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The Number of Chains of Subgroups in the Lattice of Subgroups of Group $Z_m \times A_n, n \leq 6, m \leq 3$

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Abstract: In this paper, we established the number of chains of subgroups in the subgroup lattice of the Cartesian product of the alternating group and cyclic group using computational technique induced by the set of representatives of isomorphism classes of subgroups.

Key Words: Subgroups, alternating group, chains of subgroups and fuzzy subgroups.

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§1. Introduction

All groups are finite throughout this paper.For a given group G, the lattice of subgroups is $(L(G), \leq)$ here by $L(G)$ mean all subgroups of G and the partial order $\leq$ works as set inclusion. If chain of subgroups of G contains G then it is called rooted (more precisely G-rooted) otherwise it is known as unrooted.

In this paper the study of chains of subgroups describes the set containing all chains of subgroups of G, which ends in with G.A formula of lattice of a finite cyclic group, for number of chains of subgroups was given by Tărnăuceanu and Bentea [1] by giving its one variable generating function. J.M. Oh in his paper [2] determined the number of subgroups of a finite cyclic group of 4n by giving its multi variables generating function. The problem of counting chains of subgroups in the lattice of subgroups of for any given group G got attention of researchers specially classifying fuzzy subgroups of G under a specific type of equivalence relation (see [3], [4]).

§2. Preliminaries

A set of subgroups of G fully ordered by set inclusion is a chain of subgroups of group G. In this paper, the chain of subgroups of G which end in G.

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Let us suppose a finite group G and $\mu\colon G \mapsto [0,1]$ be its fuzzy subgroup of G. By Putting $\mu(G) = \{\alpha_1, \alpha_2, \cdots, \alpha_r\}$ where $\alpha_1<\alpha_2<\cdots<\alpha_r$. Then, μ determines the chain of subgroups of G which ends in group G:

$$\mu G\alpha_1 \subset \mu G\alpha_2 \subset \cdots \subset \mu G\alpha_m = G$$

Also for any element x of group G and $i = \overline{1,r}$, we get

$$\mu(x) = \alpha_i \Leftrightarrow i = max\{j|x \in \mu G\alpha_j\} \Leftrightarrow x \in \mu G_{\alpha_i} \setminus \mu G\alpha_{1-r}$$

The authors of [7] identified the necessary and sufficient condition for equivalence of two fuzzy subgroups μ, η of G with respect to $\sim$, which is $\mu \sim \eta$ if and only if set of level subgroups of μ and η are same.For a fuzzy subgroup of group G, the corresponding equivalence classes are closely connected to the chains of subgroups in group G in this case.To determine these classes, we calculate the number of all chains of subgroups of G that terminate in G.

Let G be a finite group and $\delta(G)$ be the number of chains of subgroups of G that terminate in G. We have two kind of subgroup chains following:

(1) $G_1 \subset G_2 \subset \cdots \subset G_k = G$ with $G_1 \neq \{e\}$;

(2) $\{e\} \subset G_2 \subset \cdots \subset G_k = G$.

It is clear that the numbers of chains of types (1) and (2) are equal. So,

$$\delta(G) = 2x_k.$$

Obviously, this problem of counting all chains of subgroups of G depends entirely on the lattice of subgroups of G and not on the group itself. This leads to a more general problem.

Theorem 2.1 *Let $\delta(G)$ be the number of subgroup chains of group G that terminates in G. Then*

$$\delta(G) = \sum_{distict H \in Iso(G)} \delta(H) \times n(H),$$

where $Iso(G)$ is the set of representatives of isomorphism classes of subgroups of G, $n(H)$ denotes the size of the isomorphism class with representative H.

Proof Let fixes $\delta(H_1) = \delta(H_\alpha) = 1$, for which H_1 is the trivial group of and H_α is the improper subgroup of G for any $H_i \in Iso(G)$ and $i = \overline{1,\alpha}$. Then

$$\begin{aligned}\delta(G) &= n(H_1) * \delta(H_1) + n(H_1) * \delta(H_2) + n(H_3)\delta(H_3) + \cdots + n(H_\alpha)\delta(H_\alpha) \\ &= \sum_{H_i \in Iso(G)} \delta(H_i) \times n(H_i) \\ &= 2 + \sum_{distict H_i \in Iso(G)} \delta(H_i) \times n(H_i)\end{aligned}$$

This completes the proof. $\square$

In this work, Theorem 2.1 is used to obtain the number of chains of subgroups of G that terminates in G. It also follows that

(i) $\delta(Z_p) = 2$, where p is prime;
(ii) $\delta(Z_{pq}) = 6$, where p and q are distinct prime;
(iii) $\delta(Z_{p^2}) = 4$, where p is any prime;
(iv) $\delta(Z_p \times Z_p) = 2p + 4$, where p is any prime;
(v) $\delta(Z_p \times Z_p \times Z_p) = 2p^3 + 8p^2 + 8p + 8$, where p is any prime;
(vi) $\delta(Z_{pq} \times Z_p) = 16 + 10p$, where p and q are distinct primes.

Above result are special cases of Theorems or Corollaries $5.1, 5.2, 5.4, 5.5, 5.6$ of [6] and Section 3 of [7].

Lemma 2.1([3]) *For groups S_3, A_4 and S_4, we have $\delta(S_3) = 10$, $\delta(A_4) = 24$ and $\delta(S_4) = 232$ respectively.*

Lemma 2.2([8]) *Let G be Dihedral group of order 2p, where p is any prime then , $\delta(G) = 4+2p$*

§3. The Number of Chains of Subgroups of $Z_2 \times A_n, n \leq 6$

Theorem 3.1 *The number of chains of subgroups of $Z_2 \times A_3$ is 6*

Proof The direct product A_3 and Z_2 is isomorphic to a cyclic group of order 6. Then, $\delta(Z_2 \times A_3) = 6$. □

Theorem 3.2 *The number of chains of subgroups of $Z_2 \times A_4$ is 200.*

Proof Notice that $Z_2 \times A_4$ is a non-Abelian group of order 24, it has the following set of representatives of isomorphism classes of subgroups with their sizes:

$[e, 1]$, $[Z_2, 7]$, $[Z_3, 4]$, $[Z_2 \times Z_2, 7]$, $[Z_6, 4]$, $[(Z_2 \times Z_2 \times Z_2), 1]$, $[A_4, 1]$ and $[(Z_2 \times A_4), 1]$.

So,

$$\begin{aligned}\delta(Z_2 \times A_4) &= 1 + \delta(H_e) + 7 \times \delta(Z_2) + 4 \times \delta(Z_3) + 7 \times \delta(Z_2 \times Z_2) \\ &\quad +4 \times \delta(Z_6) + \delta(Z_2 \times Z_2 \times Z_2) + \delta(A_4) = 200.\end{aligned}$$

This completes the proof. □

Theorem 3.3 *The number of chains of subgroups of group $Z_2 \times A_5$ is 3292.*

Proof Notice that $Z_2 \times A_5$ is non-Abelian of order 120, it has the following set of representatives of isomorphism classes of subgroups with their sizes:

$[e, 1]$, $[Z_2, 31]$, $[Z_3, 10]$, $[Z_2 \times Z_2, 35]$, $[Z_5, 6]$, $[Z_6, 10]$, $[(Z_2 \times Z_2 \times Z_2), 5]$, $[Z_{10}, 6]$, $[S_3, 20]$, $[D_5, 12]$, $[D_6, 10]$, $[D_{10}, 6]$, $[Z_2 \times A_4, 5]$, $[A_4, 5]$, $[A_5, 1]$ and $[(Z_2 \times A_5), 1]$.

So,

$$\begin{aligned}\delta(Z_2 \times A_5) &= 1 + \delta(H_e) + 31 \times \delta(Z_2) + 10 \times \delta(Z_3) + 35 \times \delta(Z_2 \times Z_2) + 6 \times \delta(Z_5) \\ &\quad +10 \times \delta(Z_6) + 5 \times \delta(Z_2 \times Z_2 \times Z_2) + 6 \times \delta(Z_{10}) + 20 \times \delta(S_3) \\ &\quad +12 \times \delta(D_5) + 10 \times \delta(D_6) + 6 \times \delta(D_{10}) + 5 \times \delta(A_4) + \delta(A_5) = 3292.\end{aligned}$$

This completes the proof. □

Proposition 3.1(By Proposition 3 of [8]) *Let the wreath product of the cyclic groups : $(Z_3 \times Z_3) \ltimes Z_2$ and $\delta(Z_3 \times Z_3) \ltimes Z_4)$. Then, $\delta(Z_3 \times Z_3) \ltimes Z_2) = 158$ and $\delta(Z_3 \times Z_3) \ltimes Z_2) = 352$.*

Theorem 3.4 *Suppose that G be the Cartesian product of group $D_p \times Z_q$, where p, q are distinct prime numbers, then*

$$\delta(G) = \begin{cases} 16p + 20 & \text{if } q = 2 \\ 14p + 12 & \text{if } p = q \\ 10p + 16 & \text{if } p \neq q, q \neq 2 \end{cases}$$

Proof Our proof is divided into 3 cases following:

Case 1. $q = 2$.

Notice that $D_p \times Z_q$ has the following set of representatives of isomorphism classes of subgroups with their sizes:

$$H_e,\ Z_2(2p+1 \text{ times}),\ Z_p, Z_{pq}, D_p\ (2 \text{ times}),\ Z_2 \times Z_2 \text{ and } D_p \times Z_q.$$

Then,

$$\begin{aligned}\delta(D_p \times Z_q) &= 1 + \delta(H_e) + (2p+1) \times \delta(Z_2) + \delta(Z_p) \\ &\quad +\delta(Z_{pq}) + p \times \delta(Z_2 \times Z_2) + 2 \times \delta(D_p = 16p + 20.\end{aligned}$$

Case 2. $p = q$.

Notice that $D_p \times Z_q$ has the following set of representatives of isomorphism classes of subgroups with their sizes:

$$H_e,\ Z_2\ (p \text{ times}),\ Z_p\ (p+1 \text{ times}), Z_{pq}\ (p \text{ times}),\ D_p, Z_p \times Z_p \text{ and } D_p \times Z_q.$$

Then,

$$\begin{aligned}\delta(D_p \times Z_q) &= 1 + \delta(H_e) + p \times \delta(Z_2) + (p+1) \times \delta(Z_p) \\ &\quad +p \times \delta(Z_{pq}) + \delta(Z_p \times Z_p) + \delta(D_p = 14p + 12.\end{aligned}$$

Case 3. $p \neq q$, $q \neq 2$.

Notice that $D_p \times Z_q$ has the following set of representatives of isomorphism classes of subgroups with their sizes:

$$H_e,\ Z_2\ (p\text{ times}),\ Z_p, Z_q, Z_{pq}, Z_{2p}\ (p\text{ times}),\ D_p \text{ and } D_p\times Z_q.$$

Then,

$$\begin{aligned}\delta(D_p\times Z_q) &= 1+\delta(H_e)+p\times\delta(Z_2)+\delta(Z_p)\\ &\quad +\delta(Z_q)+\delta(Z_{pq})+p\times\delta(Z_{2p})+\delta(D_p)=10p+16.\end{aligned}$$

This completes the proof. □

Theorem 3.5 *Let G be the direct product of Z_2 and $Z_3^2\ltimes Z_2$, then, $\delta(G)=1572$.*

Proof Notice that $Z_2\times(Z_3\times Z_3)\ltimes Z_2$ has the following set of representatives of isomorphism classes of subgroups with their sizes:

$$\begin{aligned}&[e,1],\ [Z_2,19],\ [Z_2\times Z_2,9],\ [Z_3\times Z_3,1],\ [Z_3,4],\ [Z_6,4],\ [Z_6\times Z_3,1],\\ &[S_3,24],\ [D_6,12],\ [(Z_3\times Z_3)\ltimes Z_2,2]\ \text{ and }\ [Z_2\times(Z_3\times Z_3)\ltimes Z_2,1].\end{aligned}$$

So,

$$\begin{aligned}\delta(G) &= 1+\delta(H_e)+19\times\delta(Z_2)+4\times\delta(Z_3)+9\times\delta(Z_2\times Z_2)\\ &\quad +\delta(Z_3\times Z_3)+4\times\delta(Z_6)+\delta(Z_6\times Z_3)+24\times\delta(S_3)\\ &\quad +12\times\delta(D_6)+2\times\delta((Z_3\times Z_3)\ltimes Z_2)=1572.\end{aligned}$$

This completes the proof. □

Theorem 3.6 *Let G be the direct product of Z_2 and $Z_3^2\ltimes Z_4$,then, $\delta(G)=4136$.*

Proof Notice that $Z_2\times(Z_3\times Z_3)\ltimes Z_4$ is a non-Abelian, A-group of order 72. It has the following set of representatives of isomorphism classes of subgroups with their sizes:

$$\begin{aligned}&[e,1],\quad [Z_2,19],\quad [Z_2\times Z_2,9],\quad [Z_3\times Z_3,1],\quad [Z_3,4],\quad [Z_4,18],\quad [Z_6,4],\\ &[Z_4\times Z_2,9],\quad [Z_6\times Z_3,1],\quad [S_3,24],\quad [D_6,12],\quad [(Z_3\times Z_3)\ltimes Z_2,2],\\ &[(Z_3\times Z_3)\ltimes Z_4,2],\ [Z_2\times(Z_3\times Z_3)\ltimes Z_2,1] \text{ and } [Z_2\times(Z_3\times Z_3)\ltimes Z_4,1].\end{aligned}$$

So,

$$\begin{aligned}\delta(G) &= 1+\delta(H_e)+19\times\delta(Z_2)+4\times\delta(Z_3)+18\times\delta(Z_4)+9\times\delta(Z_2\times Z_2)\\ &\quad +\delta(Z_3\times Z_3)+9\times\delta(Z_4\times Z_2)+4\times\delta(Z_6)+\delta(Z_6\times Z_3)+24\times\delta(S_3)\\ &\quad +12\times\delta(D_6)+2\times\delta((Z_3\times Z_3)\ltimes Z_2)+2\times\delta((Z_3\times Z_3)\ltimes Z_4)\\ &\quad +\delta(Z_2\times(Z_3\times Z_3)\ltimes Z_2)=4136.\end{aligned}$$

This completes the proof. □

Theorem 3.7 *The number of chains of subgroups of $Z_2\times A_6$ is 301320.*

Proof Notice that $Z_2\times A_6$ is non-Abelian of order 720. It has the following set of representatives of isomorphism classes of subgroups with their sizes:

$[e,1]$, $[Z_2,91]$, $[Z_2\times Z_2,165]$, $[Z_3,40]$, $[Z_4,90]$, $[Z_5,36]$, $[Z_6,40]$, $[Z_{10},36]$, $[Z_2\times Z_2\times Z_2,30]$, $[Z_3\times Z_3,10]$, $[Z_2\times D_4,45]$, $[Z_2\times S_4,30]$, $[Z_4\times Z_2,45]$, $[(Z_6\times Z_3),10]$, $[Z_2\times(Z_3\times Z_3)\ltimes Z_4,10]$, $[(Z_3\times Z_3)\ltimes Z_2,20]$, $[Z_3\times Z_3\ltimes Z_4,20]$, $[Z_2\times(Z_3\times Z_3)\ltimes Z_2,10]$, $[Z_{10},6]$, $[S_3,240]$, $[S_4,60]$, $[D_4,180]$, $[D_5,72]$, $[D_6,120]$, $[D_{10},36]$, $[Z_2\times A_4,30]$, $[A_4,30]$, $[A_5,12]$, $[A_6,1]$, $[(Z_2\times A_5),12]$ and $[(Z_2\times A_6),1]$.

So,

$$\begin{aligned}\delta(Z_2\times A_6) &= 1+\delta(H_e)+91\times\delta(Z_2)+40\times\delta(Z_3)+90\times\delta(Z_4)\\ &+36\times\delta(Z_5)+40\times\delta(Z_6)+36\times\delta(Z_{10})+165\times\delta(Z_2\times Z_2)\\ &+20\times\delta(Z_2\times Z_2\times Z_2)+30\times\delta(Z_2\times A_4)+10\times\delta(Z_2\times(Z_3\times Z_3)\ltimes Z_2)\\ &+10\times\delta(Z_2\times(Z_3\times Z_3)\ltimes Z_4)+12\times\delta(Z_2\times A_5)+45\times\delta(Z_2\times D_4)\\ &+30\times\delta(Z_2\times S_4)+10\times\delta(Z_3\times Z_3)+45\times\delta(Z_4\times Z_2)+10\times\delta(Z_6\times Z_3)\\ &+20\times\delta((Z_3\times Z_3)\ltimes Z_2)+20\times\delta((Z_3\times Z_3)\ltimes Z_4)+30\times\delta(A_4)\\ &+12\times\delta(A_5)+\delta(A_6)+240\times\delta(S_3)+60\times\delta(S_4)+180\times\delta(D_4)\\ &+72\times\delta(D_5)+120\times\delta(D_6)+36\times\delta(D_{10})=301320.\end{aligned}$$

This completes the proof. □

Then, we get the following theorem.

Theorem 3.8 *The number of chains of subgroups $Z_2\times A_n, n\leq 6$, then*

$$\delta(Z_2\times A_n)=\begin{cases}6 & n=3\\ 200 & n=4\\ 3292 & n=5\\ 301320 & n=6\end{cases}$$

§4. The Number of Chains of Subgroups of $Z_3\times A_n, n\leq 6$

Theorem 4.1 *The number of chains of subgroups of $Z_3\times A_3$ is* 10.

Proof The direct product A_3 and Z_3 is isomorphic to $Z_3\times Z_3$. So, $\delta(Z_3\times A_3)=10$. □

Theorem 4.2 *The number of chains of subgroups of $Z_3\times A_4$ is 208.*

Proof Notice that $Z_3\times A_4$ is a non-Abelian group of order 36. It has the following set of representatives of isomorphism classes of subgroups with their sizes:

$[e,1], [Z_2,3], [Z_2\times Z_2,1], [Z_3\times Z_3,4], [Z_3,13], [Z_6,3], [Z_6\times Z_2,1], [A_4,3]$ and $[Z_3\times A_4,1]$.

So,

$$\begin{aligned}\delta(Z_3 \times A_4) &= 1 + \delta(H_e) + 3 \times \delta(Z_2) + 13 \times \delta(Z_3) + 3 \times \delta(Z_6)\\ &+\delta(Z_2 \times Z_2) + 4 \times \delta(Z_3 \times Z_3) + \delta(Z_6 \times Z_2) + 3 \times \delta(A_4) = 208.\end{aligned}$$

This completes the proof. □

Theorem 4.3 *The number of Chains of Subgroups of* $Z_3 \times A_5$ *is* 3440.

Proof Notice that $Z_3 \times A_5$ is non-Abelian group of order 180. It is isomorphic to the general linear group (2,4). The group has the following set of representatives of isomorphism classes of subgroups with their sizes:

$[e, 1]$, $[Z_2, 15]$, $[Z_2 \times Z_2, 5]$, $[Z_3 \times Z_3, 10]$, $[Z_3, 11]$, $[Z_3 \times D_5, 10]$, $[Z_3 \times S_3, 10]$, $[Z_3 \times A_4, 5]$, $[Z_5, 6]$, $[Z_6, 15]$, $[Z_{15}, 6]$, $[S_3, 10]$, $[A_4, 15]$, $[A_5, 1]$ and $[(Z_3 \times A_5), 1]$.

So,

$$\begin{aligned}\delta(Z_3 \times A_5) &= 1 + \delta(H_e) + 15 \times \delta(Z_2) + 11 \times \delta(Z_3) + 6 \times \delta(Z_5)\\ &+15 \times \delta(Z_6) + 5 \times \delta(Z_2 \times Z_2) + 10 \times \delta(Z_3 \times Z_3)\\ &+10 \times \delta(Z_3 \times D_5) + 10 \times \delta(Z_3 \times S_3) + 5 \times \delta(Z_3 \times A_4)\\ &+10 \times \delta(S_3) + 6 \times \delta(Z_{15}) + 15 \times \delta(A_4) + \delta(A_5) = 3440.\end{aligned}$$

This completes the proof. □

Theorem 4.4 *Let* G *be the direct product of* Z_3 *and* $Z_3^2 \ltimes Z_2$ *,then,* $\delta(G) = 1314$.

Proof Notice that $Z_3 \times (Z_3 \times Z_3) \ltimes Z_2$ has the following set of representatives of isomorphism classes of subgroups with their sizes:

$[e, 1]$, $[Z_2, 9]$, $[Z_3 \times Z_3, 13]$, $[Z_3, 13]$, $[Z_6, 9]$, $[Z_3 \times S_3, 12]$, $[S_3, 12]$, $[Z_3 \times Z_3 \times Z_3, 1]$, $[(Z_3 \times Z_3) \ltimes Z_2, 1]$ and $[Z_3 \times (Z_3 \times Z_3) \ltimes Z_2, 1]$.

So,

$$\begin{aligned}\delta(G) &= 1 + \delta(H_e) + 9 \times \delta(Z_2) + 13 \times \delta(Z_3) + \delta(Z_3 \times Z_3 \times Z_3)\\ &+13 \times \delta(Z_3 \times Z_3) + 9 \times \delta(Z_6) + 12 \times \delta(Z_3 \times S_3)\\ &+12 \times \delta(S_3) + \delta((Z_3 \times Z_3) \ltimes Z_2) = 1314.\end{aligned}$$

This completes the proof. □

Theorem 4.5 *Let* G *be the direct product of* Z_3 *and* $Z_3^2 \ltimes Z_4$ *,then,* $\delta(G) = 3160$.

Proof Notice that $Z_3 \times (Z_3 \times Z_3) \ltimes Z_4$ is a non-Abelian group of order 108. It has the following set of representatives of isomorphism classes of subgroups with their sizes:

$[e, 1]$, $[Z_2, 9]$, $[Z_3 \times Z_3, 13]$, $[Z_3, 13]$, $[Z_4, 9]$, $[Z_6, 9]$, $[Z_{12}, 9]$,

$$[Z_3 \times S_3, 12], \quad [S_3, 12], \quad [Z_3 \times Z_3 \times Z_3, 1], \quad [(Z_3 \times Z_3) \ltimes Z_2, 1],$$
$$[(Z_3 \times Z_3) \ltimes Z_4, 1], [Z_3 \times (Z_3 \times Z_3) \ltimes Z_2, 1] \text{ and } [Z_3 \times (Z_3 \times Z_3) \ltimes Z_4, 1].$$

So,

$$\begin{aligned}
\delta(G) &= 1 + \delta(H_e) + 9 \times \delta(Z_2) + 13 \times \delta(Z_3) + 9 \times \delta(Z_4) \\
&\quad + 9 \times \delta(Z_6) + 9 \times \delta(Z_{12}) + \delta(Z_3 \times Z_3 \times Z_3) \\
&\quad + 13 \times \delta(Z_3 \times Z_3) + 12 \times \delta(Z_3 \times S_3) + 12 \times \delta(S_3) \\
&\quad + \delta((Z_3 \times Z_3) \ltimes Z_2) + \delta((Z_3 \times Z_3) \ltimes Z_4) + \delta(Z_3 \times (Z_3 \times Z_3) \ltimes Z_2) = 3160.
\end{aligned}$$

This completes the proof. □

Theorem 4.6 *The number of Chains of Subgroups of $Z_3 \times A_6$ is 212848.*

Proof Notice that $Z_3 \times A_6$ is non-Abelian of order 1080. It has the following set of representatives of isomorphism classes of subgroups with their sizes:

$$[e, 1], [Z_2, 45], [Z_2 \times Z_2, 30], [Z_3, 121], [Z_4, 45], [Z_5, 36], [Z_6, 45], [Z_{12}, 45], [Z_{15}, 36],$$
$$[Z_3 \times Z_3 \times Z_3, 10], \ [Z_3 \times Z_3, 130], \ [Z_3 \times D_4, 45], \ [Z_3 \times D_5, 36], \ [Z_3 \times S_3, 120],$$
$$[Z_3 \times S_4, 30], \quad [(Z_3 \times A_4), 30], \quad [(Z_3 \times Z_3) \ltimes Z_4, 10], \quad [(Z_3 \times Z_3) \ltimes Z_2, 10],$$
$$[Z_3 \times (Z_3 \times Z_3) \ltimes Z_4, 10], \quad [Z_3 \times (Z_3 \times Z_3) \ltimes Z_2, 10], \quad [Z_6 \times Z_2, 30], \quad [S_3, 120],$$
$$[S_4, 30], [D_4, 45] \ , [D_5, 36], [A_4, 90], [A_5, 12], [A_6, 1], [GL(2, 4), 12] \text{ and } [(Z_3 \times A_6), 1]$$

from the isomorphism class. So,

$$\begin{aligned}
\delta(Z_3 \times A_6) &= 1 + \delta(H_e) + 45 \times \delta(Z_2) + 121 \times \delta(Z_3) + 45 \times \delta(Z_4) \\
&\quad + 36 \times \delta(Z_5) + 45 \times \delta(Z_6) + 45 \times \delta(Z_{12}) + 36 \times \delta(Z_{15}) \\
&\quad + 30 \times \delta(Z_2 \times Z_2) + 10 \times \delta(Z_3 \times Z_3 \times Z_3) + 130 \times \delta(Z_3 \times Z_3) \\
&\quad + 30 \times \delta(Z_3 \times A_4) + 10 \times \delta(Z_2 \times (Z_3 \times Z_3) \ltimes Z_2) \\
&\quad + 10 \times \delta(Z_2 \times (Z_3 \times Z_3) \ltimes Z_4) + 12 \times \delta(Z_3 \times A_5) \\
&\quad + 45 \times \delta(Z_3 \times D_8) + 120 \times \delta(Z_3 \times S_3) + 30 \times \delta(Z_3 \times S_4) \\
&\quad + 30 \times \delta(Z_6 \times Z_2) + 10 \times \delta((Z_3 \times Z_3) \ltimes Z_2) + 10 \times \delta((Z_3 \times Z_3) \ltimes Z_4) \\
&\quad + 10 \times \delta(Z_3 \times (Z_3 \times Z_3) \ltimes Z_2) + 10 \times \delta(Z_3 \times (Z_3 \times Z_3) \ltimes Z_4) \\
&\quad + 90 \times \delta(A_4) + 12 \times \delta(A_5) + \delta(A_6) + 120 \times \delta(S_3) + 30 \times \delta(S_4) \\
&\quad + 45 \times \delta(D_8) + 36 \times \delta(D_{10}) = 212848.
\end{aligned}$$

This completes the proof. □

Then, we get the following theorem.

Theorem 4.7 *The number of chains of subgroups $Z_3 \times A_n, n \leq 6$, then*

$$\delta(Z_2 \times A_n) = \begin{cases} 10 & n = 3 \\ 208 & n = 4 \\ 3440 & n = 5 \\ 212848 & n = 6 \end{cases}$$

§5. Conclusion

The study of the number of chains of subgroups in the lattice of subgroups for larger groups are interesting and give rise to potential applications to quantum computing and coding. In this paper, it is clearly observed from the results that the number of chains of subgroups of G do not depend on the order of G but the lattice of subgroups of G as in Theorems 3.8 and 4.7.

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