ASPG
American Scientific Publishing Group

INTERNATIONAL JOURNAL OF NEUTROSOPHIC SCIENCE

Volume 7, 2020

Editor in Chief: Broumi Said & Florentin Smarandache

ISSN: 2690-6805

Table of Content

International Journal of Neutrosophic Science (IJNS)

Editorial Board

Wadei faris Mohammed Al-omeri, Department of Mathematics, Faculty of Science, Al-Balqa Applied University, Salt 19117, Jordan (wadeimoon1@hotmail.com, wadeialomeri@bau.edu.jo).

Abhijit Saha, Department of Mathematics, Techno College of Engineering Agartala Maheshkhola-799004, Tripura, India (abhijit84.math@gmail.com).

D.Nagarajan, Department of Mathematics, Hindustan Institute of Technology & Science, India (dnagarajan@hindustanuniv.ac.in).

Prem Kumar Singh, Amity Institute of Information Technology, Amity University-Sector 125 Noida-201313, Uttar Pradesh-India (premsingh.csjm@gmail.com).

Avishek Chakraborty, Department of Basic Science, University/College- Narula Institute of Technology Under MAKAUT, India (tirtha.avishek93@gmail.com).

Arindam Dey, Department of Computer Science and Engineering, Saroj Mohan Institute of Technology, Hooghly 712512, West Bengal, India (arindam84nit@gmail.com).

Muhammad Gulistan, Department of Mathematics & Statistics, Hazara University Mansehra, Khyber Pakhtunkhwa, Pakistan(**gulistanmath@hu.edu.pk**).

Mohsin Khalid, The University of Lahore, Pakistan(**mk4605107@gmail.com**).

Hoang Viet Long, Faculty of Information Technology, People's Police University of Technology and Logistics, Bac Ninh, Viet Nam **(longhv08@gmail.com).**

Kishore Kumar P.K, Department of Information Technology, Al Musanna College of Technology, Sultanate of Oman (**kishore2982@gmail.com**).

Mohamed Talea, Laboratory of Information processing, Faculty of Science Ben M'Sik, Morocco (taleamohamed@yahoo.fr)

Assia Bakali, Ecole Royale Navale, Casablanca, Morocco (assiabakali@yahoo.fr).

Tahir Mahmood, Department of Mathematics and Statistics, International Islamic University Islamabad, Pakistan (**tahirbakhat@iiu.edu.pk**).

Faruk KARAASLAN, Çankırı Karatekin University, Faculty of Sciences, Department of Mathematics, 18100, Çankırı, TURKEY(fkaraaslan@karatekin.edu)

Mohamed Abdel-Basset, Department of Computer Science, Zagazig University, Egypt (**analyst_mohamed@yahoo.com, analyst_mohamed@zu.edu.eg).**

Riad Khider AlHamido, Department of mathematics Faculty of Sciences - Al- Furat University, Syria (riad-hamido1983@hotmail.com).

Fahad Mohammed Alsharari, Mathematics Department, College of Science and Human Studies at Hotat Sudair, Majmaah University, Saudi Arabia (f.alsharari@mu.edu.sa).

Maikel Leyva Vázquez, Universidad Politécnica Salesiana, Ecuador (**mleyvaz@gmail.com).**

Ir. Victor Christianto, DDiv. (associate of NSIA), Satyabhakti Advanced School of Theology - Jakarta Chapter, Indonesia(**victorchristianto@gmail.com**).

Xiaohong Zhang, Shaanxi University of Science and Technology, China (zhangxiaohong@sust.edu.cn, zxhonghz@263.net).

Huda E. Khalid, Head of the Scientific Affairs and Cultural Relations, Presidency of Telafer University ,Iraq (hodaesmail@yahoo.com, dr.huda-ismael@uotelafer.edu.iq).

Ranjan Kumar, Department of Mathematics, National Institute of Technology, Adityapur, Jamshedpur, 831014, India. (**ranjank.nit52@gmail.com**).

Choonkil Park, Department of Mathematics, Hanyang University, Republic of Korea (baak@hanyang.ac.kr**).**

Shahzaib Ashraf, Department of Mathematic Abdul Wali Khan University, Mardan 23200, Pakistan (shahzaibashraf@awkum.edu.pk).

Madeleine Al-Tahan, Lebanese International University,Lebanon (madeline.tahan@liu.edu.lb).

Rafif alhabib, Department of mathmatical statistic, Faculty of Sciences, albaath University, Syria. (rafif.alhabib85@gmail.com)

Giorgio Nordo, MIFT - Department of Mathematical and Computer Science, Physical Sciences and Earth Sciences, Messina University, Italy,(giorgio.nordo@unime.it)

Angelo de Oliveira, Departamento Academico de Ciencia da Computacao - Universidade Federal de Rondonia, Brasil, (angelo@unir.br)

A.A.A. Agboola, Department of Mathematics, Federal University of Agriculture, Abeokuta, Nigeria,(agboolaaaa@funaab.edu.ng)

A. A. Salama, Department of Mathematics and Computer Science, Faculty of Science, Port Said University, Port Said, Egypt, (drsalama44@gmail.com)

Akbar Rezaei, Department of Mathematics, Payame Noor University, P.O..Box. 19395-3697, Tehran, IRAN, (rezaei@pnu.ac.ir)

Ahmed Hatip, Department of Mathematics, Gaziantep University, Turkey, (kollnaar5@gmail.com)

Metawee Songsaeng , Department of Mathematics, School of Science, University of Phayao, Phayao 56000, Thailand, (metawee.faith@gmail.com)

Kavikumar Jacob, Department of Mathematics and Statistics, Faculty of Applied Sciences and Technology Universiti Tun Hussein Onn Malaysia, 86400 Malaysia. (kavi@uthm.edu.my)

Kifayat Ullah, Department of Mathematics and Statistics, International Islamic University Islamabad, Pakistan

M.B.Zeina, University of Aleppo, Aleppo-Syria (bisherzeina@alepuniv.edu.sy)

Aim and Scope

International Journal of Neutrosophic Science (IJNS) is a peer-review journal publishing high quality experimental and theoretical research in all areas of Neutrosophic and its Applications. IJNS is published quarterly. IJNS is devoted to the publication of peer-reviewed original research papers lying in the domain of neutrosophic sets and systems. Papers submitted for possible publication may concern with foundations, neutrosophic logic and mathematical structures in the neutrosophic setting. Besides providing emphasis on topics like artificial intelligence, pattern recognition, image processing, robotics, decision making, data analysis, data mining, applications of neutrosophic mathematical theories contributing to economics, finance, management, industries, electronics, and communications are promoted. Variants of neutrosophic sets including refined neutrosophic set (RNS). Articles evolving algorithms making computational work handy are welcome.

Topics of Interest

IJNS promotes research and reflects the most recent advances of neutrosophic Sciences in diverse disciplines, with emphasis on the following aspects, but certainly not limited to:

- Neutrosophic sets
- Neutrosophic algebra
- Neutrosophic topolog
- Neutrosophic graphs
- Neutrosophic probabilities
- Neutrosophic tools for decision making
- Neutrosophic theory for machine learning
- Neutrosophic statistics
- Neutrosophic numerical measures
- Classical neutrosophic numbers
- A neutrosophic hypothesis
- The neutrosophic level of significance
- The neutrosophic confidence interval
- The neutrosophic central limit theorem
- Neutrosophic theory in bioinformatics and medical analytics
- Neutrosophic tools for big data analytics
- Neutrosophic tools for deep learning
- Neutrosophic tools for data visualization
- Quadripartitioned single-valued neutrosophic sets
- Refined single-valued neutrosophic sets
- Applications of neutrosophic logic in image processing
- Neutrosophic logic for feature learning, classification, regression, and clustering

- Neutrosophic knowledge retrieval of medical images
- Neutrosophic set theory for large-scale image and multimedia processing
- Neutrosophic set theory for brain-machine interfaces and medical signal analysis
- Applications of neutrosophic theory in large-scale healthcare data
- Neutrosophic set-based multimodal sensor data
- Neutrosophic set-based array processing and analysis
- Wireless sensor networks Neutrosophic set-based Crowd-sourcing
- Neutrosophic set-based heterogeneous data mining
- Neutrosophic in Virtual Reality
- Neutrosophic and Plithogenic theories in Humanities and Social Sciences
- Neutrosophic and Plithogenic theories in decision making
- Neutrosophic in Astronomy and Space Sciences

Published and typeset in American Scientific Publishing Group (ASPG) is a USA academic publisher, established as LLC company on 2019 at New Orleans, Louisiana, USA. ASPG publishes online scholarly journals that are free of submission charges.

American Scientific Publishing Group (ASPG) LLC,
New Orleans, USA
Mailing Address: 625 Wright Ave, Gretna, LA 70056, USA
Phone: +1(504) 336-3385
Fax: +1-888-417-7465
e-mail: manager@americaspg.com
www.americaspg.com

Plithogenic n- Super Hypergraph in Novel Multi -Attribute Decision Making

Florentin Smarandache [1]* and Nivetha Martin [2]

[1] Department of Mathematics, University of New Mexico, Gallup, NM 87301,USA,

smarand@unm.edu

[2] Department of Mathematics, Arul Anandar College (Autonomous), Karumathur, India

nivetha.martin710@gmail.com

* Correspondence: smarand@unm.edu

Abstract

An optimal decision-making environment demands feasible Multi-Attribute Decision-Making methods. Plithogenic n – Super Hypergraph introduced by Smarandache is a novel concept and it involves many attributes. This article aims to bridge the concept of Plithogenic n-Super Hypergraph in the vicinity of optimal decision making. This research work introduces the novel concepts of enveloping vertex, super enveloping vertex, dominant enveloping vertex, classification of the dominant enveloping vertex (input, intervene, output dominant enveloping vertices), plithogenic connectors. An application of Plithogenic n-super hypergraph in making optimum decisions is discussed under various decision-making scenarios. Several insights are drawn from this research work and will certainly benefit the decision-makers to overcome the challenges in building decisions.

Keywords: Plithogenic n-super hypergraph, decision making, attributes, dominant enveloping vertex.

1.Introduction

It is quite inevitable for each one is taking up the role of decision-maker in their instances of life. Decision making isn't an activity, but a process comprising of many tasks. The desired outcomes of decisions are a success, if it fails then the process has to be revived. The cognitive contribution in choosing the best alternative with the consideration of criteria and criteria weights is not a simple task; it demands sequential steps and scientific approach. The managerial of either a start-up company or a multinational organization must possess the skills of making optimal decisions to make their companies march in the path of victory. The decision-making environment is not deterministic always and it is characterized mostly by uncertainty and impreciseness, to tackle these challenges the decision-makers are moving towards Multi-Criteria Decision Making methods (MCDM) to design optimal solutions.

DOI: 10.5281/zenodo.3876206

Received: March 07, 2020 Revised: April 30, 2020 Accepted: May 18, 2020

MCDM has been explored for the past seventy years and it has been broadly divided into MADM (Multi-Attribute Decision Making) and MODM (Multi-Objective Decision Making) [1]. The former helps in the selection of the alternatives based on attribute description and the latter is based on optimization of decision maker's multi objectives. MADM methods are gaining impetus in the decision-making environment as they are highly developed with robust mathematical principles and also these methods prevent small and medium-sized companies in purchasing expensive software or executing erudite systems of the decision process. MADM methods are more operative and the most widely used methods are Analytic Hierarchy Process (AHP) and Analytic Network Process (ANP) introduced by Satty [2]; Decision Making Trial and Evaluation Laboratory (DEMATEL) developed by Tzeng and Huang [3]; The Technique for Order Preferences by Similarity to an Ideal Solution (TOPSIS) method was proposed by Hwang and Yoon; Vlse Kriterijumska Optimizacija I Kompromisno Resenje (VIKOR) method was developed by Tzeng and Huang.

In the above described MADM methods, the major steps involved are (i) formulation of initial decision-making matrix (IDMM) comprising of values representing the degree of fulfilling the criteria by the alternatives. (ii) Normalization of the values in IDMM (iii) Determination of criterion weight (iv) Ranking of alternatives. In these MADM methods, the alternatives are ranked based only on the extent of criteria satisfaction, but the consistency of ranking is not checked as these methods do not provide space for it. The selection of alternatives is based only on attribute satisfaction and it does not consider any other input such as previous data related to the impacts or the effects of these kinds of chosen alternatives. These other inputs are not brought into the decision-making environment and the previous feedback review is also not incorporated into the decision-making environment.

Let us consider the possible situations of exercising decision making in a company, for example in the selection of personnel, methods of production, the extension of product features, the above instances of decision-making situations are not new to companies, as these processes are routine. In making decisions, certainly the managerial will be aware of the desired target to be achieved and will employ his previous experience or the feedback received by him from various sources as inputs in the selection of alternatives. The above-said MADM does not provide space for such kinds of feedback inputs. A comprehensive decision-making environment must comprise of alternative selection based on several inputs such as attribute satisfaction, feedback, and impact of attributes towards the desired output. To overcome such shortcomings, a novel MADM method is introduced in this research work with the integration of Plithogenic – super Hypergraphs introduced by Smarandache [4]. Plithogenic sets introduced by Smarandache [5] are the extension of neutrosophic sets that are characterized by truth, indeterminacy, and false functions. The robust nature of neutrosophic sets inspired several researchers to employ it in diverse fields. Gayathri et al [6] developed multiple attribute group decision making neutrosophic environments with the utilization of Jaccard index measures. Muhammad Naveed Jafar et al [7] used neutrosophic soft matrices with score function to evaluate new technology in Agriculture. Ajay et al [8] developed a single-valued triangular neutrosophic number approach of multi-objective optimization based on simple ratio analysis based on the MCDM method. Luis Andrés Crespo Berti [9] applied a neutrosophic system to tax havens with a criminal approach. Abdel-Basset [10] developed three-way decisions based on neutrosophic sets and AHP-QFD framework for supplier selection problem, also developed a hybrid neutrosophic group ANP-TOPSIS framework for supplier selection

DOI: 10.5281/zenodo.3876206

problem [11].Plithogenic sets that deal with attributes, degree of appurtenance, and degree of contradiction have been extensively used in decision making with quality function deployment for selecting supply chain sustainability metrics and for evaluating hospital medical care systems by Abdel-Basset et al [12,13]. In these decision making approaches plithogenic aggregation operations are used to make decisions based on the best and worst criteria with decision-makers' opinions as inputs. These methods of decision making focus primarily on evaluation and selection of alternatives based on combining plithogenic aggregation operators and do not provide space for any graphical representation of the relational impacts between the alternatives.

In the proposed MADM each alternative is considered as an object encompassing several attributes. The decision-making environment consists of three kinds of objects namely input, intervention, and output. The alternatives are taken as inputs, desired target as output, and intervene (intermediate) objects are the objects that combine with the input objects. A company always works on target based. Personnel design project and work on it tirelessly to achieve various sets of goals. The project never gets accomplished with the attainment of a single goal but a series of goals. The success of a project is defined in various dimensions. In the proposed MADAM, the selection of alternatives is based on the degree of association between the attributes of inputs and the attributes of outputs independent or dependent on intervening objects. This decision-making approach is more comprehensive than the conventional MADM methods as it incorporates attributes and feedback into the input system. Also, many times the company prefers collaborative works and the effects of combined initiatives are high. Conventional MADM does not provide space for it, but the proposed MADM is designed exclusively for measuring the optimal combination. Also in MADM methods, graphical representations are not made so far to represent alternatives, criteria, and their relationship. In this novel MADM, plithogenic –n super hypergraphs are used to represent the objects as enveloping vertices and the association between the vertices by plithogenic connectors.

The article is structured as follows: Section 2 introduces new concepts used in novel MADM; section 3 presents the application of novel MADM in optimal decision making; section 4 discusses the results and the last section concludes the work.

2. Preliminaries

2.1 Enveloping vertex

A vertex representing an object comprising of attributes and sub-attributes in the graphical representation of a multi attribute decision-making environment.

For instance

Let us consider Personnel (V) as an input object, this input has a vital role in target achievement, the output object. These attributes are like databases.

The attributes like Qualification (V1), Age (V2), Experience (V3) are taken into consideration

Attribute sets = {Qualification, Age, Experience}

Qualification = {Graduation, Graduation with additional degree}

Age = {25-35, 36-45}

Experience = { Local, National ,International }

DOI: 10.5281/zenodo.3876206

Local {0-5,6-10}, National{0-3,4-6}, International {0-2,2-5}

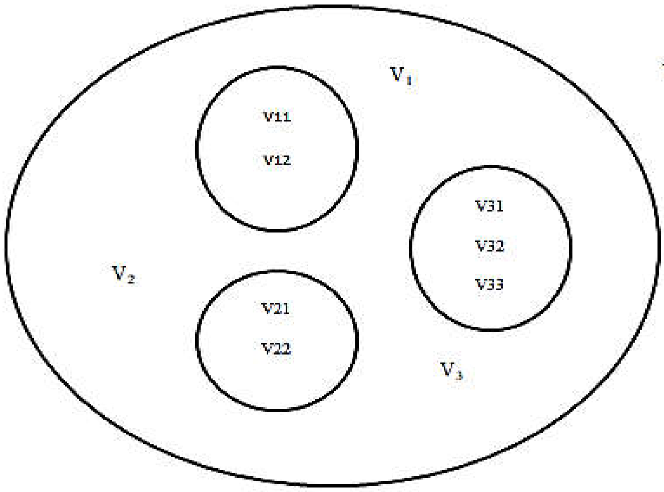

Fig.2.1Enveloping vertex

Thus an enveloping vertex comprises hyperedges, where each hyperedge represents values of the attributes.

2.2 Super Enveloping vertex

An enveloping vertex comprises of Super hyper edges

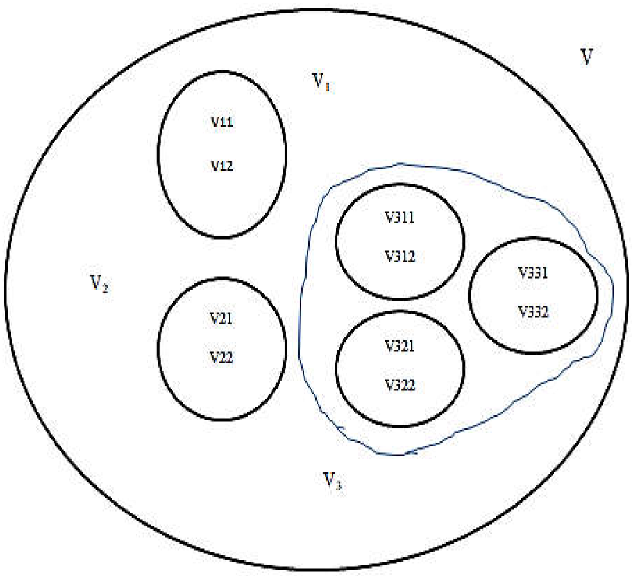

Fig.2.2 Super Enveloping vertex

DOI: 10.5281/zenodo.3876206

2.3 Dominant Enveloping Vertex

An enveloping vertex is with dominant attribute values

Attribute sets = {Qualification, Age, Experience}

The dominant attribute values

Qualification = {Graduation, **Graduation with additional degree**}

Age = {25-35, **36-45**}

Experience = { Local, National ,**International** }

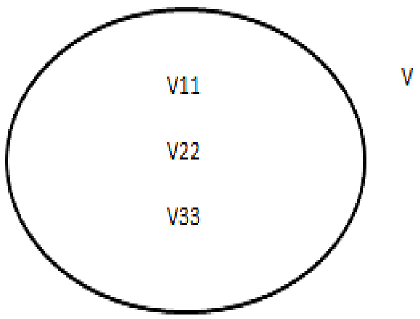

Fig.2.3 Dominant Enveloping Vertex

2.4 Dominant Super Enveloping Vertex

A super enveloping vertex with dominant attribute values

Attribute sets = {Qualification, Age, Experience}

The dominant attribute values

Qualification = {Graduation, **Graduation with additional degree**}

Age = {25-35, **36-45**}

Experience = { Local, National ,**International** }

Local {0-5,6-10}, National{0-3,4-6}, International {**0-2**,**2-5**}

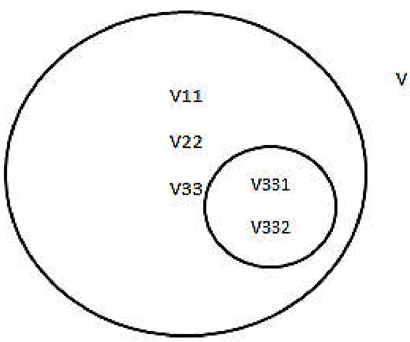

Fig.2.4 Dominant Super Enveloping Vertex

DOI: 10.5281/zenodo.3876206

2.5 Classification of Dominant Enveloping Vertex

The dominant enveloping vertex set are classified as input, intervene and output based on the nature of object's representation.

2.6 Plithogenic Connectors

The connectors associate the input enveloping vertex with output enveloping vertex. These connectors associate the effects of input attributes to output attributes and these connectors are weighted by plithogenic weights.

Let us consider the MADM environment with the product as input object, advertising as intervene object and product success as the output object

Product is the input enveloping vertex, advertising as intervening enveloping vertex and product success as the output enveloping vertex.

Input attributes = {Design, Price}

Design = {**creative**, conventional}

Price = {High, **moderate**, low}

Intervene attributes = { Target group, Medium of advertising}

Target group ={**female**, children}

Medium of advertising = { social networks, **media**}

Output Attributes = { Profit, Customer Acquisition, Product Reach}

Profit = { **Expected**, Beyond the target}

Customer Acquisition = { High, **Extremely High**}

Product Reach = { **National**, International}

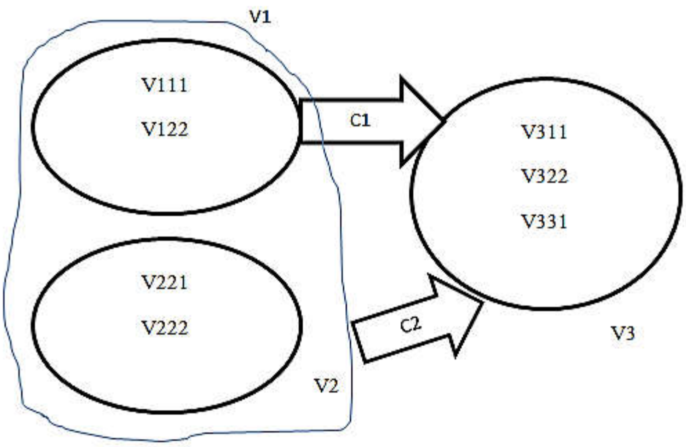

Fig.2.5 Plithogenic Connectors

C1 is the simple plithogenic connector representing the relation between the dominant input attributes to dominant output attributes.

C2 is the combined plithogenic connector representing the relation between the combined dominant input and intervene attributes to dominant output attributes.

Dominant Attribute Relational Matrix Representation

	V311	V322	V331
V111	0.5	0.2	0.3
V122	0.6	0.7	0.8
V111,V221	0.5	0.6	0.4
V111,V222	0.6	0.3	0.8
V122,V221	0.4	0.6	0.8
V122,V222	0.4	0.6	0.7

3. Application of Novel MADM method

3.1 Description of Decision-making Environment

COVID 19 has locked the academic activities to a great extent; the stratagem of Work from Home is employed by the teaching fraternity to engage the learners. One of the biggest challenges to teaching community lies in handling online learning forums and they are badly in need of exposure to the E-learning system of education. To make academicians surpass this task, educational institutions are offering various online courses and organize E-programmes to enhance the professional competency of faculty in partnership with several industries. In this period of the lockdown, the linkage between industries and institutions is getting enhanced in developing countries especially in India. The companies enter institutions as academic partners in establishing virtual laboratories and entertain many online programs in the form of webinars, online courses, and software training programs to handle online classes. The conduct of such programs will certainly contribute to the professional efficiency of faculty. Suppose if an institution decides to conduct any one of the forms of the online program, then it has to decide whether to conduct the program in partnership with industry or independently and also the decision of selecting the kind of online program is based on the feedback acquired from other institutions on the previous organization of such programs. The institution before organizing such programs should decide the component of professional efficiency to be enhanced and determine the contributing factors of the online program towards the same. An optimal solution to this decision-making situation is determined by using the representation of Plithogenic –n Super hypergraph and novel MADM method based on attributes. This decision-making method involves not only the selection process of alternatives based on criteria alike other multi-attribute decision-making methods but it provides space for the selection of alternatives independent or dependent on other alternatives based on their attributes. The outcome of decision making is also considered in the decision-making process. The selection of the alternatives is based on attributes of input objects, intervene objects and output objects.

In this decision-making environment there exist five objects [3 input objects, 1 intervene object and 1 output object] that are represented by enveloping vertices. The input enveloping vertices are Webinars, online courses, training programs on computer languages, intervene enveloping vertex is Industrial partnership and the output enveloping vertex is Professional Efficiency. The description of the attributes of the objects are presented in Table 3.1

Table 3.1 Description of Attributes

Vertex	Representation	Vertex Attributes		Vertex Sub Attributes			
V1	Webinars	V_{11}	Focus	V_{111}	General	V_{1111}	Education
						V_{1112}	Health
						V_{1113}	Psychology
				V_{112}	Specific	V_{1121}	Physics
						V_{1122}	Chemistry
						V_{1123}	Mathematics
						V_{1124}	Engineering
		V_{12}	Resource persons	V_{121}	Local	V_{1211}	within the college
						V_{1212}	neighboring colleges
				V_{122}	National	V_{1221}	AICTE affiliated
						V_{1222}	Non-AICTE affiliated
				V_{123}	International	V_{1231}	Affiliation with the host college
						V_{1232}	Non-affiliation with the host college
		V_{13}	Duration	V_{131}	Day	V_{1311}	One day
						V_{1312}	Two days
						V_{1313}	Three days
				V_{132}	Week	V_{1321}	One
						V_{1322}	Two
		V_{14}	Target Group	V_{141}	Students	V_{1411}	Engineering
						V_{1412}	Non-Engineering

				V_{142}	Research Scholars	V_{1421}	Engineering
						V_{1422}	Non-Engineering
				V_{143}	Academicians	V_{1431}	Engineering
						V_{1432}	Non-Engineering
V2	Online courses	V_{21}	Course nature	V_{211}	Basic	V_{2111}	remembrance
						V_{2112}	understanding
				V_{212}	Moderate	V_{2121}	understanding
						V_{2122}	application
				V_{213}	Advanced	V_{2131}	Analysis
						V_{2132}	Evaluation
		V_{22}	Course Delivery	V_{221}	Synchronous	V_{2211}	Zoom
						V_{2212}	Zoho
						V_{2213}	Examineer
						V_{2214}	Google meet
				V_{222}	Asynchronous	V_{2221}	Google Classroom
						V_{2222}	Youtube upload
		V_{23}	Duration	V_{231}	Day	V_{2311}	One day
						V_{2312}	Two days
						V_{2313}	Three days
				V_{232}	Week	V_{2321}	One
						V_{2322}	Two
		V_{24}	Target Group	V_{241}	Students	V_{2411}	Engineering
						V_{2412}	Non-Engineering
				V_{242}	Research Scholars	V_{2421}	Engineering
						V_{2422}	Non-Engineering
				V_{243}	Academicians	V_{2431}	Engineering
						V_{2432}	Non-Engineering
V3	Training programme on Computer languages	V_{31}	Course nature	V_{311}	Basic	V_{3111}	remembrance
						V_{3112}	understanding
				V_{312}	Moderate	V_{3121}	understanding

DOI: 10.5281/zenodo.3876206

							V_{3122}	application
				V_{313}	Advanced		V_{3131}	Analysis
							V_{3132}	Evaluation
		V_{32}	Course Delivery	V_{321}	Synchronous		V_{3211}	Zoom
							V_{3212}	Zoho
							V_{3213}	Examineer
							V_{3214}	Google meet
				V_{322}	Asynchronous		V_{3221}	Google Classroom
							V_{3222}	Youtube upload
		V_{33}	Duration	V_{331}	Day		V_{3311}	One day
							V_{3312}	Two days
							V_{3313}	Three days
				V_{332}	Week		V_{3321}	One
							V_{3322}	Two
		V_{34}	Target Group	V_{341}	Students		V_{3411}	Engineering
							V_{3412}	Non-Engineering
				V_{342}	Research Scholars		V_{3421}	Engineering
							V_{3422}	Non-Engineering
				V_{343}	Academicians		V_{3431}	Engineering
							V_{3432}	Non-Engineering
V4	Industrial Partnership	V_{41}	MOU	V_{411}	Internship		V_{4111}	Merit-based
							V_{4112}	All students
				V_{412}	Placement		V_{4121}	Merit-based
							V_{4122}	All students
		V_{42}	Financial Support	V_{421}	Equipment purchase		V_{4211}	Partial
							V_{4212}	Complete
				V_{422}	Program organization		V_{4221}	Partial
							V_{4222}	Complete
				V_{431}	Knowledge sharing		V_{4311}	Periodic

DOI: 10.5281/zenodo.3876206

		V_{43}	Technical Support			V_{4312}	Regular
				V_{432}	Experts Visit	V_{4321}	Periodic
						V_{4322}	Regular
V5	Professional Efficiency	V_{51}	Publications	V_{511}	National	V_{5111}	Scopus
						V_{5112}	ICI
				V_{512}	International	V_{5121}	Scopus
						V_{5122}	ICI
		V_{52}	Pedagogy	V_{521}	Teacher-Centered	V_{5211}	lecture
						V_{5212}	chalk & talk
				V_{522}	Learner-Centered	V_{5221}	Blended
						V_{5222}	ICT
		V_{53}	Content preparation	V_{531}	Own	V_{5311}	original
						V_{5312}	modified
				V_{532}	Experts Visit	V_{5321}	Web sources
						V_{5322}	Youtube
		V_{54}	Course Delivery	V_{541}	OER	V_{5411}	Zoom
						V_{5412}	Zoho
						V_{5413}	Google meet
						V_{5414}	Examineer
				V_{542}	Asynchronous	V_{5421}	Google Classroom
						V_{5422}	Youtube upload

In the above table, the input objects such as webinars, online courses, training programs are represented as the input enveloping vertices V1, V2, and V3 in Fig 3.1,3.2 and 3.3 respectively. The intervening object Industrial Partnership is represented as V4 in Fig 3.4. The output object Professional Efficiency is represented as V5 in Fig 3.5.

DOI: 10.5281/zenodo.3876206

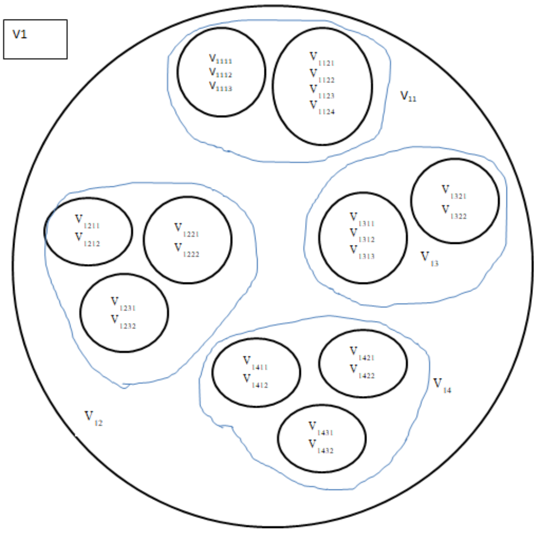

Fig.3.1 Representation of Input Object V1

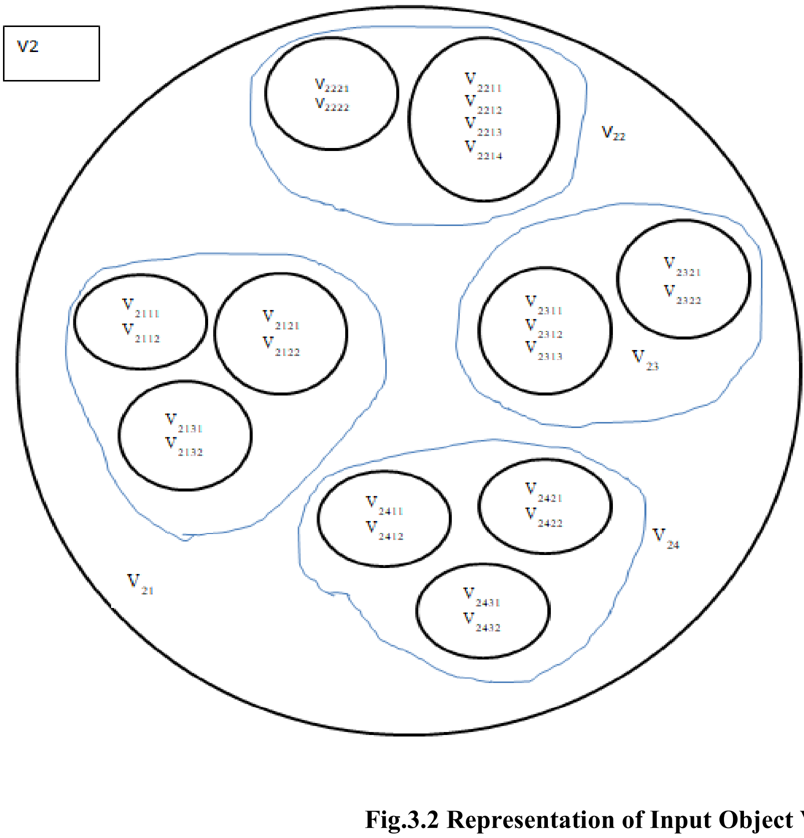

Fig.3.2 Representation of Input Object V2

DOI: 10.5281/zenodo.3876206

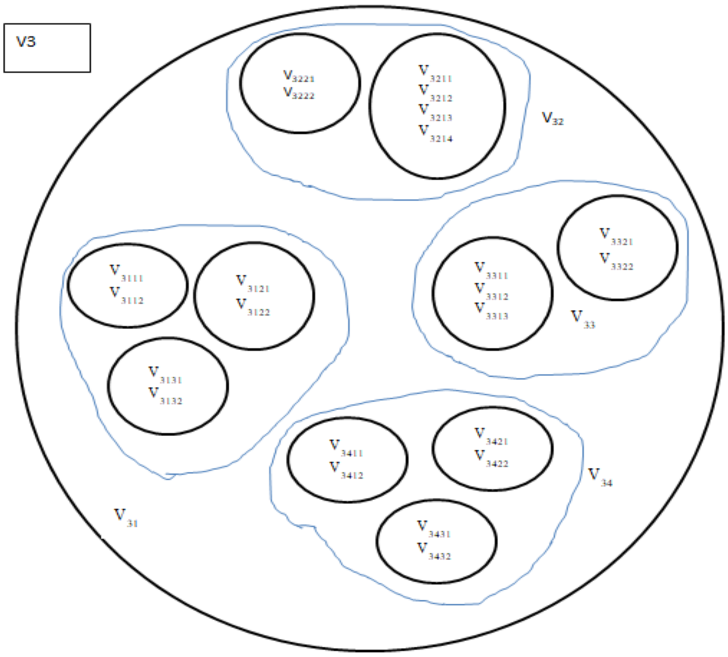

Fig.3.3 Representation of Input Object V3

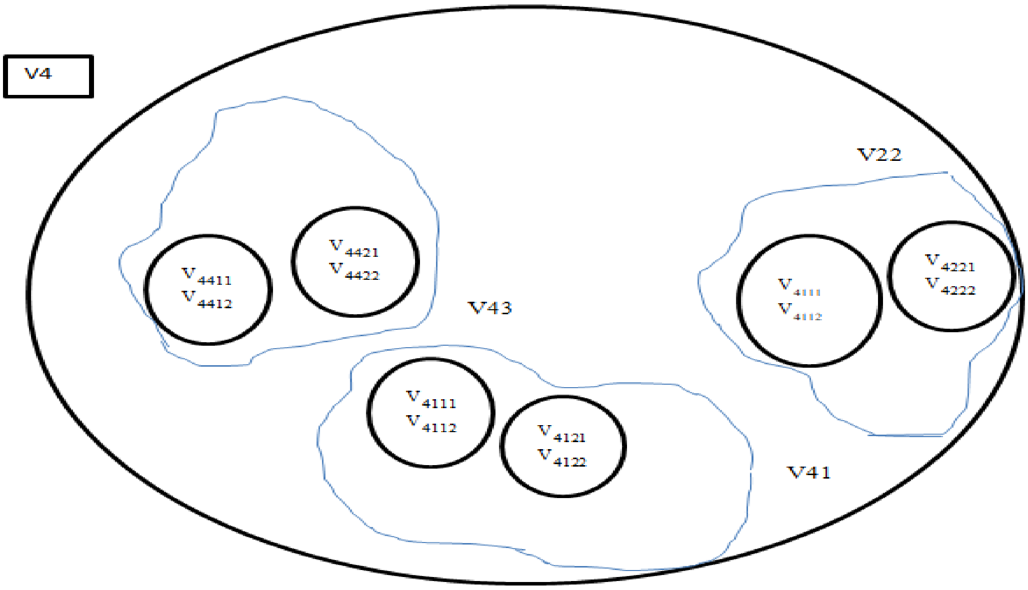

Fig.3.4 Representation of Intervening Object V4

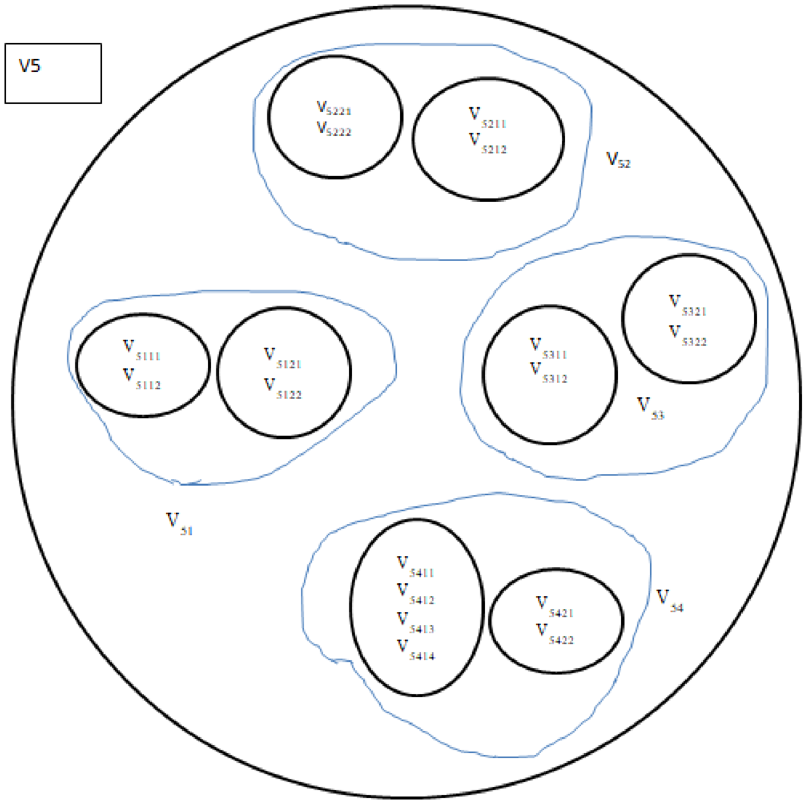

Fig.3.5 Representation of Output Object V5

Each enveloping vertices comprises of many attribute and sub-attribute values. To determine the desired output with and without the combination of input and intervene objects, the dominant attributes are chosen by the decision-makers. The dominant attribute values of the objects are represented in Table 3.2

Table 3.2 Representation of Dominant Attributes

Vertex	Representation	Vertex Attributes		Vertex Sub Attributes			
V1	Webinars	V_{11}	Focus	V_{111}	General	V_{1111}	Education
		V_{12}	Resource persons	V_{122}	National	V_{1221}	AICTE affiliated
		V_{13}	Duration	V_{131}	Day	V_{1312}	Two days
		V_{14}	Target Group	V_{143}	Academicians	V_{1431}	Engineering

DOI: 10.5281/zenodo.3876206

V2	Online courses Course Nature,	V_{21}	Course nature	V_{212}	Moderate	V_{2122}	application
		V_{22}	Course Delivery	V_{222}	Asynchronous	V_{2221}	Google Classroom
		V_{23}	Duration	V_{231}	Day	V_{2312}	Two days
		V_{24}	Target Group	V_{243}	Academicians	V_{2431}	Engineering
V3	Training program on Computer languages	V_{31}	Course nature	V_{311}	Moderate	V_{3122}	application
		V_{32}	Course Delivery	V_{321}	Synchronous	V_{3214}	Google meet
		V_{33}	Duration	V_{332}	Week	V_{3321}	One
		V_{34}	Target Group	V_{343}	Academicians	V_{3431}	Engineering
V4	Industrial Partnership	V_{41}	MOU	V_{412}	Placement	V_{4121}	Merit-based
						V_{4122}	All students
		V_{42}	Financial Support	V_{422}	Program organization	V_{4222}	Complete
		V_{43}	Technical Support	V_{432}	Experts Visit	V_{4322}	Regular
						V_{4312}	Regular
V5	Professional Efficiency	V_{51}	Publications	V_{511}	National	V_{5111}	Scopus
		V_{52}	Pedagogy	V_{522}	Learner-Centered	V_{5221}	Blended
		V_{53}	Content preparation	V_{531}	Own	V_{5311}	original
		V_{54}	Course Delivery	V_{542}	Asynchronous	V_{5421}	Google Classroom

The Dominant Enveloping vertices are presented in Fig 3.6

DOI: 10.5281/zenodo.3876206

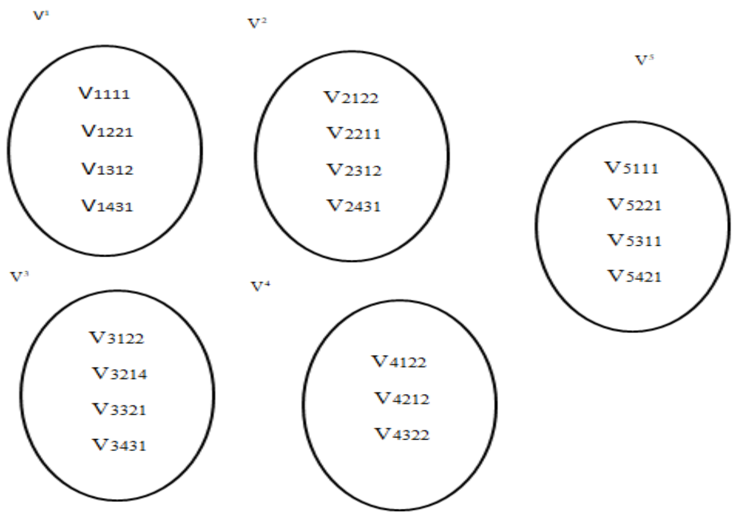

Fig. 3.6 Dominant Enveloping vertices

3.1 Decision-Making Scenario I

The institution is certain of the dominant sub-attributes and makes decisions based on dominant attributes of the input objects. The graphical representation of attribute relation between input dominant enveloping vertices and the output dominating attribute vertex with simple plithogenic fuzzy connectors is presented in Fig 3.7

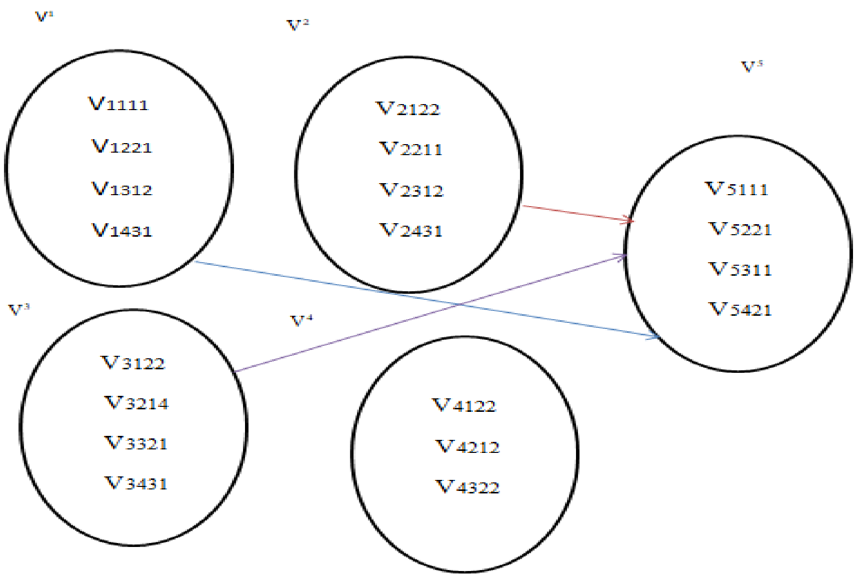

Fig.3.7 Representation of Decision-Making Scenario I

DOI: 10.5281/zenodo.3876206

The dominant attribute relational matrix representation between the input objects on the output objects is presented as follows

	V_{5111}	V_{5221}	V_{5311}	V_{5421}
V_{1111}	0.55	0.2	0.5	0.6
V_{1221}	0.8	0.5	0.5	0.8
V_{1312}	0.75	0.6	0.6	0.9
V_{1431}	0.65	0.8	0.1	0.4
V_{2122}	0.5	0.9	0.3	0.6
V_{2211}	0.3	0.5	0.6	0.8
V_{2312}	0.45	0.4	0.4	0.9
V_{2431}	0.6	0.8	0.2	0.4
V_{3122}	0.85	0.2	0.8	0.8
V_{3214}	0.9	0.3	0.9	0.9
V_{3321}	0.5	0.55	0.5	0.4
V_{3431}	0.6	0.8	0.2	0.4

The frequency matrix as discussed by [14] shall be constructed to rank the dominant attributes of input objects contributing to the dominant attribute of the output object. This is a simple decision-making environment as it does not involve the role of an intervening object.

3.2 Decision Making Scenario II

The institution is certain of the dominant sub-attributes and makes decisions based on dominant attributes of the input and intervene objects. The graphical representation of attribute relation between input and intervene dominant enveloping vertices and the output dominating attribute vertex with combined plithogenic fuzzy connectors is presented in Fig.3.8

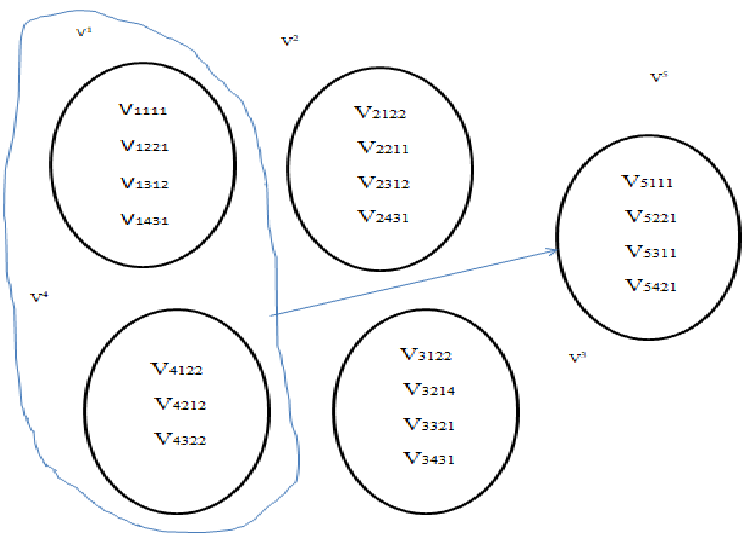

Fig.3.8 Representation of Decision-Making Scenario II

DOI: 10.5281/zenodo.3876206

The dominant attribute relational matrix representation between the input and intervene objects on the output objects is presented as follows

	V_{5111}	V_{5221}	V_{5311}	V_{5421}
V_{1111},V_{4122}	0.65	0.6	0.5	0.6
V_{1111},V_{4212}	0.8	0.65	0.7	0.8
V_{1111},V_{4322}	0.56	0.7	0.9	0.6
V_{1221},V_{4122}	0.75	0.8	0.85	0.6
V_{1221},V_{4212}	0.9	0.6	0.95	0.8
V_{1221},V_{4322}	0.6	0.8	0.45	0.9
V_{1312},V_{4122}	0.53	0.7	0.75	0.7
V_{1312},V_{4212}	0.43	0.5	0.6	0.7
V_{1312},V_{4322}	0.5	0.6	0.78	0.7
V_{1431},V_{4122}	0.62	0.85	0.8	0.69
V_{1431},V_{4212}	0.67	0.78	0.7	0.63
V_{1431},V_{4322}	0.6	0.89	0.58	0.7

The frequency matrix shall be constructed to rank the combined dominant attributes of input and intervene objects contributing to the dominant attribute of the output object. This is a little complex decision-making environment as it involves the role of an intervening object. Fig 3.9 presents the graphical representation of it.

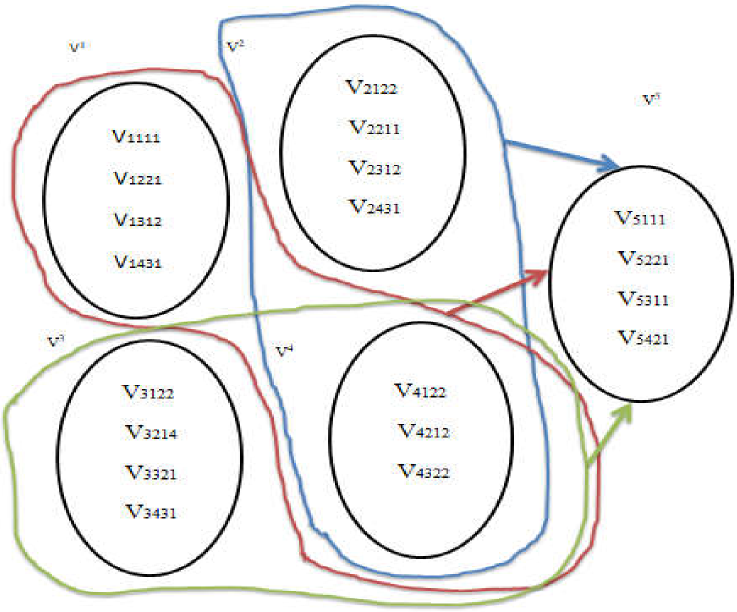

Fig.3.9 Representation of Decision-Making Scenario II with Intervening object

3.3 Decision Making Scenario III

The institution is certain of the dominant sub-attributes. Let us consider a situation, suppose if the institution decides to conduct a webinar with a focus on general, but not able to decide whether to give priority to Education, Health or Psychology, then the decision-making environment becomes more complex. The graphical representation of all sub-attribute relation between input and the output dominating attribute vertex with simple plithogenic fuzzy connectors is presented in Fig. 3.10

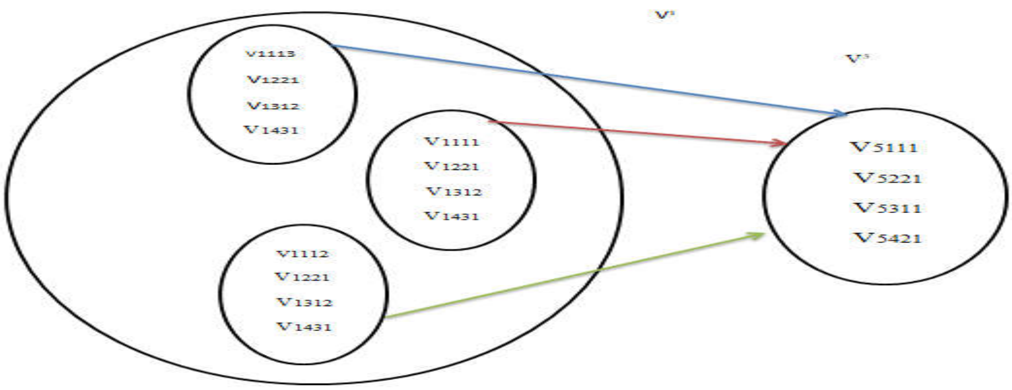

Fig. 3.10 Representation of Decision-Making Scenario III

The dominant attribute relational matrix representation is as follows

	V_{5111}	V_{5221}	V_{5311}	V_{5421}
V_{1111}	0.55	0.2	0.5	0.6
V_{1112}	0.6	0.55	0.7	0.85
V_{1113}	0.5	0.3	0.6	0.6
V_{1221}	0.8	0.4	0.4	0.8
V_{1312}	0.7	0.64	0.6	0.9
V_{1431}	0.6	0.89	0.62	0.4
V_{2122}	0.54	0.9	0.73	0.6
V_{2211}	0.83	0.5	0.6	0.8
V_{2312}	0.4	0.4	0.45	0.9
V_{2431}	0.6	0.8	0.72	0.49
V_{3122}	0.9	0.52	0.8	0.8
V_{3214}	0.95	0.63	0.9	0.9
V_{3321}	0.5	0.8	0.5	0.43
V_{3431}	0.6	0.8	0.52	0.4

3.4 Decision-Making Scenario IV

This decision-making situation is characterized when the institution is uncertain of the dominant sub-attribute values of the input object. Suppose if the institution decides to conduct a webinar with focus on general, but not able to decide whether to give priority to Education, Health or Psychology, In this case, the dominant sub-attribute value is not certain and suppose it wishes to collaborate with the industry then the decision-making environment becomes highly complex. Fig 3.11 presents this graphically

DOI: 10.5281/zenodo.3876206

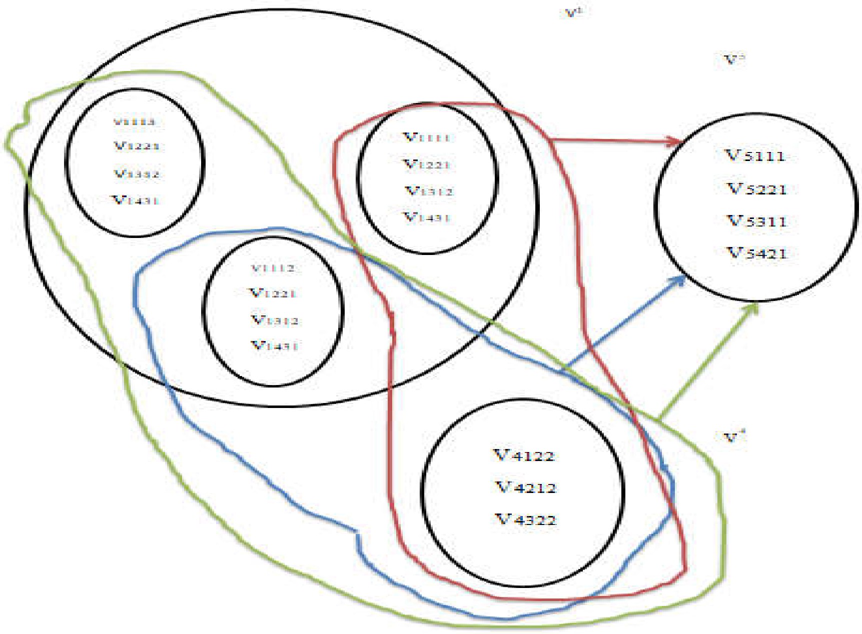

Fig.3.11 Representation of Decision-Making Scenario IV

The dominant attribute relational matrix representation is as follows

	V_{5111}	V_{5221}	V_{5311}	V_{5421}
V_{1111},V_{4122}	0.65	0.6	0.5	0.6
V_{1111},V_{4212}	0.8	0.65	0.7	0.8
V_{1111},V_{4322}	0.56	0.7	0.9	0.6
V_{1112},V_{4122}	0.52	0.68	0.57	0.66
V_{1112},V_{4212}	0.45	0.67	0.72	0.56
V_{1112},V_{4322}	0.56	0.68	0.74	0.69
V_{1113},V_{4122}	0.67	0.79	0.83	0.79
V_{1113},V_{4212}	0.57	0.82	0.74	0.68
V_{1113},V_{4322}	0.5	0.7	0.7	0.6
V_{1221},V_{4122}	0.75	0.8	0.85	0.6
V_{1221},V_{4212}	0.9	0.6	0.95	0.8
V_{1221},V_{4322}	0.6	0.8	0.45	0.9
V_{1312},V_{4122}	0.53	0.7	0.75	0.7
V_{1312},V_{4212}	0.43	0.5	0.6	0.7
V_{1312},V_{4322}	0.5	0.6	0.78	0.7
V_{1431},V_{4122}	0.62	0.85	0.8	0.69
V_{1431},V_{4212}	0.67	0.78	0.7	0.63
V_{1431},V_{4322}	0.6	0.89	0.58	0.7

The ranking of the attributes contributing to Professional efficiency corresponding to each decision-making environment is presented in Table 3.3

DOI: 10.5281/zenodo.3876206

Table 3.3 Ranking of the attributes

Decision Making Environment	**Ranking of the Attributes contributing to Professional Efficiency**
Decision Making Scenario I	$V_{1312} > V_{1221} > V_{3122} > V_{1221} > V_{2122} > V_{2211} > V_{2431} > V_{1111} > V_{2312} > V_{1431} > V_{3431} > V_{3321}$
Decision Making Scenario II	$V_{1221}, V_{4212} > V_{1221}, V_{4322} > V_{1431}, V_{4122} > V_{1111}, V_{4212}, V_{1221}, V_{4122} > V_{1431}, V_{4322} > V_{1312}, V_{4122} > V_{1312}, V_{4322} > V_{1111}, V_{4122} > V_{1312}, V_{4212}$
Decision Making Scenario III	$V_{3214} > V_{2431} > V_{1312} > V_{2122} > V_{3122} > \mathbf{V_{1111}} > V_{2211} > V_{1431} > V_{1221} > V_{3431} > V_{3321} > V_{2312} > \mathbf{V_{1112} > V_{1113}}$
Decision Making Scenario IV	$V_{1221}, V_{4212} > V_{1113}, V_{4122} > V_{1221}, V_{4122} > V_{1431}, V_{4122} > V_{1111}, V_{4212} > V_{1113}, V_{4212} > V_{1431}, V_{4212} > V_{1431}, V_{4322} > V_{1111}, V_{4322} > V_{1221}, V_{4322} > V_{1312}, V_{4122} > V_{1112}, V_{4322} > V_{1312}, V_{4322} > V_{1113}, V_{4322} > V_{1112}, V_{4122} > V_{1112}, V_{4212} > V_{1111}, V_{4122} > V_{1312}, V_{4212}$

4. Discussion

The ranking of the input attribute values of the dominant attributes contributing to the output dominant attribute values shows the significance of the individual contribution of each input attribute value. In the first decision-making scenario, the attribute values of the input object are ranked. In the second decision making a scenario the combined attribute values of input and intervene objects are ranked. This helps in finding the combined effect towards the attainment of the output attribute values. In the third decision-making scenario, the ranking of sub-attribute values are made, in this case, there was a choice to choose between Education (V_{1111}), Health (V_{1112}) or Psychology (V_{1113}), but the preference s should be given to Education based on the ranking. In the fourth decision-making scenario, the combined effects of sub-attribute values along with intervening attribute values are ranked and here also the combined effect of the sub-attribute value, Education is gaining more significance. The above decision-making scenarios were focusing on the effects of one input object and the same can be applied to other input objects and the respective results can be determined. The same method of decision making can be applied to production sectors in strategy selection which considers many attribute values and sub-attribute values and this proposed plithogenic –n superhypergraph MADM can be applied in such decision-making scenario.

5. Conclusion

This article presents the application of plithogenic n-super hypergraph in the context of optimal decision making. This research work introduces many new concepts such as enveloping vertex, dominant enveloping vertex, super enveloping vertex, and plithogenic connectors. This research work creates a new avenue in MADM by providing space for comprehensive decision making. A new approach to ranking the attribute values based on the frequency matrix is initiated. The theoretical description of plithogenic n-super hypergraph is translated into practical application in this research work and this will certainly open new vistas of research. This work can be further extended with various plithogenic sets.

References

[1] Ishizaka, A., Labib, A.. "Review of the main developments in the analytic hierarchy process", Expert Systems with Applications, Vol 38, pp.14336-14345, 2011.

[2] Saaty, T. L., "A scaling method for priorities in hierarchical structures",Journal of Mathematical Psychology. Elsevier: Amsterdam, Vol 15, pp.234-281, 1977.

[3] Tzeng, G.H.; Huang, J.J. "Multiple Attribute Decision Making: Methods and Applications"; CRC Press: Boca Raton, FL, USA, 2011.

[4] Smarandache, Florentin., "Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic –n SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-)HyperAlgebra" Neutrosophic Sets and Systems, Vol 33, pp.290-296, 2020 .

[5] Smarandache, F. "Extension of Soft set to Hypersoft Set, and then to Plithogenic Hypersoft Set", Neutrosophic set and Systems, 22, pp.68-70, 2018.

[6] Gayathri.N., Helen.M., Mounika.P., "Utilization of Jaccard Index Measures on Multiple Attribute Group Decision Making under Neutrosophic Environment", International Journal of Neutrosophic Science,Vol. 3, pp.67-77, 2020.

[7] Muhammad Naveed Jafar.,Muhammad Saqlain., Ahmad Raza Shafiq., Muhammad Khalid., Hamza Akbar., Aamir Naveed., " New Technology in Agriculture Using Neutrosophic Soft Matrices with the Help of Score Function ", International Journal of Neutrosophic Science,Vol. 3,pp.78-88, 2020.

[8] Ajay.D.,Aldring.J.,Abirami.S.,Jeni Seles Martina.D., " A SVTrN-number approach of multi-objective optimization on the basis of simple ratio analysis based on MCDM method ", International Journal of Neutrosophic Science ,Vol. 5, pp.16-28, 2020.

[9] Luis Andrés Crespo Berti.,"Application of the neutrosophic system to tax havens with a criminal approach",International Journal of Neutrosophic Science ,Vol. 5, pp.91-106, 2020.

[10] Abdel-Basset, M., Manogaran, G., Mohamed, M., Chilamkurti, N., "Three-way decisions based on neutrosophic sets and AHP-QFD framework for supplier selection problem". Future Gener. Comput. Syst. Vol 89, pp.19–30,2018.

[11] Abdel-Basset, M., Mohamed, M., Smarandache, F., "A hybrid neutrosophic group ANP-TOPSIS framework for supplier selection problems". Symmetry Vol 10, pp.226, 2018.

[12] Abdel-Basset, M., El-hoseny, M., Gamal, A., & Smarandache, F., "A novel model for evaluation Hospital medical care systems based on plithogenic sets". Artificial intelligence in medicine, 100, 101710,2019

[13] Abdel-Basset, M., Mohamed, R., Zaied, A. E. N. H., & Smarandache, F.,"A hybrid plithogenic decision-making approach with quality function deployment for selecting supply chain sustainability metrics". Symmetry, Vol 11,903,2019.

[14] Shazia Rana ., Madiha Qayyum.,Muhammad Saeed., Florentin Smarandache, Bakhtawar Ali Khan., "Plithogenic Fuzzy Whole Hypersoft Set, Construction of Operators and their Application in Frequency Matrix Multi Attribute Decision Making Technique", Neutrosophic Sets and Systems, Vol 28, pp.34-50,2019.

DOI: 10.5281/zenodo.3876206

A Note On Neutrosophic Soft Menger Topological Spaces

A.Haydar Eş

Department of Mathematics Education, Başkent University, Ankara, TURKEY; haydares@baskent.edu.tr

Abstract

In this paper, the concept of neutrosophic soft Mengerness, neutrosophic soft near Mengerness and neutrosophic soft almost Mengerness are introduced and studied. Some characterizations of neutrosophic soft almost Mengerness in terms of neutrosophic soft regular open or neutrosophic soft regular closed are given.

Keywords: Neutrosophic soft sets, Mengerness on neutrosophic soft topological space, neutrosophic soft continuous.

1. Introduction

The concept of fuzzy set was introduced by Zadeh in his classic paper [20]. C.L.Chang [6] has defined fuzzy topological spaces. Atannasov [3] introduced the notion of intuitionistic fuzzy sets, Çoker [7] defined the intuitionistic fuzzy topological spaces. Soft sets theory was proposed by Molodtsov [12] in 1999, as a new mathematical tool for handling problems which contain uncertainties. Maji et al [10] gave the first practical application of soft sets in decision-making problems. Shabir and Naz [16] presented soft topological spaces and defined some concepts of soft sets on these spaces and separation axioms. The concept of neutrosophic set (NS) was first introduced by Smarandache [17,18,19] which is the generalization of classical sets, fuzzy set, intuitionistic fuzzy set etc. Following this concept Al-Omeri and Jafari defined and investigated Neutrosophic crisp sets via Neutrosophic crisp topological spaces [1,2]. The concept of connectedness and compactness on neutrosophic soft topological space was introduced by Bera and Mahapatra [4,5]. For more applications on neutrosophic logic the refrennces are suggested [21-23]

The investigation of covering properties of topological spaces has a long history going back to papers by Menger and Rothberger [11,14]. However more recently a new theory called Selection Principles was introduced by Scheepers [15]. The theory of Selection Principles has extra ordinary connections with numerous subareas of mathematics, for example, Set theory and General topology, Uniform structures, and Ditopological texture spaces [9].

In 1999, Kocinac defined and characterized the almost Menger property [9]. Following this concept, Aqsa, Moizud Din Khan defined and investigated nearly Menger and nearly star- Menger spaces [13]. For

In this paper we are concerned with the weaker forms of the fuzzy Mengerness in neutrosophic soft topological spaces.

DOI: 10.5281/zenodo.3876208

Received: March 13, 2020 Revised: May 02, 2020 Accepted: May 19, 2020

2. Preliminaries

In this section we now state certain useful definitions, theorems, and several existing results for neutrosophic soft topological spaces that we require in the next sections.

Definition 2.1. [17] Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set A is characterized by a truth-member function T_A, an indeterminacy-membership function I_A and a falsity-membership function F_A. $T_A(x), I_A(x)$ and $F_A(x)$ are real Standard or non Standard subsets of $]^-0, 1^+[$. That is $T_A, I_A, F_A: X \to\ ]^-0, 1^+[$. There is no restriction on the sum of $T_A(x), I_A(x), F_A(x)$ and so,
$^-0 \leq sup\ T_A(x) + sup\ I_A(x) + sup\ F_A(x) \leq 3^+$.

Definition 2.2. [12] Let U be an initial universe set and E be a set of parameters. Let P(U) denote the power set of U. Then for $A \subseteq E$, a pair (F,A) is called a soft set over U, where F:A→P(U) is a mapping.

Definition 2.3. [15] Let U be an initial universe set and E be a set of parameters. Let NS(U) denote the set of neutrosophic sets (NSs) of U. Then for $A \subseteq E$, a pair (F,A) is called a neutrosophic soft set (NSS) over U, where F:A→NS(U) is a mapping.

Definition 2.4. [8] Let U be an initial universe set and E be a set of parameters. LetNS(U) denote the set of neutrosophic sets (NSs) of U. Then, a neutrosophic soft set N over U is a set defined by a set valued function F_N representing a mapping $F_N: E \to NS(U)$ where F_N is called approximate function of the neutrosophic soft set N. In other words, the neutrosophic soft set is a parametrized family of some elements of the set NS(U) and therefore it can be written as a set of ordered pairs,

$N = \{(e, \{< x, T_{fN(e)}(x), I_{fN(e)}(x), F_{fN(e)}(x) >: x \in U\}): e \in E\}$where$T_{fN(e)}(x), I_{fN(e)}(x), F_{fN(e)}(x) \in [0,1]$,

respectively the truth-membership, indeterminacy-membership, falsity-membership function obvious.

Definition 2.5. [8] The complement of a neutrosophic soft set N is denoted by N^c and is defined by
$N^c = \{(e, \{< x, F_{fN(e)}(x), 1 - I_{fN(e)}(x), T_{fN(e)}(x) >: x \in U\}): e \in E\}$,

Let N_1 and N_2 be two NSSs over the common universe (U,E). Then N_1 is said to be the neutrosophic soft subset of N_2 if for each $e \in E$ and for each $x \in U$,

$T_{fN1(e)}(x) \leq T_{fN2(e)}(x), I_{fN1(e)}(x) \geq I_{fN2(e)}(x), F_{fN1(e)}(x) \geq F_{fN2(e)}(x)$.

We write $N_1 \subseteq N_2$ and then N_2 is the is the neutrosophic soft superset of N_1.

Definition 2.6. [8] Let N_1 and N_2 be two NSSs over the common universe (U,E). Then their union is denoted by $N_1 \cup N_2 = N_3$ and is defined as:

$N_3 = \{(e, \{< x, T_{fN3(e)}(x), I_{fN3(e)}(x), F_{fN2(e)}(x) >: x \in U\}): e \in E\}$where
$T_{fN3(e)}(x) = T_{fN1(e)}(x) \diamond T_{fN2(e)}(x), I_{fN3(e)}(x) = I_{fN1(e)}(x) * I_{fN2(e)}(x), F_{fN3(e)}(x) = F_{fN1(e)}(x) * F_{fN2(e)}(x)$.

Their intersection is denoted by $N_1 \cap N_2 = N_4$ and is defined as:

$N_4 = \{(e, \{< x, T_{fN4(e)}(x), I_{fN4(e)}(x), F_{fN4(e)}(x) >: x \in U\}): e \in E\}$ where
$T_{fN4(e)}(x) = T_{fN1(e)}(x) * T_{fN2(e)}(x), I_{fN4(e)}(x) = I_{fN1(e)}(x) \diamond I_{fN2(e)}(x), F_{fN4(e)}(x) = F_{fN1(e)}(x) \diamond F_{fN2(e)}(x)$.

DOI: 10.5281/zenodo.3876208

Definition 2.7. [4] Let M and N be two NSSs over the common universe (U,E). Then M-N may be defined as, for each $e \in E$ and for each $x \in U$.

$$M - N = \left\{< x, T_{fM(e)}(x) * F_{fN(e)}(x), I_{fM(e)}(x) \diamond \left(1 - I_{fN(e)}(x)\right), F_{fM(e)}(x) \diamond T_{fN(e)}(x) >\right\};$$

A neutrosophic soft set N over (U,E) is said to be null neutrosophic soft set if $T_{fN(e)}(x) = 0, I_{fN(e)}(x) = 1, F_{fN(e)}(x)$=1 for each $e \in E$ and for each$x \in U$. It is denoted by Φ_u.

A neutrosophic soft set N over (U,E) is said to be absolute neutrosophic soft set if $T_{fN(e)}(x) = 1, I_{fN(e)}(x) = 0, F_{fN(e)}(x)$=0 for each $e \in E$ and for each $x \in U$. It is denoted by 1_u.

Clearly, $\Phi_u^c = 1_u, 1_u^c = \Phi_u$.

Definition 2.8. [4] Let NSS(U,E) be the family of all neutrosophic soft sets over U via parameters in E and $\tau_u \subseteq NSS(U,E)$. Then $\tau_u$is called neutrosophic soft topology on (U,E) if the following conditions are satisfied.

(i) $\Phi_u, 1_u \in \tau_u$,
(ii) The intersection of any finite number of members of τ_u also belongs to τ_u.
(iii) The union of any collection of members of $\tau_u$belongs toτ_u.

Then the triple (U,E, τ_u) is called a neutrosophic soft topological space. Every member of τ_u is called τ_u-open neutrosophic soft set. An NSS is called τ_u-closed iff its complement is τ_u-open.

Definition 2.9. [4] Let (U,E, τ_u) be a neutrosophic soft topological space over (U,E) and $M \in NSS(U,E)$ be arbitrary. Then the interior of M is denoted by M^0 or int(M) and is defined as:

$M^0 = \cup \{N_1 : N_1 is\ neutrosophic\ soft\ open\ and N_1 \subseteq M\}$.

Definition 2.10. [4] Let (U,E, τ_u) be a neutrosophic soft topological space over (U,E) and $A \in NSS(U,E)$ be arbitrary. Then the closure of A is denoted by $\overline{A}$ or cl(A) and is defined as:

$\bar{A} = \cap \{N_1 : N_1 is\ neutrosophic\ soft\ closed\ and\ A \subseteq N_1\}$.

Theorem 2.11. [4] Let (U,E, τ_u) be a neutrosophic soft topological space over (U,E) and $A \in NSS(U,E)$. Then, $(\bar{A})^C$=$(A^c)^0$ and $(A^0)^c = (A^c)^-$.

Proposition 2.12. [4] Let N_1 and N_2 be two neutrosophic soft sets over (U,E). Then,

(i) $(N_1 \cup N_2)^c = {N_1}^c \cap {N_2}^c$,
(ii) $(N_1 \cap N_2)^c = {N_1}^c \cup {N_2}^c$.

Definition 2.13. [4] Let (U,E, τ_u) be a neutrosophic soft topological space and $M \in \tau_u$. A family $\Omega = \{Q_i : i \in \Gamma\}$ of neutrosophic soft sets is said to be a cover of M if $M \subseteq \cup\ Q_i$.

If every member of that family which covers M is neutrosophic soft open then it is called open cover of M. A subfamily of Ω which also covers M is called a subcover of M.

Definition 2.14. [4] Let (U,E, τ_u) be a neutrosophic soft topological space and $M \in \tau_u$. Suppose Ω be a cover of M. If Ω has a finite subcover which also covers M then M is called neutrosophic soft compact.

DOI: 10.5281/zenodo.3876208

Definition 2.15. [4] Let $\varphi: U \rightarrow V$ and $\psi: E \rightarrow E$ be two functions where E is the parameter set each of the crisp sets U and V. Then the pair(φ, ψ) is called and NSS function from (U,E) to (V,E). We write, $(\varphi, \psi): (U, E) \rightarrow (V, E)$.

Definition 2.16. [4] Let (M,E) and (N,E) be two NSSs defined over U and V, respectively and (φ, ψ) be an NSS function from (U,E) to (V,E). Then,

(1) The image of (M,E) under (φ, ψ), denoted by $(\varphi, \psi)(M, E)$, is an NSS over V and is defined as:

$(\varphi, \psi)(M, E) = (\varphi(M), \psi(E)) = \{< \psi(a), f_{\varphi(M)}(\psi(a)) >: a \in E\}$ where for each $b \in \psi(E)$ and $y \in V$.

$T_{\varphi(M)(b)}(y) = \{ \begin{smallmatrix} \max\varphi(x)=y\max\psi(a)=b[Tf(M)(a)(x)], & \text{if } x\epsilon\varphi-1(y), \\ 0, & \text{otherwise.} \end{smallmatrix}$

$I_{\varphi(M)(b)}(y) = \{ \begin{smallmatrix} \min\varphi(x)=y \min \psi(a)=b[If(M)(a)(x)], & \text{if } x\epsilon\varphi-1(y), \\ 1, & \text{otherwise.} \end{smallmatrix}$

$F_{\varphi(M)(b)}(y) = \{ \begin{smallmatrix} \min\varphi(x)=y \min\psi(a)=b[Ff(M)(a)(x)], & \text{if } x\epsilon\varphi-1(y), \\ 1, & \text{otherwise.} \end{smallmatrix}$

(2) The pre-image of (N,E) under(φ, ψ), denoted by $(\varphi, \psi)^{-1}(N, E)$, is an NSS over U and is defined by:

$(\varphi, \psi)^{-1}(N, E) = (\varphi^{-1}(N), \psi^{-1}(E))$ where for each $a \in \psi^{-1}(E)$ and $x \in U$.

$T_{\varphi}^{-1}(N)(a)(x)=T_{fN(\psi(a))}(\varphi(x))$,

$I_{\varphi}^{-1}(N)(a)(x)=I_{fN(\psi(a))}(\varphi(x))$,

$F_{\varphi}^{-1}(N)(a)(x)=F_{fN(\psi(a))}(\varphi(x))$,

If ψ and φ are injective(surjective), then (φ, ψ) is injective (surjective).

Definition 2.17. [4] Let (U,E, τ_u) and (V,E, τ_v) be two neutrosophic soft topological spaces.
(φ, ψ): (U, E, τ_u) →(V,E, τ_v) is said to be a neutrosophic soft continuous mapping if for each $(N, E) \in T_v$, the inverse image $(\varphi, \psi)^{-1}(N, E) \in \tau_u$ i.e., the inverse image of each open NSS in (V,E,τ_v) is also open in (U,E,τ_u).

3. Neutrosophic Soft Mengerness

Here, the notion of Mengerness, almost Mengerness and near Mengerness on neutrosophic soft topological space is developed with some basic theorems.

Definition 3.1. (a) A neutrosophic soft topological space (U,E, τ_u)is called neutrosophic soft Menger iff every sequence $\{Q_n: n \in N\}$ of neutrosophic soft open covers of (U,E, τ_u), there exists a sequence $\{V_n: n \in N\}$ such that for every $n \in N, V_n$ is a finite subset of Q_n and $\cup_{n\in N} V_n = 1_u$.

(b) A neutrosophic soft topological space (U,E, τ_u) is called neutrosophic soft almost Menger iff every sequence $\{Q_n: n \in N\}$ of neutrosophic soft open covers of (U,E, τ_u), there exists a sequence $\{V_n: n \in N\}$ such that for every $n \in N, V_n$ is a finite subset of Q_n and $\cup_{n\in N} V_n^* = 1_u$, where $V_n^* = \{cl(V): V \subseteq V_n\}$.

DOI: 10.5281/zenodo.3876208

(c) A neutrosophic soft topological space (U,E,τ_u) is called neutrosophic soft nearly compact iff every sequence $\{Q_n: n \in N\}$ of neutrosophic soft open covers of (U,E,τ_u), there exists a sequence $\{V_n: n \in N\}$ such that for every $n \in N, V_n$ is a finite subset of Q_n and $\cup_{n\in N} V_n^* = 1_u$, where $V_n^* = \{int(cl(V)): V \subseteq V_n\}$.

It is clear that in neutrosophic soft topological spaces we have the following implications:

Neutrosophic soft Menger→neutrosophic soft nearly Menger→neutrosophic soft almost Menger.

Theorem 3.2. A neutrosophic soft topological space (U,E,τ_u) is called neutrosophic soft almost Menger iff for each family $\{Q_n: n \in N\}$ of neutrosophic soft open sets in (U,E,τ_u) having the finite intersection property we have $\cap_{n\in N} cl(Q_n) \neq \Phi_u$.

Proof. Let (U,E,τ_u) be a neutrosophic soft almost Menger topological space. Consider $\{Q_n: n \in N\}$ be a sequence of neutrosophic soft open sets in (U,E,τ_u) having the finite intersection property. Suppose the $\cap_{n\in N} cl(Q_n) = \Phi_u$. Then we have $\cup_{n\in N}$ $[\mathrm{cl}(Q_n)]^c$=$\cap_{n\in N} int({Q_n}^c) = 1_u$. Since (U,E,$\tau_u$) neutrosophic soft almost Menger, for every $n \in N$, there exists a sequence $\{H_n: n \in N\}$ such that H_n is a finite subset of int(${Q_n}^c$) and $\cup_{n\in N} H_n^* = 1_u$, where $H_n^* = \{cl(H): H \subseteq H_n\}$. But from $H_n \subseteq int({Q_n}^c)$ and $Q_n \subseteq int(cl(Q_n))$, *we see that* $\cap_{n\in N} Q_n = \Phi_u$, which is a contradiction with the finite intersection property of $\{Q_n: n \in N\}$.

Conversely, let $\{Q_n: n \in N\}$ be a neutrosophic soft open cover. If $\cup_{n\in N} H_n^* \neq 1_u$, where $H_n^* = \{cl(H): H \subseteq H_n\}$ and H_n is a finite subset of Q_n, then $\{(H_n^*)^c: n \in N\}$ is an of neutrosophic soft open sequence with the finite intersection property. Hence, from the hypothesis it follows that

$\cap_{n\in N} cl((H_n^*)^c) \neq \Phi_u =>\cup_{n\in N} [cl([cl(H_n^*)^c)]^c \neq 1_u$. Since $\cup_{n\in N} Q_n \subseteq\cup_{n\in N} [cl([cl(H_n^*)^c] \neq 1_u$,
then $\cup_{n\in N} Q_n \neq 1_u$, which is a contradiction.

Definition 3.3. A neutrosophic soft set N_1 is called a neutrosophic soft regular open set iff N_1=int(cl(N_1)); a neutrosophic soft set N_2 is called a neutrosophic soft regular closed set iff N_2=cl(int(N_2)).

Theorem 3.4. In a neutrosophic soft topological space (U,E,τ_u) the following conditions are equivalent:

(i) $(\mathrm{U}, \mathrm{E}, \tau_u)$ is neutrosophic soft almost Menger.

(ii) For each sequence $\{Q_n: n \in N\}$ of neutrosophic soft regular closed sets such that $\cap_{n\in N} Q_n = \Phi_u$, there exists a sequence $\{V_n: n \in N\}$ such that for every $n \in N, V_n$ is a finite subset of Q_n and $\cap_{n\in N} V_n^* = \Phi_u$, where $V_n^* = \{int(V): V \subseteq V_n\}$.

(iii) $\cap_{n\in N} cl(Q_n) \neq \Phi_u$ holds for each sequence $\{Q_n: n \in N\}$ of neutrosophic soft regular open sets having the finite intersection property.

(iv) For each sequence $\{Q_n: n \in N\}$ of neutrosophic soft regular open covers of $(\mathrm{U}, \mathrm{E}, \tau_u)$, there exists a sequence $\{V_n: n \in N\}$ such that for every $n \in N$, V_n is a finite subset of Q_n and $\cup_{n\in N} V_n^* = \{cl(V): V \subseteq V_n\}$.

Proof. The proof of this theorem follows a similar pattern to Theorem 3.2.

Definition 3.5. Let $(\mathrm{U}, \mathrm{E}, \tau_u)$ and $(\mathrm{V}, \mathrm{E}, \tau_v)$ be two neutrosophic soft topological spaces. Then $(\varphi, \psi): (\mathrm{U}, \mathrm{E}, \tau_u) \to (\mathrm{V}, \mathrm{E}, \tau_v)$ is said to be a neutrosophic soft almost continuous mapping if for each (N,E) neutrosophic soft regular open set of $(\mathrm{V}, \mathrm{E}, \tau_v)$, the inverse image $(\varphi, \psi)^{-1}(N, E) \in \tau_u$. The inverse image of each neutrosophic soft regular open set in $(\mathrm{V}, \mathrm{E}, \tau_v)$ is neutrosophic soft open in $(\mathrm{U}, \mathrm{E}, \tau_u)$.

Theorem 3.6. Let $(\mathrm{U}, \mathrm{E}, \tau_u)$ and $(\mathrm{V}, \mathrm{E}, \tau_v)$ be two neutrosophic soft topological spaces and $(\varphi, \psi): (\mathrm{U}, \mathrm{E}, \tau_u) \to (\mathrm{V}, \mathrm{E}, \tau_v)$ a neutrosophic soft almost continuous surjection mapping. If (M,E) is neutrosophic soft almost Menger in $(\mathrm{U}, \mathrm{E}, \tau_u)$, then (φ, ψ)(M,E) is so in $(\mathrm{V}, \mathrm{E}, \tau_v)$.

DOI: 10.5281/zenodo.3876208

Proof. Let $\{Q_n, E\}: n \in N\}$ be a neutrosophic soft open cover of (φ,ψ)(M,E) i.e., C(M,E) $\subseteq \cup_{n\in N}(Q_n, E)$. Since (φ, ψ) is neutrosophic soft almost continuous, $\{(\varphi, \psi)^{-1} int(cl((Q_n, E))): n \in N\}$ is a neutrosophic soft open cover of (M,E) . Since (M,E) is almost Menger, there is a sequence $\{(H_n, E): n \in N\}$ such that H_n is a finite subset of $\{(\varphi, \psi)^{-1} int(cl((Q_n, E))): n \in N\}$ and $\cup_{n\in N} (H_n^*, E) = 1_u$, where $(H_n^*, E) = \{cl(H, E): H \subseteq H_n\}$. For every $n \in N$ and $H \subseteq H_n$ we can choose a member $(Q_H, E) \subseteq (Q_n, E)$ such that $(H, E) = (\varphi, \psi)^{-1}(Q_H, E)$. From the surjectivity of (φ, ψ) we have $(M, E) \subseteq \cup_{n\in N} cl\left((\varphi, \psi)^{-1}\left(int\left(cl(Q_H, E)\right)\right)\right) = 1_u$.

Hence (φ, ψ)(M,E) $\subseteq (\varphi, \psi)[\cup_{n\in N} cl((\varphi, \psi)^{-1}(int(cl(Q_H, E))))]$= $\cup_{n\in N}(\varphi, \psi)[cl(\varphi, \psi)^{-1}(int(cl(Q_H, E)))] = f(1_u) = 1_v$. But from $int(cl(Q_H, E)) \subseteq cl(Q_H, E)$ and from the neutrosophic soft almost continuity of f,
$(\varphi, \psi)(\mathrm{cl}(\varphi, \psi)^{-1} int(cl((Q_H, E)))) \subseteq (\varphi, \psi)((\varphi, \psi)^{-1} cl(Q_H, E)))) \subseteq cl(Q_H, E)$ for each $n \in N$, i.e., $\cup_{n\in N} cl(Q_H, E) = 1_v$. Hence (φ, ψ)(M,E) is neutrosophic soft almost Menger also.

Definition 3.7. Let $(\mathrm{U}, \mathrm{E}, \tau_u)$ and $(\mathrm{V}, \mathrm{E}, \tau_v)$ be two neutrosophic soft topological spaces. Then $(\varphi, \psi): (\mathrm{U}, \mathrm{E}, \tau_u) \to (\mathrm{V}, \mathrm{E}, \tau_v)$ is said to be a neutrosophic soft weakly continuous mapping if for each (N,E) neutrosophic soft regular open set of $(\mathrm{V}, \mathrm{E}, \tau_v)$, $(\varphi, \psi)^{-1}(N, E) \subseteq int\left((\varphi, \psi)^{-1}\left(cl(N, E)\right)\right)$.

Theorem 3.8. Let $(\mathrm{U}, \mathrm{E}, \tau_u)$ and $(\mathrm{V}, \mathrm{E}, \tau_v)$ be two neutrosophic soft topological spaces and $(\varphi, \psi): (\mathrm{U}, \mathrm{E}, \tau_u) \to (\mathrm{V}, \mathrm{E}, \tau_v)$ is said to be a neutrosophic soft weakly continuous surjection mapping. If (M,E) is neutrosophic soft Menger in $(\mathrm{U}, \mathrm{E}, \tau_u)$, then $(\varphi, \psi)(M, E)$ is neutrosophic soft almost Menger in $(\mathrm{V}, \mathrm{E}, \tau_v)$.

Proof. By using a similar technique of the proof of Theorem 3.6, the theorem holds.

Definition 3.9. Let $(\mathrm{U}, \mathrm{E}, \tau_u)$ and $(\mathrm{V}, \mathrm{E}, \tau_v)$ be two neutrosophic soft topological spaces. Then $(\varphi, \psi): (\mathrm{U}, \mathrm{E}, \tau_u) \to (\mathrm{V}, \mathrm{E}, \tau_v)$ is said to be a neutrosophic soft strongly continuous mapping if for each (M,E) neutrosophic soft set of $(\mathrm{V}, \mathrm{E}, \tau_v)$, $(\varphi, \psi)[cl(M, E)] \subseteq (\varphi, \psi)(M, E)$.

Theorem 3.10. Let $(\mathrm{U}, \mathrm{E}, \tau_u)$ and $(\mathrm{V}, \mathrm{E}, \tau_v)$ be two neutrosophic soft topological spaces and $(\varphi, \psi): (\mathrm{U}, \mathrm{E}, \tau_u) \to (\mathrm{V}, \mathrm{E}, \tau_v)$ a neutrosophic soft strongly continuous surjection mapping. If (M,E) is neutrosophic soft almost Menger in $(\mathrm{U}, \mathrm{E}, \tau_u)$, then $(\varphi, \psi)(M, E)$ is neutrosophic soft Menger in $(\mathrm{V}, \mathrm{E}, \tau_v)$.

Proof. By using a similar technique of the proof of Theorem 3.6, the theorem holds.

Corollary 3.11. Let $(\mathrm{U}, \mathrm{E}, \tau_u)$ and $(\mathrm{V}, \mathrm{E}, \tau_v)$ be two neutrosophic soft topological spaces and $(\varphi, \psi): (\mathrm{U}, \mathrm{E}, \tau_u) \to (\mathrm{V}, \mathrm{E}, \tau_v)$ a neutrosophic soft strongly continuous surjection mapping. If (M,E) is neutrosophic soft nearly Menger in $(\mathrm{U}, \mathrm{E}, \tau_u)$, then $(\varphi, \psi)(M, E)$ is neutrosophic soft Menger in $(\mathrm{V}, \mathrm{E}, \tau_v)$.

4. Conclusions

In this paper, the concepts of neutrosophic soft Menger topological spaces, Neutrosophic topological spaces, Neutrosophic Bitopological spaces and Neutrosophic crisp supra bitopological spaces were introduced and studied. Some interesting properties were also established. It would be interesting to study similar properties for neutrosophic soft weakly Menger topological spaces, Neutrosophic crisp supra bitopological spaces, Neutrosophic Bitopological Spaces and Neutrosophic Topological Spaces.

DOI: 10.5281/zenodo.3876208

Funding: "This research received no external funding"

Conflicts of Interest: "The authors declare no conflict of interest."

References

[1] W.Al-Omeri, "Neutrosophic crisp Sets via Neutrosophic crisp Topological Spaces", Neutrosophic Sets and Systems, Vol 13, pp.96-104, 2016.
[2] W.Al-Omeri, S.Jafari, "On Generalized closed Sets and Generalized Pre-closed Sets in Neutrosophic Topological Spaces", Mathematics, Vol 7 (1), pp.1-12, 2019.
[3] K.Atanassov and S.Stoeva, "intuitionistic fuzzy sets", in:Polish Symp.on Interval and Fuzzy Mathematics, Poznan, pp.23-26, 1983.
[4] T.Bera and N.K.Mahapatatra, "On Neutrosophic Soft Topological Space", Neutrosophic Sets and Systems, Vol 19, pp.3-15, 2018.
[5] T.Bera and N.K.Mahapatatra, "Introduction to neutrosophic soft topological spaces", OPSEARCH, DOI 10. 1007/S12597-017-0308-7, (March, 2017).
[6] C.Chang, "Fuzzy topological spaces", J.Math.Anal.Appl., Vol 24, pp.182-190, 1968.
[7] D.Çoker, "An introduction to intuitionistic fuzzy topological spaces", Fuzzy Sets and Systems, Vol 88, pp.81-89, 1997.
[8] I.Deli and S.Broumi, "Neutrosophic Soft Matrices and NSM-decision Making", Journal of Intelligent and Fuzzy Systems, Vol 28 (5), pp.2233-2241, 2015.
[9] Lj.D.R.Kocinac, "Star-Menger and related spaces II", Filomat, Vol 13, pp.129-140, 1999.
[10] P.K.Maji, "Neutrosophic soft set", Annals of Fuzzy Mathematics and Informatics, Vol 5 (1), pp.157-168, 2013.
[11] K.Menger, "Einige Überdeckungssatze der Punktmengenlehre", Sitzungsberichte Abt.2a, Mathematik, Astronomie, Physik, Meteorologie und Mechanik. Wiener Akademie, Wien, Vol 133, pp.421-444, 1924.
[12] D.Molodtsov, "Soft set theory-First results", Computer and Mathematics with Applications, Vol 37, pp.19-31, 1999.
[13] A.Parvez and M.Khan, "On Nearly Menger and Nearly Star-Menger Spaces", Filomat, Vol 33 (19), pp.6219-6227, 2019.
[14] F.Rothberger, "Eine Verscharfung dar Eigenschaft G", Fund. Math., Vol 30, pp.50-55, 1938.
[15] M.Scheepers, "Combinatorics of open covers I: Ramsey Theory" , Topology Appl., Vol 73, pp.241-266, 1996.
[16] M.Shabir and M.Naz, "On soft topological spaces", Comput. Math. Appl., Vol 61, pp.1786-1799, 2011.
[17] F.Smarandache, "Neutrosophy, Neutrosophic Probability, Set and Logic," Amer.Res.Press, Rehoboth,USA., p105, 1998, http://fs.gallup.unm.edu/eBook-neutrosophics4.pdf (fourth version).
[18] F.Smarandache, "Neutrosophic set, a generalization of the intuitionistic fuzzy sets", Inter.J.Pure Appl.Math., Vol 24, pp287-297, 2005.
[19] F.Smarandache, S.Pramanik, Eds., "New Neutrosophic Sets via Neutrosophic Topological Spaces", In Neutrosophic Operational Research, Pons Editions: Brussel, Belgium, Volume 1, pp.189-209, 2017.
[20] L.A.Zadeh, "Fuzzy sets", Inf.Control, Vol 8, pp.338-353, 1965.
[21] M. Al- Tahan , Bijan Davvaz, "Neutrosophic ℵ–Ideals (ℵ-Subalgebras) of Subtraction Algebra", International Journal of Neutrosophic Science, Volume 3 , Issue 1, pp.44-53 , 2020.
[22] M. Songsaeng , A.Iampan, "Image and Inverse Image of Neutrosophic Cubic Sets in UP-Algebras under UP-Homomorphisms", International Journal of Neutrosophic Science, Volume 3 , Issue 2, pp. 89-107 , 2020
[23] T. Bera , Nirmal Kumar Mahapatra, "An Approach to Solve the Linear Programming Problem Using Single Valued Trapezoidal Neutrosophic Number", International Journal of Neutrosophic Science, Volume 3 , Issue 2, pp. 54-66 , 2020.

DOI: 10.5281/zenodo.3876208

A Short Remark on Vortex as Fluid Particle from Neutrosophic Logic perspective

(Towards "fluidicle" or "vorticle" model of QED.)

Victor Christianto[1]* & Florentin Smarandache[2]

1 Malang Institute of Agriculture (IPM), Malang – INDONESIA, email: victorchristianto@gmail.com
[2] Dept. Mathematics & Sciences, University of New Mexico, Gallup, USA. Email: smarand@unm.edu

*Corresponding author: victorchristianto@gmail.com

Abstract

In a previous paper in this journal (IJNS), it is mentioned about a possible approach to re-describe QED without *renormalization* route. As it is known that in literature, there are some attempts to reconcile vortex-based fluid dynamics and particle dynamics. Some attempts are not quite as fruitful as others. As a follow up to previous paper, the present paper will discuss two theorems for developing unification theories, and then point out some new proposals including by Simula (2020) on how to derive Maxwell equations in superfluid dynamics setting; this could be a new alternative approach towards "*fluidicle*" or "*vorticle*" model of QED. Further research is recommended in this new direction.

Keywords: Neutrosophic logic, Vortex-based fluid dynamics, Fuidicle, Vorticle, QED, Renormalization, Maxwell-Proca equations.

PACS 2010: 02, 03, 41, 98

1.Introduction

In literature, there are some attempts to reconcile between vortex-based fluid dynamics and particle dynamics, see [15-21]. Some attempts are not quite fruitful as others, concerning describing classical electrodynamics.

DOI: 10.5281/zenodo.3876241

Received: February 26, 2020 Revised: April 15, 2020 Accepted: May 20, 2020

This paper will continue our previous article, suggesting that it is possible to find a way out of the infinity problem in QED without renormalization route [14]. As in the previous paper [14], the role of neutrosophic Logic (developed by one of us, FS) here is to find a third way or intermediate solution between point particle and vortex, that is why it is suggested here a combined term: "*vorticle*" (from vortex and particle), or it may be called: "*fluidicle*" (from fluidic particle). These new words vorticle and fluidicle are intended to capture the essence of "middle way" representing the Neutrosophic Logic view.

Here three possible approaches by Tapio Simula, Lehnert's RQED, and also Carl Krafft, will also be discussed.

The present paper will point out some new papers including by Simula [7] on how to derive Maxwell equations in superfluid dynamics setting, this could be a new alternative approach towards "fluidicle" or "vorticle" model of QED.

2. A short review of progress QED theories in literature and two new theorems.

There are some progress in the literature of QED, beyond what is called "*renormalization*" route, for instance by Daywitt, using a 7-dimensional spacetime and spinor wave [22-24].

Other developments have been made by Prof. Bo Lehnert, which he calls: revised Quantum Electrodynamics. There are numerous possible ways to develop QED-like theories, and not only that some theoreticians have gone further to develop Unification Theories, SuperUnification, and even Theory of Everything (TOE).

But almost all of them boiled down to mounting complexities and ever-increasing difficult technicalities, so it appears to be more direct approach to start with writing down two theorems as follows:

2.a. Two new theorems and a corollary

Based on the above discussions, actually, it is suggested two theorems and a corollary over here:

Theorem 1:
The true unified theory between gravitation, particles, and electromagnetic (UTGPE) fields should be based on a consistent model of *vacuum*, preferably by a kind of ether fluid dynamics.

Theorem 2:
The true UTGPE, albeit it is quite difficult to find, shall be founded on no more than 3-dimensional space and 1-dimensional time (Newtonian space).

Corollary:

DOI: 10.5281/zenodo.3876241

It should be possible and indeed relatively easy to find theoretical ways to unify four fundamental forces by increasing spacetime dimensionality. Supra dimensional spacetime is one character of *anti-realism* theory of UTGPE.

2.b. **Implication.**

Therefore, a good candidate of true UTGPE, or at least a unification of gravitation and electromagnetic field in a quantum sense, should be better off based on such characteristics, as a consistent combination between a quantum feature of electrodynamics theory and/or quantum or *sub-quantum*[1] model of aether fluid.

3. Three possible alternatives on QED

Allow us to begin this section with a quote from Sonin's book [1], which can be paraphrased as follows:

> "The movement of vortices has been a region of study for over a century. During the old style time of vortex elements, from the late 1800s, many fascinating properties of vortices were found, starting with the outstanding Kelvin waves engendering along a disconnected vortex line (Thompson, 1880). The primary object of hypothetical investigations around then was a dissipationless immaculate fluid (Lamb, 1997). It was difficult for the hypothesis to find a shared opinion with try since any old style fluid shows gooey impacts. The circumstance changed after crafted by Onsager (1949) and Feynman (1955) who uncovered that turning superfluids are strung by a variety of vortex lines with quantized dissemination. With this revelation, the quantum time of vortex elements started."

Then it is possible find an expression that relates the topological and quantized vortices from the viewpoint of Bohr-Sommerfeld quantization rules, which seem to remind us to the Old Quantum Theory, albeit from a different perspective.

The quantization of circulation for nonrelativistic superfluid is given by [3]:

$$\oint v dr = N \frac{\hbar}{m_s} \tag{1}$$

Where $N, \hbar, m_s$ represents the winding number, reduced Planck constant, and superfluid particle's mass, respectively [3]. And the total number of vortices is given by [44]:

$$N = \frac{\omega . 2\pi r^2 m}{\hbar} \tag{2}$$

[1] Added note: Robert N. Boyd has suggested his sub-quantum kinetic model of aether and also electron, using some features of Kelvin-Helmholtz vortex theorem. See for instance: V. Christianto, F. Smarandache & R.N. Boyd, Electron Model Based Helmholtz's Electron Vortex & Kolmogorov's Theory of Turbulence. *Prespacetime J.* vol. 10 (1), 2019. url: https://prespacetime.com/index.php/pst/article/view/1516

DOI: 10.5281/zenodo.3876241

Some implications:

a. Simula's approach

Provided it is acceptable that there is a neat correspondence between quantized vortices in superfluid helium and Bohr-Sommerfeld quantization rules, now let us quote from abstract of a recent paper where Tapio Simula wrote, which can be rephrased as follows [7]:

> "Right now, and electromagnetism have a similar starting point and are new properties of the superfluid universe, which itself rises up out of the hidden aggregate structure of progressively basic particles, for example, atoms. The Bose-Einstein condensate is identified as the tricky dull matter of the superfluid universe with vortices and phonons, separately, comparing to huge charged particles and massless photons."[7]

In Simula's model, Maxwell equations can be re-derived right from superfluid vortices.

b. Lehnert's RQED

And one more approach is worthy to mention here. Instead of Simula's model of electromagnetic and gravitation fields in terms of superfluid vortices, we can also come up with a model of electrodynamics by Lehnert's RQED from Proca equations. As Proca equations can be used to describe the electromagnetic field of superconductor, we find it as a possible approach too.

Conventional electromagnetic theory based on Maxwell's equations and quantum mechanics has been successful in its applications in numerous problems in physics and has sometimes manifested itself in a good agreement with experiments. Nevertheless, as already stated by Feynman, there are unsolved problems which lead to difficulties with Maxwell's equations that are not removed by and not directly associated with quantum mechanics [20]. Therefore QED, which is an extension of Maxwell's equations, also becomes subject to the *typical shortcomings* of electromagnetic in its conventional form. This reasoning makes a way for Revised Quantum Electrodynamics as proposed by Bo Lehnert. [11-13]

In a series of papers, Bo Lehnert proposed a novel and revised version of Quantum Electrodynamics, which he calls as RQED. His theory is based on the hypothesis of a nonzero electric charge density in the vacuum, and it is based on *Proca-type* field equations [10, p. 23]:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)A_\mu=\mu_0 J_\mu,\mu=1,2,3,4 \qquad (3)$$

Where

$$A_\mu=\left(A,\frac{i\phi}{c}\right), \qquad (4)$$

DOI: 10.5281/zenodo.3876241

With A and ϕ standing for the magnetic vector potential and the electrostatic potential in three-space. In three dimensions, we got [20, p.23]:

$$\frac{curlB}{\mu_0} = \varepsilon_0 (divE)C + \frac{\varepsilon_0 \partial E}{\partial t}, \tag{5}$$

$$curlE = -\frac{\partial B}{\partial t}, \tag{6}$$

$$B = curlA, divB = 0, \tag{7}$$

$$E = -\nabla \phi - \frac{\partial A}{\partial t}, \tag{8}$$

$$divE = \frac{\bar{\rho}}{\varepsilon_0}. \tag{9}$$

These equations differ from the conventional form, by a nonzero electric field divergence equation (9) and by the additional space-charge current density in addition to displacement current at equation (5). The extended field equations (5)-(9) are easily found also to become invariant to a gauge transformation.[10, p.23]

The main characteristic new features of the present theory can be summarized as follows [10, p.24]:

a. The hypothesis of a nonzero electric field divergence in the vacuum introduces an additional degree of freedom, leading to new physical phenomena. The associated nonzero electric charge density thereby acts somewhat like a hidden variable.
b. This also abolishes the symmetry between the electric and magnetic fields, and then the field equations obtain the character of intrinsic linear symmetry breaking.
c. The theory is both Lorentz and gauge invariant.
d. The velocity of light is no longer a scalar quantity but is represented by a velocity vector of the modulus c.
e. Additional results: Lehnert is also able to derive the mass of Z boson and Higgs-like boson.[21] These would pave an alternative way to new physics beyond Standard Model.

Now it should be clear that Lehnert's RQED is a good alternative theory to QM/QED, and therefore it is also interesting to ask whether this theory can also explain some phenomena related to LENR and UDD reaction of Homlid (as argued by Celani *et al*).[8]

A recent paper [8] presented arguments in favor of extending RQED to become a fluidic Maxwell-Proca equations, as follows:

DOI: 10.5281/zenodo.3876241

Now it appears possible to arrive at fluidic Maxwell-Proca equations, as follows [8]

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} - \kappa^2 \phi, \tag{10}$$

$$\nabla \cdot \vec{B} = 0, \tag{11}$$

$$\dot{B} = -\nabla \times E - \nabla \times \left(\hat{\beta}\nabla \times H_0\right), \tag{12}$$

$$\varepsilon_0 \mu_0 \dot{E} = \nabla \times B - \mu_0 j - \kappa^2 A - \left(\hat{\sigma}E_0 + \rho_e v + \hat{\gamma}\nabla T\right) - \nabla \times \left(\hat{a}\nabla \times E_0\right), \tag{13}$$

where:

$$\nabla \phi = -\frac{\partial \vec{A}}{\partial t} - \vec{E}, \tag{14}$$

$$\vec{B} = \nabla \times \vec{A}, \tag{15}$$

$$\kappa = \frac{mc_0}{\hbar}. \tag{16}$$

Since according to Blackledge, the Proca equations can be viewed as a *unified wavefield* model of electromagnetic phenomena [7], therefore the fluidic Maxwell-Proca equations can be considered as a *unified wavefield* model for electrodynamics of superconductor.

Now, having defined Maxwell-Proca equations, it is possible to write down *fluidic* Maxwell-Proca-Hirsch equations using the same definition, as follows:

$$(\Box_\alpha^2 - \kappa^2)(F - F_0) = \frac{1}{\lambda_L^2}(F - F_0), \tag{17}$$

And

$$(\Box_\alpha^2 - \kappa^2)(J - J_0) = \frac{1}{\lambda_L^2}(J - J_0), \tag{18}$$

where

$$\Box_\alpha^2 = \nabla^{\alpha 2} - \frac{1}{c^2}\frac{\partial^{\alpha 2}}{\partial_t^{\alpha 2}}. \tag{19}$$

DOI: 10.5281/zenodo.3876241

In literature, the above fluidic Maxwell-Proca-Hirsch equations have never been presented elsewhere before. Provided the above equations can be verified with experiments, they can be used to describe electrodynamics of superconductors.

c. Krafft's approach

A third approach of describing elementary particles from aether vortices perspective is discussed by Carl F. Krafft [9]. See for example:

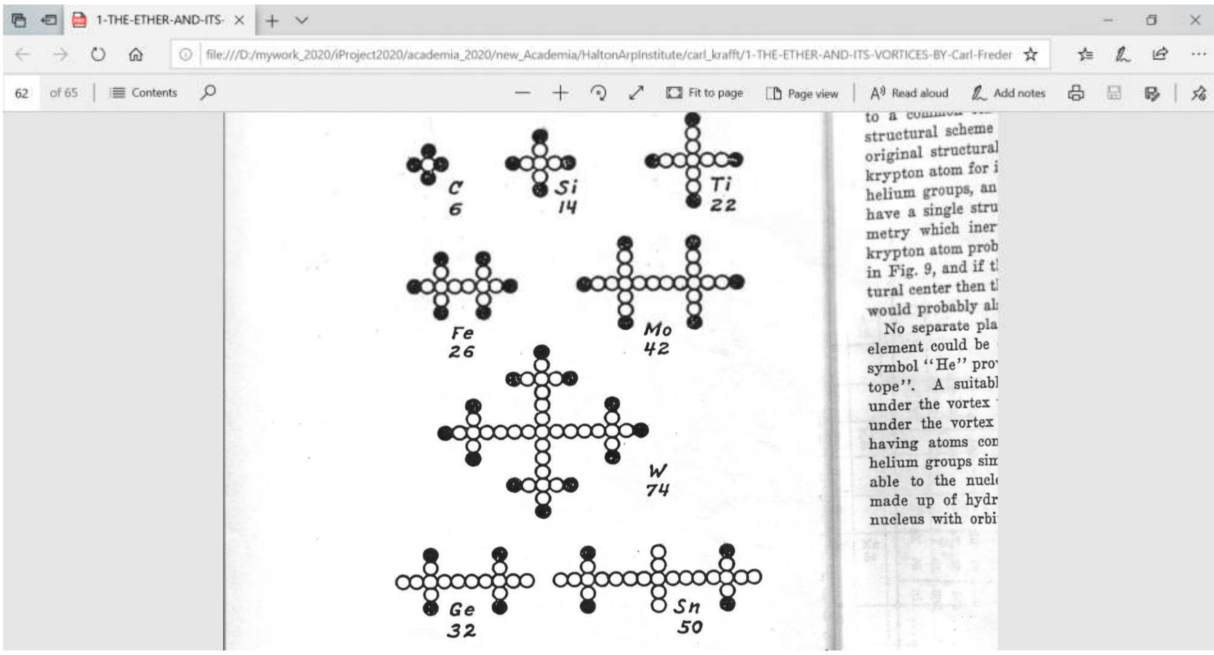

Figure 1. A few elementary particles, source: Carl Frederich Krafft [9]

4. Concluding remarks

In this paper, continuing our previous article, it is argued that it is possible to find a way out of the infinity problem in QED without renormalization route [14]. As a follow up to previous paper, in the present paper, first of all, two theorems for developing unification theories have been discussed, along with pointing out some new proposals including by Simula (2020) on how to derive Maxwell equations in superfluid dynamics setting. This could be a new alternative approach towards "*fluidicle*" or "*vorticle*" model of QED.

Three possible approaches: Tapio Simula, Lehnert's RQED and also Carl F. Krafft, have also been discussed. Nonetheless it should admitted that this article is not complete yet on possible ways to describe vorticle or fluidic as an alternative to QED.

Hopefully this article will inspire further investigations in this line of thoughts.

Acknowledgment

These authors gratefully appreciate three anonymous reviewers for their suggestions to improve this manuscript. Discussions with Robert N. Boyd, PhD. are also appreciated. Special thanks also to Said Broumi for suggesting some improvements.

References:

[1] Edouard B. Sonin, "*Dynamics of quantised vortices in superfluids",* Cambridge: Cambridge University Press, 2016.

[2] Kerson Huang, "*A superfluid Universe"*, Singapore: World Scientific Publishing Co. Pte. Ltd., 2016.

[3] Fischer, U, "Motion of quantized vortices as elementary objects", *Ann. Phys*. (N.Y.) 278, pp.62-85,1999.

[4] Victor Christianto, Florentin Smarandache, Yunita Umniyati. A Derivation of Fluidic Maxwell-Proca Equations for Electrodynamics of Superconductors & Its Implication to Chiral Cosmology Model. *Prespacetime J.,* Vol 9, No 7 (2018), url: https://prespacetime.com/index.php/pst/article/view/1471; see also [4a] V. Christianto, F. Smarandache & R.N. Boyd, Electron Model Based Helmholtz's Electron Vortex & Kolmogorov's Theory of Turbulence. *Prespacetime J.* vol. 10 (1), 2019. url: https://prespacetime.com/index.php/pst/article/view/1516

[5] F. Smarandache & V. Christianto,"Schrödinger Equation and the Quantization of Celestial Systems", *Prog. In Phys*. Vol.2, April 2006. url: http://ptep-online.com/2006/PP-05-12.PDF

[6] Siavash H. Sohrab,"Turbulence and quantum mechanics from cosmos to Planck Scale," url: http://www.mech.northwestern.edu/web/people/faculty/sohrab.php

[7] T. Simula ,"Gravitational Vortex Mass in a Superfluid", arXiv: 2001.03302 (2020), see in particular section D. Ampere–Maxwell-like law.

[8] Victor Christianto, Florentin Smarandache, Yunita Umniyati. Remark on Lehnert's Revised Quantum Electrodynamics (RQED) as an Alternative to Francesco Celani's et al's Maxwell-Clifford equations: with an Outline of Chiral Cosmology model and its role to CMNS. *JCMNS* vol. 31 (Feb. 2020), Publication of ICMNS, http://www.iscmns.org

[9] Carl F. Krafft, "*The Ether and Its vortices",*Copyright: Ken L. Wheeler, pp. 60-61, 2014.

[10] Bo Lehnert, "Mass-Radius Relations of Z and Higgs-like Bosons", *Progress in Physics*, Vol. 10 Issue 1, pp. 5-7, 2014. http://ptep-online.com/2014/PP-36-02.PDF

[11] Bo Lehnert ," A way to Revised Quantum Electrodynamics", *Progress in Physics*, April 2012, Vol. 2, pp. 21-27. url: http://ptep-online.com/2012/PP-29-05.PDF

DOI: 10.5281/zenodo.3876241

[12] Bo Lehnert,"Higgs-Like Particle due to Revised Quantum Electrodynamics", *Progress in Physics*, Vol. 3, pp. 31-32, 2013. http://ptep-online.com/2013/PP-34-10.PDF

[13] Bo Lehnert, "Potentialities of Revised Quantum Electrodynamics", *Progress in Physics*, Vol. 4, pp. 48-52, 2013. http://ptep-online.com/2013/PP-35-07.PDF

[14] V. Christianto, R.N. Boyd & F. Smarandache,"Three Possible Applications of Neutrosophic Logic in Fundamental and Applied Sciences" *International Journal of Neutrosophic Science,* Vol. 1 no. 2, pp. 90-95, 2020.

[15] Jeff D. Eldredge, Tim Colonius, and Anthony Leonard, "A Vortex Particle Method for Two-Dimensional Compressible Flow", *Journal of Computational Physics* 179, pp.371–399, 2002. doi:10.1006/jcph.2002.7060

[16] ANDREA COLAGROSSI, GIORGIO GRAZIANI AND MARIO PULVIRENTI., "Particles for fluids: SPH versus vortex methods", Mathematics and Mechanics of Complex Systems 2(1):45-70, 2014.

[17] Andrea Colagrossi, Emanuele Rossi & Salvatore Marrone. The Discrete Vortex Hydrodynamics method: similarities and di□erences with the SPH. *10th international SPHERIC workshop*. Parma, Italy, June, pp.16-18, 2015

[18] O.S, Kotsur & G.A. Shcheglov," VISCOUS FLUID SIMULATION WITH THE VORTEX ELEMENT METHOD",*31st Int. Council of. Aeoraut. Science*, 2018.

[19] Emanuele Rossi, Andrea Colagrossi, Benjamin Bouscasse and Giorgio Graziani," *Diffused Vortex hydrodynamics metho*d", Communications in Computational Physics 18(18 2), 351-379, 2015.

[20] HENRYK KUDELA AND ZIEMOWIT MIŁOSZ MALECHA," VISCOUS FLOW MODELING USING THE VORTEX PARTICLES METHOD",*TASK QUARTERLY* 13 No 1–2, pp.15–32, 2008.

[21] Peter O'Malley," *Trajectories of Inertial Particles and Fluid Elements*", Haverford College, Haverford PA 19041. May 8th, 2008.

[22] William C. Daywitt," The Anomalous Magnetic Moment of the Electron and Proton Cores According to the Planck Vacuum Theory", *EJERS, EUROPEAN JOURNAL OF ENGINEERING RESEARCH AND SCIENCE*, VOL. 4, NO. 6, 2019.

[23] William C. Daywitt," The Seven-Dimensional Spacetime in the Planck Vacuum Theory and the Structure of the Electron and Proton Cores", *EJERS, EUROPEAN JOURNAL OF ENGINEERING RESEARCH AND SCIENCE*, VOL. 5, NO. 4, 2020

[24] William C. Daywitt," The Fine Structure Constant and Radiative Corrections for the Electron and Positron Cores as Viewed in the Planck Vacuum Theory", *EJERS, EUROPEAN JOURNAL OF ENGINEERING RESEARCH AND SCIENCE*, VOL. 5, NO. 5, MAY 2020

DOI: 10.5281/zenodo.3876241

n-Refined Neutrosophic Vector Spaces

Florentin Smarandache[1] and Mohammad Abobala[2]

[1] Department of Mathematics and Science, University of New Mexico, Gallup, NM 87301, USA
[2] Tishreen University, Faculty of Science, Department of Mathematics, Lattakia, Syria
[1]e-mail: smarand@unm.edu

[2]e-mail: mohammadabobala777@gmail.com

Abstract

This paper introduces the concept of n-refined neutrosophic vector spaces as a generalization of neutrosophic vector spaces, and it studies elementary properties of them. Also, this work discusses some corresponding concepts such as weak/strong n-refined neutrosophic vector spaces, and n-refined neutrosophic homomorphisms.

Keywords: n-Refined weak neutrosophic vector space, n-Refined strong neutrosophic vector space, n-Refined neutrosophic homomorphism.

1.Introduction

Neutrosophy as a part of philosophy founded by F. Smarandache to study origin, nature, and indeterminacies became a strong tool in studying algebraic concepts. Neutrosophic algebraic structures were defined and studied such as neutrosophic modules,and neutrosopohic vector spaces, etc.See [1,2,3,4,5,6,7,8,9]. In 2013 Smarandacheintroduceda perfect idea, when he extended the neutrosophic set to refined [n-valued] neutrosophic set, i.e. the truth value T is refined/split into types of sub-truths such as (T_1, T_2, …,) similarly indeterminacy I is refined/split into types of sub-indeterminacies (I_1, I_2, …,) and the falsehood F is refined/split into sub-falsehood (F_1, F_2,..,) [10,11]. Refined neutrosophic algebraic structures were studied such as refined neutrosophic rings, refined neutrosophic modules, and n-refined neutrosophic rings [4,12].

In this article authors try to define n-refined neutrosophic vector spaces, subspaces, and homomorphisms and to present some of their elementary properties.

For our purpose we use multiplication operation (defined in [12]) between indeterminacies $I_1, I_2, \dots, I_n$ as follows:

$$I_m I_s = I_{\min(m,s)}.$$

This work is a continuation of the study on the n-refined neutrosophic structures that began in [12].

2. Preliminaries

DOI: 10.5281/zenodo.3876216
Received: March 25, 2020 Revised: May 05, 2020 Accepted: May 29, 2020

Definition 2.1: [12]

Let (R,+,×) be a ring and $I_k; 1 \leq k \leq n$ be n indeterminacies. We define R_n(I)={$a_0 + a_1 I + \cdots + a_n I_n$; $a_i \in R$} to be an n-refined neutrosophic ring.

Definition 4.3: [12]

(a) Let R_n(I) be an n-refined neutrosophic ring and P = $\sum_{i=0}^{n} P_i I_i$ = {$a_0 + a_1 I + \cdots + a_n I_n$: $a_i \in P_i$}, where P_i is a subset of R, we define P to be an AH-subring if P_i is a subring of R for all . AHS-subring is defined by the condition $P_i = P_j$ for all i , j.

(b) P is an AH-ideal if$P_i$aretwo-side ideals of R for all i, the AHS-ideal is defined by the condition $P_i = P_j$ for all i , j.

(c) The AH-ideal P is said to be null if $P_i = R or P_i = \{0\}$ for all$i$.

Definition 2.3 :[5]

Let (V , + , ·) be a vector space over the field K; then (V(I) , + , ·) is called a weak neutrosophic vector space over the field K, and it is called a strong neutrosophic vector space if it is a vector space over the neutrosophic field K(I).

Definition 2.4 : [5]

Let V(I) be a strong neutrosophic vector space over the neutrosophic field K(I) and W(I) be a non empty set of V(I) then W(I) is called a strong neutrosophic subspace if W(I) is itself a strong neutrosophic vector space.

Definition 2.6 :[5]

Let U(I) , W(I) be twostrong neutrosophic subspaces of V(I)and let $f: V(I) \to W(I)$, we say that f is a neutrosophic vector space homomorphism if

(a) f(I)=I,

(b) f is a vector space homomorphism.

We define the kernel of f by Ker(f) = { x∈ $V(I)$; f(x) = $0_{W(I)}$}.

Definition 2.7 :[5]

Let $v_1, v_2 .. v_s \in V(I)$and$x \in V(I)$; we say that x is a linear combination of { $v_i; i = 1,..,s$} if

x = $a_1 v_1 + \cdots + a_s v_s$ suchthat$a_i \in K(I)$.

The set{ $\boldsymbol{v_i; i = 1,..,s}$}is called linearly independent if $\boldsymbol{a_1 v_1 + \cdots + a_s v_s = 0}$ implies $\boldsymbol{a_i = 0}$for all i.

3. Main concepts and results

Definition 3.1:

Let (K,+,·) be a field, we say that $K_n(I) = K + KI_1 + \cdots + KI_n = \{a_0 + a_1 I_1 + \cdots + a_n I_n; a_i \in K\}$ is an n-refined neutrosophic field.

It is clear that $K_n(I)$ is an n-refined neutrosophic field, but not a field in the classical meaning.

Example 3.2 :

Let $K = Q$be the field of rationals. The corresponding 3-refined neutrosophic field is

$$Q_3(I) = \{a + bI_1 + cI_2 + dI_3; a, b, c, d \in Q\}.$$

Definition 3.3 :

Let (V,+,·) be a vector space over the field K. Then we say that $V_n(I) = V + VI_1 + \cdots + VI_n = \{x_0 + x_1I_1 + \cdots + x_nI_n;\ x_i \in V\}$ is a weak n-refined neutrosophic vector space over the field K. Elements of $V_n(I)$are called n-refined neutrosophic vectors, elements of K are called scalars.

If we take scalars from the n-refined neutrosophicfield $K_n(I)$, we say that $V_n(I)$ is a strong n-refined neutrosophic vector space over the n-refined neutrosophic field $K_n(I)$. Elements of $K_n(I)$ are called n-refined neutrosophic scalars.

Remark 3.4:

If we take n=1 we get the classical neutrosophic vector space.

Addition on $V_n(I)$is defined as:

$$\sum_{i=0}^{n} a_iI_i + \sum_{i=0}^{n} b_iI_i = \sum_{i=0}^{n} (a_i + b_i)I_i.$$

Multiplication by a scalar $m \in K$is defined as:

$$m \cdot \sum_{i=0}^{n} a_iI_i = \sum_{i=0}^{n} (m.a_i)I_i.$$

Multiplication by an n-refined neutrosophic scalar $m = \sum_{i=0}^{n} m_iI_i \in K_n(I)$is defined as:

$$(\sum_{i=0}^{n} m_iI_i) \cdot (\sum_{i=0}^{n} a_iI_i) = \sum_{i,j=0}^{n} (m_i.a_j)I_iI_j,$$

where $a_i \in V, m_i \in K, I_iI_j = I_{\min(i,j)}$.

Theorem 3.5 :

Let (V,+,·) be a vector space over the field K. Then a weak n-refined neutrosophic vector space $V_n(I)$ is a vector space over the field K. A strong n-refined neutrosophic vector space is not a vector space but a module over the n-refinedneutrosophic field K_n(I).

Proof:

It is similar to that of Theorem 2.3 in [5].

Example 3.6:

Let $V = Z_2$ be the finite vector space of integers modulo 2 over itself:

(a) The corresponding weak 2-refined neutrosophic vector space over the field Z_2 is

$V_n(I) = \{0, 1, I_1, I_2, I_1 + I_2, 1 + I_1 + I_2, 1 + I_1, 1 + I_2\}$.

Definition 3.7:

Let $V_n(I)$ be a weak n-refined neutrosophic vector space over the field K; a nonempty subset $W_n(I)$ is called a weak n-refined neutrosophic subspace of $V_n(I)$ if $W_n(I)$ is a subspace of $V_n(I)$ itself.

Definition 3.8:

Let $V_n(I)$ be a strong n-refined neutrosophic vector space over the n-refined neutrosophic field$K_n(I)$; a nonempty subset $W_n(I)$ is called a strong n-refined neutrosophic subspace of $V_n(I)$ if $W_n(I)$ is a submodule of$V_n(I)$ itself.

Theorem 3.9:

Let $V_n(I)$ be a weak n-refined neutrosophic vector space over the field K, $W_n(I)$ be a nonempty subset of $V_n(I)$. Then $W_n(I)$ is a weak n-refined neutrosophic subspace if and only if:

$x + y \in W_n(I), m \cdot x \in W_n(I)$for all $x, y \in W_n(I), m \in K$.

Proof:

It holds directly from the condition of subspace.

Theorem 3.10:

Let $V_n(I)$ be a strong n-refined neutrosophic vector space over an n-refined neutrosophic field $K_n(I)$, $W_n(I)$ be a nonempty subset of $V_n(I)$. Then $W_n(I)$ is a strong n-refined neutrosophic subspace if and only if:

$x + y \in W_n(I), m \cdot x \in W_n(I)$for all $x, y \in W_n(I), m \in K_n(I)$.

Proof:

It holds directly from the condition of submodule.

Example 3.11:

Let $V = R^2$ be a vector space over the field R, $W = <(0,1)>$ is a subspace of V, $R_2^2(I) = \{(a, b) + (m, s)I_1 + (k, t)I_2; a, b, m, s, k, t \in R\}$ is the corresponding weak/strong 2-refined neutrosophic vector space.

$W_2(I) = \{a_0 + a_1I_1 + a_2I_2\} = \{(0, x) + (0, y)I_1 + (0, z)I_2; x, y, z \in R\}$is a weak 2-refined neutrosophic subspace of the weak 2-refined neutrosophic vector space $R_2^2(I)$ over the field R.

$W_2(I) = \{a_0 + a_1I_1 + a_2I_2\} = \{(0, x) + (0, y)I_1 + (0, z)I_2; x, y, z \in R\}$is a strong 2-refined neutrosophic subspace of the strong 2-refined neutrosophic vector space $R_2^2(I)$ over the n-refined neutrosophic field $R_2(I)$.

Definition 3.12:

Let $V_n(I)$ be a weak n-refined neutrosophic vector space over the field K, x be an arbitrary element of $V_n(I)$, we say that x is a linear combination of $\{x_1, x_2, \dots, x_m\} \subseteq V_n(I)$, or$x = a_1x_1 + a_2x_2 + \cdots + a_mx_m$: $a_i \in K, x_i \in V_n(I)$.

Example 3.13:

Consider the weak 2-refined neutrosophic vector space in Example 3.11,

$x = (0,2) + (1,3)I \in R_2^2(I)$, $x = 2(0,1) + 1(1,0)I_1 + 3(0,1)I_2$, i.e x is a linear combination of the set $\{(0,1), (1,0)I_1, (0,1)I_2\}$ over the field R.

Definition 3.14:

Let $V_n(I)$ be a strong n-refined neutrosophic vector space over an n-refined neutrosophic field $K_n(I)$, x be an arbitrary element of $V_n(I)$, we say that x is a linear combination of $\{x_1, x_2, \dots, x_m\} \subseteq V_n(I)$ is $x = a_1x_1 + a_2x_2 + \dots + a_mx_m$: $a_i \in K_n(I), x_i \in V_n(I)$.

Example 3.15:

Consider the strong 2-refined neutrosophic vector space $R_2^2(I) = \{(a,b) + (m,s)I_1 + (k,t)I_2; a,b,m,s,k,t \in R\}$ over the 2-refined neutrosophic field $R_2(I)$,

$x = (0,2) + (3,3)I_1 + (-1,0)I_2 = (2 + I_1) \cdot (0,1) + (1 + I_2) \cdot (1,1)I_1 + (I_1 - I_2) \cdot (1,0)I_2$, hence x is a linear combination of the set $\{(0,1), (1,1)I_1, (1,0)I_2\}$ over the 2-refined neutrosophic field $R_2(I)$.

Definition 3.16:

Let $X = \{x_1, \dots, x_m\}$ be a subset of a weak n-refined neutrosophic vector space V_n(I) over the field K, X is a weak linearly independent set if $\sum_{i=0}^{m} a_ix_i = 0 \ implies a_i = 0;\ a_i \in K$.

Definition 3.17:

Let $X = \{x_1, \dots, x_m\}$ be a subset of a strong n-refined neutrosophic vector space V_n(I) over the n-refined neutrosophic field $K_n(I)$, X is a weak linearly independent set if $\sum_{i=0}^{m} a_ix_i = 0 \ implies a_i = 0;\ a_i \in K_n(I)$.

Definition 3.18:

Let $V_n(I), W_n(I)$ be two strong n-refined neutrosophic vector space over the n-refined neutrosophic field $K_n(I)$, let $f: V_n(I) \to U_n(I)$ be a well defined map. It is called a strong n-refined neutrosophic homomorphism if:

$f(a.x + b.y) = a.f(x) + b.f(y)$for all $x, y \in V_n(I), a, b \in K_n(I)$.

A weak n-refined neutrosophic homomorphism can be defined as the same.

We can understand the strong n-refined homomorphism as a module homomorphism, weak n-refined neutrosophic homomorphism can be understood as a vector space homomorphism.

Remark:

The previous definition of n-refined homomorphism between two strong/weak n-refined vector spaces is a classical homomorphism between two modules/spaces. We can not add a similar condition to the concept of neutrosophic homomorphism $(f(I_i) = I_i)$, since I_i is not supposed to be an element of $V_n(I)$if V has more than one dimension for example. According to our definition,*Ker(f)* will be a subspace (which is different from classical neutrosophic vector space case) sicne$f$ was defined as a classical homomorphism without any additional condition.

Definition 3.19:

Let $f: V_n(I) \to U_n(I)$ be a weak/strong n-refined neutrosophic homomorphism, we define:

(a) $Ker(f) = \{x \in V_n(I); f(x) = 0\}$.

DOI: 10.5281/zenodo.3876216

(b) $Im(f) = \{y \in U_n(I);\ \exists x \in V_n(I) \text{and} y = f(x)\}$.

Theorem 3.20:

Let $f: V_n(I) \to U_n(I)$ be a weak n-refined neutrosophic homomorphism. Then

(a) $Ker(f)$is a weak n-refined neutrosophic subspace of V_n(I).

(b) $Im(f)$is a weak n-refined neutrosophic subspace of $U_n(I)$.

Proof:

(a) $f$is a vector space homomorphism since $V_n(I), U_n(I)$ are vector spaces, hence $Ker(f)$ is a subspace of the vector space $V_n(I)$, thus $Ker(f)$ is a weak n-refined neutrosophic subspace of $V_n(I)$.

(b) It holds by similar argument.

Theorem 3.21:

Let $f: V_n(I) \to U_n(I)$ be a strong n-refined neutrosophic homomorphism. Then

(a) $Ker(f)$is a strong n-refined neutrosophic subspace of V_n(I).

(b) $Im(f)$is a strong n-refined neutrosophic subspace of $U_n(I)$.

Proof:

(a) $f$is a module homomorphism since $V_n(I), U_n(I)$ are modules over the n-refined neutrosophic field $K_n(I)$, hence $Ker(f)$ is a submodule of the vector space $V_n(I)$, thus $Ker(f)$ is a strong n-refined neutrosophic subspace of $V_n(I)$.

(b) Holds by similar argument.

Example 3.22:

Let $R_2^2(I) = \{x_0 + x_1 I_1 + x_2 I_2;\ x_0, x_1, x_2 \in R^2\}$, $R_2^3(I) = \{y_0 + y_1 I_1 + y_2 I_2;\ y_0, y_1, y_2 \in R^3\}$ be two weak 2-refined neutrosophic vector space over the field R. Consider $f: R_2^2(I) \to R_2^3(I)$, where

$f[(a,b) + (m,n)I_1 + (k,s)I_2] = (a,0,0) + (m,0,0)I_1 + (k,0,0)I_2$, f is a weak 2-refined neutrosophic homomorphism over the field R.

$Ker(f) = \{(0,b) + (0,n)I_1 + (0,s)I_2; b, n, s \in R\}$.

$Im(f) = \{(a,0,0) + (m,0,0)I_1 + (k,0,0)I_2; a, m, k \in R\}$.

Example 3.23:

Let $W_2(I) = <(0,0,1)I_1> = \{q.(0,0,a)I_1; a \in R, q \in R_2(I)\}$, $U_2(I) = <(0,1,0)I_1> = \{q.(0,a,0)I_1; a \in R; q \in R_2(I)\}$ be two strong 2-refined neutrosophic subspaces of the strong 2-refined neutrosophic vector space $R_2^3(I)$ over 2-refined neutrosophic field R_2(I). Define $f: W_2(I) \to U_2(I); f[q(0,0,a)I_1] = q(0,a,0)I_1;\ q \in R_2(I)$.

$f$is a strong 2-refinedneutrosophic homomorphism:

Let $A = q_1(0,0,a)I_1, B = q_2(0,0,b)I_1 \in W_2(I);\ q_1, q_2 \in R_2(I)$, we have

DOI: 10.5281/zenodo.3876216

$$A + B = (q_1 + q_2)(0,0,a + b)I_1, f(A + B) = (q_1 + q_2).(0,a + b,0)I_1 = f(A) + f(B).$$

Let $m = c + dI_1 + eI_2 \in R_2(I)$ be a 2-refined neutrosophic scalar, we have

$m \cdot A = c \cdot q_1(0,0,a)I_1 + d \cdot q_1(0,0,a)I_1I_1 + e \cdot q_1(0,0,a)I_2I_1 = q_1(0,0,c.a + d.a + e.a)I_1,$

$f(m.A) = q_1(0,c.a + d.a + e.a,0)I_1 = m \cdot f(A)$, hence f is a strong 2-refined neutrosophic homomorphism.

$Ker(f) = (0,0,0) + (0,0,0)I_1 + (0,0,0)I_2.$

$Im(f) = U_2(I).$

Remark 3.24:

A union of two n-refined neutrosophic vector spaces $V_n(I)$and$W_n(I)$ is not supposed to be an n-refined neutrosophic vector space, since the addition operation can not be defined. For example consider $V = R^3, W = R^2, n = 2.$

5. Conclusion

In this paper we have introducedthe concept of weak/strong n-refined neutrosophic vector space. Also, some related concepts such as weak/strong n-refined neutrosophic subspace, weak/strong n-refined neutrosophic homomorphism have been presented and studied.

Future research

Authors hope that some corresponding notions will be studied in future such as weak/strong n-refined neutrosophic basis, and AH-subspaces.

Funding:: This research received no external funding

Conflicts of Interest: The authors declare no conflict of interest

References

[1]Abobala, M., "On Some Special Substructures of Neutrosophic Rings and Their Properties", International Journal of Neutrosophic Science", Vol 4, pp.72-81, 2020.

[2]Abobala, M., "On Some Special Substructures of Refined Neutrosophic Rings", International Journal of Neutrosophic Science, Vol 5, pp.59-66, 2020.

[3]Abobala, M,. "Classical Homomorphisms Between Refined Neutrosophic Rings and Neutrosophic Rings", International Journal of Neutrosophic Science, Vol 5, pp.72-75, 2020.

[4]Adeleke, E.O., Agboola, A.A.A., and Smarandache, F., "Refined Neutrosophic Rings I", International Journal of Neutrosophic Science, Vol 2, pp.77-81, 2020.

[5]Agboola, A.A.A,.and Akinleye, S.A., "Neutrosophic Vector Spaces", Neutrosophic Sets and Systems, Vol 4,pp 9-17, 2014.

[6]Agboola, A.A.A., Akwu, A.D., and Oyebo, Y.T., "Neutrosophic Groups and Subgroups", International .J .Math.Combin, Vol 3, pp.1-9, 2012.

[7]Agboola, A.A.A., Akinola, A.D., and Oyebola, O.Y.," NeutrosophicRings I ", International J.Mathcombin, Vol 4,pp.1-14, 2011.

[8]Kandasamy, V.W.B., andSmarandache, F., "Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic Structures", Hexis, Phonex, Arizona 2006.

[9]Olgan, N., and Khatib, A., "Neutrosophic Modules", Journal of Biostatistic and Biometric Application", Vol 3, 2018.

[10]Smarandache, F., "Symbolic Neutrosophic Theory", EuropaNovaasbl, Bruxelles, 2015.

[11]Smarandache, F., *n-Valued Refined Neutrosophic Logic and Its Applications in Physics*, Progress in Physics, pp.143-146, Vol. 4, 2013.

[12]Smarandache, F., and Abobala, M., "n-RefinedNeutrosophic Rings", International Journal of Neutrosophic Science, Vol. 5, pp.83-90 , 2020.

DOI: 10.5281/zenodo.3876216

A New Approach Of Neutrosophic Topological space

Riad K. Al-Hamido

Department of Mathematics, College of Science, Al-Furat University, Deir-ez-Zor, Syria.
riad-hamido1983@hotmail.com

Abstract

In this paper, a new approach of neutrosophic topological space (NA-NTS) is going to be introduced which is more general than neutrosophic topological space. Moreover, a new kind of neutrosophic sets and neutrosophic concepts in this new space is going to be created, which may makes us created a new kind in neutrosophic topology. We prove that a new approach of neutrosophic topological space is not a classical topological space. Also, a new approach of neutrosophic topological space is neither neutrosophic topological space nor neutrosophic crisp topological space. Many examples and theories are presented..

Keywords: New approach of neutrosophic topological spaces, new approach of neutrosophic open sets, new approach of neutrosophic closed sets.

1. Introduction

Recently, as a generalization of fuzzy set was defined by Zadeh [1] and intuitionistic fuzzy set was defined by K. Atanassov[2], the concept of the neutrosophic set was first given by F. Smarandache [3,4]. A.A. Salama and S.A. Alblowi [5] presented neutrosophic topological space via neutrosophic sets. In recent years, the theory of neutrosophic theory becomes very widespread among scientists around the world. For more details about neutrosophic topological space and applications of neutrosophic set theory, the readers should see [6–13].

Recently, Agboola et al. in[14,15], presented the concept of neutrosophic ring and neutrosophic group. Then, in 2015, Agboola in[16], presented the concept of refined neutrosophic algebraic structures. Also, he introduced refined neutrosophic groups. Recently, several works have been done to generalize the neutrosophic algebraic structures to refined neutrosophic algebraic structures. In 2020, Adeleke et al. In [17,18] studied several refined concepts such as refined neutrosophic rings and introduced their basic properties, refined neutrosophic ideals and refined neutrosophic homomorphisms in details. Many researchers had many contributions to neutrosophic ring [19] and neutrosophic topology as [20], [21] and [22]. Also, F. Smarandache extended the neutrosophic set to refined [n-valued] neutrosophic set, and to refined neutrosophic logic, and to refined neutrosophic probability, See[23].

This paper is devoted to the study of a new approach of neutrosophic topology and new approach of neutrosophic topological space, and investigate its basic properties. Also, we prove that a new approach of neutrosophic

DOI: 10.5281/zenodo.3876229

Received: March 12, 2020 Revised: May 30, 2020 Accepted: June 03, 2020

topological space is not a classical neutrosophic topological space. We provide many new definitions, important results, some theories and examples.

2. Definition

In this section, we recall some basic definitions such as neutrosophic group, refined neutrosophic group and neutrosophic ring which are useful in the sequel.

Definition 2.1. [17] Let (G, *) be any group, the neutrosophic group is generated by I and G under * denoted by

$N(G)=\{<G\cup I>,*\}$.

Definition 2.2: [19]
Let R be any ring. The neutrosophic ring $\langle R \cup I\rangle$ is also a ring generated by R and I under the operations of R.
Example 2.3: [19]
Let Z be the ring of integers; $\langle Z \cup I\rangle = \{a + bI : a, b \in Z\}$. $\langle Z \cup I\rangle$ is a ring called the neutrosophic ring of integers. Also $Z \neq \subseteq \langle Z \cup I\rangle$.

Definition 2.4: [5]

A neutrosophic topology (NT) on a set $X \neq \varnothing$ is a family Γ of neutrosophic subsets in X satisfying the following axioms.

1. 1_N , $0_N \in \Gamma$.
2. Γ is closed finite intersection.
3. Γ is closed under arbitrary union.

the pair (X, Γ) is called neutrosophic topological space (NTS) in X. Moreover, elements of Γ are known as neutrosophic open sets (NOS) and their complements are neutrosophic closed sets(NCS).
For a neutrosophic set A over X, the neutrosophic interior and the neutrosophic closure of A are defined as: $Nint(A)=\cup\{G : G \subseteq A , G \in \Gamma \}$ and $Ncl (A)=\cap\{F : A \subseteq F , F^c \in \Gamma \}$.

3. A new Approach Of Neutrosophic Topological Space:

In this section, we study a new approach of neutrosophic topological space, and investigate its basic properties. We denote the indeterminacy by (I). The indeterminacy I is taken to have the properties $I.I = I^2= I$.

Definition 3.1:

Let $\chi\neq\varnothing$ be any set, then we define $(\chi)_N$ as following $(\chi)_N =\{ a\oplus bI : a\in \chi ,b\in \chi\cup \{0 \} \}$ (the set $N(\chi)$ is generated by I and G), also, bI is indeterminacy and bI = I.

- The power set of χ is denoted by $P(\chi)$.
- The power set of $(\chi)_N$ is denoted by $P[(\chi)_N]$.

Definition 3.2:

1. If $A \in P(\chi)$ and $A\neq\varnothing$ then $(A)_N \in P[(\chi)_N]$; $(A)_N =\{ a\oplus bI : a\in \chi ,b\in \chi\cup \{0 \} \}$, also, bI is indeterminacy and bI = I. But if $A=\varnothing$ then $(A)_N =\varnothing\oplus I$.
2. If $(A)_N \in P[(\chi)_N]$, then $(\varnothing\oplus I) \cup (A)_N = (A)_N$ and $(\varnothing\oplus I) \cap (A)_N = \varnothing\oplus I$.

DOI: 10.5281/zenodo.3876229

Example 3.3:
Let X={1,2,3} then $(\chi)_N$={ a⊕bI : a∈ χ ,b∈ χ∪ {0 } }={1, 2, 3, 1⊕I, 2⊕I, 3⊕I}
We remove many elements in N(χ) such as 1⊕2I, 1⊕3I, 2⊕2I, 2⊕3I, 3⊕2I, 3⊕3I because, every one of them equal to member in {1⊕I, 2⊕I, 3⊕I}.
If A={1,2}then, $(A)_N$={ a⊕bI : a∈ χ ,b∈ χ∪ {0 } }={1, 2, 1⊕I, 2⊕I}.
Example 3.4:
Let χ={x, y} then $(\chi)_N$={ a⊕bI : a∈ χ ,b∈ χ∪ {0 } }={ x,y, x⊕I, y⊕I}
We remove many elements in N(χ) as x⊕xI, x⊕yI, y⊕xI, y⊕yI. because, every one of them equals to member in { x⊕I, y⊕I}.
If B={x}then, $(B)_N$={ a⊕bI : a∈ χ ,b∈ χ∪ {0 } }={x, x⊕I}.

Definition 3.5:

Let χ≠∅, if T={A_i}$_{i\in\Delta}$ is topology on χ, then a new approach of neutrosophic topology (N_A-NT) on χ is a family

Ʈ={$(A_i)_N$}$_{i\in\Delta}$ of $(\chi)_N$.

The pair (χ,Ʈ) is called a new approach of neutrosophic topological space (N_A-NTS) in χ. Moreover, members of Ʈ are known as a new approach of neutrosophic open sets (N_A-NOS) and their complements are a new approach of neutrosophic closed sets (N_A-NCS), members of P[$(\chi)_N$] are known as a new approach of neutrosophic sets (N_A-NS).
Remark 3.6:
- N_A-NOS(χ) means the family of the new approach of neutrosophic open sets on χ.
- N_A-NCS(χ) means the family of the new approach of neutrosophic closed sets on χ.

Example 3.7:
Let χ={e , f , g}. $Ʈ = \{\emptyset, A, B, C, X\}$,
$A = \{e, f\}, B = \{e, g\}, C = \{e\}$. $Ʈ = \{\emptyset \oplus I, (A)_N, (B)_N, (C)_N, (X)_N\}$
$(A)_N = \{e, f, e\oplus I, f\oplus I\}, (B)_N = \{e, g, e\oplus I , g\oplus I\}, (C)_N = \{e, e\oplus I\}$.
Then(χ, Ʈ) is a new approach of neutrosophic space.
Remark 3.8:

New approach of neutrosophic topological space is not a classical topological space.

Proof:

Since ∅∉Ʈ, then new approach of neutrosophic topological space is not a classical topological space.
Theorem 3.9:

If I=0 then the new approach of neutrosophic topological space is a classical topological space.

Proof:

If I=0, then, $Ʈ = T$ therefore, new approach of neutrosophic topological space is a classical topological space.

Remark 3.10: Let $(A_i)_N$∈ Ʈ, for all i∈Δ, then:

1. $\cup_{i\in\Delta}(A_i)_N = (\cup_{i\in\Delta}A_i)_N$.
2. $(A_1)_N \cap (A_2)_N = (A_1 \cap A_2)_N$.

Theorem 3.11: Let χ≠∅ then if Ʈ={$(A_i)_N$}$_{i\in\Delta}$ is a new approach of neutrosophic topology (N_A-NT), then:

τ=Ʈ∪{∅,X} is N-topology on $(\chi)_N$, and (χ, τ) is N-topological space.

(N= neutrosophic, but we saied N-topology in this Theorem is for neutrosophic topology because in [4], is defined,

DOI: 10.5281/zenodo.3876229

but they don't have the same concepts).

Proof:

- It is clear that $\varnothing, X \in \tau$.
- Let $(A_1)_N$, $(A_2)_N \in \tau$ then $A_1, A_2 \in T$, but T is topology, therefore, $A_1 \cap A_2 \in T$, hence $(A_1 \cap A_2)_N \in \text{Ʈ}$.

  Therefore $(A_1)_N \cap (A_2)_N = (A_1 \cap A_2)_N \in \tau$ by remark 3.10.
- For every $i \in \Delta$, let $(A_i)_N \in \tau$ then $A_i \in T$, but T is topology, therefore, $\cup_{i\in\Delta} A_i \in T$, hence $(\cup_{i\in\Delta} A_i)_N \in \text{Ʈ}$.

  Therefore, $\cup_{i\in\Delta}(A_i)_N = (\cup_{i\in\Delta} A_i)_N \in \tau$ by remark 3.10.

Therefore τ is N-topology on $(\chi)_N$, and (χ, τ) is N-topological space.

Remark 3.12:

New approach of neutrosophic topological space is not a neutrosophic topological space.

Remark 3.13:

New approach of neutrosophic topological space is not a neutrosophic crisp topological space.

4. The interior and closure operations in a new approach of neutrosophic topological space:

In this part, we define the closure and interior via new approach of neutrosophic open (closed) set.

Definition 4.1: Let $(\chi, \text{Ʈ})$ be an N_A-NTS, and A is a new approach of neutrosophic set (N_A-NS) then :
The union of any N_A-NOS, contained in A is called the a new approach of neutrosophic interior of A (N_A-int(A)).
$N_A\text{-int}(A) = \cup\{B \ ; B \subseteq A \ ; B \in N_A\text{-NOS}\}$.

Theorem 4.2:
Let $(\chi, \text{Ʈ})$ be an N_A-NTS, A, B are a new approach of the neutrosophic set (N_A-NS) then :

1. $N_A\text{-int}(A) \subseteq A$.
2. N_A-int(A) is N_A-NOS.
3. $A \subseteq B \Rightarrow N_A\text{-int}(A) \subseteq N_A\text{-int}(B)$.

Proof :

1. Follows from the definition of N_A-int(A) as a union of any N_A-NOS, contains in A.
2. Since the union of any N_A-NOS, is N_A-NOS, then $N_A\text{-int}(A)=\cup\{B \ ; B \subseteq A \ ; B \in N_A\text{-NOS}(X)\}$ is N_A-NOS.
3. Proof is obvious.

Definition 4.3:
Let $(\chi, \text{Ʈ})$ be an N_A-NTS, and A is a new approach of neutrosophic set (N_A-NS) then :
The intersection of any N_A-NCS, including A is called a new approach of neutrosophic closure of A (N_A-cl(A)).
$N_A\text{-cl}(A) = \cap\{B \ ; B \supseteq A \ ; B \in N_A\text{-NCS}(\chi)\}$.

Theorem 4.4:
Let $(\chi, \text{Ʈ})$ be an N_A-NTS, and A is a new approach of neutrosophic set (N_A-NCS) then :

1. $A \subseteq N_A\text{-cl}(A)$.
2. N_A-cl(A) is N_A-NCS.

Proof :

1. Follow from the definition of N_A-cl(A) as an intersection of any N_A-NCS contained in A.

DOI: 10.5281/zenodo.3876229

2. Prof is obvious.

Theorem 4.5:

Let $(\chi, \mathcal{T})$ be an N_A-NTS, and A is a new approach of neutrosophic set (N_A-NCS) then :

1. N_A-cl(χ-A)= χ - (N_A-int(A)).
2. N_A-int(χ-A)= χ - (N_A-cl(A)).
3. N_A-int(A)= χ - N_A-cl(χ -A)).
4. N_A-cl(A)= χ - (N_A-int(χ -A)).

Proof :

1. χ - (N_A-int(A))= χ -[$\cup${B ; B$\subseteq$A ; B$\in$ N_A-NOS }]
   =$\cap${ χ -B ; χ -B$\supseteq$ χ -A ; χ -B$\in$ N_A-NOS(χ) }= N_A-cl(χ -A)
   =$\cup${ χ -B ; χ -B$\subseteq$ A ; X-B$\in$ N_A-NOS(X) }= χ - N_A-int(A).
2. X- N_A-c(A)= χ -[$\cap${B ; B$\supseteq$A ; B$\in$ N_A-NCS(χ) }]
   =$\cup${ χ -B ; χ -B$\subseteq$ χ -A ; χ -B$\in$ N_A-NCS(χ) }= N_A-int(χ -A).
3. Follows from (2) by put χ -A in place of A.
4. Follows from (1) by put χ -A in place of A.

Theorem 4.6:

Let $(\chi, \mathcal{T})$ be an N_A-NTS, and A is a new approach of neutrosophic set (N_A-NCS) then :

1. A is N_A-NCS, iff N_A-cl(A) = A.
2. A is N_A-NOS, iff N_A-N-int(A)= A.

Proof :

1. Follow from the definition of N_A-cl(A) and Theorem 3.4.
2. Follow from the definition of N_A-int(A) and Theorem 3.2**.**

Theorem 4.7:

Let $(\chi, \mathcal{T})$ be an N_A-NTS, and A is a new approach of neutrosophic set (N_A-NCS) then :

1. N_A-cl[N_A-cl(A)]= N_A-cl(A).
2. N_A-int[N_A-int(A)]= N_A-int(A).

Proof :

Prof is Obvious.

Remark 4.8:

Let $(\chi, \mathcal{T})$ be an N_A-NTS, A, B are a new approach of neutrosophic set (N_A-NS) then :

1. N_A-int(A$\cap$B) $\subseteq$ N_A-int(A)$\cap$$N_A$-int(B).
2. N_A-cl(A$\cap$B) $\subseteq$ N_A-cl(A)$\cap$$N_A$-cl(B).
3. N_A-int(A$\cup$B) $\supseteq$ N_A-int(A)$\cup$$N_A$-int(B).
4. N_A-cl(A$\cup$B) $\supseteq$ N_A-cl(A)$\cup$$N_A$-cl(B).

Proof:

1. Since A$\cap$B $\subseteq$ A, A$\cap$B $\subseteq$ B then N_A-int(A$\cap$B)$\subseteq$ N_A-int (A) and N_A-int(A$\cap$B)$\subseteq$ N_A-int(B), hence N_A-int(A$\cap$B)$\subseteq$ N_A-int(A)$\cap$ N_A-int(A).
2. Since A$\cap$B $\subseteq$ A, A$\cap$B $\subseteq$ B then N_A-cl(A$\cap$B)$\subseteq$ N_A-cl(A) and N_A-cl(A$\cap$B)$\subseteq$ N_A-cl(B), hence N_A-cl(A$\cap$B)$\subseteq$ N_A-cl(A)$\cap$$N_A$-cl(A).
3. Since A$\subseteq$A$\cup$B, B$\subseteq$A$\cup$B then N_A-int(A)$\subseteq$ N_A-int(A$\cup$B) and N_A-int(B)$\subseteq$$N_A$-int(A$\cup$B), hence N_A-int(A)$\cup$$N_A$-int(B)$\subseteq$ N_A-int(A$\cap$B).
4. Since A$\subseteq$A$\cup$B, B$\subseteq$A$\cup$B then N_A-cl(A)$\subseteq$ N_A-cl(A$\cup$B) and N_A-cl(B)$\subseteq$$N_A$-cl(A$\cup$B), hence N_A-cl(A)$\cup$ N_A-cl(B)$\subseteq$$N_A$-cl(A$\cap$B).

Conclusion

In this work, we have introduced a new approach of neutrosophic topology and a new approach of neutrosophic topological space. Then, we have introduced a new approach of the neutrosophic open (closed) sets in a new approach of the neutrosophic topological space. Also, we studied some of their basic properties. Finally, This paper

DOI: 10.5281/zenodo.3876229

is just a beginning of a new structure and we have studied a few ideas only. It will be necessary to carry out more theoretical research to establish a general framework for the practical application. In the future, using these notions, various classes of mappings and separation axioms on the new approach of neutrosophic topological space can be studied.

References

[1] L. A. Zadeh, "Fuzzy Sets", Inform. Control,Vol 8, pp.338-353, 1965.

[2] K. Atanassov, "Intuitionistic Fuzzy Sets", Fuzzy Sets And Systems, Vol 20, pp.87-96, 1986.

[3] F. Smarandache, "A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability", American Research Press, Rehoboth, NM, 1999.

[4] F. Smarandache, "Neutrosophy and Neutrosophic Logic, First International Conference on Neutrosophy, Neutrosophic Logic, Set, Probability, and Statistics", University of New Mexico, Gallup, NM 87301, USA 2002.

[5] A. A.Salma, S.A. Alblowi, "Neutrosophic set and neutrosophic topological spaces", IOSR J. Math., Vol 3, pp.31–35. 2012.

[6] M. Abdel-Basset, M. Mai, E. Mohamed, C. Francisco, H. Z. Abd El-Nasser, "Cosine similarity measures of bipolar neutrosophic set for diagnosis of bipolar disorder diseases", Artificial Intelligence in Medicine 101, 101735, 2019.

[7] M. Abdel-Basset, E. Mohamed, G. Abduallah, and F. Smarandache, "A novel model for evaluation Hospital medical care systems based on plithogenic sets", Artificial intelligence in medicine Vol 100, 2019, 101710.

[8] M. Abdel-Basset, G. Gunasekaran Mohamed, G. Abduallah, C. Victor, "A Novel Intelligent Medical Decision Support Model Based on Soft Computing and IoT" IEEE Internet of Things Journal, 2019.

[9] M. Abdel-Basset, G. Abduallah, G. Gunasekaran, L. Hoang Viet, "A novel group decision making model based on neutrosophic sets for heart disease diagnosis", Multimedia Tools and Applications,Vol 10, pp1-26, 2019.

[10] R.K. Al-Hamido, "Neutrosophic Crisp Bi-Topological Spaces", Neutrosophic Sets and Systems, Vol 21, pp.66-73, 2018.

[11] Al-Hamido, R. K, "A study of multi-Topological Spaces", PhD Theses, AlBaath university , Syria, 2019.

[12] A. B.AL-Nafee, R.K. Al-Hamido, F.Smarandache, "Separation Axioms In Neutrosophic Crisp Topological Spaces", Neutrosophic Sets and Systems, Vol 25, pp.25-32, 2019.

[13] Al-Hamido, R. K, "Neutrosophic Crisp Supra Bi-Topological Spaces", International Journal of Neutrosophic Science, Vol 1, pp.66-73, 2018.

[14] A.A.A. Agboola, A.D. Akinola and O.Y. Oyebola, "Neutrosophic Rings I", Int. J. of Math. Comb, Vol 4, pp.1-14, 2011.

[15] A.A.A. Agboola, A.O. Akwu and Y.T. Oyebo, "Neutrosophic Groups and Neutrosopic Subgroups", Int. J.of Math. Comb, Vol 3, pp.1-9, 2012.

[16] A.A.A. Agboola, "On Refined Neutrosophic Algebraic Structures", Neutrosophic Sets and Systems, Vol 10, pp.99-101, 2015.

[17] E.O. Adeleke, A.A.A. Agboola and F. Smarandache, "Refined Neutrosophic Rings I", International Journal of Neutrosophic Science (IJNS), Vol. 2, pp.77-81, 2020.

[18] E.O. Adeleke, A.A.A. Agboola and F. Smarandache, "Refined Neutrosophic Rings II", International Journal of Neutrosophic Science (IJNS), Vol. 2, pp.89-94, 2020.

[19] W. B. Vasantha Kandasamy, and F. Smarandache, "Neutrosophic Rings", Neutrosophic Sets and Systems, 2006.

[20] F. Smarandache, S. Pramanik, "New Neutrosophic Sets via Neutrosophic Topological Spaces", In Neutrosophic Operational Research; Pons Editions: Brussels, Belgium, Vol 1, pp.189–209, 2017.

DOI: 10.5281/zenodo.3876229

[21] W. Al-Omeri, "Neutrosophic crisp sets via neutrosophic crisp topological spaces", Neutrosophic Sets Syst, Vol 13, pp.96–104, 2016.
[22] W. Al-Omeri, S. Jafari, "On Generalized Closed Sets and Generalized Pre-Closed Sets in Neutrosophic Topological Spaces", Mathematics, Vol 7, pp.1-12, 2019. doi:doi.org/10.3390/math7010001.
[23] F. Smarandache, "*n-Valued Refined Neutrosophic Logic and Its Applications in Physics*", Progress in Physics, Vol 4, pp.143-146, 2013.

DOI: 10.5281/zenodo.3876229

Introduction to NeutroRings

Agboola A.A.A.
Department of Mathematics, Federal University of Agriculture, Abeokuta, Nigeria.
agboolaaaa@funaab.edu.ng

Abstract

The objective of this paper is to introduce the concept of NeutroRings by considering three NeutroAxioms (NeutroAbelianGroup (additive), NeutroSemigroup (multiplicative) and NeutroDistributivity (multiplication over addition)). Several interesting results and examples on NeutroRings, NeutroSubgrings, NeutroIdeals, NeutroQuotientRings and NeutroRingHomomorphisms are presented. It is shown that the 1st isomorphism theorem of the classical rings holds in the class of NeutroRings.

Keywords: Neutrosophy, NeutroGroup, NeutroSubgroup, NeutroRing, NeutroSubring, NeutroIdeal, NeutroQuotientRing and NeutroRingHomomorphism.

1 Introduction

The concept of neutrosophic logic/set introduced by Smarandache[25] is a generalization of fuzzy logic/set introduced by Zadeh[30] and intuitionistic fuzzy logic/set introduced by Atanasov.[14] In neutrosophic logic, each proposition is characterized by truth value in the set T, indeterminacy value in the set I and falsehood value in the set F where T, I, F are standard or nonstandard of the subsets of the nonstandard interval $]^{-}0, 1^{+}[$ where $^{-}0 \leq \inf T + \inf I + \inf F \leq \sup T + \sup I + \sup F \leq 3^{+}$. Statically, T, I, F are subsets, but dynamically, the components of T, I, F are set-valued vector functions/operators depending on many parameters some of which may be hidden or unknown. Neutrosophic logic/set has several real life applications in sciences, engineering, technology and social sciences. The concept has been used in medical diagnosis and multiple decision-making.[15,18,29] Neutrosophic set has been used and applied in several areas of mathematics. For instance in algebra, neutrosophic set has been used to develop neutrosophic groups, rings, vector spaces, modules, hypergroups, hyperrings, hypervector spaces, hypermodules, etc.[1,2,4,5,7–13,16,17,27,28] In analysis, neutrosophic set has been used to develop neutrosophic topological spaces[22–24] and many other areas of mathematical analysis. The concept of neutrosophic logic/set is now well known and embraced in many parts of world. Many researches have been conducted on neutrosophic logic/set and several papers have been published in many international journals by many neutrosophic researchers scattered all over the world. Neutrosophic Sets and Systems and International Journal of Neutrosophic Science are presently two international journals dedicated to publication of research articles in neutrosophic logic/set.

Smarandache[19] recently introduced new fields of research in neutrosophy called NeutroStructures and AntiStructures respectively. In,[20] Smarandache introduced the concepts of NeutroAlgebras and AntiAlgebras and in,[21] he revisited the concept of NeutroAlgebras and AntiAlgebras where he studied Partial Algebras, Universal Algebras, Effect Algebras and Boole's Partial Algebras and he showed that NeutroAlgebras are generalization of Partial Algebras. Motivated by the works of Smarandache in,[19–21] Agboola et al in[6] studied NeutroAlgebras and AntiAlgebras viz-a-viz the classical number systems $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$. Also motivated by the work of Smarandache in,[19] Agboola[3] formally introduced the concept of NeutroGroup by considering three NeutroAxioms (NeutroAssociativity, existence of NeutroNeutral element and existence of NeutroInverse element). In,[3] Agboola studied NeutroSubgroups, NeutroCyclicGroups, NeutroQuotientGroups and NeutroGroupHomomorphisms. Several interesting results and examples were presented and it was shown that generally, Lagrange's theorem and 1st isomorphism theorem of the classical groups do not hold in the class of NeutroGroups. In continuation of the work started in,[3] the present paper is devoted to the presentation of the concept of NeutroRing by considering three NeutroAxioms (NeutroAbelianGroup (additive), NeutroSemigroup (multiplicative) and

Received: March 04, 2020 Revised: May 11, 2020 Accepted: June 02, 2020

NeutroDistributivity (multiplication over addition)). Several interesting results and examples on NeutroRings, NeutroSubgrings, NeutroIdeals, NeutroQuotientRings and NeutroRingHomomorphisms are presented. It is shown that the 1st isomorphism theorem of the classical rings holds in the class of NeutroRings.

2 Preliminaries

In this section, we will give some definitions, examples and results that will be useful in other sections of the paper.

Definition 2.1. [21]

(i) A classical axiom defined on a nonempty set is an axiom that is totally true (i.e. true for all set's elements).

(ii) A NeutroAxiom (or Neutrosophic Axiom) defined on a nonempty set is an axiom that is true for some set's elements [degree of truth (T)], indeterminate for other set's elements [degree of indeterminacy (I)], or false for the other set's elements [degree of falsehood (F)], where $T, I, F \in [0,1]$, with $(T, I, F) \neq (1,0,0)$ that represents the classical axiom, and $(T, I, F) \neq (0,0,1)$ that represents the AntiAxiom.

(iii) An AntiAxiom defined on a nonempty set is an axiom that is false for all set's elements.

Therefore, we have the neutrosophic triplet: $<$ Axiom, NeutroAxiom, AntiAxiom $>$.

Definition 2.2. [3] Let G be a nonempty set and let $* : G \times G \to G$ be a binary operation on G. The couple $(G, *)$ is called a NeutroGroup if the following conditions are satisfied:

(i) $*$ is NeutroAssociative that is there exists at least one triplet $(a, b, c) \in G$ such that

$$a * (b * c) = (a * b) * c \tag{1}$$

and there exists at least one triplet $(x, y, z) \in G$ such that

$$x * (y * z) \neq (x * y) * z. \tag{2}$$

(ii) There exists a NeutroNeutral element in G that is there exists at least an element $a \in G$ that has a single neutral element that is we have $e \in G$ such that

$$a * e = e * a = a \tag{3}$$

and for $b \in G$ there does not exist $e \in G$ such that

$$b * e = e * b = b \tag{4}$$

or there exist $e_1, e_2 \in G$ such that

$$\begin{aligned} b * e_1 &= e_1 * b = b \quad \text{or} && (5)\\ b * e_2 &= e_2 * b = b \quad \text{with} \quad e_1 \neq e_2 && (6)\end{aligned}$$

(iii) There exists a NeutroInverse element that is there exists an element $a \in G$ that has an inverse $b \in G$ with respect to a unit element $e \in G$ that is

$$a * b = b * a = e \tag{7}$$

or there exists at least one element $b \in G$ that has two or more inverses $c, d \in G$ with respect to some unit element $u \in G$ that is

$$\begin{aligned} b * c &= c * b = u && (8)\\ b * d &= d * b = u. && (9)\end{aligned}$$

In addition, if $*$ is NeutroCommutative that is there exists at least a duplet $(a, b) \in G$ such that

$$a * b = b * a \tag{10}$$

and there exists at least a duplet $(c, d) \in G$ such that

$$c * d \neq d * c, \tag{11}$$

then $(G, *)$ is called a NeutroCommutativeGroup or a NeutroAbelianGroup.

If only condition (i) is satisfied, then $(G, *)$ is called a NeutroSemiGroup and if only conditions (i) and (ii) are satisfied, then $(G, *)$ is called a NeutroMonoid.

Example 2.3. [3] Let $\mathbb{U} = \{a, b, c, d, e, f\}$ be a universe of discourse and let $G = \{a, b, c, d\}$ be a subset of $\mathbb{U}$. Let $*$ be a binary operation defined on G as shown in the Cayley table below:

$*$	a	b	c	d
a	b	c	d	a
b	c	d	a	c
c	d	a	b	d
d	a	b	c	a

Then $(G, *)$ is a NeutroAbelianGroup.

Example 2.4. [3] Let $G = \mathbb{Z}_{10}$ and let $*$ be a binary operation on G defined by $x * y = x + 2y$ for all $x, y \in G$ where $''+''$ is addition modulo 10. Then $(G, *)$ is a NeutroAbelianGroup.

Definition 2.5. [3] Let $(G, *)$ be a NeutroGroup. A nonempty subset H of G is called a NeutroSubgroup of G if $(H, *)$ is also a NeutroGroup.

The only trivial NeutroSubgroup of G is G.

Example 2.6. [3] Let $(G, *)$ be the NeutroGroup of **Example** 2.3 and let $H = \{a, c, d\}$. The compositions of elements of H are given in the Cayley table below.

$*$	a	c	d
a	b	d	a
c	d	b	d
d	a	c	a

Then, H is a NeutroSubgroup of G.

Definition 2.7. [3] Let $(G, *)$ and $(H, \circ)$ be any two NeutroGroups. The mapping $\phi : G \to H$ is called a homomorphism if ϕ preserves the binary operations $*$ and $\circ$ that is if for all $x, y \in G$, we have

$$\phi(x * y) = \phi(x) \circ \phi(y). \tag{12}$$

The kernel of ϕ denoted by $Ker\phi$ is defined as

$$Ker\phi = \{x : \phi(x) = e_H\} \tag{13}$$

where $e_H \in H$ is such that $N_h = e_H$ for at least one $h \in H$.

The image of ϕ denoted by $Im\phi$ is defined as

$$Im\phi(x) = \{y \in H : y = \phi(x) \text{ for some } h \in H\}. \tag{14}$$

If in addition ϕ is a bijection, then ϕ is an isomorphism and we write $G \cong H$.

Theorem 2.8. [3] *Let $(G, *)$ and $(H, \circ)$ be NeutroGroups and let $N_x = e_G$ such that $e_G * x = x * e_G = x$ for at least one $x \in G$ and let $N_y = e_H$ such that $e_H * y = y * e_H = y$ for at least one $y \in H$. Suppose that $\phi : G \to H$ is a NeutroGroup homomorphism. Then:*

(i) $\phi(e_G) = e_H$.

(ii) $Ker\phi$ *is a NeutroSubgroup of* G.

(iii) $Im\phi$ *is a NeutroSubgroup of H.*

(iv) ϕ *is injective if and only if* $Ker\phi = \{e_g\}$.

Theorem 2.9. [3] *Let* H *be a NeutroSubgroup of a NeutroGroup* $(G, *)$*. The mapping* $\psi : G \to G/H$ *defined by*

$$\psi(x) = xH \quad \forall\, x \in G$$

is a NeutroGroup homomorphism and the $Ker\psi \neq H$.

Theorem 2.10. [3] *Let* $\phi : G \to H$ *be a NeutroGroup homomorpism and let* $K = Ker\phi$*. Then the mapping* $\psi : G/K \to Im\phi$ *defined by*

$$\psi(xK) = \phi(x) \quad \forall\, x \in G$$

is a NeutroGroup epimorphism and not an isomorphism.

3 Development of NeutroRings and their Properties

The new concept of NeutroRing is developed and studied in this section by considering three NeutroAxioms (NeutroAbelianGroup (additive), NeutroSemigroup (multiplicative) and NeutroDistributivity (multiplication over addition)). Several interesting results and examples are presented.

Definition 3.1. (a) A NeutroRing $(R, +, .)$ is a ring structure that has at least one NeutroOperation among $''+''$ and $''.''$ or at least one NeutroAxiom. Therefore, there are many cases of NeutroRing, depending on the number of NeutroOperations and NeutroAxioms, and which of them are Neutro-Sophicated.

For the purposes of this paper, the following definition of a NeutroRing will be adopted:

(b) Let R be a nonempty set and let $+, . : R \times R \to R$ be binary operations of ordinary addition and multiplication on R. The triple $(R, +, .)$ is called a NeutroRing if the following conditions are satisfied:

(i) $(R, +)$ is a NeutroAbelianGroup.

(ii) $(R, .)$ is a NeutroSemiGroup.

(iii) $''.''$ is both left and right NeutroDistributive over $''+''$ that is there exists at least a triplet $(a, b, c) \in R$ and at least a triplet $(d, e, f) \in R$ such that

$$a.(b + c) = a.b + a.c \tag{15}$$

$$(b + c).a = b.a + c.a \tag{16}$$

$$d.(e + f) \neq d.e + d.f \tag{17}$$

$$(e + f).d \neq e.d + f.d. \tag{18}$$

If $''.''$ is NeutroCommutative, then $(R, +, .)$ is called a NeutroCommutativeRing.

We will sometimes write $a.b = ab$.

Definition 3.2. Let $(R, +, .)$ be a NeutroRing.

(i) R is called a finite NeutroRing of order n if the number of elements in R is n that is $o(R) = n$. If no such n exists, then R is called an infinite NeutroRing and we write $o(R) = \infty$.

(ii) R is called a NeutroRing with NeutroUnity if there exists a multiplicative NeutroUnity element $u \in R$ such that $ux = xu = x$ that is $U_x = u$ for at least one $x \in R$.

(iii) If there exists a least positive n such that $nx = e$ for at least one $x \in R$ where e is an additive NeutroElement in R, then R is called a NeutroRing of characteristic n. If no such n exists, then R is called a NeutroRing of characteristic NeutroZero.

(iv) An element $x \in R$ is called a NeutroIdempotent element if $x^2 = x$.

(v) An element $x \in R$ is called a NeutroINilpotent element if for the least positive integer n, we have $x^n = e$ where e is an additive NeutroNeutral element in R.

(vi) An element $e \neq x \in R$ is called a NeutroZeroDivisor element if there exists an element $e \neq y \in R$ such that $xy = e$ or $yx = e$ where e is an additive NeutroNeutral element in R.

(vii) An element $x \in R$ is called a multiplicative NeutroInverse element if there exists at least one $y \in R$ such that $xy = yx = u$ where u is the multiplicative NeutroUnity element in R.

Definition 3.3. Let $(R, +, .)$ be a NeutroCommutativeRing with NeutroUnity. Then

(i) R is called a NeutroIntegralDomain if R has no at least one NeutroZeroDivisor element.

(ii) R is called a NeutroField if R has at least one NeutroInverse element.

Example 3.4. Let $\mathbb{X} = \{a, b, c, d\}$ be a universe of discourse and let $R = \{a, b, c\}$ be a subset of $\mathbb{X}$. Let $''+''$ and $''.''$ be binary operations defined on R as shown in the Cayley tables below:

$+$	a	b	c
a	a	b	b
b	c	a	c
c	b	c	a

$.$	a	b	c
a	a	a	a
b	a	c	a
c	a	c	b

It is clear from the table that:

$$\begin{aligned} c + (b + c) &= (c + b) + c = a, \\ a + (b + c) &= b, \text{ but } (a + b) + c = c \neq b. \\ a + c &= c + a = b, \\ a + b &= b, \text{ but } b + a = c \neq b. \end{aligned}$$

This shows that $''+''$ is NeutroAssociative and NeutroCommutative. Hence, $(R, +)$ is a commutative NeutroSemiGroup.

Next, let N_x and I_x represent additive neutral element and additive inverse element respectively with respect to any element $x \in R$. Then

$$\begin{aligned} N_a &= a, \\ I_a &= a. \\ N_b & \quad \text{does not exist,} \\ I_b & \quad \text{does not exist.} \\ N_c &= b, \\ I_c &= a. \end{aligned}$$

Hence, $(R, +)$ is a NeutroAbelianGroup.

Next, consider

$$\begin{aligned} b(cb) &= (bc)b = a, \\ c(bc) &= a \text{ but } (cb)c = b \neq a. \end{aligned}$$

This shows that $(R, .)$ is NeutroAssociative.

Lastly, consider

$$\begin{aligned} a.(b + c) &= a.b + a.c = a, \\ b.(c + a) &= c, \text{ but } b.c + b.a = a \neq c. \\ (b + b).b &= b.b + b.b = a, \\ (b + a).c &= b, \text{ but } b.c + a.c = a \neq b. \\ a.b &= b.a = a, \\ b.c &= a, \text{ but } c.b = c \neq a. \end{aligned}$$

This shows that $''.''$ is both left and right NeutroDistributive over $''+''$ and it is NeutroCommutative. Hence, $(R, +, .)$ is a NeutroCommutativeRing. Since $U_a = a$ that is $aa = a$, it follows that $(R, +, .)$ is a NeutroCommutativeRing with NeutroUnity.

It is observed that $''a''$ is both NeutroIdempotent and NeutroNilpotent element in R. NeutroZeroDivisor elements in R are $''a, b, c''$. Again, R is a NeutroField but not a NeutroIntegralDomain.

Theorem 3.5. *Every NeutroField R is not necessarily a NeutroIntegralDomain.*

Example 3.6. Let $X = \mathbb{Z}_{10}$ and let $\oplus$ and $\odot$ be two binary operations on X defined by $x \oplus y = 2x + y$ and $x \odot y = x + 4y$ for all $x, y \in X$ where $''+''$ is addition modulo 10. Then $(X, \oplus, \odot)$ is a NeutroCommutativeRing. To see this:

(i) $(X, \oplus)$ **is a NeutroAbelianGroup:** Let $x, y, z \in X$. Then

$$\begin{aligned} x \oplus (y \oplus z) &= 2x + 2y + z \text{ and } (x \oplus y) \oplus = 4x + 2y + z, \text{ equating these we have} \\ 2x + 2y + z &= 4x + 2y + z \\ \Rightarrow \quad 2x &= 0 \\ \therefore \quad x &= 0, 5. \end{aligned}$$

Thus, only the triplets $(0, x, y)$ and $(5, x, y)$ can verify the associativity of $\oplus$ (degree of associativity $=$ 20%) and therefore, $\oplus$ is NeutroAsociative.

(ii) **Existence of NeutroNeutral and NeutroInverse elements:** Let $e \in X$ such that $x \oplus e = 2x + e = x$ and $e \oplus x = 2e + x = x$. Then $2x + e = 2e + x$ from which we obtain $e = x$. But then, only $0 \oplus 0 = 0$ and $5 \oplus 5 = 5$ in X (degree of existence of neutral element $= 20\%$). This shows that X has a NeutroNeutral element. It can also be shown that X has a NeutroInverse element.

(iii) **NeutroComutativity of $\oplus$:** Let $x \oplus y = 2x + y$ and $y \oplus x = 2y + x$ so that $2x + y = 2y + x$ from which we obtain $x = y$. This shows that only the duplet (x, x) can verify commutativity of $\oplus$ (degree of commutativity $= 10\%$) that is, $\oplus$ is NeutroCommutative. Hence, $(X, \oplus)$ is a NeutroAbelian-Group.

(iv) $(X, \odot)$ **is a NeutroSemiGroup:** Let $x, y, z \in X$. Then

$$x \odot (y \odot z) = x + 4y + 16z \text{ and } (x \odot y) \odot z = x + 4y + 4z$$

so that $x + 4y + 16z = x + 4y + 4z$ from which we obtain $12z = 0$ so that $z = 0, 5$. Hence, only the triplets $(x, y, 0)$ and $(x, y, 5)$ can verify associativity of $\odot$ (degree of associativity $= 20\%$) and consequently, $(X, \odot)$ is a NeutroSemigroup.

(v) **NeutroDistributivity:** Let $x, y, z \in X$. Then

$$\begin{aligned} x \odot (y \oplus z) &= x + 8y + 4z, (x \odot y) \oplus (x \odot z) = 3x + 8y + 4z \text{ so that} \\ x + 8y + 4z &= 3x + 8y + 4z \\ \Rightarrow 2x &= 0 \\ \therefore \quad x &= 0, 5. \end{aligned}$$

This shows that only the triplets $(0, y, z)$ and $(5, y, z)$ can verify left distributivity of $\odot$ over $\oplus$ (degree of left distributivity $= 20\%$). Again,

$$\begin{aligned} (y \oplus z) \odot x &= 4x + 2y + z, (y \odot x) \oplus (z \odot x) = 12x + 2y + z \text{ so that} \\ 4x + 2y + z &= 12x + 2y + z \\ \Rightarrow 8x &= 0 \\ \therefore \quad x &= 0, 5. \end{aligned}$$

This shows that only the triplets $(0, y, z)$ and $(5, y, z)$ can verify right distributivity of $\odot$ over $\oplus$ (degree of right distributivity $= 20\%$). Thus, $\odot$ is both left and right NeutroDistributive over $\oplus$. Finally, let $x \odot y = x + 4y$ and $y \odot x = y + 4x$. Putting $x + 4y = y + 4x$ we have $x = y$ showing that only the duplet (x, x) can verify the commutativity of $\odot$ (degree of commutativity $= 10\%$). Hence, $\odot$ is NeutroCommutative and accordingly, $(X, \oplus, \odot)$ is a NeutroCommutativeRing.

Theorem 3.7. *Let $(R_i, +, .), i = 1, 2, \cdots, n$ be a family of NeutroRings. Then*

(i) $R = \bigcap_{i=1}^{n} R_i$ is a NeutroRing.

(ii) $R = \prod_{i=1}^{n} R_i$ is a NeutroRing.

Proof. Obvious. □

Definition 3.8. Let $(R, +, .)$ be a NeutroRing. A nonempty subset S of R is called a NeutroSubring of R if $(S, +, .)$ is also a NeutroRing.

The only trivial NeutroSubring of R is R.

Example 3.9. Let $(R, +, .)$ be the NeutroRing of **Example 3.4** and let $S = \{a, b\}$. The compositions of elements of S are given in the Cayley tables below.

$+$	a	b
a	a	b
b	c	a

$.$	a	b
a	a	a
b	a	c

Then, S is a NeutroSubring of R. To see this:

(i) $(S, +)$ **is a NeutroAbelianGroup:**

$$\begin{aligned} a + (a + b) &= (a + a) + b = b, \\ b + (a + b) &= a, \text{ but } (b + a) + b = c \neq a. \\ N_a &= a, \\ I_a &= a, \\ N_b & \quad \text{does not exist,} \\ I_b & \quad \text{does not exist.} \\ a + a &= a, \text{ but } a + b = b, b + a = c \neq b. \end{aligned}$$

Hence, $(S, +)$ is a NeutroAbelianGroup.

(ii) $(S, .)$ **is a NeutroSemigroup:**

$$\begin{aligned} a(ba) &= (ab)a = a, \\ b(bb) &= a, \text{ but } (bb)b = c \neq a. \end{aligned}$$

This shows that $(S, .)$ is a NeutroSemigroup.

(iii) **NeutroDistributivity:**

$$\begin{aligned} a(a + b) &= aa + ab = a, \\ b(b + a) &= a, \text{ but } bb + ba = b \neq a. \\ (b + a)a &= ba + aa = a, \\ (a + b)a &= c, \text{ but } aa + ba = a \neq c. \end{aligned}$$

This shows that both left and right NeutroDistributivity hold. Accordingly, $(S, +, .)$ is a NeutroRing. Since S is a subset of R, it follows that S is NeutroSubring of R.

Theorem 3.10. *Let $(R, +, .)$ be a NeutroRing and let $\{S_i\}, i = 1, 2, \cdots, n$ be a family of NeutroSubrings of R. Then*

(i) $S = \bigcap_{i=1}^{n} S_i$ is a NeutroSubring of R.

(ii) $S = \prod_{i=1}^{n} S_i$ is a NeutroSubring of R.

Proof. Obvious. □

Definition 3.11. Let $(R, +, .)$ be a NeutroRing. A nonempty subset I of R is called a left NeutroIdeal of R if the following conditions hold:

(i) I is a NeutroSubring of R.

(ii) $x \in I$ and $r \in R$ imply that at least one $xr \in I$ for all $r \in R$.

Definition 3.12. Let $(R, +, .)$ be a NeutroRing. A nonempty subset I of R is called a right NeutroIdeal of R if the following conditions hold:

(i) I is a NeutroSubring of R.

(ii) $x \in I$ and $r \in R$ imply that at least one $rx \in I$ for all $r \in R$.

Definition 3.13. Let $(R, +, .)$ be a NeutroRing. A nonempty subset I of R is called a NeutroIdeal of R if the following conditions hold:

(i) I is a NeutroSubring of R.

(ii) $x \in I$ and $r \in R$ imply that at least one $xr, rx \in I$ for all $r \in R$.

Example 3.14. Let $R = \{a, b, c\}$ be the NeutroRing of **Example 3.4** and let $I = S = \{a, b\}$ be the NeutroSubring of R given in **Example 3.9**. Consider the following:

$$aa = a, ab = a, ac = a, ba = a, bb = c \notin I, bc = a.$$

This shows that I is a left NeutroIdeal of R. Again,

$$aa = a, ba = a, ca = a, ab = a, bb = c \notin I, cb = c \notin I.$$

This also shows that I is a right NeutroIdeal of R. Hence, I is a NeutroIdeal of R.

Theorem 3.15. *Let $(R, +, .)$ be a NeutroRing and let $\{I_i\}, i = 1, 2, \cdots, n$ be a family of NeutroIdeals of R. Then*

(i) $I = \bigcap_{i=1}^{n} I_i$ is a NeutroIdeal of R.

(ii) $I = \sum_{i=1}^{n} I_i$ is a NeutroIdeal of R.

Proof. Obvious. □

Definition 3.16. Let $(R, +, .)$ be a NeutroRing and let I be a NeutroIdeal of R. The set R/I is defined by

$$R/I = \{x + I : x \in R\}. \tag{19}$$

For $x + I, y + I \in R/I$ with at least a pair $(x, y) \in R$, let $\oplus$ and $\odot$ be binary operations on R/I defined as follows:

$$\begin{aligned} (x + I) \oplus (y + I) &= (x + y) + I, && (20) \\ (x + I) \odot (y + I) &= xy + I. && (21) \end{aligned}$$

Then it can be shown that the tripple $(R/I, \oplus, \odot)$ is a NeutroRing which we call a NeutroQuotientRing.

Example 3.17. Let R be the NeutroRing of **Example 3.4** and let I be its NeutroIdeal of **Example 3.14**. Then

$$\begin{aligned} a + I &= \{a, b\} = I, \\ b + I &= \{a, c\}, \\ c + I &= \{b, c\}, \\ \therefore \quad R/I &= \{a + I, b + I, c + I\} = \{\{a, b\}, \{a, c\}, \{b, c\}\}. \end{aligned}$$

Consider the Cayley tables below:

$\oplus$	$a+I$	$b+I$	$c+I$
$a+I$	$a+I$	$b+I$	$b+I$
$b+I$	$c+I$	$a+I$	$c+I$
$c+I$	$b+I$	$c+I$	$a+I$

$\odot$	$a+I$	$b+I$	$c+I$
$a+I$	$a+I$	$a+I$	$a+I$
$b+I$	$a+I$	$c+I$	$a+I$
$c+I$	$a+I$	$c+I$	$b+I$

It easy to deduce from the tables that $(R/I, \oplus, \odot)$ is a NeutroRing.

Theorem 3.18. *Let I be a NeutroIdeal of the NeutroRing R. Then R/I is a NeutroCommutativeRing with NeutroUnity if and only if R is a NeutroCommutativeRing with NeutroUnity.*

Proof. Easy. □

Definition 3.19. Let $(R, +, .)$ and $(S, +', .')$ be any two NeutroRings. The mapping $\phi : R \to S$ is called a NeutroRingHomomorphism if ϕ preserves the binary operations of R and S that is if for at least a pair $(x, y) \in R$, we have:

$$\phi(x+y) = \phi(x) +' \phi(y), \quad (22)$$
$$\phi(x.y) = \phi(x).'\phi(y). \quad (23)$$

The kernel of ϕ denoted by $Ker\phi$ is defined as

$$Ker\phi = \{x : \phi(x) = e_R\} \quad (24)$$

where $e_R \in R$ is such that $N_r = e_R$ for at least one $r \in R$.

The image of ϕ denoted by $Im\phi$ is defined as

$$Im\phi = \{y \in S : y = \phi(x) \text{ for at least one } y \in S\}. \quad (25)$$

If in addition ϕ is a NeutroBijection, then ϕ is called a NeutroRingIsomorphism and we write $R \cong S$. NeutroRingEpimorphism, NeutroRingMonomorphism, NeutroRingEndomorphism and NeutroRingAutomorphism are defined similarly.

Example 3.20. Let R be the NeutroRing of **Example 3.4** and let $\phi : R \times R \to R$ be a mapping defined by

$$\phi((x, y)) = \begin{cases} a & \text{if } x = y, \\ d & \text{if } x \neq y. \end{cases}$$

It can be shown that ϕ is a NeutroRingHomomorphism. The $Ker\phi = \{(a, a), (b, b), (c, c)\}$ which is a NeutroSubgRing of $R \times R$ as can be seen in the Cayley tables below.

$+$	(a,a)	(b,b)	(c,c)
(a,a)	(a,a)	(b,b)	(b,b)
(b,b)	(c,c)	(a,a)	(c,c)
(c,c)	(b,b)	(c,c)	(a,a)

$.$	(a,a)	(b,b)	(c,c)
(a,a)	(a,a)	(a,a)	(a,a)
(b,b)	(a,a)	(c,c)	(a,a)
(c,c)	(a,a)	(c,c)	(b,b)

$Im\phi = \{a, d\} \not\subseteq R.$

Theorem 3.21. *Let R and S be two NeutroRings. Let $N_x = e_R$ for at least one $x \in R$ and let $N_y = e_S$ for at least one $y \in S$. Suppose that $\phi : R \to S$ is a NeutroRingHomomorphism. Then:*

(i) $\phi(e_R)$ is not necessarily equals e_S.

(ii) $Ker\phi$ is a NeutroSubring of R.

(iii) $Im\phi$ is not necessarily a NeutroSubring of S.

(iv) ϕ is NeutroInjective if and only if $Ker\phi = \{e_R\}$ for at least one $e_R \in R$.

Example 3.22. Let $R = \{a, b, c\}$ be the NeutroRing of **Example 3.4** and let $I = \{a, b\}$ be the NeutroIdeal of R given in **Example 3.14**. Let $\phi : R \to R/I$ be a mapping defined by $\phi(x) = x + I$ for at least one $x \in R$. Then $\phi(a) = a + I = \{a, b\} = I, \phi(b) = b + I = \{a, c\}$ and $\phi(c) = c + I = \{b, c\}$ from which we obtain that ϕ is a NeutroRingHomomorphism.

$$Ker\phi = \{x \in R : \phi(x) = e_{R/I}\} = \{x \in R : x + I = e_{R/I} = a + I\} = I.$$

Theorem 3.23. *Let I be a NeutroIdeal of a NeutroRing R. Then the mapping $\psi : R \to R/I$ defined by*

$$\psi(x) = x + I \quad \textit{for at least one} \quad x \in R$$

is a NeutroRingEpimomorphism and the $Ker\psi = I$.

Theorem 3.24. *Let $\phi : R \to S$ be a NeutroRingHomomorpism and let $K = Ker\phi$. Then the mapping $\psi : R/K \to Im\phi$ defined by*

$$\psi(x + K) = \phi(x) \quad \textit{for at least one} \quad x \in R$$

is a NeutroRingIsomorphism.

Proof. Let $x+K, y+K \in R/K$ with at least a pair $(x,y) \in R$. Then

$$\begin{aligned}
\psi((x+K)\oplus(y+K)) &= \psi((x+y)+K) \\
&= \phi(x+y) \\
&= \phi(x)+\phi(y) \\
&= \psi(x+K)\oplus\psi(y+K). \\
\psi((x+K)\odot(y+K)) &= \psi((xy)+K) \\
&= \phi(xy) \\
&= \phi(x)\phi(y) \\
&= \psi(x+K)\odot\psi(y+K). \\
Ker\psi &= \{x+K \in R/K : \psi(x+K) = e_{\phi(x)}\} \\
&= \{x+K \in R/K : \phi(x) = e_{\phi(x)}\} \\
&= \{e_{R/K}\}.
\end{aligned}$$

This shows that ψ is a NeutroBijectiveHomomorphism and therefore it is a NeutroRingIsomorphism that is $R/K \cong Im\phi$ which is the same as what is obtainable in the classical rings. □

Theorem 3.25. *NeutroRingIsomorphism of NeutroRings is an equivalence relation.*

Proof. The proof is the same as the classical rings. □

4 Conclusion

We have for the first time introduced in this paper the concept of NeutroRings by considering three NeutroAxioms (NeutroAbelianGroup (additive), NeutroSemigroup (multiplicative) and NeutroDistributivity (multiplication over addition)). Several interesting results and examples on NeutroRings, NeutroSubgrings, NeutroIdeals, NeutroQuotientRings and NeutroRingHomomorphisms are presented. It is shown that the 1st isomorphism theorem of the classical rings holds in the class of NeutroRings. More advanced properties of NeutroRings will be presented in our future papers. Other NeutroAlgebraicStructures such as NeutroModules, NeutroVectorSpaces etc are opened to be developed and studied by other Neutrosophic researchers.

5 Appreciation

The author is very grateful to all anonymous reviewers for their valuable observations, comments and suggestions which have contributed immensely to the quality of the paper.

References

[1] Adeleke, E.O., Agboola, A.A.A. and Smarandache, F., Refined Neutrosophic Rings I, International Journal of Neutrosophic Science, vol. 2(2), pp. 77-81, 2020. (DOI:10.5281/zenodo.3728222)

[2] Adeleke E.O., Agboola A.A.A. and Smarandache F., Refined Neutrosophic Rings II, International Journal of Neutrosophic Science, vol. 2(2), pp. 89-94, 2020. (DOI:10.5281/zenodo.3728235)

[3] Agboola, A.A.A., Introduction to NeutroGroups, International Journal of Neutrosophic Science,Volume 6 , Issue 1, pp. 41-47 , 2020

[4] Agboola, A.A.A., On Refined Neutrosophic Quotient Groups, International Journal of Neutrosophic Science, vol. 5 (2), pp. 76-82, 2020. DOI:10.5281/zenodo.3828609.

[5] Agboola, A.A.A., Ibrahim, M.A., Adeleke, E.O. and Akinleye, S.A., On Refined Neutrosophic Algebraic Hyperstructures I, International Journal of Neutrosophic Science, vol. 5 (1), pp. 29-37, 2020. DOI:10.5281/zenodo.3789007.

[6] Agboola, A.A.A., Ibrahim, M.A. and Adeleke, E.O., Elementary Examination of NeutroAlgebras and AntiAlgebras viz-a-viz the Classical Number Systems, International Journal of Neutrosophic Science, vol. 4 (1), pp. 16-19, 2020. DOI:10.5281/zenodo.3752896.

[7] Agboola, A.A.A., On Refined Neutrosophic Algebraic Structures, Neutrosophic Sets and Systems, vol.10, pp. 99-101, 2015.

[8] Agboola, A.A.A., Akinola, A.D. and Oyebola, O.Y, Neutrosophic Rings I, International Journal of Mathematical Combinatorics, vol. 4, pp. 1-14, 2011.

[9] Agboola, A.A.A., Adeleke, E.O. and Akinleye, S.A., Neutrosophic Rings II, International Journal of Mathematical Combinatorics, vol. 2, pp. 1-8, 2012.

[10] Agboola, A.A.A., Akwu, A.O. and Oyebo, Y.T., Neutrosophic Groups and Neutrosopic Subgroups, International Journal of Mathematical Combinatorics, vol. 3, pp. 1-9, 2012.

[11] Agboola, A.A.A and Davvaz, B., On Neutrosophic Canonical Hypergroups and Neutrosophic Hyperrings, Neutrosophic Sets and Systems, vol. 2, pp. 34-41, 2014.

[12] Agboola, A.A.A., Davvaz, B. and Smarandache, F., Neutrosophic Quadruple Hyperstructures, Annals of Fuzzy Mathematics and Informatics, vol. 14, pp. 29-42, 2017.

[13] Akinleye, S.A., Smarandache, F. and Agboola, A.A.A., On Neutrosophic Quadruple Algebraic Structures, Neutrosophic Sets and Systems, vol. 12, pp. 122-126, 2016.

[14] Atanassov, K., Intuitionistic fuzzy sets, Fuzzy Sets and Systems, Vol. 20, pp. 87-96, 1986.

[15] Deli, I., Broumi, S. and Smarandache, F., On neutrosophic refined sets and their applications in medical diagnosis, Journal of New Theory, Vol. 6, pp. 88-98, 2015.

[16] Gilbert, L. and Gilbert, J., Elements of Modern Algebra, Eighth Edition, Cengage Learning, USA, 2015.

[17] Ibrahim, M.A., Agboola, A.A.A., Adeleke, E.O. and Akinleye, S.A., On Refined Neutrosophic Hypervector Spaces, International Journal of Neutrosophic Science, vol. 4 (1), pp. 20-35, 2020. DOI:10.5281/zenodo.3752906.

[18] Liu, P. and Wang, Y., Multiple attribute decision-making method based on single-valued neutrosophic normalized weighted Bonferroni mean, Neural Computing and Applications, Vol. 25 (7-8), pp. 2001-2010, 2014.

[19] Smarandache, F., Introduction to NeutroAlgebraic Structures, in Advances of Standard and Nonstandard Neutrosophic Theories, Pons Publishing House Brussels, Belgium, Ch. 6, pp. 240-265, 2019.

[20] Smarandache, F., Introduction to NeutroAlgebraic Structures and AntiAlgebraic Structures (revisited), Neutrosophic Sets and Systems, vol. 31, pp. 1-16, 2020. DOI: 10.5281/zenodo.3638232.

[21] Smarandache, F., NeutroAlgebra is a Generalization of Partial Algebra, International Journal of Neutrosophic Science, vol. 2 (1), pp. 08-17, 2020.

[22] Smarandache, F. and Pramanik, S., New Neutrosophic Sets via Neutrosophic Topological Spaces, In Operational Research; Eids.; Pons Editions: Brussels, Belgium, Vol. I, pp. 189-209, 2017.

[23] Smarandache, F. and Pramanik, S., Neutrosophic crisp sets via neutrosophic crisp topological spaces, Neutrosophic Sets and Systems, Vol. 13, pp. 96-104, 2016.

[24] Smarandache, F. and Pramanik, S., On Generalized Closed Sets and Generalized Pre-Closed Sets in Neutrosophic Topological Spaces, Mathematics, Vol. 7(1), pp. 1-12, 2019. DOI:DOI.ORG/10.3390/math7010001.

[25] Smarandache, F., A Unifying Field in Logics: Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability, (3rd edition), American Research Press, Rehoboth, 2003. http://fs.gallup.unm.edu/eBook-Neutrosophic4.pdf.

[26] Smarandache, F., n-Valued Refined Neutrosophic Logic and Its Applications in Physics, Progress in Physics, USA, vol. 4, pp. 143-146, 2013.

[27] Smarandache, F., (T,I,F)- Neutrosophic Structures, Neutrosophic Sets and Systems, vol. 8, pp. 3-10, 2015.

[28] Vasantha Kandasamy, W.B. and Smarandache, F., Neutrosophic Rings, Hexis, Phoenix, Arizona, 2006. http://fs.gallup.unm.edu/NeutrosophicRings.pdf

[29] Ye, J. and Zhang, Q.S., Single-valued neutrosophic similarity measures for multiple attribute decision making, Neutrosophic Sets and Systems, Vol. 2, pp. 48-54, 2014.

[30] Zadeh, L.A., Fuzzy Sets, Information and Control, Vol. 8, pp. 338-353, 1965.

Classical Homomorphisms Between n-Refined Neutrosophic Rings

Mohammad Abobala, Faculty of Science, Tishreen University,Lattakia, Syria

e-mail: mohammadabobala777@gmail.com

Abstract

This paper studies classical homomorphisms between n-refined neutrosophic ring and m-refined neutrosophic ring. It proves that every m-refined neutrosophic ring $R_m(I)$ is a homomorphic image of n-refined neutrosophic ring $R_n(I)$, where $m \leq n$. Also, it presents a discussion of kernels and some corresponding isomorphisms between those rings.

Keywords:n-Refined neutrosophic ring, Ring homomorphism, Ring extension.

1.Introduction

Neutrosophy is a new kind of logic founded by Smarandache, concerns with origin, nature, and indeterminacy. Neutrosophic ideas found their way in algebra and its applications. Neutrosophical algebraic studies began with Smarandache and Kandasamy in [5]. They presented many neutrosophical structures such as neutrosophic rings, groups, and loops. Many generalizations came to light, such as n-refined neutrosophic structures, refined neutrosophic rings, n-refined neutrosophic rings, refined neutrosophic ideals, refined neutrosophic homomorphisms, and AH-substructures.See[1-8].

In [3], Abobala proved that each neutrosophic ring R(I) is a homomorphic image of the refined neutrosophic ring $R(I_1, I_2)$. This result means that a refined neutrosophic ring $R(I_1, I_2)$ is a ring extension of R(I). This extension can be represented by the homomorphism $f: R(I_1, I_2) \to R(I); f(a, bI_1, cI_2) = a + (b + c)I$. In this paper we generalize the previous result into n-refined neutrosophic rings. Also, we prove that each m-refined neutrosophic ring $R_m(I)$ is a homomorphic image of n-refined neutrosophic ring $R_n(I)$, where $m \leq n$.

All homomorphisms through this paper are taken by classical meaning in Ring Theory, not by neutrosophical meaning. For example, see[1, 2].

Motivation

This paper generalizes some results introduced in [3] about refined neutrosophic rings, into n-refined neutrosophic rings. Also, it clarifies that n-refined neutrosophic ideas have an algebraic origin, since n-refined neutrosophic ring $R_n(I)$ can be realized as a classical ring extension of the ring R.

2. Preliminaries

In this section, we show some concepts we used through the paper.

Theorem2.1: [3]

DOI: 10.5281/zenodo.3879508

Received: March 09, 2020 Revised: May 27, 2020 Accepted: June 04, 2020

Let (R ,+,×) be a ring and R(I), R(I_1, I_2) the related neutrosophic ring and refined neutrosophic ring respectively, we have:

(a) There is a ring homomorphism f: R$(I_1, I_2) \to R(I)$.

(b) The additive group (Ker(f),+) is isomorphic to the additive group (R,+).

Theorem 2.2: [3]

Let R be a ring, where Char(R) = 2, there is a subring of R(I_1, I_2) $say\ K$ with property $K \cong R$; R$(I_1, I_2)/K \cong R(I)$.

Definition 2.3: [6]

Let (R,+,×) be a ring and $I_k; 1 \leq k \leq n$ be n indeterminacies. We define R_n(I)={$a_0 + a_1 I + \cdots + a_n I_n$; $a_i \in R$} to be n-refined neutrosophic ring.

Addition and multiplication on R_n(I)are defined as:

$\sum_{i=0}^{n} x_i I_i + \sum_{i=0}^{n} y_i I_i = \sum_{i=0}^{n} (x_i + y_i) I_i$, $\sum_{i=0}^{n} x_i I_i \times \sum_{i=0}^{n} y_i I_i = \sum_{i,j=0}^{n} (x_i \times y_j) I_i I_j$.

Where × is the multiplication defined on the ring R.

3. Main results

In the following section, we discuss the main results and theorems.

Lemma 3.1:

Let R be a ring with unity 1, $R_n(I), R_{n-1}(I)$ be the corresponding n-refined neutrosophic ring, and

(n-1) refined neutrosophic ring respectively. Then:

(a) $R_{n-1}(I)$is a homomorphic image of $R_n(I)$.

(b) $R_n(I)/K \cong R_{n-1}(I)$; K is a ring with property $K \cong R$.

Proof:

(a) Define the map

$f: R_n(I) \to R_{n-1}(I); f(a_0 + a_1 I_1 + \cdots + a_n I_n) = a_0 + a_1 I_1 + \cdots + a_{n-2} I_{n-2} + (a_{n-1} + a_n) I_{n-1}$.

$f$is well defined. Consider$x = \sum_{i=0}^{n} a_i I_i = \sum_{i=0}^{n} b_i I_i = y$, we have $a_i = b_i$ for all i, thus

$a_0 + a_1 I_1 + \cdots + a_{n-2} I_{n-2} + (a_{n-1} + a_n) I_{n-1} = b_0 + b_1 I_1 + \cdots + b_{n-2} I_{n-2} + (b_{n-1} + b_n) I_{n-1}$, this means $f(x) = f(y)$.

$f$is a classical ring homomorphism.

Let $x = \sum_{i=0}^{n} a_i I_i, y = \sum_{i=0}^{n} b_i I_i$ be two arbitrary elements in $R_n(I)$, we have:

$x + y = \sum_{i=0}^{n} (a_i + b_i) I_i$,

$x . y = \sum_{i,j=0}^{n} (a_i . b_j) I_i I_j = \sum_{i,j=o}^{n-2} (a_i . b_j) I_i I_j + (a_{n-1} I_{n-1} + a_n I_n) . (b_{n-1} I_{n-1} + b_n I_n) =$

DOI: 10.5281/zenodo.3879508

$\sum_{i,j=o}^{n-2}(a_i.b_j)I_iI_j + (a_{n-1}.b_{n-1} + a_{n-1}.b_n + a_nb_{n-1})I_{n-1} + a_n.b_nI_n$.

$f(x+y) = \sum_{i=0}^{n-2}(a_i+b_i)I_i + (a_{n-1} + a_n + b_{n-1} + b_n)I_{n-1} = f(x) + f(y)$.

$f(x.y) = \sum_{i,j=o}^{n-2}(a_i.b_j)I_iI_j + (a_{n-1}.b_{n-1} + a_{n-1}.b_n + a_nb_{n-1} + a_n.b_n)I_{n-1}$,

$f(x).f(y) =$

$[a_0 + a_1I_1 + \cdots + a_{n-2}I_{n-2} + (a_{n-1} + a_n)I_{n-1}].[b_0 + b_1I_1 + \cdots + b_{n-2}I_{n-2} + (b_{n-1} + b_n)I_{n-1}]$.

Hence $f(x.y) = f(x).f(y)$.

(b) $Ker(f) = \{y = \sum_{i=0}^{n} b_iI_i \in R_n(I): f(y) = 0\}$, this implies $b_i = 0\ for\ all\ 0 \leq i \leq n-2$ and

$a_{n-1} = -a_n$, so $Ker(f) = \{a_n(I_n - I_{n-1}); a_n \in R\}$= K.

By first isomorphism theory we find

$R_n(I)/K \cong R_{n-1}(I)$. Consider $g: K \to R; g\big(a_n(I_n - I_{n-1})\big) = a_n$. Where $a_n \in R$.

It is easy to see that g is a well defined map, f is an isomorphism:

Let $= a_n(I_n - I_{n-1}), y = b_n(I_n - I_{n-1}); a_i, b_i \in R; i \in \{n-1, n\}$ be two arbitrary elements in K,

$x + y = (a_n + b_n)(I_n - I_{n-1}),\ g(x+y) = a_n + b_n = g(x) + g(y)$.

$x.y = a_n.b_nI_n - a_n.b_nI_{n-1} - a_n.b_nI_{n-1} + a_n.b_nI_{n-1} = a_n.b_n(I_n - I_{n-1})$,

$g(x.y) = a_n.b_n = g(x).g(y)$.

It is clear that g is bijective. Thus we get the proof.

Theorem 3.2:

Let R be a ring with unity 1, $R_n(I), R_m(I)$ be the corresponding n-refined, m-refined neutrosophic ring with $m \leq n$. Then $R_m(I)$ is a homomorphic image of $R_n(I)$.

Proof:

If $m = n$ then it is clear.

Suppose that $m < n$. Then by previous lemma, we get a series of ring homomorphisms

$R_n(I) \to_{f_n} R_{n-1}(I) \to_{f_{n-1}} R_{n-2}(I) \ldots \to_{f_{n-m+1}} R_{m+1}(I) \to_{f_{n-m}} R_m(I)$.

$f_{n-m}of_{n-m+1}o \ldots of_{n-1}of_n$is a ring homomorphism between $R_n(I), R_m(I)$ since it is a product of homomorphisms, thus our proof is complete.

Example 3.3:

Let $R = Z_6$be the ring of integers modulo 6, $R_4(I) = \{a + bI_1 + cI_2 + dI_3 + eI_4; a, b, c, d, e \in R\}$ be the corresponding 4-refined neutrosophic ring, $R_3(I) = \{a + bI_1 + cI_2 + dI_3; a, b, c, d \in R\}$ be the corresponding 3-refined neutrosophic ring. We have:

DOI: 10.5281/zenodo.3879508

(a) $f: R_4(I) \to R_3(I); f(a + bI_1 + cI_2 + dI_3 + eI_4) = a + bI_1 + cI_2 + (d + e)I_3$is a homomorphism.

(b) $Ker(f) = \{m(I_4 - I_3);\ m \in R\} \cong R$, and $R_4(I)/Ker(f) \cong R_3(I)$.

(c) $g: R_3(I) \to R_2(I); g(a + bI_1 + cI_2 + dI_3) = a + bI_1 + (c + d)I_2$is a homomorphism too.

(d) $gof: R_4(I) \to R_2(I); gof(a + bI_1 + cI_2 + dI_3 + eI_4) = a + bI_1 + (c + d + e)I_2$is a homomorphism between $R_4(I), R_2(I)$.

Result 3.4:

According to Theorem 3.2, if$R_n(I)$ is an n-refined neutrosophic ring. Then:

$R_n(I)/K_n \cong R_{n-1}(I), R_{n-1}(I)/K_{n-1} \cong R_{n-2}(I), \dots, R_1(I)/K_1 \cong R,$Where $K_i \cong R$.

So, we have the following series of ring extensions $R \to R_1(I) \to \cdots \to R_{n-1}(I) \to R_n(I)$. For each ring

$R_m(I); 1 \le m \le n$there is a subring $K \cong R$, with property $R_m(I)/K \cong R_{m-1}(I)$.

According to the previous result, we can understand the n-refined neutrosophic ring $R_n(I)$ as an extension of R with n steps. Each step can be represented by a ring homomorphism.

Remark 3.5:

The main application of Result 3.4 that it clarifies the algebraic nature of n-refined neutrosophic idea in the case of n-refined neutrosophic ring.

Splitting I into n subindeterminacies $I_1, \dots, I_n$ is a logical idea introduced by Smarandache in [7,8]. This work shows that it has an algebraic origin in rings, since the n-refined neutrosophic ring $R_n(I)$can be considered as a classical ring extention based on classical homomorphisms.

5. Conclusion

In this article we have studied classical homomorphisms between n-refined neutrosophic ring $R_n(I)$ and m-refined neutrosophic ring $R_m(I)$, where $m \le n$. Also,we haveproved the following results:
1) Each m-refined neutrosophic ring is a homomorphic image of n-refined neutrosophic ring, where $m \le n$.
2) Each n-refined neutrosophic ring $R_n(I)$ is a ring extension of the ring R by n steps. Each one can be represented by a ring homomorphism.

Funding: “This research received no external funding”

Conflicts of Interest: “The authors declare no conflict of interest.”

References

[1]Abobala, M., "On Some Special Substructures of Neutrosophic Rings and Their Properties", International Journal of Neutrosophic Science", Vol.4 , pp.72-81, 2020.

[2]Abobala, M., "On Some Special Substructures of Refined Neutrosophic Rings", International Journal of Neutrosophic Science, Vol. 5, pp.59-66, 2020.

[3]Abobala, M., "Classical HomomorphismsBetween Refined Neutrosophic Rings and Neutrosophic Rings", International Journal of Neutrosophic Science, Vol. 5, pp.72-75, 2020.

DOI: 10.5281/zenodo.3879508

[4]Adeleke, E.O., Agboola, A.A.A., and Smarandache, F., "Refined Neutrosophic Rings I", International Journal of Neutrosophic Science, Vol. 2 , pp.77-81, 2020.

[5]Kandasamy, V.W.B., and Smarandache, F., "Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic Structures", Hexis, Phonex, Arizona 2006.

[6]Smarandache, F., and Abobala, M., "n-Refined Neutrosophic Rings", International Journal of Neutrosophic Science, Vol. 6, pp.83-90 , 2020.

[7]Smarandache, F., "Symbolic Neutrosophic Theory", EuropaNovaasbl, Bruxelles, 2015.

[8].Smarandache, F., "*n-Valued Refined Neutrosophic Logic and Its Applications in Physics*", Progress in Physics, Vol. 4, pp.143-146, 2013.

DOI: 10.5281/zenodo.3879508

AH-Substructures in Neutrosophic Modules

[1]Mohammad Abobala and [2]Riad K. Alhamido

[1]Faculty of Science, Tishreen University, Lattakia, Syria

[2]Faculty of Science, Alfurat University, Deir Elzor, Syria

[1]e-mail: mohammadabobala777@gmail.com

[2]e-mail: Riad-hamido1983@hotmail.com

Abstract

This article introduces the concept of AH-submodule, AHS-submodule of a neutrosophic module, and AHS-homomorphism. This work presents some basic notions and properties of these concepts such as AH-Kernel, AH-Quotient, and dimension, and determines the algebraic structure of weak neutrosophic module over a commutative ring R.

Keywords: Neutrosophic module, AH-submodule, AHS-submodule, AH-Quotient.

1. Introduction

Neutrosophy as a new branch of philosophy founded by Smarandache has an interesting effect in real world problems, applications and algebraic studies. Recently, neutrosophic sets have been applied in the medical field such as diagnosis of bipolar disorder diseases [1], evaluation hospital medical care systems [2], intelligent medical decision support model based on soft computing and many other ares [3], and novel group decision making model for heart disease [4].

Many neutrosophic algebraic structures have been defined and studied such as neutrosophic rings, neutrosophic groups, neutrosophic vector spaces and refined neutrosophic rings. See[7, 8, 9, 10, 11, 12, 13, 14]. AH-substructures were defined for the first time in neutrosophic rings in [5], then they were introduced in refined neutrosophic rings in [2]. These structures have many symmetric properties which illustrate a line between classical algebra and neutrosophical algebra. AH-ideal in a neutrosophic ring R(I) is a set with form $P + QI$ where P, Q are ideals in R, other AH-structures can be defined to have many parts, each part has a special structure such (subspace, ideal, and submodule). AH-substructures were studied in refined neutrosophic rings, and neutrosophic vector spaces too [6,8]. In this paper we try to define AH-submodule and AHS-submodule of a neutrosophic module and introduce some of their elementary properties. Also, some interesting concepts were defined and applied in this study, such as neutrosophic module AHS-homomorphism, and AH-Quotient module.

Motivation

This work is an extension of studies about AH-substructures in neutrosophical algebraic structures in [5,6,8].

2. Preliminaries

DOI: 10.5281/zenodo.3879702

Received: March 08, 2020 Revised: April 18, 2020 Accepted: May 27, 2020

In the following part, we recall some notions which will be used in our study.

Definition 2.1: [15]

Let (N, + , .) be a module over the ring R. Then (N(I), +, .) is called a weak neutrosophic module over the ring R, and it is called a strong neutrosophic module if it is a module over the neutrosophic ring R(I).

Here elements of N(I) have the form $\boldsymbol{x+yI; x, y \in N}$, i.e N(I) can be written as $\boldsymbol{N(I) = N + NI}$.

Definition 2.2: [15]

Let M(I) be a strong neutrosophic module over the neutrosophic ring R(I) and W(I) be a nonempty set of M(I), then W(I) is called a strong neutrosophic submodule if W(I) itself is a strong neutrosophic module.

Definition 2.3: [15]

Let U(I) and W(I) be two strong neutrosophic modules and let $f: U(I) \to W(I)$, we say that f is a neutrosophic vector space homomorphism if

(a) $f(I) = I$.

(b) f is a module homomorphism.

We define the kernel of f by Ker(f) = { x$\in M(I)$; $f(x) = 0$ }.

Definition 2.4: [6]

Let R(I) be a neutrosophic ring and $P = P_0 + P_1 I = \{a_0 + a_1 I;\ a_0 \in P_0, a_1 \in P_1\}$.

(a) We say that P is an AH-ideal if $P_0$and P_1 are ideals in the ring R.

(b) We say that P is an AHS-ideal if $P_0 = P_1$.

Definition 2.5: [6]

Let (R(I_1, I_2), $+,\times$) be a refined neutrosophic ring, and P_0, P_1, P_2 be three ideals in the ring R then the set

$P = (P_0, P_1 I_1, P_2 I_2) = \{(a, bI_1, cI_2); a \in P_0, b \in P_1, c \in P_2\}$ is called a refined neutrosophic AH-ideal.

If $P_0 = P_1 = P_2$ then P is called a refined neutrosophic AHS-ideal.

3. Main concepts and discussion

In this section we introduce our main definitions, results, and we illustrate some examples.

Definition 3.1:

Let $M(I) = M + MI$ be a strong/weak neutrosophic module, the set

$S = P + QI = \{x + yI; x \in P, y \in Q\}$, where P and Q are submodules of V is called an AH-submodule of M(I).

If $P = Q$, S is called an AHS-submodule of M(I).

DOI: 10.5281/zenodo.3879702

Example 3.2:

We have M = $Z^2 = Z \times Z$ is a module over R, $P =< (0,1) >, Q =< (1,0) >$, are two submodules of M. The set

$S = P + QI = \{(0, a) + (b, 0)I; a, b \in Z\}$ is an AH-submodule of M(I).

The set $L = P + PI = \{(0, a) + (0, b)I\}; a, b \in Z$ is an AHS-submodule of M(I).

Theorem 3.3:

Let $M(I) = M + MI$ be a neutrosophic weak module over the ring R, and let $S = P + QI$ be an AH-submodule of $M(I)$, then S is a submodule.

Proof:

Suppose that $x = a + bI, y = c + dI \in S; a, c \in P,\ b, d \in Q$,

$x + y = (a + c) + (b + d)I \in S$. For each scalar $m \in R$ we obtain $m.x = m.a + (m.b)I \in S$, since P and Q are submodules; thus $S = P + QI$ is a submodule of M(I) over the ring R.

Theorem 3.4:

Let M(I) be a neutrosophic strong module over a neutrosophic ring R(I), let $S = P + PI$ be an AHS-submodule. Then S is a submodule of M(I).

Proof:

Suppose that $x = a + bI, y = c + dI \in S; a, c, b, c \in P$,

$x + y = (a + c) + (b + d)I \in S$. Let $m = t + dI \in R(I)$ be a neutrosophic scalar, we find

$m.x = (t.a) + (d.a + d.b + t.b)I \in S$, since $d.a + d.b + t.b \in P$, thus we get the desired result.

A strong AH-submodule is not supposed to be a submodule. For examples see [4].

Definition 3.5:

(a) Let M and W be two modules, $L_M: M \to W$ be a homomorphism. The AHS-homomorphism can be defined as follows:

$L: M(I) \to W(I); L(a + bI) = L_M(a) + L_M(b)I.$

(b) If $S = P + QI$ is an AH-submodule of M(I), $L(S) = L_M(P) + L_M(Q)I$.

(c) If $S = P + QI$ is an AH-submodule of W(I), $L^{-1}(S) = L_W^{-1}(P) + L_W^{-1}(Q)I$.

(d) $AH - Ker(L) = Ker(L_M) + Ker(L_M)\, I = \{x + yI; x, y \in Ker(L_M)\}$.

Theorem 3.6:

Let W(I) and M(I) be two neutrosophic strong/weak modules, and $L: M(I) \to W(I)$ be an AHS-homomorphism:

(a) $AH - Ker(L)$ is an AHS-submodule of M(I).

DOI: 10.5281/zenodo.3879702

(b) If $S = P + QI$ is an AH-submodule of M(I), $L(S)$ is an AH-submodule of W(I).

(c) If $S = P + QI$ is an AH-submodule of W(I), $L^{-1}(S)$ is an AH-submodule of M(I).

Proof:

(a) Since $Ker(L_M)$ is a submodule of M, we find that $AH - Ker(L) = Ker(L_M) + Ker(L_M)I$ is an AHS-submodule of M(I).

(b) We have $L(S) = L_M(P) + L_M(Q)I$; thus L(S) is an AH-submodule of W(I), since $L_M(P), L_M(Q)$are submodules of W.

(c) By regarding that $L^{-1}(S) = L_W^{-1}(P) + L_W^{-1}(Q)I$, $L_W^{-1}(P)$ and $L_W^{-1}(Q)$ are submodules of M, we obtain that $L^{-1}(S)$ is an AH-subModule of M(I).

Theorem 3.7:

Let W(I) and M(I) be two neutrosophic strong modules over a neutrosophic ring R(I), and $L: M(I) \to W(I)$ be an AHS-homomorphism. Then

$L(x + y) = L(x) + L(y), L(m.x) = m.L(x)$, for all $x, y \in M(I), m \in R(I)$.

Proof:

Suppose $x = a + bI, y = c + dI; a, b, c, d \in M, and\ m = s + tI \in K(I)$, we have

$L(x + y) = L([a + c] + [b + d]I) = L_M(a + c) + L_M(b + d)I = [L_M(a) + L_M(b)I] + [L_M(c) + L_M(d)I] = L(x) + L(y)$.

$m.x = (s.a) + (s.b + t.a + t.b)I, L(m.x) = L_M(s.a) + L_M(s.b + t.a + t.b)I$

$= s.L_M(a) + [s.L_M(b) + t.L_M(a) + t.L_M(b)]I = (s + tI).(L_M(a) + L_M(b)I) = m.L(x)$.

Theorem 3.8:

Let $S = P + QI$ be an AH-submodule of a neutrosophic weak module M(I) over a ring R, suppose that

$X = \{x_i; 1 \le i \le n\}$ is a bases of P and $Y = \{y_j; 1 \le j \le m\}$ is a bases of Q then $X \cup YI$ is a bases of S.

Proof:

Let $z = x + yI$ be an arbitrary element in S; $x \in P, y \in Q$. Since P and Q are submodules of M we can write

$x = a_1x_1 + a_2x_2 + \cdots + a_nx_n;\ a_i \in R\ and\ x_i \in X, y = b_1y_1 + b_2y_2 + \cdots + b_my_m;\ b_i \in K, y_i \in Y.$

Now we obtain $z = (a_1x_1 + \cdots + a_nx_n) + (b_1y_1I + \cdots + b_my_mI)$; thus $X \cup YI$ generates the subspace S.

$X \cup YI$ is linearly independent set. Assume that $(a_1x_1 + \cdots + a_nx_n) + (b_1y_1I + \cdots + b_my_mI) = 0$, this implies

$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ and $(b_1y_1 + b_2y_2 + \cdots + b_my_m)I = 0$. Since X and Y are linearly independent sets over R, we get $a_i = b_j = 0\ for\ all\ i, j$ and $X \cup YI$ is linearly independent then it is a basis of S.

Result 3.9:

DOI: 10.5281/zenodo.3879702

Let $S = P + QI$ be an AH-submodule of a neutrosophic weak module M(I) with finite dimension over a ring R, from Theorem 3.8 and the fact that $X \cap YI = \emptyset$, we find $\dim(S) = \dim(P) + \dim(Q)$.

Example 3.10:

Let M $=Z^3 = Z \times Z \times Z$ is a module over the ring Z, $P =< (0,0,1) >, Q =< (0,1,0) >$ be two submodules of M,

(a) $S = P + QI = \{(0,0,m) + (0,n,0)I;\ m,n \in Z\ \}$is an AH-submodule of M(I).

(b) The set $\{(0,0,1),(0,1,0)I\}$ is a bases of S, $\dim(S) = \dim(P) + \dim(Q) = 1 + 1 = 2$.

(c) $L_M: M \to M;\ L_M(x,y,z) = (x + y, y, z) for\ all\ x, y, z \in Z$ is a homomorphism, the corresponding AHS-homomorphism is

$L: M(I) \to M(I); L[(x,y,z) + (a,b,c)I] = L_M(x,y,z) + L_M(a,b,c)I =(x + y, y, z) + (a + b, b, c)I.$

(d) $L(S) = L_M(P) + L_M(Q) = L_M\{(0,0,m)\} + L_M\{(0,n,0)\}I = \{(0,0,m) + (n,n,0)I\}; m,n \in Z$, which is an AH-submodule of M(I).

Example 3.11 :

Let M$=Z^2 = Z \times Z$, W$=Z^3 = Z \times Z \times Z$ be two modules over the ring Z, $L_M: M \to W;\ L_M(x,y) = (x + y, x + y, x + y)$ is a homomorphism. The corresponding AHS-homomorphism is

$L: M(I) \to W(I); L[(x,y) + (a,b)I] = (x + y, x + y, x + y) + (a + b, a + b, a + b)I.$

$KerL_M =< (1,-1) >, AH - Ker(L) = Ker(L_M) + Ker(L_M)\ I =< (1,-1) > +< (1,-1) > I =$

$\{(a,-a) + (b,-b)I; a,b \in Z\}$ which is an AHS-submodule of M(I).

We find $\dim(Ker(L)) = 1 + 1 = 2$.

Definition 3.12:

Let M(I) be a neutrosophic strong/weak module, $S = P + QI$ be an AH-submodule of M(I), we define

AH-Quotient module as:

$M(I)/S = M/P + (M/Q)\ I= (x + P) + (y + Q)I; x,y \in M.$

Theorem 3.13:

Let M(I) be a neutrosophic weak module over a ring R, and $S = P + QI$ be an AH-submodule of M(I). The AH-Quotient $M(I)/S$ is a module with respect to the following operations:

Addition: $[(x + P) + (y + Q)I] + [(a + P) + (b + Q)]I = (x + a + P) + (y + b + Q)I; x, y, a, b \in M.$

Multiplication by a scalar: $(m).[(x + P) + (y + Q)I] = (m.x + P) + (m.y + Q)I;$

$x, y \in M$ and $m \in R.$

Proof:

It is easy to check that operations are well defined, and $(M(I)/S\ ,+)$ is abelian group.

DOI: 10.5281/zenodo.3879702

Let $z = [(x + P) + (y + Q)I] \in M(I)/S$, we have $1.z = z$.

Assume that $m, n \in R$, we have $m.(n.z) = m.[(n.x + P) + (n.y + Q)I] = (m.n.x + P) + (m.n.y + Q)I = (m.n).z$.

$(m + n).z = [(m + n).x + P] + [(m + n).y + Q]I = m.z + n.z.$

Let $h = [(a + P) + (b + Q)I] \in M(I)/S$, $z + h = (x + a + P) + (y + b + Q)I$,

$m.(z + h) = (m.x + m.a + P) + (m.y + m.b + Q)m.z + m.h.$

Example 3.14:

We have M = $Q^2 = Q \times Q$ is a module over the ring Z, $P =< (0,1) >, N =< (1, 0) >$ are two submodules of M,

$S = P + NI = \{(0, a) + (b, 0)I; a, b \in Q\}$ is an AH-subspace of M(I).

The AH-Quotient is $M(I)/S = \{[(x, y) + P] + [(a, b) + N]I; x, y, a, b \in Q\}$.

We clarify operations on $M(I)/S$ as follows:

$x = [(2,1) + P] + [(1,3) + N]I, y = [(2,5) + P] + [(1,1) + N]I$ are two elements in $V(I)/S$, $m = 3\ is\ a\ scalar\ in\ Z.$

$x + y = [(4,6) + P] + [(2,4) + N]I$, $3.x = [(6,3) + P] + [(3,9) + N]I.$

Remark 3.15:

If $S = P + PI$ is an AHS-submodule of a neutrosophic weak module M(I) over the ring R, then AH-Quotient $M(I)/S = M/P + M/P\ I$ is a weak neutrosophic module, since M/P is a module.

We introduce the following result, it determines the algebraic structure of neutrosophic module.

Theorem 3.16:

Let (N,+,.) be a module over the commutative ring R, $N(I)$ be the corresponding weak neutrosophic module over R. Then

$N(I) \cong N \times N.$

Proof:

Define $f: N \times N \to N(I); f(x, y) = x + yI; x, y \in N$, it is easy to see that f is well defined bijective map.

Let $(x, y), (z, t) \in M \times M, r \in R$, we have $(x, y) + (z, t) = (x + z, y + t), r.(x, y) = (r.x, r.y)$,

$f[(x, y) + (z, t)] = (x + z) + (y + t)I = (x + yI) + (z + tI) = f(x, y) + f(z, t).$

$f[r.(x, y)] = r.x + r.yI = r.(x + yI) = r.f(x)$. Hence f is a module isomorphism.

Result 3.17:

Theorem 3.16 shows that the concept of weak neutrosophic module is a rediscovering of direct product of a module with itself, thus all results in [11] can be obtained easily according to this result.

DOI: 10.5281/zenodo.3879702

According to the previous result, we can find that every submodule of a weak neutrosophic module is an AH-submodule, since every submodule of $M \times M$ has the form $W \times N$; W, N are submodules of M.

5. Conclusion

In this article, we have defined the concepts of AH-submodule, AHS-submodule, and AHS-homomorphism in neutrosophic module as new generalizaion of AH-substructures in neutrosophic vector spaces. Also, we have studied some basic properties of these concepts.
On the other hand, we have proved that every weak neutrosophic module M(I) over any commutative ring R is isomorphic to the direct product of M with itself.

Future Aspect

This study can be extended into more neutrosophic structures.

Funding: “This research received no external funding”

Conflicts of Interest: “The authors declare no conflict of interest.”

References

[1] Abdel-Basset, M., Mai M., Mohamed E., Francisco C., and Abd El-Nasser, H. Z., "Cosine Similarity Measures of Bipolar Neutrosophic Set for Diagnosis of Bipolar Disorder Diseases", Artificial Intelligence in Medicine 101, 2019 , 101735.

[2] Abdel-Basset, M., Mohamed E., Abduallah G., and Florentin S., "A Novel Model for Evaluation Hospital Medical Care Systems Based on Plithogenic Sets" Artificial Intelligence in Medicine 100 2019, 101710.

[3] Abdel-Basset, M., Gunasekaran M., Abduallah G., and Victor C., "A Novel Intelligent Medical Decision Support Model Based on Soft Computing and IoT", IEEE Internet of Things Journal , 2019.

[4] Abdel-Basset, M., Abduallah G., Gunasekaran M., and Hoang Viet L., "A Novel Group Decision Making Model Based on Neutrosophic Sets for Heart Disease Diagnosis", Multimedia Tools and Applications , 1-26, 2019.

[5] Abobala, M., "On Some Special Substructures of Neutrosophic Rings and Their Properties", International Journal of Neutrosophic Science", Vol. 4 , pp.72-81, 2020.

[6] Abobala, M., "On Some Special Substructures of Refined Neutrosophic Rings", International Journal of Neutrosophic Science, Vol. 5, pp.59-66, 2020.

[7] Abobala, M., "Classical Homomorphisms Between Refined Neutrosophic Rings and Neutrosophic Rings", International Journal of Neutrosophic Science, Vol. 5, pp.72-75, 2020.

[8] Abobala, M., " AH-Subspaces in Neutrosohic Vector Spaces", International Journal of Neutrosophic Science, Vol. 6, PP. 80-86, 2020.

[9] Adeleke, E.O., Agboola, A.A.A., and Smarandache, F., "Refined Neutrosophic Rings I", International Journal of Neutrosophic Science, Vol. 2 , pp.77-81, 2020.

[10] Agboola, A.A.A., and Akinleye, S.A., "Neutrosophic Vector Spaces", Neutrosophic Sets and Systems, Vol. 4 , pp. 9-17, 2014.

[11] Agboola, A.A.A., Akwu, A.D., and Oyebo, Y.T., "Neutrosophic Groups and Subgroups", International .J .Math.Combin, Vol. 3, pp. 1-9, 2012.

DOI: 10.5281/zenodo.3879702

[12] Agboola, A.A.A., Akinola, A.D, and Oyebola, O.Y.," NeutrosophicRings I ", International J.Mathcombin, Vol. 4, pp. 1-14, 2011.

[13] Injrou, S., "Linear Algebra 2 ", Tishreen University Press, pp.180-230, 2015.

[14] Kandasamy, V.W.B., and Smarandache, F., "Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic Structures", Hexis, Phonex, Arizona 2006.

[15] Olgan, N., and Khatib, A., "Neutrosophic Modules", Journal of Biostatistic and Biometric Application", Vol. 3, pp.1-16, 2018.

DOI: 10.5281/zenodo.3879702

On Refined Neutrosophic R-module

Necati Olgun[1], Ahmed Hatip[2]

[1,2]Department of Mathematics, Gaziantep University, Turkey
E-mail: olgun@gantep.edu.tr, kollnaar5@gmail.com

Abstract

Modules are one of the fundamental and rich algebraic structure concerning some binary operations in the study of algebra. In this paper, some basic structures of refined neutrosophic R-modules and refined neutrosophic submodules in algebra are generalized. Some properties of refined neutrosophic R-modules and refined neutrosophic submodules are presented. More precisely, classical modules and refined neutrosophic rings are utilized. Consequently, refined neutrosophic R- modules that are completely different from the classical modular in the structural properties are introduced. Also, neutrosophic R-module homomorphism is explained and some definitions and theorems are presented.

Keywords: Refined neutrosophic group, refined neutrosophic ring, refined neutrosophic R-module, weak refined neutrosophic R-module, strong refined neutrosophic R-module, refined neutrosophic R-module homomorphism.

1. Introduction

Neutrosophy is a new branch of philosophy that studies the nature, origin, and scope of neutralities as well as their interaction with ideational spectra. Neutrosophy is the base of neutrosophic logic, which is an extension of fuzzy logic in which indeterminacy is included. Florentin Smarandanche introduced the notion of neutrosophy as a new branch of philosophy in 1995. After that, he introduced the concept of neutrosophic logic and neutrosophic set where each proposition in neutrosophic logic is approximated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I and the percentage of falsity in a subset F so that this neutrosophic logic is called an extension of fuzzy logic as well as an extension of intuitionistic fuzzy logic.

Neutrosophic set is the generalization of the classical set, neutrosophic group, and neutrosophic ring the generalization of classical group and ring, etc. The same way neutrosophic R-module is the generalization of the classical R-module. By utilizing the idea of neutrosophic theory, Vasantha Kanadasamy and Florentin Smarandache [11] studied neutrosophic algebraic structures by inserting an indeterminate element I in the algebraic structure and then combining 'I' with the corresponding binary operation for corresponding binary operation.

Agboola in [1], introduced the concept of refined neutrosophic algebraic structures and studied refined neutrosophic groups in particular. Since the introduction of refined neutrosophic algebraic structures, many neutrosophic researchers have established and studied more refined neutrosophic algebraic structures. Adeleke et al. [5] studied refined neutrosophic rings and refined neutrosophic subrings and presented their fundamental properties. Also in [6], Adeleke et al. studied refined neutrosophic ideals and refined neutrosophic homomorphisms and presented their basic

DOI: 10.5281/zenodo.3881963

Received: March 07, 2020 Revised: May 18, 2020 Accepted: May 31, 2020

properties. The present paper is devoted to the study of refined neutrosophic R-module. More properties of refined neutrosophic R-module will be presented. For more details about neutrosophy, refined neutrosophic logic, neutrosophic groups, and refined neutrosophic groups, the readers should see [2-4, 7-10, 12-25].

2. Preliminaries

In this section, we present the basic definitions that are useful in this research.

Definition 2.1. [1] Let $\left(X\left(I_1,I_2\right),+,\cdot\right)$ be any refined neutrosophic algebraic structure where + and $\cdot$ are ordinary addition and multiplication respectively. I_1 *and* I_2 are the split components of the indeterminacy factor I that is $I=\alpha I_1+\beta I_2$ with $\alpha,\beta\in R$ *or* C . Also, I_1 *and* I_2 **they** are taken to have the properties $I_1^2=I_1, I_2^2=I_2$ and $I_1I_2=I_2I_1=I_1$.

For any two elements, we define

1) $x+y=\left(a,bI_1,cI_2\right)+\left(d,eI_1,fI_2\right)=\left(a+d,(b+e)I_1,(c+f)I_2\right)$

2) $x\cdot y=\left(a,bI_1,cI_2\right)\cdot\left(d,eI_1,fI_2\right)=\begin{pmatrix} ad,(ae+bd+be+bf+ce)I_1, \\ (af+cd+cf)I_2 \end{pmatrix}$.

Definition 2.2. [1] Let $(G,*)$ be any group. The couple $\left(G\left(I_1,I_2\right),*\right)$ is called a refined neutrosophic group generated by G, I_1 *and* I_2. $\left(G\left(I_1,I_2\right),*\right)$ is said to be commutative if, for all $x,y\in\left(G\left(I_1,I_2\right),*\right)$ we have $x*y=y*x$ Otherwise, we $\left(G\left(I_1,I_2\right),*\right)$ are called a non-commutative refined neutrosophic group.

Example2.3. [1] Let $\mathbb{Z}_2\left(I_1,I_2\right)=\begin{Bmatrix} (0,0,0),(1,0,0),(0,I_1,0),(0,0,I_2) \\ (0,I_1,I_2),(1,I_1,0),(1,0,I_2),(1,I_1,I_2) \end{Bmatrix}$. Then $\left(Z_2\left(I_1,I_2\right),*\right)$ it is a commutative refined neutrosophic group of integers modulo2. Generally, for a positive integer $n\geq 2$, $\left(Z_2\left(I_1,I_2\right),*\right)$ it is a finite commutative refined neutrosophic group of integers modulo n.

Theorem2.4. [1]

(1) Every refined neutrosophic group is a semigroup but not a group.

(2) Every refined neutrosophic group contains a group.

Definition 2.5. [1] Let $\left(G\left(I_1,I_2\right),*\right)$ be a refined neutrosophic group and let $H\left(I_1,I_2\right)$, be a nonempty subset of $G\left(I_1,I_2\right)$. $H\left(I_1,I_2\right)$, is called a refined neutrosophic subgroup of $G\left(I_1,I_2\right)$ if $\left(H\left(I_1,I_2\right),*\right)$, it is a refined neutrosophic group. It must contain a proper subset which is a group. Otherwise, $H\left(I_1,I_2\right)$, it will be called a pseudo refined neutrosophic subgroup $G\left(I_1,I_2\right)$.

Definition 2.6. [1] Let $\left(G\left(I_1,I_2\right),*\right)$ and $\left(H\left(I_1,I_2\right),\circ\right)$, be two refined neutrosophic groups.

DOI: 10.5281/zenodo.3881963

Then the mapping: $\psi:\left(G\left(I_1,I_2\right),*\right)\rightarrow\left(H\left(I_1,I_2\right),\circ\right)$ is called a neutrosophic homomorphism if the following conditions hold:

$$\forall x,y\in\left(G\left(I_1,I_2\right),*\right)$$
$$1)\psi\left(x*y\right)=\psi\left(x\right)\circ\psi\left(y\right)$$
$$2)\psi\left(I_k\right)=I_k:k=1,2$$

(1) The kernel of ψ denoted by $\ker\psi$ is defined by the set $\left\{g\in G\left(I_1,I_2\right):\psi\left(g\right)=\left(0,0I_1,0I_2\right)\right\}$.

(2) The image of ψ denoted by $\operatorname{Im}\psi$ is defined by the set $\left\{h\in H\left(I_1,I_2\right),\exists g\in G\left(I_1,I_2\right):\psi\left(g\right)=h\right\}$.

3. On refined Neutrosophic R-module:

Definition 3.1 Let $\left(M,+,.\right)$ be any R-module over a commutative ring R . The triple $\left(M\left(I_1,I_2\right),+,\cdot\right)$ is called a weak refined neutrosophic R-module over a ring R generated by M , I_1 *and* I_2.

If $M\left(I_1,I_2\right)$ is a refined neutrosophic R-module over a refined neutrosophic ring $R\left(I_1,I_2\right)$, then $M\left(I_1,I_2\right)$ is called a refined strong neutrosophic R-module.

Theorem 3.2. Every strong refined neutrosophic R-module is a weak refined neutrosophic R-module.

Proof: Suppose that $M\left(I_1,I_2\right)$is a strong refined neutrosophic R-module over a refined neutrosophic ring $R\left(I_1,I_2\right)$. $R\subseteq R\left(I_1,I_2\right)$ for every ring R , it follows that $M\left(I_1,I_2\right)$is a weak refined neutrosophic R-module.

Theorem 3.3. Every weak (strong) refined neutrosophic R-module is an R-module.

Proof: If we have $m=\left(a,bI_1,cI_2\right),n=\left(d,eI_1,fI_2\right)\in M\left(I_1,I_2\right)$ where $a,b,c,d,e,f\in M$ and $\alpha=\left(p,qI_1,rI_2\right),\beta=\left(s,tI_1,uI_2\right)\in R\left(I_1,I_2\right):p,q,r,s,t,u\in R$ then:

$$\begin{aligned}1)\alpha\left(m+n\right)&=\alpha\left(a+d,\left(b+e\right)I_1,\left(c+f\right)I_2\right)\\&=\left(p,qI_1,rI_2\right)\left(a+d,\left(b+e\right)I_1,\left(c+f\right)I_2\right)\end{aligned}$$

DOI: 10.5281/zenodo.3881963

$$=\begin{pmatrix} p(a+b), \\ \left(p(b+e)+q(a+d)+q(b+e)+q(c+f)+r(b+e)\right)I_1, \\ \left(p(c+f)+r(a+d),r(c+f)\right)I_2 \end{pmatrix}$$

$$=\begin{pmatrix} pa+pb, \\ \left(pb+pe+qa+qd+qb+qe+qc+qf+rb+re\right)I_1 \\ \left(pc+pf+ra+rd+rc+rf\right)I_2 \end{pmatrix}$$

$$=\left(pa,\left(pb+qa+qb+rc+re\right)I_1,\left(pc+ra+rc\right)I_2\right)$$

$$+\left(sd,\left(se+td+te+tf+ue\right)I_1,\left(sf+ud+uf\right)I_2\right)$$

$$=\alpha m+\alpha n$$

$$2)\left(\alpha+\beta\right)m=\left(\left(p,qI_1,rI_2\right)+\left(s,tI_1,uI_2\right)\right)\left(a,bI_1,cI_2\right)$$

$$=\left(\left(p+s\right),\left(q+t\right)I_1,\left(r+u\right)I_2\right)\left(a,bI_1,cI_2\right)$$

$$=\begin{pmatrix} \left(p+s\right)a, \\ \left(\left(p+s\right)b+\left(q+t\right)a+\left(q+t\right)b+\left(q+t\right)c+\left(r+u\right)b\right)I_1, \\ \left(\left(p+s\right)c+\left(r+u\right)a+\left(r+u\right)c\right)I_2 \end{pmatrix}$$

$$=\begin{pmatrix} ap+as, \\ \left(pb+bs+qa+ta+qb+tb+qc+tc+rb+ub\right)I_1, \\ \left(pc+sc+ra+ua+rc+uc\right)I_2 \end{pmatrix}$$

$$=\left(p,qI_1,rI_2\right)\left(a,bI_1,cI_2\right)+\left(s,tI_1,uI_2\right)\left(a,bI_1,cI_2\right)$$

$$=\alpha m+\beta n$$

$$3)\left(\alpha\beta\right)m=\left(\left(p,qI_1,rI_2\right)\left(s,tI_1,uI_2\right)\right)\left(a,bI_1,cI_2\right)$$

$$=\left(\left(ps\right),\left(pt+qs+qt+qu+rt\right)I_1,\left(pu+rs+ru\right)I_2\right)\left(a,bI_1,cI_2\right)$$

$$=\begin{pmatrix} \left(\left(ps\right)a\right), \\ \begin{pmatrix} \left(ps\right)b+\left(pt+qs+qt+qu+rt\right)a+\left(pt+qs+qt+qu+rt\right)b \\ \left(pt+qs+qt+qu+rt\right)c+\left(pu+rs+ru\right)b \end{pmatrix} I_1, \\ \left(\left(ps\right)c+\left(pu+rs+ru\right)a+\left(pu+rs+ru\right)c\right)I_2 \end{pmatrix}$$

DOI: 10.5281/zenodo.3881963

$$= \begin{pmatrix} p(sa), \\ \begin{pmatrix} p(sb+ta+tc+tb+ub)+qsa+q(sb+ta+tc+tb+ub)+ \\ q(sc+ua+uc)+r(sb+ta+tc+tb+ub) \end{pmatrix} I_1, \\ (p(sc+ua+uc)+rsa+r(sc+ua+uc))I_2 \end{pmatrix}$$

4) For $(1,0,0) \in R(I_1,I_2)$ we have:

$$(1,0,0)\cdot m = (1,0I_1,0I_2)(a,bI_1,cI_2)$$
$$= (1a,(1b+0a+0b+0c+0b)I_1,(1c+0a+0c)I_2)$$
$$= (a,bI_1,cI_2) = m$$

Therefore, that $M(I_1,I_2)$ is an R-module.

Lemma 3.4: If we have $M(I_1,I_2)$ as a refined neutrosophic $R-$module over a refined neutrosophic ring $R(I_1,I_2)$ and if we take $m=(a,bI_1,cI_2), n=(d,eI_1,fI_2)$ *and* $s=(x,yI_1,zI_2)\in M(I_1,I_2)$ where $a,b,c,d,e,f,x,y,z \in M$ $\alpha=(p,qI_1,rI_2)\in R(I_1,I_2): p,q,r\in R$ then:

1) $m+s=n+s \Rightarrow m=n$

2) $\alpha(0,0I_1,0I_2)=(0,0I_1,0I_2)$

3) $(0,0I_1,0I_2)m=(0,0I_1,0I_2)$

4) $(-\alpha)m=\alpha(-m)=-(\alpha m)$

Definition 3.5: Let $M(I_1,I_2)$ be a strong refined neutrosophic R- module over a refined neutrosophic ring $R(I_1,I_2)$ and let $N(I_1,I_2)$ be a nonempty subset of $M(I_1,I_2)$. $N(I_1,I_2)$ is called a strong refined neutrosophic submodule of $M(I_1,I_2)$ if $N(I_1,I_2)$ is itself a strong refined neutrosophic R- module over $R(I_1,I_2)$. It is essential that $N(I_1,I_2)$ contains a proper subset which is an R-module.

Definition 3.6: Let $M(I_1,I_2)$ be a weak refined neutrosophic R-module over a ring R and let N (I) be a nonempty subset of $M(I_1,I_2)$. $N(I_1,I_2)$ is called a weak refined neutrosophic submodule of $M(I_1,I_2)$, if $N(I_1,I_2)$ is itself a weak refined neutrosophic R-module over R. It is essential that $N(I_1,I_2)$ contains a proper subset, which is an R-module.

DOI: 10.5281/zenodo.3881963

Theorem 3.7: If we have $M\left(I_1,I_2\right)$ as a refined neutrosophic $R-$module over a ring $R\left(I_1,I_2\right)$ and if we take $N\left(I_1,I_2\right)$ as a subset of $M\left(I_1,I_2\right)$, $N\left(I_1,I_2\right)$ is a strong refined neutrosophic submodule of $M\left(I_1,I_2\right)$ if and only if the following conditions hold:

1) $n_1,n_2 \in N\left(I_1,I_2\right) \Rightarrow n_1+n_2 \in N\left(I_1,I_2\right)$

3) $for \quad all\ r=\left(\alpha,\beta I_1,\gamma I_2\right) \in R\left(I_1,I_2\right):\alpha,\beta,\gamma \in R$
$n \in N\left(I_1,I_2\right) \Rightarrow \alpha n \in N\left(I_1,I_2\right)$

3) $N\left(I_1,I_2\right)$ must have a proper subset which is a $R-$module.

Corollary 3.8: If we have $M\left(I_1,I_2\right)$ as a refined neutrosophic $R-$module over a refined neutrosophic ring $R\left(I_1,I_2\right)$ and if we take $N\left(I_1,I_2\right)$ as a subset of $M\left(I_1,I_2\right)$, then $N\left(I_1,I_2\right)$ is refined neutrosophic submodule of $M\left(I_1,I_2\right)$ if and only if the following conditions hold:

1) $for \quad all\ \alpha\left(p,qI_1,rI_2\right),\beta\left(s,tI_1,uI_2\right) \in R\left(I_1,I_2\right):p,q,r,s,t,u \in R$
$n_1+n_2 \in N\left(I_1,I_2\right) \quad implies \quad \alpha n_1+\beta n_2 \in N\left(I_1,I_2\right)$

2) $N\left(I_1,I_2\right)$ must have a proper subset which is a $R-$module.

Example 3.9. Let $M\left(I_1,I_2\right)$ be a weak (strong) refined neutrosophic R-module. $M\left(I_1,I_2\right)$ is a weak (strong) refined neutrosophic submodule called a trivial weak (strong) neutrosophic submodule.

Example 3.10. Let $M\left(I_1,I_2\right)=M_{m\times n}\left(I_1,I_2\right)=\left\{\left[a_{ij}\right]:a_{ij} \in R\left(I_1,I_2\right)\right\}$ be a strong refined neutrosophic R-module over the strong refined neutrosophic ring $R\left(I_1,I_2\right)$ and let

$$N\left(I_1,I_2\right)=N_{n\times m}\left(I_1,I_2\right)=\left\{\left[b_{ij}\right]:b_{ij} \in R\left(I_1,I_2\right),\ trace\left(N_{n\times m}\right)=\left(0,0I_1,0I_2\right)\right\}$$

Then $N\left(I_1,I_2\right)$ is a strong refined neutrosophic submodule of $M\left(I_1,I_2\right)$.

DOI: 10.5281/zenodo.3881963

Theorem 3.11: Let $M(I_1,I_2)$ be a strong refined neutrosophic R-module over a refined neutrosophic ring $R(I_1,I_2)$ and let $\{N_n(I_1,I_2)\}_{n\in\lambda}$ be a family of strong refined neutrosophic submodules of $M(I_1,I_2)$. Then $\cap N_n(I_1,I_2)$ is a strong refined neutrosophic submodule of $M(I_1,I_2)$.

Proof: Clearly $(0,0I_1,0I_2)_{M(I_1,I_2)} \in \cap N(I_1,I_2)$ and $\cap N_n(I_1,I_2) \neq \emptyset$. Since for $\forall n\in\lambda, (0,0I_1,0I_2)_{M(I_1,I_2)} \in N_n(I_1,I_2)$ Let be $x=(a,bI_1,cI_2), y\,(d,eI_1,fI_2)\in \cap N_n(I_1,I_2)$ and let be $\alpha=(p,qI_1,rI_2)\in R(I_1,I_2)$. Then $x+y, \alpha x \in \cap N_n(I_1,I_2)$. Since, for $\forall n\in\lambda, x+y\in N_n(I_1,I_2)$ and $\alpha x \in \cap N_n(I_1,I_2)$ Hence $\cap N_n(I_1,I_2)$ is a strong refined neutrosophic submodule of $M(I_1,I_2)$.

Remark: Let $M(I_1,I_2)$ be a strong refined neutrosophic R-module over a refined neutrosophic ring $R(I_1,I_2)$ and let $N_1(I_1,I_2)$ and $N_2(I_1,I_2)$ be two distinct strong refined neutrosophic submodules of $M(I_1,I_2)$. Generally, $N_1(I_1,I_2)\cup N_2(I_1,I_2)$ is not a strong refined neutrosophic submodule $M(I_1,I_2)$.

However, if $N_1(I_1,I_2)\subseteq N_2(I_1,I_2)$ or $N_1(I_1,I_2)\supseteq N_2(I_1,I_2)$ then $N_1(I_1,I_2)\cup N_2(I_1,I_2)$ is a strong neutrosophic submodule of $M(I_1,I_2)$.

Definition 3.12. If we have $M(I_1,I_2)$ and $N(I_1,I_2)$ as two refined neutrosophic $R-$modules over a ring refined neutrosophic ring $R(I_1,I_2)$, a mapping $\varphi: M(I_1,I_2)\to N(I_1,I_2)$ is said to be a refined neutrosophic homomorphism $R-$module, precisely when:

1) $\varphi(rm+r'm')=r\varphi(m)+r'\varphi(m)$ for all $m,m'\in M(I_1,I_2)$ and $r,r'\in R(I_1,I_2)$

2) $\varphi(I_1)=I_1, \varphi(I_2)=I_2$.

Endomorphism, epimorphism, monomorphism, automorphism, and isomorphism of φ have the same definitions as those of the classical cases.

Definition 3.13. Let $M(I_1,I_2)$ and $N(I_1,I_2)$ be strong refined neutrosophic R-modules over a refined neutrosophic ring $R(I_1,I_2)$ and let $\psi: M(I_1,I_2)\to N(I_1,I_2)$ be a refined neutrosophic R-module homomorphism then:

DOI: 10.5281/zenodo.3881963

(1) The kernel of ψ denoted by $\ker\psi$ is defined by the set $\{m \in M(I_1,I_2) : \psi(m) = (0,0I_1,0I_2)\}$.

(2) The image of ψ denoted by $\operatorname{Im}\psi$ is defined by the set $\{n \in N(I_1,I_2) \exists m \in M(I_1,I_2) : \psi(m) = n\}$.

Example 3.14. Let $M(I_1,I_2)$ be a strong refined neutrosophic R-module over a refined neutrosophic ring $R(I_1,I_2)$. The mapping $\psi : M(I_1,I_2) \to M(I_1,I_2)$ defined by $\psi(m) = m$ for all $m \in M(I_1,I_2)$ is refined neutrosophic R-module homomorphism and $\ker\psi = (0,0I_1,0I_2)$.

Example 3.15. The mapping $\psi : M(I_1,I_2) \to M(I_1,I_2)$ defined by $\psi(m) = (0,0I_1,0I_2)$ for all $m \in M(I_1,I_2)$ is refined neutrosophic R-module homomorphism.

Definition 3.15. Let $M(I_1,I_2)$ and $N(I_1,I_2)$ be strong refined neutrosophic R-modules over a refined neutrosophic ring $R(I_1,I_2)$ and let $\psi : M(I_1,I_2) \to N(I_1,I_2)$ be a refined neutrosophic R-module homomorphism. Then:

(1) $\ker\psi$ is not a strong refined neutrosophic submodule of $M(I_1,I_2)$ but a submodule of M.

(2) $\operatorname{Im}\psi$ is a strong refined neutrosophic submodule of $N(I_1,I_2)$.

Proof: (1) Obviously, $I_1, I_2 \in M(I_1,I_2)$ but $\varphi(I_1) = I_1 \neq 0, \varphi(I_2) = I_2 \neq 0$. That $\ker\psi$ is a submodule of M is clear.

(2) Clear.

5. Conclusion

In this paper, the refined neutrosophic R-modules and refined neutrosophic submodules which are completely different from the classical modules and submodules in the structural properties were defined. It was shown that every refined neutrosophic R-module is an R-module. Finally, refined neutrosophic R-module homomorphism was explained and some definitions and theorems were given.

References

[1] A. A. A. Agboola, "On Refined Neutrosophic Algebraic Structures", *Neutrosophic Sets and Systems*, vol.10, pp. 99-101, 2015.

[2] A. A. A. Agboola, A. D. Akinola and O.Y.Oyebola, "Neutrosophic Rings I", *Int. J. of Math*. Comb., vol. 4, pp. 1-14, 2011.

DOI: 10.5281/zenodo.3881963

[3] A. A. A. Agboola, E. O. Adeleke and S. A. Akinleye, “Neutrosophic Rings II”, *Int. J. of Math*. Comb., vol.2, pp. 1-8, 2012.

[4] A. A. A. Agboola, Akwu A. O. and Y. T. Oyebo, “Neutrosophic Groups and Neutrosopic Subgroups”, *Int. J. of Math*. Comb., vol. 3, pp. 1-9, 2012.

[5] E. O. Adeleke, A. A. A. Agboola and F. Smarandache, “Refined Neutrosophic Rings I”, *International Journal of Neutrosophic Science ,* vol. 2, pp. 77-81, 2020.

[6] E. O. Adeleke, A. A. A. Agboola and F. Smarandache, “Refined Neutrosophic Rings II”, *International Journal of Neutrosophic Science,* vol. 2, pp. 89-94, 2020.

[7] F. Smarandache, “n-Valued Refined Neutrosophic Logic and Its Applications in Physics”, *Progress in Physics*, USA, vol. 4, pp. 143-146, 2013.

[8] F. Smarandache, **“** (T,I,F)-Neutrosophic Structures**”**, Neutrosophic Sets and Systems,vol.8, pp.3-10, 2015.

[9] A.A.A. Agboola and S. A. Akinleye, “Neutrosophic Vector Space”, *Neutrosophic Sets and Systems*, Vol. 4, pp. 9-18, 2014.

[10] W. B. Vasantha Kandasamy and F. Smarandache, “Neutrosophic Rings”, Hexis, Phoenix, Arizona, 2006 http://fs.unm.edu/NeutrosophicRings.pdf

[11] W. B. Vasantha Kandasamy & Florentin Smarandache, “Basic Neutrosophic Algebraic Structures and their Applications to Fuzzy and Neutrosophic Models”, Hexis, 2004.

[12] A. B. AL-Nafee, R. K. Al-Hamido, F.Smarandache, “Separation Axioms In Neutrosophic Crisp Topological Spaces”, *Neutrosophic Sets and Systems*, vol. 25, pp.25-32, 2019.

[13] R. K. Al-Hamido, “Neutrosophic Crisp Bi-Topological Spaces”, *Neutrosophic Sets and Systems*, Vol. 21, pp.66-73, 2018.

[14] M. Al-Tahan, and B. Davvaz, “Refined neutrosophic quadruple (poly-)hypergroups and their fundamental group”, *Neutrosophic Sets and Systems*, vol. 27, pp. 138-153, 2019.

[15] A. Chakraborty, S. Broumi, P. K Singh; “Some properties of Pentagonal Neutrosophic Numbers and its Applications in Transportation Problem Environment”, *Neutrosophic Sets and Systems*, vol.28, pp.200-215, 2019.

[16] S. Broumi, M. Talea, A. Bakali, F. Smarandache, “Single Valued Neutrosophic Graphs”, *Journal of New Theory*, N 10, pp. 86-101, 2016.

[17] Nguyen Tho Thong, Luu Quoc Datc, Le Hoang Son, Nguyen Dinh Hoa, Mumtaz Alid, Florentin Smarandache, "Dynamic interval valued neutrosophic set: Modeling decision making in dynamic environments," *Computers in Industry,* vol. 108, pp. 45-52, 2019.

[18] S.Broumi, M. Talea, A. Bakali, P. K.Singh, F. Smarandache,”Energy and Spectrum Analysis of Interval Valued Neutrosophic Graph using MATLAB”, Neutrosophic Sets and Systems, vol. 24, pp. 46-60, 2019.

[19] A. Hatip, "The Special Neutrosophic Functions," *International Journal of Neutrosophic Science ,* Vol. 4, no. 2, pp. 104-116, 15 April 2020.

[20]S. Broumi, M. Talea, A.Bakali, F. Smarandache, “On Bipolar Single Valued Neutrosophic Graphs”, *Journal of New Theory*, N11, pp.84-102, 2016.

DOI: 10.5281/zenodo.3881963

[21] S. Broumi, M. Talea, A. Bakali, F. Smarandache, "Interval Valued Neutrosophic Graphs", *SISOM & ACOUSTICS*, Bucharest , Volume X, pp.79-91, 2016.

[22] S. Broumi, A. Bakali, M. Talea, and F. Smarandache, "Isolated Single Valued Neutrosophic Graphs" *Neutrosophic Sets and Systems*, Vol. 11, pp.74-78, 2016.

[23] S. Broumi, M. Talea, F. Smarandache and A. Bakali, "Single Valued Neutrosophic Graphs: Degree, Order and Size", *IEEE International Conference on Fuzzy Systems (FUZZ),* pp.2444-2451, 2016.

[24] Mohamed Bisher Zeina, "Neutrosophic Event-Based Queueing Model", *International Journal of Neutrosophic Science*, Vol. 6, No. 1, pp. 48-55, 2020.

[25] Mohamed Bisher Zeina, "Erlang Service Queueing Model with Neutrosophic Parameters", *International Journal of Neutrosophic Science*, Vol. 6, No. 2, pp.106-112, 2020.

DOI: 10.5281/zenodo.3881963

On Refined Neutrosophic Vector Spaces I

[1] M.A. Ibrahim,[2] A.A.A. Agboola,[3] B.S. Badmus,[4] S.A. Akinleye
[1,2,4]Department of Mathematics, Federal University of Agriculture, Abeokuta, Nigeria.
[3]Department of Physics, Federal University of Agriculture, Abeokuta, Nigeria.
muritalaibrahim40@gmail.com[1], agboolaaaa@funaab.edu.ng[2], badmusbs@yahoo.com[3],
sa_akinleye@yahoo.com[4]

Abstract

The objective of this paper is to present the concept of a refined neutrosophic vector space. Weak(strong) refined neutrosophic vector spaces and subspaces, and, strong refined neutrosophic quotient vector spaces are studied. Several interesting results and examples are presented. It is shown that every weak (strong) refined neutrosophic vector space is a vector space and it is equally shown that every strong refined neutrosophic vector space is a weak refined neutrosophic vector space.
Keywords: Neutrosophy, neutrosophic vector space, neutrosophic vector subspace, refined neutrosophic vector space, refined neutrosophic vector subspace, refined neutrosophic quotient vector space.

1 Introduction and Preliminaries

Neutrosophy is a new branch of philosophy that studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra. The concept of neutrosophic logic/set was introduced by Smarandache in [20,22] as a generalization of fuzzy log/set [29] and respectively intuitionistic fuzzy logic/set [9]. In neutrosophic logic, each proposition has a degree of truth (T), a degree of indeterminancy (I) and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^{-}0, 1^{+}[$. In [21], Smarandache introduced the notion of refined neutrosophic components of the form $< T_1, T_2, \cdots, T_p; I_1, I_2, \cdot, I_r; F_1, F_2, \cdots, F_s >$ of the neutrosophic components $< T, I, F >$. The refinement has given rise to the introduction of refined neutrosophic set and the extension of neutrosophic numbers $a + bI$ into refined neutrosophic numbers of the form $(a + b_1I_1 + b_2I_2 + \cdots + b_nI_n)$ where $a, b_1, b_2, \cdots, b_n$ are real or complex numbers. Agboola in [4] introduced the concept of refined neutrosophic algebraic structures and studied refined neutrosophic groups with their basic and fundamental properties. Since then, several neutrosophic researchers have studied this concept and a great deal of results have been published. Recently, Adeleke et al. studied refined neutrosophic rings and refined neutrosophic subrings in [1] and also in [2], they presented several results and examples on refined neutrosophic ideals and refined neutrosophic ring homomorphisms.

The concept of neutrosophic vector space was introduced by Vasantha Kandasamy and Florentin Smarandache in [23]. Further studies on neutrosophic vector spaces were carried out by Agboola and Akinleye in [8] where they generalized some properties of vector spaces and showed that every neutrosophic vector space over a neutrosophic field (resp. field) is a vector space. A comprehensive review of neutrosophic set, neutrosophic soft set, fuzzy set, neutrosophic topological spaces, neutrosophic vector spaces and new trends in neutrosophic theory can be found in [3,5–7,10–19,23–28].

In the present paper, we present the concept of a refined neutrosophic vector space. Weak(strong) refined neutrosophic vector spaces and subspaces, and, strong refined neutrosophic quotient vector spaces are studied. Several interesting results and examples are presented. It is shown that every weak (strong) refined neutrosophic vector space is a vector space and it is equally shown that every strong refined neutrosophic vector space is a weak refined neutrosophic vector space.

For the purposes of this paper, it will be assumed that I splits into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. It then follows logically that:

Doi :10.5281/zenodo.3884059
Received: March 27, 2020 Revised: May 03, 2020 Accepted: May 28, 2020

$$\begin{array}{lllll} I_1I_1 & = & I_1^2 & = & I_1, \\ I_2I_2 & = & I_2^2 & = & I_2, \text{ and} \\ I_1I_2 & = & I_2I_1 & = & I_1. \end{array}$$

Definition 1.1. [4] If $* : X(I_1, I_2) \times X(I_1, I_2) \mapsto X(I_1, I_2)$ is a binary operation defined on $X(I_1, I_2)$, then the couple $(X(I_1, I_2), *)$ is called a refined neutrosophic algebraic structure and it is named according to the laws (axioms) satisfied by $*$.

Definition 1.2. [4] Let $(X(I_1, I_2), +, .)$ be any refined neutrosophic algebraic structure where $+$ and . are ordinary addition and multiplication respectively.
For any two elements $(a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$, we define

$$(a, bI_1, cI_2) + (d, eI_1, fI_2) = (a + d, (b + e)I_1, (c + f)I_2),$$
$$(a, bI_1, cI_2).(d, eI_1, fI_2) = (ad, (ae + bd + be + bf + ce)I_1, (af + cd + cf)I_2).$$

Definition 1.3. [4] If $''+''$ and $''.''$ are ordinary addition and multiplication, I_k with $k = 1, 2$ have the following properties:

1. $I_k + I_k + \cdots + I_k = nI_k$.
2. $I_k + (-I_k) = 0$.
3. $I_k \cdot I_k \cdot \cdots \cdot I_k = I_k^n = I_k$ for all positive integers $n > 1$.
4. $0 \cdot I_k = 0$.
5. I_k^{-1} is undefined and therefore does not exist.

Definition 1.4. [4] Let $(G, *)$ be any group. The couple $(G(I_1, I_2), *)$ is called a refined neutrosophic group generated by G, I_1 and I_2. $(G(I_1, I_2), *)$ is said to be commutative if for all $x, y \in G(I_1, I_2)$, we have $x * y = y * x$. Otherwise, we call $(G(I_1, I_2), *)$ a non -commutative refined neutrosophic group.

Definition 1.5. [4] If $(X(I_1, I_2), *)$ and $(Y(I_1, I_2), *')$ are two refined neutrosophic algebraic structures, the mapping
$$\phi : (X(I_1, I_2), *) \longrightarrow (Y(I_1, I_2), *')$$
is called a neutrosophic homomorphism if the following conditions hold:

1. $\phi((a, bI_1, cI_2) * (d, eI_1, fI_2)) = \phi((a, bI_1, cI_2)) *' \phi((d, eI_1, fI_2))$.
2. $\phi(I_k) = I_k$ for all $(a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$ and $k = 1, 2$.

Example 1.6. [4] Let
$\mathbb{Z}_2(I_1, I_2) = \{(0, 0, 0), (1, 0, 0), (0, I_1, 0), (0, 0, I_2), (0, I_1, I_2), (1, I_1, 0), (1, 0, I_2), (1, I_1, I_2)\}$.
Then $(\mathbb{Z}_2(I_1, I_2), +)$ is a commutative refined neutrosophic group of integers modulo 2.
Generally for a positive integer $n \geq 2$, $(\mathbb{Z}_n(I_1, I_2), +)$ is a finite commutative refined neutrosophic group of integers modulo n.

Example 1.7. [4] Let $(G(I_1, I_2), *)$ and and $(H(I_1, I_2), *')$ be two refined neutrosophic groups.
Let $\phi : G(I_1, I_2) \times H(I_1, I_2) \to G(I_1, I_2)$ be a mapping defined by $\phi(x, y) = x$ and let
$\psi : G(I_1, I_2) \times H(I_1, I_2) \to H(I_1, I_2)$ be a mapping defined by $\psi(x, y) = y$. Then ϕ and ψ are refined neutrosophic group homomorphisms.

Definition 1.8. [1] Let $(R, +, .)$ be any ring. The abstract system $(R(I_1, I_2), +, .)$ is called a refined neutrosophic ring generated by R, I_1, I_2. $(R(I_1, I_2), +, .)$ is called a commutative refined neutrosophic ring if for all $x, y \in R(I_1, I_2)$, we have $xy = yx$. If there exists an element $e = (1, 0, 0) \in R(I_1, I_2)$ such that $ex = xe = x$ for all $x \in R(I_1, I_2)$, then we say that $(R(I_1, I_2), +, .)$ is a refined neutrosophic ring with unity.

Definition 1.9. [1] Let $(R(I_1, I_2), +, .)$ be a refined neutrosophic ring and let $n \in \mathbb{Z}^+$.

(i) If $nx = 0$ for all $x \in R(I_1, I_2)$, we call $(R(I_1, I_2), +, .)$ a refined neutrosophic ring of characteristic n and n is called the characteristic of $(R(I_1, I_2), +, .)$.

(ii) $(R(I_1, I_2), +, .)$ is call a refined neutrosophic ring of characteristic zero if for all $x \in R(I_1, I_2)$, $nx = 0$ is possible only if $n = 0$.

Example 1.10. [1]

(i) $\mathbb{Z}(I_1, I_2), \mathbb{Q}(I_1, I_2), \mathbb{R}(I_1, I_2), \mathbb{C}(I_1, I_2)$ are commutative refined neutrosophic rings with unity of characteristics zero.

(ii) Let $\mathbb{Z}_2(I_1, I_2) = \{(0, 0, 0), (1, 0, 0), (0, I_1, 0), (0, 0, I_2), (0, I_1, I_2), (1, I_1, 0), (1, 0, I_2), (1, I_1, I_2)\}$. Then $(\mathbb{Z}_2(I_1, I_2), +, .)$ is a commutative refined neutrosophic ring of integers modulo 2 of characteristic 2. Generally for a positive integer $n \geq 2$, $(\mathbb{Z}_n(I_1, I_2), +, .)$ is a finite commutative refined neutrosophic ring of integers modulo n of characteristic n.

Example 1.11. [1] Let $M^{\mathbb{R}}_{n\times n}(I_1, I_2) = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} : a_{ij} \in \mathbb{R}(I_1, I_2) \right\}$ be a refined neutrosophic set of all $n \times n$ matrix.
Then $(M^{\mathbb{R}}_{n\times n}(I_1, I_2), +, .)$ is a non-commutative refined neutrosophic ring under matrix multiplication.

Theorem 1.12. [7] *Let $(R(I_1, I_2), +, .)$ be any refined neutrosophic ring. Then $(R(I_1, I_2), +, .)$ is a ring.*

2 Formulation of a Refined Neutrosophic Vector Space

In this section, we develop the concept of refined neutrosophic vector space and its subspaces and also present some of their basic properties.

Definition 2.1. Let $(V, +, .)$ be any vector space over a field K. Let $V(I_1, I_2) =< V \cup (I_1, I_2) >$ be a refined neutrosophic set generated by V, I_1 and I_2. We call the triple $(V(I_1, I_2), +, .)$ a weak refined neutrosophic vector space over a field K, if $V(I_1, I_2)$ is a refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$, then $V(I_1, I_2)$ is called a strong refined neutrosophic vector space. The elements of $V(I_1I_2)$ are called refined neutrosophic vectors and the elements of $K(I_1, I_2)$ are called refined neutrosophic scalars.

If $u = a + bI_1 + cI_2$, $v = d + eI_1 + fI_2 \in V(I_1, I_2)$ where a, b, c, d, e and f are vectors in V and $\alpha = k + mI_1 + nI_2 \in K(I_1, I_2)$ where k, m and n are scalars in K, we define:

$$u + v = (a + bI_1 + cI_2) + (d + eI_1 + fI_2) = (a + d) + (b + e)I_1 + (c + f)I_2,$$

and

$$\alpha u = (k + mI_1 + nI_2).(a + bI_1 + cI_2) = k.a + (k.b + m.a + m.b + m.c + n.b)I_1 + (k.c + n.a + n.c)I_2.$$

Example 2.2. Let $\mathbb{R}^2(I_1, I_2)$ denote the refined set of all ordered pairs (x, y) where x and y are refined neutrosophic real numbers given as $x = a + bI_1 + cI_2$ and $y = d + eI_1 + fI_2$.
Define addition and scalar multiplication on $\mathbb{R}^2(I_1, I_2)$ by

$$\begin{aligned} (x, y) + (x', y') &= (a + bI_1 + cI_2, d + eI_1 + fI_2) + (a' + b'I_1 + c'I_2, d' + e'I_1 + f'I_2) \\ &= (a + a' + (b + b')I_1 + (c + c')I_2, d + d' + (e + e')I_1 + (f + f')I_2) \\ &= (x + x', y + y'). \end{aligned}$$

For $\alpha = (k + mI_1 + nI_2) \in \mathbb{R}(I_1, I_2)$

$$\begin{aligned} \alpha(x, y) &= (k + mI_1 + nI_2).(a + bI_1 + cI_2, d + eI_1 + fI_2) \\ &= ((k + mI_1 + nI_2).(a + bI_1 + cI_2), (k + mI_1 + nI_2).(d + eI_1 + fI_2)) \\ &= (k.a + (k.b + m.a + m.b + m.c + n.b)I_1 + (k.c + n.a + n.c)I_2, k.d+ \\ &\quad (k.e + m.d + m.e + m.f + n.e)I_1 + (k.f + n.d + n.f)I_2,). \end{aligned}$$

Then $\mathbb{R}^2(I_1, I_2)$ is strong refined neutrosophic vector space over $\mathbb{R}(I_1, I_2)$.

And if $\alpha \in \mathbb{R}$ with scalar multiplication defined as

$$\alpha(x, y) = \alpha(a + bI_1 + cI_2, d + eI_1 + fI_2) = (\alpha.a + \alpha.bI_1 + \alpha.cI_2, \alpha.d + \alpha.eI_1 + \alpha.fI_2) = (\alpha.x, \alpha.y),$$

then $\mathbb{R}^2(I_1, I_2)$ is weak refined neutrosophic vector space over $\mathbb{R}$.

Example 2.3. $M_{m\times n}(I_1, I_2) = \{[aij] : aij \in \mathbb{Q}(I_1, I_2)\}$ is a weak refined neutrosophic vector space over a field $\mathbb{Q}$ and it is a strong refined neutrosophic vector space over a refined neutrosophic field $Q(I_1, I_2)$.

Example 2.4. Let $V = \mathbb{Q}(I_1, I_2)(\sqrt{2}) = \{a + (bI_1 + cI_2)\sqrt{2} : a, b, c \in Q\}$. Then V is a weak refined neutrosophic vector space over $\mathbb{Q}$. If $u = a + (bI_1 + cI_2)\sqrt{2}$ and $v = d + (eI_1 + fI_2)\sqrt{2}$
then $u + v = (a + d) + (b + e)I_1\sqrt{2} + (c + f)I_2\sqrt{2}$ is again in V.
Also, for $\alpha \in \mathbb{Q}$, $\alpha u = \alpha(a + (bI_1 + cI_2)\sqrt{2}) = \alpha.a + (\alpha.bI_1 + \alpha.cI_2)\sqrt{2}$ is in V.

Proposition 2.5. *Every strong refined neutrosophic vector space is a weak refined neutrosophic vector space.*

Proof. Suppose that $V(I_1, I_2)$ is a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$ say. Since $K \subseteq K(I_1, I_2)$ for every field K, we have that $V(I_1, I_2)$ is also a weak refined neutrosophic vector space. □

Proposition 2.6. *Every weak (strong) refined neutrosophic vector space is a vector space.*

Proof. Suppose that $V(I_1, I_2)$ is a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$. That $(V(I_1, I_2), +)$ is an abelian group can be established easily.
Let $u = a + bI_1 + cI_2, v = d + eI_1 + fI_2 \in V(I_1, I_2), \alpha = k + mI_1 + nI_2, \beta = p + qI_1 + rI_2 \in K(I_1, I_2)$ where $a, b, c, d, e, f \in V$ and $k, m, n, p, q, r \in K$.
Then:

1. $\alpha(u + v) = (k + mI_1 + nI_2)(a + bI_1 + cI_2 + d + eI_1 + fI_2)$
$= (k + mI_1 + nI_2)(a + d + (b + e)I_1 + (c + f)I_2)$
$= ka+kd+[kb+ke+ma+md+mb+me+mc+mf+nb+ne]I_1+[kc+kf+na+nd+nc+nf]I_2$
$= [ka + (kb + ma + mb + mc + nb)I_1 + (kc + na + nc)I_2] + [kd + (ke + md + me + mf + ne)I_1 + (kf + nd + nf)I_2]$
$= (k + mI_1 + nI_2)(a + bI_1 + cI_2) + (k + mI_1 + nI_2)(d + eI_1 + fI_2)$
$= \alpha u + \alpha v.$

2. $(\alpha + \beta)u = (k + mI_1 + nI_2 + p + qI_1 + rI_2)(a + bI_1 + cI_2)$
$= (k + p + (m + q)I_1 + (n + r)I_2)(a + bI_1 + cI_2)$
$= ka + pa + [kb + pb + ma + qa + mb + qb + mc + qc + nb + rb]I_1 + [kc + pc + na + ra + nc + rc]I_2$
$= [ka+(kb+ma+mb+mc+nb)I_1+[kc+na+nc]I_2]+[pa+(pb+qa+qb+qc+rb)I_1+[pc+ra+rc]I_2$
$= (k + mI_1 + nI_2)(a + bI_1 + cI_2) + (p + qI_1 + rI_2)(a + bI_1 + cI_2)$
$= \alpha u + \beta u.$

3. $(\alpha\beta)u = ((k + mI_1 + nI_2)(p + qI_1 + rI_2))(a + bI_1 + cI_2)$
$= (kp + (kq + mp + mq + mr + nq)I_1 + (kr + np + nr)I_2)(a + bI_1 + cI_2)$
$= kpa + [kpb + kqa + mpa + mqa + mra + nqa + kqb + mpb + mqb + mrb + nqb + kqc + mpc + mqc + mrc + nqc + krb + npb + nrb]I_1 + [kpc + kra + npa + nra + krc + npc + nrc]I_2$
$= (k + mI_1 + nI_2)[pa + (pb + qa + qb + qc + rb)I_1 + (pc + ra + rc)I_2]$
$= (k + mI_1 + nI_2)((p + qI_1 + rI_2)(a + bI_1 + cI_2))$
$= \alpha(\beta u).$

4. For $1 = 1 + 0I_1 + 0I_2 \in K(I_1I_2)$, we have
$1u = (1 + 0I_1 + 0I_2)(a + bI_1 + cI_2)$
$= a + (b + 0 + 0 + 0 + 0)I_1 + (c + 0 + 0)I_2$
$= a + bI_1 + cI_2.$
Accordingly, $V(I_1, I_2)$ is a vector space.

□

Example 2.7. Let $P_\infty(I_1, I_2)$ be the set of refined neutrosophic formal power series in variable x of the form $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$, with $a_n \in \mathbb{R}(I_1, I_2)$ and
$a_n = p_n + q_n I_1 + r_n I_2$ for $n = 1, 2, 3 \cdots$
If addition and scalar multiplication (for $\alpha \in K(I_1, I_2)$) are defined as:

$$\begin{aligned}
(\textstyle\sum_{n=0}^{\infty} a_n x^n) + (\sum_{n=0}^{\infty} b_n x^n)) &= (\textstyle\sum_{n=0}^{\infty} (p_n + q_n I_1 + r_n I_2) x^n) + (\sum_{n=0}^{\infty} (u_n + v_n I_1 + w_n I_2) x^n) \\
&= (\textstyle\sum_{n=0}^{\infty} (p_n + u_n + (q_n + v_n) I_1 + (r_n + w_n) I_2) x^n) \\
\alpha\, (\textstyle\sum_{n=0}^{\infty} (p_n + q_n I_1 + r_n I_2) x^n) &= (\textstyle\sum_{n=0}^{\infty} \alpha (p_n + q_n I_1 + r_n I_2) x^n) \\
&= \textstyle\sum_{n=0}^{\infty} (e + f I_1 + g I_2)(p_n + q_n I_1 + r_n I_2) x^n \\
&= \textstyle\sum_{n=0}^{\infty} e p_n + (e q_n + f p_n + f q_n + f r_n + g q_n) I_1 \\
&\quad + (e r_n + g p_n + g r_n) I_2.
\end{aligned}$$

Then $P_\infty(I_1, I_2)$ is a strong refined neutrosophic vector space over $K(I_1, I_2)$.

Note 1. *A refined neutrosophic formal power series can be loosely thought of as an object that is like a refined neutrosophic polynomial. Alternatively, one may think of a refined neutrosophic formal power series as a refined neutrosophic power series in which we ignore the questions of convergence by not assuming that the variable x denotes any numerical value (not even an unknown value). Thus, we do not regard these refined neutrososphic formal power series as infinite sum in P_x of refined neutrosophic monomials.*

Example 2.8. Let $P(I_1, I_2)$ be the set of all refined neutrosophic polynomial in variable x with coefficients in $R[I_1, I_2]$. Let $p, q \in P(I_1, I_2)$ and $\alpha = (k + uI_1 + vI_2) \in K(I_1, I_2)$.

$$p = (a_0 + b_0I_1 + c_0I_2) + (a_1 + b_1I_1 + c_1I_2)x + \cdots + (a_n + b_nI_1 + c_nI_2)x^n,$$

and

$$q = (a'_0 + b'_0I_1 + c'_0I_2) + (a'_1 + b'_1I_1 + c'_1I_2)x + \cdots + (a'_m + b'_mI_1 + c'_mI_2)x^m.$$

If $m \geq n$, the sum of p and q is given by

$$\begin{aligned} p + q \;=\; & ((a_0 + a'_0) + (b_0 + b'_0)I_1 + (c_0 + c'_0)I_2) + ((a_1 + a'_1) + (b_1 + b'_1)I_1 + (c_1 + c'_1)I_2)x + \cdots + \\ & ((a_n + a'_n) + (b_n + b'_n)I_1 + (c_n + c'_n)I_2) + (a'_{n+1} + b'_{n+1}I_1 + c'_{n+1}I_2)x^{n+1} + \cdots + \\ & (a'_m + b'_mI_1 + c'_mI_2)x^m. \end{aligned}$$

A similar definition is given if $m < n$.
The product of p and a scalar α is given by

$$\alpha \cdot p = (k+uI_1+vI_2)(a_0+b_0I_1+c_0I_2)+(k+uI_1+vI_2)(a_1+b_1I_1+c_1I_2)x+\cdots+(k+uI_1+vI_2)(a_n+b_nI_1+c_nI_2)x^n.$$

Then $(P(I_1, I_2), +, \cdot)$ is a strong refined vector space over a refined neutrosophic field $K(I_1, 1_2)$.

Proposition 2.9. *Let $F(I_1, I_2)$ be a refined neutrosophic field and $(\mathbb{R}(I_1, I_2), \phi)$ a refined neutrosophic F-algebra, where $\mathbb{R}(I_1, I_2)$ is a refined neutrosophic ring with identity. Then $\mathbb{R}(I_1, I_2)$ is a strong refined neutrosophic vector space over $F(I_1, I_2)$ with addition being that in $\mathbb{R}(I_1, I_2)$ and scalar multiplication defined by $ar = \phi(a)r$. Here $\phi : F(I_1, I_2) \longrightarrow \mathbb{R}(I_1, I_2)$ is a neutrosophic homomorphism such that $\phi((1 + 0I_1 + 0I_2)) = (1 + 0I_1 + 0I_2)$ and $\phi(I_k) = I_k$.*

Proof.

1. That $(\mathbb{R}(I_1, I_2), +)$ is a neutrosophic abelian group can be easily established.

   Let $x = a + bI_1 + cI_2, y = d + eI_1 + fI_2 \in R(I_1, I_2), \alpha = k + mI_1 + nI_2$,
   $\beta = u + vI_1 + wI_2 \in F(I_1, I_2)$. Where $a, b, c, d, e, f \in \mathbb{R}$ and $k, m, n, u, v, w \in F$.

2. $\alpha(x + y) = (k + mI_1 + nI_2)((a + d + (b + e)I_1 + (c + f)I_2)$
   $= ka+kd+[kb+ke+ma+md+mb+me+mc+mf+nb+ne]I_1+[kc+kf+na+nd+nc+nf]I_2$
   $= [ka + (kb + ma + mb + mc + nb)I_1 + (kc + na + nc)I_2] + [kd + (ke + md + me + mf + ne)I_1 + (kf + nd + nf)I_2]$
   $= (k + mI_1 + nI_2)(a + bI_1 + cI_2) + (k + mI_1 + nI_2)(d + eI_1 + fI_2)$
   $= \phi((k + mI_1 + nI_2))(a + bI_1 + cI_2) + \phi((k + mI_1 + nI_2))(d + eI_1 + fI_2)$
   $= \phi(\alpha)x + \phi(\alpha)y.$

3. $(\alpha + \beta)x = (k + mI_1 + nI_2 + p + qI_1 + rI_2)(a + bI_1 + cI_2)$
   $= (k + p + (m + q)I_1 + (n + r)I_2)(a + bI_1 + cI_2)$
   $= ka + pa + [kb + pb + ma + qa + mb + qb + mc + qc + nb + rb]I_1 + [kc + pc + na + ra + nc + rc]I_2$
   $= [ka + (kb + ma + mb + mc + nb)I_1 + [kc + na + nc]I_2]$
   $+ [pa + (pb + qa + qb + qc + rb)I_1 + [pc + ra + rc]I_2$
   $= (k + mI_1 + nI_2)(a + bI_1 + cI_2) + (p + qI_1 + rI_2)(a + bI_1 + cI_2)$
   $= \phi((k + mI_1 + nI_2))(a + bI_1 + cI_2) + \beta((p + qI_1 + rI_2))(a + bI_1 + cI_2)$
   $= \phi(\alpha)x + \phi(\beta)x.$
   $= (\phi(\alpha) + \phi(\beta))x = \phi(\alpha + \beta)x,$ since ϕ is a refined neutrosophic homomorphism.

4. $(\alpha\beta)u = ((k + mI_1 + nI_2)(p + qI_1 + rI_2))(a + bI_1 + cI_2)$
   $= (kp + (kq + mp + mq + mr + nq)I_1 + (kr + np + nr)I_2)(a + bI_1 + cI_2)$
   $= kpa + [kpb + kqa + mpa + mqa + mra + nqa + kqb + mpb + mqb + mrb + nqb + kqc + mpc + mqc + mrc + nqc + krb + npb + nrb]I_1 + [kpc + kra + npa + nra + krc + npc + nrc]I_2$
   $= (k + mI_1 + nI_2)[pa + (pb + qa + qb + qc + rb)I_1 + (pc + ra + rc)I_2]$
   $= (k + mI_1 + nI_2)((p + qI_1 + rI_2)(a + bI_1 + cI_2))$
   $= \phi((k + mI_1 + nI_2))(\phi((p + qI_1 + rI_2))(a + bI_1 + cI_2))$
   $= \phi(\alpha)(\phi(\beta)x).$

5. For $1 = 1 + 0I_1 + 0I_2 \in F(I_1I_2)$, we have
$1x = (1 + 0I_1 + 0I_2)(a + bI_1 + cI_2)$
$= \phi(1 + 0I_1 + 0I_2)(a + bI_1 + cI_2)$.

Accordingly, $\mathbb{R}(I_1, I_2)$ is a strong refined neutrosophic vector space over $F(I_1, I_2)$.

□

Lemma 2.10. *Let $V(I_1, I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$ and let $x = a+bI_1+cI_2$, $y = d+eI_1+fI_2$, $z = k+mI_1+nI_2, \in V(I_1, I_2)$, $\beta = u+vI_1+wI_2 \in K(I_1, I_2)$. Then*

1. *$x + z = y + z$ implies $x = y$.*
2. *$0x = 0$.*
3. *$\beta 0 = 0$.*
4. *$(-\beta)x = \beta(-x) = -(\beta x)$.*

Proof. 1. $x + z = y + z$
$(a + bI_1 + cI_2) + (k + mI_1 + nI_2) = (d + eI_1 + fI_2) + (k + mI_1 + nI_2)$
$\Longrightarrow a + k + (b + m)I_1 + (c + n)I_2 = d + k + (e + m)I_1 + (f + n)I_2$
$\Longleftrightarrow a + k = d + k, b + m = e + m$ and $c + n = f + n$
$\Longleftrightarrow a = d, b = e$ and $c = f$
$\Longrightarrow a + bI_1 + cI_2 = d + eI_1 + fI_2 \Longrightarrow x = y$.

2. Consider $\beta x = ((u + vI_1 + wI_2) + (0 + 0I_1 + 0I_2))(a + bI_1 + cI_2)$
$= ((u + 0) + (v + 0)I_1 + (w + 0)I_2)(a + bI_1 + cI_2)$
$= (u+0)a+((u+0)b+(v+0)a+(v+0)b+(v+0)c+(w+0)b)I_1+((u+0)c+(w+0)a+(w+0)c)I_2$
$= ua + 0a + (ub + 0b + va + 0a + vb + 0b + vc + 0c + wb + 0b)I_1 + (uc + 0c + wa + 0a + wc + 0c)I_2$
$= (ua+(ub+va+vb+vc+wb)I_1+(uc+wa+wc)I_2)+(0a+(0b+0a+0b+0c+0b)I_1+(0c+0a+0c)I_2)$
$= \beta x + 0x$
$\Longrightarrow 0x = 0$.

3. Since $\beta \in K(I_1, I_2)$,
$\beta 0 + \beta 0 = (u0 + (u0 + v0 + v0 + v0 + w0)I_1 + (u0 + w0 + w0)I_2) + (u0 + (u0 + v0 + v0 + v0 + w0)I_1 + (u0 + w0 + w0)I_2)$
$= (u0+u0+(u0+u0+v0+v0+v0+v0+v0+v0+w0+w0)I_1+(u0+u0+w0+w0+w0+w0)I_2)$
$= (u(0+0)+(u(0+0)+v(0+0)+v(0+0)+v(0+0)+w(0+0))I_1+(u(0+0)+w(0+0)+w(0+0))I_2$
$= (u0 + (u0 + v0 + v0 + v0 + w0)I_1 + (u0 + w0 + w0)I_2 = \beta 0$
$\Longrightarrow \beta 0 = 0$.

4. Let $\beta \in K(I_1, I_2)$ and $x \in V(I_1, I_2)$.
So $\beta x + (-\beta)x = (u + vI_1 + wI_2)(a + bI_1 + cI_2) + (-(u + vI_1 + wI_2))(a + bI_1 + cI_2)$
$= (ua + (ub + va + vb + vc + wb)I_1 + (uc + wa + wc)I_2) + (-ua + (-ub - va - vb - vc - wb)I_1 + (-uc - wa - wc)I_2)$
$= (ua-ua+(ub-ub+va-va+vb-vb+vc-vc+wb-wb)I_1+(uc-uc+wa-wa+wc-wc)I_2)$
$= ((u-u)a+((u-u)b+(v-v)a+(v-v)b+(v-v)c+(w-w)b)I_1+((u-u)c+(w-w)a+(w-w)c)I_2)$
$= (0a + (0b + 0a + 0b + 0c + 0b)I_1 + (0c + 0a + 0c)I_2)$
$= (0 + 0I_1 + 0I_2)(a + bI_1 + cI_2) = 0x = 0$ by 2.
Then $\beta x + (-\beta)x = 0 \Longrightarrow (-\beta)x = -\beta x$.
So $\beta x + \beta(-x) = (u + vI_1 + wI_2)(a + bI_1 + cI_2) + ((u + vI_1 + wI_2))(-(a + bI_1 + cI_2)) = (ua + (ub + va + vb + vc + wb)I_1 + (uc + wa + wc)I_2) + (u(-a) + (u(-b) + v(-a) + v(-b) + v(-c) + w(-b))I_1 + (u(-c) + w(-a) + w(-c))I_2)$
$= (ua + u(-a) + (ub + u(-b) + va + v(-a) + vb + v(-b) + vc + v(-c) + wb + w(-b))I_1 + (uc + u(-c) + wa + w(-a) + wc + w(-c))I_2)$
$= (u(a-a)+(u(b-b)+v(a-a)+v(b-b)+v(c-c)+w(b-b))I_1+(u(c-c)+w(a-a)+w(c-c))I_2)$
$= (u0 + (u0 + v0 + v0 + v0 + w0)I_1 + (u0 + w0 + w0)I_2) = (u + vI_1 + wI_2)(0 + 0I_1 + 0I_2)$
$= \beta 0 = 0$ by 3.
Then $\beta x + \beta(-x) = 0 \Longrightarrow \beta(-x) = -\beta x$.

□

Definition 2.11. Let $V(I_1, I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$ and let $W(I_1, I_2)$ be a nonempty subset of $V(I_1, I_2)$. $W(I_1, I_2)$ is called a strong refined neutrosophic subspace of $V(I_1, I_2)$ if $W(I_1, I_2)$ is itself a strong refined neutrosophic vector space over $K(I_1, I_2)$. It is essential that $W(I_1, I_2)$ contains a proper subset which is a vector space.

Definition 2.12. Let $V(I_1, I_2)$ be a weak refined neutrosophic vector space over a field K and let $W(I_1, I_2)$ be a nonempty subset of $V(I_1, I_2)$. $W(I_1, I_2)$ is called a weak refined neutrosophic subspace of $V(I_1, I_2)$ if $W(I_1, I_2)$ is itself a weak refined neutrosophic vector space over K. It is essential that $W(I_1, I_2)$ contains a proper subset which is a vector space.

Proposition 2.13. *Let $V(I_1, I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$ and let $W(I_1, I_2)$ be a nonempty subset of $V(I_1, I_2)$. $W(I_1, I_2)$ is a strong refined neutrosophic subspace of $V(I_1, I_2)$ if and only if the following conditions hold:*

1. $u, v \in W(I_1, I_2)$ *implies* $u + v \in W(I_1, I_2)$.
2. $u \in W(I_1, I_2)$ *implies* $\alpha u \in W(I_1, I_2)$ *for all* $\alpha \in K(I_1, I_2)$.
3. $W(I_1, I_2)$ *contains a proper subset which is a vector space.*

Example 2.14. Let $V(I_1, I_2)$ be a weak (strong) refined neutrosophic vector space. $V(I_1, I_2)$ is a weak (strong) refined neutrosophic subspace called a trivial weak (strong) refined neutrosophic subspace.

Example 2.15. Let $K(I_1, I_2) = \mathbb{R}(I_1, I_2)$ be a refined neutrosophic field and $V(I_1, I_2) = \mathbb{R}^3(I_1, I_2)$ be a strong refined neutrosophic vector space. Take $W(I_1, I_2)$ to be the set of all vectors in $V(I_1, I_2)$ whose last component is 0 $= 0 + 0I_1 + 0I_2$. Then $W(I_1, I_2)$ is strong refined neutrosophic subspace of $V(I_1, I_2)$.

Proof. To see this, let

$$W(I_1, I_2) = \{(x = a + bI_1 + cI_2, y = d + eI_1 + fI_2, 0 = 0 + 0I_1 + 0I_2) \in V(I_1, I_2) : a, b, c, d, e, f \in V\}.$$

1. Given that $u, v \in W(I_1, I_2)$, where $u = (x, y, 0)$ and $v = (x', y', 0)$. Then
   $u + v = (x, y, 0) + (x', y', 0) = (a + bI_1 + cI_2, d + eI_1 + fI_2, 0 + 0I_1 + 0I_2) + (a' + b'I_1 + c'I_2, d' + e'I_1 + f'I_2, 0 + 0I_1 + 0I_2)$
   $= (a + a' + (b + b')I_1 + (c + c')I_2, d + d' + (e + e')I_1 + (f + f')I_2, 0 + 0 + (0 + 0)I_1 + (0 + 0)I_2)$
   $= (a + a' + (b + b')I_1 + (c + c')I_2, d + d' + (e + e')I_1 + (f + f')I_2, 0 + 0I_1 + 0I_2)$.
   Hence we have that $u + v \in W(I_1, I_2)$.
2. Given $u \in W(I_1, I_2)$ and scalar $\alpha \in K(I_1, I_2)$ with $\alpha = r + sI_1 + tI_2$.
   Then $\alpha u = (r + sI_1 + tI_2)(a + bI_1 + cI_2, d + eI_1 + fI_2, 0 + 0I_1 + 0I_2)$
   $((r + sI_1 + tI_2)(a + bI_1 + cI_2), (r + sI_1 + tI_2)(d + eI_1 + fI_2), (r + sI_1 + tI_2)(0 + 0I_1 + 0I_2))$
   $= (ra + (rb + sa + sb + sc + tb)I_1 + (rc + ta + tc)I_2, \quad rd + (re + sd + se + sf + te)I_1 + (rf + td + tf)I_2, \quad r0 + (r0 + s0 + s0 + s0 + t0)I_1 + (r0 + t0 + t0)I_2)$
   $= (ra + (rb + sa + sb + sc + tb)I_1 + (rc + ta + tc)I_2, \quad rd + (re + sd + se + sf + te)I_1 + (rf + td + tf)I_2, \quad 0 + 0I_1 + 0I_2) \in W(I_1, I_2)$.
3. Since $W \subset W(I_1, I_2)$ is a proper subset which is a vector space, $W(I_1, I_2)$ is strong refined neutrosophic subspace.

□

Example 2.16. Let $V(I_1, I_2) = \mathbb{R}^2(I_1, I_2)$ be a strong refined neutrosophic vectors space over a refined neutrosophic field $\mathbb{R}(I_1, I_2)$ let

$$W(I_1, I_2) = \{(x = a + bI_1 + cI_2, y = d + eI_1 + fI_2) \in V(I_1, I_2) : x = y \ \text{with} a, b, c, d, e, f \in V\}.$$

Then $W(I_1, I_2)$ is a strong refined neutrosophic subspace of $V(I_1, I_2)$.

Example 2.17. Let $V(I_1, I_2) = M_{n\times n}(I_1, I_2) = \{[a_{ij}] : a_{ij} \in \mathbb{R}(I_1, I_2)\}$ be a strong refined neutrosophic vector space over $R(I_1, I_2)$ and let $W(I_1, I_2) = A_{n\times n}(I_1, I_2) = \{[b_{ij}] : b_{ij} \in \mathbb{R}(I_1, I_2) \ \text{ and } \ \text{trace(A)}= 0\}$. Then $W(I_1, I_2)$ is a strong refined neutrosophic subspace of $V(I_1, I_2)$.

Example 2.18. Let $V(I_1, I_2) = \mathbb{R}^3(I_1, I_2)$ be a strong refined neutrosophic vectors space of column refined neutrosophic vectors of length 3 over a refined neutrosophic field $\mathbb{R}(I_1, I_2)$. Consider

$$W(I_1, I_2) = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}; x = a + bI_1 + cI_2, y = d + eI_1 + fI_2 \in V(I_1, I_2) \;\; a, b, c, d, e, f \in V \right\} \subseteq V(I_1, I_2).$$

$W(I_1, I_2)$ consisting of all refined neutrosophic vectors with $0 = 0 + 0I_1 + 0I_2$ in the last entry. Then $W(I_1, I_2)$ is a strong refined neutrosophic subspace of $V(I_1, I_2)$.

Proposition 2.19. *Let $V(I_1, I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_1)$ and Let $\{U_n(I_1, I_2)\}_{n\in\lambda}$ be a family of strong refined neutrosophic subspaces of $V(I_1, I_2)$. Then $\bigcap_{n\in\lambda} U_n(I_1, I_2)$ is a strong refined neutrosophic subspace of $V(I_1, I_2)$.*

Proof. Consider the collection of strong refined neutrosophic subspaces $\{U_n(I_1, I_2) : n \in \lambda\}$ of $V(I_1, I_2)$. Take $u = a + bI_1 + cI_2$, $v = d + eI_1 + fI_2$, $\alpha = k + pI_1 + qI_2$ and $\beta = r + sI_1 + tI_2$.
Let $u, v \in \bigcap_{n\in\lambda} U_n(I_1, I_2)$ then $u, v \in U_n(I_1, I_2)\ \forall\ n \in \lambda$. Now for all scalars $\alpha, \beta \in K(I_1, I_2)$ we have that $\alpha u + \beta v = (k + pI_1 + qI_2)(a + bI_1 + cI_2) + (r + sI_1 + tI_2)(d + eI_1 + fI_2)$
$= (ka+(kb+pa+pb+pc+qb)I_1+(kc+qa+qc)I_2)+(rd+(re+sd+se+sf+te)I_1+(rf+td+tf)I_2)$
$= (ka + rd) + (kb + pa + pb + pc + qb + re + sd + se + sf + te)I_1 + (kc + qa + qc + rf + td + tf)I_2$.
Therefore $\alpha u + \beta v \in U_n(I_1, I_2)\ \forall\ n \in \lambda \Longrightarrow \alpha u + \beta v \in \bigcap_{n\in\lambda} U_n(I_1, I_2)$.
Lastly, since $U_n(I_1, I_2)$ for all $n \in \lambda$ contains a proper subset U_n which is vector space, we have that $\bigcap_{n\in\lambda} U_n(I_1, I_2)$ is a strong refined neutrosophic subspace. □

Proposition 2.20. *Let $V(I_1, I_2)$ be a strong refined neutrosophic vector space over the neutrosophic field $K(I_1, I_2)$ and let $U_1(I_1, I_2), U_2(I_1, I_2)$ be any strong refined neutrosophic subspaces of $V(I_1, I_2)$. Then $U_1(I_1, I_2) \cup U_2(I_1, I_2)$ is a strong refined neutrosophic subspaces if and only if $U_1(I_1, I_2) \subseteq U_2(I_1, I_2)$ or $U_1(I_1, I_2) \supseteq U_2(I_1, I_2)$.*

Proof. Let $U_1(I_1, I_2)$ and $U_2(I_1, I_2)$ be any strong refined neutrosophic subspaces of $V(I_1, I_2)$.
$\Longrightarrow$ Now, suppose $U_1(I_1, I_2) \subseteq U_2(I_1, I_2)$ or $U_1(I_1, I_2) \supseteq U_2(I_1, I_2)$ then we shall show the $U_1(I_1, I_2) \cup U_2(I_1, I_2)$ is a strong refined neutrosophic subspaces of $V(I_1, I_2)$.
Without loss of generality, suppose that $U_1(I_1, I_2) \subseteq U_2(I_1, I_2)$.
Then we have that $U_1(I_1, I_2) \cup U_2(I_1, I_2) = U_2(I_1, I_2)$. But $U_2(I_1, I_2)$ is defined to be a strong refined neutrosophic subspace of $V(I_1, I_2)$, so we can say that $U_1(I_1, I_2) \cup U_2(I_1, I_2)$ is a strong refined neutrosophic subspace of $V(I_1, I_2)$.
$\Longleftarrow$ We want to show that if $U_1(I_1, I_2) \cup U_2(I_1, I_2)$ is a strong refined neutrosophic subspace of $V(I_1, I_2)$ then either $U_1(I_1, I_2) \subseteq U_2(I_1, I_2)$ or $U_1(I_1, I_2) \supseteq U_2(I_1, I_2)$.
Now suppose that $U_1(I_1, I_2) \cup U_2(I_1, I_2)$ is a strong refined neutrosophic subspace of $V(I_1, I_2)$ and suppose by contradiction that $U_1(I_1, I_2) \nsubseteq U_2(I_1, I_2)$ or $U_1(I_1, I_2) \nsupseteq U_2(I_1, I_2)$.
Thus there exist elements $x_1 = a_1 + b_1I_1 + c_1I_2 \in U_1(I_1, I_2)\backslash U_2(I_1, I_2)$ and $x_2 = a_2 + b_2I_1 + c_2I_2 \in U_2(I_1, I_2)\backslash U_1(I_1, I_2)$. So we have that $x_1, x_2 \in U_1(I_1, I_2) \cup U_2(I_1, I_2)$, since $U_1(I_1, I_2) \cup U_2(I_1, I_2)$ is a strong refined neutrosophic subspace, we must have that $x_1 + x_2 = x_3 \in U_1(I_1, I_2) \cup U_2(I_1, I_2)$.
Therefore $x_1 + x_2 = x_3 \in U_1(I_1, I_2)$ or $x_1 + x_2 = x_3 \in U_2(I_1, I_2)$
$\Longrightarrow x_2 = x_3 - x_1 \in U_1(I_1, I_2)$ or $x_1 = x_3 - x_2 \in U_2(I_1, I_2)$ which is a contradiction.
Hence $U_1(I_1, I_2) \subseteq U_2(I_1, I_2)$ or $U_1(I_1, I_2) \supseteq U_2(I_1, I_2)$ as required. □

Remark 2.21. Let $V(I_1, I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$ and let $W_1(I_1, I_2)$ and $W_2(I_1, I_2)$ be two distinct strong refined neutrosophic subspaces of $V(I_1, I_2)$. $W_1(I_1, I_2) \cup W_2(I)$ is a strong refined neutrosophic subspace of $V(I_1, I_2)$ iff $W_1(I_1, I_2) \subseteq W_2(I_1, I_2)$ or $W_2(I_1, I_2) \subseteq W_1(I_1, I_2)$.

Definition 2.22. Let $U(I_1, I_2)$ and $W(I_1, I_2)$ be any two strong refined neutrosophic subspaces of a strong refined neutrosophic vector space $V(I_1, I_2)$ over a neutrosophic field $K(I_1, I_2)$.

1. The sum of $U(I_1, I_2)$ and $W(I_1, I_2)$ denoted by $U(I_1, I_2) + W(I_1, I_2)$ is defined by the set $\{u + w : u \in U(I_1, I_2), w \in W(I_1, I_2)\}$.

2. $V(I_1, I_2)$ is said to be the direct sum of $U(I_1, I_2)$ and $W(I_1, I_2)$ written $V(I_1, I_2) = U(I_1, I_2) \oplus W(I_1, I_2)$ if every element $v \in V(I_1, I_2)$ can be written uniquely as $v = u + w$ where $u \in U(I_1, I_2)$ and $w \in W(I_1, I_2)$.

Proposition 2.23. *Let $U(I_1, I_2)$ and $W(I_1, I_2)$ be any two strong refined neutrosophic subspaces of a strong refined neutrosophic vector space $V(I_1, I_2)$, over a refined neutrosophic field $K(I_1, I_2)$.*
Then $U(I_1, I_2) + W(I_1, I_2)$ is a strong refined neutrosophic subspace of $V(I_1, I_2)$.

Proof. Since $U(I_1, I_2)$ and $W(I_1, I_2)$ are nonempty strong refined neutrosophic subspaces,
$U(I_1, I_2) + W(I_1, I_2) \neq \{\}$.
Obviously $U(I_1, I_2) + W(I_1, I_2)$ contains a proper subset $U + W$ which is a vector space.
Now let $x, y \in U(I_1, I_2) + W(I_1, I_2)$ and $\alpha, \beta \in K(I_1, I_2)$.
Then $x = (u_1 + u_2 I_1 + u_3 I_2) + (w_1 + w_2 I_1 + w_3 I_2)$, $y = (u_4 + u_5 I_1 + u_6 I_2) + (w_4 + w_5 I_1 + w_6 I_2)$ where $u_i \in U, w_i \in W$, with $i = 1, 2, 3, 4, 5, 6$. $\alpha = k + mI_1 + nI_2, \beta = p + qI_1 + rI_2$ where $k, m, n, p, q, r \in K(I_1, I_2)$.
Then,
$\alpha x + \beta y = (k + mI_1 + nI_2)[(u_1 + w_1) + (u_2 + w_2)I_1 + (u_3 + w_3)I_2] + (p + qI_1 + rI_2)[(u_4 + w_4) + (u_5 + w_5)I_1 + (u_6 + w_6)I_2]$
$= [(ku_1 + kw_1) + (ku_2 + kw_2 + mu_1 + mw_1 + mu_2 + mw_2 + mu_3 + mw_3 + nu_2 + nw_2)I_1 + (ku_3 + kw_3 + nu_1 + nw_1 + nu_3 + nw_3)I_2] + [(pu_4 + pw_4) + (pu_5 + pw_5 + qu_4 + qw_4 + qu_5 + qw_5 + qu_6 + qw_6 + ru_5 + rw_5)I_1 + (pu_6 + pw_6 + ru_4 + rw_4 + ru_6 + rw_6)I_2]$
$= [(ku_1 + pu_4) + (ku_2 + pu_5 + mu_1 + qu_4 + mu_2 + qu_5 + mu_3 + qu_6 + nu_2 + ru_5)I_1 + (ku_3 + pu_6 + nu_1 + ru_4 + nu_3 + ru_6)I_2] + [(kw_1 + pw_4) + (kw_2 + pw_5 + mw_1 + qw_4 + mw_2 + qw_5 + mw_3 + qw_6 + nw_2 + rw_5)I_1 + (kw_3 + pw_6 + nw_1 + rw_4 + nw_3 + rw_6)I_2] \in U(I_1, I_2) + W(I_1, I_2)$.
Accordingly $U(I_1, I_2) + W(I_1, I_2)$ is a strong refined neutrosophic subspace of $V(I_1, I_2)$. □

Proposition 2.24. *Let $U(I_1, I_2)$ and $W(I_1, I_2)$ be strong refined neutrosophic subspaces of a strong refined neutrosophic vector space $V(I_1, I_2)$ over a refined neutrosophic field $K(I_1, I_2)$.*
$V(I_1, I_2) = U(I_1, I_2) \oplus W(I_1, I_2)$ if and only if the following conditions hold:

1. *$V(I_1, I_2) = U(I_1, I_2) + W(I_1, I_2)$ and*
2. *$U(I_1, I_2) \cap W(I_1, I_2) = \{0\}$.*

Proof. The proof is similar to the proof in classical case. □

Example 2.25. Let $V(I_1, I_2) = \mathbb{R}^3(I_1, I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $R(I_1, I_2)$ and let
$U(I_1, I_2) = \{(u, 0, w) : u = a + bI_1 + cI_2, w = g + hI_1 + kI_2 \in R(I_1, I_2)\}$ and
$W(I_1, I_2) = \{(0, v, 0) : v = d + eI_1 + fI_2 \in R(I_1, I_2)\}$,
be strong refined neutrosophic subspaces of $V(I_1, I_2)$. Then $V(I_1, I_2) = U(I_1, I_2) \oplus W(I_1, I_2)$.

To see this, let $x = (u, v, w) \in V(I_1, I_2)$, then $x = (u, 0, w) + (0, v, 0)$, so $x \in U(I_1, I_2) + W(I_1, I_2)$.
Hence $V(I_1, I_2) = U(I_1, I_2) + W(I_1, I_2)$.
To show that $U(I_1, I_2) \cap W(I_1, I_2) = \{0\}$, let $x = (u, v, w) \in U(I_1, I_2) \cap W(I_1, I_2)$.
Then $v = 0$, i.e $d + eI_1 + fI_2 = 0 + 0I_1 + 0I_2$ because x lies in $U(I_1, I_2)$, and $u = w = 0$ i.e $a + bI_1 + cI_2 = g + hI_1 + kI_2 = 0 + 0I_1 + 0I_2$ because x lies in $W(I_1, I_2)$.
Thus $x = (0, 0, 0) = 0$, so $0 = 0 + 0I_1 + 0I_2$ is the only refined neutrosophic vector in $U(I_1, I_2) \cap W(I_1, I_2)$.
So $U(I_1, I_2) \cap W(I_1, I_2) = \{0 + 0I_1 + 0I_2\} = \{0\}$.
Hence, $V(I_1, I_2) = U(I_1, I_2) \oplus W(I_1, I_2)$.

Example 2.26. In the strong refined neutrosophic vector space $V(I_1, I_2) = \mathbb{R}^5(I_1, I_2)$, consider the strong refined neutrosophic subspaces

$$U(I_1, I_2) = \{(a, b, c, 0, 0) | a = x_1 + y_1 I_1 + z_1 I_2, b = x_2 + y_2 I_1 + z_2 I_2, \text{and} \quad c = x_3 + y_3 I_1 + z_3 I_2 \in V(I_1, I_2)\}$$

and

$$W = \{0, 0, 0, d, e) | d = x_4 + y_4 I_1 + z_4 I_2, \;\; e = x_5 + y_5 I_1 + z_5 I_2 \in V(I_1, I_2)\}.$$

Then $V(I_1, I_2) = U(I_1, I_2) \oplus W(I_1, I_2)$.

To see this, let $x = (a, b, c, d, e)$ be any refined neutrosophic vector in $V(I_1, I_2)$, then
$x = (a, b, c, 0, 0) + (0, 0, 0, d, e)$, so x lies in $U(I_1, I_2) + W(I_1, I_2)$.
Hence $V(I_1, I_2) = U(I_1, I_2) + W(I_1, I_2)$. To show that $U(I_1, I_2) \cap W(I_1, I_2) = \{0\}$, let $x = (a, b, c, d, e)$ lie in $U(I_1, I_2) \cap W(I_1, I_2)$.
Then $d = e = 0$ i.e $x_4 + y_4 I_1 + z_4 I_2 = x_5 + y_5 I_1 + z_5 I_2 = 0 + 0I_1 + 0I_2$ because x lies in $U(I_1, I_2)$, and

$a = b = c = 0$ i.e, $x_1 + y_1I_1 + z_1I_2 = x_2 + y_2I_1 + z_2I_2 = x_3 + y_3I_1 + z_3I_2 = 0 + 0I_1 + 0I_2$ because x lies in $W(I_1, I_2)$.
Thus $x = (0, 0, 0, 0, 0) = 0$, so $0 = 0 + 0I_1 + 0I_2$ is the only refined neutrosophic vector in $U(I_1, I_2) \cap W(I_1, I_2)$. So $U(I_1, I_2) \cap W(I_1, I_2) = \{0 + 0I_1 + 0I_2\}$.
Hence, $V(I_1, I_2) = U(I_1, I_2) \oplus W(I_1, I_2)$.

Example 2.27. Let $V(I_1, I_2) = P_{2n}(I_1, I_2)$ be the strong refined neutrosophic vector space over a neutrosophic field $K(I_1, I_2)$. Then let

$$U_1(I_1, I_2) = \{p(t) \in P_{2n} : a_0 + a_2t^2 + \cdots + a_{2n}t^{2n}, \text{with} \quad a_0, a_2 \cdots a_{2n} \in \mathbb{R}(I_1, I_2)\}$$
$$U_2(I_1, I_2) = \{p(t) \in P_{2n} : a_1 + a_3t^3 + \cdots + a_{2n-1}t^{2n-1}, \text{with} \quad a_1, a_3 \cdots a_{2n-1} \in \mathbb{R}(I_1, I_2)\}$$

be strong refined neutrosophic subspaces of $P_{2n}(I_1, I_2)$.
Then $P_{2n}(I_1, I_2) = U_1(I_1, I_2) \oplus U_2(I_1, I_2)$.

Proposition 2.28. *Let $U(I_1, I_2)$ and $V(I_1, I_2)$ be strong refined neutrosophic vector spaces over a refined neutrosophic field $K(I_1, I_2)$. Then*

$$U(I_1, I_2) \times V(I_1, I_2) = \{(u, v) : u \in U(I_1, I_2), v \in V(I_1, I_2)\}$$

is a strong refined neutrosophic vector space over $K(I_1, I_2)$ where addition and multiplication are defined by:

$$(u, v) + (u', v') = (u + u', v + v'),$$
$$\alpha(u, v) = (\alpha u, \alpha v).$$

Proof. 1. We want to show that $(U(I_1, I_2) \times V(I_1, I_2), +)$ is refined neutrosophic abelian group .

(a) Clearly $(U(I_1, I_2) \times V(I_1, I_2, +)$ is closed, since for $(u, v), (u', v') \in (U(I_1, I_2) \times V(I_1, I_2))$ where $(u, v) = ((a + bI_1 + cI_2), (d + eI_1 + fI_2))$ we have that
$(u, v)+(u', v') = [((a + bI_1 + cI_2), (d + eI_1 + fI_2)) + ((a' + b'I_1 + c'I_2), (d' + e'I_1 + f'I_2))]$
$= [(a + a' + (b + b')I_1 + (c + c')I_2), (d + d' + (e + e')I_1 + (f + f')I_2]$
$= (u + u', v + v') \in U(I_1, I_2) \times V(I_1, I_2)$.

(b) Let $(u, v), (u', v')$ and $(u'', v'') \in U(I_1, I_2) \times V(I_1, I_2)$. Then
$[(u, v) + ((u,' v') + (u'', v''))] = [((a + bI_1 + cI_2), (d + eI_1 + fI_2))$
$+ ((a' + a'' + (b' + b'')I_1 + (c' + c'')I_2), (d' + d'' + (e' + e'')I_1 + (f' + f'')I_2)]$
$= (a+a'+a''+(b+b'+b'')I_1+(c+c'+c'')I_2), (d+d'+d''+(e+e'+e'')I_1+(f+f'+f'')I_2)$
$= [(a + a' + (b + b')I_1 + (c + c')I_2), (d + d' + (e + e')I_1 + (f + f')I_2)] +$
$((a'' + b''I_1 + c''I_2), (d'' + e''I_1 + f''I_2))$
$= [(u, v) + (u', v')] + (u'', v'')$.
Then we say that $''+''$ is associative.

(c) The identity in $U(I_1, I_2) \times V(I_1, I_2))$ is $(0_{U(I_1,I_2)}, 0_{V(I_1,I_2)})$ where $0_{U(I_1,I_2)}$ is the identity in $U(I_1, I_2)$ and $0_{V(I_1,I_2)}$ is the identity in $V(I_1, I_2)$ then
$(u, v) + (0_{U(I_1,I_2)}, 0_{V(I_1,I_2)})$
$= [(a + bI_1 + cI_2), (d + eI_1 + fI_2)] + [(0 + 0I_1 + 0I_2), (0 + 0I_1 + 0I_2)]$
$= (a + 0 + (b + 0)I_1 + (c + 0)I_2), (d + 0 + (e + 0)I_1 + (f + 0)I_2)$
$= (0 + a + (0 + b)I_1 + (0 + c)I_2), (0 + d, (0 + e)I_1 + (0 + f)I_2)$
$= (a + bI_1 + cI_2), (d + eI_1 + fI_2) = (u, v)$.

(d) For each $(u, v) \in U(I_1, I_2) \times V(I_1, I_2))$ the inverse is $(-u, -v)$ where $-u \in U(I_1, I_2)$ and $-v \in V(I_1, I_2)$ is the inverse of u and v respectively. Then
$(u, v) + (-u, -v) = [((a+bI_1+cI_2), (d+eI_1+fI_2)) + ((-a-bI_1-cI_2), (-d-eI_1-fI_2))]$
$= (a - a + (b - b)I_1 + (c - c)I_2), ((d - d) + (e - e)I_1 + (f - f)I_2)$
$= (-a + a + (-b + b)I_1 + (-c + c)I_2), ((-d + d) + (-e + e)I_1 + (-f + f)I_2)$
$= (0 + 0I_1 + 0I_2), (0 + 0I_1 + 0I_2) = (0_{U(I_1,I_2)}, 0_{V(I_1,I_2)})$.

(e) For $(u, v), (u', v') \in U(I_1, I_2) \times V(I_1, I_2)$ we have that
$(u, v) + (u', v') = [((a + bI_1 + cI_2), (d + eI_1 + fI_2)) + ((a' + b'I_1 + c'I_2), (d' + e'I_1 + f'I_2))]$
$= [(a + a' + (b + b')I_1 + (c + c')I_2), ((d + d') + (e + e')I_1 + (f + f')I_2)]$
$= [(a' + a + (b' + b)I_1 + (c' + c)I_2), ((d' + d) + (e' + e)I_1 + (f' + f)I_2)]$
$= [((a'+b'I_1+c'I_2), (d'+e'I_1+f'I_2)) + ((a+bI_1+cI_2), (d+eI_1+fI_2))] = (u', v')+(u, v)$.
Then we say that $''+''$ is commutative.

Let $\alpha = k + mI_1 + nI_2$, $\beta = r + sI_1 + tI_2 \in K(I_1, I_2)$, then

2. $\alpha((u, v) + (u', v')) = \alpha[((a + bI_1 + cI_2), (d + eI_1 + fI_2)) + ((a' + b'I_1 + c'I_2), (d' + e'I_1 + f'I_2))]$
$= \alpha[(a + a' + (b + b')I_1 + (c + c')I_2), ((d + d') + (e + e')I_1 + (f + f')I_2)]$
$= (\alpha a + \alpha a' + (\alpha b + \alpha b')I_1 + (\alpha c + \alpha c')I_2), ((\alpha d + \alpha d') + (\alpha e + \alpha e')I_1 + (\alpha f + \alpha f')I_2)$
$= ((\alpha a + \alpha bI_1 + \alpha cI_2), (\alpha d + \alpha eI_1 + \alpha fI_2)) + ((\alpha a' + \alpha b'I_1 + \alpha c'I_2), (\alpha d' + \alpha e'I_1 + \alpha f'I_2))$
$= (\alpha u, \alpha v) + (\alpha u', \alpha v')$
$= \alpha(u, v) + \alpha(u', v')$.

3. $(\alpha + \beta)(u, v) = (k + r + (m + s)I_1 + (n + t)I_2)((a + bI_1 + cI_2), (d + eI_1 + fI_2))$
$= ((k+r+(m+s)I_1+(n+t)I_2)(a+bI_1+cI_2), \quad (k+r+(m+s)I_1+(n+t)I_2)(d+eI_1+fI_2))$
$= [(ka+ra+(ma+sa)I_1+(na+ta)I_2+(kb+rb+mb+sb+nb+tb)I_1+((mc+sc)I_1+(kc+rc+nc+tc)I_2)), \quad (kd+rd+(md+sd)I_1+(nd+td)I_2+(ke+re+me+se+ne+te)I_1+((mf+sf)I_1+(kf+rf+nf+tf)I_2))]$
$= [(ka + maI_1 + naI_2) + (kb + mb + nb)I_1 + (mcI_1 + (kc + nc)I_2), \quad (kd + mdI_1 + ndI_2) + (ke+me+ne)I_1+(mfI_1+(kf+nf)I_2)] + [(ra+saI_1+taI_2+(rb+sb+tb)I_1+(scI_1+(rc+tc)I_2)), \quad (rd + sdI_1 + tdI_2) + (re + se + te)I_1 + (sfI_1 + (rf + tf)I_2))]$
$= [(k+mI_1+n_2)((a+bI_1+cI_2), (d+eI_1+fI_2))]+[(r+sI_1+tI_2)((a+bI_1+cI_2), (d+eI_1+fI_2))]$
$= \alpha(u, v) + \beta(u, v)$.

4. $(\alpha\beta)(u, v) = ((k + mI_1 + nI_2)(r + sI_1 + tI_2))(u, v)$
$= (kr + (ks + mr + ms + mt + ns)I_1 + (kt + nr + nt)I_2)((a + bI_1 + cI_2), (d + eI_1 + fI_2))$
$= (kra + (krb + ksa + mra + msa + mta + nsa + ksb + mrb + msb + mtb + nsb + ksc + mrc + msc + mtc + nsc + ktb + nrb + ntb)I_1 + (krc + kta + nra + nta + ktc + nrc + ntc)I_2, \quad krd + (kre + ksd + mrd + msd + mtd + nsd + kse + mre + mse + mte + nse + ksf + mrf + msf + mtf + nsf + kte + nre + nte)I_1 + (krf + ktd + nrd + ntd + ktf + nrf + ntf)I_2)$
$= ((k + mI_1 + nI_2)(ra + (rb + sa + sb + sc + tb)I_1 + (rc + ta + tc)I_2) \quad (k + mI_1 + nI_2)(rd + (re + sd + se + sf + te)I_1 + (rf + td + tf)I_2))$
$= \alpha(\beta u, \beta v)$.

5. For $(1 + 1I_1 + 1I_2) \in K(I_1, I_2)$, we have
$(1 + 1I_1 + 1I_2) \cdot (u, v) = (1 + 1I_1 + 1I_2)(a + bI_1 + cI_2, d + eI_1 + fI_2)$
$= 1a+(1b+1a+1b+1c+1b)I_1+(1c+1a+1c)I_2 \quad 1d+(1e+1d+1e+1f+1e)I_1+(1f+1d+1f)I_2$
$= ((1 + 1I_1 + 1I_2)(a + bI_1 + cI_2), (1 + 1I_1 + 1I_2)(d + eI_1 + fI_2))$
$= (a + bI_1 + cI_2, d + eI_1 + fI_2) = (u, v)$.

□

Proposition 2.29. *Let $U(I_1, I_2)$ be weak refined neutrosophic vector spaces over a field and V be a vector space over a field K. Then*

$$U(I_1, I_2) \times V = \{(u, v) : u = (a + bI_1 + cI_2) \in U(I_1, I_2), v \in V\}$$

is a weak refined neutrosophic vector space over K where addition and multiplication are defined by:

$$(u, v) + (u', v') = (u + u', v + v'),$$

$$\alpha(u, v) = (\alpha u, \alpha v).$$

Proof. The proof follows the same approach as in the proof of Proposition 2.28 □

Definition 2.30. Let $W(I_1, I_2)$ be a strong refined neutrosophic subspace of a strong refined neutrosophic vector space $V(I_1, I_2)$ over a refined neutrosophic field $K(I_1, I_2)$. The quotient $V(I_1, I_2)/W(I_1, I_2)$ is defined by the set

$$\{v + W(I_1, I_2) : v \in V(I_1, I_2)\}.$$

Proposition 2.31. *The quotient $V(I_1, I_2)/W(I_1, I_2)$ is a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$ if addition and multiplication are defined for all $u + W(I_1, I_2), v + W(I_1, I_2) \in V(I_1, I_2)/W(I_1, I_2)$ and $\alpha \in K(I_1, I_2)$ as follows:*

$$(u + W(I_1, I_2)) + (v + W(I_1, I_2)) = (u + v) + W(I_1, I_2),$$

$$\alpha(u + W(I_1, I_2)) = \alpha u + W(I_1, I_2).$$

This strong refined neutrosophic vector space $(V(I_1, I_2)/W(I_1, I_2), +, .)$ over a neutrosophic field $K(I_1, I_2)$ is called a strong refined neutrosophic quotient space.

3 Conclusion

In this paper, we have presented the concept of refined neutrosophic vector spaces. Weak(strong) refined neutrosophic vector spaces and subspaces, and, strong refined neutrosophic quotient vector spaces were studied. Several interesting results and examples were presented. It was shown that every weak (strong) refined neutrosophic vector space is a vector space and it was equally shown that every strong refined neutrosophic vector space is a weak refined neutrosophic vector space. This work will be continued in our next paper titled "On Refined Neutrosophic Vector Spaces II".

References

[1] Adeleke, E.O, Agboola, A.A.A and Smarandache, F. Refined Neutrosophic Rings I, International Journal of Neutrosophic Science (IJNS), Vol. 2(2), pp. 77-81, 2020.

[2] Adeleke, E.O, Agboola, A.A.A and Smarandache, F. Refined Neutrosophic Rings II, International Journal of Neutrosophic Science (IJNS), Vol. 2(2), pp. 89-94, 2020.

[3] Agboola, A.A.A., Ibrahim, A.M. and Adeleke, E.O, Elementary Examination of NeutroAlgebras and AntiAlgebras Viz-a-Viz the Classical Number Systems, Vol. 4, pp. 16-19, 2020.

[4] Agboola, A.A.A. On Refined Neutrosophic Algebraic Structures, Neutrosophic Sets and Systems, Vol. 10, pp 99-101, 2015.

[5] Agboola, A.A.A., Akinola, A.D. and Oyebola, O.Y., Neutrosophic Rings I, Int. J. of Math. Comb. Vol 4, pp. 1-14, 2011.

[6] Agboola, A.A.A., Adeleke, E.O. and Akinleye, S.A., Neutrosophic Rings II, Int. J. of Math. Comb. Vol. 2, pp 1-8, 2012.

[7] Agboola, A.A.A. Akwu, A.O., and Oyebo, Y.T., Neutrosophic Groups and Neutrosopic Subgroups, Int. J. of Math. Comb. Vol. 3, pp. 1-9, 2012.

[8] Agboola, A.A.A. and Akinleye, S.A., Neutrosophic Vector Spaces, Neutrosophic Sets and Systems Vol. 4,pp. 9-18. 2014.

[9] Atanassov, K., Intuitionistic fuzzy sets, Fuzzy Sets and Systems, Vol. 20, pp. 87-96, 1986.

[10] Bera, T. and Mahapatra,N.K., Introduction to neutrosophic soft groups, Neutrosophic Sets and Systems, Vol. 13, pp, 118-127, 2016, doi.org/10.5281/zenodo.570845.

[11] Bera, T. and Mahapatra, N.K., On neutrosophic normal soft groups, International Journal of Applied and Computational Mathematics., Vol. 3, pp 3047-3066, 2017. DOI 10.1007/s40819-016-0284-2.

[12] Bera, T. and Mahapatra, N.K., On neutrosophic soft rings, OPSEARCH, Vol. 54, pp. 143-167, 2017. DOI 10.1007/ s12597-016-0273-6.

[13] Bera, T and Mahapatra, N. K., On neutrosophic soft linear spaces, Fuzzy Information and Engineering, Vol. 9, pp 299-324, 2017.

[14] Bera, T and Mahapatra, N. K., On neutrosophic soft field, IJMTT, Vol. 56(7), pp. 472-494, 2018.

[15] Hashmi, M.R., Riaz, M. and Smarandache,F., m-polar Neutrosophic Topology with Applications to Multi-Criteria Decision-Making in Medical Diagnosis and Clustering Analysis, International Journal of Fuzzy Systems, Vol.22(1), pp. 273-292, 2020. https://doi.org/10.1007/s40815-019-00763-2.

[16] Ibrahim, M.A., Agboola, A.A.A, Adeleke, E.O, Akinleye, S.A., Introduction to Neutrosophic Subtraction Algebra and Neutrosophic Subtraction Semigroup,International Journal of Neutrosophic Science (IJNS), Vol. 2(2), pp. 47-62, 2020.

[17] Riaz, M. and Hashmi,M.R., Linear Diophantine Fuzzy Set and its Applications towards Multi-Attribute Decision Making Problems, Journal of Intelligent and Fuzzy Systems, Vol.37(4),pp. 5417-5439 2019.

[18] Riaz, M., Nawa, I. and Sohail, M., Novel Concepts of Soft Multi Rough Sets with MCGDM for Selection of Humanoid Robot, Punjab University Journal of Mathematics, Vol.52(2), pp.111-137, 2020.

[19] Riaz,M. Smarandache,F., Firdous, A, and Fakhar,A., On Soft Rough Topology with Multi-Attribute Group Decision Making, Mathematics Vol.7(1),pp,1-18, 2019. Doi:10.3390/math7010067

[20] Smarandache, F., A Unifying Field in Logics: Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability, American Research Press, Rehoboth, 2003.

[21] Smarandache, F., n-Valued Refined Neutrosophic Logic and Its Applications in Physics, Progress in Physics, Vol. 4, pp. 143-146, 2013.

[22] Smarandache, F., (T,I,F)- Neutrosophic Structures, Neutrosophic Sets and Systems, Vol.8, pp.3-10, 2015.

[23] Vasantha Kandasamy, W.B and Smarandache,F., Basic Neutrosophic Algebraic Structures and Their Applications to Fuzzy and Neutrosophic Models, Hexis, Church Rock, (2004), http://fs.unm.edu/ScienceLibrary.htm

[24] Vasantha Kandasamy, W.B. and Florentin Smarandache, Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic Structures, Hexis, Phoenix, Arizona, (2006), http://fs.unm.edu/ScienceLibrary.htm

[25] Vasantha Kandasamy, W.B., Neutrosophic Rings, Hexis, Phoenix, Arizona,(2006) http://fs.unm.edu/ScienceLibrary.htm

[26] Wadei Al-Omeri and Smarandache, F., New Neutrosophic Set via Neutrosophic Topological Spaces. Excerpt from Neutrosophic Operation Research Vol I, Pons Editions: Brussels, Belgium, pp. 189-209, 2017.

[27] Wadei Al-Omeri, Neutrosophic crisp Sets Via Neutrosophic crisp Topological Spaces, Neutrosophic Set and Systems Vol 13, pp 96- 104, 2016.

[28] Wadei Al-Omeri and Saeid Jafari, On Generalized Closed Sets and Generalized Pre-Closed Sets in Neutrosophic Topological Spaces, Mathematics, Vol 7, pp 1- 12, 2019. Doi: doi.org/10.3390/math/7010001.

[29] Zadeh, L.A., Fuzzy Sets, Information and Control, Vol. 8, pp. 338-353, 1965.