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Trees with Large Roman Domination Number

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Abstract: A Roman dominating function on a graph G is a function $f : V(G) \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex $v \in V(G)$ for which $f(v) = 0$, is adjacent to at least one vertex u with $f(u) = 2$. The weight of a Roman dominating function f is the value $w(f) = \sum_{v \in V} f(v)$. The minimum weight of a Roman dominating function is called the Roman domination number of G and is denoted by $\gamma_R(G)$. In this paper, we characterize trees with $\gamma_R \geq n - \Delta$.

Key Words: Tree, domination number, Roman dominating function, Smarandache-Roman k-dominating function, Roman domination number.

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§1. Introduction

The graph $G = (V, E)$ we mean a finite, undirected, connected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. The degree of a vertex u in G is the number of edges incident with u and is denoted by $d_G(u)$, simply $d(u)$. The minimum and maximum degree of a graph G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [1] and Haynes et.al [4,5].

Let $v \in V$. The open neighborhood and closed neighborhood of v are denoted by $N(v)$ and $N[v] = N(v) \cup \{v\}$. If $S \subseteq V$ then $N(S) = \bigcup_{v \in S} N(v)$ for all $v \in S$ and $N[S] = N(S) \cup S$. If $S \subseteq V$ and $u \in S$ then the private neighbor set of u with respect to S is defined by $pn[u, S] = \{v : N[v] \cap S = \{u\}\}$. For any set $S \subseteq V$, the subgraph induced by S is the maximal subgraph of G with vertex set S and is denoted by $\langle S \rangle$.The vertex has degree one is called a pendant vertex. The set of all pendant vertices of a graph G is denoted as $l(G)$. A support is a vertex which is adjacent to a pendant vertex. A weak support is a vertex which is adjacent to exactly one pendant vertex. A strong support is a vertex which is adjacent to at least two pendant vertices. An unicyclic graph is a graph with exactly one cycle. A graph without cycle is called acyclic graph and a connected acyclic graph is called a tree.

A subset S of V is called a dominating set of G if every vertex in $V - S$ is adjacent to at least one vertex in S. The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$. E.J.Cockayne et.al [2] studied the concept of Roman domination

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first. A Roman dominating function on a graph G is a function $f : V(G) \longrightarrow \{0,1,2\}$ satisfying the condition that every vertex $v \in V$ for which $f(v) = 0$ is adjacent to at least one vertex $u \in V$ with $f(u) = 2$. Generally, if every vertex $v \in V$ for which $f(v) = 0$ is adjacent to at least k vertices $u \in V(G)$ with $f(u) = 2$ for a function $f : V(G) \longrightarrow \{0,1,2\}$, such a function f is said to be a Smarandache-Roman k-dominating function, where $k \geq 1$ is an integer. Clearly, if $k = 1$, such a Smarandache-Roman k-dominating function is nothing else but the Roman dominating function. The weight of a Roman dominating function is the value $w(f) = \sum\limits_{v \in V} f(v)$. The minimum weight of a Roman dominating function is called the roman dominating number of G and is denoted by $\gamma_R(G)$.

For a graph G, let $f : V \longrightarrow \{0,1,2\}$ and let (V_0, V_1, V_2) be the ordered partition of V induced by f, where $V_i = \{v \in V : f(v) = i\}$. Note that there exists an one to one correspondence between the function $f : V \longrightarrow \{0,1,2\}$ and the ordered partition (V_0, V_1, V_2) of V. Thus we will write $f = (V_0, V_1, V_2)$. We say that a function $f = (V_0, V_1, V_2)$ is a γ_R-function if it is a Roman dominating function and $w(f) = \gamma_R(G)$. Also $w(f) = |V_1| + 2|V_2|$.

Erin W. Chambers et.al [3] proved that $\gamma_R(G) \leq n - \Delta + 1$. In this paper we characterize the trees with $\gamma_R \geq n - \Delta$.

§2. Family of Trees $\mathscr{G}$

Notation 2.1 The family of trees $\mathscr{G}_{33}$ is obtained from $K_{1,\Delta}$ by attaching a path on three vertices twice to a pendant vertex.

Notation 2.2 The family of trees $\mathscr{G}_{23}$ is obtained from $K_{1,\Delta}$ by attaching a path on three vertices and a path on two vertices to a pendant vertex.

Notation 2.3 The family of trees $\mathscr{G}_1$ is obtained from a tree in $\mathscr{G}_{33} \cup \mathscr{G}_{23}$ by attaching a path on three vertices twice or a path on two vertices twice or a path on three vertices and a path on two vertices to at most $\Delta - 3$ pendant vertices whose support has degree Δ.

Notation 2.4 The family of trees $\mathscr{G}_{(1)}$ is obtained from a tree in $\mathscr{G}_1$ by attaching a path $P_k, k = 1$ or 2 to the pendant vertices whose support has degree Δ.

Notation 2.5 The family of trees $\mathscr{G}_{(2)}$ is obtained from $K_{1,\Delta}$ by subdividing $\Delta - 1$ edges twice.

Notation 2.6 The family of trees $\mathscr{G}_{(3)}$ is obtained from $K_{1,\Delta}$ by subdividing twice $i, 1 \leq i \leq \Delta - 2$ edges and subdividing once $k, 0 \leq k \leq \Delta - i$ edges.

Notation 2.7 The family of trees $\mathscr{G}_{(4)}$ is obtained from $K_{1,\Delta}$ by attaching twice a path on two vertices to $i, 0 \leq i \leq \Delta - 2$ pendant vertices and attaching a path on two vertices to $k, 0 \leq k \leq \Delta - i$ pendant vertices.

Notation 2.8 The family of trees $\mathscr{G} = \{K_{1,\Delta}\} \cup \mathscr{G}_{(1)} \cup \mathscr{G}_{(2)} \cup \mathscr{G}_{(3)} \cup \mathscr{G}_{(4)}$

§3. Trees with $\gamma_R = n - \Delta + 1$

Theorem 3.1 *For a tree T, $\gamma_R(T) = n - \Delta + 1$ if and only if $T \in \mathscr{G}$.*

Proof Let T be a tree with $\gamma_R(T) = n - \Delta + 1$. Let $v \in V(T)$ such that $d(v) = \Delta$. If $\Delta = n - 1$, then T is a star. Suppose $\Delta < n - 1$. Let $N(v) = \{v_1, v_2, \cdots, v_\Delta\}$ and let $T_1 = \langle V - N[v] \rangle$.

Case 1. $E(T_1) = \phi$.

Then every vertex of T_1 is adjacent to a vertex in $N(v)$. Suppose $d(v_i) \geq 4$ for some $i, 1 \leq i \leq \Delta$. Let $w_1, w_2, w_3 \in N(v_i) \cap V(T_1)$. Then $f = ([N(v) - \{v_i\}] \cup \{w_1, w_2, w_3\}, V - [N[v] \cup \{w_1, w_2, w_3\}], \{v, v_i\})$ is a Roman dominating function with $w(f) = n - (\Delta + 4) + 4 = n - \Delta$, which is a contradiction. Hence $d(v_i) \leq 3$ for all $i, 1 \leq i \leq \Delta$. Suppose $d(v_i) = 3$ for all $i, 1 \leq i \leq \Delta$. Then $f = (V - N(v), \phi, N(v))$ is a Roman dominating function with $w(f) = 2\Delta = n - \Delta - 1$, which is a contradiction. Hence $d(v_i) \leq 2$ for some $i, 1 \leq i \leq \Delta$.

Suppose $d(v_i) = 3, 1 \leq i \leq \Delta - 1$ and $d(v_\Delta) \leq 2$. Then

$$f = \begin{cases} (V - N(v), \phi, N(v)) & if\ d(v_\Delta) = 2 \\ (V - N(v), \{v_\Delta\}, N(v) - \{v_\Delta\}) & if\ d(v_\Delta) = 1 \end{cases}$$

is a Roman dominating function with $w(f) < n - \Delta + 1$ which is a contradiction. Hence at most $\Delta - 2$ vertices of $N(v)$ have degree 3.Thus T is isomorphic to a tree obtained from $K_{1,\Delta}$ by attaching twice a path on two vertices to $i, 0 \leq i \leq \Delta - 2$ pendant vertices and attaching a path on two vertices to $k, 0 \leq k \leq \Delta - i$ pendant vertices. Hence, $T \in \mathscr{G}_{(4)}$

Case 2. $E(T_1) \neq \phi$.

Let G_1 be any non trivial component of T_1 and we may assume without loss generality that $v_1 \in N(V(G_1))$. Suppose G_1 contains more than one pendant vertex of T. Let $w_1, w_2 \in V(G_1)$ such that $d(w_i) = 1$. Let $P = (w_1, u_1, u_2, \cdots, u_i, w_2), i \geq 1$ is a $w_1 - w_2$ path in G_1. Let $V_0 = N(v) \cup \{w_1, u_2\}, V_1 = V - [N(v) \cup \{v, w_1, u_1, u_2\}, V_2 = \{v, u_1\}$. Then $f = (V_0, V_1, V_2)$ is a Roman dominating function of T with $w(f) = n - (\Delta + 4) + 4 = n - \Delta$ which is a contradiction. Thus G_1 has exactly one pendant vertex of T and hence G_1 is a path. Let $G_1 = (x_1, x_2, \cdots, x_r)$ such that $v_1 \in N(x_1)$. If $r > 2$, then

$$f = (N(v) \bigcup \{x_1, x_3\}, V - [N(v) \cup \{x_1, x_2, x_3, v\}], \{v, x_2\})$$

is a Roman dominating function of T with $w(f) = n - (\Delta + 4) + 4 = n - \Delta$ which is a contradiction. Hence $r \leq 2$. Then $G_1 = P_2$. Suppose $d(v_i) \geq 4$. Let $x_1, x_2, x_3 \in N(v_1), x_i \neq v, 1 \leq i \leq 3$. Then

$$f = ([N(v) - \{v_1\}] \cup \{x_1, x_2, x_3\}, V - [N[v] \bigcup \{x_1, x_2, x_3\}], \{v, v_1\})$$

is a Roman dominating function with $w(f) = n - \Delta$, which is a contradiction. Hence $d(v_i) \leq 3$ for all i.

Suppose $d(v_i) = 3$ for all $i, 1 \leq i \leq \Delta - 1$. Then $d(v_\Delta) \leq 2$ and then

$$f = (\bigcup_{i=1}^{\Delta-1} N(v_i), V - [\bigcup_{i=1}^{\Delta-1} N[v_i]], \bigcup_{i=1}^{\Delta-1} \{v_i\})$$

is a Roman dominating function with $w(f) = n - \Delta$, which is a contradiction. Hence at most $\Delta - 2$ vertices of $N(v)$ have degree three. If at least one vertex in $N(v)$ has degree three then $T \in \mathscr{G}_{(1)}$.

Suppose $d(v_i) \leq 2$ for all $i, 1 \leq i \leq \Delta$. Then T_1 contains maximum of Δ nontrivial components. Suppose T_1 contains Δ non trivial components $G_1, G_2, \cdots, G_\Delta$. Let $V(G_i) = \{x_{i1}, x_{i2}\}$ such that $v_i \in N(x_{i1})$. Then

$$f = (N(v) \cup \{x_{i2} : 1 \leq i \leq \Delta\}, \{v\}, \{x_{i1} : 1 \leq i \leq \Delta\})$$

is a Roman dominating function of T with $w(f) = 1 + 2\Delta = n - \Delta$ which is a contradiction. Hence T_1 contains at most $\Delta - 1$ non trivial components.

Suppose T_1 contains exactly $\Delta - 1$ non trivial components. Let $G_1, G_2, \cdots, G_{\Delta-1}$ be the non trivial components of T_1. If G_Δ is trivial component of T_1, then

$$f = ((N[v] - \{v_\Delta\}) \cup \{x_{i2} : 1 \leq i \leq \Delta - 1\}, \phi, \{x_{i1} : 1 \leq i \leq \Delta - 1\})$$

is a Roman dominating function of T with $w(f) = 2(\Delta - 1) = 2\Delta - 2 = n - \Delta - 2$ which is a contradiction. Hence T is isomorphic to a tree obtained from $K_{1,\Delta}$ by subdividing $\Delta - 1$ edges twice. Thus $T \in \mathscr{G}_{(2)}$

If T_1 contains $i, 1 \leq i \leq \Delta - 2$ non trivial components, then T is isomorphic to a tree obtained from $K_{1,\Delta}$ by subdividing twice $i, 1 \leq i \leq \Delta - 2$ edges and subdividing once $k, 0 \leq k \leq \Delta - i$ edges. Hence $T \in \mathscr{G}_{(3)}$. The converse is obvious. □

§4. Family of Trees $\mathscr{F}$

Notation 4.1 The family of trees $\mathscr{T}_1$ is obtained from $K_{1,\Delta}$ by attaching thrice a path on two vertices to a pendant vertex, attaching twice a path on two vertices to $i, 0 \leq i \leq \Delta - 3$ pendant vertices and attaching a path on two vertices to $k, 0 \leq k \leq \Delta - 1 - i$ pendant vertices.

Notation 4.2 The family of trees $\mathscr{T}_2$ is obtained from $K_{1,\Delta}$ by attaching twice a path on two vertices to $\Delta - 1$ pendant vertices and attaching a path $P_k, k = 1$ or 2 to a pendant vertex.

Notation 4.3 Let v be a vertex of degree Δ in a star graph $K_{1,\Delta}$ and let $N(v) = \{v_1, v_2, \cdots, v_\Delta\}$ The family of trees $\mathscr{T}^{(a)}$ is obtained from $K_{1,\Delta}$ by subdividing $a, 2 \leq a \leq 5$ times the edge vv_1.

Notation 4.4 The family of trees $\mathscr{T}_{1i}^{(a)}$ is obtained from a tree in $\mathscr{T}^{(a)}, 2 \leq a \leq 4$ by attaching a path $P_i, 2 \leq i \leq 4$ to the vertex of distance two from center vertex v.The family of trees $\mathscr{T}_{2i}^{(a)}$ is obtained from a tree in $\mathscr{T}^{(a)}, 4 \leq a \leq 5$ by attaching a path $P_i, 1 \leq i \leq 3$ to the vertex v_1.

Notation 4.5 The family of trees $\mathscr{T}_{1(i,j,k)}^{(a)}$ is obtained from a tree in $\mathscr{T}_{1i}^{(a)}$ by attaching a path P_3 to at most two times to some or all the vertices of $v_1, v_2, \cdots, v_j, j \leq \Delta - 3$, attaching a path P_2 at most two times to some or all the vertices of $v_{j+1}, v_{j+2}, \cdots, v_k$, $k \leq \Delta - 3$ and attaching a path P_2 at most one time to the vertices $v_{k+1}, v_{k+2}, \cdots, v_\Delta$.

Notation 4.6 The family of trees $\mathscr{T}_{2(i,j,k)}^{(a)}$ is obtained from a tree in $\mathscr{T}_{2i}^{(a)}$ by attaching a path

P_3 to at most two times to some or all the vertices of $v_2, v_3, \cdots, v_j, j \leq \Delta - 3$, attaching a path P_2 at most two times to some or all the vertices of $v_{j+1}, v_{j+2}, \cdots, v_k$, $k \leq \Delta - 3$ and attaching a path P_2 at most one time to the vertices $v_{k+1}, v_{k+2}, \cdots, v_\Delta$.

Notation 4.7 The family of trees $\mathscr{T}_i^{(3)}$ is obtained from a tree in $\mathscr{T}^{(3)}$ by attaching a path $P_i, 1 \leq i \leq 3$ to the vertex v_1.The family of trees $\mathscr{T}_{(i,j,k)}^{(3)}$ is obtained from a tree in $\mathscr{T}_i^{(3)}$ by attaching a path P_3 to at most two times to some or all the vertices of $v_2, v_3, \cdots, v_j, j \leq \Delta - 3$, attaching a path P_2 at most two times to some or all the vertices of $v_{j+1}, v_{j+2}, \cdots, v_k, k \leq \Delta - 3$ and attaching a path P_2 at most one time to the vertices $v_{k+1}, v_{k+2}, \cdots, v_\Delta$.

Notation 4.8 The family of trees $\mathscr{T}_{bc}^{(3)}$ is obtained from a tree in $\mathscr{T}^{(3)}$ by attaching the paths P_b and P_c, $2 \leq b \leq 3$, $2 \leq c \leq 3$ to the vertex v_1.The family of trees $\mathscr{T}_{(bc,j,k)}^{(3)}$ is obtained from a tree in $\mathscr{T}_{bc}^{(3)}$ by attaching a path P_3 to at most two times to some or all the vertices of $v_2, v_3, \cdots, v_j, j \leq \Delta - 3$, attaching a path P_2 at most two times to some or all the vertices of $v_{j+1}, v_{j+2}, \cdots, v_k, k \leq \Delta - 3$ and attaching a path P_2 at most one time to the vertices $v_{k+1}, v_{k+2}, \cdots, v_\Delta$.

Notation 4.9 The family of trees $\mathscr{T}_{23}^{(2)}$ is obtained from the tree $\mathscr{T}^{(2)}$ by attaching the paths P_2 and P_3, to the vertex v_1.The family of trees $\mathscr{T}_{(23,j,k)}^{(2)}$ is obtained from a tree in $\mathscr{T}_{23}^{(2)}$ by attaching a path P_3 to at most two times to some or all the vertices of $v_2, v_3, \cdots, v_j, j \leq \Delta - 3$, attaching a path P_2 at most two times to some or all the vertices of $v_{j+1}, v_{j+2}, \cdots, v_k$, $k \leq \Delta - 3$ and attaching a path P_2 at most one time to the vertices $v_{k+1}, v_{k+2}, \cdots, v_\Delta$.

Notation 4.10 The family of trees

$$\mathscr{F} = \mathscr{T}_1 \cup \mathscr{T}_2 \cup \mathscr{T}_{1(i,j,k)}^{(a)} \cup \mathscr{T}_{2(i,j,k)}^{(a)} \cup \mathscr{T}_{(i,j,k)}^{(3)} \cup \mathscr{T}_{(bc,j,k)}^{(3)} \cup \mathscr{T}_{(23,j,k)}^{(2)}.$$

§5. Trees with $\gamma_R = n - \Delta$

Theorem 5.1 *For a tree T, $\gamma_R(T) = n - \Delta$ if and only if $T \in \mathscr{F}$.*

Proof Let T be a tree with $\gamma_R(G) = n - \Delta$. Let $v \in V(T)$ such that $d(v) = \Delta$. It is clear that $\Delta < n - 1$. Let $N(v) = \{v_1, v_2, \cdots, v_\Delta\}$ and let $T_1 = \langle V - N[v] \rangle$.

Case 1. $E(T_1) = \phi$.

Then every vertex of T_1 is adjacent to a vertex in $N(v)$. Suppose $d(v_i) \geq 5$ for some $i, 1 \leq i \leq \Delta$. Let $V_0 = (N(v) \cup N(v_i)) - \{v, v_i\}, V_1 = V - [N(v) \cup N(v_i)], V_2 = \{v, v_i\}$. Then $f = (V_0, V_1, V_2)$ is a Roman dominating function with $w(f) \leq n - (\Delta + 5) + 4 = n - \Delta - 1$ which is a contradiction. Hence $d(v_i) \leq 4$ for all $i, 1 \leq i \leq \Delta$. Suppose $d(v_1) = d(v_2) = 4$. Let $N(v_1) = \{v, u_1, u_2, u_3\}$ and $N(v_2) = \{v, w_1, w_2, w_3\}$. Now we assume $V_0 = (N(v) \cup N(v_1) \cup N(v_2)) - \{v, v_1, v_2\}, V_1 = V - [N(v) \cup N(v_1) \cup N(v_2)], V_2 = \{v, v_1, v_2\}$. Then $f = (V_0, V_1, V_2)$ is a Roman dominating function with $w(f) = n - (\Delta + 4 + 3) + 6 = n - \Delta - 1$ which is a contradiction. Hence at most one vertex in $N(v)$ has degree 4.

Let $d(v_1) = 4$ and $d(v_i) \leq 3, 2 \leq i \leq \Delta$. Suppose $d(v_i) = 3$ for all $i, 2 \leq i \leq \Delta$. Then

$$f = (V - N(v), \phi, N(v))$$

is a Roman dominating function with $w(f) = 2\Delta = n - \Delta - 2$ which is a contradiction. Hence $d(v_i) = 3$ for all $i, 2 \leq i \leq \Delta - 1$ and $d(v_\Delta) \leq 2$. Then

$$f = \begin{cases} (V - N(v), \phi, N(v)) & if\ d(v_\Delta) = 2 \\ (V - N(v), \{v_\Delta\}, N(v) - \{v_\Delta\}) & if\ d(v_\Delta) = 1 \end{cases}$$

is a Roman dominating function with $w(f) = n - \Delta - 1$ which is a contradiction. Hence at most $\Delta - 3$ vertices of $N(v)$ have degree 3. Thus T is isomorphic to a tree obtained from $K_{1,\Delta}$ by attaching thrice a path on two vertices to a pendant vertex, attaching twice a path on two vertices to $i, 0 \leq i \leq \Delta - 3$ pendant vertices and attaching a path on two vertices to $k, 0 \leq k \leq \Delta - 1 - i$ pendant vertices. Thus $T \in \mathscr{T}_1$.

Suppose $d(v_i) \leq 3$ for all $i, 1 \leq i \leq \Delta$. If $d(v_i) = 3$ for all $i, 1 \leq i \leq \Delta$. Then $f = (V - N(v), \phi, N(v))$ is a Roman dominating function with $w(f) = 2\Delta = n - \Delta - 1$, which is a contradiction. Hence at least one vertex in $N(v)$ has degree less than 3. If more than two vertices of $N(v)$ have degree less than 3 then by proof as in case 1 we get a contradiction. Hence $d(v_i) = 3$ for all $i, 1 \leq i \leq \Delta - 1$. Thus T is isomorphic to a tree obtained from $K_{1,\Delta}$ by attaching twice a path on two vertices to $\Delta - 1$ pendant vertices and attaching a path $P_k, k = 1$ or 2 to a pendant vertex. Thus $T \in \mathscr{T}_2$.

Case 2. $E(T_1) \neq \phi$.

Let G_1 be any nontrivial component of T_1 and we may assume without loss of generality $v_1 \in N(V(G_1))$. Suppose G_1 contains more than two pendant vertices of T. Let $w_1, w_2, w_3 \in V(G_1)$ such that $d(w_i) = 1, 1 \leq i \leq 3$. Then there is a vertex $u \in G_1$ such that $d_{G_1}(u) \geq 3$. Let $x_1, x_2, x_3 \in N(u) \cap V(G_1)$. Then

$$f = (N(v) \cup \{x_1, x_2, x_3\}, V - (N[v] \cup \{x_1, x_2, x_3\}, \{u, v\})$$

is a Roman dominating function of T with $w(f) = n - (\Delta + 1 + 4) + 4 = n - \Delta - 1$, which is a contradiction. Hence G_1 is a path.

Subcase 2.1 $|V(G_1) \cap l(T)| = 2$.

Let $w_1, w_2 \in V(G_1)$ such that $d_T(w_i) = 1$. Let $G = (w_1, u_1, u_2, \cdots, u_k, w_2)$. Suppose $d(v_1, G_1) \geq 2$. Let $(v_1, x_1, x_2, \cdots, x_i, u_j), j \leq k$, be the shortest $v_1 - G_1$ path. Then

$$f = (N(v) \cup \{x_i, u_{j-1}, u_{j+1}\}, V - (N[v] \cup \{x_i, u_{j-1}, u_{j+1}\}, \{u_j, v\})$$

is a Roman dominating function of T with $w(f) = n - \Delta - 1$, which is a contradiction. Hence $d(v_1, G_1) = 1$. Thus $v_1u_j \in E$. Suppose $d(v_1, w_i) \geq 5, i = 1$ or 2. Let $V_0 = N(v) \cup \{u_{j-1}, u_{j+1}, u_{j+2}, u_{j+4}\}, V_2 = \{v, u_j, u_{j+3}\}, V_1 = V - (V_0 \cup V_2)$. Then $f = (V_0, V_1, V_2)$ is a Roman dominating function with $w(f) = n - (\Delta + 4 + 3) + 6 = n - \Delta - 1$, which is a contradiction.

Hence $G_1 = (w_1, u_1, u_2, \cdots, u_i, w_2), i \leq 5$. If $i = 5$ then $v_1u_3 \in E$. If $i = 4$ then $v_1u_2 \in E$. If $i = 3$ then either $v_1u_1 \in E$ or $v_1u_2 \in E$. If $i = 2$ then $v_1u_1 \in E$.

Let $G_2(\neq G_1)$ be a nontrivial component of T_1. If G_2 contains more than one pendant vertex of T then there is a vertex $y_1 \in G_2$ such that $d_{G_2}(y_1) \geq 2$. Let $y_2, y_3 \in N(y_1) \cap V(G_2)$. We assume $V_0 = N(v) \cup \{u_{j-1}, u_{j+1}, y_2, y_3\}, V_2 = \{v, u_j, y_1\}$ and $V_1 = V - (V_0 \cup V_2)$. Then $f = (V_0, V_1, V_2)$ is a Roman dominating function of T with $w(f) = n - \Delta - 1$, which is a contradiction. Hence every nontrivial component of T_1 except G_1 is a path. Let $G_2 = (x_1, x_2, \cdots, x_r)$ such that $v_i \in N(x_1)$ for some i. Suppose $r \geq 3$. Let $V_0 = N(v) \cup \{u_{j-1}, u_{j+1}, x_1, x_3\}, V_2 = \{v, u_j, x_2\}$ and $V_1 = V - (V_0 \cup V_2)$. Then $f = (V_0, V_1, V_2)$ is a Roman dominating function of T with $w(f) = n - \Delta - 1$, which is a contradiction. Hence $r = 2$. If all the components of T_1 are nontrivial then by similar arguments as above we get $\gamma_R \leq n - \Delta - 1$, which is a contradiction and hence $T \in \mathscr{T}^{(a)}_{1(i,j,k)}$.

Subcase 2.2 $|V(G_1) \cap l(T)| = 1$.

Let $G_1 = (u_1, u_2, \cdots, u_r, w_1)$ with $d(w_1) = 1$ and let $v_1u_1 \in E$. If $r \geq 5$ then $f = (N(v) \cup \{u_1, u_3, u_4, u_6\}, V - (N[v] \cup \{u_1, u_2, u_3, u_4, u_5, u_6\}, \{v, u_2, u_5\})$ is a Roman dominating function with $w(f) = n - (\Delta + 1 + 6) - 6 = n - \Delta - 1$, which is a contradiction. Hence $r \leq 4$. Let $3 \leq r \leq 4$. Suppose $d(v_1) \geq 4$. Let $u_1, x_1, x_2 \in N(v_1)$ and let $V_0 = [N(v) \cup \{x_1, x_2, u_1, u_2, u_4\}] - \{v_1\}, V_1 = V - [N[v] \cup \{x_1, x_2, u_1, u_2, u_3, , u_4\}, V_2 = \{v, v_1, u_3\}$. Then $f = (V_0, V_1, V_2)$ is a Roman dominating function with $w(f) = n - [\Delta + 1 + 6] + 6 = n - \Delta - 1$, which is a contradiction. Hence $d(v_1) = 2$ or 3. If $d(v_1) = 3$ then there exists a path $P_j(\neq G_1), j \geq 1$ attached to v_1. Suppose $P_j = (v_1, x_1, x_2, \cdots, x_j), j \geq 3$. Now, let $V_0 = N(v) \cup \{u_1, u_3, x_1, x_3\}, V_1 = V - [N[v] \cup \{u_1, u_2, u_3, x_1, x_2, x_3\}], V_2 = \{v, u_2, x_2\}$. Then $f = (V_0, V_1, V_2)$ is a Roman dominating function with $w(f) = n - \Delta - 1$, which is a contradiction. Hence $j \leq 2$. Hence by similar arguments as in case 1 we have $T \in \mathscr{T}^{(a)}_{2(i,j,k)}$. If $r = 2$ then by similar arguments as above we have $T \in \mathscr{T}^{(3)}_{(i,j,k)} \cup \mathscr{T}^{(3)}_{(bc,j,k)}$. If $r = 1$ then $T \in \mathscr{T}^{(2)}_{(23,j,k)}$. The converse is obvious. □

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