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Weierstrass Formula for Minimal Surface in the Special Three-Dimensional Kenmotsu Manifold $\mathbb{K}$ with η-Parallel Ricci Tensor

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Abstract: In this paper, we study minimal surfaces for simply connected immersed minimal surfaces in the special three-dimensional Kenmotsu manifold $\mathbb{K}$ with η-parallel ricci tensor. We consider the Riemannian left invariant metric and use some results of Levi-Civita connection.

Key Words: Weierstrass representation, Kenmotsu manifold, minimal surface.

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§1. Introduction

The study of minimal surfaces played a formative role in the development of mathematics over the last two centuries. Today, minimal surfaces appear in various guises in diverse areas of mathematics, physics, chemistry and computer graphics, but have also been used in differential geometry to study basic properties of immersed surfaces in contact manifolds.

Minimal surface, such as soap film, has zero curvature at every point. It has attracted the attention for both mathematicians and natural scientists for different reasons. Mathematicians are interested in studying minimal surfaces that have certain properties, such as completedness and finite total curvature, while scientists are more inclined to periodic minimal surfaces observed in crystals or biosystems such as lipid bilayers.

Weierstrass representations are very useful and suitable tools for the systematic study of minimal surfaces immersed in n-dimensional spaces. This subject has a long and rich history. It has been extensively investigated since the initial works of Weierstrass [19]. In the literature there exists a great number of applications of the Weierstrass representation to various domains of Mathematics, Physics, Chemistry and Biology. In particular in such areas as quantum field theory [8], statistical physics [14], chemical physics, fluid dynamics and membranes [16], minimal surfaces play an essential role. More recently it is worth mentioning that works by Kenmotsu [10], Hoffmann [9], Osserman [15], Budinich [5], Konopelchenko [6,11] and Bobenko [3, 4] have

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made very significant contributions to constructing minimal surfaces in a systematic way and to understanding their intrinsic geometric properties as well as their integrable dynamics. The type of extension of the Weierstrass representation which has been useful in three-dimensional applications to multidimensional spaces will continue to generate many additional applications to physics and mathematics. According to [12] integrable deformations of surfaces are generated by the Davey–Stewartson hierarchy of 2+1 dimensional soliton equations. These deformations of surfaces inherit all the remarkable properties of soliton equations. Geometrically such deformations are characterized by the invariance of an infinite set of functionals over surfaces, the simplest being the Willmore functional.

In this paper, we study minimal surfaces for simply connected immersed minimal surfaces in the special three-dimensional Kenmotsu manifold $\mathbb{K}$ with η-parallel ricci tensor. We consider the Riemannian left invariant metric and use some results of Levi-Civita connection.

§2. Preliminaries

Let $M^{2n+1}(\phi,\xi,\eta,g)$ be an almost contact Riemannian manifold with 1-form η, the associated vector field ξ, $(1,1)$-tensor field ϕ and the associated Riemannian metric g. It is well known that [2]

$$\phi\xi=0,\quad \eta(\xi)=1,\quad \eta(\phi X)=0, \tag{2.1}$$
$$\phi^2(X)=-X+\eta(X)\xi, \tag{2.2}$$
$$g(X,\xi)=\eta(X), \tag{2.3}$$
$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y), \tag{2.4}$$

for any vector fields X, Y on M. Moreover,

$$(\nabla_X\phi)Y=-\eta(Y)\phi(X)-g(X,\phi Y)\xi,\qquad X,Y\in\chi(M), \tag{2.5}$$

$$\nabla_X\xi=X-\eta(X)\xi, \tag{2.6}$$

where ∇ denotes the Riemannian connection of g, then (M,ϕ,ξ,η,g) is called an almost Kenmotsu manifold [2].

In Kenmotsu manifolds the following relations hold [2]:

$$(\nabla_X\eta)Y=g(\phi X,\phi Y), \tag{2.7}$$
$$\eta(R(X,Y)Z)=\eta(Y)g(X,Z)-\eta(X)g(Y,Z), \tag{2.8}$$
$$R(X,Y)\xi=\eta(X)Y-\eta(Y)X, \tag{2.9}$$
$$R(\xi,X)Y=\eta(Y)X-g(X,Y)\xi, \tag{2.10}$$
$$R(\xi,X)\xi=X-\eta(X)\xi, \tag{2.11}$$
$$S(\phi X,\phi Y)=S(X,Y)+2n\eta(X)\eta(Y), \tag{2.12}$$
$$S(X,\xi)=-2n\eta(X), \tag{2.13}$$
$$(\nabla_X R)(X,Y)\xi=g(Z,X)Y-g(Z,Y)X-R(X,Y)Z, \tag{2.14}$$

where R is the Riemannian curvature tensor and S is the Ricci tensor. In a Riemannian manifold we also have

$$g\left(R\left(W,X\right)Y,Z\right)+g\left(R\left(W,X\right)Z,Y\right)=0, \tag{2.15}$$

for every vector fields X,Y,Z.

§3. Special Three-Dimensional Kenmotsu Manifold $\mathbb{K}$ with η-Parallel Ricci Tensor

Definition 3.1 *The Ricci tensor S of a Kenmotsu manifold is called η-parallel if it satisfies*

$$\left(\nabla_X S\right)\left(\phi Y,\phi Z\right)=0.$$

The notion of Ricci $\eta-$parallelity for Sasakian manifolds was introduced by M. Kon [13].

We consider the three-dimensional manifold

$$\mathbb{K}=\left\{(x,y,z)\in\mathbb{R}^3:(x,y,z)\neq(0,0,0)\right\},$$

where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$\mathbf{e}_1=x^3\frac{\partial}{\partial x^1},\quad \mathbf{e}_2=x^3\frac{\partial}{\partial x^2},\quad \mathbf{e}_3=-x^3\frac{\partial}{\partial x^3} \tag{3.1}$$

are linearly independent at each point of $\mathbb{K}$. Let g be the Riemannian metric defined by

$$\begin{aligned} g\left(\mathbf{e}_1,\mathbf{e}_1\right)&=g\left(\mathbf{e}_2,\mathbf{e}_2\right)=g\left(\mathbf{e}_3,\mathbf{e}_3\right)=1, \\ g\left(\mathbf{e}_1,\mathbf{e}_2\right)&=g\left(\mathbf{e}_2,\mathbf{e}_3\right)=g\left(\mathbf{e}_1,\mathbf{e}_3\right)=0. \end{aligned} \tag{3.2}$$

The characterizing properties of $\chi(\mathbb{K})$ are the following commutation relations:

$$[\mathbf{e}_1,\mathbf{e}_2]=0,\quad [\mathbf{e}_1,\mathbf{e}_3]=\mathbf{e}_1,\quad [\mathbf{e}_2,\mathbf{e}_3]=\mathbf{e}_2. \tag{3.3}$$

Let η be the 1-form defined by

$$\eta(Z)=g(Z,\mathbf{e}_3)\text{ for any }Z\in\chi(M).$$

Let be the (1,1) tensor field defined by

$$\phi(\mathbf{e}_1)=-\mathbf{e}_2,\ \phi(\mathbf{e}_2)=\mathbf{e}_1,\ \phi(\mathbf{e}_3)=0.$$

Then using the linearity of and g we have

$$\eta(\mathbf{e}_3)=1, \tag{3.4}$$

$$\phi^2(Z)=-Z+\ \eta(Z)\mathbf{e}_3, \tag{3.5}$$

$$g\left(\phi Z,\phi W\right)=g\left(Z,W\right)-\eta(Z)\eta(W), \tag{3.6}$$

for any $Z,W\in\chi(M)$. Thus for $\mathbf{e}_3=\xi$, (ϕ,ξ,η,g) defines an almost contact metric structure on $\mathbb{M}$.

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g\left(\nabla_X Y, Z\right) &= Xg\left(Y,Z\right) + Yg\left(Z,X\right) - Zg\left(X,Y\right) \\ &\quad -g\left(X,[Y,Z]\right) - g\left(Y,[X,Z]\right) + g\left(Z,[X,Y]\right), \end{aligned}$$

which is known as Koszul's formula.

Koszul's formula yields

$$\begin{aligned} \nabla_{\mathbf{e}_1}\mathbf{e}_1 &= 0, \ \nabla_{\mathbf{e}_1}\mathbf{e}_2 = 0, \ \nabla_{\mathbf{e}_1}\mathbf{e}_3 = \mathbf{e}_1, \\ \nabla_{\mathbf{e}_2}\mathbf{e}_1 &= 0, \ \nabla_{\mathbf{e}_2}\mathbf{e}_2 = 0, \ \nabla_{\mathbf{e}_2}\mathbf{e}_3 = \mathbf{e}_2, \\ \nabla_{\mathbf{e}_3}\mathbf{e}_1 &= 0, \ \nabla_{\mathbf{e}_3}\mathbf{e}_2 = 0, \ \nabla_{\mathbf{e}_3}\mathbf{e}_3 = 0. \end{aligned} \tag{3.7}$$

§4. Minimal Surfaces in the Special Three-Dimensional Kenmotsu Manifold $\mathbb{K}$ with η-Parallel Ricci Tensor

In this section, we obtain an integral representation formula for minimal surfaces in the special three-dimensional Kenmotsu manifold $\mathbb{K}$ with η-parallel ricci tensor.

We will denote with $\Omega \subseteq \mathbb{C} \cong \mathbb{R}^2$ a simply connected domain with a complex coordinate $z = u + iv$, $u, v \in \mathbb{R}$. Also, we will use the standard notations for complex derivatives:

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \quad \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right). \tag{4.1}$$

For $X \in \chi(\mathbb{K})$, denote by $\mathsf{ad}(X)^*$ the adjoint operator of $\mathsf{ad}(X)$, i.e., it satisfies the equation

$$g\left(\left[X,Y\right], Z\right) = g\left(Y, \mathsf{ad}(X)^*\left(Z\right)\right), \tag{4.2}$$

for any $Y, Z \in \chi(\mathbb{K})$. Let U be the symmetric bilinear operator on $\chi(\mathbb{M})$ defined by

$$U\left(X,Y\right) := \frac{1}{2}\left\{\mathsf{ad}(X)^*\left(Y\right) + \mathsf{ad}(Y)^*\left(X\right)\right\}. \tag{4.3}$$

Lemma 4.1 *Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the orthonormal basis for an orthonormal basis for $\chi(\mathbb{K})$ defined in (3.1). Then,*

$$\begin{aligned} U\left(\mathbf{e}_1,\mathbf{e}_1\right) &= \mathbf{e}_3, \ U\left(\mathbf{e}_1,\mathbf{e}_3\right) = -\frac{1}{2}\mathbf{e}_1, \\ U\left(\mathbf{e}_2,\mathbf{e}_2\right) &= \mathbf{e}_3, \ U\left(\mathbf{e}_2,\mathbf{e}_3\right) = -\frac{1}{2}\mathbf{e}_2, \\ U\left(\mathbf{e}_1,\mathbf{e}_2\right) &= U\left(\mathbf{e}_3,\mathbf{e}_3\right) = 0. \end{aligned} \tag{4.4}$$

Proof Using (4.2) and (4.3), we have

$$2g\left(U\left(X,Y\right), Z\right) = g\left(\left[X,Z\right], Y\right) + g\left(\left[Y,Z\right], X\right).$$

Thus, direct computations lead to the table of U above. Lemma 4.1 is proved. □

Lemma 4.2(see [10]) *Let D be a simply connected domain. A smooth map $\varphi : D \longrightarrow \mathbb{K}$ is harmonic if and only if*

$$\left(\varphi^{-1}\varphi_u\right)_u + \left(\varphi^{-1}\varphi_v\right)_v - \mathsf{ad}\left(\varphi^{-1}\varphi_u\right)^* \left(\varphi^{-1}\varphi_u\right) - \mathsf{ad}\left(\varphi^{-1}\varphi_v\right)^* \left(\varphi^{-1}\varphi_v\right) = 0 \tag{4.5}$$

holds.

Let $z = u + iv$. Then in terms of complex coordinates z, $\bar{z}$, the harmonic map equation (4.5) can be written as

$$\frac{\partial}{\partial \bar{z}}\left(\varphi^{-1}\frac{\partial \varphi}{\partial z}\right) + \frac{\partial}{\partial z}\left(\varphi^{-1}\frac{\partial \varphi}{\partial \bar{z}}\right) - 2U\left(\varphi^{-1}\frac{\partial \varphi}{\partial z}, \varphi^{-1}\frac{\partial \varphi}{\partial \bar{z}}\right) = 0. \tag{4.6}$$

Let $\varphi^{-1}d\varphi = Adz + \bar{A}d\bar{z}$. Then, (4.6) is equivalent to

$$A_{\bar{z}} + \bar{A}_z = 2U\left(A, \bar{A}\right). \tag{4.7}$$

The Maurer–Cartan equation is given by

$$A_{\bar{z}} - \bar{A}_z = \left[A, \bar{A}\right]. \tag{4.8}$$

(4.7) and (4.8) can be combined to a single equation

$$A_{\bar{z}} = U\left(A, \bar{A}\right) + \frac{1}{2}\left[A, \bar{A}\right]. \tag{4.9}$$

(4.9) is both the integrability condition for the differential equation $\varphi^{-1}d\varphi = Adz + \bar{A}d\bar{z}$ and the condition for φ to be a harmonic map.

Let $D(z, \bar{z})$ be a simply connected domain and $\varphi : D \longrightarrow \mathbb{K}$ a smooth map. If we write $\varphi(z) = \left(x^1(z), x^2(z), x^3(z)\right)$, then by direct calculation

$$A = \left(x^3\right)^{-1}\left(x_z^1\mathbf{e}_1 + x_z^2\mathbf{e}_2 + x_z^3\mathbf{e}_3\right). \tag{4.10}$$

It follows from the harmonic map equation (4.7) that

Theorem 4.3 *$\varphi : D \longrightarrow \mathbb{K}$ is harmonic if and only if the following equations hold:*

$$x_z^1x_{\bar{z}}^3 + x_z^3x_{\bar{z}}^1 = \left(x^3\right)^{-1}x_{\bar{z}}^3x_z^1 - 2x_{z\bar{z}}^1 + \left(x^3\right)^{-1}x_z^3x_{\bar{z}}^1, \tag{4.11}$$

$$x_z^2x_{\bar{z}}^3 + x_z^3x_{\bar{z}}^2 = \left(x^3\right)^{-1}x_{\bar{z}}^3x_z^2 - 2x_{z\bar{z}}^2 + \left(x^3\right)^{-1}x_z^3x_{\bar{z}}^2, \tag{4.12}$$

$$2\left(x_z^1x_{\bar{z}}^1 + x_z^2x_{\bar{z}}^2\right) = \left(x^3\right)^{-1}x_{\bar{z}}^3x_z^3 - 2x_{z\bar{z}}^3 + \left(x^3\right)^{-1}x_z^3x_{\bar{z}}^3. \tag{4.13}$$

Proof From (4.10), we have

$$\bar{A} = \left(x^3\right)^{-1}\left(x_{\bar{z}}^1\mathbf{e}_1 + x_{\bar{z}}^2\mathbf{e}_2 + x_{\bar{z}}^3\mathbf{e}_3\right) \tag{4.14}$$

Using (4.10) and (4.14), we obtain

$$\begin{aligned} U\left(A, \bar{A}\right) \quad = \quad & \left(x^3\right)^{-1}[-\left(\frac{1}{2}x_z^1x_{\bar{z}}^3 + \frac{1}{2}x_z^3x_{\bar{z}}^1\right)\mathbf{e}_1 - \left(\frac{1}{2}x_z^2x_{\bar{z}}^3 + \frac{1}{2}x_z^3x_{\bar{z}}^2\right)\mathbf{e}_2 \\ & + \left(x_z^1x_{\bar{z}}^1 + x_z^2x_{\bar{z}}^2\right)\mathbf{e}_3]. \end{aligned} \tag{4.15}$$

On the other hand, we have

$$A_{\bar{z}} = -\left(x^3\right)^{-2} x^3_{\bar{z}} \left(x^1_z \mathbf{e}_1 + x^2_z \mathbf{e}_2 + x^3_z \mathbf{e}_3\right) + \left(x^3\right)^{-1} \left(x^1_{z\bar{z}} \mathbf{e}_1 + x^2_{z\bar{z}} \mathbf{e}_2 + x^3_{z\bar{z}} \mathbf{e}_3\right), \quad (4.16)$$

$$\bar{A}_z = -\left(x^3\right)^{-2} x^3_z \left(x^1_{\bar{z}} \mathbf{e}_1 + x^2_{\bar{z}} \mathbf{e}_2 + x^3_{\bar{z}} \mathbf{e}_3\right) + \left(x^3\right)^{-1} \left(x^1_{z\bar{z}} \mathbf{e}_1 + x^2_{z\bar{z}} \mathbf{e}_2 + x^3_{z\bar{z}} \mathbf{e}_3\right). \quad (4.17)$$

By direct computation, we obtain

$$\begin{aligned} A_{\bar{z}} &= \left(-\left(x^3\right)^{-2} x^3_{\bar{z}} x^1_z + \left(x^3\right)^{-1} x^1_{z\bar{z}}\right) \mathbf{e}_1 + \left(-\left(x^3\right)^{-2} x^3_{\bar{z}} x^2_z + \left(x^3\right)^{-1} x^2_{z\bar{z}}\right) \mathbf{e}_2 \\ &\quad + \left(-\left(x^3\right)^{-2} x^3_{\bar{z}} x^3_z + \left(x^3\right)^{-1} x^3_{z\bar{z}}\right) \mathbf{e}_3, \end{aligned} \quad (4.18)$$

$$\begin{aligned} \bar{A}_z &= \left(-\left(x^3\right)^{-2} x^3_z x^1_{\bar{z}} + \left(x^3\right)^{-1} x^1_{z\bar{z}}\right) \mathbf{e}_1 + \left(-\left(x^3\right)^{-2} x^3_z x^2_{\bar{z}} + \left(x^3\right)^{-1} x^2_{z\bar{z}}\right) \mathbf{e}_2 \\ &\quad + \left(-\left(x^3\right)^{-2} x^3_z x^3_{\bar{z}} + \left(x^3\right)^{-1} x^3_{z\bar{z}}\right) \mathbf{e}_3. \end{aligned} \quad (4.19)$$

Hence, using (4.7) we obtain (4.11)-(4.13). This completes the proof of the Theorem. □

The exterior derivative d is decomposed as

$$d = \partial + \bar{\partial}, \quad \partial = \frac{\partial}{\partial z} dz, \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}} d\bar{z}, \quad (4.20)$$

with respect to the conformal structure of D. Let

$$\wp^1 = \left(x^3\right)^{-1} x^1_z dz, \quad \wp^2 = \left(x^3\right)^{-1} x^2_z dz, \quad \wp^3 = \left(x^3\right)^{-1} x^3_z dz. \quad (4.21)$$

Theorem 4.4 *The triplet $\{\wp^1, \wp^2, \wp^3\}$ of $(1,0)$-forms satisfies the following differential system:*

$$\bar{\partial}\wp^1 = -x^3 \left(\wp^1 \wedge \overline{\wp^3} + \wp^3 \wedge \overline{\wp^1}\right), \quad (4.22)$$

$$\bar{\partial}\wp^2 = -x^3 \left(\wp^2 \wedge \overline{\wp^3} + \wp^3 \wedge \overline{\wp^2}\right), \quad (4.23)$$

$$\bar{\partial}\wp^3 = 2x^3 \left(\wp^1 \wedge \overline{\wp^1} + \wp^2 \wedge \overline{\wp^2}\right). \quad (4.24)$$

Proof From (4.11)-(4.13), we have (4.22)-(4.24). Thus proof is complete. □

Theorem 4.5 *Let $\{\wp^1, \wp^2, \wp^3\}$ be a solution to (4.22)-(4.24) on a simply connected coordinate region D. Then*

$$\varphi(z, \bar{z}) = 2Re \int_{z_0}^{z} \left(x^3\wp^1, x^3\wp^2, x^3\wp^3\right) \quad (4.25)$$

is a harmonic map into $\mathbb{K}$.

Conversely, any harmonic map of D into $\mathbb{K}$ can be represented in this form.

Proof By theorem 4.3, we see that $\varphi(z, \bar{z})$ is a harmonic curve if and only if $\varphi(z, \bar{z})$ satisfy (4.11)-(4.13). From (4.21), we have

$$\begin{aligned} x^1(z,\bar{z}) &= 2Re\int_{z_0}^{z} x^3\wp^1,\ x^2(z,\bar{z}) = 2Re\int_{z_0}^{z} x^3\wp^2, \\ x^3(z,\bar{z}) &= 2Re\int_{z_0}^{z} x^3\wp^3, \end{aligned}$$

which proves the theorem. □

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