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Aim and Scope

International Journal of Neutrosophic Science (IJNS) is a peer-review journal publishing high quality experimental and theoretical research in all areas of Neutrosophic and its Applications. IJNS is published quarterly. IJNS is devoted to the publication of peer-reviewed original research papers lying in the domain of neutrosophic sets and systems. Papers submitted for possible publication may concern with foundations, neutrosophic logic and mathematical structures in the neutrosophic setting. Besides providing emphasis on topics like artificial intelligence, pattern recognition, image processing, robotics, decision making, data analysis, data mining, applications of neutrosophic mathematical theories contributing to economics, finance, management, industries, electronics, and communications are promoted. Variants of neutrosophic sets including refined neutrosophic set (RNS). Articles evolving algorithms making computational work handy are welcome.

Topics of Interest

IJNS promotes research and reflects the most recent advances of neutrosophic Sciences in diverse disciplines, with emphasis on the following aspects, but certainly not limited to:

- Neutrosophic sets
- Neutrosophic algebra
- Neutrosophic topolog
- Neutrosophic graphs
- Neutrosophic probabilities
- Neutrosophic tools for decision making
- Neutrosophic theory for machine learning
- Neutrosophic statistics
- Neutrosophic numerical measures
- Classical neutrosophic numbers
- A neutrosophic hypothesis
- The neutrosophic level of significance
- The neutrosophic confidence interval
- The neutrosophic central limit theorem
- Neutrosophic theory in bioinformatics and medical analytics
- Neutrosophic tools for big data analytics
- Neutrosophic tools for deep learning
- Neutrosophic tools for data visualization
- Quadripartitioned single-valued neutrosophic sets
- Refined single-valued neutrosophic sets

- Applications of neutrosophic logic in image processing
- Neutrosophic logic for feature learning, classification, regression, and clustering
- Neutrosophic knowledge retrieval of medical images
- Neutrosophic set theory for large-scale image and multimedia processing
- Neutrosophic set theory for brain-machine interfaces and medical signal analysis
- Applications of neutrosophic theory in large-scale healthcare data
- Neutrosophic set-based multimodal sensor data
- Neutrosophic set-based array processing and analysis
- Wireless sensor networks Neutrosophic set-based Crowd-sourcing
- Neutrosophic set-based heterogeneous data mining
- Neutrosophic in Virtual Reality
- Neutrosophic and Plithogenic theories in Humanities and Social Sciences
- Neutrosophic and Plithogenic theories in decision making
- Neutrosophic in Astronomy and Space Sciences

NeutroAlgebra is a Generalization of Partial Algebra

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Abstract

In this paper we recall, improve, and extend several definitions, properties and applications of our previous 2019 research referred to NeutroAlgebras and AntiAlgebras (also called NeutroAlgebraic Structures and respectively AntiAlgebraic Structures).

Let <A> be an item (concept, attribute, idea, proposition, theory, etc.). Through the process of neutrosphication, we split the nonempty space we work on into three regions {two opposite ones corresponding to <A> and <antiA>, and one corresponding to neutral (indeterminate) <neutA> (also denoted <neutroA>) between the opposites}, which may or may not be disjoint – depending on the application, but they are exhaustive (their union equals the whole space).

A NeutroAlgebra is an algebra which has at least one NeutroOperation or one NeutroAxiom (axiom that is true for some elements, indeterminate for other elements, and false for the other elements).

A Partial Algebra is an algebra that has at least one Partial Operation, and all its Axioms are classical (i.e. axioms true for all elements).

Through a theorem we prove that NeutroAlgebra is a generalization of Partial Algebra, and we give examples of NeutroAlgebras that are not Partial Algebras. We also introduce the NeutroFunction (and NeutroOperation).

Keywords: neutrosophy, algebra, neutroalgebra, neutroFunction, neutroOperation, neutroAxiom

1. Neutrosophication by Tri-Sectioning the Space

Let *X* be a given nonempty space (or simply set) included into a universe of discourse *U*.

Let *<A>* be an item (concept, attribute, idea, proposition, theory, etc.) defined on the set *X*. Through the process of neutrosphication, we split the set X into three regions [two opposite ones *<A>* and *<antiA>*, and one neutral (indeterminate) *<neutroA>* between them], regions which may or may not be disjoint – depending on the application, but they are exhaustive (their union equals the whole space).

The region denoted just by *<A>* is formed by all set's elements where *<A>* is true {degree of truth (*T*)}, the region denoted by *<antiA>* is formed by all set's elements where *<A>* is false {degree of falsehood (*F*)}, and the region denoted by *<neutroA>* is formed by all set's elements where *<A>* is indeterminate (neither true nor false) {degree of indeterminacy (*I*)}.

We further on work with the following *<A>* concepts: <u>Function</u>, <u>Operation</u>, <u>Axiom</u>, and <u>Algebra</u>.

Therefore, by tri-sectioning the set *X* with respect to each such *<A>* concept, we get the following neutrosophic triplets corresponding to *(<A>, <NeutroA>, <AntiA>)*:

<Function, NeutroFunction, AntiFunction>,

<Operation, NeutroOperation, AntiOperation>,

<Axiom, NeutroAxiom, AntiAxiom>,

<Algebra, NeutroAlgebra, AntiAlgebra>.

A NeutroAlgebra is an algebra which has at least one NeutroOperation or one NeutroAxiom (axiom that is true for some elements, indeterminate for other elements, and false for other elements).

We have proposed for the first time the NeutroAlgebraic Structures (or NeutroAlgebras), and in general the NeutroStructures, in 2019 [1] and further on in 2020 [2].

The NeutroAlgebra is a generalization of Partial Algebra, which is an algebra that has at least one Partial Operation, while all its Axioms are totally true (classical axioms).

We recall the Boole's Partial Algebras and the Effect Algebras as particular cases of Partial Algebras, and by consequence as particular cases of NeutroAlgebras.

In comparison between the Partial Algebra and the NeutroAlgebra:

- i) When the NeutroAlgebra has no NeutroAxiom, it coincides with the Partial Algebra.
- ii) There are NeutroAlgebras that have no NeutroOperations, but have NeutroAxioms. These are different from Partial Algebras.
- iii) And NeutroAlgebras that have both, NeutroOperations and NeutroAxioms. These are different from Partial Algebras too.

All the above will be proved in the following.

2-4. Partially Inner-Defined, Partially Outer-Defined, or Partially Indeterminate

Let U be a nonempty universe of discourse, and X and Y be two nonempty subsets of U.

Let's consider a function:

$$\boldsymbol{f}\colon X \rightarrow Y.$$

Let $a \in X$ be an element. Then, there are three possibilities:

i) *[Inner-defined,* or *Well-defined*; corresponding in neutrosophy to Truth (T)*]*
$f(a) \in Y;$
ii) *[Outer-defined*; corresponding in neutrosophy to Falsehood (F)*]*
$f(a) \in U\text{-}Y;$
iii) *[Indeterminacy*; corresponding in neutrosophy to Indeterminate (I)*]*
α) $f(a)$ *= indeterminacy;*
{i.e. the value of $f(a)$ does exist, but we do not know it exactly;
for example, $f(a) = c$ or d, we know that $f(a)$ may be equal to c or d (but we are not sure to which one);
or, another example, we only know that $f(a) \neq e$, where the previous $c, d, e \in U$};
β) $f(a)$ = *undefined* (i.e. the value of $f(a)$ is not defined, or it does not exist – as in Partial Function); *undefined* is considered part of *indeterminacy*;
δ) *f(indeterminacy)* $\in U$, but we either do not know the indeterminacy at all, or we only partially know some information about it
{for example we know that $f(a$ or b or $c) \in U$, where $a, b, c \in X$, but we are not sure if the argument is either a, or b, or c};
δ) more general: *f(indeterminacy1) = indeterminacy2*, where *indeterminacy1* is a vaguely known value in X and *indeterminacy2* is a vaguely known value in U*;*
ε) By the way, there are many types of indeterminacies, we only gave above some elementary examples.
Consequently we have:

5-7. Definitions of Total InnerFunction, Total OuterFunction, Total IndeterminateFunction, and Total UndefinedFunction

- i) If for any $x \in X$ one has $f(x) \in Y$ (*inner-ness,* or *well-defined*)*,* then f is called a ***Total InnerFunction*** (or classical **Total Function,** or in general **Function**).

ii) If for any $x \in X$ one has $f(x) \in U\text{-}Y$ (*outer-ness,* or *outer-defined*), then f is called a ***Total OuterFunction*** *(or **AntiFunction**)*.

iii) If for any $x \in X$ one has either $f(x) = indeterminacy,$ or $f(indeterminacy) \in U$, or $f(indeterminacy1) = indeterminacy2,$ then f is called a ***Total IndeterminateFunction***.

{ As a particular case of the Total IndeterminateFunction there is the ***Total UndefinedFunction****:* when for any $x \in X$ one has $f(x) = undefined.$ *}*

8. Definition of Partial Function

In the previous literature {[3], [4]}, the Partial Function was defined as follows:

A function $f: X \rightarrow Y$ is called a **Partial Function** if it is well-defined for some elements in *X,* and undefined for all the other elements in *X*. Therefore, there exist some elements $a \in$ X such that $f(a) \in Y$ (*well-defined*), and for all other element $b \in X$ one has $f(b) =$ (is) *undefined*.

*

We extend the partial function to NeutroFunction in order to comprise all previous *i) – iii)* situations.

9. Definition of NeutroFunction

A function $f: X \rightarrow Y$ is called a **NeutroFunction** if it has elements in X for which the function is well-defined {degree of truth (*T*)}, elements in X for which the function is indeterminate {degree of indeterminacy (*I*)}, and elements in X for which the function is outer-defined {degree of falsehood (*F*)}, where $T, I, F \in [0, 1],$ with (T, I, F) ≠ (1, 0, 0) that represents the (Total) Function, and (T, I, F) ≠ (0, 0, 1) that represents the AntiFunction.

In this definition "neutro" stands for neutrosophic, which means the existence of outer-ness, or undefined-ness, unknown-ness, or indeterminacy in general.

A NeutroFunction is more general, and it may include all three previous situations: elements in X for which the function f is well-defined, elements in X for which function f is indeterminate (including function's undefined values), and elements in X for which function f is outer-defined.

We have formed the following neutrosophic triplet:

<*Function, NeutroFunction (*that includes the *Partial Function), AntiFunction*>.

Therefore, according to the above definitions, we have the following:

10. Classification of Functions

i) (Classical) **Function**, which is a function well-defined for all the elements in its domain of definition.

ii) **NeutroFunction**, which is a function partially well-defined, partially indeterminate, and partially outer-defined on its domain of definition.

iii) **AntiFunction**, which is a function outer-defined for all the elements in its domain of definition.

11. Example of NeutroFunction

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ be a universe of discourse, and two of its nonempty subsets $X = \{1, 2, 3, 4, 5, 6\}$, $Y = \{7, 8, 9, 10, 11, 12\}$, and the function f constructed as follows:

$f: X \rightarrow Y$ such that

$f(1) = 7 \in Y$ (well-defined);

$f(2) = 8 \in U\text{-}Y$ (outer-defined);

f(3) = undefined (doesn't exist);

f(4) = 9 or *10* or *11* (it does exist, but we do not know it exactly), therefore f(4) = *indeterminate*;

f(some number greater ≥ 5) = 12, {i.e. it can be *f(5) = 12* or *f(6) = 12,* we are not sure about}, therefore *f(indeterminate) = 12*.

*

Similarly we defined the NeutroOperation.

12. Definition of NeutroOperation

An *n-ary* (for integer $n \geq 1$) operation $\omega: X^n \rightarrow Y$ is called a **NeutroOperation** if it is has *n-plets* in X^n for which the operation is well-defined {degree of truth (T)}, *n-plets* in X^n for which the operation is indeterminate {degree of indeterminacy (I)}, and *n-plets* in X^n for which the operation is outer-defined {degree of falsehood (F)}, where $T, I, F \in [0, 1]$, with $(T, I, F) \neq (1, 0, 0)$ that represents the n-ary (Total) Operation, and $(T, I, F) \neq (0, 0, 1)$ that represents the n-ary AntiOperation.

Again, in this definition "neutro" stands for neutrosophic, which means the existence of outer-ness, or undefined-ness, or unknown-ness, or indeterminacy in general.

A NeutroOperation is more general, and it may include all previous situations: elements in X^n for which the operation ω is well-defined, elements for which operation ω is outer-defined, and elements for which operation ω is indeterminate (including undefined).

13. Definition of AntiOperation

An *n-ary* (for integer $n \geq 1$) operation $\omega: X^n \rightarrow Y$ is called **AntiOperation** if for all *n-plets* $(x_1, x_2, \ldots, x_n) \in X^n$ one has $\omega(x_1, x_2, \ldots, x_n) \in U\text{-}Y$.

We have formed the neutrosophic triplet: *<Operation, NeutroOperation, AntiOperation>*.

Therefore, according to the above definitions, we have the following:

14. Classification of Operations

On a given set:

i) (Classical) **Operation** is an operation well-defined for all the set's elements.
ii) **NeutroOperation** is an operation partially well-defined, partially indeterminate, and partially outer-defined on the given set.
iii) **AntiFunction** is an operation outer-defined for all the set's elements.

*

Further, we define the NeutroHyperOperation.

15. Definition of NeutroHyperOperation

Similarly, an *n-ary* (for integer $n \geq 1$) hyperoperation $\omega: X^n \rightarrow P(Y)$ is called a **NeutroHyperOperation** if it is has *n-plets* in X^n for which the operation is well-defined $\omega(a_1, a_2, \ldots, a_n) \in P(Y)$ {degree of truth (T)}, *n-plets* in X^n for which the operation is indeterminate {degree of indeterminacy (I)}, and *n-plets* in X^n for which the operation is outer-defined $\omega(a_1, a_2, \ldots, a_n) \notin P(Y)$ {degree of falsehood (F)}, where $T, I, F \in [0, 1]$, with $(T, I, F) \neq (1, 0, 0)$ that represents the n-ary (Total) HyperOperation, and $(T, I, F) \neq (0, 0, 1)$ that represents the n-ary AntiHyperOperation.

Again, in this definition "neutro" stands for neutrosophic, which means the existence of outer-ness, or undefined-ness, or unknown-ness, or indeterminacy in general.

A NeutroOperation is more general, and it may include all previous situations: elements in X^n for which the operation ω is well-defined, elements for which operation ω is outer-defined, and elements for which operation ω is indeterminate (including undefined).

16. Definition of AntiHyperOperation

An *n-ary* (for integer $n \geq 1$) operation $\omega: X^n \rightarrow P(Y)$ is called **AntiHyperOperation** if it is outer-defined for all the *n-plets* in X^n. Or, for any *n*-plet $(x_1, x_2, \ldots, x_n) \in X^n$ one has $\omega(x_1, x_2, \ldots, x_n) \notin P(Y)$.

Again, we have formed a neutrosophic triplet:

<HyperOperation, NeutroHyperOperation, AntiHyperOperation>.

Similarly, according to the above definitions, we have the following:

17. Classification of HyperOperations

On a given set:

i) (Classical) **HyperOperation** is a hyper-operation well-defined for all the set's elements.

ii) **NeutroHyperOperation** is a hyper-operation partially well-defined, partially indeterminate, and partially outer-defined on the given set.

iii) **AntiHyperFunction** is a hyper-operation outer-defined for all the set's elements.

*

18. Definition of Universal Algebra

In the previous literature there exist the following.

The (classical) **Universal Algebra** (or **General Algebra**) is a branch of mathematics that studies classes of (classical) algebraic structures.

19. Definition of Algebraic Structure

A (classical) **Algebraic Structure** (or **Algebra**) is a nonempty set A endowed with some (totally well-defined) operations (functions) on A, and satisfying some (classical) axioms (totally true) - according to the Universal Algebra.

20. Definition of Partial Algebra

A (classical) **Partial Algebra** is an algebra defined on a nonempty set PA that is endowed with some partial operations (or partial functions: partially well-defined, and partially undefined). While the axioms (laws) defined on a Partial Algebra are all totally (100%) true.

21. Definition of Effect Algebra

A set L that contains two special elements $0, 1 \in L$, and endowed with a partially defined binary operation $\oplus$ that satisfies the following conditions (Foulis and Bennett [4]).

For all $p, q, r \in L$ one has:

i) If $p\oplus q$ is defined, then $q\oplus p$ is defined and $p\oplus q = q\oplus p$ *[Commutativity]*.

ii) If $q\oplus r$ is defined and $p\oplus(q\oplus r)$ is defined, then $p\oplus q$ is defined and $(p\oplus q)\oplus r$ is defined, and $p\oplus(q\oplus r) = (p\oplus q)\oplus r$ *[Associativity]*.

iii) For every $p \in L$ there exists a unique $q \in L$ such that $p\oplus q$ is defined and $p\oplus q = 1$ (*Orthosupplementation*).

iv) If $1\oplus p$ is defined, then $p = 0$ (*Zero-One Law*).

Clearly, the Effect Algebra is a particular case of Partial Algebra, since it has a partial operation $\oplus$, and all its (Commutative, Associative, Orthosupplementation, and Zero-One) Laws are totally true.

22. Definition of Boole's Partial Algebras

Let U be a universe of discourse, Su(U) the collection of subsets of U, and S(U) the partial algebra *(Su(U), +, ·, -, 0, 1).* Two partial operations (+ and –) were defined by George Boole (Burris and Sankappanavar [5]):

$A + B := A \cup B$, provided $A \cap B = \phi$, otherwise *undefined*;

and

$A - B := A \setminus B$, provided $B \subseteq A$, otherwise *undefined*;

one total operation:

$A \cdot B = A \cap B$;

and two constants:

$1 := U$,

$0 := \phi$.

Obviously, Boole's Partial Algebras are partial algebras since they have at least one partial operation, while its axioms are totally true.

*

Now we extend the Partial Algebra to NeutroAlgebra, but first we recall the below.

23. Classification of Axioms:

i) A (classical) **Axiom** defined on a nonempty set is an axiom that is totally true (i.e. true for all set's elements).

ii) A **NeutroAxiom** (or **Neutrosophic Axiom**) defined on a nonempty set is an axiom that is true for some set's elements {degree of truth (*T*)}, indeterminate for other set's elements {degree of indeterminacy (*I*)}, or false for the other set's elements {degree of falsehood (*F*)}, where $T, I, F \in [0, 1]$, with $(T, I, F) \neq (1, 0, 0)$ that represents the (classical) Axiom, and $(T, I, F) \neq (0, 0, 1)$ that represents the AntiAxiom.

iii) An **AntiAxiom** defined on a nonempty set is an axiom that is false for all set's elements.

Therefore, we have formed the **neutrosophic triplet**: *<Axiom, NeutroAxiom, AntiAxiom>*.

24. Classification of Algebras

i) A (classical) **Algebra** is a nonempty set CA that is endowed with total operations (or total functions, i.e. true for all set's elements) and (classical) **Axioms** (also true for all set's elements).

ii) A **NeutroAlgebra** (or **NeutroAlgebraic Structure**) is a nonempty set NA that is endowed with: at least one NeutroOperation (or NeutroFunction), or one NeutroAxiom that is referred to the set's (partial-, neutro-, or total-) operations.

iii) An **AntiAlgebra** (or **AntiAlgebraic Structure**) is a nonempty set AA that is endowed with at least one AntiOperation (or AntiFunction) or at least one AntiAxiom.

Therefore, we have formed the neutrosophic triplet:

<*Algebra, NeutroAlgebra (*which includes the *Partial Algebra), AntiAlgebra*>.

25. Definition of Universal NeutroAlgebra

The **Universal NeutroAlgebra** (or **General NeutroAlgebra**) is a branch of neutrosophic mathematics that studies classes of NeutroAlgebras and AntiAlgebras.

26. Applications of NeutroFunctions and NeutroAlgebras

Applicability of Partial Functions, when the domain is not well-known, are in computer science, computability theory, programming language, real analysis, complex analysis, charts in the atlases, recursion theory, category theory, etc.

NeutroFunctions (NeutroOperations), when the domain and/or range are/is not well-known, have a larger applicable field since, besides Partial Functions' undefined values, NeutroFunctions include functions' outer-defined and/or indeterminate values referred not only to the functions' not-well-known domain but to the functions' not-well-known range too.

NeutroAlgebras, in addition to NeutroFunctions, is equipped with NeutroAxioms that better reflect our reality where not all individuals totally agree or totally disagree with some regulation (law, rule, action, organization, idea, etc.), but each individual expresses partial degree of approval, partial degree of ignorance, and partial degree of disapproval of the regulation. NeutroAxioms are true for some elements, indeterminate for others, and false for other elements.

27. NeutroAxioms in our World

Unlike the idealistic or abstract algebraic structures, from pure mathematics, constructed on a given perfect space (set), where the axioms (laws, rules, theorems, results etc.) are totally (100%) true for all space's elements, our World and Reality consist of approximations, imperfections, vagueness, and partialities.

Most of mathematical models are too rigid to completely describe the imperfect reality. Many axioms are actually NeutroAxioms (i.e. axioms that are true for some space's elements, indeterminate for other space's elements, and false for other space's elements). See below several examples.

In Soft Sciences [2] the laws are interpreted and re-interpreted; in social, political, religious legislation the laws are flexible; the same law may be true from a point of view, and indeterminate or false from another point of view. Thus the law is partially true and partially indeterminate (neutral) or false (it is a neutrosophic law, or NeutroLaw).
Many interpretations have a *degree of objectivity,* a *degree of neutrality (indeterminacy),* and a *degree of subjectivity*. The cultural, tradition, religious, and psychological factors play important roles in interpretations and actions for or against some regulations.

a) For example, "gun control". There are people supporting it because of too many crimes and violence (and they are right), and people that oppose it because they want to be able to defend themselves and their houses (and they are right too); there also are ignorant people who do not care (so, they do not manifest for or against it).

Besides ignorant (neutral) people, we see two opposite propositions, both of them true, but from different points of view (from different criteria/parameters; plithogenic logic may better be used herein, since the truth-value of a proposition is calculated from various points of view – obtaining different results). How to solve this? Going to the middle, in between opposites (as in neutrosophy): <u>allow</u> military, police, security, registered hunters to bear arms; <u>prohibit</u> mentally ill, sociopaths, criminals, violent people from bearing arms; and <u>background check</u> on everybody that buys arms, etc.

b) Similarly for "abortion". Some people argue that by abortion one kills a life (which is true), others support the idea of the woman to be master of her body (which is true as well), and again the category of ignorants.
c) A law applying for a category of people (degree of truth), but not applying for another category of people (degree of falsehood).

For example, in India a Hindi man is allowed to marry only one wife, while a Muslim man is allowed to marry up to four wives.

d) *Double Standard*: a rule applying for some people, but not applying for other people that for example may have a higher social rank.
e) *Hypocrisy*: criticizing your enemies (but not your friends!) for what your friends do too!

Or praising your friends (but not your enemies!) for what your enemies do too!

That's why the NeutroAlgebras better model our imprecise reality and they are needed to be studied, since they are equipped with NeutroOperations (partially true, partially indeterminate, and partially false operations) and NeutroAxioms (partially true, partially indeterminate, and partially false axioms), all designed on a not-well-known space.

28. Theorem 1

The NeutroAlgebra is a generalization of Partial Algebra.

As a consequence, NeutroAlgebra is a generalization of Effect Algebra and of Boole's Partial Algebras.

Proof.

Since the Partial Algebra is equipped with partially defined operations, they are NeutroAlgebras according with the above definition of NeutroAlgebras. But the converse is not true.

Further on, the Effect Algebra and the Boole's Partial Algebras are particular cases of Partial Algebra, therefore particular cases of NeutroAlgebra.

29. Example of NeutroAlgebra that is not a Partial Algebra

Let the set $S = (0, \infty)$, endowed with the real division $\div$ of numbers. $(S, \div)$ is well defined, since

$\div$ is a total operation (there is no division by zero).

S is NeutroAssociative, because, from $x, y, z \in S$ such that

$x \div (y \div z) = (x \div y) \div z$

one gets $\dfrac{xz}{y} = \dfrac{x}{yz}$ or $xyz^2 = xy$ or $z^2 = 1$ (since both $x, y \neq 0$), whence $z = 1$ (the other solution

$z = -1$ does not belong to S).

Therefore, $(S, \div)$ is: associative for the triplets of the form $\{(x, y, 1), x, y \in S\}$, while for other triplets $\{(x, y, z), x, y, z \in S$, and $z \neq 1\}$ it is not associative. So, S is partially associative and partially nonassociative (that we call NeutroAssociative).

Thus $(S, \div)$ is a classical groupoid, it is neither a partial algebra nor an effect algebra since its operation $\div$ is not a partial operation (but a total operation), and it is a NeutroSemigroup (since it is well-defined and neutroassociative) which means part of the general NeutroAlgebra.

Thus we proved that there are NeutroAlgebras that are different from Partial Algebras.

30. Other Examples of NeutroAlgebras vs. Partial Algebras

Let $U = \{a, b, c\}$ be a universe of discourse and $S = \{a, b\}$ one of its nonempty subsets.

i) Structure $S_1 = (S, *_1)$, constructed as below using Cayley Table:

$*_1$	a	b
a	b	a
b	a	*undefined*

$*_1$ is a partially defined operation since $b *_1 b = undefined$, but for all $x \neq b$ or $y \neq b$, $x *_1 y$ is defined.
The axiom of commutativity is totally true, since $a *_1 b$ and $b *_1 a$ are defined, and they are equal: $a *_1 b = a = b *_1 a$.
Therefore, S_1 equipped with the axiom of commutativity is a partial algebra.
But S_1 equipped with the axiom of associativity is not a partial algebra, since the associativity is partially true and partially indeterminate or partially false (i.e. NeutroAssociativity):

$a*_1(b*_1a) = a*_1 a = b$, and $(a*_1b)*_1a = a*_1a = b$ *(degree of truth);*

but $a*_1(a*_1b) = a*_1 a = b$ while $(a*_1a)*_1b = b*_1b = undefined \neq b$ *(degree of falsehood).*

Therefore, S_1 equipped with the axiom of associativity is a Neutro-algebra.

ii) Structure $S_2 = (S, *_2)$, constructed as below using Cayley Table:

$*_2$	a	b
a	b	a
b	a	$c \notin S$

$*_2$ is an <u>outer-operation</u> since $b*_2b = c \in U - S$ is *outer-defined*, but for all $x \neq b$ *or* $y \neq b$, $x*_2y$ is inner-defined. Because $*_2$ is not partially defined (since $b*_2b \neq undefined$), S_2 cannot be a partial algebra.

Similarly, the axiom of commutativity is totally true, since $a*_2b$ and $b*_2a$ are defined, and they are equal: $a*_2b = a = b*_2a$.

Therefore, S_2 equipped with the axiom of commutativity is an outer-algebra (which is a particular case of NeutroAlgebra).

But S_2 equipped with the axiom of associativity is not an outer-algebra, since the associativity is partially true and partially indeterminate or partially false (i.e. NeutroAssociativity):

$a*_2(b*_2a) = a*_2 a = b$, and $(a*_2b)*_2a = a*_2a = b$ *(degree of truth);*

but $a*_2(a*_2b) = a*_2 a = b$ while $(a*_2a)*_2b = b*_2b = c \neq b$ *(degree of falsehood).*

Therefore, S_2 equipped with the axiom of associativity is a Neutro-algebra.

iii) Structure $S_3 = (S, *_3)$, constructed as below using Cayley Table:

$*_2$	a	b
a	b	a
b	a	a or b

$*_3$ is an indeterminate-operation since $b*_3b = a$ *or* b *(indeterminate)*, but for all $x \neq b$ *or* $y \neq b$, $x*_3y$ is well-defined. The same, because $*_3$ is not partially defined (since $b*_3b \neq undefined$), S_3 cannot be a partial algebra.

Similarly, the axiom of commutativity is totally true, since $a*_3b$ and $b*_3a$ are defined, and they are equal: $a*_3b = a = b*_3a$.

Therefore, S_3 equipped with the axiom of commutativity is an indeterminate-algebra (a particular case of NeutroAlgebra).

But S_3 equipped with the axiom of associativity is not an indeterminate-algebra, since the associativity is partially true and partially indeterminate or partially false (i.e. NeutroAssociativity):

$a*_3(b*_3a) = a*_3 a = b$, and $(a*_3b)*_3a = a*_3a = b$ *(degree of truth);*

but $a*_3(a*_3b) = a*_3 a = b$ while $(a*_3a)*_3b = b*_3b = (a \text{ or } b) \neq b$ *(degree of falsehood).*

Therefore, S_3 equipped with the axiom of associativity is a NeutroAlgebra.

31. The main distinction between Partial Algebra vs. NeutroAlgebra

A Partial Algebra has at least one Partial Operation, while all Axioms involving its partial and total operations (Associativity, Commutativity, etc.) are 100% true.

Whilst a NeutroAlgebra has at least one NeutroOperation (which is an extension of Partial Operation) or one NeutroAxiom:

i) When the NeutroAlgebra has no NeutroAxiom, it coincides with the Partial Algebra.

ii) There are NeutroAlgebras that have no NeutroOperations, but have NeutroAxioms. These are different from Partial Algebras.

iii) And NeutroAlgebras that have both, NeutroOperations and NeutroAxioms.
Also, these are different from Partial Algebras.

32. Remark 1

For the study of NeutroAlgebras the names of axioms (to be taken into consideration if they are partially true, partially indeterminate, partially false) and similarly for the study of AntiAlgebras the names of axioms (to be taken into consideration if they are totally false) should from the beginning be specified - since many axioms may fall in such categories.

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Uncertainty: two probabilities for the three states of neutrosophy

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Abstract

Uncertainty is inherent to the real world: everything is only probable, precision like in measurements is finite, noise is everywhere... Also, science is based on a modeling of reality that can only be approximate. Therefore we postulate that uncertainty should be considered in our models, and for making this more easy we propose a simple operational conceptualization of uncertainty. Starting from the simple model of associating a probability p to a statement supposed to be true our proposed modeling bridges the gap towards the most complex representation proposed by neutrosophy as a triplet of probabilities. The neutrosophic representation consists in using a triplet of probabilities (t,i,f) instead of just a single probability. In this triplet, t represents the probability of the statement to be true, and f it's the probability to be false. The specific point of neutrosophy it that the probability i represents the probability of the statement to be uncertain, imprecise, or neutral among other significations according to the application.

Our proposed representation uses only 2 probabilities instead of 3, and it can be easily translated into the neutrosophic representation. By being simpler we renounce to some power of representing the uncertain but we encourage the modeling of uncertainty (instead of ignoring it) by making this simpler. Briefly said, the prepare the path towards using neutrosophy. Our proposed representation of uncertainty consist, for a statement, not only to add its probability to be true p, but also a second probability pp to model the confidence we have in the first probability p. This second parameter pp represents the plausibility of p, therefore the opposite of its uncertainty. This is the confidence given to the value of p, in short pp is the probability of p (hence the name pp), This is simple to understand, and that allows calculations of combined events using classical probability such as based on the concepts of mean and variance. The stringent advantage of our modeling by the couple (p,pp) is that experts can be easily interrogated to provide their expertise by asking them simply the chance they give to an event a occur (this is p) and the confidence they have in that prediction (which is pp). We give also a formula to transform from our model to the neutrosophic representation. Finally, a short discussion on the entropy as a measure of uncertainty is done.

Keywords: Uncertainty. neutrosophy, probability, representation of uncertainty, entropy

1.Introduction

We propose a modeling of uncertainty that is intermediate between a simple probability and one of the most general representation, known under the name of neutrosophy[1]. By uncertainty we mean an event that can occur or not, an event that is not certain. This includes also the uncertainty inherent to any data such as the measure of a signal which is most often imprecise, fuzzy and noisy. This idea of an intermediate approach is based both on the desire to properly represent uncertainty for some kind of approximate calculations in an easy way, but also to facilitate later the transition to the more complete approach of neutrosophy, if this becomes required for a better estimation.

In the real world many applications have to compute from uncertain informations such values affected by the imprecision of the sensors or signals degraded by noise and simply probabilistic situations. The usual true/false logic and arithmetic numbers can only represent crisp values, leaving completely aside the uncertain. Then the statistics have been developed to be still able to make predictions for probable events, like to estimate the chance of rain tomorrow. For most real-world modeling the uncertainty it includes can not be modeled precisely enough just with a

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single probability for each data. Such highly uncertain data in complex situations should be modeled with more parameters, as proposed by fuzzy logic for example.

Multi-valued representation add the number of variables required for a finer fitting to the reality, similarly as a real number use more bits than an integer to trade in a better approximation. The final user of any application being a human, a representation adapted to humans must be envisaged in the last steps of processing to present the result. Has humans have their cognition taking place on a background of emotions, the concepts that are manipulated are often colored by a sentiment: I like it, I dislike it, or I am indifferent to this. The human by its emotional dimension has a cognition which is for the most often considering 3-state values and not boolean logical values (binary true/false). There is a characteristically human third state, the neutral state, it's not only yes or no but also I don't know, to not repeat the like/dislike/indifferent terms. So there is a 3-state based logic extending the classical binary logic. Those states are associated by the human to linguistic terms, having a semantical value to him. And more than 3 states can be considered when we make categories, but rarely more than 5 to 7, else we mix-up everything. Learning is a case of such a 3 state processing: a new situation will update the knowledge acquired from previous one by changes that are either reinforcement, inhibition or neutral (no change).

Many situations of the real-world are also intrinsically 3-state, or considered as such by the humans: for example the ambient temperature can be warm, cold, or agreeable. You can be positive, negative or indifferent about a proposition. A value can be positive, null or negative, like the one from a sensor. By driving a car you can be accelerating, braking, or just maintain a constant speed (having set on the automatic cruise-control). A regulator will add, subtract or do nothing on the value to be regulated. This extra third state is the most often a neutral state. This was roughly the inspiration that leads to the conception of neutrosophy, the taking into account of the neutral situation.

So therefore neutrosophy use 3 variables to represent these human 3 states, a value can have a truth of t percent as well as simultaneously a falsity of f percent and also a indetermination of i percent, using a triplet (t ,i, f) those values are between 0 and 1. These values are also independent, unrelated, by opposition of the classical probability were the truth is given by t and the falsity by $1 - t$. Here f is not independent of t, but dependent as $f = 1 - t$. Namely for the extreme cases: if it's 100% sure then t = 1 and therefore f = 0 (it's absolutely not false), and oppositely if it's 100% false then f = 1 and therefore t = 0 (it's not true).

Simply said, the representation by a probability uses 1 variable and the neutrosophic representation uses 3 variables, this makes the second more difficult to use, as an human expert has to decide instead for 3 values for the triplet (t,i,f). We believe there are cases or applications were something more complex than just a probability will be beneficial, and also still not requiring the full generality of neutrosophy. By using 2 variables one should be able to model in a finer way than with just one, and hopefully in way more simple than neutrosophy. This will have also the benefit to prepare the path for using neutrosophy by having with a small effort open the path to multi-valued representations.

We will start with a simple hypothesis, that we have to deal with uncertainty because it is inherent to the real-world. Then we will look at how the qualitative human cognition can be feed of information from the quantitative reality. After some considerations about the uncertainty will serve to define our representation using 2 variables, 2 probabilities instead of the usual single one. Then we will make the link between our representation and the basics of the theory of neutrosophy. Our concrete results will be, aside our 2 variables model, some extensions of it that are easy to make. Finally, we will link our representation more closely to the notion of uncertainty by evaluating it from an extension of the concept of entropy, which is closely related to the unexpected content of a signal.

2. Hypothesis

Our starting assumption H1 is that "real world situations involve some uncertainty". More precisely, this means that our measures of the real situation are both incomplete and fraught with uncertainty, in particular imprecision and noise. The understanding of the situation is necessarily incomplete and so is its perception. As a result, the modeling elaborated will be imperfect, as it is only a limited approximation of reality. Thus, what is being calculated involves an element of uncertainty in relation to reality. The objective of science has always been to reduce this gap between model and reality (to make more useful predictions).

However, the hypothesis H1 we made is demonstrated by the consideration of concrete practical cases, a few examples of which are sufficient to convince. To be short, we will retain only one example, and therefore we will continue to consider that it is just a hypothesis for the rest of this text. In the implementation of an expert system, a case of artificial intelligence, a set of calculation rules must be constructed about knowledge obtained from human experts in this field. Sometimes several experts have divergent opinions, or warn against the uncertainty of their statements. With a methodology capable of dealing with uncertainty, however, such information can be used when otherwise it should be rejected, or if nothing else is available, it can still be considered but with inconsiderate risks.

However, it is also possible to choose H1's opposite hypothesis, which claims that ¬H1 "a solution can be calculated by working on data considered accurate, without explicitly involving uncertainty". This is the preferred approach to date, except perhaps in these complex situations where it does not lead to an optimal result, or even very often to a suboptimal result acceptable in precision and certainty.

The H1 hypothesis results in a more complete one, a more precise variant H1', which is that H1' "modeling real-world situations involves some uncertainty". Indeed, the search for a numerical solution is based on a model in addition to measurements.

3. From qualitative to quantitative

Given this situation, we are coming up with the solutions that are available to us. They are of two types that can be considered together:

S1: Handle in a better way the uncertainty.

S2: Reduce uncertainty, for example by adding more sensors.

In some situations S2 is not possible, mainly because it is not feasible for economic reasons, or because to go further than what has already been done and which has proven unsatisfactory. Thus, there is always some uncertainty in the system on which a result is calculated. Then it is desirable to look for a good S1 type solution, therefore a solution able to address uncertainty explicitly.

Often the desired result is a simple "yes or no" type answer, i.e. of a binary type. It therefore seems tempting to consider the whole problem in a binary approach. In terms of human decision-makers, the desired outcome is more of a three-state type. It is a question of discriminating between three cases: for example, I am confident in the project, I do not have a clear opinion, I am rather not confident in this project. Sometimes the decision maker's choice is not only between doing or not doing, but also between doing one thing or doing the opposite, or doing nothing at all. Typically for a stock market investor, he must choose according to his current valuation of a security he owns whether he buys more or sells it or whether he thinks it is better to wait.

The way of thinking of a human expert is mainly qualitative and not quantitative. In general, the results of a calculation are quantitative, and a decision threshold is set to choose whether or not to do so. Binary logic is somehow too square, it cuts too sharply, it's not flexible enough: true or false. If a calculation is performed in binary logic, each elementary operation has this defect. In a three-state logic, certain inputs or intermediate results can be neutralized so that in specific circumstances they no longer affect the final result. This is an additional flexibility that is central to neural networks in particular.

In a numerical calculation, each value has an effect, possibly very small, on the final result. This corresponds better to the case of reality where there are many cross-influences. A particular quantitative approach is the probabilistic approach. Each entry or intermediate result is no longer either true or false, but a probability of being true. So we won't say "it will rain tomorrow", but "we have a 65% chance of rain tomorrow", or "it should fall 4 mm of water". For a specialist a numerical value speaks for itself, and for many more people a percentage will be easier to get, and even more will understand a fractional form like "there are about 3 chances out of 5 that it will rain tomorrow".

The case of this example of rainfall forecasting is interesting for two other aspects. The people most interested in this information are farmers who have to decide whether it is necessary to water their crops, or whether what comes from the sky will be enough. So they want to know the probabilities of different rainfall amounts, for the coming days, in order to make the average over the week, for example, or to determine the worst case scenario. For example, the probability of one millimeter of rain, that of 2, that of 4, that of 6 or more and also that there will be no rain. The models used for forecasting often offer this information by principle. This gives the probabilities for a whole series of values (in fact categories centred on these values), such as 0, 1, 2, 4, 6 or more mm of rain. The total of these probabilities should be 100%, for each day. This is a more complete approach, than considering only the maximum and its probability as in our previous approach, such of all situations (global) evaluation corresponds to the new extension of neutrosophy recently proposed by its creator because it is even more general: plithogenesis [2]. Indeed, these forecasting simulations are carried out using multidimensional matrix models running on a large number of vector calculation units (or matrix or tensor depending on the name we prefer to give them) in parallel. In concrete terms, the farmer wishes to reduce the uncertainty about the future that is specific to nature, to control it as good as possible in order to maintain the quantity and quality of his production with a minimum of intervention and costs.

Thus, a quantitative rather than qualitative approach, often probabilistic, is preferable.

4. Consideration of uncertainty

In the case of general public meteorological forecasts, forecasts of up to 7 days have been given since several years. So for the forecast of each coming day it is emphasized that the accuracy or certainty of the forecast decreases as the day is further away. For example, we can say that for the seventh day "the prediction is estimated to be 65% correct". So, for example, that it is estimated that "it will rain next Wednesday 4 mm of water with a probability of 65%". Here it is a question of the degree of certainty of the prediction, therefore of its uncertainty, but in the problems of the real world there are also uncertainties of measurement, modeling, calculation...

So we have a probability of a first probability, here the accuracy of the rain probability, a probability of the second order in a way.

In the case of a human expert, it is useful, and this extra parameter can be estimated also by the expert: he will be asked his prognosis, then how strongly he believes in it, what certainty he attributes to it. To do this, experts can be asked to choose between linguistic values such as "a little, medium, a lot, completely" and then convert this qualitative judgment (his confidence) into a percentage.

In this way, a model incorporating uncertainty is formed, using two variables. Which we call p and pp. The first variable p is the probability of a certain state, for example rain, the second pp is the probability of p, so the certainty that we give to the probability p. Using just two words, it's a probability and its plausibility (or confidence).

These two variables are between 0 and 1 (or in an equivalent way between 0% and 100%). If p = 1 then the expert estimates that the fact will occur. If p = 0 that it will not occur, then this corresponds to the binary logic, true and respectively false. Such a classical binary logic is therefore representable in this finer model. Since not everything is black or white, the intermediate values specify the probability that it is true, this is a very classical probability. Then the expert gives an estimate of his prognosis, the plausability (his confidence). He can consider himself certain (pp = 1) or totally uncertain (pp = 0), and with all the gradations in between.

If $pp = 1$, only the probability p is considered, and the calculation of the usual probabilities will be used. If $pp = 0$, it is not possible to say anything and the p value is not relevant.

Now uncertainty is simply the opposite of certainty pp. If the certainty pp is 0 then the uncertainty i will be $1 - pp$, and in this case will be 1, it will be total uncertainty.

So in summary we propose à simple model to consider uncertainly using only 2 variables, that can be easily and intuitively be used by human experts, considering 2 classical probabilities:

p the probability of the event estimated by the expert

pp the confidence of the expert in the above probabiliy

This second variable pp is also the opposite of the estimated uncertainty I: $pp = 1 - i$

For example, the event can be team A wins over team B in a sport game to come. The expert will express his opinion by saying there is a 75% of chances that team A is the winner, so p(teamA) = 0,75. Then he can add to be more informative that he believes so at a confidence of 9 /10 (the note he attributes to the quality of his prevision). So, here pp(p(teamA)) = 0.9

The refined prevision of the expert is given by the couple (p, pp) = (0.75,0,9). That means is strongly believe that team A has 75% of winning.

Another expert could say that he believe also that team A wins, but with of probability of 80%, but he can add he is not a specialist of team A either he knows very well team B, and that he is prudent about that prediction a little more than mildly. His opinion can be represented by the couple (p, pp) = (0.8, 0.6)

Now the goal will be to aggregate the opinions of all experts in a way taking into account the weight each one attach to his own prediction. This can be done by a simple weighted mean calculation, that is the way the center of gravity is determined.

5. Link with neutrosophy

Neutrosophy, as its name suggests, considers the case of neutrality in addition to the true and false cases, giving it a central importance in modeling. Depending on the applications, different meanings will be chosen for the three states: true, neutral, false, or positive, neutral, negative (as in the previous example of the investor), or true, uncertain, false, when modeling the existing uncertainty in the system.

In addition to considering just three states, it is possible to consider a variant called a simple neutrosophical number (SNn), that is actually a triplet of three probabilities. Such a SNn relates to a statement and is a triplet (t,i,f) where each component t, i and f is a value between 0 and 1, a probability. These probabilities are respectively the degree of belonging to (the state of) truth, the one relating to (the state of) uncertainty and also the one concerning (the state of falsity) of the statement. It should be noted that unlike the representation by a single probability, here the probability of falsity is not necessarily the complement of the probability of truth (it can be specified independently; this gives the expert additional freedom, but also this makes it is more complex to collect his opinion).

For example, true and false binary logic values are represented in an SNn by (1,0,0) and (0,0,1) respectively.

For the information described in our previously presented alternative representation of p and pp, i.e. a probability of validity p (of a statement) and an estimate of its certainty pp, we can imagine a transformation in the neutrosophical representation which is more general. On the basis of the representations of the true and false binary logical states (which can be considered as the extremes of the representation domain) we must choose a correspondence that produces, respectively for true and false, the values according to the table below.

Table 1 Correspondence between the values indicated for the first and second representation.

	p	pp	t	i	f
True	1	1	1	0	0
False	0	1	0	0	1

We can choose as a transformation satisfying these constraints, simply as;

$$t = p$$

$$i = 1 - pp$$

$$f = 1 - p$$

By the way, note that the separate processing of rules for t and f in fuzzy logic [3] with as here $f = 1 - t$ corresponds to the variant called intuitutionistic fuzzy logic [4], and therefore our proposal is an extension of it. What is also interesting, we can define, in fact arbitrarily, a transformation from the neutrosophical representation to our p and pp representation, i.e. the probability and its plausibility (or confidence).

Then, we can choose, although it is not one-to-one (bi-univoc):

$$p = \frac{(t+1-f)}{2}$$

$$pp = 1 - i$$

It should be noted that in general uncertainty, as considered in these transformations and especially in the representation by a probability and its estimate (p,pp), is not a neutral element for the combination operations that are frequently used, such as logical AND and logical OR, but rather a special kind of projection that partially eliminates a dimension. When the estimate pp is 0, the value of the probability p vanishes, it no longer makes sense. The independence of t and f is also destroyed by the conversion of the neutrosophical representation to that of a probability and its plausibility.

Except in this case of degeneration, the proposed representation has all the characteristics of probabilities. Thus it is possible to calculate the logical AND as the joint-probability per a simple arithmetic product. Here, this will be done for p and pp separately.

6. Extension

A slightly extended representation can be considered to deal with uncertainty, if not better, at least a little differently. This idea is used in fuzzy logic and it is taken up in neutrosophy also at the level of the functions of membership. For example, p or t are the value of membership to the veracity of the proposal, the probability that it is true.

Instead of giving an estimate of the probability, it is possible, which ultimately amounts to the same thing, to describe the probability by an interval, for example p is between pmin and pmax. This interval denotes a confidence (an interval of confidence), a confidence in the value of p (the narrower the interval, the higher the confidence), which can be equally expressed by a standard deviation (or variance) Δp. For example p and Δp with Δp = pmax - pmin instead of the representation of the interval type pmin and pmax.

On the assumption, quite often approximately verified in practice, of a Gaussian distribution of the value around the mean p, then we have a “normal” representation that is well treated in statistics, especially for the conjunction of two measurements of the same signal. Then the result of combining the two mean values taking into account their respective certainties (via a representation as standard deviation, as variance) will be the estimate produced by the Kalman filter [5]. This filter calculates an optimal estimate according to the rules for combining the means and standard deviations of two measurements [6]. Since the half standard deviation Δp added to or subtracted from the mean p gives the location of the probability values 0.5 = 50%, a correspondence exists with certainty and therefore uncertainty. That uncertainty is further discussed in the next section.

7. Entropy

A specific addition to this (p, pp) representation has been communicated to us by Vasile Patrascu: to calculate the uncertainty in a more precise way from the pair of probabilities (p,pp). That more precise undestanding of uncertainty is nothing else than the entropy that he has defined in [7] for various multi-valued representations of neutrosophic information. He kindly provided us the formula for our p and pp probabilistic representation. In this article Patrascu proposed a calculation formula for the entropy E (uncertainty) of the neutrosophic information given in triplet (t,i,f), that is:

$$E(t,i,f) = 1 - \frac{|t-f|}{1+|t+f-1|+i} \qquad (1.1)$$

Replacing in it t, i and f by the expressions used before, that were:

$$t = p \qquad (1.2)$$

$$i = 1 - pp \qquad (1.3)$$

$$f = 1 - p \qquad (1.4)$$

we obtain the following formula for calculating the uncertainty (or entropy) E of the pair (p,pp):

$$E(p,pp) = 1 - \frac{|2p-1|}{2-pp} \qquad (1.5)$$

In the particular case of the two boolean logical constants we obtain:

$$E(True) = E(1,1) = 0 \qquad (1.6)$$

$$E(False) = E(0,1) = 0 \qquad (1.7)$$

Further, on the basis of another of his articles [8] Patrascu propose to associate a complete neutrosophic information (t,i,f) to the probability pair (p,pp). He even provided us 2 variants.

- Variant (I)

$$t = \left(\frac{1+pp}{2}\right)p - \frac{1-pp}{4}(1-|2p-1|) \qquad (2.1)$$

$$i = \frac{1-pp}{2}(2-|2p-1|) \qquad (2.2)$$

$$f = \left(\frac{1+pp}{2}\right)(1-p) - \frac{1-pp}{4}(1-|2p-1|) \qquad (2.3)$$

with: $t + i + f = 1$

- Variant (II)

$$t = \frac{p - \frac{1-pp}{2}(1-|2p-1|)}{1+(1-pp)|2p-1|} \qquad (2.4)$$

$$i = \frac{1-pp}{1+(1-pp)|2p-1|} \qquad (2.5)$$

$$f = \frac{1-p-\frac{1-pp}{2}(1-|2p-1|)}{1+(1-pp)|2p-1|} \qquad (2.6)$$

With $t + i + f = 1$

8. Conclusion

An intermediate representation between fuzzy logic in its intuitutionistic variant and neutrosophy can be considered with more generality than the former and less complexity than the latter. It is an extension of the basic probability by considering also its plausibility (or confidence). Thus it may be appropriate in situations where a less detailed modeling of uncertainty than the one of neutrosophy is considered satisfactory. Then the treatments are simpler but, above all, it is easier to collect the opinions of the experts by simply asking them for their forecast and also to estimate he certainty they attach to their prediction. This representation is equivalent to that of an interval (of probabilities), or to that used in probability theory with the mean value and variance. So this representation allows to remain for operations at well-known numerical calculations that do not involve functions with degrading effect on information such as min and max.

Also however, on the basis of this easier questioning of the experts, it will be possible to produce a neutrosophical representation that is more general by the simple transformations proposed, and thus to spare the experts of the burden of a higher complexity of thinking while still exploiting the power of neutrosophy to better treat the uncertainty inherent in real systems

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MAGNIFICATION OF MBJ-NEUTROSOPHIC TRANSLATION ON G-ALGEBRA

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Abstract

In this article, we define the MBJ-neutrosophic magnified translation (MBJNMT) on G-algebra which is the combination of multiplication and translation and study significant results of MBJ-neutrosophic ideal and MBJ-neutrosophic subalgebra by using the notion of MBJ-neutrosophic magnified translation. We investigate the conversion of MBJ-neutrosophic ideal and MBJ-neutrosophic subalgebra with one another and use the idea of intersection and union to produce some important results of MBJ-neutrosophic magnified translation.

Keywords: G-algebra, MBJ-neutrosophic magnified translation.

1.Introduction

Zadeh [1] presented the theory of fuzzy set in 1965. Iseki and Tanaka [2] gave the idea of BCK-algebra in 1978. Iseki [3] introduced the notion of BCI-algebra in 1980 and it is obvious that BCK-algebra is a proper sub class of the class of BCI-algebra. Lee et al. [4] investigated the fuzzy translation, (normalized, maximal) fuzzy extension and fuzzy multiplication of fuzzy subalgebra in BCK/BCI-algebra. Link among fuzzy translation, fuzzy extension and fuzzy multiplication are also studied. Ansari et al. [5] introduced the idea of fuzzy translation, fuzzy extension and fuzzy multiplication of fuzzy β ideal of β-algebra and investigated important characteristic. Priya et al. [6] studied the PS-ideal and its level subsets. Bandaru et al. [7] defined the G-algebra. Lekkoksung [8] focused on fuzzy magnified translation in ternary hemirings, which is a extension of BCI / BCK/Q / KU / d-algebra. Senapati et al. [9] have done extensive research on intuitionistic fuzzy H-ideal in BCK/BCI-algebra. Jana et al. [10] interogated the intuitionistic fuzzy G-subalgebra. Senapati et al. [11] studied fuzzy translations of fuzzy H-ideals in BCK/BCI-algebra. Atanassov [12] introduced the intuitionistic fuzzy sets. Senapati [13] investigated the relationship among intuitionistic fuzzy translation, intuitionistic fuzzy extension and intuitionistic fuzzy multiplication in B-algebra. Kim et al. [14] defined the intuitionistic fuzzy structure of B-algebra. Senapati et al. [15] discussed the fuzzy dot subalgebra and fuzzy dot ideal of B-algebras. Priya et al. [16] has done research on fuzzy translation and fuzzy multiplication in PS-algebra. Chandramouleeswaran et al. [17] investigated fuzzy translation and fuzzy multiplication in BF/BG-algebra. Smarandache [18,19] extended the intuitionistic set to neutrosophic set through Several examples. Jun et al. [20] investigated the commutative falling neutrosophic ideals in BCK-algebra. C. H. Park [21] defined the neutrosophic

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ideal in subtraction algebra and studied it through several properties. Khalid et al. [22] did the research on neutrosophic soft cubic subalgebra through significant characteristic like P-union, R-intersection etc. Khalid et al. [23] interestingly worked on intuitionistic fuzzy translation and multiplication through subalgebra and ideals. Khalid et al. [24] defined the T-neutrosophic cubic set and studied this set through ideals and subalgebras and investigated many results. Takallo et al. [25] defined and studied MBJ-neutrosophic structures and its applications in BCK/BCI-algebras Khalid et al. [26] interpreted the multiplication of neutrosophic cubic set and defined the ϒ-multiplication of neutrosophic cubic set and studied it with neutrosophic cubic M-subalgebra, neutrosophic cubic normal ideal and neutrosophic cubic closed normal ideal. He also studied ϒ-multiplication under homomorphism and cartesian product through significant characteristics. Khalid et al. [27] defined and studied the MBJ-neutrosophic T-ideal through union, intersection. further he utilized important characterisitics to interrogate the MBJ-neutrosophic T-ideal under cartesian product.

The purpose of this article is to introduce the idea of MBJ-neutrosophic Magnified translation **(MBJNMT)** on G-algebra. In second section we cite some fundamental definitions which are used to develop the paper. In third section we discussed the MBJ-neutrosophic magnified translation **(MBJNMT)** of MBJ-neutrosophic ideal **(MBJNID)** and MBJ-neutrosophic subalgebra **(MBJNSU).**

2. Preliminaries

First we discuss some definitions which are used to present this article.

Definition 2.1 [3] An algebra $(Y,*,0)$ of type $(2,0)$ is called a BCI-algebra if it satisfies the following conditions:

i) $(t_1 * t_2) * (t_1 * t_3) \leq (t_3 * t_2)$,

ii) $t_1 * (t_1 * t_2) \leq t_2$,

iii) $t_1 \leq t_1$,

iv) $t_1 \leq t_2$ and $t_2 \leq t_1 \Rightarrow t_1 = t_2$,

v) $t_1 \leq 0 \Rightarrow t_1 = 0$, where $t_1 \leq t_2$ is defined by $t_1 * t_2 = 0$, for all $t_1, t_2, t_3 \in Y$.

Definition 2.2 [1] An algebra $(Y,*,0)$ of type $(2,0)$ is called a BCK-algebra if it satisfies the following conditions:

i) $(t_1 * t_2) * (t_1 * t_3) \leq (t_3 * t_2)$,

ii) $t_1 * (t_1 * t_2) \leq t_2$,

iii) $t_1 \leq t_1$,

iv) $t_1 \leq t_2$ and $t_2 \leq t_1 \Rightarrow t_1 = t_2$,

v) $0 \leq t_1 \Rightarrow t_1 = 0$, where $t_1 \leq t_2$ is defined by $t_1 * t_2 = 0$, for all $t_1, t_2, t_3 \in Y$.

Definition 2.3 [7] A non-empty set Y with a constant 0 and a binary operation $*$ is said to be G-algebra if it satisfies the following axioms.

G1: $t_1 * t_1 = 0$

G2: $t_1 * (t_1 * t_2) = t_2$, for all $t_1, t_2 \in Y$

A G-algebra is denoted by $(Y,*,0)$.

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Definition 2.4 [7] A non-empty subset S of G-algebra Y is called a G-subalgebra of Y if $t_1 * t_2 \in S\ \forall\ t_1, t_2 \in S$.

Definition 2.5 [15] A non-empty subset I of a G −algebra Y is called an ideal if for any $t_1, t_2 \in Y$,

(i) $0 \in I$,

(ii) $t_1 * t_2 \in I$ and $t_2 \in I \Rightarrow t_1 \in I$.

Definition 2.5 [6] Let Y be a PS-algebra. A fuzzy set B of Y is called a fuzzy PS ideal of Y if it satisfies the following conditions:

i) $\phi(0) \geq \phi(t_1)$,

ii) $\phi(t_1) \geq \min\{\phi(t_2 * t_1), \phi(t_2)\}$, for all $t_1, t_2 \in Y$.

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Let Y be a group of objects denoted generally by t_1. Then a fuzzy set B of Y is defined as $B = \{< t_1, \phi_B(t_1) > |t_1 \in Y\}$, where $\phi_B(t_1)$ is called the membership value of t_1 in B and $\phi_B(t_1) \in [0,1]$.

A fuzzy set B [6] of PS-algebra Y is called a fuzzy PS subalgebra of Y if $\phi(t_1 * t_2) \geq \min\{\phi(t_1), \phi(t_2)\}$, for all $t_1, t_2 \in Y$.

Let a fuzzy subset B [4], [5] of Y and $\alpha \in [0,1 - \sup\{\phi_B(t_1)|t_1 \in Y\}]$. A mapping $(\phi_B)^T_\alpha|Y \in [0,1]$ is said to be a fuzzy α translation of ϕ_B if it satisfies $(\phi_B)^T_\alpha(t_1) = \phi_B(t_1) + \alpha$, for all $t_1 \in Y$.

Let a fuzzy subset B [4], [5] of Y and $\alpha \in [0,1]$. A mapping $(\phi_B)^M_\alpha|Y \to [0,1]$ is said to be a fuzzy α multiplication of B if it satisfies $(\phi_B)^M_\alpha(t_1) = \alpha.(\phi_B)(t_1)$, for all $t_1 \in Y$.

An intuitionistic fuzzy set **(IFS)** [12] B over Y is an object having the form $B = \{\langle t_1, \phi_B(t_1), \psi_B(t_1)\rangle | t_1 \in Y\}$, where $\phi_B(t_1)|Y \to [0,1]$ and $\psi_B(t_1)|Y \to [0,1]$, with the condition $0 \leq \phi_B(t_1) + \psi_B(t_1) \leq 1$, for all $t_1 \in Y$. $\phi_B(t_1)$ and $\psi_B(t_1)$ represent the degree of existanceship and the degree of non-existanceship of the element t_1 in the set B respectively.

Let $B = \{\langle t_1, \phi_B(t_1), \psi_B(t_1)\rangle | t_1 \in Y\}$ and $B = \{\langle t_1, \phi_B(t_1), \psi_B(t_1)\rangle | t_1 \in Y\}$ be two **IFSs** [12] on Y. Then intersection and union of A and B are indicated by A ∩ B and A ∪ B respectively and are given by

$$A \cap B = \{\langle t_1, \min(\phi_A(t_1), \phi_B(t_1)), \max(\psi_A(t_1), \psi_B(t_1))\rangle | t_1 \in Y\},$$

$$A \cup B = \{\langle t_1, \max(\phi_A(t_1), \phi_B(t_1)), \min(\psi_A(t_1), \psi_B(t_1))\rangle | t_1 \in Y\}.$$

An **IFS** [14] $B = \{\langle t_1, \phi_B(t_1), \psi_B(t_1)\rangle | t_1 \in Y\}$ of Y is called an **IFSU** of Y if it satisfies these two conditions:

(i) $\phi_B(t_1 * t_2) \geq \min\{\phi_B(t_1), \phi_B(t_2)\}$,

(ii) $\psi_B(t_1 * t_2) \leq \max\{\psi_B(t_1), \psi_B(t_2)\}$, for all $t_1, t_2 \in Y$.

An **IFS** $B = \{\langle t_1, \phi_B(t_1), \psi_B(t_1)\rangle | t_1 \in Y\}$ of Y is said to be an **IFID** of Y if it satisfies these three conditions:

(i) $\phi_B(0) \geq \phi_B(t_1), \psi_B(0) \leq \psi_B(t_1)$,

(ii) $\phi_B(t_1) \geq \min\{\phi_B(t_1 * t_2), \phi_B(t_2)\}$,

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(iii) $\psi_B(t_1) \leq \max\{\psi_B(t_1 * t_2), \psi_B(t_2)\}$, for all $t_1, t_2 \in Y$.

Let ϕ be a fuzzy subset [8] of Y, $\alpha \in [0,T]$ and $q \in [0,1]$. A mapping $\phi^{M\,T}_{\beta\,\alpha}|Y \rightarrow [0,1]$ is said to be a fuzzy magnified $q\alpha$ translation of ϕ if it satisfies: $\phi^{M\,T}_{\beta\,\alpha}(t_1) = \beta.\phi(t_1) + \alpha$ for all $t_1 \in Y$.

Let Y be a non empty set. MBJ-neutrosophic set [25] in Y, is a structure of the form $C = \{\langle M_C t_1, \widehat{B}_C t_1, J_C t_1\rangle | t_1 \in Y\}$ where M_C and J_C are fuzzy sets in Y and M_C is a truth membership function,J_C is a false membership function and $\widehat{B}$ is interval valued fuzzy set in Y and is an Indeterminate Interval Valued membership function.

Let $C = (M_C, \widehat{B}_C, J_C)$ is a MBJ-neutrosphic set of B-algebra Y. Let $t \in [0,1]$, then B is called MBJ-neutrosphic T-ideal (MBJNTID) of Y if it fulfills these assertions:

i) $M_C(0) \geq M_C(t_1)$,$\widehat{B}_C(0) \geq \widehat{B}_C(t_1)$ and $J_C(0) \leq J_C(t_1)$.

ii) $M_C(t_1 * t_3) \geq \min\{M_C((t_1 * t_2) * t_3), M_C(t_2)\}$.

iii) $\widehat{B}_C(t_1 * t_3) \geq \mathrm{rmin}\{\widehat{B}_C((t_1 * t_2) * t_3), \widehat{B}_C(t_2)\}$.

iv) $J_C(t_1 * t_3) \leq \max\{J_C((t_1 * t_2) * t_3), J_C(t_2)\}$.

3. MBJ-neutrosophic Magnified Translation

In this section, we define MBJ-neutrosophic magnified translation **MBJNMT** and investigate some of its characteristics. we use $Ƕ = \inf\{J_C(t_1) | t_1 \in Y\}$ for any **MBJNS** $C = (M_C, \widehat{B}_C, J_C)$ of Y.

Definition 3.1 Let $C = (M_C, \widehat{B}_C, J_C)$ be a **MBJNS** of Y and $p, q, s \in [0, Ƕ]$, $\lambda \in [0,1]$. An object having the form $C^{M\,T}_{\lambda\,p,q,s} = \{(M_C)^{M\,T}_{\lambda\,p}, (\widehat{B}_C)^{M\,T}_{\lambda\,q}, (J_C)^{M\,T}_{\lambda\,s}\}$ is said to be a **MBJNMT** of C, when $(M_C)^{M\,T}_{\lambda\,p}(t_1) = \lambda . M_C(t_1) + p$, $(\widehat{B}_C)^{M\,T}_{\lambda\,q}(t_1) = \lambda . \widehat{B}_C(t_1) + q$, $(J_C)^{M\,T}_{\lambda\,s}(t_1) = \lambda . J_C(t_1) - s$ for all $t_1 \in Y$.

Example 3.1 Let $Y = \{0,1,2\}$ be a G-algebra. A MBJ-neutrosophic $C = (M_C, \widehat{B}_C, J_C)$ of Y is defined as

$$M_C(t_1) = \begin{cases} [0.2,0.4] & \text{if } t_1 = 0 \\ [0.5,0.7] & \text{if otherwise} \end{cases}$$

$$\widehat{B}_C(t_1) = \begin{cases} [0.4, 0.6] & \text{if } t_1 = 0 \\ [0.5, 0.8] & \text{if otherwise} \end{cases}$$

$$J_C(t_1) = \begin{cases} [0.3,0.5] & \text{if } t_1 = 0 \\ [0.6,0.8] & \text{if otherwise} \end{cases}$$

Then C is a MBJ-neutrosophic subalgebra, choose $\lambda = 0.2, p = 0.04, q = 0.05, s = 0.06$ then the mapping $C^{M\,T}_{(0.2)\,(0.04,0.05,0.06)}|Y \rightarrow [0,1]$ is given by

$$(M_C)^{M\,T}_{0.2\,0.04}(t_1) = \begin{cases} [0.08, 0.12] & \text{if } t_1 = 1 \\ [0.14, 0.18] & \text{if otherwise} \end{cases}$$

$$(\widehat{B}_C)^{M\,T}_{0.2\,0.05}(t_1) = \begin{cases} [0.13, 0.17] & \text{if } t_1 = 1 \\ [0.15, 0.21] & \text{if otherwise} \end{cases}$$

$$(J_C)^{M\,T}_{0.2\,0.06}(t_1) = \begin{cases} [0, 0.04] & \text{if } t_1 = 1 \\ [0.06, 0.1] & \text{if otherwise} \end{cases}$$

which imply $(M_C)^{M\,T}_{(0.2)(0.04)}(t_1) = (0.2).M_C(t_1) + 0.04$, $(\widehat{B}_C)^{M\,T}_{(0.2)(0.05)}(t_1) = (0.2).B_C(t_1) + 0.05$, $(J_C)^{M\,T}_{(0.2)(0.06)}$

$(t_1) = (0.2).J_C(t_1) - 0.06$ for all $t_1 \in Y$. Hence $C^{M\,T}_{(0.2)\,(0.04,0.05,0.06)}$ is a MBJ-neutrosophic magnified (0.1)(0.02,0.03,0.04) translation.

Theorem 3.1 Let C be a MBJ subset of Y such that $p, q, s \in [0, Ʊ]$, $\lambda \in [0,1]$ and a mapping $B^{M\,T}_{\lambda\,p,q,s}|Y \to [0,1]$ be a **MBJNMT** of C. If C is **MBJNSU** of Y, then $C^{M\,T}_{\lambda\,p,q,s}$ is a **MBJNSU** of Y.

Proof. Let C be a **MBJNS** of Y, $p, q, s \in [0, Ʊ]$, $\lambda \in [0,1]$ and a mapping $B^{M\,T}_{\lambda\,p,q,s}|Y \to [0,1]$ be a **MBJNMT** of C. Suppose C is a **MBJNSU** of Y. Then $M_C(t_1 * t_2) \geq \min\{M_C(t_1), M_C(t_2)\}$, $\widehat{B}_C(t_1 * t_2) \geq \mathrm{rmin}\{\widehat{B}_C(t_1), \widehat{B}_C(t_2)\}$, $J_C(t_1 * t_2) \leq \max\{J_C(t_1), J_C(t_2)\}$. Now

$$(M_C)^{M\,T}_{\lambda\,p}(t_1 * t_2) = \lambda\,.\,M_C(t_1 * t_2) + p$$

$$\geq \lambda\,.\min\{M_C(t_1), M_C(t_2)\} + p$$

$$= \min\{\lambda\,.\,M_C(t_1) + p, \lambda\,.\,M_C(t_2) + p\}$$

$$(M_C)^{M\,T}_{\lambda\,p}(t_1 * t_2) = \min\{(M_C)^{M\,T}_{\lambda\,p}(t_1), (M_C)^{M\,T}_{\lambda\,p}(t_2)\}$$

$$(M_C)^{M\,T}_{\lambda\,p}(t_1 * t_2) \geq \min\{(M_C)^{M\,T}_{\lambda\,p}(t_1), (M_C)^{M\,T}_{\lambda\,p}(t_2)\},$$

$$(\widehat{B}_C)^{M\,T}_{\lambda\,q}(t_1 * t_2) = \lambda\,.\,\widehat{B}_C(t_1 * t_2) + q$$

$$\geq \lambda\,.\,\mathrm{rmin}\{\widehat{B}_C(t_1), \widehat{B}_C(t_2)\} + q$$

$$= \mathrm{rmin}\{\lambda\,.\,\widehat{B}_C(t_1) + q, \lambda\,.\,\widehat{B}_C(t_2) + q\}$$

$$(\widehat{B}_C)^{M\,T}_{\lambda\,q}(t_1 * t_2) = \mathrm{rmin}\{(\widehat{B}_C)^{M\,T}_{\lambda\,q}(t_1), (\widehat{B}_C)^{M\,T}_{\lambda\,q}(t_2)\}$$

$$(\widehat{B}_C)^{M\,T}_{\lambda\,q}(t_1 * t_2) \geq \mathrm{rmin}\{(\widehat{B}_C)^{M\,T}_{\lambda\,q}(t_1), (\widehat{B}_C)^{M\,T}_{\lambda\,q}(t_2)\},$$

$$(J_C)^{M\,T}_{\lambda\,s}(t_1 * t_2) = \lambda\,.\,J_C(t_1 * t_2) - s$$

$$\leq \lambda\,.\max\{J_C(t_1), J_C(t_2)\} - s$$

$$= \max\{\lambda\,.\,J_C(t_1) - s, \lambda\,.\,J_C(t_2) - s\}$$

$$(J_C)^{M\,T}_{\lambda\,s}(t_1 * t_2) = \max\{(J_C)^{M\,T}_{\lambda\,s}(t_1), (J_C)^{M\,T}_{\lambda\,s}(t_2)\}$$

$$(J_C)^{M\,T}_{\lambda\,s}(t_1 * t_2) \geq \max\{(J_C)^{M\,T}_{\lambda\,s}(t_1), (J_C)^{M\,T}_{\lambda\,s}(t_2)\}.$$

Hence **MBJNMT** $C^{M\,T}_{\lambda\,p,q,s}$ is a **MBJNSU** of Y.

Theorem 3.2 Let C be a MBJ-neutrosophic set of Y such that $p, q, s \in [0, Ʊ]$, $\lambda \in [0,1]$ and a mapping $C^{M\,T}_{\lambda\,p,q,s}|Y \to [0,1]$ be a **MBJNMT** of C. If $C^{M\,T}_{\lambda\,p,q,s}$ is **MBJNSU** of Y. Then C is a **MBJNSU** of Y.

Proof. Let C be a MBJ-neutrosophic subset of Y, where $p, q, s \in [0, Ʊ]$, $\lambda \in [0,1]$ and a mapping $C^{M\,T}_{\lambda\,p,q,s}|Y \to [0,1]$ be a **MBJNMT** of C. Let $C^{M\,T}_{\lambda\,p,q,s}$ is a **MBJNSU** of Y, then

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$\lambda . M_C(t_1 * t_2) + p = (M_C)^{MT}_{\lambda p}(t_1 * t_2)$

$\geq \min\{(M_C)^{MT}_{\lambda p}(t_1), (M_C)^{MT}_{\lambda p}(t_2)\}$

$= \min\{\lambda . M_C(t_1) + p, \lambda . M_C(t_2) + p\}$

$\lambda . M_C(t_1 * t_2) + p = \lambda . \min\{M_C(t_2), M_C(t_1)\} + p,$

$\lambda . \widehat{B}_C(t_1 * t_2) + q = (\widehat{B}_C)^{MT}_{\lambda q}(t_1 * t_2)$

$\geq \mathrm{rmin}\{(\widehat{B}_C)^{MT}_{\lambda q}(t_1), (\widehat{B}_C)^{MT}_{\lambda q}(t_2)\}$

$= \mathrm{rmin}\{\lambda . \widehat{B}_C(t_1) + q, \lambda . \widehat{B}_C(t_2) + q\}$

$\lambda . \widehat{B}_C(t_1 * t_2) + q = \lambda . \mathrm{rmin}\{\widehat{B}_C(t_2), \widehat{B}_C(t_1)\} + q,$

$\lambda . J_C(t_1 * t_2) - s = (J_C)^{MT}_{\lambda s}(t_1 * t_2)$

$\leq \max\{(J_C)^{MT}_{\lambda s}(t_1), (J_C)^{MT}_{\lambda s}(t_2)\}$

$= \max\{\lambda . J_C(t_1) - s, \lambda . J_C(t_2) - s\}$

$\lambda . J_C(t_1 * t_2) - s = \lambda . \max\{J_C(t_2), J_C(t_1)\} - s,$

which imply $M_C(t_1 * t_2) \geq \min\{M_C(t_1), M_C(t_2)\}$,$\widehat{B}_C(t_1 * t_2) \geq \mathrm{rmin}\{\widehat{B}_C(t_1), \widehat{B}_C(t_2)\}$, $J_C(t_1 * t_2) \leq \max\{J_C(t_1)$ $, J_C(t_2)\}$ for all $t_1, t_2 \in Y$. Hence C is a **MBJNSU** of Y.

Theorem 3.3 If C is a **MBJNID** of Y. Then **MBJNMT** $C^{MT}_{\lambda p,q,s}$ of C is a **MBJNID** of Y for all $p, q, s \in [0, Ƕ]$ and $\lambda \in (0,1]$.

Proof. Suppose $C = (M_C, \widehat{B}_C, J_C)$ is a **MBJNID** of Y. Then

$(M_C)^{MT}_{\lambda p}(0) = \lambda . M_C(0) + p \geq \lambda . M_C(t_1) + p$

$(M_C)^{MT}_{\lambda p}(0) = (M_C)^{MT}_{\lambda p}(t_1),$

$(\widehat{B}_C)^{MT}_{\lambda q}(0) = \lambda . \widehat{B}_C(0) + q \geq \lambda . \widehat{B}_C(t_1) + q$

$(\widehat{B}_C)^{MT}_{\lambda q}(0) = (\widehat{B}_C)^{MT}_{\lambda q}(t_1),$

$(J_C)^{MT}_{\lambda s}(0) = \lambda . J_C(0) - s \leq \lambda . J_C(t_1) - s$

$(J_C)^{MT}_{\lambda s}(0) = (J_C)^{MT}_{\lambda s}(t_1)$

Now

$(M_C)^{MT}_{\lambda p}(t_1) = \lambda . M_C(t_1) + p$

$\geq \lambda . \min\{M_C(t_1 * t_2), M_C(t_2)\} + p$

$= \min\{\lambda . M_C(t_1 * t_2) + p, \lambda . M_C(t_2) + p\}$

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$$(M_C)^{M\,T}_{\lambda\,p}(t_1) = \min\{(M_C)^{M\,T}_{\lambda\,p}(t_1 * t_2), (M_C)^{M\,T}_{\lambda\,p}(t_2)\}$$

$$\Rightarrow (M_C)^{M\,T}_{\lambda\,p}(t_1) \geq \min\{(M_C)^{M\,T}_{\lambda\,p}(t_1 * t_2), (M_C)^{M\,T}_{\lambda\,p}(t_2)\},$$

$$(\widehat{B}_C)^{M\,T}_{\lambda\,q}(t_1) = \lambda . \widehat{B}_C(t_1) + q$$

$$\geq \lambda . \mathrm{rmin}\{\widehat{B}_C(t_1 * t_2), \widehat{B}_C(t_2)\} + q$$

$$= \mathrm{rmin}\{\lambda . \widehat{B}_C(t_1 * t_2) + q, \lambda . \widehat{B}_C(t_2) + q\}$$

$$(\widehat{B}_C)^{M\,T}_{\lambda\,q}(t_1) = \mathrm{rmin}\{(\widehat{B}_C)^{M\,T}_{\lambda\,q}(t_1 * t_2), (\widehat{B}_C)^{M\,T}_{\lambda\,q}(t_2)\}$$

$$\Rightarrow (\widehat{B}_C)^{M\,T}_{\lambda\,q}(t_1) \geq \mathrm{rmin}\{(\widehat{B}_C)^{M\,T}_{\lambda\,q}(t_1 * t_2), (\widehat{B}_C)^{M\,T}_{\lambda\,q}(t_2)\},$$

$$(J_C)^{M\,T}_{\lambda\,s}(t_1) = \lambda . J_C(t_1) - s$$

$$\leq \lambda . \max\{J_C(t_1 * t_2), J_C(t_2)\} - s$$

$$= \max\{\lambda . J_C(t_1 * t_2) - s, \lambda . J_C(t_2) - s\}$$

$$(J_C)^{M\,T}_{\lambda\,s}(t_1) = \max\{(J_C)^{M\,T}_{\lambda\,s}(t_1 * t_2), (J_C)^{M\,T}_{\lambda\,s}(t_2)\}$$

$$\Rightarrow (J_C)^{M\,T}_{\lambda\,s}(t_1) \leq \max\{(J_C)^{M\,T}_{\lambda\,s}(t_1 * t_2), (J_C)^{M\,T}_{\lambda\,s}(t_2)\}$$

for all $t_1, t_2 \in Y$ and all $p, q, s \in [0, Ƕ]$, $\lambda \in (0,1]$. Hence $C^{M\,T}_{\lambda\,p,q,s}$ of C is a **MBJNID** of Y.

Theorem 3.4 If C is a MBJ-neutrosophic set of Y such that **MBJNMT** $C^{M\,T}_{\lambda\,p,q,s}$ of C is a **MBJNID** of Y for all $p, q, s \in [0, Ƕ]$ and $\lambda \in (0,1]$, then C is a **MBJNID** of Y.

Proof. Suppose **MBJNMT** $C^{M\,T}_{\lambda\,p,q,s}$ is a **MBJNID** of Y for some $p, q, s \in [0, Ƕ]$, $\lambda \in (0,1]$ and $t_1, t_2 \in Y$. Then

$$\lambda . M_C(0) + p = (M_C)^{M\,T}_{\lambda\,p}(0)$$

$$\geq (M_C)^{M\,T}_{\lambda\,p}(t_1)$$

$$\lambda . M_C(0) + p = \lambda . M_C(t_1) + p,$$

$$\lambda . \widehat{B}_C(0) + q = (\widehat{B}_C)^{M\,T}_{\lambda\,q}(0)$$

$$\geq (\widehat{B}_C)^{M\,T}_{\lambda\,q}(t_1)$$

$$\lambda . \widehat{B}_C(0) + q = \lambda . \widehat{B}_C(t_1) + q,$$

$$\lambda . J_C(0) - s = (J_C)^{M\,T}_{\lambda\,s}(0)$$

$$\leq (J_C)^{M\,T}_{\lambda\,s}(t_1)$$

$$\lambda . J_C(0) - s = \lambda . J_C(t_1) - s,$$

which imply $M_C(0) \geq M_C(t_1)$,$\widehat{B}_C(0) \geq \widehat{B}_C(t_1)$,$J_C(0) \leq J_C(t_1)$. Now, we have

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$$\lambda.\mathrm{M_C}(t_1) + p = (\mathrm{M_C})^{\mathrm{M\,T}}_{\lambda\,p}(t_1)$$

$$\geq \min\{(\mathrm{M_C})^{\mathrm{M\,T}}_{\lambda\,p}(t_1 * t_2), (\mathrm{M_C})^{\mathrm{M\,T}}_{\lambda\,p}(t_2)\}$$

$$= \min\{\lambda.\mathrm{M_C}(t_1 * t_2) + p, \lambda.\mathrm{M_C}(t_2) + p\}$$

$$\lambda.\mathrm{M_C}(t_1) + p = \lambda.\min\{\mathrm{M_C}(t_1 * t_2), \mathrm{M_C}(t_2)\} + p,$$

$$\lambda.\widehat{\mathrm{B}}_{\mathrm{C}}(t_1) + q = (\widehat{\mathrm{B}}_{\mathrm{C}})^{\mathrm{M\,T}}_{\lambda\,q}(t_1)$$

$$\geq \mathrm{rmin}\{(\widehat{\mathrm{B}}_{\mathrm{C}})^{\mathrm{M\,T}}_{\lambda\,q}(t_1 * t_2), (\widehat{\mathrm{B}}_{\mathrm{C}})^{\mathrm{M\,T}}_{\lambda\,q}(t_2)\}$$

$$= \mathrm{rmin}\{\lambda.\widehat{\mathrm{B}}_{\mathrm{C}}(t_1 * t_2) + q, \lambda.\widehat{\mathrm{B}}_{\mathrm{C}}(t_2) + q\}$$

$$\lambda.\widehat{\mathrm{B}}_{\mathrm{C}}(t_1) + q = \lambda.\mathrm{rmin}\{\widehat{\mathrm{B}}_{\mathrm{C}}(t_1 * t_2), \widehat{\mathrm{B}}_{\mathrm{C}}(t_2)\} + q,$$

$$\lambda.\mathrm{J_C}(t_1) - s = (\mathrm{J_C})^{\mathrm{M\,T}}_{\lambda\,s}(t_1)$$

$$\leq \max\{(\mathrm{J_C})^{\mathrm{M\,T}}_{\lambda\,s}(t_1 * t_2), (\mathrm{J_C})^{\mathrm{M\,T}}_{\lambda\,s}(t_2)\}$$

$$= \max\{\lambda.\mathrm{J_C}(t_1 * t_2) - s, \lambda.\mathrm{J_C}(t_2) - s\}$$

$$\lambda.\mathrm{J_C}(t_1) - s = \lambda.\max\{\mathrm{J_C}(t_1 * t_2), \mathrm{J_C}(t_2)\} - s$$

which imply $\mathrm{M_C}(t_1) \geq \min\{\mathrm{M_C}(t_1 * t_2), \mathrm{M_C}(t_2)\}$, $\widehat{\mathrm{B}}_{\mathrm{C}}(t_1) \geq \mathrm{rmin}\{\widehat{\mathrm{B}}_{\mathrm{C}}(t_1 * t_2), \widehat{\mathrm{B}}_{\mathrm{C}}(t_2)\}$, $\mathrm{J_C}(t_1) \leq \max\{\mathrm{J_C}(t_1 * t_2), \mathrm{J_C}(t_2)\}$ for all $t_1, t_2 \in$ Y. Hence C is a **MBJNID** of Y.

Theorem 3.5 Intersection of any two **MBJNMT** $\mathrm{C}^{\mathrm{M\,T}}_{\lambda\,p,q,s}$ of a **MBJNID** C of Y is a **MBJNID** of Y.

Proof. Suppose $\mathrm{C}^{\mathrm{M\,T}}_{\lambda\,p,q,s}$ and $\mathrm{C}^{\mathrm{M\,T}}_{\lambda'\,p',q',s'}$ are two **MBJNMTs** of **MBJNID** C of Y, where $p, q, s, p', q', s' \in [0, \mathrm{Ƀ}]$ and $\lambda, \lambda' \in (0,1]$. Assume $p \leq p', q \leq q', s \leq s'$ and $\lambda = \lambda'$. Since $\mathrm{C}^{\mathrm{M\,T}}_{\lambda\,p,q,s}$ and $\mathrm{C}^{\mathrm{M\,T}}_{\lambda'\,p',q',s'}$ are **MBJNIDs** of Y. So

$$((\mathrm{M_C})^{\mathrm{M\,T}}_{\lambda\,p} \cap (\mathrm{M_C})^{\mathrm{M\,T}}_{\lambda'\,p'})(t_1) = \min\{(\mathrm{M_C})^{\mathrm{M\,T}}_{\lambda\,p}(t_1), (\mathrm{M_C})^{\mathrm{M\,T}}_{\lambda'\,p'}(t_1)\}$$

$$= \min\{\lambda.\mathrm{M_C}(t_1) + p, \lambda'.\mathrm{M_C}(t_1) + p'\}$$

$$= \lambda.\mathrm{M_C}(t_1) + p$$

$$((\mathrm{M_C})^{\mathrm{M\,T}}_{\lambda\,p} \cap (\mathrm{M_C})^{\mathrm{M\,T}}_{\lambda'\,p'})(t_1) = (\mathrm{M_C})^{\mathrm{M\,T}}_{\lambda\,p}(t_1),$$

$$((\widehat{\mathrm{B}}_{\mathrm{C}})^{\mathrm{M\,T}}_{\lambda\,q} \cap (\widehat{\mathrm{B}}_{\mathrm{C}})^{\mathrm{M\,T}}_{\lambda'\,q'})(t_1) = \mathrm{rmin}\{(\widehat{\mathrm{B}}_{\mathrm{C}})^{\mathrm{M\,T}}_{\lambda\,q}(t_1), (\widehat{\mathrm{B}}_{\mathrm{C}})^{\mathrm{M\,T}}_{\lambda'\,q'}(t_1)\}$$

$$= \mathrm{rmin}\{\lambda.\widehat{\mathrm{B}}_{\mathrm{C}}(t_1) + q, \lambda'.\widehat{\mathrm{B}}_{\mathrm{C}}(t_1) + q'\}$$

$$= \lambda.\widehat{\mathrm{B}}_{\mathrm{C}}(t_1) + q$$

$$((\widehat{\mathrm{B}}_{\mathrm{C}})^{\mathrm{M\,T}}_{\lambda\,q} \cap (\widehat{\mathrm{B}}_{\mathrm{C}})^{\mathrm{M\,T}}_{\lambda'\,q'})(t_1) = (\widehat{\mathrm{B}}_{\mathrm{C}})^{\mathrm{M\,T}}_{\lambda\,q}(t_1),$$

$$((\mathrm{J_C})^{\mathrm{M\,T}}_{\lambda\,s} \cap (\mathrm{J_C})^{\mathrm{M\,T}}_{\lambda'\,s'})(t_1) = \max\{(\mathrm{J_C})^{\mathrm{M\,T}}_{\lambda\,s}(t_1), (\mathrm{J_C})^{\mathrm{M\,T}}_{\lambda'\,s'}(t_1)\}$$

$$= \max\{\lambda.\mathrm{J_C}(t_1) - s, \lambda'.\mathrm{J_C}(t_1) - s'\}$$

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$= \lambda\,.\,J_C(t_1) - s$

$((J_C)^{M\,T}_{\lambda\,s} \cap (J_C)^{M\,T}_{\lambda'\,s'})(t_1) = (J_C)^{M\,T}_{\lambda\,s}(t_1).$

Hence $C^{M\,T}_{\lambda\,p,q,s} \cap C^{M\,T}_{\lambda'\,p',q',s'}$ is **MBJNID** of Y.

Theorem 3.6 Union of any two **MBJNMT** $C^{M\,T}_{\lambda\,p,q,s}$ of a **MBJNID** C of Y is a **MBJNID** of Y.

Proof. Suppose $C^{M\,T}_{\lambda\,p,q,s}$ and $C^{M\,T}_{\lambda'\,p',q',s'}$ are two **MBJNMTs** of **MBJNID** C of Y, where $p, q, s, p', q', s' \in [0, Ъ]$ and $\lambda, \lambda' \in (0,1]$. Assume $p \leq p', q \leq q', s \leq s'$ and $\lambda = \lambda'$. Since $C^{M\,T}_{\lambda\,p,q,s}$ and $C^{M\,T}_{\lambda'\,p',q',s'}$ are **MBJNIDs** of Y. Then

$((M_C)^{M\,T}_{\lambda\,p} \cup (M_C)^{M\,T}_{\lambda'\,p'})(t_1) = \max\{(M_C)^{M\,T}_{\lambda\,p}(t_1), (M_C)^{M\,T}_{\lambda'\,p'}(t_1)\}$

$= \max\{\lambda\,.\,M_C(t_1) + p, \lambda'.\,M_C(t_1) + p'\}$

$= \lambda\,.\,M_C(t_1) + p$

$((M_C)^{M\,T}_{\lambda\,p} \cup (M_C)^{M\,T}_{\lambda'\,p'})(t_1) = (M_C)^{M\,T}_{\lambda\,p}(t_1),$

$((\widehat{B}_C)^{M\,T}_{\lambda\,q} \cup (\widehat{B}_C)^{M\,T}_{\lambda'\,q'})(t_1) = \mathrm{rmax}\{(\widehat{B}_C)^{M\,T}_{\lambda\,q}(t_1), (\widehat{B}_C)^{M\,T}_{\lambda'\,q'}(t_1)\}$

$= \mathrm{rmax}\{\lambda\,.\,\widehat{B}_C(t_1) + q, \lambda'.\,\widehat{B}_C(t_1) + q'\}$

$= \lambda\,.\,\widehat{B}_C(t_1) + q$

$((\widehat{B}_C)^{M\,T}_{\lambda\,q} \cup (\widehat{B}_C)^{M\,T}_{\lambda'\,q'})(t_1) = (\widehat{B}_C)^{M\,T}_{\lambda\,q}(t_1),$

$((J_C)^{M\,T}_{\lambda\,s} \cup (J_C)^{M\,T}_{\lambda'\,s'})(t_1) = \min\{(J_C)^{M\,T}_{\lambda\,s}(t_1), (J_C)^{M\,T}_{\lambda'\,s'}(t_1)\}$

$= \min\{\lambda\,.\,J_C(t_1) - s, \lambda'.\,J_C(t_1) - s'\}$

$= \lambda\,.\,J_C(t_1) - s$

$((J_C)^{M\,T}_{\lambda\,s} \cup (J_C)^{M\,T}_{\lambda'\,s'})(t_1) = (J_C)^{M\,T}_{\lambda\,s}(t_1)$

4. Conclusion

In this article, we defined **MBJNMT** of MBJ-neutrosophic set on G-algebra. Moreover, deep study of **MBJNMT** will lead us to study the neutrosophic theory on some other framework. For future work, magnification can be applied on MBJ-neutrosophic soft set and T-MBJ-neutrosophic set.

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Some Results on Single Valued Neutrosophic (Weak) Polygroups

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Abstract

Polygroups are a generalized concept of groups and the concept of single valued neutrosophic set is a generalization of the classical notion of a set. The objective of this paper is to combine the innovative concept of single valued neutrosophic sets and polygroups. In this regard, we introduce the concepts of single valued neutrosophic polygroups and anti- single valued neutrosophic polygroups. Moreover, we investigate their properties and study the relation between level sets of single valued neutrosophic polygroups and (normal) subpolygroups.

Keywords: Polygroup, Weak polygroup, Single valued neutrosophic set, Single valued neutrosophic polygroup, Anti- single valued neutrosophic polygroup

1. Introduction

Florentin Smarandache introduced Neutrosophic sets in 1998 [9], which is the generalization of the fuzzy sets introduced by Lotfi Zadeh in 1965 [13]. Where the latter proposed fuzzy sets as mathematical model of vagueness where elements belong to a given set to some degree that is typically a number that belongs to the unit interval [0,1]. Neutrosophy is a base of Neutrosophic logic which is an extension of fuzzy logic in which indeterminancy is included. In Neutrosophic logic, each proposition is estimated to have the percentage of truth in a subset T, percentage of indeterminancy in a subset I, and the percentage of falsity in a subset F. A single valued neutrosophic set is an instance of neutrosophic set which can be used in real scientfic and engineering problems. Therefore, the study of single valued neutrosophic sets and their properties have a considerable significance in the sense of applications as well as in understanding the fundamentals of uncertainty.

Algebraic hyperstructures represent a natural generalization of classical algebraic structures and they were introduced by Frederic Marty [8] in 1934 at the eighth Congress of Scandinavian Mathematicians. Where he generalized the notion of a group to that of a hypergroup. In a group, the composition of two elements is an element whereas in a hypergroup, the composition of two elements is a set. Hypergroups have been used in algebra, geometry, convexity, automata theory, combinatorial problems of coloring, lattice theory, Boolean algebras, logic etc., over the years. Comer [5] introduced a special class of hypergroups, using the name of polygroups. He emphasized the importance of polygroups, by analyzing them in connections to graphs, relations, Boolean and cylindric algebras.

The combination between both: fuzzy sets and algebraic hyperstructures, and between neutrosophic sets and algebraic hyperstructures have attracted the attention of many researchers working in these domains and as a result, new branches of research were invented, namely, fuzzy algebraic hyperstructures and neutrosophic algebraic hyperstructures. For more details, we refer to the work done in [1-4, 11, 12].

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As a generalization of (anti-) fuzzy polygroups, this paper combines the notion of single valued neutrosophic set with polygroups to get single valued neutrosophic polygroups and it is constructed as follows: After an Introduction, in Section 2 and Section 3, we present some basic results about single valued neutrosophic sets and about (weak) polygroups. In Section 4, we introduce the notion of single valued nuetrosophic (weak) polygroup and investigate its properties. And finally in Section 5, we study the relation between level sets of single valued neutrosophic polygroups and (normal) subpolygroups.

2. Single valued neutrosophic sets

In this section, we present some basic results about single valued neutrosophic sets.

Single valued neutrosophic sets are generalization of classical sets, fuzzy sets, intuitionistic fuzzy sets and paraconsistent sets, etc.

Definition 2.1. [12] Let X be a space of points (objects), with a generic element in X denoted by x. A single valued neutrosophic set (SVNS) A in X is characterized by truth-membership T_A, indeterminacy-membership function I_A and falsity-membership function F_A. For each point x in X, $T_A(x)$, $I_A(x)$, $F_A(x) \in [0,1]$.

Example 2.1. Assume that $X = \{x_1, x_2, x_3\}$, x_1 is importance, x_2 is trustworthiness and x_3 is availability. The values of x_1, x_2 and x_3 are in [0,1]. They are obtained from the questionnaire about flu shot, their option could be a degree of "good effect", a degree of indeterminacy and a degree of "poor effect". A is a single valued neutrosophic set of X defined by $A = \langle 0.4,0.3,0.5\rangle/x_1 + \langle 0.1,0.6,0.3\rangle/x_2 + \langle 0.6,0.2,0.3\rangle/x_3$. And B is a single valued neutrosophic set of X defined by $B = \langle 0.5,0.2,0.4\rangle/x_1 + \langle 0.3,0.6,0.7\rangle/x_2 + \langle 0.5,0.4,0.5\rangle/x_3$.

Definition 2.2. [12] The complement of a single valued neutrosophic set A is denoted by $c(A)$ and is defined by $T_{c(A)}(x) = F_A(x)$, $I_{c(A)}(x) = 1 - I_A(x)$, $F_{c(A)}(x) = T_A(x)$, for all x in X.

Example 2.2. Let A be the SVNS present in Example 2.1. Then

$$c(A) = \langle 0.5,0.7,0.4\rangle/x_1 + \langle 0.3,0.4,0.1\rangle/x_2 + \langle 0.3,0.8,0.6\rangle/x_3.$$

Definition 2.3. [12] Let A and B be single valued neutrosophic sets. Then

- A is contained in B, denoted as $A \subseteq B$, if and only of $T_A(x) \le T_B(x)$, $I_A(x) \le I_B(x)$, $F_A(x) \ge F_B(x)$ for all x in X.
- A and B are equal, written as $A = B$, if and only if A ⊆ B and B ⊆ A.
- The union of A and B is a single valued neutrosophic set C, written as $C = A \cup B$, whose truth-membership, indeterminacy-membership and falsity-membership unctions are related to those of A and B by $T_C(x) = \max(T_A(x), T_B(x))$, $I_C(x) = \max(I_A(x), I_B(x))$, and $F_C(x) = \min(F_A(x), F_B(x))$ for all x in X.
- The intersection of A and B is a single valued neutrosophic set C, written as $C = A \cap B$, whose truth-membership, indeterminacy-membership and falsity-membership unctions are related to those of A and B by $T_C(x) = \min(T_A(x), T_B(x))$, $I_C(x) = \min(I_A(x), I_B(x))$, and $F_C(x) = \max(F_A(x), F_B(x))$ for all x in X.

Example 2.3. Let A and B be the SVNS present in Example 2.1. Then

$$A \cap B = \langle 0.4,0.2,0.5\rangle/x_1 + \langle 0.1,0.6,0.7\rangle/x_2 + \langle 0.5,0.2,0.5\rangle/x_3 \text{ and}$$
$$A \cup B = \langle 0.5,0.3,0.4\rangle/x_1 + \langle 0.3,0.6,0.3\rangle/x_2 + \langle 0.6,0.4,0.3\rangle/x_3.$$

3. (Weak) Polygroups

In this section, we present some definitions and examples related to (weak) polygroups that are used throughout the paper. For more details, we refer to [6, 7].

Let H be a non-empty set and $P^*(H)$ be the collection of all non-empty subsets of H. Ad define " $*$ " as follows:

$$\ast : H \times H \to P^*(H)$$
$$(x, y) \to x \ast y$$

Then " $\ast$ " is called a hyperoperation and $(H,\ast)$ is called a hypergroupoid.

Definition 3.1.[5] Let $(P,\ast)$ be a hypergroupoid. Then $(P,\ast)$ is a polygroup if the following are satisfied for all a, b, c in P.

(1) $a \ast (b \ast c) = (a \ast b) \ast c$,
(2) There exists e in P with $e \ast a = a \ast e = a$ for all a in P,
(3) $x \in y \ast z$ implies $y \in x \ast z^{-1}$ and $z \in y^{-1} \ast x$.

Weak polygroups are generalization of polygroups and they are defined in the same way as polygroups but instead of (1) in Definition 3.1, we have $a \ast (b \ast c) \cap (a \ast b) \ast c \neq \emptyset$.

In a (weak) polygroup P, $(x^{-1})^{-1} = x$ for all $x \in P$.

Remark 3.1. Every group is a (weak) polygroup.

We present examples on polygroups that are not groups.

Example 3.1. Let $P_1 = \{0, 1\}$. Then $(P_1,\ast)$ defined in Table 1 is a polygroup with 0 serving as an identity.

Table 1. The polygroup $(P_1,\ast)$

$\ast$	0	1
0	0	1
1	1	P_1

Example 3.2. [7] Let $P_2 = \{e, a, b, c\}$. Then $(P_2, .)$ defined in Table 2 is a polygroup with e serving as an identity.

Table 2. The polygroup $(P_2, .)$

.	e	a	b	c
e	e	a	b	c
a	a	e	b	c
b	b	b	{e, a, c}	{b, c}
c	c	c	{b, c}	{e, a, b}

Example 3.3. [7] Let $P_3 = \{e, a, b, c\}$. Then $(P_3,\circ)$ defined in Table 3 is a weak polygroup with e serving as an identity. Moreover, it is not a polygroup.

Table 3. The weak polygroup $(P_3,\circ)$

$\circ$	e	a	b	c
e	e	a	b	c
a	a	{e,a}	c	b
b	b	c	{e, b}	a
c	c	b	a	{e, c}

Definition 3.2. [7] Let $(P,*)$ be a polygroup. A subset S of P is subpolygroup of P if and only if $(S,*)$ is a polygroup.

Proposition 3.1. [7] Let $(P,*)$ be a polygroup. A subset S of P is subpolygroup of P if and only $x * y \subseteq S$ and $x^{-1} \in S$ for all $x, y \in S$.

Definition 3.3. [7] Let $(P,*)$ be a polygroup. A subset subpolygroup S of P is a normal subpolygroup of P if $x^{-1} * P * x \subseteq$ P for all $x \in P$.

Example 3.4. Let $(P_2, .)$ be the polygroup in Example 3.2. Then $\{e\}$ and $\{e, a\}$ are subpolygroups of P_2 that are not normal.

4. Single valued neutrosophic (weak) polygroups, A construction

In this section, we define single valued neutrosophic polygroups and investigate its properties.

Definition 4.1. [6] Let $(P,*)$ be a polygroup and A a fuzzy set over P with a fuzzy membership function μ. Then A is called a fuzzy polygroup over P if for all $x, y \in P$, the following conditions are satisfied.

(1) $\mu(z) \geq \min\{\mu(x), \mu(y)\}$ for all $z \in x * y$,
(2) $\mu(x^{-1}) \geq \mu(x)$.

Remark 4.1. [6] Intersection of fuzzy polygroups over P is a fuzzy polygroup.

Definition 4.2. Let $(P,*)$ be a (weak) polygroup and A a SVNS over P. Then A is called a single valued neutrosophic polygroup (SVNP) over P (single valued neutrosophic weak polygroup (SVNWP) over P) if for all $x, y \in P$, the following conditions are satisfied.

(1) $T_A(z) \geq \min\{T_A(x), T_A(y)\}$, $I_A(z) \geq \min\{I_A(x), I_A(y)\}$, and $F_A(z) \leq max\{F_A(x), F_A(y)\}$ for all $z \in x * y$,
(2) $T_A(x^{-1}) \geq T_A(x)$, $I_A(x^{-1}) \geq I_A(x)$, and $F_A(x^{-1}) \leq F_A(x)$.

Example 4.1. Let $(P_1,*)$ be the polygroup present in Example 3.1 and $A = \langle 0.4,0.5,0.5\rangle/0 + \langle 0.1,0.3,0.7\rangle/1$ is a SVNP over P_1.

Example 4.2. Let $P_3 = \{e, a, b, c\}$ and $(P_3,\circ)$ be the weak polygroup defined in Example 3.3. Then
$A = \langle 0.4,0.5,0.5\rangle/e + \langle 0.1,0.3,0.7\rangle/a + \langle 0.1,0.25,0.9\rangle/b + \langle 0.1,0.25,0.9\rangle/c$ is a SVNWP over P_3.

Remark 4.2. All the theorems and results in this paper that are valid for SVNP are also valid for SVNWP. So, we restrict our results to SVNP.

Proposition 4.1. Let $(P,*)$ be a polygroup and A a SVNP over P. Then the following hold for all $x \in P$.

1. $T_A(x^{-1}) = T_A(x), I_A(x^{-1}) = I_A(x),$ and $F_A(x^{-1}) = F_A(x)$;
2. $T_A(e) \geq T_A(x), I_A(e) \geq I_A(x),$ and $F_A(e) \leq F_A(x)$ where e is the identity in P.

Proof. Let $x \in P$.
Proof of 1.: Definition 4.2 implies that $T_A(x^{-1}) \geq T_A(x)$, $I_A(x^{-1}) \geq I_A(x)$, and $F_A(x^{-1}) \leq F_A(x)$. And having $(x^{-1})^{-1} = x$ implies that $T_A(x) \geq T_A(x^{-1})$, $I_A(x) \geq I_A(x^{-1})$, and $F_A(x) \leq F_A(x^{-1})$. Thus, $T_A(x^{-1}) = T_A(x)$, $I_A(x^{-1}) = I_A(x),$ and $F_A(x^{-1}) = F_A(x)$.
Proof of 2.: Since $e \in x * x^{-1}$, it follows by Definition 4.2 (1) that $T_A(e) \geq \min\,(T_A(x), T_A(x^{-1})) = T_A(x)$, $I_A(e) \geq \min\big(I_A(x), I_A(x^{-1})\big) = I_A(x),$ and $F_A(e) \leq \max\big(F_A(x), F_A(x^{-1})\big) = F_A(x)$.

Example 4.3. Let $(P_2, .)$ be the polygroup present in Example 3.2. Then $A = \langle 0.4,0.5,0.5\rangle/e + \langle 0.5,0.3,0.7\rangle/a + \langle 0.5,0.3,0.7\rangle/b$ is not a SVNP over P_2 as $T_A(e) \geq T_A(a)$ does not hold.

Proposition 4.2. Let $(P,*)$ be a polygroup, A a SVNS over P, and $A^{-1} = \{\langle T_A(x^{-1}), I_A(x^{-1}), F_A(x^{-1})\rangle/x : x \in P\}$. If A is a SVNP over P then $A^{-1} = A$.
Proof. The proof follows from Proposition 4.1.

Proposition 4.3. Let $(P,*)$ be a polygroup and t_1, t_2, t_3 be numbers in the unit interval [0, 1]. If $A = \{\langle t_1, t_2, t_3 \rangle / x : x \in P\}$. Then A is a SVNP over P.

Proof. The proof is straightforward.

Remark 4.3. The SVNP present in Proposition 4.2 is called the **constant SNVP**.

Theorem 4.1. Let $(P,*)$ be a polygroup and A a SVNS over P. Then A and $c(A)$ are SVNP over P if and only if A is the constant SVNP.

Proof. If A is the constant SVNP over P then $c(A)$ is also the constant SVNP over P.
Let A and $c(A)$ be SVNP. Then for all x in P, we have:

$$T_A(e) \geq T_A(x), I_A(e) \geq I_A(x), \text{ and } F_A(e) \leq F_A(x), \quad \text{(I)}$$
$$F_A(e) \geq F(x), 1 - I_A(e) \geq 1 - I_A(x), \text{ and } T_A(e) \leq T_A(x) \quad \text{(II)}$$

(I) and (II) implies that $T_A(e) = T_A(x), I_A(e) = I_A(x),$ and $F_A(e) = F_A(x)$. Thus, A is the constant SVNP over P.

Definition 4.3. [6] Let $(P,*)$ be a polygroup and A a fuzzy set over P with membership function μ. Then A is called anti-fuzzy polygroup over P if for all $x, y \in P$, the following conditions are satisfied.

(1) $\mu(z) \leq \max\{\mu(x), \mu(y)\}$ for all $z \in x * y$,
(2) $\mu(x^{-1}) \leq \mu(x)$.

Remark 4.4. [6] Union of anti-fuzzy polygroups over P is an anti-fuzzy polygroup.

Definition 4.4. Let $(P,*)$ be a polygroup and A a SVNS over P. Then A is called an anti-single valued neutrosophic polygroup (ASVNP) over P if for all $x, y \in P$, the following conditions are satisfied.

(1) $T_A(z) \leq \max\{T_A(x), T_A(y)\}$, $I_A(z) \leq \max\{I_A(x), I_A(y)\}$, and $F_A(z) \geq min\{F_A(x), F_A(y)\}$ for all $z \in x * y$,
(2) $T_A(x^{-1}) \leq T_A(x)$, $I_A(x^{-1}) \leq I_A(x)$, and $F_A(x^{-1}) \geq F_A(x)$.

Proposition 4.4. Let $(P,*)$ be a polygroup and A an ASVNP over P. Then the following hold for all $x \in P$.

(1) $T_A(x^{-1}) = T_A(x)$, $I_A(x^{-1}) = I_A(x)$, and $F_A(x^{-1}) = F_A(x)$;
(2) $T_A(e) \leq T_A(x)$, $I_A(e) \leq I_A(x)$, and $F_A(e) \geq F_A(x)$ where e is the identity in P.

Proof. The proof is similar to that of Proposition 4.1.

Example 4.3. Let $(P_1,*)$ be the polygroup present in Example 3.1 and $A = \langle 0.4,0.5,0.9\rangle/0 + \langle 0.5,0.5,0.7\rangle/1$ is an ASVNP over P_1.

Theorem 4.2. Let $(P,*)$ be a polygroup and A be a SVNS over P. Then A is a SVNP over P if and only if T_A and I_A are fuzzy polygroups over P and F_A is an anti-fuzzy polygroup over P.

Proof. The proof follows from the definition of SVNP, fuzzy polygroups, and anti-fuzzy polygroups.

Theorem 4.3. Let $(P,*)$ be a polygroup and A be a SVNS over P. Then A is an ASVNP over P if and only if T_A and I_A are anti-fuzzy polygroups over P and F_A is a fuzzy polygroup over P.

Proof. The proof follows from the definition of ASVNP, fuzzy polygroups, and anti-fuzzy polygroups.

Theorem 4.4. Let $(P,*)$ be a polygroup and A be a SVNS over P. Then A is a SVNP over P if and only if $c(A)$ is an ASVNP over P.

Proof. Let A be a SVNP. Theorem 4.2 asserts that T_A and I_A are fuzzy polygroups over P and F_A is an anti-fuzzy polygroup over P. We get now that $T_{c(A)} = F_A$ and $I_{c(A)} = 1 - I_A$ are anti-fuzzy polygroups over P and $F_{c(A)} = T_A$ is a fuzzy polygroup over P. Theorem 4.3. completes the proof. Similarly, we can prove that if $c(A)$ is an ASVNP over P then A is a SVNP.

Corollary 4.1. Let $(P,*)$ be a polygroup and A_α be a SVNS over P. If A_α is a SVNP over P then $\bigcap_{\alpha\in\Gamma} A_\alpha$ is SVNP over P.

Corollary 4.2. Let $(P,*)$ be a polygroup and A_α is a SVNS over P. If A_α is an ASVNP over P then $\bigcap_{\alpha\in\Gamma} A_\alpha$ is an ASVNP over P.

5. Level sets of single valued neutrosophic (weak) polygroups

In this section, we define level sets of single valued neutrosophic polygroups and relate them to (normal) subpolygroups.

Definition 5.1. Let X be any set, $t = (t_1, t_2, t_3)$ where $0 \leq t_1, t_2 < 1$ and $0 < t_3 \leq 1$, and A be a SVNS over X. Then $A_t = \{x \in X: T_A(x) \geq t_1, I_A(x) \geq t_2, F_A(x) \leq t_3\}$ is called a t-level set of A.

Theorem 5.1. Let $(P,*)$ be a polygroup and A be a SVNS over P. Then A is a SVNP over P if and only if $A_t \neq \emptyset$ is a subpolygroup of P for every $t = (t_1, t_2, t_3)$ where $0 \leq t_1, t_2 < 1$ and $0 < t_3 \leq 1$.

Proof. Let A be a SVNP over P and $x, y \in A_t \neq \emptyset$. For all $z \in x * y$, we have $T_A(z) \geq \min\{T_A(x), T_A(y)\} \geq t_1$, $I_A(z) \geq \min\{I_A(x), I_A(y)\} \geq t_2$, and $F_A(z) \leq max\{F_A(x), F_A(y)\} \leq t_3$. Thus, $x * y \subseteq A_t$. Moreover, having $T_A(x^{-1}) \geq T_A(x) \geq t_1$, $I(x^{-1}) \geq I_A(x) \geq t_2$, and $F_A(x^{-1}) \leq F_A(x) \leq t_3$ implies that $x^{-1} \in A_t$. Thus, A_t is a subpolygroup of P.

Conversely, let $A_t \neq \emptyset$ be a subpolygroup of P and $x, y \in P$. Set $t_1 = \min\{T_A(x), T_A(y)\}$, $t_2 = \min\{I_A(x), I_A(y)\}$, $t_3 = max\{F_A(x), F_A(y)\}$, and $t = (t_1, t_2, t_3)$ Since A_t is a subpolygroup of P, it follows that $x * y \subseteq A_t$ and $x^{-1} \in A_t$. The latter implies that for all $z \in x * y$, $T_A(z) \geq t_1 = \min\{T_A(x), T_A(y)\}$, $I_A(z) \geq t_2 = \min\{I_A(x), I_A(y)$, and $F_A(z) \leq t_3 = max\{F_A(x), F_A(y)\}$. Thus, Condition (1) of Definition 4.1. is satisfied. Moreover, we have $T_A(x^{-1}) \geq t_1 = T_A(x)$, $I(x^{-1}) \geq t_2 = I_A(x)$, and $F_A(x^{-1}) \leq t_3 = F_A(x)$. Thus, Condition (2) of Definition 4.1. is satisfied. Therefore, A is a SVNP over P.

Corollary 5.1. Let $(P,*)$ be a polygroup and A be a SVNP over P. Then P has no non-trivial proper subpolygroups if and only if the constant SNVP and $A = \{\langle t_1, t_2, t_3\rangle/x + \langle t_1', t_2', t_3'\rangle/e : x \neq e \in P\}$ where $t_1 \leq t_1'$, $t_2 \leq t_2'$, and $t_3 \geq t_3'$ are the only SVNP over P.

Example 5.1. Let $P_1 = \{0, 1\}$ and $(P_1,*)$be the polygroup defined in Example 3.1. Then the constant SNVP and $A = \langle t_1, t_2, t_3\rangle/1 + \langle t_1', t_2', t_3'\rangle/0$ where $t_1 \leq t_1'$, $t_2 \leq t_2'$, and $t_3 \geq t_3'$ are the only SVNP over P_1.

Notation 5.1. Let $t = (t_1, t_2, t_3)$ and A a SVNS of P. Then by $A(x) = t$, we mean that $T_A(x) = t_1$, $I_A(x) = t_2$, and $F_A(x) = t_3$. And by $A(x) \leq t$, we mean that $T_A(x) \leq t_1$, $I_A(x) \leq t_2$, and $F_A(x) \geq t_3$.

Theorem 5.2. Let $(P,*)$ be a polygroup. Then every subpolygroup of P is a level set of a SVNP over P.

Proof. Let S be a subpolygroup of P and $t = (t_1, t_2, t_3)$ where $0 < t_1, t_2 < 1$ and $0 < t_3 < 1$. Define the SVNS over P as follows:

$$A(x) = \begin{cases} (t_1, t_2, t_3) & if\ x \in S, \\ (0,0,1) & otherwise. \end{cases}$$

Let $t' = (t_1', t_2', t_3')$. Then $A_{t'} = \begin{cases} S & if\ t_1 \geq t_1', t_2 \geq t_2', and\ t_3 \leq t_3' \\ P & if\ t_1' = 0, t_2' = 0, and\ t_3' = 1, \\ \emptyset & otherwise. \end{cases}$ is either $\emptyset$ or a subpolygroup of P.

Using Theorem 5.1, we get that A is a SVNP over P.

Definition 5.2. Let $(P,*)$ be a polygroup and A be a SVNP over P. Then A is said to be a normal SVNP over P if $A(z) = A(z')$ for all $z \in x \star y, z' \in y \star x$.

Example 5.2. Let $(P,*)$ be a polygroup and A be a SVNP over P. Then the constant SNVP is a normal SNVP over P.

Theorem 5.3. Let $(P,*)$ be a polygroup and A is a SVNS over P. Then A is a normal SVNP over P if and only if $A_t \neq \emptyset$ is a normal subpolygroup of P for every $t = (t_1, t_2, t_3)$ where $0 \leq t_1, t_2 < 1$ and $0 < t_3 \leq 1$.

Proof. Let A be a normal SVNP over P and $x, y \in A_t \neq \emptyset$. Theorem 5.1 asserts that $A_t \neq \emptyset$ is a subpolygroup of P. Let $x \in P$. We need to show that $x^{-1} * A_t * x \subseteq A_t$. Let $z \in x^{-1} * A_t * x$. Then there exist y in A_t such that $z \in x^{-1} * y * x$ and hence $z \in x^{-1} * p$ where $p \in y * x$. The latter implies that $y \in p * x^{-1}$. And since A is a normal SVNP over P, it follows that $A(z) = A(y)$. Thus, $z \in A_t$.

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Conversely, let $A_t \neq \emptyset$ be a normal subpolygroup of P. Theorem 5.1 asserts that A is a SVNP over P. To show that A is a normal SVNP over P, it suffices to show that $A(z) = A(z')$ for all $z \in x \star y, z' \in y \star x$. Let $z \in x \star y, z' \in y \star x$ with $A(z') = t$. Having $z' \in y \star x$ implies that $y \in z' \star x^{-1}$. The latter implies that Let $z \in x \star z' * x^{-1}$. Since $z' \in A_t$ and $A_t \neq \emptyset$ is a normal subpolygroup of P, it follows that $z \in A_t$ and hence, $A(z) \geq A(z') = t$. Similarly, we get that $A(z') \geq A(z)$.

Corollary 5.2. Let $(P,*)$ be a polygroup and A be a SVNP over P. Then P has no proper normal subpolygroups if and only if the constant SNVP is the only normal SNVP over P.

Example 5.3. Let $P_2 = \{e, a, b, c\}$ and $(P_2, .)$ be the polygroup defined in Example 3.2. Then the constant SNVP is the only normal SVNP over P_2.

Theorem 5.4. Let $(P,*)$ be a polygroup. Then every normal subpolygroup of P is a level set of a normal SVNP over P.

Proof. The proof is the same as that of Theorem 5.2.

Corollary 5.5. Let $(P,*)$ be a polygroup and A be a SVNP over P. Then $A^* = \{x \in P: A(x) = A(e)\}$ is a subpolygroup of P. Moreover, if A is a normal SVNP over P then A^* is a normal subpolygroup of P.

Proof. Let $t = A(e)$. Then $A_t = \{x \in P: T_A(x) \geq T_A(e), I_A(x) \geq I_A(e), F_A(x) \leq F_A(e)\}$. Proposition 4.1, 2. asserts that $A_t = \{x \in P: T_A(x) = T_A(e), I_A(x) = I_A(e), F_A(x) = F_A(e)\} = A^*$. Theorem 5.1 and Theorem 5.3 complete the proof.

6. Conclusion

This paper has introduced an algebraic hyperstructure of single valued neutrosophic sets in the form of single valued neutrosophic polygroups and anti- single valued neutrosophic polygroups. Several interesting properties of the new defined notions were discussed. The results of this paper can be considered as a generalization for the work related to fuzzy polygroups.

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Introduction to Neutrosophic Subtraction Algebra and Neutrosophic Subtraction Semigroup

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Abstract

The objective of this paper is to introduce and study the notion of neutrosophic subtraction algebras and neutrosophic subtraction semigroups. It also introduces the notion of neutrosophic ideals of neutrosophic subtraction semigroup and presents some of their basic properties. In addition, the notion of neutrosophic homomorphism of neutrosophic subtraction semigroups and neutrosophic quotient subtraction semigroups was also introduced.

Keywords: Neutrosophy, neutrosophic subtraction algebra,neutrosophic subtraction semigroup, neutrosophic Ideal.

1 Introduction

B. M. Schein studied in[14] systems of the form $(\Phi, \circ, \backslash)$, where Φ is a set of functions closed under the composition " $\circ$ " of functions (and hence $(\Phi, \circ)$ is a function semi group), and closed under set theoretic subtraction "$\backslash$" (and hence $(\Phi, \backslash)$ is a subtraction algebra. He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [27] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called atomic subtraction algebras. Y. B. Jun, H.S. Kim and E. H. Roh [5] introduced the notion of ideals in subtraction algebras and discussed the characterization of such ideals. In ,[6] Y. B. Jun and H. S. Kim established the ideal generated by a set, and discussed related results. K.J. Lee, Y. B. Jun, and Y. H. Kim also introduced in,[9] the notion of weak subtraction algebras and provided a method to make a weak subtraction algebra from a quasi-ordered set.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophic set and neutrosophic logic were introduced in 1995 by Smarandache as generalizations of fuzzy set and respectively intuitionistic fuzzy logic. In neutrosophic logic, each proposition has a degree of truth (T), a degree of indeterminancy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^{-}0, 1^{+}[$, see [15,16,17].
The notion of neutrosophic algebraic structures was introduced by Kandasamy and Smarandache in 2006, see [24,25]. Since then, several researchers have studied the concepts and a great deal of literature has been produced. For example, Agboola and Davvaz introduced the concept of neutrosophic BCI/BCK-algebras in.[1] A comprehensive review of neutrosophy, neutrosophic triplet set and neutrosophic algebraic structures can be found in [3,4,10,11,12,13,18,19,20,21,22,23]. The present paper is concerned with introducing the concept of neutrosophic subtraction algebras and subtraction semigroup. Some of their elementary properties are presented.

2 Preliminaries

In this section, some basic definitions and properties that will be useful in this work are given.

Definition 2.1. [2] A pair $(A, -)$ where A is a nonempty set and '$-'$' is a binary operation on A is called a subtraction algebra if

1. $x - (y - x) = x$;
2. $x - (x - y) = y - (y - x)$ and
3. $(x - y) - z = (x - z) - y$ for all $x, y, z \in A$.

Axiom 3 of 2.1 permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on $A : x \leq y$ if and only if $x - y = 0$, where $0 = x - x$ is an element that does not depend on the choice of $x \in A$. The ordered set $(A, \leq)$ is a semi-Boolean algebra, that is, it is a meet semilattice with zero 0 in which every interval $[0, x]$ is a Boolean algebra with respect to the induced order.[7] Here $x \wedge y = x - (x - y)$; the complement of an element $y \in [0, x]$ is $x - y$; and if $y, z \in [0, x]$, then

$$y \vee z = (y' \wedge z')' = x - ((x - y) \wedge (x - z)) = x - ((x - y) - ((x - y) - (x - z))).$$

In a subtraction algebra, the following hold [7]

1. $x - 0 = x$ and $0 - x = 0$
2. $x - (x - y) \leq y$.
3. $x \leq y$ if and only if $x = y - w$ for some $w \in X$.
4. $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.
5. $x - (x - (x - y)) = x - y$.

Definition 2.2. [26] Let X be a subtraction algebra and Y a nonempty subset of X. Then Y is called a subalgebra of X if $x - y \in Y$ whenever $x, y \in Y$

Definition 2.3. [2] A triple $(A, -, \cdot)$ is called a subtraction semigroup if

1. $(A, -)$ is a subtraction algebra;
2. $(A, \cdot)$ is a semigroup and
3. $x(y - z) = xy - xz$ and $(y - z)x = yx - zx$ for all $x, y, z \in A$.

A subtraction semigroup is said to be multiplicatively abelian if multiplication is commutative.

Definition 2.4. A triple $(A, -, \cdot)$ is called a subtraction group if

1. $(A, -, \cdot)$ is a subtraction semigroup and
2. $A - \{0\}$ is a group with the multiplication inherited from A.

Example 2.5. [8] Let $X = \{0, 1\}$ be a set in which " $-$ " and " $\cdot$ " are defined by

Table 1: (a) Cayley table for the binary operation " $-$ " and (b) Cayley table for the binary operation " $\cdot$ "

$-$	0	1
0	0	0
1	1	0

(a)

$\cdot$	0	1
0	0	0
1	0	1

(b)

It is easy to check that X is a subtraction semigroup.

Definition 2.6. A non empty set X together with two binary operations $'-'$ and $'\cdot'$ is said to be a left near subtraction semigroup if

1. $(X, -)$ is a subtraction algebra
2. $(X, \cdot)$ is a semigroup and

3. $z(x-y) = zx - zy$ for every $x, y, z \in X$.

Definition 2.7. A non empty set X together with two binary operations $'-'$ and $'\cdot'$ is said to be a right near subtraction semigroup if

1. $(X, -)$ is a subtraction algebra
2. $(X, \cdot)$ is a semigroup and
3. $(x-y)z = xz - yz$ for every $x, y, z \in X$.

3 Formulation of a Neutrosophic Subtraction Algebra and Neutrosophic Subtraction Semigroup

In this section, the concept of neutrosophic subtraction algebra and subtraction semigroup are develop. Some of their basic properties are presented.

Definition 3.1. Let X be a nonempty set. A set $X(I) =< X \cup I >$generated by X and I is called a neutrosophic set. The elements of $X(I)$ are of the form (x, yI) where x and y are elements of X. I is called an indeterminate and it has the property $I^n = I$ for all positive integer n.

Definition 3.2. Let $(X, -)$ be any classical subtraction algebra and let $X(I) =< X \cup I >$ be a set generated by X and I. Consider the neutrosophic algebraic structure $(X(I), -_N)$ where for all $(a, bI), (c, dI) \in X(I)$, $-_N$ is defined by

$$(a, bI) -_N (c, dI) = (a-c, (b-d)I) \qquad \forall a, b, c, d \in X$$

We call $(X(I), -_N)$ a neutrosophic subtraction algebra.

An element $x \in X$ is represented by $(x, 0) \in X(I)$ and $(0, 0)$ represents the constant element in $X(I)$.

Example 3.3. Let $X(I) = \{(0,0), (a,0), (b,0), (1,0), (0,aI), (0,bI), (0,I)\}$ be a neutrosophic set in which "$-_N$" is defined as in the table below.

Table 2: Cayley table for the binary operation $-_N$

$-_N$	$(0,0)$	$(a,0)$	$(b,0)$	$(1,0)$	$(0,aI)$	$(0,bI)$	$(0,I)$
$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
$(a,0)$	$(a,0)$	$(0,0)$	$(a,0)$	$(0,0)$	$(a,0)$	$(a,0)$	$(a,0)$
$(b,0)$	$(b,0)$	$(b,0)$	$(0,0)$	$(0,0)$	$(b,0)$	$(b,0)$	$(b,0)$
$(1,0)$	$(1,0)$	$(b,0)$	$(a,0)$	$(0,0)$	$(1,0)$	$(1,0)$	$(1,0)$
$(0,aI)$	$(0,aI)$	$(0,aI)$	$(0,aI)$	$(0,aI)$	$(0,0)$	$(0,aI)$	$(0,0)$
$(0,bI)$	$(0,bI)$	$(0,bI)$	$(0,bI)$	$(0,bI)$	$(0,bI)$	$(0,0)$	$(0,0)$
$(0,I)$	$(0,I)$	$(0,I)$	$(0,I)$	$(0,I)$	$(0,bI)$	$(0,aI)$	$(0,0)$

Then $(X(I), -_N)$ is a neutrosophic subtraction algebra.

Proposition 3.4. *Every neutrosophic subtraction algebra $(X(I), -_N)$ is a subtraction algebra.*

Proof: Suppose that $(X(I), -_N)$ is a subtraction algebra.
Let $x = (a, bI), y = (c, dI), z = (e, fI) \in X(I)$ with $a, b, c, d, e, f \in X$.
Then

1. We show that $x -_N (y -_N x) = x$

$$\begin{aligned} x -_N (y -_N x) &= (a, bI) -_N ((c, dI) -_N (a, bI)) \\ &= (a, bI) -_N (c-a, (d-b)I) \\ &= (a-(c-a), (b-(d-b))I) \\ &= (a, bI) \qquad \text{Since a,b,c,d} \in X \\ &= x. \end{aligned}$$

2. We want to show that $x -_N (x -_N y) = y -_N (y -_N x)$

$$\begin{array}{rcll} x -_N (x -_N y) & = & (a, bI) -_N ((a, bI) -_N (c, dI)) & \\ & = & (a, bI) -_N ((a-c), (b-d)I) & \\ & = & (a-(a-c), (b-(b-d)))I & \\ & = & (c-(c-a), (d-(d-b)))I & \text{Since a,b,c,d} \in X \\ & = & (c, dI) -_N ((c-a), (d-b)I) & \\ & = & (c, dI) -_N ((c, dI) -_N (a, bI)) & \\ & = & y -_N (y -_N x). & \end{array}$$

3. We want to show that $(x -_N y) -_N z = (x -_N z) -_N y \qquad \forall x, y, z \in X(I)$

$$\begin{array}{rcll} (x -_N y) -_N z & = & ((a, bI) -_N (c, dI)) -_N (e, fI) & \\ & = & (a-c, (b-d)I) -_N (e, fI) & \\ & = & (a-c)-e, ((b-d)-f)I & \\ & = & (a-e)-c, ((b-f)-d)I & \text{Since a,b,c,d,e,f} \in X \\ & = & ((a-e), (b-f)I) -_N (c, dI) & \\ & = & ((a, bI) -_N (e, fI)) -_N (c, dI) & \\ & = & (x -_N z) -_N y. & \end{array}$$

From[24] we note that, if $x \leq y$ then we can not in general say $xI \leq yI$, it may so happen that $yI \leq xI$. Thus the neutrosophic order in general need not to preserve the order. If a set X is ordered under $' \leq'$ then the neutrosophic part of $< X \cup I >$ may or may not have the preservations of order under $\leq$; i.e., if $x \leq y$, $x, y \in X$ then $xI \leq yI$ may occur or may not occur. For the purpose of this work suppose $xI \leq yI$ occur.

Proposition 3.5. *If $(X(I), -_N)$ is a neutrosophic subtraction algebra, then the relation $\leq$ is a partial order on $X(I)$.*

Proof: Let $(a, bI), (c, dI), (e, fI) \in X(I)$ and $a, b, c, d, e, f \in X$.

1. Since $(a, bI) -_N (a, bI) = (a-a, (b-b)I) = (0, 0I)$ then $(a, bI) \leq (a, bI)$

2. Suppose that $(a, bI) \leq (c, dI)$ and $(c, dI) \leq (a, bI)$
then $(a, bI) -_N (c, dI) = (0, 0I)$ implies $(a-c, (b-d)I) = (0, 0I)$
and $(c, dI) -_N (a, bI) = (0, 0I)$ implies $(c-a, (d-b)I) = (0, 0I)$.
Now,

$$\begin{array}{rcll} (a, bI) & = & (a, bI) -_N (0, 0I) & \\ & = & (a-0, (b-0)I) & \\ & = & (a-(a-c), (b-(b-d))I) & \text{Since, a-c=0 and b-d=0} \\ & = & (c-(c-a), (d-(d-b))I) & \text{Since X is a subtraction algebra} \\ & = & (c-0, (d-0)I) & \text{Since, c-a=0 and d-b=0} \\ & = & (c, dI) & \end{array}$$

3. Suppose that $(a, bI) \leq (c, dI)$ and $(c, dI) \leq (e, fI)$
then $(a, bI) -_N (c, dI) = (0, 0I)$ implies $(a-c, (b-d)I) = (0, 0I)$
and $(c, dI) -_N (e, fI) = (0, 0I)$ implies $(c-e, (d-f)I) = (0, 0I)$.

$$\begin{array}{rcl} (a, bI) -_N (e, fI) & = & (a-e, (b-f)I) \\ & = & ((a-e)-0, ((b-f)-0)I) \\ & = & ((a-e)-(a-c), ((b-f)-(b-d))I) \\ & = & ((a-(a-c))-e, ((b-(b-d))-f)I) \\ & = & ((c-(c-a))-e, ((d-(d-b))-f)I) \\ & = & ((c-e)-(c-a), ((d-f)-(d-b))I) \\ & = & (0-(c-a), (0-(d-b))I) \\ & = & (0, 0I) \end{array}$$

Hence $(a, bI) \leq (e, fI)$. Consequently $\leq$ is a partial order.

In the next proposition some important properties of neutrosophic subtraction algebra which will be frequently used throughout this work are provided with their proofs.

Proposition 3.6. *Let $(X(I), -_N)$ be a neutrosophic subtraction algebra . If $(a, bI), (c, dI) \in X(I)$, with $a, b, c, d \in X$ then the following are true.*

1. $(a, bI) -_N (0, 0I) = (a, bI)$

2. $(0, 0I) -_N (a, bI) = (0, 0I)$

3. $((a, bI) -_N (c, dI)) -_N (a, bI) = (0, 0I)$

4. $((a, bI) -_N (c, dI)) -_N (c, dI) = (a, bI) -_N (c, dI)$

5. $((a, bI) -_N (c, dI)) -_N ((c, dI) -_N (a, bI)) = (a, bI) -_N (c, dI)$

6. $(a, bI) -_N ((a, bI) -_N ((a, bI) -_N (c, dI))) = (a, bI) -_N (c, dI)$

7. $(a, bI) -_N ((a, bI) -_N (c, dI)) \leq (c, dI)$

8. $(a, bI) = (c, dI)$ *iff* $(a, bI) -_N (c, dI) = (0, 0I)$ *and* $(c, dI) = (a, bI)$

Proof:

1. $(a, bI) -_N (0, 0I) = (a - 0, (b - 0)I) = (a - (a - a), (b - (b - b))I) = (a, bI)$.

2. $(0, 0I) -_N (a, bI) = (0 - a, (0 - b)I) = (0 - (a - 0), (0 - (b - 0))I) = (0, 0I)$.

$$\begin{aligned} 3.((a, bI) -_N (c, dI)) -_N (a, bI) &= (a - c, (b - d)I) -_N (a, bI) \\ &= ((a - c) - a, ((b - d) - b)I) \\ &= ((a - a) - c, ((b - b) - d)I) \\ &= (0 - c, (0 - d)I) \\ &= (0, 0I). \end{aligned}$$

$$\begin{aligned} 4.((a, bI) -_N (c, dI)) -_N (c, dI) &= (a - c, (b - d)I) -_N (c, dI) \\ &= ((a - c) - c, ((b - d) - d)I) \\ &= ((a - c) - (c - (a - c)), ((b - d) - (d - (b - d))I) \\ &= ((a - c), (b - d)I) \\ &= (a, bI) -_N (c, dI). \end{aligned}$$

$$\begin{aligned} 5.((a, bI) -_N (c, dI)) -_N ((c, dI) -_N (a, bI)) &= (a - c, (b - d)I) -_N (c - a, (d - b)I) \\ &= ((a - c) - (c - a), ((b - d) - (d - b))I) \\ &= ((a - (c - a)) - c, ((b - (d - b)) - d)I) \\ &= (a - c, (b - d)I) \\ &= (a, bI) -_N (c, dI). \end{aligned}$$

$$\begin{aligned} 6.(a, bI) -_N ((a, bI) -_N ((a, bI) -_N (c, dI))) &= (a, bI) -_N ((a, bI) -_N (a - c, (b - d)I)) \\ &= (a, bI) -_N (a - (a - c), (b - (b - d))I) \\ &= (a - (a - (a - c)), (b - (b - (b - d)))I) \\ &= ((a - c) - ((a - c) - a), ((b - d) - ((b - d) - b))I) \end{aligned}$$

Since from the properties of X, if $a, c \in X$ then $(a - c) - a = 0$ then we have

$$\begin{aligned} ((a - c) - ((a - c) - a), ((b - d) - ((b - d) - b))I) &= ((a - c) - 0, ((b - d) - 0)I) \\ &= (a - c, (b - d)I) \\ &= (a, bI) -_N (c, dI). \end{aligned}$$

7. $((a, bI) -_N ((a, bI) -_N (c, dI))) -_N (c, dI) = ((a, bI) -_N (c, dI)) -_N ((a, bI) -_N (c, dI)) = (0, 0I)$
$\Longrightarrow (a, bI) -_N ((a, bI) -_N (c, dI)) \leq (c, dI)$.

8. Suppose that $(a, bI) -_N (c, dI) = (0, 0I)$ and $(c, dI) -_N (a, bI) = (0, 0I)$. Then

$$\begin{aligned} (a, bI) &= (a, bI) -_N (0, 0I) \\ &= (a, bI) -_N ((a, bI) -_N (c, dI)) \\ &= (c, dI) -_N ((c, dI) -_N (a, bI)) \\ &= (c, dI) -_N (0, 0I) \\ &= (c, dI). \\ &\Longrightarrow (a, bI) = (c, dI). \end{aligned}$$

Conversely, suppose that $(a, bI) = (c, dI)$
Then $(a, bI) -_N (c, dI) = (a, bI) -_N (a, bI) = (0, 0I)$ and $(c, dI) -_N (a, bI) = (c, dI) -_N (c, dI) = (0, 0I)$
$\Longrightarrow (a, bI) -_N (c, dI) = (0, 0I)$ and $(c, dI) -_N (a, bI) = (0, 0I)$.

Definition 3.7. Let $X_1(I)$ and $X_2(I)$ be two neutrosophic subtraction algebra. The direct product of $X_1(I)$ and $X_2(I)$ denoted by $X_1(I) \times X_2(I)$ is defined by
$X_1(I) \times X_2(I) = \{((a_1, b_1I), (a_2, b_2I)) : (a_1, b_1I) \in X_1(I), (a_2, b_2I) \in X_2(I)\}$.

Proposition 3.8. *Let $(X_1(I), -_{N_1})$ and $(X_2(I), -_{N_2})$ be two neutrosophic subtraction algebra then $(X_1(I) \times X_2(I), -_N)$ is a neutrosophic subtraction algebra.*

Proof:
Let $x = ((a_1, b_1I), (a_2, b_2I)), y = ((c_1, d_1I), (c_2, d_2I)), z = ((e_1, f_1I), (e_2, f_2I)) \in X_1(I) \times X_2(I)$ for all $a_1, b_1, c_1, d_1 \in X_1$ and $a_2, b_2, c_2, d_2 \in X_2$ then

1. we shall show that $x -_N (y -_N x) = x$
$$\begin{aligned} x -_N (y -_N x) &= ((a_1, b_1I), (a_2, b_2I)) -_N [((c_1, d_1I), (c_2, d_2I)) -_N ((a_1, b_1I), (a_2, b_2I))] \\ &= ((a_1, b_1I), (a_2, b_2I)) -_N [(c_1, d_1I) -_{N_1} (a_1, b_1I), (c_2, d_2I) -_{N_2} (a_2, b_2I)] \\ &= (a_1, b_1I) -_{N_1} ((c_1, d_1I) -_{N_1} (a_1, b_1I)), (a_2, b_2) -_{N_2} ((c_2, d_2I) -_{N_2} (a_2, b_2I)) \\ &= ((a_1, b_1I), (a_2, b_2I)) \\ &= x. \end{aligned}$$

2. We want to show that $x -_N (x -_N y) = y -_N (y -_N x)$
$$\begin{aligned} x -_N (x -_N y) &= ((a_1, b_1I), (a_2, b_2I)) -_N [((a_1, b_1I), (a_2, b_2I)) -_N ((c_1, d_1I), (c_2, d_2I))] \\ &= ((a_1, b_1I), (a_2, b_2I)) -_N [((a_1, b_1I) -_{N_1} (c_1, d_1I)), ((a_2, b_2I) -_{N_2} (c_2, d_2I))] \\ &= (a_1, b_1I) -_{N_1} ((a_1, b_1I) -_{N_1} (c_1, d_1I)), (a_2, b_2I) -_{N_2} ((a_2, b_2I) -_{N_2} (c_2, d_2I)) \\ &= (c_1, d_1I) -_{N_1} ((c_1, d_1I) -_{N_1} (a_1, b_1I)), (c_2, d_2I) -_{N_2} ((c_2, d_2I) -_{N_2} (a_2, b_2I)) \\ &= ((c_1, d_1I), (c_2, d_2I) -_N [((c_1, d_1I) -_{N_1} (a_1, b_1I)), ((c_2, d_2I) -_{N_2} (a_2, b_2I))] \\ &= ((c_1, d_1I), (c_2, d_2I) -_N [((c_1, d_1I), (c_2, d_2I)) -_N ((a_1, b_1I), (a_2, b_2I))] \\ &= y -_N (x -_N y). \end{aligned}$$

3. We want to show that $(x -_N y) -_N z = (x -_N z) -_N y$
$$\begin{aligned} (x -_N y) -_N z &= [((a_1, b_1I), (a_2, b_2I)) -_N ((c_1, d_1I), (c_2, d_2I))] -_N ((e_1, f_1I), (e_2, f_2I)) \\ &= [((a_1, b_1I) -_{N_1} (c_1, d_1I)), ((a_2, b_2I) -_{N_2} (c_2, d_2I))] -_N ((e_1, f_1I), (e_2, f_2I)) \\ &= ((a_1, b_1I) -_{N_1} (c_1, d_1I)) -_{N_1} (e_1, f_1I), ((a_2, b_2I) -_{N_2} (c_2, d_2I)) -_{N_2} (e_2, f_2I) \\ &= ((a_1, b_1I) -_{N_1} (e_1, f_1I)) -_{N_1} (c_1, d_1I), ((a_2, b_2I) -_{N_2} (e_2, f_2I)) -_{N_2} (c_2, d_2I) \\ &= [((a_1, b_1I) -_{N_1} (e_1, f_1I)), ((a_2, b_2I) -_{N_2} (e_2, f_2I))] -_N ((c_1, d_1I), (c_2, d_2I)) \\ &= [((a_1, b_1I), (a_2, b_2I)) -_N ((e_1, f_1I), (e_2, f_2I))] -_N ((c_1, d_1I), (c_2, d_2I)) \\ &= (x -_N z) -_N y. \end{aligned}$$

Proposition 3.9. *Let $X_1(I)$ be a neutrosophic subtraction algebra and let A be a classical subtraction algebra then $(X_1(I) \times A, -_N)$ is a neutrosophic subtraction algebra.*

Proof : It follows similar approach as in 3.8 above.

Definition 3.10. Let $(X(I), -_N)$ be a neutrosophic subtraction algebra. A nonempty subset $A(I)$ is called a neutrosophic subtraction subalgebra of $X(I)$ if the following conditions hold:

1. If $A(I) \neq \emptyset$
2. $(a, bI) -_N (c, dI) \in A(I)$ for all $(a, bI), (c, dI) \in A(I)$.
3. $A(I)$ contains a proper subset which is a subtraction algebra.

If $A(I)$ does not contain a proper subset which is a subtraction algebra, then $A(I)$ is called a pseudo neutrosophic subtraction subalgebra of $X(I)$.

Example 3.11. Any neutrosophic subset of the neutrosophic set $X(I)$ of Example 3.3 is a neutrosophic subtraction subalgebra.

Definition 3.12. Let $(X, -, \cdot)$ be any subtraction semigroup and let $X(I) =< X \cup I >$ be a set generated by X and I. The triple $(X(I), -_N, \cdot)$ is called a neutrosophic subtraction semi-group. If (a, bI) and (c, dI) are any two elements of $X(I)$ with $a, b, c, d \in X$, we define

$$(a, bI) -_N (c, dI) = (a - c, (b - d)I)$$

and

$$(a, bI) \cdot (c, dI) = (ac, (ad + bc + bd)I).$$

Example 3.13. Let $X(I) = \{(0,0),(a,0),(b,0),(1,0),(0,aI),(0,bI),(0,I)\}$ be a neutrosophic set in which "$-_N$" and "$\cdot$" are defined as in the tables below.

Table 3: (i) Cayley table for the binary operation " $-_N$ " and (ii) Cayley table for the binary operation " $\cdot$ "

(i)

$-_N$	$(0,0)$	$(a,0)$	$(b,0)$	$(1,0)$	$(0,aI)$	$(0,bI)$	$(0,I)$
$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
$(a,0)$	$(a,0)$	$(0,0)$	$(a,0)$	$(0,0)$	$(a,0)$	$(a,0)$	$(a,0)$
$(b,0)$	$(b,0)$	$(b,0)$	$(0,0)$	$(0,0)$	$(b,0)$	$(b,0)$	$(b,0)$
$(1,0)$	$(1,0)$	$(b,0)$	$(a,0)$	$(0,0)$	$(1,0)$	$(1,0)$	$(1,0)$
$(0,aI)$	$(0,aI)$	$(0,aI)$	$(0,aI)$	$(0,aI)$	$(0,0)$	$(0,aI)$	$(0,0)$
$(0,bI)$	$(0,bI)$	$(0,bI)$	$(0,bI)$	$(0,bI)$	$(0,bI)$	$(0,0)$	$(0,0)$
$(0,I)$	$(0,I)$	$(0,I)$	$(0,I)$	$(0,I)$	$(0,bI)$	$(0,aI)$	$(0,0)$

(ii)

$\cdot$	$(0,0)$	$(a,0)$	$(b,0)$	$(1,0)$	$(0,aI)$	$(0,bI)$	$(0,I)$
$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
$(a,0)$	$(0,0)$	$(a,0)$	$(0,0)$	$(a,0)$	$(0,aI)$	$(0,0)$	$(0,aI)$
$(b,0)$	$(0,0)$	$(0,0)$	$(b,0)$	$(b,0)$	$(0,0)$	$(0,bI)$	$(0,bI)$
$(1,0)$	$(0,0)$	$(a,0)$	$(b,0)$	$(1,0)$	$(0,aI)$	$(0,bI)$	$(0,I)$
$(0,aI)$	$(0,0)$	$(0,aI)$	$(0,0)$	$(0,aI)$	$(0,aI)$	$(0,0)$	$(0,aI)$
$(0,bI)$	$(0,0)$	$(0,0)$	$(0,bI)$	$(0,bI)$	$(0,0)$	$(0,bI)$	$(0,bI)$
$(0,I)$	$(0,0)$	$(0,aI)$	$(0,bI)$	$(0,I)$	$(0,aI)$	$(0,bI)$	$(0,I)$

Then $(X(I), -_N, \cdot)$ is a neutrosophic subtraction semi-group.

Proposition 3.14. *Every neutrosophic subtraction semi-group is a subtraction semi-group.*

Proof:

1. That $(X(I), -_N)$ is a subtraction algebra follows from Proposition 3.4.
2. We need to show that $(X(I), \cdot)$ is a semi-group.
   (a) Let $(a,bI),(c,dI) \in X(I)$ with $a,b,c,d \in X$, then $(a,bI)\cdot(c,dI) = (ac,(ad+bc+bd)I) \in X(I)$.
   (b) $(a,bI),(c,dI),(e,fI) \in X(I)$ with $a,b,c,d,e,f \in X$, We want to show that $((a,bI)\cdot(c,dI))\cdot(e,fI) = (a,bI)\cdot((c,dI)\cdot(e,fI))$
$$\begin{aligned}((a,bI)\cdot(c,dI))\cdot(e,fI) &= (ac,(ad+bc+bd)I)\cdot(e,fI)\\ &= (ace,(acf+ade+bce+bde+adf+bcf+bdf)I)\\ &= (a,bI)\cdot(ce,(cf+de+df)I)\\ &= (a,bI)\cdot((c,dI)\cdot(e,fI)).\end{aligned}$$
3. Let $(a,bI),(c,dI),(e,fI) \in X(I)$ then
   (a) We shall show that $(a,bI)\cdot((c,dI) -_N (e,fI)) = (a,bI)\cdot(c,dI) -_N (a,bI)\cdot(e,fI)$
$$\begin{aligned}(a,bI)\cdot((c,dI) -_N (e,fI)) &= (a,bI)\cdot(c-e,(d-f)I)\\ &= (a(c-e),(a(d-f)+b(c-e)+b(d-f))I)\\ &= (ac-ae,((ad-af)+(bc-be)+(bd-bf))I)\\ &= (ac-ae,((ad+bc+bd)-(af+be+bf))I)\\ &= (ac,(ad+bc+bd)I) -_N (ae,(af+be+bf)I)\\ &= (a,bI)\cdot(c,dI) -_N (a,bI)\cdot(e,fI).\end{aligned}$$

(b) We shall show that $((c,dI)-_N(e,fI))\cdot(a,bI)=(c,dI)\cdot(a,bI)-_N(e,fI)\cdot(a,bI)$

$$\begin{aligned}
((c,dI)-_N(e,fI))\cdot(a,bI) &= ((c-e,(d-f)I)\cdot(a,bI)\\
&= ((c-e)a,((c-e)b+(d-f)a+(d-f)b)I)\\
&= ((ca-ea),((cb-eb)+(da-fa)+(db-fb))I)\\
&= ((ca-ea),((cb+da+db)-(eb+fa+fb))I)\\
&= (ca,(cb+da+db)I)-(ea,(eb+fa+fb)I)\\
&= (c,dI)\cdot(a,bI)-_N(e,fI)\cdot(a,bI).
\end{aligned}$$

Definition 3.15. Let $X_1(I)$ and $X_2(I)$ be two neutrosophic subtraction semigroups. The direct product of $X_1(I)$ and $X_2(I)$ denoted by $X_1(I)\times X_2(I)$ is defined by
$X_1(I)\times X_2(I)=\{((a_1,b_1I),(a_2,b_2I)):(a_1,b_1I)\in X_1(I),(a_2,b_2I)\in X_2(I)\}$.

Proposition 3.16. *Let $(X_1(I),-_{N_1},\cdot_1)$ and $(X_2(I),-_{N_2},\cdot_2)$ be two neutrosophic subtraction semigroup then $(X_1(I)\times X_2(I),-_N,\cdot)$ is a neutrosophic subtraction semigroup.*

Proposition 3.17. *Let $(X(I),-_N,\cdot)$ be a neutrosophic subtraction semigroup and let $(A,-,\cdot)$ be a classical subtraction semigroup then $(X(I)\times A,-_N,\cdot)$ is a neutrosophic subtraction semigroup.*

Proposition 3.18. *Let $(X(I),-_N,\cdot)$ be a neutrosophic subtraction semigroup.*
And let $(x,yI)\wedge(u,vI)=(x,yI)-_N((x,yI)-_N(u,vI))$ for all $(x,yI),(u,vI),(s,tI)\in X(I)$.
Then the following are true.

1. *$(x,yI)(0,0)$ and $(0,0)(x,yI)=(0,0)$*
2. *$(x,yI)\le(u,vI)$ implies $(s,tI)(x,yI)\le(s,tI)(u,vI)$ and $(x,yI)(s,tI)\le(u,vI)(s,tI)$.*
3. *$(x,yI)((u,vI)\wedge(s,tI))=(x,yI)(u,vI)\wedge(x,yI)(s,tI)$ and*
   $((x,yI)\wedge(u,vI))(s,tI)=(x,yI)(s,tI)\wedge(u,vI)(s,tI)$
4. *$((x,yI)\wedge(u,vI))-_N((x,yI)\wedge(s,tI))\le(x,yI)\wedge((u,vI)-_N(s,tI))$*

Proof :

1. Let $(x,yI),(0,0)\in X(I)$ then
$$\begin{aligned}
(x,yI)(0,0) &= (x,yI)((0,0)-_N(0,0))\\
&= (x,yI)(0,0)-_N(x,yI)(0,0)\\
&= (0,0).
\end{aligned}$$
and
$$\begin{aligned}
(0,0)(x,yI) &= ((0,0)-_N(0,0))(x,yI)\\
&= (0,0)(x,yI)-_N(0,0)(x,yI)\\
&= (0,0).
\end{aligned}$$

2. Let $(x,yI),(u,vI)\in X(I)$ and let $(x,yI)\le(u,vI)$. Then we have
$(x,yI)-_N(u,vI)=(x-u,(y-v)I)=(0,0)\Longrightarrow x-u=0$ and $y-v=0$.

   Now consider
$$\begin{aligned}
(s,tI)(x,yI)-_N(s,tI)(u,vI) &= (sx,(sy+tx+ty)I)-_N(su,(sv+tu+tv)I)\\
&= sx-su,((sy-sv)+(tx-tu)+(ty-tv))I\\
&= s(x-u),((s(y-v)+t(x-u)+t(y-v))I\\
&= s(0),(s(0)+t(0)+t(0))I\\
&= (0,0).
\end{aligned}$$
$\Longrightarrow(s,tI)(x,yI)\le(s,tI)(u,vI)$.

   Following similar approach we can also prove that $(x,yI)(s,tI)\le(u,vI)(s,tI)$.

3. Let $(x,yI),(u,vI),(s,tI)\in X(I)$ then
$$\begin{aligned}
(x,yI)((u,vI)\wedge(s,tI)) &= (x,yI)((u,vI)-_N((u,vI)-_N(s,tI)))\\
&= (x,yI)(u,vI)-_N(x,yI)((u,vI)-_N(s,tI))\\
&= (x,yI)(u,vI)-_N((x,yI)(u,vI)-_N(x,yI)(s,tI))\\
&= (x,yI)(u,vI)\wedge(x,yI)(s,tI).
\end{aligned}$$

   Following similar procedure we can show that
$((x,yI)\wedge(u,vI))(s,tI)=(x,yI)(s,tI)\wedge(u,vI)(s,tI)$.

4. Let $(x, yI), (u, vI), (s, tI) \in X(I)$
$$\begin{aligned} ((x,yI)\wedge(u,vI)) -_N ((x,yI)\wedge(s,tI)) &= [(x,yI) -_N ((x,yI) -_N (u,vI))] \\ &\quad -_N[(x,yI) -_N ((x,yI) -_N (s,tI))] \\ &= [x-(x-u),(y-(y-v))I] \\ &\quad -_N[x-(x-s),(y-(y-t))I] \\ &\leq [(x-s),(y-t)I] -_N [(x-u),(y-v))I] \\ &\leq ((x,yI) -_N (s,tI)) -_N ((x,yI) -_N (u,vI)) \\ &\leq (u,vI) -_N (s,tI) \qquad \cdots\cdots\cdots(1) \end{aligned}$$
Also, we have that
$$\begin{aligned} ((x,yI)\wedge(u,vI)) -_N ((x,yI)\wedge(s,tI)) &\leq (x,yI)\wedge(u,vI)) \\ &\leq (x,yI) \qquad \cdots\cdots\cdots(2) \end{aligned}$$
Combining 1 and 2 we have $((x,yI)\wedge(u,vI)) -_N ((x,yI)\wedge(s,tI)) \leq (x,yI)\wedge((u,vI) -_N (s,tI))$.

Proposition 3.19. *Let $X(I)$ be a neutrosophic subtraction semigroup. Then the following hold:*

1. $(x,yI)\wedge((x,yI) -_N (a,bI)) = (x,yI) -_N (a,bI)$
2. $(x,yI)\wedge((a,bI) -_N (x,yI)) = (0,0I)$
3. $((x,yI) -_N (a,bI))\wedge((a,bI) -_N (x,yI)) = (0,0I)$
4. $((x,yI) -_N (c,dI))\wedge((a,bI) -_N (c,dI)) = ((x,yI) -_N (c,dI))\wedge(a,bI)$

Proof:

1. For any $(x,yI),(a,bI) \in X(I)$ we have
$$\begin{aligned} (x,yI)\wedge((x,yI) -_N (a,bI)) &= (x,yI) -_N ((x,yI) -_N ((x,yI) -_N (a,bI))) \\ &= (x,yI) -_N (a,bI). \quad \text{From 6 of 3.6.} \end{aligned}$$

2. For any $(x,yI),(a,bI) \in X(I)$ we have
$$\begin{aligned} (x,yI)\wedge((a,bI) -_N (x,yI)) &= (x,yI) -_N ((x,yI) -_N ((a,bI) -_N (x,yI))) \\ &= (x,yI) -_N ((x,yI) -_N ((a-x,(b-y)I)) \\ &= (x,yI) -_N (x-(a-x),y-(b-y))I) \\ &= (x,yI) -_N (x,yI) \qquad \text{from 1 of 2.1} \\ &= (0,0). \end{aligned}$$

3. For any $(x,yI),(a,bI) \in X(I)$ we have
$$\begin{aligned} ((x,yI) -_N (a,bI))\wedge((a,bI) -_N (x,yI)) &= ((x,yI) -_N (a,bI)) \\ &\quad -_N(((x,yI) -_N (a,bI)) -_N ((a,bI) -_N (x,yI))) \\ &= ((x,yI) -_N (a,bI)) -_N ((x,yI) -_N (a,bI)) \\ &\quad \text{from 5 of 3.6} \\ &= (0,0I). \end{aligned}$$

4. For any $(x,yI),(a,bI),(c,dI) \in X(I)$ we have
$$\begin{aligned} ((x,yI) -_N (c,dI))\wedge((a,bI) -_N (c,dI)) &= ((x,yI) -_N (c,dI)) \\ &\quad -_N(((x,yI) -_N (c,dI)) -_N ((a,bI) -_N (c,dI))) \\ &= ((x,yI) -_N (c,dI)) -_N (((x,yI) -_N (a,bI)) -_N (c,dI)) \\ &= ((x,yI) -_N (c,dI)) -_N (((x,yI) -_N (c,dI)) -_N (a,bI)) \\ &= ((x,yI) -_N (c,dI))\wedge(a,bI). \end{aligned}$$

Definition 3.20. Let $(X(I), -_N, \cdot)$ be a neutrosophic subtraction semigroup and let $S(I)$ be a nonempty subset of $X(I)$. $S(I)$ is called a neutrosophic subtraction subsemigroup of $X(I)$ if $(S(I), -_N, \cdot)$ is itself a neutrosophic subtraction semigroup. It is essential that $S(I)$ contains a proper subset which is a subtraction semigroup. Otherwise, $S(I)$ will be called a pseudo neutrosophic subtraction subsemigroup of $X(I)$.

Example 3.21. Let $X(I) = \{(0,0),(a,0),(b,0),(1,0),(0,aI),(0,bI),(0,I)\}$ be a neutrosophic subtraction semigroup. The set $S(I) = \{(0,0),(a,0),(b,0),(1,0),(0,aI),(0,bI)\}$ is a neutrosophic subtraction subsemigroup.

Example 3.22. Let $X(I) = \{(0,0),(a,0),(b,0),(1,0),(0,aI),(0,bI),(0,I)\}$ be a neutrosophic subtraction semigroup. The set $S(I) = \{(0,0),(0,aI),(0,bI),(0,I)\}$ is a pseudo neutrosophic subtraction subsemigroup.

Definition 3.23. Let $(X(I), -_N, \cdot)$ be a neutrosophic subtraction semigroup. A nonempty subset $A(I)$ of $X(I)$ is called

1. a left neutrosophic ideal if $A(I)$ is a neutrosophic subalgebra of $(X(I), -_N)$ and $X(I)A(I) \subseteq A(I)$
2. a right neutrosophic ideal if $A(I)$ is a neutrosophic subalgebra of $(X(I), -_N)$ and $A(I)X(I) \subseteq A(I)$
3. a neutrosophic ideal if $A(I)$ is both a left neutrosophic and right neutrosophic ideal.

Example 3.24. Let $X(I)$ be a neutrososphic subtraction semigroup and $(a, bI) \in X(I)$. Then

$$X(I)(a, bI) = \{(x, yI)(a, bI) : (x, yI) \in X(I)\}$$

is a left neutrososphic ideal of $X(I)$.

Proposition 3.25. *Let $X(I)$ be a neutrosophic subtraction semigroup. If $A(I)$ is any ideal of $X(I)$. Then $\bigcap_{i=1}^{n} A_i(I)$ is a neutrosophic ideal of $X(I)$. Where $\{A_i(I)\}_{i=1}^{n}$ is a family of neutrosophic ideals of $X(I)$.*

Proof: Same as in classical sense.

Proposition 3.26. *Let $A(I)$ be a neutrosophic ideal of a neutrosophic subtraction semigroup $X(I)$. If $(x, yI) \leq (u, vI)$ and $(u, vI) \in A(I)$, then $(x, yI) \in A(I)$*

Proof: Since $(x, yI) \leq (u, vI)$ then $(x, yI) -_N (u, vI) = (x - u, (y - v)I) = (0, 0)$
$\Longrightarrow x - u = 0 \quad \text{and} \quad y - v = 0$
Now consider

$$\begin{aligned}
(u, vI) &= (u, vI) -_N ((u, vI) -_N (x, yI)) \qquad \text{from 6 of 3.6} \\
&= (x, yI) -_N ((x, yI) -_N (u, vI)) \\
&= (x, yI) -_N ((x - u), (y - v)I) \\
&= (x, yI) -_N (0, 0) \\
&= (x, y) \in A(I).
\end{aligned}$$

Proposition 3.27. *Let $A(I)$ be neutrosophic ideal of a neutrosophic subtraction semigroup $X(I)$. For any element $(a, bI) \in X(I)$ the set*

$$A(I)_{(a,bI)} = \{(u, vI) \in X(I) : (a, bI) \wedge (u, vI) \in A(I)\}$$

is a neutrosophic ideal of $X(I)$.

Proof:

1. $A(I)_{(a,bI)} \neq \emptyset$, since
$$\begin{aligned}
(a, bI) \wedge (0, 0) &= (a, bI) -_N ((a, bI) -_N (0, 0)) \\
&= (a, bI) -_N (a, bI) \\
&= (0, 0) \in A(I)
\end{aligned}.$$
Then we have that $(0, 0) \in A(I)_{(a,bI)}$.

2. Let $x = (u, vI), y = (p, qI) \in X(I)$. and $z = (r, sI) \in A(I)_{(a,bI)}$
then $(a, bI) \wedge (r, sI) \in A(I)$
Then we shall show that $xz -_N x(y -_N z) \in A(I)_{(a,bI)}$.
Now consider
$$\begin{aligned}
(a, bI) \wedge (xz -_N x(y -_N z)) &= (a, bI) \wedge ((u, vI)(r, sI)) -_N (a, bI) \wedge (u, vI)((p, qI) -_N (r, sI)) \\
&\quad \text{By 3 of 3.19} \\
&= (a, bI) \wedge ((u, vI)(r, sI)) -_N (a, bI) \wedge ((u, vI)(p, qI) -_N (u, vI)(r, sI)) \\
&= (a, bI) \wedge ((u, vI)(r, sI)) -_N \\
&\quad [(a, bI) \wedge ((u, vI)(p, qI)) -_N (a, bI) \wedge ((u, vI)(r, sI))] \\
&\quad \text{By 1 of 2.1} \\
&= (a, bI) \wedge ((u, vI)(r, sI)) \in A(I)
\end{aligned}$$
$\Longrightarrow (a, bI) \wedge (xz -_N x(y -_N z)) \in A(I)$ and hence $(xz -_N x(y -_N z)) \in A(I)_{(a,bI)}$.

3. Let $x = (u, vI) \in X(I)$ we have that
$$\begin{aligned}
A(I)_{(a,bI)}X(I) = A(I)_{(a,bI)}(u, vI) &= \{(m, nI) | (a, bI) \wedge (m, nI) \in A(I)\}(u, vI) \\
&= \{(m, nI)(u, vI) | (a, bI) \wedge (m, nI)(u, vI) \in A(I)\} \\
&\subseteq A(I)_{(a,bI)}.
\end{aligned}$$

Hence $A(I)_{(a,bI)}$ is an ideal of $X(I)$.

Proposition 3.28. *Let $X(I)$ be a neutrosophic subtraction semigroup. If $A(I)$ and $B(I)$ are any two neutrosophic ideals of $X(I)$ then*

1. $A(I) -_N B(I) = \{a -_N b : a \in A(I), b \in B(I)\}$ *is a neutrosophic ideal of* $X(I)$
2. $A(I)B(I) = \{\sum_{i=1}^{n} a_i b_i : a_i \in A(I), b_i \in B(I)\}$ *is a neutrosophic ideal of* $X(I)$

Proof:

1. To see that $A(I) -_N B(I)$ is a neutrosophic ideal of $X(I)$.
Let $x = (u, vI), y = (p, qI) \in X(I)$, and $z = a -_N b \in A(I) -_N B(I)$. Where $a = (m, nI) \in A(I)$ and $b = (r, sI) \in B(I)$
Then

$$\begin{aligned}
xz -_N x(y -_N z) &= x(a -_N b) -_N x(y -_N (a -_N b)) \\
&= (u, vI)((m, nI) -_N (r, sI)) -_N (u, vI)[(p, qI) -_N ((m, nI) -_N (r, sI))] \\
&= (u, vI)((m, nI) -_N (r, sI)) -_N [(u, vI)(p, qI) -_N (u, vI)((m, nI) -_N (r, sI))] \\
&= [(u, vI)(m, nI) -_N (u, vI)(r, sI)] \\
&\quad -_N[(u, vI)(p, qI) -_N ((u, vI)(m, nI) -_N (u, vI)(r, sI))] \\
&= [(um, (un + vm + vn)I) -_N (ur, (us + vr + vs)I)I] \\
&\quad -_N[(up, (uq + vp + vq)I) -_N [(um, (un + vm + vn)I) -_N (ur, (us + vr + vs)I)]] \\
&= [(um - ur), ((un - us) + (vm - vr) + (vn - vs))I] \\
&\quad -_N[(up, (uq + vp + vq)I) -_N [(um - ur), ((un - us) + (vm - vr) + (vn - vs))I]] \\
&= [(um - ur), ((un - us) + (vm - vr) + (vn - vs))I] \\
&\quad -_N[(up - (um - ur)), ((uq + vp + vq) - ((un - us) + (vm - vr) + (vn - vs)))I] \\
&= (um - ur) - [up - (um - ur)], ((un - us) + (vm - vr) + (vn - vs))I \\
&\quad -[(uq + vp + vq) - ((un - us) + (vm - vr) + (vn - vs))]I \\
&= (um - ur) - [up - (um - ur)], ((un - us) - [uq - (un - us)] \\
&\quad +(vm - vr) - [vp - (vm - vr)] + (vn - vs) - [vq - (vn - vs)])I \\
&= (um - ur), [(un - us) + (vm - vr) + (vn - vs)]I \qquad \text{By 1 of 2.1} \\
&= [um, (un + vm + vn)I] -_N [ur, (us + vr + vs)I] \\
&= (u, vI)(m, nI) -_N (u, vI)(r, sI) \\
&= (u, vI)[(m, nI) -_N (r, sI)] \\
&= x(a -_N b) \\
&= xz \in A(I) -_N B(I).
\end{aligned}$$

And

$$\begin{aligned}
(A(I) -_N B(I))X(I) &= \{zx : z \in A(I) -_N B(I), x \in X(I)\} \\
&\subseteq A(I) -_N B(I).
\end{aligned}$$

So, we say that $A(I) -_N B(I)$ is a neutrosophic ideal.

2. To see that $A(I)B(I)$ is a neutrosophic ideal.

- $A(I)B(I) \neq \emptyset$ since $A(I)$ and $B(I)$ are both neutrosophic ideals of $X(I)$. So $(0, 0) \in A(I)B(I)$
- Let $x = (u, vI), y = (p, qI) \in X(I)$, and $z = \sum_{i=1}^{n} a_i b_i \in A(I)B(I)$. Where $a_i = (m_i, n_i I) \in A(I)$ and $b_i = (r_i, s_i I) \in B(I)$

  Then

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$$\begin{array}{lcl}
xz -_N x(y -_N z) & = & x\sum_{i=1}^{n} a_ib_i -_N x(y -_N \sum_{i=1}^{n} a_ib_i) \\
& = & (u,vI)\sum_{i=1}^{n}(m_i,n_iI)(r_i,s_iI) -_N (u,vI)[(p,qI) -_N \sum_{i=1}^{n}(m_i - n_iI)(r_i,s_iI)] \\
& = & (u,vI)\sum_{i=1}^{n}(m_ir_i,(m_is_i+n_ir_i+n_is_i)I) \\
& & -_N[(u,vI)(p,qI) -_N (u,vI)\sum_{i=1}^{n}(m_ir_i,(m_is_i+n_ir_i+n_is_i)I)] \\
& = & \sum_{i=1}^{n}[(u,vI)(m_ir_i,(m_is_i+n_ir_i+n_is_i)I) \\
& & -_N[up,(uq+vp+vq)I -_N (u,vI)(m_ir_i,(m_is_i+n_ir_i+n_is_i)I)]] \\
& = & \sum_{i=1}^{n}[(um_ir_i,(um_is_i+un_ir_i+un_is_i+vm_ir_i+vm_is_i+vn_ir_i+vn_is_i)I) \\
& & -_N[up,(uq+vp+vq)I -_N (um_ir_i,(um_is_i+un_ir_i+un_is_i+vm_ir_i+vm_is_i \\
& & +vn_ir_i+vn_is_i)I)]] \\
\text{Take}\quad \alpha & = & (um_is_i+un_ir_i+un_is_i+vm_ir_i+vm_is_i+vn_ir_i+vn_is_i) \\
\text{and}\quad \beta & = & (uq+vp+vq) \\
\text{Then} & & \\
xz -_N x(y -_N z) & = & \sum_{i=1}^{n}[(um_ir_i,\alpha I) -_N [(up,\beta I) -_N (um_ir_i,\alpha I)]] \\
& = & \sum_{i=1}^{n}[um_ir_i - [up - (um_ir_i)],(\alpha - [\beta - \alpha])I] \\
& = & \sum_{i=1}^{n}[um_ir_i,\alpha I] \\
& = & \sum_{i=1}^{n}[um_ir_i,(um_is_i+un_ir_i+un_is_i+vm_ir_i+vm_is_i+vn_ir_i+vn_is_i)I] \\
& = & \sum_{i=1}^{n}(u,vI)[m_ir_i,(m_is_i+n_ir_i+n_is_i)I] \\
& = & (u,vI)\sum_{i=1}^{n}(m_ir_i,(m_is_i+n_ir_i+n_is_i)I) \\
& = & (u,vI)\sum_{i=1}^{n}(m_i,n_iI)(r_i,s_iI) \\
& = & (u,vI)\sum_{i=1}^{n} a_ib_i \\
& = & xz.
\end{array}$$

- For $z \in A(I)B(I)$ and $x \in X(I)$ we have that

$$\begin{array}{lcl}
zx & = & (\sum_{i=1}^{n} a_ib_i)(u,vI) \\
& = & \sum_{i=1}^{n}(m_i,n_iI)(r_i,s_iI)(u,vI) \\
& = & \sum_{i=1}^{n}(m_ir_i,(m_is_i+n_ir_i+n_is_i)I)(u,vI) \\
& = & \sum_{i=1}^{n}(m_ir_iu,(m_ir_iv+m_is_iu+n_ir_iu+n_is_iu+m_is_iv+n_ir_iv+n_is_iv)I) \in A(I)B(I)
\end{array}$$

  Then we can say that $A(I)B(I)x \subseteq A(I)B(I)$, Hence $A(I)B(I)$ is a neutrosophic ideal.

Definition 3.29. Let $(S,-,\cdot)$ be any near subtraction semigroup. The triple $(S(I),-_N,\cdot)$ is called a neutrosophic near subtraction semigroup. If (a,bI) and (c,dI) are any two elements of $S(I)$ with $a,b,c,d \in S$, we define

$$(a,bI) -_N (c,dI) = (a-c,(b-d)I)$$

and

$$(a,bI)\cdot(c,dI) = (ac,(ad+bc+bd)I)$$

Example 3.30. Let $(S(I),-_N)$ be a neutrosophic subtraction algebra. Then the neutrosophic set $M^{S(I)}$ of all mappings of $S(I)$ into $S(I)$ is a neutrososphic near subtraction semigroup under $-_N$ (Defined point-wisely) and composition of mappings. I.e For all $\phi,\psi \in M^{S(I)}$ we define

$$(\phi -_N \psi)(a,bI) = \phi(a,bI) -_N \psi(a,bI)$$

and

$$(\phi o\psi)(a,bI) = \phi o(\psi(a,bI)).$$

$M^{S(I)}$ is not a neutrosophic subtraction semigroup, because if $\phi_{(0,I)} : M^{S(I)} \longrightarrow M^{S(I)}$ is given by $\phi_{(0,I)}(\alpha) = (0,I)$ for all $\alpha \in M^{S(I)}$ ($\phi_{(0,I)}$ is a constant map). Then for any $\rho,\beta \in M^{S(I)}$,
$\phi_{(0,I)} = \phi_{(0,I)}o(\rho -_N \beta) \neq \phi_{(0,I)}o\rho -_N \phi_{(0,I)}o\beta = (0,0)$

Proposition 3.31. *Let $(S(I),-_N,\cdot)$ be a neutrosophic near subtraction semigroup. Then $S(I)$ is a near subtraction semigroup.*

Proof: Follows similar approach as the proof of Proposition 3.14.

Example 3.32. Let $X(I) = \{(0,0),(a,0),(b,0),(1,0),(0,aI),(0,bI),(0,I)\}$. For all $(u,vI) \in X(I)$ with $u,v \in X$. If $u\cdot v = u$. Then $(S(I),-_N,\cdot)$ is a neutrosophic near subtraction semigroup.

Proposition 3.33. *Let $X(I)$ be a neutrosophic left near subtraction semigroup. If $A(I)$ and $B(I)$ are any two neutrosophic ideals of $X(I)$ then $(A(I):B(I)) = \{x \in X(I) : xB(I) \subseteq A(I)\}$ is a neutrosophic left ideal of $X(I)$.*

Proof: Let $a = (m, nI), b = (r, sI) \in (A(I) : B(I))$ and let $x = (u, vI) \in X(I)$.

To show that $A(I) : B(I)$ is a neutrosophic left ideal of the neutrosophic near subtraction semigroup $X(I)$.

1. $A(I) : B(I) \neq \emptyset$ since $A(I)$ and $B(I)$ are both neutrosophic ideals of $X(I)$.
2. We shall show that $a -_N xb \in (A(I) : B(I))$. To see this it suffices to show that $(a -_N xb)B(I) \subseteq A(I)$. Let $k = (p, qI) \in B(I)$ be an arbitrary element. Then since $a = (m, nI), b = (r, sI) \in (A(I) : B(I))$, we have $ak = (mp, (mq + np + nq)I), bk = (rp, (rq + sp + sq)I) \in A(I)$.
   Since $A(I)$ is a neutrosophic ideal of $X(I)$, we have
   $$x(bk) = (u, vI)(rp, (rq + sp + sq)I) = (urp, (urq + usp + usq + vrp + vrq + vsp + vsq)I) \in A(I)$$
   as well. Thus $(a -_N xb)k = ak -_N x(bk) \in A(I)$ for any $k \in B(I)$, and hence we obtain
   $(a -_N xb)B(I) \subseteq A(I)$. By the definition of $(A(I) : B(I))$, we have $a -_N xb \in (A(I) : B(I))$.

Hence $(A(I) : B(I))$ is a neutrosophic left ideal of the neutrosophic near subtraction semigroup $X(I)$. In a right neutrosophic near subtraction semigroup $S(I)$, $(0, 0)(u, vI) = (0, 0)$ for all $(u, vI) \in S(I)$. But $(u, vI)(0, 0I)$ need not be equal to $(0, 0)$, for $(u, vI) \in S(I)$. To this end, we have the following definition:

Definition 3.34. Let $(S(I), -_N, \cdot)$ be right neutrosophic near subtraction semigroup. $S(I)$ is said to be zero-symmetric if $S(I) = S_{(0,0)}(I)$, where

$$S_{(0,0)}(I) = \{(u, vI) \in S(I) : (u, vI)(0, 0) = (0, 0)\}.$$

Example 3.35. Let $S(I) = \{(0, 0), (1, 0), (0, I)\}$, define $'-'_N$ and $'\cdot'$ as shown below:

Table 4: (i) Cayley table for the binary operation of " $-_N$ " and (ii) Cayley table for the binary operation of " . "

(i)

$-_N$	$(0,0)$	$(1,0)$	$(0,I)$
$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
$(1,0)$	$(1,0)$	$(0,0)$	$(1,0)$
$(0,I)$	$(0,I)$	$(0,I)$	$(0,0)$

(ii)

$\cdot$	$(0,0)$	$(1,0)$	$(0,I)$
$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
$(1,0)$	$(0,0)$	$(1,0)$	$(0,0)$
$(0,I)$	$(0,0)$	$(0,0)$	$(0,I)$

Then $(S(I), -_N, \cdot)$ is a zero symmetric right neutrosophic near subtraction semigroup.

Definition 3.36. Let $X(I)$ be a neutrosophic subtraction semigroup and let $A(I)$ be a neutrosophic ideal of $X(I)$. The set $X(I)/A(I)$ is defined by

$$X(I)/A(I) = \{(u, vI) + A(I) : (u, vI) \in X(I)\}$$

For all $(u, vI)+A(I), (r, sI)+A(I) \in X(I)/A(I)$, we define subtraction and multiplication in $N(I)/A(I)$ as follows:

$$((u, vI) + A(I)) \ominus_N ((r, sI) + A(I)) = (u, vI) -_N (r, sI) + A(I) = (u - r, (v - s)I) + A(I),$$

$$((u, vI) + A(I)) \odot ((r, sI) + A(I)) = (u, vI)(r, sI) + A(I) = (ur, (us + vr + vs)I) + A(I)$$

It can be shown that $\ominus_N$ and $\odot$ are well-defined on $N(I)/A(I)$ and the triple $(N(I)/A(I), \ominus_N, \odot)$ is a neutrosophic subtraction semigroup called neutrosophic quotient subtraction semigroup or neutrosophic factored subtraction semigroup. provided X/A is a subtraction semigroup.

Definition 3.37. Let $(A(I), -_N, \cdot)$ and $(B(I), -'_N, \cdot')$ be two neutrosophic subtraction semigroups. A mapping $\phi : A(I) \longrightarrow B(I)$ is called a neutrosophic subtraction semigroup homomorphism if the following conditions hold:

1. $\phi((a, bI) -_N (c, dI)) = \phi((a, bI)) -'_N \phi((c, dI)), \forall (a, bI), (c, dI) \in A(I)$.
2. $\phi((a, bI) \cdot (c, dI)) = \phi((a, bI)) \cdot' \phi((c, dI)), \forall (a, bI), (c, dI) \in A(I)$.
3. $\phi((0, I)) = (0, I)$

Epimorphism, monomorphism, isomorphism, endomorphism and automorphism of ϕ are similarly defined as in the classical cases.

Definition 3.38. Let $\phi : A(I) \longrightarrow B(I)$ be a neutrosophic subtraction semigroup homomorphism.

1. $Ker\phi = \{(a, bI) \in A(I) : \phi((a, bI)) = (0, 0)\}$.
2. $Im\phi = \{\phi((a, bI)) \in B(I) : (a, bI) \in A(I)\}$.

Example 3.39. Let $A(I)$ and $B(I)$ be two neutrosophic subtraction semigroup. Let $\phi : A(I)\times B(I) \longrightarrow A(I)$ be a mapping defined by $\phi(a, b) = a$ and let $\rho : A(I) \times B(I) \longrightarrow B(I)$ be a mapping defined by $\rho(a, b) = b$ for all $(a, b) \in A(I) \times B(I)$. Then ϕ and ρ are neutrosophic subtraction semigroup homomorphisms.

Example 3.40. Let $X(I) = \{(0, 0), (a, 0), (b, 0), (1, 0), (0, aI), (0, bI), (0, I)\}$ be a neutrosophic subtraction semigroup. Let $\phi : X(I) \times X(I) \longrightarrow X(I)$ be a neutrosophic subtraction semigroup homomorphism defined by $\phi(p, q) = p$ for all $p, q \in X(I)$. Then

1. $Im\phi = \{(0, 0), (a, 0), (b, 0), (1, 0), (0, aI), (0, bI), (0, I)\}$ which is a neutrosophic subtraction subsemigroup.
2. Also,

$$Ker\phi = \{((0, 0), (0, 0)), ((0, 0), (a, 0)), ((0, 0), (b, 0)), ((0, 0), (1, 0)), ((0, 0), (0, aI)), ((0, 0), (1, bI)), ((0, 0), (0, I))\}$$

which is just a subtraction subsemigroup, not a neutrosophic subtraction semigroup and equally not a neutrosophic ideal.

Example 3.40 leads to the following general result.

Proposition 3.41. *Let $\phi : A(I) \longrightarrow B(I)$ be a neutrosophic subtraction semigroup homomorphism. Then*

1. *$Im\phi$ is a neutrosophic subtraction subsemigroup of $B(I)$.*
2. *$Ker\phi$ is a subtraction subsemigroup of $A(I)$.*
3. *$Ker\phi$ is not a neutrosophic subtraction subsemigroup of $A(I)$.*
4. *$Ker\phi$ is not a neutrosophic ideal of $A(I)$*

Proof:

1. Same as in classical case.
2. By the definition of neutrosophic subtraction semigroup homomorphisms, we know that $\phi(I) = I$. Then, for any arbitrary $(x, yI) \in A(I)$ where $x, y \in A$, it follows that $(x, yI) \in Ker\phi$ if and only if $y = 0$ that is only elements of the form $(x, 0) \in A(I)$ can be in the kernel of ϕ. Then $Ker\phi$ is a subtraction subsemigroup of $A(I)$.
3. Since $\phi(I) = I$, it follows that $I \notin Ker\phi$ and the result follows.
4. Since $Ker\phi$ is a subtraction semigroup of $A(I)$ and not a neutrosophic subtraction semigroup of $A(I)$, it follows that $Ker\phi$ cannot be a neutrosophic ideal of A.

4 Conclusion

In this paper, we have studied subtraction algebra and subtraction semigroup in the neutrosophic environment. Their basic properties have been extended and established in the neutrosophic environment. We hope to study and establish more advanced properties of subtraction algebra and subtractions semigroup in the neutrosophic environment in our future papers.

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Thinking on Thinking: The Elementary forms of Mental Life Neutrosophical representation as enabling cognitive heuristics

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Abstract

Beyond the predominant paradigm of an essentially rational human cognition, based on the classical binary logic, we want to propose some reflections that are organized around the intuition that the representations we have of the world are weighted with appreciations, for example affective ones. resulting from our integration into a social environment. We see these connotations as essentially ternary in nature, depending on the concepts underlying neutrosophy: either positive, negative or neutral. This form of representation would then influence the very nature of the cognitive process, which in complex real-world situations, has to deal with problems of a combinatorial nature leading to a number of cases too large for our abilities. Forced to proceed by shortcuts on the basis of heuristics, cognition would use these assessments of the representations it manipulates to decide whether partial solutions are attractive for solving the problem or on the contrary are judged negative and are then quickly rejected. There is still the case of a neutral weighting that allows processing to continue. Thus a neutrosophical conception of our representations of the world explains how our cognition functions in its treatment of combinatorial problems in the form of producing processing accelerating heuristics, both in terms of partial solutions selection and processing optimization.

Keywords: Cognition, heuristics, neutrosophy, value judgment, sentiment

1. Introduction

This article offers some thoughts on human cognition, but also makes the link with artificial intelligence. Here, it is only the beginning of an opening, based on the idea that a certain way of representing information would influence how cognition unfolds. It is not an idea demonstrated by experiments, or even an elaborate theory, but rather an intuition that nevertheless has a general explanatory value.

Thus this text is more a philosophical reflection, an abstract one, the elaboration of a hypothesis and the beginning of its discussion. This hypothesis concerns the kernel of cognition and goes beyond the approach that the intelligence is above all logic, such as the classical binary logic (known as boolean logic and the logic of first order predicates). Although this vision is predominant, it may give too much importance to a deterministic and dialectical logical aspect.

2. Cognition beyond logic

The main characteristic of human beings is their ability to foresee part of their immediate future in their environment, and thus to increase their chances of survival. In this way, they can both reduce the risks of the various dangers surrounding them and make the best use of available resources. This is possible, in addition to its perceptual system, through its cognitive system (the brain) capable of processing and retaining information.

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Human cognition is the rapid search for (sub-optimal) solutions through cognitive mechanisms based on available information. The relevance of this task depends on the time and effort invested, and especially on the quality of the logical processing (intelligence) but also on the quality and quantity of the information mobilized. However, cognition is mainly studied in the Western world in terms of logic, or rationality, leaving in shadow the mechanisms for selecting information but also the forms of organization of this information and the basic forms of processing, manipulation and storage associated. In particular, little is said about how to choose one solution over another, and which may not be (sometimes by far) rationally the best.

In philosophy, Boudon refined the model taken from the economic world of the rational actor. According to the economist Becker, awarded the Nobel Prize for this paradigm, rationality would first of all be in action [1]: what a person does would be the result of a rational calculation of benefits versus costs among the possible solutions. Thus, by his reflection a person chooses the action which carries out for him the maximum of advantages for the minimum of costs (disadvantages). It is a variant of the popular law of the least effort, which perhaps however comes closer to reality.

Boudon extends the rationality [2] from a purely economic level to various aspects including those emotional, and among them those that are in fact social. The expected benefit of an action can also be immaterial: for example, emotional or symbolic. Finally, the reason for rationality can be supplemented by what the actor considers to be good reasons for him [3]: valid reasons in his own eyes.

To revisit this approach, without being significantly contradictory, here we will consider some of these elementary forms of mental life, in order of complexity, and think on their potential influence on cognition, how it could be modeled. So by looking at the how, we can better see the thinking mechanism as a whole.

3. The basis for cognition: representations

Cognition is a processing of data according to largely logical operations aiming at a certain objective: obtaining a solution to a problem. So not every problem necessarily has a solution, especially if we lack a determining element, then it is an indeterminate problem in our world of knowledge. A sub-optimal solution can however often be found, and in the case where the problem is determined, sometimes at the cost of a very great distance from the optimal solution (but most of the times not). In the worst case it is always possible to act randomly, that is to say in a sort of blind, non-rational way. For example, between two alternatives that do not seem to differ significantly from each other in terms of the cost-benefit ratio, it is possible to choose by flipping a coin. However, this is the worst solution: it is better to think, and if possible well and sufficiently.

The solution chosen, and with it the possible solutions that can be envisaged, therefore depend on the data available, then on those informations (representations) that are actually used. So, the more or less rational treatment of this whole produces in the long run one or several solutions of various qualities.

Often the actor, acting in a so-called autonomous way, therefore not a predetermined one, will choose between several solutions that he found according to some criteria of choice specific to him, according to his own good reasons.

The idea proposed here is limited to considering that these choice criteria are partly of a certain nature, and that these types of choices are made throughout cognitive processes. Even that they decide to stop the search for more solutions, considering the ones foreseen (and not yet fully developed, particularly in terms of benefits and costs) are sufficient. These are not solutions investigated in detail, but just crude preliminary views.

Before going further in the discussion, let us consider the data on which the thinking is made. A specific word is used to designate them: they are mental representations [4], and they form as a whole our vision of the world [5], i.e. all our knowledge about the world, initially understood in a synthetic way. If the representations are the processed information, the global conception of the vision of the world is also a preferential orientation in the space of possible solutions.

A representation is quite simply what in our eyes characterizes a thing, or also an abstract concept. Our representations are constructed from our senses, in compatibility with the representations we already have before, but also from the messages we receive, for example from other people. Representations are in fact the only information we have at our disposal. They are our representation of reality, but not objective ones, although it appears as such to us. Our worldview is subjective, and so are our representations, but yet they are also largely shared with the other people with whom we interact the most. These are collective representations to groups of people, also called social representations [6]. They are also social in another sense: in them are hidden social aspects [7]. Thus if the rational actor evolves

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according to his selfish interest he is never less caught up in a social world.

Deriving more or less directly from this social coloring, representations also have an evaluative component, they are seen with appreciations (affects, sentiments): what they represent is more or less liked or valuated by us. Every representation inseparably carries a value judgment: I like this, or I don't like it. To varying degrees, the appreciation/affect may be more or less strong, positive or negative (see Fig. 1). He may also be more or less consciously absent, i.e. neutral, or only in appearance. The key is to consider this neutrality as such.

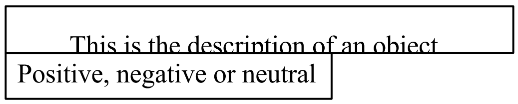

Figure **Error! No sequence specified.**. Concept of colored representation by an appreciation

4. Cognition and language

As our representations can be partly formed or adapted by information that we receive, finally off our senses and from others. There is transmission of information between people in an intersubjective way. An affective tingling of information can occur depending on the sender or the transmitter, as well as the content of the message, in particular the objects that are represented there.

Thus our representations are of such a nature that they can be transmitted, at least in part, in an intersubjective manner, allowing their sharing, which forms the culture of the group [8]. Messages (and also about representations) can be emitted via different means by a person, including gesture and speech. Thus they are mostly symbolic because of their representative nature, but also because they are transmitted by symbols. Languages being precisely a kind of organization (and coding) of symbols [9]. Remember that the appreciative and affective aspect of representations is constructed in this intersubjectivity.

So we can say that a representation is an organized collection of symbols with an emotional-affective-valuative tincture.

As a result, the cognitive treatment of representations potentially induces by their nature also an evaluative aspect of an emotional-affective-valuative type.

Before going further in the conceptualization of representations, and that which results from it on cognition we propose a detour by artificial intelligence on the basis of the symbolic content of linguistic-type representations, that is to say that they are representative in a language composed of symbols, thus provided with a semantic content carried by an arbitrary container, but conventional (shared) in a group. A strong link exists between culture and language, the first being mainly shared symbolic representations transmitted intersubjectively via symbols that the second organizes into a set of units called words (of the vocabulary representing their semantics).

5. Artificial intelligence and compression

A major axis of artificial intelligence, the computerized processing of semantic information, is constituted by learning systems. Their functionality is gradually achieved by an evolutionary process which is called learning and which consists essentially in the treatment of examples to be imitated, accompanied by the correction of the result produced. Living beings also develop their capacities by learning, with for humans in addition the transmission of symbolic information in a preponderant way in the last phases of development.

Since a few years the culmination of artificial intelligence is neural networks with so-called deep learning. In fact it is the networks that are deep, composed of many layers of neurons, and not learning that remains a classic reinforcement, as a reward in case of the expected result achieved. Such systems are able to perform complex tasks that are not achieved by algorithmic ways (detailed and deterministic designation of the steps to be performed). In particular, they manage to reproduce the major and complex aspects of living beings: sound and visual perception. Speech recognition makes it possible to dictate texts to a computer as one would to a secretary, and artificial vision makes it possible to identify objects, to classify them according to various categories with an accuracy rate exceeding that of a human.

Although these systems were originally inspired by living nature (neurons), their internal functioning is quite different.

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However some analogies can be made. The most disturbing one is that they work on representations, and even in some cases linguistic representations. They process representations by representing them with other, more representative, representations. The result is a process of compressing information to keep only the essential semantics appropriate for the task at hand, eliminating the rest which in this context is considered as disruptive noise [10]. Part of the argument in favor of a compression as the essential type of operation lies in the fact that the learning mechanism does not consist only in isolating what is important and that it is necessary to reproduce, but also in a dual way to reject all that is not relevant. The challenge is not just to find the task-significant information but rather to selectively remove everything that is not useful, and that only brings a disturbing noise. These aspects are at the heart of the problem of generalization and its opposite, over-learning, which reproduces isolated examples too closely, losing their character of generality.

Some systems are capable of processing the natural language of humans, such as a common language like English, and can perform translations into other languages with results close to a human translator. Although partial, a semantic information is treated there (that is, without any conscience), represented in linguistic form (by the equivalent of words forming a dictionary allowing the representation as DTM = Document Text Matrix [11]).

Thus, lessons from knowledge gained in one field, the living or the artificial, can be used to better understand the other.

In this way our proposal extends to both.

Recent work in the field of understanding how deep learning works is expanding our idea. While remaining at an intuitive level, here are some elements taken from a new theory [10, quoted above] explaining what happens in such artificial neural networks that were previously considered as black boxes (meaning it is impossible to know how they work internally).

An innocuous remark makes it possible to concretize at the most minimal what is fundamentally a learning which progresses: "It somehow smells right". Yet this point is essential as the beginning of the reasoning behind our approach. The learning system must at least have a sense of what is going in the right direction: this is that allows the principle of reinforcement.

Let us now turn to the last piece of our puzzle which will allow us to put everything in place to finally form the overall picture.

6. Conditioning cognition: forms of representation

This element has already been announced: if the representations are of a certain form, then their treatment is a priori organized for this type of form. The data format conditions the processor, and the processing.

Different forms of representation are possible on the basis of different conceptions of human thought. We see them as elementary forms of representations, which produce elementary forms of thought, and thus correspond to elementary forms of mental life, hence our title in reference to one of the founding books of sociology, for us the most emblematic: “The Elementary Forms of the Religious Life” by Durkheim [12].

During its development, for every human being, cognitive abilities increase, in fact in stages due to different conceptions according to the stages of mental development identified by Piaget [13] in children.

We are inspired by him here without however taking up his points, considering just what seems to us to be the most elementary. In a first stage, by babies, there is in some way no consciousness, no self or world. Then there's the self. Then there's the world, then others. We can call this an ontological conception: it is, it exists, there is X, in short it can be denoted simply by a name, like X. However, this elementary form of thought does not allow real cognition, neither real information processing because there are only constants and no variables. Nothing can change. This form of thinking is very common among all adults, it is the most elementary because it leads to a passive vision and not that of an active social actor. It does not lead to action, to carrying out acts because until the notion of time is absent: there is no future. So no possibility of change, therefore of action. This is, in a way, ad æternam. The human being is then not in touch with these elements of his environment. He can't think on them, and therefore act on them.

The next stage is characterized by the "there is", and the supplementary "there is not", with as transition link with the previous stage: "it is the same", a label is given, a name like X. So far, everything is identical with the previous elementary form of thinking that now will be extended. This can be seen as a lexical stage, different things exist. But the "there is" is now completed by the "there is not", which is its opposite. In addition to the presence of X, there is now also the non-presence of X: noted for example not-X, it is the absence of X. Not to be confused with nothing (which is more complex): X may be present or absent. The "there is no" is characteristic of disappearance, for example when closing the eyes or by occultation, one object hiding another. These two possibilities, these alternatives, are the basis of cognition, and allow choice and therefore action through the fact that a preference becomes possible: either I prefer there is X, or I prefer there is no X. Then autonomy appears. And indeed the valuation or affect too: "I like" or "I don't like", and it goes with it together.

The stages described here are not as distinct as those of Piaget, they overlap, include and extend. The "there is no" is opposed to the "there is" forming the opposite. Thus the binary appears and the logic of the same name also: either "there is", or "there is not": X or not-X, one and the other being mutually exclusive.

Although these conceptions of thought are the prerogative of the infant, they continue to exist in adults, and this in a preponderant way compared to other more evolved forms of thought. Thus, at first glance, one merely notices the existence of something without thinking anything about it. There is this and that and other things: a perception of the environment, a representation of a situation as a collection of objects. Our other most frequent and fundamental conception is opposition: there is or there is not. What also gives one thing and its opposite: day and night, hot and cold, big and small ... The importance of this simplifying binary conception of two situations sliced diametrically away in opposite is the most prominent form of mental life. It is the emblematic form of a choice. Almost all popular proverbs are representative of such an opposition. Cognition is then limited to the choice between two alternatives, whose basic variant is to do this or nothing (coming before doing this or doing that, which is actually composed of two such alternatives producing ipso-facto the triplet doing this or doing that or doing nothing, with the particular case of that being the opposite of this). We will then arrive at this ternary form of thought, following the end of the discussion already started about appreciation, affect, or feeling.

Without being possible to claim that it is after, but here it is after in our explanation, there is the appreciation, affect or feeling. Every thought, or representation, is completed by an appreciation that tints it: positive or negative, meaning I like that or I don't like that. Although it may seem binary, in our opinion it is totally different. First this is associated with the X information: I like X. Actually, it's: X plus I like X or I don't like X. In the following binary version of thought, there is X or there is not X and moreover I like X or I do not like X. The variations of X and its related appreciation can be joint or separate. This emotional aspect, this feeling, can be more or less intense, pronounced. It can also be weak, weak enough that under certain circumstances it may not appear either consciously or unconsciously. So we have three states to deal with, although it is in fact a gradation (that is precisely a more evolved representation that we will see next). But since the learning treatment consists in representing representations by more representative ones and in fact more reduced (compressed) representations, then the gradation is sometimes reduced to these three states (either positive, or neutral or negative).

Then this three-state design is called neutrosophical, we will come back to it in the next section.

We therefore consider that a representation is an description of X accompanied by an appreciation about X, at least in three states: positive, negative or neutral, either I like X, or I don't like X, or I am indifferent at X in this simple representation stage (the most elementary).

Finally it is this stage of representation, with an appreciation-affect-feeling consisting of three states, which is characteristic of human cognition in our opinion. At least one such hypothesis seems to us to be reasonably formulated on the basis of the elements partially presented here.

It follows that this type of representation conditions cognition.

Before discussing it let us show what other concepts of representation are possible in a human context that is social and therefore emotional or sentimental. A first one of those representations that associated information with an affect would be a gradation from "I don't like it at all" to "I really like it a lot". This corresponds to a weighting. A variant, perhaps more restrictive, is a quantitative valuative formulation. It's like counting, numbering. Considering quantities: for example, 5 sheep, 8 people ...

The stage stated previously would be in complexity after because a gradation can be also negative, and is continuous. It's actually a probabilistic approach. Note that it allows to represent the three-state variant. However this form with three states possesses a quality of the kind that the least in fact can do more. Some things are not continuous: there is or there is not (or in some context are considered not continuous). Thus in treatment, there is the possibility of choosing a solution, there is the possibility of abandoning a forseen solution, and by default to continue treatment (searching for better). These are not things that can be done halfway, it is either this or that, and this is all that is possible because it is all that can exist. Indeed, this ternary connotation is also binary at first: I continue or I stop my cognition. And then in more detail, I stop because I like or I stop because I don't like (that part of a potential solution). Thus the three states are in fact representative of two binary variables: there is an appreciation or not, and it is positive or negative. It is this aspect (by default) that there is no appreciation that let continue the cognitive process, showing the importance of a neutral representation.

Finally we need to see in more detail what a cognitive process is. It aims to find a solution to a problem, and that is one of the kind of operational research. To find the best solution we must examine all possible alternatives, it is combinatorics, and the reality is that in almost all cases of the real world it is a combinatorial explosion: a very large number of cases. It is impossible to examine all these cases, we must limit ourselves to a few who seem good candidates: those "that seem to smeel good". Cognition is reduced to using heuristics, i.e. indicators of potentially good solutions. For example, in the complex problem of finding the shortest path to a point in a city (the primary problem of operational research), at a crossroads, the road that will be attempted first will be the one whose next crossroads would be closest to the destination, that is using an heuristics (in the simplest example).

So what is the primary heuristic in humans? It's whether he likes it or not.

7. Neutrology as enabling the main heuristic of cognition

As cognition has to solve real-world problems that most often lead to a combinatorial explosion; the number of cases to be treated being far too large for the capacities of our brain (or the time we can devote to it). The only way out for survival (and therefore evolution) was to use shortcuts of thought in cognitive treatment. The worldview is a concentration of our practical knowledge and offers us typical behaviors or thoughts in many circumstances without us having to think much. The representations to be mobilized are probably activated more by our attraction for certain objects than their relevance to the real situation. Finally, cognitive treatment seeks to be minimal to provide a rapid response in urgent cases and also by economy (or laziness). Cognition proceeds as much as possible by heuristics that help to approach good solutions, satisfying in the concrete context. The details are left out at first, and some of the solutions foreseen are selected on the basis of experience and their appeal, or the impression that they induce to be part of the right way. Other solutions are quickly dismissed because being negative. Then cognitive processing in the search for the best path to the solution uses heuristics to not have to explore all possibilities. And the control of this treatment relies on the additional ternary information associated with the representations that indicates our appreciation for this object: positive, negative or neutral.

This three-state logic is that of neutrosophy which represents by a triplet of real numbers between 0 and 1 the belonging to each of the three states true, false, or neutral (or also with the meaning of indeterminate). Instead of a simple probability designating the veracity of an assertion, neutrosophy provides three probabilities (weights or appreciations), those of veracity, falsity and indeterminacy (or neutrality). This is much more general, and allows it to include other types of simpler logical representations, including fuzzy logic [14] and its numerous variants as well as traditional classical binary logic.

Here, we do not use the full capacity of this ternary representation for the decision to continue cognitive treatment, but simply the predominant aspect, if it is significant enough. For the rest of cognition, however, the continuous aspect is considered.

Let us now look in a few points at what neutrosophical representation more precisely is, according to the section we gave in a previous article [15].

8. Some information about neutrosophy

Recent trends in the use of neutrosophy can be found in the reference work edited by Smanradache and Pramanik entitled "New trends in neutrosophical theory and its applications" [16]. In the aforementioned article we also proposed data compression according to the physiological laws of human perception, indicating that the nature of such a compression process is fundamentally based on a three-state representation, which is an additional corroborating element for the theory that neural networks, living or artificial, elaborating their functioning by learning, rely on data compression [10, quoted above].

The inspiration for the neutrosophical representation of reality comes from the philosophy called neutrosophy. This representation is general and makes possible to unify in particular the various (apparently very distinct) variants of logic: classical logic, also called binary logic or boolean logic, fuzzy logic and its numerous varieties, and itself, a three-state logic, characterized by a neutral state [17]. Note that the three-state approach is central in natural language speech processing too, specially for sentiment analysis (which aims to discover a positive, negative or neutral connotation by the sender beyond the semantic content of his message) [18].

In summary, instead of a logical value with two states false (0) or true (1), the neutrosophical approach considers a representation by a triplet (t.i,f) where these three real values t, i and f represent the equivalent of probabilities for truth (t), indetermination or neutral state (i) and falsity (f) respectively. We prefer to speak of membership functions according to the vocabulary used in fuzzy logic. These three values are between 0 and 1. Thus the two classical binary logic values 0 and 1 are represented respectively by (0,0,1) and (1,0,0). Now a simple probability p of having the value true and therefore (1-p) of having the value false is represented by (p,0,1-p). In this particular case the neutrosophical representation mainly brings a general formulation, and thus it also makes possible to represent this conception which it encompasses in its generality.

The operations on neutrosophical triplets, preferably called simple neutrosophical numbers (for single value, in the sense of mono, a single triplet), can be defined in various ways, either using arithmetic operators (e.g. multiplication for the logical AND) or functional operators (such as minimum, maximum, etc). For example the complement of (t,i,f) can be defined as (f,1-i,t), but other conventions may be more appropriate depending on the applications. Let us return now on the preceding case of an operator with two operands, as the logical AND mentioned before, let us consider this time the logical OR, i.e. the union together. For two simple neutrosophical numbers A and B represented by the triplets (t_a, i_a,f_a) and (t_b,i_b,f_b) then their union A OR B will be the triplet (max(t_a,t_b), max(i_a,i_b), min(f_a,f_b)).

Although a neural network is organized according to the learning algorithm, cleverly chosen and parameterized to use internally representations adapted to the problem to be solved, in particular it can perform a referential change, a projection and other operations that can be geometrically illustrated (and which in fact produce appropriate data compression). This autonomous organization is however costly in terms of learning time but also in terms of the quality of the performance produced. If for a given application, it is known that a representation is generally chosen for powerful classical algorithms, then it is highly likely that the network will choose a similar representation relatively close. Indeed often for a specific application it is preferable to start from a network pre-trained on a problem either more general or rather close, which precisely means to start from a relatively appropriate representation.

9. Conclusion

Thinking on cognition reveals the importance of aspects other than the pure logical (binary) treatment that is nevertheless put forward by Western civilization. Without rejecting the successively extended model of the rational actor we propose an intuition that we think can give rise to ideas explaining the cognitive functioning of living beings and that can partly be adapted to artificial intelligence systems based on neural networks with deep learning or other training algorithms.

The rational cognition of the actor, in terms of maximizing his benefits while reducing his costs, which has been extended to consider social aspects presenting not strictly economic benefits, then expanded to include the reasons that the actor himself finds good, is completed in our hypothesis of an evaluation throughout the treatment and for all the information processed. These are our representations, social and shared collectively, which allow us to understand the essential objects and concepts related to the real world. Taken together these representations form our immediate knowledge, which can be mobilized because it is contained in our brain and also constitutes our vision of the world, which is in fact the only apprehension we have of it, and therefore we think it is objective, whereas it is only subjective, including precisely our appreciations.

The form of the representations determines how to process them, and our hypothesis is therefore that a representation of an object includes an appreciation of this object, according to a ternary weighting: positive, negative or often neutral. This corresponds to a neutrosophical representation. From this particular form, and specific to the living, corresponding to an additional valuative, emotional or sentimental content, flows the organization of their processing. Thus cognition is not only logical, but takes into account the imperatives of life in a social world, and propagates the appreciation, emotional or sentimental aspect, throughout the treatment chain, until solutions to the problems to be solved appear, which are in fact marked by values in themselves. In this way, solutions and beginnings of solutions can emerge as a function of experience (rational valuation) and affects (also indirectly contained in experience).

Moreover, this valuable content makes it possible to act on the very mechanism of the search for solutions, which is the basic scheme of operational research, having to consider the combinatorial explosion of the large number of cases to be treated in the universe of possible solutions. These beginnings of solutions are progressively pruned according to the predominance of the positive or the negative (acceptance or rejection of a partial solution) or then in the case of the neutral, the process just continues. So, this evaluation acts as a heuristic for rapid selection of solution elements, and in fact each representation carries within itself a part of these heuristic shortcut elements. In the end, the chosen solution is then naturally characterized by a good acceptance of the actor which contributes to facilitate its implementation.

Thus the aspect of ternary logic of neutrosophy is none other than the orientation for preference by the living cognition, that which allows filtering, retention of relevant characteristics, and thereby data compression in the learning process. Furthermore, in artificial neural networks with deep learning their functioning that can be explained according to this recent theory that the information is compressed to the essential necessary for the considered treatment.

Finally, the question may be asked whether this essential characteristic of an appreciative connotation of representations in the cognition of the living is not perhaps also inherent in any mechanism of learning by reinforcement, thus also in artificial intelligence.

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How we can extend the standard deviation notion with neutrosophic interval and quadruple neutrosophic numbers

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Abstract

During scientific demonstrating of genuine specialized framework we can meet any sort and rate model vulnerability. Its reasons can be incognizance of modelers or information mistake. In this way, characterization of vulnerabilities, as for their sources, recognizes aleatory and epistemic ones. The aleatory vulnerability is an inalienable information variety related with the researched framework or its condition. Epistemic one is a vulnerability that is because of an absence of information on amounts or procedures of the framework or the earth [7]. Right now, we examine fourfold neutrosophic numbers and their potential application for practical displaying of physical frameworks, particularly in the unwavering quality evaluation of engineering structures. Contribution: we propose to extend the notion of standard deviation to by using symbolic quadruple operator.

Keywords: Standard deviation, Neutrosophic Interval, Quadruple Neutrosophic Numbers.

1.Introduction

We all know about uncertainty modelling of various systems, which usually is represented by:

$$X = x' + 1.64s \tag{1}$$

Or

$$X = x' + 1.96s \tag{2}$$

Here, the constants 1.64 or 1.96 can be replaced with k. What we mean is a constant corresponding to bell curve, the number is usually assumed to be 1.96 for 95% acceptance, or 1.64 for 90% acceptance, respectively.

But since s only takes account statistical uncertainty, there is lack of measure for indeterminacy. That is why we suggest to extend from

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$$X = x' + k.\ s \tag{3}$$

To become neutrosophic quadruple numbers.

Before we move to next section, first we would mention other possibility, i.e. by expressing the relation as follow

$(X_L + X_U\ I_N) = k.\ (\sigma_L + \sigma_U\ I_N)$, where I_N is a measure of indeterminacy (4)

Actually, we we need to add some results for various I_N, for example I_N=0,0.1,0.2,0.3,0.4 etc. Nonetheless, because this paper is merely suggesting a conceptual framework, we don't explore it further here. Interested readers are suggested to consult ref. [1-2].

2. A short review on quaternions

We all know the quaternions, but quadruple neutrosophic numbers are different. In quaternions, a+bi + cj + dk you have i^2 = j^2 = k^2 = -1 = ijk, while on quadruple neutrosophic numbers we have:[3]

N = a + bT + cI + dF one has: T^2 = T, I^2 = I, F^2 = F, (5)

where a = known part of N, bT+cI+dF = unknown part of N, with T = degree of truth-membership, I = degree of indeterminate-membership, and F = degree of false-membership, and a, b, c, d are real (or complex) numbers, and an absorption law defined depending on expert and on application (so it varies); if we consider for example the neutrosophic order T > I > F, then the stronger absorbs the weaker, i.e.

TI = T, TF = T, and IF = I, TIF = T. (6)

Other orders can also be employed, for example T < I < F: (see book [1], at page 186.) Other interpretations can be given to T, I, F upon each application.

3. Application: statistical uncertainty and beyond

Designers must arrangement with dangers and vulnerabilities as a piece of their expert work and, specifically, vulnerabilities are intrinsic to building models. Models assume a focal job in designing. Models regularly speak to a dynamic and admired rendition of the scientific properties of an objective. Utilizing models, specialists can explore and gain comprehension of how an article or wonder will perform under specified conditions.[8]

Furthermore, according to Murphy & Gardoni & Harris Jr, which can be rephrased as follows: "For engineers, managing danger and vulnerability is a significant piece of their expert work. Vulnerabilities are associated with understanding the normal world, for example, knowing whether a specific occasion will happen, and in knowing the presentation of building works, for example, the conduct and reaction of a structure or foundation, the fluctuation in material properties (e.g., attributes of soil, steel, or solid), geometry, and outer limit conditions (e.g., loads or physical

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limitations). Such vulnerabilities produce dangers. In the standard record chance is the result of a lot of potential outcomes and their related probabilities of event (Kaplan and Gerrick 1981), where the probabilities measure the probability of event of the potential outcomes considering the hidden vulnerabilities. One significant utilization of models in designing danger investigation is to measure the probability or likelihood of the event of specific occasions or a lot of outcomes. Such models are regularly alluded to as probabilistic models to feature their specific capacity to represent and measure vulnerabilities."[8]

Uncertainties come in many forms, for example:

> "The uncertainties in developing a model are:
>
> • Model Inexactness. This kind of vulnerability emerges when approximations are presented in the plan of a model. There are two basic issues that may emerge: blunder as the model (e.g., a straight articulation is utilized when the real connection is nonlinear), and missing factors (i.e., the model contains just a subset of the factors that influence the amount of intrigue). …
>
> • Mistaken Assumptions. Models depend on a series of expectations. Vulnerabilities may be related with the legitimacy of such suspicions (e.g., issues emerge when a model accept typicality or homoskedasticity when these suppositions are disregarded).
>
> • Measurement Error. The parameters in a model are commonly aligned utilizing an example of the deliberate amounts of intrigue and the fundamental factors considered in the model. These watched qualities, in any case, could be inaccurate because of blunders in the estimation gadgets or systems, which at that point prompts mistakes in the alignment procedure. …
>
> • Statistical Uncertainty. Factual vulnerability emerges from the scantiness of information used to align a model. Specifically, the exactness of one's derivations relies upon the perception test size. The littler the example size, the bigger is the vulnerability in the evaluated estimations of the parameters. … However, the confidence in the model would probably increment on the off chance that it was adjusted utilizing one thousand examples. The factual vulnerability catches our level of confidence in a model considering the information used to adjust the model."[8]

With regards to statistical uncertainty, according to Ditlevsen and Madsen, which can rephrased as follows: "It is the reason for any estimating technique to produce data about an amount identified with the object of estimation. In the event that the amount is of a fluctuating nature with the goal that it requires a probabilistic model for its depiction, the estimating technique must make it conceivable to define quantitative data about the parameters of the picked probabilistic model. Clearly a deliberate estimation of a solitary result of a non-degenerate arbitrary variable X just is sufficient for giving a rough gauge of the mean estimation of X and is insufficient for giving any data about the standard deviation of X. In any case, if an example of X is given, that is, whenever estimated estimations of a specific number of freely produced results of X are given, these qualities can be utilized for figuring gauges for all parameters of the model. The reasons that such an estimation from an example of X is conceivable and bodes well are to be found in the numerical likelihood hypothesis. The most rudimentary ideas and rules of the hypothesis of insights are thought to be known to the peruser. To delineate the job of the measurable ideas in the unwavering quality examination it is beneficial to rehash the most fundamental highlights of the depiction of the data that an example of X of size n contains

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about the mean worth E[X]. It is sufficient for our motivation to make the streamlining supposition that X has a known standard deviation D[X] = σ."[5]

Now, it seems possible to extend it further to include not only statistical uncertainty but also modelling error etc. It can be a good application of Quadruple Neutrosophic Numbers.

4. Towards an improved model of standard deviation

Few days ago, we just got an idea regarding application of symbolic Neutrosophic quadruple numbers, where we can use it to extend the notion of *standard deviation.*

As we know usually people wrote:

$$X' = x + k.\sigma \tag{7}$$

Where X mean observation, σ standard deviation, and k is usually a constant to be determined by statistical bell curve, for example 1.64 for 95% accuracy.

We can extend it by using symbolic quadruple operator:

$$X' = x \pm (k.\sigma + m.i + n.f) \tag{8}$$

Where X' stands for actual prediction from a set of observed x data, σ is standard deviation, i is indeterminacy and f falsefood. That way modelling error (falsehood) and indeterminacy can be accounted for.

Alternatively, one can write a better expression:

$$X' = x \pm (T.\sigma + I.\sigma + F.\sigma) \tag{9}$$

where T = the truth degree of s (standard deviation), I = degree of indeterminacy about s, and F = degree of falsehood about s.

A slightly more general expression is the following:

$$X' = x \pm a\,(T.\sigma + I.\sigma + F.\sigma) \tag{10}$$

where T = the truth degree of s (standard deviation), I = degree of indeterminacy about s, and F = degree of falsehood about s.

Or

$$X' = x \pm (a.T.\sigma + b.I.\sigma + c.F.\sigma) \tag{11}$$

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where T = the truth degree of s (standard deviation), I = degree of indeterminacy about s, and F = degree of falsehood about s, and a, b, c are constants to be determined.

That way we reintroduce quadruple Neutrosophic numbers into the whole of statistics estimate.

For further use in engineering fields especially in reliability methods, readers can consult [5-7].

5. Conclusion

In this paper, we reviewed existing use of standard deviation in various fields of science including engineering, and then we consider a plausible extension of standad deviation based on the notion of quadruple neutrosophic numbers. More investigation is recommended.

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Refined Neutrosophic Rings I

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Abstract

The study of refined neutrosophic rings is the objective of this paper. Substructures of refined neutrosophic rings and their elementary properties are presented. It is shown that every refined neutrosophic ring is a ring.

Keywords: Neutrosophy, refined neutrosophic set, refined neutrosophic group, refined neutrosophic ring.

1 Introduction

The notion of neutrosophic ring $R(I)$ generated by the ring R and the indeterminacy component I was introduced for the first time in the literature by Vasantha Kandasamy and Smarandache in.[12] Since then, further studies have been carried out on neutrosophic ring, neutrosophic nearring and neutrosophic hyperring see.[1,3,4,6–8] Recently, Smarandache[10] introduced the notion of refined neutrosophic logic and neutrosophic set with the splitting of the neutrosophic components $< T, I, F >$ into the form $< T_1, T_2, \ldots, T_p; I_1, I_2, \ldots, I_r; F_1, F_2, \ldots, F_s >$ where T_i, I_i, F_i can be made to represent different logical notions and concepts. In,[11] Smarandache introduced refined neutrosophic numbers in the form $(a, b_1I_1, b_2I_2, \ldots, b_nI_n)$ where $a, b_1, b_2, \ldots, b_n \in \mathbb{R}$ or $\mathbb{C}$. The concept of refined neutrosophic algebraic structures was introduced by Agboola in[5] and in particular, refined neutrosophic groups and their substructures were studied. The present paper is devoted to the study of refined neutrosophic rings and their substructures. It is shown that every refined neutrosophic ring is a ring.

For the purposes of this paper, it will be assumed that I splits into two indeterminacies I_1 [**contradiction** (true (T) and false (F))] and I_2 [**ignorance** (true (T) or false (F))]. It then follows logically that:

$$I_1I_1 = I_1^2 = I_1, \quad (1)$$
$$I_2I_2 = I_2^2 = I_2, \text{and} \quad (2)$$
$$I_1I_2 = I_2I_1 = I_1. \quad (3)$$

If X is any nonempty set, then the set

$$X(I_1, I_2) =< X, I_1, I_2 >= \{(x, yI_1, zI_2) : x, y, z \in X\} \quad (4)$$

is called a refined neutrosophic set generated by X, I_1 and I_2. For $x, y, z \in X$, any element of $X(I_1, I_2)$ is of the form (x, yI_1, zI_2) and it is called a refined neutrosophic element.

If $+$ and . are the usual addition and multiplication of numbers, then I_k with $k = 1, 2$ have the following properties:

(1) $I_k + I_k + \cdots + I_k = nI_k$.

(2) $I_k + (-I_k) = 0$.

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(3) $I_k.I_k.\cdots.I_k = I_k^n = I_k$ for all positive integer $n > 1$.

(4) $0.I_k = 0$.

(5) I_k^{-1} is undefined with respect to multiplication and therefore does not exist.

For any two elements $(a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$, we define

$$\begin{aligned} (a, bI_1, cI_2) + (d, eI_1, fI_2) &= (a + d, (b + e)I_1, (c + f)I_2), && (5)\\ (a, bI_1, cI_2).(d, eI_1, fI_2) &= (ad, (ae + bd + be + bf + ce)I_1, \\ &\quad (af + cd + cf)I_2). && (6) \end{aligned}$$

For any algebraic structure $(X, *)$, the couple $(X(I_1, I_2), *)$ is called a refined neutrosophic algebraic structure and it is named according to the laws (axioms) satisfied by $*$. For instance, if $(X, *)$ is a group, then $(X(I_1, I_2), *)$ is called a refined neutrosophic group generated by X, I_1, I_2.

Given any two refined neutrosophic algebraic structures $(X(I_1, I_2), *)$ and $(Y(I_1, I_2), *')$, the mapping $\phi : (X(I_1, I_2), *) \to (Y(I_1, I_2), *')$ is called a neutrosophic homomorphism if the following conditions hold:

(1) $\phi((a, bI_1, cI_2) * (d, eI_1, fI_2)) = \phi((a, bI_1, cI_2)) *' \phi((d, eI_1, fI_2)) \quad \forall (a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$.

(2) $\phi(I_k) = I_k$ for $k = 1, 2$.

Example 1.1. [5] Let $\mathbb{Z}_2(I_1, I_2) = \{(0, 0, 0), (1, 0, 0), (0, I_1, 0), (0, 0, I_2), (0, I_1, I_2), (1, I_1, 0), (1, 0, I_2), (1, I_1, I_2)\}$. Then $(\mathbb{Z}_2(I_1, I_2), +)$ is a commutative refined neutrosophic group of integers modulo 2. Generally for a positive integer $n \geq 2$, $(\mathbb{Z}_n(I_1, I_2), +)$ is a finite commutative refined neutrosophic group of integers modulo n.

Example 1.2. [5] Let $(G(I_1, I_2), *)$ and and $(H(I_1, I_2), *')$ be two refined neutrosophic groups. Let $\phi : G(I_1, I_2) \times H(I_1, I_2) \to G(I_1, I_2)$ be a mapping defined by $\phi(x, y) = x$ and let $\psi : G(I_1, I_2) \times H(I_1, I_2) \to H(I_1, I_2)$ be a mapping defined by $\psi(x, y) = y$. Then ϕ and ψ are refined neutrosophic group homomorphisms.

For more details about refined neutrosophic sets, refined neutrosophic numbers and refined neutrosophic groups, we refer to.[5,10,11]

2 Main Results

Definition 2.1. Let $(R, +, .)$ be any ring. The abstract system $(R(I_1, I_2), +, .)$ is called a refined neutrosophic ring generated by R, I_1, I_2.

The abstract system $(R(I_1, I_2), +, .)$ is called a commutative refined neutrosophic ring if for all $x, y \in R(I_1, I_2)$, we have $xy = yx$. If there exists an element $e = (1, 0, 0) \in R(I_1, I_2)$ such that $ex = xe = x$ for all $x \in R(I_1, I_2)$, then we say that $(R(I_1, I_2), +, .)$ is a refined neutrosophic ring with unity.

Definition 2.2. Let $(R(I_1, I_2), +, .)$ be a refined neutrosophic ring and let $n \in \mathbb{Z}^+$.

(i) If for the least positive integer n such that $nx = 0$ for all $x \in R(I_1, I_2)$, then we call $(R(I_1, I_2), +, .)$ a refined neutrosophic ring of characteristic n and n is called the characteristic of $(R(I_1, I_2), +, .)$.

(ii) $(R(I_1, I_2), +, .)$ is called a refined neutrosophic ring of characteristic zero if for all $x \in R(I_1, I_2)$, $nx = 0$ is possible only if $n = 0$.

Example 2.3. (i) $\mathbb{Z}(I_1, I_2), \mathbb{Q}(I_1, I_2), \mathbb{R}(I_1, I_2), \mathbb{C}(I_1, I_2)$ are commutative refined neutrosophic rings with unity of characteristics zero.

(ii) Let $\mathbb{Z}_2(I_1, I_2) = \{(0, 0, 0), (1, 0, 0), (0, I_1, 0), (0, 0, I_2), (0, I_1, I_2), (1, I_1, 0), (1, 0, I_2), (1, I_1, I_2)\}$. Then $(\mathbb{Z}_2(I_1, I_2), +, .)$ is a commutative refined neutrosophic ring of integers modulo 2 of characteristic 2. Generally for a positive integer $n \geq 2$, $(\mathbb{Z}_n(I_1, I_2), +, .)$ is a finite commutative refined neutrosophic ring of integers modulo n of characteristic n.

Example 2.4. Let $M^{\mathbb{R}}_{n\times n}(I_1,I_2) = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} : a_{ij} \in \mathbb{R}(I_1,I_2) \right\}$ be a refined neutrosophic set of all $n \times n$ matrix. Then $(M^{\mathbb{R}}_{n\times n}(I_1,I_2),+,.)$ is a non-commutative refined neutrosophic ring under matrix multiplication.

Theorem 2.5. *Let $(R(I_1,I_2),+,.)$ be any refined neutrosophic ring. Then $(R(I_1,I_2),+,.)$ is a ring.*

Proof. It is clear that $(R(I_1,I_2),+)$ is an abelian group and and that $(R(I_1,I_2),.)$ is a semigroup. It remains to show that the distributive laws hold. To this end, let $x = (a_1,a_2I_1,a_3I_2), y = (b_1,b_2I_1,b_3I_2), z = (c_1,c_2I_1,c_3I_2)$ be any arbitrary elements of $R(I_1,I_2)$. Then

$$\begin{aligned} x(y+z) &= (a_1,a_2I_1,a_3I_2)((b_1,b_2I_1,b_3I_2)+(c_1,c_2I_1,c_3I_2)) \\ &= (a_1,a_2I_1,a_3I_2)(b_1+c_1,(b_2+c_2)I_1,b_3+c_3)I_2) \\ &= (a_1(b_1+c_1),a_1(b_2+c_2)+a_2(b_1+c_1)+a_2(b_2+c_2)+a_2(b_3+c_3)+a_3(b_2+c_2))I_1, \\ &\quad (a_1(b_3+c_3)+a_3(b_1+c_1)+a_3(b_3+c_3))I_2) \\ &= (a_1b_1+a_1c_1,(a_1b_2+a_1c_2+a_2b_1+a_2c_1+a_2b_2+a_2c_2+a_2b_3+a_2c_3+a_3b_2+a_3c_2)I_1, \\ &\quad (a_1b_3+a_1c_3+a_3b_1+a_3c_1+a_3b_3+a_3c_3)I_2). \end{aligned}$$

Also,

$$\begin{aligned} xy+xz &= ((a_1,a_2I_1,a_3I_2))((b_1,b_2I_1,b_3I_2))+((a_1,a_2I_1,a_3I_2))((c_1,c_2I_1,c_3I_2)) \\ &= (a_1b_1,(a_1b_2+a_2b_1+a_2b_2+a_2b_3+a_3b_2)I_1, \\ &\quad (a_1b_3+a_3b_1+a_3b_3)I_2)+(a_1c_1,(a_1c_2+a_2c_1+a_2c_2+a_2c_3+a_3c_2)I_1, \\ &\quad (a_1c_3+a_3c_1+a_3c_3)I_2) \\ &= (a_1b_1+a_1c_1,(a_1b_2+a_2b_1+a_2b_2+a_2b_3+a_3b_2+a_1c_2+a_2c_1+a_2c_2+a_2c_3+a_3c_2)I_1, \\ &\quad (a_1b_3+a_3b_1+a_3b_1+a_3b_3+a_1c_3+a_3c_1+a_3c_3)I_2). \end{aligned}$$

These show that $x(y+z) = xy+xz$. Similarly, it can be shown that $(y+z)x = yx+zx$. Hence $(R(I_1,I_2),+,.)$ is a ring. ☐

Definition 2.6. Let $(R(I_1,I_2),+,.)$ be a refined neutrosophic ring and let $J(I_1,I_2)$ be a nonempty subset of $R(I_1,I_2)$. $J(I_1,I_2)$ is called a refined neutrosophic subring of $R(I_1,I_2)$ if $(J(I_1,I_2),+,.)$ is itself a refined neutrosophic ring.

It is essential that $J(I_1,I_2)$ contains a proper subset which is a ring. Otherwise, $J(I_1,I_2)$ will be called a pseudo refined neutrosophic subring of $R(I_1,I_2)$.

Example 2.7. Let $(R(I_1,I_2),+,.) = (\mathbb{Z}(I_1,I_2),+)$ be the refined neutrosophic ring of integers. The set $J(I_1,I_2) = n\mathbb{Z}(I_1,I_2))$ for all positive integer n is a refined neutrosophic subring of $R(I_1,I_2)$.

Example 2.8. Let $(R(I_1,I_2),+,.) = (\mathbb{Z}_6(I_1,I_2),+)$ be the refined neutrosophic ring of integers modulo 6. The set

$$\begin{aligned} J(I_1,I_2) &= \{(0,0,0),(0,I_1,0),(0,0,I_2),(0,I_1,I_2), \\ &\quad (0,2I_1,0),(0,0,2I_2),(0,2I_1,2I_2), \\ &\quad (0,3I_1,0),(0,0,3I_2),(0,3I_1,3I_2), \\ &\quad (0,4I_1,0),(0,0,4I_2),(0,4I_1,4I_2), \\ &\quad (0,5I_1,0),(0,0,5I_2),(0,5I_1,5I_2)\}. \end{aligned}$$

is a refined neutrosophic subring of $R(I_1,I_2)$.

Theorem 2.9. *Let $\{J_k(I_1,I_2)\}_1^n$ be a family of all refined neutrosophic subrings (pseudo refined neutrosophic subrings) of a refined neutrosophic ring $(R(I_1,I_2),+,.)$. Then $\bigcap_1^n J_k(I_1,I_2)\}$ is a refined neutrosophic subring (pseudo refined neutrosophic subring) of $R(I_1,I_2)$.*

Definition 2.10. Let $A(I_1, I_2)$ and $B(I_1, I_2)$ be any two refined neutrosophic subrings (pseudo refined neutrosophic subrings) of a refined neutrosophic ring $(R(I_1, I_2), +)$. We define the sum $A(I_1, I_2) \oplus B(I_1, I_2)$ by the set

$$A(I_1, I_2) \oplus B(I_1, I_2) = \{a + b : a \in A(I_1, I_2), b \in B(I_1, I_2)\} \tag{7}$$

which is a refined neutrosophic subring (pseudo refined neutrosophic subring) of $R(I_1, I_2)$

Theorem 2.11. *Let $A(I_1, I_2)$ be any refined neutrosophic subring of a refined neutrosophic ring $(R(I_1, I_2), +)$ and let $B(I_1, I_2)$ be any pseudo refined neutrosophic subring of $(R(I_1, I_2), +)$. Then:*

(i) $A(I_1, I_2) \oplus A(I_1, I_2) = A(I_1, I_2)$.

(ii) $B(I_1, I_2) \oplus B(I_1, I_2) = B(I_1, I_2)$.

(iii) $A(I_1, I_2) \oplus B(I_1, I_2)$ *is a refined neutrosophic subring of* $R(I_1, I_2)$.

Definition 2.12. Let R be a non-empty set and let $+$ and . be two binary operations on R such that:

(i) $(R, +)$ is an abelian group.

(ii) $(R, .)$ is a semigroup.

(iii) There exists $x, y, z \in R$ such that

$$x(y + z) = xy + xz, (y + z)x = yx + zx.$$

(iv) R contains elements of the form (x, yI_1, zI_2) with $x, y, z \in R$ such that $y, z \neq 0$ for at least one value.

Then $(R, +, .)$ is called a pseudo refined neutrosophic ring.

Example 2.13. Let R be a set given by

$$R = \{(0, 0, 0), (0, 2I_1, 0), (0, 0, 2I_2), (0, 4I_1, 0), (0, 0, 4I_2), (0, 6I_1, 0), (0, 0, 6I_2)\}.$$

Then $(R, +, .)$ is a pseudo refined neutrosophic ring which is also a refined neutrosophic ring where $+$ and . are addition and multiplication modulo 8.

Example 2.14. Let $R(I_1, I_2) = \mathbb{Z}_{12}(I_1, I_2)$ be a refined neutrosophic ring of integers modulo 12 and let T be a subset of $\mathbb{Z}_{12}(I_1, I_2)$ given by

$$\begin{aligned} T \;=\; & \{(0, 0, 0), (0, 2I_1, 0), (0, 0, 2I_2), (0, 4I_1, 0), (0, 0, 4I_2), (0, 4I_1, 0), (0, 0, 4I_2), \\ & (0, 6I_1, 0), (0, 0, 6I_2)(0, 8I_1, 0), (0, 0, 8I_2), (0, 10I_1, 0), (0, 0, 10I_2)\}. \end{aligned}$$

It is clear that $(T, +, .)$ is a pseudo refined neutrosophic ring.

Since $T \subset R(I_1, I_2)$, it follows that $T \cup R(I_1, I_2) \subseteq R(I_1, I_2)$ and consequently, $(T \cup R(I_1, I_2), +, .)$ is a refined neutrosophic ring.

Theorem 2.15. *Let $(R(I_1, I_2), +, .)$ be any refined neutrosophic ring and let $(T, +, .)$ be any pseudo refined neutrosophic ring. Then $(T \cup R(I_1, I_2), +, .)$ is a refined neutrosophic ring if and only if $T \subset R(I_1, I_2)$.*

Theorem 2.16. *Let $(R(I_1, I_2), +, .)$ be any refined neutrosophic ring and let $(T, +, .)$ be any pseudo refined neutrosophic ring. Then $(T \oplus R(I_1, I_2), +, .)$ is a refined neutrosophic ring.*

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On $\alpha\omega$-closed sets and its connectedness in terms of neutrosophic topological spaces

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Abstract

The aim of this paper is to introduce the notion of neutrosophic $\alpha\omega$-closed sets and study some of the properties of neutrosophic $\alpha\omega$-closed sets. Further, we investigated neutrosophic $\alpha\omega$- continuity, neutrosophic $\alpha\omega$-irresoluteness, neutrosophic $\alpha\omega$ connectedness and neutrosophic contra $\alpha\omega$ continuity along with examples.
Keywords: neutrosophic topology, neutrosophic $\alpha\omega$-closed set, neutrosophic $\alpha\omega$-continuous function and neutrosophic contra $\alpha\omega$-continuous mappings.

1 Introduction

Zadeh [19] introduced truth (t) or the degree of membership of an object in fuzzy set theory. The falsehood (f) or the degree of non-membership of an object along with membership of an object introduced by Atanassov [4,5,6] in intuitionistic fuzzy set. Neutrosophic (i) or the degree of indeterminacy of an object along with membership and non-membership of an objects for incomplete, imprecise, indeterminate information is introduced by Smarandache [16,17] in 1998. The neutrosophic triplet set consist of three components $(t, f, i) = (truth, falsehood, indeterminacy)$. The neutrosophic topological spaces introduced and developed by Salama et al., [15]. This leads to many investigation among researchers in the field of neutrosophic topology and their application in decision making algorithms [8,11,12,13,14]. Arokiarani et al.,[3] introduced and studied α- open sets in neutrosophic topological spaces. Devi et al., [7,9,10] introduced $\alpha\omega$-closed sets in general topology, fuzzy topology and intuitionistic fuzzy topology. In this article, we introduce neutrosophic $\alpha\omega$-closed sets in neutrosophic topological spaces. Also, we introduce and investigate neutrosophic $\alpha\omega$-continuous,neutrosophic $\alpha\omega$-irresoluteness, neutrosophic $\alpha\omega$ connectedness and neutrosophic contra $\alpha\omega$-continuous mappings .

2 Preliminaries

Let (X, τ) be the neutrosophic topological space(NTS). Each neutrosophic set(NS) in (X, τ) is called a neutrosophic open set(NOS) and its complement is called a neutrosophic closed set (NCS).

We provide some of the basic definitions in neutrosophic sets. These are very useful in the sequel.

Definition 2.1. [17] A neutrosophic set (NS) A is an object of the following form

$$U = \{\langle u, \mu_U(u), \nu_U(u), \omega_U(u)\rangle : u \in X\}$$

where the mappings $\mu_U : X \to I$, $\nu_U : X \to I$ and $\omega_U : X \to I$ denote the degree of membership (namely $\mu_U(u)$), the degree of indeterminacy (namely $\nu_U(u)$) and the degree of nonmembership (namely $\omega_U(u)$) for

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each element $u \in X$ to the set U, respectively and $0 \leq \mu_U(u) + \nu_U(u) + \omega_U(u) \leq 3$ for each $u \in X$.

Definition 2.2. [17] Let U and V be NSs of the form $U = \{\langle u, \mu_U(u), \nu_U(u), \omega_U(u)\rangle : u \in X\}$ and $V = \{\langle u, \mu_V(u), \nu_V(u), \omega_V(u)\rangle : u \in X\}$. Then

(i) $U \subseteq V$ if and only if $\mu_U(u) \leq \mu_V(u)$, $\nu_U(u) \geq \nu_V(u)$ and $\omega_U(u) \geq \omega_V(u)$;

(ii) $\overline{U} = \{\langle u, \nu_U(u), \mu_U(u), \omega_U(u)\rangle : u \in X\}$;

(iii) $U \cap V = \{\langle u, \mu_U(u) \wedge \mu_V(u), \nu_U(u) \vee \nu_V(u), \omega_U(u) \vee \omega_V(u)\rangle : u \in X\}$;

(iv) $U \cup V = \{\langle u, \mu_U(u) \vee \mu_V(u), \nu_U(u) \wedge \nu_V(u), \omega_U(u) \wedge \omega_V(u)\rangle : u \in X\}$.

We will use the notation $U = \langle u, \mu_U, \nu_U, \omega_U\rangle$ instead of $U = \{\langle u, \mu_U(u), \nu_U(u), \omega_U(u)\rangle : u \in X\}$. The NSs $0_\sim$ and $1_\sim$ are defined by $0_\sim = \{\langle u, \underline{0}, \underline{1}, \underline{1}\rangle : u \in X\}$ and $1_\sim = \{\langle u, \underline{1}, \underline{0}, \underline{0}\rangle : u \in X\}$.

Let $r, s, t \in [0, 1]$ such that $0 \leq r + s + t \leq 3$. A neutrosophic point (NP) $p_{(r,s,t)}$ is neutrosophic set defined by

$$p_{(r,s,t)}(u) = \begin{cases} (r, s, t)(x) & if\ u = p \\ (0, 1, 1) & otherwise \end{cases}$$

Let f be a mapping from an ordinary set X into an ordinary set Y, If $V = \{\langle y, \mu_V(y), \nu_V(y), \omega_V(y)\rangle : y \in Y\}$ is a NS in Y , then the inverse image of V under f is a NS defined by

$$f^{-1}(V) = \{\langle u, f^{-1}(\mu_V)(u), f^{-1}(\nu_V)(u), f^{-1}(\omega_V)(u)\rangle : u \in X\}$$

The image of NS $U = \{\langle v, \mu_U(v), \nu_U(v), \omega_U(v)\rangle : v \in Y\}$ under f is a NS defined by $f(U) = \{\langle v, f(\mu_U)(v), f(\nu_U)(v), f(\omega_U)$ $v \in Y\}$ where

$$f(\mu_U)(v) = \begin{cases} \sup\limits_{u \in f^{-1}(v)} \mu_U(u), & if\ f^{-1}(v) \neq 0 \\ 0 & otherwise, \end{cases}$$

$$f(\nu_U)(v) = \begin{cases} \inf\limits_{u \in f^{-1}(v)} \nu_U(u), & if\ f^{-1}(v) \neq 0 \\ 1 & otherwise, \end{cases}$$

$$f(\omega_U)(v) = \begin{cases} \inf\limits_{u \in f^{-1}(v)} \omega_U(u), & if\ f^{-1}(v) \neq 0 \\ 1 & otherwise, \end{cases}$$

for each $v \in Y$.

Definition 2.3. [15] A neutrosophic topology (NT) in a nonempty set X is a family τ of NSs in X satisfying the following axioms:

(NT1) $0_\sim, 1_\sim \in \tau$;

(NT2) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$;

(NT3) $\cup G_i \in \tau$ for any arbitrary family $\{G_i : i \in J\} \subseteq \tau$.

Definition 2.4. [15] Let U be a NS in NTS X. Then
Nint(U) = $\cup\{O : O$ is an NOS in X and O $\subseteq U\}$ is called a neutrosophic interior of U;
Ncl(U) = $\cap\{O : O$ is an NCS in X and O $\supseteq U\}$ is called a neutrosophic closure of U.

Definition 2.5. [15] Let $p_{(r,s,t)}$ be a NP in NTS X. A NS U in X is called a neutrosophic neighborhood (NN) of $p_{(r,s,t)}$ if there exists a NOS V in X such that $p_{(r,s,t)} \in V \subseteq U$.

Definition 2.6. [3] A subset U of a neutrosophic space (X, τ) is called

1. a neutrosophic pre-open set if $U \subseteq Nint(Ncl(U))$ and a neutrosophic pre-closed set if $Ncl(Nint(U)) \subseteq U$,

2. a neutrosophic semi-open set if $U \subseteq Ncl(Nint(U))$ and a neutrosophic semi-closed set if $Nint(Ncl(U)) \subseteq U$,

3. a neutrosophic α-open set if $U \subseteq Nint(Ncl(Nint(U)))$ and a neutrosophic α-closed set if $Ncl(Nint(Ncl(U))) \subseteq U$,

The pre-closure (resp. semi-closure, α-closure) of a subset U of a neutrosophic space (X, τ) is the intersection of all pre-closed (resp. semi-closed, α-closed) sets that contain U and is denoted by $Npcl(U)$ (resp. $Nscl(U)$, $N\alpha cl(U)$).

3 On neutrosophic $\alpha\omega$-closed sets

Definition 3.1. A subset A of a neutrosophic topological space (X, τ) is called

1. a neutrosophic $N\omega$-closed set if $Ncl(U) \subseteq G$ whenever $U \subseteq G$ and G is neutrosophic semi-open in (X, τ).
2. a neutrosophic $\alpha\omega$-closed ($N\alpha\omega$-closed) set if $N\omega cl(U) \subseteq G$ whenever $U \subseteq G$ and G is an $N\alpha$-open set in (X, τ). Its complement is called a neutrosophic $\alpha\omega$-open ($N\alpha\omega$-open) set.

Definition 3.2. Let U be a NS in NTS X. Then
$N\alpha\omega int(U) = \cup\{O : O$ is an $N\alpha\omega$OS in X and O $\subseteq U\}$ is said to be a neutrosophic $\alpha\omega$-interior of U;
$N\alpha\omega cl(U) = \cap\{O : O$ is an $N\alpha\omega$CS in X and O $\supseteq U\}$ is said to be a neutrosophic $\alpha\omega$-closure of U.

Theorem 3.3. *Every $N\alpha$-closed set and N-closed set are $N\alpha\omega$-closed set.*
Proof. Let U be an $N\alpha$-closed set, then $U = N\alpha cl(U)$. Let $U \subseteq G$, G is $N\alpha$-open. Since U is $N\alpha$-closed, $N\omega cl(U) \subseteq N\alpha cl(U) \subseteq G$. Thus U is $N\alpha\omega$-closed.

Theorem 3.4. *Every neutrosophic semi-closed set in a neutrosophic set is an $N\alpha\omega$-closed.*
Proof. Let U be a Nsemi-closed set in (X, τ), then $U = Nscl(U)$. Let $U \subseteq G$, G is $N\alpha$-open in (X, τ). Since U is Nsemi-closed, $N\omega cl(U) \subseteq Nscl(U) \subseteq G$. This shows that U is $N\alpha\omega$-closed set.
The converses of the above theorems are not true as explained in Example 3.5.

Example 3.5. Let $X = \{u, v, w\}$ and neutrosophic sets A, B, C be defined by:

$$A = \langle(0.1, 0.4, 0.7), (0.9, 0.6, 0.3), (0.9, 0.6, 0.3)\rangle$$
$$B = \langle(0.6, 0.6, 0.4), (0.2, 0.7, 0.8), (1, 0.6, 0.5)\rangle$$
$$C = \langle(0.1, 0.4, 0.8), (0.2, 0.6, 0.4), (0.6, 0.5, 0.9)\rangle$$

Let $\tau = \{0_\sim, A, 1_\sim\}$. Then B is $N\alpha\omega$-closed in (X, τ) but not $N\alpha$-closed and thus it is not N-closed and C is $N\alpha\omega$-closed in (X, τ) but not $Nsemi$-closed.

Theorem 3.6. Let (X, τ) be a NTS and let $U \in NS(X)$. If U is $N\alpha\omega$-closed set and $U \subseteq V \subseteq N\omega cl(U)$, then V is $N\alpha\omega$-closed set.
Proof. Let G be a $N\alpha$-open set such that $V \subseteq G$. Since $U \subseteq V$, then $U \subseteq G$. But U is $N\alpha\omega$-closed, so $N\omega cl(U) \subseteq G$. Since $V \subseteq N\omega cl(U)$. Since $N\omega cl(V) \subseteq N\omega cl(U)$ and hence $N\omega cl(V) \subseteq G$. Therefore V is a $N\alpha\omega$-closed set.

Theorem 3.7. Let U be a $N\alpha\omega$-open set in X and $N\omega int(U) \subseteq V \subseteq U$, then V is $N\alpha\omega$-open.
Proof. Suppose U is $N\alpha\omega$-open in X and $N\omega int(U) \subseteq V \subseteq U$. Then $\overline{U}$ is $N\alpha\omega$-closed and $\overline{U} \subseteq \overline{V} \subseteq N\omega cl(\overline{U})$. Then $\overline{U}$ is a $N\alpha\omega$-closed set by theorem 3.5. Hence V is a $N\alpha\omega$-open set in X.

Theorem 3.8. A NS U in a NTS (X, τ) is a $N\alpha\omega$-open set if and only if $V \subseteq N\omega int(U)$ whenever V is a $N\alpha$-closed set and $V \subseteq U$.
Proof. Let U be a $N\alpha\omega$-open set and let V be a $N\alpha$-closed set such that $V \subseteq U$. Then $\overline{U} \subseteq \overline{V}$ and hence $N\omega cl(\overline{U}) \subseteq \overline{V}$, since $\overline{U}$ is $N\alpha\omega$-closed. But $N\omega cl(\overline{U}) = \overline{N\omega int(U)}$, thus $V \subseteq N\omega int(U)$.
Conversely, suppose that the condition is satisfied, then $\overline{N\omega int(U)} \subseteq \overline{V}$ whenever $\overline{V}$ is $N\alpha$-open set and $\overline{U} \subseteq \overline{V}$. This implies that $N\omega cl(\overline{U}) \subseteq \overline{V} = G$ where G is $N\alpha$-open set and $\overline{U} \subseteq G$. Therefore $\overline{U}$ is $N\alpha\omega$-closed set and hence U is $N\alpha\omega$-open.

Theorem 3.9. Let U be a $N\alpha\omega$-closed subset of (X, τ). Then $N\omega cl(U) - U$ does not contain any non-empty $N\alpha\omega$-closed set.
Proof. Assume that U is a $N\alpha\omega$-closed set. Let F be a non-empty $N\alpha\omega$-closed set, such that $F \subseteq$

$N\omega cl(U) - U = N\omega cl(U) \cap \overline{U}$. i.e., $F \subseteq N\omega cl(U)$ and $F \subseteq \overline{U}$. Therefore, $U \subseteq \overline{F}$. Since $\overline{F}$ is a $N\alpha\omega$-open set, $N\omega cl(U) \subseteq \overline{F} \Rightarrow F \subseteq (N\omega cl(U) - U) \cap (\overline{N\omega cl(U)}) \subseteq N\omega cl(U) \cap \overline{N\omega cl(U)}$. i.e., $F \subseteq \phi$. Therefore F is empty.

Corollary 3.10. Let U be a $N\alpha\omega$-closed set of (X, τ). Then $N\omega cl(U) - U$ does not contain no non-empty N-closed set.
Proof. The proof follows from the Theorem 3.9.

Theorem 3.11. If U is both $N\omega$-open and $N\alpha\omega$-closed set, then U is a $N\omega$-closed set.
Proof. Since U is both $N\omega$-open and $N\alpha\omega$-closed set in X, then $N\omega cl(U) \subseteq U$. Also we have $U \subseteq N\omega cl(U)$. This gives that $N\omega cl(U) = U$. Therefore U is a $N\omega$-closed set in X.

4 On neutrosophic $\alpha\omega$-continuity, connectedness and contra-continuity

Definition 4.1. Let (X, τ) and (Y, σ) be any two neutrosophic topological spaces.

1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be a neutrosophic $\alpha\omega$-continuous (briefly, $N\alpha\omega$-continuous) function if the inverse image of every open set in Y is a $N\alpha\omega$-open set in X.
   Equivalently, if the inverse image of every open set in (Y, σ) is $N\alpha\omega$-open in (X, τ);

2. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be a neutrosophic $\alpha\omega$-irresolute (briefly, $N\alpha\omega$-irresolute) function if the inverse image of every $N\alpha\omega$-open set in Y is a $N\alpha\omega$-open set in X.
   Equivalently, if the inverse image of every $N\alpha\omega$-open set in (Y, σ) is $N\alpha\omega$-open in (X, τ);

Definition 4.2. A NTS (X, τ) is said to be neutrosophic-$\alpha\omega T_{1/2}(N\alpha\omega T_{1/2}$ in short) space if every $N\alpha\omega C$ in X is an NC in X.

Definition 4.3. Let (X, τ) be any neutrosophic topological space. (X, τ) is said to be neutrosophic $\alpha\omega$-disconnected (in shortly $N\alpha\omega$-disconnected) if there exists a $N\alpha\omega$-open and $N\alpha\omega$-closed set $\overline{F}$ such that $\overline{F} \neq 0_\sim$ and $\overline{F} \neq 1_\sim$. (X, τ) is said to be neutrosophic $\alpha\omega$-connected if it is not neutrosophic $\alpha\omega$-disconnected.

Theorem 4.4. Every $N\alpha\omega$-connected space is neutrosophic connected.
Proof. For a $N\alpha\omega$-connected (X, τ) space and let (X, τ) not be neutrosophic connected. Hence, there exists a proper neutrosophic set, $\overline{F} =< \mu_{\overline{F}(x)}, \sigma_{\overline{F}(x)}, \nu_{\overline{F}(x)} >$, $\overline{F} \neq 0_\sim$ and $\overline{F} \neq 1_\sim$, such that $\overline{F}$ is both neutrosophic open and neutrosophic closed in (X, τ). Since every neutrosophic open set is $N\alpha\omega$-open and neutrosophic closed set is $N\alpha\omega$-closed, X is not $N\alpha\omega$-connected. Therefore, (X, τ) is neutrosophic connected.
However, the converse is not true.

Example 4.5. Let $X = \{u, v, w\}$ and neutrosophic sets A, B and C be defined by:

$$A = \langle(0.4, 0.5, 0.5), (0.4, 0.5, 0.5), (0.5, 0.5, 0.5)\rangle$$
$$B = \langle(0.7, 0.6, 0.5), (0.7, 0.6, 0.5), (0.3, 0.4, 0.5)\rangle$$
$$C = \langle(0.5, 0.6, 0.5), (0.5, 0.6, 0.5), (0.5, 0.6, 0.5)\rangle$$

Let $\tau = \{0_\sim, A, B, 1_\sim\}$. It is obvious that (X, τ) is NTS. Now, (X, τ) is neutrosophic connected. However, it is not a $N\alpha\omega$-connected.

Theorem 4.6. Let (X, τ) be a neutrosophic $\alpha\omega T_{1/2}$ space. (X, τ) is neutrosophic connected iff (X, τ) is $N\alpha\omega$-connected.

Proof. Let (X, τ) is neutrosophic connected. Suppose that (X, τ) is not $N\alpha\omega$-connected, and there exists a neutrosophic set $\overline{F}$ which is both $N\alpha\omega$-open and $N\alpha\omega$-closed. Since (X, τ) is neutrosophic $\alpha\omega T_{1/2}$, $\overline{F}$ is both neutrosophic open and neutrosophic closed. Therefore, (X, τ) is not a neutrosophic connected which is contradiction to our hypothesis. Hence, (X, τ) is $N\alpha\omega$-connected.
Conversely, let (X, τ) is $N\alpha\omega$-connected. Suppose that (X, τ) is not neutrosophic connected, and there exists a neutrosophic set $\overline{F}$ such that $\overline{F}$ is both NCs and NOs $\in (X, \tau)$. Since the neutrosophic open set is $N\alpha\omega$-open and the neutrosophic closed set is $N\alpha\omega$-closed, (X, τ) is not $N\alpha\omega$-connected. Hence, (X, τ) is neutrosophic connected.

Theorem 4.7. Suppose (X,τ) and (Y,σ) are any two NTSs. If $g:(X,\tau)\to(Y,\sigma)$ is $N\alpha\omega$-continuous surjection and (X,τ) is $N\alpha\omega$-connected, then (Y,σ) is neutrosophic connected.
Proof. Suppose that (Y,σ) is not neutrosophic connected, such that the neutrosophic set $\overline{F}$ is both neutrosophic open and neutrosophic closed in (Y,σ). Since g is $N\alpha\omega$-continuous, $g^{-1}(\overline{F})$ is $N\alpha\omega$-open and $N\alpha\omega$-closed in (Y,σ). Thus, (Y,σ) is not $N\alpha\omega$-connected. Hence, (Y,σ) is neutrosophic connected.

Theorem 4.8. Let $g:(X,\tau)\to(Y,\sigma)$ be a function. Then the following conditions are equivalent.

(i) g is $N\alpha\omega$-continuous;

(ii) The inverse $f^{-1}(U)$ of each N-open set U in Y is $N\alpha\omega$-open set in X.

Proof. It is clear, since $g^{-1}(\overline{U})=\overline{g^{-1}(U)}$ for each N-open set U of Y.

Theorem 4.9. If $g:(X,\tau)\to(Y,\sigma)$ be a $N\alpha\omega$-continuous mapping, then the following statements holds:

(i) $g(N\alpha\omega Ncl(U))\subseteq Ncl(g(U))$, for all neutrosophic set U in X;

(ii) $N\alpha\omega Ncl(g^{-1}(V))\subseteq g^{-1}(Ncl(V))$, for all neutrosophic set V in Y.

Proof.

(i) Since $Ncl(g(U))$ is neutrosophic closed set in Y and g is $N\alpha\omega$-continuous, then $g^{-1}(Ncl(g(U)))$ is $N\alpha\omega$-closed in X. Now, since $U\subseteq g^{-1}(Ncl(g(U)))$. So, $N\alpha\omega cl(U)\subseteq g^{-1}(Ncl(g(U)))$. Therefore, $g(N\alpha\omega Ncl(U))\subseteq Ncl(g(U))$.

(ii) By replacing U with V in (i), we obtain $g(N\alpha\omega cl(g^{-1}(V)))\subseteq Ncl(g(g^{-1}(V)))\subseteq Ncl(V)$. Hence $N\alpha\omega cl(g^{-1}(V))\subseteq g^{-1}(Ncl(V))$.

Theorem 4.10. Let g be a function from a NTS (X,τ) to a NTS (Y,σ). Then the following statements are equivalent.

(i) g is a neutrosophic $\alpha\omega$-continuous function.

(ii) For every NP $p_{(r,s,t)}\in X$ and each NN U of $g(p_{(r,s,t)})$, there exists a $N\alpha\omega$-open set V such that $p_{(r,s,t)}\in V\subseteq g^{-1}(U)$.

(iii) For every NP $p_{(r,s,t)}\in X$ and each NN U of $g(p_{(r,s,t)})$, there exists a $N\alpha\omega$-open set V such that $p_{(r,s,t)}\in V$ and $g(V)\subseteq U$.

Proof. $(i)\Rightarrow(ii)$. If $p_{(r,s,t)}$ is a NP in X and also if U be a NN of $g(p_{(r,s,t)})$, then there exists a NOS W in Y such that $g(p_{(r,s,t)})\in W\subset U$. we have g is neutrosophic $\alpha\omega$-continuous, $V=g^{-1}(W)$ is an $N\alpha\omega OS$ and

$$p_{(r,s,t)}\in g^{-1}(g(p_{(r,s,t)}))\subseteq g^{-1}(W)=V\subseteq g^{-1}(U).$$

Thus (ii) is a valid statement.
$(ii)\Rightarrow(iii)$. Let $p_{(r,s,t)}$ be a NP in X and take U be a NN of $g(p_{(r,s,t)})$. Then there exists a $N\alpha\omega OS$ U such that $p_{(r,s,t)}\in V\subseteq g^{-1}(U)$ by (ii). Thus, we have $p_{(r,s,t)}\in V$ and $g(V)\subseteq g(g^{-1}(U))\subseteq U$. Hence (iii) is valid.
$(iii)\Rightarrow(i)$. Let V be a NOS in Y and let $p_{(r,s,t)}\in g^{-1}(V)$. Then $g(p_{(r,s,t)})\in g(g^{-1}(V))\subset V$. Since V is a NOS, it follows that V is a NN of $g(p_{(r,s,t)})$ so from (iii), there exists a $N\alpha\omega OS$ U such that $p_{(r,s,t)}\in U$ and $g(U)\subseteq V$. This implies that

$$p_{(r,s,t)}\in U\subseteq g^{-1}(g(U))\subseteq g^{-1}(V).$$

Then, we know that $g^{-1}(V)$ is a $N\alpha\omega OS$ in X. Thus g is neutrosophic $\alpha\omega$-continuous.

Definition 4.11. A function is said to be a neutrosophic contra $\alpha\omega$-continuous function if the inverse image of each NOS V in Y is a N$\alpha\omega$CS in X.

Theorem 4.12. Let $g:(X,\tau)\to(Y,\sigma)$ be a function. Then, the following assertions are equivalent:

(i) g is a neutrosophic contra $\alpha\omega$-continuous function;

(ii) $g^{-1}(V)$ is a N$\alpha\omega$ CS in X, for each NOS V in Y.

Proof. $(i) \Rightarrow (ii)$ Let g be any neutrosophic contra $\alpha\omega$-continuous function and let V be any NOS in Y. Then, $\overline{V}$ is a NCS in Y. By the assumption $g^{-1}(\overline{V})$ is a $N\alpha\omega OS$ in X. Hence, we get that $g^{-1}(V)$ is a $N\alpha\omega CS$ in X.

The converse of the theorem can be done in the same sense.

Theorem 4.13. Let $g : (X, \tau) \to (Y, \sigma)$ be a bijective mapping from an NTS X into an NTS Y. The mapping g is neutrosophic contra $\alpha\omega$-continuous if $Ncl(g(U)) \subseteq g(N\alpha\omega int(U))$, for each NS U in X.
Proof. Let V be any NCS in X. Then, $Ncl(V) = V$, and also g is onto, by assumption, it shows that $g(N\alpha\omega int(g^{-1}(V))) \supseteq Ncl(g(g^{-1}(V))) = Ncl(V) = V$. Hence $g^{-1}(g(N\alpha\omega int(g^{-1}(V)))) \supseteq g^{-1}(V)$. Since g is an into mapping, we have $N\alpha\omega int(g^{-1}(V)) = g^{-1}(g(N\alpha\omega int(g^{-1}(V)))) \supseteq g^{-1}(V)$. Therefore $N\alpha\omega int(g^{-1}(V))$
$= g^{-1}(V)$, so $g^{-1}(V)$ is a $N\alpha\omega$OS in X. Hence g is a neutrosophic contra $\alpha\omega$-continuous mapping.

Theorem 4.14. Let $g : (X, \tau) \to (Y, \sigma)$ be a mapping. Then the following statements are equivalent:

(i) g is a neutrosophic contra $\alpha\omega$-continuous mapping;

(ii) for each NP $p_{(r,s,t)}$ in X and NCS V containing $g(p_{(r,s,t)})$ there exists $N\alpha\omega OS\ U$ in X containing $p_{(r,s,t)}$ such that $A \subseteq f^{-1}(B)$;

(iii) for each NP $p_{(r,s,t)}$ in X and NCS V containing $p_{(r,s,t)}$ there exists $N\alpha\omega OS\ U\ in\ X$ containing $p_{(r,s,t)}$ such that $g(U) \subseteq V$.

Proof. $(i) \Rightarrow (ii)$ Let g be an neutrosophic contra $\alpha\omega$-continuous mapping, let V be any NCS in Y and let $p_{(r,s,t)}$ be a NP in X and such that $g(p_{(r,s,t)}) \in V$. Then $p_{(r,s,t)} \in g^{-1}(V) = N\alpha\omega int(g^{-1}(V))$. Let $U = N\alpha\omega int(g^{-1}(V))$. Then U is an $N\alpha\omega OS$ and $U = N\alpha\omega int(g^{-1}(V)) \subseteq g^{-1}(V)$.
$(ii) \Rightarrow (iii)$ The results follows from the evident relations $g(U) \subseteq g(g^{-1}(V)) \subseteq V$.
$(iii) \Rightarrow (i)$ Let V be any NCS in Y and let $p_{(r,s,t)}$ be a NP in X such that $p_{(r,s,t)} \in g^{-1}(V)$. Then $g(p_{(r,s,t)}) \in V$. According to the assumption, there exists an $N\alpha\omega OS\ U$ in X such that $p_{(r,s,t)} \in U$ and $g(U) \subseteq V$. Hence $p_{(r,s,t)} \in U \subseteq g^{-1}(g(U)) \subseteq g^{-1}(V)$. Therefore $p_{(r,s,t)} \in U = \alpha\omega int(U) \subseteq N\alpha\omega int(g^{-1}(V))$. Since, $p_{(r,s,t)}$ is an arbitrary NP and $g^{-1}(V)$ is the union of all NPs in $g^{-1}(V)$, we obtain that $g^{-1}(V) \subseteq N\alpha\omega int(g^{-1}(V))$. Thus g is a neutrosophic contra $N\alpha\omega$-continuous mapping.

Corollary 4.15. Let $X,\ X_1\ and\ X_2$ be NTSs, $p_1 : X \to X_1 \times X_2\ (i = 1,\ 2)$ and $p_2 : X \to X_1 \times X_2$ are the projections of $X_1 \times X_2$ onto X_i, $(i = 1,\ 2)$. If $g : X \to X_1 \times X_2$ is a neutrosophic contra $\alpha\omega$-continuous, then $p_i g$ are also neutrosophic contra $\alpha\omega$-continuous mapping.
Proof. The proof follows from the fact that the projections are all neutrosophic continuous functions.

Theorem 4.16. Let $g : (X_1, \tau) \to (Y_1, \sigma)$ be a function. If the graph $h : X_1 \to X_1 \times Y_1$ of g is neutrosophic contra $\alpha\omega$-continuous, then g is neutrosophic contra $\alpha\omega$-continuous.
Proof. For every NOS V in Y_1 holds $g^{-1}(V) = 1 \wedge g^{-1}(V) = h^{-1}(1 \times V)$. Since h is a neutrosophic contra $\alpha\omega$-continuous mapping and $1 \times V$ is a NOS in $X_1 \times Y_1$, $g^{-1}(V)$ is a $N\alpha\omega CS$ in X_1, so g is a neutrosophic contra $\alpha\omega$-continuous mapping.

5 Conclusions

In this paper, we introduced and investigated the neutrosophic $\alpha\omega$ closed sets and its properties. Also, we investigated the continuity, irresolute, connectedness and contra-continuity in terms of neutrosophic $\alpha\omega$ closed sets.

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Refined Neutrosophic Rings II

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Abstract

This paper is the continuation of the work started in the paper titled "Refined Neutrosophic Rings I". In the present paper, we study refined neutrosophic ideals and refined neutrosophic homomorphisms along their elementary properties. It is shown that if $R = \mathbb{Z}(I_1, I_2)$ is a refined neutrosophic ring of integers and $J = n\mathbb{Z}(I_1, I_2)$ is a refined neutrosophic ideal of R, then $R/J \cong \mathbb{Z}_n(I_1, I_2)$.

Keywords: Neutrosophy, refined neutrosophic ring, refined neutrosophic ideal, refined neutrosophic ring homomorphism.

1 Preliminaries

In this section, we only state some useful definitions, examples and results. For full details about refined neutrosophic rings, the readers should see.[9]

Definition 1.1 ([9]). Let $(R, +, .)$ be any ring. The abstract system $(R(I_1, I_2), +, .)$ is called a refined neutrosophic ring generated by R, I_1, I_2. $(R(I_1, I_2), +, .)$ is called a commutative refined neutrosophic ring if for all $x, y \in R(I_1, I_2)$, we have $xy = yx$. If there exists an element $e = (1, 0, 0) \in R(I_1, I_2)$ such that $ex = xe = x$ for all $x \in R(I_1, I_2)$, then we say that $(R(I_1, I_2), +, .)$ is a refined neutrosophic ring with unity.

Definition 1.2 ([9]). Let $(R(I_1, I_2), +, .)$ be a refined neutrosophic ring and let $n \in \mathbb{Z}^+$.

(i) If for the least positive integer n such that $nx = 0$ for all $x \in R(I_1, I_2)$, we call $(R(I_1, I_2), +, .)$ a refined neutrosophic ring of characteristic n and n is called the characteristic of $(R(I_1, I_2), +, .)$.

(ii) $(R(I_1, I_2), +, .)$ is call a refined neutrosophic ring of characteristic zero if for all $x \in R(I_1, I_2)$, $nx = 0$ is possible only if $n = 0$.

Example 1.3 ([9]). (i) $\mathbb{Z}(I_1, I_2), \mathbb{Q}(I_1, I_2), \mathbb{R}(I_1, I_2), \mathbb{C}(I_1, I_2)$ are commutative refined neutrosophic rings with unity of characteristics zero.

(ii) Let $\mathbb{Z}_2(I_1, I_2) = \{(0, 0, 0), (1, 0, 0), (0, I_1, 0), (0, 0, I_2), (0, I_1, I_2), (1, I_1, 0), (1, 0, I_2), (1, I_1, I_2)\}$. Then $(\mathbb{Z}_2(I_1, I_2), +, .)$ is a commutative refined neutrosophic ring of integers modulo 2 of characteristic 2. Generally for a positive integer $n \geq 2$, $(\mathbb{Z}_n(I_1, I_2), +, .)$ is a finite commutative refined neutrosophic ring of integers modulo n of characteristic n.

Example 1.4 ([9]). Let $M^{\mathbb{R}}_{n\times n}(I_1, I_2) = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} : a_{ij} \in \mathbb{R}(I_1, I_2) \right\}$ be a refined neutrosophic set of all $n \times n$ matrix. Then $(M^{\mathbb{R}}_{n\times n}(I_1, I_2), +, .)$ is a non-commutative refined neutrosophic ring under matrix multiplication.

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Theorem 1.5 ([9]). *Let* $(R(I_1, I_2), +, .)$ *be any refined neutrosophic ring. Then* $(R(I_1, I_2), +, .)$ *is a ring.*

Definition 1.6 ([9]). Let $(R(I_1, I_2), +, .)$ be a refined neutrosophic ring and let $J(I_1, I_2)$ be a nonempty subset of $R(I_1, I_2)$. $J(I_1, I_2)$ is called a refined neutrosophic subring of $R(I_1, I_2)$ if $(J(I_1, I_2), +, .)$ is itself a refined neutrosophic ring. It is essential that $J(I_1, I_2)$ contains a proper subset which is a ring. Otherwise, $J(I_1, I_2)$ will be called a pseudo refined neutrosophic subring of $R(I_1, I_2)$.

Example 1.7 ([9]). Let $(R(I_1, I_2), +, .) = (\mathbb{Z}(I_1, I_2), +)$ be the refined neutrosophic ring of integers. The set $J(I_1, I_2) = n\mathbb{Z}(I_1, I_2))$ for all positive integer n is a refined neutrosophic subring of $R(I_1, I_2)$.

Example 1.8 ([9]). Let $(R(I_1, I_2), +, .) = (\mathbb{Z}_6(I_1, I_2), +)$ be the refined neutrosophic ring of integers modulo 6. The set

$$\begin{aligned} J(I_1, I_2) \quad = \quad & \{(0,0,0), (0, I_1, 0), (0, 0, I_2), (0, I_1, I_2), \\ & (0, 2I_1, 0), (0, 0, 2I_2), (0, 2I_1, 2I_2), \\ & (0, 3I_1, 0), (0, 0, 3I_2), (0, 3I_1, 3I_2), \\ & (0, 4I_1, 0), (0, 0, 4I_2), (0, 4I_1, 4I_2), \\ & (0, 5I_1, 0), (0, 0, 5I_2), (0, 5I_1, 5I_2)\}. \end{aligned}$$

is a refined neutrosophic subring of $R(I_1, I_2)$.

Definition 1.9 ([9]). Let R be a non-empty set and let $+$ and . be two binary operations on R such that:

(i) $(R, +)$ is an abelian group.

(ii) $(R, .)$ is a semigroup.

(iii) There exists $x, y, z \in R$ such that

$$x(y + z) = xy + xz, (y + z)x = yx + zx.$$

(iv) R contains elements of the form (x, yI_1, zI_2) with $x, y, z \in \mathbb{R}$ such that $y, z \neq 0$ for at least one value.

Then $(R, +, .)$ is called a pseudo refined neutrosophic ring.

Example 1.10 ([9]). Let R be a set given by

$$R = \{(0,0,0), (0, 2I_1, 0), (0, 0, 2I_2), (0, 4I_1, 0), (0, 0, 4I_2), (0, 6I_1, 0), (0, 0, 6I_2)\}.$$

Then $(R, +, .)$ is a pseudo refined neutrosophic ring where $+$ and . are addition and multiplication modulo 8.

Example 1.11 ([9]). Let $R(I_1, I_2) = \mathbb{Z}_{12}(I_1, I_2)$ be a refined neutrosophic ring of integers modulo 12 and let T be a subset of $\mathbb{Z}_{12}(I_1, I_2)$ given by

$$\begin{aligned} T \quad = \quad & \{(0,0,0), (0, 2I_1, 0), (0, 0, 2I_2), (0, 4I_1, 0), (0, 0, 4I_2), (0, 4I_1, 0), (0, 0, 4I_2), \\ & (0, 6I_1, 0), (0, 0, 6I_2)(0, 8I_1, 0), (0, 0, 8I_2), (0, 10I_1, 0), (0, 0, 10I_2)\}. \end{aligned}$$

It is clear that $(T, +, .)$ is a pseudo refined neutrosophic ring.

2 Main Results

In this section except if otherwise stated, all refined neutrosophic rings $R(I_1, I_2)$ will be assumed to be commutative refined neutrosophic rings with unity.

Definition 2.1. Let $R(I_1, I_2)$ be a refined neutrosophic ring.

(i) An element $x \in R(I_1, I_2)$ is called an idempotent element if $x^2 = x$.

(ii) A nonzero element $x \in R(I_1, I_2)$ is called a zero divisor if there exists a nonzero element $y \in R(I_1, I_2)$ such that $xy = 0$.

(ii) A nonzero element $x \in R(I_1, I_2)$ is said to be invertible if there exists an element $y \in R(I_1, I_2)$ such that $xy = 1$.

Example 2.2. Consider the refined neutrosophic rings $\mathbb{Z}_2(I_1, I_2)$ and $\mathbb{Z}_3(I_1, I_2)$ of integers modulo 2 and 3 respectively. The element $x = (1, I_1, I_2)$ is idempotent in $\mathbb{Z}_2(I_1, I_2)$ and the element $x = (1, 0, I_2)$ is invertible in $\mathbb{Z}_3(I_1, I_2)$. The elements $x = (0, I_1, 0)$ and $y = (1, I_1, 0)$ are zero divisors in $\mathbb{Z}_2(I_1, I_2)$ because $xy = (0, 0, 0)$.

Definition 2.3. Let $R(I_1, I_2)$ be a refined neutrosophic ring. Then $R(I_1, I_2)$ is called a refined neutrosophic integral domain if it has no zero divisors.

Theorem 2.4. *$\mathbb{Z}_n(I_1, I_2)$ is not a refined neutrosophic integral domain for all n.*

Proof. For nonzero integers α, β, let $x = (0, \alpha I_1, 0)$ and $y = (0, \beta(1 - I_1), 0)$ be arbitrary elements in $\mathbb{Z}_n(I_1, I_2)$. It is clear that x and y are zero divisors since $xy = (0, 0, 0)\ \forall\ \alpha, \beta \in \mathbb{Z}^+$ and therefore, $\mathbb{Z}_n(I_1, I_2)$ is not a refined neutrosophic integral domain for all n. $\square$

Corollary 2.5. *Let $R(I_1, I_2)$ be a refined neutrosophic ring where R is an integral domain. Then $R(I_1, I_2)$ is not necessarily a refined neutrosophic integral domain.*

Theorem 2.6. *If $R = \mathbb{Z}_n$ is a ring of integers modulo n, then $R(I_1, I_2)$ is a finite refined neutrosophic ring of order n^3.*

Definition 2.7. Let F be a field. A refined neutrosophic field is a set $F(I_1, I_2)$ generated by F, I_1, I_2 defined by

$$F(I_1, I_2) = \{(x, yI_1, zI_2) : x, y, z \in F\}.$$

Example 2.8. (i) $\mathbb{Q}(I_1, I_2), \mathbb{R}(I_1, I_2)$ and $\mathbb{C}(I_1, I_2)$ of rational, real and complex numbers are examples of refined neutrosophic fields.

(ii) $\mathbb{Z}_p(I_1, I_2)$ for a prime p is a refined neutrosophic field.

It is worthy of noting that refined neutrosophic fields are not fields in the classical sense since not every element of refined neutrosophic fields is invertible.

Definition 2.9. Let $R(I_1, I_2)$ be a refined neutrosophic ring and let J be a nonempty subset of $R(I_1, I_2)$. Then J is called a refined neutrosophic ideal of $R(I_1, I_2)$ if the following conditions hold:

(i) J is a refined neutrosophic subring of $R(I_1, I_2)$.

(ii) For every $x \in J$ and $r \in R(I_1, I_2)$, we have $xr \in J$.

If J is a pseudo refined neutrosophic subring of $R(I_1, I_2)$, and, for every $x \in J$ and $r \in R(I_1, I_2)$, we have $xr \in J$, then J is called a pseudo refined neutrosophic ideal of $R(I_1, I_2)$.

Example 2.10. In the refined neutrosophic ring $\mathbb{Z}_4(I_1, I_2)$ of integers modulo 4, the set $J = \{(0, 0, 0), (2, 0, 0), (0, 2I_1, 0), (0, 0, 2I_2), (2, 2I_1, 2I_2)\}$ is a refined neutrosophic ideal.

Example 2.11. Consider

$$\begin{aligned} \mathbb{Z}_3(I_1, I_2) \quad = \quad & \{(0,0,0), (1,0,0), (2,0,0), (0,0,I_2), (0,0,2I_2), (0,I_1,0), \\ & (0,I_1,I_2), (0,I_1,2I_2), (0,2I_2,0), (0,2I_1,I_1), (0,2I_1,2I_2), \\ & (1,0,I_2), (1,0,2I_2), (1,I_1,0), (1,I_1,I_2), (1,I_1,2I_2), (1,2I_2,0), \\ & (1,2I_1,I_1), (1,2I_1,2I_2), (2,0,I_2), (2,0,2I_2), (2,I_1,0), \\ & (2,I_1,I_2), (2,I_1,2I_2), (2,2I_2,0), (2,2I_1,I_1), (2,2I_1,2I_2)\} \end{aligned}$$

the refined neutrosophic ring of integers modulo 3. The set

$$J = \{(0, 0, 0), (0, I_1, 0), (0, 0, I_2), (0, 2I_1, 0), (0, 0, 2I_2)\}$$

is a pseudo refined neutrosophic ideal. Consider the set

$$K = \{(0, 0, 0), (2, 0, 0), (0, 2I_1, 0), (0, 0, 2I_2), (2, 2I_1, 2I_2)\}.$$

It can easily be shown that K is not a refined neutrosophic ideal of $\mathbb{Z}_3(I_1, I_2)$ and J is the only pseudo refined neutrosophic ideal.

Theorem 2.12. *Let $\{J_k(I_1, I_2)\}_1^n$ be a family of refined neutrosophic ideals (pseudo refined neutrosophic ideals) of a refined neutrosophic ring $R(I_1, I_2)$. Then $\bigcap_1^n J_k(I_1, I_2)\}$ is a refined neutrosophic ideal (pseudo refined neutrosophic ideal) of $R(I_1, I_2)$.*

Definition 2.13. Let $J(I_1, I_2)$ and $K(I_1, I_2)$ be any two refined neutrosophic ideals (pseudo refined neutrosophic ideals) of a refined neutrosophic ring $R(I_1, I_2)$. We define the sum $J(I_1, I_2) \oplus K(I_1, I_2)$ by the set

$$J(I_1, I_2) \oplus K(I_1, I_2) = \{x + y : x \in J(I_1, I_2), y \in K(I_1, I_2)\}$$

which can easily be shown to be a refined neutrosophic ideal (pseudo refined neutrosophic ideal) of $R(I_1, I_2)$

Theorem 2.14. *Let $J(I_1, I_2)$ be any refined neutrosophic ideal of a refined neutrosophic ring $R(I_1, I_2)$ and let $K(I_1, I_2)$ be any pseudo refined neutrosophic ideal of $R(I_1, I_2)$. Then:*

(i) $J(I_1, I_2) \oplus J(I_1, I_2) = J(I_1, I_2)$.

(ii) $K(I_1, I_2) \oplus K(I_1, I_2) = K(I_1, I_2)$.

(iii) $J(I_1, I_2) \oplus K(I_1, I_2)$ *is a pseudo refined neutrosophic ideal of* $R(I_1, I_2)$.

(iv) $x + J = J \;\forall\; x \in J$.

Definition 2.15. Let J be a refined neutrosophic ideal of the refined neutrosophic ring $R(I_1, I_2)$. The set $R(I_1, I_2)/J$ is defined by

$$R(I_1, I_2)/J = \{r + J : r \in R(I_1, I_2)\}.$$

If $\bar{x} = r_1 + J$ and $\bar{y} = r_2 + J$ are two arbitrary elements of $R(I_1, I_2)/J$ and $\oplus, \odot$ are two binary operations on $R(I_1, I_2)/J$ defined by

$$\begin{aligned} \bar{x} \oplus \bar{y} &= (x + y) + J, \\ \bar{x} \odot \bar{y} &= (xy) + J. \end{aligned}$$

It can be shown that $(R(I_1, I_2)/J, \oplus, \odot)$ is a refined neutrosophic ring with the additive identity J. $(R(I_1, I_2)/J, \oplus, \odot)$ is called a refined quotient neutrosophic ring.

Example 2.16. (i) Let $R = \mathbb{Z}(I_1, I_2)$ be a refined neutrosophic ring of integers and let $J = 2\mathbb{Z}(I_1, I_2)$. It is clear that J is a refined neutrosophic ideal of R. Now, R/J is obtained as follows:

$$\begin{aligned} R/J = \{&J, (1, 0, 0) + J, (0, I_1, 0) + J, (0, 0, I_2) + J, (0, I_1, I_2) + J, \\ &(1, I_1, 0) + J, (1, 0, I_2) + J, (1, I_1, I_2) + J\} \end{aligned}$$

which is a refined neutrosophic ring of order 8.

(ii) Let $S = \mathbb{Z}(I_1, I_2)$ be a refined neutrosophic ring of integers and let $K = 3\mathbb{Z}(I_1, I_2)$. It is also clear that K is a refined neutrosophic ideal of S. Now, S/K is obtained as follows:

$$\begin{aligned} S/K = \{&K, (1, 0, 0) + K, (2, 0, 0) + K, (0, I_1, 0) + K, (0, 2I_1, 0) + K, (0, 0, I_2) + K, \\ &(0, 0, 2I_2) + K, (0, 2I_1, I_2) + K, (0, 2I_1, 2I_2) + K, (0, I_1, I_2) + K, (0, I_1, 2_2) + K, \\ &(1, 0, I_2) + K, (1, I_1, 0) + K, (1, I_1, I_2) + K, (1, 2I_1, 0) + K, (1, 0, 2I_2) + K, \\ &(1, 2I_1, 2I_2) + K, (1, 2I_1, I_2) + K, (1, I_1, 2_2) + K, (2, 0, I_2) + K, \\ &(2, 0, 2I_2) + K, (2, I_1, 0) + K, (2, I_1, I_2) + K, (2, I_1, 2I_2) + K, (2, 2I_1, 0) + K, \\ &(2, 2I_1, I_2) + K, (2, 2I_2, 2I_2) + K\} \end{aligned}$$

which is a refined neutrosophic ring of order 27.

These two examples lead to the following general result:

Theorem 2.17. *Let $R = \mathbb{Z}(I_1, I_2)$ be a refined neutrosophic ring of integers and let $J = n\mathbb{Z}(I_1, I_2)$ be a refined neutrosophic ideal of R. Then*

$$R/J \cong \mathbb{Z}_n(I_1, I_2).$$

Definition 2.18. Let $(R(I_1, I_2), +, .)$ and and $(S(I_1, I_2), +, .)$ be two refined neutrosophic rings. The mapping $\phi : (R(I_1, I_2), +, .) \to (S(I_1, I_2), +, .)$ is called a refined neutrosophic ring homomorphism if the following conditions hold:

(i) $\phi(x+y) = \phi(x) + \phi(y)$.

(ii) $\phi(x.y) = \phi(x).\phi(y)$.

(iii) $\phi(I_k) = I_k \ \forall x, y \in R(I_1, I_2)$ and $k = 1, 2$.

The image of ϕ denoted by $Im\phi$ is defined by the set

$$Im\phi = \{y \in S(I_1, I_2) : y = \phi(x) \text{ for some } x \in R(I_1, I_2)\}.$$

The kernel of ϕ denoted by $Ker\phi$ is defined by the set

$$Ker\phi = \{x \in R(I_1, I_2) : \phi(x) = (0, 0, 0)\}.$$

Epimorphism, monomorphism, isomorphism, endomorphism and automorphism of ϕ are similarly defined as in the classical cases.

Example 2.19. Let $R_1(I_1, I_2)$ and $R_2(I_1, I_2)$ be two refined neutrosophic rings. Let $\phi : R_1(I_1, I_2) \times R_2(I_1, I_2) \to R_1(I_1, I_2)$ be a mapping defined by $\phi(x, y) = x$ and let $\psi : R_1(I_1, I_2) \times R_2(I_1, I_2) \to R_2(I_1, I_2)$ be a mapping defined by $\psi(x, y) = y$ for all $(x, y) \in R_1(I_1, I_2) \times R_2(I_1, I_2)$. Then ϕ and ψ are refined neutrosophic ring homomorphisms.

Example 2.20. Let $\phi : \mathbb{Z}_2(I_1, I_2) \times \mathbb{Z}_2(I_1, I_2) \to \mathbb{Z}_2(I_1, I_2)$ be a refined neutrosophic ring homomorphism defined by $\phi(x, y) = x$ for all $x, y \in \mathbb{Z}_2(I_1, I_2)$. Then

(i)
$$\begin{aligned} Im\phi &= \{(0,0,0), (1,0,0), (0,I_1,0), (0,0,I_2), \\ &\quad (0,I_1,I_2), (1,I_1,0), (1,0,I_2), (1,I_1,I_2)\} \end{aligned}$$

which is a refined neutrosophic subring.

(ii) Also,
$$\begin{aligned} Ker\phi &= \{((0,0,0),(0,0,0)), ((0,0,0),(1,0,0)), ((0,0,0),(0,I_1,0)), \\ &\quad ((0,0,0),(0,I_1,I_2)), ((0,0,0),(0,0,I_2)), ((0,0,0),(1,I_1,0)), \\ &\quad ((0,0,0),(1,0,I_2)), ((0,0,0),(1,I_1,I_2))\} \end{aligned}$$

which is just a subring, not a refined neutrosophic subring and equally not a refined neutrosophic ideal.

This example leads to the following general results:

Theorem 2.21. *Let $\phi : R_1(I_1, I_2) \to R_2(I_1, I_2)$ be a refined neutrosophic ring homomorphism. Then*

(i) $Im\phi$ is a refined neutrosophic subring $R_2(I_1, I_2)$.

(ii) $Ker\phi$ is a subring of R_1.

(iii) $Ker\phi$ is not a refined neutrosophic subring of R_1.

(iv) $Ker\phi$ is not a refined neutrosophic ideal of R_1.

Theorem 2.22. *Let $R = R(I_1, I_2)$ be a refined neutrosophic rings and let $J = J(I_1, I_2)$ be a refined neutrosophic ideal. Then the mapping $\phi : R \to R/J$ defined by $\phi(r) = r + J \ \forall r \in R$ is not a refined neutrosophic ring homomorphism.*

Proof. It is clear that $\phi(r+s) = (r+s)+J = (r+J)+(s+J) = \phi(r)+\phi(s)$ and $\phi(rs) = (rs)+J = (r+J)(s+J) = \phi(r)\phi(s)$. But then, $\phi(I_k) \neq I_k$ for $k = 1, 2$ and so, ϕ is not a refined neutrosophic ring homomorphism. □

This is different from what is obtainable in the classical rings and consequently, classical isomorphism theorems cannot hold in refined neutrosophic rings.

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