \\
\dot{M}_2 &= k_2 T, \\
\kappa(t) &= \sqrt{k_1^2 + k_2^2},\ \varphi(t) = \arctan\left(\frac{k_1}{k_2}\right),\ \tau(t) = \dot{\varphi}, \\
k_1 &= \kappa \cos\varphi,\ k_2 = \kappa \sin\varphi, \\
T &= T, \\
M_1 &= \cos\varphi N - \sin\varphi B, \\
M_2 &= \sin\varphi N + \cos\varphi B
\end{aligned}$$

where the differentiable functions k_1 and k_2 are the Bishop curvatures.

Definition 2.2([1]) *Let M be a hypersurface in $\mathbb{R}^3$ and let $\alpha : I \longrightarrow M$ be a parameterized curve. γ is called an integral curve of X if*

$$\frac{d}{ds}(\gamma(s)) = X(\gamma(s))\ \ (\textit{for all } t \in I),$$

where X is a smooth tangent vector field on M. We have

$$TM = \bigcup_{P \in M} T_P M = \chi(M)$$

where $T_P M$ is the tangent space of M at P and $\chi(M)$ is the space of vector fields on M.

Definition 2.3([4]) *For any parameterized curve $\gamma : I \longrightarrow M$, $\overline{\gamma} : I \longrightarrow TM$ given by*

$$\overline{\gamma}(s) = \left(\gamma(s), \gamma'(s)\right) = \gamma'(s)|_{\gamma(s)}$$

is called the natural lift of γ on TM. Thus, we can write

$$\frac{d\overline{\gamma}}{ds} = \frac{d}{ds}\left(\gamma'(s)|_{\gamma(s)}\right) = D_{\gamma'(s)}\gamma'(s),$$

where D is the Levi-Civita connection on $\mathbb{R}^3$.

Definition 2.4([4]) *A $X \in \chi(TM)$ is called a geodesic spray if for $V \in TM$*

$$X(V) = -\langle S(V), V\rangle N. \tag{1}$$

Theorem 2.5([4]) *The natural lift $\overline{\gamma}$ of the curve γ is an integral curve of geodesic spray X if and only if γ is a geodesic on M.*

§3 The Natural Lift Curve of the Spherical Indicatrix of a Curve According to Bishop Frame

Let ∇, $\overline{\nabla}$ be Levi-Civita connections on $\mathbb{R}^3$ and S^2 and ξ be a unit normal vector field of S^2. Then Gauss equations are given by the followings

$$\nabla_X Y = \overline{\nabla}_X Y + \langle S(X), Y\rangle \xi,$$

where S is the shape operator of S^2 and

$$S = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let γ_T be the spherical indicatrix of tangent vectors of γ and $\overline{\gamma}_T$ be the natural lift of the curve γ_T. In this case the equation for γ_T is $\gamma_T = T$. If $\overline{\gamma}_T$ is an integral curve of the geodesic spray, then from Theorem 2.5 we have

$$\overline{\nabla}_{\dot{\gamma}_T} \dot{\gamma}_T = 0,$$

that is,

$$\nabla_{\dot{\gamma}_T}\dot{\gamma}_T = \bar{\nabla}_{\dot{\gamma}_T}\dot{\gamma}_T - \left\langle S\left(\dot{\gamma}_T\right), \dot{\gamma}_T\right\rangle \xi$$

$$\nabla_{\dot{\gamma}_T}\dot{\gamma}_T = -\left\langle S\left(\dot{\gamma}_T\right), \dot{\gamma}_T\right\rangle T$$

$$\nabla_{\dot{\gamma}_T}\dot{\gamma}_T = -\left(k_1^2 + k_2^2\right) T$$

$$\frac{d}{ds_T}\left(k_1 M_1 + k_2 M_2\right) = -\left(k_1^2 + k_2^2\right) T$$

$$\frac{d}{ds}\left(k_1 M_1 + k_2 M_2\right)\frac{ds}{ds_T} + \left(k_1^2 + k_2^2\right) T = 0$$

$$\left(\dot{k}_1 M_1 - k_1^2 T + \dot{k}_2 M_2 - k_2^2 T\right)\frac{ds}{ds_T} + \left(k_1^2 + k_2^2\right) T = 0$$

$$\left(-\frac{k_1^2 + k_2^2}{\sqrt{k_1^2 + k_2^2}} + \left(k_1^2 + k_2^2\right)\right) T + \left(\frac{\dot{k}_1}{\sqrt{k_1^2 + k_2^2}}\right) M_1 + \left(\frac{\dot{k}_2}{\sqrt{k_1^2 + k_2^2}}\right) M_2 = 0$$

Since $\{T, M_1, M_2\}$ bishop frame is linearly independent system, we have

$$\begin{aligned}
\left(\frac{k_1^2 + k_2^2}{\sqrt{k_1^2 + k_2^2}} - \left(k_1^2 + k_2^2\right)\right) &= 0, \\
\left(\frac{\dot{k}_1}{\sqrt{k_1^2 + k_2^2}}\right) &= 0, \\
\left(\frac{\dot{k}_2}{\sqrt{k_1^2 + k_2^2}}\right) &= 0.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
k_1^2 + k_2^2 &= 1, \\
i)\ k_1 &= 1,\ k_2 = 0 \\
ii)\ k_1 &= 0,\ k_2 = 1
\end{aligned}$$

Proposition 3.1 *If the natural lift $\overline{\gamma}_T$ of γ_T is an integral curve of the geodesic on the tangent bundle $T\left(S^2\right)$, then there is a relationship between frames $\{T, N, B\}$ and $\{T, M_1, M_2\}$ as follows,*

$$\begin{cases}
T = T,\ M_1 = N,\ \ M_2 = B,\ (k_1 = 1,\ k_2 = 0) \\
\\
T = T,\ M_1 = -B,\ \ M_2 = N, (\ k_1 = 0,\ k_2 = 1)
\end{cases}$$

Let γ_{M_1} be the spherical indicatrix of γ relative to M_1 and $\overline{\gamma}_{M_1}$ be the natural lift of the curve γ_{M_1}. In this case the equation for γ_{M_1} is $\gamma_{M_1} = M_1$. If $\overline{\gamma}_{M_1}$ is an integral curve of the

geodesic spray, then from Theorem 2.5 we have

$$\bar{\nabla}_{\dot{\gamma}_{M_1}} \dot{\gamma}_{M_1} = 0,$$

that is,

$$\nabla_{\dot{\gamma}_{M_1}} \dot{\gamma}_{M_1} = \bar{\nabla}_{\dot{\gamma}_{M_1}} \dot{\gamma}_{M_1} - \left\langle S\left(\dot{\gamma}_{M_1}\right), \dot{\gamma}_{M_1} \right\rangle \xi$$

$$\nabla_{\dot{\gamma}_{M_1}} \dot{\gamma}_{M_1} = -\left\langle S\left(\dot{\gamma}_{M_1}\right), \dot{\gamma}_{M_1} \right\rangle M_1$$

$$\nabla_{\dot{\gamma}_{M_1}} \dot{\gamma}_{M_1} = -k_1^2 M_1$$

$$\frac{d}{ds_{M_1}} (-k_1 T) = -k_1^2 M_1$$

$$\frac{d}{ds} (-k_1 T) \frac{ds}{ds_{M_1}} + k_1^2 M_1 = 0$$

$$\left(-\dot{k}_1 T - k_1^2 M_1 - k_1 k_2 M_2\right) \frac{ds}{ds_{M_1}} + k_1^2 M_1 = 0$$

$$\left(-\frac{\dot{k}_1}{k_1^2}\right) T + \left(\frac{-k_1 + k_1^2}{k_1^2}\right) M_1 + \left(-\frac{k_2}{k_1}\right) M_2 = 0$$

Since $\{T, M_1, M_2\}$ bishop frame is linearly independent system, we have

$$\begin{aligned} \frac{\dot{k}_1}{k_1^2} &= 0, \\ \left(-k_1 + k_1^2\right) &= 0, \\ k_2 &= 0. \end{aligned}$$

Hence, we have

$$k_1 = constant \ \ (k_1 \neq 0), \ k_2 = 0$$

Proposition 3.2 *If the natural lift $\overline{\gamma}_{M_1}$ of γ_{M_1} is an integral curve of the geodesic on the tangent bundle $T\left(S^2\right)$, then there is a relationship between frames $\{T, N, B\}$ and $\{T, M_1, M_2\}$ as follows,*

$$T = T, \ M_1 = N, \quad M_2 = B.$$

Let γ_{M_2} be the spherical indicatrix of γ relative to M_2 and $\overline{\gamma}_{M_2}$ be the natural lift of the curve γ_{M_2}. In this case the equation for γ_{M_2} is $\gamma_{M_2} = M_2$. If $\overline{\gamma}_{M_2}$ is an integral curve of the geodesic spray, then from Theorem 2.5 we have

$$\bar{\nabla}_{\dot{\gamma}_{M_2}} \dot{\gamma}_{M_2} = 0,$$

that is,

$$\nabla_{\dot{\gamma}_{M_2}}\dot{\gamma}_{M_2} = \bar{\nabla}_{\dot{\gamma}_{M_2}}\dot{\gamma}_{M_2} - \left\langle S\left(\dot{\gamma}_{M_2}\right), \dot{\gamma}_{M_2}\right\rangle \xi$$

$$\nabla_{\dot{\gamma}_{M_2}}\dot{\gamma}_{M_2} = -\left\langle S\left(\dot{\gamma}_{M_2}\right), \dot{\gamma}_{M_2}\right\rangle M_2$$

$$\nabla_{\dot{\gamma}_{M_2}}\dot{\gamma}_{M_2} = -k_2^2 M_2$$

$$\frac{d}{ds_{M_2}}\left(-k_2 T\right) = -k_2^2 M_2$$

$$\frac{d}{ds}\left(-k_2 T\right)\frac{ds}{ds_{M_2}} + k_2^2 M_2 = 0$$

$$\left(-\dot{k}_2 T - k_1 k_2 M_1 - k_2^2 M_2\right)\frac{ds}{ds_{M_2}} + k_2^2 M_2 = 0$$

$$\left(-\frac{\dot{k}_2}{k_2}\right)T + \left(-\frac{k_1}{k_2}\right)M_1 + \left(-1 + k_2^2\right)M_2 = 0$$

Since $\{T, M_1, M_2\}$ Bishop frame is linearly independent system, we have

$$\frac{\dot{k}_2}{k_2} = 0, \quad \frac{k_1}{k_2} = 0, \quad -1 + k_2^2 = 0.$$

Hence, we have

$$k_1 = 0, \; k_2 = \pm 1$$

Proposition 3.3 *If the natural lift $\overline{\gamma}_{M_2}$ of γ_{M_2} is an integral curve of the geodesic on the tangent bundle $T\left(S^2\right)$, then there is a relationship between frames $\{T, N, B\}$ and $\{T, M_1, M_2\}$ as follows*

$$\begin{aligned} T &= T, \; M_1 = -B, \quad M_2 = N, \\ T &= T, \; M_1 = B, \quad M_2 = -N, \\ k_1 &= 0, \; k_2 = \pm 1. \end{aligned}$$

Example 3.4 Given the arclength timelike curve $\gamma(s) = \left(\frac{\sqrt{5}}{3}s, \frac{2}{9}\cos 3s, \frac{2}{9}\sin 3s\right)$.In this trihedron, it is easy to show that its Frenet apparatus are

$$\begin{aligned} T(s) &= \left(\frac{\sqrt{5}}{3}, -\frac{2}{3}\sin 3s, \frac{2}{3}\cos 3s\right), \\ N(s) &= (0, -\cos 3s, -\sin 3s), \\ B(s) &= \left(\frac{2}{3}, \frac{\sqrt{5}}{3}\sin 3s, -\frac{\sqrt{5}}{3}\cos 3s\right), \end{aligned}$$

If the natural lift $\overline{\gamma}_T$ of γ_T is an integral curve of the geodesic on the tangent bundle $T\left(S^2\right)$,

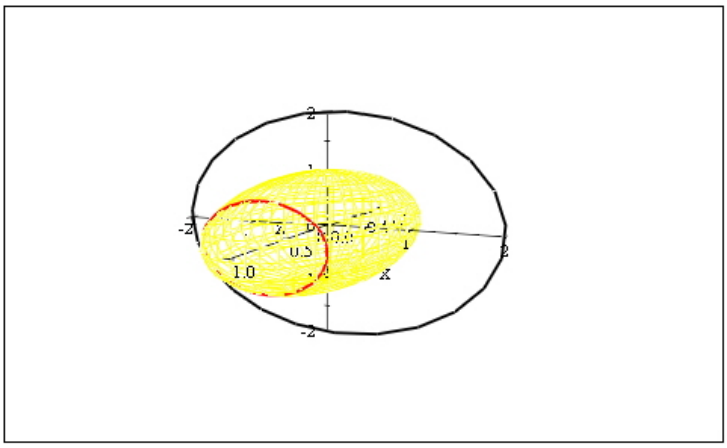
then there is a relationship between frames $\{T, N, B\}$ and $\{T, M_1, M_2\}$ as follows

$$\begin{cases} T = T,\ M_1 = N,\ \ M_2 = B,\ (k_1 = 1,\ k_2 = 0) \\ T = T,\ M_1 = -B,\ \ M_2 = N, (\ k_1 = 0,\ k_2 = 1) \end{cases}$$

with

$$\begin{aligned} T(s) &= \left(\frac{\sqrt{5}}{3}, -\frac{2}{3}\sin 3s, \frac{2}{3}\cos 3s\right), \\ M_1(s) &= (0, -\cos 3s, -\sin 3s), \\ M_2(s) &= \left(\frac{2}{3}, \frac{\sqrt{5}}{3}\sin 3s, -\frac{\sqrt{5}}{3}\cos 3s\right), \\ & (k_1 = 1,\ k_2 = 0) \end{aligned}$$

$$P(s,t) = (\sin 3s \cos 3t, \sin 3s \sin 3t, \cos 3s)$$



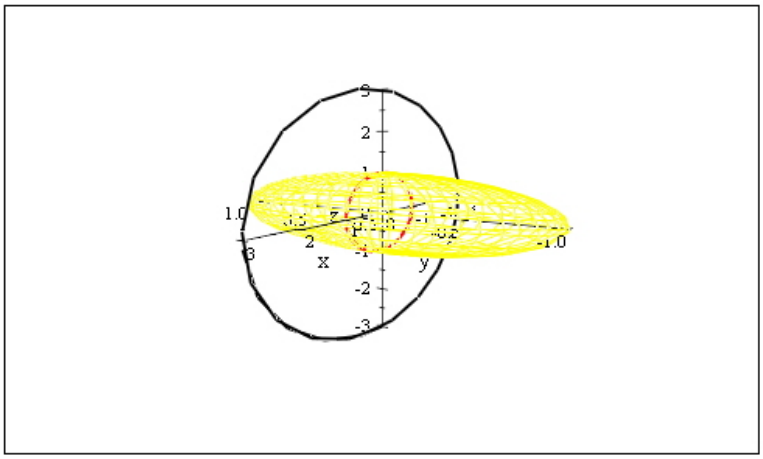
$\gamma_T = \left(\frac{\sqrt{5}}{3}, -\frac{2}{3}\sin 3s, \frac{2}{3}\cos 3s\right)$ and $\gamma_T = (0, -2\cos 3s, -2\sin 3s)$

If the natural lift $\overline{\gamma}_{M_1}$ of γ_{M_1} is an integral curve of the geodesic on the tangent bundle $T\left(S^2\right)$, then there is a relationship between frames $\{T, N, B\}$ and $\{T, M_1, M_2\}$ as follows

$$T = T,\ M_1 = N,\ \ M_2 = B.$$

with

$$\begin{aligned} T(s) &= \left(\frac{\sqrt{5}}{3}, -\frac{2}{3}\sin 3s, \frac{2}{3}\cos 3s\right), \\ M_1(s) &= (0, -\cos 3s, -\sin 3s), \\ M_2(s) &= \left(\frac{2}{3}, \frac{\sqrt{5}}{3}\sin 3s, -\frac{\sqrt{5}}{3}\cos 3s\right), \\ P(s,t) &= (\cos 3s \sin 3t, \cos 3s \cos 3t, \sin 3s). \end{aligned}$$

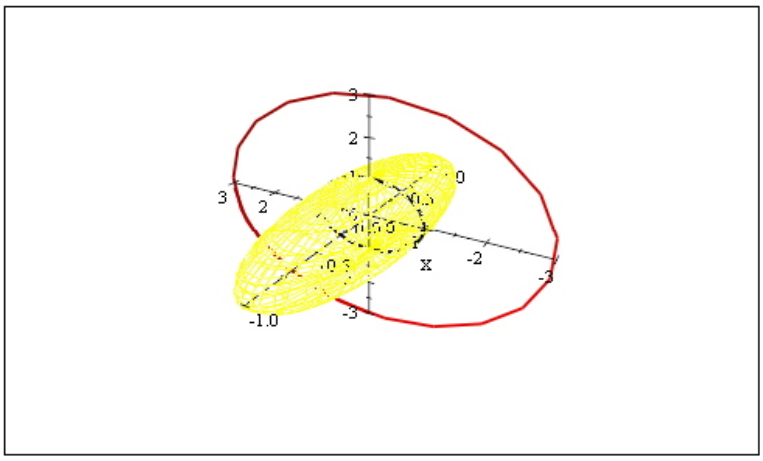


$\gamma_{M_1} = (0, -\cos 3s, -\sin 3s)$ and $\gamma_{M_1} = (0, 3\sin 3s, -3\cos 3s)$

If the natural lift $\overline{\gamma}_{M_2}$ of γ_{M_2} is an integral curve of the geodesic on the tangent bundle $T\left(S^2\right)$, then there is a relationship between frames $\{T, N, B\}$ and $\{T, M_1, M_2\}$ as follows

$$\begin{aligned}
T &= T,\ M_1 = -B,\quad M_2 = N, \\
T &= T,\ M_1 = B,\quad M_2 = -N, \\
k_1 &= 0,\ k_2 = \pm 1, \\
T(s) &= \left(\frac{\sqrt{5}}{3}, -\frac{2}{3}\sin 3s, \frac{2}{3}\cos 3s\right), \\
M_1(s) &= \left(-\frac{2}{3}, -\frac{\sqrt{5}}{3}\sin 3s, \frac{\sqrt{5}}{3}\cos 3s\right), \\
M_2(s) &= (0, -\cos 3s, -\sin 3s).
\end{aligned}$$

$$P(s,t) = (\cos 3s \sin 3t, \cos 3s \cos 3t, \sin 3s)$$



$\gamma_{M_2} = (0, -\cos 3s, -\sin 3s)$ and $\gamma_{M_2} = (0, 3\sin 3s, -3\cos 3s)$

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