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Aim and Scope

International Journal of Neutrosophic Science (IJNS) is a peer-review journal publishing high quality experimental and theoretical research in all areas of Neutrosophic and its Applications. IJNS is published quarterly. IJNS is devoted to the publication of peer-reviewed original research papers lying in the domain of neutrosophic sets and systems. Papers submitted for possible publication may concern with foundations, neutrosophic logic and mathematical structures in the neutrosophic setting. Besides providing emphasis on topics like artificial intelligence, pattern recognition, image processing, robotics, decision making, data analysis, data mining, applications of neutrosophic mathematical theories contributing to economics, finance, management, industries, electronics, and communications are promoted. Variants of neutrosophic sets including refined neutrosophic set (RNS). Articles evolving algorithms making computational work handy are welcome.

Topics of Interest

IJNS promotes research and reflects the most recent advances of neutrosophic Sciences in diverse disciplines, with emphasis on the following aspects, but certainly not limited to:

- Neutrosophic sets
- Neutrosophic algebra
- Neutrosophic topolog
- Neutrosophic graphs
- Neutrosophic probabilities
- Neutrosophic tools for decision making
- Neutrosophic theory for machine learning
- Neutrosophic statistics
- Neutrosophic numerical measures
- Classical neutrosophic numbers
- A neutrosophic hypothesis
- The neutrosophic level of significance
- The neutrosophic confidence interval
- The neutrosophic central limit theorem
- Neutrosophic theory in bioinformatics
- and medical analytics
- Neutrosophic tools for big data analytics
- Neutrosophic tools for deep learning
- Neutrosophic tools for data visualization
- Quadripartitioned single-valued
- neutrosophic sets
- Refined single-valued neutrosophic sets
- Applications of neutrosophic logic in image processing
- Neutrosophic logic for feature learning, classification, regression, and clustering

- Neutrosophic knowledge retrieval of medical images
- Neutrosophic set theory for large-scale image and multimedia processing
- Neutrosophic set theory for brain-machine interfaces and medical signal analysis
- Applications of neutrosophic theory in large-scale healthcare data
- Neutrosophic set-based multimodal sensor data
- Neutrosophic set-based array processing and analysis
- Wireless sensor networks Neutrosophic set-based Crowd-sourcing
- Neutrosophic set-based heterogeneous data mining
- Neutrosophic in Virtual Reality
- Neutrosophic and Plithogenic theories in Humanities and Social Sciences
- Neutrosophic and Plithogenic theories in decision making
- Neutrosophic in Astronomy and Space Sciences

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Concepts of Neutrosophic Complex Numbers

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Abstract

In this paper, concept of neutrosophic complex numbers and its properties were presented inculding the conjugate of neutrosophic complex number, division of neutrosophic complex numbers, the inverted neutrosophic complex number and the absolute value of a neutrosophic complex number. Theories related to the conjugate of neutrosophic complex numbers are proved, the product of a neutrosophic complex number by its conjugate equals the absolute value of number is also proved. This is an important introduction to define neutrosophic complex numbers in polar.

Keywords: Classical neutrosophic numbers, Neutrosophic complex numbers, Conjugate.

1. Introduction

As an alternative to the existing logics, Smarandache proposed the neutrosophic Logic to represent a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction, where the concept of neutrosophy is a new branch of philosophy introduced by Smarandache [3][7]. He presented the definition of the standard form of neutrosophic real number and conditions for the division of two neutrosophic real numbers to exist, he defined the standard form of neutrosophic complex number, and found root index $n \geq 2$ of a neutrosophic real and complex number [2][4], studying the concept of the Neutrosophic probability [3][5], the Neutrosophic statistics [4][6], and professor Smarandache entered the concept of preliminary calculus of the differential and integral calculus, where he introduced for the first time the notions of neutrosophic mereo-limit, mereo-continuity, mereoderivative, and mereo-integral [1][8]. Madeleine Al- Taha presented results on single valued neutrosophic (weak) polygroups [9]. Edalatpanah proposed a new direct algorithm to solve the neutrosophic linear programming where the variables and right hand side represented with triangular neutrosophic numbers [10]. Chakraborty used pentagonal neutrosophic number in networking problem, and Shortest Path Problem [11][12].

This paper aims to study neutrosophic logic in the complex numbers by defining the conjugate of neutrosophic complex number, division of neutrosophic complex numbers, the inverted neutrosophic complex number, the absolute value of a neutrosophic complex number, I also have proven theories related to the conjugate of neutrosophic complex numbers, and finally we proved the product of a neutrosophic complex number by its conjugate equals the absolute value of number.

2. Preliminaries

2.1 Neutrosophic Real Number [4]

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Suppose that w is a neutrosophic number, then it takes the following standard form: $w = a + bI$ where a , b are real coefficients, and I represent indeterminacy, such $0.\mathrm{I} = 0$ and $\mathrm{I}^n = \mathrm{I}$, for all positive integers n.

2.2 Neutrosophic Complex Number [4]

Suppose that z is a neutrosophic complex number, then it takes the following standard form: $z = a + cI + bi + diI$ where a , b, c, d are real coefficients, and I indeterminacy, such that $i^2 = -1 \Rightarrow i = \sqrt{-1}$.

Note: we can say that any real number can be considered a neutrosophic number.

For example: $2 = 2 + 0.I$, $or: 2 = 2 + 0.I + 0.i + 0.i.I$

2.3 Division of neutrosophic real numbers [4]

Suppose that w_1, w_2 are two neutrosophic numbers, where

$$w_1 = a_1 + b_1 I \ , \qquad w_2 = a_2 + b_2 I$$

To find $(a_1 + b_1 I) \div (a_2 + b_2 I)$, we can write:

$$\frac{a_1 + b_1 I}{a_2 + b_2 I} \equiv x + yI$$

where x and y are real unknowns.

$$a_1 + b_1 I \equiv (a_2 + b_2 I)(x + yI)$$

$$a_1 + b_1 I \equiv a_2 x + (b_2 x + a_2 y + b_2 y)I$$

by identifying the coefficients, we get

$$a_1 = a_2 x$$

$$b_1 = b_2 x + (a_2 + b_{2)} y$$

We obtain unique one solution only, provided that:

$$\begin{vmatrix} a_2 & 0 \\ b_2 & a_2 + b_2 \end{vmatrix} \neq 0 \quad \Rightarrow \quad a_2(a_2 + b_2) \neq 0$$

Hence: $a_2 \neq 0 \quad and \quad a_2 \neq -b_2$ are the conditions for the division of two neutrosophic real numbers to exist.

Then:

$$\frac{a_1 + b_1 I}{a_2 + b_2 I} = \frac{a_1}{a_2} + \frac{a_2 b_1 - a_1 b_2}{a_2(a_2 + b_2)}.I$$

2.4 Root index $n \geq 2$ of a neutrosophic real number [4]

1) Case: $n = 2$

Let $w = a + \mathrm{b}\mathrm{I}$ be a neutrosophic real number, then

$$\sqrt{a + \mathrm{bI}} = x + y.I$$

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$$a + \mathrm{bI} \equiv (x + y.I)^2$$

$a + \mathrm{bI} \equiv x^2 + 2xy.I + y^2 I$

by identifying the coefficients, we get:

$x^2 = a$

$y^2 + 2xy = b$

Hence $\quad x = \pm\sqrt{a}$

$y^2 \pm 2\sqrt{a}y - b = 0$

By solving the second equation in respect to y we find:

$$y = \frac{\mp 2\sqrt{a} \pm \sqrt{4a + 4\mathrm{b}}}{2} = \mp\sqrt{a} \pm \sqrt{a + \mathrm{b}}$$

Then we fined four solutions of $\sqrt{a + \mathrm{bI}}$:

$\sqrt{a + \mathrm{bI}} = \sqrt{a} + \left(-\sqrt{a} + \sqrt{a + \mathrm{b}}\right).I$

Or: $\quad \sqrt{a + \mathrm{bI}} = \sqrt{a} - \left(-\sqrt{a} + \sqrt{a + \mathrm{b}}\right).I$

Or: $\quad \sqrt{a + \mathrm{bI}} = -\sqrt{a} + \left(\sqrt{a} + \sqrt{a + \mathrm{b}}\right).I$

Or: $\quad \sqrt{a + \mathrm{bI}} = -\sqrt{a} + \left(\sqrt{a} - \sqrt{a + \mathrm{b}}\right).I$

particular case: $\sqrt{I} = \pm I$

2) Case: $n > 2$

$$\sqrt[n]{a + \mathrm{bI}} = x + y.I$$

$$a + \mathrm{bI} \equiv (x + y.I)^n$$

$$a + \mathrm{bI} \equiv x^n + \left(\sum_{k=0}^{n-1} C_n^k \, y^{n-k} \, x^k\right).I$$

$$x^n = a \;\Rightarrow \qquad x = \begin{cases} \sqrt[n]{a} & ;\ n\ odd \\ \pm\sqrt[n]{a} & ;\ n\ even \end{cases}$$

$$\sum_{k=0}^{n-1} C_n^k \, y^{n-k} \, a^{\frac{k}{n}} = b$$

Solve it in respect to y, we can distinguish two cases:

When the x and y solutions are real, we get neutrosophic real solutions,

When x and y solutions are complex, we get neutrosophic complex solutions.

2.5 Multiplying two neutrosophic complex numbers [2]

Let z_1 , z_2 are two neutrosophic complex numbers, where

$\mathrm{z_1 = a_1 + c_1 I + b_1 i + d_1 iI}$, $\mathrm{z_2 = a_2 + c_2 I + b_2 i + d_2 iI}$

Then:

$$z_1 . z_2 = (a_1 + c_1 I + b_1 i + d_1 iI\,)(a_2 + c_2 I + b_2 i + d_2 iI)$$

$$= (a_1 a_2 - b_1 b_2) + (a_1 c_2 + a_2 c_1 + c_1 c_2 - b_1 d_2 - d_1 b_2 - d_1 d_2)I$$

$$+(a_1 b_2 + a_2 b_1)i + (a_1 d_2 + c_1 b_2 + c_1 d_2 + b_1 c_2 + a_2 d_1 + d_1 c_2)i.I$$

Example 2.1:

$(3 + 5i + 2iI\,)(1 + 3iI) = 3 + 9iI + 5i - 15I + 2iI - 6I$

$= 3 - 21I + 5i + 11iI$

3. Conjugate of a neutrosophic complex number

Definition 3.1:

Suppose that z is a neutrosophic complex number, where $z = a + cI + bi + d.iI$. We denote the conjugate of a neutrosophic complex number by $\bar{z}$ and define it by the following form:

$$\bar{z} = a + cI - bi - d.iI$$

Example 3.1:

$$z = 4 + 5i - 7iI \quad \Rightarrow \quad \bar{z} = 4 - 5i + 7iI$$

$$z = -2I - i + 8iI \quad \Rightarrow \quad \bar{z} = -2I + i - 8iI$$

$$z = iI \quad \Rightarrow \quad \bar{z} = -iI$$

As consequences, we have:

1. the conjugate of neutrosophic complex number $\bar{z}$ is the same neutrosophic complex number z.

   $\overline{(\bar{z})} = z$
2. If $z = a + cI + bi + d.iI$

   then

   $z + \bar{z} = 2(a + cI) = 2\mathrm{Re}(z) \quad and \quad z - \bar{z} = 2(b + d.I)i = 2Im(z)$

where $Re(Z)$ is the real part of the complex number and $Im(Z)$ is the imagine

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3. We conclude from this, that the neutrosophic complex number is real if and only if $= \bar{z}$, and it is imaginary if and only if $z = -\bar{z}$.

Remark 3.1:

The conjugate of the sum of two neutrosophic complex numbers is equal to the sum of their two conjugates.

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

Proof:

Suppose that z_1 , z_2 are two neutrosophic complex numbers, where

$z_1 = a_1 + c_1 I + b_1 i + d_1 iI \quad , \quad z_2 = a_2 + c_2 I + b_2 i + d_2 iI$

Then:

$$z_1 + z_2 = (a_1 + a_2) + (c_1 I + c_2 I) + (b_1 + b_2) i + (d_1 + d_2) iI$$

$$\overline{z_1 + z_2} = (a_1 + a_2) + (c_1 I + c_2 I) - (b_1 + b_2) i - (d_1 + d_2) iI$$

$$= a_1 + c_1 I - b_1 i - d_1 iI + a_2 + c_2 I - b_2 i - d_2 iI$$

$$= \overline{z_1} + \overline{z_2}$$

Theorem 3.1

The conjugate of multiplication two neutrosophic complex numbers is equal to the multiplication of their two conjugates.

$$\overline{z_1 . z_2} = \overline{z_1} . \overline{z_2}$$

Proof:

Suppose that z_1 , z_2 are two neutrosophic complex numbers, where

$z_1 = a_1 + c_1 I + b_1 i + d_1 iI \quad , \quad z_2 = a_2 + c_2 I + b_2 i + d_2 iI$

Then:

$$z_1 . z_2 = (a_1 + c_1 I + b_1 i + d_1 iI)(a_2 + c_2 I + b_2 i + d_2 iI)$$

$$= (a_1 a_2 - b_1 b_2) + (a_1 c_2 + a_2 c_1 + c_1 c_2 - b_1 d_2 - d_1 b_2 - d_1 d_2) I$$

$$+ (a_1 b_2 + a_2 b_1) i + (a_1 d_2 + c_1 b_2 + c_1 d_2 + b_1 c_2 + a_2 d_1 + d_1 c_2) i . I$$

$$\overline{z_1 . z_2} = (a_1 a_2 - b_1 b_2) + (a_1 c_2 + a_2 c_1 + c_1 c_2 - b_1 d_2 - d_1 b_2 - d_1 d_2) I$$

$$- (a_1 b_2 + a_2 b_1) i - (a_1 d_2 + c_1 b_2 + c_1 d_2 + b_1 c_2 + a_2 d_1 + d_1 c_2) i . I$$

$$\overline{z_1} . \overline{z_2} = (a_1 + c_1 I - b_1 i - d_1 iI)(a_2 + c_2 I - b_2 i - d_2 iI)$$

$$= (a_1 a_2 - b_1 b_2) + (a_1 c_2 + a_2 c_1 + c_1 c_2 - b_1 d_2 - d_1 b_2 - d_1 d_2) I$$

$$- (a_1 b_2 + a_2 b_1) i - (a_1 d_2 + c_1 b_2 + c_1 d_2 + b_1 c_2 + a_2 d_1 + d_1 c_2) i . I$$

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$$\Rightarrow \quad \overline{z_1 . z_2} = \overline{z_1} . \overline{z_2}$$

4. Division of neutrosophic complex numbers

Suppose that z_1 , z_2 are two neutrosophic complex numbers, where

$z_1 = a_1 + c_1 I + b_1 i + d_1 iI$, $z_2 = a_2 + c_2 I + b_2 i + d_2 iI$; $z_2 \neq 0$

Then:

$$\frac{z_1}{z_2} = \frac{a_1 + c_1 I + b_1 i + d_1 iI}{a_2 + c_2 I + b_2 i + d_2 iI}$$

multiply the numerator and denominator by conjugate of z_2 we get:

$$\frac{z_1}{z_2} = \frac{(a_1 + c_1 I + b_1 i + d_1 iI)(a_2 + c_2 I - b_2 i - d_2 iI)}{(a_2 + c_2 I + b_2 i + d_2 iI)\,(a_2 + c_2 I - b_2 i - d_2 iI)}$$

$$= \frac{(a_1 + c_1 I + b_1 i + d_1 iI)(a_2 + c_2 I - b_2 i - d_2 iI)}{(a_2 + c_2 I)^2 + (b_2 + d_2 I)^2}$$

$$= \frac{(a_1 a_2 + b_1 b_2) + (a_1 c_2 + a_2 c_1 + c_1 c_2 + b_1 d_2 + d_1 b_2 + d_1 d_2) I}{(a_2 + c_2 I)^2 + (b_2 + d_2 I)^2} + \frac{(a_2 b_1 - a_1 b_2) i + (b_1 c_2 + a_2 d_1 + d_1 c_2 - a_1 d_2 - c_1 b_2 - c_1 d_2) i.I}{(a_2 + c_2 I)^2 + (b_2 + d_2 I)^2}$$

$$= \frac{(a_1 a_2 + b_1 b_2) + (a_1 c_2 + a_2 c_1 + c_1 c_2 + b_1 d_2 + d_1 b_2 + d_1 d_2) I}{(a_2 + c_2 I)^2 + (b_2 + d_2 I)^2} + \frac{(a_2 b_1 - a_1 b_2) + (b_1 c_2 + a_2 d_1 + d_1 c_2 - a_1 d_2 - c_1 b_2 - c_1 d_2) . I}{(a_2 + c_2 I)^2 + (b_2 + d_2 I)^2} . i$$

Example4.1:

$$\frac{3 + 5i + 2iI}{1 + 3iI}$$

Solution:

multiply the numerator and denominator by conjugate of $(1 - 3iI)$ we get:

$$\frac{3 + 5i + 2iI}{1 + 3iI} = \frac{(3 + 5i + 2iI)(1 - 3iI)}{(1 + 3iI)(1 - 3iI)} = \frac{3 - 21I + 5i + 11iI}{1 + 9I}$$

$$= \frac{3 - 21I}{1 + 9I} + \frac{5 + 11I}{1 + 9I} i \qquad (1)$$

Let us find:

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$$\frac{3-21I}{1+9I} \equiv x + yI$$

$3-21I \equiv (1+9I)(x+yI)$

$3-21I \equiv x+9xI+10yI$

$3-21I \equiv x+(9x+10y)I$

$$\Rightarrow \begin{cases} x=3 \\ 9x+10y=-21 \end{cases}$$

$$\Rightarrow \begin{cases} x=3 \\ 9(3)+10y=-21 \end{cases} \quad \Rightarrow \begin{cases} x=3 \\ y=-\dfrac{48}{10}=-4.8 \end{cases}$$

$$\Rightarrow \frac{3-21I}{1+9I} \equiv 3-4.8I$$

Let us find:

$$\frac{5+11I}{1+9I} \equiv x+yI$$

$$5+11I \equiv (1+9I)(x+yI)$$

$$5+11I \equiv x+9xI+10yI$$

$$5+11I \equiv x+(9x+10y)I$$

$$\Rightarrow \begin{cases} x=5 \\ 9x+10y=11 \end{cases}$$

$$\Rightarrow \begin{cases} x=3 \\ 9(5)+10y=11 \end{cases} \quad \Rightarrow \begin{cases} x=3 \\ y=-\dfrac{34}{10}=-3.4 \end{cases}$$

$$\Rightarrow \frac{5+11I}{1+9I} \equiv 5-3.4I$$

By substitution in (1):

$$\frac{3+5i+2iI}{1+3iI} = 3-4.8I+(5-3.4I)i$$

$$= 3-4.8I+5i-3.4iI$$

5. Inverted Neutrosophic complex number

Suppose that z is a neutrosophic complex number, where $z=a+cI+bi+d.iI$

Then:

$$\frac{1}{z} = \frac{1}{a+cI+bi+d.iI}$$

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$$= \frac{a + cI}{(a + cI)^2 + (b + dI)^2} - \frac{b - dI}{(a + cI)^2 + (b + dI)^2}\, i$$

Example 5.1:

$$\frac{1}{1 - 2iI} = \frac{1}{1 + 4I} + \frac{2I}{1 + 4I}\, i$$

$$= 1 - \frac{4}{5}I + \frac{2}{5}I.i$$

6. The absolute value of a neutrosophic complex number

Suppose that $z = a + cI + bi + d.iI$ is a neutrosophic complex number, the absolute value of a neutrosophic complex number defined by the following form:

$$|Z| = \sqrt{(a + cI)^2 + (b + dI)^2}$$

Example 6.1:

Let $z = 1 + 2I + iI$, find the absolute value of z .

Solution:

$$|Z| = \sqrt{(a + cI)^2 + (b + dI)^2}$$

$$= \sqrt{(1 + 2I)^2 + (I)^2}$$

$$= \sqrt{1 + 4I + 4I + I}$$

$$= \sqrt{1 + 10I}$$

$$\sqrt{1 + 10I} \equiv x + yI$$

$$1 + 10I \equiv x^2 + 2xyI + y^2$$

by identifying we get:

$$\begin{cases} x^2 = 1 \\ y^2 + 2xy = 10 \end{cases}$$

Since the absolute value is positive, we take: $x = 1$

By substitution in the second equation:

$$y^2 + 2y = 10 \implies y^2 + 2y - 10 = 0$$

$$y = \frac{-2 + 2\sqrt{11}}{2} = -1 + \sqrt{11} \approx 2.3$$

Therefore,

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$$|Z| = |1 + 2I + iI| = 1 + 2.3I$$

Theorem 6.1:

Suppose that $z = a + cI + bi + d.iI$ is a neutrosophic complex number, multiplication the absolute value of z by its conjugate equals to square of the absolute value of .

$$z.\bar{z} = |Z|^2$$

Proof:

$$z = a + cI + bi + d.iI \quad \Rightarrow \quad \bar{z} = a + cI - bi - d.iI$$

$$z.\bar{z} = (a + cI + bi + d.iI)(\,a + cI - bi - d.iI)$$

$$= a^2 + acI - abi - adiI + acI + c^2I - bciI - cdIi + abi$$

$$+bciI + b^2 + bdI + adiI + cdiI + bdI + d^2I$$

$$= (a^2 + 2acI + c^2I) + (b^2 + 2bdI + d^2I)$$

$$= (a + cI)^2 + (b + dI)^2 = |Z|^2$$

$$\Rightarrow \qquad z.\bar{z} = |Z|^2$$

Example 6.2:

Let $z = 4 - I + 2i + 3iI$, find $z.\bar{z}$.

Solution:

$$z.\bar{z} = |Z|^2$$

$$= (a + cI)^2 + (b + dI)^2$$

$$= (4 - I)^2 + (2 + 3I)^2$$

$$= 16 - 8I + I + 4 + 12I + 9I$$

$$= 20 + 14I$$

5. Conclusions

In this paper, conjugate of neutrosophic complex number was defiend and used to find the division of neutrosophic complex numbers, the inverted neutrosophic complex number and the absolute value of a neutrosophic complex number. This research has proven theorems related to the conjugate of neutrosophic complex numbers. This approach can be applied to define the polar form and exponential form of the neutrosophic complex number.

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Octagonal Neutrosophic Number: Its Different Representations, Properties, Graphs and De-neutrosophication with the application of Personnel Selection

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Abstract

To deal with fluctations in decision-making, fuzzy / neutrosophic numbers are used. The problem having more fluctuations are difficult to sovle. Thus it is a dire need to define higher order number, also It is a very curious question by researchers all around the world that how octagonal neutrosophic number can be represented and how to be graphed? In this research article, the primarily focused on the representation and graphs of octagonal neutrosophic number. at last, a case study is done using VIKOR method based on octagonal neutrosophic number. These representations will be helpful in multi-criteria decision making problems in the case that there are large number of fluctuations. Finally, concluded the present work with future directions.

Keywords: Neutrosophic Number, Octagonal Number, VIKOR Method, MCDM, Uncertainty, Indeterminacy, Accuracy Function, De-neutrosophication.

1. Introduction

The theory of uncertainty plays a very important role to solve different issues like modelling in engineering domain. To deal with uncertainty the first concept was given by [1], extended by [2] named as intuitionistic fuzzy numbers. In year 1995, Smarandache proposed the idea of neutrosophic set, and the idea was published in 1998 [3], they have three distinct logic components i) truthfulness ii) indeterminacy iii) falsity. This idea also has a concept of hesitation component the research gets a high impact in different research domain. In neutrosophic, truth membership is noted by T, indeterminacy membership is noted by I, falsity membership is noted by F, These are all independent and their sum is between $0 \leq T + I + F \leq 3$. While when talking about intuitionistic fuzzy sets, uncertainty depends on the degree of membership and non-membership, but in neutrosophic sets then indeterminacy factor does not depend on the truth and falsity value. Neutrosophic fuzzy number can describe about the uncertainty, falsity and hesitation information of real-life problem.

Researchers from different fields developed triangular, trapezoidal and pentagonal neutrosophic numbers, and presented the notions, properties along with applications in different fields [4-6]. The de-neutrosophication technique of pentagonal number and its applications are presented by [7-10].

Scientists from different areas investigated the various properties and fluctuations of neutrosophic numbers and the properties of correlation between these numbers [6-7]. The applications in decision-making in different fields like phone selection [11-12], games prediction [13], supplier selection [14-16], medical [17], personnel selection [18-19].

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Octagonal neutrosophic number and its types are presented by [20] in his recent work. The graphical representation and properties are yet to be defined while dealing with the concept of octagonal neutrosophic number a decision-maker can solve more fluctuations because they have more edges as compare to pentagonal. Table:1 represents different numbers and their applicability.

Edge Parameter	Uncertainty Measurement	Hesitation Measurement	Vagueness Measurement	Fluctuations
Crisp number	*	*	*	*
Fuzzy number	determinable	*	*	*
Intuitionistic Fuzzy number	determinable	determinable	*	*
Neutrosophic number	determinable	determinable	determinable	determinable

Table 1: Fuzzy numbers, their extensions and applicability

1.1 Motivation

From the literature, it is found that octagonal neutrosophic numbers (ONN) their notations, graphs and properties are not yet defined. Since it is not yet defined so also it will be a question that how and where it can be applied? For this purpose, is de-neutrosophication important? How should we define membership, indeterminacy and non-membership functions? From this point of view ONN is a good choice for a decision maker in a practical scenario.

1.2 Novelties

The work contributed in this research is;

- Membership, Non-membership and Indeterminacy functions
- Graphical Representation of ONN.
- De-neutrosophication technique of ONN.
- Case study of personnel selection having octagonal fluctuations.

1.3 Structure of Paper

The article is structured as follows as shown in the Figure 1:

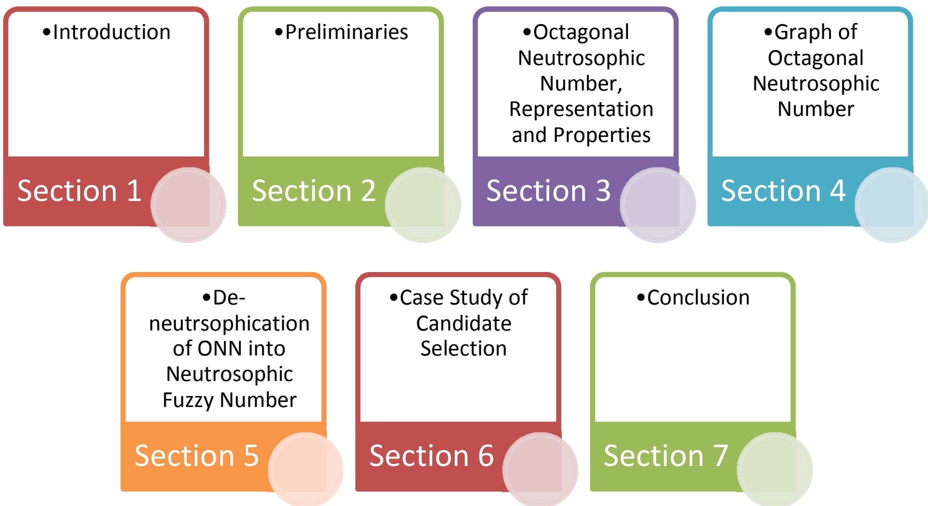

Figure 1: Pictorial view of the structure of the article

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2. Preliminaries

Definition 2.1: Fuzzy Number [1]

A fuzzy number is generalized form of a real number. It doesn't represent a single value, instead a group of values, where each entity has its membership value between $[0, 1]$. Fuzzy number $\bar{S}$ is a fuzzy set in R if it satisfies the given conditions.

- ∃ relatively one $y \in R$ with $\mu_{\bar{S}}$ (y) = 1.
- $\mu_{\bar{S}}$ (y) is piecewise continuous.
- $\bar{S}$ should be convex and normal.

Definition 2.2: Neutrosophic Fuzzy Number [3]

Let U be a universe of discourse then the neutrosophic set A is an object having the form
A = {< x: $\mathrm{T_A}\ (x), \mathrm{I_A}(x), \mathrm{F_A}\ (x),$ >; x ∈ U}
where the functions T, I, F : U→ [0,1] define respectively the degree of membership, the degree of indeterminacy, and the degree of non-membership of the element x ∈ X to the set A with the condition. $0 \le \mathrm{T_A}\ (x) + \mathrm{I_A}\ (x) + \mathrm{F_A}\ (x) \le 3$.

Definition 2.3: Accuracy Function [21]

Accuracy function is used to convert neutrosophic number NFN into fuzzy number (De-neutrosophication using $\boldsymbol{A_F}$).
$\mathrm{A(F)} = \{ x = \frac{[T_x + I_x + F_x]}{3} \}$

$\boldsymbol{A_F}$ represents the De-neutrosophication of neutrosophic number into fuzzy number.

Definition 2.4: Pentagonal Neutrosophic Number [6]

Pentagonal Neutrosophic Number PNN is defined as,

$$PNN = \langle [(\Omega, \square, \text{ƹ}, \text{ɤ}, \varepsilon) : \Theta], [(\Omega^1, \square^1, \text{ƹ}^1, \text{ɤ}^1, \varepsilon^1) : \Psi], [(\Omega^2, \square^2, \text{ƹ}^2, \text{ɤ}^2, \varepsilon^2) : \quad] \rangle$$

Where Θ, Ψ , ∈ [0,1].

The truth membership function (Θ): ℝ → [0, б],

the indeterminacy membership function (Ψ):□ → [ȥ ,1],

and the falsity membership function ( ):ℝ→ [,1].

3. Octagonal Neutrosophic Number [ONN] Representation and Properties

In this section, we define ONN, representations and properties along with suitable examples.

Definition 3.1: Side Conditions of Octagonal Neutrosophic Number [ONN]

An Octagonal Neutrosophic Number denoted by;

$\hat{S}$ $\langle [(\Omega, \square, \text{ƹ}, \text{ɤ}, \varepsilon, \ , \text{ό}, \text{з}) : \Theta], [(\Omega^1, \square^1, \text{ƹ}^1, \text{ɤ}^1, \varepsilon^1, \ ^1, \text{ό}^1, \text{з}^1) : \Psi], [(\Omega^2, \square^2, \text{ƹ}^2, \text{ɤ}^2, \varepsilon^2, \ ^2, \text{ό}^2, \text{з}^2) : \quad] \rangle$ should satisfy the following conditions:

Condition 1:

1. $\Theta_{\hat{s}}$: truth membership function ($\Theta_{\hat{s}}$): ℝ→ [0,1],
2. $\Psi_{\hat{s}}$: indeterminacy membership function ($\Psi_{\hat{s}}$):ℝ→ [ȥ ,1],
3. $_{\hat{s}}$: falsity membership function ($_{\hat{s}}$):ℝ→ [,1].

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Condition 2:

1. $\Theta_{ŝ}$: truth membership function is strictly non-decreasing continuous function on the intervals $[\Omega, \varepsilon]$.
2. $\Psi_{ŝ}$: indeterminacy membership function is strictly non-decreasing continuous function on the intervals $[\Omega^1, \varepsilon^1]$.
3. $_{ŝ}$: falsity membership function is strictly non-decreasing continuous function on the intervals $[\Omega^2, \varepsilon^2]$.

Condition 3:

1. $\Theta_{ŝ}$: truth membership function is strictly non-increasing continuous function on the intervals $[\varepsilon, з]$.
2. $\Psi_{ŝ}$: indeterminacy membership function is strictly non-increasing continuous function on the intervals $[\varepsilon^1, з^1]$.
3. $_{ŝ}$: falsity membership function is strictly non-increasing continuous function on the intervals $[\varepsilon^2, з^2]$.

Definition 3.2 : Octagonal Neutrosophic Number [ONN] A Neutrosophic Number denoted by $\widehat{S}$ is defined as,

$\widehat{S}$=⟨[(Ω, □, ɛ̧, ɤ, ε, , ó, з): Θ], [(Ω¹, □¹, ɛ̧¹, ɤ¹, ε¹, ¹, ó¹, з¹): Ψ], [(Ω², □², ɛ̧², ɤ², ε², ², ó², з²):] ⟩

Where Θ, Ψ ,   ∈ [0,1].

The truth membership function ($\Theta_{ŝ}$): ℝ→ [0,1],

the indeterminacy membership function ($\Psi_{ŝ}$):ℝ→ [ƺ ,1],

and the falsity membership function ($_{ŝ}$):ℝ→ [,1] are given as:

$$\Theta_{ŝ}(x)=\begin{cases} \Theta_{ŝ0}(x) & \Omega \le x < □ \\ \Theta_{ŝ1}(x) & □ \le x < ɛ̧ \\ \Theta_{ŝ2}(x) & ɛ̧ \le x < ɤ \\ \Theta_{ŝ3}(x) & ɤ \le x < \varepsilon \\ б & x = \varepsilon \\ \Theta_{ŝ3}(x) & \varepsilon \le x < \\ \Theta_{ŝ2}(x) & \le x < ó \\ \Theta_{ŝ1}(x) & ó \le x < з \\ 0 & otherwise \end{cases}$$

$$\Psi_{ŝ}(x)=\begin{cases} \Psi_{ŝ0}(x) & \Omega^1 \le x < □^1 \\ \Psi_{ŝ1}(x) & □^1 \le x < ɛ̧^1 \\ \Psi_{ŝ2}(x) & ɛ̧^1 \le x < ɤ^1 \\ \Psi_{ŝ3}(x) & ɤ^1 \le x < \varepsilon^1 \\ ƺ & x = \varepsilon^1 \\ \Psi_{ŝ3}(x) & \varepsilon^1 \le x < {}^1 \\ \Psi_{ŝ2}(x) & {}^1 \le x < ó^1 \\ \Psi_{ŝ1}(x) & ó^1 \le x < з^1 \\ 1 & otherwise \end{cases}$$

$$
{}_{\hat{s}}(x)=\begin{cases} {}_{\hat{s}0}(x) & \Omega^2 \le x < \square^2 \\ {}_{\hat{s}1}(x) & \square^2 \le x < \text{ɛ̧}^2 \\ {}_{\hat{s}2}(x) & \text{ɛ̧}^2 \le x < \text{ꝟ}^2 \\ {}_{\hat{s}3}(x) & \text{ꝟ}^2 \le x < \varepsilon^2 \\ & x = \varepsilon^2 \\ {}_{\hat{s}3}(x) & \varepsilon^2 \le x < {}^2 \\ {}_{\hat{s}2}(x) & {}^2 \le x < \text{ó}^2 \\ {}_{\hat{s}1}(x) & \text{ó}^2 \le x < \text{з}^2 \\ 1 & otherwise \end{cases}
$$

Where $\hat{S}=\langle [(\Omega < \square < \text{ɛ̧} < \text{ꝟ} < \varepsilon < \quad < \text{ó} < \text{з}) : \Theta], \left[\left(\Omega^1 < \square^1 < \text{ɛ̧}^1 < \text{ꝟ}^1 < \varepsilon^1 < \quad {}^1 < \text{ó}^1 < \text{з}^1\right) : \Psi\right], [\left(\Omega^2 < \square^2 < \text{ɛ̧}^2 < \text{ꝟ}^2 < \varepsilon^2 < \quad {}^2 < \text{ó}^2 < \text{з}^2\right) : \quad] \rangle$

4. Graphical Representation of Octagonal Neutrosophic Number [ONN]

In this section, graphs of truthiness, indeterminacy and falsity function are presented.

Definition 4.1: Octagonal Neutrosophic Number [ONN]

$$
\Theta_{\hat{s}}(x)=\begin{cases} \Theta_{\hat{s}0}(0) & 0.1 \le x < 0.2 \\ \Theta_{\hat{s}1}(0) & 0.2 \le x < 0.3 \\ \Theta_{\hat{s}2}(0.1) & 0.3 \le x < 0.4 \\ \Theta_{\hat{s}3}(0.1) & 0.4 \le x < 0.5 \\ 1 & x = 0.5 \\ \Theta_{\hat{s}3}(1) & 0.5 \le x < 0.6 \\ \Theta_{\hat{s}2}(0.1) & 0.6 \le x < o.7 \\ \Theta_{\hat{s}1}(0.1) & 0.7 \le x < 0.8 \\ 0 & otherwise \end{cases}
$$

$$
\Psi_{\hat{s}}(x)=\begin{cases} \Psi_{\hat{s}0}(1) & 0.1 \le x < 0.2 \\ \Psi_{\hat{s}1}(1) & 0.2 \le x < 0.3 \\ \Psi_{\hat{s}2}(0.9) & 0.3 \le x < 0.4 \\ \Psi_{\hat{s}3}(0.9) & 0.4 \le x < 0.5 \\ 0 & x = 0.5 \\ \Psi_{\hat{s}3}(0) & 0.5 \le x < 0.6 \\ \Psi_{\hat{s}2}(0.9) & 0.6 \le x < o.7 \\ \Psi_{\hat{s}1}(0.9) & 0.7 \le x < 0.8 \\ 1 & otherwise \end{cases}
$$

$$
{}_{\hat{s}}(x)=\begin{cases} {}_{\hat{s}0}(1) & 0.1 \le x < 0.2 \\ {}_{\hat{s}1}(1) & 0.2 \le x < 0.3 \\ {}_{\hat{s}2}(0.9) & 0.3 \le x < 0.4 \\ {}_{\hat{s}3}(0.9) & 0.4 \le x < 0.5 \\ 0 & x = 0.5 \\ {}_{\hat{s}3}(0) & 0.5 \le x < 0.6 \\ {}_{\hat{s}2}(0.9) & 0.6 \le x < o.7 \\ {}_{\hat{s}1}(0.9) & 0.7 \le x < 0.8 \\ 1 & otherwise \end{cases}
$$

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4.1 Graphical Representation of Membership, Non-membership, Indeterminacy and ONN

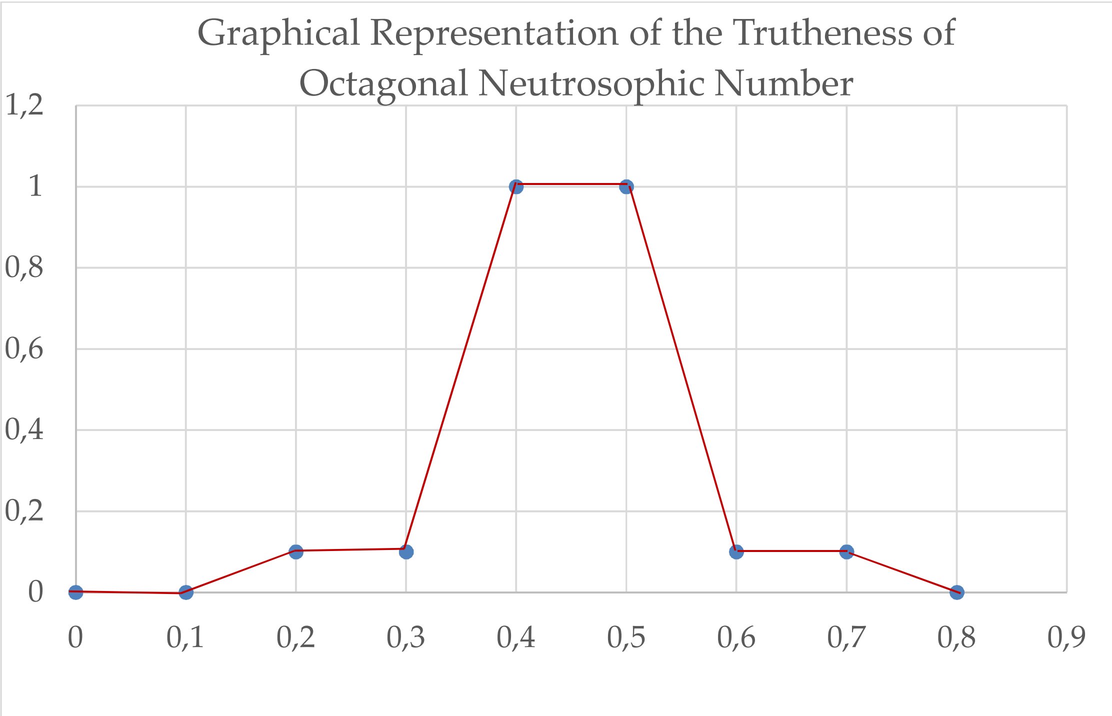

Figure 2: Graphical representation of the truthiness of ONN

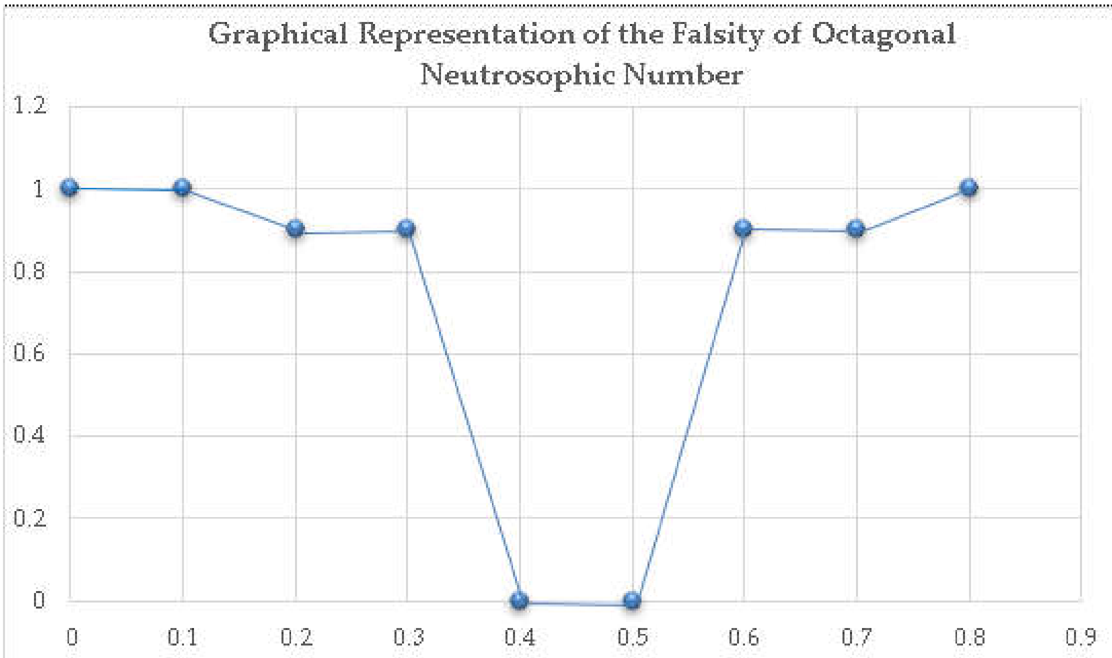

Figure 3: Graphical representation of the Falsity of ONN

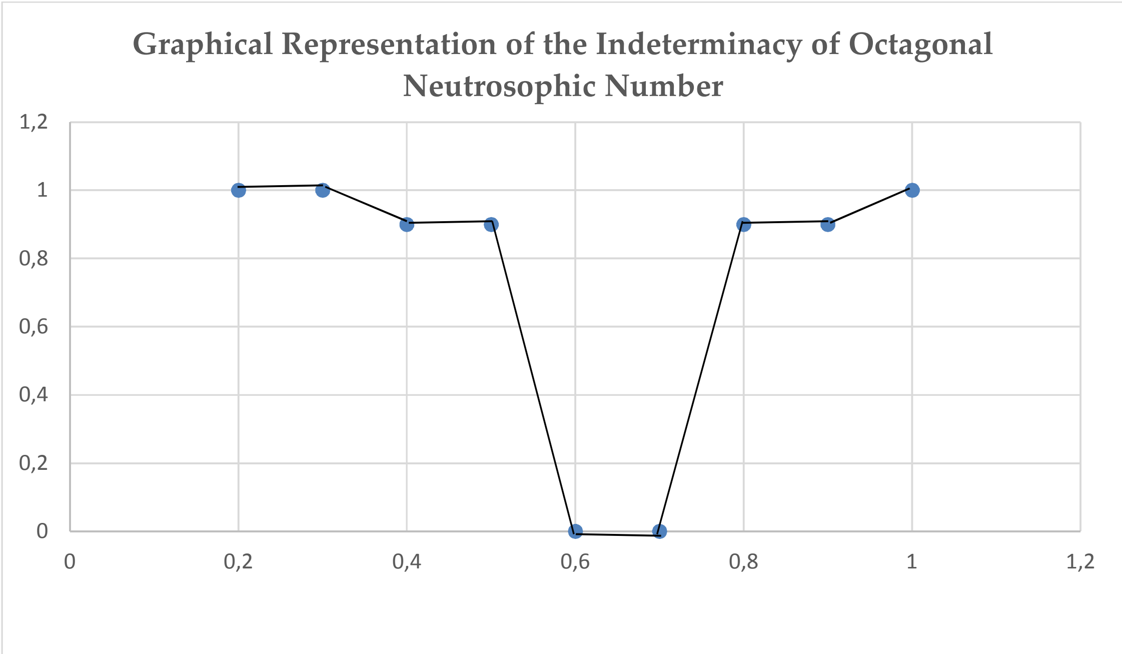

Figure 4: Graphical representation of the Indeterminacy of ONN

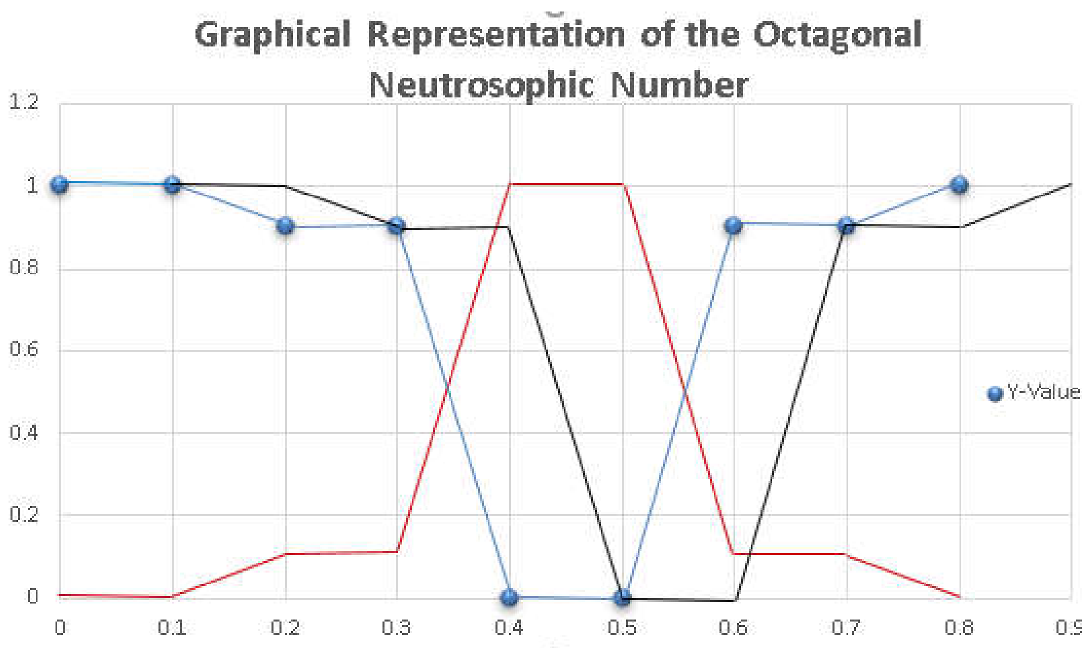

Figure 5: Graphical representation of the Octagonal Neutrosophic Number

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5. Accuracy Function for De-neutrosophication of Octagonal Neutrosophic Number (ONN)

5.1 De-neutrosophication of ONN into Neutrosophic Number

On the way of development of De-neutrosophication technique, we can generate results into neutrosophic number according to the result of octagonal neutrosophic number and its membership functions.

$$D^{T_{NO_N}}=\left(\frac{\Omega+\square+\text{ȩ}+\text{ꝟ}+\varepsilon+\ \ +\text{ó}+\text{з}}{8}\right),$$

$$D^{I_{NO_N}} = \left(\frac{\Omega^1+\square^1+\text{ȩ}^1+\text{ꝟ}^1+\varepsilon^1+\ ^1+\text{ó}^1+\text{з}^1}{8}\right),$$

$$D^{F_{NO_N}} = \left(\frac{\Omega^2+\square^2+\text{ȩ}^2+\text{ꝟ}^2+\varepsilon^2+\ ^2+\text{ó}^2+\text{з}^2}{8}\right)$$

$$D_{NO_N} = \left\{\frac{\Omega+\square+\text{ȩ}+\text{ꝟ}+\varepsilon+\ \ ,\text{ó},\text{з}}{8},\frac{\Omega^1+\square^1+\text{ȩ}^1+\text{ꝟ}^1+\varepsilon^1+\ ^1+\text{ó}^1+\text{з}^1}{8},\frac{\Omega^2+\square^2+\text{ȩ}^2+\text{ꝟ}^2+\varepsilon^2+\ ^2+\text{ó}^2+\text{з}^2}{8}\right\}$$

$$D_{NO_F}=\frac{D^{T_{NO_N}}+D^{I_{NO_N}}+D^{F_{NO_N}}}{3},$$

- $D^{T_{NO_N}}$ represents the de-neutrosophication of trueness of neutrosophic octagonal number into neutrosophic.
- $D^{I_{NO_N}}$ represents the de-neutrosophication of indeterminacy of neutrosophic octagonal number into neutrosophic.
- $D^{F_{NO_N}}$ represents the de-neutrosophication of falsity of neutrosophic octagonal number into neutrosophic.
- D_{NO_N} represents the de-neutrosophication of octagonal number into neutrosophic number.

Example 1: In Table: 3 five octagonal neutrosophic numbers ONN are defuzzified into Neutrosophic Number.

	Octagonal Neutrosophic Number	D_{NO_N}
1	(0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8;0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9; 0.1,0.3,0.4,0.5,0.6,0.7,0.8,0.9)	(0.45,0.55,0.5375)
2	(0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9;0.1,0.3,0.4,0.5,0.6,0.7,0.8,0.9; 0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9)	(0.55,0.5375,0.55)
3	(0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.9;0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8; 0.1,0.2,0.4,0.5,0.6,0.7,0.8,0.9)	(0.4625,0.45,0.525)
4	(0.1,0.2,0.4,0.5,0.6,0.7,0.8,0.9; 0.1,0.3,0.4,0.5,0.6,0.7,0.8,0.9; 0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9)	(0.45,0.5375,0.55)
5	(0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9; 0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8; 0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.9)	(0.55,0.45,0.4625)

Table 2: De-neutrosophication of ONN into Neutrosophic number using Accuracy Function.

5.2 De-neutrosophication of Neutrosophic Number

On the way of development of de-Neutrosophication technique, we can generate results into fuzzy number according to the result of neutrosophic number.

$$D_{NO_F}=\frac{D^{T_{NO_N}}+D^{I_{NO_N}}+D^{F_{NO_N}}}{3},$$

D_{NO_F} represents the de-neutrosophication of octagonal number into fuzzy number.

Example 2: In Table: 3 five octagonal neutrosophic numbers are defuzzified into Fuzzy.

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	Octagonal Neutrosophic Number	D_{NO_N}	D_{NO_F}
1	(0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8;0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9; 0.1,0.3,0.4,0.5,0.6,0.7,0.8,0.9)	(0.45,0.55,0.5375)	0.5125
2	(0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9;0.1,0.3,0.4,0.5,0.6,0.7,0.8,0.9; 0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9)	(0.55,0.5375,0.55)	0.54583
3	(0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.9;0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8; 0.1,0.2,0.4,0.5,0.6,0.7,0.8,0.9)	(0.4625,0.45,0.525)	0.47916
4	(0.1,0.2,0.4,0.5,0.6,0.7,0.8,0.9; 0.1,0.3,0.4,0.5,0.6,0.7,0.8,0.9; 0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9)	(0.45,0.5375,0.55)	0.5125
5	(0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9; 0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8; 0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.9)	(0.55,0.45,0.4625)	0.4875

Table 3: De-neutrosophication of ONN using Accuracy Function.

6. Case Study

To demonstrate the;

- Feasibility
- Productiveness

of the proposed method, here is the most useful real-life candidate selection problem is presented.

6.1 Problem Formulation

Suppose we have three candidates which have different degree, experience and number of publications, the thing which matter the most to select one which have more potential to deal with situation. The potential of person depends upon degree, experience and number of publications they have. To improve the competitiveness capability, the best selection plays an important role, and to select the best one. Due to octagonal we can deal with more fluctuations. The background of formal education comparison also necessary. Same case for experience because it illustrates the personality and also mention that person is capable to handle the circumstances. Same as publications is also important for selection. With the concept of octagonal we have more expanse to deal with more edges. Suppose we are talking about degree we can mention his all necessary degrees with grades.

6.2 Parameters

Selection is a complex issue, to resolve this problem criteria and alternative plays an important role. Following criteria and alternatives are considered in this problem formulation.

6.2.1 Alternatives

Candidates are considered as the set of alternatives represented with $\check{S} = < \zeta, \varpi, \upsilon >$

6.2.2. Criteria

Following three criteria are considered for the selection

- Degree
- Experience
- Publications

6.3 Assumptions

The decision makers $\{\mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_3, \mathfrak{D}_4, \mathfrak{D}_5, \mathfrak{D}_6, \mathfrak{D}_7, \mathfrak{D}_8\}$ will assign ONN, according to his own interest, knowledge and experience, to the above-mentioned criteria and alternatives.

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- Assigning Octagonal Neutrosophic Number ONN, by decision makers to the candidate ζ.

Sr # No	Criteria	Octagonal Neutrosophic Number (ONN)
1	Degree	< (0.72,0.35,0.71,0.77,0.41,0.73,0.77,0.81), (0.93,0.83,0.93,0.88 ,0.94,0.99,0.96,0.90), (0.86,0.95,0.99,0.97,0.94,0.93,0.95,0.91) >
2	Experience	< (0.75,0.65,0.96,0.54,0.73,0.65,0.83,0.56), (0.75,0.45,0.95,0.38,0.68,0.79,0.57,0.13), (0.36,0.59,0.68,0.79,0.47,0.36,0.47,0.95) >
3	Publications	< (0.74,0.73,0.64,0.75,0.96,0.34,0.85,0.89), (0..35,0.46,0.58,0.79,0.85,0.71,0.64,0.96), (0.84,0.73,0.85,0.75,0.98,0.84,0.66,0.94)>

Table 4(a): ONN by decision makers to each criterion to the candidate ζ.

- Assigning Octagonal Neutrosophic Number ONN, by decision makers to the candidate ϖ.

Sr # No	Criteria	Octagonal Neutrosophic Number (ONN)
1	Degree	< (0.73,0.73,0.94,0.85,0.96,0.74,0.95,0.89), (0.33,0.46,0.59,0.79 ,0.85,0.79,0.74,0.86), (0.48,0.33,0.55,0.75,0.68,0.64,0.36,0.70) >
2	Experience	< (0.75,0.55,0.96,0.54,0.93,0.65,0.73,0.56), (0.93,0.83,0.83,0.58,0.84,0.69,0.76,0.80), (0.66,0.59,0.68,0.99,0.47,0.46,0.87,0.95) >
3	Publications	< (0.94,0.93,0.74,0.95,0.96,0.94,0.85,0.99), (0.28,0.26,0.58,0.35,0.45,0.61,0.64,0.36), (0.28,0.23, 0.25, 0.45, 0.68, 0.44, 0.26, 0.34)>

Table 4(b): ONN by decision makers to each criterion to the candidate ϖ.

- Assigning Octagonal Neutrosophic Number ONN, by decision makers to the candidate υ.

Sr # No	Criteria	Octagonal Neutrosophic Number (ONN)
1	Degree	< (0.73,0.83,0.93,0.56,0.95,0.95,0.73,0.88), (0.76,0.95,0.69,0.94,0.94,0.63,0.55,0.61), (0.74,0.73,0.85,0.75,0.48,0.34,0.66,0.74) >
2	Experience	< (0.73,0.65,0.96,0.54,0.63,0.65,0.81,0.59), (0.75,0.45,0.85,0.38,0.78,0.79,0.67,0.13), (0.38,0.59,0.68,0.79,0.97,0.36,0.67,0.85) >
3	Publications	< (0.74,0.73,0.64,0.75,0.96,0.34,0.85,0.89), (0.35,0.44,0.58,0.79,0.75,0.71,0.54,0.96), (0.74,0.63,0.35,0.35,0.98,0.34,0.28,0.64)>

Table 4(c): ONN by decision makers to each criterion to the candidate υ.

6.4 VIKOR Method

Vikor method is best for solve the problem of multi criteria decision making.it is used to drive on ranking and for selection of a set of possibilities and solve consolation solution for a problem with aggressive criteria. Opricovic [12] introduced the idea of Vikor method in 1998. It is related with both positive and the negative ideal solution, it can change the variable into two or more alternative variables to find out the best compromise solution. By the help of Vikor method we can put new ideas for group decision making problem under the certain criteria.

Vikor Method consist of following steps;

Step 1. Normalization of decision matrix and weight assigning.

Step 2. Now we will calculate the group unity value $\breve{H}_i$=$[H_i^L, H_i^U]$ and the individual regard value $\breve{\mathbb{S}}_i$=$[\mathbb{S}_i^L, \mathbb{S}_i^U]$, where;

$$H_i^L = \sum_j w_j \frac{\mathbb{S}_i^L - \mathbb{S}_j^+}{\mathbb{S}_i^- - \mathbb{S}_j^+}, \; H_i^U = \sum_j w_j \frac{\mathbb{S}_i^U - \mathbb{S}_j^+}{\mathbb{S}_i^- - \mathbb{S}_j^+}$$

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And

$S_i^L = \max_{1\le j\le n} w_j \left\{\left(\frac{s_{ij}^L - s_j^+}{s_j^- - s_j^+}\right)\right\}$, $\mathbb{S}_i^U = \max_{1\le j\le n} w_j \left\{\left(\frac{\mathbb{S}_{ij}^U - \mathbb{S}_j^+}{\mathbb{S}_j^- - \mathbb{S}_j^+}\right)\right\}$

Step 3. Here we will Calculate the comprehensive sorting index $\widetilde{W}_i$=[W_i^L, W_i^U], where

$\widetilde{W}_i$=σ$\frac{\widetilde{H}_i - H^*}{H^- - H^*}$+(1- σ) $\frac{\widetilde{\mathbb{S}}_i - \mathbb{S}^*}{\mathbb{S}^- - \mathbb{S}^*}$

Now by using algorithm of interval fuzzy number:

W_i^L=σ$\frac{H_i^L - H^*}{H^- - H^*}$+(1- σ) $\frac{\mathbb{S}_i^L - \mathbb{S}^*}{\mathbb{S}^- - \mathbb{S}^*}$

and

W_i^U=σ$\frac{H_i^U - H^*}{H^- - H^*}$+(1- σ) $\frac{\mathbb{S}_i^L - \mathbb{S}^*}{\mathbb{S}^- - \mathbb{S}^*}$

Here $H^* = \min_i H_i^L$, $H^- = \max_i H_i^U$, $\mathbb{S}^* = \min_i \mathbb{S}_i^L$, $\mathbb{S}^* = \max_i \mathbb{S}_i^U$. Parameter σ is called decision mechanism index, and it lies between [0,1]. If σ >0.5, it is the decision making in the light of maximum group benefit (i.e., if σ is big, group utility is emphasized); if σ=0.5, here decision making in accordance with compromise. If σ <0.5, it is the decision making in the light of minimum individual regret value. In VIKOR, we take σ =0.5 generally, that is called compromise makes maximum group benefit and minimum individual regret value.

Step 4. The rank of fuzzy numbers is $\widetilde{\mathbb{S}}_i$, $\widetilde{W}_i$ *and* $\widetilde{H}_i$.

Since $\widetilde{\mathbb{S}}_i$, $\widetilde{W}_i$ *and* $\widetilde{H}_i$ are all still individual numbers, now to compare the two-interval value we use the possible degree theory.

Here number of interval number $\breve{A}_i$= [A_i^L , A_i^U], (i=1,2, 3,…,m), the comparison steps are given of these interval numbers;

(a) For any two intervals numbers $\breve{A}_i$=[A_i^L , A_i^U] and $\breve{A}_j$=[A_j^L , A_j^U], now we will calculate the possible degree ρ_{ij}= $\rho(\breve{A}_i \ge \breve{A}_j)$ and now we will construct the possible degree matrix $\rho = (\rho_{ij})_{m\times m}$, and the product by comparison of any two interval numbers $\breve{A}_i$=[A_i^L , A_i^U] and $\breve{A}_j$=[A_j^L , A_j^U], where i,j=1,2,3,….,m. Xu [18] proved that matrix $\rho = (\rho_{ij})_{m\times m}$ satisfies ($\rho_{ij} \ge 0, \rho_{ij}$ + (ρ_{ji}=1, ρ_{ii} = 0.5 (i,j=1,2,3,…,m)

The matrix $\rho = (\rho_{ij})_{m\times m}$ is called the fuzzy complementary judgement matrix, and we can rank the alternatives as follow.

(b) The rank of interval numbers $\breve{A}_i$= [A_i^L , A_i^U], (i=1,2,3,,m)
Ranking formula is given below
$\mathcal{U}_i = \frac{1}{m(m-1)}\left(\sum_{j=1}^{m}\rho_{ij} + \frac{m}{2} - 1\right)$,i=1,2,3,…,m
The smaller $\mathcal{U}_i$, is the smaller $\breve{A}_i$= [A_i^L , A_i^U] is.

Step 5. Now we will rank the alternatives based on $\widetilde{\mathbb{S}}_i$, $\widetilde{W}_i$ *and* $\widetilde{H}_i$(i=1,2,3,…,m).here the smaller of interval number $\widetilde{\mathbb{S}}_i$ is, and the better alternative x_i is. propose as a min {$\widetilde{\mathbb{S}}_i$ | i=1,2,3,…,m} if these two condition are satisfied[16]:

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(i) $\mathbb{S}\,(A^{(2)}) - \mathbb{S}\,(A^{(1)}) \geq 1/(m-1)$, where $A^{(2)}$ called the second alternative with second position in the ranking list by $\mathcal{R}$; m is the number of alternatives.

(ii) $A^{(1)}$alternative also must be best ranked by $\{\breve{\mathbb{S}}_i or/and\, \mathcal{R}_i \mid I = 1,2,3\ldots m\}$.

Figure 6: Flowchart of VIKOR algorithm

6.5 Numerical Analysis

Suppose that U is the universal set. Let HR which is responsible for recruiting and interviewing, and wants to hire a new candidate in company. Three candidates $\breve{S} = < \zeta, \varpi, \upsilon >$ apply for this opportunity, which have different degrees, experiences and publications. On the base of choice parameters $\{\mathbb{C}_1 = \text{Dergre}, \mathbb{C}_2 = \text{Experience}, \mathbb{C}_3 = \text{Publication}\}$ we apply the algorithm to find the potential candidate.

Step 1. Associated Decision Matrix

$$
\begin{array}{ccc}
\text{Candidate}=\zeta & \text{Candidate}=\varpi & \text{Candidate}=\upsilon
\end{array}
$$

$$
\begin{pmatrix}
\{\mathbb{C}_1(0.72,0.35,0.71,0.77,0.41,0.73,0.77,0.81) & \{\mathbb{C}_1(0.73,0.73,0.94,0.85,0.96,0.74,0.95,0.89) & \{\mathbb{C}_1(0.73,0.83,0.93,0.56,0.95,0.95,0.73,0.88) \\
(0.93,0.83,0.93,0.88\ ,0.94,0.99,0.96,0.90) & (0.33,0.46,0.59,0.79,0.85,0.79,0.74,0.86) & (0.76,0.95,0.69,0.94,0.94,0.63,0.55,0.61)\} \\
(0.86,0.95,0.99,0.97,0.94,0.93,0.95,0.91)\} & (0.48,0.33,0.55,0.75,0.68,0.64,0.36,0.70)\} & (0.74,0.73,0.85,0.75,0.48,0.34,0.66,0.74)\} \\
\{\mathbb{C}_2(0.75,0.65,0.96,0.54,0.73,0.65,0.83,0.56) & \{\mathbb{C}_2(0.75,0.55,0.96,0.54,0.93,0.65,0.73,0.56) & \{\mathbb{C}_2(0.73,0.65,0.96,0.54,0.63,0.65,0.81,0.59) \\
(0.75,0.45,0.95,0.38,0.68,0.79,0.57,0.13) & (0.93,0.83,0.83,0.58\ ,0.84,0.69,0.76,0.80) & (0.75,0.45,0.85,0.38,0.78,0.79,0.67,0.13) \\
(0.36,0.59,0.68,0.79,0.47,0.36,0.47,0.95)\} & (0.66,0.59,0.68,0.99,0.47,0.46,0.87,0.95)\} & (0.38,0.59,0.68,0.79,0.97,0.36,0.67,0.85)\} \\
\{\mathbb{C}_3(0.74,0.73,0.64,0.75,0.96,0.34,0.85,0.89) & \{\mathbb{C}_3(0.94,0.93,0.74,0.95,0.96,0.94,0.85,0.99) & \{\mathbb{C}_3(0.74,0.73,0.64,0.75,0.96,0.34,0.85,0.89) \\
(0..35,0.46,0.58,0.79,0.85,0.71,0.64,0.96) & (0.28,0.26,0.58,0.35,0.45,0.61,0.64,0.36) & (0.35,0.44,0.58,0.79,0.75,0.71,0.54,0.96) \\
(0.84,0.73,0.85,0.75,0.98,0.84,0.66,0.94)\} & (0.28,0.23,0.25,0.45,0.68,0.44,0.26,0.34)\} & (0.74,0.63,0.35,0.35,0.98,0.34,0.28,0.64)\}
\end{pmatrix}
$$

De-Neutrosophication of Octagonal Neutrosophic number by,

$$\boldsymbol{D^{T}}_{\boldsymbol{NO_N}} = \left(\frac{\Omega+\square+\varepsilon+\vartheta+\varepsilon+\square+\acute{o}+3}{\mathbf{8}}\right), \boldsymbol{D^{I}}_{\boldsymbol{NO_N}} = \left(\frac{\Omega^1+\square^1+\varepsilon^1+\vartheta^1+\varepsilon^1+\square^1+\acute{o}^1+3^1}{\mathbf{8}}\right), \boldsymbol{D^{F}}_{\boldsymbol{NO_N}} = \left(\frac{\Omega^2+\square^2+\varepsilon^2+\vartheta^2+\varepsilon^2+\square^2+\acute{o}^2+3^2}{\mathbf{8}}\right)$$

The associated neutrosophic matrix is,

$$X = \begin{pmatrix} (0.65,0.92,0.93) & (0.84,0.67,0.56) & (0.82,0.88,0.66) \\ (0.70,0.59,0.58) & (0.70,0.78,0.70) & (0.69,0.60,0.66) \\ (0.86,0.66,0.82\ ) & (0.91,0.44,0.36) & (0.73,0.64,0.49) \end{pmatrix}$$

The associated fuzzy matrix is,

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$$X = \begin{pmatrix} (0.8333) & (0.6900) & (0.7866) \\ (0.6233) & (0.7266) & (0.6500) \\ (0.7800\,) & (0.5700) & (0.6200) \end{pmatrix}$$

After calculating normalized decision matrix, we determine the positive ideal solution as well as negative ideal solution

$\mathbb{r}^{+}$={(0.65,0.92,0.93)} $\mathbb{r}^{-}$={(0.91,0.44,0.36)}

Step 2. Calculate the group utility value as $\breve{H}_i = [H_i^L, H_i^U]$ and $\breve{\mathbb{S}}_i = [\mathbb{S}_i^L, \mathbb{S}_i^U]$

$\breve{H}_1$= [0.2769,0.2000] $\breve{H}_2$ =[0.1076,0.3846] $\breve{H}_3$ =[0.4230,0.2230]

And $\breve{\mathbb{S}}_1$ =[0.1461,0.1615] $\breve{\mathbb{S}}_2$ =[0.0384,0.2000] $\breve{\mathbb{S}}_3$= [0.2000,0.1307]

Step 3. Now we will calculate the comprehensive sorting index $\widetilde{W}_i = [W_i^L, W_i^U]$

W1 = 0.0506

W2 = 0.0275

W3 = 0.0163

Step 4. Calculation of $\boldsymbol{H_i}$,$\boldsymbol{W_i}$ *and* $\boldsymbol{S_i}$

S1 = 0.2767 H1 = 0.1088 W1 = 0.0506

S2 = 0.2394 H2 = 0.1165 W2 = 0.0275

S3 = 0.2530 H3 = 0.1066 W3 = 0.0163

Step 5. Ordering of $\boldsymbol{H_i}$,$\boldsymbol{W_i}$ *and* $\boldsymbol{S_i}$

Order the alternatives, listed by the values Si; Hi and Wi:

S2 = 0.2394 H3 = 0.1066 W3 = 0.0163

S3 = 0.2530 H1 = 0.1088 W2 = 0.0275

S1 = 0.2767 H2 = 0.1165 W1 = 0.0506

According to the ranking S3 is the potential candidate for the company.

7. Conclusion

The concept of octagonal neutrosophic number has sufficient scope of utilization in different studies in various domain. In this paper, we proposed a new concept of octagonal neutrosophic number ONN, notion and graphical representation. The de-neutrosophication technique is carried out by implementing accuracy function and following points were concluded.

- The octagonal neutrosophic number, function and graph add a new tool for modeling different aspects of daily life issues, science and environment.
- Since this study has not yet been studied yet, the comparative study cannot be done with the existing methods.

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- Detailed illustrations of truthiness, indeterminacy, falsity and de-neutrosophication techniques will provide all the required information in one platform to model any real-world problem.

In forthcoming work, authors will define the types Symmetric, Asymmetric, along with their α-cuts. Proposed work can be used to model different dynamics, of applied sciences, such as MCDM and networking problems, etc.

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Weakly b-Closed Sets and Weakly b-Open Sets based of Fuzzy Neutrosophic bi-Topological Spaces

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Abstract

In this paper, we present and study some of the basic properties of the new class of sets called weakly b-closed sets and weakly b- open sets in fuzzy neutrosophic bi-topological spaces. We referred to some results related to the new definitions, which we taked the case of equal in the definition of b-sets instead of subset. Then, we discussed the relations between the new defined sets by hand and others fuzzy neutrosophic sets which were studied before us on the other hand on fuzzy neutrosophic bi-topological spaces. Then, we have studied some of characteristics and some relations are compared with necessary examples.

Keywords: Fuzzy neutrosophic sets, Fuzzy neutrosophic bi-topology, Fuzzy neutrosophic weakly b-closed sets, Fuzzy neutrosophic weakly b-open sets.

1.Introduction

The notion of fuzzy set "FS" proved was show by L. Zadeh [1] where the membership of any element to this set "FS" be a single falue between 0 and 1. After that K Atanassov [2-4] introduced the notion of intuitionistic fuzzy sets "IFS" which was generalization of "FS", where the elements have membership and non-membership value between the same interval 0 and 1. As proved and indulged F. Smarandache [5] introduced the concept of neutrosophic sets "NSs" where he added the independed value between the value of "IFS" also in the same value, 0 and 1, then in the next papers [6], studied the neutrosophic topological spaces "NTSs" on the non standard interval which made many important consequences and theorems so as so, foundation for all family of new mathematical theories generalizing both their fuzzy topology counterparts and old classical topology, A. A. Salama [7] studied the term of neutrosophic topology "NT". Finally, Y.Veereswari [8] gave an introduction of fuzzy neutrosophic topological spaces "FNTSs".

The concept of fuzzy neutrosophic weakly-α^m generalized closed set and fuzzy neutrosophic b-closed sets "FN-b-CS" was introduced and studied by F. Mohammed [9,10]. The term of bi-topological spaces was studied in neutrosophic topology by R. Al-Hamido [11-13], so in this paper we will show the idea of fuzzy neutrosophic weakly b-closed and fuzzy neutrosophic weakly b-closed sets also investigated some their properties on fuzzy neutrosophic bi-topological spaces and we getting some neccesarly properties as generalized of many authors studying, for more details and information about the applications of neutrosophic theory in new trends see [14-21].

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2. Preliminaries:

In this section of our study, we will refer to some basic definition and operations which are necessarily in our work.

Definition 2.1 [7]: "Let U_N be a non-empty fixed set. The fuzzy neutrosophic set (FNS) μ_N is an object having the form $\mu_N = \{< u, \lambda_{\mu N}(u), \gamma_{\mu N}(u), V_{\mu N}(u)>: u \in U_N\}$ where the functions $\lambda_{\mu N}(u), \gamma_{\mu N}(u), V_{\mu N}(u): U_N \to [0,1]$ denote the degree of membership function (namely $\lambda_{\mu N}(u)$), the degree of indeterminacy function (namely $\gamma_{\mu N}(u)$) and the degree of non-membership function (namely $V_{\mu N}(u)$) respectively of each element $u \in U_N$ to the set μ_N and $0 \leq \lambda_{\mu N}(u) + \gamma_{\mu N}(u) + V_{\mu N}(u) \leq 3$, for each $u \in U_N$".

Remark 2.2: "FNS $\mu_N = \{< u, \lambda_{\mu N}(u), \sigma_{\mu N}(u), V_{\mu N}(u) >: u \in U_N\}$ can be identified to an ordered triple $< u, \lambda_{\mu N}, \sigma_{\mu N}, V_{\mu N} >$ *in* $[0,1]$ on U_N".

Lemma 2.3 [8]: "Let U_N be a non-empty set and the FNSs μ_N and γ_N be in the form:

$\mu_N = \{<u, \lambda_{\mu N}, \sigma_{\mu N}, V_{\mu N}> \}$ and $\gamma_N = \{<u, \lambda_{\gamma N}, \sigma_{\gamma N}, V_{\gamma N}>\}$ on U_N. Then,

(i) $\mu_N \subseteq \gamma_N$ *iff* $\lambda_{\mu N} \leq \lambda_{\gamma N}$, $\sigma_{\mu N} \leq \sigma_{\gamma N}$ *and* $V_{\mu N} \geq V_{\gamma N}$,

(ii) $\mu_N = \gamma_N$ *iff* $\mu_N \subseteq \gamma_N$ *and* $\gamma_N \subseteq \mu_N$,

(iii) $(\mu_N)^c = \{< u, V_{\mu N}, 1-\sigma_{\mu N}, \lambda_{\mu N} >\}$,

(iv) $\mu_N \cup \gamma_N = \{< u, Mx(\lambda_{\mu N}, \lambda_{\gamma N}), Mx(\sigma_{\mu N}, \sigma_{\gamma N}), Mn(V_{\mu N}, V_{\gamma N}) >\}$,

(v) $\mu_N \cap \gamma_N = \{< u, Mn(\lambda_{\mu N}, \lambda_{\gamma N}), Mn(\sigma_{\mu N}, \sigma_{\gamma N}), Mx(V_{\mu N}, V_{\gamma N}) >\}$,

(vi) $0_N = \{< u, 0, 0, 1>\}$ *and* $1_N = \{ < u, 1, 1, 0 >\}$."

Definition 2.4 [8]: "A Fuzzy neutrosophic topology (For short, FNT) on a non-empty set U_N is a family T_N of fuzzy neutrosophic subset in U_N satisfying the following axioms.

(i) $0_N, 1_N \in T_N$,

(ii) $\mu_{N1} \cap \mu_{N2} \in T_N \ \forall \mu_{N1}, \mu_{N2} \in T_N$,

(iii) $\cup \mu_{Nj} \in T_N, \ \forall \{ \mu_{Nj} : j \in J\} \subseteq T_N$.

In this case the pair (U_N, T_N) is called fuzzy neutrosophic topological space (for short, FNTS). The elements of T_N are called fuzzy neutrosophic-open sets (for short, FN-OS). The complement of FN-OS in the FNTS (U_N, T_N) is called fuzzy neutrosophic- closed set (for short, FN-CS)."

Definition 2.5 [8]: "Let (U_N, T_N) is FNTS and $\mu_N = < u, \lambda_{\mu N}, \sigma_{\mu N}, V_{\mu N} >$ is FNS in U_N. Then the fuzzy neutrosophic-closure (for short, *FN-Cl*) and the fuzzy neutrosophic -interior (for short, FN-In) of μ_N are defined by:

$$FN\text{-}Cl(\mu_N) = \cap \{ \gamma_N : \gamma_N \text{ is } FN\text{-}CS \text{ set in } U \text{ and } \mu_N \subseteq \gamma_N \},$$

$$FN\text{-}In(\mu_N) = \cup \{ \gamma_N : \gamma_N \text{ is } FN\text{-}OS \text{ set in } U \text{ and } \gamma_N \subseteq \mu_N \}.$$

Now, the $FN\text{-}Cl(\mu_N)$ is FN-CS set and $FN\text{-}In(\mu_N)$ is FN-OS. set in U_N.

Further,

(i) μ_N is *FN-CS* in U iff $FN\text{-}Cl(\mu_N) = \mu_N$,

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(ii) μ_N is FN-OS in U iff $FN\text{-}In(\mu_N) = \mu_N$."

Definition 2.6 [9,10]: "The FNS λ_N in FNTS (U_N, T_N) is called:

(i) Fuzzy neutrosophic regular-open set (FNR-OS) iff $\mu_N = FN\text{-}In(FN\text{-}Cl(\mu_N))$,
(ii) Fuzzy neutrosophic regular-closed set (FNR-CS.) iff $\mu_N = FN\text{-}Cl(FN\text{-}In(\mu_N))$,
(iii) Fuzzy neutrosophic semi-open set (FNS-OS) iff $\mu_N \subseteq FN\text{-}Cl(FN\text{-}In(\mu_N))$,
(iv) Fuzzy neutrosophic semi-closed set (FNS-CS) iff $FN\text{-}In(FN\text{-}Cl(\mu_N)) \subseteq \mu_N$,
(v) Fuzzy neutrosophic pre-open set (FNP-OS) iff $\mu_N \subseteq FN\text{-}In(FN\text{-}Cl(\mu_N))$,
(vi) Fuzzy neutrosophic pre-closed set (FNP-CS) iff $FN\text{-}Cl(FN\text{-}In(\mu_N)) \subseteq \mu_N$,
(vii) Fuzzy neutrosophic α-open set (FNα-OS) iff $\mu_N \subseteq FN\text{-}In(FN\text{-}Cl(FN\text{-}In(\mu_N)))$,
(viii) Fuzzy neutrosophic α-closed set (FNα-CS) iff $FN\text{-}Cl(FN\text{-}In(FN\text{-}Cl(\mu_N))) \subseteq \mu_N$,
(ix) Fuzzy neutrosophic β-open set (FNβ-OS) iff $\mu_N \subseteq FN\text{-}Cl(FN\text{-}In(FN\text{-}Cl(\mu_N)))$,
(x) Fuzzy neutrosophic β-closed set (FNβ-CS) iff $FN\text{-}In(FN\text{-}Cl(FN\text{-}In(\mu_N))) \subseteq \mu_N$."

Definition 2.7 [7]: "A fuzzy neutrosophic set K in FNTS U_N is called fuzzy neutrosophic b-open set (for short, FNb-OS) if and only if $K \leq In(Cl(K)) \vee Cl(In\ (K))$."

Definition 2.8 [15]: "A fuzzy neutrosophic set K in FNTS U_N is called fuzzy neutrosophic b-closed (for short, FNb-CS) set iff $In(Cl\ K)\ \vee Cl(In\ K) \leq K$."

Definition 2.9 [11]: "Let U_N be a non-empty set and (U, T_{N1}), (U, T_{N2}) be two topological spaces then, the triple (U_N, T_{N1}, T_{N2}) is a fuzzy neutrosophic bi-topological space (for short, FN-bi-TS)."

Definition 2.10 [11]: "Let U_N be a non-empty set and T_{N1}, T_{N2} be two topologies on U_N. A subset A of U_N is called fuzzy neutrosophic bi-open set (for short, FN-OS) if $A \in T_{N1} \cup T_{N2}$. A is called fuzzy neutrosophic bi-closed set (for short, FN-CS) if $1_N\text{-}A$ is FN-OS.

Definition 2.11 [18]: "A FNS K in FN-bi-TS (U, T_{N1}, T_{N2}) is called fuzzy neutrosophic nowhere dense set if there exists no FN-OS. set V such that $V \subseteq FN\text{-}Cl(K)$. That is $FN\text{-}In(FN\text{-}Cl(K)) = 0_N$."

Remark 2.12 [15]: "Let K be a FNS in FN-bi-TS (U, T_{N1}, T_{N2}). If K is a fuzzy neutrosophic nowhere dense set in (U_N, T_{N1}, T_{N2}), then, $FN\text{-}In(K) = 0_N$. "

3.**Weakly b.Closed Sets and Weakly b.open Sets in Fuzzy Neutrosophic bi-Topological Spaces**

In this section, we introduce the concepts of weakly b-closed sets and weakly b-open sets and study some of their characteristics on fuzzy neutrosophic bi-topological spaces.

Definition 3.1: A FNS K in a FN-bi-TS (U_N,T_{N1}, T_{N2}) is said to be a fuzzy neutrosophic weakly b.closed set (for short, FNWb-CS) if $FN\text{-}In(FN\text{-}Cl(K)) \cap FN\text{-}Cl(FN\text{-}In(K)) = K$. The complement $1_N\text{-}K$ of a FNWb-CS in a FN-bi-TS (U_N, T_{N1}, T_{N2}) is called a fuzzy neutrosophic weakly b.open set (for short, FNWb-OS) in U_N. The family of all FNWb-CS of a FN-bi-TS (U_N, T_{N1}, T_{N2}) is denoted by FNWb-CS(U_N).

Example 3.2: Let $U_N = \{a, b\}$ on $T_{N1} = \{0_N, E_1, 1_N\}$ and $T_{N2} = \{0_N, 1_N\}$ where $E_1 = < u, (0.4_a, 0.4_b), (0.4_a, 0.6_b), (0.2_a, 0.3_b) >$ is a FN-bi-TS on U_N.

Let $K = < u, (0.4_a, 0.4_b), ((0.0_a, 0.0_b), (0.2_a, 0.3_b) >$.

Now, $FN\text{-}In(FN\text{-}Cl(K)) \cap FN\text{-}Cl(FN\text{-}In(K)) = E_1 \cap 1_N\text{-}\ E_1 = E_1 = K$.

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Then, K is a FNWb -CS in FN-bi-TS (U_N, T_{N1}, T_{N2}).

Remark 3.3: The following cases are independent to each other in general in a FN-bi-TS (U_N, T_{N1}, T_{N2}).

(i) FN-CS and FNWb-CS.

(ii) FNR-CS and FNWb-CS.

(iii) FNP-CS and FNWb-CS.

(iv) FNα-CS and FNWb-CS.

Example 3.4: In Example 3.2,we have:

(1) K is a FNWb-CS but not a FN-CS in U as $FN\text{-}Cl(K) = 1_N\text{-}E_1 \neq K$.

(2) $K = < u, (0.4_a, 0.6_b), (0.5_a, 0.5_b), (0.4_a, 0.4_b) >$ is a FN-CS as FN-$Cl(K) = 1_N\text{-}E_1 = K$, but not a FNWb-CS in U_N as FN-$In(FN\text{-}Cl(K)) \cap FN\text{-}Cl(FN\text{-}In(K)) = E_1 \neq K$.

(3) $K = < u, (0.4_a, 0.4_b), (0.5_a, 0.5_b), (0.4_a, 0.6_b) >$ is a FNWb-CS in U_N, but not a FNR-CS in U_N as $FN\text{-}Cl(FN\text{-}In(K)) = 1_N\text{-}E_1 \neq K$.

(4) $K = < u, (0.4_a, 0.6_b), (0.5_a, 0.5_b), (0.4_a, 0.4_b) >$ is a FNR-CS in U_N as FN-$Cl(FN\text{-}In(K)) = 1_N\text{-}E_1 = K$, but not a FNWb-CS in U_N as $FN\text{-}In(FN\text{-}Cl(K)) \cap FN\text{-}Cl(FN\text{-}In(K)) \neq K$.

(5) $K = < u, (0.4_a, 0.6_b), (0.5_a, 0.5_b), (0.4_a, 0.4_b) >$ is a FNP-CS in U_N as $FN\text{-}Cl(FN\text{-}In(K)) = 1_N\text{-}E_1 \subseteq K$, but not a FNWb-CS in U_N as $FN\text{-}In(FN\text{-}Cl(K)) \cap FN\text{-}Cl(FN\text{-}In(K)) \neq K$.

(6) $K = < u, (0.4_a, 0.4_b), (0.5_a, 0.5_b), (0.4_a, 0.6_b) >$ is a FNWb-CS in U_N but not a FNP-CS in U_N as $FN\text{-}Cl(FN\text{-}In(K)) = 1_N\text{-}E_1 \not\subseteq K$.

(7) $K = < u, (0.4_a, 0.6_b), (0.5_a, 0.5_b), (0.4_a, 0.4_b) >$ is a FN-αclos. set in U_N as $FN\text{-}Cl(FN\text{-}In(FN\text{-}Cl(K))) = 1_N\text{-}E_1 \subseteq K$, but not a FNWb-CS in U_N as $FN\text{-}In(FN\text{-}Cl(K)) \cap FN\text{-}Cl(FN\text{-}In(K)) = E_1 \neq K$.

(8) $K = < u, (0.4_a, 0.4_b), (0.5_a, 0.5_b), (0.4_a, 0.6_b) >$ is a FNWb-CS in U_N as $FN\text{-}In(FN\text{-}Cl(K)) \cap FN\text{-}Cl(FN\text{-}In(K)) = E_1 = K$, but not a FNα-CS in U_N as $FN\text{-}Cl(FN\text{-}In(FN\text{-}Cl(K))) = 1_N\text{-} E_1 \not\subseteq K$.

Theorem 3.5 : Let (U_N, T_{N1}, T_{N2}) be a FN-bi-TS, then:

(i) Every FNWb-CS is a FNb-CS.

(ii) Every FNWb-CS is an FNS-CS.

(iii) Every FNWb-CS is a FNβ-CS.

Proof : (i) Let K be a FNWb-CS in FN-bi-TS $((U_N, T_{N1}, T_{N2})$,

Then, $FN\text{-}In(FN\text{-}Cl(K)) \cap FN\text{-}Cl(FN\text{-}In(K)) = K$.

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Now, as $K \subseteq K$, we get $FN\text{-}In(FN\text{-}Cl(K)) \cap FN\text{-}Cl(FN\text{-}In(K)) \subseteq K$.

Therefore, K is a FNb-CS in (U_N, T_{N1}, T_{N2}).

(ii): Let K be a FNWb-CS in FN-bi-TS (U_N, T_{N1}, T_{N2}).

Then, $FN\text{-}In(FN\text{-}Cl(K)) \cap FN\text{-}Cl(FN\text{-}In(K)) = K$.

Now, as $FN\text{-}In(FN\text{-}Cl(K)) = FN\text{-}In(FN\text{-}Cl(FN\text{-}In(FN\text{-}Cl(K) \cap FN\text{-}Cl(FN\text{-}In(K))))$

$\subseteq FN\text{-}In(FN\text{-}Cl(FN\text{-}Cl(K) \cap FN\text{-}Cl(FN\text{-}In(K))))$

$\subseteq FN\text{-}In(FN\text{-}Cl(K)) \cap FN\text{-}Cl(FN\text{-}In(K))) = K$.

Hence, K is a FNS-CS in (U_N, T_{N1}, T_{N2}).

(iii): Let K be a FNWb-CS in FN-bi-TS (U_N, T_{N1}, T_{N2}).

Then, $FN\text{-}In(FN\text{-}Cl(K)) \cap FN\text{-}Cl(FN\text{-}In(K)) = K$.

Now, $FN\text{-}In(FN\text{-}Cl(FN\text{-}In(K))) = FN\text{-}In(FN\text{-}Cl(FN\text{-}In(K))) \cap FN\text{-}Cl(FN\text{-}In(K))$

$\subseteq FN\text{-}In(FN\text{-}Cl(K)) \cap FN\text{-}Cl(FN\text{-}In(K)) = K$.

Therefore, we have $FN\text{-}In(FN\text{-}Cl(FN\text{-}In(K))) \subseteq K$.

Hence, K is a FNβ-CS in (U_N, T_{N1}, T_{N2}).

Note: The converse of Theorem 3.5 is not true in general.

Example 3.6: In Example 3.2

(1) Let $K = < u, (0.4_a, 0.6_b), (0.5_a, 0.5_b), (0.4_a, 0.4_b) >$.

Then, $FN\text{-}In(FN\text{-}Cl(K)) \cap FN\text{-}Cl(FN\text{-}In(K)) = E_1 \neq K$.

Therefore, K is a FNb-CS but, not a FNWb-CS U_N as $FN\text{-}In(FN\text{-}Cl(K)) \cap FN\text{-}Cl(FN\text{-}In(K)) \neq K$.

(2) If $K = < u, (0.4_a, 0.6_b), (0.5_a, 0.5_b), (0.4_a, 0.4_b)>$ is a FNS-CS in U_N as $FN\text{-}In(FN\text{-}Cl(K)) = E_1 \subseteq K$, but not a FNWb-CS.

(3) *If* $K = < u, (0.4_a, 0.6_b), (0.5_a, 0.5_b), (0.4_a, 0.4_b) >$ is a FNβ-CS in U_N as $FN\text{-}In(FN\text{-}Cl(FN\text{-}In(K))) = E_1 \subseteq 1_N\text{-}E_{1'}$ but, not a FNWb-CS in U_N as $FN\text{-}In(FN\text{-}Cl(K)) \cap FN\text{-}Cl(FN\text{-}In(K)) \neq K$.

Definition 3.7: Let (U_N, T_{N1}, T_{N2}) be a FN-bi-TS, then the subset K of U_N is called FN-clopen set (for shortly, FN-CLOP) iff K is FN-CS and FN-OS in the same time.

Remark 3.8 : Every FN- CLOP set is both FNR-OS and FNR-CS.

Proposition 3.9: Let (U_N, T_{N1}, T_{N2}) be a FN-bi-TS, then:

(i) If K is both a FNR-OS and a FNR-CS then, K is a FNWb-CS.

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(ii) If K is both a FN-CLOP set then, K is a FNWb-CS.

Proof: (i) Let K be both FNR-OS and FNR-CS in FN-bi-TS (U_N, T_{N1}, T_{N2}).

Then, $FN\text{-}In(FN\text{-}Cl(K)) \cap FN\text{-}Cl(FN\text{-}In(K)) = K \cap K = K \Longrightarrow K$ is a FNWb-CS.

(ii) Let K be a FN-CLOP set in FN-bi-TS (U_N, T_{N1}, T_{N2}).

Then, $FN\text{-}In(FN\text{-}Cl(K)) \cap FN\text{-}Cl(FN\text{-}In(K)) = FN\text{-}In(K) \cap FN\text{-}Cl(K) = FN\text{-}In(K) = K.$

Therefore, K is a FNWb-CS.

Theorem 3.10: If K is both a FNWb-CS and a FN-CS in FN-bi-TS (U_N, T_{N1}, T_{N2}) then, K is a FN-OS.

Proof: Let K be both a FNWb-CS and a FN-CS in FN-bi-TS (U_N, T_{N1}, T_{N2}).

Then, $K = FN\text{-}In(FN\text{-}Cl(K)) \cap FN\text{-}Cl(FN\text{-}In(K)).$

Now, $K = FN\text{-}In(FN\text{-}Cl(K)) \cap FN\text{-}Cl(FN\text{-}In(K))$

$= FN\text{-}In(K) \cap FN\text{-}Cl(FN\text{-}In(K)) = FN\text{-}In(K).$

Therefore, K is a FN-OS in FN-bi-TS ((U_N, T_{N1}, T_{N2}).

Definition 3.11: Let (U_N, T_{N1}, T_{N2}) is FNTS and $\mu_N =< u, \lambda_{\mu N}, \sigma_{\mu N}, V_{\mu N} >$ is FNS in U_N. Then the fuzzy neutrosophic semi-closure (for short, *FN-sCl*) and the fuzzy neutrosophic –semi interior (for short, *FN-sIn*) of μ_N are defined by:

$$FN\text{-}sCl(\mu_N) = \cap \{ \gamma_N : \gamma_N \text{ is FNS-CS in U and } \mu_N \subseteq \gamma_N \} = \mu_N \cup FN\text{-}In(FN\text{-}Cl(\mu_N)),$$

$$FN\text{-}sIn(\mu_N) = \cup \{ \gamma_N : \gamma_N \text{ is FNS-OS in U and } \gamma_N \subseteq \mu_N \} = .$$

Theorem 3.12: For any FNWb-CS K in a FN-bi-TS (U_N, T_{N1}, T_{N2}), the following conditions hold:

(i) If K is a FNR-OS then, FN-sCl(K) is a FNWb-CS.

(ii) If K is a FNR-CS then, FN-sIn(K) is a FNWb-CS.

Proof:(i) Let K be a FNR-OS in FN-bi-TS (U_N, T_{N1}, T_{N2}).Then, $FN\text{-}In(FN\text{-}Cl(K)) = K.$

By definition, we have $FN\text{-}sCl(K) = K \cup FN\text{-}In(FN\text{-}Cl(K)) = K \cup K = K$, by hypothesis.

Since, K is a FNWb-CS in (U_N, T_{N1}, T_{N2}), FN-sCl(K) is a FNWb-CS in U_N,.

(ii) Let K be a FNR-CS in (U_N, T_{N1}, T_{N2}), then, $FN\text{-}Cl(FN\text{-}In(K)) = K.$

By definition, we have $FN\text{-}sIn(K) = K \cap FN\text{-}Cl(FN\text{-}In(K)) = K \cap K = K,$ by hypothesis.

Since, K is a FNWb-CS in(U_N, T_{N1}, T_{N2}), *FN-sIn(K)* is a FNWb-CS.

Theorem 3.13: For a FNS K in FN-bi-TS (U_N, T_{N1}, T_{N2}), the following conditions are equivalent:

(i) K is both a FN-OS and a FNWb-CS.

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(ii) K is a FNR-OS.

Proof: (i) ⇒ (ii) Let K be a FN-OS and a FNWb-CS in FN-bi-TS (U_N, T_{N1}, T_{N2}).

Then, $FN\text{-}Cl(FN\text{-}In(K)) \cap FN\text{-}In(FN\text{-}Cl(K)) = K$ *and* $FN\text{-}In(FN\text{-}Cl(K)) \cap FN\text{-}Cl(K) = K$,

Therefore, $K = FN\text{-}In(FN\text{-}Cl(K))$.

Hence, K is a FNR-OS in (U_N, T_{N1}, T_{N2}).

(ii)⇒ (i) Let K be a FNR-OS in FN-bi-TS(U_N, T_{N1}, T_{N2}).

Since every FNR-OS is a FN-OS, K is a FN-OS in (U_N, T_{N1}, T_{N2}) and $K = FN\text{-}In(FN\text{-}Cl(K))$.

Now, $FN\text{-}In(FN\text{-}Cl(K)) \cap FN\text{-}Cl(FN\text{-}In(K)) = K \cap FN\text{-}Cl(FN\text{-}In(K)) = K \cap FN\text{-} Cl(K) = K$.

Hence, K is a FNWb-CS in FN-bi-TS (U_N, T_{N1}, T_{N2}).

Definition 3.14: A FNS K in a FN-bi-TS (U_N, T_{N1}, T_{N2}) is said to be a fuzzy neutrosophic weakly b-open set (for short, FNWb-OS) iff $K = FN\text{-}In(FN\text{-}Cl(K)) \cup FN\text{-}Cl(FN\text{-}In(K))$.

The family of all FNWb-OS of a FN-bi-TS (U_N, T_{N1}, T_{N2}) is denoted by FNWb-OS(U_N).

Example 3.15: In Example 3.2,

Let $K = < u, (0.4_a, 0.6_b), (0.5_a, 0.5_b), (0.4_a, 0.4_b) >$ be a FNS in (U_N, T_{N1}, T_{N2}).

Now, $FN\text{-}In(FN\text{-}Cl(1_N\text{-}K)) \cap Fn\text{-}Cl(FN\text{-}In(1_N\text{-}K)) = E_1 \cap 1_N\text{-}E_1 = E_1 = 1_N\text{-}K \Longrightarrow 1_N\text{-}K$ is a FNWb-CS

$\Longrightarrow 1_N\text{-}K$ is a FNWb-CS in (U_N, T_{N1}, T_{N2}).

Hence, K is a FNWb-OS in (U_N, T_{N1}, T_{N2}).

Theorem 3.16: Every FNWb-OS are FNb-OS (FNS-OS., FNβ-OS) but not conversely in general.

Proof: Straight forward.

Example 3.17: Obvious from Example 3.6 (1), (2), (3) by taking complement of K in the respective examples.

Theorem 3.18: Every FN-OS, FNR-OS, FNP-OS and FNα-OS are independent to FNWb-OS in FN-bi-TS (U_N, T_{N1}, T_{N2}) and vice versa in general.

Example 3.19: Obvious from Example 3.4 (1), (2), (3), (4), (5), (6), (7), (8), by taking the complement of A in the respective examples.

Theorem 3.20: If K is a FNWb-OS and fuzzy neutrosophic nowhere dense in FN-bi-TS ((U_N, T_{N1}, T_{N2}), then:

(i) K is a FNR-CS.

(ii) K is a FNS-CS.

(iii) K is a FNα-CS.

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(iv) K is a FNβ-CS.

Proof: (i) Let K be a FNWb-OS and fuzzy neutrosophic nowhere dense in FN-bi-TS (U_N, T_{N1}, T_{N2}).

Then, $K = FN\text{-}In(FN\text{-}Cl(K)) \cup FN\text{-}Cl(FN\text{-}In(K)) = 0_N \cup FN\text{-}Cl(FN\text{-}In(K)) = FN\text{-}Cl(FN\text{-}In(K))$.

Therefore, $FN\text{-}Cl(FN\text{-}In(K)) = K$.

Hence, K is a FNR-CS in (U_N, T_{N1}, T_{N2}).

(ii) Let K be a FNWb-OS and fuzzy neutrosophic nowhere dense in FN-bi-TS (U_N, T_{N1}, T_{N2}).

Then, $K \subseteq FN\text{-}In(FN\text{-}Cl(K)) \cup FN\text{-}Cl(FN\text{-}In(K)) = 0_N \cup FN\text{-}Cl(FN\text{-}In(K)) \subseteq FN\text{-}Cl(FN\text{-}In(K))$.

Therefore, $K \subseteq FN\text{-}Cl(FN\text{-}In(K))$.

Hence, K is a FNS-CS in (U_N, T_{N1}, T_{N2}).

(iii) Let K be a FNWb-OS and fuzzy neutrosophic nowhere dense in FN-bi-TS (U_N, T_{N1}, T_{N2}).

Then, $FN\text{-}In(FN\text{-}Cl(K)) \cup FN\text{-}In(FN\text{-}Cl(FN\text{-}In(K))) \subseteq K$

Now, $0_N \cup FN\text{-}In(FN\text{-}Cl(FN\text{-}In(K))) \subseteq K$.

Therefore, $FN\text{-}In(FN\text{-}Cl(FN\text{-}In(K))) \subseteq K$.

Hence, K is a FNα-CS in (U_N, T_{N1}, T_{N2}).

(iv) Let K be a FNWb-OS and fuzzy neutrosophic nowhere dense in FN-bi-TS (U_N, T_{N1}, T_{N2}).

Then, $FN\text{-}In(FN\text{-}Cl(K)) \cup FN\text{-}In(FN\text{-}Cl(FN\text{-}In(K))) \subseteq K$.

Now, $0_N \cup FN\text{-}In(FN\text{-}Cl(FN\text{-}In(K))) \subseteq K$.

Therefore, $FN\text{-}In(FN\text{-}Cl(FN\text{-}In(K))) \subseteq K$.

Hence, K is a FNβ-CS in (U_N, T_{N1}, T_{N2}).

Theorem 3.21: If K is both a FNWb-OS and a FN-OS then, K is a FN-CS in FN-bi-TS (U_N, T_{N1}, T_{N2}).

Proof: Let K be both a FNWb-OS and a FN-OS in FN-bi-TS (U_N, T_{N1}, T_{N2}).

Then, $K = FN\text{-}In(FN\text{-}Cl(K)) \cup FN\text{-}Cl(FN\text{-}In(K))$.

Now, $K = FN\text{-}In(FN\text{-}Cl(K)) \cup FN\text{-}Cl(FN\text{-}In(K)) = FN\text{-}In(FN\text{-}Cl(K)) \cup FN\text{-}Cl(K) = FN\text{-}Cl(K)$.

Hence, K is a FN-CS in (U_N, T_{N1}, T_{N2}).

Theorem 3.22: Let K be a FNWb-OS in a FN-bi-TS (U_N, T_{N1}, T_{N2}), such that $FN\text{-}In(K) = 0_N$, then, K is a FNP-OS.

Proof: Let K be a FNWb-OS then, K is a FNb-OS in FN-bi-TS ((U_N, T_{N1}, T_{N2}).

Now, $K \subseteq FN\text{-}In(FN\text{-}Cl(K)) \cup FN\text{-}Cl(FN\text{-}In(K)) \subseteq FN\text{-}In(FN\text{-}Cl(K)) \cup 0_N \subseteq FN\text{-}In(FN\text{-}Cl(K))$.

Hence, $K \subseteq FN\text{-}In(FN\text{-}Cl(K))$.

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So, K is a FNP-OS in FN-bi-TS (U_N, T_{N1}, T_{N2}).

4.Conclusions

In this paper, we have introduced a new class of sets called fuzzy neutrosophic weakly b-closed sets and fuzzy neutrosophic weakly b-open sets via fuzzy nutrosophic topological spaces. Many results have been studied as generalization of classical fuzzy topology and intuitionistic fuzzy topology and applied in the field of fuzzy bi-neutrosophic topology. After giving some ideas to compared the already existing new sets with the others existing closed sets by definitions, theorems and propositions. Some interesting properties were investigated in addition, we have provided some examples where such properties failed to be preserved via fuzzy neutrosophic bi-topological spaces.We think, our studied class of sets belongs to the important class of fuzzy neutrosophic closed sets which is very useful not only in the deepening of our understanding of some special features of the already well-known notions of fuzzy neutrosophic topology but also proves useful in neutrosophic control theory.

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Neutrosophic Ideal layers & Some Generalizations for GIS Topological Rules

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ABSTRACT.

This paper aims to introduce and study some new neutrosophic fuzzy pairwise notions via neutrosophic fuzzy ideals. We, also generalize the notion of FPL-open sets. In addition to generalizing the concept of FPL-closed sets and NPL-open function, the relationship between the above new neutrosophic Fuzzy pairwise notions and there other relevant classes are investigated. In Geographical information systems (GIS) there is a need to statistically model spatial regions with indeterminate boundary and under indeterminacy. Possible applications to GIS rules are touched upon.

Keywords: Neutrosophic set; Neutrosophic topology; Neutrosophic ideal open set; Neutrosophic closed set; GIS Neutrosophic Rules; Neutrosophic Statistical Model

Introduction

Since the world is full of indeterminacy, the neutrosophic found their place into contemporary research. The neutrosophic set was introduced by Smarandache in [3,4,7] and Salama et al. [5, 6, 8,10, 11, 12, 13, 14] introduced the neutrosophic crisp set, neutrosophic topological spaces and many applications in statistics, computer science and information systems. Neutrosophy has laid the foundation for a whole family of new mathematical theories generalizing both their classical and fuzzy counterparts, such as neutrosophic set theory, in this paper is to introduce and study some new neutrosophic fuzzy pairwise notion via neutrosophic fuzzy pairwise ideals. We, also generalize the notion of FPL-open sets due to Abd El-Monsef, et. al [1, 2]. In addition to generalizing the concept of FPL-closed sets. In Geographical information systems (GIS) there is a need to statistically model spatial regions with indeterminate boundary and under indeterminacy. Possible applications to GIS rules are touched upon.

PRELIMINARIES

We recollect some relevant basic preliminaries, and in particular, the work of Smarandache in [3, 4, 7], and Salama et al. [8-13].

2. Neutrosophic Fuzzy Pairwise Set and its Neutrosophic Fuzzy Pairwise Local Function.

Definition 2.1. Given (X,τ_i), $i\in\{1,2\}$ be an NFTS with neutrosophic fuzzy ideal L on X, $\mu, \sigma, u \in I^X$. Then <μ,σ,u> is said to be :

(i) Neutrosophic fuzzy pairwise τ^*_i-closed, $i\in\{1,2\}$ may have two types

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Type 1: if $\mu^* \le \mu$, $u^* \ge u$, $\sigma^* \le \sigma$.

Type 2: if $\mu^* \le \mu$, $u^* \ge u$, $\sigma^* \ge \sigma$

(or PN-*closed) if $<\mu,\sigma,u>^* \le <\mu,\sigma,u>$

(ii) Fuzzy neutrosophic pairwise NPL-dense – in – itself may have two types

Type 1: if $\mu^* \le \mu, u^* \ge u$, $\sigma^* \le \sigma$.

Type 2: if $\mu^* \le \mu, u^* \ge u$, $\sigma^* \ge \sigma$

(or PN*-dense- in – itself) if $<\mu,\sigma,u> \subseteq <\mu,\sigma,u>^*$.

(iii) Neutrosophic fuzzy pairwise*-perfect if $<\mu,\sigma,u>$ is PN*-closed and PN*- dense – in itself.

Theorem 2.1: Given $(X,\tau_i), i\in\{1,2\}$ be a nfbts with neutrosophic fuzzy ideal L on X , $\mu,\sigma,u \in I^x$ then $<\mu,\sigma,u>$ is

(i) PN*- closed iff $Ncl^*(<\mu,\sigma,u>) = <\mu,\sigma,u>$.

(ii) PN*- dense – in – itself iff $Ncl^*(<\mu,\sigma,u>) = <\mu,\sigma,u>^*$.

(iii) PN*- perfect iff $Ncl^*(<\mu,\sigma,u>) = <\mu,\sigma,u>^* = <\mu,\sigma,u>$.

Proof: Follows directly from the neutrosophic fuzzy pairwise closure operator Ncl* for a neutrosophic fuzzy pairwise $\tau^*{}_i(L), i \in \{1,2\}$ in and Definition 2.1.

Remark 2.1: One can deduce that

(i) Every PN*-dense- in – itself is a neutrosophic fuzzy pairwise dense set.

(ii) Every neutrosophic fuzzy pairwise closed (resp. neutrosophic fuzzy pairwise open) set is PN*-closed (resp. $PN\tau^*{}_i - \text{open}, i \in \{1,2\}$).

Corollary 2.1: Given $(X,\tau_i), i\in\{1,2\}$ be a nfbts with neutrosophic fuzzy ideal L on X , $<\mu,\sigma,u> \in \tau_i$ then we have :

(i) If $<\mu,\sigma,u>$ is PN*-closed then $<\mu,\sigma,u>^* \le Nint(<\mu,\sigma,u>) \le Ncl(<\mu,\sigma,u>)$.

(ii) If $<\mu,\sigma,u>$ is PN*-dense- itself then $Nint(<\mu,\sigma,u>) \le <\mu,\sigma,u>^*$.

(iii) If $<\mu,\sigma,u>$ is PN*- perfect then $Nint(<\mu,\sigma,u>) = Ncl(<\mu,\sigma,u>) = <\mu,\sigma,u>^*$.

Proof: Obvious.

Theorem 2.2: Given $(X,\tau_i), i\in\{1,2\}$ be a nfbts with neutrosophic fuzzy ideal L_n on X , $\mu,\sigma,u \in I^x$ then we have the following: $<\mu,\sigma,u>$ is neutrosophic fuzzy pairwise α - closed iff $<\mu,\sigma,u>$ is PN*- closed.

Proof: It's clear.

Corollary 2.2: For a nbts $(X,\tau_i), i\in\{1,2\}$ with neutrosophic fuzzy ideal L on X , $\mu,\sigma,u \in I^x$, the following holds:

(i) If $<\mu,\sigma,u> \in PNC(X)$ then $<\mu,\sigma,u>$ is PN*- closed.

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(ii) If <μ,σ,u> ∈PNβC(X) then Nint (Nint(<μ,σ,u>*)) ≤ <μ,σ,u>.

(iii) If <μ,σ,u> ∈PNSC(X) then Nint(<μ,σ,u>*) ≤ <μ,σ,u>.

Proof: Obvious.

3. Neutrosophic Fuzzy Pairwise L-Open and Neutrosophic Fuzzy Pairwise L- Closed Sets.

Definition 3.1. Given (X,τ_i), $i\epsilon\{1,2\}$ be a NFBTS with neutrosophic fuzzy ideal L on X , $\mu,\sigma,u\epsilon I^x$ and <μ,σ,u> is called a neutrosophic fuzzy pairwise NL -open set iff there exists $<\xi,\rho,\theta>\epsilon\tau_i, i\epsilon\{1,2\}, P=<p_1,p_2,p_3>$ such that $<\mu,\sigma,u>\subseteq<\xi,\rho,\theta>\subseteq P(<\mu,\sigma,u>^*)(L,\tau_i), i\epsilon\{1,2\}$.

We will denote the family of all neutrosophic fuzzy pairwise NL –open $(X,\tau_i)=\mu,\sigma,u\epsilon I^x:<\mu,\sigma,u>\subseteq \tau_1-\text{Nint}[P(<\mu,\sigma,u>^*)(L,\tau_1)]$ and $<\mu,\sigma,u>\subseteq\tau_2-\text{Nint}[P(<\mu,\sigma,u>^*)(L,\tau_2)], i\epsilon\{1,2\}$ (simplify NPLO(X)) when there is no chance for confusion.

Theorem 3.1: Let (X,τ_i), $i\epsilon\{1,2\}$ be a nfbts with neutrosophic fuzzy ideal L, then $<\mu,\sigma,u>\epsilon\text{NPOL}(X)$

iff $<\mu,\sigma,u>\subseteq\tau_i-\text{int}\big(P(<\mu,\sigma,u>^*)(L,\tau_i)\big)$ for $i\epsilon\{1,2\}, P=<p_1,p_2,p_3>$.

Proof: Assume that $<\mu,\sigma,u>\epsilon\text{NPOL}(X)$ then Definition 3.1.1. there exists $<\xi,\rho,\theta>\epsilon\tau_i$

such that $<\mu,\sigma,u>\subseteq<\xi,\rho,\theta>\subseteq P(<\mu,\sigma,u>^*)(L,\tau_i), i\epsilon\{1,2\}$. But $\text{Nint}\big(P(<\mu,\sigma,u>^*)\big)\subseteq P(<\mu,\sigma,u>^*)$,

put $<\xi,\rho,\theta>=\text{Nint}\big(P(<\mu,\sigma,u>^*)\big)$. Hence $<\mu,\sigma,u>\subseteq\text{Nint}\big(P(<\mu,\sigma,u>^*)\big)$.

Conversely $<\mu,\sigma,u>\ \subseteq\ \text{Nint}(P(<\mu,\sigma,u>^*))\subseteq P(<\mu,\sigma,u>^*)$.

Then there exists $<\xi,\rho,\theta>=\text{Nint}(P(<\mu,\sigma,u>^*))\epsilon\tau_i$. Hence $<\mu,\sigma,u>$ in NPLO(X).

Definition 3.2. The largest $\tau_i-\text{NPL}-\text{open}(\text{simply }\tau_i-\text{NPLO}(X))$ set contained in $<\mu,\sigma,u>$ is called a $\tau_i-\text{NPL}-\text{neutrosophic interior of }<\mu,\sigma,u>$. The complement of the neutrosophic fuzzy pairwise NL-open subset of X is a neutrosophic fuzzy pairwise NL-closed subset of X (simply NPLC(X)).

We denoted by NPL-Nint($<\mu,\sigma,u>$).

Theorem 3.2. Let (X,τ_i), $i\epsilon\{1,2\}$ be a nfbts with neutrosophic fuzzy ideal L, $\mu,\sigma,u\epsilon I^x$ and j is an arbitrary set then

i- The union of neutrosophic fuzzy pairwise NL-open subsets may be neutrosophic fuzzy neutrosophic pairwise NL-open.

ii- If $\nu=<\nu_1,\nu_2,\nu_3>$ is neutrosophic fuzzy pairwise open and <μ,σ,u> may be neutrosophic fuzzy pairwise NL-open subset of X. Then $<\mu,\sigma,u>\cap\nu$ may be pairwise NL-open subset.

Proof. (i) Let $\{<\mu,\sigma,u>_J : J\epsilon j\}$ be a family of NPLO(X). Then for each

$J\epsilon j, <\mu,\sigma,u>_J\subseteq\tau_i-\text{Nint}(p<\mu,\sigma,u>^*{}_J)$

and so $Y_J<\mu,\sigma,u>_J\subseteq Y_J\tau_i-\text{Nint}\left(p<\mu,\sigma,u>^*{}_J\right)\subseteq\tau_i-\text{Nint}(Y_J p<\mu,\sigma,u>^*{}_J)^*$.

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(ii) Assume that $\nu =< \nu_1, \nu_2, \nu_3 >$ is neutrosophic fuzzy pairwise open and <μ,σ,u> may be neutrosophic fuzzy pairwise NL-open subsets of X.

Then $< \mu, \sigma, u > \cap \nu \subseteq \nu \cap \left(\tau_i - \mathrm{Nint}\left(P(< \mu, \sigma, u >^*)\right)\right) \subseteq \tau_i - \mathrm{Nint}\left(\nu \cap P(< \mu, \sigma, u >^*)\right) \subseteq$

$\tau_i - \mathrm{Nint}P(\nu \cap < \mu, \sigma, u >)^*$.

Definition 3.3. Let (X,τ_i), $i\epsilon\{1,2\}$ be a nfbts with neutrosophic fuzzy ideal L on X, $\mu, \sigma, u\epsilon I^X$. Then <μ,σ,u> is said to be neutrosophic fuzzy.

i- $\tau^*{}_i$ - closed iff $\tau^*{}_i - N\mathrm{cl}^*(< \mu, \sigma, u >) =< \mu, \sigma, u >$.

ii- $\tau^*{}_i - \mathrm{dense} - \mathrm{in} - \mathrm{itself}$ if $< \mu, \sigma, u > \subseteq P(< \mu, \sigma, u >^*)(L, \tau_i)$.

iii- $\tau^*{}_i - \mathrm{perfect}$ if $< \mu, \sigma, u >$ is $\tau^*{}_i -$ closed and $\tau^*{}_i - \mathrm{dense} - \mathrm{in\ itself}$.

Proof. Follows directly from the neutrosophic fuzzy closure operator for $\tau^*{}_i$ and Definition 3.1.

Theorem 3.3: Given (X,τ_i), $i\epsilon\{1,2\}$ a nfbts with neutrosophic fuzzy ideal L on X, $\mu, \sigma, u\epsilon I^X$, then the following holds:

(i) If <μ,σ,u> is both neutrosophic fuzzy pairwise NL-open and $\tau^*{}_i$ -perfect then <μ,σ,u> may be neutrosophic fuzzy pairwise open.

(ii) If <μ,σ,u> is both neutrosophic fuzzy pairwise open and $\tau^*{}_i$ dense – in – itself then <μ,σ,u> may be neutrosophic fuzzy pairwise NL-open.

Proof. Follows from the definitions.

Corollary 3.1. For a neutrosophic fuzzy subset <μ,σ,u> of a nfbts (X,τ_i), $i\epsilon\{1,2\}$ with neutrosophic fuzzy ideal L on X, we have:

(i) If <μ,σ,u> is $\tau^*{}_i -$ closed and $\mathrm{NPL} -$ open then $\mathrm{Nint}(< \mu, \sigma, u >) = \mathrm{Nint}(P(< \mu, \sigma, u >))^*$.

(ii) If <μ,σ,u> is $\tau^*{}_i -$ perfect and $\mathrm{NPL} -$ open then $< \mu, \sigma, u > = \mathrm{Nint}(P(< \mu, \sigma, u >^*))$.

Theorem 3.4: If (X,τ_i), $i\epsilon\{1,2\}$ a nfbts with neutrosophic fuzzy ideal L and $\mu, \sigma, u\epsilon I^X$ then

(i) $< \mu, \sigma, u > \cap\ \mathrm{Nint}(P(< \mu, \sigma, u >^*))$ may be a neutrosophic fuzzy NL-open set.

(ii) NPL $\tau_i - \mathrm{Nint}(< \mu, \sigma, u >) = 0_N$ iff $\mathrm{Nint}(P(< \mu, \sigma, u >^*)) = 0_N$.

Proof. (i) Since $\mathrm{Nint}\left(P(< \mu, \sigma, u >^*)\right) = P(< \mu, \sigma, u >^*) \cap \mathrm{Nint}\left(P(< \mu, \sigma, u >^*)\right)$, then

$\mathrm{Nint}(P(< \mu, \sigma, u >^*)) = P(< \mu, \sigma, u >^*) \cap \mathrm{Nint}\,P(< \mu, \sigma, u >^*) \subseteq P((< \mu, \sigma, u >) \cap (< \mu, \sigma, u >^*))^*$. Thus

$< \mu, \sigma, u > \cap\ P(< \mu, \sigma, u >^*) \subseteq (< \mu, \sigma, u > \cap\ (< \mu, \sigma, u > \cap\ \mathrm{Nint}\,P\left(P(< \mu, \sigma, u >^*)\right)^* \subseteq \mathrm{Nint}\,P(< \mu, \sigma, u >) \cap$ $\mathrm{Nint}\,P(P(< \mu, \sigma, u >^*))^*$

Hence $< \mu, \sigma, u > \cap\ \mathrm{Nint}\left(P(< \mu, \sigma, u >^*)\right)$ in $\mathrm{NPLO}(X)$.

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(ii) Let $\mathrm{NPL}\tau_i - \mathrm{Nint}(<\mu,\sigma,u>) = 0_N$, then $<\mu,\sigma,u> \cap P(<\mu,\sigma,u>^*) = 0_N$, implies

$\mathrm{Ncl}\big(<\mu,\sigma,u> \cap \mathrm{Nint}\, P(<\mu,\sigma,u>^*)\big) = 0_N$ and so

$<\mu,\sigma,u> \cap \mathrm{Nint}\, P(<\mu,\sigma,u>^*) = 0_N$. Conversely assume that $\mathrm{Nint}\, P(<\mu,\sigma,u>^*) = 0_N$,

then $<\mu,\sigma,u> \cap \mathrm{Nint}\, P(<\mu,\sigma,u>^*) = 0_N$. Hence $\mathrm{NPL}\tau_i - \mathrm{Nint}(<\mu,\sigma,u>) = 0_N$.

Theorem 3.5: If (X,τ_i), $i\epsilon\{1,2\}$ a nfbts with neutrosophic fuzzy ideal L on X, $\mu,\sigma,u\epsilon I^X$,

then $\mathrm{NPL}\tau_i - \mathrm{Nint}(<\mu,\sigma,u>) =<\mu,\sigma,u> \wedge \mathrm{Nint}\, P(<\mu,\sigma,u>^*)$.

Proof. The first implication follows from Theorem 3.1.1 that is

$$<\mu,\sigma,u> \wedge p <\mu,\sigma,u>^* \leq \mathrm{NPL} - \mathrm{Nint}(<\mu,\sigma,u>) \qquad (1)$$

For the reverse inclusion, if $<\xi,\rho,\theta>$ in NPLO(X) and $<\xi,\rho,\theta> \leq <\mu,\sigma,u>$ then $P(<\xi,\rho,\theta>^*) \leq P(<\mu,\sigma,u>^*)$ and hence $\mathrm{Nint}\, P(<\xi,\rho,\theta>^*) \leq \mathrm{Nint}\, P(<\mu,\sigma,u>^*)$. This implies $<\xi,\rho,\theta> = <\xi,\rho,\theta> \wedge \mathrm{Nint}\, P(<\xi,\rho,\theta>^*) \leq <\mu,\sigma,u> \wedge P(<\mu,\sigma,u>^*)$.

Thus $\mathrm{NPL}\tau_i - \mathrm{Nint}(<\mu,\sigma,u>) \leq <\mu,\sigma,u> \wedge P(<\mu,\sigma,u>^*) \qquad (2)$

From (1) and (2) we have the result.

Definition 3.4: Given (X,τ_i), $i\epsilon\{1,2\}$ a nfbts with neutrosophic fuzzy ideal L and $\xi,\rho,\theta\epsilon I^X$, $<\xi,\rho,\theta>$ is called neutrosophic fuzzy pairwise NL-closed set if its complement is neutrosophic fuzzy NL-open set. We will denote the family of neutrosophic fuzzy NL-closed sets by NPLC(X).

Theorem 3.6: Given (X,τ_i), $i\epsilon\{1,2\}$ a nfbts with neutrosophic fuzzy ideal L and $\xi,\rho,\theta\epsilon I^X$, $<\xi,\rho,\theta>$ η is neutrosophic fuzzy $\mathrm{NL}-\text{closed}$, then $P((\mathrm{Nint}\ <\xi,\rho,\theta>)^*) \leq <\xi,\rho,\theta>$.

Proof. It' s clear.

Theorem 3.7: Given (X,τ) be a nfbts with neutrosophic fuzzy ideal L on X and $\xi,\rho,\theta\epsilon I^X$ such that

$$P((\mathrm{Nint}\ <\xi,\rho,\theta>)^{*c}) = \mathrm{Nint}\, P\left(\left(<\xi,\rho,\theta>^{c^*}\right)\right),$$

then $<\xi,\rho,\theta>$ in NPLC(X) iff $P((\mathrm{Nint}\ <\xi,\rho,\theta>)^*) \leq <\xi,\rho,\theta>$.

Proof. (Necessity) Follows immediately from the above theorem. (Sufficiency).

Let $P((\mathrm{Nint}\ <\xi,\rho,\theta>)^*) \leq <\xi,\rho,\theta>$, then $<\xi,\rho,\theta>^c \leq (P(\ \mathrm{Nint}\ <\xi,\rho,\theta>))^{*c} = \mathrm{Nint} <\xi,\rho,\theta>^{c^*}$, from the hypothesis.

Hence $<\xi,\rho,\theta>^c$ in NPLO(X). Thus $<\xi,\rho,\theta>$ in NPLC(X).

Corollary 3.2: For a nfbts (X,τ_i), $i\epsilon\{1,2\}$ with neutrosophic fuzzy ideal L on X the following holds:

(i) The union of neutrosophic fuzzy NPL-closed set and neutrosophic fuzzy NP-closed set may be a neutrosophic fuzzy NPL-closed set.

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(ii) The union of neutrosophic fuzzy NPL-closed and neutrosophic fuzzy NPL-closed may be neutrosophic fuzzy NPL-closed.

Conclusions

The notions of the sets and functions in neutrosophic fuzzy bitopological spaces are highly developed and several characterizations have already been obtained. These are used extensively in many practical and engineering problems, the computational topology for geometric design, computer-aided geometric design, engineering design research, Geographic Information System (GIS) and mathematical sciences. We are in the process of preparing a statistical model for neutrosophic bitopological layers for geographic information systems.

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On Refined Neutrosophic Hypervector Spaces

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Abstract

This paper presents the refinement of neutrosophic hypervector spaces and studies some of its basic properties. Some basic definitions and important results are presented. The paper also establishes the existence of a good linear transformation between a weak refined neutrosophic hypervector space $V(I_1, I_2)$ and a weak neutrosophic hypervector space $V(I)$.
Keywords: Neutrosophy, neutrosophic hypervector space, neutrosophic subhypervector space, refined neutrosophic hypervector space, refined neutrosophic subhypervector space , refined neutrosophic hypervector space homomorphism.

1 Introduction and Preliminaries

The concept of algebraic hyperstructure was first introduced by Marty [25]. He presented the definition of a hypergroup, studied its properties and applied them to study the groups of rational algebraic functions. Also, Marty used the new approach to solve several problems of the non-commutative algebra. Since then, several researchers have been working on this new field of modern algebra and developed it to a very large extent.
M. Krasner [26], introduced the notions of hyperring and hyperfield and used them as technical tools in the study of the approximation of valued fields. There exist several types of hyperrings, some of which are: additive hyperring, multiplicative hyperring and general hyperrings. An important class of additive hyperrings is Krasner hyperrings [23,29,34].
A class of hyperrings $(R, +, .)$ where $''+''$ and $''\cdot''$ are hyperoperations was introduced by De Salvo [24]. This class of hyperrings has been further studied by Asokkumar [9], Asokkumar and Velrajan [10,11,28] and Davvaz and Leoreanu-Fotea [23]. Mittas in [27] introduced the theory of canonical hypergroups. J. Mittas was the first who studied them independently from their operations. Some connected hyperstructures with canonical hypergroups were introduced and analyzed by P. Corsini [21,22], P. Bonansinga [19,20], and K. Serafimidis in [32,33]. Further contributions to the theory of hyperstructures can be found in the books of P. Corsini [21], T. Vougiouklis , P. Corsini and V. Leoreanu [22], and Davvaz and V. Leoreanu [23]. The notion of hypervector spaces was introduced by M. Scafati Tallini . In the definition [31] of hypervector spaces, M. Scafati Tallini considered the field as the usual field. In [30], Sanjay Roy and T. K. Samanta generalized the notion of hypervector space by considering the hyperfield and considering the multiplication structure of a vector by a scalar as a hyperoperation like M. Scafati Tallini and they both called the hyperstructure a hypervector space. They established basic properties of hypervector space and thereafter the notions of linear combinations, linearly dependence, linearly independence, Hamel basis were introduced and several important properties like deletion theorem, extension theorem were developed.

Neutrosophy is a new branch of philosophy that studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophic set and neutrosophic logic were introduced in 1995 by Smarandache as generalizations of fuzzy logic/set [43] and respectively intuitionistic fuzzy logic/set [13]. In neutrosophic logic, each proposition has a degree of truth (T), a degree of indeterminancy (I) and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$ as can be seen in

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[35,36]. A comprehensive review of neutrosophic set, neutrosophic soft set, neutrosophic topological spaces, neutrosophic algebraic structures and new trends in neutrosophic theory can be found in [3,14–18,37–42].

Agboola and Davvaz introduced and studied neutrosophic hypergroups and presented some of their elementary properties in [7] and in [8], they studied and presented basic properties of canonical hypergroups and hyperrings in a neutrosophic environment, Quotient neutrosophic canonical hypergroups and neutrosophic hyperrings were also presented. In [5], Agboola and Akinleye studied neutrosophic hypervector spaces and they presented their basic properties.

In [36], Smarandache introduced the concept of refined neutrosophic logic and neutrosophic set which allows for the splitting of the components $< T, I, F >$ into the form $< T_1, T_2, \cdots, T_p; I_1, I_2, \cdot, I_r;$
$F_1, F_2, \cdots, F_s >$. This refinement has given rise to the extension of neutrosophic numbers $a + bI$ into refined neutrosophic numbers of the form $(a + b_1I_1 + b_2I_2 + \cdots + b_nI_n)$ are real or complex numbers which has led to the introduction of refined neutrosophic set. Refined neutrosophic set has been applied in the development of refined neutrosophic algebraic structures and refined neutrosophic hyperstructures. Agboola in [4] introduced the concept of refined neutrosophic algebraic structures and studied refined neutrosophic groups in particular. Since then, several researchers in this field have studied this concept and a great deal of results have been published. Recently for instance, Adeleke et al published results on refined neutrosophic rings,refined neutrosophic subring in [1] and in [2], they presented some results on refined neutrosophic ideals and refined neutrosophic homomorphism. The present paper is devoted to the study of refined neutrosophic hypervector space and presents some elementary properties of this structure.
For the purposes of this paper, it will be assumed that I splits into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. It then follows logically that:

$$\begin{array}{rcccl} I_1I_1 & = & I_1^2 & = & I_1, \\ I_2I_2 & = & I_2^2 & = & I_2, \text{ and} \\ I_1I_2 & = & I_2I_1 & = & I_1. \end{array}$$

Definition 1.1. Let $(F, +, .)$ be any field. The triple $(F(I), +, \cdot)$ is called a neutrosophic field generated by F and I. $(\mathbb{Q}(I), +, \cdot)$ and $(\mathbb{R}(I), +, \cdot)$ are examples of neutrosophic fields.

Definition 1.2. [6] Let $(V, +, .)$ be any vector space over a field K and let $V(I) =< V \cup I >$ be a neutrosophic set generated by V and I. The triple $(V(I), +, \cdot)$ is called a weak neutrosophic vector space over a field K. If $V(I)$ is a neutrosophic vector space over a neutrosophic field $K(I)$, then $V(I)$ is called a strong neutrosophic vector space. The elements of $V(I)$ are called neutrosophic vectors and the elements of $K(I)$ are called neutrosophic scalars.
If $u = a + bI, v = c + dI \in V(I)$ where a, b, c and d are vectors in V and $\alpha = k + mI \in K(I)$ where k and m are scalars in K, then :

$$u + v = (a + bI) + (c + dI) = (a + c) + (b + d)I,$$

and

$$\alpha u = (k + mI) \cdot (a + bI) = k \cdot a + (k \cdot b + m \cdot a + m \cdot b)I.$$

Definition 1.3. [23] Let H be a non-empty set and $\circ : H \times H \longrightarrow P^*(H)$ be a hyperoperation. The couple $(H, \circ)$ is called a hypergroupoid. For any two non-empty subsets A and B of H and $x \in H$, we define

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \quad A \circ x = A \circ \{x\} \quad \text{and} \quad x \circ B = \{x\} \circ B.$$

Definition 1.4. [23] A hypergroupoid $(H, \circ)$ is called a semihypergroup if for all a, b, c of H we have $(a \circ b) \circ c = a \circ (b \circ c)$, which means that

$$\bigcup_{u \in a \circ b} u \circ c = \bigcup_{v \in b \circ c} a \circ v.$$

A hypergroupoid $(H, \circ)$ is called a quasihypergroup if for all $a \in H$ we have $a \circ H = H \circ a = H$. This condition is also called the reproduction axiom.

Definition 1.5. [23] A hypergroupoid $(H, \circ)$ which is both a semihypergroup and a quasi- hypergroup is called a hypergroup.

Definition 1.6. [23] Let $(H, \circ)$ and $(H', \circ')$ be two hypergroupoids. A map $\phi : H \longrightarrow H'$, is called

1. an inclusion homomorphism if for all x, y of H, we have $\phi(x \circ y) \subseteq \phi(x) \circ' \phi(y)$;
2. a good homomorphism if for all x, y of H, we have $\phi(x \circ y) = \phi(x) \circ' \phi(y)$.

Definition 1.7. [23] Let $(H_1, \star_1)$ and $(H_2, \star_2)$ be any two refined hypergroupoids and let $f : H_1 \longrightarrow H_2$ be a map. We say that :

1. f is a homomorphism if for all x, y of H_1,
$$f(x \star_1 y) \subset f(x) \star_2 f(y);$$
2. f is a good homomorphism if for all x, y of H_1,
$$f(x \star_1 y) = f(x) \star_2 f(y);$$
3. f is a strong homomorphism on the left if
$$f(x) \in f(y) \star_2 f(z) \Longrightarrow \exists\, y' \in H_1 \ni f(y) = f(y') \text{ and } x \in y' \star_1 z.$$
Similarly, we can define a homomorphism , which is strong on the right. If f is strong on the right and on the left we say that f is a strong homomorphism.

Definition 1.8. [23] Let H be a non-empty set and let + be a hyperoperation on H. The couple $(H, +)$ is called a canonical hypergroup if the following conditions hold:

1. $x + y = y + x$, for all $x, y \in H$,
2. $x + (y + z) = (x + y) + z$, for all $x, y, z \in H$,
3. there exist a neutral element $0 \in H$ such that $x + 0 = \{x\} = 0 + x$, for all $x \in H$,
4. for every $x \in H$, there exist a unique element $-x \in H$ such that $0 \in x + (-x) \cap (-x) + x$,
5. $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$, for all $x, y, z \in H$.

Definition 1.9. [23] A hyperring is a triple $(R, +, \cdot)$ satisfying the following axioms:

1. $(R, +)$ is a canonical hypergroup.
2. $(R, \cdot)$ is a semihypergroup such that $x \cdot 0 = 0 \cdot x = 0$ for all $x \in R$, that is, 0 is a bilaterally absorbing element.
3. For all $x, y, z \in R$
   (a) $x \cdot (y + z) = x \cdot y + x \cdot z$ and
   (b) $(x + y) \cdot z = x \cdot z + y \cdot z$.

   That is, the hyperoperation $\cdot$ is distributive over the hyperoperation $+$.

Definition 1.10. [5] Let $P(V)$ be the power set of a set V, $P^*(V) = P(V) - \{\emptyset\}$ and let K be a field.
The quadruple $(V, +, \bullet, K)$ is called a hypervector space over a field K if:

1. $(V, +)$ is an abelian group.
2. $\bullet : K \times V \longrightarrow P^*(V)$ is a hyperoperation such that for all $k, m \in K$ and $u, v \in V$, the following conditions hold:
   (a) $(k + m) \bullet u \subseteq (k \bullet u) + (m \bullet u)$,
   (b) $k \bullet (u + v) \subseteq (k \bullet u) + (k \bullet v)$,
   (c) $k \bullet (m \bullet u) = (km) \bullet u$, where $k \bullet (m \bullet u) = \{k \bullet v : v \in m \bullet u\}$,
   (d) $(-k) \bullet u = k \bullet (-u)$,
   (e) $u \in 1 \bullet u$.

A hypervector space is said to be strongly left distributive (resp. strongly right distributive) if equality holds in (a) (resp. in (b)). $(V, +, \bullet, K)$ is called a strongly distributive hypervector space if it is both strongly left and strongly right distributive.

Definition 1.11. [12] Let V and W be hypervector spaces over K. A mapping $T : V \longrightarrow W$ is called

1. weak linear transformation iff

$$T(x + y) = T(x) + T(y) \text{ and } T(a \circ x) \cap a \circ T(x) \neq \emptyset,\ \forall\, x, y \in V,\ a \in K,$$

2. linear transformation iff

$$T(x + y) = T(x) + T(y) \text{ and } T(a \circ x) \subseteq a \circ T(x),\ \forall\, x, y \in V,\ a \in K,$$

3. good linear transformation iff

$$T(x + y) = T(x) + T(y) \text{ and } T(a \circ x) = a \circ T(x),\ \forall\, y \in V,\ a \in K.$$

Definition 1.12. [7] Let $(H, \star)$ be any hypergroup and let $< H \cup I >= \{x = (a, bI) : a, b \in H\}$.
The couple $N(H) = (< H \cup I >, \star)$ is called a neutrosophic hypergroup generated by H and I under the hyperoperation $\star$. The part a is called the non-neutrosophic part of x and the part b is called the neutrosophic part of x.
If $x = (a, bI)$ and $y = (c, dI)$ are any two elements of $N(H)$, where $a, b, c, d \in H$, then
$x \star y = (a, bI) \star (c, dI) = \{(u, vI)|u \in a \star c, v \in a \star d \cup b \star c \cup b \star d\} = (a \star c, (a \star d \cup b \star c \cup b \star d)I)$.
Note that $a \star c \subseteq H$ and $(a \star d \cup b \star c \cup b \star d) \subseteq H$.

Definition 1.13. [8] A neutrosophic hyperring is a triple $(N(R), +, .)$ satisfying the following axioms :

1. $(N(R), +)$ is a neutrosophic canonical hypergroup.
2. $(N(R), .)$ is a neutrosophic semihypergroup.
   For all $(a, bI), (c, dI), (e, fI) \in N(R)$,
   (a) $(a, bI).((c, dI) + (e, fI)) = (a, bI).(c, dI) + (a, bI).(e, fI)$ and
   (b) $((c, dI) + (e, fI)).(a, bI) = (c, dI).(a, bI) + (e, fI).(a, bI)$.

Definition 1.14. [6] Let $(V, +, \bullet, K)$ be any strongly distributive hypervector space over a field K and let $V(I) =< V \cup I >= \{u = (a, bI) : a, b \in V\}$ be a set generated by V and I.
The quadruple $(V(I), +, \bullet, K)$ is called a weak neutrosophic strongly distributive hypervector space over a field K.
For every $u = (a, bI), v = (c, dI) \in V(I)$ and $k \in K$, then

$$u + v = (a + c, (b + d)I) \in V(I),$$

$$k \bullet u = \{(x, yI) : x \in k \bullet a, y \in k \bullet b\}.$$

If K is a neutrosophic field, that is, $K = K(I)$, then the quadruple $(V(I), +, \bullet, K(I))$ is called a strong neutrosophic strongly distributive hypervector space over a neutrosophic field $K(I)$. For every $u = (a, bI), v = (c, dI) \in V(I)$ and $\alpha = (k, mI) \in K(I)$, we define

$$u + v = (a + c, (b + d)I) \in V(I),$$

$$\alpha \bullet u = \{(x, yI) : x \in k \bullet a, y \in k \bullet b \cup m \bullet a \cup m \bullet b\}.$$

The zero neutrosophic vector of $V(I)$, $(0, 0I)$, is denoted by θ, the zero element $0 \in K$ is represented by $(0, 0I)$ in $K(I)$ and $1 \in K$ is represented by $(1, 0I)$ in $K(I)$.

Definition 1.15. [4] If $* : X(I_1, I_2) \times X(I_1, I_2) \mapsto X(I_1, I_2)$ is a binary operation defined on $X(I_1, I_2)$, then the couple $(X(I_1, I_2), *)$ is called a refined neutrosophic algebraic structure and it is named according to the laws (axioms) satisfied by $*$.

Definition 1.16. [4] Let $(X(I_1, I_2), +, .)$ be any refined neutrosophic algebraic structure where $''+''$ and $''.''$ are ordinary addition and multiplication respectively.
For any two elements $(a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$, we define

$$(a, bI_1, cI_2) + (d, eI_1, fI_2) = (a + d, (b + e)I_1, (c + f)I_2),$$
$$(a, bI_1, cI_2).(d, eI_1, fI_2) = (ad, (ae + bd + be + bf + ce)I_1, (af + cd + cf)I_2).$$

Definition 1.17. [4] If $''+''$ and $''.''$ are ordinary addition and multiplication, I_k with $k = 1, 2$ have the following properties:

1. $I_k + I_k + \cdots + I_k = nI_k$.
2. $I_k + (-I_k) = 0$.
3. $I_k \cdot I_k \cdot \cdots \cdot I_k = I_k^n = I_k$ for all positive integers $n > 1$.
4. $0 \cdot I_k = 0$.
5. I_k^{-1} is undefined and therefore does not exist.

2 Formulation of Refined Neutrosophic Hypervector Space

This section shows the formulation of refined neutrosophic hypervector space and present some of its properties.

Definition 2.1. Let $(V, +, \bullet, K)$ be any strongly distributive hypervector space over a field K and let

$$V(I_1, I_2) =< V \cup (I_1, I_2) >= \{u = (a, bI_1, cI_2) : a, b, c \in V\}$$

be a set generated by V, I_1 and I_2. The quadruple $(V(I_1, I_2), +, \bullet, K)$ is called a weak refined neutrosophic strongly distributive hypervector space over a field K.
For every element $u = (a, bI_1, cI_2), v = (d, eI_1, fI_2) \in V(I_1, I_2)$, and $k \in K$ we define

$$u + v = (a + d, (b + e)I_1, (c + f)I_2) \in V(I_1, I_2),$$
$$k \bullet u = \{(x, yI_1, zI_2) : x \in k \bullet a, y \in k \bullet b, z \in k \bullet c\}.$$

If K is a refined neutrosophic field, that is, $K = K(I_1, I_2)$, then the quadruple $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ is called a strong refined neutrosophic strongly distributive hypervector space over a refined neutrosophic field $K(I_1, I_2)$.
For every element $u = (a, bI_1, cI_2), v = (d, eI_1, fI_2) \in V(I_1, I_2)$, and $\alpha = (k, mI_1, nI_2) \in K(I_1, I_2)$, we define

$$u + v = (a, bI_1, cI_2) + (d, eI_1, fI_2) = (a + d, (b + e)I_1, (c + f)I_2),$$
$$\alpha \bullet u = \{(x, yI_1, zI_2) : (x \in k \bullet a, y \in k \bullet b \cup m \bullet a \cup m \bullet b \cup m \bullet c \cup n \bullet b, z \in k \bullet c \cup n \bullet a \cup n \bullet c)\}.$$

The elements of $V(I_1, I_2)$ are called refined neutrosophic vectors and the elements of $K(I_1, I_2)$ are called refined neutrosophic scalars. The zero refined neutrosophic vector of $V(I_1, I_2)$, $(0, 0I_1, 0I_2)$, is denoted by θ, the zero element $0 \in K$ is represented by $(0, 0I_1, 0I_2)$ in $K(I_1, I_2)$ and $1 \in K$ is represented by $(1, 0I_1, 0I_2) \in K(I_1, I_2)$.

Example 2.2. 1. Let r be a fixed positive integer and let $V = \mathbb{Q}(I_1, I_2)(\sqrt{r}) = \{(a, b\sqrt{r}I_1, c\sqrt{r}I_2) : a, b, c \in Q, r \in \mathbb{Z}^+\}$. Then V is a weak refined neutrosophic strongly distributive hypervector space over $\mathbb{Q}$. If $u = (a, b\sqrt{r}I_1, c\sqrt{r}I_2)$ and $v = (d, e\sqrt{r}I_1, f\sqrt{r}I_2)$ then $u + v = (a + d), (b + e)\sqrt{r}I_1, (c + f)\sqrt{r}I_2$ is again in V.
Also, for $\alpha \in \mathbb{Q}$, then

$$\alpha \bullet u = \{(x, y\sqrt{r}I_1, z\sqrt{r}I_2) : x \in \alpha \bullet a, y \in \alpha \bullet b, z \in \alpha \bullet c\} \in V.$$

2. Let $V(I_1, I_2) = \mathbb{R}(I_1, I_2)$ and let $K = \mathbb{R}$. For all $u = (a, bI_1, cI_2)$, $v = (d, eI_1, fI_2) \in V(I_1, I_2)$ and $k \in K$, define:
$$u + v = (a + d, (b + e)I_1, (c + f)I_2)$$
$$k \bullet u = \{(x, yI_1, zI_2) : x \in k \bullet a, y \in k \bullet b, z \in k \bullet c\}.$$
Then $(V(I_1, I_2), +, \bullet, K)$ is a weak neutrosophic strongly distributive hypervector space over the field K.

Example 2.3. 1. Let $V(I_1, I_2) = \mathbb{R}^3(I_1, I_2)$ and let $K = \mathbb{R}(I_1, I_2)$. For all $u = ((a, bI_1, cI_2), (d, eI_1, fI_2), (g, hI_1, jI_2)), v = ((a', b'I_1, c'I_2), (d', e'I_1, f'I_2), (g', h'I_1, j'I_2)) \in V(I_1, I_2)$ and $\alpha = (k, mI_1, nI_2) \in K(I_1, I_2)$, define :
$$u+v = ((a+a', (b+b')I_1, (c+c')I_2), (d+d', (e+e')I_1, (f+f')I_2), (g+g', (h+h')I_1, (j+j')I_2)),$$
$$\begin{aligned} \alpha \bullet u \ = \ & \{((x_1, y_1I_1, z_1I_2), (x_2, y_2I_1, z_2I_2), (x_3, y_3I_1, z_3I_2)) : \\ & x_1 \in k \bullet a, \\ & y_1 \in k \bullet b \cup m \bullet a \cup m \bullet b \cup m \bullet c \cup n \bullet b, \\ & z_1 \in k \bullet c \cup n \bullet a \cup n \bullet c, \\ & x_2 \in k \bullet d, \\ & y_2 \in k \bullet e \cup m \bullet d \cup m \bullet e \cup m \bullet f \cup n \bullet e, \\ & z_2 \in k \bullet f \cup n \bullet d \cup n \bullet f \\ & x_3 \in k \bullet g, \\ & y_3 \in k \bullet h \cup m \bullet g \cup m \bullet h \cup m \bullet j \cup n \bullet h, \\ & z_3 \in k \bullet j \cup n \bullet g \cup n \bullet j\}. \end{aligned}$$
Then $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ is a strong refined neutrosophic hypervector space over the refined neutrosophic field $K(I_1, I_2)$.

2. Let $V(I_1, I_2) = \mathbb{R}^2(I_1, I_2)$ and $K = \mathbb{R}$ define for all $x = (u, v) \in V(I_1, I_2)$ with $u = (a, bI_1, cI_2)$, $v = (d, eI_1, fI_2)$ and $\alpha \in K$
$$\begin{cases} \bullet : \mathbb{R} \times \mathbb{R}^2(I_1, I_2) \longrightarrow P^*(\mathbb{R}^2(I_1, I_2)), \\ \alpha \bullet (u, v) = \alpha \bullet u \times \mathbb{R}(I_1, I_2). \end{cases} \quad \text{OR} \quad \begin{cases} \bullet : \mathbb{R} \times \mathbb{R}^2(I_1, I_2) \longrightarrow P^*(\mathbb{R}^2(I_1, I_2)), \\ \alpha \bullet (u, v) = \mathbb{R}(I_1, I_2) \times \alpha \bullet v. \end{cases}$$
Then $(V(I_1, I_2), +, \bullet, K)$ is a weak refined neutrosophic strongly distributive hypervector space.
From now on, every weak(strong) refined neutrosophic strongly distributive hypervector space will simply be called a weak(strong) refined neutrosophic hypervector space.

Lemma 2.4. *Let $V(I_1, I_2)$ be a weak refined neutrosophic hypervector space over a field K. Then for all $k \in K$ and $u = (a, bI_1, cI_2) \in V(I_1, I_2)$, we have*

1. $k \bullet \theta = \{\theta\}$.
2. $k \bullet u = \{\theta\}$ *implies that* $k = \theta$ *or* $u = \theta$.
3. $-u \in (-1) \bullet u$

Proof. 1. $k \bullet \theta = k \bullet (0 \bullet \theta) = (k.0) \bullet \theta = 0 \bullet \theta = \theta$

2. Let $k \in K$ and $u \in V$ be such that $k \bullet u = \{\theta\}$.
If $k = 0$, then $0 \bullet u = \theta$.
If $k \neq 0$, then $k^{-1} \in K$. Therefore $k \bullet u = \theta \Longrightarrow k^{-1} \bullet (k \bullet u) = k^{-1} \bullet \theta$
$\Longrightarrow (k^{-1}.k) \bullet u = \theta \Longrightarrow 1_K \bullet u = \theta \Longrightarrow u = \theta$.

□

Proposition 2.5. *Every strong refined neutrosophic hypervector space is a weak refined neutrosophic hypervector space.*

Proof. Suppose that $V(I_1, I_2)$ is a strong refined neutrosophic hypervector space over a refined neutrosophic field $K(I_1, I_2)$ say. Since $K \subseteq K(I_1, I_2)$ for every field K, then we have that $V(I_1, I_2)$ is also a weak refined neutrosophic hypervector space. □

Proposition 2.6. *Every weak refined neutrosophic hypervector space is a strongly distributive hypervector space.*

Proof. Suppose that $V(I_1, I_2)$ is a weak refined neutrosophic hypervector space over a field K. Obviously, $(V(I_1, I_2), +)$ is an abelian group. Let $u = (a, bI_1, cI_2), v = (d, eI_1, fI_2) \in V(I_1, I_2)$ and $k, m \in K$ be arbitrary. Then

$$\begin{aligned}
(1)\quad k \bullet u + m \bullet u &= \{(p, qI_1, rI_2) : p \in k \bullet a, q \in k \bullet b, r \in k \bullet c\}+\\
&\quad \{(s, tI_1, wI_2) : s \in m \bullet a, t \in m \bullet b, w \in m \bullet c\}\\
&= \{(p + s, (q + t)I_1, (r + w)I_2) : p + s \in k \bullet a + m \bullet a, q + t \in k \bullet b + m \bullet b,\\
&\quad r + w \in k \bullet c + m \bullet c\}.
\end{aligned}$$

And,

$$\begin{aligned}
(k + m) \bullet u &= \{(x, yI_1, zI_2) : x \in (k + m) \bullet a, y \in (k + m) \bullet b, z \in (k + m) \bullet c\}\\
&= \{(x, yI_1, zI_2) : x \in k \bullet a + m \bullet a, y \in k \bullet b + m \bullet b, z \in k \bullet c + m \bullet c\}\\
&= k \bullet u + m \bullet u.
\end{aligned}$$

$$\begin{aligned}
(2)\quad k \bullet u + k \bullet v &= \{(p, qI_1, rI_2) : p \in k \bullet a, q \in k \bullet b, r \in k \bullet c\}+\\
&\quad \{(s, tI_1, wI_2) : s \in k \bullet d, t \in k \bullet e, w \in k \bullet f\}\\
&= \{(p + s, (q + t)I_1, (r + w)I_2) : p + s \in k \bullet a + k \bullet d, q + t \in k \bullet b + k \bullet e,\\
&\quad r + w \in k \bullet c + k \bullet f\}.
\end{aligned}$$

And,

$$\begin{aligned}
k \bullet (u + v) &= k \bullet (a + d, (b + e)I_1, (c + f)I_2)\\
&= \{(x, yI_1, zI_2) : x \in k \bullet (a + d), y \in k \bullet (b + e), z \in k \bullet (c + f)\}\\
&= \{(x, yI_1, zI_2) : x \in k \bullet a + k \bullet d, y \in k \bullet b + k \bullet e, z \in k \bullet c + k \bullet f\}\\
&= k \bullet u + k \bullet v.
\end{aligned}$$

$$\begin{aligned}
(3)\quad k \bullet (m \bullet u) &= k \bullet \{(x, yI_1, zI_2) : x \in m \bullet a, y \in m \bullet b, z \in m \bullet c\}\\
&= \{(p, qI_1, rI_2) : p \in k \bullet x, q \in k \bullet y, r \in k \bullet z\}\\
&= \{(p, qI_1, rI_2) : p \in k \bullet (m \bullet a), q \in k \bullet (m \bullet b), r \in k \bullet (m \bullet c)\}\\
&= \{(p, qI_1, rI_2) : p \in (km) \bullet a, q \in (km) \bullet b, r \in (km) \bullet c\}\\
&= (km) \bullet (a, bI_1, cI_2)\\
&= (km) \bullet u.
\end{aligned}$$

$$\begin{aligned}
(4)\quad (-k) \bullet u &= \{(x, yI_1, zI_2) : x \in (-k) \bullet a, y \in (-k) \bullet b, z \in (-k) \bullet c\}\\
&= \{(x, yI_1, cI_2) : x \in k \bullet (-a), y \in k \bullet (-b), z \in k \bullet (-c)\}\\
&= k \bullet (-a, -bI_1, -cI_2)\\
&= k \bullet (-u).
\end{aligned}$$

$$\begin{aligned}
(5)\quad 1 \bullet u &= \{(x, yI_1, zI_2) : x \in 1 \bullet a, y \in 1 \bullet b, z \in 1 \bullet c\}\\
&= \{(x, yI_1, zI_2) : x \in \{a\}, y \in \{b\}, z \in \{c\}\}\\
&= \{(a, bI_1, cI_2)\}.\\
&\Longrightarrow u \in 1 \bullet u.
\end{aligned}$$

Accordingly, $V(I_1, I_2)$ is a strongly distributive hypervector space.

□

Corollary 2.7. *Every weak refined neutrosophic hypervector space which is strongly right distributive is strongly left distributive .*

Proof. The proof follows from the proof of Proposition 2.6 . □

Proposition 2.8. *Let $(V_1(I_1, I_2), +_1, \bullet_1, K(I_1, I_2))$ and $(V_2(I_1, I_2), +_2, \bullet_2, K(I_1, I_2))$ be two strong refined neutrosophic hypervector spaces over a refined neutrosophic field $K(I_1, I_2)$. Let*
$V_1(I_1, I_2) \times V_2(I_1, I_2) =$
$\{((a_1, b_1I_1, c_1I_2), (a_2, b_2I_1, c_2I_2)) : (a_1, b_1I_1, c_1I_2) \in V_1(I_1, I_2), (a_2, b_2I_1, c_2I_2) \in V_2(I_1, I_2)\}$ *and for all* $u = ((a_1, b_1I_1, cI_2), (a_2, b_2I_1, c_2I_2)), v = ((a'_1, b'_1I_1, c'_1I_2), (a'_2, b'_2I_2, c'_2I_2)) \in V_1(I_1, I_2) \times V_2(I_1, I_2)$ *and* $\alpha = (k, mI_1, nI_2) \in K(I_1, I_2)$, *define:*
$u + v = (((a_1 + a'_1), (b_1 + b'_1)I_1, (c_1 + c'_1)I_2), ((a_2 + a'_2), (b_2 + b'_2)I_1, (c_2 + c'_2)I_2)),$
$\alpha \bullet u = \{((x, yI_1, zI_2), (p, qI_1, rI_2))\}.$
$x \in k \bullet a_1,$
$y \in k \bullet b_1 \cup m \bullet a_1 \cup m \bullet b_1 \cup m \bullet c_1 \cup n \bullet b_1,$
$z \in k \bullet c_1 \cup n \bullet a_1 \cup n \bullet c_1,$

$p \in k \bullet a_2$,
$q \in k \bullet b_2 \cup m \bullet a_2 \cup m \bullet b_2 \cup m \bullet c_2 \cup n \bullet b_2$,
$z \in k \bullet c_2 \cup n \bullet a_2 \cup n \bullet c_2$.
Then $(V_1(I_1, I_2) \times V_2(I_1, I_2), +, \bullet, K(I_1, I_2))$ *is a strong neutrosophic hypervector space.*

Proof. Suppose that $V_1(I_1, I_2)$ and $V_2(I_1, I_2)$ are strong refined neutrosophic hypervector spaces over a refined neutrosophic field $K(I_1, I_2)$.
Let $u = ((a_1, b_1I_1, c_1I_2), (a_2, b_2I_1, c_2I_2)), v = ((a'_1, b'_1I_1, c'_1I_2), (a'_2, b'_2I_2, c'_2I_2)) \in V_1(I_1, I_2) \times V_2(I_1, I_2)$ and $\alpha = (k, mI_1, nI_2), \beta = (k', m'I_1, n'I_2) \in K(I_1, I_2)$ be arbitrary.

1. We can easily show that $(V_1(I_1, I_2) \times V_2(I_1, I_2), +)$ is an abelian group.

2. Now we want to show that $(\alpha + \beta) \bullet u \subseteq \alpha \bullet u + \beta \bullet u$.

   Consider
   $(\alpha + \beta) \bullet u = (k + k', (m + m')I_1, (n + n')I_2) \bullet ((a_1, b_1I_1, c_1I_2), (a_2, b_2I_1, c_2I_2))$
   $\subseteq \{((x, yI_1, zI_2), (p, qI_1, rI_2)) : x \in (k + k') \bullet a_1, y \in (k + k') \bullet b_1 \cup (m + m') \bullet a_1 \cup (m + m') \bullet b_1 \cup (m + m') \bullet c_1 \cup (n + n') \bullet b_1, z \in (k + k') \bullet c_1 \cup (n + n') \bullet a_1 \cup (n + n') \bullet c_1,$
   $p \in (k+k') \bullet a_2, q \in (k+k') \bullet b_2 \cup (m+m') \bullet a_2 \cup (m+m') \bullet b_2 \cup (m+m') \bullet c_2 \cup (n+n') \bullet b_2, z \in (k + k') \bullet c_2 \cup (n + n') \bullet a_2 \cup (n + n') \bullet c_2\}$

   $= \{((x, yI_1, zI_2), (p, qI_1, rI_2)) : x \in k \bullet a_1 + k' \bullet a_1, y \in k \bullet b_1 + k' \bullet b_1 \cup m \bullet a_1 + m' \bullet a_1 \cup m \bullet b_1 + m' \bullet b_1 \cup m \bullet c_1 + m' \bullet c_1 \cup n \bullet b_1 + n' \bullet b_1, z \in k \bullet c_1 + k' \bullet c_1 \cup n \bullet a_1 + n' \bullet a_1 \cup n \bullet c_1 + n' \bullet c_1,$
   $p \in k \bullet a_2 + k' \bullet a_2, q \in k \bullet b_2 + k' \bullet b_2 \cup m \bullet a_2 + m' \bullet a_2 \cup m \bullet b_2 + m' \bullet b_2 \cup m \bullet c_2 + m' \bullet c_2 \cup n \bullet b_2 + n' \bullet b_2, z \in k \bullet c_2 + k' \bullet c_2 \cup n \bullet a_2 + n' \bullet a_2 \cup n \bullet c_2 + n' \bullet c_2\}$.
   Now if we take $x = s_1 + s'_1, y = t_1 + t'_1, z = w_1 + w'_1, p = s_2 + s'_2, q = t_2 + t'_2$ and $r = w_1 + w'_2$ then we have
   $\{((s_1+s'_1, (t_1+t'_1)I_1, (w_1+w'_1)I_2), (s_2+s'_2, (t_2+t'_2)I_1, (w_1+w'_1)I_2)) : s_1+s'_1 \in k \bullet a_1 + k' \bullet a_1, t_1 + t'_1 \in k \bullet b_1 + k' \bullet b_1 \cup m \bullet a_1 + m' \bullet a_1 \cup m \bullet b_1 + m' \bullet b_1 \cup m \bullet c_1 + m' \bullet c_1 \cup n \bullet b_1 + n' \bullet b_1, w_1 + w'_1 \in k \bullet c_1 + k' \bullet c_1 \cup n \bullet a_1 + n' \bullet a_1 \cup n \bullet c_1 + n' \bullet c_1,$
   $s_2+s'_2 \in k \bullet a_2 + k' \bullet a_2, t_2 + t'_2 \in k \bullet b_2 + k' \bullet b_2 \cup m \bullet a_2 + m' \bullet a_2 \cup m \bullet b_2 + m' \bullet b_2 \cup m \bullet c_2 + m' \bullet c_2 \cup n \bullet b_2 + n' \bullet b_2, w_2 + w'_2 \in k \bullet c_2 + k' \bullet c_2 \cup n \bullet a_2 + n' \bullet a_2 \cup n \bullet c_2 + n' \bullet c_2\} = \{((s_1, t_1I_1, w_1I_2), (s_2, t_2I_1, w_2I_2)) : s_1 \in k \bullet a_1, t_1 \in k \bullet b_1 \cup m \bullet a_1 \cup m \bullet b_1 \cup m \bullet c_1 \cup n \bullet b_1, w_1 \in k \bullet c_1 \cup n \bullet a_1 \cup n \bullet c_1, \quad s_2 \in k \bullet a_2, t_2 \in k \bullet b_2 \cup m \bullet a_2 \cup m \bullet b_2 \cup m \bullet c_2 \cup n \bullet b_2, w_2 \in k \bullet c_2 \cup n \bullet a_2 \cup n \bullet c_2\} + \{((s'_1, t'_1I_1, w'_1I_2), (s'_2, t'_2I_1, w'_2I_2)) :$
   $s'_1 \in k' \bullet a_1, t'_1 \in k' \bullet b_1 \cup m' \bullet a_1 \cup m' \bullet b_1 \cup m' \bullet c_1 \cup n' \bullet b_1, w'_1 \in k' \bullet c_1 \cup n' \bullet a_1 \cup n' \bullet c_1,$
   $s'_2 \in k' \bullet a_2, t'_2 \in k' \bullet b_2 \cup m' \bullet a_2 \cup m' \bullet b_2 \cup m' \bullet c_2 \cup n' \bullet b_2, w'_2 \in k' \bullet c_2 \cup n' \bullet a_2 \cup n' \bullet c_2\}$
   $\subseteq \alpha \bullet u + \beta \bullet u$.
   Then $(\alpha + \beta) \bullet u \subseteq \alpha \bullet u + \beta \bullet u$.

3. Now we want to show that $\alpha \bullet (u + v) \subseteq \alpha \bullet u + \alpha \bullet v$
   $\alpha \bullet (u + v) = (k, mI_1, nI_2) \bullet ((a_1 + a'_1, (b_1 + b'_1)I_1, (c_1 + c'_1)I_2), (a_2 + a'_2, (b_2 + b'_2)I_1, (c_2 + c'_2)I_2))$
   $\subseteq \{(x, yI_1, zI_2), (p, qI_1, rI_2) : x \in k \bullet (a_1 + a'_1), y \in k \bullet (b_1 + b'_1) \cup m \bullet (a_1 + a'_1) \cup m \bullet (b_1 + b'_1) \cup m \bullet (c_1 + c'_1) \cup n \bullet (b_1 + b'_1), z \in k \bullet (c_1 + c'_1) \cup n \bullet (a_1 + a'_1) \cup n \bullet (c_1 + c'_1),$
   $p \in k \bullet (a_2 + a'_2), q \in k \bullet (b_2 + b'_2) \cup m \bullet (a_2 + a'_2) \cup m \bullet (b_2 + b'_2) \cup m \bullet (c_2 + c'_2) \cup n \bullet (b_2 + b'_2),$
   $r \in k \bullet (c_2 + c'_2) \cup n \bullet (a_2 + a'_2) \cup n \bullet (c_2 + c'_2)\}$

   $= \{(x, yI_1, zI_2), (p, qI_1, rI_2) : x \in k \bullet a_1 + k \bullet a'_1, y \in k \bullet b_1 + k \bullet b'_1 \cup m \bullet a_1 + m \bullet a'_1 \cup m \bullet b_1 + m \bullet b'_1 \cup m \bullet c_1 + m \bullet c'_1 \cup n \bullet b_1 + n \bullet b'_1, z \in k \bullet c_1 + k \bullet c'_1 \cup n \bullet a_1 + n \bullet a'_1 \cup n \bullet c_1 + n \bullet c'_1,$
   $p \in k \bullet a_2 + k \bullet a'_2, q \in k \bullet b_2 + k \bullet b'_2 \cup m \bullet a_2 + m \bullet a'_2 \cup m \bullet b_2 + m \bullet b'_2 \cup m \bullet c_2 + m \bullet c'_2 \cup n \bullet b_2 + n \bullet b'_2, r \in k \bullet c_2 + k \bullet c'_2 \cup n \bullet a_2 + n \bullet a'_2 \cup n \bullet c_2 + n \bullet c'_2\}$.
   If we take $x = s_1 + s'_1, y = t_1 + t'_1, z = w_1 + w'_1, p = s_2 + s'_2, q = t_2 + t'_2$ and $r = w_1 + w'_2$ then we have
   $\{(s_1+s'_1, (t_1+t'_1)I_1, (w_1+w'_1)I_2), (s_2+s'_2, (t_2+t'_2)I_1, (w_2+w'_2)I_2) : s_1+s'_1 \in k \bullet a_1 + k \bullet a'_1, t_1+t'_1 \in k \bullet b_1 + k \bullet b'_1 \cup m \bullet a_1 + m \bullet a'_1 \cup m \bullet b_1 + m \bullet b'_1 \cup m \bullet c_1 + m \bullet c'_1 \cup n \bullet b_1 + n \bullet b'_1, w_1 + w'_1 \in k \bullet c_1 + k \bullet c'_1 \cup n \bullet a_1 + n \bullet a'_1 \cup n \bullet c_1 + n \bullet c'_1, s_2 + s'_2 \in k \bullet a_2 + k \bullet a'_2, t_2 + t'_2 \in k \bullet b_2 + k \bullet b'_2 \cup m \bullet a_2 + m \bullet a'_2 \cup m \bullet b_2 + m \bullet b'_2 \cup m \bullet c_2 + m \bullet c'_2 \cup n \bullet b_2 + n \bullet b'_2, w_2 + w'_2 \in k \bullet c_2 + k \bullet c'_2 \cup n \bullet a_2 + n \bullet a'_2 \cup n \bullet c_2 + n \bullet c'_2\}$
   $= \{(s_1, t_1I_1, w_1I_2), (s_2, t_2I_1, w_2I_2) : s_1 \in k \bullet a_1, t_1 \in k \bullet b_1 \cup m \bullet a_1 \cup m \bullet b_1 \cup m \bullet c_1 \cup n \bullet b_1, w_1 \in k \bullet c_1 \cup n \bullet a_1 \cup n \bullet c_1, s_2 \in k \bullet a_2, t_2 \in k \bullet b_2 \cup m \bullet a_2 \cup m \bullet b_2 \cup m \bullet c_2 \cup n \bullet b_2, w_2 \in k \bullet c_2 \cup n \bullet a_2 \cup n \bullet c_2\} + \{(s'_1, t'_1I_1, w'_1I_2), (s'_2, t'_2I_1, w'_2I_2) : s'_1 \in k \bullet a'_1, t'_1 \in k \bullet b'_1 \cup m \bullet a'_1 \cup m \bullet b'_1 \cup m \bullet c'_1 \cup n \bullet b'_1, w'_2 \in k \bullet c'_1 \cup n \bullet a'_1 \cup n \bullet c'_1, s'_2 \in k \bullet a'_2, t'_2 \in k \bullet b'_2 \cup m \bullet a'_2 \cup m \bullet b'_2 \cup m \bullet c'_2 \cup n \bullet b'_2, w'_2 \in k \bullet c'_2 \cup n \bullet a'_2 \cup n \bullet c'_2\} \subseteq \alpha \bullet u + \alpha \bullet v$.
   Then we have that $\alpha \bullet (u + v) \subseteq \alpha \bullet u + \alpha \bullet v$.

4. $\alpha \bullet (\beta \bullet u) = \alpha \bullet \{((x, yI_1, zI_2), (p, qI_1, rI_2)) : x \in k' \bullet a_1, y \in k' \bullet b_1 \cup m' \bullet a_1 \cup m' \bullet b_1 \cup m' \bullet c_1 \cup n' \bullet b_1, z \in k' \bullet c_1 \cup n' \bullet a_1 \cup n' \bullet c_1, p \in k' \bullet a_2, q \in k' \bullet b_2 \cup m' \bullet a_2 \cup m' \bullet b_2 \cup m' \bullet c_2 \cup n' \bullet b_2, r \in k' \bullet c_2 \cup n' \bullet a_2 \cup n' \bullet c_2\}$
$= \{((x', y'I_1, z'I_2), (p', q'I_1, r'I_2)) : x' \in k \bullet x, y' \in k \bullet y \cup m \bullet x \cup m \bullet y \cup m \bullet z \cup n \bullet y,$
$z' \in k \bullet z \cup n \bullet x \cup n \bullet z, \quad p' \in k \bullet p, q' \in k \bullet q \cup m \bullet p \cup m \bullet q \cup m \bullet r \cup n \bullet q,$
$z' \in k \bullet r \cup n \bullet p \cup n \bullet r\}$
$= \{((x', y'I_1, z'I_2), (p', q'I_1, r'I_2)) : x' \in k \bullet (k' \bullet a_1), y' \in k \bullet (k' \bullet b_1 \cup m' \bullet a_1 \cup m' \bullet b_1 \cup m' \bullet c_1 \cup n' \bullet b_1) \cup m \bullet (k' \bullet a_1) \cup m \bullet (k' \bullet b_1 \cup m' \bullet a_1 \cup m' \bullet b_1 \cup m' \bullet c_1 \cup n' \bullet b_1) \cup m \bullet (k' \bullet c_1 \cup n' \bullet a_1 \cup n' \bullet c_1) \cup n \bullet (k' \bullet b_1 \cup m' \bullet a_1 \cup m' \bullet b_1 \cup m' \bullet c_1 \cup n' \bullet b_1),$
$z' \in k \bullet (k' \bullet c_1 \cup n' \bullet a_1 \cup n' \bullet c_1) \cup n \bullet (k' \bullet a_1) \cup n \bullet (k' \bullet c_1 \cup n' \bullet a_1 \cup n' \bullet c_1), \quad p' \in k \bullet (k' \bullet a_2), q' \in k \bullet (k' \bullet b_2 \cup m' \bullet a_2 \cup m' \bullet b_2 \cup m' \bullet c_2 \cup n' \bullet b_2) \cup m \bullet (k' \bullet a_2) \cup m \bullet (k' \bullet b_2 \cup m' \bullet a_2 \cup m' \bullet b_2 \cup m' \bullet c_2 \cup n' \bullet b_2) \cup m \bullet (k' \bullet c_2 \cup n' \bullet a_2 \cup n' \bullet c_2) \cup n \bullet (k' \bullet b_2 \cup m' \bullet a_2 \cup m' \bullet b_2 \cup m' \bullet c_2 \cup n' \bullet b_2),$
$z' \in k \bullet (k' \bullet c_2 \cup n' \bullet a_2 \cup n' \bullet c_2) \cup n \bullet (k' \bullet a_2) \cup n \bullet (k' \bullet c_2 \cup n' \bullet a_2 \cup n' \bullet c_2)\}$
$= \{((x', y'I_1, z'I_2), (p', q'I_1, r'I_2)) : x' \in (kk') \bullet a_1, y' \in (kk') \bullet b_1 \cup (km') \bullet a_1 \cup (km') \bullet b_1 \cup (km') \bullet c_1 \cup (kn') \bullet b_1 \cup (mk') \bullet a_1 \cup (mk') \bullet b_1 \cup (mm') \bullet a_1 \cup (mm') \bullet b_1 \cup (mm') \bullet c_1 \cup (mn') \bullet b_1 \cup (mk') \bullet c_1 \cup (mn') \bullet a_1 \cup (mn') \bullet c_1 \cup (nk') \bullet b_1 \cup (nm') \bullet a_1 \cup (nm') \bullet b_1 \cup (nm') \bullet c_1 \cup (nn') \bullet b_1, z' \in (kk') \bullet c_1 \cup (kn') \bullet a_1 \cup (kn') \bullet c_1 \cup (nk') \bullet a_1 \cup (nk') \bullet c_1 \cup (nn') \bullet a_1 \cup (nn') \bullet c_1, \quad p' \in (kk') \bullet a_2, q' \in (kk') \bullet b_2 \cup (km') \bullet a_2 \cup (km') \bullet b_2 \cup (km') \bullet c_2 \cup (kn') \bullet b_2 \cup (mk') \bullet a_2 \cup (mk') \bullet b_2 \cup (mm') \bullet a_2 \cup (mm') \bullet b_2 \cup (mm') \bullet c_2 \cup (mn') \bullet b_2 \cup (mk') \bullet c_2 \cup (mn') \bullet a_2 \cup (mn') \bullet c_2 \cup (nk') \bullet b_2 \cup (nm') \bullet a_2 \cup (nm') \bullet b_2 \cup (nm') \bullet c_2 \cup (nn') \bullet b_2,$
$z' \in (kk') \bullet c_2 \cup (kn') \bullet a_2 \cup (kn') \bullet c_2 \cup (nk') \bullet a_2 \cup (nk') \bullet c_2 \cup (nn') \bullet a_2 \cup (nn') \bullet c_2\}$
$= ((k, mI_1, nI_2)(k', m'I_1, nI_2)) \bullet ((a_1, b_1I_1, c_1I_2)(a_2, b_2I_1, c_2I_2))$
$= (\alpha\beta) \bullet u.$

5. $(-\alpha)u = \{(x, yI_1, zI_2)(p, qI_1, rI_2)) : x \in -k \bullet a_1, y \in -k \bullet b_1 \cup -m \bullet a_1 \cup -m \bullet b_1 \cup -m \bullet c_1 \cup -n \bullet b_1, z \in -k \bullet c_1 \cup -n \bullet a_1 \cup -n \bullet c_1, p \in -k \bullet a_2, q \in -k \bullet b_2 \cup -m \bullet a_2 \cup -m \bullet b_2 \cup -m \bullet c_2 \cup -n \bullet b_2, r \in -k \bullet c_2 \cup -n \bullet a_2 \cup -n \bullet c_2\}$
$= \{(x, yI_1, zI_2(p, qI_1, rI_2)) : x \in k \bullet (-a_1), y \in k \bullet (-b_1) \cup m \bullet (-a_1) \cup m \bullet (-b_1) \cup m \bullet (-c_1) \cup n \bullet (-b_1), z \in k \bullet (-c_1) \cup n \bullet (-a_1) \cup n \bullet (-c_1), p \in k \bullet (-a_2), q \in k \bullet (-b_2) \cup m \bullet (-a_2) \cup m \bullet (-b_2) \cup m \bullet (-c_2) \cup n \bullet (-b_2), r \in k \bullet (-c_2) \cup n \bullet (-a_2) \cup n \bullet (-c_2)\}$
$= (k, mI_1, nI_2)(-((a_1, b_1I_1, c_1I_2)(a_2, b_2I_1, c_2I_2))) = \alpha(-u).$

6. $1 \bullet u = \{(x, yI_1, zI_2)(p, qI_1, rI_2)) : x \in 1 \bullet a_1,\ y \in 1 \bullet b_1, z \in 1 \bullet c_1,\ p \in 1 \bullet a_2, q \in 1 \bullet b_2, r \in 1 \bullet c_2\}$
$= \{(a_1, b_1I_1, c_1I_2)(a_2, b_2I_1, c_2I_2)) : a_1 \in 1 \bullet a_1,\ b_1 \in 1 \bullet b_1,\ c_1 \in 1 \bullet c_1,\ a_2 \in 1 \bullet a_2, b_2 \in 1 \bullet b_2,\ c_2 \in 1 \bullet c_2\}$, which shows that $u \in 1 \bullet u$.
Accordingly, $V_1(I_1, I_2) \times V_2(I_1, I_2)$ is a strong refined neutrosophic hypervector space.

□

Proposition 2.9. *Let $(V(I_1, I_2), \oplus, \bullet_1, K)$ and $(H, +_H, \bullet_H, K)$ be a weak refined neutrosophic hypervector spaces and a hypervector space, respectively. Let*
$V(I_1, I_2) \times H = \{((a, bI_1, cI_2), h) : (a, bI_1, cI_2) \in V_1(I_1, I_2), h \in H\}$.
For all $u = ((a, bI_1, cI_2), h)$, $v = ((a', b'I_1, c'I_2), g) \in V(I_1, I_2) \times H$ and $k \in K$, define:
$u + v = ((a \oplus a', (b \oplus b')I_1, (c \oplus c')I_2), h +_H g)$,
$k \bullet u = \{((x, yI_1, zI_2), p) : x \in k \bullet_1 a, y \in k \bullet_1 b \in k \bullet_1 c, p \in k \bullet_H h\}$.
Then $(V_1(I_1, I_2) \times H, +, \bullet, K)$ is a weak neutrosophic hypervector space.

Proof. The proof follows from the same pattern as the proof of Proposition 2.8 . □

Definition 2.10. Let $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ be a strong refined neutrosophic hypervector space over a refined neutrosophic field $K(I_1, I_2)$ and let $W[I_1, I_2]$ be a nonempty subset of $V(I_1, I_2)$. $W[I_1, I_2]$ is said to be a subhypervector space of $V(I_1, I_2)$ if $(W[I_1, I_2], +, \bullet, K(I_1, I_2))$ is also a refined neutrosophic hypervector space over the refined neutrosophic field $K(I_1, I_2)$. It is essential that $W[I_1, I_2]$ contains a proper subset which is a hypervector space over a field K.

Example 2.11. Let $V(I_1, I_2) = \mathbb{R}^2(I_1, I_2)$ and $K = \mathbb{R}(I_1, I_2)$ then $(\mathbb{R}^2(I_1, I_2), +, \bullet, K(I_1, I_2))$ is a strong refined neutrosophic hypervector space over refined neutrosophic field $K = R(I_1, 1_2)$, where the hyperoperations + and $\bullet$ are defined $\forall\ u = ((a_1, b_1I_1, c_1I_1), (a_2, b_2I_1, cI_2))$,
$v = ((a_1', b_1'I_1, c_1'I_1), (a_2', b_2I_1, c'I_2)) \in V(I_1, I_2)$ by :
$u + v = ((a_1 + a_1', (b_1 + b_1')I_1, (c_1 + c_1')I_2), (a_2 + a_2', (b_2 + b_2')I_1, (c_2 + c_2')I_2))$,

$\alpha \bullet u = \{((x, yI_1, zI_2), (p, qI_1, rI_2)) : x \in k \bullet a_1, y \in k \bullet b_1 \cup m \bullet a_1 \cup m \bullet b_1 \cup m \bullet c_1 \cup n \bullet b_1,$
$z \in k \bullet c_1 \cup n \bullet a_1 \cup n \bullet c_1,\ p \in k \bullet a_2, q \in k \bullet b_2 \cup m \bullet a_2 \cup m \bullet b_2 \cup m \bullet c_2 \cup n \bullet b_2,$
$r \in k \bullet c_2 \cup n \bullet a_2 \cup n \bullet c_2\}$.
Let $W(I_1, I_2) = K(I_1, I_2) \times \{(0, 0I_1, 0I_2)\} \subseteq V(I_1, I_2)$.
Then $W(I_1, I_2)$ is a strong refined neutrosophic subhypervector space.

Proof. Since $\theta = ((0, 0I_1, 0I_2), (0, 0I_1, 0I_2)) \in W(I_1, I_2)$. Then $W(I_1, I_2) \neq \emptyset$.
Now let $u_1 = ((a_1, b_1I_1, c_2I_2), (0, 0I_1, 0I_2))$, $v_1 = ((a_1', b1I_1, c_1I_2)(0, 0I_1, 0I_2)) \in W(I_1, I_2)$, and $\alpha = (k, mI_1, nI_2), \beta = (k', m'I_1, n'I_2) \in K(I_1, I_2)$ with $a_1, b_1, c_1, a_1', b_1', c_1',\ k, m, n, k', m', n' \in \mathbb{R}$.

$$\begin{aligned}
\alpha \bullet u_1 + \beta \bullet v_1 &= (k, mI_1, nI_2) \bullet [(a_1, b_1I_1, c_1I_2), (0, 0I_1, 0I_2)] + \\
&\quad (k', m'I_1, n'I_2) \bullet [(a_1', b1I_1, c_1I_2)(0, 0I_1, 0I_2)] \\
&\subseteq \{((x, yI_1, zI_2), (p, qI_1, rI_2)) : x \in k \bullet a_1, y \in k \bullet b_1 \cup m \bullet a_1 \cup m \bullet b_1 \cup \\
&\quad m \bullet c_1 \cup n \bullet b_1, z \in k \bullet c_1 \cup n \bullet a_1 \cup n \bullet c_1, p \in k \bullet 0, \\
&\quad q \in k \bullet 0 \cup m \bullet 0 \cup m \bullet 0 \cup m \bullet 0 \cup n \bullet 0, r \in k \bullet 0 \cup n \bullet 0 \cup n \bullet 0\} \\
&\quad + \{((x', y'I_1, z'I_2), (p', q'I_1, r'I_2)) : x' \in k' \bullet a_1', \\
&\quad y' \in k' \bullet b_1' \cup m' \bullet a_1' \cup m' \bullet b_1' \cup m' \bullet c_1' \cup n' \bullet b_1', \\
&\quad z' \in k' \bullet c_1' \cup n' \bullet a_1' \cup n' \bullet c_1', \\
&\quad p' \in k' \bullet 0, q' \in k' \bullet 0 \cup m' \bullet 0 \cup \\
&\quad m' \bullet 0 \cup m' \bullet 0 \cup n' \bullet 0, r' \in k' \bullet 0 \cup n' \bullet 0 \cup n' \bullet 0\} \\
&= \{((s, tI_1, wI_2), (s', t'I_1, w'I_2)) : s \in k \bullet a_1 + k' \bullet a_1', \\
&\quad t \in k \bullet b_1 + k' \bullet b_1' \cup m \bullet a_1 + m' \bullet a_1' \cup m \bullet b_1 + m' \bullet b_1' \cup m \bullet c_1 + \\
&\quad m' \bullet c_1' \cup n \bullet b_1 + \cup n' \bullet b_1', \\
&\quad w \in k \bullet c_1 + k' \bullet c_1' \cup n \bullet a_1 + n' \bullet a_1' \cup n \bullet c_1 + \cup n' \bullet c_1', \\
&\quad s' \in 0, t' \in 0, w' \in 0\} \subseteq W(I_1, I_2). \\
&\Longrightarrow \alpha \bullet u_1 + \beta \bullet v_1 \subseteq W(I_1, I_2).
\end{aligned}$$

Lastly, we can see from the definition of $W(I_1, I_2)$ that $W(I_1, I_2)$ contains a proper subset which is a hypervector space over K.
To this end we can conclude that $W(I_1, I_2)$ is a strong refined neutrosophic hypervector space. □

Proposition 2.12. *Let $W_1[I_1, I_2], W_2[I_1, I_2], \cdots, W_n[I_1, I_2]$ be refined neutrosophic subhypervector spaces of a strong refined neutrosophic hypervector space $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ over a refined neutrosophic field $K(I_1, I_2)$. Then $\bigcap_{i=1}^{n} W_i[I_1, I_2]$ is a refined neutrosophic subhypervector space of $V(I_1, I_2)$.*

Proof. Consider the collection of refined neutrosophic subhypervector space
$\{W_i(I_1, I_2) : i = 1, 2, \cdots n\}$ of a strong refined neutrosophic hypervector space $V(I_1, I_2)$.
Take $u = (a, bI_1, cI_2), v = (d, eI_1, fI_2), \alpha = (k, pI_1, qI_2)$ and $\beta = (r, sI_1, tI_2)$.
Let $u, v \in \bigcap_{i=1}^{n} W_i(I_1, I_2)$ then $u, v \in W_i(I_1, I_2)$ for all $i = 1, 2, \cdots n$.
Now for all scalars $\alpha, \beta \in K(I_1, I_2)$ we have that
$\alpha \bullet u + \beta \bullet v = (k, pI_1, qI_2) \bullet (a, bI_1, cI_2) + (r, sI_1, tI_2) \bullet (d, eI_1, fI_2)$
$\subseteq \{(x, yI_1, zI_2) : x \in k \bullet a, y \in k \bullet b \cup p \bullet a \cup p \bullet b \cup p \bullet c \cup q \bullet b, z \in k \bullet c \cup q \bullet a \cup q \bullet c\} +$
$\{(x', y'I_1, z'I_2) : x' \in r \bullet d, y' \in r \bullet e \cup s \bullet d \cup s \bullet e \cup s \bullet f \cup t \bullet e, z' \in r \bullet f \cup t \bullet d \cup t \bullet f\}$
$= \{(x + x', (y + y')I_1, (z + z')I_2) : x + x' \in k \bullet a + r \bullet d,$
$y + y' \in k \bullet b + r \bullet e \cup p \bullet a + s \bullet d \cup p \bullet b + s \bullet e \cup p \bullet c + s \bullet f \cup q \bullet b + t \bullet e,$
$z + z' \in k \bullet c + r \bullet f \cup q \bullet a + t \bullet d \cup q \bullet c + t \bullet f\} \subseteq W_i(I_1, I_2)\ \forall i = 1, 2, 3 \cdots, n.$
$\Longrightarrow \alpha \bullet u + \beta \bullet v \subseteq \bigcap_{i=1}^{n} W_i(I_1, I_2)$.
Lastly, since $W_i(I_1, I_2)\ \forall\ i = 1, 2, 3, \cdots, n$ contain proper subsets W_i which are hypervector space,
$\bigcap_{i=1}^{n} W_i(I_1, I_2)$ is a strong refined neutrosophic subhyperspace. □

Proposition 2.13. *Let $W[I_1, I_2]$ be a subset of a strong refined neutrosophic hypervector space $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ over a refined neutrosophic field $K(I_1, I_2)$. Then $W[I_1, I_2]$ is a refined neutrosophic subhypervector space of $V(I_1, I_2)$ if and only if for all $u = (a, bI_1, cI_2), v = (d, eI_1, I_2) \in V(I_1, I_2)$ and $\alpha = (k, mI_1, nI_2) \in K(I_1, I_2)$, the following conditions hold:*

1. $W[I_1, I_2] \neq \emptyset$,
2. $u + v \in W[I_1, I_2]$,
3. $\alpha \bullet u \subseteq W[I_1, I_2]$,
4. $W[I_1, I_2]$ *contains a proper subset which is a hypervector space over* K.

Proposition 2.14. *Let $V(I_1, I_2)$ be a strong refined neutrosophic hypervector space over $K(I_1, I_2)$ and let $U_1(I_1, I_2), U_2(I_1, I_2)$ be any strong refined neutrosophic subhypervector spaces of $V(I_1, I_2)$. Then $U_1(I_1, I_2) \cup U_2(I_1, I_2)$ is a strong refined neutrosophic subhypervector space if and only if $U_1(I_1, I_2) \subseteq U_2(I_1, I_2)$ or $U_1(I_1, I_2) \supseteq U_2(I_1, I_2)$.*

Proof. Let $U_1(I_1, I_2)$ and $U_2(I_1, I_2)$ be any strong refined neutrosophic subhypervector spaces of $V(I_1, I_2)$.
$\Longrightarrow$ Now, suppose $U_1(I_1, I_2) \subseteq U_2(I_1, I_2)$ or $U_1(I_1, I_2) \supseteq U_2(I_1, I_2)$ then we shall show the $U_1(I_1, I_2) \cup U_2(I_1, I_2)$ is a strong refined neutrosophic subhypervector space of $V(I_1, I_2)$.
Without loss of generality, suppose that $U_1(I_1, I_2) \subseteq U_2(I_1, I_2)$.
Then we have that $U_1(I_1, I_2) \cup U_2(I_1, I_2) = U_2(I_1, I_2)$. But $U_2(I_1, I_2)$ is defined to be a strong refined neutrosophic subhypervector space of $V(I_1, I_2)$, so $U_1(I_1, I_2) \cup U_2(I_1, I_2)$ is a strong refined neutrosophic subhypervector space of $V(I_1, I_2)$.
$\Longleftarrow$ We want to show that if $U_1(I_1, I_2) \cup U_2(I_1, I_2)$ is a strong refined neutrosophic subhypervector space of $V(I_1, I_2)$ then either $U_1(I_1, I_2) \subseteq U_2(I_1, I_2)$ or $U_1(I_1, I_2) \supseteq U_2(I_1, I_2)$.
Now suppose that $U_1(I_1, I_2) \cup U_2(I_1, I_2)$ is a strong refined neutrosophic subhypervector space of $V(I_1, I_2)$ and suppose by contradiction that $U_1(I_1, I_2) \not\subseteq U_2(I_1, I_2)$ or $U_1(I_1, I_2) \not\supseteq U_2(I_1, I_2)$.
Thus there exist elements $x_1 = (a_1 + b_1I_1 + c_1I_2) \in U_1(I_1, I_2) \backslash U_2(I_1, I_2)$ and $x_2 = (a_2 + b_2I_1 + c_2I_2) \in U_2(I_1, I_2) \backslash U_1(I_1, I_2)$. So we have that $x_1, x_2 \in U_1(I_1, I_2) \cup U_2(I_1, I_2)$, since $U_1(I_1, I_2) \cup U_2(I_1, I_2)$ is a strong refined neutrosophic subhypervector space, we must have that $x_1 + x_2 = x_3 \in U_1(I_1, I_2) \cup U_2(I_1, I_2)$.
Therefore $x_1 + x_2 = x_3 \in U_1(I_1, I_2)$ or $x_1 + x_2 = x_3 \in U_2(I_1, I_2)$
$\Longrightarrow x_2 = x_3 - x_1 \in U_1(I_1, I_2)$ or $x_1 = x_3 - x_2 \in U_2(I_1, I_2)$ which is a contradiction.
Hence $U_1(I_1, I_2) \subseteq U_2(I_1, I_2)$ or $U_1(I_1, I_2) \supseteq U_2(I_1, I_2)$ as required. □

Remark 2.15. If $W_1[I_1, I_2]$ and $W_2[I_1, I_2]$ are refined neutrosophic subhypervector spaces of a strong refined neutrosophic hypervector space $V(I_1, I_2)$ over a refined neutrosophic field $K(I_1, I_2)$, then generally, $W_1[I_1, I_2] \bigcup W_2[I_1, I_2]$ is not a refined neutrosophic subhypervector space of $V(I_1, I_2)$ except if $W_1[I_1, I_2] \subseteq W_2[I_1, I_2]$ or $W_2[I_1, I_2] \subseteq W_1[I_1, I_2]$.

Definition 2.16. Let $W_1[I_1, I_2]$ and $W_2[I_1, I_2]$ be two refined neutrosophic subhypervector spaces of a strong refined neutrosophic hypervector space $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ over a refined neutrosophic field $K(I_1, I_2)$. The sum of $W_1[I_1, I_2]$ and $W_2[I_1, I_2]$ denoted by $W_1[I_1, I_2] + W_2[I_1, I_2]$ is defined by the set

$$\bigcup\{w + x : w = (a, bI_1, cI_2) \in W_1[I_1, I_2], x = (d, eI_1, fI_2) \in W_2[I_1, I_2]\}.$$

If $W_1[I_1, I_2] \cap W_2[I_1, I_2] = \{\theta\}$, then the sum of $W_1[I_1, I_2]$ and $W_2[I_1, I_2]$ is denoted by $W_1[I_1, I_2] \oplus W_2[I_1, I_2]$ and it is called the direct sum of $W_1[I_1, I_2]$ and $W_2[I_1, I_2]$.

Proposition 2.17. *Let $W_1[I_1, I_2]$ and $W_2[I_1, I_2]$ be two refined neutrosophic subhypervector spaces of a strong refined neutrosophic hypervector space $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ over a refined neutrosophic field $K(I_1, I_2)$.*

1. *$W_1[I_1, I_2] + W_2[I_1, I_2]$ is a refined neutrosophic subhypervector space of $V(I_1, I_2)$.*
2. *$W_1[I_1, I_2] + W_2[I_1, I_2]$ is the least refined neutrosophic subhypervector space of $V(I_1, I_2)$ containing $W_1[I_1, I_2]$ and $W_2[I_1, I_2]$.*

Proof. 1. Since $\theta \in W_1[I_1, I_2]$ and $\theta \in W_2[I_1, I_2]$, $\{\theta + \theta\} \subseteq W_1[I_1, I_2] + W_2[I_1, I_2]$.
So,$\{\theta\} \subseteq W_1[I_1, I_2] + W_2[I_1, I_2] \Longrightarrow \theta \in W_1[I_1, I_2] + W_2[I_1, I_2]$, therefore $W_1[I_1, I_2] + W_2[I_1, I_2]$ is non-empty.
Let $u = (a, bI_1, cI_2), v = (d, eI_1, fI_2) \in W_1[I_1, I_2] + W_2[I_1, I_2]$, then $\exists\ u_1 = (a_1, b_1I_1, c_1I_2)$, $u_2 = (a_2, b_2I_1, c_2I_2) \in W_1[I_1, I_2]$ and $v_1 = (d_1, e_1I_1, f_1I_2), v_2 = (d_2, e_2I_1, f_2I_2) \in W_2[I_1, I_2]$ such that $u \in u_1 + v_1$ and $v \in u_2 + v_2$.
Let $\alpha = (k, mI_1, nI_2), \beta = (k', m'I_1, n'I_2) \in K(I_1, I_2)$.

$$\begin{aligned}
& \text{Now } \alpha \bullet u + \beta \bullet v \subseteq \alpha \bullet (u_1 + v_1) + \beta \bullet (u_2 + v_2) \\
= \ & (k, mI_1, nI_2) \bullet (a_1 + d_1, (b_1 + e_1)I_1, (c_1 + e_1)I_2)+ \\
& (k', m'I_1, n'I_2) \bullet (a_2 + d_2, (b_2 + e_2)I_1, (c_2 + e_2)I_2) \\
\subseteq \ & \{(x_1, y_1I_1, z_1I_2) : x_1 \in k \bullet (a_1 + d_1), \\
& y_1 \in k \bullet (b_1 + e_1) \cup m \bullet (a_1 + d_1) \cup m \bullet (b_1 + e_1) \cup m \bullet (c_1 + f_1) \cup n \bullet (b_1 + e_1), \\
& z_1 \in k \bullet (c_1 + f_1) \cup n \bullet (a_1 + d_1) \cup n \bullet (c_1 + f_1)\} \\
& +\{(x_2, y_2I_1, z_2I_2) : x_2 \in k' \bullet (a_2 + d_2), \\
& y_2 \in k' \bullet (b_2 + e_2) \cup m' \bullet (a_2 + d_2) \cup m' \bullet (b_2 + e_2) \cup m' \bullet (c_2 + f_2) \cup n' \bullet (b_2 + e_2), \\
& z_2 \in k' \bullet (c_2 + f_2) \cup n' \bullet (a_2 + d_2) \cup n' \bullet (c_2 + f_2)\}
\end{aligned}$$

$$\begin{aligned}
&= \{(x, yI_1, zI_2) : x \in (k \bullet a_1 + k \bullet d_1 + k' \bullet a_2 + k' \bullet d_2),\\
&\quad y \in (k \bullet b_1 + k \bullet e_1 + k' \bullet b_2 + k' \bullet e_2) \cup (m \bullet a_1 + m \bullet d_1 + m' \bullet a_2 + m' \bullet d_2) \cup\\
&\quad (m \bullet b_1 + m \bullet e_1 + m' \bullet b_2 + m' \bullet e_2) \cup (m \bullet c_1 + m \bullet f_1 + m' \bullet c_2 + m' \bullet f_2) \cup\\
&\quad (n \bullet b_1 + n \bullet e_1 + n' \bullet b_2 + n' \bullet e_2),\\
&\quad z \in (k \bullet c_1 + k \bullet f_1 + k' \bullet c_2 + k' \bullet f_2) \cup (n \bullet a_1 + n \bullet d_1 + n' \bullet a_2 + n' \bullet d_2) \cup\\
&\quad (n \bullet c_1 + n \bullet f_1 + n' \bullet c_2 + n' \bullet f_2)\}\\
&= \{(s_1, t_1I_1, w_1I_2) : s_1 \in (k \bullet a_1 + k' \bullet a_2),\\
&\quad t_1 \in (k \bullet b_1 + k' \bullet b_2) \cup (m \bullet a_1 + m' \bullet a_2) \cup (m \bullet b_1 + m' \bullet b_2) \cup (m \bullet c_1 + m' \bullet c_2) \cup\\
&\quad (n \bullet b_1 + n' \bullet b_2), w_1 \in (k \bullet c_1 + k' \bullet c_2) \cup (n \bullet a_1 + n' \bullet a_2) \cup (n \bullet c_1 + n' \bullet c_2)\}+\\
&\quad \{(s_2, t_2I_1, w_2I_2) : s_2 \in (k \bullet d_1 + k' \bullet d_2),\\
&\quad t_2 \in (k \bullet e_1 + k' \bullet e_2) \cup (m \bullet d_1 + m' \bullet d_2) \cup (m \bullet e_1 + m' \bullet e_2) \cup (m \bullet f_1 + m' \bullet f_2) \cup\\
&\quad (n \bullet e_1 + n' \bullet e_2),\\
&\quad w_2 \in (k \bullet f_1 + k' \bullet f_2) \cup (n \bullet d_1 + n' \bullet d_2) \cup (n \bullet f_1 + n' \bullet f_2)\}\\
&\subseteq W_1[I_1, I_2] + W_2[I_1, I_2].
\end{aligned}$$

Hence $\alpha \bullet u + \beta \bullet v \subseteq W_1[I_1, I_2] + W_2[I_1, I_2]$.
Now since W_1, W_2 are proper subsets of $W_1[I_1, I_2]$ and $W_2[I_1, I_2]$ respectively, with both W_1 and W_2 being hypervector space. Then $W_1 + W_2$ is a hypervector space which is properly contained in $W_1[I_1, I_2] + W_2[I_1, I_2]$. Then we can conclude that $W_1[I_1, I_2] + W_2[I_1, I_2]$ is a refined neutrososphic subhypervector space.

2. Let $W[I_1, I_2]$ be refined neutrosophic subhypervector space of $V[I_1, I_2]$ such that $W_1[I_1, I_2] \subseteq W[I_1, I_2]$ and $W_2[I_1, I_2] \subseteq W[I_1, I_2]$.
Let $u = (a, bI_1, cI_2) \in W_1[I_1, I_2] + W_2[I_1, I_2]$, then $\exists u_1 = (a_1, b_1I_1, c_1I_2) \in W_1[I_1, I_2]$ and $u_2 = (a_2, b_2I_1, c_2I_2) \in W_2[I_1, I_2]$ such that $u \in u_1 + u_2$.
Since $W_1[I_1, I_2] \subseteq W[I_1, I_2]$ and $W_2[I_1, I_2] \subseteq W[I_1, I_2]$, then $u_1, u_2 \in W[I_1, I_2]$.
Again since $W[I_1, I_2]$ is a refined neutrosophic subhypervector space of $V[I_1, I_2]$, then we have that $u_1 + u_2 \subseteq W[I_1, I_2] \Longrightarrow u \in W[I_1, I_2]$.
Hence $W_1[I_1, I_2] + W_2[I_1, I_2] \subseteq W[I_1, I_2]$ and the proof follows.

□

Remark 2.18. If $V(I_1, I_2)$ is a weak refined neutrosophic strongly left distributive hypervector space over a field K, then

1. $W[I_1, I_2] = \bigcup\{k \bullet u : k \in K\}$ forms a weak refined neutrosophic subhypervector space of $V(I_1, I_2)$, where $u = (a, bI_1, cI_2) \in V(I_1, I_2)$. This refined neutrosophic subhypervector space is said to be generated by the refined neutrosophic vector u and it is called a refined neutrosophic hyperline span by the refined neutrosophic vector u.

2. If $u = (a, bI_1, cI_2), v = (d, eI_1, fI_2) \in V(I_1, I_2)$, then the set $W = \bigcup\{\alpha \bullet u + \beta \bullet v, \alpha, \beta \in K\}$ is a weak refined neutrosophic subhypervector space of $V(I_1, I_2)$. This refined neutrosophic subhypervector space is called refined neutrosophic hyperlinear span of the refined neutrosophic vectors u and v.

Proposition 2.19. *Let $V(I_1, I_2)$ be a weak refined neutrosophic strongly left distributive hypervector space over the field K and $u_1, u_2, \cdots, u_n \in V(I_1, I_2)$, with $u_i = (a_i, b_iI_1, c_iI_2)$ for $i = 1, 2, 3 \cdots n$. Then*

1. *$W(I_1, I_2) = \bigcup\{\alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \cdots + \alpha_n \bullet u_n : \alpha_1, \alpha_2, \cdots, \alpha_n \in K\}$ is a weak refined neutrosophic subhypervector space of $V(I_1, I_2)$.*

2. *$W(I_1, I_2)$ is the smallest weak refined neutrosophic subhypervector space of $V(I_1, I_2)$ containing $u_1, u_2, \cdots, u_n$.*

Proof. 1. The proof follows from similar approach as 1 of Proposition 2.17 .

2. Suppose that $M(I_1, I_2)$ is a weak refined neutrosophic subhypervector space of $V(I_1, I_2)$ containing $u_1 = (a_1, b_1I_1, c_1I_2), u_2 = (a_2, b_2I_1, c_2I_2), \cdots, u_n = (a_n, b_nI_1, c_nI_2)$. Let $t \in W(I_1, I_2)$, then there exist $\alpha_1, \alpha_2, \cdots, \alpha_n \in K$ such that

$$t \in \alpha_1 \bullet (a_1, b_1I_1, c_1I_2) + \alpha_2 \bullet (a_2, b_2I_1, c_2I_2) + \cdots + \alpha_n \bullet (a_n, b_nI_1, c_nI_2).$$

Therefore $t \in M(I_1, I_2) \Longrightarrow W(I_1, I_2) \subseteq M(I_1, I_2)$.
Hence $W(I_1, I_2)$ is the smallest weak refined neutrosophic subhypervector space of $V(I_1, I_2)$ containing $u_1, u_2, \cdots, u_n$.

□

Proposition 2.20. *Let $V(I_1, I_2)$ be a strong refined neutrosophic hypervector space over a refined neutrosophic field $K(I_1, I_2)$, and let*
$u_1 = (a_1, b_1I_1, c_1I_2), u_2 = (a_2, b_2I_1, c_2I_2), \cdots, u_n = (a_n, b_nI_1, c_nI_2) \in V(I_1, I_2)$,
$\alpha_1 = (k_1, m_1I_1, t_1I_2), \alpha_2 = (k_2, m_2I_1, t_2I_2) \cdots, \alpha_n = (k_n, m_nI_1, t_nI_2)$.
Then:

1. $W(I_1, I_2) = \bigcup\{\alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \cdots + \alpha_n \bullet u_n : \alpha_1, \alpha_2, \cdots, \alpha_n \in K(I_1, I_2)\}$ *is a refined neutrosophic subhypervector space of* $V(I_1, I_2)$.
2. $W(I_1, I_2)$ *is the smallest refined neutrosophic subhypervector space of* $V(I_1, I_2)$ *containing* $u_1, u_2, \cdots, u_n$.

Proof: The proof follows from similar approach as that of Proposition 2.19 .

Remark 2.21. The refined neutrosophic subhypervector space $W(I_1, I_2)$ of the strong refined neutrosophic hypervector space $V(I_1, I_2)$ over a refined neutrosophic field $K(I_1, I_2)$ of Proposition 2.20 is said to be generated by the refined neutrosophic vectors $u_1, u_2, \cdots, u_n$ and we write $W(I_1, I_2) = span\{u_1, u_2, \cdots, u_n\}$.

Definition 2.22. Let $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ be a strong refined neutrosophic hypervector space over a refined neutrosophic field $K(I_1, I_2)$ and let
$B(I_1, I_2) = \{u_1 = (a_1, b_1I_1, c_1I_2), u_2 = (a_2, b_2I_1, c_2I_2), \cdots, u_n = (a_n, b_nI_1, c_nI_2)\}$ be a subset of $V(I_1, I_2)$. $B(I_1, I_2)$ is said to generate or span $V(I_1, I_2)$ if $V(I_1, I_2) = span(B(I_1, I_2))$.

Example 2.23. Let $V(I_1, I_2) = \mathbb{R}^3(I_1, I_2)$ be a strong refined neutrosophic hypervector space over a neutrosophic field $R(I_1, I_2)$ and let $B(I_1, I_2) = \{u_1 = ((1, 0I_1, 0I_2), (0, 0I_1, 0I_2), (0, 0I_1, 0I_2))$,
$u_2 = ((0, 0I_1, 0I_2), (1, 0I_1, 0I_2), (0, 0I_1, 0I_2)), u_3 = ((0, 0I_1, 0I_2), (0, 0I_1, 0I_2), (1, 0I_1, 0I_2))\}$.
Then $B(I_1, I_2)$ spans $V(I_1, I_2)$.

Example 2.24. Let $V(I_1, I_2) = \mathbb{R}^2(I_1, I_2)$ be a weak refined neutrosophic hypervector space over a field $\mathbb{R}$ and let $B(I_1, I_2) = \{u_1 = ((1, 0I_1, 0I_2), (0, 0I_1, 0I_2)), u_2 = ((0, 0I_1, 0I_2), (1, 0I_1, 0I_2))$,
$u_3 = ((0, I_1, 0I_2), (0, 0I_1, 0I_2)), u_4 = ((0, 0I_1, 0I_2), (0, I_1, 0I_2)), u_5 = ((0, 0I_1, I_2), (0, 0I_1, 0I_2))$,
$u_6 = ((0, 0I_1, 0I_2), (0, 0I_1, I_2))\}$. Then $B(I_1, I_2)$ spans $V(I_1, I_2)$.

Definition 2.25. Let $W[I_1, I_2]$ and $X[I_1, I_2]$ be two refined neutrosophic subhypervector spaces of a strong refined neutrosophic hypervector space $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ over a refined neutrosophic field $K(I_1, I_2)$. $V(I_1, I_2)$ is said to be the direct sum of $W[I_1, I_2]$ and $X[I_1, I_2]$ written $V(I_1, I_2) = W[I_1, I_2] \oplus X[I_1, I_2]$ if every element $v \in V(I_1, I_2)$ can be written uniquely as $v = w + x$ where $w \in W[I_1, I_2]$ and $x \in X[I_1, I_2]$.

Proposition 2.26. *Let $W[I_1, I_2]$ and $X[I_1, I_2]$ be two refined neutrosophic subhypervector spaces of a strong refined neutrosophic hypervector space $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ over a refined neutrosophic field $K(I_1, I_2)$. $V(I_1, I_2) = W[I_1, I_2] \oplus X[I_1, I_2]$ if and only if the following conditions hold:*

1. $V(I_1, I_2) = W[I_1, I_2] + X[I_1, I_2]$.
2. $W[I_1, I_2] \cap X[I_1, I_2] = \{\theta\}$.

Proof. Same as in classical case. □

Definition 2.27. Let $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ be a strong refined neutrosophic hypervector space over a refined neutrosophic field $K(I_1, I_2)$. The refined neutrosophic vector
$u = (a, bI_1, cI_2) \in V(I_1, I_2)$ is said to be a linear combination of the refined neutrosophic vectors
$u_1 = (a_1, b_1I_1, c_1I_2), u_2 = (a_2, b_2I_1, c_2I_1), \cdots, u_n = (a_n, b_nI_1, c_nI_2) \in V(I_1, I_2)$ if there exist refined neutrosophic scalars $\alpha_1 = (k_1, m_1I_1, t_1I_2), \alpha_2 = (k_2, m_2I_1, t_2I_2), \cdots, \alpha_n = (k_n, m_nI_1, t_nI_2) \in K(I_1, I_2)$ such that

$$u \in \alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \cdots + \alpha_n \bullet u_n.$$

Definition 2.28. Let $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ be a strong refined neutrosophic hypervector space over a refined neutrosophic field $K(I_1, I_2)$ and let
$B(I_1, I_2) = \{u_1 = (a_1, b_1I_1, c_1I_2), u_2 = (a_2, b_2I_1, c_2I_2), \cdots, u_n = (a_n, b_nI_1, c_nI_2)\}$ be a subset of $V(I_1, I_2)$.

1. $B(I_1, I_2)$ is called a linearly dependent set if there exist refined neutrosophic scalars
$\alpha_1 = (k_1, m_1I_1, t_1I_2), \alpha_2 = (k_2, m_2I_1, t_2I_2), \cdots, \alpha_n = (k_n, m_nI_1, t_nI_2)$ (not all zero) such that

$$\theta \in \alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \cdots + \alpha_n \bullet u_n.$$

2. $B(I_1, I_2)$ is called a linearly independent set if

$$\theta \in \alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \cdots + \alpha_n \bullet u_n$$

implies that $\alpha_1 = \alpha_2 = \cdots = \alpha_n = (0, 0I_1, 0I_2)$.

Proposition 2.29. *Let $(V(I_1, I_2), +, \bullet, K)$ be a weak refined neutrosophic hypervector space over a field K. Any singleton set of non-null refined neutrosophic vector of the weak refined neutrosophic hypervector space $V(I_1, I_2)$ is linearly independent.*

Proof. Suppose that $\theta \neq v = (a, bI_1, cI_2) \in V(I_1, I_2)$. Let $\theta \in k \bullet v$ and suppose that $\theta \neq k \in K$. Then $k^{-1} \in K$ and therefore, $k^{-1} \bullet \theta \subseteq k^{-1} \bullet (k \bullet v)$ so that

$$\begin{aligned} \theta &\in (k^{-1}k) \bullet v \\ &= 1 \bullet v \\ &= \{(x, yI_1, zI_2) : x \in 1 \bullet a, y \in 1 \bullet b, z \in 1 \bullet c\} \\ &= \{(x, yI_1, zI_2) : x \in \{a\}, y \in \{b\}, z \in \{c\}\} \\ &= \{(a, bI_1, cI_2)\} \\ &= \{v\}. \end{aligned}$$

This shows that $v = \theta$ which is a contradiction. Hence, $k = \theta$ and thus, the singleton $\{v\}$ is a linearly independent set. □

Proposition 2.30. *Let $(V(I_1, I_2), +, \bullet, K)$ be a weak refined neutrosophic hypervector space over a field K. Any set of refined neutrosophic vectors of the weak refined neutrosophic hypervector space $V(I_1, I_2)$ containing the null refined neutrosophic vector is always linearly dependent.*

Proposition 2.31. *Let $(V(I_1, I_2), +, \bullet, K)$ be a weak refined neutrosophic hypervector space over a field K and let $\mathbb{B}(I_1, I_2) = \{u_1 = (a_1, b_1I_1, c_1I_2), u_2 = (a_2, b_2I_1, c_2I_2), \cdots, u_n = (a_n, b_nI_1, c_nI_2)\}$ be a subset of $V(I_1, I_2)$. Then $B(I_1, I_2)$ is a linearly independent set if and only if at least one element of $\mathbb{B}(I_1, I_2)$ can be expressed as a linear combination of the remaining elements of $\mathbb{B}(I_1, I_2)$.*

Proof : This can be easily established.

Proposition 2.32. *Let $(V(I_1, I_2), +, \bullet, K)$ be a weak refined neutrosophic hypervector space over a field K and let*

$$B(I_1, I_2) = \{u_1 = (a_1, b_1I, c_1I_1), u_2 = (a_2, b_2I_1, c_2I_2), \cdots, u_n = (a_n, b_nI_1, cI_2)\}$$

be a subset of $V(I_1, I_2)$. Then $B(I_1, I_2)$ is a linearly dependent set if and only if at least one element of $B(I_1, I_2)$ can be expressed as a linear combination of the remaining elements of $B(I_1, I_2)$.

Proof : Suppose that $B(I_1, I_2)$ is a linearly dependent set. Then there exist scalars $k_1, k_2, \cdots, k_n$ not all zero in K such that

$$\theta \in k_1 \bullet u_1 + k_2 \bullet u_2 + \cdots + k_n \bullet u_n.$$

Suppose that $k_1 \neq 0$, then $k_1^{-1} \in K$ and therefore

$$\begin{aligned} k_1^{-1} \bullet \theta &\subseteq k_1^{-1} \bullet (k_1 \bullet u_1 + k_2 \bullet u_2 + \cdots + k_n \bullet u_n) \\ &= (k_1^{-1}k_1) \bullet u_1 + (k_2^{-1}k_2) \bullet u_2 + \cdots + (k_n^{-1}k_n) \bullet u_n \\ &= 1 \bullet u_1 + (k_1^{-1}k_2) \bullet u_2 + \cdots + (k_1^{-1}k_n) \bullet u_n \end{aligned}$$

so that

$$\theta \in 1 \bullet u_1 + \{u\}$$

where $u = (a, bI_1, cI_2) \in (k_1^{-1}k_2) \bullet u_2 + \cdots + (k_n^{-1}k_n) \bullet u_n$.
Thus $\theta \in \{(a+a_1, (b+b_1)I_1, (c+c_1)I_2)\}$ from which we obtain $u_1 = (a_1, b_1I_1, c_1I_2) = -u = -(a, bI_1, cI_2)$ so that

$$\begin{aligned} u_1 &\in (-1) \bullet u \\ &\subseteq (-1) \bullet ((k_1^{-1}k_2) \bullet u_2 + \cdots + (k_n^{-1}k_n) \bullet u_n) \\ &\subseteq (-k_1^{-1}k_2) \bullet u_2 + (-k_1^{-1}k_3) \bullet u_3 + \cdots + (-k_1^{-1}k_n) \bullet u_n. \end{aligned}$$

This shows that $u_1 \in \text{span}\{u_2, u_3, \cdots, u_n\}$.
Conversely, suppose that $u_1 \in \text{span}\{u_2, u_3, \cdots, u_n\}$ and suppose that $0 \neq -1 \in K$.
Then there exist $k_2, k_3, \cdots, k_n \in K$ such that

$$u_1 \in k_2 \bullet u_2 + k_3 \bullet u_3 + \cdots + k_n \bullet u_n$$

and we have

$$u_1 + (-u_1) \in (-1) \bullet u_1 + k_2 \bullet u_2 + k_3 \bullet u_3 + \bullet + k_n \bullet u_n.$$

From which

$$\theta \in (-1) \bullet u_1 + k_2 \bullet u_2 + k_3 \bullet u_3 + \cdots + k_n \bullet u_n.$$

Since $-1 \neq 0 \in K$, it follows that $B(I_1, I_2)$ is a linearly dependent set.

Proposition 2.33. *Let $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ be a strong refined neutrosophic hypervector space over a refined neutrosophic field $K(I_1, I_2)$ and let $B_1(I_1, I_2)$ and $B_2(I_1, I_2)$ be subsets of $V(I_1, I_2)$ such that $B_1(I_1, I_2) \subseteq B_2(I_1, I_2)$. If $B_1(I_1, I_2)$ is linearly dependent, then $B_2(I_1, I_2)$ is linearly dependent.*

Proposition 2.34. *Let $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ be a strong refined neutrosophic hypervector space over a refined neutrosophic field $K(I_1, I_2)$ and let $B_1(I_1, I_2)$ and $B_2(I_1, I_2)$ be subsets of $V(I_1, I_2)$ such that $B_1(I_1, I_2) \subseteq B_2(I_1, I_2)$. If $B_1(I_1, I_2)$ is linearly independent, then $B_2(I_1, I_2)$ is linearly independent.*

Definition 2.35. Let $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ be a strong refined neutrosophic hypervector space over a refined neutrosophic field $K(I_1, I_2)$ and let $B(I_1, I_2) = \{u_1 = (a_1, b_1I_1, c_1I_1), u_2 = (a_2, b_2I_1, c_2I_2), \cdots\}$ be a subset of $V(I_1, I_2)$. $B(I_1, I_2)$ is said to be a basis for $V(I_1, I_2)$ if the following conditions hold:

1. $B(I_1, I_2)$ is a linearly independent set
2. $V(I_1, I_2) = \text{span}(B(I_1, I_2))$.

If $B(I_1, I_2)$ is finite and its cardinality is n, then $V(I_1, I_2)$ is called an n-dimensional strong refined neutrosophic hypervector space and we write $dim_s(V(I_1, I_2)) = n$. If $B(I_1, I_2)$ is not finite, then $V(I_1, I_2)$ is called an infinite-dimensional strong refined neutrosophic hypervector space.

Definition 2.36. Let $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ be a weak refined neutrosophic hypervector space over a field K and let $B(I_1, I_2) = \{u_1 = (a_1, b_1I_1, c_1I_1), u_2 = (a_2, b_2I_1, c_2I_2), \cdots\}$ be a subset of $V(I_1, I_2)$. $B(I_1, I_2)$ is said to be a basis for $V(I_1, I_2)$ if the following conditions hold:

1. $B(I_1, I_2)$ is a linearly independent set
2. $V(I_1, I_2) = \text{span}(B(I_1, I_2))$.

If $B(I_1, I_2)$ is finite and its cardinality is n, then $V(I_1, I_2)$ is called an n-dimensional weak refined neutrosophic hypervector space and we write $dim_w(V(I_1, I_2)) = n$. If $B(I_1, I_2)$ is not finite, then $V(I_1, I_2)$ is called an infinite-dimensional weak refined neutrosophic hypervector space.

Example 2.37. In Example 2.23 $B(I_1, I_2)$ is a basis for $V(I_1, I_2)$ and $dim_s(V(I_1, I_2)) = 3$.

Example 2.38. In Example 2.24 $B(I_1, I_2)$ is a basis for $V(I_1, I_2)$ and $dim_W(V(I_1, I_2)) = 6$.

Proposition 2.39. *Let $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ be a strong refined neutrosophic hypervector space over a refined neutrosophic field $K(I_1, I_2)$ and let*
$B(I_1, I_2) = \{u_1 = (a_1, b_1I_1, c_1I_2), u_2 = (a_2, b_2I_1, c_2I_2), \cdots, u_n = (a_n, b_nI_1, c_nI_2)\}$ be a subset of $V(I_1, I_2)$. Then $B(I_1, I_2)$ is a basis for $V(I_1, I_2)$ if and only if each refined neutrosophic vector
$u = (a, bI_1, cI_2) \in V(I_1, I_2)$ can be expressed uniquely as a linear combination of the elements of $B(I_1, I_2)$.

Proof. Suppose that each refined neutrosophic vector $u = (a, bI, cI_2) \in V(I_1, I_2)$ can be expressed uniquely as a linear combination of the elements of $B(I_1, I_2)$. Then $u \in \text{span}(B(I_1, I_2)) = V(I_1, I_2)$.
Since such a representation is unique, it follows that $B(I_1, I_2)$ is a linearly independent set and since $u \in V(I_1, I_2)$ is arbitrary, it follows that $B(I_1, I_2)$ is a basis for $V(I_1, I_2)$.
Conversely, suppose that $B(I_1, I_2)$ is a basis for $V(I_1, I_2)$, then $V(I_1, I_2) = \text{span}(B(I_1, I_2))$ and $B(I_1, I_2)$ is linearly independent. Now it remains to show that $u = (a, bI_1, cI_2) \in V(I_1, I_2)$ can be expressed uniquely as a linear combination of the elements of $B(I_1, I_2)$.
To this end, for $\alpha_1 = (k_1, m_1I_1, p_1I_2), \alpha_2 = (k_2, m_2I_1, p_2I_2), \cdots, \alpha_n = (k_n, m_nI_1, p_nI_2)$,
$\beta_1 = (r_1, s_1I_1, t_1I_2), \beta_2 = (r_2, s_2I_1, t_2I_2), \cdots, \beta_n = (r_n, s_nI_1, t_nI_2) \in K(I_1, I_2)$,
let us express u in two ways as follows:

$$u \in \alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \cdots + \alpha_n \bullet u_n, \quad (1)$$

$$u \in \beta_1 \bullet u_1 + \beta_2 \bullet u_2 + \cdots + \beta_n \bullet u_n. \quad (2)$$

From equation(2), we have

$$\begin{array}{rcl} -u \in (-1)\bullet u & \subseteq & (-1)\bullet(\beta_1\bullet u_1+\beta_2\bullet u_2+\cdots+\beta_n\bullet u_n)\\ & = & ((-1)\beta_1)\bullet u_1+((-1)\beta_2)\bullet u_2+\cdots+((-1)\beta_n)\bullet u_n\\ & = & (-\beta_1)\bullet u_1+(-\beta_2)\bullet u_2+\bullet+(-1\beta_n)\bullet u_n. \end{array} \qquad (3).$$

From equations (1) and (3), we have

$$u+(-u)\in(\alpha_1+(-\beta_1))\bullet u_1+(\alpha_2+(-\beta_2))\bullet u_2+\cdots+(\alpha_n+(-\beta_n))\bullet u_n$$

$$\Longrightarrow \theta\in(\alpha_1-\beta_1)\bullet u_1+(\alpha_2-\beta_2)\bullet u_2+\cdots+(\alpha_n-\beta_n)\bullet u_n.$$

Since $B(I_1,I_2)$ is linearly independent, it follows that

$$\alpha_1-\beta_1=\alpha_2-\beta_2=\cdots=\alpha_n-\beta_n=(0,0I_1,0I_2)$$

and therefore,

$$\alpha_1=\beta_1,\alpha_2=\beta_2,\cdots,\alpha_n=\beta_n.$$

This shows that u has been expressed uniquely as a linear combination of the elements of $B(I_1,I_2)$. The proof is complete. □

Proposition 2.40. *Let $(V(I_1,I_2),+,\bullet,K)$ be a weak refined neutrosophic hypervector space over a field K and let* $B_1(I_1,I_2)=\{u_1=(a_1,b_1I_1,c_1I_2),u_2=(a_2,b_2I_1,c_2I_2),\cdots,u_n=(a_n,b_nI_1,cI_2)\}$ *be a linearly independent subset of $V(I_1,I_2)$. If $u\in V(I_1,I_2)\backslash B_1(I_1,I_2)=V(I_1,I_2)\cap(B(I_1,I_2))^c$ is arbitrary, then* $B_2(I_1,I_2)=\{u_1=(a_1,b_1I,c_1I_2),u_2=(a_2,b_2I_1,c_2I_2),\cdots,u_n=(a_n,b_nI_1,c_nI_2),u\}$ *is a linearly dependent set if and only if $u\in span((B_1(I_1,I_2))$.*

Proof. Suppose that $B_2(I_1,I_2)$ is a linearly dependent set. Then there exist scalars $k_1,k_2,\cdots,k_n,k$ not all zero such that

$$\theta\in k_1\bullet u_1+k_2\bullet u_2+\cdots+k_n\bullet u_n+k\bullet u. \qquad (4)$$

Suppose that $k=0$, then there exist at least one of the $k_i's$ say $k_1\neq0$ and equation (4) becomes

$$\theta\in k_1\bullet u_1+k_2\bullet u_2+\cdots+k_n\cdots u_n \qquad (5)$$

from which it follows that the set
$B_1(I_1,I_2)=\{u_1=(a_1,b_1I_1,c_1I_2),u2=(a_2,b_2I_1,c_2I_2),\cdots,u_n=(a_n,b_nI_1,cI_2)\}$ is linearly dependent. This contradicts the hypothesis that $B_1(I_1,I_2)$ is linearly independent. Hence $k\neq0$ and therefore $k^{-1}\in K$. From equation (4), we have

$$\begin{array}{rcl} k^{-1}\bullet\theta & \subseteq & k^{-1}\bullet(k_1\bullet u_1+k_2\bullet u_2+\cdots+k_n\bullet u_n+k\bullet u)\\ \Longrightarrow\theta & \in & (k^{-1}k_1)\bullet u_1+(k^{-1}k_2)\bullet u_2+\cdots+(k^{-1}k_n)\bullet u_n+(k^{-1}k)\bullet u\\ \Longrightarrow\theta & = & v+u\,\big(\text{where}\quad v\in(k^{-1}k_1)\bullet u_1+(k^{-1}k_2)\bullet u_2+\cdots+(k^{-1}k_n)\bullet u_n\big)\\ \Longrightarrow u & = & -v\in(-1)\bullet((k^{-1}k_1)\bullet u_1+(k^{-1}k_2)\bullet u_2+\cdots+(k^{-1}k_n)\bullet u_n)\\ \Longrightarrow u & \in & (-1)\bullet[(k^{-1}k_1)\bullet u_1+(k^{-1}k_2)\bullet u_2+\cdots+(k^{-1}k_n)\bullet u_n]\\ \Longrightarrow u & \in & (-k^{-1}k_1)\bullet u_1+(-k^{-1}k_2)\bullet u_2+\cdots+(-k^{-1}k_n)\bullet u_n\\ \Longrightarrow u & \in & \text{span}(B_1(I_1,I_2)). \end{array}$$

Conversely, suppose that $u\in\text{span}(B_1(I_1,I_2))$. Then there exist $k_1,k_2,\cdots,k_n\in K$ such that

$$\begin{array}{rcl} u & \in & k_1\bullet u_1+k_2\bullet u_2+\cdots+k_n\bullet u_n\\ \Longrightarrow u+(-u) & \in & k_1\bullet u_1+k_2\bullet u_2+\cdots+k_n\bullet u_n+(-1)\bullet u\\ \Longrightarrow\theta & \in & k_1\bullet u_1+k_2\bullet u_2+\cdots+k_n\bullet u_n+(-1)\bullet u. \end{array}$$

Since $u\notin B_1(I_1,I_2)$ and $B_1(I_1,I_2)$ is linearly independent, it follows that $\{u_1,u_2,\cdots,u_n,u\}=B_2(I_1,I_2)$ is a linearly dependent set. The proof is complete. □

Definition 2.41. Let $W[I_1,I_2]$ be a refined neutrosophic subhypervector space of a strong refined neutrosophic hypervector space $(V(I_1,I_2),+,\bullet,K(I_1,I_2))$ over a refined neutrosophic field $K(I_1,I_2)$. The quotient $V(I_1,I_2)/W[I_1,I_2]$ is defined by the set

$$\{[v]=v+W[I_1,I_2]:v\in V(I_1,I_2)\}.$$

Proposition 2.42. *Let $V(I_1,I_2)/W[I_1,I_2]=\{[v]=v+W[I_1,I_2]:v\in V(I_1,I_2)\}$.*
If for every $[u],[v]\in V(I_1,I_2)/W[I_1,I_2]$ and $\alpha\in K(I_1,I_2)$ we define:

$$[u]\oplus[v]=(u+v)+W[I_1,I_2]$$

and

$$\alpha\odot[u]=[\alpha\bullet u]=\{[x]:x\in\alpha\bullet u\}.$$

$(V(I_1,I_2)/W[I_1,I_2],\oplus,\odot,K(I_1,I_2))$ is a strong refined neutrosophic hypervector space over a refined neutrosophic field $K(I_1,I_2)$ called a strong refined neutrosophic quotient hypervector space.

Proof. The proof is similar to the proof in classical case. □

Proposition 2.43. *Let $W[I_1,I_2]$ be a refined neutrosophic subhypervector space of a strong refined neutrosophic hypervector space $V(I_1,I_2)$ over a refined neutrosophic field $K(I_1,I_2)$, let $(V(I_1,I_2)/W[I_1,1_2])$ be as defined in Proposition 2.42 , then the following hold:*

1. *$W[I_1,I_2]$ is finite dimensional and $dim_sW[I_1,I_2]\leq dim_sV(I_1,I_2)$.*
2. *$dim_s(V(I_1,I_2)/W[I_1,1_2])=dim_sV(I_1,I_2)-dim_sW[I_1,I_2]$.*

Proof:

1. Let $B_1(I_1,I_2)$ be the basis for $W[I_1,I_2]$ and let $B_2(I_1,I_2)$ be a basis for $V(I_1,I_2)$. Since $W[I_1,I_2]\subseteq V(I_1,I_2)$ then $B_1(I_1,I_2)$ is contained in $B_2(I_1,I_2)$. Therefore $B_1(I_1,I_2)$ is a linearly independent subset of $V(I_1,I_2)$. Then we have that
$|B_1(I_1,I_2)|\leq|B_2(I_1,I_2)|$. Now, since $|B_1(I_1,I_2)|\leq|B_2(I_1,I_2)|$ and $V(I_1,I_2)$ is finite dimensional we can conclude that $W(I_1,I_2)$ is finite dimensional and
$$dim_sW(I_1,I_2)=|B_1(I_1,I_2)|\leq|B_2(I_1,I_2)|=dim_sV(I_1,I_2).$$

2. Let $\{u_1,u_2,\cdots,u_m\}$ be a basis of $W[I_1,I_2]$. Then this can be filled out to a basis, $\{u_1,u_2\cdots,u_m,v_1,v_2,\cdots,v_n\}$ of $V(I_1,I_2)$, where $m+n=dim_sV(I_1,I_2)$ and $m=dim_sW[I_1,I_2]$. Let $[v_1],[v_2],\cdots,[v_n]$ be the images in $V(I_1,I_2)/W[I_1,I_2]$, of $v_1,v_2,\cdots,v_n$.
Since any vector $v\in V(I_1,I_2)$ is in a linear combination of $u_1,u_2,\cdots,u_m,v_1,v_2,\cdots,v_n$, we have that
$$v\in\alpha_1\bullet u_1+\alpha_2\bullet u_2+\cdots+\alpha_m\bullet u_m+\beta_1\bullet v_1+\beta_2\bullet v_2+\cdots+\beta_n\bullet v_n,$$
then
$v\in[\alpha_1\bullet u_1]\oplus[\alpha_2\bullet u_2]\oplus\cdots\oplus[\alpha_m\bullet u_m]\oplus[\beta_1\bullet v_1]\oplus[\beta_2\bullet v_2]\oplus\cdots\oplus[\beta_n\bullet v_n]$
$\subseteq[\beta_1\bullet v_1]\oplus[\beta_2\bullet v_2]\oplus\cdots\oplus[\beta_n\bullet v_n]$ (since $[\alpha_i\bullet u_i]\subseteq(\alpha_i\bullet u_i)+W[I_1,I_2]\subseteq W[I_1,I_2]$)
$=\beta_1\bullet[v_1]\oplus\beta_2\bullet[v_2]\oplus\cdots\oplus\beta_n\bullet[v_n]$.
Thus $[v_1],[v_2]\cdots,[v_n]$ span $V(I_1,I_2)/W[I_1,I_2]$. We claim that they are linearly independent, for if
$$\theta\in\lambda_1\bullet[v_1]\oplus\lambda_2\bullet[v_2]\oplus\cdots\oplus\lambda_n\bullet[v_n]$$
then
$$\theta\in\lambda_1\bullet[v_1]\oplus\lambda_2\bullet[v_2]\oplus\cdots\oplus\lambda_n\bullet[v_n]\oplus W[I_1,I_2]$$
$$\theta\subseteq\lambda_1\bullet[v_1]\oplus\lambda_2\bullet[v_2]\oplus\cdots\oplus\lambda_n\bullet[v_n]\oplus\gamma_1\bullet[u_1]\oplus\gamma_2\bullet[u_2]\oplus\cdots\oplus\gamma_m\bullet[u_m]$$
which by the linear independence of the set $\{u_1,u_2\cdots,u_m,v_1,v_2\cdots,v_n\}$ forces
$\lambda_1=\lambda_2=\cdots=\lambda_n=\gamma_1=\gamma_2=\cdots=\gamma_m=0$.
This shows that $V(I_1,I_2)/W[I_1,I_2]$ has a basis of n elements, and
$$dim_s(V(I_1,I_2)/W[I_1,I_2])=n=(n+m)-m=dim_sV(I_1,I_2)-dim_sW[I_1,I_2].$$

Proposition 2.44. *Let $W_1(I_1,I_2)$ and $W_2(I_1,I_2)$ be finite dimensional weak refined neutrosophic subhypervector spaces of a weak refined neutrosophic vector space $V(I_1,I_2)$ over a field K. Then $W_1(I_1,I_2)+W_2(I_1,I_2)$ is a finite dimensional refined neutrosophic subhypervector space of $V(I_1,I_2)$ and*

$$dim_w(W_1(I_1,I_2)+W_2(I_1,I_2))=dim_w(W_1(I_1,I_2))+dim_w(W_2(I_1,I_2))-dim_w(W_1(I_1,I_2)\cap W_2(I_1,I_2)).$$

If $V(I_1,I_2)=W_1(I_1,I_2)\oplus W_2(I_1,I_2)$ then

$$dim_w(W_1(I_1,I_2)+W_2(I_1,I_2))=dim_w(W_1(I_1,I_2))+dim_w(W_2(I_1,I_2)).$$

Proof: We know that $W_1(I_1, I_2) \cap W_2(I_1, I_2)$ is a refined neutrosophic subhypervector space of both $W_1(I_1, I_2)$ and $W_2(I_1, I_2)$. So $W_1(I_1, I_2) \cap W_2(I_1, I_2)$ is a finite dimensional refined neutrosophic subhypervector space of $V(I_1, I_2)$.
Suppose that $dim_w(W_1(I_1, I_2) \cap W_2(I_1, I_2)) = k$, $dim_w(W_1(I_1, I_2)) = m$ and $dim_w(W_2(I_1, I_2)) = n$ then we have that $k \leq m$ and $k \leq n$.
Now, let $\{u_1, u_2, \cdots, u_k\}$ be a basis of $W_1(I_1, I_2) \cap W_2(I_1, I_2)$. Then we have that $\{u_1, u_2, \cdots, u_k\}$ is a linearly independent set of refined neutrosophic vectors in $W_1[I_1, I_2]$ and $W_2[I_1, I_2]$ with $k \leq m$ and $k \leq n$, then it follows that either $\{u_1, u_2, \cdots, u_k\}$ is a basis of $W_1[I_1, I_2]$ and $W_2[I_1, I_2]$ or it can be extended to a basis for $W_1[I_1, I_2]$ and $W_2[I_1, I_2]$.
Let $\{u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_{m-k}\}$ be a basis for $W_1[I_1, I_2]$, and let $\{u_1, u_2, \cdots, u_k, w_1, w_2, \cdots, w_{n-k}\}$ be a basis of $W_1[I_1, I_2]$.
Then the refined neutrosophic subhypervector space $W_1[I_1, I_2] + W_2[I_1, I_2]$ is spanned by the refined neutrosophic vectors $\{u_1, u_2, \cdots, u_k, v_1, v_2, \cdots v_{m-k}, w_1, w_2, \cdots, w_{n-k}\}$ and these refined neutrosophic vectors form an independent set. For suppose

$$\theta \in \sum_{i=1}^{k} \alpha_i u_i + \sum_{j=1}^{m} \beta_j v_j + \sum_{r=1}^{n} \gamma_r w_r.$$

Then

$$-\sum_{r=1}^{n} \gamma_r w_r \in \sum_{i=1}^{k} \alpha_i u_i + \sum_{j=1}^{m} \beta_j v_j$$

$$\Longrightarrow (-1) \bullet (-\sum_{r=1}^{n} \gamma_r w_r) \subseteq \sum_{i=1}^{k} (-1) \bullet \alpha_i u_i + \sum_{j=1}^{m} (-1) \bullet \beta_j v_j$$

$$\Longrightarrow \sum_{r=1}^{n} \gamma_r w_r \in \sum_{i=1}^{k} (-\alpha_i) u_i + \sum_{j=1}^{m} (-\beta_j) v_j$$

which shows that $\sum_{r=1}^{n} \gamma_r w_r$ belongs to $W_1[I_1, I_2]$. As $\sum_{r=1}^{n} \gamma_r w_r$ also belongs to $W_2[I_1, I_2]$, it follows that

$$\sum_{r=1}^{n} \gamma_r w_r = \sum_{i=1}^{k} \lambda_i u_i$$

for certain scalars $\lambda_1, \lambda_2, \cdots, \lambda_k$.
Because the set $\{u_1, u_2, \cdots, u_k, w_1, w_2, \cdots, w_{n-k}\}$ is independent, each of the scalars $\gamma_r = 0$. Thus

$$\theta \in \sum_{i=1}^{k} \alpha_i u_i + \sum_{j=1}^{m} \beta_j v_j$$

and since $\{u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_{m-k}\}$ is also an independent set, each $\alpha_i = 0$ and each $\beta_j = 0$. Thus, $\{u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_{m-k}, w_1, w_2, \cdots, w_{n-k}\}$ is a basis for $W_1[I_1, I_2] + W_2[I_1, I_2]$.
Finally,
$dim_w(W_1(I_1, I_2) + W_2(I_1, I_2)) = k + m - k + n - k$
$= m + n - k$
$= dim_w(W_1(I_1, I_2)) + dim_w(W_2(I_1, I_2)) - dim_w(W_1(I_1, I_2) \cap W_2(I_1, I_2))$.

Definition 2.45. Let $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ and $W(I_1, I_2), +', \bullet', K(I_1, I_2))$ be two strong refined neutrosophic hypervector spaces over a neutrosophic field $K(I_1, I_2)$.
A mapping $\phi : V(I_1, I_2) \longrightarrow W(I_1, I_2)$ is called a strong refined neutrosophic hypervector space homomorphism if the following conditions hold:

1. ϕ is a strong hypervector space homomorphism.
2. $\phi(0, I_1, I_2) = (0, I_1, I_2)$.

If in addition ϕ is a bijection, we say that $V(I_1, I_2)$ is isomorphic to $W(I_1, I_2)$ and we write $V(I_1, I_2) \cong W(I_1, I_2)$.

Proposition 2.46. *Let $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ and $(W(I_1, I_2), +, \bullet, K(I_1, I_2))$ be two strong refined neutrosophic hypervector spaces over a refined neutrosophic field $K(I_1, I_2)$ and let $\phi : V(I_1, I_2) \longrightarrow W(I_1, I_2)$ be a bijective strong refined neutrosophic hypervector space homomorphism.*
If $B(I_1, I_2) = \{u_1 = (a_1, b_1I_1, c_1I_2), u_2 = (a_2, b_2I_1, c_2I_2), \cdots, u_n = (a_n, b_nI_1, c_nI_n)\}$ is a basis for $V(I_1, I_2)$, then $B'(I_1, I_2) = \phi(B(I_1, I_2)) = \{\phi(u_1), \phi(u_2), \cdots, \phi(u_n)\}$ is a basis for $W(I_1, I_2)$.

Proof. Suppose that $B(I_1, I_2)$ is a basis for $V(I_1, I_2)$. Then for an arbitrary $u = (a, bI_1, cI_2) \in V(I_1, I_2)$, there exist refined neutrosophic scalars
$\alpha_1 = (k_1, m_1I_1, t_1I_2), \alpha_2 = (k_2, m_2I_1, t_2I_2), \cdots, \alpha_n = (k_n, m_nI_1, t_nI_2) \in K(I_1, I_2)$ such that

$$u \in \alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \cdots + \alpha_n \bullet u_n$$

$$\Longrightarrow \phi(u) \in \phi(\alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \cdots + \alpha_n \bullet u_n)$$
$$= \alpha_1 \bullet' \phi(u_1) +' \alpha_2 \bullet' \phi(u_2) +' \cdots +' \alpha_n \bullet' \phi(u_n).$$

Since ϕ is surjective, it follows that $\phi(u), \phi(u_1), \phi(u_2), \cdots, \phi(u_n) \in W(I_1, I_2)$ and therefore
$\phi(u) \in \text{span}(B'(I_1, I_2))$. To complete the proof, we must show that $B'(I_1, I_2)$ is linearly independent.
To this end, suppose that

$$\phi(\theta) \in \beta_1 \bullet' \phi(u_1) +' \beta_2 \bullet' \phi(u_2) +' \cdots +' \beta_n \bullet' \phi(u_n)$$

where $\beta_1 = (p_1, q_1I_1, s_1I_2), \beta_2 = (p_2, q_2I_1, s_2I_2), \cdots, \beta_n = (p_n, q_nI_1, s_nI_2) \in K(I_1, I_2)$, then

$$\phi(\theta) \in \phi(\beta_1 \bullet u_1) +' \phi(\beta_2 \bullet u_2) +' \bullet +' \phi(\beta_n \bullet u_n)$$

$$= \phi(\beta_1 \bullet u_1 + \beta_2 \bullet u_2 + \cdots + \beta_n \bullet u_n).$$

Since ϕ is injective, we must have

$$\theta \in \beta_1 \bullet u_1 + \beta_2 \bullet u_2 + \cdots + \beta_n \bullet u_n.$$

Also, since $B(I_1, I_2)$ is linearly independent, we must have
$\beta_1 = \beta_2 = \cdots = \beta_n = (0, 0I_1, 0I_2)$.
Hence $B'(I_1, I_2) = \{\phi(u_1), \phi(u_2), \cdots, \phi(u_n)\}$ is linearly independent and therefore a basis for $W(I_1, I_2)$.
□

Remark 2.47. Suppose we wish to transform a refined neutrosophic hypervector space into a neutrosophic hypervector space, an interesting question to ask will be, can we find a mapping that will help us achieve this? The answer to this is Yes.
The mapping $\phi : V(I_1, I_2) \longrightarrow V(I)$ defined by

$$\phi((x, yI_1, zI_2)) = (x, (y + z)I) \ \ \forall \, x, y, z \in V$$

will make such transformation possible. This mapping is a non-neutrosophic one. This make sense since every refined neutrosophic hypervector space and neutrosophic hypervector spaces are hypervector spaces.

Proposition 2.48. *Let $(V(I_1, I_2), +, \bullet)$ be a weak refined neutrosophic vector space over a field K and let $V(I)$ be a weak neutrosophic vector space over K. The mapping $\phi : V(I_1, I_2) \longrightarrow V(I)$ defined by*

$$\phi((x, yI_1, zI_2)) = (x, (y + z)I) \ \ \forall \, x, y, z \in V$$

is a good linear transformation.

Proof. ϕ is well defined. Suppose $(x, yI_1, zI_2) = (x'y'I_1, z'I_2)$ then we that $x = x', y = y'$ and $z' = z'$. So,

$$\phi((x, yI_1, zI_2)) = (x, (y + z)I) = x' + (y' + z')I = \phi(x', y'I_1, z'I_2).$$

Now, suppose $(x, yI_1, zI_2), (x', y'I_1, 1z'I_2) \in V(I_1, I_2)$ then

$$\begin{aligned}
\phi((x, yI_1, zI_2) + (x', y'I_1, z'I_2)) &= \phi((x + x'), (y + y')I_1, (z + z')I_2) \\
&= (x + x'), (y + y' + z + z')I \\
&= (x + x'), ((y + z) + (y' + z'))I \\
&= (x + x'), ((y + z)I + (y' + z')I) \\
&= (x, (y + z)I) + (x', (y' + z')I) \\
&= \phi(x, yI_1, zI_2) + \phi(x, yI_1, zI_2).
\end{aligned}$$

$$\begin{aligned}
\phi(k\circ(x,yI_1,zI_2)) &= \phi\{(a,bI_1,cI_2): a\in k\circ x, b\in k\circ y, c\in k\circ z\}\\
&= \{\phi(a,bI_1,cI_2): a\in k\circ x, b\in k\circ y, c\in k\circ z\}\\
&= \{(u,vI): u\in a, v\in b+c\}\\
&= \{(u,vI): u\in k\circ x, v\in k\circ y+k\circ z\}\\
&= \{(u,vI): u\in k\circ x, v\in k\circ(y+z)\}\\
&= k\circ(x,(y+z)I)\\
&= k\circ\phi(x,yI_1,zI_2).
\end{aligned}$$

Hence ϕ is a good linear transformation. □

Proposition 2.49. *Let $L_k(V(I_1,I_2),V(I))$ be the set of good linear transformation from a weak refined neutrosophic vector space $V(I_1,I_2)$ over a field K into a weak neutrosophic vector space $V(I)$ over a field K. Define addition and scalar multiplication as below;*

$$(\phi+\psi)(x,yI_1,zI_2)=\phi((x,yI_1,zI_2))+\psi((x,yI_1,zI_2))$$

and for $k\in K$

$$(k\phi)((x,yI_1,zI_2))=k\phi(x,yI_1,zI_2).$$

Then, it can be shown that $(L_k(V(I_1,I_2),V(I)),+,\cdot)$ is a weak neutrosophic strongly distributive hypervector space.

Definition 2.50. Let $\phi:V(I_1,I_2)\longrightarrow V(I)$ be a good linear transformation, then

$$\begin{aligned}
\ker\phi &= \{(x,yI_1,zI_2):\phi((x,yI_1,zI_2))=(0,0I)\}\\
&= \{(x,yI_1,zI_2):(x,(y+z)I)=(0,0I)\}\\
&= \{(0,yI_1,(-y)I_2)\}.
\end{aligned}$$

Proposition 2.51. *Let $\phi:V(I_1,I_2)\longrightarrow V(I)$ be a good linear transformation.*

1. $\ker\phi$ *is a subhyperspace of $V(I_1,I_2)$.*
2. *If $W[I_1,I_2]$ is a refined neutrosophic subhyperspace of $V(I_1,I_2)$, then the image of $W[I_1,I_2]$, $\phi(W[I_1,I_2])$ is a neutrosophic subhyperspace of $V(I)$.*

3 Conclusion

This paper studied refinement of neutrosophic hypervector space, linear dependence, independence, bases and dimension of refined neutrosophic hypervector spaces and presented some of their basic properties. Also, the paper established the existence of a good linear transformation between a weak refined neutrosophic hypervector space $V(I_1,I_2)$ and a weak neutrosophic hypervector space $V(I)$. We hope to present and study more properties of refined neutrosophic Hypervector spaces in our future papers.

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Pre Separation Axioms In Neutrosophic Crisp Topological Spaces

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Abstract

In this paper, A new type of separation axioms in the neutrosophic crisp Topological space named neutrosophic crisp pre separation axioms is going to be defined , in which neutrosophic crisp pre open set and neutrosophic crisp point are to be depended on. Also, relations among them and the other type are going to be found.

Keywords: Neutrosophic crisp pre separation axiom, neutrosophic crisp separation axiom, neutrosophic crisp point, Neutrosophic crisp semi separation axiom.

1. Introduction

In 1995, F.Samarandache generalized the fuzzy logic concept into the neutrosophic logic which presents a more detaied and concise description than the fuzzy logic and classical logic; then several researches emerged in this logic, in all branches of mathematics, especially Topology.

In 2012 A. A Salama et al. defined the concept of the neutrosophic set. Also, in 2020 A. Al-Nafey, R. Al-Hamido and F. Smarandache define the neutrosophic crisp separation axioms[2]. Also, in 2020 R. K. Al-Hamido, L.A. Salha and T. Gharibah define neutrosophic crisp semi separation axioms[13].

Recently, the neutrosophic crisp set theory may have applications in image processing [3-4]and possible applications to database[6]. Also, neutrosophic sets [7] have applications in the medical field [8-11], [9], [10], [11] and in the field of geographic information systems[5].Many researchers studied topology, and they had many contributions to neutrosophic toplogy as [14], [15], [16], [17] and [18] and in neutrosophic bitopology in [19], [20], [21] and [22], and in neutrosophic algebra in [23], [24], [25], [26] and [27].

In this paper, neutrosophic crisp pre separation axioms via neutrosophic crisp pre open set and neutrosophic crisp point are going to be studied.

Lastly, the definition of separation axioms is as follows $T_i, i = 0,1,2$ and the relations among them.

2. Preliminaries

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In this paper, the symbol (χ,Ţ) means a neutrosophic crisp topological space (N_cTS), Also

$N_c.$OS ($N_c.$CS) means a neutrosophic crisp open(closed) sets and neutrosophic crisp pre open set in N_cTS mean a N_cP.OS .

Some important definitions to this paper will be shown.

Definition 2.1. [1]

Suppos that $X \neq \emptyset$ be a fixed set. A neutrosophic crisp set ($N_c.$S) *U* is an object with the *U*=< *U₁*, *U₂*, *U₃*> shape; U_1, U_2 *and* U_3 are subsets of X.

Definition 2.5. [12]

Suppos that χ be a non-empty set. And x,y,z∈ χ, then:

a. x_{N_1}= <{x},∅,∅ > is called a neutrosophic crisp point ($N_cP_{N_1}$) in χ.

b. y_{N_2}= <∅,{y},∅ > is called a neutrosophic crisp point ($N_cP_{N_2}$) in χ.

c. z_{N_3}= <∅,∅,{z} > is called a neutrosophic crisp point ($N_cP_{N_3}$) in χ .

The set of all neutrosophic crisp points ($N_cP_{N_1}$, $N_cP_{N_2}$, $N_cP_{N_3}$) is denoted by NCP_N.

Definition 2.6. [12]

Suppos that (χ,Ţ) be an $NcTS$. Then χ is called:

a. N_1Ţo-space for every two diffrant points from χ there exists neutrosophic crisp open set in χ containing one of them but not the other.

b. N_2Ţo-space for every two diffrant points from χ there exists neutrosophic crisp open set in χ containing one of them but not the other.

c. N_3Ţo-space for every two diffrant points from χ there exists neutrosophic crisp open set in χ containing one of them but not the other.

Definition 2.7. [12]

Suppos that (χ,Ţ) be an $NcTS$. Then χ is called:

a. N_1Ţ₁-space for every two diffrant points from χ are x_{N_1}, y_{N_1} there exists two neutrosophic crisp open set M_1, M_2 in χ such that $x_{N_1} \in M_1$, $y_{N_1} \notin M_1$ and $x_{N_1} \notin M_2$, $y_{N_1} \in M_2$.

b. N_2Ţ₁-space for every two diffrant points from χ are x_{N_1}, y_{N_1} there exists two neutrosophic crisp open set M_1, M_2 in χ such that $x_{N_2} \in G_1$, $y_{N_2} \notin G_1$ and $x_{N_2} \notin G_2$, $y_{N_2} \in G_2$.

c. N_3Ţ₁-space for every two diffrant points from χ are x_{N_1}, y_{N_1} there exists two neutrosophic crisp open set M_1, M_2 in χ such that $x_{N_3} \in M_1$, $y_{N_3} \notin M_1$ and $x_{N_3} \notin M_2$, $y_{N_3} \in$ M_2.

Definition 2.8. [12]

Suppos that (χ,Ţ) be an $NcTS$. Then χ is called:

a. N_1Ţ₂-space for every two diffrant points from χ are x_{N_1}, y_{N_1} there exists two neutrosophic crisp open set M_1, M_2 in χ such that $x_{N_1} \in M_1$, $y_{N_1} \notin M_1$ and $x_{N_1} \notin M_2$, $y_{N_1} \in M_2$. with $M_1 \cap M_2 = \emptyset$.

b. N_2Ţ₂-space for every two diffrant points from χ are x_{N_1}, y_{N_1} there exists two neutrosophic crisp open set M_1, M_2 in χ such that $x_{N_2} \in M_1$, $y_{N_2} \notin M_1$ and $x_{N_2} \notin M_2$, $y_{N_2} \in M_2$ with $M_1 \cap M_2 = \emptyset$.

c. N_3Ţ₂-space for every two diffrant points from χ are x_{N_1}, y_{N_1} there exists two neutrosophic crisp open set M_1, M_2 in χ such that $x_{N_3} \in M_1$, $y_{N_3} \notin M_1$ and $x_{N_3} \notin M_2$, $y_{N_3} \in M_2$ with $M_1 \cap M_2 = \emptyset$.

Definition 2.9. [13]

Suppos that (χ,Ţ) be an $NcTS$. Then χ is called:

a. N_1semiŢo-space if for every $x_{N_1} \neq y_{N_1} \in$ χ there exists $NcS.OS$ M in χ containing one of them but not the other.

b. N_2semiŢo-space if ∀ $x_{N_2} \neq y_{N_2} \in \mathcal{X}$ there exists $NcS.OS$ M in χ containing one of them but not the other.

c. N_3semiŢo-space if ∀ $x_{N_3} \neq y_{N_3} \in$ χ there exists $NcS.OS$ M in χ containing one of them but not the other.

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Definition 2.10. [13]

Suppos that (χ,Ţ) be an $NcTS$. Then χ is called:

a. N_1 semiŢ₁-space if for every $x_{N_1} \neq y_{N_1} \in \chi$ there exist $NcS.OS$ M_1, M_2 in χ such that $x_{N_1} \in M_1$, $y_{N_1} \notin M_1$ and $x_{N_1} \notin M_2$, $y_{N_1} \in M_2$.

b. N_2semiŢ₁-space if ∀ $x_{N_2} \neq y_{N_2} \in \chi$ there exist $NcS.OS$ M_1, M_2 in χ such that $x_{N_2} \in G_1$, $y_{N_2} \notin G_1$ and $x_{N_2} \notin G_2$, $y_{N_2} \in G_2$

c. N_3semiŢ₁-space if ∀ $x_{N_3} \neq y_{N_3} \in \chi$ there exist $NcS.OS$ M_1, M_2 in χ such that $x_{N_3} \in M_1$, $y_{N_3} \notin M_1$ and $x_{N_3} \notin M_2$, $y_{N_3} \in M_2$

Definition 2.11. [13]

Suppos that (χ,Ţ) be an $NcTS$. Then χ is called:

a. N_1semiŢ₂-space if for every $x_{N_1} \neq y_{N_1} \in \chi$ there exists $NcS.OS$ M_1, M_2 in χ such that $x_{N_1} \in M_1$, $y_{N_1} \notin M_1$ and $x_{N_1} \notin M_2$, $y_{N_1} \in M_2$. with $M_1 \cap M_2 = \emptyset$.

b. N_2semiŢ₂-space if ∀ $x_{N_2} \neq y_{N_2} \in \chi$ there exists $NcS.OS$ M_1, M_2 in χ such that $x_{N_2} \in M_1$, $y_{N_2} \notin M_1$ and $x_{N_2} \notin M_2$, $y_{N_2} \in M_2$ with $M_1 \cap M_2 = \emptyset$.

c. N_3semiŢ₂-space if ∀ $x_{N_3} \neq y_{N_3} \in \chi$ there exists $NcS.OS$ M_1, M_2 in χ such that $x_{N_3} \in M_1$, $y_{N_3} \notin M_1$ and $x_{N_3} \notin M_2$, $y_{N_3} \in M_2$ with $M_1 \cap M_2 = \emptyset$.

3. Separation axioms via pre open sets

This section introuduces a new type of separation axioms in the neutrosophic crisp Topological space named neutrosophic crisp pre separation axioms.

Definition 3.1.

Suppos that (χ,Ţ) be an NcTS. Then χ is called:

d. N_1preŢo-space if for every $x_{N_1} \neq y_{N_1} \in \chi$ there exists $N_cP.OS$ M in χ containing one of them but not the other.

e. N_2preŢo-space if ∀ $x_{N_2} \neq y_{N_2} \in \mathcal{X}$ there exists $N_cP.OS$ M in χ containing one of them but not the other.

f. N_3preŢo-space if ∀ $x_{N_3} \neq y_{N_3} \in \chi$ there exists $N_cP.OS$ M in χ containing one of them but not the other.

Definition 3.2.

Suppos that (χ,Ţ) be an N_cTS. Then χ is called:

d. N_1Ţ₁-space if for every $x_{N_1} \neq y_{N_1} \in \chi$ there exist $NcP.OS$ M_1, M_2 in χ such that $x_{N_1} \in M_1$, $y_{N_1} \notin M_1$ and $x_{N_1} \notin M_2$, $y_{N_1} \in M_2$.

e. N_2preŢ₁-space if ∀ $x_{N_2} \neq y_{N_2} \in \chi$ there exist $NcP.OS$ M_1, M_2 in χ such that $x_{N_2} \in G_1$, $y_{N_2} \notin G_1$ and $x_{N_2} \notin G_2$, $y_{N_2} \in G_2$

f. N_3preŢ₁-space if ∀ $x_{N_3} \neq y_{N_3} \in \chi$ there exist $NcP.OS$ M_1, M_2 in χ such that $x_{N_3} \in M_1$, $y_{N_3} \notin M_1$ and $x_{N_3} \notin M_2$, $y_{N_3} \in M_2$

Definition 3.3.

Suppos that (χ,Ţ) be an N_cTS. Then χ is called:

d. N_1preŢ₂-space if for every $x_{N_1} \neq y_{N_1} \in \chi$ there exists $NcP.OS$ M_1, M_2 in χ such that $x_{N_1} \in M_1$, $y_{N_1} \notin M_1$ and $x_{N_1} \notin M_2$, $y_{N_1} \in M_2$. with $M_1 \cap M_2 = \emptyset$.

e. N_2preŢ₂-space if ∀ $x_{N_2} \neq y_{N_2} \in \chi$ there exists $NcP.OS$ M_1, M_2 in χ such that $x_{N_2} \in M_1$, $y_{N_2} \notin M_1$ and $x_{N_2} \notin M_2$, $y_{N_2} \in M_2$ with $M_1 \cap M_2 = \emptyset$.

f. N_3preŢ₂-space if ∀ $x_{N_3} \neq y_{N_3} \in \chi$ there exists $NcP.OS$ M_1, M_2 in χ such that $x_{N_3} \in M_1$, $y_{N_3} \notin M_1$ and $x_{N_3} \notin M_2$, $y_{N_3} \in M_2$ with $M_1 \cap M_2 = \emptyset$.

Theorem 3.4.

Suppos that (χ,Ţ) be an N_cTS, then :

1. Every N_1Ţ₀-space is N_1preŢ₀-space.
2. Every N_2Ţ₀-space is N_2preŢ₀-space.
3. Every N_3Ţ₀-space is N_3preŢ₀-space.

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Proof:

1. Suppose that (χ,Ʈ) is an N_1Ʈ0-space, therefore for every two $x_{N_1} \neq y_{N_1}$, there exists an N_c.OS M in χ containing one of them to which the other does not belong. So there exists an $NcP.OS$ M in χ containing one of them to which the other does not belong, therefore X is N_1preƮ0-space.
2. Suppose that (χ,Ʈ) is an N_2Ʈ0-space, therefore for every two $x_{N_2} \neq y_{N_2}$, there exists an N_c.OS M in χ containing one of them to which the other does not belong. So there exists an $NcP.OS$ M in χ containing one of them to which the other does not belong, therefore X is N_2preƮ0-space.
3. Suppose that (χ,Ʈ) is an N_3Ʈ0-space, therefore for every two $x_{N_3} \neq y_{N_3}$, there exists an N_c.OS M in χ containing one of them to which the other does not belong. So there exists an $NcP.OS$ M in χ containing one of them to which the other does not belong, therefore X is N_3preƮ0-space.

Remark 3.5.
The converse of theorem 3.4 is not true., as it is shown in the following examples.

Example 3.6.
Let $\chi = \{a, b, c\}$, $Ʈ = \{\emptyset_N, X_N, A\}$, $A = \{<\{a\}, \emptyset, \emptyset>\}$
$N_cP.OS = Ʈ \cup \{B = \{<\{a, c\}, \emptyset, \emptyset>\}, C = \{<\{a, b\}, \emptyset, \emptyset>\}\}$.
Let $x_{N_1} = \{<\{b\}, \emptyset, \emptyset>\} \neq y_{N_1} = \{<\{c\}, \emptyset, \emptyset>\} \in \chi$ there is no a N_c.OS M in χ containing one of them but not the other. Therefore (χ,Ʈ) is not N_1Ʈ0-space.
Let $x_{N_1} = \{<\{b\}, \emptyset, \emptyset>\} \neq y_{N_1} = \{<\{a\}, \emptyset, \emptyset>\} \in \chi$ there is a N_cP.OS B in χ containing one of them but not the other.
Let $x_{N_1} = \{<\{a\}, \emptyset, \emptyset>\} \neq y_{N_1} = \{<\{c\}, \emptyset, \emptyset>\} \in \chi$ there is a N_cP.OS A in χ containing one of them but not the other.
Let $x_{N_1} = \{<\{b\}, \emptyset, \emptyset>\} \neq y_{N_1} = \{<\{c\}, \emptyset, \emptyset>\} \in \chi$ there is a N_c.OS B in χ containing one of them but not the other. Therefore (χ,Ʈ) $N_1 preƮ_0$-space.
Then (χ,Ʈ) $N_1 preƮ_0$-space, But (χ,Ʈ) is not N_1Ʈ0-space.

Example 3.7.
Let $\chi = \{a, b, c\}$, $Ʈ = \{\emptyset_N, X_N, A\}$, $A = \{<\{a\}, \emptyset, \emptyset>\}$
$N_cP.OS = Ʈ \cup \{B = \{<\{a, c\}, \emptyset, \emptyset>\}, C = \{<\{a, b\}, \emptyset, \emptyset>\}\}$.
Let $x_{N_2} = \{<\emptyset, \{b\}, \emptyset>\} \neq y_{N_1} = \{<\emptyset, \{c\}, \emptyset>\} \in \chi$ there is no a N_c.OS M in χ containing one of them but not the other. Therefore (χ,Ʈ) is not N_2Ʈ0-space.
Let $x_{N_2} = \{<\emptyset, \{b\}, \emptyset>\} \neq y_{N_2} = \{<\emptyset, \{c\}, \emptyset>\} \in \chi$ there is a N_cP.OS B in χ containing one of them but not the other.
Let $x_{N_2} = \{<\emptyset, \{a\}, \emptyset>\} \neq y_{N_2} = \{<\emptyset, \{c\}, \emptyset>\} \in \chi$ there is a N_cP.OS A in χ containing one of them but not the other.
Let $x_{N_2} = \{<\emptyset, \{b\}, \emptyset>\} \neq y_{N_2} = \{<\emptyset, \{a\}, \emptyset>\} \in \chi$ there is a N_cP.OS A in χ containing one of them but not the other. Therefore (χ,Ʈ) $N_2 preƮ_0$-space.
Then (χ,Ʈ) $N_2 preƮ_0$-space, But (χ,Ʈ) is not N_2Ʈ0-space.

Example 3.8.
Let $\chi = \{a, b, c\}$, $Ʈ = \{\emptyset_N, X_N, A\}$, $A = \{<\{a\}, \emptyset, \emptyset>\}$
$N_cP.OS = Ʈ \cup \{B = \{<\{a, c\}, \emptyset, \emptyset>\}, C = \{<\{a, b\}, \emptyset, \emptyset>\}\}$.
Let $x_{N_3} = \{<\emptyset, \emptyset, \{b\}>\} \neq y_{N_3} = \{<\emptyset, \emptyset, \{c\}>\} \in \chi$ there is no a N_c.OS M in χ containing one of them but not the other. Therefore (χ,Ʈ) is not N_3Ʈ0-space.
Let $x_{N_3} = \{<\emptyset, \emptyset, \{b\}>\} \neq y_{N_3} = \{<\emptyset, \emptyset, \{c\}>\} \in \chi$ there is a N_cP.OS B in χ containing one of them but not the other.
Let $x_{N_3} = \{<\emptyset, \emptyset, \{a\}>\} \neq y_{N_3} = \{<\emptyset, \emptyset, \{c\}>\} \in \chi$ there is a N_cP.OS A in χ containing one of them but not the other.
Let $x_{N_3} = \{<\emptyset, \emptyset, \{b\}>\} \neq y_{N_3} = \{<\emptyset, \emptyset, \{a\}>\} \in \chi$ there is a N_c.OS A in χ containing one of them but not the other. Therefore (χ,Ʈ) $N_3 preƮ_0$-space.
Then (χ,Ʈ) $N_3 preƮ_0$-space, But (χ,Ʈ) is not N_3Ʈ0-space.

Theorem 3.9.

Let (χ,Ʈ) be an N_cTS, then :

1. Every N_1Ʈ1-space is N_1preƮ1-space.

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2. Every N_2Ꚍ1-space is N_2preꚌ1-space.
3. Every N_3Ꚍ1-space is N_3preꚌ1-space.

Proof:

1. Suppose that (χ,Ꚍ) is an N_1Ꚍ1-space , therefore for every two $x_{N_1} \neq y_{N_1}$, there exist an Nc.OS Ɱ 1, Ɱ2 in χ such that $x_{N_1} \in$ Ɱ 1, $y_{N_1} \notin$ Ɱ 1 and $x_{N_1} \notin$ Ɱ2 , $y_{N_1} \in$ Ɱ2. So there exists an $NcP.OS$ Ɱ 1, Ɱ2 in χ such that $x_{N_1} \in$ Ɱ 1, $y_{N_1} \notin$ Ɱ 1 and $x_{N_1} \notin$ Ɱ2 , $y_{N_1} \in$ Ɱ2.
   Therefore X is N_1preꚌ1-space.
2. Suppose that (χ,Ꚍ) is an N_2Ꚍ1-space , therefore for every two $x_{N_2} \neq y_{N_2}$, there exist an Nc.OS Ɱ 1, Ɱ2 in χ such that $x_{N_2} \in$ Ɱ 1, $y_{N_2} \notin$ Ɱ 1 and $x_{N_2} \notin$ Ɱ2 , $y_{N_2} \in$ Ɱ2. So there exists an $NcP.OS$ Ɱ 1, Ɱ2 in χ such that $x_{N_2} \in$ Ɱ 1, $y_{N_2} \notin$ Ɱ 1 and $x_{N_2} \notin$ Ɱ2 , $y_{N_2} \in$ Ɱ2.
   Therefore X is N_2preꚌ1-space.
3. Suppose that (χ,Ꚍ) is an N_3Ꚍ1-space , therefore for every two $x_{N_3} \neq y_{N_3}$, there exist an Nc.OS Ɱ 1, Ɱ2 in χ such that $x_{N_3} \in$ Ɱ 1, $y_{N_3} \notin$ Ɱ 1 and $x_{N_3} \notin$ Ɱ2 , $y_{N_3} \in$ Ɱ2. So there exists an $NcP.OS$ Ɱ 1, Ɱ2 in χ such that $x_{N_3} \in$ Ɱ 1, $y_{N_3} \notin$ Ɱ 1 and $x_{N_3} \notin$ Ɱ2 , $y_{N_3} \in$ Ɱ2.
   Therefore X is N_3preꚌ1-space.

Remark 3.10.
The converse of a theorem 3.9 is not true, as it is shown in the following example.

Example 3.11.
Let $\chi = \{a, b, c\}$, Ꚍ $= \{\emptyset_N , X_N , A, B\}, A = \{< \{ a\}, \emptyset, \emptyset >\}, B = \{< \{ b, c\}, \emptyset, \emptyset >\}$.
$N_cP.OS =$ Ꚍ $\cup \{G = \{< \{ b\}, \emptyset, \emptyset >\}, C = \{< \{ c\}, \emptyset, \emptyset >\}, E = \{< \{ a, b\}, \emptyset, \emptyset >\}, H = \{< \{ a, c\}, \emptyset, \emptyset >\} \}$.
Let $x_{N_1} = \{< \{ b\}, \emptyset, \emptyset >\} \neq y_{N_1} = \{< \{ c\}, \emptyset, \emptyset >\} \in \chi$ there is no $Nc.OS$ Ɱ 1, Ɱ2 in χ such that $x_{N_1} \in$ Ɱ1, $y_{N_1} \notin$ Ɱ1 and $x_{N_1} \notin$ Ɱ2 , $y_{N_1} \in$ Ɱ2. Therefore (χ,Ꚍ) is not N_1Ꚍ1-space.

Then (χ,Ꚍ) N_1preꚌ1-space, but (χ,Ꚍ) is not N_1Ꚍ1-space.

Example 3.12.
Let $\chi = \{a, b, c\}$, Ꚍ $= \{\emptyset_N , X_N , A, B\}, A = \{< \emptyset, \{ a\}, \emptyset >\}, B = \{< \emptyset, \{ b, c\}, \emptyset >\}$.
$N_cS.OS =$ Ꚍ $\cup \{G = \{< \emptyset, \{ b\}, \emptyset >\}, C = \{< \emptyset, \{ c\}, \emptyset >\}, E = \{< \emptyset, \{ a, b\}, \emptyset >\}, H = \{< \emptyset, \{ a, c\}, \emptyset >\} \}$.
Let $x_{N_2} = \{< \emptyset, \{ b\}, \emptyset >\} \neq y_{N_2} = \{< \emptyset, \{ c\}, \emptyset >\} \in \chi$ there is no $Nc.OS$ Ɱ 1, Ɱ2 in χ such that $x_{N_2} \in$ Ɱ1, $y_{N_2} \notin$ Ɱ1 and $x_{N_2} \notin$ Ɱ2 , $y_{N_2} \in$ Ɱ2. Therefore (χ,Ꚍ) is not N_2Ꚍ1-space.

Then (χ,Ꚍ) N_2preꚌ1-space, but (χ,Ꚍ) is not N_2Ꚍ1-space.

Example 3.13.
Let $\chi = \{a, b, c\}$, Ꚍ $= \{\emptyset_N , X_N , A, B\}, A = \{< \emptyset, \emptyset, \{ a\} >\}, B = \{< \emptyset, \emptyset, \{ b, c\} >\}$.
$N_cS.OS =$ Ꚍ $\cup \{G = \{< \emptyset, \emptyset, \{ b\} >\}, C = \{< \emptyset, \emptyset, \{ c\} >\}, E = \{< \emptyset, \emptyset, \{ a, b\} >\}, H = \{< \emptyset, \emptyset, \{ a, c\} >\} \}$.
Let $x_{N_3} = \{< \emptyset, \emptyset, \{ b\} >\} \neq y_{N_3} = \{< \emptyset, \emptyset, \{ c\} >\} \in \chi$ there is no $Nc.OS$ Ɱ 1, Ɱ2 in χ such that $x_{N_3} \in$ Ɱ1, $y_{N_3} \notin$ Ɱ1 and $x_{N_3} \notin$ Ɱ2 , $y_{N_3} \in$ Ɱ2. Therefore (χ,Ꚍ) is not N_3Ꚍ1-space.

Then (χ,Ꚍ) N_3preꚌ1-space, but (χ,Ꚍ) is not N_3Ꚍ1-space.

Theorem 3.14.

Let (χ,Ꚍ) be an NcTS, then :

1. Every N_1Ꚍ2-space is N_1preꚌ2-space.
2. Every N_2Ꚍ2-space is N_2preꚌ2-space.
3. Every N_3Ꚍ2-space is N_3preꚌ2-space.

Proof:

1. Suppose that (χ,Ꚍ) is an N_1Ꚍ2-space , therefore for every two $x_{N_1} \neq y_{N_1}$, there exists an $Nc.OS$ Ɱ 1, Ɱ2 in χ such that $x_{N_1} \in$ Ɱ 1, $y_{N_1} \notin$ Ɱ 1 and $x_{N_1} \notin$ Ɱ2 , $y_{N_1} \in$ Ɱ2. with Ɱ1∩Ɱ2= ∅. So there exists $NcP.OS$ Ɱ 1, Ɱ2 in χ such that $x_{N_1} \in$ Ɱ 1, $y_{N_1} \notin$ Ɱ 1 and $x_{N_1} \notin$ Ɱ2 , $y_{N_1} \in$ Ɱ2. with Ɱ1∩Ɱ2= ∅.
   Therefore X is N_1preꚌ2-space.

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2. Suppose that (χ,Ʈ) is an N_2Ʈ$_2$-space , therefore for every two $x_{N_2} \neq y_{N_2}$, there exists an $Nc.OS$ M$_1$, M$_2$ in χ such that $x_{N_2} \in$ M$_1$, $y_{N_2} \notin$ M$_1$ and $x_{N_2} \notin$ M$_2$, $y_{N_2} \in$ M$_2$. with M$_1$∩M$_2$= ∅. So there exists $NcP.OS$ M$_1$, M$_2$ in χ such that $x_{N_2} \in$ M$_1$, $y_{N_2} \notin$ M$_1$ and $x_{N_2} \notin$ M$_2$, $y_{N_2} \in$ M$_2$. with M$_1$∩M$_2$= ∅.
   Therefore X is N_2preƮ$_2$-space.
3. Suppose that (χ,Ʈ) is an N_3Ʈ$_2$-space , therefore for every two $x_{N_3} \neq y_{N_3}$, there exists an $Nc.OS$ M$_1$, M$_2$ in χ such that $x_{N_3} \in$ M$_1$, $y_{N_3} \notin$ M$_1$ and $x_{N_3} \notin$ M$_2$, $y_{N_3} \in$ M$_2$. with M$_1$∩M$_2$= ∅. So there exists $NcP.OS$ M$_1$, M$_2$ in χ such that $x_{N_3} \in$ M$_1$, $y_{N_3} \notin$ M$_1$ and $x_{N_3} \notin$ M$_2$, $y_{N_3} \in$ M$_2$. with M$_1$∩M$_2$= ∅.
   Therefore X is N_3preƮ$_2$-space.
4.

Remark 3.15.
The converse of the Theorem 3.14 is not true., as it is shown in the following example.

Example 3.16.
In example 3.11. (χ,Ʈ) Ɲ$_1$$pre$Ʈ$_2$-space, but (χ,Ʈ) is not Ɲ$_1$Ʈ$_2$-space.

Example 3.17.
In example 3.12. (χ,Ʈ) Ɲ$_2$$pre$Ʈ$_2$-space, but (χ,Ʈ) is not Ɲ$_2$Ʈ$_2$-space.

Example 3.18.
In example 3.13. (χ,Ʈ) Ɲ$_3$$pre$Ʈ$_2$-space, but (χ,Ʈ) is not Ɲ$_3$Ʈ$_2$-space.

Theorem 3.19.

Let (χ,Ʈ) be an N$_c$TS, then :

1. N_1preƮ$_2$-space ⇒ N_1preƮ$_1$-space ⇒ N_1preƮ$_0$-space.
2. N_2preƮ$_2$-space ⇒ N_2preƮ$_1$-space ⇒ N_2preƮ$_0$-space.
3. N_3preƮ$_2$-space ⇒ N_3preƮ$_1$-space ⇒ N_3preƮ$_0$-space.

The converse of the Theorem 3.19 is not true.

Remark. 3.21.
Relations among the different types of neutrosophic crisp separation axioms which were studied in this paper, appear in the following diagram.

$$\begin{array}{ccccc}
\text{Ni-ST}_2\text{-space} & \overset{\longrightarrow}{\not\longleftarrow} & \text{Ni-ST}_1\text{-space} & \overset{\longrightarrow}{\not\longleftarrow} & \text{Ni-ST}_o\text{-space.} \\
\not\downarrow\ \uparrow & & \not\downarrow\ \uparrow & & \not\downarrow\ \uparrow \\
\text{Ni-T}_2\text{-space} & \overset{\longrightarrow}{\not\longleftarrow} & \text{Ni-T}_1\text{-space} & \overset{\longrightarrow}{\not\longleftarrow} & \text{Ni-T}_o\text{-space.} \\
\downarrow\ \not\uparrow & & \downarrow\ \not\uparrow & & \downarrow\ \not\uparrow \\
\text{Ni-PT}_2\text{-space} & \overset{\longrightarrow}{\not\longleftarrow} & \text{Ni-PT}_1\text{-space} & \overset{\longrightarrow}{\not\longleftarrow} & \text{Ni-PT}_o\text{-space.}
\end{array}$$

i=0,1,2.

Conclusion

In this paper, a new type of neutrosophic crisp separation axioms has been defined by using neutrosophic crisp pre open sets and certain point in the neutrosophic crisp topological spaces. Moreover, the connections between neutrosophic crisp pre separation axioms and the existing neutrosophic crisp separation axioms are studied. And many examples are presented to clear the concepts introduced. Also, proof their basic properties. Also, investigate their fundamental properties and characterizations.

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A short remark on Bong Han duct system (PVS) as a Neutrosophic bridge between Eastern and Western Medicine paradigms

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Abstract

In a previous paper in this journal (IJNS), it is mentioned about a possible approach of "*curemony*" as a middle way in order to reconcile Eastern and Western's paradigms of medicine [1]. Although it is known in literature that there are some attempts to reconcile between Eastern and Western medicine paradigms, known as "*integrative medicine,*" here a new viewpoint is submitted, i.e. Bong Han *duct* system (PVS), which is a proof of Meridian system, can be a bridge between those two medicine paradigms in neutrosophic sense. This can be considered as a Neutrosophic Logic way to bridge or reconcile the age-old debates over the Western and Eastern approach to medicine. It is also hoped that there will be further research in this direction, especially to clarify the distinction between Pasteur's *germ theory* and Bechamp's *microzyma theory*. More research is obviously recommended. **Motivation of this paper:** to prove that Neutrosophic Logic offers a reconciliation towards better dialogue between Western and Eastern medicine systems. **Novelty aspect:** it is discussed here how Bong Han Duct system offers a proven and observable way to Meridian system, which in turn it can be a good start to begin meaningful dialogue between Western and Eastern systems.

Keywords: Pasteur, microzyma, Bechamp theory, meridian system, Bong Han Kim, Bong Han duct system, neutrosophic logic

1.Introduction

In the light of recent advancements on the use of Neutrosophic Logic in various branches of science and mathematics, this paper discuss possible application in medicine philosophy. See for instance [13-19].

This paper is inspired partly by the movie, *Jewel in the Palace* (Dae Jang Geum). One of these authors (VC) has a younger brother who likes to watch that movie. He already completed watching the entire series (more than 70 episodes) more than three times. According to a good documentary on that movie [11]:

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A history book courageous woman is reawakened in a hit TV dramatization. In 1392, the Joseon Dynasty appeared. The rulers of Joseon would lead the Korean landmass until the administration fell, to be supplanted by a Japanese provincial system, in 1910. All things considered, Joseon's heritage suffers: It was one of the world's longest-running imperial administrations. In the "Joseon-Wangjo-Sillok" - "The Annals of the Joseon Dynasty;" the official record of the realm - a lady named "Daejanggeum" is referenced. She lived during the rule of King Jungjong (1506~1544), and the archives disclose to us that she had been a low-positioning court woman who picked up the ruler's trust and was elevated to the most noteworthy positioned woman in the kitchen, and furthermore to regal doctor. In one notice in the archives, the ruler states, "*I have nearly recuperated from the sickness of a couple of months. So I should offer honors to the individuals who put forth bunches of attempts to fix me. Give the imperial doctors and euinyeo (female associate) Daejanggeum blessings.*"

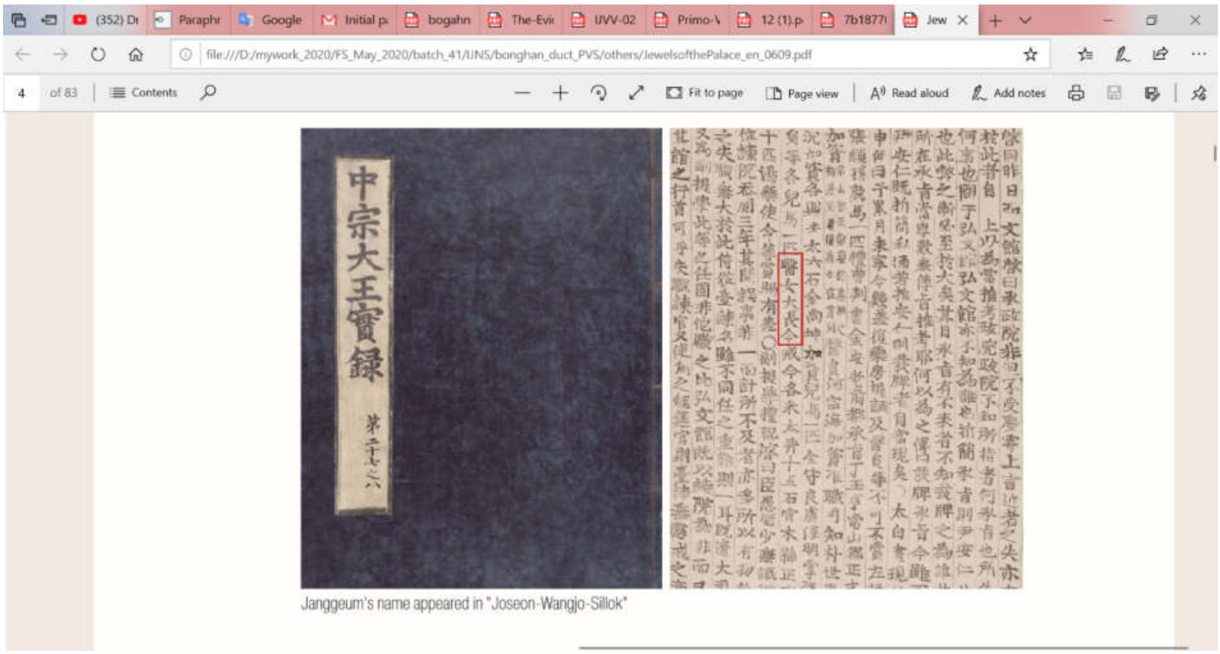

Figure 1. Jang Geum name was recorded in the "*Joseon-Wangjo-Sillok*" - "The Annals of the Joseon Dynasty." After Kang Min Su [11]

What is more interesting to these authors, is not only the depiction of royal palace at the time, but also the use of royal cuisine as medication, beside the use of acupuncture methods.[11]

Now it seems obvious for Western scholars to pause at this point and ask: "*What? Acupuncture? Are you joking?*"

This short review paper is discussing that approach: whether it is possible to reconcile both Eastern and Western medicine paradigms from the view point of Bong Han Kim's duct system (PVS) and its relation to Bechamp's microzyma.

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As it is brought up in [1], it is notable by most medication experts, that Western way to deal with medication depends on "*assaulting*" an infection, individually. This is called *germ* hypothesis: one remedy for one ailment (Pasteur). On the contrary side, Eastern medication is situated specifically on old knowledge of *restoring* the parity (harmonious functions) of the body, at the end of the day: to blend our body and our live with nature. In spite of the fact that those two methodologies in medication and social insurance have caused such a large number of contentions and false impressions, really it is conceivable to do an exchange between them. From Neutrosophic Logic's point of view, a goal to the above clashing ideal models can be found in creating novel methodologies which acknowledge the two conventions in medication, or it is conceivable to call such a methodology: "*curemony*," for example by simultaneously relieving an infection and reestablishing harmony and returning concordance in one's body-mind-soul all in all.

Now it is known that one of the objections by Western scholars about the Eastern medicine (based on meridian points) is the unobservability of meridian vascular/duct system. This makes meridian system neglected in almost all textbooks taught in Western medicine schools. Therefore, here a new viewpoint is submitted, i.e. Bong Han *duct* system (PVS), which is a *proof of Meridian system,* can be a bridge between those two medicine paradigms in neutrosophic sense. This can be considered as a Neutrosophic Logic way to bridge or reconcile the age old debates over the Western and Eastern approach to medicine.

It would be a lot easier to merge both the eastern (ancient) and the modern western curative system in terms of neutrosophy.These neutrosophic intermediates will help further to boost dialogues between those Western and Eastern system and their useful information. This neutrosophic intermediator is actually dealing with conscious of both non-matter and matter in terms of ancients and modern techniques.

2. Introduction to Bong Han duct system

Nonetheless, in literature it is recorded that Bong Han Kim is a Professor in Biology in Korea. Around 1950-1960 he found the vessel which is a *"duct"* to known Eastern Meridian system, which is already known in acupuncture medicine system. Therefore it seems like a bridge between Western and Eastern medicine paradigms. As it is mentioned in previous paper [1], this paper will discuss how those paradigms can be reconciled in Neutrosophic Logic, using a degree of Western medicine and a degree of Eastern medicine, as the neutral part of neutrosophy. To us, Bong Han duct system is a good way to start a healthy and meaningful dialogue between those two paradigms in medicine.

As Vitaly Vodanoy wrote, which can be rephrased as follows:

> "In the 1960's Bong Han Kim found and described another vascular framework. He had the option to separate it unmistakably from vascular blood and lymph frameworks by the utilization of an assortment of techniques, which were accessible to him in the mid-twentieth century. He gave nitty gritty portrayal of the framework and made thorough graphs and photos in his distributions. He showed that this framework is made out of hubs and vessels, and it was answerable for tissue recovery. In any case, he didn't reveal in subtleties his

> techniques. Thus, his outcomes are moderately dark from the vantage purpose of contemporary researchers. The stains that Kim utilized had been idealized and being used for over 100 years. In this manner, the names of the stains coordinated to the unequivocal conventions for the use with the specific cells or particles. Generally, it was not typically important to portray the strategy utilized except if it is altogether strayed from the first technique."[9]

Although his method was almost forgotten until recently, it has been recovered again in the past decade. It is clear therefore, that Bonghan Kim's work, who essentially (and without being aware of the work previously done by Bechamp) discovered that what we call the 'Meridian System' (known as the *Kyungrak System* in the Korean tongue) which exists in the body as an actual third anatomical vascular system, comprised of ducts, ductules, corpuscles, and a unique type of fluid, the contents of which tie directly back to Bechamp's own discoveries (work is still being done today on the mapping out of this anatomical system, as it is far more extensive than the old Oriental texts gave it credit.) See [4].

Remark on terminology:

> "In a matter of seconds before the primary International Symposium on Primo Vascular System, which was held in Jecheon, Korea during September 17–18, 2010, Dr. Kwang-Sup Soh, recommended that it is critical to concur upon a solitary phrasing for the Bonghan framework. It was concurred that following terms would be embraced: Bong-Han System (BHS) - Primo Vascular System (PVS); Bonghan Duct (BHD)- Primo Vessel (PV); Bonghan Corpuscle (BHC)- Primo Node (PN); Bonghan Ductule - P-Subvessel; Bonghan Liquor-Primo Fluid (P-liquid); Sanalp-Microcell"[9].

Now in the next section, it will be discussed virus research, especially at their beginning.

Hidden the introduction of virology is a conviction that infections are monomorphism, they are fixed species, unchangeable; that each neurotic kind produces (typically) just a single explicit illness; that microforms never emerge endogenously, i.e., have supreme source with the host. Thus the worldview prompts conviction called "germ hypothesis" of Pasteur: for example one remedy for one disease.[6-7]

Bechamp recorded standard as the premise of another hypothesis about "infections." Briefly, this guideline holds that in every single living life form are organically indestructible anatomical components, which he called microzymas. They are freely living sorted out matures, equipped for creating compounds and fit for advancement into increasingly complex microforms, for example, microbes. Bechamp's proposition is that infection is a state of one's interior condition (landscape); that ailment (and its indications) are "conceived of us and in us"; and that malady isn't created by an assault of microentities yet considers forward their endogenous cause. [8]

All things considered, it is realized that Pasteur duplicated whatever he discovered Bechamp thoughts would fit in his own hypothesis. Consequently, Bechamp was unmistakably increasingly unique researcher contrasted with Pasteur.

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3. A re-interpretation of diseases and virus from Bechamp's theory

This section begin by citing [4], which can be paraphrased as follows:

> "Through a cautious perception of the wonders of the thickening of the blood just as the procedure of maturation; and as a methods for all the more accurately deciphering the basic idea of these marvels; Bechamp straightforwardly saw that there exist a layers of subcellular, miniaturized scale natural living structures known as 'microzymas', a word which when interpreted signifies 'minor ages'. These structures were alluded to without anyone else and by other people (who came later, and mentioned a similar objective facts) as some type of 'atomic granulations' (more on this beneath). These microzyma are littler in size than some other known types of small scale natural life, and fill in as the base establishment for the development of every other type of such life."

Moreover, on a more recent setting, see Andrew Kaufmann's report on WHO's early investigation of the corona virus, before it was declared globally as an epidemic.[1]

According to Dr. Andrew Kaufman's report, a "virus" as observed is actually an *exosome*. That is not impossible. Even if you want to be more assertive. It's not just the PCR test that is inaccurate. So the so-called virus is indeed questionable. Because it relates to the *germ theory* of Pasteur, meaning each disease will need one type of medicine [1][2].

That's not right. Pasteur's theory draws a lot from the real expert at the time: Bechamp.[4]

In essence, according to Bechamp, the source of the disease is most likely to be endogenous. Meaning from within the body when adjusting itself to the environment.

What is interesting to ask here is what kind of the changes in the environment that triggers the emergence of symptoms such as excessive thirs? Actually, it is known as one of the symptoms known for electromagnetic radiation. Therefore, it is no surprise that there are some allegations by experts: severe radiation disturbances arise in Wuhan and Italy and also the USA because of they are the locations where the massive 5G network has begun to be installed (see also Firstenberg's report [5]).

But this short paper is not intended to discuss more detailed about relation between 5G and covid-19, so this problem will be left to others to take up this matter and investigate further.

4. Concluding remarks

This paper continued our previous article, where possible approach of "*curemony*" is discussed as a middle Neutrosophic way in order to reconcile Eastern and Western's paradigms of medicine [1]. Although it is known in literature, that there are some attempts to reconcile between Eastern and Western medicine paradigms, known as

[1] Dr. Andrew Kaufman's interview on corona virus test. url: https://www.youtube.com/watch?v=f9mzdvOEjBc

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"*integrative medicine*," here it is submitted a viewpoint that Bong Han duct system (PVS) which is a proof of meridian system, can be a neutrosophic bridge between those two medicine paradigms.

Here a new viewpoint is submitted, i.e. Bong Han *duct* system (PVS), which is a *proof of Meridian system,* can be a bridge between those two medicine paradigms in neutrosophic sense. This can be considered as a Neutrosophic Logic way to bridge or reconcile the age old debates over the Western and Eastern approach to medicine.

It would be a lot easier to merge both the eastern (ancient) and the modern western curative system in terms of neutrosophy.These neutrosophic intermediates will help further to boost dialogues between those Western and Eastern system and their useful information. This neutrosophic intermediator is actually dealing with conscious of both non-matter and matter in terms of ancients and modern techniques.

As mentioned in our previous paper [1], it is also discussed how those paradigms can be reconciled in Neutrosophic Logic. To us, Bong Han duct system (PVS) is a good way to start a healthy and meaningful dialogue between those two paradigms in medicine.

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Online Analytical Processing Operations via Neutrosophic Systems

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Abstract
In this paper, a neutrosophic fuzzy data warehouse modelling approach is presented to support the neutrosophic analysis of the publishing house for books which allows integration of neutrosophic concept in dimensions and facts without affecting the core of a crisp data warehouse. Also we describe a method is presented which includes guidelines that can be used to convert a crisp data warehouse into a neutrosophic fuzzy domain. Finally we have presented an OLAP system that implements a neutrosophic multidimensional model to represent imprecision using neutrosophic concept in hierarchies and facts and achieving knowledge discovery from imperfect data. The use of neutrosophic structures and the definition of the OLAP operations (roll-up, drill-down, slice, and dice) enable the model to manage the imprecision of data and hide the complexity of the model and provide the user with a more understandable result.

Keywords: Fuzzy Sets, Neutrosophic Fuzzy Data Warehouse, Neutrosophic Fuzzy Cube, Neutrosophic Fuzzy OLAP Operation.

1. Introduction

In business scenarios, where some of the data or the business attributes are neutrosophic, it may be useful to construct a warehouse that can support the analysis of neutrosophic data. Accurate information about an organization's state is necessary in order to make strategic decisions. Information contains historical data derived from transaction data, but It usually include data from other sources such as relational databases, spreadsheets, mainframes, mail systems or even paper files Each of these data stores tends to serve a subset of the enterprise for decision making. An increasing number of heterogeneous information systems makes retrieving meaningful information more difficult. In order to gather, store and process this information, various information systems are used. The enterprise information system map shows often numerous, heterogeneous and complex information system constellations. Often, for operational use relational database systems are used and for analytical purposes a data warehouse is used. Bill Inmon [1] is cited very often and seems to be the father of the term Data Warehouse. In fact, Inmon's definition goes back to the first edition of his book "Building The Data Warehouse" from 1993, Wolfgang Lehner [2], a researcher in data warehousing, has recently published a profound and comprehensive book on data warehouse systems in German. He references Bill Inmon, but his book also contains a more elaborate definition of data warehouse systems, In addition to a relational database, a data warehouse environment includes an Extraction, Transformation, and Loading (ETL) solution, an Online Analytical Processing (OLAP) engine, client analysis tools, and other applications that manage the process of gathering data and delivering it to business users. This analytical view on data finally enables the enterprise to have a more global sight on its business environment than operational systems can provide. Therefore, data warehouses are often used as systems for decision making [3]. Besides positive aspects of centralized processing of business information such as decision making support, difficulties occur in maintaining and analyzing data warehouses. The amount of data that has to be processed in a data warehouse increases every day and turns into challenging tasks for administration and analysis. Next to the problem of high quantity, data from operational systems are often incomplete, vague or uncertain. This quality issue cannot be completely eliminated in the pre-processing stage of the data. Consequently, a certain amount of vagueness directly impacts the analysis and decision making that is based on the information of a data warehouse [4]. Data warehousing and on-line analytical processing (OLAP) are essential elements of decision support. "In [24-29] OLAP is computer processing that enables a user to easily and selectively extract and view data from different points of view. Data warehouse and OLAP tools provide an efficient framework for data mining. Besides, data from real world are often imperfect, either because they are uncertain, or because they are imprecise. To solve this problem, We have presented a structure that manages imprecision by means of

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neutrosophic techniques [7, 8]. The use of neutrosophic set theory in systems enhances the understand ability of the discovered knowledge; this is the reason why we have proposed an approach to perform OLAP-based on neutrosophic concept. The ability to analyze large amounts of data for the extraction of valuable information presents a competitive advantage for any organization. . The managers need information about their business and insight into the existing data so as to make decision more efficiently without interrupting the daily work of an On-Line Transaction Processing (OLTP) system. The technologies of data warehousing, OLAP, and data classification support that ability. The data warehouse is a central data pool which integrates heterogeneous data sources and provides strategic information for analysis and decision support. The special needs of the OLAP technology was the main cause of the use of a multidimensional view of the data. On-Line Analytical Processing (OLAP) presents an approach to data analysis where data is consolidated and aggregated with respect to multiple dimensions of interest. The idea is to consolidate large amounts of data by summarizing and aggregating data elements for every cell of a data cube. Classification of data elements reduces an arbitrarily high number of data elements into an arbitrarily small set of classes, which highly reduces the granularity of data. In OLAP, classification is used for the consolidation of dimensional attributes." In Many complicated problems like, engineering problems, social, economic, computer science, medical science…etc, the data associated are not necessarily crisp, precise, and deterministic because of their vague nature. Most of these problems were solved by different theories. One of these theories was the fuzzy set theory discovered by Zadeh in [17-20] to handle vagueness , uncertainty and imprecision ,In fuzzy logics the two-point set of classical truth values {0, 1} is replaced by the real unit interval [0, 1] each real value in [0, 1] is intended to represent a different degree of truth, ranging from 0, corresponding to false in classical logic, to 1, corresponding to true. . A fuzzy set A in M can be represented as an ordered set of tuples $\{(m , \boldsymbol{\mu_A}(m))\}$. But for some applications it is not enough to satisfy to consider only the membership-function supported by the evident but also have to consider the non-membership-function against by the evident Atanassov [6] introduced another type of fuzzy sets that is called Intuitionistic Fuzzy Set (IFS) which is more practical in real life situations. The main novelty of neutrosophic logic, as we shall see, is that we do not even assume that the incompleteness or "indeterminacy degree" is always given by $1 - (t + f)$. Smarandache and A.A.Salama [7, 8] introduced another concept of imprecise data called neutrosophic crisp sets. Neutrsophic set is a powerful general formal framework that has been recently proposed. Let N be a set defined as follows: $N = \{(T, I, F) : T, I, F \subseteq [0, 1]\}$. Where (T) the Truth degree, (F) the falsehood degree and (I) the indeterminacy degree, $I \subseteq [0, 1]$ may represent not only indeterminacy but also vagueness, uncertainty, imprecision, error etc. Note also that T, I, F, called the neutrosophic components [9].Several researchers dealing with the concept of neutrosophic set such as M. Bhowmik and M.Pal in [10] and A.A.Salama in [11-15]. For more information on the application of neutrosophic theory, the readers can referes to [30-33] . In this paper we aim to construct a neutrosophic data warehouse. The key benefit of integrating neutrosophic logic in data warehouse it allows analysis of data in both classical and neutrosophic manners. The use of the proposed approach is demonstrated through a case study of a published housing for books. Finally we have presented an OLAP system that implements a neutrosophic multidimensional model to represent imprecision using neutrosophic concept in hierarchies and facts and achieving knowledge discovery from imperfect data.

2. Crisp Data Warehouse Concept

A data warehouse **[1]** is a database, which is kept separate from the organization's operational database. There is no frequent updating done in a data warehouse, it possesses consolidated historical data, which helps the organization to analyze, organize, understand, and use their data to take strategic decisions. This analytical view on data finally enables the enterprise to have a more global sight on its business environment than operational systems can provide. Therefore, data warehouses are often used as systems for decision making. The term "Data Warehouse" was first coined by Bill Inmon. he describe the data warehouse as "subject-oriented, integrated, non volatile, and time-variant collection of data in support of management's decision support. The components of his definition in the following way:

2.1. Subject-Oriented: Subject-Oriented means that the main objective of data warehouse is to facilitate decision process of a data company, and within any company data naturally concentrates around subject areas, so information gathering in warehouse is aiming for a specific subject rather than for the functions of a company.

2.2. Integrated: Being integrated means that the data is collected within the data warehouse, that can come from different tables, databases or even servers, but can be combined into one unit that is relevant and logical for convenience of making strategic decision.

2.3. Non-volatile: Non-volatile means the previous data is not erased when new data is added to it. A data warehouse is kept separate from the operational database and therefore frequent changes in operational database is not reflected in the data warehouse.

2.4. Time-variant: The content of the data warehouse grows over time, where on regular basis snapshot of current data is entered into the data pool. The key structure of the data warehouse always contains time.

3. Linguistic variables

Linguistic variables are collect elements into similar groups where we can deal with less precisely and hence we can handle more complex systems. it's is an important concept in fuzzy logic and plays a key role in its applications, especially in the fuzzy expert system Linguistic variable is a variable whose values are words in a natural language , For example, “speed” is a linguistic variable, which can take the values as “slow”, “fast”, “very fast” and so on. Zadeh developed on top of the fuzzy set theory a means for mathematically representing natural language [16]. Therefore, he defined a linguistic variable values [17, 18, 19] . The values of the linguistic variable called linguistic terms, are projected on a universe of discourse. Fuzzy sets are used to define the degree of membership with which a value might belong to a linguistic term. Zadeh defines a linguistic variable as follows:

3.1. Definition (Linguistic variable [20]). A linguistic variable is a quintuple (X; T(X); G;M; F) defined as follows:

X is the name of the linguistic variable, T(X) is the set linguistic terms of X , G represents a syntactic rule that generates the set of linguistic terms, M is the universe of discourse and F is a semantic rule that defines for each linguistic term its meaning in the sense of a fuzzy subset on U.

4. Concept of Neutrosophic Fuzzy Sets

The main idea of Neutrosophic Sets is to characterize each logical statement in a 3D Neutrosophic Space, where each dimension of the space represents respectively the truth (T), the falsehood (F), and the indeterminacy (I) of the statement. Neutrosophic Logic (NL) is a generalization of Zadeh’s fuzzy logic (FL), and especially of Atanassov’s intuitionistic fuzzy logic (IFL), and of other logics For example, suppose there are 10 voters during a voting process In time t1, fivevote \yes", three vote \no" and two are undecided, using neutrosophic notation, it can be expressed as $\boldsymbol{x_1}$(0.5, 0.2, 0.3) In time t2, four vote \yes", two vote \no", and three are undecided, it then can be expressed as (0.4 0.3, 0.2). the notion of neutrosophic set is more general and overcomes the aforementioned issues. In neutrosophic set, indeterminacy is quantified explicitly and truth-membership, indeterminacy- membership and falsity-membership are independent. This assumption is very important in many applications such as information fusion in which we try to combine the data from different sensors. The neutrosophic set takes the value from real standard or non-standard subsets of $]^-0,1^+[$. So instead of $]^-0,1^+[$ we need to take the interval [0,1] for technical applications, because $]^-0,1^+[$ will be difficult to apply in the real applications such as in scientific and engineering problems. For software engineering proposals the classical unit interval [0, 1] is used. For single valued neutrosophic logic, the sum of the components is:
case (1) $0 \leq t+i+f \leq 3$ when all three components are independent;
case (2) $0 \leq t+i+f \leq 2$ when two components are dependent, while the third one is independent from them.
case (3) $0 \leq t+i+f \leq 1$ when all three components are dependent.

5. Case Study

The case study discusses a The Publishing house for books , It currently offers a collection of books for purchase, Each customer is asked to rate the book when he read it, When the publishing house makes statistical survey To measure the performance of their business such that Publishing house for books analyzes the revenue of the books based on the age of the customers or stores or measure the performance based on rating of customers it is found that the proportion of persons did not give a specific answer (undecided). Their answer is not belong to a certain class or not belonging to this category, This percentage has not been taken into account for it found that there is ambiguity in the data became unclear ,for example the book "Scientific Miracles in the Holy Quran " some of people classify this book to scientific category and some of people classify it to religious category and others not decided(not sure) if this book belong to scientific category or religious category. so they didn't give a specific answer. Now we have three answers membership, Non Membership and Indeterminacy. Neutrosophic Sets to solve this ambiguity in the data and taking the opinion of indeterminacy into account and gave them the degree. for example: the Neutrosophic set " scientific books " might contain the following tuples:" scientific books " = {" Scientific Miracles in the Holy Quran " , < 0.7,0.1,0.2 >}< 0.7,0.1,0.2,> which 0.7 is represented the membership degree of this book to scientific books genre, 0.2 is represented the non membership degree of this book to scientific books genre and 0.1 is represented the indeterminacy degree of this book to scientific books genre. so we must integration neutrosophic fuzzy concept to data warehouse.

6. Neutrosophic Fuzzy Data Warehouse

In order to create a neutrosophic data warehouse, a method is presented that can guide the transformation of a classical data warehouse into a neutrosophic data warehouse. The input to the method is a classical data warehouse and the output is a neutrosophic data warehouse. This approach is allows integrating neutrosophic concepts without the need for redesigning the core of a data warehouse. By using this neutrosophic data warehousing approach, it is possible to extract and analyze the data simultaneously in a classical manner and in a neutrosophic manner.

For example, books might be classified into different genres. In the classical data warehouse, a book always belongs or not belong fully to one or more genres the numbers of the interval [0, 1] where 1 implies full belonging and 0 implies no belonging at all. In reality, books can often be categorized into several genres while belonging or not belonging more to one genre than to another with different degrees. A book "can be a scientific book, a religious book, a political book, a social book or a literary book and so on. In this classification it belonging at the same time to one or more genre but with different degrees and it not belonging with different degrees, and also has indeterminacy degree. In order to truly represent this ambiguity in classification the neutrosophic set theory can be applied therefore, the Publishing house classifies the books with a neutrosophic concept.

The following figure show convert classical data warehouse into neutrosophic data warehouse:

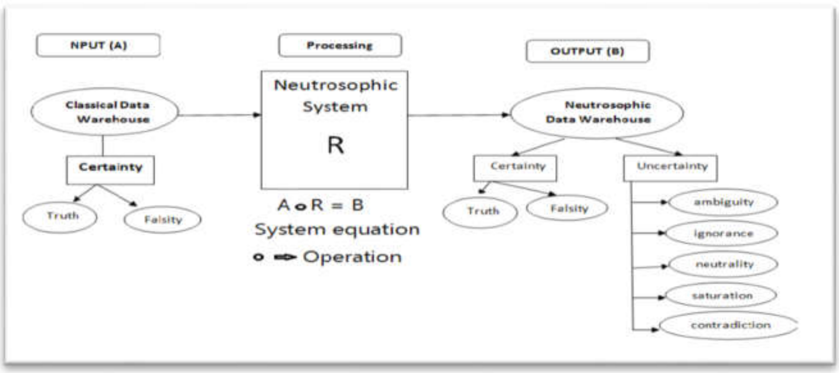

Figure 1: convert classical data warehouse into Neutrosophic Fuzzy Data warehouse

6.1. Basic Definitions of Neutrosophic Fuzzy Data Warehouse

In this section we introduce and study the following definitions of Neutrosophic Data Warehouse.

6.1.1 Definition (Neutrosophic fuzzy data warehouse (NDW)).

A neutrosophic fuzzy data warehouse model is a set of combination of four types of tables .these are (Dimension tables (D), Fact tables (F), Neutrosophic Classification Tables (NCT) and Neutrosophic Degree Tables (NDT)) and it is represented by NDW. NDW = {D, F, NCT, NDT}

6.1.2 Definition (Neutrosophic Fuzzy Table (NT)).

Neutrosophic Table is the table which contain a neutrosophic target element and the table may be dimension table or Fact table.

6.1.3 Definition (Neutrosophic Fuzzy Target Element (NTE)).

Neutrosophic Target Element (NTE) is the elemet may be in in a Fact table or a dimension table which required to be classified in neutrosophic.

6.1.4 Definition (Class Neutrosophic Fuzzy Target Element (CNTE)).

A class neutrosophic Target element (CNTE) for a neutrosophic Target element (NTE), it's all possible values (linguistic terms) for a neutrosophic Target element.

6.1.5 Definition (Neutrosophic Fuzzy Degree (ND)).

All values for neutrosophic Target element belong to a certain neutrosophic degree to a class neutrosophic which neutrosophic Target element belong . The degree of belonging to a value to class neutrosophic is called neutrophic degree.

6.1.7 Definition (Neutrosophic Classification Table (NCT)).

A table that holds linguistic terms (neutrosophic classes and it consists of two attribute (a primary key of the table and a class neutrosophic target element which can be classified neutrosophic), NCT = {PK, CNTE}

6.1.8 Definition (Neutrosophic Fuzzy Degree Table (NDT)).

A table that stores the degree of each linguistic term is called neutrosophic degree table,it has contain four attributes: (the primary key of the table, the foreign key of neutrosophic table which contain neutrosophic target, the foreign key of neutrosophic classification table and neutrosophic degree of each linguistic terms for linguistic variable (neutrosophic target)). NDT= {PK, FK_NT, FK_NCT, FK_NDT}

6.2. Neutrosophic Date Warehouse Model

In addition to the classical analysis in a data warehouse, The Publishing house for books needs some features that are available using neutrosophic concepts. For integrating neutrosophic concepts into a data warehouse, one must first analyze which elements in the data warehouse should be classified neutrosophic. The element may be an element in the fact table or an element in a dimension table. An element that has to be classified neutrosophic is called the neutrosophic target element (NTE). The steps are the follow:
1) First step: identify what should be classified to identify the neutrosophic target element.
2) Second step: identify the set of linguistics terms that are used for classifying the neutrosophic target element. Repeat this step for all neutrosophic target elements.
3) Third step: define a neutrosophic function for each linguistic term. Repeat this step for each linguistic term.
4) Forth step : create Neutrosophic classification table which holds classes of neutrosophic target element (linguistic terms) and it contain two attribute one is the primary key of the table and second is the class neutrosophic target element.
5) Fifth step : create Neutrosophic degree table which holds neutrosophic degrees fo each linguistic term and it contain four attribute which the first attribute is the primary key of the table, the second attribute is the foreign key of the neutrosophic table, the third attribute is the foreign key of Neutrosophic classification table and the fourth attribute is the neutrosophic degree (ND) attribute for the neutrosophic target element ,The values of neutrosophic degree attribute are calculated by neutrosophic functions (represented by $< T, I, F >$ where T is the membership degree of element to a set A , F is the non membership degree of element to a set A and I is the degree of indeterminacy element to a set A.
6) sixth step : Relate neutrosophic table with neutrosophic classification table and neutrosophic degree table with each other.

The following figure represented Neutrosophic Date Warehouse Model:

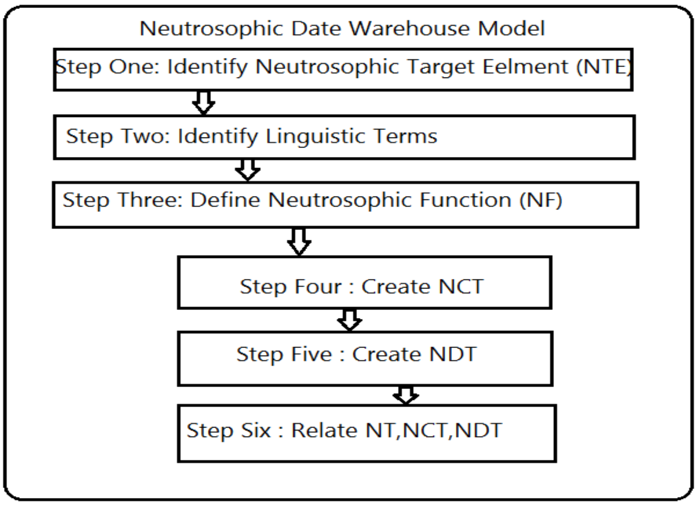

Figure 2: Neutrosophic fuzzy data warehouse Model

6.2.1 Dimension Book

The books are classified into different genres. In the classical data warehouse, a book always belongs to interval [0,1] where 1 is implies to belonging degree of the book fully to one or more genres and 0 is implies to not belonging degree of the book to one or more genres. In reality, books can often be categorized into several genres while (belonging, indeterminacy, not belonging) more to one genre than to another at the same time. For Example, the book "Scientific Miracles in the Holy Quran" can be classified scientific and Religious, but more strongly scientific (membership) ,and can be classified not belonging to one or more genres by different degrees (non membership) and also the book have the indeterminacy degree. With the classification in the classical data warehouse approach, the published house cannot classify the books into different genres. Therefore, the published house classifies the books with a neutrosophic concept

- First the book genre is defined as neutrosophic target element.
- The second step is to identify the linguistic terms. In this case, the linguistic terms are the different genres to which the books belong. These genres can be extracted from the dimension category in the book dimension.
- In the third task is to identify the neutrosophic functions for each genre has to be defined

After identifying neutrosophic target element, linguistic terms and their neutrosophic functions,

- forth step is to create neutrosophic classification table holds the genres as class neutrosophic element.
- Fifth step is to create neutrosophic degree table contains neutrosophic degrees for each neutrosophic target element corresponding to class neutrosophic elements.

-finally, sixth step is to relate the neutrosophic classification table, the neutrosophic degree table and the neutrosophic table to each other.

The following Figure show neutrosophic concept in book dimension:

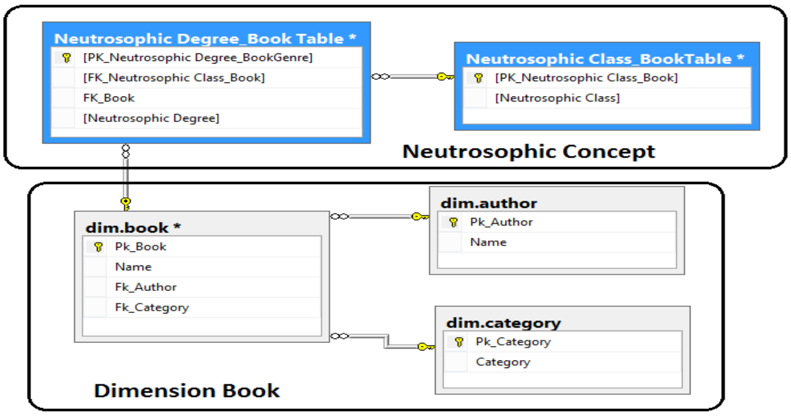

Figure 3: Neutrosophic Concept Book Genre

The following figure shows result sets of apply neutrosophic concept in book dimension

Book ID	Name	Neutrosophic Class	Membership Degree
1	Scientific Miracles in the Holy Quran	scientific ...	< 0.6 , 0.1 , 0.3>
1	Scientific Miracles in the Holy Quran	religious ...	< 0.5 , 0.2 , 03 >
2	Political Islam and the coming battle	political ...	< 0.7 , 0.1 , 0.2 >
2	Political Islam and the coming battle	religious ...	< 0.6 , 0.1 , 0.3 >
3	perhaps you laugh	social ...	< 0.4 , 0.3 , 0.3 >
3	perhaps you laugh	Comic ...	< 0.6 , 0.2 , 0.2 >
5	Soft hands	romantic ...	< 0.7 , 0 , 0.3>
5	Soft hands	Comic ...	< 0.3 , 0.2 , 0.5 >
6	Astronomical calculations and scientific applications in the service of Islamic law	scientific ...	<0.7 , 0.1 , 0.1 >
6	Astronomical calculations and scientific applications in the service of Islamic law	Fiction ...	<0.4 , 0.1 , 0.5 >

Figure 4: Result Set of Apply Neutrosophic Concept Book Genre

6.2.2 Dimension Customers A data warehouse contains the dimension customer, each customer has the attributes name, address and birthday. From the attribute birthday, the age of the customer can be calculated using the function today birthday

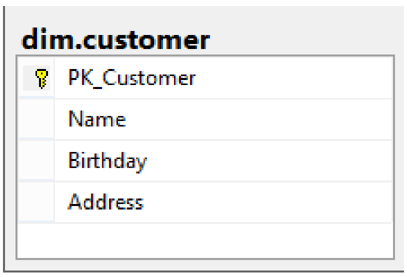

Figure 5: Dimension Customer

The published housing is interested in analyzing the revenue based on customers ages therefore, the publishing house classifies the customers ages with a neutrosophic concept in the following steps:

- The first step "The neutrosophic fuzzy target element is the customer age.

- The second step is to identify the linguistic terms for customer age. In this case, the linguistic terms for linguistic variable (customer age) are (old, middle, young) where old: customers more than 60

Middle: customers between 20 and 60, Young: customers less than 20
- The third step is to identify the neutrosophic functions for each linguistic term.

the publishing house defines the neutrosophic function that transform the age of customer into neutrosophic degrees by calculating neutrosophic function which represented by N =< μ_A, σ_A, ν_A >

Where μ_A is the membership degree (belonging degree), σ_A is the indeterminacy degree and ν_A is the non-membership degree (not belonging). The membership function depends on the customer age is the following
For example, if the customer age is 26 years old, it is transformed to term Young N_{young} (26) = <0.4,0.3 ,0.3> and term Middle N_{Midle} (26) = <0.6,0.2 ,0.2> and term Old N(26) = <0, 0.1, 0.9 >

- membership degree of age 26 years old to Linguistic term young as fellow$\boldsymbol{\mu_{young}}(26) =$ 0.4 and linguistic term Middle$\boldsymbol{\mu_{middleg}}$((26) = 0.6 and linguistic term Old $\boldsymbol{\mu_{old}}(26)$ = 0.0
- Non membership degree of age 26 years old to linguistic term young as fellow $\boldsymbol{\vartheta_{young}}(26) =$ 0.3 and linguistic term Middle $\boldsymbol{\vartheta_{middle}}(26)$ = 0.2 and linguistic term Old$\boldsymbol{\vartheta_{old}}(26)$ = 0.9
- indeterminacy degree of age 26 years old to linguistic term young as fellow $\boldsymbol{\sigma_{young}(26)}$ = 0.3

and linguistic term Middle $\boldsymbol{\sigma_{middle}(26)}$ = 0.2 and linguistic term Old $\boldsymbol{\sigma_{old}(26)}$ = 0.1
After identifying neutrosophic target element, linguistic terms and their neutrosophic functions,
one neutrosophic classification table (NCT) which holds category of customers ages (set of linguistic terms (old , middle , young) as class neutrosophic target element.
-Fifth step is to create neutrosophic degree table (NDT) contains neutrosophic degrees for each linguistic term corresponding to class neutrosophic target elements.
- Sixth step and final task, is to relate the neutrosophic classification table, the neutrosophic degree table and the neutrosophic table to each other.
The following figure gives dimension customer in neutrosophic concept

Book ID	Book Name	Category	Rating
1	Scientific Miracles in the Holy Quran	scientific	9.00
2	Political Islam and the coming battle	political	7.00
3	perhaps you laugh	Comic	5.00
4	Genius Omar	religious	8.00
5	Soft hands	romantic	2.00
6	Astronomical calculations and scientific applications in the service of Islamic law	scientific	3.00

Figure 6: Neutrosophic Concept Customer age

The following figure show the input data in customer dimension in classical data warehouse

Customer ID	Customer Name	Age	Class"Old"	Class"Middle"	Class"Young"
1	Samy Ghareeb Bondok	59	< 0.9 , 0.1 , 0>	< 0.1 , 0.2 , 0.7>	< 0 , 0.3 , 0.7>
2	Walaa Samy Bondok	20	< 0.0 , 0.1 , 0.9>	< 0 , 0.2 , 0.8>	< 1 , 0 , 0>
3	Marwa Fawzy Abbas	36	< 0 , 0.5 , 0.5>	< 1 , 0 , 0>	< 0 , 0.3 , 0.7>
4	Mohamed Samy Bondok	25	< 0 , 0.1 , 0.9>	< 0.5 , 0.2 , 0.3>	< 0.5 , 0.3 , 0.2>
5	Rana Alaa Hamoda	66	< 1 , 0 , 0>	< 0 , 0.2 , 0.8>	< 0 , 0.1 , 0.9>
6	Ayoub Ahmed	6	< 0 , 0.1 , 0.9>	< 0 , 0.2 , 0.9>	< 1 , 0 , 0>
7	Omnia Hassan Osman	42	< 0 , 0.5 , 0.5>	< 1 , 0 , 0 >	< 0 , 0.3 , 0.7>
8	Ali Mohsen Ali	33	< 0 , 0.5 , 0.5>	< 1 , 0 , 0 >	< 0 , 0.3 , 0.7>
9	Brouj Ahmed	9	< 0 , 0.1 , 0.9>	< 0 , 0.2 , 0.8>	< 1 , 0 , 0>

Figure 7: Input Date in Classical Data Warehouse in Dimension Customer

The following figure show how to construct neutrosophic analysis of dimension customer

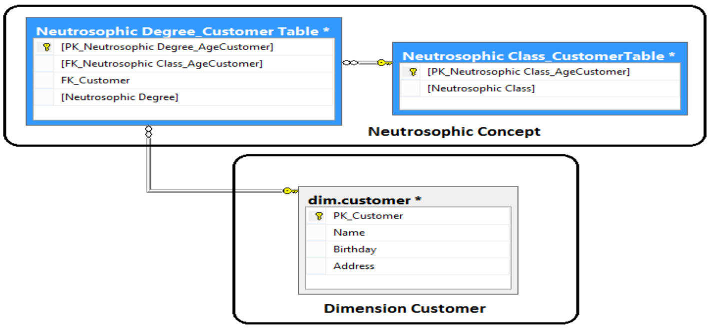

Figure 8: Result Set of Apply Neutrosophic Concept Customer Age

6.2.3 Fact

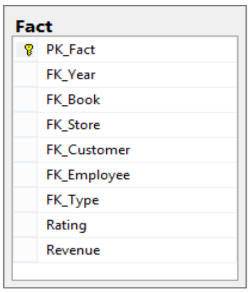

Figure 9: Fact Table

Customers are asked to rate every book when they read it. The rating of the book in the fact table As the rating is always between 0 and 10. The published housing for books uses this customer rating to evaluate the books into good, bad books. For this neutrosophic concept, the steps are the following:

The first step "The neutrosophic target element is the rating attribute in the fact table.

The second step is to identify the linguistic terms. In this case, the linguistic terms for customer rating are (good , bad) rate .

In the third task is to identify the neutrosophic functions for each linguistic term.

The publishing house defines the neutrosophic function as follows:
For example, if a customer rate a certain book 6 from 10 transformed to term good $\boldsymbol{\mathcal{N}}_{\boldsymbol{good}}$ = < 0.6 , 0.1 , 0.3 > and term bad $\boldsymbol{\mathcal{N}}_{\boldsymbol{bad}}$ = < 0.4 , 0.2 , 0.4 > and
Membership degree of rate (6) to Linguistic term good as fellow $\boldsymbol{\mu}_{\boldsymbol{good}}(26) = 0.6$ and linguistic term bad $\boldsymbol{\mu}_{\boldsymbol{bed}}(26) = 0.4$
Non membership degree of rate (6) to linguistic term good as fellow $\boldsymbol{\vartheta}_{\boldsymbol{good}}(6) = 0.1$ and linguistic term bad $\boldsymbol{\vartheta}_{\boldsymbol{bad}}(6) = 0.2$

Indeterminacy degree of rate (6) to linguistic term good as fellow $\boldsymbol{\sigma}_{good}(\mathbf{6})$ = 0.3 and linguistic term bad $\boldsymbol{\sigma}_{bad}(\mathbf{6})$ = 0.4
After identifying neutrosophic target element, linguistic terms and their neutrosophic functions,
one neutrosophic classification table which holds the linguistic terms as class neutrosophic target element (good , bad) is created then The neutrosophic degree table contains neutrosophic degrees for each neutrosophic target element corresponding to class neutrosophic elements. For final step, the neutrosophic classification table, the neutrosophic degree table and the neutrosophic table have to be related to each other
. The following figure show how to apply neutrosophic concept in fact table:

Book ID	Book Name	Rating	Class"Good"	Class"Bad"
1	Scientific Miracles in the Holy Quran	9	$<1, 0, 0>$	$<0, 0.1, 0.9>$
2	Political Islam and the coming battle	7	$<0.8, 0.1, 0.1>$	$<0.2, 0.2, 0.6>$
3	perhaps you laugh	5	$<0.4, 0.1, 0.5>$	$<0.6, 0.2, 0.2>$
5	Soft hands	2	$<0, 0.1, 0.9>$	$<1, 0, 0>$
6	Astronomical calculations and scientific applications in the service of Islamic law	3	$<0, 0.1, 0.9>$	$<1, 0, 0>$

Figure 10: Neutrosophic Concept Fact Rating

The following figure show input data in fact table in classical data warehouse

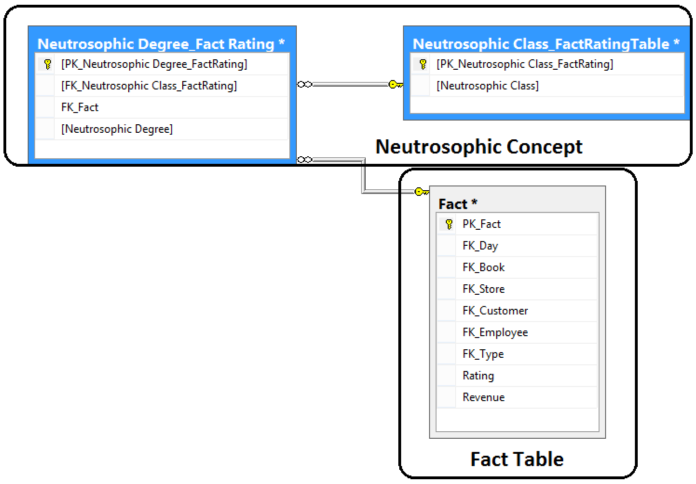

Figure 11: Input Data In Classical Data Warehouse in Fact Table

The following figure shows result set of apply neutrosophic concept in Fact table:

Book ID	Book Name	Rating	Class"Good"	Class"Bad"
1	Scientific Miracles in the Holy Quran	9	<1 , 0 , 0>	<0 , 0.1 , 0.9>
2	Political Islam and the coming battle	7	<0.8 , 0.1 , 0.1>	<0.2 , 0.2 , 0.6>
3	perhaps you laugh	5	<0.4 , 0.1 , 0.5>	<0.6 , 0.2 , 0.2>
5	Soft hands	2	<0 , 0.1 , 0.9>	<1 , 0 , 0>
6	Astronomical calculations and scientific applications in the service of Islamic law	3	<0 , 0.1 , 0.9>	<1 , 0 , 0>

Figure 12: Result Set of Apply Neutrosophic Concept in fact rating

7. Olap Operations in Crisp Data Warehouse:

Codd, E. F. defined [8] the specialized OLAP operations drill-down, roll-up, slice and dice. According Codd and the OLAP Council [27] the OLAP operations called classical data warehouse operations, can be described shortly as follows:

- The Roll-up operation consolidates the values of a dimension hierarchy to a value on the next higher level.
- The Drill-down operation is used to navigate from top to bottom in a dimension. It is the opposite operation of the Roll-up operation.
- The Slice operation extracts a subset of values based on one or more dimensions using one dimension attribute to define the subset.
- Dice operation extracts a subset of values based on more than one dimension using more than one dimension attribute to define the subset.

Vassiliadis [12] proposed a notation of basic cube, cube and multidimensional database as follows :

7.1 Definition (Basic Cube):

A basic cube [27] is as a 3-tuple (D, L, R_b); where D = (D_1,.........,D_n, M) is a list of dimensions D and a measure M in a fact, L =(DL_1,.........,DL_n, ML) is a list of dimension levels DL, aggregated measures ML and R_b is a set of cells x = {x_1,.........., x_n, m}, where $x_n \in$ dom(DL_i) and m $\in$ dom(ML) represents the instance values of the basic cube.

Example 1, The example considers a data warehouse with a fact table containing the fact revenue and three dimensions region, Product and time. The hierarchies for the dimensions are as follows:

- region: store → city → region
- Product: book
- time: day → month → year

a basic cube may take the following form: (< region ,Book, time, revenue >,< city, book, day, aggregated revenue >, R) where D is the dimensions region, Book, time, revenue > , L is the dimension levels < city, book, day, aggregated revenue > and R : set of cells represented by Figure 2.

Book	Region	Time	Revenue
book A	cairo	jan 1	620
book A	cairo	jan 2	420
book A	Alex	Feb 1	300
book A	Alex	Feb 2	450
book B	cairo	jan 1	310

Figure 13: R_b of Basic Cube

7.2 Definition (Cube):

A cube [28] is a 4-tuple < D, L, C_b, R > where D = < D_1,..,D_n, M > is a list of dimensions D and a measure M in a fact, L = < DL_1,..,DL_n , ML > is a list of dimension levels DL and aggregated measure ML, C_b is a basic cube and R is a set of cells x = {x_1,..., x_n, m}, where $x_n \in$ dom(DL) and m $\in$ dom(ML) represents the instance values of the cube. A cube can therefore be denoted as (< region, product, time, revenue >, < city, book, month, aggregated revenue >, (< region, product, time, revenue >,< city, book, day, aggregated revenue > , R_b), R) where R_b is represented in Figure 12 and R in Figure 13.

book	Region	Time	Revenue
book A	cairo	January	1040
book A	Alex	February	750
book B	cairo	January	310

Figure 14: R of Cube

The aim of defining a cube and a basic cube is the traceability of operations. Suppose that a data warehouse operation will calculate the average yearly revenue of books based on the cube with monthly revenues. It is necessary to go back to daily revenue, the lowest granularity, in order to give a meaningful result on yearly level. If the basic cube of the cube monthly revenue is not known, it is not possible to go to a lower level. No prediction can be made how the daily revenues have been aggregated to monthly
revenues. Therefore, one can say that every cube representing a data collection in the data warehouse owns a basic cube which contain the lowest granularity of a dimension [23].

7.3 Definition (Multidimensional database):
A multidimensional database is a couple < D, C > where D is a set of dimensions and C is the basic cube representing the lowest granularity [12] .
Every OLAP operation defined below will have the following characteristics

- To identify the level *l* of the dimension *d* we will use *d.l* such as time.year
- The dimension and dimension level are merged using a dot notation into one variable in order to simplify the operation. Therefore, a dimension can be specified as D = time or with including a dimension level as D = time. month.
- The dot notation can be extended in order to integrate other category attributes. A full path to a dimensional attribute month in dimension level month of dimension time can be specified as D = time. month. month.

The definition of a cube is adapted as follows:
7.4 Definition (cube (C))):
A basic cube [13] is a 3-tuple < D , M , R > where $D = (D_1,........,Dn)$ is a list of dimensions, dimensions levels separated by a dot , M is an aggregated measure (in a fact) and R is a set of data tuples $x = \{x\ ,..............., x_k, m\}$, where $x_k \in dom(D)$ and $m \in dom(M)$ representing the instance values of the cube.the following figure represent a cube for three dimension year , books , city and measure revenue:

Figure 15: Cube

Once we have the structure of the multidimensional model, we need the operations to analyze the data in the Data Cube. Over this structure we have defined the normal operations of the multidimensional model such as Roll-up, Drill-down, Slice and Dice .

7.5 Roll up Operation in Classical Data Warehouse:

The roll-up operation is used to navigate a dimension upwards. Agrawal, Gupta and Sarawagi [29] defined a roll-up operation as a special case of their merge operation that it is executed on one dimension. For the dimension region a roll-up operation is executed when the dimension level store is aggregated to the next higher level city, and dimension level day is aggregated to the next higher level month. An example would be a cube C that is a subset representing the daily revenues, the revenues aggregated to the lowest hierarchy level of dimension time. Cube C can then be merged into a cube C′ that is a data set representing the monthly revenues.

7.5.1 Definition (Roll-up):

Roll up (C,D′, f_D, f_m) = C′; where C is a basic cube, C′ is anew cube after applying roll up operation, D′ is the dimension of higher level , f_D is the dimension merge function and f_m is the measure aggregation function. The domain (dom_D) of D $dom_D(C)$ is a set containing the dimension instancesd, $dom_{D'}(C')$ is calculated by applying the function f_D on the dimension d of $dom_D(C)$. $dom_{D'}(C') = \{f_D(d) = d' \mid d \in dom_D(C)\}$, d ∈ D, ∈d′∈€ D′. The measure $m_{C'}(d)$ of each instance d is calculated by applying the function f_m to each $m_C(d)$ in regards to the aggregation function f_D.

d ={jan1, jan2,........jan31} ∈ D, D = {d_1, d_2, d_3, ...}, d′ = {jan} ∈ D′, D′ = {jan, feb, mar, ...} = {d'_1, d'_2, d'_3,........} $m_{C'}(d) = f_m(\{t \mid t = m_C(d) \wedge f_D(d) = d'\})$. dom $_{d'}$(C′) = $dom_{time:month}(C')$ might be the set {January, February}, The instances of $dom_{time:month}(C')$ are composed of instances of $dom_d(C) = dom_{time:day}(C)$ = {January 1,......, January 31, February 1,....., February 28},The function f_D defines {January 1,......., January 31} →January and {February 1,....., February 28}→February.

C′ = roll up(< time.day, revenue, S >, time.month, f_d, Sum Revenue) rollup the daily revenue to monthly revenue where C = < time.day, revenue, S >, D′ = time.month, f_d = function on dimension time to transform lower level day to higher level month ex {January 1,......., January 31} →January and f_m = sum of revenue per month for example The revenue of January 1 might be m_C (January1) = 6000 and January 31 m_C (January31) = 5000, The function f_m is defined as summation the revenue of January is calculated as $m_{C'}$ (January) = 6000 + 5000 = 11000.

7.6 Drill-down Operation in Crisp Data Warehouse:

With a Drill-down operation, values in a dimension level will be decomposed in values of the lower dimension level. This operation is used to reveal more detailed levels of data in a dimension. Drill-down is the opposite operation of roll-up. To be able to perform a drill-down, how the category attribute instances are compound from the lower hierarchy level instances must be known in advance. Considering the revenue of the year 2010 cannot drilled-down to monthly revenue, if the revenue of every month in 2010 is not known in advance. Otherwise, one can decompose the revenue of 2010 in infinite ways. A roll-up operation defining the aggregation functions and the value instances of the higher hierarchy level has to be executed before a drill-down operation [27].

Hence, the drill-down operation can be considered as a binary operation and formally be defined as:

7.6.1 Definition (Drill-down).

Drill down(C′,D″, f^{-1}_D ,f^{-1}_m) =C″; where C″ is anew cube after applying the operation drill down , D″ is the dimension of lower , C′ = roll up(C,D′, f_D, f_m) and f^{-1}_D, f^{-1}_m are the inverse function of f_D respectively f_m. The following figure shows the roll-up and drill-down operation in a cube:

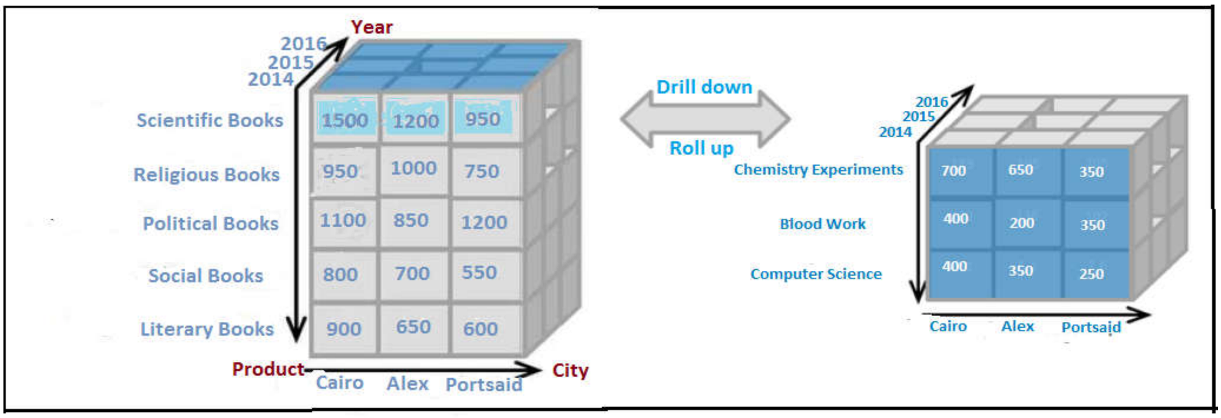

Figure 16: Roll-up and Drill-down Operation in a Cube

7.7 Slice Operation in Crisp Data Warehouse:

Slice is the act of picking a rectangular subset of a cube by choosing a single value for one of its dimensions**.** The slice operation cuts out a slice from a data cube in the multidimensional space of a data warehouse. For example, the cube C =(< time.year, product.book, region.city >, revenue, R) can be sliced using the value 2014 for the dimensional attribute year. This will extract the revenue by all books category achieved in 2014 in each city .A slice operation can formally be defined as:

7.7.1 Definition (Slice):

slice(C, d_m) = C′; where C is a cube ,C′ is anew cube after applying the slice operation and $d_m \in dom(D_m)$ is the element instance that slices the cube. For extracting the revenue of all books category in all cities in 2014, the slice operation would be defined as follows: slice (C = < time.year, product.book, region.city >, revenue, S >, d_m = time.year.year =" 2014"). The following figure 6 shows the slice operation in a cube:

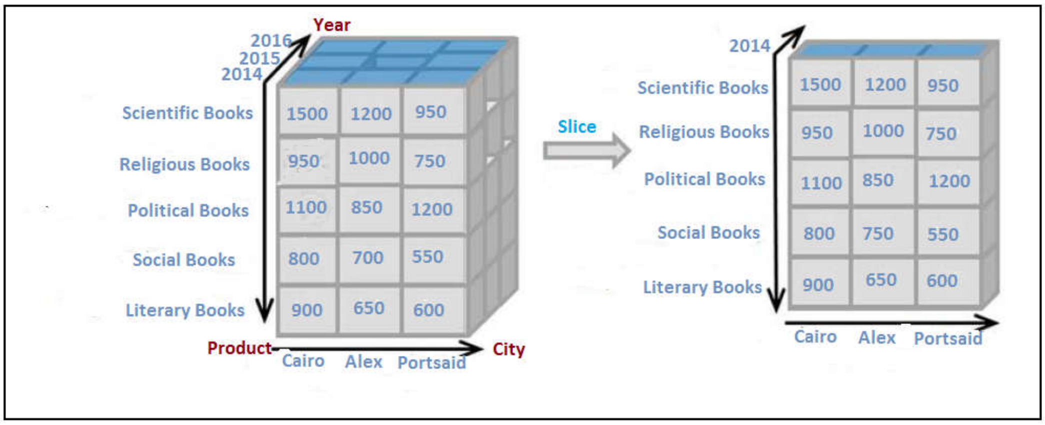

Figure 17: Slice Operation

7.8 Dice Operation in Crisp Data Warehouse:

The dice operation produces a sub cube by allowing the analyst to pick specific values of multiple dimensions. The Dice operation cuts out a dice from a data cube in the multidimensional space of a data warehouse. Slicers in a dice are combined using the logical operations AND, OR or NOT. The dice operation can formally be defined as follows:

7.8.1 Definition (Dice).

Dice (C,{d_m,........., d_k}, {f_m,........., f_{k-1}}) =C′ ; where C is the cube, C′ is acube after applying the dice operation, $\forall n \in \{m,........, k\}: d_n \in dom(Dn)$ are the element instances that slice the cube and $\forall x \in$ m,......., k-1 : $f_x \in$ {AND, OR , NOT} are the logical operator that combine the slicers in a way that d_m f_m f_{m+1} ,......, d_{k-1} f_{k-1} d_k. As an example of a dice operation, the cube C =(< time.year , product.book , region.city >, revenue, R) can be diced in order to show the revenue of category of books(scientific ,religious and political) in city cairo and, dice = (< time.year, product.book, region.city >, revenue,S >, < product.book = "Scientific" or " religious" or "political", regon.city = "Cairo"or"Alex">, {AND,AND}).The following figure 18 shows the dice operation:

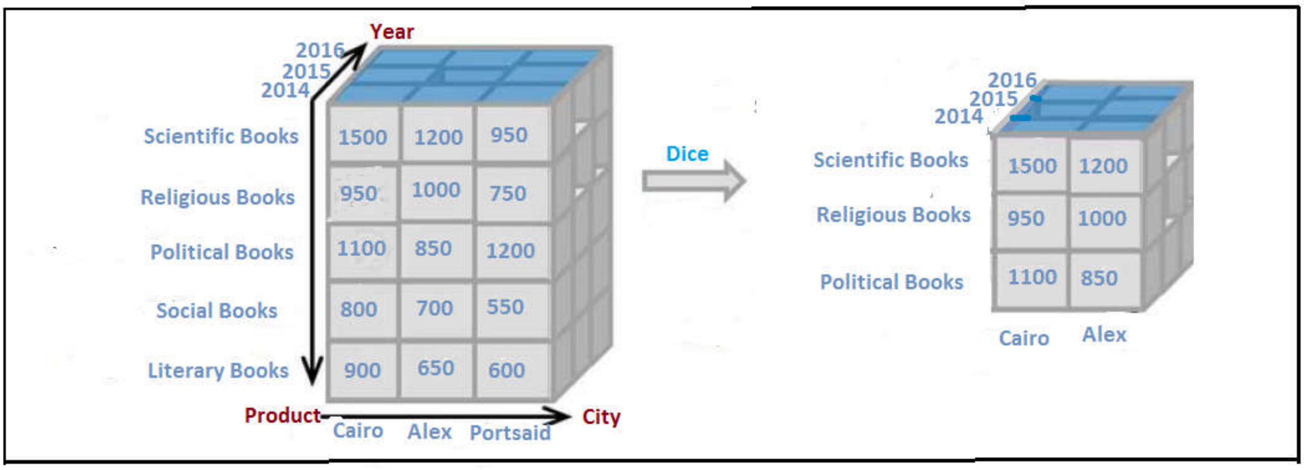

Figure 18: Dice Operation

8. Aggregation of Neutrosophic Fuzzy Concepts:

Aggregating of data in a data warehouse affects the neutrosophic concepts that classify data. Data is grouped together in a more dense view or split to reveal a more detailed view. The grouping is defined by the aggregation function that is often a summation, a maximization, a minimization, a count or an average function. This aggregation function is fundamental to the standard data warehouse operations. In order to be able to classify aggregated data, the neutrosophic concepts have to be aggregated too. In the next sections, two methods for aggregation of neutrosophic concepts are discussed. The first method is redefines the neutrosophic concept with the aggregated data instances as the new neutrosophic target element. for each dimension level, one neutrosophic degree table is created for the aggregate neutrosophic concepts. The class neutrosophic target elements are reused from the base neutrosophic neutrosophic concept containing the class neutrosophic target elements and a neutrosophic function is specified Therefore, this method is described not aggregating values. the second method is aggregates the neutrosophic degree instances of the neutrosophic concept to a more dense view. Each method is illustrated using an example.

7.5.1 First Method "Redefine of Neutrosophic Fuzzy Concepts for Aggregation Data":

A possible solution for aggregation of the neutrosophic concept onto another dimension hierarchy level. Redefining of a neutrosophic concept does not take the neutrosophic degree values into consideration. The linguistic terms and the neutrosophic functions are applied to a new hierarchy level. The neutrosophic degrees are recalculated based on the new neutrosophic target element. Neutrosophic degrees from the neutrosophic concept on the lower hierarchy level are not taken into consideration. for example , To redefine the neutrosophic concept store surface from dimension hierarchy level store, the neutrosophic concept is created on the level city. The neutrosophic classification table (NCT) is taken from the original neutrosophic concept. Whereas, the neutrosophic degree table (NDT) has to be newly created for the new neutrosophic concept. This is due to that new neutrosophic functions are calculating the neutrosophic degree based on the new neutrosophic target elements. These newly calculated neutrosophic degree are stored in the new NDT.
for example 2. To the dimension store a fact table is added. The fact table contains a measure revenue and the primary key of fact table and the foreign key relation to the store table (FK_store). A neutrosophic concept is added having revenue as the neutrosophic target element. Store A and B earned multiple revenues of 2500 for store A and 4500 for store B. For every revenue a new instance is stored in the fact table. The total revenue of a city is the sum of all revenues earned by stores. Each revenue has a neutrosophic degree for each linguistic term in the neutrosophic concept revenue ($\mathcal{N}$high, $\mathcal{N}$middle, $\mathcal{N}$low). For the city hierarchy level the revenues are aggregated to the city Cairo and we want to classify revenues of level city. For do that the neutrosophic concept including the neutrososphic degree table must be defined on city level. The neutrosophic target element for this neutrosophic concept is the city revenue and neutrosophic class for store revenue is reused for city revenue

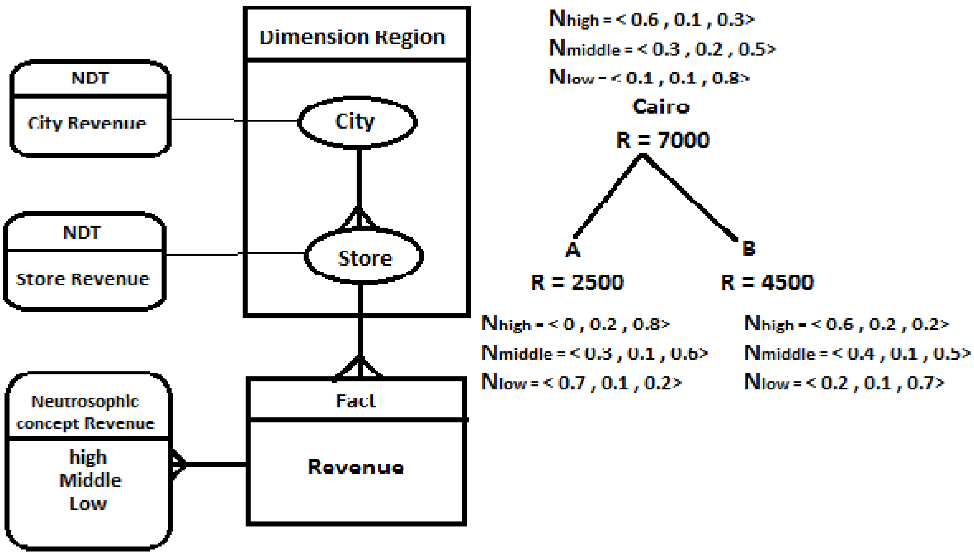

Figure 19: Dimension Store and Fact Revenue with Neutrosophic Concepts

the idea of reuse the neutrosophic classification table of the base concept to reduce the amount of extra tables and to limit the resources in the neutrosophic data

7.5.2 Second method: Aggregation of Neutrosophic Concepts

The second method is to aggregate the neutrosophic degree of each neutrosophic target element instance into a next higher hierarchy level. Each value of the next higher hierarchy level is composed of a set of value from the neutrosophic target element .a neutrosophic degree for each instance of the next higher hierarchy level can be considered as aggregation of the neutrosophic degree of the lower level. In order to illustrate the aggregation of neutrosophic concepts, consider the following Example 2. A data warehouse contains a dimension store with two category: store and city. All stores are aggregated to the corresponding city. For all stores their area is measured and added to the dimension as dimensional attributes on level store. Considering Lehner [16], dimensional attributes can be aggregated on higher hierarchy levels, similarly as it would happen for measures. Therefore, the store area can be aggregated to the level city. a neutrosophic concept with store area as neutrosophic target element is defined. The neutrosophic concept classifies the area as big, medium and small. In example 1, the average store area of a city can be calculated by aggregating the area of all stores in a city with an average function. In order to apply the neutrosophic concept store area on the level city, the neutrosophic degree have to be aggregated. By the foreign key relation of stores to cities, it is known which store areas are aggregated to a distinct city area. the neutrosophic degree on level store can be identified that aggregate to a neutrosophic degree on level city. An additional aggregation function can then be defined that aggregates the neutrosophic degree of the stores to the neutrosophic degree of the city. In the case of store area, the arithmetic average of the neutrosophic degree of each class neutrosophic can be used to generate the corresponding neutrosophic degree for the cities. The dimensional attribute area is aggregated using an average function and therefore an aggregation of the neutrosophic concept using the arithmetic average . In this case, no additional tables have to be created. For example: A city Cairo contain 2 store A and B, A neutrosophic concept apply in store area ,the area of store A is 90 meter square with neutrosophic degree $\mathcal{N}_{big} = \langle 0, 0.4, 0.6\rangle$, $\mathcal{N}_{\text{medium}} = \langle 0, 0.6, 0.4\rangle$, $\mathcal{N}_{\text{small}} = \langle 1, 0, 0\rangle$ and the area of store B is 270 meter square with neutrosophic degree $\mathcal{N}_{big}$= < 0.7 , 0.1 , 0.2 >, $\mathcal{N}_{\text{medium}}$ = < 0.7 , 0.2 , 0.1 >, $\mathcal{N}_{\text{small}}$ = < 0 , 0.6 , 0.4 >, here need to do roll up operation from level store to a higher level city ,this is done by two steps:

1) the city (cairo) area is aggregated using an average function = (90+270) /2 = 180 meter square
2) the neutrosophic degree for city area is aggregation of the neutrosophic concept using the arithmetic average such as:

$$\mathcal{N}_{big} = < \frac{\mu\text{big}*90 + \mu\text{big}*270}{360}, \frac{\sigma\text{big}*90 + \sigma\text{big}*270}{360}, \frac{\vartheta\text{big}*90 + \vartheta\text{big}*270}{360} >$$

$$= < \frac{0*90 + 0.7*270}{360}, \frac{0.4*90 + 0.1*270}{360}, \frac{0.6*90 + 0.2*270}{360} > = <0.525 \ , 0175 \ , 0.3>$$

$$\mathcal{N}_{\text{medium}} = <\frac{\mu\text{medium}*90 + \mu\text{medium}*270}{360}, \frac{\sigma\text{medium}*90 + \sigma\text{medium}*270}{360}, \frac{\vartheta\text{medium}*90 + \vartheta\text{medium}*270}{360}>$$
$$= <\frac{0*90 + 0.7*270}{360}, \frac{0.6*90 + 0.2*270}{360}, \frac{0.4*90 + 0.1*270}{360}> = <0.525 , 0.3 , 0.175>$$
$$\mathcal{N}_{\text{small}} = <\frac{\mu\text{small}*90 + \mu\text{small}*270}{360}, \frac{\sigma\text{small}*90 + \sigma\text{small}*270}{360}, \frac{\vartheta\text{small}*90 + \vartheta\text{small}*270}{360}>$$
$$= <\frac{1*90 + 0*270}{360}, \frac{0*90 + 0.6*270}{360}, \frac{0*90 + 0.4*270}{360}> = <0.25 , 0.45 , 0.3>$$

The following figure 8 shows that the second method for aggregation neutrosophic concept in dimension store :

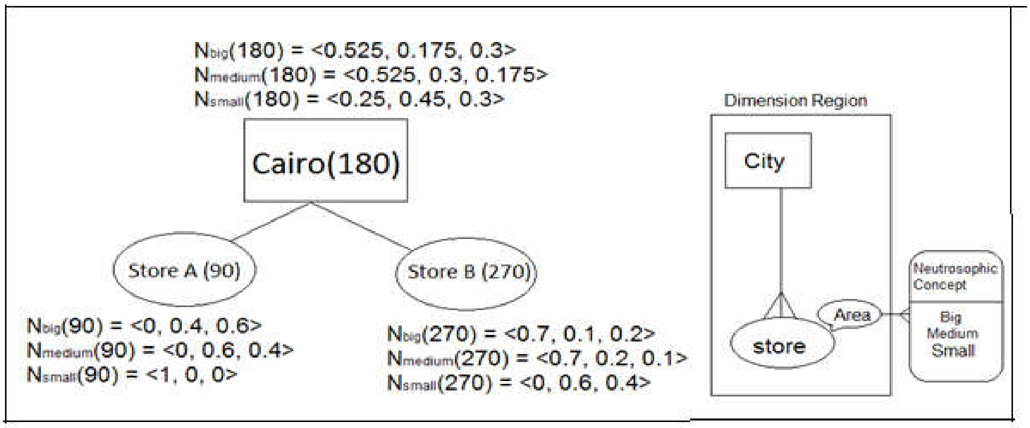

Figure20: Aggregation of a Neutrosophic Concept

8. Olap Operations in Neutrosophic Data Warehouse :

the classical data warehouse operation can be extended to support neutrosophic concept. neutrosophic concepts can be treated as dimensional attributes. Lehner [2] defines multidimensional objects that are capable of aggregating dimensional attributes and takes them into consideration as segments for slice and dice operations. it is possible to aggregate neutrosophic concepts and to use them as slicers for slice and dice operations. a fact can be aggregated over a dimension hierarchy, the neutrosophic concept with the fact as neutrosophic target can be aggregated . The neutrosophic concepts on facts can be considered as segmentation in slice and dice operation just as neutrosophic concepts on dimensional attributes. When segmenting a cube with a neutrosophic concept, the class neutrosophic degree are the delimiter of a segment. In order to discuss the classical operations, these characteristics of neutrosophic concepts have to be taken into consideration. The definition of crisp cube is adapted to integrate neutrosophic concepts.

8.1 Definition (Neutrosophic Fuzzy Cube (NC)):

A cube in neutrosophic data warehouse is composed of 4-tuple (D,N,M,S); where D = (D_1, D_2,........,D_n) is a list of dimensions, dimensions levels including the dimension attribute separated by a dot , N = (N_1, N_2,...........,N_k) is a list of neutrosophic concepts with neutrosophic target that are either in (facts or dimension) or class neutrosophic target element of neutrosophic concept separated by a dot, M is a measure in a fact and S is a set of data tuples x = {x_1,......x_n, n_1,..........n_k ,m}, where $x_n \in$ dom(D), $\forall n_k \in$ dom(N) and m $\in$ dom(M) representing the instance values of the cube. For example: A neutrosophic concept is added to the fact revenue as neutrosophic target element. The neutrosophic concept revenue contains three classes “low”, “middle” and “high” revenue and the book decomposed into several genre such as (scientific, political, religious,.... so on) and the dimension city with neutrosophic target area contain three neutrosophic class (Big , Medium , Small).

A Neutrosophic Cube is a binary operation which can be involved two steps as fellow:

1) select Crisp Dimensions.
2) apply the neutrosophic concept on the neutrosophic target element.

A neutrosophic cube can be: (< time.month, Region.city, Product.book>,< time.month.revenue, Region.city.area, Product.book.genre >, revenue ,S). The neutrosophic concept revenue is propagated on dimension time on level month and on dimension Region on level city and on dimension Product on level book. The result set of the neutrosophic cube are tuples containing the aggregated revenue as follow:

Book Name	NC.Book	Neutrosophic Degree Book	City	NC.City Area	Neutrosophic Degree City Area	Month	Revenue	NC.Revenue	Neutrosophic Degree Revenue
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Big	< 0.9 , 0.1 , 0 >	February	6200	High	< 0.7 , 0.1 , 0.2 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Big	< 0.9 , 0.1 , 0 >	February	6200	Middle	< 0.3 , 0.2 , 0.5 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Big	< 0.9 , 0.1 , 0 >	February	6200	Low	< 0 , 0.3 , 0.7 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Medium	< 0.1 , 0.2 , 0.7 >	February	6200	High	< 0.7 , 0.1 , 0.2 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Medium	< 0.1 , 0.2 , 0.7 >	February	6200	Middle	< 0.3 , 0.2 , 0.5 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Medium	< 0.1 , 0.2 , 0.7 >	February	6200	Low	< 0 , 0.3 , 0.7 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Small	< 0 , 0.2 , 0.8 >	February	6200	High	< 0.7 , 0.1 , 0.2 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Small	< 0 , 0.2 , 0.8 >	February	6200	Middle	< 0.3 , 0.2 , 0.5 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Small	< 0 , 0.2 , 0.8 >	February	6200	Low	< 0 , 0.3 , 0.7 >
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Cairo	Big	< 0.9 , 0.1 , 0 >	February	6200	High	< 0.7 , 0.1 , 0.2 >
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Cairo	Big	< 0.9 , 0.1 , 0 >	February	6200	Middle	< 0.3 , 0.2 , 0.5 >
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Cairo	Big	< 0.9 , 0.1 , 0 >	February	6200	Low	< 0 , 0.3 , 0.7 >
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Cairo	Medium	< 0.1, 0.2 , 0.7 >	February	6200	High	< 0.7 , 0.1 , 0.2 >
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Cairo	Medium	< 0.1, 0.2 , 0.7 >	February	6200	Middle	< 0.3 , 0.2 , 0.5 >
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Cairo	Medium	< 0.1, 0.2 , 0.7 >	February	6200	Low	< 0 , 0.3 , 0.7 >
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Cairo	Small	< 0 , 0.2 , 0.8 >	February	6200	High	< 0.7 , 0.1 , 0.2 >
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Cairo	Small	< 0 , 0.2 , 0.8 >	February	6200	Middle	< 0.3 , 0.2 , 0.5 >
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Cairo	Small	< 0 , 0.2 , 0.8 >	February	6200	Low	< 0 , 0.3 , 0.7 >

Figure 21: S of Neutrosophic Cube

If N contains a class neutrosophic target element, By applying the third step (class neutrosophic target element) in the binary neutrosophic cube. A Neutrosophic Cube is a trinary which can be involved three steps as fellow :

1) Select Crisp Dimensions.
2) Apply the neutrosophic concept on the neutrosophic target element.
3) Apply the class neutrosophic target element.

For example : (< time.month, Region.city , Product.book>,< time.month.revenue, Region.city.area.big, Product.book.genre.scientific >, revenue ,S)

The following figure 10 shows the neutrosophic cube with class neutrosophic target (big city area) :

Book Name	NC.Book	Neutrosophic Degree Book	City	NC.City Area	Neutrosophic Degree City Area	Month	Revenue	NC.Revenue	Neutrosophic Degree Revenue
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Big	< 0.9 , 0.1 , 0 >	February	6200	High	< 0.7 , 0.1 , 0.2 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Big	< 0.9 , 0.1 , 0 >	February	6200	Middle	< 0.3 , 0.2 , 0.5 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Big	< 0.9 , 0.1 , 0 >	February	6200	Low	< 0 , 0.3 , 0.7 >

Figure 22: Neutrosophic Cube with Class Neutrosophic Target.

8.2 Roll-up in Neutrosophic Data Warehouse:

A roll-up operation can be applied to a neutrosophic cube, the roll up operation on a neutrosophic data warehouse involving neutrosophic concepts can be defined as:

8.2.1 Definition (Roll-up involving Neutrosophic Concept (RNC):

RNC = roll-up (NC, DH, NH, f_D, f_m, f_N); where NC is a Neutrosophic cube , DH is the dimension of higher level, NH is the neutrosophic concept of higher level, f_D is the dimension merge function, f_m is the measure aggregation function , f_N is the method how to aggreagate neutrosophic concept on the next level and RNC is the result cube on the higher level after applying roll up operation. This roll-up operation is a binary operation that first aggregates the crisp fact revenue. In the second step, it applies the new neutrosophic concept to the data collection with the following steps:

1) roll-up of the crisp cube.
2) apply the neutrosophic concept on the new dimensional level.

For example: roll up (NC, < time.month >, < time.month.revenue >, revenue, S >, time.year , time.year.revenue, $f_{time.year}$, Revenue, f_N) = (< time.year > ,<time.year.revenue>, revenue , NS >. where DH is the dimension of higher level year = time.year , NH is the neutrosophic concept of higher level year = time.year.revenue, f_d is the dimension merge function such as { Jan_{2015},Feb_{2015},.........Dec_{2015}}→ 2015, f_m is the aggregation revenue per year and RNS is the result set of apply the roll up operation on neutrosophic cube. the following figure 11, 12 shows the roll-up operation in neutrosophic cube :

Book Name	NC.Book	Neutrosophic Degree Book	City	NC.City Area	Neutrosophic Degree City Area	year	Revenue	NC.Revenue	Neutrosophic Degree Revenue
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Big	< 0.9 , 0.1 , 0 >	2015	60500	High	< 08 , 0.1 , 0.1 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Big	< 0.9 , 0.1 , 0 >	2015	60500	Middle	< 0.2 , 0.2 , 0.6 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Big	< 0.9 , 0.1 , 0 >	2015	60500	Low	< 0 , 0.3 , 0.7 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Medium	< 0.1 , 0.2 , 0.7 >	2015	60500	High	< 08 , 0.1 , 0.1 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Medium	< 0.1 , 0.2 , 0.7 >	2015	60500	Middle	< 0.2 , 0.2 , 0.6 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Medium	< 0.1 , 0.2 , 0.7 >	2015	60500	Low	< 0 , 0.3 , 0.7 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Small	< 0 , 0.2 , 0.8 >	2015	60500	High	< 08 , 0.1 , 0.1 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Small	< 0 , 0.2 , 0.8 >	2015	60500	Middle	< 0.2 , 0.2 , 0.6 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Small	< 0 , 0.2 , 0.8 >	2015	60500	Low	< 0 , 0.3 , 0.7 >
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Cairo	Big	< 0.9 , 0.1 , 0 >	2016	40000	High	< 0.3 , 0.1 , 0.6>
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Cairo	Big	< 0.9 , 0.1 , 0 >	2016	40000	Middle	< 0.4 , 0.3 , 0.3 >
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Cairo	Big	< 0.9 , 0.1 , 0 >	2016	40000	Low	< 0 , 0.2 , 0.8 >
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Cairo	Medium	< 0.1 , 0.2 , 0.7 >	2016	40000	High	< 0.3 , 0.1 , 0.6>
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Cairo	Medium	< 0.1 , 0.2 , 0.7 >	2016	40000	Middle	< 0.4 , 0.3 , 0.3 >
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Cairo	Medium	< 0.1 , 0.2 , 0.7 >	2016	40000	Low	< 0 , 0.2 , 0.8 >
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Cairo	Small	< 0 , 0.2 , 0.8 >	2016	40000	High	< 0.3 , 0.1 , 0.6>
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Cairo	Small	< 0 , 0.2 , 0.8 >	2016	40000	Middle	< 0.4 , 0.3 , 0.3 >
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Cairo	Small	< 0 , 0.2 , 0.8 >	2016	40000	Low	< 0 , 0.2 , 0.8 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Alex	Big	< 0.7 , 0.2 , 0.1 >	2015	50800	High	< 0.7 , 0.1 , 0.2 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Alex	Big	< 0.7 , 0.2 , 0.1 >	2015	50800	Middle	< 0.6 , 0.2 , 0.2 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Alex	Big	< 0.7 , 0.2 , 0.1 >	2015	50800	Low	< 0.1 , 0.2 , 0.7 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Alex	Medium	< 0.5 , 0.1 , 0.4 >	2015	50800	High	< 0.7 , 0.1 , 0.2 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Alex	Medium	< 0.5 , 0.1 , 0.4 >	2015	50800	Middle	< 0.6 , 0.2 , 0.2 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Alex	Medium	< 0.5 , 0.1 , 0.4 >	2015	50800	Low	< 0.1 , 0.2 , 0.7 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Alex	Small	< 0.3 , 0.2 , 0.5 >	2015	50800	High	< 0.7 , 0.1 , 0.2 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Alex	Small	< 0.3 , 0.2 , 0.5 >	2015	50800	Middle	< 0.6 , 0.2 , 0.2 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Alex	Small	< 0.3 , 0.2 , 0.5 >	2015	50800	Low	< 0.1 , 0.2 , 0.7 >
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Alex	Big	< 0.7 , 0.2 , 0.1 >	2016	28000	High	< 0.1 , 0.2 , 0.7 >
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Alex	Big	< 0.7 , 0.2 , 0.1 >	2016	28000	Middle	< 0.4 , 0.3 , 0.3 >
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Alex	Big	< 0.7 , 0.2 , 0.1 >	2016	28000	Low	< 0.8 , 0.1 , 0.1 >
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Alex	Medium	< 0.5 , 0.1 , 0.4 >	2016	28000	High	< 0.1 , 0.2 , 0.7 >
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Alex	Medium	< 0.5 , 0.1 , 0.4 >	2016	28000	Middle	< 0.4 , 0.3 , 0.3 >
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Alex	Medium	< 0.5 , 0.1 , 0.4 >	2016	28000	Low	< 0.8 , 0.1 , 0.1 >
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Alex	Small	< 0.3 , 0.2 , 0.5 >	2016	28000	High	< 0.1 , 0.2 , 0.7 >
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Alex	Small	< 0.3 , 0.2 , 0.5 >	2016	28000	Middle	< 0.4 , 0.3 , 0.3 >
Scientific Miracles in the Holy Quran	Political Books	< 0.3 , 0.2 , 0.5 >	Alex	Small	< 0.3 , 0.2 , 0.5 >	2016	28000	Low	< 0.8 , 0.1 , 0.1 >

Figure 23: Roll-up Operation in Neutrosophic Cube

Year	Cairo	Alex	Portsaid
2015	60500	31000	19000
ND	Nhigh < 0.8 , 0.1 , 0.1> Nmiddle < 0.2 , 0.2 , 0.6> Nlow< 0 , 0.3, 0.7>	Nhigh < 0.7 , 0.1 , 0.2> Nmiddle < 0.5 , 0.1 , 0.4> Nlow< 0 , 0.3, 0.7>	Nhigh < 0.4 , 0.2 , 0.4> Nmiddle <0.3 , 0.3 , 0.4> Nlow< 0.7 , 0.1, 0.2>
2016	40000	10000	8000
ND	Nhigh < 0.3 , 0.1 , 0.6> Nmiddle <0.4 , 0.3 , 0.3> Nlow< 0 , 0.2, 0.8 >	Nhigh < 0.3 , 0.2 , 0.5> Nmiddle <0.3 , 0.1 , 0.6> Nlow< 0.8 , 0.1, 0.1>	Nhigh < 0.1 , 0.1 , 0.8> Nmiddle <0.2 , 0.2 , 0.6> Nlow< 0.8 , 0.1, 0.1>

Figure 24: Roll-up Operation from month to year

If N contains a class neutrosophic target element, the third step is apply class neutrosophic target element to above steps in binary operation ,the roll up operation is a trinary operation in neutrosophic concept:
1) roll-up of the crisp cube
2) apply the neutrosophic concept on the new dimensional level.
3) select the class neutrosophic target element.
roll up (< time.month >, < time.month.revenue >, revenue, S >, time.year, time.year.revenue.middle, $f_{time.year}$, Revenue) = (< time.year > ,<time.year.revenue>, revenue , NS >
The following figure 13 , 14 shows the roll-up operation in neutrosophic cube with neutrosophic target element (middle revenue):

Book Name	NC.Book	Neutrosophic Degree Book	City	NC.City Area	Neutrosophic Degree City Area	year	Revenue	NC.Revenue	Neutrosophic Degree Revenue
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Big	< 0.9 , 0.1 , 0 >	2015	60500	Middle	< 0.2 , 0.2 , 0.6 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Medium	< 0.1 , 0.2 , 0.7 >	2015	60500	Middle	< 0.6 , 0.1 , 0.3 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.3 , 0.2 , 0.5 >	Cairo	Small	< 0 , 0.2 , 0.8 >	2015	60500	Middle	< 0.3 , 0.2 , 0.5 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Big	< 0.9 , 0.1 , 0 >	2016	40000	Middle	< 0.4 , 0.3 , 0.3 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Medium	< 0.1 , 0.2 , 0.7 >	2016	40000	Middle	< 0.4 , 0.3 , 0.3 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.3 , 0.2 , 0.5 >	Cairo	Small	< 0 , 0.2 , 0.8 >	2016	40000	Middle	< 0.4 , 0.3 , 0.3 >

Figure 25: Roll-up Operation with Class Neutrosophic Target Element

Year	Cairo	Alex	Portsaid
2015	60500	31000	19000
ND	Nmiddle < 0.2 , 0.2 , 0.6>	Nmiddle < 0.5 , 0.1 , 0.4>	Nmiddle <0.3 , 0.3 , 0.4>
2016	40000	10000	8000
ND	Nmiddle <0.4 , 0.3 , 0.3>	Nmiddle <0.3 , 0.1 , 0.6>	Nmiddle <0.2 , 0.2 , 0.6>

Figure 26: Roll-up Operation with Class Neutrosophic Target Element

The original cube contains the aggregated revenue of month in all cities ordered by the neutrosophic concept revenue. The cube resulting from the roll-up operation contains the aggregated revenue of year in all cities ordered by the neutrosophic concept revenue

8.3 Drill-down in Neutrosophic Data Warehouse:
A drill-down operation is the opposite operation of a roll-up. It is not a valid operation if the roll-up operation is not defined in an earlier step. Therefore, a drill-down operation in the neutrosophic data warehouse can be defined as given in § 6.3.1.
6.3.1 Definition (Drill-down Involving Neutrosophic Concepts (DRNC)):
DRNC = drill down (RNC, DL, NL, f^{-1}_D, f^{-1}_m, $f\text{-}^1_N$); where RNC is ROLL UP (NC, DH, NH, f_d, f_m, f_N), DL is the dimension of level lower ,NL is the neutrosophic concept of lower level, f^{-1}_D is the inverse dimension merge function of f_D, f^{-1}_m is the inverse measure aggregation function of f_m and $f\text{-}^1_N$ is the inverse of f_N, DRNC = result cube on the lower level after applying the drill down operation .The following figure 15 shows that the function and inverse function on dimension:

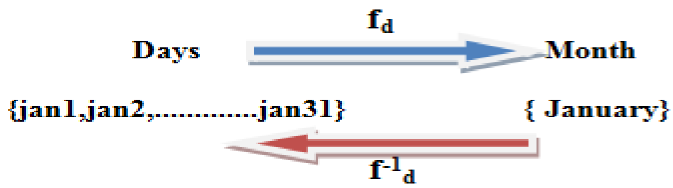

Figure 27.: Function merge on Dimension

The following example shows a drill-down operation with a neutrosophic cube:
DRNC = drill down ((< time.year, region.city > ,<time.year.revenue, region.city.area >, revenue , S >), < time.month, region.store > , < time.month.revenue, region.store.area > , f^{-1}_{d}, f^{-1}m); where DL = < time.month, region.store >, NL = time.month.revenue, region.store.area > , f^{-1}_{d} is the inverse function from year to month and from city to store and , f^{-1}_{m} is the aggregate revenue per month and RNC = rollup (< time.month, rgion.store >, < time.month.revenue, region.store.area > , revenue , S >, < time.year.revenue, Region.city.area >, f_D, Revenue)= (< time.year, region.city > ,<time.year.revenue, region.city.area > , revenue , NS >. The following figure16 shows the drill down operation in neutrosophic cube:

Year	Month	Cairo			Alex		Portsaid
		A	B	C	D	E	F
2015	January	3200	4100	3900	3800	2300	3600
	ND	Nhigh < 0.7 , 0.1 , 0.2> Nmiddle <0.5 , 0.2 , 0.3> Nlow<0.2 , 0.2, 0.6>	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree
	February	4200	3800	3200	4100	3200	4200
	ND	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree
	....	9900	8500	8700	5300	6200	7100
	ND	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree
	December	3600	3800	3600	3700	2400	4100
	ND	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree
2016	January	3200	3700	2900	1100	900	2100
	ND	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree
	February	3000	2500	2100	1200	850	1500
	ND	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree
	......	6700	4600	3800	2000	1600	3100
	ND	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree
	December	2300	2700	2500	1200	1150	1300
	ND	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree	Neutrosophic Degree

Figure 28: Drill-down Operation in Neutrosophic Cube

The cube resulting from the drill down operation contains the aggregated revenue per month in all stores ordered by the neutrosophic concept revenue.

9.4 Slice in Neutrosophic Data Warehouse:

The classical slice operation extracts a subset of values of a neutrosophic cube depend on select one dimension. A definition of a slice operation in the neutrosophic data warehouse is as follows:

9.4.1 Definition (Slice in Neutrosophic Concepts(SNC)):

SNC= slice (NC, s) where NC is neutrosophic cube and s is the element instance that slices cube NC
A slice operation of a neutrosophic cube will always result in a neutrosophic cube. A slice operation on a crisp cube will also always result in a crisp cube. In order to illustrate, the following operation slices a neutrosophic cube according its class neutrosophic target element:
slice(<< time.year > , < time.year.revenue >, revenue, R >, time.year.year = "2015") = << time.year, > , < time.year.year > , revenue , S >.The following figure 17 show the slice operation in neutrosophic cube:

Book Name	NC.Book	NC.Books	City	NC.City Area	osophic Degree City	year	Revenue	NC.Revenue	Neutrosophic Degree Revenue
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Big	< 0.9 , 0.1 , 0 >	2015	60500	High	< 08 , 0.1 , 0.1 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Big	< 0.9 , 0.1 , 0 >	2015	60500	Middle	< 0.2 , 0.2 , 0.6 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Big	< 0.9 , 0.1 , 0 >	2015	60500	Low	< 0 , 0.3 , 0.7 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Medium	< 0.1 , 0.2 , 0.7 >	2015	60500	High	< 08 , 0.1 , 0.1 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Medium	< 0.1 , 0.2 , 0.7 >	2015	60500	Middle	< 0.2 , 0.2 , 0.6 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Medium	< 0.1 , 0.2 , 0.7 >	2015	60500	Low	< 0 , 0.3 , 0.7 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Small	< 0 , 0.2 , 0.8 >	2015	60500	High	< 08 , 0.1 , 0.1 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Small	< 0 , 0.2 , 0.8 >	2015	60500	Middle	< 0.2 , 0.2 , 0.6 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Small	< 0 , 0.2 , 0.8 >	2015	60500	Low	< 0 , 0.3 , 0.7 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Alex	Big	< 0.7 , 0.2 , 0.1 >	2015	50800	High	< 0.7 , 0.1 , 0.2 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Alex	Big	< 0.7 , 0.2 , 0.1 >	2015	50800	Middle	< 0.6 , 0.2 , 0.2 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Alex	Big	< 0.7 , 0.2 , 0.1 >	2015	50800	Low	< 0.1 , 0.2 , 0.7 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Alex	Medium	< 0.5 , 0.1 , 0.4 >	2015	50800	High	< 0.7 , 0.1 , 0.2 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Alex	Medium	< 0.5 , 0.1 , 0.4 >	2015	50800	Middle	< 0.6 , 0.2 , 0.2 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Alex	Medium	< 0.5 , 0.1 , 0.4 >	2015	50800	Low	< 0.1 , 0.2 , 0.7 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Alex	Small	< 0.3 , 0.2 , 0.5 >	2015	50800	High	< 0.7 , 0.1 , 0.2 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Alex	Small	< 0.3 , 0.2 , 0.5 >	2015	50800	Middle	< 0.6 , 0.2 , 0.2 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Alex	Small	< 0.3 , 0.2 , 0.5 >	2015	50800	Low	< 0.1 , 0.2 , 0.7 >

Figure 29: Slice Operation in Neutrosophic Cube

In the resulting cube contain all revenue in city for year 2015 and the year 2016 and others not found

9.5 Dice Operation in Neutrosophic Data Warehouse:

A dice operation in a neutrosophic can restrict neutrosophic concepts and dimensions. Furthermore,
The slice operation can be extended to a dice definition by including logical operators to combine multiple slicers.

9.5.1 Definition (Dice involving neutrosophic fuzzy concepts (DNC)):

DNC = Dice (NC,{s_m,.........., s_k},{f_m,............, f_{K-1}}); where NC is a neutrosophic cubes, s_m,........., s_k $\in$ dom (D), f_m,............,f_{k-1} $\in$ {AND, OR,NOT} are the logical operators that combine the slicers in a way that combines s_m with s_{m+1}.
The dice operation works similarly and performs a selection on two or more dimensions. For example:
A dice operation on a neutrosophic cube with two neutrosophic concepts is illustrated as follows:
dice (<< time.year , region.city > , < time.year.revenue, region.city.area >, revenue , S >, {time.year.year = "2015", region.city.area = "big"},{AND}) .The following figure 18 shows that the dice operation in neutrosophic cube.

Book Name	NC.Book	NC.Books	City	NC.City Area	osophic Degree City	year	Revenue	NC.Revenue	Neutrosophic Degree Revenue
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Big	< 0.9 , 0.1 , 0 >	2015	60500	High	< 08 , 0.1 , 0.1 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Big	< 0.9 , 0.1 , 0 >	2015	60500	Middle	< 0.2 , 0.2 , 0.6 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Cairo	Big	< 0.9 , 0.1 , 0 >	2015	60500	Low	< 0 , 0.3 , 0.7 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Alex	Big	< 0.7 , 0.2 , 0.1 >	2015	50800	High	< 0.7 , 0.1 , 0.2 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Alex	Big	< 0.7 , 0.2 , 0.1 >	2015	50800	Middle	< 0.6 , 0.2 , 0.2 >
Scientific Miracles in the Holy Quran	Scientific Books	< 0.8 , 0.1 , 0.1 >	Alex	Big	< 0.7 , 0.2 , 0.1 >	2015	50800	Low	< 0.1 , 0.2 , 0.7 >

Figure 30: Dice Operation in Neutrosophic Cube

The resulting cube shows only the revenue of the year2015" that belong to city class "big"

Conclusion

Using a neutrosophic approach in data warehouse concepts improves information quality for the business process. this approach include neutrosophic concept into structure of dimensions or into fact tables of the data warehouse model, then we construct truth degree, falsity degree and indeterminacy degree which close to natural language. Added , We have presented a structure that manages imprecision by means of neutrosophic logic. Most of the methods previously documented give a neutrosophic set as a result. we have presented an OLAP system that implements a neutrosophic multidimensional model to achieve knowledge discovery from imperfect data and to enhance the system performance.

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NEUTRO-BCK-ALGEBRA

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Abstract

This paper introduces the novel concept of Neutro-BCK-algebra. In Neutro-BCK-algebra, the outcome of any given two elements under an underlying operation (neutro-sophication procedure) has three cases, such as: appurtenance, non-appurtenance, or indeterminate. While for an axiom: equal, non-equal, or indeterminate. This study investigates the Neutro-BCK-algebra and shows that Neutro-BCK-algebra are different from BCK-algebra. The notation of Neutro-BCK-algebra generates a new concept of NeutroPoset and Neutro-Hass-diagram for NeutroPosets. Finally, we consider an instance of applications of the Neutro-BCK-algebra.
Keywords: Neutro-BCK-algebra, NeutroPoset, Neutro-Hass diagram.

1 Introduction

Neutrosophy, as a newly-born science, is a branch of philosophy that studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra. It can be defined as the incidence of the application of a law, an operation, an axiom, an idea, a conceptual accredited construction on an unclear, indeterminate phenomenon, contradictory to the purpose of making it intelligible. Neutrosophic Sets and Systems international journal (which is in Scopus and Web of Science) is a tool for publications of advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics, neutrosophic measure, neutrosophic integral, and so on, studies that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc. Recently, Florentin Smarandache [2019] generalized the classical Algebraic Structures to NeutroAlgebraic Structures NeutroAlgebras) and AntiAlgebraic Structures (AntiAlgebras) and he proved that the NeutroAlgebra is a generalization of Partial Algebra.[7] He considered $< A >$ as an item (concept, attribute, idea, proposition, theory, etc.). Through the process of neutrosphication, he split the nonempty space and worked onto three regions two opposite ones corresponding to $< A >$ and $< antiA >$, and one corresponding to neutral (indeterminate) $< neutA >$ (also denoted $< neutroA >$) between the opposites, regions that may or may not be disjoint - depending on the application, but they are exhaustive (their union equals the whole space). A NeutroAlgebra is an algebra which has at least one NeutroOperation operation that is well-defined (also called inner-defined) for some elements, indeterminate for others, and outer-defined for the others or one NeutroAxiom (axiom that is true for some elements, indeterminate for other elements, and false for the other elements). A Partial Algebra is an algebra that has at least one partial operation (well-defined for some elements, and indeterminate for other elements), and all its axioms are classical (i.e., the axioms are true for all elements). Through a theorem he proved that NeutroAlgebra is a generalization of Partial Algebra, and examples of NeutroAlgebras that are not partial algebras were given. Also, the NeutroFunction and NeutroOperation were introduced.[7]

Regarding these points, we now introduce the concept of Neutro-BCK-algebras based on axioms of BCK-algebras, but having a different outcome. In the system of BCK-algebras, the operation is totally well-defined for any two given elements, but in Neutro-BCK-algebras its outcome may be well-defined, outer-defined, or indeterminate. Any BCK-algebra is a system which considers that all its axioms are true; but we weaken the conditions that the axioms are not necessarily totally true, but also partially false, and partially indeterminate. So, one of our main motivation is a weak coverage of the classical axioms of BCK-algebras. This causes new partially ordered relations on a non-empty set, such as NeutroPosets and Neutro-Hass Dia-

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grams. Indeed Neutro-Hass Diagrams of NeutroPosets contain relations between elements in the set that are true, false, or indeterminate.

2 Preliminaries

In this section, we recall some definitions and results from,[7] which are needed throughout the paper.

Let $n \in \mathbb{N}$. Then an n-ary operation $\circ : X^n \to Y$ is called a NeutroOperation if it has $x \in X^n$ for which $\circ(x)$ is well-defined (degree of truth (T)), $x \in X^n$ for which $\circ(x)$ is indeterminate (degree of indeterminacy (I)), and $x \in X^n$ for which $\circ(x)$ is outer-defined (degree of falsehood (F)), where $T, I, F \in [0, 1]$, with $(T, I, F) \neq (1, 0, 0)$ that represents the n-ary (total, or classical) Operation, and $(T, I, F) \neq (0, 0, 1)$ that represents the n-ary AntiOperation. Again, in this definition "neutro" stands for neutrosophic, which means the existence of outer-ness, or undefined-ness, or unknown-ness, or indeterminacy in general. In this regards, for any given set X, we classify n-ary operation on X^n by (i); (**classical**) **Operation** is an operation well-defined for all set's elements, (ii); **NeutroOperation** is an operation partially well-defined, partially indeterminate, and partially outer-defined on the given set and (iii); **AntiOperation** is an operation outer-defined for all set's elements.

Moreover, we have (i); a (**classical**) **Axiom** defined on a non-empty set is an axiom that is totally true (i.e. true for all set's elements), (ii); **NeutroAxiom (or neutrosophic axiom)** defined on a non-empty set is an axiom that is true for some elements (degree of true = T), indeterminate for other elements (degree of indeterminacy = I), and false for the other elements (degree of falsehood = F), where T, I, F are in $[0, 1]$ and (T, I, F) is different from $(1, 0, 0)$ i.e., different from totally true axiom, or classical Axiom and (T, I, F) is different from $(0, 0, 1)$ i.e., different from totally false axiom, or AntiAxiom. (iii); an **AntiAxiom** of type $\mathcal{C}$ defined on a non-empty set is an axiom that is false for all set's elements.

Based on the above definitions, there is a classification of algebras as follows. Let X be a non-empty set and $\mathcal{O}$ be a family of binary operations on X. Then $(A, \mathcal{O})$ is called

(i) a (**classical**) **Algebra** of type $\mathcal{C}$, if $\mathcal{O}$ is the set of all total Operations (i.e. well-defined for all set's elements) and $(A, \mathcal{O})$ is satisfied by (classical) Axioms of type $\mathcal{C}$(true for all set's elements).

(ii) a **NeutroAlgebra (or neutro-algebraic structure)** of type $\mathcal{C}$, if $\mathcal{O}$ has at least one NeutroOperation (or NeutroFunction), or $(A, \mathcal{O})$ is satisfied by at least one NeutroAxiom of type $\mathcal{C}$ that is referred to the set's (partial-, neutro-, or total-) operations or axioms;

(iii) an **AntiAlgebra (or anti-algebraic structure)** of type $\mathcal{C}$, if $\mathcal{O}$ has at least one AntiOperation or $(A, \mathcal{O})$ is satisfied by at least one AntiAxiom of type $\mathcal{C}$.

3 Neutro-BCK-algebra

3.1 Concept of Neutro-BCK-algebra

In this section, we introduce several concepts suc has: Neutro-BCK-algebra, Neutro-BCK-algebra of type 5, NeutroPoset and Neutro-Hass Diagram and investigate the properties of these concepts.

Definition 3.1. [2] Let X be a non-empty set with a binary operation "$*$" and a constant "0". Then, $(X, *, 0)$ is called a *BCK-algebra* if it satisfies the following conditions:

$(BCI$-1$)$ $((x * y) * (x * z)) * (z * y) = 0$,

$(BCI$-2$)$ $(x * (x * y)) * y = 0$,

$(BCI$-3$)$ $x * x = 0$,

$(BCI$-4$)$ $x * y = 0$ and $y * x = 0$ imply $x = y$,

$(BCK$-5$)$ $0 * x = 0$.

Now, we define Neutro-BCK-algebras as follows.

Definition 3.2. Let X be a non-empty set, $0 \in X$ be a constant and "$*$" be a binary operation on X. An algebra $(X, *, 0)$ of type $(2, 0)$ is said to be a Neutro-BCK-algebra, if it satisfies at least one of the following NeutroAxioms (while the others are classical BCK-axioms):

$(NBCI\text{-}1)$ $\big(\exists\, x, y, z \in X$ such that $((x * y) * (x * z)) * (z * y) = 0)\big)$ and $\big(\exists\, x, y, z \in X$ such that $((x * y) * (x * z)) * (z * y) \neq 0$ or indeterminate $\big)$;

$(NBCI\text{-}2)$ $\big(\exists\, x, y \in X$ such that $(x * (x * y)) * y = 0\big)$ and $\big(\exists\, x, y \in X$ such that $(x * (x * y)) * y \neq 0$ or indeterminate $\big)$;

$(NBCI\text{-}3)$ $\big(\exists\, x \in X$ such that $x * x = 0\big)$ and $\big(\exists\, x \in X$ such that $x * x \neq 0$ or indeterminate $\big)$;

$(NBCI\text{-}4)$ $\big(\exists\, x, y \in X$, such that if $x*y = y*x = 0$, we have $x = y\big)$ and $\big(\exists\, x, y \in X$, such that if $x*y = y*x = 0$, we have $x \neq y$ $\big)$;

$(NBCK\text{-}5)$ $\big(\exists\, x \in X$ such that $0 * x = 0\big)$ and $\big(\exists\, x \in X$ such that $0 * x \neq 0$ or indeterminate $\big)$. Each above NeutroAxiom has a degree of equality (T), degree of non-equality (F), and degree of indeterminacy (I), where $(T, I, F) \notin (1, 0, 0), (0, 0, 1)$.

If $(X, *, 0)$ is a NeutroAlgebra and satisfies the conditions $(NBCI\text{-}1)$ to $(NBCI\text{-}4)$ and $(NBCK\text{-}5)$, then we will call it is a Neutro-BCK-algebra of type 5 (i.e. it satisfies 5 NeutroAxioms).

Example 3.3. Let $X = \mathbb{Z}$. Then
(i) $(X, *, 0)$ is a Neutro-BCK-algebra, where for all $x, y \in X$, we have $x * y = x - y + xy$.
(ii) $(X, *, 1)$ is a Neutro-BCK-algebra, where for all $x, y \in X$, we have $x * y = xy$.
(iii) $(X, *, 1)$ is a Neutro-BCK-algebra, where for all $x, y \in X$, we have $x * y = \begin{cases} 1 & \text{if } x \text{ an even} \\ xy & \text{if } x \text{ an odd} \end{cases}$.

Let $X \neq \emptyset$ be a finite set. We denote $\mathcal{N}_{BCK}(X)$ and $\mathcal{N}_{NBCK}(X)$ by the set of all Neutro-BCK-algebras and Neutro-BCK-algebras of type 5 that are constructed on X, respectively.

Theorem 3.4. *Let $(X, *, 0)$ be a Neutro BCK-algebra. Then*

(i) *If $|X| = 1$, then $(X, *, 0)$ is a trivial BCK-algebra.*

(ii) *If $|X| = 2$, then $|\mathcal{N}_{BCK}(X)| = 2$ and $|\mathcal{N}_{NBCK}(X)| = \infty$.*

(iii) *If $|X| = 3$, then there exists $\emptyset \neq Y \subseteq X$, such that $(Y, *', 0)$ is a nontrivial or trivial BCK-algebra.*

Proof. We consider only the cases $(ii), (iii)$, because the others are immediate.
(ii) Let $X = \{0, x\}$. Then we have 2 trivial Neutro-BCK-algebras $(X, *_1), (X, *_2)$ and an infinite number of trivial Neutro-BCK-algebras of type 5 $(X, *, 0)$ in Table 1, where $w \notin X$.
(iii) Let $X = \{0, x, y\}$. Now consider $Y = \{0, x\}$ and define a Neutro-BCK-algebra $(X, *', 0)$ in Table 1. Clearly $(Y, *', 0)$ is a non-trivial BCK-algebra. If $Y = \{0\}$, it is a trivial BCK-algebra. $\square$

Table 1: Neutro-BCK-algebras of order 2

$$\begin{array}{c|cc} *_1 & 0 & x \\ \hline 0 & 0 & x \\ x & 0 & x \end{array}\ ,\quad \begin{array}{c|cc} *_2 & 0 & x \\ \hline 0 & x & 0 \\ x & x & 0 \end{array}\ ,\quad \begin{array}{c|cc} * & 0 & x \\ \hline 0 & x & 0 \\ x & w & 0 \end{array}\quad \text{and} \quad \begin{array}{c|ccc} *' & 0 & x & y \\ \hline 0 & 0 & 0 & y \\ x & x & 0 & 0 \\ y & 0 & y & x \end{array}\ .$$

Theorem 3.5. *Every BCK-algebra, can be extended to a Neutro-BCK-algebra.*

Proof. Let $(X, *, 0)$ be a BCK-algebra and $\alpha \notin X$, and U be the universe of discourse that strictly includes $X \cup \alpha$. For all $x, y \in X \cup \{\alpha\}$, define $*_\alpha$ on $X \cup \{\alpha\}$ by $x *_\alpha y = x * y$ where, $x, y \in X$ and if $\alpha \in \{x, y\}$, define $x *_\alpha y$ as indeterminate or $x *_\alpha y \notin X \cup \alpha$. Then $(X \cup \{\alpha\}, *_\alpha, 0)$ is a Neutro-BCK-algebra. $\square$

Example 3.6. Let $X = \{0, 1, 2, 3, 4, 5\}$. Consider Table 3.
Then
(i) If $a = 0$, then $(X, *_1, 0)$ is a Neutro-BCK-algebra and if $a = 1$, then $(X \setminus \{3, 4, 5\}, *_1, 0)$ is a BCK-algebra.
(ii) $(X, *_2, 0)$ is a Neutro-BCK-algebra and $(X \setminus \{4, 5\}, *_2, 0)$ is a BCK-algebra.
(iii) If $s = t = y = z = 0, w = 3$, then $(X, *_3, 0)$ is a Neutro-BCK-algebra and for $s = t = 1, y = 2, z = 3$, $(X \setminus \{5\}, *_3, 0)$ is a BCK-algebra. If $s = t = y = z = 0, w = \sqrt{2}$, then $(X, *_3, 0)$ is a Neutro-BCK-algebra of type 5 where $s, t \in \{0, 1\}, x \in \{4, 5\}, y \in \{2, 0\}, z \in \{3, 0\}$ and $w \in \{3, \sqrt{2}\}$.

Table 2: Neutro-BCK-algebras and Neutro-BCK-algebra of type 5

$*_1$	0	1	2	3	4	5
0	0	0	0	0	2	0
1	1	0	a	2	0	3
2	2	2	0	0	2	0
3	3	0	1	2	0	5
4	0	4	0	1	4	0
5	4	0	1	0	2	3

,

$*_2$	0	1	2	3	4	5
0	0	0	0	0	2	0
1	1	0	0	0	0	5
2	2	1	0	0	5	0
3	3	2	1	0	0	2
4	0	1	0	4	1	2
5	5	0	4	0	0	x

and

$*_3$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	t	0	s	0
2	2	2	0	y	0	3
3	3	1	3	0	z	0
4	4	4	4	4	0	1
5	0	2	0	2	0	w

,

Remark 3.7. In Neutro-BCK-algebra $(X, *_3, 0)$, which is defined as in Example 3.6, we have $(1, 5) \in\leq$ and $(5, 0) \in\leq$, but $(1, 0) \notin\leq$, where $(x, y) \in\leq$ means $x *_3 y = 0$. Thus $\leq$, necessarily, is not a transitive relation. So we have the following definition of neutro-partially ordered relation on Neutro-BCK-algebra.

Definition 3.8. Let X be a non-empty set and R be a binary relation on X. Then R is called a

(i) neutro-reflexive, if $\exists\, x \in X$ such that $(x, x) \in R$ (degree of truth T), and $\exists\, x \in X$ such that (x, x) is indeterminate (degree of indeterminacy I), and $\exists\, x \in X$ such that $(x, x) \notin R$ (degree of falsehood F);

(ii) neutro-antisymmetric, if $\exists\, x, y \in X$ such that $(x, y) \in R$ and $(x, y) \in R$ imply that $x = y$ (degree of truth T), and $\exists\, x, y \in X$ such that (x, y) or (y, x) are indeterminate in R (degree of indeterminacy I), and $\exists\, x, y \in X$ such that $(x, y) \in R$ and $(y, x) \in R$ imply that $x \neq y$ (degree of falsehood F);

(iii) neutro-transitive, if $\exists\, x, y, z \in X$ such that $(x, y) \in R, (y, z) \in R$ imply that $(x, z) \in R$ (degree of truth T),and $\exists\, x, y, z \in X$ such that (x, y) or (y, z) are indeterminate in R (degree of indeterminacy I), and $\exists\, x, y, z \in X$ such that $(x, y) \in R, (y, z) \in R$, but $(x, z) \notin R$ (degree of falsehood F). In all above neutro-axioms $(T, I, F) \notin (1, 0, 0), (0, 0, 1)$.

(iv) neutro-partially ordered binary relation, if the relation satisfies at least one of the above neutro-axioms neutro-reflexivity, neutro-antisymmetry, neutro-transitivity, while the others (if any) are among the classical axioms reflexivity, antisymmetry, transitivity.

If R is a neutro-partially ordered relation on X, we will call (X, R) by neutro-poset. We will denote, the related diagram with to neutro-poset (X, R) by neutro-Hass diagram.

We define binary relations " $\leq_1, \leq_2$ " on X by $\big(x \leq_1 y$ if or only if $x * y = 0$ or $x \leq_1 x\,\big)$ and $\big(x \leq_2 y$ if and only if $(x * y \neq 0$ or indeterminate $)$ or $x \leq_2 x\,\big)$. So we have the following theorem.

Theorem 3.9. *An algebra $(X, *, 0)$ is a Neutro-BCK-algebra if and only if it satisfies the following conditions:*

$(NBCI\text{-}1')$ $\big(\exists\, x, y \in X$ *such that* $((x*y)*(x*z)) \leq_1 (z*y)\big)$ *and* $\big(\exists\, x, y \in X$ *such that* $((x*y)*(x*z)) \leq_2 (z*y)\big)$,

$(NBCI\text{-}2')$ $\big(\exists\, x, y \in X$ *such that* $(x * (x * y)) \leq_1 y\big)$ *and* $\big(\exists\, x, y \in X$ *such that* $(x * (x * y)) \leq_2 y\big)$,

$(NBCI\text{-}3')$ $\big(\exists\, x, y \in X$ *such that* $x \leq_1 x\big)$ *and* $\big(\exists\, x, y \in X$ *such that* $x \leq_2 x\big)$,

$(NBCI\text{-}4')$ $\big(\forall\, x, y \in X$, *if* $x \leq_1 y$ *and* $y \leq_1 x$, *we get* $x = y\big)$ *and* $\big(\forall\, x, y \in X$, *if* $x \leq_2 y$ *and* $y \leq_2 x$, *we get* $x = y\big)$,

$(NBCK\text{-}5')$ $\big(\exists\, x, y \in X$ *such that* $0 \leq_1 x\big)$ *and* $\big(\exists\, x, y \in X$ *such that* $0 \leq_2 x\big)$.

Proof. Let $(X, *, 0)$ be a Neutro-BCK-algebra. We prove only the item $(NBCI\text{-}1')$, other items are similar to. Since $(X, *, 0)$ be a Neutro-BCK-algebra, $\big(\exists\, x, y \in X$ such that $(x*(x*y))*y = 0\big)$ and $\big(\exists\, x, y \in X$ such that $(x*(x*y))*y \neq 0$ or indeterminate$\big)$. By definition, $\big(\exists\, x, y \in X$ such that $((x*y)*(x*z)) \leq_1 (z*y)\big)$ and $\big(\exists\, x, y \in X$ such that $((x*y)*(x*z)) \leq_2 (z*y)\big)$. Conversely, let the items $(NBCI\text{-}1')$ to $(NBCI\text{-}4')$ and $(NBCK\text{-}4')$. Just prove $(NBCI\text{-}1)$ and other items are similar to. Since $\big(\exists\, x, y \in X$ such that $((x*y)*(x*z)) \leq_1 (z*y)\big)$ and $\big(\exists\, x, y \in X$ such that $((x*y)*(x*z)) \leq_2 (z*y)\big)$, we get that $\big(\exists\, x, y \in X$ such that $((x*y)*(x*z))*(z*y) = 0)\big)$ and $\big(\exists\, x, y \in X$ such that $((x*y)*(x*z))*(z*y) \neq 0$ or indeterminate $\big)$. □

Let $(X, *, 0)$ be a Neutro-BCK algebra. Define binary relation $\leq$ on X, by $x \leq y$ if and only $x \leq_1 y$ and $y \leq_2 x$. So we have the following results.

Theorem 3.10. *Let $(X, *, 0)$ be a Neutro-BCK algebra and $x, y, z \in X$. Then*

(i) *if $x \neq y$ and $x \leq y$, then $y \leq x$;*

(ii) *$\leq$ is a reflexive and symmetric relation on X;*

(iii) *$\leq$ is a neutro-transitive algebra relation on X.*

Proof. (i) Let $x \neq y \in X$ and $x \leq y$. If $y \leq x$, by definition we obtain $(x * y = y * x = 0)$ and $(x * y = y * x \neq 0)$ and so $x = y$.
$(ii), (iii)$ It is clear by item (i) and Remark 3.7.
(iii) It is obtained by (ii). □

Corollary 3.11. *Let $(X, *, 0)$ be a Neutro-BCK algebra. Then $(X, *, 0, \leq_1), (X, *, 0, \leq_2)$ and $(X, *, 0, \leq)$ are neutro-posets.*

Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be BCK-algebras, where $X_1 \cap X_2 = \emptyset$. For some $x, y \in X$, define an operation $*$ as follows: $x * y = \begin{cases} x *_1 y & \text{if if } x, y \in X_1 \setminus X_2 \\ x *_2 y & \text{if if } x, y \in X_2 \setminus X_1 \\ 0_1 & \text{if if } x \in X_1, y \in X_2 \\ 0_2 & \text{if if } x \in X_2, y \in X_1 \end{cases}$, where $0_1 * 0_2 = 0_2$ and $0_2 * 0_1 = 0_1$.

Theorem 3.12. *Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be BCK-algebras, where $X_1 \cap X_2 = \emptyset$ and $X = X_1 \cup X_2$. Then*

(i) *$(X, *, 0_1)$ is a Neutro-BCK-algebra;*

(ii) *$(X, *, 0_2)$ is a Neutro-BCK-algebra;*

Proof. (i) We only prove $(NBCI$-$4)$. Let $x * y = 0_1$. It follows that $x \in X_1$ and $y \in X_2$ or $x, y \in X_1$. If $x, y \in X_1$, because $(X_1, *_1, 0_1)$ is a BCK-algebra, $y * x = 0_1$ implies that $x = y$. But for $x \in X_1$ and $y \in X_2$, we have $y * x \neq 0_1$ so $(NBCI$-$4)$ is valid in any cases. Other items are clear.
(ii) It is similar to item (i). □

Example 3.13. Let $X_1 = \{a, b\}$ and $X_2 = \{w, x, y, z\}$. Then $(X_1, *, a)$ and $(X_2, *, w)$ are BCK-algebras. So by Theorem 3.12, $(X_1 \cup X_1, *, a)$ and $(X_1 \cup X_1, *, w)$ are Neutro-BCK-neutralgebras in Table 3.

Table 3: BCK-algebras and Neutro-BCK-algebra

$*$	a	b	w	x	y	z
a	a	a	w	a	a	a
b	b	a	a	a	a	a
w	a	w	w	w	w	w
x	w	w	x	w	w	w
y	w	w	y	x	w	w
z	w	w	z	x	x	w

.

Corollary 3.14. *Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be BCK-algebras. Then*

(i) *$(X, *, 0_1, \leq_1), (X, *, 0_2, \leq_2)$ and $(X, *, 0_2, \leq_2)$ are posets.*

(ii) *$(X, *, 0_1, \leq_2), (X, *, 0_2, \leq_1)$ are neutro-posets.*

Example 3.15. Consider the Neutro-BCK-algebra in Example 3.13. Then we have neutro-posets $(X, *, w, \leq_1$ $), (X, *, a, \leq_2)$ and $(X, *, 0_2, \leq)$ in Table 4, where $-$ means that elements are not comparable and I means that are indeterminates.

Definition 3.16. Let $(X, *, 0)$ be a Neutro-BCK-algebra, $\theta \in X$ and $Y \subseteq X$. Then

Table 4: neutro-posets

$\leq_1$	a	b	w	x	y	z
a	a	$-$	a	x	y	z
b	$-$	b	w	x	y	z
w	a	w	w	w	w	w
x	x	x	w	x	x	x
y	y	y	w	x	y	y
z	z	z	w	x	y	z

,

$\leq_2$	a	b	w	x	y	z
a	a	b	a	x	y	z
b	b	b	w	x	y	z
w	a	w	w	I	I	I
x	x	x	I	x	I	I
y	y	y	I	I	y	I
z	z	z	I	I	I	z

,

$\leq$	a	b	w	x	y	z
a	a	a	w	a	a	a
b	a	b	b	b	b	b
w	w	b	w	$-$	$-$	$-$
x	a	b	$-$	x	$-$	$-$
y	a	b	$-$	$-$	y	$-$
z	a	b	$-$	$-$	$-$	z

.

(i) Y is called a Neutro-BCK-subalgebra, if (1) $0 \in Y$, (2) for all $x, y \in Y$, we have $x * y \in Y$, (3) satisfies in conditions $(NBCI$-3), $(NBCI$-4) and $(NBCK$-5).

(ii) $\theta \in X$ is called a source element, if it is a minimum or maximum element in neutro-Hass diagram of $(X, *, 0)$.

Theorem 3.17. *Let $(X, *, 0)$ be a Neutro-BCK-algebra and $Y \subseteq X$. If Y is a Neutro-BCK-subalgebra of X, then*

(i) *$(Y, *, 0)$ is a Neutro-BCK-algebra.*

(ii) *X is a Neutro-BCK-subalgebra of X.*

Proof. They are clear. □

Corollary 3.18. *Let $(X, *, 0)$ be a Neutro-BCK-algebra and $|X| = n$. Then there exist $m \leq n$ and $x_1, x_2, \ldots, x_m \in X$ such that $(\{0, x_1, x_2, \ldots, x_m\}, *, 0)$ is a Neutro-BCK-algebra of X.*

Theorem 3.19. *Let X be a non-empty set. Then there exists a binary operation "$\bullet$" on X and $0 \in X$ such that*

(i) *$(X, \bullet, x_0)$ is a Neutro-BCK-algebra.*

(ii) *For all $\emptyset \neq Y \subseteq X$, $Y \cup \{x_0\}$ is a Neutro-BCK-subalgebra of X.*

(iii) *If X is a countable set, then in neutro-Hass diagram $(X, \bullet, x_0)$, we have $|Maximal(X)| = 1$ and $Minimal(X) = |X| - 1(|X|$ is cardinal of $X)$.*

(iv) *neutro-Hass diagram $(X, \bullet, x_0)$ has a source element.*

Proof. Let $x, y \in X$. Fixed $x_0 \in X$ and define $x * y = y$.
(i) Some modulations show that $(X, *, x_0)$ is a Neutro-BCK-algebra.
(ii) By Theorem 3.4 and definition, it is clear.
(iii) Let $X = \{x_0, x_1, x_2, x_3, \ldots\}$. Then by Corollary 3.11, $(X, \leq, x_0)$ is a neutro-poset and so has a neutro-Hass diagram as Figure 1. □

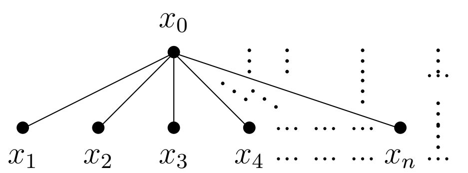

Figure 1: neutro-Hass diagram $(X, \leq, x_0)$ with source x_0.

Theorem 3.20. *Let $(X, \leq_X)$ be a chain. Then*

(i) *there exists $*_X$ on X and $0 \in X$ such that $(X, *_X, 0)$ is a Neutro-BCK-algebra.*

(ii) *for all $x, y \in X$, we have $x \leq y$ if and only if $y \leq_X x$.*

(iii) *In neutro-Hass diagram $(X, \bullet, x_0)$, 0 is source element.*

there exists $*_X$ on X and $0 \in X$ such that $(X, *_X, 0)$ is a Neutro-BCK-algebra.

Proof. Let $0, x, y \in X$, where $0 = Min(X)$.

(i) Define $x *_X y = \begin{cases} x \vee y & \text{if } x \leq_X y \\ x \wedge y & \text{otherwise} \end{cases}$. Some modulations show that $(X, *_X, 0)$ is a Neutro-BCK-algebra.

(ii) Let $x, y \in X$. Clearly $x * x = x$, then by definition $x \leq y$ if and only if $x * y = 0$ and $y * x \neq 0$ if and only if $y = 0$ if and only if $y \leq_X x$.

(iii) By item (ii), we get the neutro-Hass diagram $(X, \leq_X, 0)$ in Figure 1, so 0 is source element. □

Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be two Neutro-BCK-algebras, where $X_1 \cap X_2 = \emptyset$. Define $*$ on $X_1 \cup X_2$, by $x * y = \begin{cases} x *_1 y & \text{if } x, y \in X_1 \setminus X_2 \\ x *_2 y & \text{if } x, y \in X_2 \setminus X_1 \\ y & \text{otherwise} \end{cases}$.

Theorem 3.21. *Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be two Neutro-BCK-algebras. Then*

(i) $(X_1 \cup X_2, *, 0_1)$ *is a Neutro-BCK-algebra.*

(ii) $(X_1 \cup X_2, *, 0_2)$ *is a Neutro-BCK-algebra.*

Proof. It is obvious. □

Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be two Neutro-BCK-algebras. Define $*$ on $X_1 \times X_2$, by $(x, y) * (x', y') = (x *_1 x', y *_2 y')$, where $(x, y), (x', y') \in X_1 \times X_2$.

Theorem 3.22. *Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be two Neutro-BCK-algebras. Then $(X_1 \times X_2, *, (0_1, 0_2))$ is a Neutro-BCK-algebra.*

Proof. We prove only the item $(NBCI$-$4)$. Let $(x, y), (x', y') \in X_1 \times X_2$. If $(x, y) * (x', y') = (x', y') * (x, y) = (0_1, 0_2)$, then $(x *_1 x', y *_2 y') = (0_1, 0_2)$ and $(x' *_1 x, y' *_2 y) = (0_2, 0_1)$. It follows that $(x, y) = (x', y')$. In a similar way, $(x, y) * (x', y') = (x', y') * (x, y) \neq (0_1, 0_2)$, we get that $(x, y) = (x', y')$. Thus, $(X_1 \times X_2, *, (0_1, 0_2))$ is a Neutro-BCK-algebra. □

3.2 Application of Neutro-BCK-algebra

In this subsection, we describe some applications of Neutro-BCK-algebra.

In the following example, we describe some applications of Neutro-BCK-algebra. We discuss applications of Neutro-BCK-algebra for studying the competition along with algorithms. The Neutro-BCK-algebra has many utilizations in different areas, where we connect Neutro-BCK-algebra to other sciences such as economics, computer sciences and other engineering sciences. We present an example of application of Neutro-BCK-algebra in COVID-19.

Example 3.23. (COVID-19) Let $X = \{a = China, b = Italy, c = USA, d = Spain, e = Germany, f = Iran\}$ be a set of top six COVID-19 affected countries. There are many relations between the countries of the world. Suppose $*$ is one of relations on X which is described in Table 5. This relation can be economic impact, political influence, scientific impact or other chasses. For example $x * y = z$, means that the country z influences the relationship $*$ from country x to country y. Clearly $(X, *, China)$ is a Neutro-BCK-algebra.

Table 5: Neutro-BCK-algebra

$*$	$China$	$Italy$	USA	$Spain$	$Germany$	$Iran$
$China$	$China$	$Iran$	$Spain$	$Germany$	$Italy$	USA
$Italy$	$China$	$Italy$	$Iran$	$Germany$	$Spain$	$Germany$
USA	$China$	$Italy$	USA	USA	$Iran$	$Iran$
$Spain$	$China$	$China$	$China$	$Spain$	USA	$Italy$
$Germany$	$China$	$Germany$	$Italy$	$Spain$	$Germany$	$Italy$
$Iran$	$China$	$Spain$	USA	USA	$China$	$Iran$

And so we obtain neutro-Hass diagram as Figure 2. Applying Figure 2, we obtain that China is main source of COVID-19 to top five affected countries and Iran, Spain, Italy are indeterminated countries in COVID-19 affection together, USA effects Spain and Germany effects Iran.

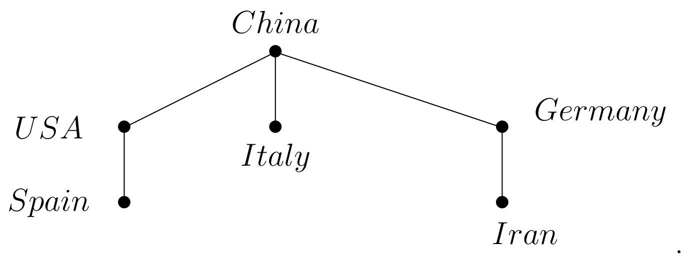

Figure 2: neutro-Hass diagram $(X, *, China)$ associated to infected COVID-19 .

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A Note on Neutrosophic Submodule of an R-module M

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Abstract

The paper focuses on the applications of neutrosophic set theory in the domain of classical algebraic structures, especially R-module. This study discusses some algebraic operations of neutrosophic sets of an R-module M, induced by the operations in M and demonstrates certain properties of the neutrosophic submodules of an R-module. The ideas of R module's non-empty arbitrary family of neutrosophic submodules are characterized, and related outcomes are proved. The last section of this paper also derives a necessary and sufficient condition for a neutrosophic set of an R-module M.
Keywords: R-module, Neutrosophic Set, Neutrosophic Submodule, Support, Neutrosophic Point

1 Introduction

Traditionally the set theory deals with sets of objects, their features and framework models of axioms. Practical problems in research, science and economics cannot be solved in the current environment due to the inadequacy of the ideas of parameterisation techniques. In 1965, Lotfi Aliaskar Zadeh [1] published a paper describing the concept of imprecise boundaries of sets which led to the emergence of fuzzy set theory. After the implementation of fuzzy sets by Zadeh, this basic notion has been generalized in several ways. In 1986 Atanassov [2] put forward intuitionistic fuzzy set theory as a stereotype illustration of a set in which each component is concomitant with membership and non-membership grades. In 1995, the University of New Mexico's scientist and mathematics professor Florentin Smarandache [3] inspired by sport matches (winning / defeating / tie scores), votes (pro / counter / null or black votes), decision making (accept / reject / pending) and control theory (yes / no / not relevant) coined a new idea and a branch of philosophy called neutrosophy. Neutrosophy means understanding neutral concepts and extending of tri-valued logic by non-standard analysis [4, 5].

The main objective of the neutrosophic set is to narrow the gap between the vague, ambiguous and imprecise real-world situations. Neutrosophic set theory gives a thorough scientific and mathematical model knowledge in which speculative and uncertain hypothetical phenomena can be managed by hierarchal membership of the components " truth / indeterminacy / falsehood " [4, 6]. Among the different branches of applied and pure mathematics, abstract algebra was one of the first few areas where research was conducted using the concept of neutrosophic set. Some authors have studied the algebraic structure associated in pure mathematics with uncertainty. In 1971, Azriel Rosenfeld [7] presented a seminal paper on fuzzy subgroup and W.J. Liu [8] developed the idea of fuzzy normal subgroup and fuzzy subring. It was a significant milestone in the area of mathematics research and fuzzy algebra. Mordeson's and Malik's book [9] gives an account of all these concepts upto 1998. Negoita and Ralescu [10] launched the notion of a fuzzy module. Then fuzzy module was further developed by Mashinchi & Zahedi [11]. The idea of a direct sum of fuzzy modules was investigated by P. Isaac [12]. In 2011 P. Isaac, P.P.John [13] studied about the algebraic nature of intuitionistic fuzzy submodule of a classical module.

The consolidation of the neutrosophic set hypothesis with algebraic structures is a growing trend in mathematical research. One of the key developments in the neutrosophic set theory is the hybridization of the neutrosophic set with various potential algebraic structures such as bipolar set, soft set and hesitant fuzzy set [14–16]. W. B. Vasantha Kandasamy and Florentin Smarandache [17] initially presented basic neutrosophic algebraic structures and their application. Vidan Cetkin [18, 19] consolidated the neutrosophic set theory and algebraic

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structures, creating neutrosophic subgroups and neutrosophic submodules. The basic features of single valued neutrosophic submodules of an R-module (classical module) are studied by Cetkin and Olgun N [19,20]. Neutrosophic algebraic structures have higher expressive power than classical crisp set-based structures. This paper explains certain elementary properties of neutrosophic set of an R-module M and characteristics of neutrosophic submodule of an R-module M.

Neutrosophic set generalizes a classical set, fuzzy set, interval-valued fuzzy set and intuitionistic fuzzy set that can be used to make a mathematical model for the real problems of science and engineering. The remaining of the paper is structured as follows. The section 2 briefs about neutrosophic set operations and neutrosophic sub-modules of an R-module M. Section 3 provides some elementary properties of neutrosophic set of an R-module M and related results. The findings and description of the related work are also briefed in section 4. Finally section 5 presents a valid summary and future work of the proposed study.

2 Preliminaries

This section presents some of the preliminary definitions and results which are basic for a better and clear cognizance of next chapters.

Definition 2.1. [21] Let R be a commutative ring with unity. A *module* M over R is an abelian group with a law of composition written '$+$'and a scalar multiplication $R \times M \to M$, written $(r, x) \rightsquigarrow rx$, that satisfy these axioms

1. $1x = x$
2. $(rs)x = r(sx)$
3. $(r + s)x = rx + sx$
4. $r(x + y) = rx + ry \ \forall\, r, s \in R$ and $x, y \in M$.

Definition 2.2. [21] A *submodule* N of an R-module M is a nonempty subset that is closed under addition and scalar multiplication, i.e., $x_1 + x_2 \in N,\ rx \in N\, \forall\, r \in R, x_1, x_2, x \in N$.

Definition 2.3. [22] Let $A \; and \; B$ be submodules of an R-module M. The sum of A and B, denoted as a set

$$A + B = \{x + y : x \in A, y \in B\}$$

which is also a submodule and smallest submodule which contains both $A \; and \; B$.

Theorem 2.1. [22] The intersection of any non empty collection of submodules of an R-module is a sub module .

Definition 2.4. [4] A *neutrosophic set* P of the universal set X is defined as $P = \{(x, t_P(x), i_P(x), f_P(x)) : x \in X\}$ where $t_P, i_P, f_P : X \to (^-0, 1^+)$. The three components $t_P, i_P \; and \; f_P$ represent membership value (Percentage of truth), indeterminacy (Percentage of indeterminacy) and non membership value (Percentage of falsity) respectively. These components are functions of non standard unit interval $(^-0, 1^+)$.

Remark 2.1. [23,24] "If $t_P, i_P, f_P : X \to [0, 1]$, then P is known as Single Valued Neutrosophic Set (SVNS).

Remark 2.2. This paper considers only SVNS. For simplicity SVNS will be called neutrosophic set.

Remark 2.3. U^X denotes the set of all neutrosophic subsets of X or neutrosophic power set of X.

Definition 2.5. [4,25] Let P and Q be two neutrosophic sets of X. Then P is contained in Q, denoted as $P \subseteq Q$ if and only if $P(x) \leqslant Q(x)\ \forall x \in X$, this means that $t_P(x) \leq t_Q(x),\ i_P(x) \leq i_Q(x),\ f_P(x) \geq f_Q(x),\ \forall\, x \in X$

Definition 2.6. [4,26] The complement of a neutrosophic set $P = \{x, t_P(x), i_P(x), f_P(x) : x \in X\}$ of X is denoted and defined as $P^c = \{x, f_P(x), 1 - i_P(x), t_P(x) : x \in X\}$.

Definition 2.7. [4,27] Let $P, Q \in U^X\ \forall\, x \in X$. Then

1. The union C of P and Q is denoted by $C = P \cup Q$ and defined as $C(x) = P(x) \vee Q(x)$ where $C(x) = \{x, t_C(x), i_C(x), f_C(x) : x \in X\}$ is given by

$$t_C(x) = t_P(x) \vee t_Q(x)$$
$$i_C(x) = i_P(x) \vee i_Q(x)$$
$$f_C(x) = f_P(x) \wedge f_Q(x)$$

2. The intersection C of P and Q is denoted by $C = P \cap Q$ and is defined as $C(x) = P(x) \wedge Q(x)$ where $C(x) = \{x, t_C(x), i_C(x), f_C(x) : x \in X\}$ is given by

$$t_C(x) = t_P(x) \wedge t_Q(x)$$
$$i_C(x) = i_P(x) \wedge i_Q(x)$$
$$f_C(x) = f_P(x) \vee f_Q(x)$$

Definition 2.8. [28] Let P and Q be neutrosophic sets of an R-Module M . Then their sum $P + Q$ is a neutrosophic set of M, defined as follows

$$P + Q(x) = \{x, t_{P+Q}(x), i_{P+Q}(x), f_{P+Q}(x) : x \in M\} \text{ where}$$
$$t_{P+Q}(x) = \vee\{t_P(y) \wedge t_Q(z) | x = y + z, y, z \in M\}$$
$$i_{P+Q}(x) = \vee\{i_P(y) \wedge i_Q(z) | x = y + z, y, z \in M\}$$
$$f_{P+Q}(x) = \wedge\ f_P(y) \vee f_Q(z) | x = y + z, y, z \in M\}.$$

Definition 2.9. [19] Let P be a neutrosophic set of an R-module M and $r \in R$. Define neutrosophic set $rP = \{x, t_{rP}(x), i_{rP}(x), f_{rP}(x) : x \in M\}$ of M as follows $t_{rP}(x) = \begin{cases} \vee\{t_P(y)\} & if\ y \in M, x = ry \\ 0 & otherwise \end{cases}$; $i_{rP}(x) = \begin{cases} \vee\{i_P(y)\} & if\ y \in M, x = ry \\ 0 & otherwise \end{cases}$ and $f_{rP}(x) = \begin{cases} \wedge\{f_P(y)\} & if\ y \in M, x = ry \\ 0 & otherwise \end{cases}$

3 Neutrosophic Set of an R-module M

In this section a few algebraic properties of neutrosophic submodules of an R-module M are demonstrated and evaluated using three different membership grade values of neutrosophic submodules.

Definition 3.1. [19] Let M be an R module. Let $P \in U^M$ where U^Mdenotes the neutrosophic power set of R-module M. Then a neutrosophic subset $P = \{x, t_P(x), i_P(x),$
$f_P(x) : x \in M\}$ in M is called neutrosophic submodule of M if it satisfies the following;

1. $t_P(0) = 1, i_P(0) = 1, f_P(0) = 0$
2. $t_P(x + y) \geq t_P(x) \wedge t_P(y)$
   $i_P(x + y) \geq i_P(x) \wedge i_P(y)$
   $f_P(x + y) \leq f_P(x) \vee f_P(y),\ \forall x, y \in M$
3. $t_P(rx) \geq t_P(x),\ i_P(rx) \geq i_P(x),\ f_P(rx) \leq f_P(x), \forall\ x \in M, \forall\ r \in R$.

Remark 3.1. The set of all neutrosophic submodules of R-module M represented by $U(M)$.

Example 3.1. [19] Consider the classical ring $R = \mathbb{Z}_4 = \{0, 1, 2, 3\}$. Since each ring is a module on itself, take $R = \mathbb{Z}_4$ as a classical module. Define the single valued neutrosophic set P as follows
$P = \{(0, \langle 1, 1, 0\rangle), (1, \langle 0.6, 0.3, 0.6\rangle), (2, \langle 0.8, 0.1, 0.4\rangle), (3, \langle 0.6, 0.3, 0.6\rangle)\}$. Then the neutrosophic set P is a neutrosophic submodule of M.

Definition 3.2. [19] Let M be an R module. Let $P \in U^M$ where U^Mdenotes the neutrosophic power set of R-module M. Then a neutrosophic subset $P = \{x, t_P(x), i_P(x),$
$f_P(x) : x \in M\}$ in M is called neutrosophic submodule of M if it satisfies the following;

1. $t_P(0) = 1, i_P(0) = 1, f_P(0) = 0$
2. $t_P(x + y) \geq t_P(x) \wedge t_P(y)$
   $i_P(x + y) \geq i_P(x) \wedge i_P(y)$
   $f_P(x + y) \leq f_P(x) \vee f_P(y),\ \forall x, y \in M$
3. $t_P(rx) \geq t_P(x),\ i_P(rx) \geq i_P(x),\ f_P(rx) \leq f_P(x), \forall\ x \in M, \forall\ r \in R.$

Remark 3.2. The set of all neutrosophic submodules of R-module M represented by $U(M)$.

Theorem 3.1. [19] Let P be a neutrosophic set of M. Then $P \in U(M)$ if and only if the following properties are satisfied $\forall\ x, y \in M,\ r, s \in R$
$i)\ t_P(0) = 1, i_P(0) = 1, f_P(0) = 0$
$ii)\ t_P(rx + sy) \geq t_P(x) \wedge t_P(y);\ i_P(rx + sy) \geq i_P(x) \wedge i_P(y);$
$f_P(rx + sy) \leq f_P(x) \vee f_P(y)$".

Definition 3.3. For any neutrosophic subset $P = \{(x, t_P(x), i_P(x), f_P(x)) : x \in X\}$ of X, the support P^* of the neutrosophic set P can be defined as

$$P^* = \{x \in X, t_P(x) > 0, i_P(x) > 0, f_P(x) < 1\}.$$

Proposition 3.1. Let $P,\ Q \in U^X$. If $P \subseteq Q$, then $P^* \subseteq Q^*$.

Proof. Given that $P \subseteq Q$, then $t_P(x) \leq t_Q(x) : i_P(x) \leq i_Q(x) : f_P(x) \geq f_Q(x)\ \forall x \in X$. Consider $x \in P^*$, then , $t_P(x) > 0, i_P(x) > 0, f_P(x) < 1$. So it can conclude that

$$t_Q(x) \geq t_P(x) > 0$$
$$i_Q(x) \geq i_P(x) > 0$$
$$f_Q(x) \leq f_P(x) < 1$$

So that $x \in Q^*$, $\therefore\ P^* \subseteq Q^*$. Hence proved. ☐

Proposition 3.2. Let $P = \{x, t_P(x), i_P(x), f_P(x); x \in M\}$ be a neutrosophic set of M, then $t_{rP}(rx) \geq t_P(x), i_{rP}(rx) \geq i_P(x)$ and $f_{rP}(rx) \leq f_P(x)$.

Proof. Consider

$$t_{rP}(rx) = \vee\{t_P(y) : y \in M, rx = ry\} \geq t_P(x), \forall\ x \in M$$

Similarly $i_{rP}(rx) \geq i_P(x)$. Also

$$f_{rP}(rx) = \wedge\{f_P(y) : y \in M, rx = ry\} \leq f_P(x), \forall\ x \in M$$

Hence the proof. ☐

Proposition 3.3. If $P, Q \in U^M$, then $\forall x, y \in M,\ r, s \in R$

1. $t_{(rP+sQ)}(rx + sy) \geq t_P(x) \wedge t_Q(y)$
2. $i_{(rP+sQ)}(rx + sy) \geq i_P(x) \wedge i_Q(y)$
3. $f_{(rP+sQ)}(rx + sy) \leq f_P(x) \wedge f_Q(y)$

Proof. 1. Consider

$$\begin{aligned} t_{(rP+sQ)}(rx + sy) &= \bigvee\{t_{rP}(\vartheta_1) \wedge t_{sQ}(\vartheta_2) : \vartheta_1, \vartheta_2 \in M, \vartheta_1 + \vartheta_2 = rx + sy\} \\ &\geq t_{rP}(rx) \wedge t_{sQ}(sy) \\ &\geq t_P(x) \wedge t_Q(y)\ \forall\ x, y \in M,\ r, s \in R. \end{aligned}$$

2. Same as above.
3. Consider

$$\begin{aligned} f_{(rP+sQ)}(rx + sy) &= \bigwedge\{f_{rP}(\vartheta_1) \vee f_{sQ}(\vartheta_2) : \vartheta_1, \vartheta_2 \in M, \vartheta_1 + \vartheta_2 = rx + sy\} \\ &\leq f_{rP}(rx) \vee f_{sQ}(sy) \\ &\leq f_P(x) \vee f_Q(y)\ \forall\ x, y \in M,\ r, s \in R. \end{aligned}$$

Hence the proof. ☐

Definition 3.4. Let $P_i,\ i \in J$ be an arbitrary non empty family of U^M where $P_i = \{x, t_{P_i}(x), i_{P_i}(x), f_{P_i}(x) : x \in M\}$ for each $i \in J$. Then

$$\sum_{i \in J} P_i = \{x, t_{\sum\limits_{i \in J} P_i}(x), i_{\sum\limits_{i \in J} P_i}(x), f_{\sum\limits_{i \in J} P_i}(x) : x \in M\} \text{ where}$$
$$t_{\sum\limits_{i \in J} P_i}(x) = \vee\{\bigwedge_{i \in J} t_{P_i}(x_i) : x_i \in M, \sum_{i \in J} x_i = x\}\ \forall x \in M$$
$$i_{\sum\limits_{i \in J} P_i}(x) = \vee\{\bigwedge_{i \in J} i_{P_i}(x_i) : x_i \in M, \sum_{i \in J} x_i = x\}\ \forall x \in M$$
$$f_{\sum\limits_{i \in J} P_i}(x) = \wedge\{\bigvee_{i \in J} f_{P_i}(x_i) : x_i \in M, \sum_{i \in J} x_i = x\}\ \forall x \in M$$

in $\sum\limits_{i \in J} x_i$, at most finitely x_i's are not equal to zero.

Proposition 3.4. Let $P_i,\ i \in J$ be an arbitrary non empty family of U^M, then $r(\bigcup_{i \in J} P_i) = \bigcup_{i \in J}(rP_i)\ for\ r \in R$.

Proof. Consider $r\bigcup_{i\in J} P_i = \{x, t_{r\bigcup_{i\in J} P_i}(x), i_{r\bigcup_{i\in J} P_i}(x), f_{r\bigcup_{i\in J} P_i}(x) : x \in M,\ r \in R\}$
Now

$$\begin{aligned} t_{r\bigcup_{i\in J} P_i}(x) &= \begin{cases} \vee\{t_{\bigcup_{i\in J} P_i}(y) & if\ y \in M, x = ry \\ 0 & otherwise \end{cases} \\ &= \begin{cases} \vee\{\underset{i\in J}{\vee} t_{P_i}(y)\} & if\ : y \in M, x = ry \\ 0 & otherwise \end{cases} \\ &= \underset{i\in J}{\vee} t_{rP_i}(x) \\ &= t_{\bigcup_{i\in J} rP_i}(x) \end{aligned}$$

Similarly $i_{r\bigcup_{i\in J} P_i}(x) = i_{\bigcup_{i\in J} rP_i}(x)$
Now

$$\begin{aligned} f_{r\bigcup_{i\in J} P_i}(x) &= \begin{cases} \wedge\{f_{\bigcup_{i\in J} P_i}(y)\} & if\ y \in M, x = ry \\ 1 & otherwise \end{cases} \\ &= \begin{cases} \wedge\{\underset{i\in J}{\wedge} f_{P_i}(y)\} & if\ y \in M, x = ry \\ 1 & otherwise \end{cases} \\ &= \underset{i\in J}{\wedge} f_{rP_i}(x) \\ &= f_{\bigcup_{i\in J} rP_i}(x) \end{aligned}$$

Hence $r(\bigcup_{i\in J} P_i) = \bigcup_{i\in J}(rP_i)\ for\ r \in R.$ □

Definition 3.5. For any $x \in X$, the neutrosophic point $\hat{N}_{\{x\}}$ is defined as $\hat{N}_{\{x\}}(s) = \{s, t_{\hat{N}_{\{x\}}}(s), i_{\hat{N}_{\{x\}}}(s), f_{\hat{N}_{\{x\}}}(s) : s \in X\}$ where

$$\hat{N}_{\{x\}}(s) = \begin{cases} (1,1,0) & x = s \\ (0,0,1) & x \neq s \end{cases}$$

Remark 3.3. Let X be a non empty set. The neutrosophic point $\hat{N}_{\{0\}}$ in X is $\hat{N}_{\{0\}}(x) = \{x, t_{\hat{N}_{\{0\}}}(x), i_{\hat{N}_{\{0\}}}(x), f_{\hat{N}_{\{0\}}}(x) : x \in X\}$ where

$$\hat{N}_{\{0\}}(x) = \begin{cases} (1,1,0) & x = 0 \\ (0,0,1) & x \neq 0 \end{cases}$$

Proposition 3.5. Let $\hat{N}_{\{0\}}$ be the neutrosophic point in X. Then $r\hat{N}_{\{0\}} = \hat{N}_{\{0\}}\ \forall r \in R$.

Proof. Consider $r\hat{N}_{\{0\}}(x) = \{x, t_{r\hat{N}_{\{0\}}}(x), i_{r\hat{N}_{\{0\}}}(x), f_{r\hat{N}_{\{0\}}}(x) : x \in M\}$, where $\forall r \in R$ and

$$\begin{aligned} t_{r\hat{N}_{\{0\}}}(x) &= \vee\{t_{\hat{N}_{\{0\}}}(y) : y \in M, x = ry\} \\ &= \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases} \\ &= t_{\hat{N}_{\{0\}}}(x)\ \forall x \in M \end{aligned}$$

Similarly it can prove that, $i_{r\hat{N}_{\{0\}}}(x) = i_{\hat{N}_{\{0\}}}(x)$

$$\begin{aligned} f_{r\hat{N}_{\{0\}}}(x) &= \wedge\{t_{r\hat{N}_{\{0\}}}(y) : y \in M, x = ry\} \\ &= \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases} \\ &= f_{\hat{N}_{\{0\}}}(x)\ \forall x \in M \end{aligned}$$

Hence for any $r \in R,\ r\hat{N}_{\{0\}} = \hat{N}_{\{0\}}.$ □

Theorem 3.2. Let $P \in U^X$. $P = \hat{N}_{\{0\}}$ if and only if $P^* = \{0\}$.

Proof. If $P = \hat{N}_{\{0\}}$, then $P^* = (x \in X, t_P(x) > 0, i_P(x) > 0, f_P(x) < 1) = \{0\}$.
Conversely, if $P^* = \{0\} \Rightarrow t_P(0) > 0, i_P(0) > 0, f_P(x) < 1$ and $t_P(x) = 0,\ i_P(x) = 0\ and\ f_P(x) = 1\ \forall\ x \neq 0$.Therefore

$$P(x) = \begin{cases} (1,1,0) & x = 0 \\ (0,0,1) & x \neq 0 \end{cases} = \hat{N}_{\{0\}}$$

Hence the proof. □

Definition 3.6. For any neutrosophic subset $P = \{(x, t_P(x), i_P(x), f_P(x)) : x \in X\}$ of X, the support P^* of the neutrosophic set P can be defined as

$$P^* = \{x \in X, t_P(x) > 0, i_P(x) > 0, f_P(x) < 1\}.$$

Proposition 3.6. Let $P,\ Q \in U^X$. If $P \subseteq Q$, then $P^* \subseteq Q^*$.

Proof. Given that $P \subseteq Q$, then $t_P(x) \leq t_Q(x) : i_P(x) \leq i_Q(x) : f_P(x) \geq f_Q(x)\ \forall x \in X$. Consider $x \in P^*$, then , $t_P(x) > 0, i_P(x) > 0, f_P(x) < 1$. So it can conclude that

$$t_Q(x) \geq t_P(x) > 0$$
$$i_Q(x) \geq i_P(x) > 0$$
$$f_Q(x) \leq f_P(x) < 1$$

So that $x \in Q^*$, $\therefore\ P^* \subseteq Q^*$. Hence the proof. □

Definition 3.7. Let $P \in U^X$. If for all $\beta \in [0,1]$, the β-level sets of of P, can be denoted and defined as $P_\beta = \{x \in X : t_P(x) \geq \beta, i_P(x) \geq \beta, f_P(x) \leq \beta\}$ and the strict β level sets of P can be denoted and defined as $P^*_\beta = \{x \in X : t_P(x) > \beta, i_P(x) > \beta, f_P(x) < \beta\}$.

Proposition 3.7. Let $P_i,\ i \in J$ be an arbitrary non empty family of U^X. Then for any $\beta \in [0,1]$, then

1. $\bigcap_{i \in J} (P_i)_\beta = (\bigcap_{i \in J} P_i)_\beta$
2. $\bigcup_{i \in J} (P_i)_\beta \subseteq (\bigcup_{i \in J} P_i)_\beta$

Proof. 1. Consider

$$\begin{aligned} x \in \bigcap_{i \in J} (P_i)_\beta &\Leftrightarrow x \in (P_i)_\beta \forall i \in J \\ &\Leftrightarrow t_{P_i}(x) \geq \beta,\ i_{P_i}(x) \geq \beta,\ f_{P_i}(x) \leq \beta\ \forall\ i \in J \\ &\Leftrightarrow \underset{i \in J}{\wedge} t_{P_i}(x) \geq \beta,\ \underset{i \in J}{\wedge} i_{P_i}(x) \geq \beta,\ \underset{i \in J}{\vee} f_{P_i}(x) \leq \beta \\ &\Leftrightarrow t_{\bigcap_{i \in J} P_i}(x) \geq \beta,\ i_{\bigcap_{i \in J} P_i}(x) \geq \beta,\ f_{\bigcap_{i \in J} P_i}(x) \leq \beta \\ &\Leftrightarrow x \in (\bigcap_{i \in J} P_i)_\beta \end{aligned}$$

2. Consider

$$\begin{aligned} x \in \bigcup_{i \in J} (P_i)_\beta &\Rightarrow x \in (P_j)_\beta \text{ for some } j \in J \\ &\Rightarrow t_{P_j}(x) \geq \beta,\ i_{P_j}(x) \geq \beta,\ f_{P_j}(x) \leq \beta \\ &\Rightarrow \underset{i \in J}{\vee} t_{P_i}(x) \geq \beta,\ \underset{i \in J}{\vee} i_{P_i}(x) \geq \beta,\ \underset{i \in J}{\wedge} f_{P_i}(x) \leq \beta \\ &\Rightarrow x \in (\bigcup_{i \in J} P_i)_\beta \end{aligned}$$

Hence the proof. □

4 Neutrosophic Submodule of an R-module M

This section explains the characteristics of neutrosophic submodules of an R-module and some associated results and theorems.

Theorem 4.1. Let $P \in U(M)$. Then P^* is a submodule of M.

Proof. Given $P \in U(M)$ and $P^* = \{x \in M, t_P(x) > 0, i_P(x) > 0, f_P(x) < 1\}$. Let $x, \theta \in P^*$. Then

$$t_P(x) > 0, i_P(x) > 0, f_P(x) < 1$$
$$t_P(\theta) > 0, i_P(\theta) > 0, f_P(\theta) < 1$$

To prove that $rx + s\theta \in P^*$ $where$ $r, s \in R$.
i.e. to prove that $t_P(rx + s\theta) > 0,\ i_P(rx + s\theta) > 0,\ and\ f_P(rx + s\theta) < 1$.
Now

$$\begin{aligned} t_P(rx + s\theta) &\geq t_P(rx) \wedge t_P(s\theta) \\ &\geq t_P(x) \wedge t_P(\theta) > 0. \end{aligned}$$

The remaining two inequalities can be proved in the same way.
Hence P^* is a submodule of M . □

Theorem 4.2. Let $P_i,\ i \in J$ be an arbitrary non empty family of $U(M)$, then $\bigcap_{i\in J} P_i \in U(M)$.

Proof. Consider $\bigcap_{i\in J} P_i = \{x, t_{\bigcap_{i\in J} P_i}(x), i_{\bigcap_{i\in J} P_i}(x), f_{\bigcap_{i\in J} P_i}(x) : x \in M\}$ and $t_{\bigcap_{i\in J} P_i}(0) = \underset{i\in J}{\wedge} t_{P_i}(0) = 1;\ i_{\bigcap_{i\in J} P_i}(0) = \underset{i\in J}{\wedge} i_{P_i}(0) = 1;\ f_{\bigcap_{i\in J} P_i}(0) = \underset{i\in J}{\vee} f_{P_i}(0) = 0$ Now, $\forall\ x, y \in M, r, s \in R$

$$\begin{aligned} t_{\bigcap_{i\in J} P_i}(rx + sy) &= \underset{i\in J}{\wedge} t_{P_i}(rx + sy) \\ &\geq \underset{i\in J}{\wedge}(t_{P_i}(x) \wedge t_{P_i}(y)) \\ &= [\underset{i\in J}{\wedge} t_{P_i}(x)] \wedge [\underset{i\in J}{\wedge} t_{P_i}(y)] \\ &= t_{\bigcap_{i\in J} P_i}(x) \wedge t_{\bigcap_{i\in J} P_i}(y) \end{aligned}$$

in the same way it can derive

$$i_{\bigcap_{i\in J} P_i}(rx + sy) \geq i_{\bigcap_{i\in J} P_i}(x) \wedge i_{\bigcap_{i\in J} P_i}(y)$$
$$f_{\bigcap_{i\in J} P_i}(rx + sy) \leq f_{\bigcap_{i\in J} P_i}(x) \vee f_{\bigcap_{i\in J} P_i}(y)$$

Hence $\bigcap_{i\in J} P_i \in U(M)$ □

Remark 4.1. If $P, Q \in U(M)$, then $P \cap Q \in U(M)$.

Definition 4.1. Let $P_i,\ i \in J$ be an arbitrary non empty family of U^M where $P_i = \{x, t_{P_i}(x), i_{P_i}(x), f_{P_i}(x) : x \in M\}$ for each $i \in J$. Then

$$\sum_{i\in J} P_i = \{x, t_{\underset{i\in J}{\sum} P_i}(x), i_{\underset{i\in J}{\sum} P_i}(x), f_{\underset{i\in J}{\sum} P_i}(x) : x \in M\} \text{ where}$$
$$t_{\underset{i\in J}{\sum} P_i}(x) = \vee\{\underset{i\in J}{\wedge} t_{P_i}(x_i) : x_i \in M, \textstyle\sum_{i\in J} x_i = x\}\ \forall x \in M$$
$$i_{\underset{i\in J}{\sum} P_i}(x) = \vee\{\underset{i\in J}{\wedge} i_{P_i}(x_i) : x_i \in M, \textstyle\sum_{i\in J} x_i = x\}\ \forall x \in M$$
$$f_{\underset{i\in J}{\sum} P_i}(x) = \wedge\{\underset{i\in J}{\vee} f_{P_i}(x_i) : x_i \in M, \textstyle\sum_{i\in J} x_i = x\}\ \forall x \in M$$

in $\underset{i\in J}{\sum} x_i$, at most finitely x_i's are not equal to zero.

Theorem 4.3. If $P, Q \in U(M)$, then $P + Q \in U(M)$.

Proof. It is enough to prove that $P + Q$ satisfies the following conditions $\forall x, y \in M, r, s \in R$

1. $t_{P+Q}(0) = 1, i_{P+Q}(0) = 1, f_{P+Q}(0) = 0$
2. $t_{P+Q}(rx + sy) \geq t_{P+Q}(x) \wedge t_{P+Q}(y),\ i_{P+Q}(rx + sy) \geq i_{P+Q}(x) \wedge i_{P+Q}(y),$
   $f_{P+Q}(rx + sy) \leq f_{P+Q}(x) \vee f_{P+Q}(y)$

Since $P, Q \in U(M),$ from the definition 2.8, condition 1 is obvious.

Consider

$$\begin{aligned} t_{P+Q}(x) \wedge t_{P+Q}(y) &= \bigvee\{t_P(x_1) \wedge t_Q(x_2) : x = x_1 + x_2\} \wedge \\ &\quad \bigvee\{t_P(y_1) \wedge t_Q(y_2) : y = y_1 + y_2\} \\ &\leq \bigvee\{t_P(rx_1) \wedge t_Q(rx_2) : rx = rx_1 + rx_2\} \wedge \\ &\quad \bigvee\{t_P(sy_1) \wedge t_Q(sy_2) : sy = sy_1 + sy_2\} \\ &= \bigvee\{[t_P(rx_1) \wedge t_P(sy_1)] \wedge [t_Q(rx_2) \wedge t_Q(sy_2)] \\ &\quad : rx = rx_1 + rx_2, sy = sy_1 + sy_2\} \\ &\leq \bigvee\{t_P(rx_1 + sy_1) \wedge t_Q(rx_2 + sy_2) \\ &\quad : rx + sy = rx_1 + sy_1 + rx_2 + sy_2\} \\ &= t_{P+Q}(rx + sy) \; where \; rx + sy = r(x_1 + x_2) + s(y_1 + y_2) \\ &\quad \forall\, x_1, x_2, y_1, y_2 \in M \end{aligned}$$

Similarly, $i_{P+Q}(rx + sy) \geq i_{P+Q}(x) \wedge i_{P+Q}(y)$; $f_{P+Q}(rx + sy) \leq f_{P+Q}(x) \vee f_{P+Q}(y)$
$\therefore\ P + Q \in U(M)$. □

Corollary 4.3.1. Let $P_i,\ i \in J$ be a family of neutrosophic submodules of an R-module M. Then $\sum_{i\in J} P_i \in U(M)$.

Corollary 4.3.2. Let $P, Q \in U(M)$, then
1. $(P + Q)^* = P^* + Q^*$
2. $(P \cap Q)^* = P^* \cap Q^*$

Theorem 4.4. Let $P \in U^M$. Then $P \in U(M) \Leftrightarrow P$ hold the following

1. $\hat{N}_{\{0\}} \subseteq P$
2. $rP \subseteq P\ \forall r \in R$
3. $P + P \subseteq P$

Proof. Consider $P \in U(M)$

1. Consider $\hat{N}_{\{0\}}(x) = \{x, t_{\hat{N}_{\{0\}}}(x), i_{\hat{N}_{\{0\}}}(x), f_{\hat{N}_{\{0\}}}(x) : x \in M\}$ where

$$\hat{N}_{\{0\}}(x) = \begin{cases} (1,1,0) & x = 0 \\ (0,0,1) & x \neq 0 \end{cases}$$

Then obviously, $t_{\hat{N}_{\{0\}}}(x) \leq t_P(x),\ i_{\hat{N}_{\{0\}}}(x) \leq i_P(x)\ and\ f_{\hat{N}_{\{0\}}}(x) \geq f_P(x)\ \forall\, x \in M$
Hence $\hat{N}_{\{0\}} \subseteq P$.

2. Consider $rP = \{x, t_{rP}(x), i_{rP}(x), f_{rP}(x) : x \in M\}$ where

$$\begin{aligned} t_{rP}(x) &= \begin{cases} \vee\{t_P(y)\} & if\ y \in M, x = ry \\ 0 & otherwise \end{cases} \\ &\leq t_P(x)\ \forall x \in M\ [t_P(x) = t_P(ry) \geq t_P(y)] \end{aligned}$$

Similarly $i_{rP}(x) \leq i_P(x),\ f_{rP}(x) \geq f_P(x), \forall x \in M$.
Hence $rP \subseteq P$.

3. Comsider $x \in M, r \in R$

$$\begin{aligned} t_{P+P}(x) &= \vee\{t_P y \wedge t_P(z) : y, z \in M, x = y + z\} \\ &\leq t_P(x)\ \forall x \in M\ [t_A(x) = t_A(y + z) \geq t_A(y) \wedge t_A(z)] \end{aligned}$$

Similarly, $i_{P+P}(x) \leq i_P(x)\ and\ f_{P+P}(x) \geq f_P(x)$
Therefore $P + P \subseteq P$.

Conversely, assume $P \in U^M$ satisfies the given three conditions and to show that $P \in U(M)$.
From the condition 1,
$\hat{N}_{\{0\}} \subseteq P \Rightarrow t_{\hat{N}_{\{0\}}}(x) \leq t_P(x),\ i_{\hat{N}_{\{0\}}}(x) \leq i_P(x)\ a.nd\ f_{\hat{N}_{\{0\}}}(x) \geq f_P(x)\ \forall\ x \in M$
$\Rightarrow t_P(0) = 1, i_P(0) = 1\ and\ f_P(0) = 0.$
Now for $x, y \in M$,
From condition 3, $P + P \subseteq P$

$$\begin{aligned} t_P(x+y) &\geq t_{P+P}(x+y) \\ &= \vee\{t_P(z_1) \wedge t_P(z_2) : z_1, z_2 \in M, x+y = z_1 + z_2\} \\ &\geq t_P(x) \wedge t_P(y) \end{aligned}$$

Similarly, $i_P(x+y) \geq i_P(x) \wedge i_P(y),\ f_P(x+y) \leq f_P(x) \vee f_P(y)\ \forall\ x, y \in M.$
Also from condition 2, $\forall\ r \in R,\ x \in M,$

$$\begin{aligned} t_P(rx) &\geq t_{rP}(rx) \\ &= \begin{cases} \vee\{t_P(y)\} & if\ y = rx, rx = ry \\ 0 & otherwise \end{cases} \\ &\geq t_P(x) \end{aligned}$$

Similarly, $i_P(rx) \geq i_P(x)\ and\ f_P(rx) \leq f_P(x).$
Therefore it can conclude, $P \in U(M)$. □

Corollary 4.4.1. Let $P \in U^M$, then $P \in U(M) \Leftrightarrow P$ hold the following

1. $\hat{N}_{\{0\}} \subseteq P$
2. $rP + sP \subseteq P$, $\forall r, s \in R$

5 Conclusion

Neutrosophic submodule is one of the generalizations of the algebraic structure "module" that supplements the classic structure by assigning three diverse level graded features of each module component. This paper presented numerous operations of neutrosophic sets of an R-module M, instigated by the operation addition in M and an action of a ring R on M. The scope and intent of this study are to generalize algebraic structures and to create algebraic neutrosophic structures and find their application. The present study leads to explore the concept of injective and projective neutrosophic submodules of an R-module, semi-simple neutrosophic submodule of an R-module and a quasi neutrosophic submodule of an R-module.

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