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Aim and Scope

International Journal of Neutrosophic Science (IJNS) is a peer-review journal publishing high quality experimental and theoretical research in all areas of Neutrosophic and its Applications. IJNS is published quarterly. IJNS is devoted to the publication of peer-reviewed original research papers lying in the domain of neutrosophic sets and systems. Papers submitted for possible publication may concern with foundations, neutrosophic logic and mathematical structures in the neutrosophic setting. Besides providing emphasis on topics like artificial intelligence, pattern recognition, image processing, robotics, decision making, data analysis, data mining, applications of neutrosophic mathematical theories contributing to economics, finance, management, industries, electronics, and communications are promoted. Variants of neutrosophic sets including refined neutrosophic set (RNS). Articles evolving algorithms making computational work handy are welcome.

Topics of Interest

IJNS promotes research and reflects the most recent advances of neutrosophic Sciences in diverse disciplines, with emphasis on the following aspects, but certainly not limited to:

- Neutrosophic sets
- Neutrosophic algebra
- Neutrosophic topolog
- Neutrosophic graphs
- Neutrosophic probabilities
- Neutrosophic tools for decision making
- Neutrosophic theory for machine learning
- Neutrosophic statistics
- Neutrosophic numerical measures
- Classical neutrosophic numbers
- A neutrosophic hypothesis
- The neutrosophic level of significance
- The neutrosophic confidence interval
- The neutrosophic central limit theorem
- Neutrosophic theory in bioinformatics and medical analytics
- Neutrosophic tools for big data analytics
- Neutrosophic tools for deep learning
- Neutrosophic tools for data visualization
- Quadripartitioned single-valued neutrosophic sets
- Refined single-valued neutrosophic sets
- Applications of neutrosophic logic in image processing
- Neutrosophic logic for feature learning, classification, regression, and clustering

- Neutrosophic knowledge retrieval of medical images
- Neutrosophic set theory for large-scale image and multimedia processing
- Neutrosophic set theory for brain-machine interfaces and medical signal analysis
- Applications of neutrosophic theory in large-scale healthcare data
- Neutrosophic set-based multimodal sensor data
- Neutrosophic set-based array processing and analysis
- Wireless sensor networks Neutrosophic set-based Crowd-sourcing
- Neutrosophic set-based heterogeneous data mining
- Neutrosophic in Virtual Reality
- Neutrosophic and Plithogenic theories in Humanities and Social Sciences
- Neutrosophic and Plithogenic theories in decision making
- Neutrosophic in Astronomy and Space Sciences

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Plithogenic Cognitive Maps in Decision Making

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Abstract

Plithogenic sets introduced by Smarandache (2018) have disclosed new research vistas and this paper introduces the novel concept of plithogenic cognitive maps (PCM) and its applications in decision making. The new approach of defining instantaneous state neutrosophic vector with the confinement of indeterminacy to (0,1] is proposed to quantify the degree of indeterminacy. The resultant vector is obtained by applying instantaneous state vector through the connection matrix together with plithogenic operators comprising the contradiction degrees. The connection matrix is represented as fuzzy matrix and neutrosophic matrix and the resultant vector is determined by applying plithogenic fuzzy operators and plithogenic neutrosophic operators respectively. The obtained results are highly feasible in making the decision as it incorporates the contradiction degree of the conceptual nodes with respect to the dominant node. This research work will certainly pave the way for developing new approaches in decision making using PCM.

Keywords: Plithogenic set, cognitive maps, plithogenic cognitive maps, confinement of indeterminacy, plithogenic fuzzy operators, plithogenic neutrosophic operators.

1.Introduction

Robert Axelrod [1] developed cognitive maps, a decision making tool primarily used in handling the system of making decisions related to political and social frameworks. A cognitive map is a directed graph with nodes and edges representing the concept variables or factors, and it's causal relationships respectively. The intensity of the relationship between two concepts say Ci and Cj is represented by edge weights eij, where eij∈ {-1,0,1}. The value 1 represents the positive influence of Ci over Cj; 0 represents no influence and -1 represents negative influence. The causal relationship between the nodes is represented as a connection matrix. Cognitive maps have a wide range of applications in diverse fields. Nakamura et al [2] used cognitive maps in decision support systems; Chaib-draa [3] developed multi agent system model using cognitive maps; Klein et al [4] developed

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cognitive maps in decision makers and other broad spectrum of its applications in student modeling whilst knowledge management are discussed by Alejandro Pena [5]. One of the limitations of cognitive maps is modeling decision making in uncertain environment. The concept of fuzzy sets introduced by Zadeh [6] was integrated with cognitive maps by Kosko [7]. Fuzzy cognitive maps (FCM) introduced by Kosko [7], he handled the aspects of uncertainty and impreciseness. In FCM, the edge weights eij ∈ [-1,1] and the connection matrix has fuzzy values. The comprehensive nature of FCM has several applications such as but not limited to the pattern recognition see Papakostas et al [8],in the medicine see Abdollah et al [9], in large manufacturing system see Chrysostomos et al [10], in the field of decision making on farming scenarios see Asmaa Mourhir et al [11]. Atannsov [12] introduced intuitionistic fuzzy sets that deal with membership, non-membership and hesitancy values. Elpiniki Papageorgiou[13] extended FCM to Intuitionistic FCM models to apply in medical diagnosis and this gained momentum in the domain of FCM. IFCM are the extension of FCM models , that are highly applied in diverse fields. The connection matrix of iFCM models has intuitionistic values. Hajek et al [14-15] extended iFCM models into interval –valued IFCM for stock index forecasting and supplier selection.

Smarandache [16] introduced neutrosophic sets that deal with truth, indeterminacy and falsity membership functions. Neutrosophic sets are applied in various domain of the natural science. Mohamed Bisher Zeina [17] applied neutrosophic parameters in Erlang Service Queuing Model and developed neutrosophic event-based queuing model. Malath [18]studied the integration of neutrosophic thick function. Salma [19] developed online analytical processing operations via neutrosophic systems. Neutrosophy is also extended to explore new algebraic structures and concepts. Agboola [20] proposed the introduction of neutrogroups and neutrorings, Riad et al [21] constructed neutrosophic crisp semi separation axioms in neutrosophic crisp topological spaces. Necati Olgun [22] discussed refined neutrosophic R-module, Ibrahim [23] explored the concepts of n Refined Neutrosophic Vector Spaces. Mohammad Hamidi [24] discussed Neutro – BCK-algebra. Neutrosophic research is gaining momentum and it has wide spectrum of applications in decision making. Abdel-Baset [25] developed a novel neutrosophic approach in green supplier selection and a novel decision making approach was developed to diagnose heart diseases using neutrosophic sets. Interval – valued neutrosophic sets are also used in decision making. Neutrosophic cognitive maps (NCM) introduced by Vasantha Kandasamy [26] has incorporated the concept of indeterminacy into edge weights. NCMs are also applied to diverse decision making scenarios by many social researchers. NCMs are widely applied to analyze the causal relationship between the concepts of decision making problems. Nivetha et al [27] developed decagonal linguistic neutrosophic FCM to analyze the risk factors of lifestyle diseases. Nivetha et al [28] made a case study on the problems faced by entrepreneurs using NCM.

In NCM models, the influence of one factor over another is represented by either -1,0,1,I, where I represents indeterminacy. Let us consider a decision making problem comprising of five factors and the respective 5×5 connection matrix has the values {-1,0,1,I}. The initial state vector X, [X= (10000)] has first of the factors in ON position and other factors in off position. X is passed into connection matrix and the resulting vector Z , [Z = (a1,a2,a3,a4,a5)] is updated using threshold operation by replacing ai by 1 if ai ≥ g and ai by 0 if ai < g, (g is an integer) and ai by I, if ai is not an integer. The process is repeated until two updated resultant vectors obtained are

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same, which is called as the fixed point or limit cycle. The process ends when the fixed point is obtained. In this NCM procedure suppose the fixed point is (1 0 1 1 I), the inference is the first factor has positive influence on third and fourth factors, no impact on second factor and the fifth factor is indeterminant to it. The existence of indeterminacy in the connection matrix and the fixed point does not give us the complete picture of the decision making, but if indeterminacy is quantified then the decision making will be feasible. To make so, the approach of indeterminacy confinement is introduced in this research work. As the connection matrix is based on expert's opinion, the indeterminacy can also be confined to (0,1] based on the expert's opinion. Also in the instantaneous vector any of the factors are in ON position or combination of factors are in ON position say, X = (1 0 1 01) to see the combined effect of the factors. In this article the factors are kept in indeterminate position and it is confined to give a numerical value. This is a new kind of approach in neutrosophic cognitive maps and the NCMs of this kind can be labeled as novel neutrosophic cognitive maps (NNCM).

The NCM and NNCM can also be extended to plithogenic cognitive maps (PCM). Plithogenic sets introduced by Smarandache [29] are the generalization of crisp sets, fuzzy sets, intuitionistic fuzzy sets and neutrosophic sets. The membership values are mainly used to quantify the qualitative aspects. This principle of quantification of qualitative aspects is used as the underlying principle in the construction of plithogenic sets. The degree of appurtenance and the contradiction degree are the two distinctive aspects of plithogenic sets. The concept of plithogeny is extended to plithogenic hypersoft sets by Smarandache [30]. Plithogenic sets are widely used in decision making. Shazia Rana et al [31] extended plithogenic fuzzy hypersoft set to plithogenic fuzzy whole hypersoft set and developed plithogenic ranking model. Nivetha and Smarandache [32] developed concentric plithogenic hypergraphs. Smarandache [33] developed plithogenic n super hypergraph and a novel decision making approach is proposed by Smarandache and Nivetha [34]. Abdel - Baset [35] framed a hybrid plithogenic decision-making approach with quality function deployment for selecting supply chain sustainability. The compatibility of the plithogenic sets motivated us to incorporate the concept of plithogeny to coginitve maps.

This research work proposes the approach of integrating plithogeny to cognitive maps to develop PCM models as the extension of NNCM, NCM, IFCM and FCM. The PCM model follows the underlying methodology of FCM but it incorporates contradiction degree to the factors of the decision making problem. If any of the factors is in ON position, then it becomes the dominant factor and the contradiction degree of the dominant factor with respect to other factors is considered. The instantaneous vector is passed into connection matric and the resultant vector is obtained by applying plithogenic operators. The resultant vector is updated by using the conventional threshold function. PCM models are classified as cognitive maps if the connection matrix is crisp; fuzzy cognitive maps if the connection matrix has fuzzy values; intuitionistic fuzzy cognitive maps if the connection matrix has intuitionistic values and neutrosophic cognitive maps if the connection matrix has neutrosophic values. Thus the proposed PCM models are the generalization of the earlier forms of FCM models. The incorporation of the contradiction degrees will certainly give us new insights in decision making.

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The paper is organized as follows: section 2 presents the outlook of PCM; section 3 consists of the methodology of PCM; section 4 comprises of application of PCM in decision making; section 5 discusses the results and concludes the work.

2. Plithogenic Cognitive Maps

A plithogenic cognitive map is a directed graph consisting of nodes and edges representing the concepts and its causal relationship respectively . The contradiction degree of the nodes with respect to the dominant node is determining the fixed point.

Let P_1, P_2,..P_n denotes n nodes of PCM. The directed edge from P_i to P_j represents the association between the two nodes and the edge weights illustrate the intensity of the association between the nodes. If the edge weight $e_{ij} \in \{-1,0,1\}$ then it is plithogenic crisp cognitive maps; if $e_{ij} \in [-1,1]$ then it is plithogenic fuzzy cognitive maps; if $e_{ij} \in \rho([0,1]^2)$ then it is plithogenic intuitionistic cognitive maps , if $e_{ij} \in \rho([0,1]^3)$ then it is plithogenic neutrosophic cognitive maps. Plithogenic connection matrix or adjacency matrix $P(E) = (e_{ij})$ represents the relation between the nodes. An instantaneous state vector in PCM of the form $A = (a_1, a_2,..a_n)$ represents the ON-OFF-indeterminate position of the node at an instant of time. If $a_i = 1$ refer to (ON state); $a_i = 0$ refer to (OFF state) and $a_i = I$ means (Indeterminate state). In PCM, the indeterminate state I is confined to a value belonging to (0,1], which is the extension of NCM.

Let P1,P2,P3 be three nodes of PCM. Let P1 be in ON position and P2, P3 be in off state, then the node P1 is considered to be dominant. The contradiction degrees of other nodes with respect to dominant node are

P1	P2	P3
0	1/3	2/3

The contradiction degree represents the extent of distinctiveness between the two concepts .The value 1/3 and 2/3 is assigned to P2 or P3 based on the perception of decision makers who have choosen the factors of PCM.

The instantaneous vector X = (1 0 0) is passed into P(E) and the vector which is obtained by applying plithogenic operators is Y. The assumed threshold operation is applied to Y and the resultant vector G is obtained. The recurrence of passing the resultant vector G to P(E) if results in repetition of resultant vectors then the limit cycle of the PCM is obtained and the resultant vector is called as fixed point.

3. Methodology of Plithogenic Cognitive Maps

This section presents the algorithm of obtaining the limit cycle of PCM.

Step 1: The factors P_1, P_2,..P_n or the concepts of the decision making problem are decided based on the expert's opinion.

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Step 2: The plithogenic connection matrix P(E) of dimension n (the number of concepts) is obtained from the causal relationship between the concepts.

Step 3: The edge weight e_{ij} may belong to {1,0,1}, [-1,1], $\rho([0,1]^2)$, $\rho([0,1]^3)$. The nature of the edge weights determines the type of plithogenic cognitive maps.

Step 4: To determine the effect of one concept say P1, is kept in ON position and the contradiction degree with respect to other concepts are determined.

Step 5: The instantaneous state vector X = (1 0 0 0 0 …0) is passed into connection matrix and by applying the plithogenic operators, a resultant vector is obtained, and updated by applying the threshold operation by assigning 1 to the values (a_i) greater than k, 0 to the values (a_i) lesser than k, where k - a is a suitable positive integer. In this proposed approach the on position of the concepts is threshold with 1 and the indeterminate position of the concepts will be confined with the value C. The value 0 is assigned to the values lesser than 1 and the value 1 is assigned to the values greater than 1.

The plithogenic operators are defined as

$a \wedge_p b = (1-c).[a \wedge_F b] + c.[a \vee_F b]$, where c represents contradiction degree, $a \wedge_F b$ is the t_{norm} defined by ab and $a \vee_F b$ is the t_{conorm} defined by a + b - ab

The plithogenic new neutrosophic operators are defined as

$$a \wedge_p b = \prec a_1 \wedge_p b_1, \frac{1}{2}\left[(a_2 \wedge_p b_2) + (a_2 \vee_p b_2)\right], a_3 \vee_p b_3 \succ,$$

where $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$, $a \vee_p b = (1-c).[a \vee_p b] + c.[a \wedge_p b]$

Step 6 : The updated vector is passed into P(E) and the process is repeated until the fixed point is arrived. The fixed point is the limit cycle of PCM.

4. Application of Plithogenic Cognitive Maps in Decision Making

This section presents the application of plithogenic cognitive maps in decision making. Let us consider a decision making environment where the expert's opinion is constructed to promote the farming sectors to a progressive phase with their suggestive strategies. The following proposed strategies of the experts are taken as the nodes of the PCM.

P1 Encouraging the reverse migration by helping the socially mobilized groups with credit flow.

P2 Supporting vulnerable farming areas with community driven approach

P3 Promoting Farmer's Productive Organizations as transformative agents

P4 Perpetuating gender equalities to create new opportunities for women

P5 Effective use of modern ICT to connect farmers with extension, market and continuous learning

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The causal association between the concepts is represented as linguistic variables and it is quantified by triangular fuzzy numbers and the kind of PCM is plithogenic fuzzy cognitive map which is used to determine the fixed point of the dynamical system.

The plithogenic fuzzy connection matrix with linguistic variables is presented as below

$$\begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{array} \begin{pmatrix} P_1 & P_2 & P_3 & P_4 & P_5 \\ 0 & M & M & L & L \\ L & 0 & H & M & L \\ H & H & 0 & M & H \\ M & M & L & 0 & M \\ H & H & VH & M & 0 \end{pmatrix}$$

The linguistic variables are quantified by using triangular fuzzy numbers as in Table 4.1

Table 4.1 Quantification of Linguistic variable

Linguistic Variable	**Triangle Fuzzy Number**	**Crisp value**
Very Low	(0,0.1,0.2)	0.1
Low	(0.2,0.3,0.4)	0.3
Medium	(0.4,0.5,0.6)	0.5
High	(0.6,0.7,0.8)	0.7
Very High	(0.8,0.9,1)	0.9

The modified plithogenic fuzzy connection matrix P(E) is

$$\begin{array}{c} \\ P1 \\ P2 \\ P3 \\ P4 \\ P5 \end{array} \begin{pmatrix} P1 & P2 & P3 & P4 & P5 \\ 0 & 0.5 & 0.5 & 0.3 & 0.3 \\ 0.3 & 0 & 0.7 & 0.5 & 0.3 \\ 0.7 & 0.7 & 0 & 0.5 & 0.7 \\ 0.5 & 0.5 & 0.3 & 0 & 0.5 \\ 0.7 & 0.7 & 0.9 & 0.5 & 0 \end{pmatrix}$$

The graphical representation of the causal association between the concepts are represented in Fig.4.1

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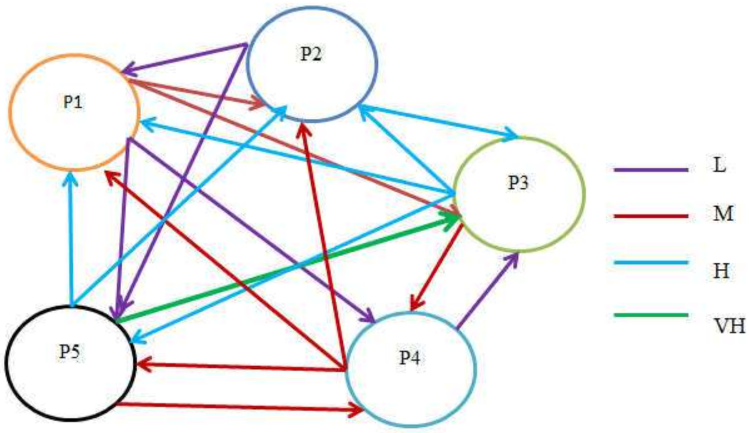

Fig.4.1 Graphical representation of the causal relationship

Case (i) Conventional FCM [7]

Let us consider the conventional approach of FCM without the incorporation of contradiction degree. Let X = (1 0 0 0 0)

X* P(E) = (0 0.5 0.5 0.3 0.3) → (1 0.5 0.5 0.3 0.3) = $X1$

X1 * P(E) = (1.65 2 2.14 1.6 1.25) → (1 1 1 1 1) = X2

X2* P(E) = (2.2 2.4 2.4 1.8 1.8) → ((1 1 1 1 1) = X3

X2 = X3 ------------------------------ (1)

Case (ii) Plithogenic Fuzzy FCM with ON/OFF state of vectors

Let us consider the concept P1 to be in ON position and other factors in off position. The contradiction degrees of the dominant node with respect to other nodes are

P1	P2	P3	P4	P5
0	1/5	2/5	3/5	4/5

Let us consider the instantaneous state vector as X = (1 0 0 0 0)

X *p P(E) = Y , where Y = (a b c d e)

a = Max [1 $\wedge_p$ 0 , 0 $\wedge_p$ 0.3 , 0 $\wedge_p$ 0.7, 0 $\wedge_p$ 0.5, 0 $\wedge_p$ 0.7]

b = Max [1 $\wedge_p$ 0.5 , 0 $\wedge_p$ 0 , 0 $\wedge_p$ 0.7, 0 $\wedge_p$ 0.5, 0 $\wedge_p$ 0.7]

c = Max [1 $\wedge_p$ 0.5 , 0 $\wedge_p$ 0.7 , 0 $\wedge_p$ 0, 0 $\wedge_p$ 0.3, 0 $\wedge_p$ 0.9]

d= Max [1 $\wedge_p$ 0.3 , 0 $\wedge_p$ 0.5 , 0 $\wedge_p$ 0.5, 0 $\wedge_p$ 0, 0 $\wedge_p$ 0.5]

e= Max [1 $\wedge_p$ 0.3 , 0 $\wedge_p$ 0.3 , 0 $\wedge_p$ 0.7, 0 $\wedge_p$ 0.5, 0 $\wedge_p$ 0]

X*p P(E) = (0 0.6 0.7 0.72 0.94) → (1 0.6 0.7 0.72 0.94) = X1

X1 *p P(E) = (0.602 0.6732 0.8588 0.73 0.86) → (1 0.67 0.86 0.73 0.86) = X2

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X2 *p P(E) = (0.602 0.6732 0.8588 0.73 0.8868) → (1 0.67 0.86 0.73 0.89) = X3

X3 *p P(E) = (0.623 0.6918 0.8762 0.745 0.8868) → (1 0.69 0.88 0.75 0.89) = X4

X4 *p P(E) = (0.623 0.6918 0.8762 0.745 0.8944) → (1 0.69 0.88 0.75 0.89) = X5

X4 = X5 ----------------------- (2)

Case (iii) Plithogenic Fuzzy FCM with ON/OFF and confined indeterminate I_C state of vectors

Let us consider the concept P1 to be in ON position, P2 be in indeterminate state and other factors in off position. The indeterminate state of the vector here reflects the impact on the concept P2.The contradiction degrees of the dominant node with respect to other nodes are considered as the same.

Let us consider the instantaneous state vector as X = (1 I_C 0 0 0), C = 0.25, the value of indeterminacy is 0.25. i.e X = (1 0.25 0 0 0)

X *p P(E) = (0.075 0.6 0.7 0.72 0.86) → (1 0.25 0.7 0.72 0.86) = X1

X1*p P(E) = (0.602 0.6732 0.8588 0.73 0.86) → (1 0.25 0.86 0.73 0.86) = X2

X2*p P(E) = (0.602 0.673 0.859 0.73 0.887) → (1 0.25 0.86 0.73 0.89) = X3

X3*p P(E) = (0.623 0.6918 0.8762 0.745 0.8868) → (1 0.25 0.88 0.75 0.89) = X4

X4 *p P(E) = (0.623 0.6918 0.8762 0.745 0.8944) → (1 0.25 0.88 0.75 0.89) = X5

X4 = X5 ----------------------- (3)

Let X = (1 I_C 0 0 0), C = 0.5, the value of indeterminacy is 0.5. i.e X = (1 0.5 0 0 0)

The fixed point is (1 0.25 0.88 0.75 0.89) ------------ (4)

Let X = (1 I_C 0 0 0), C = 0.75, the value of indeterminacy is 0.5. i.e X = (1 0.5 0 0 0)

The fixed point is (1 0.75 0.88 0.75 0.89) ------------ (5)

Let X = (1 I_C 0 0 0), C = 0.95, the value of indeterminacy is 0.5. i.e X = (1 0.5 0 0 0)

The fixed point is (1 0.75 0.86 0.78 0.86) ------------ (6)

Eq. (1) states that the concept P1 has influence on the other concepts, but Eq. (2) states the extent of influence of the concept P1 on other concepts which is the added advantage of using contradiction degree. The confined indeterminate state for various values of c results in different fixed points. The confinement of indeterminacy to the values C= 0.25, 0.5,0.75 have same impact on other factors and also it produce the same effect as P2 concept in OFF state, but as the value of indeterminacy is enhanced , slight variation in the impact are found in Eq. (6). This shows that the OFF state and the indeterminate state of concept P2, when the concept P1 being in ON position produce no much difference.

Case (iv). Plithogenic Neutrosophic FCM with new neutrosophic state.

The new neutrosophic instantaneous state vector is of the form ((T_{P1}, I_{P1}, F_{P1}), 0, 0, 0, 0), where the truth (T_{P1}), indeterminacy (I_{P1}) and falsity (F_{P1}) of the concept P1 to be in ON position is expressed. This representation of the ON position of the vector is highly a pragmatic representation. If I_{P1} and F_{P1} are zero then the concept P1 is highly certain to be in ON position. The indeterminate and off state of the concept P1 can be expressed by keeping the values of I_{P1} and F_{P1} to 1 and keeping the other respective set of value to be zero.

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The new neutrosophic state vector (X_N) when the On state of the concept P1 is considered.((1,0,0) (0,1,1) (0,1,1) (0,1,1) (0,1,1) (0,1,1)).The plithogenic neutrosophic matrix $P_N(E)$ is

	P1	P2	P3	P4	P5
P1	(0,1,1)	(0.6,0.3,0.4)	(0.6,0.3,0.4)	(0.3,0.4,0.7)	(0.3,0.4,0.7)
P2	(0.3,0.4,0.7)	(0,1,1)	(0.7,0.2,0.2)	(0.6,0.3,0.4)	(0.3,0.4,0.7)
P3	(0.7,0.2,0.2)	(0.7,0.2,0.2)	(0,1,1)	(0.6,0.3,0.4)	(0.7,0.2,0.2)
P4	(0.6,0.3,0.4)	(0.6,0.3,0.4)	(0.3,0.4,0.7)	(0,1,1)	(0.6,0.3,0.4)
P5	(0.7,0.2,0.2)	(0.7,0.2,0.2)	(0.9,0.1,0.1)	(0.6,0.3,0.4)	(0,1,1)

The plithogenic representation of the causal relationship between the concepts is presented in Fig.4.2. The positive sign indicates the positive impacts of the concepts, and it is represented as neutrosophic values in $P_N(E)$.

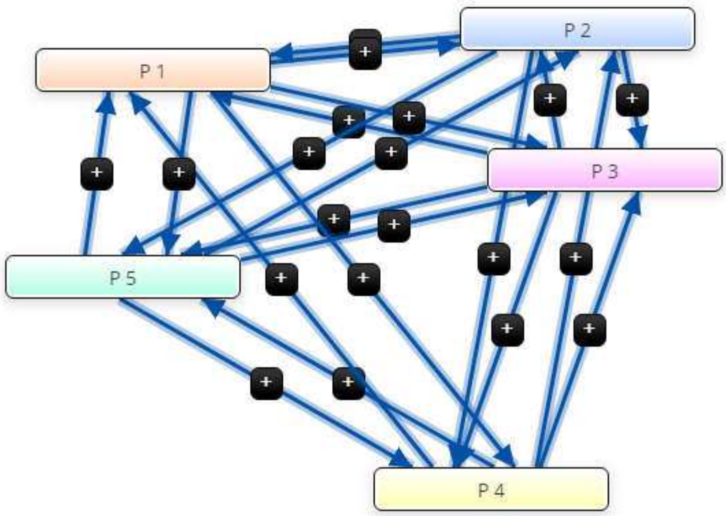

Fig.4.2 Representation of Plithogenic association between the concepts

The plithogenic operators are used to obtain the resultant vector.

Let X_N = ((1,0,0), (0,1,1), (0,1,1), (0,1,1), (0,1,1))

X_N*p P_N (E) = ((0, 0 ,0) (0.68,0.19,0),(0.76,0.18,0) (0.72,0.26,0) (0.86,0.23,0))

→ ((1, 0 ,0) (0.68,0.19,0),(0.76,0.18,0) (0.72,0.26,0) (0.86,0.23,0)) = X_{N1}

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X_{N1}*p P_N (E) = ((0.602, 0.118,0) (0.68,0.1288,0.1088) (0.76,0.1396,0.1632) (0.84,0.1504,0.1632) (0.92 0.16 0.1088)) → ((1, 0 ,0) (0.68,0.1288,0.1088) (0.76,0.1396,0.1632) (0.84,0.1504,0.1632) (0.92 0.16 0.1088)) = X_{N2}

X_{N2}*p P_N (E) = ((0.644 0.15952 0) (0.7104 0.162592 0.1088) (0.7768, 0.165664 0.1632) (0.8432 0.168736 0.1632) (0.92 0.171808 0.1088)) →((1, 0 ,0) (0.7104 0.162592 0.1088) (0.7768, 0.165664 0.1632) (0.8432 0.168736 0.1632) (0.92 0.171808 0.1088)) = X_{N3}

X_{N3}*p P_N (E) = ((0.644 0.1607008 0) (0.7104 0.16448128 0.113664) (0.7768 0.16826176 0.113664) (0.7768 0.16826176 0.170496) (0.8432 0.17204224 0.170496) (0.92 0.17582272 0.113664)) → ((1, 0 ,0) (0.710 0.164 0.114) (0.777 0.168 0.114) (0.777 0.168 0.170) (0.843 0.172 0.170) (0.92 0.176 0.114)) = X_{N4}

By repeating in the same fashion,

X_{N5} = ((1, 0 ,0) (0.71, 0.167,0.114) (0.777, 0.171, 0.17) (0.843, 0.174, 0.17) (0.91 ,0.178, 0.114)

X_{N6} = ((1, 0 ,0) (0.71, 0.167,0.114) (0.777, 0.171 ,0.17) (0.843, 0.174, 0.17) (0.91, 0.178 ,0.114)

$X_{N5} = X_{N6}$

Thus the neutrosophic impact of the concept P1 on other factors is determined. The various kinds of plithogenic cognitive maps are discussed in different cases and in each case, the impact of the concept P1 over the other concepts is determined. In section 4 various cases are discussed and the differences between cognitive maps (CM), fuzzy cognitive maps (FCM), intuitionistic cognitive maps(IFCM) , neutrosophic cognitive maps(NCM) and plithogenic cognitive maps (PCM) based on edge weights (e_{ij}) are presented in Table 4.2.

Table 4.2. Differences beteween CM, FCM, IFCM,NCM and PCM

Cognitive Maps	$e_{ij} \in \{-1,0,1\}$ with no contradiction degree between the concepts
Fuzzy cognitive maps	$e_{ij} \in [-1,1]$ with no contradiction degree between the concepts
Intuitionistic cognitive maps	e_{ij} = (μ, ν)where μ, the membership value and ν , the non-membership value with no contradiction degree between the concepts
Neutrosophic Cognitive Maps	$e_{ij} \in \{-1,0,1, I\}$ with no contradiction degree between the concepts
Plithogenic Cognitive Maps	$e_{ij} \in \{-1,0,1\}$ or $e_{ij} \in [-1,1]$ or $e_{ij} \in \rho([0,1]^2)$ or $e_{ij} \in \rho([0,1]^3)$ with contradiction degree between the concepts

5. Conclusion

This research work proposes the concept of plithogenic cognitive maps and new neutrosophic maps. The integration of contradiction degree with the plithogenic operators is applied to determine the resultant vector. Several kinds of plithogenic cognitive maps are discussed in this article and it is validated with applications in decision making. The proposed plithogenic cognitive maps can be applied in decision making scenario. The state of

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indeterminacy of the concept is quantified by various confinement values which will certainly assist in making optimal decisions. Plithogenic cognitive maps can be also developed to various representations based on the characterisation of the decision making environment. PCM decision making models can be extended to interval-valued plithogenic cognitive maps and also it can be integrated to multi criteria decision making. The association between the concepts of decision making can be represented in terms of plithogenic hypersoft set. Plithogenic hypergraph can also be integrated with plithogenic cognitive maps to formulate novel and feasible decision making models.

References

[1] Axelrod, R., Structure of Decision: The Cognitive Maps of Political Elites. Princeton,1976.

[2] Nakamura, K., S. Iwai and T. Sawaragi., "Decision Support Using Causation Knowledge Base", J. IEEE Transactions on Systems, Man and Cybernetics, Vol.12, pp.765-777,1982.

[3] Chaib-draa, B., "Causal Maps: Theory, Implementation, and Practical Applications in Multiagent Environments", J. IEEE Transactions on Knowledge and Data Engineering, Vol. 14, 2002.

[4] Klein, J.L., D.F. Cooper, "Cognitive Maps Decision Markers in a Complex Game", J. of Operational Research Soc., Vol.2, pp -377-393,1982.

[5] Alejandro Pena1.,Humberto Sossa., Agustin Gutierrez., "Cognitive Maps: an Overview and their Application for Student Modeling", Computacion y Sistemas Vol. 10 No. 3, pp -230-250, 2007.

[6] Zadeh, L. A. "Fuzzy set", Inform and Control 8, 338-353,1965.

[7] Kosko, B., Fuzzy Cognitive Maps. Int. J. of Man-Machine Studies, V. 24, 1986, 65-75.

[8] Papakostas, G. A. , Boutalis, Y. S., Koulouriotis, D. E. ,Mertzios, B. G., "Fuzzy cognitive maps for pattern recognition applications", International Journal of Pattern Recognition and Artificial IntelligenceVol. 22, No. 08, pp. 1461-1486,2008.

[9] Abdollah Amirkhania.,Elpiniki I.Papageorgioub.,AkramMohsenia.,Mohammad R.Mosavia., "A review of fuzzy cognitive maps in medicine: Taxonomy, methods, and applications'', Computer Methods and Programs in Biomedicine,Vol.142, pp – 129-145,2017.

[10] Chrysostomos ,D.,StyliosPeter P.Groumpos, "Application of Fuzzy Cognitive Maps in Large Manufacturing Systems", IFAC Proceedings Volumes,Vol. 31, pp-521-526,1998.

[11] Asmaa Mourhir.,Elpiniki I Papageorgiou., Konstantinos Nikolaos Kokkinos.,Tajje-eddine Rachidi., "Exploring Precision Farming Scenarios Using Fuzzy Cognitive Maps", Sustainability , Vol.9,pp-1-23,2017.

[12] Atanassov, K.T. "Intuitionistic fuzzy sets". Fuzzy Sets Syst. 20, 87–96,1983.

[13] Elpiniki I Papageorgiou.,Dimitris Iakovidis., "Intuitionistic Fuzzy Cognitive Maps", IEEE Transactions on Fuzzy Systems. Vol 21, pp -342-354,2013.

DOI:10.5281/zenodo.3950063

[14] Hajek.P., Prochazka,O., Froelich,W., "Interval-valued intuitionistic fuzzy cognitive maps for stock index forecasting," 2018 IEEE Conference on Evolving and Adaptive Intelligent Systems (EAIS), Rhodes, 2018, pp. 1-7, doi: 10.1109/EAIS.2018.8397170.

[15] Hajek.P., Prochazka,O., "Interval-valued intuitionistic fuzzy cognitive maps for supplier selection", Intelligent Decision Technologies 2017: Proceedings of the 9th KES International Conference on Intelligent Decision Technologies (KES-IDT 2017) - Part I, pp. 207-217, 2018.

[16]Smarandache,F.,"Neutrosophic set, a generalization of the Intuitionistic Fuzzy Sets", International Journal of Pure and Applied Mathematics, Vol.24,pp-, 287-297,2005.

[17] Mohamed Bisher Zeina., "Erlang Service Queueing Model with Neutrosophic Parameters", International Journal of Neutrosophic Science, Vol.6,pp- 106-112,2020.

[18] Malath F. Alaswad., "A Study of the Integration of Neutrosophic Thick Function", International Journal of Neutrosophic Science, Vol.6,pp- 97-105,2020.

[19] Salama, A. A., Bondok Henawy, M.S, Rafif Alhabib, "Online Analytical Processing Operations via Neutrosophic Systems", International Journal of Neutrosophic Science, Vol.8,pp- 87-109,2020.

[20] Agboola A.A.A., "Introduction to Neutro Rings", ", International Journal of Neutrosophic Science, Vol.7,pp-62-73,2020.

[21] Riad .K. Al-Hamido , Luai Salha , Taleb Gharibah, "Neutrosophic Crisp Semi Separation Axioms In Neutrosophic Crisp Topological Spaces", International Journal of Neutrosophic Science, Vol.6,pp- 32-40,2020.

[22] Necati Olgun., Ahmed Hatip., "On Refined Neutrosophic R-module", International Journal of Neutrosophic Science, Vol.7,pp- 87-96,2020.

[23] Ibrahim, M.A.,Agboola, A.A.A. , Badmus, B.S.,Akinleye, S.A., "n Refined Neutrosophic Vector Spaces", International Journal of Neutrosophic Science, Vol.7,pp- 97-109,2020.

[24] Mohammad Hamidi, Florentin Smarandache, "NEUTRO-BCK-algebra", International Journal of Neutrosophic Science, Vol.8,pp- 110-117,2020.

[25] Abdel-Baset, M., Chang, V., Gamal, A. Evaluation of the green supply chain management practices: A novel neutrosophic approach. Computers in Industry,Vol. 108,pp- 210-220,2019.

[26]Vasantha Kandasamy,W. B. ,Florentin Smarandache, "Fuzzy Cognitive Maps and Neutrosophic Cognitive Maps", Xiquan Publishers, Phoenix, 2003.

[27] Nivetha Martin., Aleeswari.A., Lilly Merline,W., "Risk Factors of Lifestyle Diseases – Analysis by Decagonal Linguistic Neutrosophic Fuzzy Cognitive Map", Materials Today: Proceedings,Vol. 24, pp-1939-1943,2020.

[28]Nivetha Martin., Lilly Merline,W., "A Case Study on the problems faced by the Entrepreneurs of Karumathur in Madurai District", Global Journal of Finance and Management, Vol. 9, pp. 23-28,2017.

[29]Smarandache, F. Plithogeny, Plithogenic Set, Logic, Probability, and Statistics. 2018,arXiv preprint arXiv:1808.03948.

[30]Smarandache, F. Extension of Soft set to Hypersoft Set, and then to Plithogenic Hypersoft Set, Neutrosophic set and Systems, Vol.22, pp-68-70.,2018.

DOI:10.5281/zenodo.3950063

[31] Shazia Rana ., Madiha Qayyum.,Muhammad Saeed., Florentin Smarandache ., Bakhtawar Ali Khan., "Plithogenic Fuzzy Whole Hypersoft Set, Construction of Operators and their Application in Frequency Matrix Multi Attribute Decision Making Technique", Neutrosophic Sets and Systems, Vol 28, pp.34-50,2019.

[32] Martin, Nivetha and Florentin Smarandache. "Concentric Plithogenic Hypergraph based on Plithogenic Hypersoft sets – A Novel Outlook." Neutrosophic Sets and Systems Vol.33, pp-78-91,2020.

[33] Smarandache, Florentin., "Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-)HyperAlgebra" Neutrosophic Sets and Systems, Vol 33, pp.290-296, 2020.

[34] Florentin Smarandache., Nivetha Martin., "Plithogenic n- Super Hypergraph in Novel Multi – Attribute Decision Making", International Journal of Neutrosophic Science, Vol.7,pp-8-30,2020.

[35] Abdel-Basset, M., Mohamed, R., Zaied, A. E. N. H., & Smarandache, F. A hybrid plithogenic decision-making approach with quality function deployment for selecting supply chain sustainability metrics. Symmetry, Vol.11,903,2019.

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On Refined Neutrosophic Vector Spaces II

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Abstract

The concept of refined neutrosophic vector spaces was introduced by Ibrahim et al. in [20] and the present paper is the continuation of the work. In the present paper, further studies on neutrosophic vector spaces are presented. Specifically, linear dependence, independence, bases and dimensions of refined neutrosophic vector spaces are studied with several results and examples presented. Also, refined neutrosophic homomorphisms of refined neutrosophic vector spaces are studied and existence of linear maps between weak refined neutrosophic vector spaces and weak neutrosophic vector spaces are established.

Keywords: Neutrosophy, neutrosophic vector space, refined neutrosophic vector space, refined neutrosophic vector space homomorphism.

1 Introduction and Preliminaries

Neutrosophy is a new branch of philosophy introduced by Florentin Smarandache in 1995. Neutrosophic logic/set introduced by Smarandache in [28] is an extension of fuzzy logic/set introduced by Zadeh [38] and intuitionistic fuzzy logic/set introduced by Atanassov [13]. In neutrosophic logic/set, each proposition is characterized by the degree of truth in the set (T), degree if indeterminacy in the set (I) and the degree of falsehood in the set (F) where (T, I, F) are not necessarily intervals, but may be any real sub-unitary subsets: discrete or continuous; single-element, finite, or (countable or uncountable) infinite; union or intersection of various subsets; etc. Neutrosophic logic/set has many applications in mathematics, computer science, engineering, technology, decision making, medical diagnosis, social sciences and many other fields. For full details, the reader should see [19,23–27], [14–18], [31]-[33], [35]-[37].

Smarandache recently introduced the concept of refined neutrosophic logic/set in [29] where it was shown that the neutrosophic components (T, I, F) can be split into refined neutrosophic components of the form $< T_1, T_2, \cdots, T_p; I_1, I_2, \cdot, I_r; F_1, F_2, \cdots, F_s >$ with applications in physics and other sciences and mathematics. In [30], Smarandache presented (T, I, F) structures and this motivated Agboola to introduce the concept of refined neutrosophic algebraic structures in[8] where he studied refined neutrosophic groups. Since the introduction of refined neutrosophic algebraic structures, many researchers have further studied the concepts and several results have been published as can be found in [1–7,9–11,21].

The concept of a neutrosophic vector space $V(I)$ generated by a vector space V and indeterminacy factor I was introduced by Vasantha Kandasamy and Florentin Smarandache in [31]. Since then, several researchers have studied the concept and a great deal of literature have been published. Recently, Agboola and Akinleye in [12] studied classical vector spaces in a neutrosophic environment and they showed that every neutrosophic vector space over a neutrosophic field (resp. field) is a vector space. In [34], Vasantha Kandasamy, et al. introduced for the first time the concept of neutrosophic quadruple vector spaces over the classical fields $\mathbb{R}, \mathbb{C}$ and $\mathbb{Z}_p$ and they presented several interesting results. Further studies on neutrosophic quadruple vector spaces were carried out in [22] by Ibrahim et al. where several results and examples were presented. The notion of refined neutrosophic vector spaces and their properties was introduces by Ibrahim et al. in [20]. They studied Weak(strong) refined neutrosophic vector spaces and subspaces, and also, they studied strong refined

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neutrosophic quotient vector spaces. Several interesting results and examples were presented. It was shown that every weak (strong) refined neutrosophic vector space is a vector space and it was equally shown that every strong refined neutrosophic vector space is a weak refined neutrosophic vector space. In the present paper however, further studies on refined neutrosophic vector spaces are presented. Specifically, linear dependence, independence, bases and dimensions of refined neutrosophic vector spaces are studied and several results and examples are presented. Refined neutrosophic homomorphisms of refined vector spaces are studied and existence of linear maps between weak refined neutrosophic vector spaces $V(I_1, I_2)$ and weak neutrosophic vector spaces $V(I)$ are established.

For the purposes of this paper, it will be assumed that I splits into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. It then follows logically that:

$$\begin{array}{lllll} I_1I_1 & = & I_1^2 & = & I_1, \\ I_2I_2 & = & I_2^2 & = & I_2, \text{ and} \\ I_1I_2 & = & I_2I_1 & = & I_1. \end{array}$$

Definition 1.1. [8] If $* : X(I_1, I_2) \times X(I_1, I_2) \mapsto X(I_1, I_2)$ is a binary operation defined on $X(I_1, I_2)$, then the couple $(X(I_1, I_2), *)$ is called a refined neutrosophic algebraic structure and it is named according to the laws (axioms) satisfied by $*$.

Definition 1.2. [8] Let $(X(I_1, I_2), +, .)$ be any refined neutrosophic algebraic structure where $+$ and . are ordinary addition and multiplication respectively.
For any two elements $(a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$, we define

$$(a, bI_1, cI_2) + (d, eI_1, fI_2) = (a + d, (b + e)I_1, (c + f)I_2),$$
$$(a, bI_1, cI_2).(d, eI_1, fI_2) = (ad, (ae + bd + be + bf + ce)I_1, (af + cd + cf)I_2).$$

Definition 1.3. [8] If $''+''$ and $''\cdot''$ are ordinary addition and multiplication, I_k with $k = 1, 2$ have the following properties:

1. $I_k + I_k + \cdots + I_k = nI_k$.
2. $I_k + (-I_k) = 0$.
3. $I_k \cdot I_k \cdot \cdots \cdot I_k = I_k^n = I_k$ for all positive integers $n > 1$.
4. $0 \cdot I_k = 0$.
5. I_k^{-1} is undefined and therefore does not exist.

Definition 1.4. [8] Let $(G, *)$ be any group. The couple $(G(I_1, I_2), *)$ is called a refined neutrosophic group generated by G, I_1 and I_2. $(G(I_1, I_2), *)$ is said to be commutative if for all $x, y \in G(I_1, I_2)$, we have $x * y = y * x$. Otherwise, we call $(G(I_1, I_2), *)$ a non -commutative refined neutrosophic group.

Definition 1.5. [8] If $(X(I_1, I_2), *)$ and $(Y(I_1, I_2), *')$ are two refined neutrosophic algebraic structures, the mapping

$$\phi : (X(I_1, I_2), *) \longrightarrow (Y(I_1, I_2), *')$$

is called a neutrosophic homomorphism if the following conditions hold:

1. $\phi((a, bI_1, cI_2) * (d, eI_1, fI_2)) = \phi((a, bI_1, cI_2)) *' \phi((d, eI_1, fI_2))$.
2. $\phi(I_k) = I_k$ for all $(a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$ and $k = 1, 2$.

Example 1.6. [8] Let
$\mathbb{Z}_2(I_1, I_2) = \{(0, 0, 0), (1, 0, 0), (0, I_1, 0), (0, 0, I_2), (0, I_1, I_2), (1, I_1, 0), (1, 0, I_2), (1, I_1, I_2)\}$.
Then $(\mathbb{Z}_2(I_1, I_2), +)$ is a commutative refined neutrosophic group of integers modulo 2.
Generally for a positive integer $n \geq 2$, $(\mathbb{Z}_n(I_1, I_2), +)$ is a finite commutative refined neutrosophic group of integers modulo n.

Example 1.7. [8] Let $(G(I_1, I_2), *)$ and and $(H(I_1, I_2), *')$ be two refined neutrosophic groups.
Let $\phi : G(I_1, I_2) \times H(I_1, I_2) \to G(I_1, I_2)$ be a mapping defined by $\phi(x, y) = x$ and let
$\psi : G(I_1, I_2) \times H(I_1, I_2) \to H(I_1, I_2)$ be a mapping defined by $\psi(x, y) = y$. Then ϕ and ψ are refined neutrosophic group homomorphisms.

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Definition 1.8. [4] Let $(R, +, .)$ be any ring. The abstract system $(R(I_1, I_2), +, .)$ is called a refined neutrosophic ring generated by R, I_1, I_2. $(R(I_1, I_2), +, .)$ is called a commutative refined neutrosophic ring if for all $x, y \in R(I_1, I_2)$, we have $xy = yx$. If there exists an element $e = (1, 0, 0) \in R(I_1, I_2)$ such that $ex = xe = x$ for all $x \in R(I_1, I_2)$, then we say that $(R(I_1, I_2), +, .)$ is a refined neutrosophic ring with unity.

Definition 1.9. [4] Let $(R(I_1, I_2), +, .)$ be a refined neutrosophic ring and let $n \in \mathbb{Z}^+$.

(i) If $nx = 0$ for all $x \in R(I_1, I_2)$, we call $(R(I_1, I_2), +, .)$ a refined neutrosophic ring of characteristic n and n is called the characteristic of $(R(I_1, I_2), +, .)$.

(ii) $(R(I_1, I_2), +, .)$ is call a refined neutrosophic ring of characteristic zero if for all $x \in R(I_1, I_2)$, $nx = 0$ is possible only if $n = 0$.

Example 1.10. [4]

(i) $\mathbb{Z}(I_1, I_2), \mathbb{Q}(I_1, I_2), \mathbb{R}(I_1, I_2), \mathbb{C}(I_1, I_2)$ are commutative refined neutrosophic rings with unity of characteristics zero.

(ii) Let $\mathbb{Z}_2(I_1, I_2) = \{(0, 0, 0), (1, 0, 0), (0, I_1, 0), (0, 0, I_2), (0, I_1, I_2), (1, I_1, 0), (1, 0, I_2), (1, I_1, I_2)\}$. Then $(\mathbb{Z}_2(I_1, I_2), +, .)$ is a commutative refined neutrosophic ring of integers modulo 2 of characteristic 2. Generally for a positive integer $n \geq 2$, $(\mathbb{Z}_n(I_1, I_2), +, .)$ is a finite commutative refined neutrosophic ring of integers modulo n of characteristic n.

Example 1.11. [4] Let $M^{\mathbb{R}}_{n\times n}(I_1, I_2) = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} : a_{ij} \in \mathbb{R}(I_1, I_2) \right\}$

be a refined neutrosophic set of all $n \times n$ matrix. Then $(M^{\mathbb{R}}_{n\times n}(I_1, I_2), +, .)$ is a non-commutative refined neutrosophic ring under matrix multiplication.

Theorem 1.12. [4] *Let $(R(I_1, I_2), +, .)$ be any refined neutrosophic ring. Then $(R(I_1, I_2), +, .)$ is a ring.*

2 Linear dependence, independence, bases and dimensions of a refined neutrosophic vector space

Definition 2.1. Let $(V, +, .)$ be any vector space over a field K. Let $V(I_1, I_2) =< V \cup (I_1, I_2) >$ be a refined neutrosophic set generated by V, I_1 and I_2. We call the triple $(V(I_1, I_2), +, .)$ a weak refined neutrosophic vector space over a field K, if $V(I_1, I_2)$ is a refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$, then $V(I_1, I_2)$ is called a strong refined neutrosophic vector space.

The elements of $V(I_1I_2)$ are called refined neutrosophic vectors and the elements of $K(I_1, I_2)$ are called refined neutrosophic scalars.

If $u = a + bI_1 + cI_2$, $v = d + eI_1 + fI_2 \in V(I_1, I_2)$ where a, b, c, d, e and f are vectors in V and $\alpha = k + mI_1 + nI_2 \in K(I_1, I_2)$ where k, m and n are scalars in K, we define:

$$u + v = (a + bI_1 + cI_2) + (d + eI_1 + fI_2) = (a + d) + (b + e)I_1 + (c + f)I_2,$$

and

$$\alpha u = (k + mI_1 + nI_2).(a + bI_1 + cI_2) = k.a + (k.b + m.a + m.b + m.c + n.b)I_1 + (k.c + n.a + n.c)I_2.$$

Definition 2.2. Let $V(I_1, I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$ and let $v_1, v_2, \cdot, v_n \in V(I_1, I_2)$.

1. An element $v \in V(I_1, I_2)$ is said to be a linear combination of the $v_i's$ if

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n, \text{where} \quad \alpha_i \in K(I_1, I_2).$$

2. $v_i's$ are said to be linearly independent if

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

implies that $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.
In this case, the set $\{v_1, v_2, \cdots, v_n\}$ is called a linearly independent set.

3. $v_i's$ are said to be linearly dependent if

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

implies that not all α_i are equal to zero.
In this case, the set $\{v_1, v_2, \cdots, v_n\}$ is called a linearly dependent set.

Definition 2.3. Let $V(I_1, I_2)$ be a weak refined neutrosophic vector space over a field K and let $v_1, v_2, \cdots, v_n \in V(I_1, I_2)$.

1. An element $v \in V(I_1, I_2)$ is said to be a linear combination of the $v_i's$ if

$$v = k_1 v_1 + k_2 v_2 + \cdots + k_n v_n, \text{where} \quad k_i \in K.$$

2. $v_i's$ are said to be linearly independent if

$$k_1 v_1 + k_2 v_2 + \cdots + k_n v_n = 0$$

implies that $k_1 = k_2 = \cdots = k_n = 0$.
In this case, the set $\{v_1, v_2, \cdots, v_n\}$. is called a linearly independent set.

3. $v_i's$ are said to be linearly dependent if

$$k_1 v_1 + k_2 v_2 + \cdots + k_n v_n = 0.$$

implies that not all k_i are equal to zero.
In this case, the set $\{v_1, v_2, \cdots, v_n\}$ is called a linearly dependent set.

Example 2.4. Let $V(I_1, I_2) = \mathbb{R}(I_1, I_2)$ be a weak refined neutrosophic vector space over a field $K = \mathbb{R}$. An element $v = 8 + 19I_1 + 18I_2 \in V(I_1, I_2)$ is a linear combination of the elements $v_1 = 2 + 5I_1 + 4I_2$, $v_2 = 1 + 2I_1 + 3I_2 \in V(I_1, I_2)$, since $8 + 19I_1 + 18I_2 = 3(2 + 5I_1 + 4I_2) + 2(1 + 2I_1 + 3I_2)$.

Example 2.5. Let $V(I_1, I_2) = \mathbb{R}(I_1, I_2)$ be a weak refined neutrosophic vector space over a field $K = \mathbb{R}$. An element $v = 3 + 15I_1 + 7I_2 \in V(I_1, I_2)$ is a linear combination of the elements $v_1 = 2 + 5I_1 + 3I_2$, $v_2 = 1 + I_1 + I_2 \in V(I_1, I_2)$, since $3 + 15I_1 + 7I_2 = 4(2 + 5I_1 + 3I_2) - 5(1 + I_1 + I_2)$.

Example 2.6. Let $V(I_1, I_2) = R(I_1, I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2) = \mathbb{R}(I_1, I_2)$. An element $v = 8 + 19I_1 + 18I_2 \in V(I_1, I_2)$ is a linear combination of the elements $v_1 = 1 + 2I_1 + 3I_2, v_2 = 2 + 5I_1 + 4I_2 \in V(I_1, I_2)$, since

$$\begin{aligned} 8 + 19I_1 + 18I_2 &= (2 + 5I_1 + 6I_2)(1 + 2I_1 + 3I_2) + (3 - 2I_1 - 4I_2)(2 + 5I_1 + 4I_2) \\ &= (2 + 8I_1 + 3I_2)(1 + 2I_1 + 3I_2) + (3 - 4I_1 - 2I_2)(2 + 5I_1 + 4I_2) \\ &= (4 + 11I_1 - 2I_2)(1 + 2I_1 + 3I_2) + (2 - 6I_1 + I_2)(2 + 5I_1 + 4I_2) \\ &= (4 + 8I_1 + I_2)(1 + 2I_1 + 3I_2) + (2 - 4I_1 - I_2)(2 + 5I_1 + 4I_2). \end{aligned}$$

Here $(2 + 5I_1 + 6I_2), (3 - 2I_1 - 4I_2), (2 + 8I_1 + 3I_2), (3 - 4I_1 - 2I_2), (4 + 11I_1 - 2I_2)$, $(2 - 6I_1 + I_2), (4 + 8I_1 + I_2), (2 - 4I_1 - I_2) \in K(I_1, I_2)$.

This example shows that the element $v = 8+19I_1+18I_2$ can be infinitely expressed as a linear combination of the elements $v_1 = 1 + 2I_1 + 3I_2$, $v_2 = 2 + 5I_1 + 4I_2 \in V(I_1, I_2)$. This observation is recorded in the next proposition.

Proposition 2.7. *Let $V(I_1, I_2)$ be a strong refined neutrosophic vector space over a neutrosophic field $K(I_1, I_2)$ and let $v_1, v_2, \cdots, v_n \in V(I_1, I_2)$. An element $v \in V(I_1, I_2)$ can be infinitely expressed as a linear combination of the $v_i s$.*

Proof: Suppose that $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ where $v = a + bI_1 + cI_2$, $v_1 = a_1 + b_1 I_1 + c_1 I_2$, $v_2 = a_2 + b_2 I_1 + cI_2, \cdots, v_n = a_n + b_n I_1 + cI_n$ and $\alpha_1 = k_1 + m_1 I_1 + tI_2, \alpha_2 = k_2 + m_2 I_1 + tI_2, \cdots, \alpha_n = k_n + m_n I_1 + tI_2 \in K(I_1, I_2)$.
Then
$a + bI_1 + cI_2 = (k_1 + m_1 I_1 + t_1 I_2)(a_1 + b_1 I_1 + c_1 I_2) + (k_2 + m_2 I_1 + t_2 I_2)(a_2 + b_2 I_1 + c_2 I_2) + \cdots + (k_n + m_n I_1 + t_n I_2)(a_n + b_n I_1 + c_n I_2)$
from which we obtain
$a_1 k_1 + a_2 k_2 + \cdots + a_n k_n = a$,
$b_1 k_1 + a_1 m_1 + b_1 m_1 + c_1 m_1 + b_1 t_1 + b_2 k_2 + a_2 m_2 + b_2 m_2 + c_2 m_2 + b_2 t_2 + \cdots + b_n k_n + a_n m_n + b_n m_n + c_n m_n + b_n t_n = b$,
$c_1 k_1 + a_1 t_1 + c_1 t_1 + c_2 k_2 + a_2 t_2 + c_2 t_2 + \cdots + c_n k_n + a_n t_n + c_n t_n = c$.

This is a linear system in unknowns $k_i, m_i, t_i \quad i = 1, 2, \cdots, n$.
Since the system is consistent and have infinitely many solutions, it follows that the $v_i s$ can be infinitely combined to produce v.
But if $V(I_1, I_2)$ and $K(I_1, I_2)$ are finite the $v_i s$ will be finitely combined to produce v.

Proposition 2.8. *Let $V(I_1, I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$ and let $U[I_1, I_2]$ and $W[I_1, I_2]$ be subsets of $V[I_1, I_2]$ such that $U[I_1, I_2] \subseteq W[I_1, I_2]$. If $U[I_1, I_2]$ is linearly dependent, then $W[I_1, I_2]$ is linearly dependent.*

Proposition 2.9. *Let $V(I_1, I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$ and let $U[I_1, I_2]$ and $W[I_1, I_2]$ be subsets of $V[I_1, I_2]$ such that $U[I_1, I_2] \subseteq W[I_1, I_2]$. If $W[I_1, I_2]$ is linearly independent, then $U[I_1, I_2]$ is linearly independent.*

Proof : Let $W[I_1, I_2] = \{v_1 = (a_1 + b_1 I_1 + c_1 I_2), v_2 = (a_2 + b_2 I_1 + c_2 I_2), \cdots, v_n = (a_n + b_n I_1 + c_n I_2)\}$, be a linearly independent set. Let, if possible,
$U[I_1, I_2] = \{v_1 = (a_1 + b_1 I_1 + c_1 I_2), v_2 = (a_2 + b_2 I_1 + c_2 I_2), \cdots, v_z = (a_z + b_z I_1 + c_z I_2)\}, z < n$, be a linearly dependent subset of $\{v_1 = (a_1 + b_1 I_1 + c_1 I_2), v_2 = (a_2 + b_2 I_1 + c_2 I_2), \cdots, v_n = (a_n + b_n I_1 + c_n I_2)\}$.
Then there exist some scalars $(k_1 + m_1 I_1 + t_1 I_2), (k_2 + m_2 I_1 + t_2 I_2), \cdots (k_z + m_z I_1 + t_z I_2) \in K(I_1, I_2)$, not all zero, such that
$(k_1 + m_1 I_1 + t_1 I_2)(a_1 + b_1 I_1 + c_1 I_2) + (k_2 + m_2 I_1 + t_2 I_2)(a_2 + b_2 I_1 + c_2 I_2) + \cdots +$
$(k_z + m_z I_1 + t_z I_2)(a_z + b_z I_1 + c_z I_2) = 0.$
$\Longrightarrow (k_1 + m_1 I_1 + t_1 I_2)(a_1 + b_1 I_1 + c_1 I_2) + (k_2 + m_2 I_1 + t_2 I_2)(a_2 + b_2 I_1 + c_2 I_2) + \cdots +$
$(k_z + m_z I_1 + t_z I_2)(a_z + b_z I_1 + c_z I_2) + (0_{z+1} + 0_{z+1} I_1 + 0_{z+1} I_2)(a_{z+1} + b_{z+1} I_1 + c_{z+1} I_2) +$
$(0_{z+2} + 0_{z+2} I_1 + 0_{z+2} I_2)(a_{z+2} + b_{z+2} I_1 + c_{z+2} I_2) + \cdots + (0_n + 0_n I_1 + 0_n I_2)(a_n + b_n I_1 + c_n I_2) = 0.$
The scalars $(k_1, m_1 I_1, t_1 I_2), (k_2, m_2 I_1, t_2 I_2), \cdots, (k_z, m_z I_1, t_z I_2), (0_{z+1} + 0_{z+1} I_1 + 0_{z+1} I_2)$,
$(0_{z+2} + 0_{z+2} I_1 + 0_{z+2} I_2), \cdots, (0_n + 0_n I_1 + 0_n I_2)$ are not all zero.
Thus, the vectors $v_1 = (a_1 + b_1 I_1 + c_1 I_2), v_2 = (a_2 + b_2 I_1 + c_2 I_2), \cdots, v_n = (a_n + b_n I_1 + c_n I_2)$ are linearly dependent. This contradiction the assumption that the vectors
$v_1 = (a_1 + b_1 I_1 + c_1 I_2), v_2 = (a_2 + b_2 I_1 + c_2 I_2), \cdots, v_n = (a_n + b_n I_1 + c_n I_2)$ are linearly independent.
Hence, the set $\{v_1 = (a_1 + b_1 I_1 + c_1 I_2), v_2 = (a_2 + b_2 I_1 + c_2 I_2), \cdots, v_z = (a_n + b_z I_1 + c_z I_2)\}$ is a linearly independent set.

Proposition 2.10. *Let $V(I_1, I_2)$ be a weak refined neutrosophic vector space over a field K. The set $W(I_1, I_2) = \{v_1, v_2, \cdots v_n\} \subseteq V(I_1, I_2)$ is linearly dependent, if and only if at least one vector v_i is a linear combination of the other vectors .*

Proof : The proof is similar to the proof in classical case.

Proposition 2.11. *Let $V(I_1, I_2)$ be a strong refined neutrosophic vector space over a neutrosophic field $K(I_1, I_2)$ and let $v_1 = k_1 + k_1 I_1 + k_1 I_2, v_2 = k_2 + k_2 I_1 + k_2 I_2, \cdots, v_n = k_n + k_n I_1 + k_n I_2$ be elements of $V(I_1, I_2)$ where $0 \neq k_i \in K$. Then $\{v_1, v_2, \cdots, v_n\}$ is a linearly dependent set.*

Proof :
Let $\alpha_1 = p_1 + q_1 I_1 + r_1 I_2, \alpha_2 = p_2 + q_2 I_1 + r_2 I_2, \cdots, \alpha_n = p_n + q_n I_1 + r_n I_2$ be elements of $K(I_1, I_2)$.
Then $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$ which implies that
$(p_1 + q_1 I_1 + r_1 I_2)(k_1 + k_1 I_1 + k_2 I_2) + (p_2 + q_2 I_1 + r_2 I_2)(k_2 + k_2 I_1 + k_2 I_2) + \cdots + (p_n + q_n I_1 + r_n I_2)(k_n + k_n I_1 + k_n I_2) = 0$
from which we obtain
$k_1 p_1 + k_2 p_2 + \cdots + k_n p_n = 0.$

This is a homogeneous linear system in unknowns $p_i, i = 1, 2, \cdots, n$.
This system has infinitely many nontrivial solutions. Hence $\alpha_i s$ are not all zero and therefore, $\{v_1, v_2, \cdots, v_n\}$ is a linearly dependent set.

Note 1. *If in Proposition 2.11 we consider a single vector $v \in V(I_1, I_2)$, the statement still hold. For instance, let $0 \neq v = a + bI_1 - aI_2 \in V(I_1, I_2)$ and $0 \neq \beta = pI_1 - pI_2 \in K(I_1, I_2)$, we have*

$$\beta \cdot v = (a + bI_1 - aI_2) \cdot (pI_1 - pI_2) = apI_1 + bpI_1 - bpI_1 - apI_1 - apI_2 + apI_2 = 0.$$

Definition 2.12. Let $V(I_1, I_2)$ be weak(strong) refined neutrosophic vector space. If $\{v_1, v_2, \cdots, v_n\}$ is any set of refined neutrosophic vectors in $V(I_1, I_2)$, the set of all linear combinations of these refined neutrosophic vectors is called their span, and is denoted by

$$\text{span}\{v_1, v_2, \cdots, v_n\}.$$

If it happens that $V(I_1, I_2) = \text{span}\{v_1, v_2 \cdots, v_n\}$, then these vectors are called a spanning set for $V(I_1, I_2)$.

Proposition 2.13. *Let $U(I_1, I_2) = span\{v_1, v_2, \cdots, v_n\}$ be in a strong refined neutrosophic vector space $V(I_1, I_2)$ over a refined neutrosophic field $K(I_1, I_2)$ then*

1. *$U(I_1, I_2)$ is a strong refined neutrosophic subspace of $V(I_1, I_2)$ containing $v_1, v_2, \cdots, v_n$.*
2. *$U(I_1, I_2)$ is the smallest subspace containing $v1, v_2, \cdots, v_n$ in the sense that any strong refined neutrosophic subspace of $V(I_1, I_2)$ that contains each of these refined neutrosophic vectors, must contain $U(I_1, I_2)$.*

Proof. 1. (a) $U(I_1, I_2) \neq \emptyset$, since we can find $0 = 0 + 0I_1 + 0I_2 \in K(I_1, I_2)$ such that $0 = 0v_1 + \cdots + 0v_n$ belongs to $U(I_1, I_2)$.

(b) Let $v, u \in U(I_1, I_2)$ where $u = s_1v_1 + s_2v_2 + \cdots + s_nv_n$ and $v = t_1v_1 + t_2v_2 + \cdots + t_nv_n$ and $\alpha = p + p_1I_1 + p_2I_2 \in K(I_1, I_2)$ then

$$u + v = (s_1 + t_1)v_1 + (s_2 + t_2)v_2 + \cdots + (s_n + t_n)v_n,$$

$$\alpha u = (\alpha s_1)v_1 + (\alpha s_2)v_2 + \cdots + (\alpha s_n)v_n.$$

So both $u + v$ and αu lie in $U(I_1, I_2)$.
Finally, since $U \subseteq U(I)$, where U is a vector space we conclude that $U(I_1, I_2)$ is a refined neutrosophic subspace.

2. Let $W(I_1, I_2)$ be a refined neutrosophic subspace of $V(I_1, I_2)$ that contains each of $v_1, v_2, \cdots, v_n$. Since $W(I_1, I_2)$ is closed under scalar multiplication, each of $\alpha_1v_1, \alpha_2v_2, \cdots, \alpha_nv_n$ lies in $W(I_1, I_2)$ for any choice of

$$\alpha_1 = p_1 + q_1I_1 + r_1I_2,\ \alpha_2 = p_2 + q_2I_1 + r_2I_2,\ \cdots, \alpha_n = p_n + q_nI_1 + r_nI_2 \in K(I_1, I_2).$$

But then $\alpha v_1 + \alpha_2v_2 + \cdots + \alpha_nv_n$ lies in $W(I_1, I_2)$ since $W(I_1, I_2)$ is closed under addition.
This means that $W(I_1, I_2)$ contains every member of $U(I_1, I_2)$, which proves (2).

□

Example 2.14. Let $P_n(I_1, I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$. Then $P_n(I_1, I_2) = \text{span}\{1, x, x^2, \cdots, x^n\}$.

We need only show that each neutrosophic polynomial $p(x)$ in $P_n(I_1, I_2)$ is a linear combination of $1, x, \cdots, x^n$. But this is clear because $p(x)$ has the form $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^2$.
With $a_0, a_1, \cdots, a_n \in K(I_1, I_2)$.

Example 2.15. Let $\mathbb{R}^3(I_1, I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$. Then $\mathbb{R}^3(I_1, I_2) = \text{span}\{(1 + I_1 + I_2), (1 + I_1 + 0I_2), (0 + I_1 + I_2)\}$.

Write $v_1 = (1 + I_1 + I_2), v_2 = (1 + I_1 + 0I_2), v_3 = (0 + I_1 + I_2)$, and $U(I_1, I_2) = \text{span}\{v_1, v_2, v_3\}$. Obviously $U(I_1, I_2)$ is contained in $\mathbb{R}^3(I_1, I_2)$.
We have $\mathbb{R}^3(I_1, I_2) = \text{span}\{(1 + 0I_1 + 0I_2), (0 + I_1 + 0I_2), (0 + 0I_1 + I_2)\}$.
So to prove that $\mathbb{R}^3(I_1, I_2)$ is contained in $U(I_1, I_2)$, it is enough by Proposition 2.13 to show that each of

$(1+0I_1+0I_2), (0+I_1+0I_2), (0+0I_1+I_2)$ lies in span$\{v_1, v_2, v_3\}$. But they can be given explicitly as linear combinations of v_1, v_2, and v_3:

$$(1+0I_1+0I_2) = (1+I_1+I_2) - (0+I_1+I_2) = v_1 - v_3,$$

$$(0+0I_1+I_2) = (1+I_1+I_2) - (1+I_1+0I_2) = v_1 - v_2$$

and then, using the first of these, we have

$$(0+I_1+0I_2) = (1+I_1+0I_2) - (1+0I_1+0I_2) = v_2 - (v_1 - v_3) = v_2 - v_2 + v_3.$$

Proposition 2.16. *Let $x = a + bI_1 + cI_2$ and $y = d + eI_1 + fI_2$ be two refined neutrosophic vectors in a strong refined neutrosophic vector space $V(I_1, I_2)$ over refined neutrosophic field $K(I_1, I_2)$.*
Then span $\{x, y\} =$ span$\{x + y,\ x - y\}$, i.e.,
span$\{a + bI_1 + cI_2, d + eI_1 + fI_2\} =$ span$\{a + d + (b + e)I_1 + (c + f)I_2, a - d + (b - e)I_1 + (c - f)I_2\}$.

Proof. We have
span$\{a+d+(b+e)I_1+(c+f)I_2, a-d+(b-e)I_1+(c-f)I_2\} \subseteq$ span$\{a+bI_1+cI_2, d+eI_1+fI_2\}$ because both $a+d+(b+e)I_1+(c+f)I_2$ and $a-d+(b-e)I_1+(c-f)I_2$ lie in span$\{a+bI_1+cI_2, d+eI_1+fI_2\}$.
On the other hand,

$$a + bI_1 + cI_2 = \frac{1}{2}[a + d + (b + e)I_1 + (c + f)I_2] + \frac{1}{2}[a - d + (b - e)I_1 + (c - f)I_2]$$

$$d + eI_1 + fI_2 = \frac{1}{2}[a + d + (b + e)I_1 + (c + f)I_2] - \frac{1}{2}[a - d + (b - e)I_1 + (c - f)I_2],$$

so
span$\{a + bI_1 + cI_2, d + eI_1 + fI_2\} \subseteq$ span$\{a + d + (b + e)I_1 + (c + f)I_2, a - d + (b - e)I_1 + (c - f)I_2\}$
by Proposition 2.13 . Hence the prove. □

Proposition 2.17. *Let $U(I_1, I_2)$ and $W(I_1, I_2)$ be strong refined neutrosophic subspaces of as strong refined neutrosophic vector space $V(I_1, I_2)$ over a refined neutrosophic field $K(I_1, I_2)$. Then*

1. $U(I_1, I_2) \subseteq W(I_1, I_2) \Longrightarrow$ *span*$(U(I_1, I_2)) \subseteq$ *span*$(W(I_1, I_2))$.
2. *span*(*span*$(U(I_1, I_2))) =$ *span*$(U(I_1, I_2))$.
3. *span*$(U(I_1, I_2) \cup W(I_1, I_2)) =$ *span*$(U(I_1, I_2)) +$ *span*$(W(I_1, I_2))$.

Proof. The proof of 1, 2 and 3 are the same as in classical case. □

Definition 2.18. Let $V(I_1, I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$. A linearly independent subset $\mathbb{B}[I_1, I_2] = \{v_1, v_2, \cdots, v_n\}$ of $V(I_1, I_2)$ is called a basis for $V(I_1, I_2)$ if $B[I_1, I_2]$ spans $V(I_1, I_2)$.

Proposition 2.19. *Let $V(I_1, I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$. The bases of $V(I_1, I_2)$ are the same as the bases of V over a field K.*

Proof:
Suppose that $\mathbb{B} = \{v_1, v_2, \cdots, v_n\}$ is an arbitrary basis for V over the field K. Let $v = a+bI_1+cI_2$ be an arbitrary element of $V(I_1, I_2)$ and let $\alpha_1 = k_1+m_1I_1+t_1I_2, \alpha_2 = k_2+m_2I_1+t_2I_2 \cdots, \alpha_n = k_n+m_nI_1+t_nI_2$ be elements of $K(I_1, I_2)$. Then from $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$,
we obtain

$$k_1 v_1 + k_2 v_2 + \cdots + k_n v_n = 0,$$

$$m_1 v_1 + m_2 v_2 + \cdots + m_n v_n = 0,$$

$$t_1 v_1 + t_2 v_2 + \cdots + t_n v_n = 0.$$

Since $v_i s$ are linearly independent, we have $k_i = 0,\ m_j = 0$ and $t_z = 0$ where $i, j, z = 1, 2, \cdots, n$. Hence, $\alpha_i = 0, i = 1, 2, \cdots, n$. This shows that $\mathbb{B}$ is also a linearly independent set in $V(I_1, I_2)$.
To show that B spans $V(I_1, I_2)$, let $v = a + bI_1 + cI_2 = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$.
Then we have

$$a = k_1 v_1 + k_2 v_2 + \cdots + k_n v_n,$$

$$b = m_1v_1 + m_2v_2 + \cdots + m_nv_n,$$

$$c = t_1v_1 + t_2v_2 + \cdots + t_nv_n = 0.$$

Since $a, b, c \in V$, it follows that $v = a + bI_1 + cI_2$ can be written uniquely as a linear combination of v_is. Hence, $\mathbb{B}$ is a basis for $V(I_1, I_2)$. Since $\mathbb{B}$ is arbitrary, the required result follows;

Proposition 2.20. *Let $V(I_1, I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$ which is spanned by a finite set of neutrosophic vectors $v_1, v_2, \cdots, v_m$. Then any independent set of refined neutrosophic vectors in $V(I_1, I_2)$ is finite and contains no more than m elements.*

Proof: Let $v = a + bI_1 + cI_2$, $u = d + eI_1 + fI_2$.
To prove this it suffices to show that every refined neutrosophic subset $S(I_1, I_2)$ of $V(I_1, I_2)$ which contains more than m refined neutrosophic vectors is linearly dependent.
Let $S(I_1, I_2)$ be such a set. In $S(I_1, I_2)$ there are distinct refined neutrosophic vectors $u_1, u_2, \cdots, u_n$ where $n > m$.
Since $v_1, v_2, \cdots, v_m$ span $V(I_1, I_2)$, there exist scalars C_{ij} with $C = r + sI_1 + tI_2 \in K(I_1, I_2)$ such that

$$u_j = \sum_{i=1}^{m} C_{ij}v_i = \sum_{i=1}^{m} (r_{ij}a_i + (r_{ij}b_i + s_{ij}a_i + s_{ij}b_i + s_{ij}c_i + t_{ij}b_i)I_1 + (r_{ij}c_i + t_{ij}a_i + t_{ij}c_i)I_2).$$

For any n scalars $x_1, x_2, \cdots, x_n$ with $x = p + qI_1 + zI_2 \in K(I_1, I_2)$ we have

$$\begin{aligned}
x_1u_1 \;+\;& x_2u_2 + \cdots + x_nu_n = \textstyle\sum_{j=1}^{n} x_ju_j \\
=\;& \textstyle\sum_{j=1}^{n}(p_j + q_jI_1 + z_jI_2)u_j \\
=\;& \textstyle\sum_{j=1}^{n}(p_j + q_jI_1 + z_jI_2)\sum_{i=1}^{m} C_{ij}v_i \\
=\;& \textstyle\sum_{j=1}^{n}(p_j + q_jI_1 + z_jI_2)\sum_{i=1}^{m}(r_{ij}a_i + (r_{ij}b_i + s_{ij}a_i + s_{ij}b_i + s_{ij}c_i+ \\
& t_{ij}b_i)I_1 + (r_{ij}c_i + t_{ij}a_i + t_{ij}c_i)I_2) \\
=\;& \textstyle\sum_{j=1}^{n}\sum_{i=1}^{m}(p_j + q_jI_1 + z_jI_2)(r_{ij}a_i + (r_{ij}b_i + s_{ij}a_i + s_{ij}b_i + s_{ij}c_i + t_{ij}b_i)I_1+ \\
& (r_{ij}c_i + t_{ij}a_i + t_{ij}c_i)I_2) \\
=\;& \textstyle\sum_{j=1}^{n}\sum_{i=1}^{m}(p_jr_{ij}a_i + (p_jr_{ij}b_i + p_js_{ij}a_i + p_js_{ij}b_i + p_js_{ij}c_i + p_jt_{ij}b_i + q_jr_{ij}a_i + q_jr_{ij}b_i+ \\
& q_js_{ij}a_i + q_js_{ij}b_i + q_js_{ij}c_i + q_jt_{ij}b_i + q_jr_{ij}c_i + q_jt_{ij}a_i + q_jt_{ij}c_i + z_jr_{ij}b_i + z_js_{ij}a_i + z_js_{ij}b_i \\
& +z_js_{ij}c_i + z_jt_{ij}b_i)I_1 + (p_jr_{ij}c_i + p_jt_{ij}a_i + p_jt_{ij}c_i + z_jr_{ij}a_i + z_jr_{ij}c_i + z_jt_{ij}a_i + z_jt_{ij}c_i)I_2) \\
=\;& \textstyle\sum_{j=1}^{n}\sum_{i=1}^{m}((p_j + q_jI_1 + z_jI_2)(r_{ij} + s_{ij}I_1 + t_{ij}I_2)(a_i + bI_1 + bI_2)) \\
=\;& \textstyle\sum_{j=1}^{n}\sum_{i=1}^{m}((p_j + q_jI_1 + z_jI_2)(r_{ij} + s_{ij}I_1 + t_{ij}I_2))(a_i + bI_1 + bI_2) \\
=\;& \textstyle\sum_{j=1}^{n}\sum_{i=1}^{m}(C_{ij}x_j)v_i \\
=\;& \textstyle\sum_{i=1}^{m}(\sum_{j=1}^{n} C_{ij}x_j)v_i.
\end{aligned}$$

Since $n > m$, there exist scalars $x_1, x_2, \cdots, x_n$ not all 0 such that

$$\sum_{j=1}^{n} C_{ij}x_j = 0 \qquad\qquad 1 \leq i \leq m.$$

Hence $x_1u_1 + x_2u_2 + \cdots + x_nu_n = 0$. This shows that $S(I_1, I_2)$ is a linearly dependent set.

Definition 2.21. Let $V(I_1, I_2)$ be a strong refined neutrosophic vector space over a neutrosophic field $K(I_1, I_2)$. The number of elements in the basis for $V(I_1, I_2)$ is called the dimension of $V(I_1, I_2)$ and it is denoted by $dim_s(V(I_1, I_2))$. If the number of elements in the basis for $V(I_1, I_2)$ is finite, $V(I_1, I_2)$ is called a finite dimensional strong refined neutrosophic vector space. Otherwise, $V(I_1, I_2)$ is called an infinite dimensional strong refined neutrosophic vector space.

Definition 2.22. Let $V(I_1, I_2)$ be a weak refined neutrosophic vector space over a field K. The number of elements in the basis for $V(I_1, I_2)$ is called the dimension of $V(I_1, I_2)$ and it is denoted by $dim_w(V(I_1, I_2))$. If the number of elements in the basis for $V(I_1, I_2)$ is finite, $V(I_1, I_2)$ is called a finite dimensional weak refined neutrosophic vector space. Otherwise, $V(I_1, I_2)$ is called an infinite dimensional weak refined neutrosophic vector space.

Example 2.23. The strong refined neutrosophic vector space of Example 2.14 is finite dimensional and $dim_s(V(I_1, I_2)) = n + 1$.

Proposition 2.24. *Let $V(I_1, I_2)$ be a finite dimensional strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$. Then every basis of $V(I_1, I_2)$ has the same number of elements.*

Proof. The proof is similar to the proof in classical case.

Proposition 2.25. *Let $V(I_1, I_2)$ be a finite dimensional weak (strong) refined neutrosophic vector space over a field K (resp. over a refined neutrosophic field $K(I_1, I_2)$). If $dim_s(V(I_1, I_2)) = n$, then $dim_w(V(I_1, I_2)) = 2n$.*

This can be easily seen in the examples given below.

Example 2.26. Let $V(I_1, 1_2) = \mathbb{R}^n(I_1, I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $R(I_1, 1_2)$.
The set $\mathbb{B} = \{v_1 = (1, 0, 0, \cdots, 0), v_2 = (0, 1, 0, \cdots, 0), \cdots, v_n = (0, 0, 0, \cdots, 1)\}$ is a basis for $V(I_1, I_2)$.

Example 2.27. Let $V(I_1, I_2) = \mathbb{R}^n(I_1, I_2)$ be a weak refined neutrosophic vector space over $\mathbb{R}$. The set $\mathbb{B} = \{v_1 = (1, 0, 0, \cdots, 0), v_2 = (0, 1, 0, \cdots, 0), \cdots, v_k = (0, 0, 0, \cdots, 1)$,
$v_{k+1} = (I_1 + I_2, 0, 0, \cdots, 0), v_{k+2} = (0, I_1 + I_2, 0, \cdots, 0), \cdots, v_n = (0, 0, 0, \cdots, I_1 + I_2)\}$ is a basis for $V(I_1, I_2)$.

Note 2. *The bases of the strong refined neutrosophic vector space of Example 2.26 is contained in the bases of the weak refined neutrosophic vector space of Example 2.27. This observation is recorded in the next proposition.*

Proposition 2.28. *Let $V(I_1, I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$. Then the bases of $V(I_1, I_2)$ over $K(I_1, I_2)$ are contained in the bases of the weak refined neutrosophic vector space $V(I_1, I_2)$ over a field K.*

Proof. The proof follows from Examples 2.26 and 2.27 . □

Proposition 2.29. *Let $W(I_1, I_2)$ be a strong refined neutrosophic subspace of a finite dimensional strong refined neutrosophic vector space $V(I_1, I_2)$ over a neutrosophic field $K(I_1, I_2)$. Then $W(I_1, I_2)$ is finite dimensional and $dim_s(W(I_1, I_2)) \leq dim_s(V(I_1, I_2))$. If $dim_s(W(I_1, I_2)) = dim_s(V(I_1, I_2))$, then $W(I_1, I_2) = V(I_1, I_2)$.*

Proof. If $W(I_1, I_2) = \{\}$, $dim_s W(I_1, I_2) = 0$. So assume $W(I_1, I_2) \neq \{\}$, and choose $u_1 \neq 0$ in $W(I_1, I_2)$. If $W(I_1, I_2)$ = span$\{u_1\}$, then $dim_s W(I_1, I_2) = 1$. If $W(I_1, I_2) \neq$ span$\{u_1\}$, choose u_2 in $W(I_1, I_2)$ outside span$\{u_1\}$. Then $\{u_1, u_2\}$, is linearly independent.
If $W(I_1, I_2)$ = span $\{u_1, u_2\}$, then $dim_s W(I_1, I_2) = 2$. If not, repeat the process to find u_3 in $W(I_1, I_2)$ such that $\{u_1, u_2, u_3\}$ is linearly independent. Continue in this way. The process must terminate because the refined neutrosophic space $V(I_1, I_2)$ (having dimension n) cannot contain more than n independent vectors. Hence $W(I_1, I_2)$ has a basis of at most n refined neutrosophic vectors.
Secondly, Let $dim_s W(I_1, I_2) = dim_s V(I_1, I_2) = m$. Then any basis $\{u_1, \cdots, u_m\}$ of $W(I_1, I_2)$ is an independent set of m refined neutrosophic vectors in $V(I_1, I_2)$ and so is a basis of $V(I_1, I_2)$.
In particular, $\{u_1, \cdots, u_m\}$ spans $V(I_1, I_2)$ so, because it also spans $W(I_1, I_2)$,
$V(I_1, I_2) =$ span$\{u_1, \cdots, u_m\} = W(I_1, I_2)$. □

Proposition 2.30. *Let $U(I_1, I_2)$ and $W(I_1, I_2)$ be finite dimensional strong refined neutrosophic subspaces of a strong refined neutrosophic vector space $V(I_1, I_2)$ over a refined neutrosophic field $K(I_1, I_2)$. Then $U(I_1, I_2) + W(I_1, I_2)$ is a finite dimensional strong refined neutrosophic subspace of $V(I_1, I_2)$ and*

$$dim_s(U(I_1, I_2) + W(I_1, I_2)) = dim_s(U(I_1, I_2)) + dim_s(W(I_1, I_2)) - dim_s(U(I_1, I_2) \cap W(I_1, I_2)).$$

If $V(I_1, I_2) = U(I_1, I_2) \oplus W(I_1, I_2)$ then

$$dim_s(U(I_1, I_2) + W(I_1, I_2)) = dim_s(U(I_1, I_2)) + dim_s(W(I_1, I_2)).$$

Definition 2.31. Let $V(I_1, I_2)$ and $W(I_1, I_2)$ be strong refined neutrosophic vector spaces over a refined neutrosophic field $K(I_1, I_2)$ and let $\phi : V(I_1, I_2) \longrightarrow W(I_1, I_2)$ be a mapping of $V(I_1, I_2)$ into $W(I_1, I_2)$. ϕ is called a refined neutrosophic vector space homomorphism if the following conditions hold:

1. ϕ is a vector space homomorphism.
2. $\phi(I_k) = I_k$ for $k = 1, 2$.

If ϕ is a bijective refined neutrosophic vector space homomorphism, then ϕ is called a refined neutrosophic vector space isomorphism and we write $V(I_1, I_2) \cong W(I_1, I_2)$.

Definition 2.32. Let $V(I_1, I_2)$ and $W(I_1, I_2)$ be strong refined neutrosophic vector spaces over a refined neutrosophic field $K(I_1, I_2)$ and let $\phi : V(I_1, I_2) \longrightarrow W(I_1, I_2)$ be a refined neutrosophic vector space homomorphism.

1. The kernel of ϕ denoted by Kerϕ is defined by the set $\{v \in V(I_1, I_2) : \phi(v) = 0\}$.
2. The image of ϕ denoted by Imϕ is defined by the set
   $\{w \in W(I_1, I_2) : \phi(v) = w \quad \text{for some} \quad v \in V(I_1, I_2))\}$.

Example 2.33. Let $V(I_1, I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$

1. The mapping $\phi : V(I_1, I_1) \longrightarrow V(I_1, I_1)$ defined by $\phi(v) = v$ for all
   $v = a + bI_1 + cI_2 \in V(I_1, I_2)$ is a refined neutrosophic vector space homomorphism and Ker$\phi = \{0\}$.
2. The mapping $\phi : V(I_1, I_1) \longrightarrow V(I_1, I_1)$ defined by $\phi(v) = 0$ for all $v = a + bI_1 + cI_2 \in V(I_1, I_2)$ is not a refined neutrosophic vector space homomorphism. Since for $I_k \in V(I_1, I_2),\ \phi(I_k) \neq 0$.

Proposition 2.34. *Let $V(I_1, I_2)$ and $W(I_1, I_2)$ be strong refined neutrosophic vector spaces over a neutrosophic field $K(I_1, I_2)$ and let $\phi : V(I_1, I_2) \longrightarrow W(I_1, I_2)$ be a refined neutrosophic vector space homomorphism. Then*

1. *Kerϕ is not a strong refined neutrosophic subspace of $V(I_1, I_2)$ but a subspace of $V(I_1, I_2)$.*
2. *Imϕ is a strong refined neutrosophic subspace of $W(I_1, I_2)$.*

Proof. That Kerϕ is a subspace of $V(I_1, I_2)$, and Imϕ is a strong refined neutrosophic subspace of $W(I_1, I_2)$ follows easily .
Now, to show that Kerϕ is not a strong refined neutrosophic subspace of $V(I_1, I_2)$, we note that for $I_k \in V(I_1, I_2)$ we have that $\phi(I_k) = I_k \neq 0$, this implies that $I_k \notin ker\phi$.
Hence, $\ker \phi$ is not a strong refined neutrosophic subspace. □

Proposition 2.35. *Let $V(I_1, I_2)$ and $W(I_1, I_2)$ be strong refined neutrosophic vector spaces over a refined neutrosophic field $K(I_1, I_2)$ and let $\phi : V(I_1, I_2) \longrightarrow W(I_1, I_2)$ be a refined neutrosophic vector space homomorphism. If $\mathbb{B} = \{v_1, v_2, \cdots, v_n\}$ is a basis for $V(I_1, I_2)$, then $\phi(\mathbb{B}) = \{\phi(v_1), \phi(v_2), \cdots, \phi(v_n)\}$ is a basis for $W(I_1, I_2)$.*

Proof. Since $\mathbb{B} = \{v_1, v_2, \cdots, v_n\}$ is a basis for $V(I_1, I_2)$, it spans $V(I_1, I_2)$, so for every $v \in V(I_1, I_2)$, there exist $\alpha_i \in K(I_1, I_2)$, with $i = 1, 2, 3, \cdots n$ such that $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$.

$$\begin{aligned} \phi(v) &= \phi(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) \\ &= \phi(\alpha_1 v_1) + \phi(\alpha_2 v_2) + \cdots + \phi(\alpha_n v_n) \\ &= \alpha_1 \phi(v_1) + \alpha_2 \phi(v_2) + \cdots + \alpha_n \phi(v_n). \\ \text{Thus} \quad & \phi(v) = \alpha_1 \phi(v_1) + \alpha_2 \phi(v_2) + \cdots + \alpha_n \phi(v_n). \end{aligned}$$

Then for every $v \in V(I_1, I_2)$, its image $\phi(v) \in W(I_1, I_2)$ can be written as a linear combination of $\{\phi(v_1), \phi(v_2), \cdots \phi(v_n)\}$. Hence $\{\phi(v_1), \phi(v_2), \cdots \phi(v_n)\}$ spans $W(I_1, I_2)$.
Now if $\alpha_1 \phi(v_1) + \alpha_2 \phi(v_2) + \cdots + \alpha_n \phi(v_n) = 0$ then $\phi(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = 0$.
But then each $\alpha_i = 0$ by the independence of the v_i so $\{\phi(v_1), \phi(v_2), \cdots \phi(v_n)\}$ is linearly independent.
To this end we can conclude that $\phi(\mathbb{B}) = \{\phi(v_1), \phi(v_2), \cdots \phi(v_n)\}$ is a basis for $W(I_1, I_2)$. □

Proposition 2.36. *Let $W(I_1, I_2)$ be a strong refined neutrosophic subspace of a strong refined neutrosophic vector space $V(I_1, I_2)$ over a neutrosophic field $K(I_1, I_2)$. Let $\phi : V(I_1, I_2) \longrightarrow V(I_1, I_2)/W(I_1, I_2)$ be a mapping defined by $\phi(v) = v + W(I_1, I_2)$ for all $v \in V(I_1, I_2)$. Then ϕ is not a neutrosophic vector space homomorphism.*

Proof. It is easily seen, since for $k = 1, 2,\ \phi(I_k) = I_k + W(I_1, I_2) = W(I_1, I_2) \neq I_k$.

Remark 2.37. One of the natural questions would be if $V(I_1, I_2)$ and $W(I_1, I_2)$ are strong (weak) refined neutrosophic vector spaces over a refined neutrosophic field $K(I_1, I_2)(respectively(K))$. Suppose Hom$(V(I_1, I_2), W(I_1, I_2))$ is the collection of all refined neutrosophic vector space homomorphisms from $V(I_1, I_2)$ into $W(I_1, I_2)$, then by defining $+$ and scalar multiplication on Hom$(V(I_1, I_2), W(I_1, I_2))$ can we obtain a refined neutrosophic vector? The answer to this is given in the next proposition .

Proposition 2.38. *Let $V(I_I, I_2)$ and $W(I_1, I_2)$ be any two strong refined neutrosophic vector spaces over the refined neutrosophic field $K(I_1, I_2)$. Let Hom$(V(I_1, I_2), W(I_1, I_2))$ be the collection of all refined neutrosophic vector space homomorphisms from $V(I_1, I_2)$ into $W(I_1, I_2)$, then the triple $(Hom(V(I_1, I_2), W(I_1, I_2)), +, \cdot)$ is not a refined neutrosophic vector space over $K(I_1, I_2)$.*

Proof. $\phi, \psi \in Hom(V(I_1, I_2), W(I_1, I_2))$ then $(\phi + \psi)$ and $(\psi\phi) \in Hom(V(I_1, I_2), W(I_1, I_2))$, since $(\phi + \psi)(I_k) = \phi(I_k) + \psi(I_k) = I_k + I_k = 2I_k \neq I_k$ and $(\alpha\phi)(I_k) = \alpha\phi(I_k) = \alpha\phi(I_k) \neq I_k$ for all $\alpha \in K(I_1, I_2)$ and $k = 1, 2$. □

Definition 2.39. Let $U(I_1, I_2), V(I_1, I_2)$ and $W(I_1, I_2)$ be strong refined neutrosophic vector spaces over a refined neutrosophic field $K(I_1, I_2)$ and let $\phi : U(I_1, I_2) \longrightarrow V(I_1, I_2)$, $\psi : V(I_1, I_2) \longrightarrow W(I_1, I_2)$ be refined neutrosophic vector space homomorphisms.
The composition $\psi\phi : U(I_1, I_2) \longrightarrow W(I_1, I_2)$ is defined by $\psi\phi(u) = \psi(\phi(u))$ for all $u \in U(I_1, I_2)$.

Proposition 2.40. *Let $U(I_1, I_2), V(I_1, I_2)$ and $W(I_1, I_2)$ be strong refined neutrosophic vector spaces over a refined neutrosophic field $K(I_1, I_2)$ and let $\phi : U(I_1, I_2) \longrightarrow V(I_1, I_2)$, $\psi : V(I_1, I_2) \longrightarrow W(I_1, I_2)$ be refined neutrosophic vector space homomorphisms. Then the composition $\psi\phi : U(I_1, I_2) \longrightarrow W(I_1, I_2)$ is a refined neutrosophic vector space homomorphism.*

Proof: That $\psi\phi$ is a vector space homomorphism is clear. Then for $u = I_k \in U(I_1, I_2)$, we have

$$\psi\phi(I_k) = \psi(\phi(I_k)) = \phi(I_k) = I_k \text{ with } k = 1, 2.$$

Hence $\psi\phi$ is a neutrosophic vector space homomorphism.

Let $V(I_1, I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$ and let $\beta : V(I_1, I_2) \longrightarrow V(I_1, I_2)$ be a refined neutrosophic vector space homomorphism. If $\mathbb{B} = \{v_1, v_2 \cdots, v_n\}$ is a basis for $V(I_1, I_2)$, then each $\beta(v_i) \in V(I_1, I_2)$ and thus for $\beta_{ij} \in K(I_1, I_2)$, we can write

$$\begin{array}{lcllll} \beta(v_1) & = & \beta_{11}v_1 + & \beta_{12}v_2 + & \cdots + & \beta_{1n}v_n \\ \beta(v_2) & = & \beta_{21}v_1 + & \beta_{22}v_2 + & \cdots + & \beta_{2n}v_n \\ \vdots & = & \vdots & \vdots & \cdots & \vdots \\ \beta(v_n) & = & \beta_{n1}v_1 + & \beta_{n2}v_2 + & \cdots + & \beta_{nn}v_n. \end{array}$$

Let

$$[\beta]_{\mathbb{B}} = \begin{bmatrix} \beta_{11} & \beta_{21} & \cdots & \beta_{n1} \\ \beta_{12} & \beta_{22} & \cdots & \beta_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ \beta_{1n} & \beta_{2n} & \cdots & \beta_{nm} \end{bmatrix}.$$

$[\beta]_{\mathbb{B}}$ is called the matrix representation of β relative to the basis $\mathbb{B}$.

Proposition 2.41. *Let $V(I_1, I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$ and let $\beta : V(I_1, I_2) \longrightarrow V(I_1, I_2)$ be a refined neutrosophic vector space homomorphism. If $\mathbb{B}$ is a basis for $V(I_1, I_2)$ and v is any element of $V(I_1, I_2)$, then*

$$[\beta]_{\mathbb{B}}[v]_{\mathbb{B}} = [\beta(v)]_{\mathbb{B}}.$$

We give an example to help establish this proposition.

Example 2.42. Let $V(I_1, I_2) = \mathbb{R}^3(I_1, I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2) = \mathbb{R}(I_1, I_2)$ and
let $v = (2 + 3I_1 + I_2, 4 + 3I_1 - I_2, 2 + 4I_1 + 4I_2) \in V(I_1, I_2)$. If $\beta : V(I_1, I_2) \longrightarrow V(I_1, I_2)$ is a refined neutrosophic vector space homomorphism defined by $\beta(v) = v$ for all $v \in V(I_1, I_2)$, then relative to the basis $\mathbb{B} = \{v_1 = (1, 1, 0), v_2 = (1, 0, 1), v_3 = (0, 1, 1)\}$ for $V(I_1, I_2)$, the matrix of β is obtained as

$$[\beta]_{\mathbb{B}} = \begin{bmatrix} 1 + 0I_1 + 0I_2 & 1 + 0I_1 + 0I_2 & 0 + 0I_1 + 0I_2 \\ 1 + 0I_1 + 0I_2 & 0 + 0I_1 + 0I_2 & 1 + 0I_1 + 0I_2 \\ 0 + 0I_1 + 0I_2 & 1 + 0I_1 + 0I_2 & 1 + 0I_1 + 0I_2 \end{bmatrix}.$$

For $v = (2+3I_1+I_2, 4+3I_1-I_2, 2+4I_1+4I_2) \in V(I_1,I_2)$, we have
$\beta(v) = v = (2+I_1-2I_2)v_1 + (2I_1+3I_2)v_2 + (2+2I_2+I_2)v_3$
So that

$$[v]_{\mathbb{B}} = \begin{bmatrix} 2+I_1-2I_2 \\ 2I_1+3I_2 \\ 2+2I_1+I_2 \end{bmatrix} = [\beta(v)]_{\mathbb{B}}$$

and we have

$$[\beta]_{\mathbb{B}}[v]_{\mathbb{B}} = [\beta(v)]_{\mathbb{B}}.$$

Example 2.43. Let $V(I_1,I_2) = \mathbb{R}^2(I_1,I_2)$ be a weak refined neutrosophic vector space over a field $K = R$ and let $v = (1-3I_1+2I_2, 3+I_1-4I_2) \in V(I_1,I_2)$.
If $\beta : V(I_1,I_2) \longrightarrow V(I_1,I_2)$ is a refined neutrosophic vector space homomorphism defined by $\beta(v) = v$ for all $v \in V(I_1,I_2)$, then relative to the basis
$\mathbb{B} = \{v_1 = (1,0), v_2 = (0,1), v_3 = (I_1,0), v_4 = (0,I_1), v_5 = (I_2,0), v_6 = (0,I_2)\}$ for $V(I_1,I_2)$, the matrix of β is obtained as

$$[\beta]_{\mathbb{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

For $v = (1-3I_1+2I_2, 3+I_1-4I_2) \in V(I_1,I_2)$, we have

$$\beta(v) = v = v_1 + 3v_2 - 3v_3 + v_4 + 2v_5 - 4v_6.$$

Therefore,

$$[v]_{\mathbb{B}} = \begin{bmatrix} 1 \\ 3 \\ -3 \\ 1 \\ 2 \\ -4 \end{bmatrix} = [\beta(v)]_{\mathbb{B}}$$

and thus

$$[\beta]_{\mathbb{B}}[v]_{\mathbb{B}} = [\beta(v)]_{\mathbb{B}}.$$

One interesting question to ask will be, can we find a mapping that will transform a refined neutrosophic vector space into a neutrosophic vector space? The answer to this is positive. Since every refined neutrosophic vector space and every neutrosophic vector space are vector spaces, then by relaxing the second axiom in Definition 2.31, the mapping ϕ becomes a classical vector space homomorphism which can be use for such transformation.

Proposition 2.44. *Let $V(I_1,I_2)$ be a weak refined neutrosophic vector space over a field K and let $V(I)$ be a weak neutrosophic vector space over K. Let $\phi : V(I_1,I_2) \longrightarrow V(I)$ be a mapping defined by*

$$\phi((x+yI_1+zI_2)) = (x+(y+z)I) \ \forall (x+yI_1+zI_2) \in V(I_1,I_2) \ \textit{with } x,y,z \in V.$$

Then ϕ is a linear map.

Proof. 1. ϕ is well defined. Suppose $x_1+y_1I_1+z_1I_2 = x_2+y_2I_1+z_2I_2$ then we that $x_1 = x_2, y_1 = y_2$ and $z_1 = z_2$. So,

$$\phi((x_1+y_1I_1+z_1I_2)) = (x_1+(y_1+z_1)I) = x_2+(y_2+z_2)I = \phi(x_2+y_2I_1+z_2I_2).$$

2. For additivity, suppose $(x_1+y_1I_1+z_1I_2), (x_2+y_2I_1+z_2I_2) \in V(I_1,I_2)$ then

$$\begin{aligned} \phi((x_1+y_1I_1+z_1I_2)+(x_2+y_2I_1+z_2I_2)) &= \phi((x_1+x_2)+(y-1+y_2)I_1+(z_1+z_2)I_2) \\ &= (x_1+x_2)+(y_1+y_2+z_1+z_2)I \\ &= (x_1+x_2)+((y_1+z_1)+(y_2+z_2))I \\ &= (x_1+x_2)+((y_1+z_1)I+(y_2+z_2)I) \\ &= (x_1+(y_1+z_1)I)+(x_2+(y_2+z_2)I) \\ &= \phi(x_1+y_1I_1+z_1I_2)+\phi(x_2+y_2I_1+z_2I_2). \end{aligned}$$

3. For homogeneity, let $(x + yI_1 + zI_2) \in V(I_1, I_2)$ and $k \in K$, then

$$\begin{aligned}\phi(k(x_1 + y_1I_1 + z_1I_2)) &= \phi(kx_1 + ky_1I_1 + kz_1I_2)\\ &= kx_1 + (ky_1 + kz_1)I\\ &= kx_1 + k(y_1 + z_1)I\\ &= k(x_1 + (y_1 + z_1)I) = k\phi((x_1 + y_1I_1 + z_1I_2)).\end{aligned}$$

Hence ϕ is a linear map. □

Note 3. *The kernel of this linear map is given by*

$$\begin{aligned}\ker\phi &= \{(x + yI_1 + zI_2) : \phi((x + yI_1 + zI_2)) = (0 + 0I)\}\\ &= \{(x + yI_1 + zI_2) : (x + (y + z)I) = (0 + 0I)\}\\ &= \{(0 + yI_1 + (-y)I_2)\}.\end{aligned}$$

1. *It can be shown that* $\ker\phi$ *is a linear subspace of* $V(I_1, I_2)$.
2. *It can also be shown that* $(ker\phi, +) \cong (V(I_1, I_2), +)$.

Proposition 2.45. *Let $L_k(V(I_1, I_2), V(I))$ be the set of linear maps from a weak refined neutrosophic vector space $V(I_1, I_2)$ over a field K into a weak neutrosophic vector space $V(I)$ over a field K. Define addition and scalar multiplication as below;*

$$(\phi + \psi)(x + yI_1 + zI_2) = \phi((x + yI_1 + zI_2)) + \psi((x + yI_1 + zI_2))$$

and for $k \in K$

$$(k\phi)((x + yI_1 + zI_2)) = k\phi(x + yI_1 + zI_2).$$

Then, it can be shown that $(L_k(V(I_1, I_2), V(I)), +, \cdot)$ is a weak neutrosophic vector space.

Proposition 2.46. *Let $\phi \in L_k(V(I_1, I_2), V(I))$ and $dimV(I_1, I_2),\ dimV(I) < \infty$.*

1. *If $dimV(I_1, I_2) > dimV(I)$, then, no linear map of $V(I_1, I_2)$ to $V(I)$ is one to one.*
2. *If $dimV(I_1, I_2) < dimV(I)$, then, no linear map of $V(I_1, I_2)$ to $V(I)$ is onto.*

Proof. 1. Suppose there exist a function $\phi \in L_k(V(I_1, I_2), V(I))$ which is one to one . Then

$$dimV(I_1, I_2) = dimker\phi + dimIm\phi.$$

Thus, $dimV(I_1, I_2) = dimIm\phi = dimV(I)$ $(dimker\phi = 0$, since ϕ is one to one).
This gives a contradiction. Hence there exist no such function.

2. Suppose there exist a function $\phi \in L_k(V(I_1, I_2), V(I))$ which is onto . Then $Im\phi = V(I)$. Thus,

$$dimV(I_1, I_2) = dimker\phi + dimIm\phi$$

and also

$$dimV(I_1, I_2) \geq dimV(I).$$

Thus

$$dimV(I) > \ dimV(I_1, I_2) \geq dimV(I).$$

This is not possible. Hence there exist no such function.

□

3 Conclusion

This paper studied linear dependence, independence, bases and dimensions of refined neutrosophic vector spaces and presented some of their basic properties. Also, the paper studied refined neutrosophic vector space homomorphisms and established the existence of linear maps between weak refined neutrosophic vector spaces $V(I_1, I_2)$ and weak neutrosophic vector spaces $V(I)$. We hope to present more properties of refined neutrosophic vector spaces in our future papers.

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References

[1] Abobala, M., On some Special Substructures of Neutrosophic Rings and their Properties, International Journal of Neutrosophic Science, Vol. 4 (2), pp. 72-81, 2020.

[2] Abobala, M., On some Special Substructures of Refined Neutrosophic Rings, International Journal of Neutrosophic Science, Vol. 5 (1), pp. 59-66, 2020.

[3] Abobala, M., Classical Homomorphisms between Refined Neutrosophic Rings and Neutrosophic Rings, International Journal of Neutrosophic Science, Vol. 5 (2), pp. 72-75, 2020.

[4] Adeleke, E.O, Agboola, A.A.A and Smarandache, F. Refined Neutrosophic Rings I, International Journal of Neutrosophic Science (IJNS), Vol. 2(2), pp. 77-81, 2020.

[5] Adeleke, E.O, Agboola, A.A.A and Smarandache, F. Refined Neutrosophic Rings II, International Journal of Neutrosophic Science (IJNS), Vol. 2(2), pp. 89-94, 2020.

[6] Agboola, A.A.A., On Refined Neutrosophic Quotient Groups, International Journal of Neutrosophic Science, vol. 5 (2), pp. 76-82, 2020. DOI:10.5281/zenodo.3828609.

[7] Agboola, A.A.A., Ibrahim, A.M. and Adeleke, E.O, Elementary Examination of NeutroAlgebras and AntiAlgebras Viz-a-Viz the Classical Number Systems, Vol. 4, pp. 16-19, 2020.

[8] Agboola, A.A.A. On Refined Neutrosophic Algebraic Structures, Neutrosophic Sets and Systems, Vol. 10, pp 99-101, 2015.

[9] Agboola, A.A.A., Akinola, A.D. and Oyebola, O.Y., Neutrosophic Rings I, Int. J. of Math. Comb. Vol 4, pp. 1-14, 2011.

[10] Agboola, A.A.A., Adeleke, E.O. and Akinleye, S.A., Neutrosophic Rings II, Int. J. of Math. Comb. Vol. 2, pp 1-8, 2012.

[11] Agboola, A.A.A. Akwu, A.O., and Oyebo, Y.T., Neutrosophic Groups and Neutrosopic Subgroups, Int. J. of Math. Comb. Vol. 3, pp. 1-9, 2012.

[12] Agboola, A.A.A. and Akinleye, S.A., Neutrosophic Vector Spaces, Neutrosophic Sets and Systems Vol. 4, pp. 9-18. 2014.

[13] Atanassov, K., Intuitionistic fuzzy sets, Fuzzy Sets and Systems, Vol. 20, pp. 87-96, 1986.

[14] Bera, T. and Mahapatra,N.K., Introduction to neutrosophic soft groups, Neutrosophic Sets and Systems, Vol. 13, pp, 118-127, 2016, doi.org/10.5281/zenodo.570845.

[15] Bera, T. and Mahapatra, N.K., On neutrosophic normal soft groups, International Journal of Applied and Computational Mathematics., Vol. 3, pp 3047-3066, 2017. DOI 10.1007/s40819-016-0284-2.

[16] Bera, T. and Mahapatra, N.K., On neutrosophic soft rings, OPSEARCH, Vol. 54, pp. 143-167, 2017. DOI 10.1007/ s12597-016-0273-6.

[17] Bera, T and Mahapatra, N. K., On neutrosophic soft linear spaces, Fuzzy Information and Engineering, Vol. 9, pp 299-324, 2017.

[18] Bera, T and Mahapatra, N. K., On neutrosophic soft field, IJMTT, Vol. 56(7), pp. 472-494, 2018.

[19] Deli, I., Broumi, S. and Smarandache, F., On neutrosophic refined sets and their applications in medical diagnosis, Journal of New Theory, Vol. 6, pp. 88-98, 2015.

[20] Ibrahim, M.A., Agboola, A.A.A, Badmus, B.S, Akinleye, S.A., On Refined Neutrosophic Vector Spaces I, (submitted for publication) in International Journal of Neutrosophic Science.

[21] Ibrahim, M.A., Agboola, A.A.A, Adeleke, E.O, Akinleye, S.A., Introduction to Neutrosophic Subtraction Algebra and Neutrosophic Subtraction Semigroup, International Journal of Neutrosophic Science (IJNS), Vol. 2(2), pp. 47-62, 2020.

[22] Ibrahim, M.A., Agboola, A.A.A, Adeleke, E.O, Akinleye, S.A., On Neutrosophic Quadruple Hypervector Spaces, International Journal of Neutrosophic Science (IJNS), Vol. 4(1), pp. 20-35, 2020.

[23] Liu, P. and Wang, Y., Multiple attribute decision-making method based on single-valued neutrosophic normalized weighted Bonferroni mean, Neural Computing and Applications, Vol. 25 (7-8), pp. 2001-2010, 2014.

[24] Smarandache, F., Introduction to Neutrosophic Sociology (NeutroSociology), Pons Publishing House/Pons asb1, Quai du Batelage, 5, 1000-Bruxelles, Belgium, 2019.

[25] Smarandache, F. and Pramanik, S., New Neutrosophic Sets via Neutrosophic Topological Spaces, In Operational Research; Eids.; Pons Editions: Brussels, Belgium, Vol. I, pp. 189-209, 2017.

[26] Smarandache, F. and Pramanik, S., Neutrosophic crisp sets via neutrosophic crisp topological spaces, Neutrosophic Sets and Systems, Vol. 13, pp. 96-104, 2016.

[27] Smarandache, F. and Pramanik, S., On Generalized Closed Sets and Generalized Pre-Closed Sets in Neutrosophic Topological Spaces, Mathematics, Vol. 7(1), pp. 1-12, 2019. DOI:DOI.ORG/10.3390/math7010001.

[28] Smarandache, F., A Unifying Field in Logics: Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability, American Research Press, Rehoboth, 2003.

[29] Smarandache, F., n-Valued Refined Neutrosophic Logic and Its Applications in Physics, Progress in Physics, Vol. 4, pp. 143-146, 2013.

[30] Smarandache, F., (T,I,F)- Neutrosophic Structures, Neutrosophic Sets and Systems, Vol.8, pp.3-10, 2015.

[31] Vasantha Kandasamy, W.B and Smarandache,F., Basic Neutrosophic Algebraic Structures and Their Applications to Fuzzy and Neutrosophic Models, Hexis, Church Rock, (2004), http://fs.unm.edu/ScienceLibrary.htm

[32] Vasantha Kandasamy, W.B. and Florentin Smarandache, Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic Structures, Hexis, Phoenix, Arizona, (2006), http://fs.unm.edu/ScienceLibrary.htm

[33] Vasantha Kandasamy, W.B., Neutrosophic Rings, Hexis, Phoenix, Arizona,(2006) http://fs.unm.edu/ScienceLibrary.htm

[34] Vasantha Kandasamy, W.B., Ilanthenral Kandasamy and Smarandache, F., Neutrosophic Quadruple Vector Spaces and Their Properties, Mathematics, Vol. 7, pp. 758-765, 2019. http://fs.unm.edu/ScienceLibrary.htm

[35] Wadei Al-Omeri and Smarandache, F., New Neutrosophic Set via Neutrosophic Topological Spaces. Excerpt from Neutrosophic Operation Research Vol I, Pons Editions: Brussels, Belgium, pp. 189-209, 2017.

[36] Wadei Al-Omeri, Neutrosophic crisp Sets Via Neutrosophic crisp Topological Spaces, Neutrosophic Set and Systems Vol 13, pp 96- 104, 2016.

[37] Wadei Al-Omeri and Saeid Jafari, On Generalized Closed Sets and Generalized Pre-Closed Sets in Neutrosophic Topological Spaces, Mathematics, Vol 7, pp 1- 12, 2019. Doi: doi.org/10.3390/math/7010001.

[38] Zadeh, L.A., Fuzzy Sets, Information and Control, Vol. 8, pp. 338-353, 1965.

Aggregation Operators of Bipolar Neutrosophic Soft Sets and It's Applications In Auto Car Selection

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Abstract: In this paper, it is intended to study the concept of bipolar Neutrosophic soft set ($BNSS$). It is aimed to defined bipolar Neutrosophic soft set. Definitions and operations have been presented the BNSS. Then we present an aggregation BNSS operator and decision making algorithm depend on the BNSS. Number-based examples discussed to show (ability to be done) and efficiency of the advanced method.

Keywords: Bipolar Soft Sets, Neutrosophic Sets, Aggregation Operators, Soft Sets, Car Selection

1. Introduction

The uncommon theory of fuzzy set Zadeh [1] was presented as the addition of a crisp set by expanding the truthfulness value set to [0,1].In fuzzy set theory, if the association membership of an element is $\mu(x)$ then the non-association membership $1-\mu(x)$ and so it fixed Intuitionistic fuzzy set presented by Atanassov in 1986 [2] and form an addition of fuzzy set by expanding the truth value to the lattice $[0,1] \times [0,1]$. In daily life, problems, engineering, medical diagnosis, in the economy, and social science in many areas have to do with facts that have in its uncertainties. This problem may not be positively demonstrated by the existing methodologies in Greek and Latin math. There are approximately well- known methodological theories for discuss with vagueness such as FSS [3], IFSS [2], rough set [4], etc. To try to deal with a problem which is pointed out in [5]. To grip with these complications, Molodtsov coined the concept of a SS as a new methodical implement for studying with hesitancies.

BSS and basic operations defined by Shabir and Naz [6] in 2013.Lee [7], the idea of bipolar FS as a simplification of a FS. A bipolar fuzzy set (BFS) is an addition of fuzzy sets whose association degree range is [1,-1]. In a BFS , the association degree of a component means that the components are immaterial to the correspondent property, the association degree [0, 1] of a component shows that the elements somewhat fulfils the property and the association degree [-1, 0] of an element shows that the elements somewhat fulfil the implied counter property.

Neutrosophic sets suggested by Smarandache [8] (1998,1999,2002,2005,2006,2010) which is a extension of fuzzy set and the IFS is a great instrument to deal with incomplete information which happen in the real world Neutrosophic sets are categorized. Neutrosophic set proposed by Smarandache in by truthfulness (T), Indeterminacy (I), and falseness (F). This concept is very significant in many applications areas since indeterminacy is computed explicitly and truthfulness, Indeterminacy, and falseness are independent.

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Wang et al [9] coined the idea of single-valued neutrosophic sets. The SVNS can express the truthness, degree, indeterminate and stable information. All the reason described by the SVNS is very suitable for human approach due to the deficiencies of knowledge that human accept.

Then Deli et al [10] advanced multi-criteria bipolar $\mathcal{NS}$ and study their applications in DM. Ali et al [11] suggested the concept of cubic set with applications in specimen identification. Broumi et al [12,13], proposed BSVNS, graph theory and its shortest path problem.

Molodtsov in [14] has some possible applications of soft set theory. Furthermore, Maji in [4] presented some results as an application of neutrosophic SS in DM. Also, some writer planning the ideas of neutrosophic SS. Soft set to deal undeserving in a parameterized way. The soft set is a mathematical arrangement which has the capability of an independent state of parameterizations lack, conditions of fuzzy set, rough set, expectations, etc. Also, some writers studied the theory of neutrosophic soft set in [10, 15, 16, 4, and 17]. Jafar et al [20-24] worked on decision making using soft sets , fuzzy soft sets, intuitionistic soft sets and NS. Saqlain at el [28-30] worked a lot in Neutrosophic environment which help us a lot.Jafar et al [31] discussed a new technology in agriculture using Neutrosophic sot sets.

This article is committed to suggested bipolar neutrosophic sets which are a mixture of a SS and BNSS. Initially, then we launch the BNSS and discuss some fundamentals things with instructive examples. Moreover, we defined some algebraic functions of the BNSS such as the complement, union, intersection, etc. Then, we defined aggregation bipolar neutrosophic sets operators are bipolar neutrosophic soft set residents decision-making algorithms based on bipolar $\mathcal{NSS}$. Furthermore, we offer the DM method for DM difficulties including the bipolar neutrosophic SS and present a sample associated with this technique.

Motivation and Objectives

Motivation behind this paper is to present decision making issues in some different notions. Generally every one is discussing decision Making issues only with one direction we extened the work of Deli et al [10] and discussed the concept of NSS in bipolar environment. We discussed Agrigation operator and then use it to the selection of best car in market.

2. Prelimanaries

2.1 Definition-1: [8]

If $\mathfrak{U}$ is a universal set. Neutrosophic sets $(\mathcal{NS})$ $\mathcal{K}$ in $\mathfrak{U}$ is categorized by truthfulness $\mathcal{T}_{\mathcal{K}}$, an indetermination $\mathcal{I}_{\mathcal{K}}$ and a falseness $\mathcal{F}_{\mathcal{K}}$. Standard or non- standard elements are $\mathcal{T}_{\mathcal{K}}, \mathcal{I}_{\mathcal{K}}, \mathcal{F}_{\mathcal{K}}$ of $]0^-, 1^+[$ is defined as:

$$\mathcal{K} = \{< \wp, \left(\mathcal{T}_{\mathcal{K}}(\wp), \mathcal{I}_{\mathcal{K}}(\wp), \mathcal{F}_{\mathcal{K}}(\wp)\right) >: \wp\, \mathcal{E}\, \mathfrak{U}\ , \mathcal{T}_{\mathcal{K}}(\wp), \mathcal{I}_{\mathcal{K}}(\wp), \mathcal{F}_{\mathcal{K}}(\wp)\ \in\ ]0^-, 1^+[\ \}.$$

So that there is no confinement to the sum of $\mathcal{T}_{\mathcal{K}}(\wp), \mathcal{I}_{\mathcal{K}}(\wp)$ *and* $\mathcal{F}_{\mathcal{K}}(\wp)$, so $0^- \leq \mathcal{T}_{\mathcal{K}}(\wp) + \mathcal{I}_{\mathcal{K}}(\wp) + \mathcal{F}_{\mathcal{K}}(\wp) \leq 3^+$.

2.2 Definition-2: [9]

If $\mathcal{C}$ is a universal set. A single-valued neutrosophic sets $(\mathcal{SVNS})$ $\mathcal{B}$, that is applicable in engineering and scientific field, in $\mathcal{C}$ is categorized by a truthfulness $\mathcal{T}_{\mathcal{B}}$, an indetermination $\mathcal{I}_{\mathcal{B}}$ and a falseness $\mathcal{F}_{\mathcal{B}}$. Standard elements are $\mathcal{T}_{\mathcal{B}}(\wp), \mathcal{I}_{\mathcal{B}}(\wp), \mathcal{F}_{\mathcal{B}}(\wp)$ of [0, 1] is defined as:

$$\mathcal{B} = \left\{< \wp, \left(\mathcal{T}_{\mathcal{B}}(\wp), \mathcal{I}_{\mathcal{B}}(\wp), \mathcal{F}_{\mathcal{B}}(\wp)\right) >: \wp\ \in\ \mathcal{C}, \mathcal{T}_{\mathcal{B}}(\wp), \mathcal{I}_{\mathcal{B}}(\wp), \mathcal{F}_{\mathcal{B}}(\wp)\ \in [0,1]\right\}.$$

2.3 Definition-3: [14]

If $\mathfrak{U}$ be a universal set, $\mathbb{E}$ be a constraint that expresses the elements of $\mathfrak{U}$, $\widetilde{\mathbb{A}} \subseteq \mathbb{E}$. A function $f_{\widetilde{\mathbb{A}}}$ is known as soft set $\mathcal{F}_{\widetilde{\mathbb{A}}}$ w.r.t the universal set $\mathfrak{U}$ and represented by:

$$f_{\widetilde{\mathbb{A}}}: \mathbb{E} \rightarrow \mathcal{P}(\mathfrak{U}) \text{ s.t } f_{\widetilde{\mathbb{A}}}(\wp) = \emptyset \text{ if } \wp \in \mathbb{E} - \widetilde{\mathbb{A}}$$

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Everywhere $\not{f}_{\widetilde{\mathbb{A}}}$ is known as an approximation function$\mathcal{F}_{\widetilde{\mathbb{A}}}$. Also, soft set is categorized by the family of the subset of the set $\mathfrak{U}$, and consequently, it can be inscribed a set of well-ordered pairs.

$$\mathcal{F}_{\widetilde{\mathbb{A}}} = \left\{ \left(\grave{\wp}, \not{f}_{\widetilde{\mathbb{A}}}(\grave{\wp}) \right) : \ \grave{\wp} \in \mathbb{E}, \not{f}_{\widetilde{\mathbb{A}}}(\grave{\wp}) = \emptyset \ if \ \grave{\wp} \in \grave{\mathbb{E}} - \widetilde{\mathbb{A}} \right\}$$

2.4 Definition-4: [15]

If $\mathfrak{U}$ is a universal set,$\mathcal{N}(\mathfrak{U})$ be a group of all Neutrosophic sets on $\mathfrak{U}$, $\mathbb{E}$ be constraint that express the components of $\mathfrak{U}$.Then a Neutrosophic soft set $\mathcal{N}$ above $\mathfrak{U}$ is clear by set-valued function represented by:

$$\not{f}_{\mathcal{N}} : \mathbb{E} \longrightarrow \mathcal{N}(\mathfrak{U})$$

The set $\not{f}_{\mathcal{N}}(\grave{\wp})$ is the set of $\grave{\wp}$ elements of($\mathcal{NSS}$) that may be arbitrary few of them have empty and non-empty intersection.

$$\mathcal{N} = \{ (\grave{\wp}, \{ < \mathfrak{U}, \mathcal{T}_{\not{f}_{\mathcal{N}}(\grave{\wp})}(\mathfrak{u}), \mathcal{I}_{\not{f}_{\mathcal{N}}(\grave{\wp})}(\mathfrak{u}), \mathcal{F}_{\not{f}_{\mathcal{N}}(\grave{\wp})}(\mathfrak{u}) >: \grave{\wp} \in \mathfrak{U} \} : \grave{\wp} \in \mathbb{E}) \}$$

Everywhere $\quad \mathcal{T}_{\not{f}_{\mathcal{N}}(\grave{\wp})}(\mathfrak{u}), \mathcal{I}_{\not{f}_{\mathcal{N}}(\grave{\wp})}(\mathfrak{u}), \mathcal{F}_{\not{f}_{\mathcal{N}}(\grave{\wp})}(\mathfrak{u}) \in [0,1]$

2.5 Definition-5: [15]

If $\breve{\mathcal{N}}_{\dot{1}}$and $\breve{\mathcal{N}}_{\dot{2}}$ be two ($\mathcal{NSS}$) over universal set$\mathfrak{U}$, correspondingly.

i. $\breve{\mathcal{N}}_{\dot{1}}$ is known as neutrosophic soft subset of $\breve{\mathcal{N}}_{\dot{2}}$ is $\widetilde{\mathbb{A}} \subseteq \widetilde{\mathbb{B}}$ and $\mathcal{T}_{\not{f}\widetilde{\mathcal{N}_{\dot{1}}(\grave{\wp})}}(\mathfrak{u}) \leq \mathcal{T}_{\not{f}\widetilde{\mathcal{N}_{\dot{2}}(\grave{\wp})}}(\mathfrak{u})$, $\mathcal{I}_{\not{f}\widetilde{\mathcal{N}_{\dot{1}}(\grave{\wp})}}(\mathfrak{u}) \leq \mathcal{I}_{\not{f}\widetilde{\mathcal{N}_{\dot{2}}(\grave{\wp})}}(\mathfrak{u}), \mathcal{F}_{\not{f}\widetilde{\mathcal{N}_{\dot{1}}(\grave{\wp})}}(\mathfrak{u}) \geq \mathcal{F}_{\not{f}\widetilde{\mathcal{N}_{\dot{2}}(\grave{\wp})}}(\mathfrak{u})$, $\forall \grave{\wp} \in \widetilde{\mathbb{A}}$, $\mathfrak{u} \in \mathfrak{U}$.

ii. $\breve{\mathcal{N}}_{\dot{1}}$ And $\breve{\mathcal{N}}_{\dot{2}}$ are equal if $\breve{\mathcal{N}}_{\dot{1}} \subseteq \breve{\mathcal{N}}_{\dot{2}}$ and$\breve{\mathcal{N}}_{\dot{2}} \subseteq \breve{\mathcal{N}}_{\dot{1}}$.

2.6 Definition-6: [15]

If $\breve{\mathcal{N}}_{\dot{1}}$and $\breve{\mathcal{N}}_{\dot{2}}$ be two ($\mathcal{NSS}$). Now,

1) $\mathcal{N}_{\dot{1}}{}^{\mathbb{C}}$ is said to be a complement of $\mathcal{NSS}$ is defined by:

$$\mathcal{N}_{\dot{1}}{}^{\mathbb{C}} = \left\{ \left(\grave{\wp}, \left\{ < u, \mathcal{F}_{\not{f}\widetilde{\mathcal{N}_{\dot{1}}(\grave{\wp})}}(\mathfrak{u}), 1 - \mathcal{I}_{\not{f}\widetilde{\mathcal{N}_{\dot{1}}(\grave{\wp})}}(\mathfrak{u}), \mathcal{T}_{\not{f}\widetilde{\mathcal{N}_{\dot{1}}(\grave{\wp})}}(\mathfrak{u}) >: \grave{\wp} \in \mathfrak{u} \right\} : \grave{\wp} \in \mathbb{E} \right) \right\}$$

2) The union of $\breve{\mathcal{N}}_{\dot{1}}$and $\breve{\mathcal{N}}_{\dot{2}}$ can be defined as $\breve{\mathcal{N}}_{\dot{3}}$=$\breve{\mathcal{N}}_{\dot{1}} \cup \breve{\mathcal{N}}_{\dot{2}}$ and written as:

$$\breve{\mathcal{N}}_{\dot{3}} = \left\{ \left(\grave{\wp}, \left\{ < u, \mathcal{T}_{\not{f}\widetilde{\mathcal{N}_{\dot{3}}(\grave{\wp})}}(\mathfrak{u}), \mathcal{I}_{\not{f}\widetilde{\mathcal{N}_{\dot{3}}(\grave{\wp})}}(\mathfrak{u}), \mathcal{F}_{\not{f}\widetilde{\mathcal{N}_{\dot{3}}(\grave{\wp})}}(\mathfrak{u}) >: \grave{\wp} \in \mathfrak{u} \right\} : \grave{\wp} \in \mathbb{E} \right) \right\}$$

Everywhere

$$\mathcal{T}_{\not{f}\widetilde{\mathcal{N}_{\dot{3}}(\grave{\wp})}}(\mathfrak{u}) = max \left(\mathcal{T}_{\not{f}\widetilde{\mathcal{N}_{\dot{1}}(\grave{\wp})}}(\mathfrak{u}), \mathcal{T}_{\not{f}\widetilde{\mathcal{N}_{\dot{2}}(\grave{\wp})}}(\mathfrak{u}) \right)$$

$$\mathcal{I}_{\not{f}\widetilde{\mathcal{N}_{\dot{3}}(\grave{\wp})}}(\mathfrak{u}) = min \left(\mathcal{I}_{\not{f}\widetilde{\mathcal{N}_{\dot{1}}(\grave{\wp})}}(\mathfrak{u}), \mathcal{I}_{\not{f}\widetilde{\mathcal{N}_{\dot{2}}(\grave{\wp})}}(\mathfrak{u}) \right)$$

$$\mathcal{F}_{\not{f}\widetilde{\mathcal{N}_{\dot{3}}(\grave{\wp})}}(\mathfrak{u}) = min \left(\mathcal{F}_{\not{f}\widetilde{\mathcal{N}_{\dot{1}}(\grave{\wp})}}(\mathfrak{u}), \mathcal{F}_{\not{f}\widetilde{\mathcal{N}_{\dot{2}}(\grave{\wp})}}(\mathfrak{u}) \right)$$

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3) The intersection of $\breve{\mathcal{N}}_{\dot{1}}$ and $\breve{\mathcal{N}}_{\dot{2}}$ can be defined as $\breve{\mathcal{N}}_{\dot{4}}=\breve{\mathcal{N}}_{\dot{1}} \ddot{\cap}\, \breve{\mathcal{N}}_{\dot{2}}$ and written as:

$$\breve{\mathcal{N}}_{\dot{4}} = \left\{\left(\grave{\wp}, \left\{< \ell, \mathcal{T}_{\not{f}\widetilde{\mathcal{N}_4(\grave{\wp})}}(\ell), \mathcal{I}_{\not{f}\widetilde{\mathcal{N}_4(\grave{\wp})}}(\ell), \mathcal{F}_{\not{f}\widetilde{\mathcal{N}_4(\grave{\wp})}}(\ell) >: \grave{\wp} \in \ell \right\} : \grave{\wp} \in \mathbb{E}\right)\right\}$$

Everywhere

$$\mathcal{T}_{\not{f}\widetilde{\mathcal{N}_4(\grave{\wp})}}(\ell) = min\left(\mathcal{T}_{\not{f}\widetilde{\mathcal{N}_{\dot{1}}(\grave{\wp})}}(\ell), \mathcal{T}_{\not{f}\widetilde{\mathcal{N}_{\dot{2}}(\grave{\wp})}}(\ell)\right)$$

$$\mathcal{I}_{\not{f}\widetilde{\mathcal{N}_4(\grave{\wp})}}(\ell) = max\left(\mathcal{I}_{\not{f}\widetilde{\mathcal{N}_{\dot{1}}(\grave{\wp})}}(\ell), \mathcal{I}_{\not{f}\widetilde{\mathcal{N}_{\dot{2}}(\grave{\wp})}}(\ell)\right)$$

$$\mathcal{F}_{\not{f}\widetilde{\mathcal{N}_4(\grave{\wp})}}(\ell) = max\left(\mathcal{F}_{\not{f}\widetilde{\mathcal{N}_{\dot{1}}(\grave{\wp})}}(\ell), \mathcal{F}_{\not{f}\widetilde{\mathcal{N}_{\dot{2}}(\grave{\wp})}}(\ell)\right)$$

2.7 Definition-7: [7]

If $\mathfrak{U}$ is a universal set. Ω be Bipolar FS in $\mathfrak{U}$. It can be describe as:

$$\Omega = \left\{\left(\ell, \mathcal{T}^{+}(\ell), \mathcal{T}^{-}(\ell)\right) : \ell \in \mathfrak{U}\right\}$$

Everywhere $\mathcal{T}^{+} \longrightarrow [0,1]$ and $\mathcal{T}^{-} \longrightarrow [-1,0]$.The positive truthfulness $\mathcal{T}^{+}(\ell)$ correspondent to bipolar fuzzy set and negative truthfulness $\mathcal{T}^{-}(\ell)$ of a component $\ell \in \mathfrak{U}$ to about implied counter-property correspondent toΩ.

2.8 Definition-8: [13]

If $\mathfrak{U}$ be a universal set.$\mathbb{E}$ be a constraint that express the element of $\mathfrak{U}$. A bipolar fuzzy soft set $\circledast$ in $\mathfrak{U}$. It can be written as: $\circledast = \left\{\left(\ddot{e}, \left\{\left(\ell, \mathcal{T}^{+}(\ell), \mathcal{T}^{-}(\ell)\right) : \ell \in \mathfrak{U}\right\}\right) : \ddot{e} \in \mathbb{E}\right\}$

Everywhere $\mathcal{T}^{+} \longrightarrow [0,1]$ and $\mathcal{T}^{-} \longrightarrow [-1,0]$.The positive truthfulness$\mathcal{T}^{+}(\ell)$, correspondent to bipolar fuzzy set $\circledast$ and negative truthfulness $\mathcal{T}^{-}(\ell)$ of a component $\ell \in \mathfrak{U}$ to about implied counter-property correspondent to $\circledast$.

2.9 Definition-9: [10]

If $\mathfrak{U}$ is a universal set. A Bipolar neutrosophic set in $\mathfrak{U}$. It is denoted by $\hat{\mathcal{A}}$.It can be written as:

$$\hat{\mathcal{A}} = \left\{\left(\ell, \mathcal{T}^{+}(\ell), \mathcal{I}^{+}(\ell), \mathcal{F}^{+}(\ell), \mathcal{T}^{-}(\ell), \mathcal{I}^{-}(\ell), \mathcal{F}^{-}(\ell)\right) : \ell \in \mathfrak{U}\right\}$$

Everywhere $\mathcal{T}^{+}, \mathcal{I}^{+}, \mathcal{F}^{+} \longrightarrow [,1]$ and$\mathcal{T}^{-}, \mathcal{I}^{-}, \mathcal{F}^{-} \longrightarrow [-1,0]$. The positive degrees truthfulness, indeterminacy and falseness are denoted by $\mathcal{T}^{+}(\ell), \mathcal{I}^{+}(\ell), \mathcal{F}^{+}(\ell)$ correspondent to bipolar $\mathcal{NS}$ $\hat{\mathcal{A}}$ and negative degrees truthfulness, indeterminacy and falseness are denoted by $\mathcal{T}^{-}(\ell), \mathcal{I}^{-}(\ell), \mathcal{F}^{-}(\ell)$ of a component $\ell \in \mathfrak{U}$ to about implied counter- property correspondent to bipolar neutrosophic set $\hat{\mathcal{A}}$.

3. Bipolar Neutrosophic Soft Sets

In this segment, we propose the concept of $\mathcal{NSS}$ and its operations.

3.1 Definition-10

If $\mathfrak{U}$ is a universal set. $\mathbb{E}$ be a constraint that expresses the element of $\mathfrak{U}$. A bipolar neutrosophic soft set $\mathfrak{B}$ in $\mathfrak{U}$.It can be describe as:

$$\mathfrak{B} = \left\{\left(\ddot{e}, \left\{\left(\ell, \mathcal{T}^{+}(\ell), \mathcal{I}^{+}(\ell), \mathcal{F}^{+}(\ell), \mathcal{T}^{-}(\ell), \mathcal{I}^{-}(\ell), \mathcal{F}^{-}(\ell)\right) : \ell \in \mathfrak{U}\right\}\right) : \ddot{e} \in \mathbb{E}\right\}$$

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Everywhere $\mathcal{T}^+, \mathcal{I}^+, \mathcal{F}^+ \longrightarrow [,1]$ and $\mathcal{T}^-, \mathcal{I}^-, \mathcal{F}^- \longrightarrow [-1,]$. The positive degrees truthfulness, indeterminacy and falseness are denoted by $\mathcal{T}^+(\mathit{li}), \mathcal{I}^+(\mathit{li}), \mathcal{F}^+(\mathit{li})$ corresponding to bipolar $\mathcal{NSS}\ \mathfrak{B}$ and negative degrees truthfulness, indeterminacy, and falseness are denoted by $\mathcal{T}^-(\mathit{li}), \mathcal{I}^-(\mathit{li}), \mathcal{F}^-(\mathit{li})$ of a component $\mathit{li} \in \mathfrak{U}$ to about implied counter- property correspondent to bipolar $\mathcal{NSS}\ \mathfrak{B}$.

Example-1: If $\mathfrak{U} = \{\mathit{li}_1, \mathit{li}_2, \mathit{li}_3\}$, $\mathbb{E} = \{\ddot{e}_1, \ddot{e}_2\}$.After that bipolar $\mathcal{NSS}\ \mathfrak{B}_1 and\ \mathfrak{B}_2$ above $\mathfrak{U}$ is given as, respectively;

$$\mathfrak{B}_1 = \{(\ddot{e}_1, \{(\mathit{li}_1, .6, .9, .2, -.6, -.8, -.4), (\mathit{li}_2, .7, .9, .8, -.6, -.8, -.3), (\mathit{li}_3, .7, .8, .2, -.6, -.9, -.9))\},$$

$$(\ddot{e}_2, \{(\mathit{li}_1, .9, .6, .7, -.6, -.8, -.4), (\mathit{li}_2, .5, .4, .8, -.6, -.8, -.3), (\mathit{li}_3, .8, .6, .2, -.5, -.8, -.7))\}$$

And

$$\mathfrak{B}_2 = \{(\ddot{e}_1, \{(\mathit{li}_1, .3, .8, .6, -.7, -.8, -.3), (\mathit{li}_2, .4, .6, .8, -.2, -.8, -.3), (\mathit{li}_3, .7, .3, .6, -.6, -.5, -.1))\},$$

$$(\ddot{e}_2, \{(\mathit{li}_1, .2, .6, .7, -.1, -.8, -.3), \mathit{li}_2, .3, .9, .8, -.6, -.4, -.6), \mathit{li}_3, .8, .5, .2, -.5, -.8, -.1))\}$$

3.2 Definition-11

An empty bipolar Neutrosophic soft set $\mathfrak{B}^{\emptyset}$ with respect to universal set $\mathfrak{U}$. it can be written as:

$$\mathfrak{B}^{\emptyset} = \{(\ddot{e}, \{(\mathit{li}, 0, 0, 1, -1, 0,)): \mathit{li} \in \mathfrak{U}\}): \ddot{e} \in \mathbb{E}\}$$

3.3 Definition-12

An absolute bipolar Neutrosophic soft set $\mathfrak{B}^{\mathcal{U}}$ with respect to universal set $\mathfrak{U}$. it can be written as:

$$\mathfrak{B}^{\mathcal{U}} = \{(\ddot{e}, \{(\mathit{li}, 1, 1, 0, 0, -1, -1)): \mathit{li} \in \mathfrak{U}\}): \ddot{e} \in \mathbb{E}\}$$

Example-2: If $\mathfrak{U} = \{\mathit{li}_1, \mathit{li}_2, \mathit{li}_3\}$, $\mathbb{E} = \{\ddot{e}_1, \ddot{e}_2\}$

1. Empty bipolar NSS $\mathfrak{B}^{\emptyset}$ in $\mathfrak{U}$ is given as:

$$\mathfrak{B}^{\emptyset} = \{(\ddot{e}_1, \{(\mathit{li}_1, 0,0, 1, -1,0,), (\mathit{li}_2, 0,0, 1, -1,0, 0), (\mathit{li}_3, 0,0, 1, -1,0, 0))\},$$

$$(\ddot{e}_2, \{(\mathit{li}_1, 0,0, 1, -1,0, 0), (\mathit{li}_2, 0,0, 1, -1, 0,0), (\mathit{li}_3, 0, 0, 1, -1, 0,0))\}$$

2. Absolute bipolar NSS $\mathfrak{B}^{\mathcal{U}}$ in $\mathfrak{U}$ is given as:

$$\mathfrak{B}^{\mathcal{U}} = \{(\ddot{e}_1, \{(\mathit{li}_1, 1, 1,0, 0, -1, -1), (\mathit{li}_2, 1, 1, 0, 0, -1, -1), (\mathit{li}_3, 1, 1, 0,0, -1, -1))\},$$

$$(\ddot{e}_2, \{(\mathit{li}_1, 1, 1,0, 0, -1, -1), (\mathit{li}_2, 1, 1,0, 0, -1, -1), (\mathit{li}_3, 1, 1, 0,0, -1, -1))\}$$

3.4 Definition-13

If $\mathfrak{B}_{j} = \left\{\left(\ddot{e}, \left\{\left(\mathit{li}, \mathcal{T}_{j}^+(\mathit{li}), \mathcal{I}_{j}^+(\mathit{li}), \mathcal{F}_{j}^+(\mathit{li}), \mathcal{T}_{j}^-(\mathit{li}), \mathcal{I}_{j}^-(\mathit{li}), \mathcal{F}_{j}^-(\mathit{li})\right): \mathit{li} \in \mathfrak{U}\right\}\right): \ddot{e} \in \mathbb{E}\right\}$ for $j = 1,2, \ldots\ldots, n$ be two bipolar NSS with respect to universal set. Then, $\mathfrak{B}_1$ is a bipolar neutrosophic soft subset $\mathfrak{B}_2$ and It is represented by $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$, if $\mathcal{T}_1^+(\mathit{li}) \lesssim \mathcal{T}_2^+(\mathit{li})$, $\mathcal{I}_1^+(\mathit{li}) \gtrsim \mathcal{I}_2^+(\mathit{li})$, $\mathcal{F}_1^+(\mathit{li}) \gtrsim \mathcal{F}_2^+(\mathit{li})$, $\mathcal{T}_1^-(\mathit{li}) \gtrsim \mathcal{T}_2^-(\mathit{li})$, $\mathcal{I}_1^-(\mathit{li}) \lesssim \mathcal{I}_2^-(\mathit{li})$, $\mathcal{F}_1^-(\mathit{li}) \lesssim \mathcal{F}_2^-(\mathit{li})$ for all $(\ddot{e}, \mathit{li}) \in \mathbb{E} \times \mathfrak{U}$.

Example-3: If $\mathfrak{U} = \{\mathit{li}_1, \mathit{li}_2\}$, $\mathbb{E} = \{\ddot{e}_1, \ddot{e}_2\}$. If

$$\mathfrak{B}_1 = \{(\ddot{e}_1, \{(\mathit{li}_1, .8, .7, .2, -.4, -.8, -.3), (\mathit{li}_2, .7, .6, .7, -.4, -.8, -.3)\}),$$

$$(\ddot{e}_2, \{(\mathit{li}_1, .6, .7, .5, -.4, -.6, -.3), (\mathit{li}_2, .3, .8, .4, -.4, -.7, -.2)\})$$

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And $\mathfrak{B}_2 = \{(\ddot{e}_1, \{(\,ll_1, .9, .3, .2, -.6, -.7, -.3), (\,ll_2, .9, .2, .4, -.8, -.6, -.1)\}),$

$(\ddot{e}_2, \{(\,ll_1, .8, .7, .5, -.4, -.6, -.3),\ ll_2, .5, .8, .4, -.9, -.7, -.1)\})$

Then, we have $\mathbf{\mathfrak{B}_1 \subseteq \mathfrak{B}_2}$.

3.5 Definition-14

If $\mathfrak{B}_{j} = \left\{\left(\ddot{e}, \left\{\left(\,ll, \mathcal{T}_{j}^{+}(\,ll), \mathcal{I}_{j}^{+}(\,ll), \mathcal{F}_{j}^{+}(\,ll), \mathcal{T}_{j}^{-}(\,ll), \mathcal{I}_{j}^{-}(\,ll), \mathcal{F}_{j}^{-}(\,ll)\right):\ ll \in \mathfrak{U}\right\}\right) : \ddot{e} \in \mathbb{E}\right\}$ for $j = 1,2, \dots \dots, n$ stand two bipolar NSS with respect to universal set. Then, $\mathfrak{B}_1$ is bipolar Neutrosophic soft equal to $\mathfrak{B}_2$, is denoted by $\mathfrak{B}_1 = \mathfrak{B}_2$, if $\mathcal{T}_1^{+}(\,ll) = \mathcal{T}_2^{+}(\,ll)$, $\mathcal{I}_1^{+}(\,ll) = \mathcal{I}_2^{+}(\,ll), \mathcal{F}_1^{+}(\,ll) = \mathcal{F}_2^{+}(\,ll)$, $\mathcal{T}_1^{-}(\,ll) = \mathcal{T}_2^{-}(\,ll)$, $\mathcal{I}_1^{-}(\,ll) = \mathcal{I}_2^{-}(\,ll), \mathcal{F}_1^{-}(\,ll) = \mathcal{F}_2^{-}(\,ll)$ for all $(\ddot{e},\ ll) \in \mathbb{E} \times \mathfrak{U}$.

3.6 Definition-15

If $\mathfrak{B}$ is a bipolar NSS with respect to universal set $\mathfrak{U}$. then, $\mathfrak{B}^{\mathcal{C}}$ be a complement of a bipolar $\mathcal{NSS}$. It can be written as;

$$\mathfrak{B}^{\mathcal{C}} = \left\{\left(\ddot{e}, \left\{\left(\,ll, \mathcal{F}^{+}(\,ll), 1 - \mathcal{I}^{+}(\,ll), \mathcal{T}^{+}(\,ll), \mathcal{F}^{-}(\,ll), -1 - \mathcal{I}^{-}(\,ll), \mathcal{T}^{-}(\,ll)\right):\ ll \in \mathfrak{U}\right\}\right) :\ \ddot{e} \in \mathbb{E}\right\}$$

Example-4: consider the example 1

$$\mathfrak{B}^{\mathcal{C}} = \{(\ddot{e}_1, \{(\,ll_1, .2, .1, .6, -.4, -.2, -.6), (\,ll_2, .8, .1, .7, -.3, -.2, -.6), (\,ll_3, .2, .2, .7, -.9, -.1, -.6)\}),$$

$$(\ddot{e}_2, \{(\,ll_1, .7, .4, .9, -.4, -.2, -.6), (\,ll_2, .8, .6, .5, -.3, -.2, -.6), (\,ll_3, .2, .4, .8, -.7, -.2, -.5))\}$$

3.7 Definition-16

If $\mathfrak{B}_{j} = \left\{\left(\ddot{e}, \left\{\left(\,ll, \mathcal{T}_{j}^{+}(\,ll), \mathcal{I}_{j}^{+}(\,ll), \mathcal{F}_{j}^{+}(\,ll), \mathcal{T}_{j}^{-}(\,ll), \mathcal{I}_{j}^{-}(\,ll), \mathcal{F}_{j}^{-}(\,ll)\right):\ ll \in \mathfrak{U}\right\}\right) : \ddot{e} \in \mathbb{E}\right\}$ for $j = 1,2, \dots \dots, n$ be two bipolar NSS above $\mathfrak{U}$. After that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are union also denoted by $\mathfrak{B}_1 \uplus \mathfrak{B}_2$. It can be written as;

$$\mathfrak{B}_1 \uplus \mathfrak{B}_2 = \left\{\left(\ddot{e}, \left\{\left(\,ll, \max_{j}\{\mathcal{T}_{j}^{+}(\,ll)\}, \min_{j}\{\mathcal{I}_{j}^{+}(\,ll)\}, \min_{j}\{\mathcal{F}_{j}^{+}(\,ll)\}, \min_{j}\{\mathcal{T}_{j}^{-}(\,ll)\}, \max_{j}\{\mathcal{I}_{j}^{-}(\,ll)\}, \max_{j}\{\mathcal{F}_{j}^{-}(\,ll)\}\right)\right.\right.\right.$$
$$\left.\left.:\ ll \in \mathfrak{U}\right\}\right) :\ \ddot{e} \in \mathbb{E}, and\ j = 1,2, \dots, n\Big\}$$

Example-5: consider the example 1

$$\mathfrak{B}_1 \uplus \mathfrak{B}_2 = \{(\ddot{e}_1, \{(\,ll_1, .6, .8, .2, -.7, -.8, -.3), (\,ll_2, .7, .6, .8, -.6, -.8, -.3), (\,ll_3, .7, .3, .2, -.6, -.5, -.1)\}),$$

$$(\ddot{e}_2, \{(\,ll_1, .9, .6, .7, -.6, -.8, -.3), (\,ll_2, .5, .4, .8, -.6, -.4, -.3), (\,ll_3, .8, .5, .2, -.5, -.8, -.1))\}$$

3.8 Definition-17

If $\mathfrak{B}_{j} = \left\{\left(\ddot{e}, \left\{\left(\,ll, \mathcal{T}_{j}^{+}(\,ll), \mathcal{I}_{j}^{+}(\,ll), \mathcal{F}_{j}^{+}(\,ll), \mathcal{T}_{j}^{-}(\,ll), \mathcal{I}_{j}^{-}(\,ll), \mathcal{F}_{j}^{-}(\,ll)\right):\ ll \in \mathfrak{U}\right\}\right) : \ddot{e} \in \mathbb{E}\right\}$ for $j = 1,2, \dots \dots, n$ stand n Bipolar NSS above $\mathfrak{U}$. After that $\mathfrak{B}_{j}$ are the union of n bipolar NSS is denoted by $\uplus_{j=1}^{n} \mathfrak{B}_{j}$, can be written as;

$$\uplus_{j=1}^{n} \mathfrak{B}_{j} = \left\{\left(\ddot{e}, \left\{\left(\,ll, \max_{j}\{\mathcal{T}_{j}^{+}(\,ll)\}, \min_{j}\{\mathcal{I}_{j}^{+}(\,ll)\}, \min_{j}\{\mathcal{F}_{j}^{+}(\,ll)\}, \min_{j}\{\mathcal{T}_{j}^{-}(\,ll)\}, \max_{j}\{\mathcal{I}_{j}^{-}(\,ll)\}, \max_{j}\{\mathcal{F}_{j}^{-}(\,ll)\}\right)\right.\right.\right.$$
$$\left.\left.:\ ll \in \mathfrak{U}\right\}\right) :\ \ddot{e} \in \mathbb{E}, and\ j = 1,2, \dots, n\Big\}$$

3.9 Definition-18

If $\mathfrak{B}_{j} = \left\{\left(\ddot{e}, \left\{\left(\,ll, \mathcal{T}_{j}^{+}(\,ll), \mathcal{I}_{j}^{+}(\,ll), \mathcal{F}_{j}^{+}(\,ll), \mathcal{T}_{j}^{-}(\,ll), \mathcal{I}_{j}^{-}(\,ll), \mathcal{F}_{j}^{-}(\,ll)\right):\ ll \in \mathfrak{U}\right\}\right) : \ddot{e} \in \mathbb{E}\right\}$ for $j = 1,2, \dots \dots, n$ be two bipolar NSS above $\mathfrak{U}$. After that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are union also denoted by $\mathfrak{B}_1 \dot{\cap} \mathfrak{B}_2$. It can be written as;

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$$\mathfrak{B}_1 \dot{\cap} \mathfrak{B}_2 = \{(\ddot{e}, \{(\mathcal{U}, \min_j\{\mathcal{T}_j^+(\mathcal{U})\}, \max_j\{\mathcal{I}_j^+(\mathcal{U})\}, \max_j\{\mathcal{F}_j^+(\mathcal{U})\}, \max_j\{\mathcal{T}_j^-(\mathcal{U})\}, \min_j\{\mathcal{I}_j^-(\mathcal{U})\}, \min_j\{\mathcal{F}_j^-(\mathcal{U})\}) : \mathcal{U} \in \mathfrak{U}\}) : \ddot{e} \in \mathbb{E}, and\ j = 1,2, \dots, n\}$$

Example-6: consider the example 1

$$\mathfrak{B}_1 \dot{\cap} \mathfrak{B}_2 = \{(\ddot{e}_1, \{(\mathcal{U}_1, .3, .9, .6, -.6, -.8, -.4), (\mathcal{U}_2, .4, .9, .8, -.2, -.8, -.3), (\mathcal{U}_3, .7, .8, .6, -.6, -.9, -.9)\}),$$

$$(\ddot{e}_2, \{(\mathcal{U}_1, .2, .6, .7, -.1, -.8, -.4), (\mathcal{U}_2, .3, .9, .8, -.6, -.4, -.6), (\mathcal{U}_3, .8, .6, .2, -.5, -.8, -.7))\}$$

3.10 Definition-19

If $\mathfrak{B}_j = \left\{\left(\ddot{e}, \left\{\left(\mathcal{U}, \mathcal{T}_j^+(\mathcal{U}), \mathcal{I}_j^+(\mathcal{U}), \mathcal{F}_j^+(\mathcal{U}), \mathcal{T}_j^-(\mathcal{U}), \mathcal{I}_j^-(\mathcal{U}), \mathcal{F}_j^-(\mathcal{U})\right) : \mathcal{U} \in \mathfrak{U}\right\}\right) : \ddot{e} \in \mathbb{E}\right\}$ for $j = 1,2, \dots \dots, n$ be n bipolar NSS above $\mathfrak{U}$. After that $\mathfrak{B}_j$ are the intersection of n bipolar NSS is denoted by $\dot{\cap}_{j=1}^{n} \mathfrak{B}_j$, can be written as;

$$\dot{\cap}_{j=1}^{n} \mathfrak{B}_j = \left\{\ddot{e}, \left\{\begin{array}{c}(\mathcal{U}, \min_j\{\mathcal{T}_j^+(\mathcal{U})\}, \max_j\{\mathcal{I}_j^+(\mathcal{U})\}, \max_j\{\mathcal{F}_j^+(\mathcal{U})\}, \max_j\{\mathcal{T}_j^-(\mathcal{U})\}, \min_j\{\mathcal{I}_j^-(\mathcal{U})\}, \min_j\{\mathcal{F}_j^-(\mathcal{U})\}) \\ : \mathcal{U} \in \mathfrak{U} \ddot{e} \in \mathbb{E}, and\ j = 1,2, \dots, n\end{array}\right\}\right\}$$

4. Methodology

AGGREGATION BIPOLAR NEUTROSOPHIC SOFT OPERATOR

The following portion presents an aggregation bipolar soft operator for implementing in bipolar $\mathcal{NSS}$ furthermore demonstration of an algorithm developed based on bipolar $\mathcal{NSS}$ is given by arithmetical example to apply the appropriate usage and applicability of proposition.

Definition

If $\mathfrak{B} = \{(\ddot{e}, \{(\mathcal{U}, \mathcal{T}^+(\mathcal{U}), \mathcal{I}^+(\mathcal{U}), \mathcal{F}^+(\mathcal{U}), \mathcal{T}^-(\mathcal{U}), \mathcal{I}^-(\mathcal{U}), \mathcal{F}^-(\mathcal{U})) : \mathcal{U} \in \mathfrak{U}\}) : \ddot{e} \in \mathbb{E}\} =$ $\left\{\left\{\left(\mathcal{U}, \mathcal{T}_{\ddot{e}}^+(\mathcal{U}), \mathcal{I}_{\ddot{e}}^+(\mathcal{U}), \mathcal{F}_{\ddot{e}}^+(\mathcal{U}), \mathcal{T}_{\ddot{e}}^-(\mathcal{U}), \mathcal{I}_{\ddot{e}}^-(\mathcal{U}), \mathcal{F}_{\ddot{e}}^-(\mathcal{U})\right) : \mathcal{U} \in \mathfrak{U}\right\} : \ddot{e} \in \mathbb{E}\right\}$ be a bipolar NSS above $\mathfrak{U}$. After that, the aggregation bipolar NSS operator is denoted by $\mathfrak{B}_{agg}$, can be written as;

$$\mathfrak{B}_{agg} = \{\eta_{\mathfrak{B}}(\mathcal{U}) / \mathcal{U} : \mathcal{U} \in \mathfrak{U}\}$$

$$\eta_{\mathfrak{B}}(\mathcal{U}) = \frac{1}{2|\mathbb{E} \times \mathfrak{U}|} \sum_{e \in \mathbb{E}} \left(\left|1 - \mathcal{I}_{\ddot{e}}^+(\mathcal{U})(\mathcal{T}_{\ddot{e}}^+(\mathcal{U}) - \mathcal{F}_{\ddot{e}}^+(\mathcal{U})) + \mathcal{I}_{\ddot{e}}^-(\mathcal{U})\left(\mathcal{T}_{\ddot{e}}^-(\mathcal{U}) - \mathcal{F}_{\ddot{e}}^-(\mathcal{U})\right)\right|\right)$$

Everywhere $|\mathbb{E} \times \mathfrak{U}|$ is the cardinality of $\mathbb{E} \times \mathfrak{U}$.

5. Algorithm

1. Make the bipolar NSS on $\mathfrak{U}$.
2. Calculate the aggregation bipolar NS operator.
3. Find an alternative set on$\mathfrak{U}$.

Example-7: Bipolar condition is a serious psychological disease especially if not treated early that can exceed to dangerous performance, challenging careers etc. A bipolar mood chard representing the condition of patient's every month. Bipolar teenagers and their families will greatly advantage from mood plotting and can suppose initial finding the signs and purpose of proper cures by their doctors. We make mood plan according to algorithm. Let $\mathfrak{U} = \{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4\}$ present the set of day in which data has been maintain and $\mathbb{E} = \{\ddot{e}_1 = \text{severe depression}, "\ddot{e}_2 = anxiety, \ddot{e}_3 = medication"\}$ be set of parameters. Now we apply a set of rules as follows

1- Decision making of bipolar NS $\mathfrak{B}$ above another set $\mathfrak{U}$ as:

$\mathfrak{B}$
$= \{(\ddot{e}_1, \{(\ \mathscr{l}_1, .3, .5, .6, -.8, -.9, -.8)(\ \mathscr{l}_2, .4, .2, .9, -.7, -.3, -.1), (\ \mathscr{l}_3, .7, .8, .3, -.9, -.6, -.2), (\ \mathscr{l}_4, .2, .9, .1, -.6, -.9, -.8)\}\},$

$\{(\ddot{e}_2, \{(\ \mathscr{l}_1, .9, .6, .8, -.2, -.7, -.9), (\ \mathscr{l}_2, .6, .2, .9, -.7, -.1, -.4), (\ \mathscr{l}_3, .1, .7, .4, -.3, -.6, -.9), (\ \mathscr{l}_4, .6, .9, .3, -.1, -.3, -.7)\}\},$

$\{(\ddot{e}_3, \{(\ \mathscr{l}_1, .7, .1, .3, -.9, -.2, -.8), (\ \mathscr{l}_2, .8, .6, .9, -.1, -.5, -.8), (\ \mathscr{l}_3, .1, .5, .9, -.6, -.9, -.7), (\ \mathscr{l}_4, .8, .4, .7, -.3, -.5, -.1)\}\}$

2- find the decision making of aggregation bipolar NS operator $\mathfrak{B}_{\mathfrak{agg}}$ of $\mathfrak{B}$ as:

$$\mathfrak{B}_{\mathfrak{agg}} = \{.1075/\ \mathscr{l}_1, .1283/\ \mathscr{l}_2\ , .1358/\ \mathscr{l}_3\ , .0891/\ \mathscr{l}_4\}$$

3- Choose the maximum degree of $\mathscr{l}_3$ is .1358 from $\mathfrak{B}_{\mathfrak{agg}}$ amongst the other.

Example-8: Let $\mathfrak{U} = \{\ a_1, a_2,\ a_3, a_{\ 4}, a_5\}$ present the set of AC inverters and $\mathbb{E} = \{\ddot{e}_1, \ddot{e}_2, \ddot{e}_3, \ddot{e}_4\}$ be the set of constraints in which" $\ddot{e}_1 = company$", " $\ddot{e}_2 = functions$ ", " $\ddot{e}_3 = cheap$", " $\ddot{e}_4 = AC\ capacity$". Now we apply a set of rules as follows:

1. **Decision making of bipolar NS $\mathfrak{B}$ above another set $\mathfrak{U}$ as:**

$\mathfrak{B} = \{(\ddot{e}_1, \{(\ a_1, .2, .3, .7, -.5, -.8, -.6)(a_2, .6, .8, .9, -.3, -.9, -.7), (a_3, .5, .4, .8, -.6, -.8, -.3),$

$(a_4, .3, .9, .6, -.6, -.8, -.4), (a_5, .2, .5, .6, -.8, -.5, -.7)\}\}, \{(\ddot{e}_2, \{(a_1, .3, .7, .5, -.9, -.1, -.9), (a_2, .5, .9, .8, -.6, -.8, -.1),$

$(a_3, .8, .6, .2, -.5, -.8, -.7), (a_4, .9, .6, .7, -.6, -.7, -.9), (a_5, .8, .5, .2, -.5, -.8, -.8)\}\},$

$\ddot{e}_3, \{(a_1, .3, .5, .8, -.9, -.7, -.1), (a_2, .6, .3, .1, -.8, -.9, -.9), (a_3, .9, .9, .5, -.3, -.2, -.1),$

$(a_4, .3, .1, .7, -.2, -.7, -.3), (a_5, .4, .6, .8, -.1, -.9, -.7)\}\}\{(\ddot{e}_4, \{(a_1, .9, .8, .3, -.1, -.2, -.7)$

$(a_2, .3, .7, .8, -.3, -.5, -.4)(a_3, .5, .1, .3, -.7, -.4, -.3)(a_4, .6, .9, .7, -.9, -.8, -.4)(a_5, .9, .5, .2, -.1, -.1, -.6)\}\}$

2. **find the decision making of aggregation bipolar NS operator $\mathfrak{B}_{\mathfrak{agg}}$ of $\mathfrak{B}$ as:**

$$\mathfrak{B}_{\mathfrak{agg}} = \{.1105/a_{\ 1}, .11525/a_{\ 2}\ , .0915/a_{\ 3}\ , .114/a_{\ 4}, .07525/a_{\ 5}\}$$

3. **Choose the maximum degree of a_2 is .11525 from $\mathfrak{B}_{\mathfrak{agg}}$ amongst the other.**

Example-9: Let $\mathfrak{U} = \{\ c_1, c_2,\ c_3\}$ present the set of auto car and $\mathbb{E} = \{"\ddot{e}_1 = cheap", "\ddot{e}_2 = features", "\ddot{e}_3 = metallic\ colour", \}$ be a set of constraints. Now we set of rules as follows:

1. **Decision making of bipolar NS $\mathfrak{B}$ above another set $\mathfrak{U}$ as:**

$\mathfrak{B} = \{(\ddot{e}_1, \{(\ c_1, .1, .3, .5, -.4, -.2, -.5)(c_2, .3, .1, .6, -.3, -.5, -.6), (c_3, .2, .4, .5, -.3, -.1, -.2)\},$

$\{(\ddot{e}_2, \{(c_1, .6, .1, .3, -.2, -.1, -.4), (c_2, .7, .3, .1, -.2, -.3, -.5), (c_3, .1, .5, .4, -.3, -.2, -.1)\},$

$\{(\ddot{e}_3, \{(c_1, .5, .2, .1, -.7, -.1, -.3), (c_2, .3, .4, .1, -.5, -.2, -.1), (c_3, .4, .1, .6, -.3, -.2, -.5)\}$

2. **Find the decision making of aggregation bipolar NS operator $\mathfrak{B}_{\mathfrak{agg}}$ of$\mathfrak{B}$ as:**

$$\mathfrak{B}_{\mathfrak{agg}} = \mathfrak{B}_{\mathfrak{agg}} = \{.1672/c_{\ 1}, .145/c_{\ 2}\ , .1833/c_{\ 3}\}$$

3. **Choose the maximum degree of c_3 is .1833 from $\mathfrak{B}_{\mathfrak{agg}}$ amongst the other.**

6. Conclusion

In this paper, we discussed about the concept of bipolar neutrosophic soft sets and redefined some features of that particular concepts. Then we have employed bipolar neutrosophic soft sets in auto car selection with the help of aggregation operators. So we reached at the decision that what type of car is selected on what characteristics.

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References

[1] Zadeh.L.A,.Fuzzy set, Inform, and control, Vol 8 , pp. 338-353, 1965.

[2] Atanassov.K.T, Intuitionistic fuzzy sets, Phsica-Verlag A Springer-Verlag Company, 1999.

[3] Smarandache.F,Neutrosophic overset, Neutrosophic underset, and Neutrosophic offset. Similarly for Neutrosophic over-/under-/off-logic, Probability, and Statistics, arXiv preprint arXiv: 1607.00234. 2016.

[4] Maji.P.K,A neutrosophic soft set approach to a decision-making problem, Annuals of fuzzy mathematics and informatics, Vol 3(2), pp. 313-319, 2012.

[5] Lee.K.J,Bipolar fuzzy sub-algebras and bipolar fuzzy ideals of BCK/BCL-algebras, Bull Malays Math Sci Soc, Vol 32(3), pp. 361-373, 2009.

[6] Shabir.M and Naz.M, On bipolar soft set,arXiv:pp. 1303.1344. 2013

[7] Lee.K.M, Bipolar-valued fuzzy sets and their operations, Proc Int conf on intelligent technologies pp. 307-312, 2000.

[8] Smarandache.F, A unifying field in logics, Neutrosophy: Neutrosophic probability, set, and logic, Rehoboth: American research press. 1998.

[9] Wang.H, Smarandache.F, Zhang. Q and Sunderraman.R, Single valued neutrosophic sets, Multispace and Multistructure Vol 4 pp. 410-413, 2010

[10] Deli.I, Ali.M, and Smarandache.F, Bipolar Neutrosophic sets and their application based on multi- criteria decision-making problems, Proceedings of the 2015 international conference on advanced mechanics system.2015.

[11] Ali.M, Smarandache.F, The theory of neutrosophic cubic sets and their applications in pattern recognition, Journal of intelligent and fuzzy system, Vol 30(4), pp. 1957-1963,2016.

[12] Broumi.S, Smarandache.F, Talea.M and Bakali.A, An introduction to bipolar single-valued neutrosophic graph theory, applied mechanics and materials, Vol 841, pp. 184-191, 2016.

[13] Broumi.S, Bakali.A, Talea.M, Smarandache. F and Ali.M,shortest path problem under bipolar Neutrosophic setting, applied mechanics and materials**,** Vol 859, pp. 59-66, 2016.

[14] Molodtsov.D,Soft sets theory- the first result, Computer math, Vol 37(4/5), pp. 19-31, 1999.

[15] Deli.I and Broumi.S,Neutrosophic soft matrices and NSM decision making, Journal of intelligent and fuzzy systems,Vol 28(5), pp. 2233-2241, 2015.

[16] Deli.I and Broumi.S, Neutrosophic soft relations and some properties, Annuals of fuzzy mathematics and informatics, Vol 9(1) pp. 169-182, 2015.

[17] Maji.P.K,Neutrosophic soft set, Annuals of fuzzy mathematics and informatics,Vol 5(1), pp. 157-168, 2013

[18] Kang.M.K, and Kang.J.G, Bipolar fuzzy set theory applied to sub-semigroups with operators in semigroup, J Korean soc math Educ ser b pure appl math, Vol 19(1), pp. 23-35, 2012.

[19] Aslam.M, Abdullad.S and Ullah.K, Bipolar fuzzy soft sets and its applications in decision-making problem, arXiv: 1303.6932, 2013

DOI: 10.5281/zenodo.3957384

[20] Jafar.N.M, Saqlain.M, Saeed.M, Abbas.Q , Application of Soft-Set Relations and Soft Matrices in Medical Diagnosis using Sanchez's Approach , International Journal of Computer Applications, Vol 177(32), pp. 7-11, 2020

[21] Jafar.N.M, Muniba.K, Saeed.A, Abbas.S, Bibi.I , Application of Sanchez's Approach to Disease Identification Using Trapezoidal Fuzzy Numbers, International Journal of Latest Engineering Research and Applications, Vol 4 (9), pp. 51-57, 2019

[22] Jafar.N.M, Faizullah, Shabbir.S, Alvi.F.M.S, Shaheen.L , Intuitionistic Fuzzy Soft Matrices, Compliments and Their Relations with Comprehensive Study of Medical Diagnosis, International Journal of Latest Engineering Research and Applications, Vol 5(1), pp. 23-30, 2020

[23] Jafar.N.M, Saeed.A, Waheed.M, Shafiq.A , A Comprehensive Study of Intuitionistic Fuzzy Soft Matrices and its Applications in Selection of Laptop by Using Score Function, International Journal of Computer Applications, Vol 177(38): 8-17, 2020

[24] Jafar.N.M, Khan.R.M, , Sultan.H, Ahmad.N , Interval Valued Fuzzy Soft Sets and Algorithm of IVFSS Applied to the Risk Analysis of Prostate Cancer, International Journal of Computer Applications, Vol 177(38), pp. 18-26, 2020

[25] Riaz.M, Saeed.M, Saqlain.M, Jafar.N , Impact of Water Hardness in Instinctive Laundry System Based on Fuzzy Logic Controller, Punjab University Journal of Mathematics, Vol 51(4), pp. 73-84, 2019

[26] Saeed, M., Zulqarnain, M. and Dayan, F.TOPSIS analysis for the prediction of diabetes based on general characteristics of humans. Int. J. of Pharm. Sci. and Research. Vol 9, pp. 2932-2939, 2018.

[27] Saqlain.M.Jafar.N, Riffat.A Smart Phone Selection by Consumers' In Pakistan: FMCGDM Fuzzy Multiple Criteria Group Decision Making Approach, Gomal University Journal of Research, Vol 34 (1), pp. 27-31.2018

[28] Saqlain.M, Jafar.N, Hamid.R,Shahzad.A.Prediction of Cricket World Cup 2019 by TOPSIS Technique of MCDM-A Mathematical Analysis, International Journal of Scientific & Engineering Research, Vol 10(2), pp. 789-792, 2019

[29] Saqlain.M, Naz.K, Ghaffar.K, Jafar.N.M , Fuzszy Logic Controller: The Impact of Water pH on Detergents, Scientific Inquiry of Review Vol 3(3), pp. 16–29, 2019.

[30] Saqlain M, Saeed M, Ahmad M. R, Smarandache F, Generalization of TOPSIS for Neutrosophic Hypersoft set using Accuracy Function and its Application, Neutrosophic Sets and Systems ,Vol 27 , pp. 131-137, 2020

[31] Jafar.N.M,Saqlain.M,Shafique.A.R, Khalid.M, Akbar.H, Naveed.A, New Technology in Agriculture Using Neutrosophic Soft Matrices with the Help of Score Function, International Journal of Neutrosophic ScienceVol 3 (2), pp. 78-88, 2020.

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Neutrosophic Environment for Traffic Control Management

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Abstract

Neutrosophic along with its environment development over the past decades. Neutrosophic environment is apply to various applications in logic,statstics,albebra, neural networks and several other fields. Neutrosophic sets has been presented to handle the indeterminacy in real-world decision-making problem. Real world problems have some kind of uncertainty in nature and one of the influential problem in environment. Neutrosophic environment results are apply to a new dimension in traffic control. Neutrosophic is the vital role on traffic flow control . It is deal with membership , non membership and also indeterminacy of the data as well. The advantage of the neutrosophic environment is to find the optimized result of the system choosing the best alternative.In this paper, traffic flow control is analyzed under neutrosophic environment using MATLAB.

Keywords: Traffic flow ,Neutrosophic environment, Neutrosophic network

1.Introduction

If the number of vehicles are increased and having lower phase of highways then there will be a traffic congestion problem. The general factors for traffic problems are density of the vehicles, human behavior, traffic light system and social behavior. The complex and changing traffic situations cannot be dealt by conventional traffic control methodologies or control systems. Analyzed the Parametric and nonparametric traffic volume forecasting[1]. Deals with Spectral and cross-spectral analysis of urban traffic flows [2]. To introduced the traffic forecast using simulations of large scale networks[3]. Analyzed the multivariate state space approach for urban traffic flow modeling and prediction[4].Introduced Interval neutrosophic Sets and Logic[5].Introduced A unifying field in logic. Neutrosophy: Neutrosophic probability, set, logic[6]. Analyzed A Bayesian Network Approach to Traffic Flow Forecasting[7]. Introduced a novel Fuzzy Neural Approach to Road Traffic Analysis and Prediction[8]. Introduced the Type-2 fuzzy logic approach for short term traffic forecasting[9]. Analyzed the Ensemble learning approach for freeway short-term traffic flow prediction[10]. Introduced the Fuzzy Neural Network model Applied in the Traffic Flow Prediction[11]. Introduced an Aggregation Approach to Short-Term Traffic Flow Prediction[12].Predicting traffic flow, speed, length of the queue and travel time are necessary for the transportation management applications

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[13]. Predicting and modeling traffic flow has drawn attention from literature as it is very important for formatting intelligent transportation system theoretically and practically. The area of transportation studies has attracted interest among the researchers [14]. Fuzzy logic is a powerful tool to used in unncertainties in measurements and information used to calculate the parameters; here the membership value for a particular traffic state is not a crisp value Any number of intersections and lanes can be handled using fuzzy and interval fuzzy logic in traffic control management. Generally there are two types of signal control available, namely, fixed time control (the traffic conditions are fixed) and adaptive time control (the traffic conditions may be refined over a period of time control [15-16]. Introduced an intelligent Traffic Light Control System for Isolated Intersection Using Fuzzy Logic[17]. The complex and changing traffic situations cannot be dealt by conventional traffic control methodologies or control systems

If the number of vehicles are increased and having lower phase of highways then there will be a traffic congestion problem. The general factors for traffic problems are density of the vehicles, human behavior, traffic light system and social behavior. The complex and changing traffic situations cannot be dealt by conventional traffic control methodologies or control systems.

As the flow of traffic varies from hour to hour in morning and evening. Especially during office timing the traffic flow will be high in general. The one of the advantage of fuzzy logic is, there is a possibility of computing with words [18].

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Neutrosophic logic was proposed by Smarandache can express determinate as well as indeterminate of the information by neutrosophic numbers. Solving traffic flow problem is a difficult one for certain parameters as the real-time situations are uncertain in nature and can be solved easily by considering neutrosophic logic [19]. Introduced a Traffic signal control using fuzzy logic[20]. Deal with Interval neutrosophic multiple attribute decision-making method with credibility information[21]. An Improved score function for ranking neutrosophic Sets and Its Application to decsion making process Fuzzy and neutrosophic logic are playing a vital role in dealing with uncertainties. Introduced the traffic control management in triangular interval type-2 fuzzy and interval neutrosophic environments [26-34]. Introducted the traffic control management using Gauss Jordan method under neutrosophic Environment[35].

2. Definition

Neutrosophic number

Neutrosophic linear equations and solving method for traffic flow control under neutrosophic number (z=a +bI) environment. A system consists of linear equations (LEs) can be solved . By finding an augmented matrix form the given system, one can find the inverse of the matrix and using MATLAB.

3. Applications

At Analyze of traffic flow

The roads are considering as Road1, Road2, Road3 and Road4. FIGURE 1 shows the traffic flow on the four roads. Here z is a neutrosophic variable, and y_1, y_2 and y_3 are the unknown variables. In this junctions falsity considered as zero. Where I is the indeterminacy of the traffic flow.

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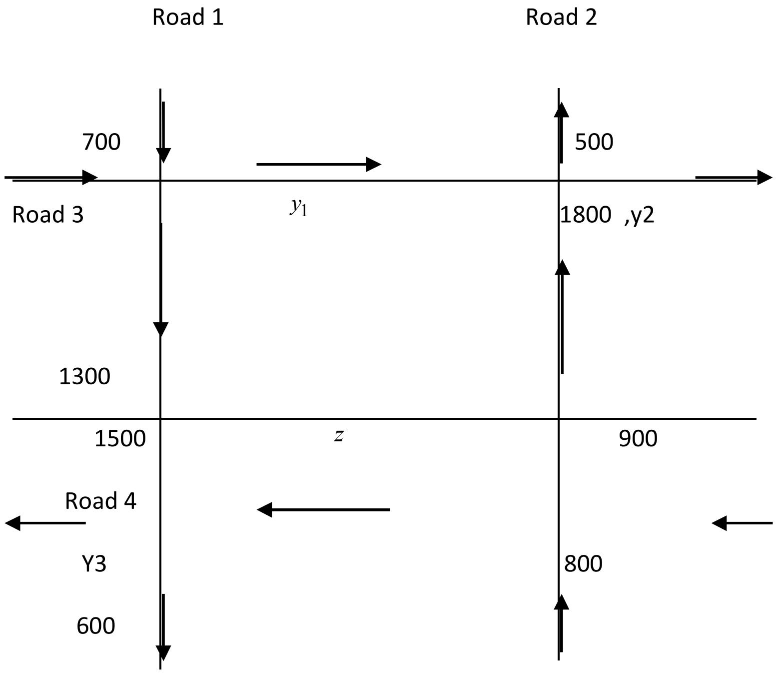

FIGURE 1. The traffic flows of four roads

The traffic in the junction as follows:

2000=y+z

2300=y1+y2

1700=y2+y3

2100=y3+z

Rewrite the equation as,

y1 = 2000-z

y1+y2 = 2300

y2+2y3 = 3800-z

Based on, z = 500+I, the system can also be described by the following three NLEs:

Then, the neutrosophic equations are:

y_1=1500-I

y1+y2 = 2300

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y2+2y3 = 3300-I

Thus, the neutrosophic matrices are:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1500 - I \\ 2300 \\ 3300 - I \end{bmatrix}$$

For the system consists of NLEs, apply the MATLAB software, shown in the following program and the solution of the system is :

For the system consists of NLEs, apply the MATLAB software, shown in the following program

```
clc

syms I;

a=[1 0 0;1 1 0;0 1 2]

b=[1500-I;2300;3300-I]

[v,j]=jordan(a)

j=inv (v)*a*v

y=a\b
```

Output of the program are as follows:

```
a=[1 0 0;1 1 0;0 1 2]

 b=[1500-I;2300;3300-I]

 [v,j]=jordan(a)

y=a\b
```

$$Y = \begin{bmatrix} 1500 - I \\ I + 800 \\ 1250 - I \end{bmatrix}$$

The values of Y are NNs.

In some of the real-time situations, when I ∈[0, 100] is the possible range, the solution of the system is:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} [1400,1500] \\ [800,900] \\ [1150,1250] \end{bmatrix}$$

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Corresponding to the possible traffic flow z = [200,250]

Thus the ranges of the three traffic flows are

$$y_1 = [1400,1500]$$
$$y_2 = [800,900]$$
$$y_3 = [1150,1250]$$

In table 1 represent the various range of indeterminacy.

TABLE 1. Traffic flo ws according to various ranges of Indeterminacy

I	z	y_1	y_2	y_3
$I = 0$	200	1500	800	1250
$I \in [100,200]$	$[300,400]$	$[1300,1400]$	$[900,1000]$	$[1050,1150]$
$I \in [200,300]$	$[400,500]$	$[1200,1300]$	$[1000,1100]$	$[950,1050]$
$I \in [300,400]$	$[500,600]$	$[1100,1200]$	$[1100,1200]$	$[850,950]$
$I \in [400,500]$	$[600,700]$	$[1000,1100]$	$[1200,1300]$	$[750,850]$

In table 2 shows that the advantages and disadvantage in fuzzy and Neutrosophic traffic control management.

Table :2 Comparison of Traffic Control Management using Crisp , Fuzzy and Neutrosophic

Traffic Control Management	Advantages	Limitations
Crisp	• Traffic density for all fixed time period.	• It is not possible to act in varing traffic density. • Not able to solve quickly in uncertainty behaviour
Fuzzy	• Traffic density in different time can be consider. • Intelligent • Act the best security	• Not able to use in stability. • Not able to use flexibility. • Not able to use in on line planning.
Neutrosophic	• Deals not only uncertainty but also indeterminacy due to unpredictable environmental disturbances	• Not able to calculate the error.

4. CONCLUSION

Traffic control management is an essential task which insure the safty of the people . In this paper, fundamental concepts of traffic control have been reviewed. Triangular and Trapezoidal Fuzzy numbers are widely used in many of the real world problems as it deals with the problems which having less number of membership values with covering the linguistic parameters of the system effectively. Traffic flow management has been analyzed with respect to various

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ranges of indeterminacy under neutrosophic environment using MATLAB program Also compared the traffic control management for crisp sets,fuzzy and neutrosophic.

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References

[1] R. K. Oswald, B.L. Smith and B.M. Williams, “Parametric and nonparametric traffic volume forecasting,” in Transportation Research Board Annual Meeting, (Washington, DC, 2000).
[2] Stathopoulos and M.G. Karlaftis “Spectral and cross-spectral analysis of urban traffic flows,” in Intelligent Transportation Systems, (2001 IEEE Proceedings 2001), pp. 820 –825.
[3] R. Chrobok, J. Wahle and M. Schreckenberg, “Traffic forecast using simulations of large scale networks,” in Intelligent Transportation Systems, IEEE Proceedings pp. 434 –439, 2001
[4] A. Stathopoulos and M. G. Karlaftis, A multivariate state space approach for urban traffic flow modeling and prediction. Transportation Research Part C: Emerging Technologies, 11(2), 121 – 135, 2003.
[5] H. Wang, F. Smarandache, Y. Q. Zhang, R. Sunderraman, Interval neutrosophic Sets and Logic: Theory and Applications in Computing (Hexis, Phoenix, 2005.
[6] F. Smarandache, A unifying field in logic. Neutrosophy: Neutrosophic probability, set, logic (American Research Press, Rehoboth, fourth edition, 2005
[7] Shiliang, Chnagshui and Guoqiang, A Bayesian Network Approach to Traffic Flow Forecasting. IEEE Transactions on Intelligent Transportation Systems 7(1), 2006.
[8] Chai, Michel, Bernard and Seng, POP-TRAFFIC: A Novel Fuzzy Neural Approach to Road Traffic Analysis and Prediction. IEEE Transactions on Intelligent Transportation systems 7(2), 2006.
[9] Li, Lin and Liu, Type-2 fuzzy logic approach for short term traffic forecasting. Intelligent Transportation System, 153(1), 2006.
[10] L. Chen and P. Chen, “Ensemble learning approach for freeway short-term traffic flow prediction,” in IEEE 2007.
[11] Tong, Fan, Cui and Meng, “Fuzzy Neural Network model Applied in the Traffic Flow Prediction,” in IEEE 2007.
[12] M. C. Tan, S. C. Wong, J. M. Xu, Z. R. Guan, and P. Zhang, An Aggregation Approach to Short-Term Traffic Flow Prediction. IEEE Transactions On Intelligent Transportation Systems 10(1) 60-69, 2009.
[13] W. Ge and Yan, “Research on Traffic Flow Fuzzy Time Series Forecasting Algorithm of Single Intersection,” in Fourth International Conference on Innovative Computing, Information and Control, 2009.
[14] T. Zhang, L. Hu, Z. Liu, and Y. Zhang, “Nonparametric regression for the short-term traffic flow forecasting,” in 2010 International Conference on Mechanic Automation and Control Engineering (MACE), pp. 2850 –2853, 2010.
[15] H. Wang, F. Smarandache, Y. Zhang and R. Sunderraman, Single Valued Neutrosophic Sets. Multispace and Multisrtucture 4, 410-413, 2010.
[16] M. A. Tahaa, L. Ibrahimb, Traffic Simulation System Based on Fuzzy Logic. Procedia Computer Science 12, 356 – 360, 2012.
[17] J. Alam, M. K. Pandey and H. Ahmed, “Intelligent Traffic Light Control System for Isolated Intersection Using Fuzzy Logic,” Conference on Advances in Communication and Control Systems 2013 (CAC2S 2013), 209-215
[18] B. Sharma, Gupta, Fuzzy Logic model for the prediction of traffic volume in week days. International Journal of computer Applications 107(17), 2014.
[19] E. Pasetiyo, O. Wahyunggoro and S. Sulistyo, Design and Simulation of Adaptive Traffic Light Controller Using Fuzzy Logic Control Sugeno Method. International Journal of Scientific and Research Publications 5(4), 2015.
[20] S. Prontri, P. Wuttidittachotti and S. Thajchayapong, “Traffic signal control using fuzzy logic,” in International conference on, Electrical Engineering/ Electronics, Computer, Telecommunications and Information Technology(ECTI-CON), 2015.
[21] J. Ye, Interval neutrosophic multiple attribute decision-making method with credibility information. International Journal of Fuzzy Systems 18(5), 914–923, 2016.

DOI: 10.5281/zenodo.3957460

[22] S. Broumi, T. Mohamed, A. Bakali and F. Smarandache, Single Valued Neutrosophic Graphs. Journal of New Theory 10, 86-101, 2016.
[23] Nancy, and G. Harish, An Improved score function for ranking neutrosophic Sets and Its Application to decsion making process. International Journal for Uncertainty Quantification 6 (5), 377–385, 2016.
[24] M. Alodat and I. Addullah, "Surveillance Rapid Detection of Signs of Traffic Services in Real Time," 4 th international conference on E-Commerce ICoEC, pp. 51-55,2017.
[25] S. Banerjee, S. Biswas and T. K. Roy, Intuitionistic Fuzzy Linear System, Advances in Fuzzy Mathematics. 12 (13), 475-487, 2017.
[26] J. Ye, Neutrosophic linear equations and application in traffic flow problems, Algorithms 10(4), 133, 2017.
[27] D. Nagarajan, M. Lathamaheswari, R. Sujatha and J. Kavikumar, Edge Detection on DIOM Image using Triangular Norms in Type-2 Fuzzy. International Journal of Advanced Computer Science and Applications 9(11), 462-475, 2018.
[28] M. Lathamaheswari, D. Nagarajan, A. Udayakumar and J. Kavikumar, Review on Type-2 Fuzzy in Biomedicine. Indian Journal of Public Health Research & Development 9(12), 322-326,2018. DOI Number: 10.5958/0976-5506.2018.01855.7.
[29] S. Broumi, A. Bakali, T. Mohamed, F. Samarandache, L. H. Son, P. K. K. Kumar, New matrix algorithm for minimum spanning tree with undirected interval valued Neutrosophic Graph. Neutrosophic Operational Research 2, 54-69, 2018.
[30] M. Lathamaheswari, D. Nagarajan, J. Kavikumar and C. Phang. A Review on Type-2 Fuzzy Controller on Control System. Journal of Advanced Research in Dynamical and Control Systems 10(11), 430-435, 2018.
[31] S. Broumi S, A. Bakali, M. Talea, F. Smarandache, V. Ulucay, M. Sahin, A. Dey, M. Dhar, R. P. Tan, A. Bahnasse and S. Pramanik, Neutrosophic Sets: On overview, Project: New Trends in Neutrosophic Theory and Applications 2, pp.403-434, 2018.
[32] D. Nagarajan, M. Lathamaheswari, R. Sujatha and J. Kavikumar, A Type- 2 Fuzzy in Image Extraction for DICOM Image. International Journal of Advanced Computer Science and Applications 9(12), 351-362, 2018.
[33] S. Broumi, K. Ullah, A. Bakali, M. Talea, P. K. Singh, T. Mahmood, F. Samarandache, A. Bahnasse, S. K. Patro and A. D. Oliveira, Novel System and Method for Telephone Network Planing based on Neutrosophic Graph. Global Journal of Computer Science and Technology: E Network, Web & Security 18(2), 1-11, 2018.
[34] D. Nagarajan, M. Lathamaheswari, S. Broumi and J. Kavikumar, A new perspective on traffic control management using triangular interval type-2 fuzzy sets and interval neutrosophic sets. Operations Research perspectives, Article in Press. https://doi.org/10.1016/j.orp.2019.100099
[35] D. Nagarajan, T. Tamizhi, M. Lathamaheswari, and J. Kavikumar" Traffic control management using Gauss Jordan method under neutrosophic Environment" AIP Conference Proceedings 2112, 020060 (2019); https://doi.org/10.1063/1.5112245

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A Suggested Diagnostic System of Corona Virus based on the Neutrosophic Systems and Deep Learning

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Abstract

The idea for this paper is based on the use of a *computer-connected microscope* associated with Deep Learning, using Convolutional Neural Network (CNN). CNN is a mathematical type of Deep Learning used to recognize and diagnose images. After that, we photograph blood samples, as well as samples, were taken from the mouth and nose, as well as it is possible to photograph the throat from the inside of a large number of injured and uninfected people as well as suspected of infection and provide a large number of references for this program for each type of those different samples. It is possible to perform this process in few minutes, save time and money, make analyzes for the largest possible number of people, and provide results in an accurate and documented manner, which is through the Neutrosophic time series. The basis and analysis of dealing with all data, whether specific or not, that can be taken by time series values, then we present the linear model for the neutrosophic time series, and we test the significance of its coefficient based on patients distribution. Finally, from the above, we can provide a patient neutrosophic time series according to the linear model through which we can accurately predict the program will give degrees of verification and degrees of the uncertainty of the data.

Keywords: COVID-19, Corona Virus, Neutrosophic Systems, Neutrosophic Domain, Deep Learning, Convolutional Neural Network

Background: Coronavirus disease has widely spread all over the world since the beginning of 2020. It is desirable to develop automatic and accurate detection of COVID-19 using chest CT.

1. Introduction

Medical workers deal with a vast amount of information, which has arisen from the womb of laboratory tests and clinical and physiological observations. As doctors began a shift in clinical practice from an accidental analysis to relying on the accuracy of their observation to analyzing different evidence and structured algorithms, relying on groups of constantly updated to improve the ability to diagnose a disease or predict patient outcomes. From this standpoint, that study, which relies on the use of artificial intelligence and Neutrosophic data to identify people with this virus in degrees of certainty, uncertainty, impartiality, and to exploit this in the medical aspect to provide the greatest amount of time and money. Coronavirus, also known as COVID-19, which is among the viruses is a large series of viruses that cause many diseases ranging from the common cold and acute respiratory diseases, and the disease was first discovered in 2012 [10], who are infected with children and the elderly, the cause of their immunodeficiency and those who have heart diseases are more vulnerable to this virus. This virus has invaded many countries in the world now, caused a recession in the world economy, left many deaths, and injured. Neutrosophic is a new view of modeling, designed to effectively deal with underlying doubts in the real world, as it came to replace binary logic that recognized right and wrong by introducing a third neutral case which

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could be interpreted as non-specific or uncertain. Founded by Florentin Smarandache in 1999 [2,3], as a generalization of fuzzy logic . As an extension of this, A. A. Salama introduced the neutrosophic crisp sets Theory as a generalization of crisp sets [1] and developed, inserted and formulated new concepts in the fields of mathematics, statistics, computer science and information systems [4-9]. Neutrosophic has grown significantly in recent years through its application in measurement, sets and graphs and in many scientific and practical fields.

2. Scientific Experiment:

The first step of the experiment is defining the pre-dataset of images of the available samples about people who are infected with the Coronavirus, to make image references for comparison with modern intraoral pictures. There is four sections of data set are used, as follows:

- The first section is images of blood samples.
- The second section is images of saliva samples.
- The third section is images of nose samples.
- The fourth section is images of nose samples.

The system needs a microscope imaging connected to a computer as well as a photographic camera connected to the same device and a computer device that uses CNN to match and identify similarities of images with the pre-dataset of images that were previously placed on the program, by enlarging the modern images that are pulled from the microscope and the camera. The pre-dataset of images is the reference to diagnose each section separately so that the system gives an expectation to each section through analyzing and processing data and images as well as determining the probability of a person being injured or not, as well as the risk of injury.

The following experiment package:

Fig.1. Experiment Tools

Below are the stages and mechanism for implementing the idea and requirements:

1- Stages:

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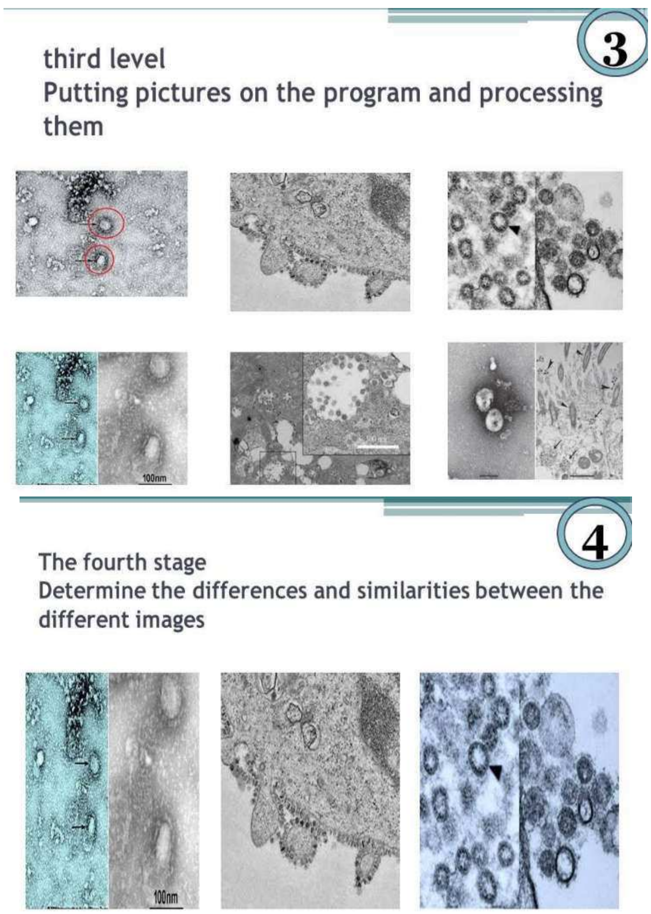
Fig.2. Implementation stages (1, 2, 3 and 4)



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In the last step of the system, the principles of neutrosophic computing useful to system implementation for a large plethora of applications.

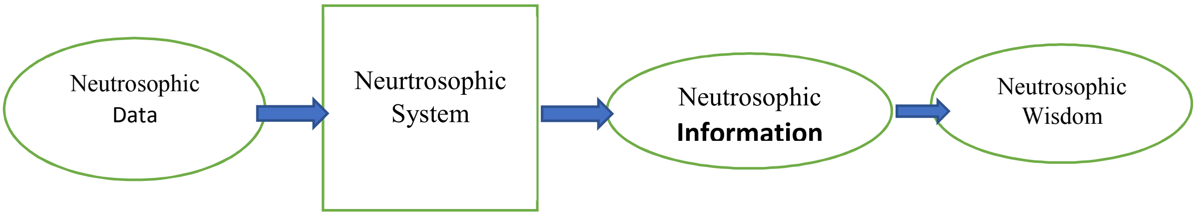

Fig.3. Neutrosophic Information Systems

The following figure represents the Neutrosophic COVID-19 image classifier Architecture

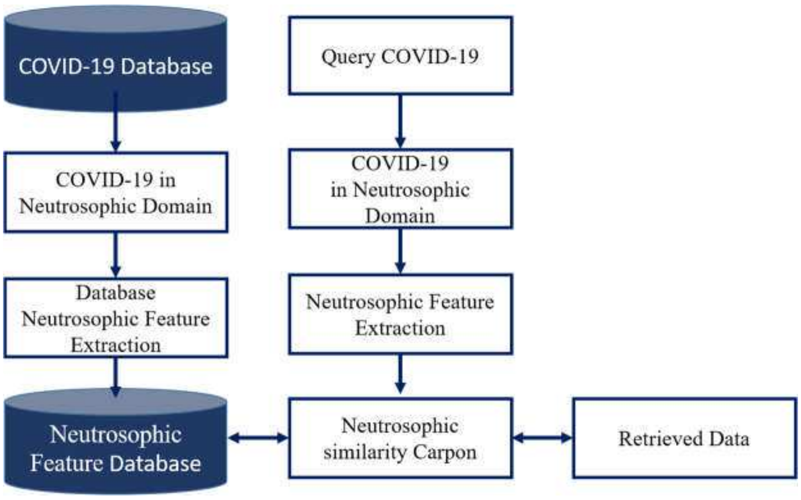

Figure 4. Neutrosophic COVID-19 image classifier Architecture [11]

The algorithm for the proposed system is given below which presented in Figure 4:

1. Convert each image in the database from the spatial domain to the neutrosophic domain.
2. Create a database containing various COVID-19.
3. Extract the Texture feature of COVID-19 in the database.
4. Construct a combined feature vector for T, I, F and Stored in another database called Featured Database.
5. Find the distance between feature vectors of query COVID-19 and that of featured databases.
6. Sort the distance and Retrieve the N-top most similar.

The RNN structure replaces the traditional neuron by two neurons (lower neuron, upper neuron) to represent lower and upper approximations of each attribute in the CTG data set, its structure formed from 4 layers input, 2 hidden and output layers. The hidden layers have rough neurons, which overlap and exchange information between each other, While the input and output layers consist of traditional neurons

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The following figure gives an example of a patient's neutrosophic time series

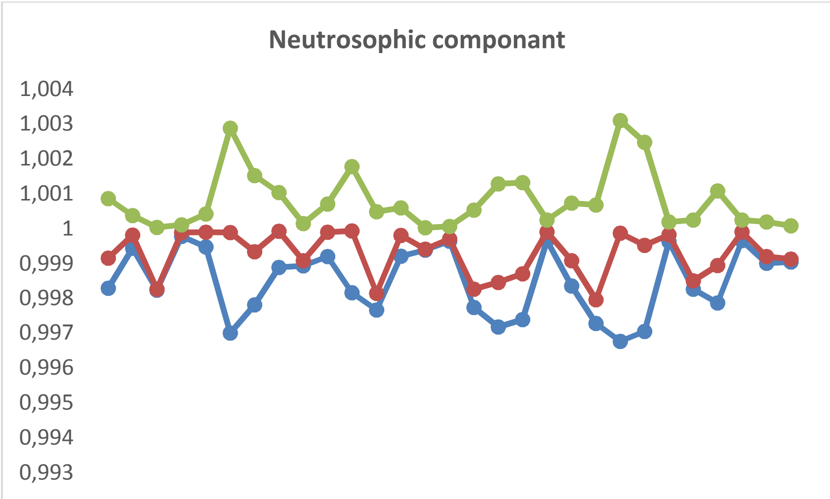

Fig.5. Patient neutrosophic time series

Recommendations:

The implementation steps require:

1) Providing modern computers inside the isolation hospitals equipped with a program for matching pictures, processing data, and issuing expectations and recommendations.
2) Provide a special microscope for imaging samples connected to a computer, as well as a high-quality camera connected to the computer as well.
3) Place blood samples, as well as samples, were taken from the mouth and nose, on the microscope, take pictures and place them on the aforementioned program.
4) Take pictures of patients' mouths from the inside (the beginning of the throat) and put them on the program.
5) Use all images for samples and mouth and put them as references for the system and divide the pictures into patterns and each reference pattern is different.
6) Determine the different virus patterns from the pictures and put all the data related to that pattern of symptoms, health status, and development in the case.
7) Take samples from the persons suspected of being infected, as well as those in contact with patients, and provide them with the program.
8) The start of applying this system in hospitals to identify the injured and the savings of time, effort and money.

Conclusion and Future Works

The idea for this study is based on the use of a computer-connected microscope associated with Deep Learning, using Convolutional Neural Network. The basis and analysis of dealing with all data, whether specific or not, that can be taken by time series values, then we present the linear model for the neutrosophic time series, and we test the significance of its coefficient based on patients distribution. Finally, from the above, we can provide a patient neutrosophic time series according to the linear model through which we can accurately predict the program will give degrees of verification and degrees of the uncertainty of the data. Furthermore, the proposed framework can also be extended towards other important domains of healthcare such as diabetes, cancer, and hepatitis, which can provide efficient services to corresponding patients.

References

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[1] Salama, A. A., Smarandache, F. .Neutrosophic Crisp Set Theory, Educational. Education Publishing 1313 Chesapeake, Avenue, Columbus, Ohio 43212, 2015.

[2] Smarandache, Florentin., Neutrosophy and Neutrosophic Logic , First International Conference on Neutrosophy , NeutrosophicLogic , Set, Probability, and Statistics University of New Mexico, Gallup, NM 87301, USA, 2002.

[3] Smarandache,F. A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability, American Research Press, Rehoboth, NM, 1999.

[4] Salama, A. A., Haitham A. El-Ghareeb, Ayman M. Manie, Florentin Smarandache. Introduction to Develop Some Software Programs for Dealing with Neutrosophic Sets, Neutrosophic Sets and Systems, vol. 3, pp. 51-52, 2014.

[5] Salama, A. A., Abdelfattah, M., El-Ghareeb, H. A., & Manie, A. M. Design and implementation of neutrosophic data operations using object oriented programming. International Journal of Computer Application, 4(10), 2014.

[6] Elwahsh, H., Gamal, M., Salama, A. A., & El-Henawy, I. M. Modeling neutrosophic data by self-organizing feature map: MANETs data case study. Procedia computer science, 121, 152-159, 2017.

[7] Salama, A. A., Eisa, M., ElGhawalby, H., & Fawzy, A. E. A New Approach in Content-Based Image Retrieval Neutrosophic Domain. In Fuzzy Multi-criteria Decision-Making Using Neutrosophic Sets (pp. 361-369). Springer, Cham, 2019.

[8] Alhabib, R., & Salama, A. A. The Neutrosophic Time Series-Study Its Models (Linear-Logarithmic) and test the Coefficients Significance of Its linear model. Neutrosophic Sets and Systems, 33, pp105-115, 2020.

[9] Elwahsh, H., Gamal, M., Salama, A. A., & El-Henawy, I. M. A novel approach for classifying Manets attacks with a neutrosophic intelligent system based on genetic algorithm. Security and Communication Networks, 2018.

[10] Coronavirus disease (COVID-19) pandemic, https://www.who.int/emergencies/diseases/novel-coronavirus-2019

[11] Ibrahim Yasser, Abeer Twakol, A. A. Abd El-Khalek, Ahmed Samrah, A. A. Salama, 5COVID-X: Novel Health-Fog Framework Based on Neutrosophic Classifier for Confrontation Covid-19, Neutrosophic Sets and Systems, 35 (Accpeted)

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Decision Support Modeling For Agriculture Land Selection Based On Sine Trigonometric Single Valued Neutrosophic Information

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Abstract

A single valued neutrosophic set (SVNS) is a useful tool to portray uncertainty in multi attribute decision-making. In this article, we develop hybrid averaging and hybrid geometric aggregation operator using sine trigonometric function to handle uncertainty in single valued neutrosophic information, which are, sine trigonometric-single valued neutrosophic hybrid weighted averaging (ST-SVNHWA) operator and , sine trigonometric-single valued neutrosophic hybrid weighted geometric (ST-SVNHWG) operator. We investigate properties, namely, idempotancy, monotonicity and boundedness for the proposed operators. Moreover, we give an algorithm to solve multi-criteria decision-making issues which involve SVN information with ST-SVNHWA and ST-SVNHWG operators. Finally, an illustrative example of agricultural land selection is provided to verify the effectiveness. Sensitivity and comparative analyses are also implemented to assess the stability and validity of our method.

Keywords:

Single valued neutrosophic set, Sine trigonometric single valued neutrosophic information,Agriculture land selection,Decision Support.

1 Introduction

Multi-criteria decision-making (MCDM) is performing a vital role in different areas, including social, physical, medical and environmental sciences. MCDM methods use not only to evaluate an appropriate object but also to rank the objects in a given problem. To solve different uncertain problems for decision-making, Atanassov[13] presented the concept of intuitionistic fuzzy set (IFS) to include both membership and non membership parts, an extension of fuzzy set[47] in which simply membership part is characterized. After that, various hybrid models of fuzzy sets (FSs) have been presented and investigated such as, Pythagorean fuzzy sets (PyFSs),[46] Picture fuzzy sets (PFSs),[17] Spherical fuzzy sets (SFSs)[1,2] and single-valued neutrosophic set (SVNS).[38,40]

Aggregation operators (AOs) perform an important role in order to combine data into a single form and solve MCDM problems. Aggregation implies the invention of a numeral of things to a cluster or a bunch of objects that have come or been taken together. In the past few years, aggregation operators based on FSs and its various hybrid compositions have made very much attention and become attractive because they can quickly execute functional areas of diverse regions. For example, Yager[45] introduced weighted aggregation operators (AOs). Khan et al.,[23] presented the probabilistic hesitant based DM technique. Xu[41] proposed some new AOs under IFSs. Khan et al.,[25] established the novel decision making (DM) methodology under generalized intuitionistic fuzzy soft information. Khan et al.,[24] established the Dombi AOs under PyFSs. Ashraf et al.,[5] proposed the fuzzy decision support modelling for internet finance soft power evaluation based on sine trigonometric Pythagorean fuzzy information. Batool et al., established the entropy based DM method under probabilistic Pythagorean hesitant fuzzy information. Sajjad et al.,[39] established the TOPSIS approach under PyFSs. Ashraf et al.,[6] presented the AOs based on algebraic norm under PFSs. Khan et al.,[26] presented the AOs based on Einstein norm under PFSs. Ashraf et al.,[7] introduced the DM method under picture cubic fuzzy sets (PCFSs). Qiyas et al.,[34,35] established the AOs based on the Dombi and algebraic norm using

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linguistic picture fuzzy information respectively. Rafiq et al.,[37] proposed the cosine trigonometric function based similarity measure under SFSs. Zang et al.,[48] proposed the DM approach using TOPSIS under spherical fuzzy rough set. Ashraf et al.,[8] introduced the norm representation under SFSs. Ashraf et al.,[9] presented the AOs using Dombi norm under SFSs. Jin et al.,[20] introduced the AOs for hybrid structure of linguistic and SFSs. Ashraf et al.,[10] presented the AOs for SFSs using the symmetric sum technique and discussed their application in DM. Jin et al.,[21] established the logarithmic function based AOs under SFSs. Ashraf et al.,[11] presented the DM technique using distance measures under SFSs. Barukab et al.,[14] introduced the entropy measure based extended TOPSIS under SFSs. Ashraf et al.,[12] established the AOs using logarithmic function under SVNSs. Ye[42] established the correlation coefficient based DM approach under SVNSs. Liu et al.,[28] proposed the Dombi norm based DM methodology under linguistic SVNSs. Liu et al.,[29] established the muirhead mean based power AOs under SVNSs. Liu et al.,[30, 31] presented the Heronian mean based power AOs using linguistic SVNSs and cubic neutrosophic information respectively. Ji et al.,[22] presented the Bonferroni mean AOs using Frank norm under SVNSs. Ye[43] introduced the list of novel AOs using exponential function under SVNSs and discussed their application to tackle the uncertainty in decision making problems.

Since single valued neutrosophic model is more general than the fuzzy sets, IFS, PyFS, PFS and SFSs due to the wider range of applicability over different complex problems. The SVN sets can explain uncertainties concurrently more precisely than the other existing methods, like fuzzy set and existing structures.[16, 27, 36]

The motivation of developed AOs is summarized as below.

(1) A very difficult MCDM problem is the estimation of the supreme option in a single valued neutrosophic environment due to the involvement of several imprecise factors. Assessment of information in different MCDM techniques is simply depicted through fuzzy and their existing structures which may not consider all the data in a real-world problem.

(2) As a general theory, single valued neutrosophic numbers describes efficient execution in the assessment process about uncertain, imprecise and vague information. Thus, single valued neutrosophic theory provide an excellent approach for the assessment of objects under multinary data.

(3) In view of the fact that sine trigonometric hybrid AOs are simple but provide a pioneering tool for solving MCDM problems when combine with other powerful mathematical tools, this article aims to develop sine trigonometric hybrid AOs in a single valued neutrosophic environment to handle complex problems.

(4) A single valued neutrosophic model is different from the mathematical tools like fuzzy sets and their extensions. Because the fuzzy set and their extensions can only handle one dimensional data, two dimensional data and three dimensional data, respectively, which may prompt a loss in data. Nevertheless, in many daily life problems, we handle the situations having higher dimension to sort out all the attributes.

(5) The sine trigonometric hybrid AOs employed in the construction of SVN sine trigonometric hybrid AOs are more suitable than all other aggregation approaches to tackle the MCDM situations as developed AOs have ability to consider all the information within the aggregation procedure.

(6) sine trigonometric AOs make the optimal outcomes more accurate and definite when utilized in practical MCDM problems under single valued neutrosophic environment. However, the proposed single valued neutrosophic operators handle the drawbacks of AOs present in the literature.

Therefore, some single valued neutrosophic sine trigonometric hybrid AOs are developed to choose the best option in different decision-making situations. The developed operators has some advantages over other approaches which are given as below:

(1) Our proposed methods explain the problems more accurately which involve multiple attributes because they consider single valued neutrosophic numbers.

(2) The developed AOs are more precise and efficient with single attribute.

(3) To solve practical problems by using sine trigonometric hybrid AOs with single valued neutrosophic numbers is very significant.

The rest of this article is structured as follows: Section 2 recalls some fundamental definitions and operations of the SVNSs. Section 3 presents novel sine hybrid aggregation operators. Section 4 develops a methodology of these AOs to MCDM problems under single valued nuetrosophic environment. Section 5 discusses a application of the selection of best agricultural land. Section 6 provides comparative analysis of developed approaches with different aggregating methodologies. Section 7 discusses the conclusions and future directions.

2 Preliminaries

In this section some essential notions of PFS, SFS, and SVNS are examined.

Definition 2.1. [17]A PFS $\mathcal{U}$ in fixed universe set $\gimel$ is defined as

$$\mathcal{U} = \{\langle \flat, \widetilde{\rho}_{\wp}(\flat), \daleth_{\wp}(\flat), \tilde{n}_{\wp}(\flat)\rangle \, | \flat \in \gimel\},$$

where the positive, neutral and negative membership grades, $\widetilde{\rho}_{\wp} : \gimel \to \Theta$, $\daleth_{\wp} : \gimel \to \Theta$ and $\tilde{n}_{\wp} : \gimel \to \Theta$, respectively and also, $\Theta = [0, 1]$ is the unit interval. Furthermore, $0 \leq \widetilde{\rho}_{\wp}(\flat) + \daleth_{\wp}(\flat) + \tilde{n}_{\wp}(\flat) \leq 1$, for each $\flat \in \gimel$.

Definition 2.2. [1,2]A SFS $\mathcal{U}$ in fixed universe set $\gimel$ is defined as

$$\mathcal{U} = \{\langle \flat, \widetilde{\rho}_{\wp}(\flat), \daleth_{\wp}(\flat), \tilde{n}_{\wp}(\flat)\rangle \, | \flat \in \gimel\},$$

where the positive, neutral and negative membership grades, $\widetilde{\rho}_{\wp} : \gimel \to \Theta$, $\daleth_{\wp} : \gimel \to \Theta$ and $\tilde{n}_{\wp} : \gimel \to \Theta$, respectively and also, $\Theta = [0, 1]$ is the unit interval. Furthermore, $0 \leq \widetilde{\rho}^2_{\wp}(\flat) + \daleth^2_{\wp}(\flat) + \tilde{n}^2_{\wp}(\flat) \leq 1$, for each $\flat \in \gimel$.

Definition 2.3. [38] A neutrosophic set $\mathcal{U}$ in a fixed universe set $\gimel$ is defined as

$$\mathcal{U} = \{\langle \flat, \widetilde{\rho}_{\wp}(\flat), \daleth_{\wp}(\flat), \tilde{n}_{\wp}(\flat)\rangle \, | \flat \in \gimel\},$$

where the positive, neutral and negative membership grades, $\widetilde{\rho}_{\wp} : \gimel \to]0^-, 1^+[$, $\daleth_{\wp} : \gimel \to]0^-, 1^+[$ and $\tilde{n}_{\wp} : \gimel \to]0^-, 1^+[$, respectively. Furthermore, $0^- \leq \widetilde{\rho}_{\wp}(\flat) + \daleth_{\wp}(\flat) + \tilde{n}_{\wp}(\flat) \leq 3^+$, for each $\flat \in \gimel$.

Definition 2.4. [40] A SVNS $\mathcal{U}$ in a fixed universe set $\gimel$ is defined as

$$\mathcal{U} = \{\langle \flat, \widetilde{\rho}_{\wp}(\flat), \daleth_{\wp}(\flat), \tilde{n}_{\wp}(\flat)\rangle \, | \flat \in \gimel\},$$

where the positive, neutral and negative membership grades, $\widetilde{\rho}_{\wp} : \gimel \to \Theta$, $\daleth_{\wp} : \gimel \to \Theta$ and $\tilde{n}_{\wp} : \gimel \to \Theta$, respectively and also, $\Theta = [0, 1]$ is the unit interval. Furthermore, $0 \leq \widetilde{\rho}_{\wp}(\flat) + \daleth_{\wp}(\flat) + \tilde{n}_{\wp}(\flat) \leq 3$, for each $\flat \in \gimel$.

In simplification, throughout the whole work, the triplet $\mathcal{U} = \{\widetilde{\rho}_{\wp}, \daleth_{\wp}, \tilde{n}_{\wp}\}$ called single valued neutrosophic number (SVNN) and their collection denoted by $SVNN(\gimel)$.

Here we highlight the some basic operations for SVNNs,[4044] &[50] as follows:

Definition 2.5. [50] Let $\mathcal{U}_1 = \{\widetilde{\rho}_{\wp_1}, \daleth_{\wp_1}, \tilde{n}_{\wp_1}\}$ and $\mathcal{U}_2 = \{\widetilde{\rho}_{\wp_2}, \daleth_{\wp_2}, \tilde{n}_{\wp_2}\} \in SVNN(\gimel)$. than,
(1) $\mathcal{U}_1 \subseteq \mathcal{U}_2$ if and only if $\widetilde{\rho}_{\wp_1} \leq \widetilde{\rho}_{\wp_2}$, $\daleth_{\wp_1} \geq \daleth_{\wp_2}$ and $\tilde{n}_{\wp_1} \geq \tilde{n}_{\wp_2}$ for each $\flat \in \gimel$.
(2) $\mathcal{U}_1 = \mathcal{U}_2$ if and only if $\mathcal{U}_1 \subseteq \mathcal{U}_2$ and $\mathcal{U}_2 \subseteq \mathcal{U}_1$.
(2) $\mathcal{U}_1 \cap \mathcal{U}_2 = \{\min(\widetilde{\rho}_{\wp_1}, \widetilde{\rho}_{\wp_2}), \max(\daleth_{\wp_1}, \daleth_{\wp_2}), \max(\tilde{n}_{\wp_1}, \tilde{n}_{\wp_2})\}$,
(3) $\mathcal{U}_1 \cup \mathcal{U}_2 = \{\max(\widetilde{\rho}_{\wp_1}, \widetilde{\rho}_{\wp_2}), \min(\daleth_{\wp_1}, \daleth_{\wp_2}), \min(\tilde{n}_{\wp_1}, \tilde{n}_{\wp_2})\}$,
(4) $\mathcal{U}_1^c = \{\tilde{n}_{\wp_1}, \daleth_{\wp_1}, \widetilde{\rho}_{\wp_1}\}$.

Definition 2.6. [40,44] Let $\mathcal{U}_1 = \{\widetilde{\rho}_{\wp_1}, \daleth_{\wp_1}, \tilde{n}_{\wp_1}\}$ and $\mathcal{U}_2 = \{\widetilde{\rho}_{\wp_2}, \daleth_{\wp_2}, \tilde{n}_{\wp_2}\} \in SVNN(\gimel)$ with $\eth > 0$. than,
(1) $\mathcal{U}_1 \otimes \mathcal{U}_2 = \{\widetilde{\rho}_{\wp_1}\widetilde{\rho}_{\wp_2}, \daleth_{\wp_1} + \daleth_{\wp_2} - \daleth_{\wp_1} \cdot \daleth_{\wp_2}, \tilde{n}_{\wp_1} + \tilde{n}_{\wp_2} - \tilde{n}_{\wp_1} \cdot \tilde{n}_{\wp_2}\}$;
(2) $\mathcal{U}_1 \oplus \mathcal{U}_2 = \{\widetilde{\rho}_{\wp_1} + \widetilde{\rho}_{\wp_2} - \widetilde{\rho}_{\wp_1}\widetilde{\rho}_{\wp_2}, \daleth_{\wp_1}\daleth_{\wp_2}, \tilde{n}_{\wp_1}\tilde{n}_{\wp_2}\}$;
(3) $(\mathcal{U}_1)^{\eth} = \left\{(\widetilde{\rho}_{\wp_1})^{\eth}, 1 - (1 - \daleth_{\wp_1})^{\eth}, 1 - (1 - \tilde{n}_{\wp_1})^{\eth}\right\}$;
(4) $\eth \cdot \mathcal{U}_1 = \left\{1 - (1 - \widetilde{\rho}_{\wp_1})^{\eth}, (\daleth_{\wp_1})^{\eth}, (\tilde{n}_{\wp_1})^{\eth}\right\}$.

Definition 2.7. [4] Let $\mathcal{U}_{\hbar} = \{\widetilde{\rho}_{\wp_{\hbar}}, \daleth_{\wp_{\hbar}}, \tilde{n}_{\wp_{\hbar}}\} \in SVNN(\gimel)$ $(\hbar = 1, 2, 3, ..., n)$. Then, the Algebraic averaging aggregation operator for $SVNN(\gimel)$ is denoted by $SVNWA$ and defined as follows:

$$\begin{aligned} SVNWA(\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, ..., \mathcal{U}_n) &= \sum_{\hbar=1}^{n} \eth_{\hbar}\mathcal{U}_{\hbar}, \\ &= \left\{1 - \Pi_{\hbar=1}^{n}(1 - \widetilde{\rho}_{\wp_{\hbar}})^{\eth_{\hbar}}, \Pi_{\hbar=1}^{n}(\daleth_{\wp_{\hbar}})^{\eth_{\hbar}}, \Pi_{\hbar=1}^{n}(\tilde{n}_{\wp_{\hbar}})^{\eth_{\hbar}}\right\} \end{aligned}$$

where the weights of $\mathcal{U}_{\hbar}$ represented by $\eth_{\hbar}$ $(\hbar = 1, 2, ..., n)$ having $\eth_{\hbar} \geq 0$ and $\sum_{\hbar=1}^{n} \eth_{\hbar} = 1$.

Definition 2.8. [4] Let $\mathcal{U}_{\hbar} = \{\widetilde{\rho}_{\wp_{\hbar}}, \daleth_{\wp_{\hbar}}, \tilde{n}_{\wp_{\hbar}}\} \in SVNN(\gimel)$ $(\hbar = 1, 2, 3, ..., n)$. Then, the Algebraic geometric aggregation operator for $SVNN(\gimel)$ is denoted by $SVNWG$ and defined as follows:

$$\begin{aligned} SFWG(\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, ..., \mathcal{U}_n) &= \prod_{\hbar=1}^{n} (\mathcal{U}_{\hbar})^{\eth_{\hbar}}, \\ &= \left\{\Pi_{\hbar=1}^{n}(\widetilde{\rho}_{\wp_{\hbar}})^{\eth_{\hbar}}, 1 - \Pi_{\hbar=1}^{n}(1 - \daleth_{\wp_{\hbar}})^{\eth_{\hbar}}, 1 - \Pi_{\hbar=1}^{n}(1 - \tilde{n}_{\wp_{\hbar}})^{\eth_{\hbar}}\right\} \end{aligned}$$

where $\eth_{\hbar}$ $(\hbar = 1, 2, ..., n)$ represents the weights of $\mathcal{U}_{\hbar}$ $(\hbar = 1, 2, 3, ..., n)$ with $\eth_{\hbar} \geq 0$ and $\sum_{\hbar=1}^{n} \eth_{\hbar} = 1$.

Ashraf et al.,[4] introduced the sine trigonometric operational laws for single valued neutrosophic environments.

Definition 2.9. [4] Let $\mathcal{U} = \{\widetilde{\rho}_{\wp}, \urcorner_{\wp}, \tilde{n}_{\wp}\} \in SVNN(\daleth)$. If

$$\sin(\mathcal{U}) = \left\{ \begin{pmatrix} \daleth, \sin\left(\frac{\pi}{2}\widetilde{\rho}_{\wp}(\daleth)\right), 1-\sin\left(\frac{\pi}{2}1-\urcorner_{\wp}(\daleth)\right), \\ 1-\sin\left(\frac{\pi}{2}1-\tilde{n}_{\wp}(\daleth)\right) \end{pmatrix} | \flat \in \daleth \right\}$$

then, $\sin(\mathcal{U})$ is said to be sine trigonometric operator and their value known to be sine trigonometric SVNN (ST-SVNN).

Definition 2.10. [4] Let $\sin(\mathcal{U}_1) = \left\{ \begin{pmatrix} \sin\left(\frac{\pi}{2}\widetilde{\rho}_{\wp_1}\right), \\ 1-\sin\left(\frac{\pi}{2}1-\urcorner_{\wp_1}\right), \\ 1-\sin\left(\frac{\pi}{2}1-\tilde{n}_{\wp_1}\right) \end{pmatrix} \right\}$ and $\sin(\mathcal{U}_2) = \left\{ \begin{pmatrix} \sin\left(\frac{\pi}{2}\widetilde{\rho}_{\wp_2}\right), \\ 1-\sin\left(\frac{\pi}{2}1-\urcorner_{\wp_2}\right), \\ 1-\sin\left(\frac{\pi}{2}1-\tilde{n}_{\wp_2}\right) \end{pmatrix} \right\}$ be two ST-SVNNs. The basic operations for SVNNs are as follows

(1) $\sin(\mathcal{U}_1) \oplus \sin(\mathcal{U}_2) = \begin{pmatrix} 1-\left(1-\sin\left(\frac{\pi}{2}\widetilde{\rho}_{\wp_1}\right)\right)\left(1-\sin\left(\frac{\pi}{2}\widetilde{\rho}_{\wp_2}\right)\right), \\ \left(1-\sin\left(\frac{\pi}{2}1-\urcorner_{\wp_1}\right)\right)\left(1-\sin\left(\frac{\pi}{2}1-\urcorner_{\wp_2}\right)\right), \\ \left(1-\sin\left(\frac{\pi}{2}1-\tilde{n}_{\wp_1}\right)\right)\left(1-\sin\left(\frac{\pi}{2}1-\tilde{n}_{\wp_2}\right)\right) \end{pmatrix}$,

(2) $\psi \cdot \sin(\mathcal{U}_1) = \begin{pmatrix} 1-\left(1-\sin\left(\frac{\pi}{2}\widetilde{\rho}_{\wp_1}\right)\right)^{\psi}, \left(1-\sin\left(\frac{\pi}{2}1-\urcorner_{\wp_1}\right)\right)^{\psi}, \\ \left(1-\sin\left(\frac{\pi}{2}1-\tilde{n}_{\wp_1}\right)\right)^{\psi} \end{pmatrix}$,

(3) $\sin(\mathcal{U}_1) \otimes \sin(\mathcal{U}_2) = \begin{pmatrix} \sin\left(\frac{\pi}{2}\widetilde{\rho}_{\wp_1}\right)\sin\left(\frac{\pi}{2}\widetilde{\rho}_{\wp_2}\right), \\ 1-\left(\sin\left(\frac{\pi}{2}1-\urcorner_{\wp_1}\right)\right)\left(\sin\left(\frac{\pi}{2}1-\urcorner_{\wp 2}\right)\right), \\ 1-\left(\sin\left(\frac{\pi}{2}1-\tilde{n}_{\wp_1}\right)\right)\left(\sin\left(\frac{\pi}{2}1-\tilde{n}_{\wp 2}\right)\right) \end{pmatrix}$,

(4) $\left(\sin(\mathcal{U}_1)\right)^{\psi} = \begin{pmatrix} \left(\sin\left(\frac{\pi}{2}\widetilde{\rho}_{\wp_1}\right)\right)^{\psi}, 1-\left(\sin\left(\frac{\pi}{2}1-\urcorner_{\wp_1}\right)\right)^{\psi}, \\ 1-\left(\sin\left(\frac{\pi}{2}1-\tilde{n}_{\wp_1}\right)\right)^{\psi} \end{pmatrix}$.

Definition 2.11. Let $\mathcal{U} = \{\widetilde{\rho}_{\wp}, \urcorner_{\wp}, \tilde{n}_{\wp}\} \in SVNN(\daleth)$. Then, the score and accuracy of $\mathcal{U}$ is denoted and defined as
(1) $\check{s}\check{c}(\mathcal{U}) = \widetilde{\rho}_{\wp} - \urcorner_{\wp} - \tilde{n}_{\wp}$, and
(2) $\tilde{a}\check{c}(\mathcal{U}) = \widetilde{\rho}_{\wp} + \urcorner_{\wp} + \tilde{n}_{\wp}$.

Definition 2.12. Let $\mathcal{U}_1 = \{\widetilde{\rho}_{\wp_1}, \urcorner_{\wp_1}, \tilde{n}_{\wp_1}\}$ and $\mathcal{U}_2 = \{\widetilde{\rho}_{\wp_2}, \urcorner_{\wp_2}, \tilde{n}_{\wp_2}\} \in SVNN(\daleth)$. Then,
(1) If $\check{s}\check{c}(\mathcal{U}_1) < \check{s}\check{c}(\mathcal{U}_2)$ then $\mathcal{U}_1 < \mathcal{U}_2$,
(2) If $\check{s}\check{c}(\mathcal{U}_1) > \check{s}\check{c}(\mathcal{U}_2)$ then $\mathcal{U}_1 > \mathcal{U}_2$,
(3) If $\check{s}\check{c}(\mathcal{U}_1) = \check{s}\check{c}(\mathcal{U}_2)$ then
(a) $\tilde{a}\check{c}(\mathcal{U}_1) < \tilde{a}\check{c}(\mathcal{U}_2)$ then $\mathcal{U}_1 < \mathcal{U}_2$,
(b) $\tilde{a}\check{c}(\mathcal{U}_1) > \tilde{a}\check{c}(\mathcal{U}_2)$ then $\mathcal{U}_1 > \mathcal{U}_2$,
(c) $\tilde{a}\check{c}(\mathcal{U}_1) = \tilde{a}\check{c}(\mathcal{U}_2)$ then $\mathcal{U}_1 = \mathcal{U}_2$.

3 Novel Sine Trigonometric Hybrid AOs for SVNNs

This section propose the novel sine trigonometric hybrid AOs under SVN information.

Definition 3.1. Let $\mathcal{U}_{\hbar} = \{\widetilde{\rho}_{\wp_{\hbar}}(\flat), \urcorner_{\wp_{\hbar}}(\flat), \tilde{n}_{\wp_{\hbar}}(\flat)\} \in SVNN(\daleth)$ $(\hbar \in \mathbb{N})$. The sine trigonometric hybrid weighted averaging AO for $SVNN(\daleth)$ is represented by $ST-SVNHWA$ and discribed as follows:

$$\begin{aligned} ST-SVNHWA(\mathcal{U}_1, \mathcal{U}_2, ..., \mathcal{U}_n) &= \sigma_1 \sin\left(\mathcal{U}'_{\upsilon(1)}\right) \oplus \sigma_2 \sin\left(\mathcal{U}'_{\upsilon(2)}\right) \oplus ... \oplus \sigma_n \sin\left(\mathcal{U}\prime_{\upsilon(n)}\right) \\ &= \sum_{\hbar=1}^{n} \sigma_{\hbar} \sin\left(\mathcal{U}'_{\upsilon(\hbar)}\right), \end{aligned}$$

where the weights of $\mathcal{U}_{\hbar}$ is represented by $\eth_{\hbar}$ $(\hbar = 1, 2, ..., n)$ having $\eth_{\hbar} \geq 0$ and $\sum_{\hbar=1}^{n} \eth_{\hbar} = 1$ and gth biggest weighted value is $\mathcal{U}'_{\upsilon(\hbar)}$ $\left(\mathcal{U}'_{\upsilon(\hbar)} = n\eth_{\hbar}\mathcal{U}_{\upsilon(\hbar)} | \hbar = 1, 2, ..., n\right)$ consequently by total order $(\upsilon(1), \upsilon(2), \upsilon(3), ..., \upsilon(n))$. Also the associated weight vector $\sigma = \sigma_{\hbar}$ $(\hbar = 1, 2, ..., n)$ with $\sigma_{\hbar} \geq 0$ and $\sum_{\hbar=1}^{n} \sigma_{\hbar} = 1$.

Theorem 3.2. *Let* $\mathcal{U}_{\hbar} = \{\widetilde{\rho}_{\wp_{\hbar}}(\flat), \urcorner_{\wp_{\hbar}}(\flat), \tilde{n}_{\wp_{\hbar}}(\flat)\} \in SVNN(\daleth)$ $(\hbar \in \mathbb{N})$ *and the weights of* $\mathcal{U}_{\hbar}$ *represented by* $(\eth_1, \eth_2, ..., \eth_n)^T$ *subject to* $\sum_{\hbar=1}^{n} \eth_{\hbar} = 1$. *The* $ST-SVNHWA$ *AO is defined by a mapping* $G^n \longrightarrow G$

with associated weight vector $\sigma_\hbar$ $(\hbar = 1, 2, ..., n)$ *having* $\sigma_\hbar \geq 0$ *and* $\sum_{\hbar=1}^{n} \sigma_\hbar = 1$*:*

$$\begin{aligned} ST - SVNHWA\left(\mathcal{U}_1, \mathcal{U}_2, ..., \mathcal{U}_n\right) &= \sum_{\hbar=1}^{n} \sigma_\hbar \sin\left(\mathcal{U}'_{\upsilon(\hbar)}\right) \\ &= \begin{pmatrix} 1 - \prod_{\hbar=1}^{n} \left(1 - \sin\left(\frac{\pi}{2}\widetilde{\rho}'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar}, \\ \prod_{\hbar=1}^{n} \left(1 - \sin\left(\frac{\pi}{2} 1 - \daleth'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar}, \\ \prod_{\hbar=1}^{n} \left(1 - \sin\left(\frac{\pi}{2} 1 - \tilde{n}'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar} \end{pmatrix} \end{aligned} \quad (1)$$

Proof. By using mathematical induction on n we prove Theorem 3.2. Then the mathematical induction steps below were implemented.

Step-1: For $n = 2$, we obtained

$$ST - SVNHWA\left(\mathcal{U}_1, \mathcal{U}_2\right) = \sigma_1 \sin\left(\mathcal{U}'_{\upsilon(1)}\right) \oplus \sigma_2 \sin\left(\mathcal{U}'_{\upsilon(2)}\right).$$

Since by the Definition 2.9, $\sin\left(\mathcal{U}_1\right)$ and $\sin\left(\mathcal{U}_2\right)$ are SVNNs and hence $\sigma_1 \sin\left(\mathcal{U}'_{\upsilon(1)}\right) \oplus \sigma_2 \sin\left(\mathcal{U}'_{\upsilon(2)}\right)$ is also SVNN. Further, for $\mathcal{U}_1$ and $\mathcal{U}_2$, we have

$$\begin{aligned} & ST - SVNHWA\left(\mathcal{U}_1, \mathcal{U}_2\right) \\ =\ & \sigma_1 \sin\left(\mathcal{U}'_{\upsilon(1)}\right) \oplus \sigma_2 \sin\left(\mathcal{U}'_{\upsilon(2)}\right) \\ =\ & \begin{pmatrix} 1 - \left(1 - \sin\left(\frac{\pi}{2}\widetilde{\rho}'_{\wp_{\upsilon(1)}}\right)\right)^{\sigma_1}, \\ \left(1 - \sin\left(\frac{\pi}{2} 1 - \daleth'_{\wp_{\upsilon(1)}}\right)\right)^{\sigma_1}, \\ \left(1 - \sin\left(\frac{\pi}{2} 1 - \tilde{n}'_{\wp_{\upsilon(1)}}\right)\right)^{\sigma_1} \end{pmatrix} \oplus \begin{pmatrix} 1 - \left(1 - \sin\left(\frac{\pi}{2}\widetilde{\rho}'_{\wp_{\upsilon(2)}}\right)\right)^{\sigma_2}, \\ \left(1 - \sin\left(\frac{\pi}{2} 1 - \daleth'_{\wp_{\upsilon(2)}}\right)\right)^{\sigma_2}, \\ \left(1 - \sin\left(\frac{\pi}{2} 1 - \tilde{n}'_{\wp_{\upsilon(2)}}\right)\right)^{\sigma_2} \end{pmatrix} \\ =\ & \begin{pmatrix} 1 - \prod_{\hbar=1}^{2} \left(1 - \sin\left(\frac{\pi}{2}\widetilde{\rho}'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar}, \\ \prod_{\hbar=1}^{2} \left(1 - \sin\left(\frac{\pi}{2} 1 - \daleth'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar}, \\ \prod_{\hbar=1}^{2} \left(1 - \sin\left(\frac{\pi}{2} 1 - \tilde{n}'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar} \end{pmatrix} \end{aligned}$$

Step-2: Suppose that Equation 1 is holds for $n = \kappa$, we have

$$ST - SVNWA\left(\mathcal{U}_1, \mathcal{U}_2, ...\mathcal{U}_\kappa\right) = \begin{pmatrix} 1 - \prod_{\hbar=1}^{\kappa} \left(1 - \sin\left(\frac{\pi}{2}\widetilde{\rho}'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar}, \\ \prod_{\hbar=1}^{\kappa} \left(1 - \sin\left(\frac{\pi}{2} 1 - \daleth'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar}, \\ \prod_{\hbar=1}^{\kappa} \left(1 - \sin\left(\frac{\pi}{2} 1 - \tilde{n}'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar} \end{pmatrix}$$

Step-3: Now we have to prove that Equation 1 is holds for $n = \kappa + 1$.

$$\begin{aligned}
& ST - SVNHWA\left(\mathcal{U}_1, \mathcal{U}_2, ...\mathcal{U}_{\kappa+1}\right) \\
= & \sum_{\hbar=1}^{\kappa} \sigma_\hbar \sin\left(\mathcal{U}'_{\upsilon(\hbar)}\right) \oplus \sigma_{\kappa+1} \sin\left(\mathcal{U}'_{\upsilon(\kappa+1)}\right) \\
= & \begin{pmatrix} 1 - \prod_{\hbar=1}^{\kappa}\left(1 - \sin\left(\frac{\pi}{2}\widetilde{\rho}'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar}, \\ \prod_{\hbar=1}^{\kappa}\left(1 - \sin\left(\frac{\pi}{2}1 - \urcorner'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar}, \\ \prod_{\hbar=1}^{\kappa}\left(1 - \sin\left(\frac{\pi}{2}1 - \tilde{n}'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar} \end{pmatrix} \oplus \begin{pmatrix} 1 - \left(1 - \sin\left(\frac{\pi}{2}\widetilde{\rho}'_{\wp_{\upsilon(\kappa+1)}}\right)\right)^{\sigma_{\kappa+1}}, \\ \left(1 - \sin\left(\frac{\pi}{2}1 - \urcorner'_{\wp_{\upsilon(\kappa+1)}}\right)\right)^{\sigma_{\kappa+1}}, \\ \left(1 - \sin\left(\frac{\pi}{2}1 - \tilde{n}'_{\wp_{\upsilon(\kappa+1)}}\right)\right)^{\sigma_{\kappa+1}} \end{pmatrix} \\
= & \begin{pmatrix} 1 - \prod_{\hbar=1}^{\kappa+1}\left(1 - \sin\left(\frac{\pi}{2}\widetilde{\rho}'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar}, \\ \prod_{\hbar=1}^{\kappa+1}\left(1 - \sin\left(\frac{\pi}{2}1 - \urcorner'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar}, \\ \prod_{\hbar=1}^{\kappa+1}\left(1 - \sin\left(\frac{\pi}{2}1 - \tilde{n}'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar} \end{pmatrix}
\end{aligned}$$

that is, when $n = z + 1$, Equation 1 also holds.

Hence, Equation 1 holds for any n. The proof is completed. □

Next, we give some properties that are apparently carried by the proposed ST-SVNHWA aggregation operator.

(1) Let $\mathcal{U}_\hbar = \{\widetilde{\rho}_{\wp_\hbar}, \urcorner_{\wp_\hbar}, \tilde{n}_{\wp_\hbar}\} \in SVNN(\beth)$ $(\hbar = 1, 2, 3, ..., n)$ such that $\mathcal{U}_\hbar = \mathcal{U}$. Then

$$ST - SVNHWA\left(\mathcal{U}_1, \mathcal{U}_2, ..., \mathcal{U}_n\right) = \sin\left(\mathcal{U}\right).$$

(2) Let $\mathcal{U}_\hbar = \{\widetilde{\rho}_{\wp_\hbar}, \urcorner_{\wp_\hbar}, \tilde{n}_{\wp_\hbar}\}, \mathcal{U}_\hbar^- = \{\min\left(\widetilde{\rho}_{\wp_\hbar}\right), \max\left(\urcorner_{\wp_\hbar}\right), \max\left(\tilde{n}_{\wp_\hbar}\right)\}$ and $\mathcal{U}_\hbar^+ = \{\max\left(\widetilde{\rho}_{\wp_\hbar}\right), \min\left(\urcorner_{\wp_\hbar}\right), \min\left(\tilde{n}_{\wp_\hbar}\right)\}$ $\in SVNN(\beth)$ $(\hbar = 1, 2, 3, ..., n)$. Then,

$$\sin\left(\mathcal{U}_\hbar^-\right) \leq ST - SVNHWA\left(\mathcal{U}_1, \mathcal{U}_2, ..., \mathcal{U}_n\right) \leq \sin\left(\mathcal{U}_\hbar^+\right).$$

(3) Let $\mathcal{U}_\hbar = \{\widetilde{\rho}_{\wp_\hbar}, \urcorner_{\wp_\hbar}, \tilde{n}_{\wp_\hbar}\}, \mathcal{U}_\hbar^* = \left\{\widetilde{\rho}^*_{\wp_\hbar}, \urcorner^*_{\wp_\hbar}, \tilde{n}^*_{\wp_\hbar}\right\} \in SVNN(\beth)$ $(\hbar = 1, 2, 3, ..., n)$. If $\widetilde{\rho}_{\wp_\hbar} \leq \widetilde{\rho}^*_{\wp_\hbar}, \urcorner_{\wp_\hbar} \leq \urcorner^*_{\wp_\hbar}$ and $\tilde{n}_{\wp_\hbar} \leq \tilde{n}^*_{\wp_\hbar}$, then

$$ST - SVNHWA\left(\mathcal{U}_1, \mathcal{U}_2, ..., \mathcal{U}_n\right) \leq ST - SVNHWA\left(\mathcal{U}_1^*, \mathcal{U}_2^*, ..., \mathcal{U}_n^*\right).$$

Definition 3.3. Let $\mathcal{U}_\hbar = \{\widetilde{\rho}_{\wp_\hbar}(\flat), \urcorner_{\wp_\hbar}(\flat), \tilde{n}_{\wp_\hbar}(\flat)\} \in SVNN(\beth)$ $(\hbar \in \mathbb{N})$. The sine trigonometric hybrid weighted geometric AO for $SVNN(\beth)$ is represented by $ST - SVNHWG$ and discribed as follows:

$$\begin{aligned}
ST - SVNHWG\left(\mathcal{U}_1, \mathcal{U}_2, ..., \mathcal{U}_n\right) &= \left(\sin\left(\mathcal{U}'_{\upsilon(1)}\right)\right)^{\sigma_1} \otimes \left(\sin\left(\mathcal{U}'_{\upsilon(2)}\right)\right)^{\sigma_2} \otimes ... \otimes \left(\sin\left(\mathcal{U}'_{\upsilon(n)}\right)\right)^{\sigma_n} \\
&= \prod_{\hbar=1}^{n}\left(\sin\left(\mathcal{U}'_{\upsilon(\hbar)}\right)\right)^{\sigma_\hbar}
\end{aligned}$$

where the weights of $\mathcal{U}_\hbar$ is represented by $\eth_\hbar$ $(\hbar = 1, 2, ..., n)$ having $\eth_\hbar \geq 0$ and $\sum_{\hbar=1}^{n}\eth_\hbar = 1$ and gth biggest weighted value is $\mathcal{U}'_{\upsilon(\hbar)}$ $\left(\mathcal{U}'_{\upsilon(\hbar)} = n\eth_\hbar\mathcal{U}_{\upsilon(\hbar)}|\hbar = 1, 2, ..., n\right)$ consequently by total order $(\upsilon(1), \upsilon(2), \upsilon(3), ..., \upsilon(n))$. Also the associated weight vector $\sigma = \sigma_\hbar$ $(\hbar = 1, 2, ..., n)$ with $\sigma_\hbar \geq 0$ and $\sum_{\hbar=1}^{n}\sigma_\hbar = 1$.

Theorem 3.4. *Let $\mathcal{U}_\hbar = \{\widetilde{\rho}_{\wp_\hbar}(\flat), \urcorner_{\wp_\hbar}(\flat), \tilde{n}_{\wp_\hbar}(\flat)\} \in SVNN(\beth)$ $(\hbar \in \mathbb{N})$ and the weights of $\mathcal{U}_\hbar$ represented by $(\eth_1, \eth_2, ..., \eth_n)^T$ subject to $\sum_{\hbar=1}^{n}\eth_\hbar = 1$. The $ST - SVNHWG$ AO is defined by a mapping $G^n \longrightarrow G$ with associated weight vector $\sigma_\hbar$ $(\hbar = 1, 2, ..., n)$ having $\sigma_\hbar \geq 0$ and $\sum_{\hbar=1}^{n}\sigma_\hbar = 1$:*

$$\begin{aligned}
ST - SVNHWG\left(\mathcal{U}_1, \mathcal{U}_2, ..., \mathcal{U}_n\right) &= \prod_{\hbar=1}^{n}\left(\sin\left(\mathcal{U}'_{\upsilon(\hbar)}\right)\right)^{\sigma_\hbar} \\
&= \begin{pmatrix} \prod_{\hbar=1}^{n}\left(\sin\left(\frac{\pi}{2}\widetilde{\rho}'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar}, \\ 1 - \prod_{\hbar=1}^{n}\left(\sin\left(\frac{\pi}{2}1 - \urcorner'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar}, \\ 1 - \prod_{\hbar=1}^{n}\left(\sin\left(\frac{\pi}{2}1 - \tilde{n}'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar} \end{pmatrix} \qquad (2)
\end{aligned}$$

Proof. By using mathematical induction on n we prove Theorem 3.4. Then the mathematical induction steps below were implemented.

Step-1: For $n = 2$, we get

$$ST-SVNHWG\left(\mathcal{U}_1,\mathcal{U}_2\right)=\left(\sin\left(\mathcal{U}'_{\upsilon(1)}\right)\right)^{\sigma_1}\otimes\left(\sin\left(\mathcal{U}'_{\upsilon(2)}\right)\right)^{\sigma_2}.$$

Since by the Definition 2.9, $\sin\left(\mathcal{U}_1\right)$ and $\sin\left(\mathcal{U}_2\right)$ are SFNs and hence $\left(\sin\left(\mathcal{U}'_{\upsilon(1)}\right)\right)^{\sigma_1}\otimes\left(\sin\left(\mathcal{U}'_{\upsilon(2)}\right)\right)^{\sigma_2}$ is also SVNN. Further, for $\mathcal{U}_1$ and $\mathcal{U}_2$, we have

$$\begin{aligned}
& ST-SVNHWG\left(\mathcal{U}_1,\mathcal{U}_2\right)\\
= & \left(\sin\left(\mathcal{U}'_{\upsilon(1)}\right)\right)^{\sigma_1}\otimes\left(\sin\left(\mathcal{U}'_{\upsilon(2)}\right)\right)^{\sigma_2}\\
= & \begin{pmatrix} \left(\sin\left(\frac{\pi}{2}\widetilde{\rho}'_{\wp_{\upsilon(1)}}\right)\right)^{\sigma_1}, \\ 1-\left(\sin\left(\frac{\pi}{2}1-\daleth'_{\wp_{\upsilon(1)}}\right)\right)^{\sigma_1}, \\ 1-\left(\sin\left(\frac{\pi}{2}1-\tilde{n}'_{\wp_{\upsilon(1)}}\right)\right)^{\sigma_1} \end{pmatrix}\otimes\begin{pmatrix} \left(\sin\left(\frac{\pi}{2}\widetilde{\rho}'_{\wp_{\upsilon(1)}}\right)\right)^{\sigma_1}, \\ 1-\left(\sin\left(\frac{\pi}{2}1-\daleth'_{\wp_{\upsilon(1)}}\right)\right)^{\sigma_1}, \\ 1-\left(\sin\left(\frac{\pi}{2}1-\tilde{n}'_{\wp_{\upsilon(1)}}\right)\right)^{\sigma_1} \end{pmatrix}\\
= & \begin{pmatrix} \prod_{\hbar=1}^{2}\left(\sin\left(\frac{\pi}{2}\widetilde{\rho}'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar}, \\ 1-\prod_{\hbar=1}^{2}\left(\sin\left(\frac{\pi}{2}1-\daleth'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar}, \\ 1-\prod_{\hbar=1}^{2}\left(\sin\left(\frac{\pi}{2}1-\tilde{n}'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar} \end{pmatrix}
\end{aligned}$$

Step-2: Suppose that Equation 2 is holds for $n = \kappa$, we have

$$ST-SVNHWG\left(\mathcal{U}_1,\mathcal{U}_2,...\mathcal{U}_\kappa\right)=\begin{pmatrix} \prod_{\hbar=1}^{\kappa}\left(\sin\left(\frac{\pi}{2}\widetilde{\rho}'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar}, \\ 1-\prod_{\hbar=1}^{\kappa}\left(\sin\left(\frac{\pi}{2}1-\daleth'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar}, \\ 1-\prod_{\hbar=1}^{\kappa}\left(\sin\left(\frac{\pi}{2}1-\tilde{n}'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar} \end{pmatrix}$$

Step-3: Now we have to prove that Equation 2 is holds for $n = \kappa + 1$.

$$\begin{aligned}
& ST-SVNHWG\left(\mathcal{U}_1,\mathcal{U}_2,...\mathcal{U}_{\kappa+1}\right)\\
= & \prod_{\hbar=1}^{\kappa}\left(\sin\left(\mathcal{U}'_{\upsilon(\hbar)}\right)\right)^{\sigma_\hbar}\otimes\left(\sin\left(\mathcal{U}'_{\upsilon(\kappa+1)}\right)\right)^{\sigma_{\kappa+1}}\\
= & \begin{pmatrix} \prod_{\hbar=1}^{\kappa}\left(\sin\left(\frac{\pi}{2}\widetilde{\rho}'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar}, \\ 1-\prod_{\hbar=1}^{\kappa}\left(\sin\left(\frac{\pi}{2}1-\daleth'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar}, \\ 1-\prod_{\hbar=1}^{\kappa}\left(\sin\left(\frac{\pi}{2}1-\tilde{n}'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar} \end{pmatrix}\otimes\begin{pmatrix} \left(\sin\left(\frac{\pi}{2}\widetilde{\rho}'_{\wp_{\upsilon(\kappa+1)}}\right)\right)^{\sigma_{\kappa+1}}, \\ 1-\left(\sin\left(\frac{\pi}{2}1-\daleth'_{\wp_{\upsilon(\kappa+1)}}\right)\right)^{\sigma_{\kappa+1}}, \\ 1-\left(\sin\left(\frac{\pi}{2}1-\tilde{n}'_{\wp_{\upsilon(\kappa+1)}}\right)\right)^{\sigma_{\kappa+1}} \end{pmatrix}\\
= & \begin{pmatrix} \prod_{\hbar=1}^{\kappa+1}\left(\sin\left(\frac{\pi}{2}\widetilde{\rho}'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar}, \\ 1-\prod_{\hbar=1}^{\kappa+1}\left(\sin\left(\frac{\pi}{2}1-\daleth'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar}, \\ 1-\prod_{\hbar=1}^{\kappa+1}\left(\sin\left(\frac{\pi}{2}1-\tilde{n}'_{\wp_{\upsilon(\hbar)}}\right)\right)^{\sigma_\hbar} \end{pmatrix}
\end{aligned}$$

that is, when $n = z + 1$, Equation 2 also holds.

Hence, Equation 2 holds for any n. The proof is completed. □

Next, we give some properties that are apparently carried by the proposed ST-SVNHWG aggregation operator.

(1) Let $\mathcal{U}_\hbar = \{\widetilde{\rho}_{\wp_\hbar}, \daleth_{\wp_\hbar}, \tilde{n}_{\wp_\hbar}\} \in SVNN\left(\beth\right)$ $(\hbar = 1, 2, 3, ..., n)$ such that $\mathcal{U}_\hbar = \mathcal{U}$. Then

$$ST-SVNHWG\left(\mathcal{U}_1,\mathcal{U}_2,...,\mathcal{U}_n\right)=\sin\left(\mathcal{U}\right).$$

(2) Let $\mathcal{U}_\hbar = \{\widetilde{\rho}_{\wp_\hbar}, \urcorner_{\wp_\hbar}, \tilde{n}_{\wp_\hbar}\}$, $\mathcal{U}_\hbar^- = \{\min(\widetilde{\rho}_{\wp_\hbar}), \max(\urcorner_{\wp_\hbar}), \max(\tilde{n}_{\wp_\hbar})\}$ and $\mathcal{U}_\hbar^+ = \{\max(\widetilde{\rho}_{\wp_\hbar}), \min(\urcorner_{\wp_\hbar}), \min(\tilde{n}_{\wp_\hbar})\}$ $\in SVNN(\beth)$ $(\hbar = 1, 2, 3, ..., n)$. Then,

$$\sin\left(\mathcal{U}_\hbar^-\right) \leq ST - SVNHWG(\mathcal{U}_1, \mathcal{U}_2, ..., \mathcal{U}_n) \leq \sin\left(\mathcal{U}_\hbar^+\right).$$

(3) Let $\mathcal{U}_\hbar = \{\widetilde{\rho}_{\wp_\hbar}, \urcorner_{\wp_\hbar}, \tilde{n}_{\wp_\hbar}\}$, $\mathcal{U}_\hbar^* = \left\{\widetilde{\rho}_{\wp_\hbar}^*, \urcorner_{\wp_\hbar}^*, \tilde{n}_{\wp_\hbar}^*\right\} \in SVNN(\beth)$ $(\hbar = 1, 2, 3, ..., n)$. If $\widetilde{\rho}_{\wp_\hbar} \leq \widetilde{\rho}_{\wp_\hbar}^*$, $\urcorner_{\wp_\hbar} \leq \urcorner_{\wp_\hbar}^*$ and $\tilde{n}_{\wp_\hbar} \leq \tilde{n}_{\wp_\hbar}^*$, then

$$ST - SVNHWG(\mathcal{U}_1, \mathcal{U}_2, ..., \mathcal{U}_n) \leq ST - SVNHWG(\mathcal{U}_1^*, \mathcal{U}_2^*, ..., \mathcal{U}_n^*).$$

4 Decision Making Algorithm

This section describes a decision-making algorithm to address the uncertainty of decision-making problems (DMPs) in the SVN environment, supported by an illustrative example. The following notions are utilized to tackle the MCDM situations having SVN information. Suppose that $\{\beth_1, \beth_2, ..., \beth_k\}$ is a universal set and $\{r_1, r_2, r_3, ..., r_n\}$ is the universe of attributes. Assume $\eth = \{\eth_1, \eth_2, ..., \eth_n\}$ is a weigh-vector with $\eth_\hbar \in [0, 1]$ such that $\sum_{\hbar=1}^{n} \eth_\hbar = 1$. Consider $D^{(k)} = \left(\mathcal{U}_{ij}^{(k)}\right)_{k \times n}$ is a single valued nuetrosophic decision-matrix, which represents the membership values evaluated by the experts.

We construct an algorithmic method to handle MCDM problems by proposed aggregation operators. The algorithm includes steps below:

Step-1 Summarize the values of each alternative in term of decision matrix $D^{(k)} = \left(\mathcal{U}_{ij}^{(k)}\right)_{n \times m}$ with SVNNs.

Step-2 Construct the normalized decision matrix $P = (p_{ij})$ from $D = (\mathcal{U}_{ij})$, where p_{ij} is calculated as

$$p_{ij} = \begin{cases} (\widetilde{\rho}_{ij}, \urcorner_{ij}, \tilde{n}_{ij}) & \text{If criteria are benefit type} \\ \\ (\tilde{n}_{ij}, \urcorner_{ij}, \widetilde{\rho}_{ij}) & \text{If criteria are cost type} \end{cases} \tag{3}$$

Step-3 If the attribute weights are known as a prior then utilize them. Otherwise, we compute them by utilizing the concept of the entropy measure. For it, the information entropy of criteria t_j is computed as

$$En_j = \frac{1 + \frac{1}{n}\sum_{i=1}^{n}\left(\widetilde{\rho}_{\wp_{ij}} \log\left(\widetilde{\rho}_{\wp_{ij}}\right) + \urcorner_{\wp_{ij}} \log\left(\urcorner_{\wp_{ij}}\right) + \tilde{n}_{\wp_{ij}} \log\left(\tilde{n}_{\wp_{ij}}\right)\right)}{\sum_{j=1}^{m}\left(1 + \frac{1}{n}\sum_{i=1}^{n} \widetilde{\rho}_{\wp_{ij}} \log\left(\widetilde{\rho}_{\wp_{ij}}\right) + \urcorner_{\wp_{ij}} \log\left(\urcorner_{\wp_{ij}}\right) + \tilde{n}_{\wp_{ij}} \log\left(\tilde{n}_{\wp_{ij}}\right)\right)}$$

Step-4 Using proposed aggregation operators defined in Theorem 3.2 & 3.4 and attributes weight vector, the aggregated single valued neutrosophic numbers of the each alternative $\{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, ..., \mathcal{U}_n\}$ are obtained.

Step-6 Evaluate the scores values $\check{s}\check{c}(\mathcal{U}_\hbar)$ using the Definition 2.11 of collective spherical fuzzy numbers $\mathcal{U}_\hbar$ $(\hbar = 1, 2, ..., n)$ and rank using the Definition 2.12 according the maximum score values.

Step-6 Select the optimal alternative according the maximum score value or accuracy degree.

5 Application Decision Making Algorithm

In this segment, the numerical implementation of agricultural land selection is used to demonstrate the MCDM methodology developed.

5.1 Practical case study

Agriculture is an important component of the Economic System of Pakistan. This area directly supports the population of the country and accounts for 26% of gross domestic product (GDP). The major agricultural crops are sugarcane, wheat , rice, cotton, vegetables and fruit. A businessman wants to invest in the agriculture sector and to look for appropriate land. The options in his brain are $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4$ and $\mathcal{Y}_5$. He consults to an expert to get his suggestion about the alternatives based on the following desired parameters:

(1) Location (r_1) ,
(2) Climate (r_2) ,
(3) Fertility (r_3) ,
(4) Price (r_4) and
(5) Water Availability (r_5) .

The expert was asked to use SVN data in this assessment and weights of the attributes are $(0.15, 0.28, 0.20, 0.22, 0.15)^T$. The expert's findings are summarized in Table-1:

Table-2.: Expert Information

$$\begin{array}{cc} & \begin{array}{ccccc} r_1 & r_2 & r_3 & r_4 & r_5 \end{array} \\ D^1 = \begin{array}{c} \mathcal{Y}_1 \\ \mathcal{Y}_2 \\ \mathcal{Y}_3 \\ \mathcal{Y}_4 \\ \mathcal{Y}_5 \end{array} & \begin{pmatrix} (0.5,0.3,0.4) & (0.3,0.2,0.5) & (0.2,0.2,0.6) & (0.4,0.2,0.3) & (0.3,0.3,0.4) \\ (0.7,0.1,0.3) & (0.3,0.2,0.7) & (0.6,0.3,0.2) & (0.2,0.4,0.6) & (0.7,0.1,0.2) \\ (0.5,0.3,0.4) & (0.4,0.2,0.6) & (0.6,0.1,0.2) & (0.3,0.1,0.5) & (0.6,0.4,0.3) \\ (0.7,0.3,0.2) & (0.2,0.2,0.7) & (0.4,0.5,0.2) & (0.2,0.2,0.5) & (0.4,0.5,0.4) \\ (0.4,0.1,0.3) & (0.2,0.1,0.5) & (0.4,0.1,0.5) & (0.6,0.3,0.4) & (0.3,0.2,0.4) \end{pmatrix} \end{array}$$

Step-2 The r_1, r_3 & r_5 are benefits type and r_2 & r_4 are cost type criteria. According to defined formula 3, the normalized decision matrix is summarized in Table-2:

Table-2.: Normalized Information

$$\begin{array}{cc} & \begin{array}{ccccc} r_1 & r_2 & r_3 & r_4 & r_5 \end{array} \\ P = \begin{array}{c} \mathcal{Y}_1 \\ \mathcal{Y}_2 \\ \mathcal{Y}_3 \\ \mathcal{Y}_4 \\ \mathcal{Y}_5 \end{array} & \begin{pmatrix} (0.5,0.3,0.4) & (0.5,0.2,0.3) & (0.2,0.2,0.6) & (0.3,0.2,0.4) & (0.3,0.3,0.4) \\ (0.7,0.1,0.3) & (0.7,0.2,0.3) & (0.6,0.3,0.2) & (0.6,0.4,0.2) & (0.7,0.1,0.2) \\ (0.5,0.3,0.4) & (0.6,0.2,0.4) & (0.6,0.1,0.2) & (0.5,0.1,0.3) & (0.6,0.4,0.3) \\ (0.7,0.3,0.2) & (0.7,0.2,0.2) & (0.4,0.5,0.2) & (0.5,0.2,0.2) & (0.4,0.5,0.4) \\ (0.4,0.1,0.3) & (0.5,0.1,0.2) & (0.4,0.1,0.5) & (0.4,0.3,0.6) & (0.3,0.2,0.4) \end{pmatrix} \end{array}$$

Step-3 The Expert provides the following parameters weights

$$\kappa = \{\kappa_1 = 0.15, \kappa_2 = 0.28, \kappa_3 = 0.20, \kappa_4 = 0.22, \kappa_5 = 0.15\}$$

Step-4 In this step, we used proposed AOp namely, ST-SVHWA and ST-SVHWG to aggregate the single valued neutrosophic information as follows:

Firstly, we find out the weighted matrix shown in Table-3:

Table-3.: $\left(n\eth_{\hbar}\mathcal{U}_{\upsilon(\hbar)}|\hbar = 1, 2, ..., n\right)$

$$\begin{pmatrix} (0.60,0.09,0.14) & (0.8,0.004,0.01) & (0.30,0.02,0.17) & (0.48,0.01,0.06) & (0.36,0.09,0.14) \\ (0.81,0.01,0.09) & (0.9,0.004,0.01) & (0.80,0.04,0.02) & (0.83,0.06,0.01) & (0.81,0.01,0.05) \\ (0.60,0.09,0.14) & (0.9,0.004,0.02) & (0.8,0.004,0.02) & (0.7,0.002,0.03) & (0.71,0.14,0.09) \\ (0.81,0.09,0.05) & (0.9,0.004,0.004) & (0.58,0.12,0.02) & (0.74,0.01,0.01) & (0.48,0.20,0.14) \\ (0.48,0.01,0.09) & (0.8,0.005,0.04) & (0.5,0.004,0.12) & (0.62,0.03,0.14) & (0.36,0.05,0.14) \end{pmatrix}$$

Table-4.: Corresponding Score Values

$$\begin{pmatrix} (0.3557) & (0.8037) & (0.1144) & (0.4113) & (0.1187) \\ (0.6943) & (0.9380) & (0.7444) & (0.7634) & (0.7384) \\ (0.3557) & (0.8687) & (0.7840) & (0.7052) & (0.4650) \\ (0.6601) & (0.9467) & (0.4454) & (0.7140) & (0.1296) \\ (0.3696) & (0.8160) & (0.4603) & (0.4433) & (0.1628) \end{pmatrix}$$

Now, according to the score values, the $\left(\mathcal{U}'_{\upsilon(\hbar)} : \hbar = 1, 2, ..., n\right)$ values are shows as follows:

Table-5.: $\left(\mathcal{U}'_{\upsilon(\hbar)} : \hbar = 1, 2, ..., n\right)$

$\mathcal{U}'_{\upsilon(11)} = \mathcal{U}_{12} = (0.5, 0.2, 0.3)$	$\mathcal{U}'_{\upsilon(12)} = \mathcal{U}_{14} = (0.3, 0.2, 0.4)$	$\mathcal{U}'_{\upsilon(13)} = \mathcal{U}_{11} = (0.5, 0.3, 0.4)$
$\mathcal{U}'_{\upsilon(14)} = \mathcal{U}_{15} = (0.3, 0.3, 0.4)$	$\mathcal{U}'_{\upsilon(15)} = \mathcal{U}_{13} = (0.2, 0.2, 0.6)$	
$\mathcal{U}'_{\upsilon(21)} = \mathcal{U}_{22} = (0.7, 0.2, 0.3)$	$\mathcal{U}'_{\upsilon(22)} = \mathcal{U}_{24} = (0.6, 0.4, 0.2)$	$\mathcal{U}'_{\upsilon(23)} = \mathcal{U}_{23} = (0.6, 0.3, 0.2)$
$\mathcal{U}'_{\upsilon(24)} = \mathcal{U}_{25} = (0.7, 0.1, 0.2)$	$\mathcal{U}'_{\upsilon(25)} = \mathcal{U}_{21} = (0.7, 0.1, 0.3)$	
$\mathcal{U}'_{\upsilon(31)} = \mathcal{U}_{32} = (0.6, 0.2, 0.4)$	$\mathcal{U}'_{\upsilon(32)} = \mathcal{U}_{33} = (0.6, 0.1, 0.2)$	$\mathcal{U}'_{\upsilon(33)} = \mathcal{U}_{34} = (0.5, 0.1, 0.3)$
$\mathcal{U}'_{\upsilon(34)} = \mathcal{U}_{35} = (0.6, 0.4, 0.3)$	$\mathcal{U}'_{\upsilon(35)} = \mathcal{U}_{31} = (0.5, 0.3, 0.4)$	
$\mathcal{U}'_{\upsilon(41)} = \mathcal{U}_{42} = (0.7, 0.2, 0.2)$	$\mathcal{U}'_{\upsilon(42)} = \mathcal{U}_{44} = (0.5, 0.2, 0.2)$	$\mathcal{U}'_{\upsilon(43)} = \mathcal{U}_{41} = (0.7, 0.3, 0.2)$
$\mathcal{U}'_{\upsilon(44)} = \mathcal{U}_{43} = (0.4, 0.5, 0.2)$	$\mathcal{U}'_{\upsilon(45)} = \mathcal{U}_{45} = (0.4, 0.5, 0.2)$	
$\mathcal{U}'_{\upsilon(51)} = \mathcal{U}_{52} = (0.5, 0.1, 0.2)$	$\mathcal{U}'_{\upsilon(52)} = \mathcal{U}_{53} = (0.4, 0.1, 0.5)$	$\mathcal{U}'_{\upsilon(53)} = \mathcal{U}_{54} = (0.4, 0.3, 0.6)$
$\mathcal{U}'_{\upsilon(54)} = \mathcal{U}_{51} = (0.4, 0.1, 0.3)$	$\mathcal{U}'_{\upsilon(55)} = \mathcal{U}_{55} = (0.3, 0.2, 0.4)$	

Aggregated information of the alternatives are calculated in Table-6 as follows:

Table-6.: Aggregated Information

	$ST-SVNHWA$	$ST-SVNHWG$
$\mathcal{Y}_1$	$(0.54515, 0.02797, 0.08164)$	$(0.50042, 0.03039, 0.08899)$
$\mathcal{Y}_2$	$(0.85733, 0.02063, 0.02539)$	$(0.85066, 0.03635, 0.02741)$
$\mathcal{Y}_3$	$(0.77818, 0.01567, 0.04227)$	$(0.77177, 0.02991, 0.04829)$
$\mathcal{Y}_4$	$(0.76483, 0.04584, 0.02450)$	$(0.71602, 0.06397, 0.02901)$
$\mathcal{Y}_5$	$(0.59151, 0.00952, 0.07508)$	$(0.58134, 0.01529, 0.09549)$

Step-6 Score value of the aggregated alternatives are calculated in Table-7:

Table-7.: Score Values

	$\check{s}\check{c}(\mathcal{Y}_1)$	$\check{s}\check{c}(\mathcal{Y}_2)$	$\check{s}\check{c}(\mathcal{Y}_3)$	$\check{s}\check{c}(\mathcal{Y}_4)$	$\check{s}\check{c}(\mathcal{Y}_5)$
$ST-SVNHWA$	0.43553	0.81130	0.72023	0.69448	0.50690
$ST-SVNHWG$	0.38103	0.78688	0.69357	0.62302	0.47055

Step-7 Ranking of the alternatives are shown in Table-8:

Table-8.: Ranking

	Score Ranking	Best Alternative
$ST-SVNHWA$	$\check{s}\check{c}(\mathcal{Y}_2) > \check{s}\check{c}(\mathcal{Y}_3) > \check{s}\check{c}(\mathcal{Y}_4) > \check{s}\check{c}(\mathcal{Y}_5) > \check{s}\check{c}(\mathcal{Y}_1)$	$\mathcal{Y}_2$
$ST-SVNHWG$	$\check{s}\check{c}(\mathcal{Y}_2) > \check{s}\check{c}(\mathcal{Y}_3) > \check{s}\check{c}(\mathcal{Y}_4) > \check{s}\check{c}(\mathcal{Y}_5) > \check{s}\check{c}(\mathcal{Y}_1)$	$\mathcal{Y}_2$

From this above computational process, we can conclude that the alternative $\mathcal{Y}_2$ is the best among the others and hence it is highly recommendable to select for the required task/plan.

5.2 Comparison Study

In the section, we include some appropriate examples to demonstrate the feasibility and efficacy of the established decision-making approach and make a comparison with existing studies. Here, we presented the different existing aggregated information of the SVNNs namely, SVNWA,[33] SVNOWA,[33] NWA,[44] SVNFWA,[18] SVNHWA,[32] L-SVNWA,[19] L-SVNOWA,[19] ST-SVNWA[4] and ST-SVNWG[4] in Table-9, 10 and 11:

Table-9: Aggregated Information

	SVNWA[33]	SVNOWA[33]	NWA[44]	SVNFWA[18]
$\mathcal{Y}_1$	$(0.37, 0.22, 0.40)$	$(0.38, 0.24, 0.40)$	$(0.37, 0.23, 0.42)$	$(0.37, 0.22, 0.40)$
$\mathcal{Y}_2$	$(0.66, 0.20, 0.23)$	$(0.66, 0.18, 0.24)$	$(0.66, 0.24, 0.24)$	$(0.66, 0.20, 0.23)$
$\mathcal{Y}_3$	$(0.56, 0.17, 0.31)$	$(0.55, 0.18, 0.31)$	$(0.56, 0.21, 0.32)$	$(0.56, 0.17, 0.31)$
$\mathcal{Y}_4$	$(0.57, 0.29, 0.22)$	$(0.57, 0.31, 0.22)$	$(0.57, 0.33, 0.23)$	$(0.56, 0.29, 0.22)$
$\mathcal{Y}_5$	$(0.41, 0.14, 0.36)$	$(0.39, 0.13, 0.36)$	$(0.41, 0.16, 0.41)$	$(0.41, 0.14, 0.36)$

Table-10: Aggregated Information

	SVNHWA[32] $\gamma = 2$	SVNHWA[32] $\gamma = 3$	L-SVNWA[19]	L-SVNOWA[19]
$\mathcal{Y}_1$	$(0.37, 0.22, 0.40)$	$(0.36, 0.22, 0.40)$	$(0.31, 0.17, 0.35)$	$(0.32, 0.19, 0.36)$
$\mathcal{Y}_2$	$(0.66, 0.20, 0.23)$	$(0.66, 0.20, 0.23)$	$(0.64, 0.19, 0.23)$	$(0.65, 0.17, 0.23)$
$\mathcal{Y}_3$	$(0.56, 0.17, 0.31)$	$(0.56, 0.18, 0.31)$	$(0.49, 0.17, 0.33)$	$(0.48, 0.18, 0.33)$
$\mathcal{Y}_4$	$(0.56, 0.29, 0.22)$	$(0.56, 0.30, 0.22)$	$(0.55, 0.27, 0.19)$	$(0.55, 0.29, 0.19)$
$\mathcal{Y}_5$	$(0.41, 0.14, 0.36)$	$(0.41, 0.14, 0.37)$	$(0.28, 0.12, 0.37)$	$(0.24, 0.12, 0.38)$

Table-11: Aggregated Information

	ST-SVNWA[4]	ST-SVNWA[4]
$\mathcal{Y}_1$	$(0.56, 0.02, 0.07)$	$(0.50, 0.02, 0.08)$
$\mathcal{Y}_2$	$(0.86, 0.02, 0.02)$	$(0.85, 0.03, 0.03)$
$\mathcal{Y}_3$	$(0.77, 0.01, 0.04)$	$(0.76, 0.02, 0.05)$
$\mathcal{Y}_4$	$(0.78, 0.04, 0.02)$	$(0.73, 0.06, 0.02)$
$\mathcal{Y}_5$	$(0.60, 0.009, 0.06)$	$(0.59, 0.01, 0.08)$

Also, their corresponding ranking are calculated in Table-12:

Table-12: Ranking

Existing Operators	Ranking	Best Alternative
NWA[43]	$\mathcal{Y}_2 > \mathcal{Y}_3 > \mathcal{Y}_4 > \mathcal{Y}_5 > \mathcal{Y}_1$	$\mathcal{Y}_2$
SVNWA[33]	$\mathcal{Y}_2 > \mathcal{Y}_3 > \mathcal{Y}_4 > \mathcal{Y}_5 > \mathcal{Y}_1$	$\mathcal{Y}_2$
SVNOWA[33]	$\mathcal{Y}_2 > \mathcal{Y}_3 > \mathcal{Y}_4 > \mathcal{Y}_5 > \mathcal{Y}_1$	$\mathcal{Y}_2$
SVNWG[33]	$\mathcal{Y}_2 > \mathcal{Y}_3 > \mathcal{Y}_4 > \mathcal{Y}_5 > \mathcal{Y}_1$	$\mathcal{Y}_2$
SVNOWG[33]	$\mathcal{Y}_2 > \mathcal{Y}_3 > \mathcal{Y}_4 > \mathcal{Y}_5 > \mathcal{Y}_1$	$\mathcal{Y}_2$
SVNFWA[18]	$\mathcal{Y}_2 > \mathcal{Y}_3 > \mathcal{Y}_4 > \mathcal{Y}_5 > \mathcal{Y}_1$	$\mathcal{Y}_2$
SVNHWA[32] $\gamma = 2$	$\mathcal{Y}_2 > \mathcal{Y}_3 > \mathcal{Y}_4 > \mathcal{Y}_5 > \mathcal{Y}_1$	$\mathcal{Y}_2$
SVNHWA[32] $\gamma = 3$	$\mathcal{Y}_2 > \mathcal{Y}_3 > \mathcal{Y}_4 > \mathcal{Y}_5 > \mathcal{Y}_1$	$\mathcal{Y}_2$
NWG[43]	$\mathcal{Y}_2 > \mathcal{Y}_3 > \mathcal{Y}_4 > \mathcal{Y}_5 > \mathcal{Y}_1$	$\mathcal{Y}_2$
SVNFWG[18]	$\mathcal{Y}_2 > \mathcal{Y}_3 > \mathcal{Y}_4 > \mathcal{Y}_5 > \mathcal{Y}_1$	$\mathcal{Y}_2$
SVNHWG[32] $\gamma = 2$	$\mathcal{Y}_2 > \mathcal{Y}_3 > \mathcal{Y}_4 > \mathcal{Y}_5 > \mathcal{Y}_1$	$\mathcal{Y}_2$
SVNHWG[32] $\gamma = 3$	$\mathcal{Y}_2 > \mathcal{Y}_3 > \mathcal{Y}_4 > \mathcal{Y}_5 > \mathcal{Y}_1$	$\mathcal{Y}_2$
SNWEA[44]	$\mathcal{Y}_2 > \mathcal{Y}_3 > \mathcal{Y}_5 > \mathcal{Y}_4 > \mathcal{Y}_1$	$\mathcal{Y}_2$
L-SVNWA[19]	$\mathcal{Y}_2 > \mathcal{Y}_4 > \mathcal{Y}_3 > \mathcal{Y}_5 > \mathcal{Y}_1$	$\mathcal{Y}_2$
L-SVNOWA[19]	$\mathcal{Y}_2 > \mathcal{Y}_4 > \mathcal{Y}_3 > \mathcal{Y}_5 > \mathcal{Y}_1$	$\mathcal{Y}_2$
L-SVNWG[19]	$\mathcal{Y}_2 > \mathcal{Y}_4 > \mathcal{Y}_3 > \mathcal{Y}_1 > \mathcal{Y}_5$	$\mathcal{Y}_2$
L-SVNOWG[19]	$\mathcal{Y}_2 > \mathcal{Y}_3 > \mathcal{Y}_4 > \mathcal{Y}_5 > \mathcal{Y}_1$	$\mathcal{Y}_2$
L-SVNWA[4]	$\mathcal{Y}_2 > \mathcal{Y}_4 > \mathcal{Y}_3 > \mathcal{Y}_5 > \mathcal{Y}_1$	$\mathcal{Y}_2$
L-SVNWG[4]	$\mathcal{Y}_2 > \mathcal{Y}_3 > \mathcal{Y}_4 > \mathcal{Y}_5 > \mathcal{Y}_1$	$\mathcal{Y}_2$
L-SVNOWA[4]	$\mathcal{Y}_2 > \mathcal{Y}_4 > \mathcal{Y}_3 > \mathcal{Y}_5 > \mathcal{Y}_1$	$\mathcal{Y}_2$
L-SVNOWG[4]	$\mathcal{Y}_2 > \mathcal{Y}_3 > \mathcal{Y}_4 > \mathcal{Y}_5 > \mathcal{Y}_1$	$\mathcal{Y}_2$

Table-9.: Overall ranking of the alternatives

Proposed Operators	Ranking	Best Alternative
L-SVNHWA	$\mathcal{Y}_2 > \mathcal{Y}_3 > \mathcal{Y}_4 > \mathcal{Y}_5 > \mathcal{Y}_1$	$\mathcal{Y}_2$
L-SVNHWG	$\mathcal{Y}_2 > \mathcal{Y}_3 > \mathcal{Y}_4 > \mathcal{Y}_5 > \mathcal{Y}_1$	$\mathcal{Y}_2$

It is evident that the above conversation confirms the effectiveness and applicability of the proposed decision-making methodology based on new sine trigonometric aggregation operators under SVN environments.

6 Conclusion

Due to the existence of multiple attributes/criteria in many real-world problems, classical MCDM methods are not useful to tackle complicated decision-making situations. To overcome the difficulties of existing models, we have combined SVNSs with sine trigonometric AOs.

In this article, we have discussed MCDM issues based on single valued neutrosophic information. Motivated by sine trigonometric function based operational laws, we have proposed different AOs, namely, ST-SVNHWA and ST-SVNHWG aggregation operators. We have investigated different features of these operators. We have employed these AOs to enlarged the applicability scope of MCDM. We have given real-life application for the selection of best agricultural land for best investment. At the end, we have provided a comparison of developed AOs with existing aggregation techniques in the literatures and authenticate the proposed strategy by effectiveness comparison study. In the future, we are extending our work in interval valued single valued neutrosophic environment.

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References

[1] Ashraf, S., Abdullah, S. and Mahmood, T., "GRA method based on spherical linguistic fuzzy Choquet integral environment and its application in multi-attribute decision-making problems", Mathematical Sciences, 12(4), pp.263-275, 2018.

[2] Ashraf, S. and Abdullah, S., "Spherical aggregation operators and their application in multiattribute group decision-making", International Journal of Intelligent Systems, 34(3), pp.493-523, 2019.

[3] Ashraf, S., Abdullah, S., Mahmood, T., Ghani, F. and Mahmood, T., "Spherical fuzzy sets and their applications in multi-attribute decision making problems", Journal of Intelligent & Fuzzy Systems, 36(3), pp.2829-2844, 2019.

[4] Ashraf, S., Abdullah, S., Zeng, S., Jin, H. and Ghani, F., "Fuzzy Decision Support Modeling for Hydrogen Power Plant Selection Based on Single Valued Neutrosophic Sine Trigonometric Aggregation Operators", Symmetry, 12(2), p.298, 2020.

[5] Ashraf, S., Abdullah, S., Khan, S., "Fuzzy Decision support modeling for internet finance soft power evaluation Based on sine trigonometric Pythagorean fuzzy information", Journal of Ambient Intelligence and Humanized Computing, 2020. (Pre print)

[6] Ashraf, S., Mahmood, T., Abdullah, S. and Khan, Q., "Different approaches to multi-criteria group decision making problems for picture fuzzy environment", Bulletin of the Brazilian Mathematical Society, New Series, 50(2), pp.373-397, 2019.

[7] Ashraf, S., Abdullah, S., Mahmood, T. and Aslam, M., "Cleaner production evaluation in gold mines using novel distance measure method with cubic picture fuzzy numbers", International Journal of Fuzzy Systems, 21(8), pp.2448-2461, 2019.

[8] Ashraf, S., Abdullah, S., Aslam, M., Qiyas, M. and Kutbi, M.A., "Spherical fuzzy sets and its representation of spherical fuzzy t-norms and t-conorms", Journal of Intelligent & Fuzzy Systems, 36(6), pp.6089-6102, 2019.

[9] Ashraf, S., Abdullah, S. and Mahmood, T., "Spherical fuzzy Dombi aggregation operators and their application in group decision making problems", Journal of Ambient Intelligence and Humanized Computing, 11, pp.2731-2749, 2020.

[10] Ashraf, S., Abdullah, S. and Aslam, M., "Symmetric sum based aggregation operators for spherical fuzzy information: Application in multi-attribute group decision making problem", Journal of Intelligent & Fuzzy Systems, 38(4), pp.5241-5255, 2020.

[11] Ashraf, S., Abdullah, S. and Abdullah, L., "Child development influence environmental factors determined using spherical fuzzy distance measures", Mathematics, 7(8), p.661, 2019.

[12] Ashraf, S., Abdullah, S. and Smarandache, F., "Logarithmic hybrid aggregation operators based on single valued neutrosophic sets and their applications in decision support systems", Symmetry, 11(3), p.364, 2019.

[13] Atanassov, K.T., "Intuitionistic fuzzy sets", Fuzzy Set and systems, 20, pp.87-96, 1986.

[14] Barukab, O., Abdullah, S., Ashraf, S., Arif, M. and Khan, S.A., ”A New Approach to Fuzzy TOPSIS Method Based on Entropy Measure under Spherical Fuzzy Information”, Entropy, 21(12), p.1231, 2019.

[15] Batool, B., Ahmad, M., Abdullah, S., Ashraf, S. and Chinram, R., ”Entropy Based Pythagorean Probabilistic Hesitant Fuzzy Decision Making Technique and Its Application for Fog-Haze Factor Assessment Problem”, Entropy, 22(3), p.318, 2020.

[16] Berti, L.A.C., ”Application of the neutrosophic system to tax havens with a criminal approach”, International Journal of Neutrosophic Science, 5(2), pp.91-106, 2020.

[17] Cuong, B.C. and Kreinovich, V., ”Picture Fuzzy Sets-a new concept for computational intelligence problems”, In 2013 Third World Congress on Information and Communication Technologies (WICT 2013) (pp. 1-6), IEEE, 2013.

[18] Garg, H., ”Novel single-valued neutrosophic aggregated operators under Frank norm operation and its application to decision-making process”, International Journal for Uncertainty Quantification, 6(4), pp.361-375, 2016.

[19] Garg, H., ”New logarithmic operational laws and their applications to multiattribute decision-making for single-valued neutrosophic numbers”, Cogn. Syst. Res., 52, pp.931-946, 2018.

[20] Jin, H., Ashraf, S., Abdullah, S., Qiyas, M., Bano, M. and Zeng, S., ”Linguistic spherical fuzzy aggregation operators and their applications in multi-attribute decision making problems”, Mathematics, 7(5), p.413, 2019.

[21] Jin, Y., Ashraf, S. and Abdullah, S., ”Spherical fuzzy logarithmic aggregation operators based on entropy and their application in decision support systems”, Entropy, 21(7), p.628, 2019.

[22] Ji, P., Wang, J.Q. and Zhang, H.Y., ”Frank prioritized Bonferroni mean operator with single-valued neutrosophic sets and its application in selecting third-party logistics providers”, Neural Computing and Applications, 30(3), pp.799-823, 2018.

[23] Khan, M.A., Ashraf, S., Abdullah, S. and Ghani, F., ”Applications of probabilistic hesitant fuzzy rough set in decision support system”, Soft Computing, 2020. https://doi.org/10.1007/s00500-020-04971-z

[24] Khan, A.A., Ashraf, S., Abdullah, S., Qiyas, M., Luo, J. and Khan, S.U., ”Pythagorean fuzzy Dombi aggregation operators and their application in decision support system”, Symmetry, 11(3), p.383, 2019.

[25] Khan, M.J., Kumam, P., Liu, P., Kumam, W. and Ashraf, S., ”A novel approach to generalized intuitionistic fuzzy soft sets and its application in decision support system”, Mathematics, 7(8), p.742, 2019.

[26] Khan, S., Abdullah, S. and Ashraf, S., ”Picture fuzzy aggregation information based on Einstein operations and their application in decision making”, Mathematical Sciences, 13(3), pp.213-229, 2019.

[27] Khan, M., Beg, I. and Gulistan, M., ”Exponential Laws and Aggregation Operators on Neutrosophic Cubic Sets”, International Journal of Neutrosophic Science, 4(1), pp.47-71, 2020.

[28] Liu, P., Khan, Q., Mahmood, T., Smarandache, F. and Li, Y., ”Multiple attribute group decision making based on 2-tuple linguistic neutrosophic Dombi power Heronian mean operators”/, IEEE Access, 7, pp.100205-100230, 2019.

[29] Liu, P., Khan, Q. and Mahmood, T., ”Some single-valued neutrosophic power muirhead mean operators and their application to group decision making”, Journal of Intelligent & Fuzzy Systems, 37(2), pp.2515-2537, 2019.

[30] Liu, P., Mahmood, T. and Khan, Q., ”Group decision making based on power Heronian aggregation operators under linguistic neutrosophic environment”, International Journal of Fuzzy Systems, 20(3), pp.970-985, 2018.

[31] Liu, P., Khan, Q. and Mahmood, T., ”Group decision making based on power Heronian aggregation operators under neutrosophic cubic environment”, Soft Computing, 24(3), pp.1971-1997, 2020.

[32] Liu, P., Chu, Y., Li, Y. and Chen, Y., ”Some generalized neutrosophic number Hamacher aggregation operators and their application to group decision making”, International Journal of fuzzy systems, 16(2), pp.242-255, 2014.

[33] Peng, J.J., Wang, J.Q., Wang, J., Zhang, H.Y. and Chen, X.H., "Simplified neutrosophic sets and their applications in multi-criteria group decision-making problems", International journal of systems science, 47(10), pp.2342-2358, 2016.

[34] Qiyas, M., Abdullah, S., Ashraf, S. and Abdullah, L., "Linguistic Picture Fuzzy Dombi Aggregation Operators and Their Application in Multiple Attribute Group Decision Making Problem", Mathematics, 7(8), p.764, 2019.

[35] Qiyas, M., Abdullah, S., Ashraf, S. and Aslam, M., "Utilizing linguistic picture fuzzy aggregation operators for multiple-attribute decision-making problems", International Journal of Fuzzy Systems, 22(1), pp.310-320, 2020.

[36] Rana, S., Saeed, M., Qayyum, M. and Smarandache, F., "Plithogenic Subjective Hyper-Super-Soft Matrices with New Definitions & Local, Global, Universal Subjective Ranking Model", International Journal of Neutrosophic Science, 6(2), pp.56-79, 2020.

[37] Rafiq, M., Ashraf, S., Abdullah, S., Mahmood, T. and Muhammad, S., "The cosine similarity measures of spherical fuzzy sets and their applications in decision making", Journal of Intelligent & Fuzzy Systems, 36(6), pp.6059-6073, 2019.

[38] Smarandache, F., "Neutrosophy Neutrosophic Probability, Set and Logic", American Research Press, (1998) Rehoboth, USA.

[39] Sajjad Ali Khan, M., Abdullah, S., Yousaf Ali, M., Hussain, I. and Farooq, M., "Extension of TOPSIS method base on Choquet integral under interval-valued Pythagorean fuzzy environment", Journal of Intelligent & Fuzzy Systems, 34(1), pp.267-282, 2018.

[40] Wang, H., Smarandache, F., Zhang, Y.Q. and Sunderraman, R., "Single valued neutrosophic sets", Multispace and Multistructure, 4, pp.410-413, 2010.

[41] Xu, Z., "Intuitionistic fuzzy aggregation operators", IEEE Transactions on fuzzy systems, 15(6), pp.1179-1187, 2007.

[42] Ye, J., "Multicriteria decision-making method using the correlation coefficient under single-valued neutrosophic environment", International Journal of General Systems, 42(4), pp.386-394, 2013.

[43] Ye, J., "Exponential operations and aggregation operators of interval neutrosophic sets and their decision-making methods", SpringerPlus 2016, 5, 1488.

[44] Ye, J., "Subtraction and division operations of simplified neutrosophic sets", Information, 8(51), 2017.

[45] Yager, R.R., "On ordered weighted averaging aggregation operators in multi criteria decision making", IEEE Transactions on systems, Man, and Cybernetics, 18(1), pp.183-190, 1988.

[46] Yager, R.R., "Pythagorean membership grades in multicriteria decision making", IEEE Transactions on Fuzzy Systems, 22(4), pp.958-965, 2013.

[47] Zadeh, L.A., "Fuzzy sets", Information and control, 8(3), pp.338-353, 1965.

[48] Zeng, S., Hussain, A., Mahmood, T., Irfan Ali, M., Ashraf, S. and Munir, M., "Covering-based spherical fuzzy rough set model hybrid with TOPSIS for multi-attribute decision-making", Symmetry, 11(4), p.547, 2019.

[49] Zeng, S., Chen, S.M. and Kuo, L.W., "Multiattribute decision making based on novel score function of intuitionistic fuzzy values and modified VIKOR method", Information Sciences, 488, pp.76-92, 2019.

[50] Zhang, X.; Bo, C.; Smarandache, F.; Dai, J., "New inclusion relation of neutrosophic sets with applications and related lattice structure", Int. J. Mach. Learn. Cybern., 9, pp.1753-1763, 2018.

A Study of AH-Substructures in n-Refined Neutrosophic Vector Spaces

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Abstract

The aim of this paper is to define and study for the first time AH-substructures in n-refined neutrosophic vector spaces such as weak/strong AH-subspaces, and weak/strong AH-linear transformations between two n-refined neutrosophic vector spaces. Also, this paper introduces some elementary properties of these concepts.

Keywords: n-Refined Neutrosophic vector space , AH-subspace, AHS-subspace, AH-linear transformation.

1. Introduction

Neutrosophy as a new kind of logic, concerns with nature, origin, and scope of neutralities became a rich material in algebra. Many algebraic structures have been defined and handled such as neutrosophic rings, neutrosophic modules, and neutrosophic vector spaces. See [8,9,10,11,12,14]. More generalizations came to light such as refined neutrosophic rings, n-refined neutrosophic rings, and n-refined neutrosophic vector spaces. See [3,5,6,7,15,16].

AH-substructures were defined for the first time in neutrosophic rings [1]. Then they were defined in n-refined neutrosophic rings, and neutrosophic vector spaces in [2,4]. AH-structures consist of similar objects, each object has the same structure. For example in a neutrosophic vector space $V(I) = V + VI$, AH-subspace is a non empty subset with form $T = M + NI$, where M, N are two classical subspaces in V. Also, AHS-linear transformations were defined by similar aspect [4]. AH-substructures illustrate a bridge between neutrosophic structures and classical algebraic structures and help us to use classical methods in neutrosophical studies.

This article defines some AH-substructures in n-refined neutrosophic vector spaces. Concepts such as AH-subspaces, and AH-linear transformations. Also, it presents some interesting properties and theorems concerning these concepts.

2. Preliminaries

Definition 2.1: [15]

Let (R, +, ×) be a ring and $I_k; 1 \leq k \leq n$ be n indeterminacies. We define R_n(I)={$a_0 + a_1 I + \cdots + a_n I_n$; $a_i \in R$} to be n-refined neutrosophic ring.

Definition 2.2: [15]

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(a) Let R_n(I) be an n-refined neutrosophic ring and P = $\sum_{i=0}^{n} P_i I_i$ = { $a_0 + a_1 I + \cdots + a_n I_n$: $a_i \in P_i$}, where P_i is a subset of R, we define P to be an AH-subring if P_i is a subring of R for all i . AHS-subring is defined by the condition $P_i = P_j$ for all i , j.

(b) P is an AH-ideal if P_i are two sided ideals of R for all i, the AHS-ideal is defined by the condition $P_i = P_j$ for all i , j.

Definition 2.3 :[8]

Let (V, +, .) be a vector space over the field K then (V(I),+,.) is called a weak neutrosophic vector space over the field K, and it is called a strong neutrosophic vector space if it is a vector space over the neutrosophic field K(I).

Definition 2.4 : [8]

Let V(I) be a strong neutrosophic vector space over the neutrosophic field K(I) and W(I) be a non empty set of V(I), then W(I) is called a strong neutrosophic subspace if W(I) is itself a strong neutrosophic vector space.

Definition 2.5: [14]

Let (K, +, .) be a field, we say that $K_n(I) = K + KI_1 + \cdots + KI_n = \{a_0 + a_1 I_1 + \cdots + a_n I_n; a_i \in K\}$ is an n-refined neutrosophic field.

It is clear that $K_n(I)$ is an n-refined neutrosophic ring but not a field in classical meaning**.**

Definition 2.6 : [14]

Let (V, +, .) be a vector space over the field K. Then we say that $V_n(I) = V + VI_1 + \cdots + VI_n = \{x_0 + x_1 I_1 + \cdots + x_n I_n;\ x_i \in V\}$ is a weak n-refined neutrosophic vector space over the field K. Elements of $V_n(I)$ are called n-refined neutrosophic vectors, elements of K are called scalars.

If we take scalars from the n-refined neutrosophic field $K_n(I)$, we say that $V_n(I)$ is a strong n-refined neutrosophic vector space over the n-refined neutrosophic field $K_n(I)$. Elements of $K_n(I)$ are called n-refined neutrosophic scalars.

Definition 2.7: [14]

Let $V_n(I)$ be a weak n-refined neutrosophic vector space over the field K, a nonempty subset $W_n(I)$ is called a weak n-refined neutrosophic subspace of $V_n(I)$ if $W_n(I)$ is a subspace of $V_n(I)$ itself.

Definition 2.8: [14]

Let $V_n(I)$ be a strong n-refined neutrosophic vector space over the n-refined neutrosophic field $K_n(I)$, a nonempty subset $W_n(I)$ is called a strong n-refined neutrosophic subspace of $V_n(I)$ if $W_n(I)$ is a submodule of $V_n(I)$ itself.

Definition 2.9: [4]

Let V(I) = V+VI be a strong/weak neutrosophic vector space, the set

$S = P + QI = \{x + yI; x \in P, y \in Q\}$, where P and Q are subspaces of V is called an AH-subspace of V(I).

If $P = Q$ then S is called an AHS-subspace of V(I).

Definition 2.10: [4]

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(a) Let V and W be two vector spaces, $L_V: V \to W$ be a linear transformation. The AHS-linear transformation can be defined as follows:

$L: V(I) \to W(I); L(a + bI) = L_V(a) + L_V(b)I.$

(b) If $S = P + QI$ is an AH-subspace of V(I), $L(S) = L_V(P) + L_V(Q)I.$

3. Main discussion

Definition 3.1:

Let (V,+,.) be a vector space over a field K, $V_n(I)$ be the corresponding weak n-refined neutrosophic vector space over K. Consider the set $\{M_i; 0 \leq i \leq n\}$, where M_i is a subspace of V. We say:

$M_n(I) = M_0 + M_1I_1 + \cdots + M_nI_n = \{m_0 + m_1I_1 + \cdots + m_nI_n;\ m_i \in M_i\}$ is a weak n-refined AH-subspace of the weak n-refined vector space $V_n(I)$.

We say that $M_n(I)$ is a weak n-refined AH-subspace if $M_j = M_i\ for\ all\ i, j.$

Definition 3.2:

Let (V, +, .) be a vector space over a field K, $V_n(I)$ be the corresponding strong n-refined neutrosophic vector space over the n-refined neutrosophic field $K_n(I)$. Consider the set $\{M_i; 0 \leq i \leq n\}$, where M_i is a subspace of V. We say:

$M_n(I) = M_0 + M_1I_1 + \cdots + M_nI_n = \{m_0 + m_1I_1 + \cdots + m_nI_n;\ m_i \in M_i\}$ is a strong n-refined AH-subspace of the strong n-refined vector space $V_n(I)$.

We say that $M_n(I)$ is a strong n-refined AH-subspace if $M_j = M_i\ for\ all\ i, j.$

Theorem 3.3:

Let (V, +, .) be a vector space over a field K, $V_n(I)$ be the corresponding weak n-refined neutrosophic vector space over K, $M_n(I) = M_0 + M_1I_1 + \cdots + M_nI_n$ be a weak n-refined AH-subspace. Then

(a) $M_n(I)$ is a vector subspace of $V_n(I)$.

(b) If X_i is a bases of M_i, $X = \bigcup_{i=0}^{n} X_iI_i$ is a bases of $M_n(I)$.

(c) $\dim\big(M_n(I)\big) = \sum_{i=0}^{n} \dim(M_i)$.

Proof:

(a) Let $x = \sum_{i=0}^{n} a_iI_i$, $y = \sum_{i=0}^{n} b_iI_i;\ b_i, a_i \in M_i$ be two arbitrary elements in $M_n(I)$, r be an arbitrary element in K, we have:

$x + y = \sum_{i=0}^{n}(a_i + b_i)I_i \in M_n(I)$, since $a_i + b_i \in M_i$ because M_i is a subspace of V.

$r.x = \sum_{i=0}^{n} ra_iI_i \in M_n(I)$, since $ra_i \in M_i$ because M_i is a subspace of V. Thus $M_n(I)$ is a vector subspace of $V_n(I)$.

(b) Suppose that $X_0 = \left\{x_1^{(0)}, \dots, x_{s_0}^{(0)}\right\}, X_1 = \left\{x_1^{(1)}, \dots, x_{s_1}^{(1)}\right\}, \dots \dots, X_n = \{x_1^{(n)}, \dots, x_{s_n}^{(n)}\}$, let $x = \sum_{i=0}^{n} a_iI_i$ be an arbitrary element of $M_n(I)$, since X_i is a basis of M_i for all i. We can write:

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$a_i = \sum_{j=0}^{s_i} t_j^{(i)}\, x_j^{(i)};\ t_j \in K$, so $x = \sum_{j=0}^{s_0} t_j^{(0)}\, x_j^{(0)} + \sum_{j=0}^{s_1} t_j^{(1)}\, x_j^{(1)} I_1 + \cdots + \sum_{j=0}^{s_n} t_j^{(n)}\, x_j^{(n)} I_n$. This implies that X is a generating set of $M_n(I)$.

Now we prove that X is linearly independent. For our purpose we assume

$\sum_{j=0}^{s_0} t_j^{(0)}\, x_j^{(0)} + \sum_{j=0}^{s_1} t_j^{(1)}\, x_j^{(1)} I_1 + \cdots + \sum_{j=0}^{s_n} t_j^{(n)}\, x_j^{(n)} I_n = 0$, by definition of n-refined vector space we find

$\sum_{j=0}^{s_i} t_j^{(i)}\, x_j^{(i)}$ for all i, hence $t_j^{(i)} = 0$ for all i,j, since each X_i is linearly independent itself. Thus our proof is complete.

(c) It holds directly from (b).

Example 3.4:

Let $V = R^2$ be a vector space over the field R, $V_2(I) = R_2^2(I) = \{(a_0, b_0) + (a_1, b_1)I_1 + (a_2, b_2)I_2;\ a_i, b_i \in R\}$ be the corresponding weak 2-refined neutrosophic vector space over the field R, we have:

(a) $M =< (1,0) >= \{(m, 0); m \in R\}$, $N =< (0,1) >= \{(0, n); n \in R\}$ are two subspaces of $V = R^2$.

(b) $T = M + NI_1 + MI_2 = \{(m, 0) + (0, n)I_1 + (s, 0)I_2; m, n, s \in R\}$ is a weak AH-subspace of $V_2(I)$.

(c) The set $X = \{(1,0), (0,1)I_1, (1,0)I_2\}$ is a bases of T, $\dim(T) = \dim(M) + \dim(N) + \dim(M) = 3$.

(d) $D = N + NI_1 + NI_2 = \{(0, a) + (0, b)I_1 + (0, c)I_2; a, b, c \in R\}$ is a weak AHS-subspace.

Theorem 3.5:

Let V be a vector space with $\dim(V) = n + 1$. Then V is isomorphic to a weak AHS-subspace of the corresponding weak n-refined neutrosophic vector space.

Proof:

Let M be any one dimensional subspace of V, $T = M + MI_1 + \cdots + MI_n$ is a weak AHS-subspace of the weak n-refined neutrosophic vector space $V_n(I)$. As a result of Theorem 3.3 , we find $\dim(T) = n + 1 = \dim(V)$, thus V is isomorphic to T.

Example 3.6:

Let $V = R^3$ be a vector space over the field R, $V_3(I) = \{a + bI_1 + cI_2 + dI_3; a, b, c, d \in V\}$ is the corresponding weak 3-refined neutrosophic vector space, $M =< (1,0,0) >$ is a subspace of V.

$T = M + MI_1 + MI_2 = \{(a, 0,0) + (b, 0,0)I_1 + (c, 0,0)I_2; a, b, c \in R\}$ is a weak AHS-subspace of $V_3(I)$ with $\dim(T) = 3$, this implies $T \cong V$.

Theorem 3.7:

Let (V, +, .) be a vector space over a field K, $V_n(I)$ be the corresponding strong n-refined neutrosophic vector space over the n-refined neutrosophic field $K_n(I)$, $M_n(I) = M + MI_1 + \cdots + MI_n$ be a strong n-refined AHS-subspace. Then:

(a) $M_n(I)$ is a submodule of $V_n(I)$.

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(b) If Y is a bases of M, $X = \bigcup_{i=0}^{n} YI_i$ is a bases of $M_n(I)$.

(c) $\dim\big(M_n(I)\big) = \sum_{i=0}^{n} \dim(M) = n.\dim(M)$.

Proof:

(a) Let $x = \sum_{i=0}^{n} a_i I_i$, $y = \sum_{i=0}^{n} b_i I_i$; $b_i, a_i \in M_i$ be two arbitrary elements in $M_n(I)$, $r = \sum_{i=0}^{n} r_i I_i$ be an arbitrary element in $K_n(I)$, we have:

$x + y = \sum_{i=0}^{n} (a_i + b_i) I_i \in M_n(I)$, since $a_i + b_i \in M_i$ because M_i is a subspace of V.

$r.x = \sum_{i,j=0}^{n} r_i a_j I_i I_j \in M_n(I)$, since $r_i a_j \in M$ because M is a subspace of V. Thus $M_n(I)$ is a vector subspace of $V_n(I)$.

(b),(c) They are similar to that of Theorem 3.5 .

Remark 3.8:

If $V_n(I)$ is a strong n-refined neutrosophic vector space over the n-refined neutrosophic field $K_n(I)$, and

$M_n(I) = M_0 + M_1 I_1 + \cdots + M_n I_n$ is a strong n-refined AH-subspace, then it is not supposed to be a submodule.

We clarify it by the following example.

Example 3.9:

Let $V = R^2$ be a vector space over R, $V_2(I) = R_2^2(I) = \{(a,b) + (c,d)I_1 + (e,f)I_2; a,b,c,d,e,f \in R\}$ be the corresponding strong 2-refined neutrosophic vector space over the neutrosophic field $R_2(I)$.

$M =< 0,1 >, N =< (1,0) >$ are two subspaces of V, $T = M + NI_1 + NI_2$ is a strong AH-subspace of $V_2(I)$.

$x = (0,1) + (2,0)I_1 + (1,0)I_2 \in T, r = 1 + 1.I_1 + 1.I_2 \in R_2(I)$,

$r.x = 1.(0,1) + 1.(0,1)I_1 + 1.(0,1)I_2 + 1.(2,0)I_1 I_1 + 1.(2,0)I_1 + 1.(1,0)I_1 I_2 + 1.(0,1)I_2 + 1.(2,0)I_1 I_2 + 1.(2,0)I_2 I_2 = (0,1) + [(0,1) + (2,0) + (1,0) + (2,0)]I_1 + [(0,1) + (0,1) + (2,0)]I_2 =$

$(0,1) + (5,1)I_1 + (2,2)I_2$, $r.x$ does not belong to T, thus T is not a submodule.

Definition 3.10:

Let $V_n(I)$ be a weak/strong n-refined neutrosophic vector space, $M_n(I) = M_0 + M_1 I_1 + \cdots + M_n I_n$,

$W_n(I) = W_0 + W_1 I_1 + \cdots + W_n I_n$ be two weak/strong AH-subspaces of $V_n(I)$, we define:

(a) $M_n(I) \cap W_n(I) = (M_0 \cap W_0) + (M_1 \cap W_1)I_1 + \cdots + (M_n \cap W_n)I_n$.

(b) $M_n(I) + W_n(I) = (M_0 + W_0) + (M_1 + W_1)I_1 + \cdots + (M_n + W_n)I_n$.

Theorem 3.11:

Let $V_n(I)$ be a weak n-refined neutrosophic vector space, $M_n(I) = M_0 + M_1 I_1 + \cdots + M_n I_n$,

$W_n(I) = W_0 + W_1 I_1 + \cdots + W_n I_n$ be two weak AH-subspaces of $V_n(I)$. Then:

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$M_n(I) \cap W_n(I), M_n(I) + W_n(I)$ are two weak AH-subspaces of $V_n(I)$.

Proof:

Since $M_i \cap W_i, M_i + W_i$ are subspaces of V for all i, we obtain the proof.

Theorem 3.12:

Let $V_n(I)$ be a strong n-refined neutrosophic vector space, $M_n(I) = M_0 + M_1 I_1 + \cdots + M_n I_n$,

$W_n(I) = W_0 + W_1 I_1 + \cdots + W_n I_n$ be two strong AH-subspaces of $V_n(I)$. Then:

(a) $M_n(I) \cap W_n(I)$ is a strong AH-subspaces of $V_n(I)$.

(b) $M_n(I) + W_n(I)$ is not supposed to be a strong AH-subspace of $V_n(I)$.

Proof:

The proof is similar to that of Theorem 3.11 .

Definition 3.13:

Let V,W be two vector spaces over the field K, $f_i: V \to W; 0 \le i \le n+1$ be $n+1$ linear transformations, $V_n(I), W_n(I)$ be the corresponding weak n-refined neutrosophic vector spaces over the field K respectively. We say:

(a) $f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$ is a weak AH-linear transformation.

(b) If $f_i = f_j \; for \; all \; i,j$, we call f a weak AHS-linear transformation.

Example 3.14:

(a) Let $V = R^3, W = R^2$ be two vector spaces over the field R, $V_2(I) = R_2^3(I) = \{ (x_0, y_0 z_0) + (x_1, y_1, z_1)I_1 + (x_2, y_2, z_2)I_2;\ x_i, y_i, z_i \in R\}$,

$W_2(I) = \{(x_0, y_0) + (x_1, y_1)I_1 + (x_2, y_2)I_2;\ x_i, y_i \in R\}$ be the corresponding weak 2-refined neutrosophic vector spaces. We have $g: V \to W; g(a,b,c) = (b,c), h: V \to W; h(a,b,c) = (2a, 0)$

$s: V \to W; s(a,b,c) = (2b, 3c)$ are three linear transformations.

(b) $f: V_2(I) \to W_2(I); f(m + nI_1 + qI_2) = g(m) + h(n)I_1 + s(q)I_2; m,n,q \in V$ is a weak AH-linear transformation.

(c) We clarify f as follows:

$x = (1,2,2) + (1,0,1)I_1 + (3,-1,0)I_2 \in V_2(I)$,

$f(x) = g(1,2,2) + [h(1,0,1)]I_1 + [s(3,-1,0)]I_2 = (2,2) + (2,0)I_1 + (-2,0)I_2$.

(d) $k: V_2(I) \to W_2(I); k(m + nI_1 + qI_2) = g(m) + g(n)I_1 + g(q)I_2; m,n,q \in V$ is a weak AHS-linear transformation.

Definition 3.15:

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Let V,W be two vector spaces over the field K, $f_i: V \to W; 0 \leq i \leq n+1$ be $n+1$ linear transformations, $V_n(I), W_n(I)$ be the corresponding strong n-refined neutrosophic vector spaces over the n-refined neutrosophic field $K_n(I)$ respectively. We say:

(a) $f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$ is a strong AH-linear transformation.

(b) If $f_i = f_j \; for \; all \; i, j$, we call f a strong AHS-linear transformation.

Example 3.16:

(a) Let $V = R^3, W = R^2$ be two vector spaces over the field R, $V_2(I) = R_2^3(I) = \{(x_0, y_0 z_0) + (x_1, y_1, z_1)I_1 + (x_2, y_2, z_2)I_2; \; x_i, y_i, z_i \in R\}$,

$W_2(I) = \{(x_0, y_0) + (x_1, y_1)I_1 + (x_2, y_2)I_2; \; x_i, y_i \in R\}$ be the corresponding strong 2-refined neutrosophic vector spaces over the 2-refined neutrosophic field $R_2(I)$. We have $g: V \to W; g(a,b,c) = (b,c), h: V \to W; h(a,b,c) = (2a, 0)$,

$s: V \to W; s(a,b,c) = (2b, 3c)$ are three linear transformations.

(b) $f: V_2(I) \to W_2(I); f(m + nI_1 + qI_2) = g(m) + h(n)I_1 + s(q)I_2; m, n, q \in V$ is a strong AH-linear transformation.

(c) We clarify f as follows:

$x = (1,2,2) + (1,0,1)I_1 + (3,-1,0)I_2 \in V_2(I)$,

$f(x) = g(1,2,2) + [h(1,0,1)]I_1 + [s(3,-1,0)]I_2 = (2,2) + (2,0)I_1 + (-2,0)I_2$.

(d) $k: V_2(I) \to W_2(I); k(m + nI_1 + qI_2) = g(m) + g(n)I_1 + g(q)I_2; m, n, q \in V$ is a strong AHS-linear transformation.

Definition 3.17:

Let $V_n(I), W_n(I)$ be two weak/strong n-refined neutrosophic vector spaces,

$f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$ be a weak/strong AH-linear transformation. We define:

(a) $AH - Ker(f) = Ker(f_0) + Ker(f_1)I_1 + \cdots + Ker(f_n)I_n$.

(b) $AH - Im(f) = Im(f_0) + Im(f_1)I_1 + \cdots + Im(f_n)I_n$.

Theorem 3.18:

Let $V_n(I), W_n(I)$ be two weak n-refined neutrosophic vector spaces,

$f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$ be a weak AH-linear transformation. Then:

(a) $AH - Ker(f)$ is a weak AH-subspace of $V_n(I)$.

(b) $AH - Im(f)$ is a weak AH-subspace of $W_n(I)$.

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(c) If $M_n(I) = M_0 + M_1 I_1 + \cdots + M_n I_n$ is a weak AH-subspace of $V_n(I)$, $f(M_n(I))$ is a weak AH-subspace of $W_n(I)$.

Proof:

(a) Since $Ker(f_i)$ is a subspace of V, we find that

$AH - Ker(f) = Ker(f_0) + Ker(f_1)I_1 + \cdots + Ker(f_n)I_n$ is a weak AH-subspace of $V_n(I)$.

(b) Since $Im(f_i)$ is a subspace of W, we find that $AH - Im(f) = Im(f_0) + Im(f_1)I_1 + \cdots + Im(f_n)I_n$ is a weak AH-subspace of $W_n(I)$.

(c) It is known that $f_i(M_i)$ is a subspace of W, hence

$f\big(M_n(I)\big) = f_0(M_0) + f_1(M_1)I_1 + \cdots + f_n(M_n)I_n$ is a weak AH-subspace of $W_n(I)$.

Theorem 3.19:

Let $V_n(I), W_n(I)$ be two strong n-refined neutrosophic vector spaces over the n-refined neutrosophic field $K_n(I)$,

$f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$ be a strong AH-linear transformation. Then:

(a) $AH - Ker(f)$ is a strong AH-subspace of $V_n(I)$.

(b) $AH - Im(f)$ is a strong AH-subspace of $W_n(I)$.

(c) If $M_n(I) = M_0 + M_1 I_1 + \cdots + M_n I_n$ is a strong AH-subspace of $V_n(I)$, $f(M_n(I))$ is a strong AH-subspace of $W_n(I)$.

Proof:

The proof is similar to that of Theorem 3.18 .

Example 3.20:

Let $V_2(I), W_2(I)$ be the two weak 2-refined neutrosophic vector spaces defined in Example 3.16 .

(a) $M =< (1,0,0) >, N =< (0,1,0) >, L =< (0,0,1) >$ are three subspaces of V,

$T = M + NI_1 + LI_2 = \{(a,0,0) + (0,b,0)I_1 + (0,0,c)I_2; a,b,c \in R\}$ is a weak AH-subspace of $V_2(I)$.

Consider $f: V_2(I) \to W_2(I)$ the weak AH-linear transformation defined in Example 3.16 .

(b) $AH - Ker(f) = Ker(g) + Ker(h)I_1 + Ker(s)I_2 = \{(a,0,0) + (0,b,c)I_1 + (d,0,0)I_2; a,b,c,d \in R$.

(c) $AH - Im(f) = Im(g) + Im(h)I_1 + Im(s)I_2 = R^2 +< (1,0) > I_1 + R^2 I_2$.

(d) $f(T) = g(M) + h(N)I_1 + s(L)I_2 =< (0,0) > +< (0,0) > I_1 +< (0,1)I_2 =$

$\{(0,0) + (0,0).I_1 + (0,a)I_2; a \in R\}$, which is a weak AH-subspace of $W_2(I)$.

Theorem 3.21:

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Let $V_n(I), W_n(I)$ be two weak n-refined neutrosophic vector spaces over the field K,

$f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$ be a weak AH-linear transformation. Then:

$f(x+y) = f(x) + f(y), f(r.x) = r.f(x)$ for all $x, y \in V_n(I), r \in K$.

Proof:

Let $x = \sum_{i=0}^{n} a_i I_i$, $y = \sum_{i=0}^{n} b_i I_i$ be two arbitrary elements in $V_n(I)$, $r \in K$ be any element in the field K, we have:

$f(x+y) = f(\sum_{i=0}^{n}(a_i + b_i)I_i) = \sum_{i=0}^{n} f_i(a_i + b_i)I_i = \sum_{i=0}^{n} f_i(a_i)I_i + \sum_{i=0}^{n} f_i(b_i)I_i = f(x) + f(y)$.

$f(r.x) = f(\sum_{i=0}^{n} r a_i I_i) = \sum_{i=0}^{n} f_i(r a_i)I_i = r.\sum_{i=0}^{n} f_i(a_i)I_i = r.f(x)$.

Theorem 3.22:

Let $V_n(I), W_n(I)$ be two strong n-refined neutrosophic vector spaces over the n-refined neutrosophic field $K_n(I)$,

$f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$ be a strong AH-linear transformation. Then:

$f(x+y) = f(x) + f(y), f(r.x) = r.f(x)$ for all $x, y \in V_n(I), r \in K_n(I)$.

Proof:

Let $x = \sum_{i=0}^{n} a_i I_i$, $y = \sum_{i=0}^{n} b_i I_i$ be two arbitrary elements in $V_n(I)$, $r = \sum_{i=0}^{n} r_i I_i \in K_n(I)$ be any element in the n-refined neutrosophic field $K_n(I)$, we have:

$f(x+y) = f(\sum_{i=0}^{n}(a_i + b_i)I_i) = \sum_{i=0}^{n} f_i(a_i + b_i)I_i = \sum_{i=0}^{n} f_i(a_i)I_i + \sum_{i=0}^{n} f_i(b_i)I_i = f(x) + f(y)$.

For the proof of the second proposition we use induction on n. If n=0, the theorem is true clearly.

Suppose that it is true for n-1, we must prove it for n.

$f(r.x) = f\left(\sum_{i,j=0}^{n} r_i a_j I_i I_j\right) = f(\sum_{i,j=0}^{n-1} r_i a_j I_i I_j + (\sum_{i=0}^{n} r_i I_i)a_n I_n)$, we can write

$\sum_{i,j=0}^{n-1} r_i a_j I_i I_j = m_0 + m_1 I_1 + \cdots + m_{n-1} I_{n-1}$,

$(\sum_{i=0}^{n} r_i I_i)a_n I_n = r_1 a_n I_1 + r_2 a_n I_2 + \cdots + (r_0 a_n + r_n a_n)I_n$,

$r.x = \sum_{i,j=0}^{n-1} r_i a_j I_i I_j + (\sum_{i=0}^{n} r_i I_i)a_n I_n = \mathrm{m}_0 + (\mathrm{m}_1 + \mathrm{r}_1 \mathrm{a}_\mathrm{n})\mathrm{I}_1 + (\mathrm{m}_2 + \mathrm{r}_2 \mathrm{a}_\mathrm{n})\mathrm{I}_2 + \cdots + (r_0 a_n + r_n a_n)I_n$,

$f(r.x) = f_0(m_0) + f_1(m_1 + r_1 a_n)I_1 + f_2(m_2 + r_2 a_n)I_2 + \cdots f_n(r_0 a_n + r_n a_n)I_n =$

$f_0(m_0) + [f_1(m_1) + r_1 f_1(a_n)]I_1 + \cdots + [r_0 f_n(a_n) + r_n f_n(a_n)]I_n = r.f(x)$.

Theorem 3.23:

Let $V_n(I), W_n(I), U_n(I)$ be three weak n-refined neutrosophic vector spaces over the field K,

$f: W_n(I) \to U_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$,

$g: V_n(I) \to W_n(I); g(\sum_{i=0}^{n} a_i I_i) = g_0(a_0) + g_1(a_1)I_1 + \cdots + g_n(a_n)I_n = \sum_{i=0}^{n} g_i(a_i)I_i$,

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be two weak AH-linear transformations. Then:

(a) $fog = \sum_{i=0}^{n}(f_i o g_i)$.

(b) fog is a weak AH-linear transformation between $V_n(I), U_n(I)$.

Proof:

(a) Let $x = \sum_{i=0}^{n} a_i I_i \in V_n(I)$, $fog(x) = f(\sum_{i=0}^{n} g_i(a_i)I_i) = f(g_0(a_0) + g_1(a_1)I_1 + \cdots + g_n(a_n)I_n)$=

$f_0(g_0(a_0)) + f_1(g_1(a_1))I_1 + \cdots + f_n(g_n(a_n))I_n = \sum_{i=0}^{n}(f_i o g_i)(a_i)I_i$.

(b) Since $f_i o g_i$ is a linear transformation for all i, then we get the proof.

Theorem 3.24:

Let $V_n(I), W_n(I), U_n(I)$ be three strong n-refined neutrosophic vector spaces over the n-refined neutrosophic field K,

$f: W_n(I) \to U_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$,

$g: V_n(I) \to W_n(I); g(\sum_{i=0}^{n} a_i I_i) = g_0(a_0) + g_1(a_1)I_1 + \cdots + g_n(a_n)I_n = \sum_{i=0}^{n} g_i(a_i)I_i$,

be two strong AH-linear transformations. Then:

(a) $fog = \sum_{i=0}^{n}(f_i o g_i)$.

(b) fog is a strong AH-linear transformation between $V_n(I), U_n(I)$.

Proof:

The proof is similar to that of Theorem 3.23.

Example 3.25:

(a) Let $V = R^3$ be a vector spaces over the field R, $V_2(I) = R_2^3(I) = \{(x_0, y_0 z_0) + (x_1, y_1, z_1)I_1 + (x_2, y_2, z_2)I_2;\ x_i, y_i, z_i \in R\}$,

be the corresponding weak 2-refined neutrosophic vector space. We have $g: V \to V; g(a,b,c) = (2b, 2c, 0)$,

$h: V \to V; h(a,b,c) = (2a, c, c)$,

$s: V \to V; s(a,b,c) = (2b, 3c, a)$ are three linear transformations.

(b) $f: V_2(I) \to V_2(I); f(m + nI_1 + qI_2) = g(m) + h(n)I_1 + s(q)I_2; m, n, q \in V$ is a weak AH-linear transformation, $j: V_2(I) \to V_2(I); j(m + nI_1 + qI_2) = g(m) + g(n)I_1 + h(q)I_2; m, n, q \in V$ is a weak AH-linear transformation.

(c) $foj(m + nI_1 + qI_2) = gog(m) + hog(n)I_1 + soh(q)I_2$.

(d) Put $m = (2,1,0), n = (-1,0,0), q = (3,2,2)$, we compute foj as follows:

$foj(m + nI_1 + qI_2) = gog[(2,1,0)] + hog[(-1,0,0)]I_1 + soh[(3,2,2)]I_2$=

$g(2,0,0) + h(0,0,0)I_1 + s(6,2,2)I_2 = (0,0,0) + (0,0,0)I_1 + (4,6,6)I_2$.

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5. Conclusion

In this paper we have defined and studied weak/strong AH-subspaces and weak/strong AH-linear transformations in n-refined neutrosophic vector spaces. Also, we have presented some elementary properties and theorems about these concepts.

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References

[1] Abobala, M., "On Some Special Substructures of Neutrosophic Rings and Their Properties", International Journal of Neutrosophic Science", Vol. 4 , pp. 72-81, 2020.

[2] Abobala, M., "On Some Special Substructures of Refined Neutrosophic Rings", International Journal of Neutrosophic Science, Vol. 5, pp. 59-66, 2020.

[3] Abobala, M,. "Classical Homomorphisms Between Refined Neutrosophic Rings and Neutrosophic Rings", International Journal of Neutrosophic Science, Vol. 5, pp. 72-75, 2020.

[4] Abobala, M., " AH-subspaces in Neutrosophic Vector Spaces", International Journal of neytrosophic Science, Vol. 6 , pp. 80-86, 2020.

[5] Abobala, M., " Classical Homomorphisms Between n-Refined neutrosophic Rings", International Journal of Neutrosophic Science, Vol. 7 , pp. 74-78, 2020.

[6] Adeleke, E.O., Agboola, A.A.A., and Smarandache, F., "Refined Neutrosophic Rings II", International Journal of Neutrosophic Science, Vol. 2, pp. 89-94, 2020.

[7] Adeleke, E.O., Agboola, A.A.A., and Smarandache, F., "Refined Neutrosophic Rings I", International Journal of Neutrosophic Science, Vol. 2 , pp. 77-81, 2020.

[8] Agboola, A.A.A,. and Akinleye, S.A,. "Neutrosophic Vector Spaces", Neutrosophic Sets and Systems, Vol. 4 , pp. 9-17, 2014.

[9] Agboola, A.A.A,. Akwu, A.D,. and Oyebo, Y.T,. "Neutrosophic Groups and Subgroups", International .J .Math.Combin, Vol. 3, pp. 1-9, 2012.

[10] Agboola, A.A.A., Akinola, A.D., and Oyebola, O.Y.," NeutrosophicRings I ", International J.Mathcombin, Vol. 4, pp. 1-14, 2011.

[11] T. Bera., and N. K. Mahapatra., On Neutrosophic Soft Field, IJMTT, 56(7), pp. 472-494, 2018.

[12] T. Bera., and N. K. Mahapatra., On Neutrosophic Soft Linear Spaces, Fuzzy Information and Engineering., 9, 299-324, (2017).

[13] Injrou, S., "Linear Algebra 2 ", Tishreen University Press, pp 180-230, 2015.

DOI: 10.5281/zenodo.3958086

[14] Kandasamy, V.W.B., and Smarandache, F., "Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic Structures", Hexis, Phonex, Arizona 2006.

[15] Smarandache, F., and Abobala, M., "n-Refined Neutrosophic Rings", International Journal of Neutrosophic Science, Vol. 6, pp. 83-90 , 2020.

[16] Smarandache F., and Abobala, M., "n-Refined Neutrosophic Vector Spaces", International Journal of Neutrosophic Science, Vol. 7, pp. 47-54, 2020.

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On Refined Neutrosophic Hypergroup

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Abstract

This paper presents the refinement of neutrosophic hypergroup and studies some of its properties. Several interesting results and examples are presented. The existence of a good homomorphism between a refined neutrosophic hypergroup $H(I_1, I_2)$ and a neutrosophic hypergroup $H(I)$ is established.
Keywords: Neutrosophy, neutrosophic hypegroup, neutrosophic subhypergroup, refined neutrosophic hypergroup, refined neutrosophic subhypergroup.

1 Introduction and Preliminaries

Neutrosophy is a new branch of philosophy that studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophic set and neutrosophic logic were introduced in 1995 by Smarandache as generalizations of fuzzy set [21] and respectively intuitionistic fuzzy logic [10]. In neutrosophic logic, each proposition has a degree of truth (T), a degree of indeterminancy (I) and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$, for more detailed information, the reader should see [15,17]. In 2013, Florentin Smarandache in [16] introduced refined neutrosophic components of the form $< T_1, T_2, \cdots, T_p; I_1, I_2, \cdot, I_r; F_1, F_2, \cdots, F_s >$. The birth of refinement of the neutrosophic components $< T, I, F >$ has led to the extension of neutrosophic numbers $a + bI$ into refined neutrosophic numbers of the form $(a+b_1I_1+b_2I_2+\cdots+b_nI_n)$ where $a, b_1, b_2, \cdots, b_n$ are real or complex numbers. Using these refined neutrosophic numbers, the concept of refined neutrosophic set was introduced and this paved way for the development of refined neutrosophic algebraic structures. Agboola in [4] introduced the concept of refined neutrosophic structure and he studied refined neutrosophic groups in particular and presented their fundamental properties. Since then, several researchers in this field have studied this concept and a great deal of results have been published. In [1], Adeleke et al. presented results on refined neutrosophic rings, refined neutrosophic subrings and in [2], they presented results on refined neutrosophic ideals and refined neutrosophic homomorphisms. A comprehensive review of neutrosophy, neutrosophic triplet set and neutrosophic algebraic structures can be found in [12,18–20].

In [13], Marty, introduced the concept of hypergroups by considering the quotient of a group by its subgroup. And this was the birth of an interesting new branch of Mathematics known as “Algebraic hyperstructures” which is considered as a generalization of classical algebraic structures. In the classical algebraic structure, the composition of two elements is an element whereas in algebraic hyperstructure, the composition of two elements is a non-empty set. Since then, many different kinds of hyperstructures (hyperrings, hypermodules, hypervector spaces,$\cdots$) have been introduced and studied. Also, many theories of algebraic hyperstructures have been propounded as well as their applications to various areas of sciences and technology. For comprehensive details on hyperstructures, the reader should see [11,14]. The concept of neutrosophic hypergroup and their properties was introduced by Agboola and Davvaz in [7]. More connections between algebraic hyperstructures and neutrosophic set can be found in many recent publications, see [3,5,6,8,9].

The present paper is concerned with the development of connections between algebraic hyperstructures and neutrosphic algebraic structures and again concerned with studying the refinement of neutrosophic hypergroups in particular and present some of their basic properties.

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For the purposes of this paper, it will be assumed that I splits into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. It then follows logically that:

$$\begin{array}{lllll} I_1 I_1 & = & I_1^2 & = & I_1, \\ I_2 I_2 & = & I_2^2 & = & I_2, \text{ and} \\ I_1 I_2 & = & I_2 I_1 & = & I_1. \end{array}$$

Definition 1.1. [4] If $* : X(I_1, I_2) \times X(I_1, I_2) \mapsto X(I_1, I_2)$ is a binary operation defined on $X(I_1, I_2)$, then the couple $(X(I_1, I_2), *)$ is called a refined neutrosophic algebraic structure and it is named according to the laws (axioms) satisfied by $*$.

Definition 1.2. [4] Let $(X(I_1, I_2), +, \cdot)$ be any refined neutrosophic algebraic structure where $+$ and . are ordinary addition and multiplication respectively.
For any two elements $(a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$, we define

$$(a, bI_1, cI_2) + (d, eI_1, fI_2) = (a + d, (b + e)I_1, (c + f)I_2),$$
$$(a, bI_1, cI_2) \cdot (d, eI_1, fI_2) = (ad, (ae + bd + be + bf + ce)I_1, (af + cd + cf)I_2).$$

Definition 1.3. [4] If $''+''$ and $''.''$ are ordinary addition and multiplication, I_k with $k = 1, 2$ have the following properties:

1. $I_k + I_k + \cdots + I_k = nI_k$.
2. $I_k + (-I_k) = 0$.
3. $I_k \cdot I_k \cdot \cdots \cdot I_k = I_k^n = I_k$ for all positive integers $n > 1$.
4. $0 \cdot I_k = 0$.
5. I_k^{-1} is undefined and therefore does not exist.

Definition 1.4. [4] Let $(G, *)$ be any group. The couple $(G(I_1, I_2), *)$ is called a refined neutrosophic group generated by G, I_1 and I_2. $(G(I_1, I_2), *)$ is said to be commutative if for all $x, y \in G(I_1, I_2)$, we have $x * y = y * x$. Otherwise, we call $(G(I_1, I_2), *)$ a non -commutative refined neutrosophic group.

Definition 1.5. [4] If $(X(I_1, I_2), *)$ and $(Y(I_1, I_2), *')$ are two refined neutrosophic algebraic structures, the mapping

$$\phi : (X(I_1, I_2), *) \longrightarrow (Y(I_1, I_2), *')$$

is called a neutrosophic homomorphism if the following conditions hold:

1. $\phi((a, bI_1, cI_2) * (d, eI_1, fI_2)) = \phi((a, bI_1, cI_2)) *' \phi((d, eI_1, fI_2))$.
2. $\phi(I_k) = I_k$ for all $(a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$ and $k = 1, 2$.

Example 1.6. [4] Let
$\mathbb{Z}_2(I_1, I_2) = \{(0, 0, 0), (1, 0, 0), (0, I_1, 0), (0, 0, I_2), (0, I_1, I_2), (1, I_1, 0), (1, 0, I_2), (1, I_1, I_2)\}$.
Then $(\mathbb{Z}_2(I_1, I_2), +)$ is a commutative refined neutrosophic group of integers modulo 2.
Generally for a positive integer $n \geq 2$, $(\mathbb{Z}_n(I_1, I_2), +)$ is a finite commutative refined neutrosophic group of integers modulo n.

Example 1.7. [4] Let $(G(I_1, I_2), *)$ and and $(H(I_1, I_2), *')$ be two refined neutrosophic groups.
Let $\phi : G(I_1, I_2) \times H(I_1, I_2) \to G(I_1, I_2)$ be a mapping defined by $\phi(x, y) = x$ and let
$\psi : G(I_1, I_2) \times H(I_1, I_2) \to H(I_1, I_2)$ be a mapping defined by $\psi(x, y) = y$. Then ϕ and ψ are refined neutrosophic group homomorphisms.

Definition 1.8. [11] Let H be a non-empty set and $\circ : H \times H \longrightarrow P^*(H)$ be a hyperoperation. The couple $(H, \circ)$ is called a hypergroupoid. For any two non-empty subsets A and B of H and $x \in H$, we define

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \quad A \circ x = A \circ \{x\} \quad \text{and} \quad x \circ B = \{x\} \circ B.$$

Definition 1.9. [11] A hypergroupoid $(H, \circ)$ is called a semihypergroup if for all a, b, c of H we have $(a \circ b) \circ c = a \circ (b \circ c)$, which means that

$$\bigcup_{u \in a \circ b} u \circ c = \bigcup_{v \in b \circ c} a \circ v.$$

A hypergroupoid $(H, \circ)$ is called a quasihypergroup if for all a of H we have $a \circ H = H \circ a = H$. This condition is also called the reproduction axiom.

Definition 1.10. [11] A hypergroupoid $(H, \circ)$ which is both a semihypergroup and a quasi- hypergroup is called a hypergroup.

Definition 1.11. [11] Let $(H, \circ)$ and $(H', \circ')$ be two hypergroupoids. A map $\phi : H \longrightarrow H'$, is called

1. an inclusion homomorphism if for all x, y of H, we have $\phi(x \circ y) \subseteq \phi(x) \circ' \phi(y)$;
2. a good homomorphism if for all x, y of H, we have $\phi(x \circ y) = \phi(x) \circ' \phi(y)$.

Definition 1.12. [11] Let H be a non-empty set and let + be a hyperoperation on H. The couple $(H, +)$ is called a canonical hypergroup if the following conditions hold:

1. $x + y = y + x$, for all $x, y \in H$,
2. $x + (y + z) = (x + y) + z$, for all $x, y, z \in H$,
3. there exists a neutral element $0 \in H$ such that $x + 0 = \{x\} = 0 + x$, for all $x \in H$,
4. for every $x \in H$, there exists a unique element $-x \in H$ such that $0 \in x + (-x) \cap (-x) + x$,
5. $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$, for all $x, y, z \in H$.

Definition 1.13. [7] Let $(H, \star)$ be any hypergroup and let $< H \cup I >= \{x = (a, bI) : a, b \in H\}$. The couple $N(H) = (< H \cup I >, \star)$ is called a neutrosophic hypergroup generated by H and I under the hyperoperation $\star$. The part a is called the non-neutrosophic part of x and the part b is called the neutrosophic part of x. If $x = (a, bI)$ and $y = (c, dI)$ are any two elements of $N(H)$, where $a, b, c, d \in H$, we define

$$x \star y = (a, bI) \star (c, dI) = \{(u, vI) | u \in a \star c, v \in a \star d \cup b \star c \cup b \star d\} = (a \star c, (a \star d \cup b \star c \cup b \star d)I).$$

Note that $a \star c \subseteq H$ and $(a \star d \cup b \star c \cup b \star d) \subseteq H$.

2 Formulation of Refined Neutrosophic Hypergroup

Definition 2.1. Let $(H, \star)$ be any hypergroup and let $< H \cup (I_1, I_2) >= \{x = (a, bI_1, cI_2) : a, b, c \in H\}$. The couple $(H(I_1, I_2), \star)$, is called a refined neutrosophic hypergroup generated by H, I_1 and I_2 under the hyperoperation $\star$. The part a is called the non-neutrosophic part of x and the part b and c are called the neutrosophic parts of x.

If $x = (a, bI_1, cI_2)$ and $y = (d, eI_1, fI_2)$ are any two elements of $H(I_1, I_2)$, where $a, b, c, d \in H$, we define

$$\begin{aligned} x \star y &= (a, bI_1, cI_2) \star (d, eI_1, fI_2) \\ &= \{(u, vI_1, wI_2) : u \in a \star d, v \in a \star e \cup b \star d \cup b \star e \cup b \star f \cup c \star e, w \in a \star f \cup c \star d \cup c \star f\} \\ &= (a \star d, (a \star e \cup b \star d \cup b \star e \cup b \star f \cup c \star e)I_1, (a \star f \cup c \star d \cup c \star f)I_2). \end{aligned}$$

Note that $a \star d \subseteq H$, $(a \star e \cup b \star d \cup b \star e \cup b \star f \cup c \star e) \subseteq H$ and $(a \star f \cup c \star d \cup c \star f) \subseteq H$.

Note 1. *If the operation on $H(I_1, I_2)$ is hyperaddition $(+')$ then for all $x = (a, bI_1, cI_2)$ and $y = (d, eI_1, fI_2)$ elements of $H(I_1, I_2)$, with $a, b, c, d \in H$, we define*

$$x +' y = (a, bI_1, cI_2) +' (d, eI_1, fI_2) = (a + d, (b + e)I_1, (c + f)I_2).$$

Here the addition on the right is the hyperaddition in H.

Proposition 2.2. *Let $(H, +)$ be a hypergroupoid, then, the refined neutrosophic hypergroup $(H(I_1, I_2), +')$ is a hypergroup with identity element $\theta = (0, 0I_1, 0I_2)$ iff $(H, +)$ is a hypergroup with identity element 0.*

Proof. Suppose $(H,+)$ is a hypergroup and $x=(a,bI_1,cI_2), y=(d,eI_1,fI_2), z=(g,hI_1,kI_2)\in H(I_1,I_2)$.
Then we show that $(H(I_1,I_2),+')$ is a hypergroup.
First, we shall show that $(H(I_1,I_2),+')$ is a semihypergroup.

$$\begin{aligned} x+'(y+'z) &= (a,bI_1,cI_2)+'((d,eI_1,fI_2)+'(g,hI_1,kI_2)\\ &= (a,bI_1,cI_2)+'((d+g,(e+h)I_1,(f+k)I_2))\\ &= (a+(d+g),(b+(e+h))I_1,(c+(f+k))I_2)\\ &= ((a+d)+g,((b+e)+h)I_1,((c+f)+k)I_2)\\ &= (a+d,(b+e)I_1,(c+f)I_2)+'(g,hI_1,kI_2)\\ &= ((a,bI_1,cI_2)+'(d.eI_1,fI_2))+'(g,hI_1,kI_2)\\ &= (x+'y)+'z. \end{aligned}$$

Secondly, we shall show that $(H(I_1,I_2),+')$ is a quasihypergroup.
That is , we want to show that $x+'H(I_1,I_2)=H(I_1,I_2)+'x=H(I_1,I_2)$.

$$\begin{aligned} x+'H(I_1,I_2) &= (a,bI_1,cI_2)+'\{(d,eI_1,fI_2):(d,eI_1,fI_2)\in H(I_1,I_2)\}\\ &= (a+d,(b+e)I_1,(c+f)I_2)\\ &\subseteq H(I_1,I_2).\\ &\Longrightarrow x+'H(I_1,I_2)\subseteq H(I_1,I_2). \end{aligned}$$

Now we show that $H(I_1,I_2)\subseteq x+'H(I_1,I_2)$, let $z=(g,hI_1,kI_2)\in H(I_1,I_2)$ with $g,h,k\in H$.
There exist $a_1,a_2,a_3\in H$ such that $g\in a_1+H, h\in a_2+H$ and $k\in a_3+H$, since H is a hypergroup.
Hence $(g,hI_1,kI_2)\in(a_1,a_2I_1,a_3I_2)+H$, which implies that $H(I_1,I_2)\subseteq x+'H(I_1,I_2)$.
Accordingly, $H(I_1,I_2)=x+'H(I_1,I_2)$. Similarly, we can show that $H(I_1,I_2)=H(I_1,I_2)+'x$.
$\therefore$ We can conclude that $(H(I_1,I_2),+')$ is a hypergroup.
Conversely, suppose $(H(I_1,I_2),+')$ is a hypergroup and $x=(a,bI_1,cI_2), y=(d,eI_1,fI_2)$,
$z=(g,hI_1,kI_2)\in H(I_1,I_2)$, with $a,d,g,b=c=e=f=h=k=0\in H$.
Then we show that $(H,+)$ is a hypergroup.
Since $H(I_1,I_2)$ is a hypergroup, $\;x+'(y+'z)=(x+'y)+'z$.
But

$$\begin{aligned} x+'(y+'z) &= (a,0I_1,0I_2)+'((d,0I_1,0I_2)+'(g,0I_1,0I_2))\\ &= (a,0I_1,0I_2)+'((d+g),(0+0)I_1,(0+0)I_2)\\ &= (a+(d+g),0I_1,0I_2)\\ &= a+(d+g))\ \text{ and}\\ (x+'y)+'z &= ((a,0I_1,0I_2)+'(d,0I_1,0I_2))+'(g,0I_1,0I_2)\\ &= ((a+d,(0+0)I_1,(0+0)I_2)+'((d,0I_1,0I_2)\\ &= ((a+d)+g,0I_1,0I_2)=(a+d)+g. \end{aligned}$$

$$\therefore x+'(y+'z)=(x+'y)+'z\Longrightarrow a+(d+g)=(a+d)+g.$$

Thus $(H,+)$ is a semihypergroup.

Since $H(I_1,I_2)$ is a quasihypergroup, for $x=(a,bI_1,cI_2)\in H(I_1,I_2)$ with $a,b=c=0\in H$
we have that $x+'H(I_1,I_2)=H(I_1,I_2)+'x=H(I_1,I_2)$.
But

$$\begin{aligned} x+'H(I_1,I_2) &= (a,0I_1,0I_2)+'H(I_1,I_2)\\ &= (a,0I_1,0I_2)+'\{(h,0I_1,0I_2):h\in H\}\\ &= \{(a+h,0I_1,0I_2):h\in H\}\\ &= \{a+h:h\in H\}=a+H\ \text{ and}\\ H(I_1,I_2)+'x &= H(I_1,I_2)+'(a,0I_1,0I_2)\\ &= \{(h,0I_1,0I_2):h\in H\}+'(a,0I_1,0I_2)\\ &= \{(h+a,0I_1,0I_2):h\in H\}\\ &= \{h+a:h\in H\}=H+a. \end{aligned}$$

$\therefore x+'H(I_1,I_2)=H(I_1,I_2)+'x\Longrightarrow a+H=H+a$.
Since $a\in H,\quad a+H=H$ and $H+a=H$ which implies that $a+H=H+a=H$.
Hence, we can conclude that $(H,+)$ is a hypergroup.

□

Proposition 2.3. *Every refined neutrosophic hypergroup is a semihypergroup.*

Proof. Let $(H(I_1,I_2),\star)$ be any refined neutrosophic hypergroup and let $x=(a,bI_1,cI_2)$,
$y=(d,eI_1,fI_2), z=(g,hI_1,kI_2)$ be arbitrary elements of $H(I_1,I_2)$, where $a,b,c,d,e,f,g,h,k\in H$.

Then,

$$\begin{aligned} x\star y &= (a,bI_1,cI_2)\star(d,eI_1,fI_2)\\ &= \{(u,vI_1,wI_2): u\in a\star d, v\in a\star e\cup b\star d\cup b\star e\cup b\star f\cup c\star e, w\in a\star f\cup c\star d\cup c\star f\}\\ &= (a\star d,(a\star e\cup b\star d\cup b\star e\cup b\star f\cup c\star e)I_1,(a\star f\cup c\star d\cup c\star f)I_2).\\ &\subseteq H(I_1,I_2). \end{aligned}$$

Hence, $(H(I_1,I_2),\star)$ is a hypergroupoid.
Next

$$\begin{aligned} x\star(y\star z) &= (a,bI_1,cI_2)\star((d,eI_1,fI_2)\star(g,hI_1,kI_2))\\ &= (a,bI_1,cI_2)\star((d\star g,(d\star h\cup e\star g\cup e\star h\cup e\star k\cup f\star h)I_1,\\ &\quad (d\star k\cup f\star g\cup f\star k)I_2)\\ &= a\star(d\star g),(a\star(d\star h)\cup a\star(e\star g)\cup a\star(e\star h)\cup a\star(e\star k)\cup a\star(f\star h)\\ &\quad \cup b\star(d\star g)\cup b\star(d\star h)\cup b\star(e\star g)\cup b\star(e\star h)\cup b\star(e\star k)\\ &\quad \cup b\star(f\star h)\cup b\star(d\star k)\cup b\star(f\star g)\cup b\star(f\star k)\cup c\star(d\star h)\cup c\star(e\star g)\\ &\quad \cup c\star(e\star h)\cup c\star(e\star k)\cup c\star(f\star h))I_1,\\ &\quad (a\star(d\star k)\cup a\star(f\star g)\cup a\star(f\star k)\cup c\star(d\star g)\cup c\star(d\star k)\cup c\star(f\star g)\\ &\quad \cup c\star(f\star k))I_2)\\ &= (a\star d)\star g,((a\star d)\star h\cup(a\star e)\star g\cup(a\star e)\star h\cup(a\star e)\star k\cup(a\star f)\star h\\ &\quad \cup(b\star d)\star g\cup(b\star d)\star h\cup(b\star e)\star g\cup(b\star e)\star h\cup(b\star e)\star k\cup(b\star f)\star h\\ &\quad \cup(b\star d)\star k\cup(b\star f)\star g\cup(b\star f)\star k\cup(c\star d)\star h\cup(c\star e)\star g\cup(c\star e)\star h\\ &\quad \cup(c\star e)\star k\cup(c\star f)\star h)I_1,\\ &\quad ((a\star d)\star k\cup(a\star f)\star g\cup(a\star f)\star k\cup(c\star d)\star g\cup(c\star d)\star k\cup(c\star f)\star g\\ &\quad \cup(c\star f)\star k)I_2)\\ &= ((a,bI_1,cI_2)\star(d,eI_1,fI_2))\star(g,hI_1,kI_2)\\ &= (x\star y)\star z. \end{aligned}$$

Accordingly, $(H(I_1,I_2),\star)$ is a semihypergroup. □

Proposition 2.4. *A refined neutrosophic hypergroup is not always a quasihypergroup.*

Proof. To see this, consider a refined neutrosophic hypergroup, say $(H(I_1,I_2),\star)$, where $(0,0I_1,0I_2)\notin H(I_1,I_2)$. Then for $x=(a,bI_1,cI_2)\in H(I_1,I_2)$ we have that

$$\begin{aligned} x\star H(I_1,I_2) &= (a,bI_1,cI_2)\star H(I_1,I_2)\\ &= (a,bI_1,cI_2)\star\{(h_1,h_2I_1,h_3I_2): h_1,h_2,h_3\in H\}\\ &= \{a\star h_1,(a\star h_2\cup b\star h_1\cup b\star h_2\cup b\star h_3\cup c\star h_2)I_1,(a\star h_3\cup c\star h_1\cup c\star h_3)I_2\}\\ &= \{(u,vI_1,wI_2): u\in a\star h_1, v\in(a\star h_2\cup b\star h_1\cup b\star h_2\cup b\star h_3\cup c\star h_2),\\ &\quad w\in(a\star h_3\cup c\star h_1\cup c\star h_3)\}\\ &\subset H(I_1,I_2). \end{aligned}$$

$$\begin{aligned} H(I_1,I_2)\star x &= H(I_1,I_2)\star(a,bI_1,cI_2)\\ &= \{(h_1,h_2I_1,h_3I_2): h_1,h_2,h_3\in H\}\star(a,bI_1,cI_2)\\ &= \{h_1\star a,(h_1\star b\cup h_2\star a\cup h_2\star b\cup h_2\star c\cup h_3\star b)I_1,(h_1\star c\cup h_3\star a\cup h_3\star c)I_2\}\\ &= \{(u,vI_1,wI_2): u\in h_1\star a,, v\in(h_1\star b\cup h_2\star a\cup h_2\star b\cup h_2\star c\cup h_3\star b),\\ &\quad w\in(h_1\star c\cup h_3\star a\cup h_3\star c)\}\\ &\subset H(I_1,I_2). \end{aligned}$$

We can see that $x\star H(I_1,I_2)=H(I_1,I_2)\star x\neq H(I_1,I_2)$. This implies that reproduction axioms fails to hold in this case. □

We note that the reproduction axioms fails to hold in some refined neutrosophic hypergroup, hence there exist some neutrosophic hypergroups that are not hypergroup. This observation is recorded in the next proposition.

Proposition 2.5. *Let $(H(I_1,I_2),\star)$ be a refined neutrosophic hypergroup, then*

1. *$(H(I_1,I_2),\star)$ in general is not a hypergroup ;*
2. *$(H(I_1,I_2),\star)$ always contain a hypergroup.*

Proof. 1. From Proposition 2.4 above, we can see that the reproduction axiom is not always satisfied. Then the proof follows.

2. It follows from the definition of a neutrosophic hypergroup.

□

Example 2.6. Let $H = \{a, b\}$ be a set with the hyperoperation defined as follows

$$a \star a = a, \quad a \star b = b \star a = b \text{ and } b \star b = \{a, b\}.$$

Let
$H(I_1, I_2) = \{a, b, \alpha_1 = (a, aI_1, aI_2), \alpha_2 = (a, aI_1, bI_2), \alpha_3 = (a, bI_1, aI_2), \alpha_4 = (a, bI_1, bI_2), \beta_1 = (b, bI_1, bI_2), \beta_2 = (b, bI_1, aI_2), \beta_3 = (b, aI_1, bI_2), \beta_4 = (b, aI_1, aI_2)\}$ be a refined neutrosophic set and let $\star'$ be a hyperoperation on $H(I_1, I_2)$ defined in the table below.

Take $\alpha = \{\alpha_1 = (a, aI_1, aI_2), \alpha_2 = (a, aI_1, bI_2), \alpha_3 = (a, bI_1, aI_2), \alpha_4 = (a, bI_1, bI_2)\}$ and $\beta = \{\beta_1 = (b, bI_1, bI_2), \beta_2 = (b, bI_1, aI_2), \beta_3 = (b, aI_1, bI_2), \beta_4 = (b, aI_1, aI_2)\}$.

Table 1: Cayley table for the binary operation " $\star'$ "

$\star'$	a	b	α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4
a	a	b	α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4
b	b	$\{a, b\}$	β_1	$\{\beta_1, \beta_2\}$	$\{\beta_1, \beta_3\}$	β	$\{\alpha, \beta\}$	$\{\alpha_2, \alpha_4, \beta_1, \beta_3\}$	$\{\alpha_3, \alpha_4, \beta_1, \beta_2\}$	$\{\alpha_4, \beta_1\}$
α_1	α_1	β_1	α_1	α	$\{\alpha_1, \alpha_3\}$	α	β_1	β	$\{\beta_1, \beta_3\}$	β
α_2	α_2	$\{\beta_1, \beta_2\}$	α	α	α	α	β	β	β	β
α_3	α_3	$\{\beta_1, \beta_3\}$	$\{\alpha_1, \alpha_3\}$	α	$\{\alpha_1, \alpha_3\}$	α	$\{\beta_1, \beta_3\}$	β	$\{\beta_1, \beta_3\}$	β
α_4	α_4	β	α	α	α	α	β	β	β	β
β_1	β_1	$\{\alpha, \beta\}$	β_1	β	$\{\beta_1, \beta_3\}$	β	$\{\alpha, \beta\}$	$\{\alpha, \beta\}$	$\{\alpha, \beta\}$	$\{\alpha, \beta\}$
β_2	β_2	$\{\alpha_2, \alpha_4, \beta_1, \beta_3\}$	β	β	β	β	$\{\alpha, \beta\}$	$\{\alpha, \beta\}$	$\{\alpha, \beta\}$	$\{\alpha, \beta\}$
β_3	β_3	$\{\alpha_3, \alpha_4, \beta_1, \beta_2\}$	$\{\beta_1, \beta_3\}$	β	$\{\beta_1, \beta_3\}$	β	$\{\alpha, \beta\}$	$\{\alpha, \beta\}$	$\{\alpha, \beta\}$	$\{\alpha, \beta\}$
β_4	β_4	$\{\alpha_4, \beta_1\}$	β	β	β	β	$\{\alpha, \beta\}$	$\{\alpha, \beta\}$	$\{\alpha, \beta\}$	$\{\alpha, \beta\}$

It is clear from the table that $(H(I_1, I_2), \star)$ is a refined neutrosophic hypergroup since it contains a proper subset $\{a, b\}$ which is a hypergroup under $\star$.

Example 2.7. Let $H = \{a, b, c\}$ and define " $\star$ " on H as follows

Table 2: Cayley table for the binary operation " $\star$ "

$\star$	a	b	c
a	a	b	c
b	b	$\{a, b\}$	$\{b, c\}$
c	c	$\{b, c\}$	$\{a, b, c\}$

Let $H(I_1, I_2) = \{a, b, c, \alpha_1 = (a, aI_1, aI_2), \alpha_2 = (a, aI_1, bI_2), \alpha_3 = (a, aI_1, cI_2), \alpha_4 = (a, bI_1, aI_2), \alpha_5 = (a, cI_1, aI_2), \alpha_6 = (a, bI_1, cI_2), \alpha_7 = (a, cI_1, bI_2), \alpha_8 = (a, bI_1, bI_2), \alpha_9 = (a, cI_1, cI_2), \beta_1 = (b, bI_1, bI_2), \beta_2 = (b, bI_1, aI_2), \beta_3 = (b, bI_1, cI_2), \beta_4 = (b, aI_1, bI_2), \beta_5 = (b, cI_1, bI_2), \beta_6 = (b, aI_1, cI_2), \beta_7 = (b, cI_1, aI_2), \beta_8 = (b, aI_1, aI_2), \beta_9 = (b, cI_1, cI_2), \tau_1 = (c, cI_1, cI_2), \tau_2 = (c, cI_1, aI_2), \tau_3 = (c, cI_1, bI_2), \tau_4 = (c, aI_1, cI_2), \tau_5 =$

$(c, bI_1, cI_2), \tau_6 = (c, aI_1, bI_2), \tau_7 = (c, bI_1, aI_2), \tau_8 = (c, aI_1, aI_2), \tau_9 = (c, bI_1, bI_2)\}$ be a refined neutrosophic set and let $\star'$ be a hyperoperation on $H(I_1, I_2)$, then using the definition of $\star$ in Table 2 , $(H(I_1, I_2), \star')$ is a refined neutrosophic hypergroup.

Example 2.8. Let $H(I_1, I_2) = \{e, a, b, c, (I_1, I_2), (aI_1, aI_2), (bI_1, bI_2), (cI_1, cI_2)\}$ be a refined neutrosophic semi group where $a^2 = b^2 = c^2 = e, ab = ba = c$ and $ac = ca = b$ and let $P(I_1, I_2) = \{e, a, (aI_1, aI_2)\}$ be a refined neutrosophic subset of $H(I_1, I_2)$.Then for all $x, y \in H(I_1, I_2)$ define

$$xoy = xP(I_1, I_2)y.$$

Then $(H(I_1, I_2)), \circ)$ is a refined neutrosophic hypergroup.

Example 2.9. Let $V(I_1, I_2)$ be a weak refined neutrosophic vector space over a field K. Then for all $x = (a, bI_1, cI_2), y = (d, eI_1, fI_2) \in V(I_1, I_2)$ define

$$\begin{aligned} x \circ y &= \{k \bullet (x + y) : k \in K\} \\ &= \{k \bullet (a + d), k \bullet (b + e)I_1, k \in \bullet(c + f)I_2 : k \in K\} \\ &= \{k \bullet a + k \bullet d, (k \bullet b + k \bullet e)I_1, (k \bullet c + k \bullet f)I_2 : k \in K\}. \end{aligned}$$

Then $(V(I_1, I_2), \circ)$ is a hypergroup.

To see this we proceed as follows :
Firstly we show that $(V(I_1, I_2), \circ)$ is a hypergroupoid.
So, for $x = (a, bI_1, cI_2), y = (d, eI_1, fI_2) \in (V(I_1, I_2), \circ)$ and $k \in K$we have that

$$x \circ y = \{k \bullet (x + y) : k \in K\} = \{k \bullet (a + d), k \bullet (b + e)I_1, k \bullet (c + f)I_2 : k \in K\}$$
$$= \{(u, vI_1, wI_2) : u \in k \bullet (a + d), v \in k \bullet (b + e), w \in k \bullet (c + f)\} \in V(I_1, I_2).$$

Next we show $(V(I_1, I_2), \circ)$ is a semi-hypergroup, i.e., $\circ$ is associative.
Let $x = (a, bI_1, cI_2), y = (d, eI_1, fI_2)$ and $z = (g, hI_1, jI_2) \in V(I_1, I_2)$ then we want to show that $x \circ (y \circ z) = (x \circ y) \circ z$.
Consider $x \circ (y \circ z) = (a, bI_1, cI_2) \circ ((d, eI_1, fI_2)o(g, hI_1, jI_2))$
$= (a, bI_1, cI_2) \circ \{(u, vI_1, wI_2) : u \in k \bullet (d + g), v \in k \bullet (e + h), w \in k \bullet (f + j)\}$
$= \{(p, qI_1, rI_2) : p \in k \bullet (a + u), q \in k \bullet (b + v), r \in k \bullet (c + w)\}$
$= \{(p, qI_1, rI_2) : p \in k \bullet (a + k \bullet (d + g)), q \in k \bullet (b + k \bullet (e + h)), r \in k \bullet (c + k \bullet (f + j))\}$
$= \{(p, qI_1, rI_2) : p \in k \bullet (a + (d + g)), q \in k \bullet (b + (e + h)), r \in k \bullet (c + (f + j))\}$
$= \{(p, qI_1, rI_2) : p \in k \bullet ((a + d) + g), q \in k \bullet ((b + e) + h), r \in k \bullet ((c + f) + j)\}$
$= \{(p, qI_1, rI_2) : p \in k \bullet (k \bullet (a + d) + g), q \in k \bullet (k \bullet (b + e) + h), r \in k \bullet (k \bullet (c + f) + j)\}$
$= \{(p', q'I_1, r'I_2) : p' \in k \bullet (a + d), q' \in k \bullet (b + e), r' \in k \bullet (c + f)\} \circ (g, hI_1, jI_2)$
$= ((a, bI_1, cI_2) \circ (d, eI_1, fI_2)) \circ (g, hI_1, jI_2)$
$= (x \circ y) \circ z$.
Next, we show that $\circ$ satisfies the reproduction axiom.
Let $x = (a, bI_1, cI_2) \in V(I_1, I_2)$ with $a, b, c \in V$ then
$(a, bI_1, cI_2) \circ V(I_1, I_2) = \{(a, bI_1, cI_2) \circ (v_1, v_2I_1, v_3I_2) : v_1, v_2, v_3 \in V\}$
$= \{(p, qI_1, rI_2) : p \in k \bullet (a + v_1), q \in k \bullet (b + v_2)I_1, k \bullet (c + v_3)I_3\}$
$= \{(p, qI_1, rI_2) : p \in k \bullet (v_1 + a), q \in k \bullet (v_2 + b)I_1, k \bullet (v_3 + c)I_3\}$
$= \{(v_1, v_2I_1, v_3I_2) \circ (a, bI_1, cI_2) : v_1, v_2, v_3 \in V\} = V(I_1, I_2) \circ (a, bI_1, cI_2) = V(I_1, I_2)$.
Therefore $V(I_1, I_2) \circ (a, bI_1, cI_2) = (a, bI_1, cI_2) \circ V(I_1, I_2) = V(I_1, I_2)$.

Example 2.10. Let $V(I_1, I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$. For all $x = (a, bI_1, cI_2), y = (d, eI_1, fI_2) \in V(I_1, I_2)$ and $k = (p, qI_1, rI_2) \in K(I_1, I_2)$ define

$$\begin{aligned} x \circ y &= \{(p, qI_1, rI_2) \bullet (a + d, (b + e)I_1, (c + f)I_2)\} \\ &= \{(u, vI_1, wI_2) : \\ &\quad u \in p \bullet (a + d), \\ &\quad v \in p \bullet (b + e) \cup q \bullet (a + d) \cup q \bullet (b + e) \cup q \bullet (c + f) \cup r \bullet (b + e), \\ &\quad w \in p \bullet (c + f) \cup r \bullet (a + d) \cup r \bullet (c + f)\}. \end{aligned}$$

Then $V(I_1, I_2, o)$ is refined neutrosophic hypergroup.

Proposition 2.11. *Let $(H(I_1, I_2), \star_1)$ and $(K(I_1, I_2), \star_2)$ be any two refined neutrosophic hypergroups. Then, $(H(I_1, I_2) \times K(I_1, I_2), \star)$ is a refined neutrosophic hypergroup, where*

$$(x_1, x_2) \star (y_1, y_2) = \{(x, y) : x \in x_1 \star_1 y_1, y \in x_2 \star_2 y_2, \forall (x_1, x_2), (y_1, y_2) \in H(I_1, I_2) \times K(I_1, I_2)\}.$$

Proof. Let $(x_1,x_2),(y_1,y_2)\in H(I_1,I_2)\times K(I_1,I_2)$, where $x=(a,bI_1,cI_2)$ and $y=(d,eI_1,fI_2)$ then

$$\begin{aligned}(x_1,x_2)\star(y_1,y_2) &= ((a_1,b_1I_1,c_1I_2),(a_2,b_2I_1,c_2I_2))\star((d_1,e_1I_1,f_1I_2),(d_2,e_2I_1,f_1I_2))\\ &= \{((k_1,m_1I_1,t_1I_2),(k_2,m_2I_1,t_2I_2)):\\ &\quad (k_1,m_1I_1,t_1I_2)\in(a_1,b_1I_1,c_1I_2)\star(d_1,e_1I_1,f_1I_2),\\ &\quad (k_2,m_2I_1,t_2I_2)\in(a_2,b_2I_1,c_2I_2)\star(d_2,e_2I_1,f_2I_2)\}\\ &= \{k_1\in a_1\star d_1, m_1\in a_1\star e_1\cup b_1\star d_1\cup b_1\star e_1\cup b_1\star f_1\cup c_1\star e_1,\\ &\quad t_1\in a_1\star f_1\cup c_1\star d_1\cup c_1\star f_1,\\ &\quad k_2\in a_2\star d_2, m_2\in a_2\star e_2\cup b_2\star d_2\cup b_2\star e_2\cup b_2\star f_2\cup c_2\star e_2,\\ &\quad t_2\in a_2\star f_2\cup c_2\star d_2\cup c_2\star f_2\}\\ &= \{((a_1\star d_1,(a_1\star e_1\cup b_1\star d_1\cup b_1\star e_1\cup b_1\star f_1\cup c_1\star e_1)I_1\\ &\quad (a_1\star f_1\cup c_1\star d_1\cup c_1\star f_1)I_2),\\ &\quad (a_2\star d_2,(a_2\star e_2\cup b_2\star d_2\cup b_2\star e_2\cup b_2\star f_2\cup c_2\star e_2)I_2,\\ &\quad (a_2\star f_2\cup c_2\star d_2\cup c_2\star f_2)I_2))\}\subseteq H(I_1,I_2)\times K(I_1,I_2).\end{aligned}$$

Then $(H(I_1,I_2))\times K(I_1,I_2),\star)$ is a refine neutrosophic hypergroupoid.
Let, $(x_1,x_2),(y_1,y_2),(z_1,z_2)\in H(I_1,I_2)\times K(I_1,I_2)$, where $x=(a,bI_1,cI_2)$, $y=(d,eI_1,fI_2)$ and $z=(g,hI_1jI_2)$ then
$((x_1,x_2)\star(y_1,y_2))\star(z_1,z_2)=(((a_1,b_1I_1,c_1I_2),(a_2,b_2I_1,c_2I_2))\star((d_1,e_1I_1,f_1I_2),(d_2,e_2I_1,f_2I_2)))$
$\star((g_1,h_1I_1,j_1I_1),(g_2,h_2I_1,j_2I_2))$
$=\{((k_1,m_1I_1,t_1I_2),(k_2,m_2I_1,t_2I_2)):k_1\in a_1\star d_1,m_1\in a_1\star e_1\cup b_1\star d_1\cup b_1\star e_1\cup b_1\star f_1\cup c_1\star e_1,$
$t_1\in a_1\star f_1\cup c_1\star d_1\cup c_1\star f_1,\quad k_2\in a_2\star d_2,m_2\in a_2\star e_2\cup b_2\star d_2\cup b_2\star e_2\cup b_2\star f_2\cup c_2\star e_2,$
$t_2\in a_2\star f_2\cup c_2\star d_2\cup c_2\star f_2\}\star((g_1,h_1I_1,j_1I_1),(g_2,h_2I_1,j_2I_2))$
$=((k_1,m_1I_1,t_1I_2),(k_2,m_2I_1,t_2I_2))\star((g_1,h_1I_1,j_1I_1),(g_2,h_2I_1,j_2I_2))$
$=\{((p_1,q_1I_1,r_1I_2),(p_2,q_2I_1,r_2I_2)):p_1\in k_1\star g_1,q_1\in k_1\star h_1\cup m_1\star g_1\cup m_1\star h_1\cup m_1\star j_1\cup t_1\star h_1,$
$r_1\in k_1\star j_1\cup t_1\star g_1\cup t_1\star j_1,p_2\in k_2\star g_2,q_2\in k_2\star h_2\cup m_2\star g_2\cup m_2\star h_2\cup m_2\star j_2\cup t_2\star h_2,$
$r_2\in k_2\star j_2\cup t_2\star g_2\cup t_2\star j_2\}$
$=\{((p_1,q_1I_1,r_1I_2),(p_2,q_2I_1,r_2I_2)):p_1\in(a_1\star d_1)\star g_1,$
$q_1\in(a_1\star d_1)\star h_1\cup(a_1\star e_1)\star g_1\cup(b_1\star d_1)\star g_1\cup(b_1\star e_1)\star g_1\cup(b_1\star f_1)\star g_1\cup(c_1\star e_1)\star g_1\cup(a_1\star e_1)\star h_1\cup(b_1\star d_1)\star h_1\cup(b_1\star e_1)\star h_1\cup(b_1\star f_1)\star h_1\cup(c_1\star e_1)\star h_1\cup(a_1\star e_1)\star j_1$
$\cup(b_1\star d_1)\star j_1\cup(b_1\star e_1)\star j_1\cup(b_1\star f_1)\star j_1\cup(c_1\star e_1)\star j_1\cup(a_1\star f_1)\star h_1\cup(c_1\star d_1)\star h_1\cup(c_1\star f_1)\star h_1,$
$r_1\in(a_1\star d_1)\star j_1\cup(a_1\star f_1)\star g_1\cup(c_1\star d_1)\star g_1\cup(c_1\star f_1)\star g_1\cup(a_1\star f_1)\star j_1\cup(c_1\star d_1)\star j_1\cup(c_1\star f_1)\star j_1$
$p_2\in(a_2\star d_2)\star g_2,$
$q_2\in(a_2\star d_2)\star h_2\cup(a_2\star e_2)\star g_2\cup(b_2\star d_2)\star g_2\cup(b_2\star e_2)\star g_2\cup(b_2\star f_2)\star g_2\cup(c_2\star e_2)\star g_2\cup(a_2\star e_2)\star h_2\cup(b_2\star d_2)\star h_2\cup(b_2\star e_2)\star h_2\cup(b_2\star f_2)\star h_2\cup(c_2\star e_2)\star h_2\cup(a_2\star e_2)\star j_2\cup(b_2\star d_2)\star j_2\cup$
$(b_2\star e_2)\star j_2\cup(b_2\star f_2)\star j_2\cup(c_2\star e_2)\star j_2\cup,(a_2\star f_2)\star h_2\cup(c_2\star d_2)\star h_2\cup(c_2\star f_2)\star h_2,$
$r_2\in(a_2\star d_2)\star j_2\cup(a_2\star f_2)\star g_2\cup(c_2\star d_2)\star g_2\cup(c_2\star f_2)\star g_2\cup(a_2\star f_2)\star j_2\cup(c_2\star d_2)\star j_2\cup(c_2\star f_2)\star j_2\}$
$=\{((p_1,q_1I_1,r_1I_2),(p_2,q_2I_1,r_2I_2)):p_1\in a_1\star(d_1\star g_1),$
$q_1\in a_1\star(d_1\star h_1)\cup a_1\star(e_1\star g_1)\cup b_1\star d_1(\star g_1)\cup b_1\star(e_1\star g_1)\cup b_1\star(f_1\star g_1)\cup c_1\star(e_1\star g_1)\cup a_1\star(e_1\star h_1)\cup b_1\star(d_1\star h_1)\cup b_1\star(e_1\star h_1)\cup b_1\star(f_1\star h_1)\cup c_1\star(e_1\star h_1)\cup a_1\star(e_1\star j_1)$
$\cup b_1\star(d_1\star j_1)\cup b_1\star(e_1\star j_1)\cup b_1\star(f_1\star j_1)\cup c_1\star(e_1\star j_1)\cup a_1\star(f_1\star h_1)\cup c_1\star(d_1\star h_1)\cup c_1\star(f_1\star h_1),$
$r_1\in a_1\star(d_1\star j_1)\cup a_1\star(f_1\star g_1)\cup c_1\star(d_1\star g_1)\cup c_1\star(f_1\star g_1)\cup a_1\star(f_1\star j_1)\cup c_1\star(d_1\star j_1)\cup c_1\star(f_1\star j_1)$
$p_2\in a_2\star(d_2\star g_2),$
$q_2\in a_2\star(d_2\star h_2)\cup a_2\star(e_2\star g_2)\cup b_2\star(d_2\star g_2)\cup b_2\star(e_2\star g_2)\cup b_2\star(f_2\star g_2)\cup c_2\star(e_2\star g_2)\cup a_2\star(e_2\star h_2)\cup b_2\star(d_2\star h_2)\cup b_2\star(e_2\star h_2)\cup b_2\star(f_2\star h_2)\cup c_2\star(e_2\star h_2)\cup a_2\star(e_2\star j_2)\cup b_2\star(d_2\star j_2)\cup$
$b_2\star(e_2\star j_2)\cup b_2\star(f_2\star j_2)\cup c_2\star(e_2\star j_2)\cup,a_2\star(f_2\star h_2)\cup c_2\star(d_2\star h_2)\cup c_2\star(f_2\star h_2),$
$r_2\in a_2\star(d_2\star j_2)\cup a_2\star(f_2\star g_2)\cup c_2\star(d_2\star g_2)\cup c_2\star(f_2\star g_2)\cup a_2\star(f_2\star j_2)\cup c_2\star(d_2\star j_2)\cup c_2\star(f_2\star j_2)\}$
$=((a_1,b_1I_1,c_1I_2),(a_2,b_2I_1,c_2I_2))\star\{((u_1,v_1I_1,w_1I_2),(u_2,v_2I_1,w_2I_2)):$
$u_1\in d_1\star g_1,v_1\in d_1\star h_1\cup e_1\star g_1\cup e_1\star h_1\cup e_1\star j_1\cup f_1\star h_1,w_1\in d_1\star j_1\cup f_1\star g_1\cup f_1\star j_1,$
$u_2\in d_2\star g_2,v_2\in d_2\star h_2\cup e_2\star g_2\cup e_2\star h_2\cup e_2\star j_2\cup f_2\star h_2,w_2\in d_2\star j_2\cup f_2\star g_2\cup f_2\star j_2\}$
$=((a_1,b_1I_1,c_1I_2),(a_2,b_2I_1,c_2I_2))\star(((d_1,e_1I_1,f_1I_2),(d_2,e_2I_1f_2I_2))\star((g_1,h_1I_1,j_1I_2),(g_2,h_2I_1j_2I_2)))$
$=(x_1,x_2)\star((y_1,y_2)\star(z_1,z_2)).$
Hence, $(H(I_1,I_2)\times K(I_1,I_2),\star)$ is a refined neutrosophic semi-hypergroup.
Lastly, let $(x_1,x_2)\in H(I_1,I_2)\times K(I_1,I_2)$ then
$((a_1,b_1I_1,c_1I_2),(a_2,b_2I_1,c_2I_2))\star(H(I_1,I_2)\times K(I_1,I_2))$
$=\{((a_1,b_1I_1,c_1I_2),(a_2,b_2I_1,c_2I_2))\star((d_1,e_1I_1,e_2I_2),(d_2,e_2I_1,f_2I_2)):(d_1,e_1I_1,e_2I_2)\in H(I_1,I_2),$
$(d_2,e_2I_1,f_2I_2)\in K(I_1,I_2)\}$
$=\{(a_1\star d_1,(a_1\star e_1\cup b_1\star d_1\cup b_1\star e_1\cup b_1\star f_1\cup c_1\star e_1)I_1,(a_1\star f_1\cup c_1\star d_1\cup c_1\star f_1)I_2,$
$(a_2\star d_2,(a_2\star e_2\cup b_2\star d_2\cup b_2\star e_2\cup b_2\star f_2\cup c_2\star e_2)I_1(a_2\star f_2\cup c_2\star d_2\cup c_2\star f_2)I_2\}$

$= (a_1, b_1I_1, c_1I_2) \star H(I_1, I_2), (a_2, b_2I_1, c_2I_2) \star K(I_1, I_2)$
$= H(I_1, I_2)\star(a_1, b_1I_1, c_1I_2), K(I_1, I_2)\star(a_2, b_2I_1, c_2I_2)$ since $H(I_1, I_2)$ and $K(I_1, I_2)$ are hypergroups.
$= H(I_1, I_2) \times K(I_1, I_2)$.
Hence $(H(I_1, I_2)) \times K(I_1, I_2), \star)$ is a refined neutrosophic quasi hypergroup.
Then we can conclude that $(H(I_1, I_2)) \times K(I_1, I_2), \star)$ is a refined neutrososphic hypergroup. □

Proposition 2.12. *Let $(H(I_1, I_2), \star)$ be a refined neutrosophic hypergroup and let (K, o) be a hypergroup. Then, $(H(I_1, I_2) \times K, \star')$ is a refined neutrosophic hypergroup, where*

$$(h_1, k_1) \star' (h_2, k_2) = \{(h, k) : h \in h_1 \star' h_2, k \in k_1 \circ k_2, \forall (h_1, k_1), (h_2, k_2) \in N(H) \times K\}.$$

Proof : It follows from similar approach to the proof of Proposition 2.11 .

Proposition 2.13. *Let $(H(I_1, I_2), \star)$ be a refined neutrosophic hypergroup, then for all elements of $H(I_1, I_2)$ no two elements combine to give empty set.*

Proof. Let $(a, bI_1, cI_2), (x, yI_1, zI_2) \in H(I_1, I_2)$. Suppose $(a, bI_1, I_2) \star (x, yI_1, zI_2) = \emptyset$.
Then since $H(I_1, I_2)$ is a neutrosophic hypergroup, by reproduction axiom we have

$$\begin{aligned} H(I_1, I_2) &= (a, bI_1, cI_2) \star H(I_1, I_2) \\ &= (a, bI_1, cI_2) \star ((x, yI_1, I_2) \star H(I_1, I_2)) \\ &= ((a, bI_1, cI_2) \star (x, yI_1, I_2)) \star H(I_1, I_2) \\ &= \emptyset \star H(I_1, I_2) \\ &= \emptyset. \end{aligned}$$

This is absurd, hence there exist no two elements of $H(I_1, I_2)$ that combine to give empty set. □

Definition 2.14. Let $H(I_1, I_2)$ be a refined neutrosophic hypergroup and let $K[I_1, I_2]$ be a proper subset of $H(I_1, I_2)$. Then $K[I_1, I_2]$ is said to be a refined neutrosophic semi-subhypergroup of $H(I_1, I_2)$ if $x \star y \subseteq K[I_1, I_2]$ for all $x, y \in K[I_1, I_2]$.

Definition 2.15. Let $H(I_1, I_2)$ be a refined neutrosophic hypergroup and let $K[I_1, I_2]$ be a proper subset of $H(I_1, I_2)$. Then

1. $K[I_1, I_2]$ is said to be a refined neutrosophic subhypergroup of $H(I_1, I_2)$ if $K[I_1, I_2]$ is a refined neutrosophic hypergroup, that is, $K[I_1, I_2]$ must contain a proper subset which is a hypergroup.
2. $K[I_1, I_2]$ is said to be a refined pseudo neutrosophic subhypergroup of $H(I_1, I_2)$ if $K[I_1, I_2]$ is a refined neutrosophic hypergroup which contains no proper subset which is a hypergroup.

Note 2. *A refined neutrosophic hypergroup is a much more complicated structure than the structure of a refined neutrosophic group. In a refined neutrosophic group, the intersection of any two refined neutrosophic subgroups is a refined neutrosophic subgroup, this is not usually so in the case of a refined neutrosophic hypergroups, since the reproductive axioms fails to hold in this case. This has led to the consideration of different kinds of refined neutrosophic subhypergroups, which are ; Closed, Ultraclosed and Conjugable.*

Proposition 2.16. *Let $M(I_1, I_2)$ and $N(I_1, I_2)$ be any refined neutrosophic subhypergroups of a refined neutrososphic hypergroup $H(I_1, I_2)$, then $M(I_1, I_2) \cap N(I_1, I_2)$ is a refined neutrosophic semi-subhypergroup.*

Proof. $M(I_1, I_2) \cap N(I_1, I_2) \neq \emptyset$, since $M(I_1, I_2)$ and $N(I_1, I_2)$ are non-empty subhypergroups of $H(I_1, I_2)$.
Now, let $x = (a_1, b_1I_1, c_1I_2), y = (a_2, b_2I_1, c_2I_2) \in M(I_1, I_2) \cap N(I_1, I_2)$.
Then $(a_1, b_1I_1, c_1I_2), (a_2, b_2I_1, c_2I_2) \in M(I_1, I_2)$ and $(a_1, b_1I_1, c_1I_2), (a_2, b_2I_1, c_2I_2) \in N(I_1, I_2)$. Since, $M(I_1, I_2)$ and $N(I_1, I_2)$ are refined neutrosophic subhypergroup, we have that
$(a_1, b_1I_1, c_1I_2) \star (a_2, b_2I_1, c_2I_2) \subseteq M(I_1, I_2)$ and
$(a_1, b_1I_1, c_1I_2)\star(a_2, b_2I_1, c_2I_2) \subseteq N(I_1, I_2) \Longrightarrow (a_1, b_1I_1, c_1I_2)\star(a_2, b_2I_1, c_2I_2) \subseteq M(I_1, I_2) \cap N(I_1, I_2)$.
Hence, $M(I_1, I_2) \cap N(I_1, I_2)$ is a refined neutrosophic semi-subhypergroup. □

Proposition 2.17. *Let $M(I_1, I_2)$ and $N(I_1, I_2)$ be any refined neutrosophic semi-subhypergroups of a refined neutrososphic commutative hypergroup $H(I_1, I_2)$, then the set*

$$M(I_1, I_2)N(I_1, I_2) = \{xy : x \in M(I_1, I_2), y \in N(I_1, I_2)\}$$

is a refined neutrosophic semi-subhypergroup of $H(I_1, I_2)$.

Definition 2.18. Let $K(I_1, I_2)$ be a refined neutrosophic subhypergroup of a refined neutrosophic hypergroup $(H(I_1, I_2), \star)$. Then,

1. $K(I_1, I_2)$ is said to be closed on the left (right) if for all $k_1, k_2 \in K(I_1, I_2), x \in H(I_1, I_2)$ we have $k_2 \in x \star k_1 (k_2 \in k_1 \star x)$ implies that $x \in K(I_1, I_2)$;
2. $K(I_1, I_2)$ is said to be ultraclosed on the left (right) if for all $x \in H(I_1, I_2)$ we have $x \star K(I_1, I_2) \cap x \star (H(I_1, I_2) \backslash K(I_1, I_2)) = \emptyset \;\; (K(I_1, I_2) \star x \cap (H(I_1, I_2) \backslash K(I_1, I_2)) \star x = \emptyset)$;
3. $K(I_1, I_2)$ is said to be left (right) conjugable if $K(I_1, I_2)$ is left (right) closed and if for all $x \in H(I_1, I_2)$, there exists $h \in H(I_1, I_2)$ such that $x \star h \subseteq K(I_1, I_2) \;\; (h \star x \subseteq K(I_1, I_2))$;
4. $K(I_1, I_2)$ is said to be (closed, ultraclosed, conjugable) if it is left and right (closed, ultraclosed, conjugable).

Proposition 2.19. *Let $K[I_1, I_2]$ be a refined neutrosophic subhypergroup of $H(I_1, I_2)$, $A[I_1, I_2] \subseteq K[I_1, I_2]$ and $B[I_1, I_2] \subseteq H(I_1, I_2)$, then*

1. $A[I_1, I_2](B[I_1, I_2] \cap K[I_1, I_2]) \subseteq A[I_1, I_2]B[I_1, I_2] \cap K[I_1, I_2]$ *and*
2. $(B[I_1, I_2] \cap K[I_1, I_2])A[I_1, I_2] \subseteq B[I_1, I_2]A[I_1, I_2] \cap K[I_1, I_2]$.

Proof. The proof is similar to the proof in classical case. □

Proposition 2.20.
1. *If $K[I_1, I_2]$ is a left closed refined neutrososphic subhypergroup in $H[I_1, I_2]$, $A[I_1, I_2] \subseteq K[I_1, I_2]$ and $B[I_1, I_2] \subseteq H[I_1, I_2]$, then $(B[I_1, I_2] \cap K[I_1, I_2])/A[I_1, I_2] = (B[I_1, I_2]/A[I_1, I_2]) \cap K[I_1, I_2]$.*
2. *If $K[I_1, I_2]$ is a right closed subhypergroup in $H[I_1, I_2]$, $A[I_1, I_2] \subseteq K[I_1, I_2]$ and $B[I_1, I_2] \subseteq H[I_1, I_2]$, then $(B[I_1, I_2] \cap K[I_1, I_2]) \backslash A[I_1, I_2] = B[I_1, I_2] \backslash A[I_1, I_2] \cap K[I_1, I_2]$.*

Proof. The proof is similar to the proof in classical case. □

Proposition 2.21. *Let $K[I_1, I_2], M[I_1, I_2]$ be two refined neutrosophic subhypergroups of a refined neutrosophic hypergroup $H[I_1, I_2]$ and suppose that $K[I_1, I_2]$ is left (or right) closed in $H[I_1, I_2]$.
Then $K[I_1, I_2] \cap M[I_1, I_2]$ is left (or right) closed in $M[I_1, I_2]$.*

Proof. The proof is similar to the proof in classical case. □

Proposition 2.22. *Let $(H(I_1, I_2)), \star)$ be a refined neutrosophic hypergroup and let ρ be an equivalence relation on $H(I_1, I_2)$.*

1. *If ρ is regular, then $H(I_1, I_2)/\rho$ is a refined neutrosophic hypergroup.*
2. *If ρ is strongly regular, then $H(I_1, I_2)/\rho$ is a refined neutrosophic group.*

The proposition will be proved with the example provided below.

Example 2.23. If $(G(I_1, I_2), +)$ is a refined neutrosophic abelian hypergroup, ρ is an equivalence relation in $G(I_1, I_2)$, which has classes $\bar{x} = \{x, -x\}$, then for all $\bar{x}, \bar{y}$ of $G(I_1, I_2)/\rho$, we define

$$\bar{x} o \bar{y} = \{\overline{x+y}, \overline{x-y}\}.$$

Then $(G(I_1, I_2)/\rho, o)$ is a refined neutrosophic hypergroup.

Proof. Let $\bar{x}, \bar{y} \in G(I_1, I_2)/\rho$, where $\bar{x} = \overline{(a, bI_1, cI_2)}$, and $\bar{y} = \overline{(d, eI_1, fI_2)}$ then
$\bar{x} o \bar{y} = \overline{(a, bI_1, cI_2)} o \overline{(d, eI_1, fI_2)} = \{\overline{(a+d, (b+e)I_1, (c+f)I_2)}, \overline{(a-d, (b-e)I_1, (c-f)I_2)}\}$
$= \{\overline{x+y}, \overline{x-y}\} \in G(I_1, I_2)/\rho$.
Then $(G(I_1, I_2)/\rho, o)$ is a refined neutrosophic hypergroupoid.
Next we show that o satisfies the associative law. Let $\bar{x}, \bar{y}, \bar{z} \in G(I_1, I_2)/\rho$, where $\bar{x} = \overline{(a, bI_1, cI_2)}$, $\bar{y} = \overline{(d, eI_1, fI_2)}$ and $\bar{z} = \overline{(g, hI_1, jI_2)}$ then
$\bar{x} o (\bar{y} o \bar{z}) = \overline{(a, bI_1, cI_2)} o \left(\overline{(d, eI_1, fI_2)} o \overline{(g, hI_1, jI_2)}\right)$
$= \overline{(a, bI_1, cI_2)} o \left\{\overline{(d+g, (e+h)I_1, (f+j)I_2)}, \;\; \overline{(d-g, (e-h)I_1, (f-j)I_2)}\right\}$
$= \left\{\overline{(a, bI_1, cI_2)} o \overline{(d+g, (e+h)I_1, (f+j)I_2)}, \;\; \overline{(a, bI_1, cI_2)} o \overline{(d-g, (e-h)I_1, (f-j)I_2)}\right\}$

$= \Big\{\{\overline{(a+(d+g),(b+(e+h))I_1,(c+(f+j))I_2)}, \overline{(a-(d+g),(b-(e+h))I_1,(c-(f+j))I_2)}\},$
$\{\overline{(a+(d-g),(b+(e-h))I_1,(c+(f-j))I_2)}, \overline{(a-(d-g),(b-(e-h))I_1,(c-(f-j))I_2)}\}\Big\}$
$= \bigcup\Big\{\overline{(a+(d+g),(b+(e+h))I_1,(c+(f+j))I_2)}, \overline{(a-(d+g),(b-(e+h))I_1,(c-(f+j))I_2)},$
$\overline{(a+(d-g),(b+(e-h))I_1,(c+(f-j))I_2)}, \overline{(a-(d-g),(b-(e-h))I_1,(c-(f-j))I_2)}\Big\}$
$= \bigcup\Big\{\overline{(a+(d+g),(b+(e+h))I_1,(c+(f+j))I_2)}, \overline{(a+(d-g),(b+(e-h))I_1,(c+(f-j))I_2)},$
$\overline{(a-(d-g),(b-(e-h))I_1,(c-(f-j))I_2)}, \overline{(a-(d+g),(b-(e+h))I_1,(c-(f+j))I_2)}\Big\}$
$= \bigcup\Big\{\overline{((a+d)+g,((b+e)+h)I_1,((c+f)+j)I_2)}, \overline{((a+d)-g,((b+e)-h)I_1,((c+f)-j)I_2)}$
$\overline{((a-d)+g,((b-e)+h)I_1,((c-f)+j)I_2)}\ , \overline{((a-d)-g,((b-e)-h)I_1,((c-f)-j)I_2)}\Big\}$
$= \Big\{\overline{((a+d),(b+e)I_1,(c+f)I_2)}o\overline{(g,hI_1,jI_2)}, \overline{((a-d),(b-e)I_1,(c-f)I_2)}o\overline{(g,hI_1,jI_2)}\Big\}$
$= \Big\{\overline{((a+d),(b+e)I_1,(c+f)I_2)}, \overline{((a-d),(b-e)I_1,(c-f)I_2)}\Big\}\ o\overline{(g,hI_1,jI_2)}$
$= \Big(\overline{(a,bI_1,cI_2)}o\overline{(d,eI_1,fI_2)}\Big)\ o\overline{(g,hI_1,jI_2)}$
$= (\bar{x}o\bar{y})o\bar{z}.$
Now we show that o satisfies the reproduction axiom. Let $\bar{x} \in G(I_1,I_2)/\rho$ then
$\overline{(a,bI_1,cI_2)}oG(I_1,I_2)/\rho = \{\overline{(a,bI_1,cI_2)}o\{\overline{(d,eI_1,fI_2)} : (d,eI_1,fI_2) \in G(I_1,I_2)\}$
$= \{\overline{(a,bI_1,cI_2)}o\overline{(d,eI_1,fI_2)} : (a,bI_1,cI_2),(d,eI_1,fI_2) \in G(I_1,I_2)\}$
$= \{\overline{((a+d),(b+e)I_1,(c+f)I_2)}, \overline{((a-d),(b-e)I_1,(c-f)I_2)}\}$
$= \{((a+d),(b+e)I_1,(c+f)I_2), -((a+d),(b+e)I_1,(c+f)I_2),$
$((a-d),(b-e)I_1,(c-f)I_2), -((a-d),(b-e)I_1,(c-f)I_2)\}$
$= \{((d+a),(e+b)I_1,(f+e)I_2), -((d+a),(e+b)I_1,(f+e)I_2),$
$-((d-a),(e-b)I_1,(f-c)I_2), ((d-a),(e-b)I_1,(f-c)I_2)\}$
$= \{((d+a),(e+b)I_1,(f+e)I_2), -((d+a),(e+b)I_1,(f+e)I_2),$
$\underline{((d-a),(e-b)I_1,(f-c)I_2), -((d-a),(e-b)I_1,(f-c)I_2)\}}$
$= \{\overline{((d+a),(e+b)I_1,(f+e)I_2)}, \overline{((d-a),(e-b)I_1,(f-c)I_2)}\}$
$= \{\overline{(d,eI_1,fI_2)}o\overline{(a,bI_1,cI_2)}\}$
$= \{\overline{(d,eI_1,fI_2)}o\overline{(a,bI_1,cI_2)} : (d,eI_1,fI_2) \in G(I_1,I_2)\}$
$= G(I_1,I_2)/\rho \ \ o \ \ \overline{(a,bI_1,cI_2)}$
$= G(I_1,I_2)/\rho.$
Hence we say that $(G(I_1,I_2)/\rho, o)$ is a refined neutrosophic hypergroup. □

Definition 2.24. Let $(H_1(I_1,I_2),\star_1)$ and $(H_2(I_1,I_2)),\star_2)$ be any two refined neutrosophic hypergroups and let $f : H_1(I_1,I_2) \longrightarrow H_2)(I_1,I_2)$ be a map. Then

1. f is called a refined neutrosophic homomorphism if:
   (a) for all x,y of $H_1(I_1,I_2)$, $f(x \star_1 y) \subseteq f(x) \star_2 f(y)$,
   (b) $f(I_k) = I_k$ for $\quad k = 1,2$.
2. f is called a good refined neutrosophic homomorphism if:
   (a) for all x,y of $H_1(I_1,I_2)$, $f(x \star_1 y) = f(x) \star_2 f(y)$,
   (b) $f(I_k) = I_k$ for $\quad k = 1,2$.
3. f is called a refined neutrosophic isomorphism if f is a refined neutrosophic homomorphism and f^{-1} is also a refined neutrosophic homomorphism.
4. f is called a 2-refined neutrosophic homomorphism if for all x,y of $H_1(I_1,I_2)$,
   $f^{-1}(f(x) \star_2 f(y)) = f^{-1}(f(x \star_1 y))$.
5. f is called an almost strong refined neutrosophic homomorphism if for all x,y of $H_1(I_1,I_2)$,
   $f^{-1}(f(x) \star_2 f(y)) = f^{-1}(f(x)) \star_1 f^{-1}(f(y))$.

Proposition 2.25. *Let $(H(I_1,I_2)),\star)$ be a refined neutrosophic hypergroup and let ρ be a regular equivalence relation on $H(I_1,I_2)$. Then, the map $\phi : H(I_1,I_2) \longrightarrow H(I_1,I_2)/\rho$ defined by*
$\phi(x) = \bar{x}$ is not a refined neutrosophic homomorphism (good refined neutrosophic homomorphism).

Proof. It is clear since $I \in H(I_1, I_2)$ but $\phi(I_k) \neq I_k$. □

Note 3. *Suppose we wish to establish any relationship between the refined neutrosophic hypergroups and the parent neutrosophic hypergroups, or any other neutrosophic hypergroup. Then, our task will be to find a mapping ϕ say, such that*

$$\phi : H(I_1, I_2) \longrightarrow H(I).$$

For all $(x, yI_1, zI_2) \in H(I_1, I_2)$ define ϕ by

$$\phi((x, yI_1, zI_2)) = (x, (y + z)I). \tag{1}$$

In what follows we present some of the basic properties of such mapping.

Proposition 2.26. *Let $(H(I_1, I_2), +')$ be a refined neutrosophic hypergroup and let $(H(I), +)$ be a neutrosophic hypergroup. The mapping ϕ defined in 1 above is a good homomorphism.*

Proof. ϕ is well defined. Suppose $(x, yI_1, zI_2) = (x'y'I_1, z'I_2)$ then we that $x = x', y = y'$ and $z' = z'$. So,

$$\phi((x, yI_1, zI_2)) = (x, (y + z)I) = x' + (y' + z')I = \phi(x', y'I_1, z'I_2).$$

Now, suppose $(x, yI_1, zI_2), (x', y'I_1, 1z'I_2) \in H(I_1, I_2)$ then

$$\begin{aligned}
\phi((x, yI_1, zI_2) +' (x', y'I_1, z'I_2)) &= \phi((x + x'), (y + y')I_1, (z + z')I_2) \\
&= (x + x'), (y + y' + z + z')I \\
&= (x + x'), ((y + z) + (y' + z'))I \\
&= (x + x'), ((y + z)I + (y' + z')I) \\
&= (x, (y + z)I) + (x', (y' + z')I) \\
&= \phi(x, yI_1, zI_2) + \phi(x, yI_1, zI_2).
\end{aligned}$$

Hence ϕ is a good homomorphism. □

Definition 2.27. Let $(H(I_1, I_2), +')$ be a refined neutrosophic hypergroup with identity element $(0, 0I_1, 0I_2)$ and $(H(I_1, I_2), +)$ be a neutrosophic hypergroup with identity element $(0, 0I)$. Let $\phi : H(I_1, I_2) \longrightarrow H(I)$ be a good homomorphism, then

$$\begin{aligned}
\ker\phi &= \{(x, yI_1, zI_2) : \phi((x, yI_1, zI_2)) = (0, 0I)\} \\
&= \{(x, yI_1, zI_2) : (x, (y + z)I) = (0, 0I)\} \\
&= \{(0, yI_1, (-y)I_2)\}.
\end{aligned}$$

Proposition 2.28. *Let $\phi : H(I_1, I_2) \longrightarrow H(I)$ be a good homomorphism.*

1. $\ker\phi$ *is a semi-subhypergroup of* $H(I_1, I_2)$.
2. $Im\phi$ *is a subhypergroup of* $H(I)$.

Proof. 1. Let $(a, bI_1, cI_2), (x, yI_1, zI_2) \in ker\ \phi$, then

$$\begin{aligned}
\phi((a, bI_1, cI_2) +' (x, yI_1, zI_2)) &= \phi((a, bI_1, cI_2)) + \phi((x, yI_1, zI_2)) \\
&= (0, 0I) + (0, 0I) \\
&= (0, 0I) \\
&\Longrightarrow (a, bI_1, cI_1) +' (x, yI_1, zI_2) \subseteq ker\phi.
\end{aligned}$$

Hence, $ker\phi$ is a semi-subhypergroup.

2. Let $(a, bI_1, cI_2) \in H(I_1, I_2)$, then

$$\begin{aligned}
\phi((a, bI_1, cI_2)) + \phi(H(I_1, I_2)) &= \textstyle\bigcup_{(x, yI_1, zI_2) \in H(I_1, I_2)} \phi((a, bI_1, cI_2) +' (x, yI_1, zI_2)) \\
&= \phi((a, bI_1, cI_2) +' H(I_1, I_2)) \\
&= \phi(H(I_1, I_2)).
\end{aligned}$$

Following similar approach we can show that $\phi(H(I_1, I_2)) + \phi((a, bI_1, cI_2)) = \phi(H(I_1, I_2))$.
Thus, $Im\phi$ is a subhypergroup of $H(I)$.

□

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3 Conclusion

In this paper, we have studied the refinement of neutrosophic hyperstructures. In particular, we have studied refined neutrosophic hypergroups and presented several results and examples. Also, we have established the existence of a good homomorphism between a refined neutrosophic hypergroup $H(I_1, I_2)$ and a neutrosophic hypergroup $H(I)$. We hope to present and study more advance properties of refined neutrosophic Hypergroups in our future papers.

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References

[1] Adeleke, E.O, Agboola, A.A.A and Smarandache, F. Refined Neutrosophic Rings I, International Journal of Neutrosophic Science (IJNS), Vol. 2(2), pp. 77-81, 2020.

[2] Adeleke, E.O, Agboola, A.A.A and Smarandache, F. Refined Neutrosophic Rings II, International Journal of Neutrosophic Science (IJNS), Vol. 2(2), pp. 89-94, 2020.

[3] Agboola, A.A.A., Ibrahim, A.M. and Adeleke, E.O, Elementary Examination of NeutroAlgebras and AntiAlgebras Viz-a-Viz the Classical Number Systems, Vol. 4, pp. 16-19, 2020.

[4] Agboola, A.A.A. On Refined Neutrosophic Algebraic Structures, Neutrosophic Sets and Systems 10, pp 99-101, 2015.

[5] Agboola, A.A.A. and Akinleye, S.A., Neutrosophic Hypervector Spaces, ROMAI Journal, Vol.11 , pp. 1-16, 2015.

[6] Agboola, A.A.A and Davvaz, B., On Neutrosophic Canonical Hypergroups and Neutrosophic Hyperrings, Neutrosophic Sets and Systems. Vol. 2, pp. 34-41, 2014.

[7] Agboola, A.A.A and Davvaz, B., Introduction to Neutrosophic Hypergroups, ROMAI J., Vol. 9, no.2, pp. 1-10, 2013.

[8] Agboola,A.A.A., Davvaz, B. and Smarandache,F. Neutrosophic quadruple algebraic hyperstructures, Ann. Fuzzy Math.Inform. Vol. 14, pp. 29-42, 2017.

[9] Al-Tahan, M. and Davvaz, B., Refined neutrosophic quadruple (po-)hypergroups and their fundamental group, Neutrosophic Sets and Systems, Vol. 27, pp. 138-153, 2019.

[10] Atanassov, K., Intuitionistic fuzzy sets, Fuzzy Sets and Systems, Vol. 20, pp. 87-96, 1986.

[11] Davvaz, B. and Leoreanu-Fotea,V., Hyperring Theory and Applications, International Academic Press, Palm Harber, USA, 2007.

[12] Ibrahim, M.A.,Agboola, A.A.A, Adeleke, E.O, Akinleye, S.A., Introduction to Neutrosophic Subtraction Algebra and Neutrosophic Subtraction Semigroup,International Journal of Neutrosophic Science (IJNS), Vol. 2(2), pp. 47-62, 2020.

[13] Marty, F. Role de la notion dhypergroupe dans letude des groupes non abeliens, Comptes Renclus Acad. Sci. Paris Math, Vol. 201, pp 636-638, 1935.

[14] Massouros, C., Some properties of certain Subhypergroups, RATIO MATHEMATICA Vol.25, pp.67-76, 2013.

[15] Smarandache, F., A Unifying Field in Logics: Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability, American Research Press, Rehoboth, 2003.

[16] Smarandache, F., n-Valued Refined Neutrosophic Logic and Its Applications in Physics, Progress in Physics, Vol. 4, pp. 143-146, 2013.

[17] Smarandache, F., (T,I,F)- Neutrosophic Structures, Neutrosophic Sets and Systems,Vol. 8 , pp.3-10, 2015.

[18] Vasantha Kandasamy W.B and Smarandache, F., Basic Neutrosophic Algebraic Structures and Their Applications to Fuzzy and Neutrosophic Models, Hexis, Church Rock,(2004), http://fs.unm.edu/ScienceLibrary.htm

[19] Vasantha Kandasamy, W.B. and Smarandache, F., Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic Structures, Hexis, Phoenix, Arizona, (2006), http://fs.unm.edu/ScienceLibrary.htm

[20] Vasantha Kandasamy, W.B., Neutrosophic Rings, Hexis, Phoenix, Arizona,(2006) http://fs.unm.edu/ScienceLibrary.htm

[21] Zadeh, L.A., Fuzzy Sets, Information and Control, Vol. 8, pp. 338-353, 1965.

The Natural Bases of Neutrosophy

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Abstract

Neutrosophy began as a branch of philosophy that considered neutrality in addition to the positive and negative. It consists of the addition consideration of a neutral state to complement the binary approach of true or false. Its creator quickly extended it to the field of mathematics and it was gradually applied to all fields of science. Here, we present a reverse approach that highlights the importance of neutrality in all fields of study and application, citing some revealing examples. Furthermore, we explain that this importance of neutrality is intrinsic to all sciences because it is based on natural foundations. Indeed, neutrality is a forming part first of all of the human conception of things, of our way of thinking, of cognition in general but also of living things, matter and even particles. In addition to these most real-world physical concrete aspects, neutrality is inherent to mathematics, to logic first of all, but also to probabilities and statistics where neutrality which simply results from a large number of objects, the universe. Thus neutrosophy is well adapted to the majority of applied problems because its modeling is inspired by reality and that it allows, in particular, to deal with the component of uncertainty and indeterminacy that the real world comprises intrinsically.

Keywords: Neutrosophy, three-state , neutral state , undetermined , incertitude , natural basis.

1-Introduction

We will first point out that neutrosophy [1] rests on several bases that are natural: these bases result directly from the temporal aspect of the real world. Then, in the second part, we will describe some examples of situations that are intrinsically three-state, and which are therefore areas where the neutrosophic approach is essential.

Neutrosophy is a modeling based on three states and not just two as in classical logic. In addition to the true and false states, which define the classical logic known as Aristotelian [2] or Cartesian (according to Descartes, [3]), neutrosophy introduces a third state: the neutral state (we will see below that it also represents indetermination). This neutral state gives its name to neutrosophy; it extends the dialectic [4] of the the positive and negative by also

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considering neutrality. Neutrosophy is originally a branch of philosophy [5], which was introduced into the various fields of science, especially mathematics with first logic, then sets, [6] probabilities, and statistics [7,13-14], etc.

2.These natural bases derive from the temporality of the real world

Neutrosophy is closely related to reality, the material world, because it is only a reflective representation of reality in its many natural bases, and this is due to the temporal aspect of the world (the most fundamental one). Thus temporality introduces the possibility of change in addition to the constancy that characterizes timelessness. If change is possible, if it can exist, then constancy can also exist. Absence of change (no-change) is often observable at a certain scale and in a certain time interval. In a temporal view of the world, neutrality is this aspect of constancy, of non-change. In that case of constancy, the change is then neutral, there is no change, neither in one direction nor in the opposite. Therefore, neither positive nor negative, it is the neutral state. In neutrosophy this third state can also be seen as indeterminacy, it then also serves to model phenomena that are not perfectly determined or known.

We will come back to this aspect of indeterminacy linked to neutrality, which results from temporality. Before doing so, we wish to mention the other fundamental bases that imply neutrality, and therefore the need to consider this concept. In physical reality, any instantaneous situation is characterized by a set of real values that measure it quantitatively: they are not discrete states, although quantum mechanics has discovered this, but in a first approximation and in a more macroscopic way, continuous values, continuously variable, without discontinuous jumps. In this simplified paradigm, which is that of the usual sciences, the instantaneous value of a descriptive parameter is expressed as a real number. This value can be changing or approximately stable during a time interval. Although everything is constantly changing, at least infinitely, we can consider as a first approximation that there are sometimes more or less long moments of stability.

Mathematically it is customary to model according to the simplest approach by a polynomial representation and from this representation has derived differential and integral mathematics (8). From a temporal signal represented by a polynomial, one can calculate its variation: this is the derivative, i.e. the difference (and therefore the change) it presents in a time interval. Dually, from a derivative we can reconstitute the signal, it is the integration. When the derivative is approximately zero then the signal is approximately constant. The neutral derivative implies the absence of variation of the temporal signal. Accessorily a positive derivative represents an increase of the signal and a negative derivative corresponds to a decrease.

This mathematical basis comes from a physical basis: the notion of the trajectory (in mechanics, 9). Considering a particle, its velocity is the (first) derivative of its temporal position, and so on, its acceleration is its second derivative, and the curvature of the trajectory is linked to it. In the absence of an external force causing an acceleration of the particle, its speed is unchanged (in value and direction), and its trajectory is reduced to its simplest expression: a straight, rectilinear trajectory. This is a physical reality that corresponds to a neutrosophic vision: positive, zero, or negative tangent force corresponding to acceleration, neutrality, deceleration, and also a perpendicular force producing a positive curbature, a straight line, a negative curvature of the trajectory.

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Above all, neutrosophy is a model adapted to human thought (more qualitative than quantitative), which most crudely way will perceive the parameters a situation either as constant, changing in one direction (increasing, positive), or changing in the other direction (decreasing, negative).

Finally, in simple arithmetic, we consider integers. In this discontinuous case the situation can also change or not. For example, to a collection of 5 objects we can add 1, and then we have 6, or on the contrary remove 1 and there are 4 left, or finally be neutral, not acting so that the number does not change (the Brownian motion of the thermal agitation of the molecules gave rise to the theoretical development of Markov chain [12], where the simplest case consists precisely in a variation at each step of -1, 0 or +1, see reference [10]).

In connection with the simple model inspired by human perception of the evolution of a three-state situation, a three-state control can be defined: increment, hold constant or decrement. For example, it was the control lever of the first elevators that either turned on the motor in one direction or the other, or stopped it. This type of control can be found in many devices because it is so intuitive and efficient, like the joysticks in video games. Although only two push-buttons are needed to perform this function, in a human interface design a three-state toggle lever is often preferred as it is more in line with the human three-state apprehension.

3-Transition to situations characterized by 3 states

The real world includes a large number of situations that are characterized by 3 states, and these are therefore advantageously modeled by neutrosophy because of its design around the positive, neutral and negative states.

The bases that we have seen in the previous section are derived from the temporal essence of the physical world, therefore they are characterized by its change but also by the possibility of constancy at some scale of observation and for some duration.

Another fundamental aspect of the physical world is its plurality, at least apparent, it is made up of several objects, together with linked by forces of interaction. This number of objects is large, so a statistical approach can be envisaged. The states of objects, especially those that are similar or close, are often governed by statistical distribution laws, such as the so-called precisely normal (or Gauss') law. In an interpretation of a world seen from a statistical perspective one can then classify the objects in the first approximation among 3 main categories: those whose situation is characterized by a value close to the statistical mean of all these objects, those below the mean and those above the mean. Thus the statistical modeling at its coarsest level is consistent with the simplified human perception of the three states: similar to the average, less and more than the average.
In this way, the physical world as seen through human eyes at the most immediate level is organized into three categories: (in vincinity of the) average, above average, and below average. This basic perception can be applied to any observation. Typically in sociology, one of the basic criteria of (socio-professional) classification is to consider 3 levels of wealth: the middle, upper and lower classes.

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4-Situations characterized by 3 states

Now we can give some examples of situations intrinsically characterized by 3 states, some of which stem from, or are similar to, this elementary statistical vision.

4.1. Perception : temperature

As an example of human-related perception and thus also of this simplistic and qualitative assessment of a reference value, we would like to start with temperature. Human beings generally consider several different situations concerning their body temperature: room (ambient) temperature, food temperature, and water temperature for bathing. These cases come from whether we consider it to be a pleasant temperature or not, and if not, whether the temperature is too hot or too cold. The first case that corresponds to our comfort zone is perceived as neutral. For example, our body temperature is 37°C, and we find the room temperature pleasant when it is between 19 and 22 degrees.

The appreciation of the temperature can vary according to the circumstances, for example the ideal temperature while standing still (21 degrees Celsius) is not the same as for physical activity (18°C), nor is the ideal temperature for a cool drink (8°C) or on the contrary for warming up (50°C), but with more difference. The pleasant, or neutral zone is generally relatively narrow compared to the zones that appear to us to be either too little or too much (before reaching the pain zones). This remark can be made in many situations that we will present as intrinsically three-state, which probably led to the oversimplified two-state representation which then gradually imposed itself as the only one that can be thought of, for example day or night, while there is also twilight which can be quite long (if we are located far from the equator).

4.2.Chemistry: acidity

Also related to life and the conditions it imposes, inorganic chemistry we have the measurement of the pH (hydrogen potential, hydrogen ion concentration) of a solution and its representation in 3 classes, neutral, acidic and basic. Here this is due to the primordial role of water in life, and pH 0, therefore neutral, is defined as that of water.

4.3.Linearization at the working point

Many phenomena in the real world are non-linear, however for simplification we wish to use a linear approximation to make approximate calculations easily. For example, a transistor working with small signals will have its transfer characteristic approximated by a straight line tangent (to the response characteristic curve) at the bias point (operating point).

For slightly larger signals, three linearly approximated zones are then considered: a zone around the operating point with a certain range, a zone below the range and another zone above it.

4.4. Neurons

Another example of a non-linear situation is the neuron of the nervous system or its artificial equivalent, as used in computer science for artificial intelligence, which exhibits strong non-linearity due to its saturation characteristic.

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This characteristic stems from its learning function which proceeds by reinforcement, and symmetrically inhibition (in case of good results, respectively bad results). We will then use a modeling by three zones: lower (negative saturation), central or neutral (transmission), and upper (positive saturation). Reinforcement as a learning method consists in treating the difference between what the neuron produces as an output value from its inputs and what it should do in this situation. If the difference is small, then the neuron has already learned well and only a small adaptation is required, this is the central, linear, and relatively neutral zone. On the other hand if the difference is large either in the right, or in the opposite, directions then the retroaction proper to the reinforcement should be stronger in positive or negative, but easily saturated, i.e. be almost the same in all cases, whether the error is medium or large. Here saturation consists of limiting the correction in case of large deviations in order to progress towards the result in small steps for a large number of successful successive examples, so as not to give too much importance to a single example, to obtain generalized learning of the examples treated.

4.5.Human Conceptions of the World

The real world often consists of 3 categories, or the human being perceives it for a particular area as consisting of objects of three main types.

For example, our capacity for consciousness implies a conception of temporality. We distinguish between the present in which we are and act, the future which is die to come and that we wish to influence by our present actions, and the past which we no longer have control over but that is valuable as information, reusable knowledge to determine our actions. Here the present is only a small thing in the infinite time of both past and future, however it is considered as a specific category. A binary vision of only the existence of a past and the arrival of a future would deprive us of any awareness of a possibility of action on the world.

Our world is made up of physical objects and living beings. Some of these with which we consciously relate are the result of sexual reproduction and so are we. Thus we have a conception of gender that is in three states: neutral for the inanimate, and male and female for the living. Note that the French language gives very little importance to the neutral gender compared to English and German, perhaps as a result of a more anthropomorphic conception of the world.

Finally, conditioning all our cognition, and therefore our way of thinking, we attach to people, living beings, objects and even abstract concepts an affective coloration, and this in a typical three-state way: we appreciate, like love, something or on the contrary we consider it negatively or for less known things we are relatively indifferent to it. Concerning a large part of things, we affect them, often unconscious, which can be of varying degrees of intensity and which influences all our thoughts about them. As a result, our cognition is weighted by the effects of the three-state type. Moreover, it would seem that the very mechanism of reflection is three-state, particularly in the rationality used to decide on a choice and to seek solutions, which is done step by step. Either a partial solution is seen as positive or negative and then tends to end the reflection, or it is rather neutral and then allows the reflection to continue.

An emblematic case of choice is that of voting where, for a subject as well as for a candidate, one will have in addition to the rational aspects a positive, negative or neutral attraction, and also one will vote accordingly respectively either yes, no or blank (for example by abstention) if rational considerations are not clearly not preponderant.

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A concrete application is the technique of sentiment analysis, which is used in political science and also in finance to determine the positive, negative or neutral content of e.g. speeches of politicians or financial news for investors in a computerized way.

When a person has to make a decision, often in addition to accepting or rejecting what is proposed he may also decide to postpone his decision, to prefer to wait, which corresponds to a third state, in a certain neutral way (and expressing his uncertainty).

Another area where human conceptions are naturally in three states is that of evaluation, one can also speak of appreciation, and which therefore relates to effects and feelings, and even, as a result, to accounting. For example, the budget consumed by a project or the time allotted to carry it out can be: exceeded, within acceptable limits, or the forecasts were too pessimistic. A project can be judged as qualitatively good, satisfactory, or bad. Similarly, an objective may be exceeded, achieved, or not achieved. The result of a learning outcome is considered according to the first approach in a classical way, but according to the second in an approach by objectives.

For example, the cost of a project may be within the forecasts, or it may exceed or fall short of the forecasts. In a not-for-profit association one will try to balance income and expenditure, whereas in a commercial company one will aim at a profit, fearing the deficit, the alarm of which is the appearance of relatively balanced accounts.

4.6.Chemistry: phase change

In each discipline many examples can be found, here is another in chemistry. During the phenomenon of phase change, as between solid and liquid, matter does not only have two states, the original and the final one, but also a transition state (viscous matter infusion).

4.7. Physics: electrical charges

Any particle in quantum physics has an electric charge or not, and this charge can be positive or negative. This produces 3 states for the electric charge characteristic of particles: positive, neutral, negative.

Similarly molecules also have a charge that is likewise either positive, neutral or negative. If they are charged then they are called ions, subdivided into positively charged cations and negatively charged anions.

4.8 Neutral or indetermination

In the introduction, we explained that neutrosophy is characterized by the addition of a third state to allow a better (comprehensive and qualitative) representation of reality. This additional state is generally considered as a neutral state. It can also be given other meanings, leading to other fields of application. This state is often considered to represent uncertain information. Then neutrosophy will be used to treat in a better way real-world situations where uncertainty exists and is not negligible. For example, several simultaneous measurements of the same quantity generally produce different values when the sensors are sensitive. Or different measurement procedures with different accuracies have to be combined, which implies considering the uncertainty (here the measurement uncertainty). Many phenomena are inherently uncertain, especially at the atomic level, because they are governed by probabilistic laws.

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For example, the radioactivity of a radioactive isotope decreases with time at the macroscopic level according to an exponential law characterized by a halving in a constant period, called the half-life. However, each atom during a time interval decays or not, independently of the others and only in a probable manner. All atoms are taken together, however, follow the macroscopic statistical law of exponential decay,

Already with logical information, such a problem arises. When a large number of facts have been manipulated for different questions, for example in an expert system, some facts will be irrelevant for certain questions and therefore have to be considered neutraly. This corresponds to the neutrosophic logic mentioned at the beginning, which is more flexible in dealing with large numbers of facts than a classical logic that knows only true or false. Thus, composition with a non-relevant or neutral value will not change the value of a fact, whereas composition in classical logic can only be done with true or false, which in either case will not consider the other information. Typically, a counterexample is enough to cancel a law in classical logic, but it can be a bad counterexample, not very relevant, not very frequent, obtained by oversimplification in true or false. Or a counterexample that is impressive but misleading. Such binary logic is easier to handle for humans: it is black or white, and for computers; but it often goes too fast. The brain manipulates preferences very well, so why not consider them more. In any case, the best of reasons will remain ours: we ultimately think about what we desire to think.

Let's end this section with another example, taken from meteorology, which brings out the concepts of uncertainty and neutral zone: a prognosis of future weather conditions may be that the weather will be uncertain and not just good or bad. In a simplified forecast, for example, the air pressure will be considered to be either rising, stable or falling. Figure 1 of the weather station shows that the same indication, depending on the context, can mean the opposite: it says that in summer a rising temperature indicates an improvement in the weather, while it also says that in winter a rising temperature indicates a deterioration in the weather. If the indications of the three types of sensors agree then the forecast is more plausible.

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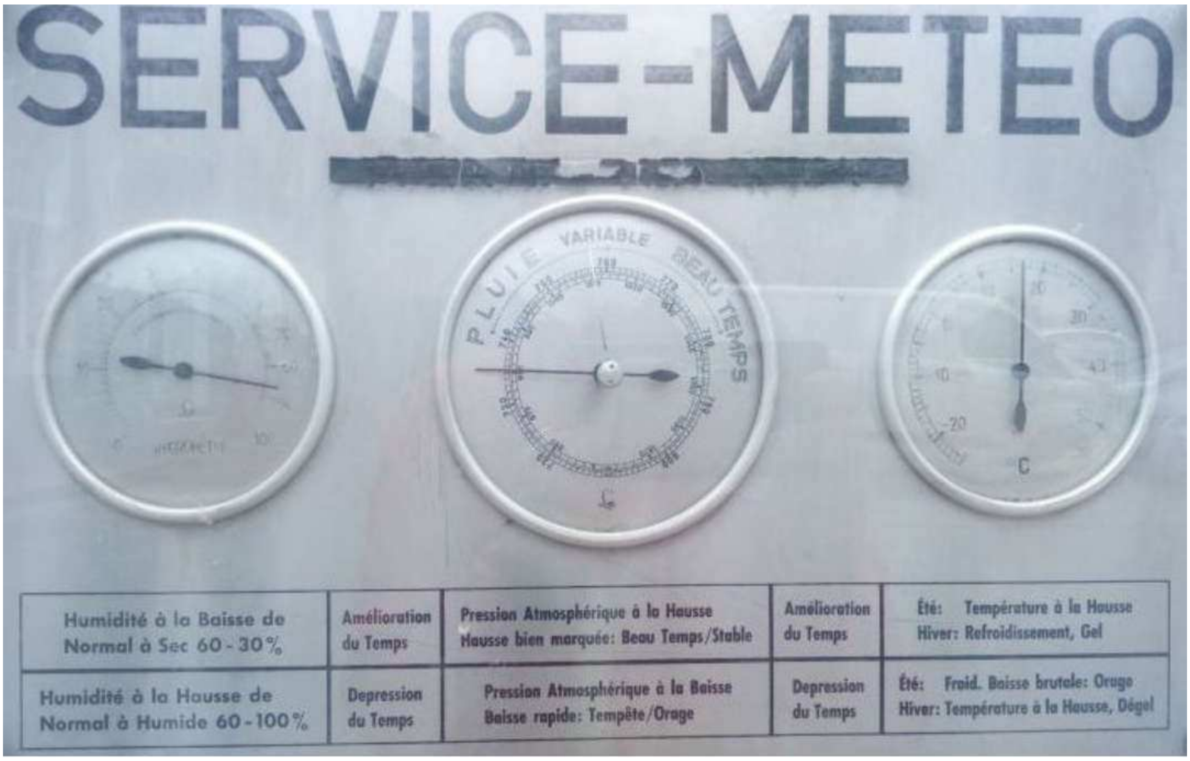

Figure 1 : Weather station

For information at the time of the photo it was almost the beginning of summer, and there was a big storm half an hour later, which lasted only three minutes. It would have been good to have taken a second photograph a few minutes apart to be able to determine the changes in the measurements, because according to the written indications it is the upward or downward variation that is indicative and not the current value. It is not explicitly stated that if, for example, the pressure is stable, then this information is neutral, non-relevant. Thus it appears that in this example we are considering 3 cases: rising, stable, falling.

4.9. Communications: bipolar coding

This may seem paradoxical that bipolar coding is actually in three states. On a simple electrical wire, such as the telegraph wire, a message can be sent using Morse code: with a switch, a current can be sent or not (in fact by imposing a voltage, coming for example from a battery, connected to earth on the other pole). It is a unipolar coding: voltage or nothing. You can also flip the battery, thus inverting its poles, so you have the opposite voltage, negative if it was positive before. So instead of transmitting two states 0 and 1, we can now transmit three: -1, 0 and +1, using the two poles of the battery, one after the other. By working at the same speed it is now possible to transmit more information, so this coding with two polarities voltage is more efficient. The receiver can be made with a device consisting of a magnetized needle that deflects when a current passes through the wire wound in a coil (forming an electromagnet), and it deflects in the opposite direction when the current is reversed. A spring is used to return the needle to its rest (neutral) position when there is no current. A cable is usually limited or defined by its insulation, so it can also work as a bipolar cable.

5. Conclusion

We have given some examples of situations where it is usual to consider three states in very different fields. These situations appear to us first of all because of the habit of naturally having three states, and therefore neutrosophy is well suited to model them.

By considering these cases, which are numerous, become perceptible some rather deep reasons that make us consider these situations as intrinsically of three-state type. The most apparent reason is subjective, we see such a situation according to our intellectual perception as having three states. It is in a way a privileged way for humans to see the world and to conceive a representation of it.

Then similarities between these situations show that more profoundly they are of a three-state nature, following the more general way of functioning, by increasing degrees of universality and decreasing degrees of evidence: first of cognition, then of living beings and finally of the physical world with its temporal aspect in particular. According to the first approach, a reflection is a progression of stages ending in success or failure. Each stage is colored by our preferences in positive, negative or neutral. If it is neutral, then we must evaluate other stages, otherwise our preferences will lead us to a conclusion. Our motivation for satisfying needs or for pleasure implies that some things are irrelevant, unimportant, some positive and some negative. The physical world is also governed by laws of attraction and repulsion, as with the electric charges of ions, atoms and particles: positive, neutral, or negative.

Finally, we find the three great universal categories of perception or representation: increase, relative stability, decrease that derive from the existence of time, intrinsically made of the present but which produces the past and consumes the future, inexorably. This time allows variation and also relative constancy over a certain time. More abstractly, any phenomenon, any collection, has an average value, and any situation or element can be approximately in the average, above or below it. Everything is thus essentially of a three-state type and not binary, a too limited mode of representation, considering only that something exists or not: static whereas everything is dynamic, a too over-simplifying view and therefore a misleading one.

References

[1] Smarandache, F.,"New Trends in Neutrosophic Theory and Applications", Pons asbi, Brussels.1995.

[2] Parry and Hacker,. Aristotelian Logic. State University of New York Press, 1991

[3] Bruno, Leonard C. "Math and Mathematicians: The History of Math Discoveries Around the World", Baker, Lawrence W. Detroit, Mich.: U X L. p. 99. ISBN 978-0-7876-3813-9. OCLC 41497065.2003.

[4] Smarandache, F.,"A Unifying Field in Logics: Neutrosophic Logic", - 6th ed. USA: InfoLearnQuest, 2007

[5] Smarandache, F.& Osman, S.,"Neutrosophy in Arabic Philosophy", Renaissance High Press, Ann Arbor, MI, 2017.

[6] Smarandache, F.,"Neutrosophic set, a generalization of the Intuitionistic Fuzzy Sets", International Journal of Pure and Applied Mathematics, Vol.24,pp-, 287-297,2005.

DOI: 10.5281/zenodo.3960783

[7] Smarandache, F.,"Neutrosophy / Neutrosophic probability, set, and logic", American Re-search Press: Santa Fe, NM, USA. 1998.

[8] Smarandache, F.,"Introduction to the neutrosophic measure, neutrosophic integral and neutrosophic probability", Sitech, Craiova, Romania, 2013.

[9] Leibnitz, Descartes, Fermat, Newton UE 4M039 : Histoire des mathématiques (Alexandre Guilbaud et Laurent Mazliak, Semaine 4 La naissance du calcul différentiel et intégral, https://www.lpsm.paris/pageperso/mazliak/semaine4.pdf

[10] Wikipedia, entry Kinematics, https://en.wikipedia.org/wiki/Kinematics

[11] Wikipedia, entry Brownian motion, https://en.wikipedia.org/wiki/Brownian_motion

[12] Wikipedia, entry Markov chain, https://en.wikipedia.org/wiki/Markov_chain

[13] Johana Cristina Sierra Morán, Jenny Fernanda Enríquez Chuga, Wilmer Medardo Arias Collaguazo And Carlos Wilman Maldonado Gudiño:Neutrosophic statistics applied to the analysis of socially responsible participation in the community , Neutrosophic Sets and Systems, vol. 26, pp. 19-28, 2019. DOI: 10.5281/zenodo.3244232

[14] Leny Cecilia Campaña Muñoz, Holman Steven Sánchez Ramos, and Johanna Rocío Cabrera Granda: Use of neutrosophy for the analysis of the social reintegration factors of released prisoners in Ecuador : Neutrosophic Sets and Systems, vol. 26, pp. 145-152, 2019. DOI: 10.5281/zenodo.3244814

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AH-Substructures in Strong Refined Neutrosophic Modules

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Abstract

The objective of this paper is to define and study the concepts of strong AH-submodule, and AH-homomorphism in a refined neutrosophic module. Also, this work describes the algebraic structure of all AH-endomorphisms defined over a refined neutrosophic module.

Keywords: Refined neutrosophic module, Strong AH-submodule, AH-homomorphism

1. Introduction

A neutrosophic set is a powerful general formal framework which generalizes the concept of the classic set, fuzzy set [13], interval valued fuzzy set [12], intuitionistic fuzzy set [9] etc. A neutrosophic set A defined on a universe U . $x = x(T, I, F) \in A$ with T , I and F being the real standard or non-standard subsets of $\left]0^-, 1^+\right[$. T is the degree of truth membership function in the set A, I is the indeterminacy-membership function in the set A and F is the falsity-membership function in the set A . Agboola introduced the concept of refined neutrosophic algebraic structures and studied refined neutrosophic groups in particular [6]. Adeleke et al. in [7,8] studied refined neutrosophic rings and refined neutrosophic subrings and presented their fundamental properties. Recently, Hatip et al. studied refined neutrosophic modules and refined neutrosophic homomorphisms modules and presented their basic properties [10,11]. Abobala et al. in [1,2] studied some special substructures of refined neutrosophic rings. Also in [3], Abobala et al. studied classical homomorphisms between refined neutrosophic rings and neutrosophic rings and presented their basic properties. Abobala and Alhamido studied AH-substructures in neutrosophic modules and AH-subspaces in neutrosophic vector spaces [4,5].

The present paper is devoted to the study of AH-strong refined neutrosophic modules. Also, the strong AH-homomorphism modules will be established.

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2. Preliminaries

In this section, we present the basic definitions that are useful in this research.

Definition 2.1: [10] Let $(M,+,.)$ be any R-module over aneutrosophic ring $R(I)$, The triple $(M(I),+,\cdot)$ is called a strong neutrosophic R-module over a neutrosophic ring $R(I)$, generated by M *and* I.

Definition 2.2: [6] Let $(X(I_1,I_2),+,\cdot)$ be any refined neutrosophic algebraic structure where + and $\cdot$ are ordinary addition and multiplication respectively. I_1 *and* I_2 are the split components of the indeterminacy factor I that is $I=\alpha I_1+\beta I_2$ with $\alpha,\beta\in R$ *or* C. Also, I_1 *and* I_2 are taken to have the properties $I_1^2=I_1, I_2^2=I_2$ and $I_1I_2=I_2I_1=I_1$.

For any two elements, we define

1) $x+y=(a,bI_1,cI_2)+(d,eI_1,fI_2)=(a+d,(b+e)I_1,(c+f)I_2)$

2) $x\cdot y=(a,bI_1,cI_2)\cdot(d,eI_1,fI_2)=\begin{pmatrix} ad,(ae+bd+be+bf+ce)I_1, \\ (af+cd+cf)I_2 \end{pmatrix}$

Definition 2.3: [10] Let $(M,+,.)$ be any R-module over a refined neutrosophic ring $R(I_1,I_2)$, The triple $(M(I_1,I_2),+,\cdot)$ is called a strong refined neutrosophic R-module over a refined neutrosophic ring $R(I_1,I_2)$, generated by M, I_1 *and* I_2.

Definition 2.4: Let $M(I)$ be a strongneutrosophicR-module, the set $S=P+QI=\{x+yI : x\in P, y\in Q\}$ where P and Q are submodules of M is called an AH-submodule of $M(I)$ and If $P=Q$ then S is called an AHS-submodule of $M(I)$.

3. Main discussion

Definition 3.1:

Let $M(I_1,I_2)$ be a strong refined neutrosophic module over the refined neutrosophic ring $R(I_1,I_2)$, *P, Q, S* be three submodules of M. The set N = $(P,QI_1,SI_2)=\{(a,bI_1,cI_2); a\in P, b\in Q, c\in S\}$ is called a strong AH-submodule of the strong refined neutrosophic module $M(I_1,I_2)$.

If P = Q = S, we call N a strong AHS-submodule.

Theorem 3.2:

Let $M(I_1,I_2)$ be a strong refined neutrosophic module over the refined neutrosophic ring $R(I_1,I_2)$,

N = (P,PI_1,PI_2) be a strong AHS-submodule. Then N is a submodule by classical meaning.

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Proof:

The proof is similar to that of theorem 3.4 in [13].

An AH-submodule is not supposed to be a submodule of $M(I_1, I_2)$ in general. See the following example.

Example 3.3:

Let $M = Z_6$ be a module over the ring of integers Z, the corresponding refined neutrosophic module is

$M(I_1, I_2) = \{(a, bI_1, cI_2); a, b, c \in M\}$ over the refined neutrosophic ring $Z(I_1, I_2)$, we have $P = \{0,3\}, Q = \{0,2,4\}$ as two submodules of M.

$N = (P, QI_1, PI_2)$ is a strong AH-submodule of $M(I_1, I_2)$, $x = (2,3I_1, 0) \in N$, $r = (1, I_1, I_2) \in Z(I_1, I_2)$

$r.x = (2, [2 + 3 + 3 + 3 + 0]I_1, [2 + 0 + 0]I_2) = (2,5I_1, 2I_2)$, which is not in N, thus N is not a submodule.

Theorem 3.4: Let $M\left(I_1,I_2\right)$ be a strong refinedneutrosophic R-module over a refined neutrosophic ring $R\left(I_1,I_2\right)$ and let $\{N_n\}_{n\in\lambda}$ be a family of a strong AH-submoduleof $M\left(I_1,I_2\right)$. Then $\cap\{N_n\}_{n\in\lambda}$ is a strong AH-submodule of $M\left(I_1,I_2\right)$.

Proof: Clearly $\cap\{N_n\}_{n\in\lambda} \neq \varnothing$, $\forall n \in \lambda$ let we have $x = \left(a, bI_1, cI_2\right), y\left(d, eI_1, fI_2\right) \in \cap\{N_n\}_{n\in\lambda}$ for a,b,c,d,e,f belong to P,Q,S,T,V,K respectively where P,Q,S,T,V,K are asubmodules of M and let be $\alpha = \left(p, qI_1, rI_2\right) \in R\left(I_1,I_2\right)$. Then $x + y, \alpha x \in \cap\{N_n\}_{n\in\lambda}$. Since, for $\forall n \in \lambda, x + y \in \cap\{N_n\}_{n\in\lambda}$ and $\alpha x \in \cap\{N_n\}_{n\in\lambda}$ Hence $\cap\{N_n\}_{n\in\lambda}$ is a strong AH-submoduleof $M\left(I_1,I_2\right)$.

Remark 3.5: Let $M\left(I_1,I_2\right)$ be a strong refinedneutrosophic R-module over a refinedneutrosophic ring $R\left(I_1,I_2\right)$ and let N_1 and N_2 be two distinct strong AH-submoduleof $M\left(I_1,I_2\right)$. Generally, $N_1 \cup N_2$ is not a strongAH-submoduleof $M\left(I_1,I_2\right)$.

However, if $N_1 \subseteq N_2$ or $N_1 \supseteq N_2$ then $N_1 \cup N_2$ is a AH-submoduleof $M\left(I_1,I_2\right)$.

Definition 3.6:

Let M, W be two modules over the ring R, $M(I_1, I_2)$ and $W(I_1, I_2)$ be the corresponding strong refined neutrosophic modules over the refined neutrosophic ring $R(I_1, I_2)$. Let $f, g, h: M \to W$ be three homomorphisms, then

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$[f,g,h]: M(I_1,I_2) \rightarrow W(I_1,\mathrm{I}_2)$; $[f,g,h](a,bI_1,cI_2) = (f(a),g(b)I_1,h(c)I_2)$ is called a strong AH-homomorphism. If $f = g = h$, we get the strong AHS-homomorphism.

Definition 3.7:

Let $M(I_1,I_2), W(I_1,I_2)$ be two strong refined neutrosophic modules over the refined neutrosophic ring $R(I_1,I_2)$, $[f,g,h]: M(I_1,I_2) \rightarrow W(I_1,\mathrm{I}_2)$ be a strong AH-homomorphism, we define

(a) $AH - Ker[f,g,h] = (Ker(f), Ker(g)I_1, Ker(h)I_2) = \{(a,bI_1,cI_2); a \in Ker(f), b \in Ker(g), c \in Ker(h)\}$.

(b) $AH - Im[f,g,h] = (Im(f), Im(g)I_1, Im(h)I_2)$.

Theorem 3.8:

Let $M(I_1,I_2), W(I_1,I_2)$ be two strong refined neutrosophic modules over the refined neutrosophic ring $R(I_1,I_2)$, $[f,g,h]: M(I_1,I_2) \rightarrow W(I_1,\mathrm{I}_2)$ be a strong AH-homomorphism.

(a) If N = (P,QI_1,SI_2) is a strong AH-submodule of $M(I_1,I_2)$, then $[f,g,h](N)$ is a strong AH-submodule of $W(I_1,I_2)$.

(b) $[f,g,h]$ is a classical module homomorphism.

(c) $AH - Ker[f,g,h]$ is a strong AH-submodule of $M(I_1,I_2)$.

(d) $AH - Im[f,g,h]$ is a strong AH-submodule of $W(I_1,I_2)$.

Proof:

(a) Since $f(P), g(Q), h(S)$ are submodules of N, we find that $[f,g,h](N) = (f(P),g(Q)I_1,h(S)I_2)$ is a strong AH-submodule of $W(I_1,I_2)$.

(b) Let $m = (x,yI_1,zI_2), n = (a,bI_1,cI_2)$ be two arbitrary elements in $M(I_1,I_2)$, $r = (t,uI_1,vI_2)$ be any element in $R(I_1,I_2)$,

$m + n = (x + a,[y + b]I_1,[z + c]I_2), r.m = (tx,[xu + yt + yu + yv + zu]I_1,[xv + zt + zv]I_2)$,

$[f,g,h](m + n) = (f(x + a), g([y + b])I_1, h([z + c])I_2)$=$(f(x),g(y)I_1,h(z)I_2) + (f(a),g(b)I_1,h(c)I_2) = [f,g,h](m) + [f,g,h](n)$.

$[f,g,h](r.m) = (f(tx), g([xu + yt + yu + yv + zu])I_1, h([xv + zt + zv])I_2)$=

$(t,uI_1,vI_2).(f(x),g(y)I_1,h(z)I_2) = r.[f,g,h](m)$. Thus $[f,g,h]$ is a classical homomorphism.

(c) Since $Ker(f), Ker(g), Ker(h)$ are submodules of M, then $AH - Ker[f,g,h] = (Ker(f), Ker(g)I_1, Ker(h)I_2)$ as a strong AH-submodule of $M(I_1,I_2)$.

(d) Since $Im(f), Im(g), Im(h)$ are submodules of W, we get $AH - Im[f,g,h] = (Im(f), Im(g)I_1, Im(h)I_2)$ as a strong AH-submodule of $W(I_1,I_2)$.

Example 3.9:

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(a) Let $M = R^2, W = R$ be two modules over the ring R,

$f: M \to W; f(x,y) = 2x, g: M \to W; g(x,y) = 3y, h: M \to W; h(x,y) = x + y$ are three homomorphisms.

(b) $[f,g,h]: M(I_1,I_2) \to W(I_1,I_2); [f,g,h]\big((x,y),(z,t)I_1,(s,m)I_2\big) = (f(x,y), g(z,t)I_1, h(s,m)I_2)$=

$(2x, 3tI_1, [s+m]I_2)$ is a strong AH-homomorphism, where $x,y,z,t,s,m \in R$.

(c) $P = \{(0,x); x \in R\}, Q = \{(x,0); x \in R\}$ are two submodules of M,

$N = (P, PI_1, QI_2) = \{((0,x),(0,y)I_1,(z,0)I_2; x,y,z \in R\}$ is a strong AH-submodule of $M(I_1,I_2)$.

(d) $f(P) = \{0\}, g(P) = \{3y; y \in R\} = R, h(Q) = \{z; z \in R\} = R$,

$[f,g,h](N) = (f(P), g(P)I_1, h(Q)I_2) = (0, RI_1, RI_2) = \{(0, xI_1, yI_2); x,y \in R\}$ is a strong AH-submodule of $W(I_1,I_2)$.

(e) $Ker(f) = \{(0,x); x \in R\}, Ker(g) = \{(x,0); x \in R\}, Ker(h) = \{(y,-y); y \in R\}$,

$AH - Ker[f,g,h] = (\ker(f), Ker(g)I_1, Ker(h)I_2) = \{(0,x),(y,0)I_1,(z,-z)I_2); x,y,z \in R\}$.

Remark 3.10:

We denote to the set of all strong AH-homomorphisms from a strong refined neutrosophic module $M(I_1,I_2)$ to itself by $AH - END(M(I_1,I_2))$.

Definition 3.11:

Let $M(I_1,I_2)$ be a strong refined neutrosophic module over the refined neutrosophic ring $R(I_1,I_2)$,

$AH - END(M(I_1,I_2))$ be the set of all strong AH-endomorphisms, we define operations on $AH - END(M(I_1,I_2))$ as follows:

Let $f_i, g_i; i \in \{0,1,2\}$ be any homomorphisms from M to itself, we define

Addition: $[f_0,f_1,f_2] + [g_0,g_1,g_2] = [f_0+g_0, f_1+g_1, f_2+g_2]$.

Multiplication by a scalar, if $r = (r_0, r_1I_1, r_2I_2)$ is any element in $R(I_1,I_2)$, then

$r = (r_0, r_1I_1, r_2I_2).[f_0,f_1,f_2]$=

$[r_0f_0, (r_0f_1 + r_1f_0 + r_1f_1 + r_1f_2 + r_2f_1), (r_0f_2 + r_2f_2 + r_2f_0)]$.

Multiplication: $[f_0,f_1,f_2]o[g_0,g_1,g_2] =$

$[f_0og_0, f_0og_1 + f_1og_0 + f_1og_1 + f_1og_2 + f_2og_1, f_0og_2 + f_2og_0 + f_2og_2]$.

Theorem 3.12:

$(AH - END\big(M(I_1,I_2)\big), +, o)$ is a refined neutrosophic ring.

Proof:

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Since $L = \{f: M \to M; f$ is a homomorphism$\}$ is a ring with respect to addition and multiplication, then $L(I_1, I_2)$ is a refined neutrosophic ring as a result of the definition of neutrosophic rings. It is easy to see that $L(I_1, I_2) = AH - END(M(I_1, I_2)$, thus we get the desired proof.

Theorem 3.13:

$(AH - END\big(M(I_1, I_2)\big), +, .)$ is a refined neutrosophic module.

Proof:

Since $L = \{f: M \to M; f$ is a homomorphism$\}$ is a module with respect to addition and multiplication by a scalar taken from the ring R, we regard that $L(I_1, I_2) = AH - END(M(I_1, I_2)$ is a strong refined neutrosophic module over the refined neutrosophic ring $R(I_1, I_2)$ as a simple result from the definition of strong neutrosophic modules.

4. Conclusion

In this research, we have defined the AH- Strong refined neutrosophic modules, and established the definition of AH-homomorphisms in refined neutrosophic modules. We have proved some theories related to these issues and given some clarifying examples.

5. Future Research Directions

As a future work, this article can be extended to include semi AH-homomorphism in modules as well as the definition of semi refiend homomorphism in general.

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References

[1] Abobala, M., "On Some Special Substructures of Neutrosophic Rings and Their Properties", International Journal of Neutrosophic Science", Vol. 4 , pp. 72-81, 2020.

[2] Abobala, M., "On Some Special Substructures of Refined Neutrosophic Rings", International Journal of Neutrosophic Science, Vol. 5, pp. 59-66, 2020.

[3] Abobala, M,. "Classical Homomorphisms Between Refined Neutrosophic Rings and Neutrosophic Rings", International Journal of Neutrosophic Science, Vol. 5, pp. 72-75, 2020.

[4] Abobala, M., "AH-Subspaces in Neutrosophic Vector Spaces", International Journal of Neutrosophic Science, Vol. 6 , pp. 80-86, 2020.

[5] Abobala, M., and Alhamido, R., "AH-Substructures in Neutrosophic Modules", International Journal of Neutrosophic Science, Vol. 7, pp. 79-86 , 2020.

[6] Agboola, A.A.A., “On Refined Neutrosophic Algebraic Structures”, Neutrosophic Sets and Systems,Vol.10, pp. 99-101, 2015.

[7]Adeleke, E.O., Agboola, A.A.A.,and Smarandache, F., "RefinedNeutrosophicRings I",International Journal of Neutrosophic Science, Vol. 2(2), pp. 77-81, 2020.

[8] Adeleke, E.O., Agboola, A.A.A., and Smarandache, F.,"Refined Neutrosophic Rings II",International Journal of Neutrosophic Science, Vol. 2(2), pp. 89-94, 2020

[9] Atanassov, K., “Intuitionistic Fuzzy Sets”, Fuzzy Sets and Systems, 20, pp.87-96, 1986.

[10] Olgun, N., and Khatib, A., "Neutrosophic Modules", Journal of Biostatistic and Biometric Application", Vol. 3, 2018.

[11] Olgun, N., and Hatip, A., " On Refined Neutrosophic R-Module", International Journal of Neutrosophic Science, Vol. 7, pp.87-96, 2020.

[12] Turksen, I., “Interval valued fuzzy sets based on normal forms”, Fuzzy Sets and Systems, 20, pp.191-210, 1986.

[13] Zadeh, L., “Fuzzy Sets”, Inform and Control, 8, pp.338-353, 1965.

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