Introduction to NeutroRings

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Abstract

The objective of this paper is to introduce the concept of NeutroRings by considering three NeutroAxioms (NeutroAbelianGroup (additive), NeutroSemigroup (multiplicative) and NeutroDistributivity (multiplication over addition)). Several interesting results and examples on NeutroRings, NeutroSubgrings, NeutroIdeals, NeutroQuotientRings and NeutroRingHomomorphisms are presented. It is shown that the 1st isomorphism theorem of the classical rings holds in the class of NeutroRings.

Keywords: Neutrosophy, NeutroGroup, NeutroSubgroup, NeutroRing, NeutroSubring, NeutroIdeal, NeutroQuotientRing and NeutroRingHomomorphism.

1 Introduction

The concept of neutrosophic logic/set introduced by Smarandache[25] is a generalization of fuzzy logic/set introduced by Zadeh[30] and intuitionistic fuzzy logic/set introduced by Atanasov.[14] In neutrosophic logic, each proposition is characterized by truth value in the set T, indeterminacy value in the set I and falsehood value in the set F where T, I, F are standard or nonstandard of the subsets of the nonstandard interval $]^{-}0, 1^{+}[$ where $^{-}0 \leq \inf T + \inf I + \inf F \leq \sup T + \sup I + \sup F \leq 3^{+}$. Statically, T, I, F are subsets, but dynamically, the components of T, I, F are set-valued vector functions/operators depending on many parameters some of which may be hidden or unknown. Neutrosophic logic/set has several real life applications in sciences, engineering, technology and social sciences. The concept has been used in medical diagnosis and multiple decision-making.[15,18,29] Neutrosophic set has been used and applied in several areas of mathematics. For instance in algebra, neutrosophic set has been used to develop neutrosophic groups, rings, vector spaces, modules, hypergroups, hyperrings, hypervector spaces, hypermodules, etc.[1,2,4,5,7–13,16,17,27,28] In analysis, neutrosophic set has been used to develop neutrosophic topological spaces[22–24] and many other areas of mathematical analysis. The concept of neutrosophic logic/set is now well known and embraced in many parts of world. Many researches have been conducted on neutrosophic logic/set and several papers have been published in many international journals by many neutrosophic researchers scattered all over the world. Neutrosophic Sets and Systems and International Journal of Neutrosophic Science are presently two international journals dedicated to publication of research articles in neutrosophic logic/set.

Smarandache[19] recently introduced new fields of research in neutrosophy called NeutroStructures and AntiStructures respectively. In,[20] Smarandache introduced the concepts of NeutroAlgebras and AntiAlgebras and in,[21] he revisited the concept of NeutroAlgebras and AntiAlgebras where he studied Partial Algebras, Universal Algebras, Effect Algebras and Boole's Partial Algebras and he showed that NeutroAlgebras are generalization of Partial Algebras. Motivated by the works of Smarandache in,[19–21] Agboola et al in[6] studied NeutroAlgebras and AntiAlgebras viz-a-viz the classical number systems $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$. Also motivated by the work of Smarandache in,[19] Agboola[3] formally introduced the concept of NeutroGroup by considering three NeutroAxioms (NeutroAssociativity, existence of NeutroNeutral element and existence of NeutroInverse element). In,[3] Agboola studied NeutroSubgroups, NeutroCyclicGroups, NeutroQuotientGroups and NeutroGroupHomomorphisms. Several interesting results and examples were presented and it was shown that generally, Lagrange's theorem and 1st isomorphism theorem of the classical groups do not hold in the class of NeutroGroups. In continuation of the work started in,[3] the present paper is devoted to the presentation of the concept of NeutroRing by considering three NeutroAxioms (NeutroAbelianGroup (additive), NeutroSemigroup (multiplicative) and

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NeutroDistributivity (multiplication over addition)). Several interesting results and examples on NeutroRings, NeutroSubgrings, NeutroIdeals, NeutroQuotientRings and NeutroRingHomomorphisms are presented. It is shown that the 1st isomorphism theorem of the classical rings holds in the class of NeutroRings.

2 Preliminaries

In this section, we will give some definitions, examples and results that will be useful in other sections of the paper.

Definition 2.1. [21]

(i) A classical axiom defined on a nonempty set is an axiom that is totally true (i.e. true for all set's elements).

(ii) A NeutroAxiom (or Neutrosophic Axiom) defined on a nonempty set is an axiom that is true for some set's elements [degree of truth (T)], indeterminate for other set's elements [degree of indeterminacy (I)], or false for the other set's elements [degree of falsehood (F)], where $T, I, F \in [0,1]$, with $(T, I, F) \neq (1,0,0)$ that represents the classical axiom, and $(T, I, F) \neq (0,0,1)$ that represents the AntiAxiom.

(iii) An AntiAxiom defined on a nonempty set is an axiom that is false for all set's elements.

Therefore, we have the neutrosophic triplet: $<$ Axiom, NeutroAxiom, AntiAxiom $>$.

Definition 2.2. [3] Let G be a nonempty set and let $* : G \times G \rightarrow G$ be a binary operation on G. The couple $(G, *)$ is called a NeutroGroup if the following conditions are satisfied:

(i) $*$ is NeutroAssociative that is there exists at least one triplet $(a, b, c) \in G$ such that

$$a * (b * c) = (a * b) * c \tag{1}$$

and there exists at least one triplet $(x, y, z) \in G$ such that

$$x * (y * z) \neq (x * y) * z. \tag{2}$$

(ii) There exists a NeutroNeutral element in G that is there exists at least an element $a \in G$ that has a single neutral element that is we have $e \in G$ such that

$$a * e = e * a = a \tag{3}$$

and for $b \in G$ there does not exist $e \in G$ such that

$$b * e = e * b = b \tag{4}$$

or there exist $e_1, e_2 \in G$ such that

$$b * e_1 = e_1 * b = b \quad \text{or} \tag{5}$$

$$b * e_2 = e_2 * b = b \quad \text{with} \quad e_1 \neq e_2 \tag{6}$$

(iii) There exists a NeutroInverse element that is there exists an element $a \in G$ that has an inverse $b \in G$ with respect to a unit element $e \in G$ that is

$$a * b = b * a = e \tag{7}$$

or there exists at least one element $b \in G$ that has two or more inverses $c, d \in G$ with respect to some unit element $u \in G$ that is

$$b * c = c * b = u \tag{8}$$

$$b * d = d * b = u. \tag{9}$$

In addition, if $*$ is NeutroCommutative that is there exists at least a duplet $(a, b) \in G$ such that

$$a * b = b * a \tag{10}$$

and there exists at least a duplet $(c, d) \in G$ such that

$$c * d \neq d * c, \tag{11}$$

then $(G, *)$ is called a NeutroCommutativeGroup or a NeutroAbelianGroup.

If only condition (i) is satisfied, then $(G, *)$ is called a NeutroSemiGroup and if only conditions (i) and (ii) are satisfied, then $(G, *)$ is called a NeutroMonoid.

Example 2.3. [3] Let $\mathbb{U} = \{a, b, c, d, e, f\}$ be a universe of discourse and let $G = \{a, b, c, d\}$ be a subset of $\mathbb{U}$. Let $*$ be a binary operation defined on G as shown in the Cayley table below:

$*$	a	b	c	d
a	b	c	d	a
b	c	d	a	c
c	d	a	b	d
d	a	b	c	a

Then $(G, *)$ is a NeutroAbelianGroup.

Example 2.4. [3] Let $G = \mathbb{Z}_{10}$ and let $*$ be a binary operation on G defined by $x * y = x + 2y$ for all $x, y \in G$ where $''+''$ is addition modulo 10. Then $(G, *)$ is a NeutroAbelianGroup.

Definition 2.5. [3] Let $(G, *)$ be a NeutroGroup. A nonempty subset H of G is called a NeutroSubgroup of G if $(H, *)$ is also a NeutroGroup.

The only trivial NeutroSubgroup of G is G.

Example 2.6. [3] Let $(G, *)$ be the NeutroGroup of **Example 2.3** and let $H = \{a, c, d\}$. The compositions of elements of H are given in the Cayley table below.

$*$	a	c	d
a	b	d	a
c	d	b	d
d	a	c	a

Then, H is a NeutroSubgroup of G.

Definition 2.7. [3] Let $(G, *)$ and $(H, \circ)$ be any two NeutroGroups. The mapping $\phi : G \to H$ is called a homomorphism if ϕ preserves the binary operations $*$ and $\circ$ that is if for all $x, y \in G$, we have

$$\phi(x * y) = \phi(x) \circ \phi(y). \tag{12}$$

The kernel of ϕ denoted by $Ker\phi$ is defined as

$$Ker\phi = \{x : \phi(x) = e_H\} \tag{13}$$

where $e_H \in H$ is such that $N_h = e_H$ for at least one $h \in H$.

The image of ϕ denoted by $Im\phi$ is defined as

$$Im\phi(x) = \{y \in H : y = \phi(x) \text{ for some } h \in H\}. \tag{14}$$

If in addition ϕ is a bijection, then ϕ is an isomorphism and we write $G \cong H$.

Theorem 2.8. [3] *Let $(G, *)$ and $(H, \circ)$ be NeutroGroups and let $N_x = e_G$ such that $e_G * x = x * e_G = x$ for at least one $x \in G$ and let $N_y = e_H$ such that $e_H * y = y * e_H = y$ for at least one $y \in H$. Suppose that $\phi : G \to H$ is a NeutroGroup homomorphism. Then:*

(i) $\phi(e_G) = e_H$.

(ii) $Ker\phi$ is a NeutroSubgroup of G.

(iii) $Im\phi$ *is a NeutroSubgroup of H.*

(iv) ϕ *is injective if and only if* $Ker\phi = \{e_g\}$.

Theorem 2.9. [3] *Let* H *be a NeutroSubgroup of a NeutroGroup* $(G, *)$*. The mapping* $\psi : G \to G/H$ *defined by*

$$\psi(x) = xH \quad \forall\, x \in G$$

is a NeutroGroup homomorphism and the $Ker\psi \neq H$.

Theorem 2.10. [3] *Let* $\phi : G \to H$ *be a NeutroGroup homomorpism and let* $K = Ker\phi$*. Then the mapping* $\psi : G/K \to Im\phi$ *defined by*

$$\psi(xK) = \phi(x) \quad \forall\, x \in G$$

is a NeutroGroup epimorphism and not an isomorphism.

3 Development of NeutroRings and their Properties

The new concept of NeutroRing is developed and studied in this section by considering three NeutroAxioms (NeutroAbelianGroup (additive), NeutroSemigroup (multiplicative) and NeutroDistributivity (multiplication over addition)). Several interesting results and examples are presented.

Definition 3.1. (a) A NeutroRing $(R, +, .)$ is a ring structure that has at least one NeutroOperation among $''+''$ and $''.''$ or at least one NeutroAxiom. Therefore, there are many cases of NeutroRing, depending on the number of NeutroOperations and NeutroAxioms, and which of them are Neutro-Sophicated.

For the purposes of this paper, the following definition of a NeutroRing will be adopted:

(b) Let R be a nonempty set and let $+, . : R \times R \to R$ be binary operations of ordinary addition and multiplication on R. The triple $(R, +, .)$ is called a NeutroRing if the following conditions are satisfied:

(i) $(R, +)$ is a NeutroAbelianGroup.

(ii) $(R, .)$ is a NeutroSemiGroup.

(iii) $''.''$ is both left and right NeutroDistributive over $''+''$ that is there exists at least a triplet $(a, b, c) \in R$ and at least a triplet $(d, e, f) \in R$ such that

$$a.(b + c) = a.b + a.c \quad (15)$$
$$(b + c).a = b.a + c.a \quad (16)$$
$$d.(e + f) \neq d.e + d.f \quad (17)$$
$$(e + f).d \neq e.d + f.d. \quad (18)$$

If $''.''$ is NeutroCommutative, then $(R, +, .)$ is called a NeutroCommutativeRing.

We will sometimes write $a.b = ab$.

Definition 3.2. Let $(R, +, .)$ be a NeutroRing.

(i) R is called a finite NeutroRing of order n if the number of elements in R is n that is $o(R) = n$. If no such n exists, then R is called an infinite NeutroRing and we write $o(R) = \infty$.

(ii) R is called a NeutroRing with NeutroUnity if there exists a multiplicative NeutroUnity element $u \in R$ such that $ux = xu = x$ that is $U_x = u$ for at least one $x \in R$.

(iii) If there exists a least positive n such that $nx = e$ for at least one $x \in R$ where e is an additive NeutroElement in R, then R is called a NeutroRing of characteristic n. If no such n exists, then R is called a NeutroRing of characteristic NeutroZero.

(iv) An element $x \in R$ is called a NeutroIdempotent element if $x^2 = x$.

(v) An element $x \in R$ is called a NeutroINilpotent element if for the least positive integer n, we have $x^n = e$ where e is an additive NeutroNeutral element in R.

(vi) An element $e \neq x \in R$ is called a NeutroZeroDivisor element if there exists an element $e \neq y \in R$ such that $xy = e$ or $yx = e$ where e is an additive NeutroNeutral element in R.

(vii) An element $x \in R$ is called a multiplicative NeutroInverse element if there exists at least one $y \in R$ such that $xy = yx = u$ where u is the multiplicative NeutroUnity element in R.

Definition 3.3. Let $(R, +, .)$ be a NeutroCommutativeRing with NeutroUnity. Then

(i) R is called a NeutroIntegralDomain if R has no at least one NeutroZeroDivisor element.

(ii) R is called a NeutroField if R has at least one NeutroInverse element.

Example 3.4. Let $\mathbb{X} = \{a, b, c, d\}$ be a universe of discourse and let $R = \{a, b, c\}$ be a subset of $\mathbb{X}$. Let $''+''$ and $''.''$ be binary operations defined on R as shown in the Cayley tables below:

$+$	a	b	c
a	a	b	b
b	c	a	c
c	b	c	a

$.$	a	b	c
a	a	a	a
b	a	c	a
c	a	c	b

It is clear from the table that:

$$\begin{aligned} c + (b + c) &= (c + b) + c = a, \\ a + (b + c) &= b, \text{ but } (a + b) + c = c \neq b. \\ a + c &= c + a = b, \\ a + b &= b, \text{ but } b + a = c \neq b. \end{aligned}$$

This shows that $''+''$ is NeutroAssociative and NeutroCommutative. Hence, $(R, +)$ is a commutative NeutroSemiGroup.

Next, let N_x and I_x represent additive neutral element and additive inverse element respectively with respect to any element $x \in R$. Then

$$\begin{aligned} N_a &= a, \\ I_a &= a. \\ N_b & \quad \text{does not exist,} \\ I_b & \quad \text{does not exist.} \\ N_c &= b, \\ I_c &= a. \end{aligned}$$

Hence, $(R, +)$ is a NeutroAbelianGroup.

Next, consider

$$\begin{aligned} b(cb) &= (bc)b = a, \\ c(bc) &= a \text{ but } (cb)c = b \neq a. \end{aligned}$$

This shows that $(R, .)$ is NeutroAssociative.

Lastly, consider

$$\begin{aligned} a.(b + c) &= a.b + a.c = a, \\ b.(c + a) &= c, \text{ but } b.c + b.a = a \neq c. \\ (b + b).b &= b.b + b.b = a, \\ (b + a).c &= b, \text{ but } b.c + a.c = a \neq b. \\ a.b &= b.a = a, \\ b.c &= a, \text{ but } c.b = c \neq a. \end{aligned}$$

This shows that $''.''$ is both left and right NeutroDistributive over $''+''$ and it is NeutroCommutative. Hence, $(R, +, .)$ is a NeutroCommutativeRing. Since $U_a = a$ that is $aa = a$, it follows that $(R, +, .)$ is a NeutroCommutativeRing with NeutroUnity.

It is observed that $''a''$ is both NeutroIdempotent and NeutroNilpotent element in R. NeutroZeroDivisor elements in R are $''a, b, c''$. Again, R is a NeutroField but not a NeutroIntegralDomain.

Theorem 3.5. *Every NeutroField R is not necessarily a NeutroIntegralDomain.*

Example 3.6. Let $X = \mathbb{Z}_{10}$ and let $\oplus$ and $\odot$ be two binary operations on X defined by $x \oplus y = 2x + y$ and $x \odot y = x + 4y$ for all $x, y \in X$ where $''+''$ is addition modulo 10. Then $(X, \oplus, \odot)$ is a NeutroCommutativeRing. To see this:

(i) $(X, \oplus)$ **is a NeutroAbelianGroup:** Let $x, y, z \in X$. Then

$$\begin{aligned} x \oplus (y \oplus z) &= 2x + 2y + z \text{ and } (x \oplus y) \oplus = 4x + 2y + z, \text{ equating these we have} \\ 2x + 2y + z &= 4x + 2y + z \\ \Rightarrow \quad 2x &= 0 \\ \therefore \quad x &= 0, 5. \end{aligned}$$

Thus, only the triplets $(0, x, y)$ and $(5, x, y)$ can verify the associativity of $\oplus$ (degree of associativity $=$ 20%) and therefore, $\oplus$ is NeutroAsociative.

(ii) **Existence of NeutroNeutral and NeutroInverse elements:** Let $e \in X$ such that $x \oplus e = 2x + e = x$ and $e \oplus x = 2e + x = x$. Then $2x + e = 2e + x$ from which we obtain $e = x$. But then, only $0 \oplus 0 = 0$ and $5 \oplus 5 = 5$ in X (degree of existence of neutral element $= 20\%$). This shows that X has a NeutroNeutral element. It can also be shown that X has a NeutroInverse element.

(iii) **NeutroComutativity of $\oplus$:** Let $x \oplus y = 2x + y$ and $y \oplus x = 2y + x$ so that $2x + y = 2y + x$ from which we obtain $x = y$. This shows that only the duplet (x, x) can verify commutativity of $\oplus$ (degree of commutativity $= 10\%$) that is, $\oplus$ is NeutroCommutative. Hence, $(X, \oplus)$ is a NeutroAbelian-Group.

(iv) $(X, \odot)$ **is a NeutroSemiGroup:** Let $x, y, z \in X$. Then

$$x \odot (y \odot z) = x + 4y + 16z \text{ and } (x \odot y) \odot z = x + 4y + 4z$$

so that $x + 4y + 16z = x + 4y + 4z$ from which we obtain $12z = 0$ so that $z = 0, 5$. Hence, only the triplets $(x, y, 0)$ and $(x, y, 5)$ can verify associativity of $\odot$ (degree of associativity $= 20\%$) and consequently, $(X, \odot)$ is a NeutroSemigroup.

(v) **NeutroDistributivity:** Let $x, y, z \in X$. Then

$$\begin{aligned} x \odot (y \oplus z) &= x + 8y + 4z, (x \odot y) \oplus (x \odot z) = 3x + 8y + 4z \text{ so that} \\ x + 8y + 4z &= 3x + 8y + 4z \\ \Rightarrow 2x &= 0 \\ \therefore \quad x &= 0, 5. \end{aligned}$$

This shows that only the triplets $(0, y, z)$ and $(5, y, z)$ can verify left distributivity of $\odot$ over $\oplus$ (degree of left distributivity $= 20\%$). Again,

$$\begin{aligned} (y \oplus z) \odot x &= 4x + 2y + z, (y \odot x) \oplus (z \odot x) = 12x + 2y + z \text{ so that} \\ 4x + 2y + z &= 12x + 2y + z \\ \Rightarrow 8x &= 0 \\ \therefore \quad x &= 0, 5. \end{aligned}$$

This shows that only the triplets $(0, y, z)$ and $(5, y, z)$ can verify right distributivity of $\odot$ over $\oplus$ (degree of right distributivity $= 20\%$). Thus, $\odot$ is both left and right NeutroDistributive over $\oplus$. Finally, let $x \odot y = x + 4y$ and $y \odot x = y + 4x$. Putting $x + 4y = y + 4x$ we have $x = y$ showing that only the duplet (x, x) can verify the commutativity of $\odot$ (degree of commutativity $= 10\%$). Hence, $\odot$ is NeutroCommutative and accordingly, $(X, \oplus, \odot)$ is a NeutroCommutativeRing.

Theorem 3.7. *Let $(R_i, +, .), i = 1, 2, \cdots, n$ be a family of NeutroRings. Then*

(i) $R = \bigcap_{i=1}^{n} R_i$ *is a NeutroRing.*

(ii) $R = \prod_{i=1}^{n} R_i$ *is a NeutroRing.*

Proof. Obvious. □

Definition 3.8. Let $(R, +, .)$ be a NeutroRing. A nonempty subset S of R is called a NeutroSubring of R if $(S, +, .)$ is also a NeutroRing.

The only trivial NeutroSubring of R is R.

Example 3.9. Let $(R, +, .)$ be the NeutroRing of **Example 3.4** and let $S = \{a, b\}$. The compositions of elements of S are given in the Cayley tables below.

$+$	a	b
a	a	b
b	c	a

$.$	a	b
a	a	a
b	a	c

Then, S is a NeutroSubring of R. To see this:

(i) $(S, +)$ **is a NeutroAbelianGroup:**

$$\begin{aligned} a + (a + b) &= (a + a) + b = b, \\ b + (a + b) &= a, \text{ but } (b + a) + b = c \neq a. \\ N_a &= a, \\ I_a &= a, \\ N_b & \quad \text{does not exist,} \\ I_b & \quad \text{does not exist.} \\ a + a &= a, \text{ but } a + b = b, b + a = c \neq b. \end{aligned}$$

Hence, $(S, +)$ is a NeutroAbelianGroup.

(ii) $(S, .)$ **is a NeutroSemigroup:**

$$\begin{aligned} a(ba) &= (ab)a = a, \\ b(bb) &= a, \text{ but } (bb)b = c \neq a. \end{aligned}$$

This shows that $(S, .)$ is a NeutroSemigroup.

(iii) **NeutroDistributivity:**

$$\begin{aligned} a(a + b) &= aa + ab = a, \\ b(b + a) &= a, \text{ but } bb + ba = b \neq a. \\ (b + a)a &= ba + aa = a, \\ (a + b)a &= c, \text{ but } aa + ba = a \neq c. \end{aligned}$$

This shows that both left and right NeutroDistributivity hold. Accordingly, $(S, +, .)$ is a NeutroRing. Since S is a subset of R, it follows that S is NeutroSubring of R.

Theorem 3.10. *Let $(R, +, .)$ be a NeutroRing and let $\{S_i\}, i = 1, 2, \cdots, n$ be a family of NeutroSubrings of R. Then*

(i) $S = \bigcap_{i=1}^{n} S_i$ is a NeutroSubring of R.

(ii) $S = \prod_{i=1}^{n} S_i$ is a NeutroSubring of R.

Proof. Obvious. □

Definition 3.11. Let $(R, +, .)$ be a NeutroRing. A nonempty subset I of R is called a left NeutroIdeal of R if the following conditions hold:

(i) I is a NeutroSubring of R.

(ii) $x \in I$ and $r \in R$ imply that at least one $xr \in I$ for all $r \in R$.

Definition 3.12. Let $(R, +, .)$ be a NeutroRing. A nonempty subset I of R is called a right NeutroIdeal of R if the following conditions hold:

(i) I is a NeutroSubring of R.

(ii) $x \in I$ and $r \in R$ imply that at least one $rx \in I$ for all $r \in R$.

Definition 3.13. Let $(R, +, .)$ be a NeutroRing. A nonempty subset I of R is called a NeutroIdeal of R if the following conditions hold:

(i) I is a NeutroSubring of R.

(ii) $x \in I$ and $r \in R$ imply that at least one $xr, rx \in I$ for all $r \in R$.

Example 3.14. Let $R = \{a, b, c\}$ be the NeutroRing of **Example 3.4** and let $I = S = \{a, b\}$ be the NeutroSubring of R given in **Example 3.9**. Consider the following:

$$aa = a, ab = a, ac = a, ba = a, bb = c \notin I, bc = a.$$

This shows that I is a left NeutroIdeal of R. Again,

$$aa = a, ba = a, ca = a, ab = a, bb = c \notin I, cb = c \notin I.$$

This also shows that I is a right NeutroIdeal of R. Hence, I is a NeutroIdeal of R.

Theorem 3.15. *Let $(R, +, .)$ be a NeutroRing and let $\{I_i\}, i = 1, 2, \cdots, n$ be a family of NeutroIdeals of R. Then*

(i) $I = \bigcap_{i=1}^{n} I_i$ is a NeutroIdeal of R.

(ii) $I = \sum_{i=1}^{n} I_i$ is a NeutroIdeal of R.

Proof. Obvious. □

Definition 3.16. Let $(R, +, .)$ be a NeutroRing and let I be a NeutroIdeal of R. The set R/I is defined by

$$R/I = \{x + I : x \in R\}. \tag{19}$$

For $x + I, y + I \in R/I$ with at least a pair $(x, y) \in R$, let $\oplus$ and $\odot$ be binary operations on R/I defined as follows:

$$(x + I) \oplus (y + I) = (x + y) + I, \tag{20}$$
$$(x + I) \odot (y + I) = xy + I. \tag{21}$$

Then it can be shown that the tripple $(R/I, \oplus, \odot)$ is a NeutroRing which we call a NeutroQuotientRing.

Example 3.17. Let R be the NeutroRing of **Example 3.4** and let I be its NeutroIdeal of **Example 3.14**. Then

$$\begin{aligned} a + I &= \{a, b\} = I, \\ b + I &= \{a, c\}, \\ c + I &= \{b, c\}, \\ \therefore \quad R/I &= \{a + I, b + I, c + I\} = \{\{a, b\}, \{a, c\}, \{b, c\}\}. \end{aligned}$$

Consider the Cayley tables below:

$\oplus$	$a+I$	$b+I$	$c+I$
$a+I$	$a+I$	$b+I$	$b+I$
$b+I$	$c+I$	$a+I$	$c+I$
$c+I$	$b+I$	$c+I$	$a+I$

$\odot$	$a+I$	$b+I$	$c+I$
$a+I$	$a+I$	$a+I$	$a+I$
$b+I$	$a+I$	$c+I$	$a+I$
$c+I$	$a+I$	$c+I$	$b+I$

It easy to deduce from the tables that $(R/I, \oplus, \odot)$ is a NeutroRing.

Theorem 3.18. *Let I be a NeutroIdeal of the NeutroRing R. Then R/I is a NeutroCommutativeRing with NeutroUnity if and only if R is a NeutroCommutativeRing with NeutroUnity.*

Proof. Easy. □

Definition 3.19. Let $(R, +, .)$ and $(S, +', .')$ be any two NeutroRings. The mapping $\phi : R \rightarrow S$ is called a NeutroRingHomomorphism if ϕ preserves the binary operations of R and S that is if for at least a pair $(x, y) \in R$, we have:

$$\phi(x + y) = \phi(x) +' \phi(y), \tag{22}$$

$$\phi(x.y) = \phi(x).'\phi(y). \tag{23}$$

The kernel of ϕ denoted by $Ker\phi$ is defined as

$$Ker\phi = \{x : \phi(x) = e_R\} \tag{24}$$

where $e_R \in R$ is such that $N_r = e_R$ for at least one $r \in R$.

The image of ϕ denoted by $Im\phi$ is defined as

$$Im\phi = \{y \in S : y = \phi(x) \text{ for at least one } y \in S\}. \tag{25}$$

If in addition ϕ is a NeutroBijection, then ϕ is called a NeutroRingIsomorphism and we write $R \cong S$. NeutroRingEpimorphism, NeutroRingMonomorphism, NeutroRingEndomorphism and NeutroRingAutomorphism are defined similarly.

Example 3.20. Let R be the NeutroRing of **Example 3.4** and let $\phi : R \times R \rightarrow R$ be a mapping defined by

$$\phi((x, y)) = \begin{cases} a & \text{if } x = y, \\ d & \text{if } x \neq y. \end{cases}$$

It can be shown that ϕ is a NeutroRingHomomorphism. The $Ker\phi = \{(a, a), (b, b), (c, c)\}$ which is a NeutroSubgRing of $R \times R$ as can be seen in the Cayley tables below.

$+$	(a,a)	(b,b)	(c,c)
(a,a)	(a,a)	(b,b)	(b,b)
(b,b)	(c,c)	(a,a)	(c,c)
(c,c)	(b,b)	(c,c)	(a,a)

$.$	(a,a)	(b,b)	(c,c)
(a,a)	(a,a)	(a,a)	(a,a)
(b,b)	(a,a)	(c,c)	(a,a)
(c,c)	(a,a)	(c,c)	(b,b)

$Im\phi = \{a, d\} \nsubseteq R$.

Theorem 3.21. *Let R and S be two NeutroRings. Let $N_x = e_R$ for at least one $x \in R$ and let $N_y = e_S$ for at least one $y \in S$. Suppose that $\phi : R \rightarrow S$ is a NeutroRingHomomorphism. Then:*

(i) $\phi(e_R)$ is not necessarily equals e_S.

(ii) $Ker\phi$ is a NeutroSubring of R.

(iii) $Im\phi$ is not necessarily a NeutroSubring of S.

(iv) ϕ is NeutroInjective if and only if $Ker\phi = \{e_R\}$ for at least one $e_R \in R$.

Example 3.22. Let $R = \{a, b, c\}$ be the NeutroRing of **Example 3.4** and let $I = \{a, b\}$ be the NeutroIdeal of R given in **Example 3.14**. Let $\phi : R \rightarrow R/I$ be a mapping defined by $\phi(x) = x + I$ for at least one $x \in R$. Then $\phi(a) = a + I = \{a, b\} = I, \phi(b) = b + I = \{a, c\}$ and $\phi(c) = c + I = \{b, c\}$ from which we obtain that ϕ is a NeutroRingHomomorphism.

$$Ker\phi = \{x \in R : \phi(x) = e_{R/I}\} = \{x \in R : x + I = e_{R/I} = a + I\} = I.$$

Theorem 3.23. *Let I be a NeutroIdeal of a NeutroRing R. Then the mapping $\psi : R \rightarrow R/I$ defined by*

$$\psi(x) = x + I \quad \text{for at least one } x \in R$$

is a NeutroRingEpimomorphism and the $Ker\psi = I$.

Theorem 3.24. *Let $\phi : R \rightarrow S$ be a NeutroRingHomomorpism and let $K = Ker\phi$. Then the mapping $\psi : R/K \rightarrow Im\phi$ defined by*

$$\psi(x + K) = \phi(x) \quad \text{for at least one } x \in R$$

is a NeutroRingIsomorphism.

Proof. Let $x + K, y + K \in R/K$ with at least a pair $(x, y) \in R$. Then

$$\begin{aligned}
\psi((x+K)\oplus(y+K)) &= \psi((x+y)+K) \\
&= \phi(x+y) \\
&= \phi(x)+\phi(y) \\
&= \psi(x+K)\oplus\psi(y+K). \\
\psi((x+K)\odot(y+K)) &= \psi((xy)+K) \\
&= \phi(xy) \\
&= \phi(x)\phi(y) \\
&= \psi(x+K)\odot\psi(y+K). \\
Ker\psi &= \{x+K \in R/K : \psi(x+K) = e_{\phi(x)}\} \\
&= \{x+K \in R/K : \phi(x) = e_{\phi(x)}\} \\
&= \{e_{R/K}\}.
\end{aligned}$$

This shows that ψ is a NeutroBijectiveHomomorphism and therefore it is a NeutroRingIsomorphism that is $R/K \cong Im\phi$ which is the same as what is obtainable in the classical rings. □

Theorem 3.25. *NeutroRingIsomorphism of NeutroRings is an equivalence relation.*

Proof. The proof is the same as the classical rings. □

4 Conclusion

We have for the first time introduced in this paper the concept of NeutroRings by considering three NeutroAxioms (NeutroAbelianGroup (additive), NeutroSemigroup (multiplicative) and NeutroDistributivity (multiplication over addition)). Several interesting results and examples on NeutroRings, NeutroSubgrings, NeutroIdeals, NeutroQuotientRings and NeutroRingHomomorphisms are presented. It is shown that the 1st isomorphism theorem of the classical rings holds in the class of NeutroRings. More advanced properties of NeutroRings will be presented in our future papers. Other NeutroAlgebraicStructures such as NeutroModules, NeutroVectorSpaces etc are opened to be developed and studied by other Neutrosophic researchers.

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