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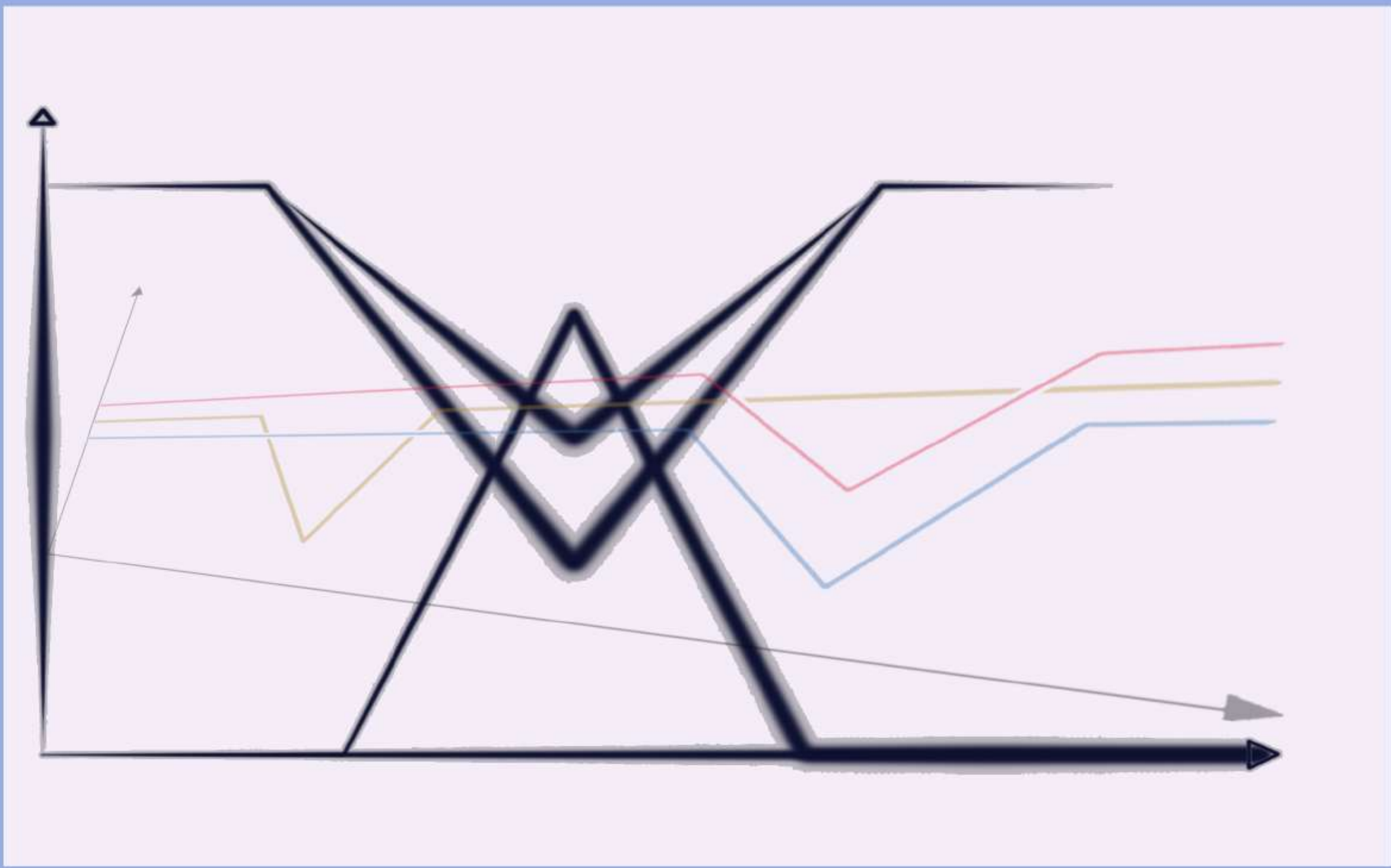

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Supplier Selection Using Fuzzy AHP Method and D-Numbers

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A B S T R A C T

Success in supply begins with the right choice of suppliers and in the long run is directly related to how suppliers are managed, because suppliers have a significant impact on the success or failure of a company. Multi-criteria decisions are approaches that deal with ranking and selecting one or more suppliers from a set of suppliers. Multi-criteria decisions provide an effective framework for comparing suppliers based on the evaluation of different criteria. The present research is applied based on the purpose and descriptive-survey based on the nature and method of the research. In the present study, two library and field methods have been used to collect information. According to the objectives of this study, suppliers will be evaluated using two methods of fuzzy hierarchical analysis with D-numbers. In order to better understand these two methods, a case study is presented in which suppliers are ranked using two methods and then the results are compared with each other. For manufacturing companies, 4 categories of parts were considered and based on the classification, the suppliers of the manufacturing company were evaluated and analyzed. In the results of suppliers of type A and B components in hierarchical analysis, D and fuzzy methods have many differences in the evaluation and ranking of suppliers, and this shows the lack of expectations of experts in D and fuzzy analysis. On the other hand, in type C and D components, the classification and ranking of suppliers have been matched in two ways and shows that the opinions in the evaluation of these suppliers are the same.

1. Introduction

In recent years, much attention has been paid to the importance of selecting suppliers and supply chain management to allocate orders. Thus, in this regard, it focuses on identifying the key factors affecting the optimal selection of suppliers in the supply chain in industries [1]. Also, with the acceleration of the process of globalization and the increasing facilitation of communication, the manager's perception of the environment becomes more complex, uncertain and ambiguous [2]. Existence of numerous and unstable information and variables affecting the consequences of the decision, challenges the manager to make the right and fast decision. Although human beings have always faced the challenge of decision making, it is no exaggeration to say that the subject of decision making has never been so complicated [3]. Therefore, along with the growth of human knowledge, various thinkers have addressed the issue of decisions and methods that can make this process easier and safer. One of the most important multi-criteria decisions that has attracted the attention of researchers in the organization is the choice of

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supplier in the supply chain of the organization [4]. This is due to the fact that in the current competitive environment, the process of effective selection of suppliers is very important in the success of any production organization [5]. In fact, success in procurement begins with the right choice of suppliers and in the long run depends directly on how suppliers are managed, because suppliers have a significant impact on the success or failure of a company [6]. Choosing the right supplier requires consideration of several criteria. Many decision makers or experts choose suppliers based on their own experiences and tastes, which are purely subjective and personal. Multi-criteria decisions are approaches that deal with ranking and selecting one or more suppliers from a set of suppliers. Multi-criteria decisions provide an effective framework for comparing suppliers based on the evaluation of different criteria [7]. At present, in order to solve the problem of evaluating supplier performance according to one criterion or determining the importance of a number of criteria with high accuracy, multi-criteria decision-making vocabulary is used by both researchers and experts [8]. On the other hand, multiple criteria decision making technique and besides multiple objective decision making can consider several goals in order of the decision maker's priority. In multi-objective planning, the decision maker has the ability to formulate conflicting goals in the form of a linear equation under the objective function and on the other hand to formulate real constraints such as purchase budget, capacity, etc. under the constraints of suppliers. Solving this model can determine the amount of materials received from each supplier in a way that provides the maximum amount of optimization and also covers the amount of aspirations for each goal [9]. The combination of these two techniques can create a model that takes into account different ideals while considering different criteria. For more than two decades, supply chain management and the supplier selection process have received considerable attention in the literature. Many factories and industry owners have been looking for ways to partner with suppliers to increase their management performance and competitiveness on the global stage. The quality of the supplier base affects the competitiveness of companies. The continuity of the relationship between suppliers and industry owners causes the company's supply chain to be a serious and strong obstacle in the way of competitors. Also, establishing a long-term relationship with the supplier will reduce the costs of the supplier and reduce the costs of the supplier will lead to a reduction in the costs of the organization (employer) (mutual benefit). On the other hand, a stable relationship causes the supplier to follow the rules and standards of the employer and the organization uses the facilities available to the suppliers such as engineering technical facilities (benefit to the organization). Therefore, the decision to select the best supplier for supply chain management is essential [10]. One of the most important issues in designing a supply chain is the issue of supplier selection. The complexity of this issue is in fact because each of the suppliers meets part of the buyer criteria, and the choice between them is in fact a Multiple Criteria Decision Making (MCDM) that requires a structured and systematic approach, and without it an important decision is likely to fail. With the help of computers, decision-making techniques have become very acceptable in all areas of the decision-making process. Therefore, the application of multi-criteria decision making methods for users, due to the mathematical complexity, has become very easy to implement. Decision making is the process of finding the best option from a range of available options. In fact, choosing the right set of suppliers to work with is crucial to a company's success, and the emphasis on supplier selection has been emphasized for many years [11]. There are different techniques and methods for making multiple fuzzy criteria that have different advantages and disadvantages over each other. A supply chain is a series of organizations involved in the production and delivery of a product or service. This chain starts with raw material suppliers and continues to the end customer. Supply chain management is one of the effective and efficient approaches that reduces production costs and waiting time. This attitude facilitates the provision of better customer service and ensures the opportunity for effective monitoring of transportation systems, inventory and distribution networks. In this way, the organization can exceed the expectations and demands of customers. Today, organizations are facing customers who want high product diversity, low costs, high quality and fast response. Organizations are well aware that they need an efficient supply chain to be able to compete in today's global marketplace and interconnected network economy [12]. Many experienced companies believe that choosing a supplier is the most important activity of an organization. Also, since the performance of suppliers has a major impact on the success or failure of a chain, selecting a supplier is now considered a strategic task. As a result, wrong decisions in choosing suppliers will have many negative consequences and losses for the company. Therefore, finding the right methods to select the right suppliers, which are the most important components of the supply chain, is very important. On the other hand, because raw materials and parts are the most important part of a company's costs, proper purchasing management is of considerable importance to the efficiency, effectiveness and profitability of an organization. On the other hand, today, due to new concepts of supply chain management and similar cases that lead to partnerships with suppliers and the company's close relationship with suppliers, suppliers and customers are no longer recognized as competitors of the organization. Rather, they are members of a core set called the supply chain, each of which aims to maximize profits and increase the productivity of the entire chain. Nosrati and Jafari Eskandari [13] in their research to design a supply chain network considering sustainability. The supply chain network model is considered to be uncertain and includes uncertain parameters (demand, shipping costs, and operating activity) that exist, which is a pessimistic possibility to control the model through robust optimization method. Therefore, by considering the conflicting goals of the supply chain network, including minimizing the total network costs and minimizing the amount of greenhouse gas emissions, the community-based multi-objective decision-making methods and refrigeration simulation algorithm have been used to solve the model. The results of T-Test statistical test on the means of the first, second, and computational objective functions show that there is a significant difference between the means of computational time. Sensitivity analysis performed on some parameters of the model also shows that reducing network costs and reducing greenhouse gas emissions increases the supply capacity and reduces the discount period for the purchase of raw materials.

Qasbeh [14] in his research stated that the key to success in the competition scene is to focus more on the main activities and goals of the organization. Since the 1980s, many managers of large organizations have decided to outsource activities that are not of strategic importance to the organization.

In their research, Shafia et al. [15] presented a new framework for evaluating suppliers by considering risk factors using decision-making techniques and two-level data envelopment analysis approach. In the first step, the criteria of the hierarchical analysis process were weighted with the opinion of experts and then used the data envelopment analysis approach to evaluate.

Mardani [16] stated in their research that frequent discussions related to supply chain sustainability events show that companies with a global presence are trying to improve the environmental, social and economic outcomes of global supply chains. They proposed sustainable supply chain management to improve the results of sustainability in supply chains.

Ghadimi [17] stated in his research that in the last two decades, the issue of sustainable supply chain has attracted the attention of many academics and professionals. In this regard, resources, maintenance and recycling, as well as their pairs (i.e., resources and maintenance, maintenance and recycling) have provided a platform for the exchange of technical, economic, institutional and policy aspects to help move societies towards sustainability.

Rifaki [18] stated in his research that the supply chain plays an important role in today's global economy. Therefore, in order to closely pursue sustainable business, a dynamic understanding of the issues affecting sustainability in supply chains must be formed. However, this field of research is still unknown due to limited theoretical knowledge and practical application.

2. Research Method

The approach of the present research is quantitative and qualitative according to the intended objectives. Therefore, the present research is of an applied type. Also, the present research is a field research in terms of implementation. Because in this research, the relationships between variables are expressed in the form of decision model, using fuzzy techniques and D numbers and variables are observed, measured and described, so the type of research method is descriptive-analytical. According to the objectives of this study, supplier evaluation will be evaluated using two methods of fuzzy hierarchical analysis and D-numbers. First, using the common and widely used method of multi-criteria decision making, namely the Analytical Hierarchy Process (AHP) using mathematics based on fuzzy sets, a method has been proposed to select the suppliers of a supply chain. Then this problem is evaluated again by combining the two methods of AHP and D numbers. Finally, in order to achieve the desired results, the results of these two methods will be compared with each other. In order to better understand these two methods, a case study is presented in which suppliers are ranked using two methods and then the results are compared with each other.

2.1. Basic Concepts in Dempster-Scheffer Theory or Belief Function

The detection framework in Dempster-Scheffer theory is a set of two by two separated elements or propositions, and if the set $x=\{x_1, x_2,\ldots,x_n\}$is a set of elements or propositions, the sample space or detection framework is displayed as $\Omega = 2^x$. This set is a set of all sub-sets of X as follows:
$\Omega = \{\{X_1\}, \{X_2\},\ldots,\{X_n\},\{X_1, X_2\},\ldots, \{X_1, X_2,\ldots,X_n\}\}$.
If A1 = {X1}, A2 = {X2},... are sets belonging to the detection frame, the probability mass function or the detection function of the set Ai on the detection frame is displayed as m (Ai), which has the following conditions:

$$\mathrm{m(Ai)} \geq 0\ , \qquad \mathrm{Ai} \in \Omega$$

$$\mathrm{m}(\emptyset) = 0$$

$$\sum_{A\in 2^x} \mathrm{m(Ai)} = 1\ .$$

The most accurate belief that can be obtained from the correctness or occurrence of set A from the framework of diagnosis and based on the available evidence is called belief function. This function is the sum of the mass of probabilities determined for the elements in set A and is calculated as follows:

$$\mathrm{bel(A)} = \sum_{\mathrm{A_i}} \mathrm{m(b)}.$$

Contrary to the probability theory, *bel* (A) = 0 means lack of evidence about set A; While p (A) = 0 means the impossibility of this set, while bel (A) = 1 means the certainty of the occurrence of event A and it is similar to the probability p (A) = 1, which means the certainty of the correctness of the set A. The maximum possible belief for the correctness of set A, which is determined on the basis of evidence, is called the possibility function. This function is the sum of the total probability masses of the existing elements of the detection framework with zero inter section by set A. It is defined as follows:

$$pl(A) = \sum_{A_i} m(b).$$

The probability value of set A can be defined as the complement of not being belief of A, or in other words, lack of evidence showing A is true:

$$pl(A) = 1 - bel(\sim A).$$

$pl(A) = 0$ means that the set A is impossible or similarly $p(A) = 0$. Also $pl(A) = 0$is equal to $bel(\sim A) = 1$. This means that if event A is impossible based on the evidence, then A is certainly not true. The degree of uncertainty or degree of doubt in determining the magnitude of belief and possibility based on available evidence is the distance between belief in the occurrence or correctness of set A and unbelief in the occurrence or inaccuracy of set A in the context of diagnosis and is defined as follows:

$$U = 1 - bel(A) - bel(\sim A).$$

Suppose A∈Ω, set A is defined according to the above definitions and using the belief sizes $bel(A), U(A)$ and $bel(\sim A)$ as follows:

$$s = \{(bel(A), u(A), bel(\sim A)/A \in \Omega\}.$$

So that for each set A of the detection framework, and $bel(\sim A)\in [0,1]$ and U (A) and $bel(A)$and their sum for A∈Ω is as follows:

$$0 \leq bel(A) + u(A) + bel(\sim A) \leq 1.$$

Hence, according to Dempster-Schaffer theory, the generated D numbers will be as follows:
For the discrete set Ω= {b1, b2, …, bi, …, bn} such that bi ϵ R and bi ≠ bj if i ≠ j, a special form of numbers is expressed as follows:

$$D(\{b1\}) = v1,$$
$$D(\{b2\}) = v2,$$
$$D(\{bi\}) = vi,$$
$$D(\{bn\}) = vn.$$

Or more simply D = {(b1, v1), (b2, v2),…, (bi, vi),… (bn, vn)} such that vi> 0 and if two numbers D, D1 and D2 exist, they will be as follows:

$$D_1 = \left\{\left(b_1^1, v_1^1\right), \ldots, \left(b_i^1, v_i^1\right), \ldots, \left(b_n^1, v_n^1\right)\right\},$$
$$D_2 = \left\{\left(b_1^2, v_1^2\right), \ldots, \left(b_j^2, v_j^2\right), \ldots, \left(b_m^2, v_m^2\right)\right\}.$$

The combination of D1 and D2 is shown and calculated as follows:

$$b(b) = v,$$
$$b = \frac{b_i^1 + b_j^2}{2},$$
$$v = \frac{V_i^1 + V_j^2}{2} / C,$$
$$\begin{cases} \sum_{j=1}^{m}\sum_{i=1}^{n}\left(\frac{V_i^1 + V_j^2}{2}\right), & \sum_{i=1}^{n} V_i^1 = 1 \quad \text{and} \sum_{j=1}^{m} V_j^2 = 1; \\ \sum_{j=1}^{m}\sum_{i=1}^{n}\left(\frac{V_i^1 + V_j^2}{2}\right) + \sum_{j=1}^{m}\left(\frac{V_c^1 + V_j^2}{2}\right), & \sum_{i=1}^{n} V_i^1 < \text{ and } \sum_{j=1}^{m} V_j^2 = 1; \\ \sum_{j=1}^{m}\sum_{i=1}^{n}\left(\frac{V_i^1 + V_j^2}{2}\right) + \sum_{i=1}^{n}\left(\frac{V_i^1 + V_c^2}{2}\right), & \sum_{i=1}^{n} V_i^1 < \text{ and } \sum_{j=1}^{m} V_j^2 = 1; \\ \sum_{j=1}^{m}\sum_{i=1}^{n}\left(\frac{V_i^1 + V_j^2}{2}\right) + \sum_{j=1}^{m}\left(\frac{V_c^1 + V_j^2}{2}\right) & \\ +\sum_{i=1}^{n}\left(\frac{V_i^1 + V_j^2}{2}\right) + \frac{V_c^1 + V_c^2}{2} & \sum_{i=1}^{n} V_i^1 < \text{ and } \sum_{j=1}^{m} V_j^2 < 1; \end{cases}$$

such that $V_c^1 = 1 - \sum_{i=1}^{n} V_i^1, \; V_c^2 = 1 - \sum_{i=1}^{m} V_i^2$.

It should be noted that hybrid operations do not maintain corporate property, so D numbers can be combined correctly and efficiently:

$$(D1 \oplus D2) \oplus D3 \neq (D1 \oplus D3) \oplus D2 \neq (D2 \oplus D3) \oplus D1.$$

If D = {(b1, v1), (b2, v2),…, (bi, vi),… (bn, vn)} is a D number, the consensus operator D is defined as follows:

$$I(D) = \sum_{i=1}^{n} b_i v_i.$$

3. Findings

3.1. Evaluation of Suppliers Based on AHP Method with Theory D

To evaluate suppliers based on approach D in AHP method, we perform the following steps: In this section, 8 expert opinions will be evaluated and analyzed based on three criteria of cost, time and quality, and based on the collected opinions; first the opinions will be evaluated. We will scale and then formulate a decision matrix in which experts present their views to each supplier at a brainstorming session. According to the evaluation of suppliers for the classification of parts, this section evaluates and weighs the indicators based on the average opinions of experts, which is the final weight from the experts' point of view (*Table 1* values are calculated based on the percentage of importance).

Table 1. *Weight of criteria from the perspective of experts.*

	C1	C2	C3		C1	C2	C3
Expert 1	*0.0778*	*0.4868*	*0.4355*	**Expert 5**	*0.0993*	*0.5109*	*0.3897*
Expert 2	*0.1694*	*0.4431*	*0.3875*	**Expert 6**	*0.2411*	*0.2101*	*0.5488*
Expert 3	*0.3278*	*0.2611*	*0.4111*	**Expert 7**	*0.1186*	*0.6123*	*0.2691*
Expert 4	*0.1146*	*0.4798*	*0.4057*	**Expert 8**	*0.4002*	*0.233*	*0.3668*

Hence the display of D numbers for A1 is as shown in *Table 2*.

According to *Table 2*, evaluations are performed for the other 25 suppliers. According to the evaluation, in the next step, the combination of D numbers will be done. Therefore, based on the following relation, the numbers will be combined as *Table 3*.

$$DA1 = D11 + D12 + D13 + D14 + D15 + D16 + D17 + D18.$$

Table 2. *Display of D numbers.*

A1	D numbers
Expert 1	*D11= [(0.56,0.4355), (0.66,0.4868), (0.28,0.0778)]*
Expert 2	*D12= [(0.25,0.3875), (0.4,0.4431), (0.2,0.1694)]*
Expert 3	*D13= [(0.09,0.4111), (0.71,0.2611), (0.41,0.3278)]*
Expert 4	*D14= [(0.46,0.4057), (0.61,0.4798), (0.24,0.1146)]*
Expert 5	*D15= [(0.33,0.3897), (0.65,0.5109), (0.43,0.0993)]*
Expert 6	*D16= [(0.34,0.5488), (0.45,0.2110), (0.46,0.2411)]*
Expert 7	*D17= [(0.08,0.2691), (0.72,0.6123), (0.25,0.1186)]*
Expert 8	*D18= [(0.43,0.3668), (0.82,0.2330), (0.45,0.4002)]*

According to the accepted evaluation, the suppliers' ranking for category A parts is as shown in *Table 3*.

Table 3. *Ranking of suppliers of category A parts.*

Suppliers	A1	A2	A3	A4	A5	A6
I (D)	*0.3869*	*0.2886*	*0.3420*	*0.2024*	*0.2032*	*0.3640*
ranking	*4*	*8*	*7*	*12*	*11*	*5*
Suppliers	**A7**	**A8**	**A9**	**A10**	**A11**	**A12**
I (D)	*0.2483*	*0.3981*	*0.4378*	*0.3908*	*0.2716*	*0.3616*
ranking	*10*	*2*	*1*	*3*	*9*	*6*

As can be seen, supplier A9 with a weight of 0.4378 was in the first place and supplier A8 with a weight of 0.3981 were in the second place. Also, the ranking of suppliers of category B parts is as shown in *Table 4*.

Table 4. *Ranking of suppliers of parts category B.*

Suppliers	A13	A14	A15	A16	A17	A18
I (D)	*0.3526*	*0.3077*	*0.3377*	*0.2326*	*0.3826*	*0.3829*
ranking	*3*	*5*	*4*	*6*	*1*	*2*

According to the evaluation performed on the category B parts, supplier A17 with a weight of 0.3826 was in the first place and supplier a18 with a weight of 0.3829 were in the second place. The rating of suppliers of category C parts is as shown in *Table 5*.

Table 5. *Ranking of suppliers of parts category C.*

Suppliers	A19	A20	A21	A22
I (D)	*0.4916*	*0.4892*	*0.3038*	*0.3731*
ranking	*1*	*2*	*4*	*3*

For category C, suppliers A19 with a weight of 0.4916 were ranked first and A20 with a weight of 0.4892 were ranked second. The ranking of suppliers of D parts is as shown in *Table 6*.

Also, suppliers were classified for type D components and all suppliers were evaluated and analyzed by D numbers in the hierarchical analysis method. In the next step, suppliers were ranked using the fuzzy AHP method approach.

Table 6. *Ranking of suppliers of parts category D.*

Suppliers	A23	A24	A25
I (D)	*0.5851*	*0.5400*	*0.4441*
ranking	*1*	*2*	*3*

3.2. Evaluation and Ranking of Suppliers Based on F-AHP Method

In this section, 25 suppliers identified for 4 types of parts required for supply in manufacturing companies will be evaluated and analyzed based on the fuzzy AHP method, which are in three steps as shown in *Figs. (1)-(3)*.

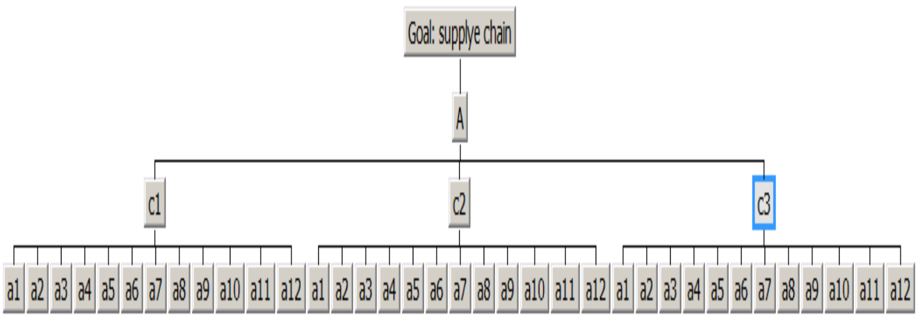

Fig. 1. *Step1: cluster the levels in expert choice software.*

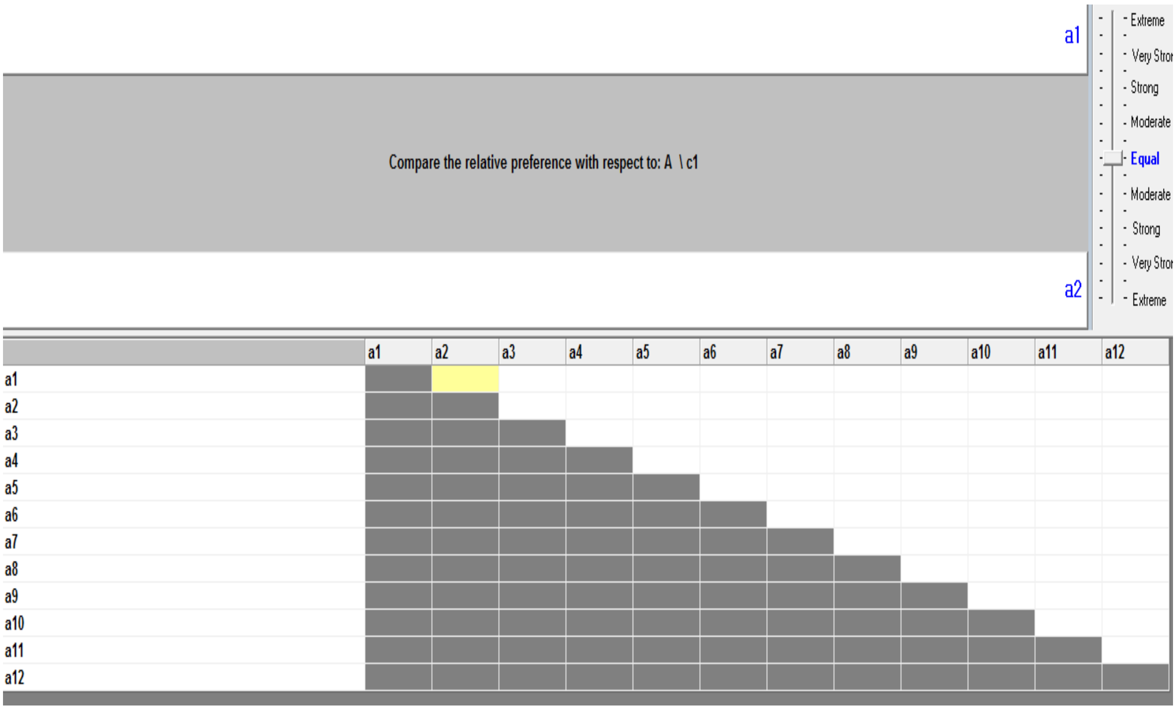

Fig. 2. *Step 2: matrix of pairwise comparison of indicators based on the mode of expert opinions.*

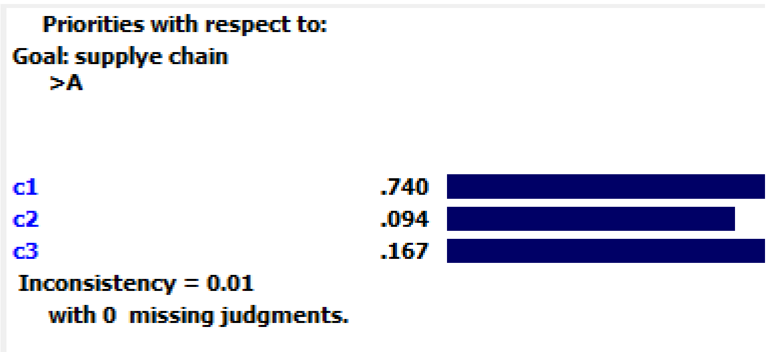

Fig. 3. *Step 3: obtain the weight of the indicators.*

As can be seen, the cost index with a weight of 0.740 was in the first place and the delivery time index with a weight of 0.167 was in the second place and the quality index with a weight of 0.094 were in the third place.

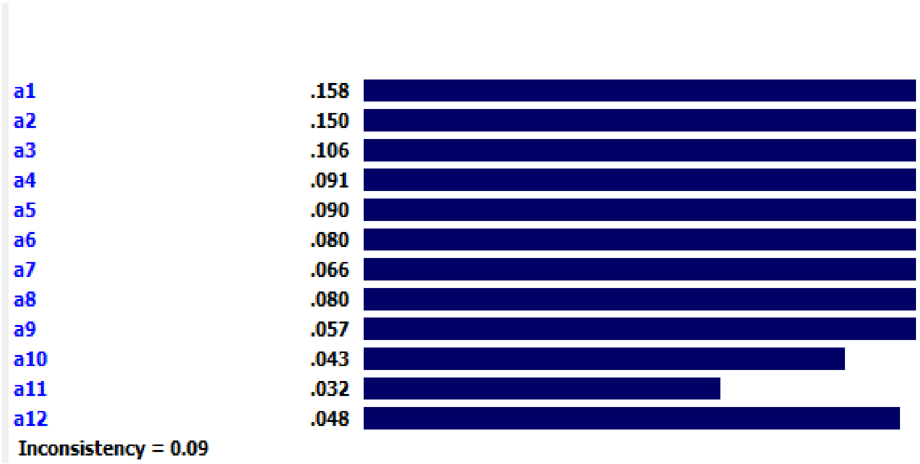

Fig. 4. *Supplier ratings for category A components.*

According to *Fig. 4*, Supplier A1 with a weight of 0.158 was in the first place and supplier A2 with a weight of 0.150 was in the third place and A3 with a weight of 0.106 was in the third place. Also, the sensitivity analysis of indicators and suppliers is as shown in *Fig. 5*.

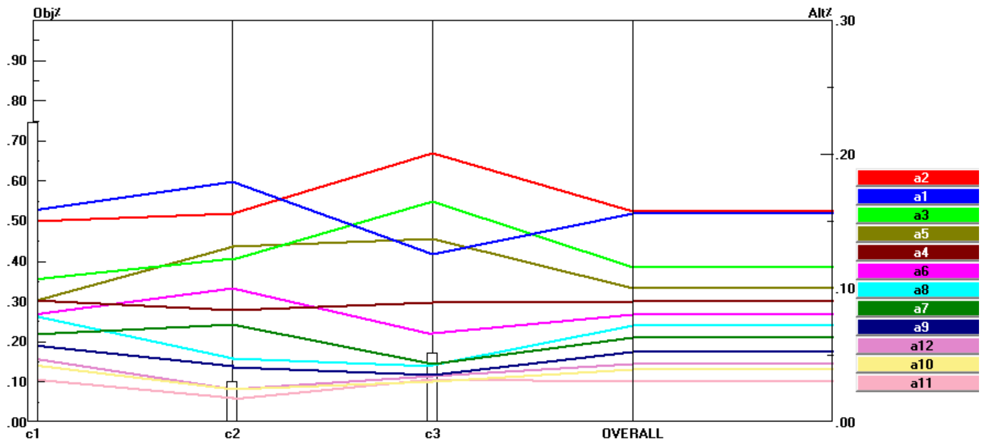

Fig. 5. *The sensitivity analysis of indicators and suppliers for category A.*

Supplier ratings for Type B components are shown in *Fig. 6*.

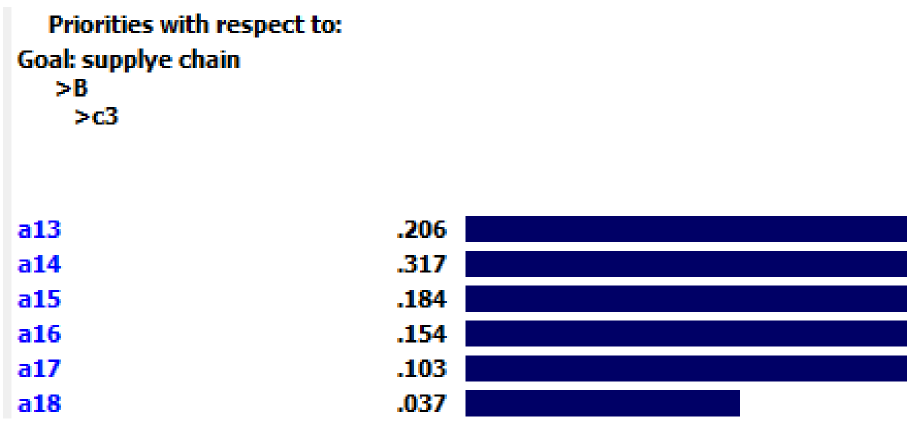

Fig. 6. *Supplier ratings for category B components.*

As can be seen, supplier 14 with a weight of 0.317 was in the first place and supplier A13 with a weight of 0.206 were in the second place. The sensitivity analysis of the assessment is as shown in *Fig. 7*.

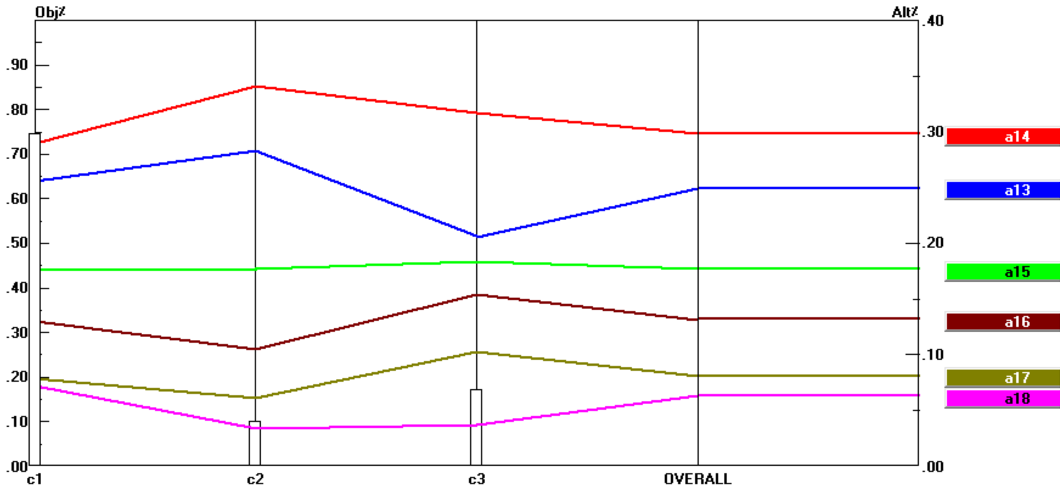

Fig. 7. *The sensitivity analysis of indicators and suppliers for category B.*

The evaluation for category C parts is shown in *Fig. 8*.

Fig. 8. *The evaluation for category C parts.*

According to the evaluation, supplier A19 with a weight of 0.473 was in the first place and A20 with a weight of 0.332 were in the second place. The sensitivity analysis indicators and suppliers for category C is as shown in *Fig. 9*.

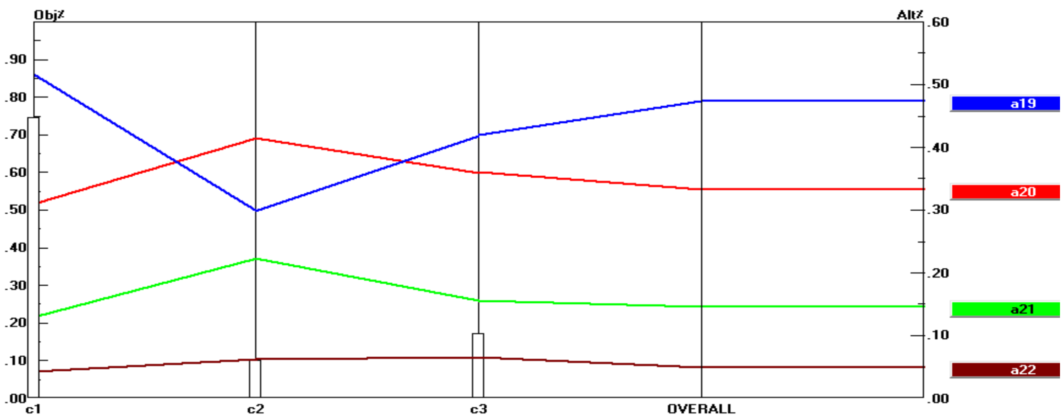

Fig. 9. *The sensitivity analysis of indicators and suppliers for category C.*

Then, for the category of type D parts, the evaluation is as *Fig. 10*.

Fig. 10. *The evaluation for category D parts.*

We also have a sensitivity analysis performed for category D which is shown in *Fig. 11*.

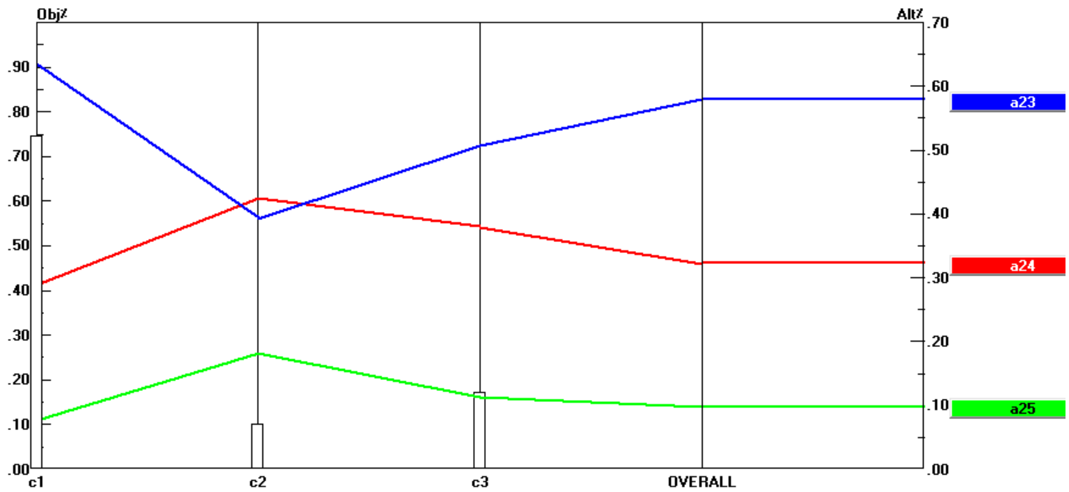

Fig. 11. *The sensitivity analysis of indicators and suppliers for category D.*

As shown in *Fig. 11*, suppliers A23 with a weight of 0.579 were in first place and A24 with a weight of 0.322 were in third place.

4. Conclusion

Today, the demands of customers along with the advancement of technology, are widely and constantly changing. This has caused the life cycle of products to be shorter and business organizations must launch a variety of products with desirable features to attract customer attention and satisfaction [19]. For this reason, in order to stay competitive, most organizations consider outsourcing the product parts to suppliers who have the technology and special ability in that field in their management, and design and produce the main parts themselves. They pay. This solution requires effective communication with suppliers and has made the issue of selecting and evaluating suppliers an important principle in the supply chain [20]. In evaluating suppliers, the most important criteria that have the greatest impact on this process must first be identified. In previous studies, criteria and indicators such as price, quality, and delivery time have been considered important in evaluating and selecting suppliers [21].Wang [22] concluded from customer research that price and quality, delivery time, and performance history are important factors. Therefore, based on two models of hierarchical analysis with D and fuzzy numbers in the evaluation of the supply chain of the manufacturing company was discussed. Therefore, 4 categories of parts were considered for manufacturing companies and based on the classification; the suppliers of the manufacturing company were evaluated and analyzed. In the results obtained from suppliers of type A and B components in the hierarchical analysis of D and fuzzy methods, there are many differences in the evaluation and ranking of suppliers, and this shows the lack of expectations of experts in D and fuzzy analysis. On the other hand, in type C and D components, the classification and ranking of suppliers have been matched in two ways and it has been shown that the opinions in the evaluation of these suppliers are the same. Like any other research, conducting this research was faced with many obstacles and problems, some of which were eliminated and some others changed the direction of the research or limited the application of the results. These restrictions include:

- Some of the contracts between the manufacturing company and the suppliers of raw materials are related to previous years, which make the price and other influential factors of these suppliers different from other suppliers that have signed a contract this year and makes it influential in choosing suppliers.
- Due to price fluctuations and market demand, it is possible to change the company's production volume. Therefore, what is considered in this study as the technical requirements of the product is without considering the product development.
- Due to the current currency situation, some suppliers are not willing to cooperate with the company due to the export of their products, which can complicate the research process and affect the choice of supplier by the company.

Considering that so far, the selection of suppliers has been done according to the needs of the company and in order to meet it, based on the intuitive judgments of experts, and the experts used to compare suppliers based on their own judgments. It is suggested that from now on, using the results of this study, the selection of suppliers in this company and other similar companies be done by collecting the required information of the models in a systematic and scientific manner. During the different stages of this research, new points were realized and as the research progressed, more ambiguities were created in front of the researcher, which due to the existing limitations, their study requires more research. Therefore, for the research of future researchers who intend to work in this field, some topics are suggested:

- To increase accuracy and reduce uncertainty in prioritizing criteria and suppliers and allocating the optimal order amount to each supplier, it is suggested to combine this model with neural network models and fuzzy logic and compare it with the results of this study.
- It is suggested to provide a comprehensive model related to similar organizations and large companies by examining other similar companies that covers all the criteria of the companies involved.
- It is suggested that the indicators be tested in similar companies based on the conceptual model or structural model in order to identify the supply framework.
- Using the gray approach to develop the accuracy of the answers obtained
- Using the heuristic factor analysis approach to identify customers' technical requirements.
- Use of fuzzy Delphi approach in order to identify the technical requirements of the product.

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A New Approach For Ranking Of Intuitionistic Fuzzy Numbers

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P A P E R I N F O	A B S T R A C T
Chronicle: *Received: 08 October 2019* *Revised: 16 December 2019* *Accepted: 08 January 2020* **Keywords:** *Intuitionistic Fuzzy Sets.* *Intuitionistic Fuzzy Numbers.* *Trapezoidal Intuitionistic.* *Fuzzy Numbers.* *Triangular Intuitionistic Fuzzy Numbers.* *Magnitude of Intuitionistic Fuzzy Number.*	The concept of an intuitionistic fuzzy number (I F N) is of importance for representing an ill-known quantity. Ranking fuzzy numbers plays a very important role in the decision process, data analysis and applications. The concept of an intuitionistic fuzzy number (IFN) is of importance for quantifying an ill-known quantity. Ranking of intuitionistic fuzzy numbers plays a vital role in decision making and linear programming problems. Also, ranking of intuitionistic fuzzy numbers is a very difficult problem. In this paper, a new method for ranking intuitionistic fuzzy number is developed by means of magnitude for different forms of intuitionistic fuzzy numbers. In Particular ranking is done for trapezoidal intuitionistic fuzzy numbers, triangular intuitionistic fuzzy numbers, symmetric trapezoidal intuitionistic fuzzy numbers, and symmetric triangular intuitionistic fuzzy numbers. Numerical examples are illustrated for all the defined different forms of intuitionistic fuzzy numbers. Finally some comparative numerical examples are illustrated to express the advantage of the proposed method.

1. Introduction

Atanassov [1] introduced the concept of Intuitionistic Fuzzy Sets (IFS) which is a generalization of the concept of fuzzy set. In IFS the degree of non-membership denoting the non-belonging of an element to a set is explicitly specified along with the degree of membership.

In many real world problems, due to insufficiency in the information available, the evaluation of membership values is not possible up to our satisfaction. Also the evaluation of non –membership values is not always possible and there remains an indeterministic part in which hesitation survives. A fuzzy number plays a vital role in representation of such unknown quantity. Following this concept, the generalized concept of intuitionistic Fuzzy Number (IFN) introduced by Grzegrorzewski [5] in 2003 receives high attention and different definitions of IFN's have been proposed. Grzegrorzewski [6] defined two families of metrics in the space of IFNs and proposed a ranking method of IFNs.

Mitchell [9] interpreted an IFN as an ensemble of fuzzy numbers and introduced a ranking method. Wang [18] gave the definition of intuitionistic trapezoidal fuzzy number and interval intuitionistic

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trapezoidal fuzzy number. Based on expected values, score functions and accuracy function of intuitionistic trapezoidal fuzzy numbers a new kind of ranking was proposed by Wang et al. in 2009. They also developed the Hamming distance of intuitionistic trapezoidal fuzzy numbers and Intuitionistic Trapezoidal Fuzzy Weighted Arithmetic Averaging (ITFWAA) operator, then proposed multi-criteria decision-making method with incomplete certain information based on intuitionistic trapezoidal fuzzy number.

In 2011, Salim Rezvani defined a new ranking technique for trapezoidal intuitionistic fuzzy numbers based on value-index and ambiguity –index of trapezoidal intuitionistic fuzzy numbers. Similar value-index and ambiguity – index based ranking method for triangular intuitionistic fuzzy numbers was given by Li et al. [7] in 2010. Li [8] proposed a ranking order relation of TIFN using lexicographic technique. Nayagam et al. [12] introduced TIFNs of special type and described a method to rank them which seems to be unrealistic. Nehi [11] put forward a new ordering method for TIFNs in which two characteristic values for IFN.

Symmetric trapezoidal intuitionistic fuzzy numbers are ranked with a special ranking function which has been applied to solve a class of linear programming problems in which the data parameters are symmetric trapezoidal intuitionistic fuzzy number by Parvathi et al. [14] in 2012. Dubey et al. in 2011 developed a ranking technique for special form of triangular intuitionistic fuzzy numbers.

This paper is organized as follows. In Section 2 some preliminary definitions and concepts regarding intuitionistic fuzzy numbers were presented. In Section 3, we define the magnitude of different forms of trapezoidal and triangular intuitionistic fuzzy numbers. Section 4 is devoted to the illustration of some numerical examples for the concepts defined in the Section 3 and also contains the comparative study of results obtained by the proposed method with other existing ranking methods. Section 5 concludes the paper by giving some advantages of the proposed method over other methods.

2. Preliminaries

Definition 1. [1] An IFS A in X is given by

$$A = \{(x, \mu_A(x), \nu_A(x)), x \in X\},$$

where the functions $\mu_A(x)$: $X \to [0,1]$ and $\nu_A(x)$: $X \to [0,1]$ define, respectively, the degree of membership and degree of non-membership of the element $x \in X$ to the set A, which is a subset of X, and for every $x \in X$, $0 \le \mu_A(x) + \nu_A(x) \le 1$.

Obviously, every fuzzy set has the form $\{(x, \mu_A(x), \mu_{A^c}(x)), x \in X\}$.

For each IFS A in X, we will call $\Pi_A(x) = 1 - \mu(x) - \nu(x)$ the intuitionistic fuzzy index of x in A. It is obvious that $0 \le \Pi_A(x) \le 1, \forall\, x \in X$.

Definition 2. [11]. An IFS $A = \{(x, \mu_A(x), \gamma_A(x) | x \in X)\}$ is called IF-normal, if there exist at least two points $x_0, x_1 \in X$ such that $\mu_A(x_0) = 1, \gamma_A(x_1) = 1$, It is easily seen that given intuitionistic fuzzy set A is IF-normal if there is at least one point that surely belongs to A and at least one point which does not belong to A.

Definition 3. [11]. An IFS $A = \{(x, \mu_A(x), \gamma_A(x) | x \in X)\}$ of the real line is called IF-convex, if

$$\forall x_1, x_2 \in \mathbb{R}, \forall \lambda \in [0,1],$$
$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \mu_A(x_1) \wedge \mu_A(x_2),$$
$$\gamma_A(\lambda x_1 + (1-\lambda)x_2) \geq \gamma_A(x_1) \wedge \gamma_A(x_2).$$

Thus A is IF –convex if its membership function is fuzzy convex and its non-membership function is fuzzy concave.

Definition 4. [11]. An IFS $A = \{(x, \mu_A(x), \gamma_A(x) | x \in X)\}$of the real line is called an IFN if

- A is IF-normal,
- A is IF-convex,
- μ_A is upper semicontinuous and γ_A is lower semicontinuous,
- $A = \{(x \in X \mid \gamma_A(x) < 1\}$ is bounded.

Definition 5. [11]. A is a trapezoidal intuitionistic fuzzy number with parameters $b_1 \leq a_1 \leq b_2 \leq a_2 \leq a_3 \leq b_3 \leq a_4 \leq b_4$ and denoted by $A = (b_1, a_1, b_2, a_2, a_3, b_3, a_4, b_4)$. In this case we will give

$$\mu_A(x) = \begin{cases} 0 & ; x < a_{1,} \\ \dfrac{x - a_1}{a_2 - a_1} & ; a_1 \leq x \leq a_2 \\ 1 & ; a_2 \leq x \leq a_3 \\ \dfrac{x - a_4}{a_3 - a_4} & ; a_3 \leq x \leq a_4 \\ 0 & ; a_4 < x. \end{cases}$$

$$\gamma_A(x) = \begin{cases} 0 & ; x < b_{1,} \\ \dfrac{x - b_2}{b_1 - b_2} & ; b_1 \leq x \leq b_2 \\ 1 & ; b_2 \leq x \leq b_3 \\ \dfrac{x - b_3}{b_4 - b_3} & ; b_3 \leq x \leq b_4 \\ 0 & ; b_4 < x. \end{cases}$$

In the above definition, if we let$b_2 = b_3$ (*and hence* $a_2 = a_3$), then we will get a triangular intuitionistic fuzzy number with parameters $b_1 \leq a_1 \leq (b_2 = a_2 = a_3 = b_{3)} \leq a_4 \leq b_4$ and denoted by $A = (b_1, a_1, b_2, a_4, b_4)$.

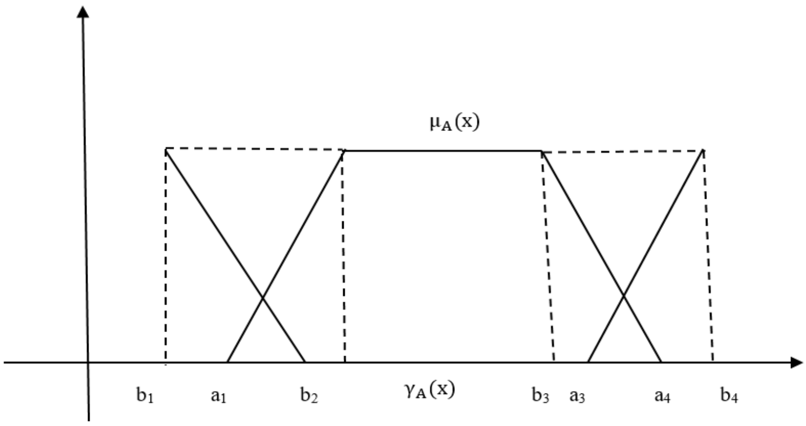

***Fig. 1.** Trapezoidal intuitionistic fuzzy number.*

Definition 6. [7]. A TIFN $\tilde{a} = (\underline{a}, a, \overline{a}\,; w_{\tilde{a}}, u_{\tilde{a}})$ is a special IF set on the real number set R, whose membership function and non-membership function are defined as follows:

$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{w_{\tilde{a}}(x-\underline{a})}{(a-\underline{a})} & \text{if } \underline{a} \le x < a \\ w_{\tilde{a}} & \text{if } x = a \\ \dfrac{w_{\tilde{a}}(\overline{a}-x)}{(\overline{a}-a)} & \text{if } a < x \le \overline{a} \\ 0 & \text{if } x < \underline{a} \text{ or } x > \overline{a} \end{cases}$$

$$v_{\tilde{a}}(x) = \begin{cases} \dfrac{[a-x+u_{\tilde{a}}(x-\underline{a})]}{(a-\underline{a})} & \text{if } \underline{a} \le x < a \\ u_{\tilde{a}} & \text{if } x = a \\ \dfrac{[x-a+u_{a}(\overline{a}-x)]}{(\overline{a}-a)} & \text{if } a < x \le \overline{a} \\ 0 & \text{if } x < \underline{a} \text{ or } x > \overline{a} \end{cases}$$

Where the values $w_{\tilde{a}}$ and $u_{\tilde{a}}$ represent the maximum degree of membership and the minimum degree of non-membership, respectively, such that they satisfy the conditions $0 \le w_{\tilde{a}} \le 1, 0 \le u_{\tilde{a}} \le 1, 0 \le w_{\tilde{a}} + u_{\tilde{a}} \le 1$.

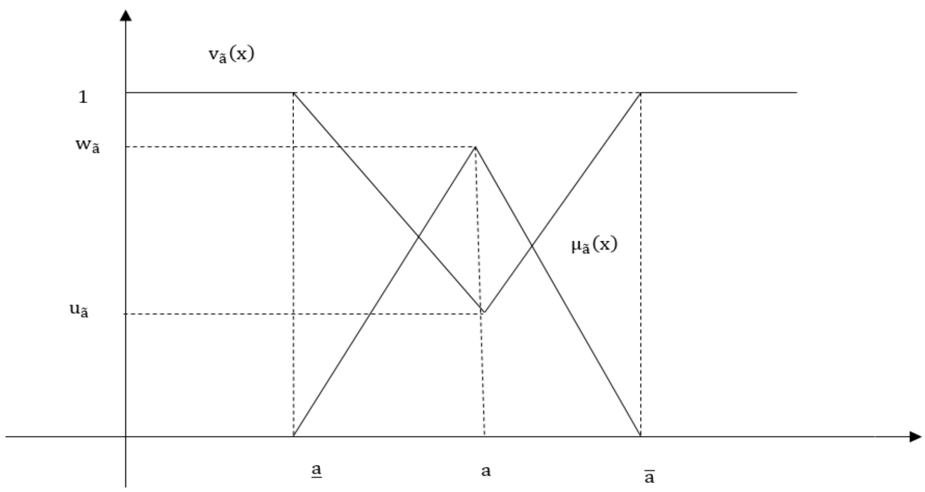

***Fig. 2.** Triangular intuitionistic fuzzy number.*

Definition 7. [14]. An IFN $\tilde{A}$ in R is said to be a symmetric trapezoidal intuitionistic fuzzy numbers if there exists real numbers a_1, a_2, h, h' where $a_1 \le a_2, h \le h'$ and $h, h' > 0$ such that the membership and non-membership functions are as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-(a_1-h)}{h} & ; x \in [a_1-h, a_1] \\ 1 & ; x \in [a_1, a_2] \\ \dfrac{a_2+h-x}{h} & ; x \in [a_2, a_2+h] \\ 0 & ; \text{otherwise} \end{cases}$$

$$v_{\tilde{A}}(x) = \begin{cases} \dfrac{(a_1-x)}{h'} & ; x \in [a_1-h', a_1] \\ 0 & ; x \in [a_1, a_2] \\ \dfrac{x-a_2}{h'} & ; x \in [a_2, a_2+h'] \\ 1 & ; \text{otherwise} \end{cases}$$

Definition 8. [17]. A Generalized Triangular Intuitionistic Fuzzy Number (GTIFN) $\tilde{\tau}_a = \langle (a, l_\mu, r_\mu; w_a), (a, l_\gamma, r_\gamma; u_a) \rangle$ is a special intuitionistic fuzzy set on a real number set $\mathfrak{R}$, whose membership function and non-membership functions are defined as follows:

$$\mu_{\tilde{\tau}_a}(x) = \begin{cases} \dfrac{x-a+l_\mu}{l_\mu} w_a & ; a-l_\mu \le x < a \\ w_a & ; x = a \\ \dfrac{a+r_\mu-x}{r_\mu} w_a & ; a < x \le a+r_\mu \\ 0 & ; \text{otherwise} \end{cases}$$

$$v_{\tilde{\tau}_a}(x) = \begin{cases} \dfrac{(a-x)+u_a(x-a+l_\gamma)}{l_\gamma} & ; a-l_\gamma \le x < a \\ u_a & ; x = a \\ \dfrac{(x-a)+u_a(a+r_\gamma-x)}{r_\gamma} & ; a < x \le a+r_\gamma \\ 1 & ; \text{otherwise} \end{cases}$$

Where $l_\mu, r_\mu, l_\gamma, r_\gamma$ are called the spreads of membership and non-membership functions, respectively and a is called mean value. w_a and u_a represent the maximum degree of membership and minimum degree of non-membership respectively such that they satisfy the conditions $0 \le w_a \le 1, 0 \le u_a \le 1$ and $0 \le w_a + u_a \le 1$.

Definition 9. [13]. A TIFN is an intuitionistic fuzzy set in R with the following membership function $\mu_A(x)$ and non-membership function $\vartheta_A(x)$

$$\mu_A(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1} & , a_1 \le x \le a_2 \\ \dfrac{x-a_3}{a_2-a_3} & , a_2 \le x \le a_3 \\ 0, & \text{otherwise} \end{cases}$$

$$\vartheta_A(x) = \begin{cases} \dfrac{a_2 - x}{a_2 - a_1'} & , a_1' \le x \le a_2 \\ \dfrac{x - a_2}{a_3' - a_2} & , a_2 \le x \le a_3' \\ 1 & , \text{otherwise.} \end{cases}$$

Where $a_1' \le a_1 \le a_2 \le a_3 \le a_3'$ and $\mu_A(x) + \vartheta_A(x) \le 1$ or $\mu_A(x) = \vartheta_A(x)$ for all $x \in R$. This TIFN is denoted by $A = (a_1, a_2, a_3; a_1', a_2, a_3')$.

Definition 10. [18]. Let $\tilde{a} = \langle([a, b, c, d]; \mu_{\tilde{a}}), ([a_1, b, c, d_1]; \gamma_{\tilde{a}})\rangle$ be a trapezoidal intuitionistic fuzzy number whose membership and non-membership is given by

$$\mu_{\tilde{a}} = \begin{cases} \dfrac{x - a}{b - a}\mu_{\tilde{a}}, & a \le x < b \\ 1 & , \quad b \le x \le c \\ \dfrac{d - x}{d - c}\mu_{\tilde{a}}, & c < x \le d \\ 0 & , \text{otherwise} \end{cases}$$

$$\gamma_{\tilde{a}} = \begin{cases} \dfrac{b - x + \gamma_{\tilde{a}}\,(x - a_1)}{b - a} & , a_1 \le x < b \\ 0 & , \quad b \le x \le c \\ \dfrac{x - c + \gamma_{\tilde{a}}\,(d_1 - x)}{d_1 - c} & , \quad c < x \le d_1 \\ 1 & , \text{otherwise.} \end{cases}$$

Where $0 \le \mu_{\tilde{a}} \le 1, 0 \le \gamma_{\tilde{a}} \le 1, \mu_{\tilde{a}} + \gamma_{\tilde{a}} \le 1, a, b, c, d \in R$. When $b = c$, the intuitionistic trapezoidal fuzzy number becomes intuitionistic triangular fuzzy number.

3. New Approach for Ranking of Intuitionistic Fuzzy Numbers

In this section we define the concept of magnitude of an intuitionistic fuzzy number and discussed various methods for ranking the different forms of triangular intuitionistic fuzzy numbers and trapezoidal intuitionistic fuzzy numbers by means of magnitude.

Definition 11. Let $A = (b_1, a_1, b_2, a_2, a_3, b_3, a_4, b_4)$ be a Trapezoidal intuitionistic fuzzy number we define magnitude as follows:

$$\mathrm{Mag}(A) = \frac{1}{2}\int_0^1 (f_A(x) + g_A(x) + h_A(x) + k_A(x))\, f(r)dr. \qquad (1)$$

where (r) is a non-negative and increasing weighting function on $[0,1]$ with

$$f(0) = 0, \qquad f(1) = 1 \text{ and } \int_0^1 f(r)dr = \frac{1}{2}.$$

In this paper we assume $f(r) = r$ for our convenience, we get magnitude of A as

$$\mathrm{Mag(A)} = \frac{1}{12}\left(\mathrm{a_1 + b_1 + a_4 + b_4 + 2(a_2 + a_3 + b_2 + b_3)}\right). \quad (2)$$

Using this definition of $Mag(A)$, we define the ranking procedure of any two trapezoidal intuitionistic fuzzy numbers as follows:

- $Mag(A) > Mag\ (B) iff\ A \succ B$.
- $Mag(A) < Mag\ (B) iff\ A \prec B$.
- $Mag(A) = Mag\ (B) iff\ A \backsim B$.

Remark 1. If $A = (b_1, a_1, b_2, a_2, a_3, b_3, a_4, b_4)\ and\ B = (b_1', a_1', b_2', a_2', a_3', b_3', a_4', b_4')$ be any two trapezoidal intuitionistic fuzzy numbers, then $Mag(A + B) = Mag\ A + Mag\ B$.

Definition 12. We define magnitude of a symmetric trapezoidal intuitionistic fuzzy number, $A = (b_1, a_1, a_2, a_2, a_3, a_3, a_4, b_4)$ using *Eq. (1)* as

$$\mathrm{Mag(A)} = \frac{1}{12}\left(\mathrm{a_1 + b_1 + a_4 + b_4 + 4(a_2 + a_3)}\right). \quad (3)$$

Remark 2. For any two symmetric trapezoidal intuitionistic fuzzy numbers $A = (a_1, a_2, h, h, a_1, a_2, h', h'), B = (a_1, a_2, k, k, a_1, a_2, k', k')$, we have

$$\mathrm{Mag(A) = Mag(B)}. \quad (4)$$

Remark 3. For any symmetric trapezoidal intuitionistic fuzzy number

$$\mathrm{A = (-a_1, a_1, h, h, -a_1, a_1, h', h'),\ Mag(A) = 0}. \quad (5)$$

Definition 13. For a trapezoidal intuitionistic fuzzy number $A = (a_1, a_2, \alpha, \beta, a_1, a_2, \alpha', \beta')$

$$\mathrm{Mag(A)} = \frac{1}{12}\left(\beta - \alpha + 6(\mathrm{a_1 + a_2}) + 2(\beta' - \alpha'\right). \quad (6)$$

Definition 14. Let $A = (a_1, a_2, h, h, a_1, a_2, h', h')$ be a symmetric trapezoidal intuitionistic fuzzy number. Then its magnitude defined by

$$\mathrm{Mag}(A) = \frac{1}{2}(a_1 + a_2).. \tag{7}$$

Definition 15. Let $A = (a_1, b_1, c_1, d_1; a'_1, b_1, c_1, d'_1)$ be a trapezoidal intuitionistic fuzzy number, then

$$\mathrm{Mag}(A) = \frac{1}{12}(a_1 + d_1 + 2(a'_1 + d'_1) + 3(b_1 + c_1). \tag{8}$$

Definition 16. If $A = (\overline{a}, a, \underline{a}; w_a, u_a)$ is a triangular intuitionistic fuzzy number , then

$$\mathrm{Mag}(A) = \frac{1}{12}\left[\frac{4a-2(\overline{a}+\underline{a})+3w_a(\underline{a}+\overline{a})}{w_a} + \frac{2(\overline{a}+a+\underline{a})-3u_a(a+\overline{a})}{(1-u_a)}\right] \tag{9}$$

Definition 17. Let $A = ((a, l_\mu, r_\mu; w_a), (a, l_\gamma, r_\gamma; u_a))$ be a triangular intuitionistic fuzzy number. Then

$$\mathrm{Mag}\ (A) = \frac{1}{12}\left\{\left(\frac{6aw_a - 3w_a(l_\mu - r_\mu) + 2(l_\mu - r_\mu)}{w_a} + \frac{6(a - au_a) + 3u_a(l_\gamma - r_\gamma) + 2(r_\gamma - l_\gamma)}{(1 - u_a)}\right)\right\}. \tag{10}$$

Definition 18. Let $A = \langle [a, b, c, d]; \mu_a, \gamma_a \rangle$ be a trapezoidal intuitionistic fuzzy number, then

$$\mathrm{Mag}(A) = \frac{1}{12}\left\{\frac{2(b - a + c - d) + 3\mu_a(a + d)}{\mu_a} + \frac{2(a + d) + (b + c) - 3\gamma_a(a + d)}{1 - \gamma_a}\right\}. \tag{11}$$

Definition 19. Consider a triangular intuitionistic fuzzy number of the form $A = (a_1, a_2, a_3; a'_1, a_2, a'_3)$, then

$$\mathrm{Mag}(A) = \frac{1}{12}\{a_1 + a_3 + 6a_2 + 2(a'_1 + a'_3)\} \tag{12}$$

4. Numerical Examples

This section illustrates some examples for comparative analysis of various existing ranking methods

Example 1. Consider two trapezoidal intuitionistic fuzzy numbers as follows: A= (0.2,0.4,0.6,0.8,0.11,0.12,0.13,0.15) and B=(0,0.1,0.2,0.3,0.4,0.5,0.6,0.7). In [11], Nehi used characteristic values of membership or non-membership functions to rank trapezoidal intuitionistic fuzzy numbers. The ranking procedure depends on the value of 'k'. As 'k' varies in the interval$(0, \infty)$, the ranking also varies which leads to an unreasonable result. This can be seen from the following example.

Table 1. *Calculation of* $c_{\mu}^{k}(A)$.

a1	**a2**	**a3**	**a4**	**b1**	**b2**	**b3**	**b4**	**k**	$\mathbf{c_{\mu}^{k}(A)}$
0.4	*0.8*	*0.11*	*0.13*	*0.2*	*0.6*	*0.12*	*0.15*	*1*	*0.392*
0.4	*0.8*	*0.11*	*0.13*	*0.2*	*0.6*	*0.12*	*0.15*	*2*	*0.408*

Table 2. *Calculation of* $c_{\mu}^{k}(B)$.

a1	**a2**	**a3**	**a4**	**b1**	**b2**	**b3**	**b4**	**k**	$\boldsymbol{c_{\mu}^{k}(B)}$
0.1	*0.3*	*0.4*	*0.6*	*0*	*0.2*	*0.5*	*0.7*	*1*	*0.350*
0.2	*0.4*	*0.5*	*0.7*	*0.1*	*0.3*	*0.6*	*0.8*	*2*	*0.450*

From the table, we see that when k=1, $A > B$ and when k=2, $B > A$

Example 2. Consider two symmetric trapezoidal intuitionistic fuzzy numbers A=(23,25,1,1;23,25,3,3) and B = (5,7,2,2; 5,7,4,4) as in [15]. Here the ranking of STIFNs are obtained by a special ranking function by considering all the parameters of both membership and non- membership functions of given STIFNs. The values obtained by this method are similar to the proposed method.

Example 3. Consider two trapezoidal intuitionistic fuzzy numbers of the forms A = (0.2,0.3,0.4,0.5; 0.1,0.3,0.4,0.6) and B = (0.1,0.2,0.3,0.4; 0,0.2,0.3,0.5) discussed in [16]. Rezvani used value index of membership and non-membership functions separately to rank trapezoidal intuitionistic fuzzy numbers.

Example 4. Consider three triangular intuitionistic fuzzy numbers as below A=(0.592,0.774,0.910;0.6,0.4) , B=(0.769,0.903,1;0.4,0.5) and C=(0.653,0.849,0.956;0.5,0.2) as given in [7]. In the paper [7] Li used ratio ranking method to rank triangular intuitionistic fuzzy numbers and applied it to multi attribute decision making problem In the case of ration ranking method, the raking differs on the choice of λ. For the above IFN's we have

Table 3. *Ranking of IFN's for values of* λ.

S.No	$\boldsymbol{\lambda}$	**Ranking results**
1	$[0, 0.1899)$	$A > C > B$
2	$(0.1899,0.9667)$	$C > A > B$
3	$(0.9667,1]$	$C > B > A$

So this leads to a conflicted state which yields an unreasonable result.

Example 5. Consider the same IFN's as in *example 4* and ranking developed in [8]. Here the ranking is done by the extended additive weighted method using the value-index and ambiguity-index. For the above numbers, we have the following ranking results as tabulated below from [8].

Table 4. *Ranking of IFN's for values of* λ.

S.No.	$\boldsymbol{\lambda}$	**Ranking results**
1	*[0,0.793]*	$C > A > B >$
2	*(0.793,1]*	$A > C > B$

From the above table, we see that the ranking differs on the basis of given weight λ.

Example 6. Consider the two Generalized triangular fuzzy intuitionistic numbers $\tilde{\tau}_a = \big((5,1,2;0.6),(5,1.5,2.6;0.3)\big)$ and $\tilde{\tau}_b = \big((6,2,1;0.6),(6,2.1,1.5;0.4)\big)$ in [17]. If we use $R_{\mu}(\tilde{\tau}_a)$ to rank these numbers we obtain $\tilde{\tau}_a < \tilde{\tau}_b$. But when we rank in terms of $R_{\gamma}(\tilde{\tau}_a)$, we get $\tilde{\tau}_a > \tilde{\tau}_b$. Hence the ranking of generalized triangular intuitionistic fuzzy numbers varies with the use of membership and non-membership value in ranking. This is an unreasonable result. Therefore the proposed method which uses both membership and non-membership values as a whole is suitable for ranking such GTIFN's.

Example 7. Consider the two triangular intuitionistic fuzzy numbers as follows: A = {(14,15,17;0.9),(10,15,18;0)} and B = {(25,30,34;0.9),(23,30,38;0)}as in [4]. In this paper, Dubey used the concept of value and ambiguity of a triangular intuitionistic fuzzy numbers to rank the above numbers. The ranking obtained in [4] is similar to the proposed method.

Example 8. Consider 5 set of trapezoidal intuitionistic fuzzy number as in [18].

$$\widetilde{a_1} = \langle [0.407,0.539,0.683,0.814]; 0.727,0.21 \rangle.$$

$$\widetilde{a_2} = \langle [0.547,0.679,0.810,0.942]; 0.705,0.230 \rangle.$$

$$\widetilde{a_3} = \langle [0.424,0.572,0.704,0.868]; 0.697,0.252 \rangle.$$

$$\widetilde{a_4} = \langle [0.392,0.557,0.724,0.902]; 0.639,0.280 \rangle.$$

$$\widetilde{a_5} = \langle [0.411,0.555,0.699,0.831]; 0.812,0.137 \rangle.$$

In [18] ranking is done based on the comparison of score function values and accuracy function values of integrated intuitionistic fuzzy numbers. The ranking here in [18] differs from our proposed method.

Example 9. Consider two triangular intuitionistic fuzzy numbers as below $\tilde{A} = \{(2.68,3,3.71); (2.2,3,4.67)\}$ and $B = \{(2.75,6,9.375); (2.38,6,16.2)\}$ as in [13]. In this paper ranking is done by using the score function and the result obtained is similar to the proposed method.

The following table gives a comparative analysis of various ranking methods so far defined in intuitionistic fuzzy setting with the proposed method.

Table 5. *Comparative analysis of different ranking methods.*

S.No	Intuitionistic Fuzzy Numbers	Existing Method	Proposed Method
1	*A=(0.2,0.4,0.6,0.8,0.11,0.12,0.13,0.15)* *B=(0,0.1,0.2,0.3,0.4,0.5,0.6,0.7)*	$c_\mu^k(A) = 0.392$; $c_\mu^k(B) = 0.35$, $A \succ B$ [11]	$Mag(A) = 0.35$ $Mag(B) = 0.35$ $A \sim B$
2	*A=(23,25,1,1;23,25,3,3)* *B=(5,7,2,2;5,7,4,4)*	$\Re(A) = 49$ $\Re(B) = 13$, $A \succ B$ [15]	$Mag\ (A) = 24$; $Mag(B) = 6$, $A \succ B$
3	*A=(0.2,0.3,0.4,0.5;0.1,0.3,0.4,0.6)* *B= (0.1,0.2,0.3,0.4;0,0.2,0.3,0.5)*	$v_\mu(A) = 0.35$; $v_\mu(B) = 0.25$, $A \succ B$ [16]	$Mag(A) = 0.35$; $Mag(B) = 0.25$ $A \succ B$
4	*A=(0.592,0.774,0.910;0.6,0.4)* *B=(0.769,0.903,1;0.4,0.5)* *C=(0.653,0.849,0.956;0.5,0.2)*	$R(A,\lambda) = 0.4321$; $R(B,\lambda) = 0.3455$; $R(C,\lambda) = 0.3858$ $A \succ C \succ B$ [7]	$Mag(A) = 0.8282$; $Mag(B) = 0.9688$; $Mag(C) = 0.9322$, $B \succ C \succ A$
5	*A=(0.592,0.774,0.910;0.6,0.4)* *B=(0.769,0.903,1;0.4,0.5)* *C=(0.653,0.849,0.956;0.5,0.2)*	$v_\lambda(A) = 0.276$; $v_\lambda(B) = 0.224$; $v_\lambda(C) = 0.534$, $C > A > B$ [8]	$Mag(A) = 0.828$; $Mag(B) = 0.969$; $Mag(C) = 0.932$, $B > C > A$
6	$\tau_a = \big((5,1,2;0.6),(5,1.5,2.6;0.3)\big)$ $\tau_b = \big((6,2,1;0.6),(6,2.1,1.5;0.4)\big)$	$R_\gamma(\tau_a) = 3.98$, $R_\gamma(\tau_b) = 3.51$, $A \succ B$ [17]	$Mag(\tau_a) = 5.12$, $Mag\ (\tau_b) = 5.96$, $A \prec B$
7	$A = \{(14,15,17;0.9),(10,15,18;0)\}$ $B = \{(25,30,34;0.9),(23,30,38;0)\}$	$F(A,\lambda) = 13.76$; $F(B,\lambda) = 27.47$; $A \succ B$ [4]	$Mag(A) = 15.28$; $Mag(B) = 31.74$ $A \succ B$
8	$\widetilde{a_1} = \langle[0.407,0.539,0.683,0.814];0.727,0.21\rangle$ $\widetilde{a_2} = \langle[0.547,0.679,0.810,0.942];0.705,0.230\rangle$ $\widetilde{a_3} = \langle[0.424,0.572,0.704,0.868];0.697,0.252\rangle$ $\widetilde{a_4} = \langle[0.392,0.557,0.724,0.902];0.639,0.280\rangle$ $\widetilde{a_5} = \langle[0.411,0.555,0.699,0.831];0.812,0.137\rangle$	$S(\widetilde{a_1}) = 0.236$; $S(\widetilde{a_2}) = 0.261$; $S(\widetilde{a_3}) = 0.206$; $S(\widetilde{a_4}) = 0.153$; $S(\widetilde{a_5}) = 0.353$; $a_5 \succ a_2 \succ a_1 \succ a_3 \succ a_4$ [18]	$Mag(\widetilde{a_1}) = 0.611$; $Mag(\widetilde{a_2}) = 0.745$; $Mag(\widetilde{a_3}) = 0.640$; $Mag(\widetilde{a_4}) = 0.642$; $Mag(\widetilde{a_5}) = 0.625$; $a_2 \succ a_4 \succ a_3 \succ a_5 \succ a_1$
9	$\tilde{A} = \{(2.68,3,3.71);(2.2,3,4.67)\}$ *and* $\tilde{B} = \{(2.75,6,9.375);(2.38,6,16.2)\}$	$S(\tilde{A}) = 3.2175$ $S(\tilde{B}) = 7.645$ $\tilde{A} \prec \tilde{B}$ [13]	$Mag(\tilde{A}) = 3.1772$ $Mag(\tilde{B}) = 7.12$ $\tilde{A} \prec \tilde{B}$

5. Conclusions

In many of the existing ranking methods, ranking is done either by considering the membership or non-membership values only. But in the newly proposed method the ranking is done directly by taking both membership and non-membership values in a single formula. This ranking procedure is very simple and time consuming compared to the existing methods. We also illustrated the advantages of our method by means of suitable examples. The proposed ranking technique can be applied to multi-criteria decision making problems, linear programming problems, assignment problems, transportation, some management problems and industrial problems which are our future research works.

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Application of Transportation Problem under Pentagonal Neutrosophic Environment

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P A P E R I N F O	A B S T R A C T
Chronicle: *Received: 04 September 2019* *Revised: 09 December 2019* *Accepted: 15 January 2020* **Keywords:** *Transportation Problem.* *Pentagonal Neutrosophic Numbers.* *Linear Programming*.	The paper talks about the pentagonal Neutrosophic sets and its operational law. The paper presents the $\alpha_N, \beta_N, \lambda_N -$ cut of single valued pentagonal Neutrosophic numbers and additionally introduced the arithmetic operation of single-valued pentagonal Neutrosophic numbers. Here, we consider a transportation problem with pentagonal Neutrosophic numbers where the supply, demand and transportation cost is uncertain. Taking the benefits of the properties of ranking functions, our model can be changed into a relating deterministic form, which can be illuminated by any method. Our strategy is easy to assess the issue and can rank different sort of pentagonal Neutrosophic numbers. To legitimize the proposed technique, some numerical tests are given to show the adequacy of the new model.

1. Introduction

For survival of our life there is a need to move the item from various sources to various goals. Due to high population, it is very challenging to company, how to send the product to numbers of costumers or origins to a numbers of warehouse or store in a minimizing cost. This kind of issue is called Transportation Problem (TP) and it is an exceptional sort of Linear Programming (LP) problem where the organization's primary goal is limiting the expense. On account of wide application i.e. production planning, health sector, inventory control, network system etc., TP have consistently made separate space for analysts. An outline has attracted *Fig.1* which is speaks to connection among supply and demand.

Hitchcock [1] pioneered the basic transportation problem. This kind of traditional issue can be as a direct programming issue and afterward tackled simplex strategy. This type of classical problem can be modelled as a linear programming problem and then solved simplex method. A primal simplex method to transportation problem was solved by Dantzig & Thapa [2]. Transportation Problem is a different type of structure therefore simplex method is not suitable for finding the objectives. Due to some drawback in simplex method for solving TP, a new Initial Basic Feasible Solution (IBFS) method was developed. By using the IBFS, there are three type of methods (1) north-west corner (NWC), (2) least-

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cost method, (3) vogel's approximation method. In classical TP the decision makers knows the values of supply, demand and transportation cost i.e. the decision makers consider the crisp numbers. However, in our day to day life applications, the decision makers may not be known precisely to all the parameters of transportation problem due to some uncontrolled factor. To overcome this uncontrolled factor, fuzzy decision making method is introduced.

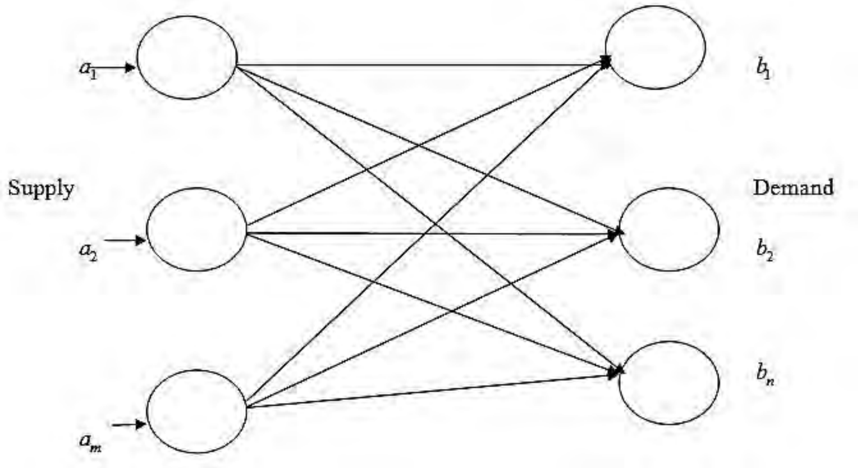


Fig. 1. *LP Transportation problem.*

The basic concepts of fuzzy set theory was introduced by Zadeh [3] in 1965. Since, several researchers have carried out investigation on fuzzy transportation problem (FTP). A Fuzzy Linear Programming (FLP) problem was proposed by Zimmermann [4] and he has proved that the method was always very effective. Subsequently, Zimmermann FLP model has developed to solve different fuzzy transportation problems. Chanas et al. [5] considered FTP where supply and demand are fuzzy numbers and transportation cost is crisp number. Das et al. [6] proposed a fully fuzzy LP problem where all the parameters are trapezoidal fuzzy numbers and that method extend to solve Fully Fuzzy TP (FFTP). Dinagar and Palanivel [7] proposed a FFTP where demand, supply, and transportation costs are trapezoidal fuzzy numbers. Kaur and Kumar [8] have introduced an algorithmic for solving the fuzzy transportation problem. Fuzzy zero pint method for solving FFTP problem was proposed by Pandian and Natrajan [9] in which supply, demand and transportation cost are trapezoidal fuzzy numbers. Kundu et al. [10] introduced a solid transportation model with crisp and rough costs. Some other researchers [11-19] also have studied this general transportation problem in a fuzzy environment. Maheswari and Ganesan [20] proposed fully fuzzy transportation problem using pentagonal fuzzy numbers.

Due to some complications, insufficient information, multiple sources of data arises in our real-life problem; it is not always possible to use fuzzy numbers. The fuzzy sets mainly consider the membership functions. Intuitionistic Fuzzy Sets (IFS) is an extension version of fuzzy sets and it can be used to solve them. IFS has been proposed by Atanassov [24] and it's take care both mixture of membership function and non-membership function. Since, several researchers [25-30] considered the IFS for solving TP. Aggarwal and Gupta [31] studied the sensitivity analysis of intuitionistic fuzzy solid transportation problem. Singh and Yadav [32] introduced a novel solution for solving fully Intuitionistic Fuzzy Transportation Problem (IFTP) in which demand, supply and transportation cost are considered intuitionistic triangular fuzzy numbers. In that paper, they obtaining both negative solutions for variables and objective functions instead of positive transportations cost. After the shortcomings of

Singh and Yadav paper, Mahmoodirad et al. [33] proposed a method for fully IFTP in which demand, supply and transportation cost are considered intuitionistic fuzzy numbers.

In genuine situation, we regularly experience with deficient and uncertain data where it isn't conceivable to speak to the data just by the methods for membership function and non-membership function. To manage such circumstances, Smarandache [36] in 1988, introduced the structure of Neutrosophic Set (NS) which is higher version of both fuzzy and intuitionistic fuzzy. Neutrosophy set might be described by three autonomous degrees, i.e. (i) truth-membership degree (T), (ii) indeterminacy membership degree (I), and (iii) falsity membership degree (F). Later, Wang et al. [37] introduced a Single Value Neutrosophic Set (SVNS) problem for solving a practical problem. Ye [38] introduced the Trapezoidal Neutrosophic Set (TrNS) by combining the concept of Trapezoidal Fuzzy Numbers (TrFN) and SVNS. To take the advantages of beauty of NS, several researchers [39-45] proposed different method for solving LP problem under Neutrosophic environment. Das and Dash [46] proposed a modified solution of Neutrosophic LP problem. Recently, Das and Chakraborty [47] proposed a new approach for solving LP problem in pentagonal Neutrosophic environment.

Motivation. Neutrosophic sets always plays a vital role in uncertainty environment. Before going to discussion the motivation of our paper, we demonstrate the different author's research work towards the TP under mixed constraints.

Table 1. *Significance influences of the different authors towards under various environment.*

Author	Year	Main Contribution
Korukoglu and Balli [50]	*2011*	*Crisp environment.*
Bharati [51]	*2019*	*Trapezoidal intuitionistic fuzzy.*
environment		
Ahmad and Adhami [52]	*2018*	*Neutrosophic fuzzy environment.*
Singh and Yadav [32]	*2016*	*Traingular fully intuitionistic fuzzy.*
Ebrahimnejad & Verdegay [25]	*2017*	*Trapezoidal intuitionistic fuzzy.*
Environment		
Mahaswari & Ganesan[20]	*2018*	*Pentagonal fuzzy number.*
Srinivasan et al. [53]	*2020*	*Triangular fuzzy number.*

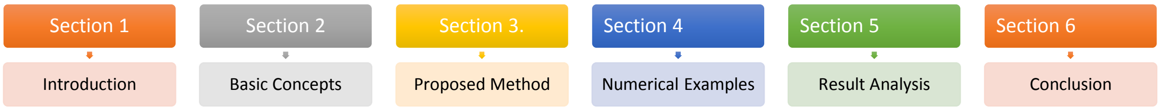
From the above discussion on TP which are readily available in our literature, there are no current techniques for solving pentagonal TP under Neutrosophic condition. In this way, there is have to setup another technique for pentagonal Neutrosophic transportation issue. This total situation has persuaded us to build up another strategy for illuminating TP with pentagonal Neutrosophic numbers. Just because, we build up a calculation and applied, all things considered, issue. The primary commitments of the paper as follows:

We characterize Neutrosophic Transportation Problem (NTP) issue in which the supply, demand and transportation cost are taken as pentagonal Neutrosophic numbers.

- This model assists with settling another arrangement of issue with pentagonal Neutrosophic numbers.
- In our literature of pentagonal Neutrosophic numbers, we will in general present a recently evolved scoring function.
- By using our recently scoring function, the pentagonal Neutrosophic TP is changing over into its crisp TP.

– To best our insight, it would be the primary strategy to unravel the PNTP. Consequently, in our paper direct relationship with relative system doesn't rise.

The rest of the paper is organized as the following way:



2. Preliminaries

In this section, we call back the some definitions and basic concepts which are pivotal in this paper. The well-defined definitions are referred [47, 49] throughout the paper.

2.1. Definition: Neutrosophic Set (NS)

A set $\tilde{N}_M$ is identified as a Neutrosophic set if $\tilde{N}_M = \{\langle x; [\theta_{\tilde{N}_M}(x), \varphi_{\tilde{N}_M}(x), \sigma_{\tilde{N}_M}(x)]\rangle : x \in X\}$, where $\theta_{\tilde{N}_M}(x): X \rightarrow]-0,1+[$ is declared as the truthness function, $\varphi_{\tilde{N}_M}(x): X \rightarrow]-0,1+[$ is declared as the hesitation function, and $\sigma_{\tilde{N}_M}(x): X \rightarrow]-0,1+[$ is declared as the falseness function.

$\theta_{\tilde{N}_M}(x), \varphi_{\tilde{N}_M}(x)$ & $\sigma_{\tilde{N}_M}(x)$ displays the following relation:

$$-0 \leq \mathrm{Sup}\,\{\theta_{\tilde{N}_M}(x)\} + \mathrm{Sup}\,\{\varphi_{\tilde{N}_M}(x)\} + \mathrm{Sup}\,\{\sigma_{\tilde{U}_M}(x)\} \leq 3+.$$

2.2. Definition: Single-Valued Neutrosophic Set (SNS)

A set $\tilde{N}_M$ in the definition 2.3 is called as a SNS $(\widetilde{SN}_M)$ if x is a single-valued independent variable. $\widetilde{SNA} = \{\langle x; [\theta_{\widetilde{SN}_M}(x), \varphi_{\widetilde{SN}_M}(x), \sigma_{\widetilde{SN}_M}(x)]\rangle : x \in X\}$, $\theta_{\widetilde{SN}_M}(x), \varphi_{\widetilde{SN}_M}(x)$& $\sigma_{\widetilde{SN}_M}(x)$ signified the notion of correct, indefinite and incorrect memberships function, respectively.

2.3. Definition: Single-Valued Pentagonal Neutrosophic Number (SPNN)

A SPNN $(\tilde{M})$ is defined as $\widetilde{SPN} = \langle[(m^1, n^1, o^1, p^1, q^1); \mu], [(m^2, n^2, o^2, p^2, q^2); \theta], [(m^3, n^3, o^3, p^3, q^3); \eta]\rangle$, where $\mu, \theta, \eta \in [0,1]$. The truth membership function$(\mu_{\widetilde{SPN}}): \mathbb{R} \rightarrow [0, \mu]$, the hesitant membership function $(\theta_{\widetilde{SPN}}): \mathbb{R} \rightarrow [\theta, 1]$ and the false membership function $(\eta_{\widetilde{SPN}}): \mathbb{R} \rightarrow [\eta, 1]$ are given as:

$$\mu_{\widetilde{SPN}}(x) = \begin{cases} \mu_{\widetilde{Ssl1}}(x) m^1 \leq x < n^1 \\ \mu_{\widetilde{Ssl2}}(x)\, n^1 \leq x < o^1 \\ \mu \qquad x = o^1 \\ \mu_{\widetilde{Ssr2}}(x)\, o^1 \leq x < p^1 \\ \mu_{\widetilde{Ssr1}}(x) p^1 \leq x < q^1 \\ 0 \quad otherwise \end{cases}, \quad \theta_{\widetilde{SPN}}(x) = \begin{cases} \theta_{\widetilde{Ssl1}}(x) \quad m^2 \leq x < n^2 \\ \theta_{\widetilde{Ssl2}}(x) \;\; n^2 \leq x < o^2 \\ \vartheta \qquad x = o^2 \\ \theta_{\widetilde{Ssr2}}(x) \;\; o^2 \leq x < p^2 \\ \theta_{\widetilde{Ssr1}}(x)\, p^2 \leq x < q^2 \\ 1 \quad otherwise \end{cases}.$$

$$\eta_{\widetilde{SPN}}(x)=\begin{cases}\eta_{\widetilde{Ssl1}}(x) m^3\leq x<n^3\\ \eta_{\widetilde{Ssl2}}(x)\, n^3\leq x<o^3\\ \quad\vartheta \qquad x=o^3\\ \eta_{\widetilde{Ssr2}}(x)\, o^3\leq x<p^3\\ \eta_{\widetilde{Ssr1}}(x) p^3\leq x<q^3\\ \quad 1 \quad otherwise\end{cases}.$$

2.4. Score and Accuracy Function

Let us consider a single valued Pentagonal Neutrosophic Numbers (PNN) as $\tilde{F}_{Pen}=(F_1,F_2,F_3,F_4,F_5;\pi,\sigma,\rho)$, The primary application of score function is to drag the judgment of conversion of PNN to crisp number. Also, the mean of the PNN components is $\frac{(F_1+F_2+F_3+F_4+F_5)}{5}$ and score value of the membership portion is $\frac{\{2+\pi-\rho-\sigma\}}{3}$.

Thus, for a P.N.N $\tilde{F}_{Pen}=(F_1,F_2,F_3,F_4,F_5;\pi,\sigma,\rho)$. Score function is scaled as $\widetilde{SC}_{Pen}=\frac{1}{15}(F_1+F_2+F_3+F_4+F_5)\times\{2+\pi-\rho-\sigma\}$. Accuracy function is scaled as $\widetilde{AC}_{Pen}=\frac{1}{15}(F_1+F_2+F_3+F_4+F_5)\times\{2+\pi-\sigma\}$. Here, $\widetilde{SC}_{Pen}\in R$, $\widetilde{AC}_{Pen}\in R$.

2.5. Relationship between any Two PNN

Let us consider any two pentagonal Neutrosophic number defined as follows:

$F_{Pen1}=(\pi_{Pen1},\sigma_{Pen1},\rho_{Pen1})$,$F_{Pen2}=(\pi_{Pen2},\sigma_{Pen2},\rho_{Pen2})$.

$SC_{Pen1}>SC_{Pen2}$,$F_{Pen1}>F_{Pen2}$,

$SC_{Pen1}<SC_{Pen2}$,$F_{Pen1}<F_{Pen2}$,

$SC_{Pen1}=SC_{Pen2}$,$F_{Pen1}=F_{Pen2}$.

Then,

$AC_{Pen1}>AC_{Pen2}$,$F_{Pen1}>F_{Pen2}$,

$AC_{Pen1}<AC_{Pen2}$,$F_{Pen1}<F_{Pen2}$,

$AC_{Pen1}=AC_{Pen2}$,$F_{Pen1}=F_{Pen2}$.

2.6. Basic Operations

Let $\widetilde{F_1}=<(c_1,c_2,c_3,c_4,c_5);\pi_{\widetilde{p_1}},\mu_{\widetilde{p_1}},\sigma_{\widetilde{p_1}}>$ and $\widetilde{F_2}=<(d_1,d_2,d_3,d_4,d_5);\pi_{\widetilde{p_2}},\mu_{\widetilde{p_2}},\sigma_{\widetilde{p_2}}>$ be two IPFNs and $\alpha\geq 0$. Then the following operational relations hold:

$\widetilde{F_1}+\widetilde{F_2}=<(c_1+d_1,c_2+d_2,c_3+d_3,c_4+d_4,c_5+d_5);\max\{\pi_{\widetilde{p_1}},\pi_{\widetilde{p_2}}\},\min\{\mu_{\widetilde{p_1}},\mu_{\widetilde{p_2}}\},\min\{\sigma_{\widetilde{p_1}},\sigma_{\widetilde{p_2}}\}>$,

$\widetilde{F_1}$- $\widetilde{F_2}$= < (c_1- d_5, c_2-d_4, c_3-d_3, c_4-d_2, c_5-d_1); max{$\pi_{\widetilde{p_1}}$, $\pi_{\widetilde{p_2}}$}, min{$\mu_{\widetilde{p_1}}$, $\mu_{\widetilde{p_2}}$},min{ $\sigma_{\widetilde{p_1}}$, $\sigma_{\widetilde{p_2}}$} >,

$\widetilde{F_1} \times \widetilde{F_2}$= < ($c_1d_1, c_2d_2, c_3d_3, c_4d_4, c_5d_5$); $\pi_{\widetilde{p_1}}$ $\pi_{\widetilde{p_2}}$, $\mu_{\widetilde{p_1}}$+, $\mu_{\widetilde{p_2}} - \mu_{\widetilde{p_1}}$ $\mu_{\widetilde{p_2}}$, $\sigma_{\widetilde{p_1}} + \sigma_{\widetilde{p_2}} - \sigma_{\widetilde{p_1}}\sigma_{\widetilde{p_2}}$ >,

$\alpha\widetilde{F_1}$= <($\alpha c_1, \alpha c_2, \alpha c_3, \alpha c_4, \alpha c_5$);$1-(1-\pi_{\widetilde{p_1}})^{\alpha}$, ${\mu_{\widetilde{p_1}}}^{\alpha}$, ${\sigma_{\widetilde{p_1}}}^{\alpha}$)>,

$\widetilde{F_1}^{\alpha}$= <(${c_1}^{\alpha}$, ${c_2}^{\alpha}$, ${c_3}^{\alpha}$, ${c_4}^{\alpha}$, ${c_5}^{\alpha}$);${\pi_{\widetilde{p_1}}}^{\alpha}$, $(1-\mu_{\widetilde{p_1}})^{\alpha}$, $(1-\sigma_{\widetilde{p_1}})^{\alpha}$ >.

3. Neutrosophic Transportation Problem

Assume that there are *s* number of sources and *t* destinations. Mathematically, the NTP may be stated as:

$$\text{Min}\, Z = \sum_{i=1}^{s}\sum_{j=1}^{t}\tilde{c}_{ij}^{N} y_{ij}. \tag{1}$$

Subject to constraints

$$\sum_{j=1}^{t} y_{ij} = \tilde{p}_i^{N}, i = 1, 2, ..., s. \tag{2}$$

$$\sum_{i=1}^{s} y_{ij} = \tilde{q}_i^{N}, j = 1, 2, ..., t. \tag{3}$$

$$y_{ij} \geq 0 \quad \forall i, j.$$

For the above mathematical model of PNTP, we defined the following notations:

- *s & t* is the number of sources and destination being indexed by *i & j*.
- $\tilde{p}_i^N = (p_i^{N1}, p_i^{N2}, p_i^{N3}, p_i^{N4}, p_i^{N5}; \theta_i^N, \sigma_i^N, \mu_i^N)$ is the PNN for the items supplied by source i .
- $\tilde{q}_j^N = (q_j^{N1}, q_j^{N2}, q_j^{N3}, p_j^{N4}, p_j^{N5}; \theta_j^N, \sigma_j^N, \mu_j^N)$ is the PNN for the items demanded by destination j .
- $\tilde{c}_{ij}^N = (c_{ij}^{N1}, c_{ij}^{N2}, c_{ij}^{N3}, c_{ij}^{N4}, c_{ij}^{N5}; \theta_{ij}^N, \sigma_{ij}^N, \mu_{ij}^N)$ is the PNN for the items sending one unit from source i to destination j .
- $\tilde{y}_{ij}^N = (y_{ij}^{N1}, y_{ij}^{N2}, y_{ij}^{N3}, y_{ij}^{N4}, y_{ij}^{N5}; \theta_{ij}^N, \sigma_{ij}^N, \mu_{ij}^N)$ is the PNN cost from sources to destination.

The steps of the proposed method are as follows:

Step 1. Considering the pentagonal Neutrosophic parameters and variables, the *problem (3)* may be written as:

$$\text{Min}\, Z=\sum_{i=1}^{s}\sum_{j=1}^{t}(c_{ij}^{N1},c_{ij}^{N2},c_{ij}^{N3},c_{ij}^{N4},c_{ij}^{N5};\theta_{ij}^{N},\sigma_{ij}^{N},\mu_{ij}^{N})\otimes(y_{ij}^{N1},y_{ij}^{N2},y_{ij}^{N3},y_{ij}^{N4},y_{ij}^{N5};\theta_{ij}^{N},\sigma_{ij}^{N},\mu_{ij}^{N}) \quad (4)$$

Subject to constraints

$$\sum_{j=1}^{t}(y_{ij}^{N1},y_{ij}^{N2},y_{ij}^{N3},y_{ij}^{N4},y_{ij}^{N5};\theta_{ij}^{N},\sigma_{ij}^{N},\mu_{ij}^{N})=(p_{i}^{N1},p_{i}^{N2},p_{i}^{N3},p_{i}^{N4},p_{i}^{N5};\theta_{i}^{N},\sigma_{i}^{N},\mu_{i}^{N}), i=1,2,\ldots,s. \quad (5)$$

$$\begin{aligned}&\sum_{i=1}^{s}(y_{ij}^{N1},y_{ij}^{N2},y_{ij}^{N3},y_{ij}^{N4},y_{ij}^{N5};\theta_{ij}^{N},\sigma_{ij}^{N},\mu_{ij}^{N})=(q_{j}^{N1},q_{j}^{N2},q_{j}^{N3},p_{j}^{N4},p_{j}^{N5};\theta_{j}^{N},\sigma_{j}^{N},\mu_{j}^{N}), j=1,2,\ldots,t\\&y_{ij}^{N1},y_{ij}^{N2},y_{ij}^{N3},y_{ij}^{N4},y_{ij}^{N5};\theta_{ij}^{N},\sigma_{ij}^{N},\mu_{ij}^{N}\geq 0 \quad \forall i,j.\end{aligned} \quad (6)$$

Step 2. Here, we confirmed whether the available model is balanced or not, i.e., demand=supply (or) $\sum_{i=1}^{s}\tilde{q}_{j}^{N}=\sum_{j=1}^{t}\tilde{p}_{i}^{N}$. If not, then add dummy variables on row or column and make it balanced model.

Step 3. With the help of accuracy function $\Re$, we transform the supply, demand and transportation cost as the following model:

$$\text{Min}\, Z=\Re(\sum_{i=1}^{s}\sum_{j=1}^{t}(c_{ij}^{N1},c_{ij}^{N2},c_{ij}^{N3},c_{ij}^{N4},c_{ij}^{N5};\theta_{ij}^{N},\sigma_{ij}^{N},\mu_{ij}^{N})\otimes(y_{ij}^{N1},y_{ij}^{N2},y_{ij}^{N3},y_{ij}^{N4},y_{ij}^{N5};\theta_{ij}^{N},\sigma_{ij}^{N},\mu_{ij}^{N})). \quad (7)$$

Subject to constraints

$$\Re(\sum_{j=1}^{t}(y_{ij}^{N1},y_{ij}^{N2},y_{ij}^{N3},y_{ij}^{N4},y_{ij}^{N5};\theta_{ij}^{N},\sigma_{ij}^{N},\mu_{ij}^{N}))=\Re((p_{i}^{N1},p_{i}^{N2},p_{i}^{N3},p_{i}^{N4},p_{i}^{N5};\theta_{i}^{N},\sigma_{i}^{N},\mu_{i}^{N})), i=1,2,\ldots,s. \quad (8)$$

$$\begin{aligned}&\Re(\sum_{i=1}^{s}(y_{ij}^{N1},y_{ij}^{N2},y_{ij}^{N3},y_{ij}^{N4},y_{ij}^{N5};\theta_{ij}^{N},\sigma_{ij}^{N},\mu_{ij}^{N}))=\Re(q_{j}^{N1},q_{j}^{N2},q_{j}^{N3},p_{j}^{N4},p_{j}^{N5};\theta_{j}^{N},\sigma_{j}^{N},\mu_{j}^{N}), j=1,2,\ldots,t\\&y_{ij}^{N1},y_{ij}^{N2},y_{ij}^{N3},y_{ij}^{N4},y_{ij}^{N5};\theta_{ij}^{N},\sigma_{ij}^{N},\mu_{ij}^{N}\geq 0 \quad \forall i,j.\end{aligned} \quad (9)$$

Step 4. After using our new accuracy function, we get PNTP into crisp transportation problem.

Step 5. Now find initial basic feasible solution.

- To determine the penalty, subtraction between smallest unit and next to smallest unit in the row (column).
- Identify the largest penalty in row/column, and make the allotment in the cell having the least unit cost.
- If the largest penalty arises in more than one row/column, then select topmost row/left side column.
- When the rows (column) have zero supply and demand until (m+n-1), then stop. Otherwise go to first line.

Step 6. Substitute the all y_{ij} in the objective function, we get the transportation cost.

4. Numerical Example

In our literature study, we got there is no method to solve PNTP. There is a lot of scope in this area to develop new method. We take the advantages in the field of pentagonal Neutrosophic area and we focus to start a developing new algorithm for solving PNTP. The main limitations in between fuzzy and Neutrosophic numbers is that fuzzy numbers taken only membership function (truth degree) however, Neutrosophic number taken truth, indeterminacy & falsity degree. In this segment, we consider another strategy to solve PNLP problem and compare with fuzzy pentagonal LP problem. To prove the relevance and proficiency of our proposed strategy, we consider the fuzzy problem which introduced by [20, 47].

Example 1. (Real-life problem) [47]. In Odisha, India have a company named M/s. Ashirivad dress pvt. Ltd. and the organisation has three plants for delivering dress. The dresses ought to be transport to three ware house under pentagonal Neutrosophic numbers. The conditions of transportation problem are presented in *Table 2*. As the problem should be PNN therefore, the decision-maker considers the confirmation degree of pentagonal number is (1,0,0).

Table 2. *Cost of unit for pentagonal Neutrosophic transportation problem.*

Ware house **Factories**	**Bhubaneswar**	**Cuttack**	**Rourkela**	**Supply**
Asha	*(5,10,13,14,18;1,0,0)*	*(1,2,3,4,5;1,0,0)*	*(2,6,8,10,14;1,0,0)*	*(2,11,23,34,45;1,0,0)*
Omm	*(3,4,5,6,7;1,0,0)*	*(1,5,6,7,11;1,0,0)*	*(1,4,5,9,16;1,0,0)*	*(10,47,52,65,76;1,0,0)*
Disha	*(3,6,9,12,15;1,0,0)*	*(2,5,7,8,8;1,0,0)*	*(1,1,1,1,1;1,0,0)*	*(3,18,56,76,87;1,0,0)*
Demand	*(11,16,51,67,75;1,0,0)*	*(20,40,60,80,100;1,0,0)*	*(15,30,45,75,110;1,0,0)*	

Step 1. Now using our new ranking function, the issue of PNTP is converting into crisp transportation problem. The model is now available in *Table 3*.

Table 3. *The defuzzified pentagonal Neutrosophic transportation problem.*

Ware house **Factories**	**Bhubaneswar**	**Cuttack**	**Rourkela**	**Supply**
Asha	*12*	*3*	*8*	*25*
Omm	*5*	*6*	*7*	*50*
Disha	*9*	*6*	*1*	*48*
Demand	*44*	*60*	*55*	

Step 2. To check whether the model is balanced or not.

Supply $\sum a_i = 23 + 50 + 48 = 121.$

Demand $\sum b_i = 44 + 60 + 55 = 159.$

Table 4. *Balanced transportation problem.*

Ware house Factories	Bhubaneswar	Cuttack	Rourkela	Supply
Asha	*12*	*3*	*8*	*23*
Omm	*5*	*6*	*7*	*50*
Disha	*9*	*6*	*1*	*48*
Demand	*44*	*60*	*55*	

Step 3. We use our algorithm (Step 5, Line 1) for finding the initial basic feasible solution.

Table 5. *Initial penalties allocation.*

Ware house Factories	Bhubaneswar	Cuttack	Rourkela	Supply	Penalty
Asha	*12*	*3*	*8*	*23*	*5*
Omm	*5*	*6*	*7*	*50*	*1*
Disha	*9*	*6*	*1*	*48*	*5*
Dummy	*0*	*0*	*0*	*38*	*0*
Demand	*44*	*60*	*55*		
Penalty	*5*	*3*	*1*		

Table 6. *After strike out of 3rd Row the penalties allocation.*

Ware house Factories	Bhubaneswar	Cuttack	Rourkela	Supply	Penalty
Asha	*12*	*3*	*8*	*23*	*5*
Omm	*5*	*6*	*7*	*50*	*1*
Disha	*9*	*6*	*1*	*48*	*5*
Dummy	*0*	*0*	*0*	--	--
Demand	*6*	*60*	*55*		
Penalty	*4*	*3*	*6*		

Table 7. *3rd Penalties allocation.*

Ware house Factories	Bhubaneswar	Cuttack	Rourkela	Supply	Penalty
Asha	*12*	*3*	*8*	*23*	*5*
Omm	*5*	*6*	*7*	*50*	*1*
Disha	*9*	*6*	*1*	--	--
Dummy	*0*	*0*	*0*	--	--
Demand	*6*	*60*	*7*		
Penalty	*7*	*3*	*1*		

Table 8. *4th Penalties allocation.*

Ware house Factories	Bhubaneswar	Cuttack	Rourkela	Supply	Penalty
Asha	*12*	*3*	*8*	*23*	*5*
Omm	*5*	*6*	*7*	*44*	*1*
Disha	*9*	*6*	*1*	--	--
Dummy	*0*	*0*	*0*	--	--
Demand	--	*60*	*7*		
Penalty	--	*3*	*1*		

Table 9. *Final allocation.*

Ware house Factories	Bhubaneswar	Cuttack	Rourkela	Supply	Penalty
Asha	*12*	*3*	*8*	--	--
Omm	*5*	*6*	*7*	*7*	*7*
Disha	*9*	*6*	*1*	--	--
Dummy	*0*	*0*	*0*	--	--
Demand	--	--	*7*		
Penalty	--	--	*7*		

The maximum penalty occurs, 7 in row 2.

The minimum c_{ij} in this row is c_{23} =7.

The maximum allocation in this cell is min (7,7)=7.

It is also satisfy the supply of row 2 (Omm) and demand in column 3 (Rourkela).

Table 10. *Initial basic feasible solution.*

Ware house Factories	Bhubaneswar	Cuttack	Rourkela	Supply
Asha	*12*	*3*	*8*	*23*
Omm	*5*	*6*	*7*	*50*
Disha	*9*	*6*	*1*	*48*
Dummy	*0*	*0*	*0*	*38*
Demand	*44*	*60*	*55*	

The minimum transportation cost is obtained as:

$$\text{Min} = (23\times 3) + (6\times 5) + (37\times 6) + (7\times 7) + (48\times 1) + (38\times 0) = 418.$$

Here, the number of allocated cells=6 which is equal to m+n-1=4+3-1=6

Therefore, this solution is non-degenerate.

Table 11. *Comparison of proposed method with existing method of example 1.*

	Transportation Cost
Proposed Method	*418*
Existing Method [20]	*418*

Example 2 [47]. Consider the pentagonal Neutrosophic numbers (supply, demand & transportation cost) are presented in *Table 12*.

Table 12. *Input data for pentagonal transportation problem.*

Factories	Bhubaneswar	Cuttack	Supply
Protein	*(0.2,0.4,0.5,0.6,0.7)*	*(0.3,0.2,0.6,0.5,0.1)*	*(0.1,0.2,0.5,0.4,0.3)*
Calories	*(0.7,0.8,0.6,0.9,0.1)*	*(0.2,0.3,0.5,0.7,0.1)*	*(0.2,0.3,0.5,0.8,0.9)*
Demand	*(0.8,0.7,0.5,0.3,0.2)*	*(0.2,0.3,0.4,0.1,0.2)*	

Here, the decision-makers consider the degree of each pentagonal number is (1,0,0).

After executing the steps of our algorithm, we get the initial basic feasible solutions presented in *Table 13*. This is balanced transportation problem.

Table 13. *Final initial basic feasible solution.*

Factories	Bhubaneswar	Cuttack	Supply
Protein	*0.48*	*0.34*	*0.3*
Calories	*0.62*	*0.36*	*0.54*
Demand	*0.5*	*0.34*	

The minimum transportation cost is obtained as:

$$\text{Min} = (0.3\times 0.48) + (0.2\times 0.62) + (0.34\times 0.36) = 0.3904.$$

Here, the number of allocated cells=3 which is equal to m+n-1=2+2-1=3

Therefore, this solution is non-degenerate.

Table 14. *Comparison of proposed method with existing method of example 2.*

	Transportation cost
Proposed Method	*0.3904*
Existing Method [20]	*0.41*

5. Analysis and Observation of the Proposed Model

5.1. Observation

Due to non-availability of pentagonal transportation problem under Neutrosophic environment, there is no direct comparison made in this paper. Therefore, we consider pentagonal Neutrosophic transportation problem under fuzzy environment for comparison our result. Hence, we compared our proposed method with the existing method [20].

For *Example 1* the pentagonal Neutrosophic transportation cost of IBFS is 418, which is exactly the transportation cost of fuzzy pentagonal numbers. In *Example 2*, the pentagonal Neutrosophic transportation problem is 0.3904, which is not exactly the cost of fuzzy pentagonal transportation problem i.e. 0.41. The decision-makers always want to minimize the cost of transportation when supplying the materials. Thusly, we can say that our proposed technique under Neutrosophic environment is always better than the other existing method. We also depicted our result along with the existing method results in graphical representation i.e. *Figs. (1)-(2)*.

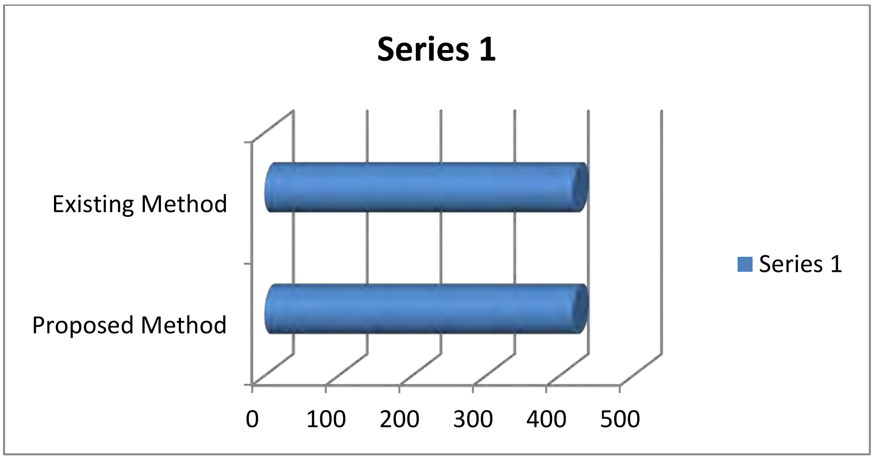


Fig. 1. *Graphical analysis of our proposed method with existing method of example 1.*

From the above analysis of both tabular form and graphical form, we can finalise that our proposed method is better to the existing method. Further, we can also claimed that our transportation cost obtained by our proposed method always lie within region of Neutrosophic sets.

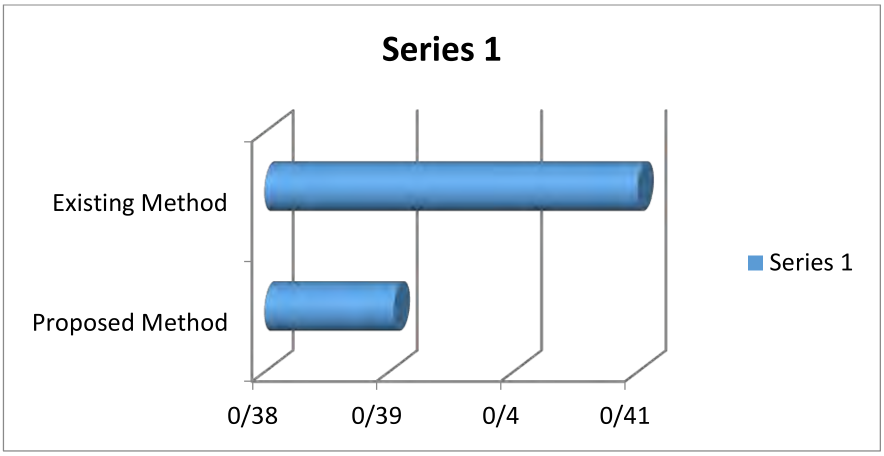


Fig. 2. *Graphical analysis of proposed work with existing work of example 2.*

5.2. Advantages of the Proposed Model

The pentagonal fuzzy numbers were widely applied in transportation problem to get the minimum cost. However, the decision-makers always consider the truth degree of pentagonal fuzzy numbers, which is the main drawback. In real-life problem, the decision-makers always want the clarity data means truth degree, indeterminacy degree and falsity degree. Neutrosophic sets consider the degree of truth, indeterminacy and falsity and we take the advantage of the properties of Neutrosophic sets, we develop a new algorithm for solving pentagonal Neutrosophic transportation problem. We proposed a new score function of Neutrosophic pentagonal numbers and also developed a new technique for getting initial basic feasible solutions. In our problem, our transportation cost is always minimizing then other existing method and minimization the cost is required for decision-makers. We also solved our problem in LP model by using LINGO 18 version or MATLAB, we get the same result.

In the above conversation, we can infer that our proposed calculation is another approach to deal with the vulnerability and indeterminacy in the transportation issue.

6. Conclusions

The transportation problem is one of the most popular optimization problems in operation research. The main objective of this problem is finding the minimum cost of transportation to supplier and demand. In this paper, Neutrosophic transportation problem has been solved under pentagonal Neutrosophic numbers. We also developed a score function and applied to find the IBFS. In the computation point of view, our proposed method is very easier when applied in real-life problem. Further comparative analysis is done with fuzzy pentagonal transportation problem. Also, the proposed algorithm has less computational complexity and saves time. By comparing our method with fuzzy method, we can conclude that our method can handle any type of uncertainties arise in real-life situation and it is very simple & efficient than other uncertainty. In addition to our proposed method, it will be extend in application of pentagonal assignment problem, pentagonal linear fractional programming and pentagonal job scheduling problem.

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Precise Services and Supply Chain Prioritization in Manufacturing Companies Using Cost Analysis Provided in a Fuzzy Environment

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ABSTRACT

In recent years, management and, consequently, supply chain performance measurement, has attracted the attention of a large number of managers and researchers in the field of production and operations management. In parallel with the evolution of organizations from a single approach to a network and supply chain approach, performance measurement systems have also changed and moved towards network and supply chain performance measurement. Therefore, in order to face the storm of great change and transformation and not give in to the wave of competitive aggression, organizations have long had one thing in common, and that is to focus approaches and focus efforts towards achieving results. Results that lead to a competitive advantage and are more effective and decisive in the performance indicators of the organization, including earning more. In this study, in order to identify and prioritize the factors affecting the supply chain in manufacturing companies, using indicators such as cost, timely delivery and procurement time to evaluate the supply chain efficiency is considered. And performance evaluation was performed at the manufacturer level. Therefore, in order to evaluate the performance of the supply chain using the AHP integration approach and the DEA method approach in the fuzzy environment, the suppliers and suppliers of the manufacturing company were evaluated and ranked in terms of performance.

1. Introduction

In recent years, management and, consequently, supply chain performance measurement, has attracted the attention of a large number of managers and researchers in the field of production and operations management [1]. In parallel with the evolution of organizations from a single approach to network approach and supply chain, performance measurement systems have also changed and moved towards measuring network performance and supply chain [2]. This attitude is rooted in the thinking of a system in which the efficiency of any production system does not depend only on the optimal functioning of a subsystem, and all subsystems must work diligently to achieve the pre-drawn goals [1]. Supply chain management is one of the components of competitive strategies for organizational productivity and profitability. Managers in many industries, especially those in the manufacturing sector, try to better

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manage the supply chain and evaluate its performance [3]. Therefore, it is important to evaluate and track the performance of its supply chain because several organizations are involved in this chain [4]. Many critical and complex barriers may distract current performance measurement systems from providing significant assistance to improve and expand supply chain management. Due to this inherent complexity, it is necessary to select the appropriate criteria for evaluating the performance of the supply chain [5]. In this chapter, while presenting the problem statement, the topics related to the necessity of conducting research, the theoretical framework of research (model and definition of variables), research objectives, research questions, research scope and limitations are also described. To deal with the storm of change and massive transformation and not to give in to the wave of competitive aggression, organizations have long had one thing in common, and that is to focus approaches and focus all efforts on achieving results; Results that lead to competitive advantage and are more effective and decisive in the performance indicators of the organization, including earning more revenue. Knowing that we are in the age of information and competition between organizations, and every organization to create a new way to transform its organization to surpass its competitors and maintain and gain a competitive advantage. As well as the important role that efficiency plays in the development of societies; examining all its dimensions, especially in the form of mathematical analysis, as a criterion for measuring performance is inevitable [6]. Manufacturing organizations need a high degree of flexibility in order to maintain a competitive advantage as well as to operate in an ever-changing dynamic environment. The success of organizations depends on their ability to deliver outputs. Optimal presentation of products according to criteria such as cost, quality, performance, delivery, flexibility and innovation depends on the ability of the organization to manage the flow of materials, information, etc. inside and outside the organization [7]. Supply chain evaluation is done using different methods. Data envelopment analysis as a non-parametric method is based on linear programming technique and compares the efficiency of different units. Wen et al. [8] provide evidence and reasons that data envelopment analysis is a good way to manage supply chain. Data envelopment analysis can have multi-cell inputs and outputs and uses quantitative and qualitative indicators [8]. Data envelopment analysis is a method to evaluate the performance of organizations in the private and public sectors. The reason for using data analysis as a way to evaluate performance is the complex nature of the relationships between multiple inputs and outputs in activities [9].

In this paper, indicators such as cost, timely delivery and procurement time are considered to evaluate the efficiency of the supply chain and performance evaluation is done at the manufacturer level, while usually looking at the supply chain as a system and overview. This means that performance appraisal indicators are measured for the manufacturer (second level of the chain) and in relation to the supplier and the customer, and the overall supply chain is maintained. Therefore, in this study, we seek to answer the question, how to identify and prioritize the factors affecting the supply chain in manufacturing companies using Data Envelopment Analysis (DEA) in a fuzzy environment? The analytical models proposed to evaluate supply chain efficiency include a variety of techniques, from simple rhythmic scoring methods to complex mathematical scheduling, and from definitive evaluation models to models under uncertainty conditions. Recently, various methods have been proposed to address the issue of supply chain efficiency assessment. 16 categories of these methods are presented by Estampe et al. [10]. To evaluate the efficiency of the supply chain, various indicators are measured in categories such as cost, time, profit, level of service and. Thomas and Griffin [11] equated transportation with more than half the cost of a supply chain and used it for evaluation.

Lee and Billington [12] consider the level of customer satisfaction in companies with customers from all over the world as an important factor and point out that the strategies adopted will not be very costly in order to achieve customer satisfaction. Most existing studies are based on evaluating the supply chain efficiency of a comprehensive evaluation index system. However, most of these methods use the individuals themselves to calculate the weight of the indicators in the evaluation process. Due to personal opinions, the weight of the indicators cannot be measured accurately [13].

To reduce the inaccuracy of the index weight, which is increased by the decision maker's personal opinion, data envelopment analysis is used as a non-parametric method to evaluate supply chain efficiency. The main feature of overlay analysis is that it can measure performance when there are multiple inputs and outputs. Wang and Wang [14] presented a data envelopment analysis model using indicators such as cost, on-time delivery, profit, and production time cycle. Given that some of the indicators measured in the supply chain, especially costs are not definitive and indices of uncertainty are seen in them, the use of uncertainty methods such as fuzzy logic seems appropriate [15]. Until now, uncertainty methods have rarely been used to evaluate the supply chain [14]. The methods of uncertainty used can be referred to uneven sets [15]. In this reference, by developing a rugged set of indicators such as cost, number of employees, production flexibility and level of service have been used to evaluate efficiency. In this research, fuzzy data envelopment analysis is used to evaluate the efficiency of supply chains, which has not been used in previous research. In this paper, the supply chain is considered as a whole and a system that the inputs and outputs of the fuzzy data envelopment analysis model are the same as the manual inputs and outputs of the supply chain. Evaluation indicators are measured at the manufacturer level and to maintain the integrity of the supply chain, the indicators are measured for the manufacturer and by maintaining its relationship with suppliers and customers. In this paper, the cost index is considered fuzzy due to the uncertainty present during the measurement. To deal with the uncertainty environment created by the cost index area, fuzzy set theory is used as a method to deal with uncertainty environments. Considering that no conceptual model is presented in this research, then there is no hypothesis in this research, but the assumptions for conducting the research are as follows:

- Information received from suppliers is fuzzy uncertainty.
- Suppliers are evaluated in the company's supply chain list and are accepted in the initial and technical evaluation.

Saleh and Shafiei [16], in a study entitled "Performance evaluation using envelopment analysis of three-level data" state that, attention to organizational performance evaluation in recent years has led to the development of several frameworks and methodologies, each of which has provided a wide range of benefits. One of the appropriate methods in calculating the efficiency of data envelopment analysis is that despite some limitations, it is a powerful methodology that allows managers to determine the efficiency of the organization under their management compared to other units. In the real world, we encounter different situations that follow a hierarchical structure with decentralized decisions. In this research, the efficiency of supply chains that have a hierarchical structure will be evaluated and a three-level model of data envelopment analysis will be presented by selecting appropriate indicators.

Koushki and Mashayekhi Nezamabadi [17], in a study entitled "A method of network data envelopment analysis to evaluate supply chains and its application in pharmacy" state that, data envelopment analysis is a non-parametric technique based on mathematical programming to evaluate the performance of heterogeneous decision-making units. Many units have a multi-stage structure in which the output of one stage is the input of the next stage. A supply chain, which includes several members such as supplier and manufacturer, has a multi-step process. In this paper, for the first time, network methods for achieving maximum productivity in supply chains, which are considered as a multi-stage system, are introduced. Such a view provides management concepts to improve the efficiency of the supply chain as well as the productivity of each member.

Mousavi and Ahmadzadeh [18], in a study entitled "Study and evaluation of supply chain efficiency using data envelopment analysis (Case study: Amol paper companies)" state that, rapid progress and development, rapid environmental changes, and awareness of new developments and approaches to achieve efficiency and effectiveness in organizations have become essential. In recent years, supply chain management and performance evaluation has received more attention in the business administration of organizations. In this study, the operational performance of seven supply chains operating in the same industry and having different key companies and relatively similar suppliers and distributors has been evaluated using data envelopment analysis method. In order to evaluate the supply chain in this research, indicators such as direct costs, manpower, and depreciation have been considered.

Hosseinzadeh Seljooghi and Rahimi [19], in a study entitled "Evaluation of efficiency and efficiency at the scale of the supply chain of the Iranian resin industry with the model of definitive and fuzzy data envelopment analysis" state, the fuzzy DEA model is used based on the cut-off approach to measure efficiency and determine the supply chain scale efficiency. The proposed ideas have been used to evaluate the efficiency and efficiency of the supply chain scale of 27 resin production companies. In evaluating with definitive data, 6 companies are network efficient; while in the case of fuzzy data, three companies are network efficient. These companies have managed and coordinated the flow of materials between several organizations and within the organization in the most optimal way and with regard to environmental issues.

Samuelinko stated in 2013 that the competitive nature of the business environment requires the awareness of productivity-based organizations of the relative level of effectiveness and efficiency of their competitors. This indicates, firstly, the need for an effective mechanism that allows the discovery of appropriate productivity models to improve overall organizational performance, and secondly, the need for a feedback mechanism that allows the evaluation of different productivity models to select the most appropriate model. In this article, we focus on organizations that consider the state of the internal organizational environment (for example, likely to represent a resource-oriented perspective) and external (for example, likely to represent a positioning perspective) in formulating their strategies. We propose and test a DEA-based Decision Support System (DSS) that aims to evaluate and manage the relative performance of such organizations [20].

Singh stated in 2014 that manpower in an organization is an important and fundamental asset. Qualified personnel have unique academic and managerial abilities in specific disciplines and individual capabilities that can perform many of the different marketing and research tasks required in any organization because they are, in fact, the creditors of the organization's performance. They forgive. Therefore, designing rational methods for assessing the capability of personnel during employment is crucial. The methods that are commonly used for decision making in identifying functional characteristics, including their heavy tasks, include methods such as Delphi and decision matrix, Hierarchical Analysis Process (AHP), and so on. The AHP method converts experts' qualitative theory into quantitative values and creates a decision matrix. In this paper, in this study, the Data Push Analysis method is investigated to establish the internal weights of alternative methods by comparing two-by-two comparison matrices in AHP for a three-property system to measure personnel performance at levels. Login to the management hierarchy is used. Several expert judgments have been made to determine the weight of the features. In conclusion, the SUPER EFFICIENCY DEA (or DEA-AHP combination method) is proposed in this paper as an alternative to traditional weight derivation methods in AHP [21].

Comelli et al. [22] have proposed an approach for evaluating production planning in supply chains. They noted that production planning evaluations are usually based on physical parameters such as inventory level and demand satisfaction. They found it useful to add financial valuation to classical models. They applied an ABC method to estimate the cash flow of supply chain production planning.

In 2016, Lim stated that supplier selection is an important issue that supply chain managers have faced for many years. Choosing the right suppliers is no longer as easy as choosing (based on the price) they offer. There are many quantitative and qualitative criteria that must be considered. Therefore, there is an urgent need for an approach that can meet these criteria. In addition, as supply chains become increasingly important today, it is important to consider the risks of inadequate supply in evaluating suppliers. This research presents an approach that focuses mainly on data envelopment analysis to analyze and compare the relative performance of suppliers. Because data envelopment analysis can only cover quantitative features, the Analytic Hierarchy Process (AHP) is used to aid qualitative analysis. Risks are also considered in the evaluation of suppliers. The purpose of the proposed approach is to provide a comprehensive approach to addressing the issue of supplier selection [23].

Liang et al. [24] identified two barriers to supply chain evaluation and its members in the form of multiple indicators that determine member performance and the existence of conflict between chain members. They showed that the classical DEA model could not perform as well as the mosque due to the presence of intermediate indicators, so in their research they have developed several DEA-based models in which intermediate indicators are integrated in performance evaluation. They developed their model as a two-chain, seller-buyer model. They considered two different modes. The first mode is that one chain acts as the leader and the second chain follows it. The leader is evaluated using member results. The second case is in the form of a partnership in which an attempt is made to maximize the joint efficiency of the two chains, which is considered as their average efficiency. In this case, both supply chains are evaluated simultaneously.

In his research, Chen [25] divided supply chain evaluation criteria into two main categories: quantitative and qualitative. Quantitative indicators include cost and resource use, and qualitative indicators include quality, flexibility, visibility, trust and innovation. He then states the measurement criteria for each of these seven categories of indicators and then uses the AHP technique to identify the most important indicators for the electronics industry. He also made suggestions for other industries.

Easton et al. [26] evaluated the evaluation of purchasing sector efficiency in the supply chain. They pointed out that it is very difficult to measure the efficiency of the procurement department and compare that efficiency with other departments of procurement, and attributed this difficulty to the lack of acceptable measurement criteria and appropriate methods to integrate these criteria and provide an overall efficiency. They developed a DEA model to evaluate purchasing efficiency in the petrochemical industry.

2. Methodology

In this study, according to the parts intended to provide an efficient supply chain, first, according to the conditions governing the production of these parts, all suppliers in this field are identified and we put one of the basic and serious principles in the list of suppliers with contract priority. In the supply of these parts, the reduction of risk arises from the selection of the supplier, which in the event of a mistake will incur irreparable losses, which will lead to the failure of the project. In order to conclude a contract for the supply of these parts, it is necessary to prove the efficiency of the supplier in the first stage and to be ranked according to the rank in which they are placed in the next step. In order to evaluate the efficiency of suppliers, it is necessary to measure the input to output ratio of each supplier, and for this issue, according to the main source of this research, the Super Efficiency DEA method has been used. Therefore, it can be said that this research is applied based on the purpose and descriptive-survey based on the nature and method of research. The data collected to solve the research model are related to the years 2019-2020. In the present study, two library and field methods have been used to collect information. In order to collect information in this research, first the documentary method will be used. In order to study and obtain more information in order to know more precisely the subject of research and use the findings of research in this field, the researcher to study and study academic dissertations, foreign and Iranian books, Persian and English journals and textbooks Some professors pay. This research is in the field of measuring the efficiency of supply chains of a manufacturing company and examines the separation of efficient and inefficient chains, determining the appropriate pattern for inefficient units, as well as how to allocate resources optimally. The present study is conducted to investigate the efficiency of supply chains in manufacturing and industrial companies.

2.1. Identify Supplier Evaluation Indicators

In the first step of the research, after reviewing and identifying the suppliers, the first phase of the evaluation begins by selecting appropriate indicators for evaluation. In this section, after reviewing the written scientific texts, the evaluation indicators were identified as *Table 1*.

Table 1. *Supplier survey indicators.*

Row	Description of the index
1	*Price product*
2	*Place of delivery*
3	*Quality systems certifications*
4	*After sales service indicators*
5	*Customization capability*
6	*Product quality*
7	*Ability to reduce costs*
8	*Packing*

In this study, the verbal variables to determine the importance of the indicators are fuzzified according to the triangular fuzzy numbers in *Table 2* and *Fig. 1*.

Table 2. *Fuzzification of verbal variables in Delphi technique.*

Verbal Variables	Triangular Fuzzy Numbers
Very little importance	*(0.25,0,0)*
Low importance	*(0.5,0.25,0)*
Medium importance	*(0.75,0.5,0.25)*
Important	*(1,0.75,0.5)*
Very important	*(1,1,0.75)*

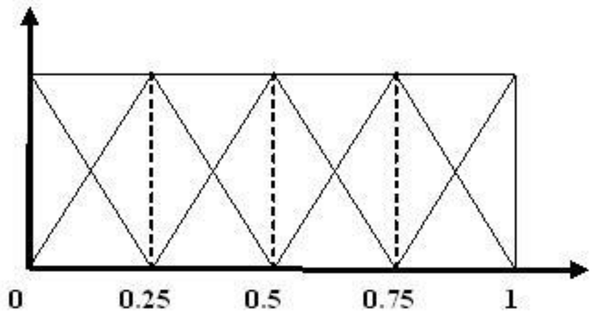

Fig. 1. *Triangular fuzzy numbers.*

3. Findings

3.1. Introducing the Company's Suppliers

According to the scope of work of the manufacturing company and also the studies carried out in accordance with the executive instructions of the company, 10 suppliers have been selected as the final candidate supply chain for evaluation and transfer of supply of parts. Suppliers are as follows:

- Sepehr Ryan Sanhat Company (Symbol: A).
- Cheese Company (symbol: B).
- Parsian Sazeh Sepahan Company (Symbol: C).
- Techno Sanat Company (symbol: D).
- Peyman Sanat Company (Symbol: E).
- Tractor Manufacturing Company (Symbol: F).
- Ataco Company (Symbol: G).
- Sarco Company (symbol: H).
- Beshl Motor Company (Symbol: I).
- Iran Casting Company (Symbol: K).

3.2. Introducing the Experts of the Research

In this research, in order to evaluate the indicators and select them, using the opinion of the company's experts, the specifications of the experts are as *Table 3*.

Table 3. *The Specifications of the experts.*

Row	Side	Work Experience	Education	Age
1	*plan and program manager*	*18 years*	*MA*	*54 years*
2	*Supply management*	*20 years*	*Bachelor*	*48 years*
3	*Procurement manager*	*10 years*	*Bachelor*	*38 years*
4	*Quality assurance management*	*20 years*	*Bachelor*	*48 years*
5	*Market research and development management*	*10 years*	*Doctorate*	*35 years*
6	*Engineering management*	*23 years*	*Bachelor*	*45 years*
7	*Laboratory management*	*20 years*	*MA*	*55 years*

3.3. Identification, Refining and Screening of Input and Output Indicators with Fuzzy Technique

First, based on the research literature and specialized interviews, a set of input and output indicators of DMUs has been identified. Fuzzy technique was used for screening and final confirmation of the indicators. The indicators are symbolized in *Table 4*.

Table 4. *Symbolization of indicators.*

Symbol	Description of the Index
i1	*price product*
i2	*Place of delivery*
i3	*Quality systems certifications*
i4	*After sales service indicators*
i5	*Customization capability*
O1	*Product quality*
O2	*Ability to reduce costs*
O3	*Packing*

The views of seven experts to measure the importance of the indicators related to each of the input and output indicators are as *Table 5*.

Table 5. *Experts' views about each indicator.*

Symbol	Expert 1	Expert 2	Expert 3	Expert 4	Expert 5	Expert 6	Expert 7
i1	*very much*	*much*	*much*	*very much*	*medium*	*medium*	*medium*
i2	*very much*	*medium*	*medium*	*very much*	*much*	*very much*	*much*
i3	*very much*	*very much*	*much*	*very much*	*much*	*much*	*much*
i4	*very much*	*very much*	*medium*	*very much*	*much*	*medium*	*much*
i5	*much*	*very much*	*medium*	*much*	*very much*	*medium*	*very much*
O1	*medium*	*medium*	*medium*	*medium*	*very much*	*much*	*very much*
O2	*much*	*very much*	*much*	*much*	*very much*	*much*	*very much*
O3	*very much*	*much*	*medium*	*very much*	*very much*	*much*	*very much*

The collected data are fuzzy evaluated according to the *Table 5*. The fuzzy values of the experts' point of view are shown in *Table 6*.

Table 6. *Fuzzified values of the seven experts' views about each indicator.*

Symbol	Expert 1			Expert 2			Expert 3			Expert 4		
i1	*1*	*1*	*0.75*	*1*	*0.75*	*0.5*	*1*	*0.75*	*0.5*	*1*	*1*	*0.75*
i2	*1*	*1*	*0.75*	*0.75*	*0.5*	*0.25*	*0.75*	*0.5*	*0.25*	*1*	*1*	*0.75*
i3	*1*	*1*	*0.75*	*1*	*1*	*0.75*	*1*	*0.75*	*0.5*	*1*	*1*	*0.75*
i4	*1*	*1*	*0.75*	*1*	*1*	*0.75*	*0.75*	*0.5*	*0.25*	*1*	*1*	*0.75*
i5	*1*	*0.75*	*0.5*	*1*	*1*	*0.75*	*0.75*	*0.5*	*0.25*	*1*	*0.75*	*0.5*
O1	*1*	*0.75*	*0.5*	*1*	*1*	*0.75*	*0.75*	*0.5*	*0.25*	*1*	*0.75*	*0.5*
O2	*1*	*1*	*0.75*	*1*	*0.75*	*0.5*	*1*	*0.75*	*0.5*	*1*	*1*	*0.75*
O3	*0.75*	*0.5*	*0.25*	*1*	*0.75*	*0.5*	*0.75*	*0.5*	*0.25*	*0.75*	*0.5*	*0.25*

Continue of table 6

Symbol	Expert 5			Expert 6			Expert 7		
i1	*0.75*	*0.5*	*0.25*	*0.75*	*0.5*	*0.25*	*0.75*	*0.5*	*0.25*
i2	*1*	*0.75*	*0.5*	*1*	*1*	*0.75*	*1*	*0.75*	*0.5*
i3	*1*	*0.75*	*0.5*	*1*	*0.75*	*0.5*	*1*	*0.75*	*0.5*
i4	*1*	*0.75*	*0.5*	*0.75*	*0.5*	*0.25*	*1*	*0.75*	*0.5*
i5	*1*	*1*	*0.75*	*0.75*	*0.5*	*0.25*	*1*	*1*	*0.75*
O1	*1*	*1*	*0.75*	*1*	*0.75*	*0.5*	*1*	*1*	*0.75*
O2	*1*	*1*	*0.75*	*1*	*1*	*0.75*	*1*	*1*	*0.75*
O3	*1*	*0.75*	*0.5*	*1*	*0.75*	*0.5*	*1*	*0.75*	*0.5*

In the next step, the fuzzy average of expert opinions is calculated. In the following work is used to defuzzificate and determine the importance of input and output indicators. The fuzzy mean and the definite value of the values related to the indicators are shown in *Table 7*. Since the definite value of all values is greater than 0.5, all indices are confirmed.

Table 7. *The fuzzy average of experts' opinions and the definite amounts of the indicators' values.*

Symbol	Description of the index	Fuzzy average	Definite amount
i1	*Price product*	*(0.46,0.71,0.89)*	*0.70*
i2	*Place of delivery*	*(0.54.0.79.0.93)*	*0.77*
i3	*Quality systems certifications*	*(0.61,0.86,1)*	*0.84*
i4	*After sales service indicators*	*(0.54.0.79.0.93)*	*0.77*
i5	*Customization capability*	*(0.54.0.79.0.93)*	*0.77*
O1	*Product quality*	*(0.57,0.82,0.96)*	*0.80*
O2	*Ability to reduce costs*	*(0.68,0.93,1)*	*0.90*
O3	*Packing*	*(0.39,0.64,0.89)*	*0.64*

3.4. Pairewise Comparison of Suppliers Based on Input and Output Indicators

In this step, according to the identification of input and output indicators of each supplier, we prioritize suppliers using pairwise comparison based on each indicator.

3.4.1. Prioritization of suppliers based on product quality index

According to the identified quality index, by forming a pairwise comparison matrix by 7 experts, the matrix shown in *Table 8* was formed.

Table 8. *Prioritization of suppliers based on product quality index.*

	A	B	C	D	E	F	G	H	I	j
A	*1.00*	*2.50*	*0.75*	*1.33*	*0.40*	*0.40*	*0.75*	*0.75*	*6.00*	*0.40*
B	*0.40*	*1.00*	*0.17*	*1.33*	*2.50*	*1.33*	*0.40*	*1.33*	*0.75*	*0.40*
C	*1.33*	*6.00*	*1.00*	*0.40*	*0.75*	*1.33*	*2.50*	*6.00*	*1.33*	*0.40*
D	*0.75*	*0.75*	*2.50*	*1.00*	*0.40*	*2.50*	*0.75*	*0.75*	*1.33*	*0.17*
E	*2.50*	*0.40*	*1.33*	*2.50*	*1.00*	*0.40*	*0.17*	*6.00*	*0.75*	*0.75*
F	*2.50*	*0.75*	*0.75*	*0.40*	*2.50*	*1.00*	*0.17*	*0.40*	*1.33*	*1.33*
G	*1.33*	*2.50*	*0.40*	*1.33*	*6.00*	*6.00*	*1.00*	*0.75*	*0.75*	*2.50*
H	*1.33*	*0.75*	*0.17*	*1.33*	*0.17*	*2.50*	*1.33*	*1.00*	*0.75*	*2.50*
I	*0.17*	*1.33*	*0.75*	*0.75*	*1.33*	*0.75*	*1.33*	*1.33*	*1.00*	*0.40*
j	*2.50*	*2.50*	*2.50*	*6.00*	*1.33*	*0.75*	*0.40*	*0.40*	*2.50*	*1.00*

Table 9. *Ranking of suppliers based on the product quality index.*

A	B	C	D	E	F	G	H	I	j
0.940241	*0.737199*	*1.383006*	*0.840422*	*0.971642*	*0.835959*	*1.568282*	*0.863876*	*0.785027*	*1.436823*

3.4.2. Prioritization of suppliers based on cost reduction capability index

According to the cost reduction capability index, by forming a pair comparison matrix by 7 experts, the matrix shown in *Table 10* was formed.

Table 10. *Prioritization of suppliers based on cost reduction capability index.*

	A	B	C	D	E	F	G	H	I	j
A	*1.00*	*0.75*	*1.33*	*0.17*	*2.50*	*1.33*	*0.40*	*0.40*	*1.33*	*0.75*
B	*1.33*	*1.00*	*0.40*	*0.75*	*0.17*	*2.50*	*1.33*	*6.00*	*1.33*	*0.40*
C	*0.75*	*2.50*	*1.00*	*0.75*	*2.50*	*0.75*	*0.75*	*1.33*	*2.50*	*2.50*
D	*6.00*	*1.33*	*1.33*	*1.00*	*0.75*	*1.33*	*1.33*	*1.33*	*1.33*	*2.50*
E	*0.40*	*6.00*	*0.40*	*1.33*	*1.00*	*0.40*	*2.50*	*6.00*	*1.33*	*1.33*
F	*0.75*	*0.40*	*1.33*	*0.75*	*2.50*	*1.00*	*0.75*	*1.33*	*2.50*	*2.50*
G	*2.50*	*0.75*	*1.33*	*0.75*	*0.40*	*1.33*	*1.00*	*6.00*	*1.33*	*2.50*
H	*2.50*	*0.17*	*0.75*	*0.75*	*0.17*	*0.75*	*0.17*	*1.00*	*2.50*	*2.50*
I	*0.75*	*0.75*	*0.40*	*0.75*	*0.75*	*0.40*	*0.75*	*0.40*	*1.00*	*0.75*
j	*1.33*	*2.50*	*0.40*	*0.40*	*0.75*	*0.40*	*0.40*	*0.40*	*1.33*	*1.00*

Table 11. *Ranking of suppliers based on cost reduction capability index.*

A	B	C	D	E	F	G	H	I	j
0.785027	*0.966482*	*1.32341*	*1.513835*	*1.298745*	*1.167063*	*1.349283*	*0.705432*	*0.639226*	*0.713375*

3.4.3. Prioritization of suppliers based on the packaging index

According to the packing index, by forming a pairwise comparison matrix by 7 experts, the matrix shown in *Table 12* was formed.

Table 12. *Prioritization of suppliers based on packing index.*

	A	B	C	D	E	F	G	H	I	j
A	*1.00*	*0.40*	*1.33*	*0.17*	*1.33*	*0.75*	*2.50*	*0.40*	*0.75*	*6.00*
B	*2.50*	*1.00*	*0.40*	*1.33*	*0.75*	*1.33*	*1.33*	*0.40*	*2.50*	*2.50*
C	*0.75*	*2.50*	*1.00*	*0.75*	*1.33*	*1.33*	*2.50*	*2.50*	*6.00*	*0.75*
D	*6.00*	*0.75*	*1.33*	*1.00*	*0.75*	*1.33*	*1.33*	*1.33*	*1.33*	*2.50*
E	*0.75*	*1.33*	*0.75*	*1.33*	*1.00*	*2.50*	*1.33*	*1.33*	*0.40*	*0.17*
F	*1.33*	*0.75*	*0.75*	*0.75*	*0.40*	*1.00*	*0.75*	*1.33*	*2.50*	*2.50*
G	*0.40*	*0.75*	*0.40*	*0.75*	*0.75*	*1.33*	*1.00*	*0.17*	*0.75*	*0.40*
H	*2.50*	*2.50*	*0.40*	*0.75*	*0.75*	*0.75*	*6.00*	*1.00*	*0.75*	*0.75*
I	*1.33*	*0.40*	*0.17*	*0.75*	*2.50*	*0.40*	*1.33*	*1.33*	*1.00*	*1.33*
j	*0.17*	*0.40*	*1.33*	*0.40*	*6.00*	*0.40*	*2.50*	*1.33*	*0.75*	*1.00*

Table 13. *Ranking of suppliers based on the packaging index.*

A	B	C	D	E	F	G	H	I	j
0.912444	*1.160865*	*1.530042*	*1.429193*	*0.885467*	*1.03468*	*0.582534*	*1.135376*	*0.83152*	*0.856852*

3.4.4. Prioritization of suppliers based on product price index

According to the input indices identified for each supplier, based on the price index of the pairwise comparison by 7 experts, the matrix shown in *Table 14* was formed.

Table 14. *Prioritization of suppliers based on product price index.*

	A	B	C	D	E	F	G	H	I	G
A	1.00	0.40	1.33	1.33	0.17	2.50	2.50	1.33	1.33	0.40
B	2.50	1.00	0.75	0.75	2.50	0.75	1.33	6.00	2.50	2.50
C	0.75	1.33	1.00	2.50	0.75	1.33	0.17	1.33	1.33	0.75
D	0.75	1.33	0.40	1.00	2.50	0.40	0.40	6.00	2.50	2.50
E	6.00	0.40	1.33	0.40	1.00	0.75	1.33	1.33	1.33	1.33
F	0.40	1.33	0.75	2.50	1.33	1.00	2.50	2.50	1.33	1.33
G	0.40	0.75	6.00	2.50	0.75	0.40	1.00	6.00	2.50	1.33
H	0.75	0.17	0.75	0.17	0.75	0.40	0.17	1.00	2.50	1.33
I	0.75	0.40	0.75	0.40	0.75	0.75	0.40	0.40	1.00	2.50
j	2.50	0.40	1.33	0.40	0.75	0.75	0.75	0.75	0.40	1.00

Table 15. *Ranking of suppliers based on the product price Index.*

A	B	C	D	E	F	G	H	I	j
0.937908	1.629309	0.942915	1.196231	1.117384	1.309392	1.390389	0.551527	0.677084	0.763713

3.4.5. Prioritization of suppliers based on the place of delivery index

Table 16. *Prioritization of suppliers based on the place of delivery index.*

	A	B	C	D	E	F	G	H	I	j
A	1.00	0.75	2.50	1.33	0.75	1.33	1.33	2.50	6.00	1.33
B	1.33	1.00	2.50	1.33	2.50	2.50	1.33	0.17	1.33	0.75
C	0.40	0.40	1.00	0.75	1.33	2.50	1.33	6.00	1.33	2.50
D	0.75	0.75	1.33	1.00	1.33	1.33	1.33	6.00	0.40	2.50
E	1.33	0.40	0.75	0.75	1.00	0.75	0.40	0.40	0.75	6.00
F	0.75	0.40	0.40	0.75	1.33	1.00	0.50	0.75	0.75	2.50
G	0.75	0.75	0.75	0.75	2.50	2.00	1.00	6.00	0.75	1.33
H	0.40	6.00	0.17	0.17	2.50	1.33	0.17	1.00	2.50	1.33
I	0.17	0.75	0.75	2.50	1.33	1.33	1.33	0.40	1.00	1.33
j	0.75	1.33	0.40	0.40	0.17	0.40	0.75	0.75	0.75	1.00

Table 17. *Supplier rating.*

A	B	C	D	E	F	G	H	I	j
1.521917	1.199633	1.267077	1.267077	0.833588	0.780947	1.252381	0.811244	0.885467	0.582534

3.4.6. Prioritization of suppliers based on the quality system certification index

Table 18. *Prioritization of suppliers based on the quality system certification index.*

	A	B	C	D	E	F	G	H	I	G
A	*1.00*	*0.40*	*0.40*	*0.40*	*0.75*	*0.75*	*0.17*	*0.40*	*0.17*	*0.75*
B	*2.50*	*1.00*	*1.33*	*0.75*	*0.75*	*2.50*	*0.40*	*0.40*	*0.75*	*1.33*
C	*2.50*	*0.75*	*1.00*	*1.33*	*0.40*	*0.17*	*0.17*	*0.17*	*0.40*	*0.75*
D	*2.50*	*1.33*	*0.75*	*1.00*	*0.75*	*2.50*	*2.50*	*2.50*	*6.00*	*1.33*
E	*1.33*	*1.33*	*2.50*	*1.33*	*1.00*	*1.33*	*1.33*	*2.50*	*2.50*	*6.00*
F	*1.33*	*0.40*	*6.00*	*0.40*	*0.75*	*1.00*	*1.33*	*1.33*	*0.75*	*2.50*
G	*6.00*	*2.50*	*6.00*	*0.40*	*0.75*	*0.75*	*1.00*	*6.00*	*0.40*	*1.33*
H	*2.50*	*2.50*	*6.00*	*0.40*	*0.40*	*0.75*	*0.17*	*1.00*	*0.40*	*1.33*
I	*6.00*	*1.33*	*2.50*	*0.17*	*0.40*	*1.33*	*2.50*	*2.50*	*1.00*	*1.33*
j	*1.33*	*0.75*	*1.33*	*0.75*	*0.17*	*0.40*	*0.75*	*0.75*	*0.75*	*1.00*

Table 19. *Supplier rating.*

A	B	C	D	E	F	G	H	I	j
0.444337	*0.971642*	*0.517925*	*1.725803*	*1.818304*	*1.12335*	*1.517601*	*0.912444*	*1.309392*	*0.699696*

3.4.7. Prioritization of suppliers based on after-sales service indicators

Table 20. *Prioritization of suppliers based on after-sales service indicators.*

	A	B	C	D	E	F	G	H	I	G
A	*1.00*	*0.40*	*1.33*	*1.33*	*0.17*	*2.50*	*2.50*	*1.33*	*1.33*	*0.40*
B	*2.50*	*1.00*	*0.75*	*0.75*	*2.50*	*0.75*	*1.33*	*6.00*	*2.50*	*2.50*
C	*0.75*	*1.33*	*1.00*	*2.50*	*0.75*	*1.33*	*0.17*	*1.33*	*1.33*	*0.75*
D	*0.75*	*1.33*	*0.40*	*1.00*	*2.50*	*0.40*	*0.40*	*6.00*	*2.50*	*2.50*
E	*6.00*	*0.40*	*1.33*	*0.40*	*1.00*	*0.75*	*1.33*	*1.33*	*1.33*	*1.33*
F	*0.40*	*1.33*	*0.75*	*2.50*	*1.33*	*1.00*	*2.50*	*2.50*	*1.33*	*1.33*
G	*0.40*	*0.75*	*6.00*	*2.50*	*0.75*	*0.40*	*1.00*	*6.00*	*2.50*	*1.33*
H	*0.75*	*0.17*	*0.75*	*0.17*	*0.75*	*0.40*	*0.17*	*1.00*	*2.50*	*1.33*
I	*0.75*	*0.40*	*0.75*	*0.40*	*0.75*	*0.75*	*0.40*	*0.40*	*1.00*	*2.50*
j	*2.50*	*0.40*	*1.33*	*0.40*	*0.75*	*0.75*	*0.75*	*0.75*	*0.40*	*1.00*

Table 21. *Supplier rating.*

A	B	C	D	E	F	G	H	I	j
0.937908	*1.629309*	*0.942915*	*1.196231*	*1.117384*	*1.309392*	*1.390389*	*0.551527*	*0.677084*	*0.763713*

3.4.8. Prioritization of suppliers based on customization indicators

Table 22. *Prioritization of suppliers based on customization indicators.*

	A	B	C	D	E	F	G	H	I	j
A	*1.00*	*0.40*	*1.33*	*0.17*	*1.33*	*0.75*	*2.50*	*0.40*	*0.75*	*6.00*
B	*2.50*	*1.00*	*0.40*	*1.33*	*0.75*	*1.33*	*1.33*	*0.40*	*2.50*	*2.50*
C	*0.75*	*2.50*	*1.00*	*0.75*	*1.33*	*1.33*	*2.50*	*2.50*	*6.00*	*0.75*
D	*6.00*	*0.75*	*1.33*	*1.00*	*0.75*	*1.33*	*1.33*	*1.33*	*1.33*	*2.50*
E	*0.75*	*1.33*	*0.75*	*1.33*	*1.00*	*2.50*	*1.33*	*1.33*	*0.40*	*0.17*
F	*1.33*	*0.75*	*0.75*	*0.75*	*0.40*	*1.00*	*0.75*	*1.33*	*2.50*	*2.50*
G	*0.40*	*0.75*	*0.40*	*0.75*	*0.75*	*1.33*	*1.00*	*0.17*	*0.75*	*0.40*
H	*2.50*	*2.50*	*0.40*	*0.75*	*0.75*	*0.75*	*6.00*	*1.00*	*0.75*	*0.75*
I	*1.33*	*0.40*	*0.17*	*0.75*	*2.50*	*0.40*	*1.33*	*1.33*	*1.00*	*1.33*
J	*0.17*	*0.40*	*1.33*	*0.40*	*6.00*	*0.40*	*2.50*	*1.33*	*0.75*	*1.00*

Table 23. *Supplier rating.*

A	B	C	D	E	F	G	H	I	j
0.912444	*1.160865*	*1.530042*	*1.429193*	*0.885467*	*1.03468*	*0.582534*	*1.135376*	*0.83152*	*0.856852*

According to the final evaluation of suppliers based on input and output indicators, the final matrix of suppliers based on indicators will be as *Tables (24)-(25)*. Supplier scores based on output indicators.

Table 24. *Supplier scores based on output indicators.*

	Product quality	Reduce costs	Packaging
A	*0.94*	*0.79*	*0.91*
B	*0.74*	*0.97*	*1.16*
C	*1.38*	*1.32*	*1.53*
D	*0.84*	*1.51*	*1.43*
E	*0.97*	*1.30*	*0.89*
F	*0.84*	*1.17*	*1.03*
G	*1.57*	*1.35*	*0.58*
H	*0.86*	*0.71*	*1.14*
I	*0.79*	*0.64*	*0.83*
j	*1.44*	*0.71*	*0.86*

Table 25. *Scores of suppliers based on input indicators.*

	Price Product	Place of Delivery	Quality Systems	After Sales Service	Customization
A	*0.94*	*1.52*	*0.44*	*0.94*	*0.91*
B	*1.63*	*1.20*	*0.97*	*1.63*	*1.16*
C	*0.94*	*1.27*	*0.52*	*0.94*	*1.53*
D	*1.20*	*1.27*	*1.73*	*1.20*	*1.43*
E	*1.12*	*0.83*	*1.82*	*1.12*	*0.89*
F	*1.31*	*0.78*	*1.12*	*1.31*	*1.03*
G	*1.39*	*1.25*	*1.52*	*1.39*	*0.58*
H	*0.55*	*0.81*	*0.91*	*0.55*	*1.14*
I	*0.68*	*0.89*	*1.31*	*0.68*	*0.83*
j	*0.76*	*0.58*	*0.70*	*0.76*	*0.86*

4. Conclusion

With the increase in the number of suppliers in the supply sector of manufacturing companies, the need to have information about the capabilities, capabilities and executive records of suppliers for companies is felt more than ever. In the meantime, having a procedure and instructions that can evaluate suppliers from several different criteria and angles and can select the best supplier is more important. Therefore, in this study, after initial screening of supplier review indicators, the most important indicators were evaluated and selected. Due to the quality of the evaluation indicators, at first, all suppliers were ranked and weighted based on each index using the AHP method. Then, according to the evaluation, all suppliers were evaluated using the Super Efficiency DEA method, all suppliers, based on which the suppliers were ranked among the efficient suppliers, and an accurate evaluation can be provided in this regard. The results of comparing the manufacturing company supplier chain rankings based on AHP, FAHP, and Super Efficiency DEA methods are as *Table 26*.
According to the points obtained, the ranking of suppliers with the methods introduced is as *Table 27*.

All With the increase in the number of suppliers in the supply sector of manufacturing companies, the need to have information about the capabilities, capabilities and executive records of suppliers for companies is felt more than ever. In the meantime, having a procedure and instructions that can evaluate suppliers from several different criteria and angles and can select the best supplier is more important. Therefore, in this study, after initial screening of supplier review indicators, the most important indicators were evaluated and selected. Due to the quality of the evaluation indicators, at first, all suppliers were ranked and weighted based on each index using the AHP method. Then, according to the evaluation, all suppliers were evaluated using the Super Efficiency DEA method, all suppliers, based on which the suppliers were ranked among the efficient suppliers, and an accurate evaluation can be provided in this regard. The results of comparing the manufacturing company supplier chain rankings based on AHP, FAHP, and Super Efficiency DEA methods are as *Table 26*.

Table 26. *Comparison of supplier chain rankings.*

Row	Supplier	AHP	FAHP	Super Efficiency DEA
1	**A**	*4.75*	*0.124*	*0.992*
2	**B**	*6.59*	*0.11*	*0.969*
3	**C**	*5.2*	*0.114*	*1.693*
4	**D**	*6.83*	*0.1*	*0.998*
5	**E**	*5.78*	*0.093*	*1.187*
6	**F**	*5.55*	*0.097*	*1.122*
7	**G**	*6.13*	*0.093*	*1.992*
8	**H**	*3.96*	*0.098*	*1.273*
9	**I**	*4.39*	*0.087*	*0.972*
10	**j**	*3.66*	*0.085*	*2.066*

According to the points obtained, the ranking of suppliers with the methods introduced is as *Table 27*.

Table 27. *Scores of suppliers ranking.*

Row	Supplier	AHP	FAHP	Super Efficiency DEA
1	**A**	*7*	*1*	*8*
2	**B**	*2*	*3*	*10*
3	**C**	*6*	*2*	*3*
4	**D**	*1*	*4*	*7*
5	**E**	*4*	*7*	*5*
6	**F**	*5*	*6*	*6*
7	**G**	*3*	*8*	*2*
8	**H**	*9*	*5*	*4*
9	**I**	*8*	*9*	*9*
10	**j**	*10*	*10*	*1*

According to the study, AHP and FAHP methods in the ranking of suppliers had closer answers than data envelopment analysis. And according to the computational accuracy of data envelopment analysis methods, which is based on the input and output information of each supplier, so supplier number 10 is declared the best supplier. According to the assessments made in this study, first, key indicators regarding supply risks using the articles [23, 24, 27-29] using selection of experts from seven experts of the company, based on the risks of selecting suppliers, the most appropriate indicators have been identified using fuzzy, which in the meantime, article [23] was accepted with the highest selection of indicators and then we evaluated the suppliers. Due to the very high sensitivity in supply chain development, it is necessary for suppliers to be evaluated and selected based on all strategic indicators of the organization, so to develop this research, the following suggestions are provided:

- It is suggested that the production company form a working group consisting of executive units for accurate evaluation of suppliers and all evaluations be reviewed and selected in a multi-purpose working group.
- It is suggested that the executive instructions of the organization be updated and rewritten in accordance with the context of this research.
- It is recommended to conduct periodic evaluations of suppliers to maintain efficiency.

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Using Interval Arithmetic for Providing A MADM Approach

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ABSTRACT

The VIKOR method was developed for Multi-Criteria Decision Making (MCDM). It determines the compromise ranking list and the compromise solution obtained with the initial weights. This method focuses on ranking and selecting from a set of alternatives in the presence of conflicting criteria. It introduces the multi-criteria ranking index based on the particular measure of ‘‘closeness” to the “Ideal” solution. The aim of this paper is to extend the VIKOR method for decision making problems with interval number. The extended VIKOR method’s ranking is obtained through comparison of interval numbers and for doing the comparisons between intervals. In the end, a numerical example illustrates and clarifies the main results developed in this paper.

1. Introduction

The MCDM is the process of determining the best feasible solution according to the established criteria. Practical problems are often characterized by several non-commensurable and conflicting criteria and there may be no solution satisfying all criteria simultaneously. Thus, the solution is a set of non-inferior solutions, or a compromise solution according to the decision maker’s preferences. The compromise solution was established by Yu [1] and Zeleny [2] for a problem with conflicting criteria and it can be helping the decision makers to reach a final solution. The compromise solution is a feasible solution, which is the closest to the Ideal, and compromise means an agreement established by mutual concessions.

A MADM problem can be defined as:

	$\mathbf{C_1}$	$\mathbf{C_2}$	…	$\mathbf{C_n}$
$\mathbf{A_1}$	f_{11}	f_{12}	…	f_{1n}
$\mathbf{A_2}$	f_{21}	f_{22}	…	f_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{A_m}$	f_{m1}	f_{m2}	…	f_{mn}
$\boldsymbol{w}$	w_1	w_2	…	w_n

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where $A_1, A_2, \ldots, A_m$ are possible alternatives among which decision makers have to choose, $C_1, C_2, \ldots, C_n$ are criteria with which alternative performance is measured, fij is the rating of alternative A_i with respect to criterion C_j, w_j is the weight of criterion C_j [3–5].

In classical MCDM methods, the ratings and the weights of the criteria are known precisely, whereas in the real world, in an imprecise and uncertain environment, it is an unrealistic assumption that the knowledge and representation of a decision maker or expert are so precise. For example, human judgment including preferences is often vague and Decision Maker (DM) cannot estimate his preference with exact numerical values. In these situations, determining the exact value of the attributes is difficult or impossible. So, to describe and treat imprecise and uncertain elements present in a decision problem, fuzzy and stochastic approaches are frequently used. In the literature, in the works of fuzzy decision making [6–8], fuzzy parameters are assumed to be with known membership functions and in stochastic decision making [9–12] parameters are assumed to have known probability distributions. However, in reality to a DM it is not always easy to specify the membership function or probability distribution in an inexact environment. At least in some of the cases, the use of interval numbers may serve the purpose better. An interval number can be thought as an extension of the concept of a real number and also as a subset of the real line R [13]. However, in decision problems its use is not much attended as it merits. Recently, Jahanshahloo et al. [14] have extended TOPSIS method to solve decision making problems with interval data.

According to a comparative analysis of VIKOR and TOPSIS written by Opricovic and Tzeng [15], VIKOR method and TOPSIS method use different aggregation functions and different normalization methods. TOPSIS method is based on the principle that the optimal point should have the shortest distance from the Positive Ideal Solution (PIS) and the farthest from the Negative Ideal Solution (NIS). Therefore, this method is suitable for cautious (risk avoider) decision maker(s), because the decision maker(s) might like to have a decision which not only makes as much profit as possible, but also avoids as much risk as possible. Besides, computing the optimal point in the VIKOR is based on the particular measure of ‘‘closeness” to the PIS. Therefore, it is suitable for those situations in which the decision maker wants to have maximum profit and the risk of the decisions is less important for him. Therefore, we extend the concept of VIKOR method to develop a methodology for solving MADM problems with interval numbers. The VIKOR method is presented in the next section. In Section 3, extended VIKOR method is introduced and a new method is proposed for interval ranking on the basis of decision maker's optimistic level. In Section 4, an illustrative example is presented to show an application of extended VIKOR method. Finally, conclusion is presented.

2. VIKOR Method

The VIKOR method was introduced as one applicable technique to be implemented within MCDM problem and it was developed as a multi attribute decision making method to solve a discrete decision making problem with non-commensurable and conflicting criteria [15, 16]. This method focuses on ranking and selecting from a set of alternatives, and determines compromise solution for a problem with conflicting criteria, which can help the decision makers to reach a final solution. The multi-criteria measure for compromise ranking is developed from the LP-metric used as an aggregating function in a compromise programming method [1, 2].

Assuming that each alternative is evaluated according to each criterion function, the compromise ranking could be performed by comparing the measure of closeness to the Ideal alternative. The various m alternatives are denoted as $A_1, A_2, \ldots, A_m$. For alternative A_i, the rating of the jth aspect is denoted by

f_{ij}, i.e. f_{ij} is the value of jth criterion function for the alternative A_i; n is the number of criteria. Development of the VIKOR method is started with the following form of LP-metric:

$$L_{pi} = \left\{\sum_{j=1}^{n}\left(\frac{f_j^* - f_{ij}}{f_j^* - f_j^-}\right)^p\right\}^{\frac{1}{p}} \quad 1 \le p \le \infty; \ i = 1,2,\dots,m \tag{1}$$

In the VIKOR method $L_{1,i}$ (as S_i) and $L_{\infty,i}$ (as R_i) are used to formulate ranking measure. The solution obtained by $\min S_i$ is with a maximum group utility, and the solution obtained by min R_i is with a minimum individual regret of the ‘‘opponent”.The compromise ranking algorithm of the VIKOR method has the following steps:

<u>*Step 1.*</u> Determine the best f_j and the worst f_j values of all criterion functions $j = 1,2,\dots,n$. If the jth function represents a benefit then:

$$f_j^* = \max_i f_{ij}, f_j^- = \min_i f_{ij} \tag{2}$$

<u>*Step 2.*</u> Compute the values S_i and R_i; $i = 1,2,\dots,m$, by these relations:

$$S_i = \sum_{j=1}^{n} w_j\left(\frac{f_j^* - f_{ij}}{f_j^* - f_j^-}\right). \tag{3}$$

$$R_i = \max_i w_j\left(\frac{f_j^* - f_{ij}}{f_j^* - f_j^-}\right). \tag{4}$$

Where w_j are the weights of criteria, expressing their relative importance.

<u>*Step 3.*</u> Compute the values Q_i; $i = 1,2,\dots,m$, by the following relation:

$$Q_i = v\left(\frac{S_i - S^*}{S^- - S^*}\right) + (1 - v)\left(\frac{R_i - R^*}{R^- - R^*}\right). \tag{5}$$

Where

$$S^* = \mathrm{Min}_i S_i \, , \ S^- = \mathrm{Max}_i S_i. \tag{6}$$

$$R^* = \mathrm{Min}_i R_i \, , \ R^- = \mathrm{Max}_i R_i. \tag{7}$$

v is introduced as weight of the strategy of ‘‘the majority of criteria” (or ‘‘the maximum group utility”), here suppose that $v = 0.5$.

<u>*Step 4.*</u> Rank the alternatives, sorting by the values S, R and Q in decreasing order. The results are three ranking lists.

Step 5. Propose as a compromise solution the alternative$'$, which is ranked the best by the measure Q (Minimum Value) if the following two conditions are satisfied:

- Acceptable advantage, $Q(A") - Q(A') \geq DQ$. where $A"$ is the alternative with second position in the ranking list by Q; $DQ = \frac{1}{m-1}$; m is the number of alternatives.
- Acceptable stability in decision making. Alternative A' must also be the best ranked by S or/and R. This compromise solution is stable within a decision making process, which could be ''voting by majority rule" (when $v > 0.5$ is needed), or ''by consensus" $v = 0.5$, or ''with veto"($v < 0.5$). Here, v is the weight of the decision making strategy ''the majority of criteria" (or ''the maximum group utility").

If one of the conditions is not satisfied, then a set of compromise solutions is proposed, which consists of:

- Alternatives A' and $A"$ if only condition C_2 is not satisfied.
- AlternativesA', $A"$, ... ,$A^{(M)}$ if condition C_1 is not satisfied; $A^{(M)}$ is determined by the relation $Q(A^{(M)})\_Q(A') < DQ$ for maximum M (the positions of these alternatives are ''in closeness").

The best alternative, ranked by Q, is the one with the minimum value of Q. The main ranking result is the compromise ranking list of alternatives, and the compromise solution with the ''advantage rate". VIKOR is an effective tool in multi-criteria decision making, particularly in a situation where the decision maker is not able, or does not know to express his/her preference at the beginning of system design. The obtained compromise solution could be accepted by the decision makers because it provides a maximum ''group utility" (represented by min S) of the ''majority", and a minimum of the ''individual regret" (represented by min$R$) of the ''opponent". The compromise solutions could be the basis for negotiations, involving the decision maker's preference by criteria weights.

3. VIKOR Method with Interval Numbers

As it was said in the introduction, the interval numbers are more suitable to deal with the decision making problems in the imprecise and uncertain environment, because they are the simplest form of representing uncertainty in the decision matrix. The interval numbers require the minimum amount of information about the values of attributes. Specifying an interval for a parameter in decision matrix indicates that the parameter can take any value within the interval. Note that, the interval numbers does not indicate how probable it is to the value to be in the interval, nor does it indicate which of the many values in the interval is the most likely to occur [17]. In other way, an interval number can be thought as:

- An extension of the concept of a real number and also as a subset of the real line.
- A degenerate flat fuzzy number or fuzzy interval with zero left and right spreads.
- An α-cut of a fuzzy number [18].

So an interval number signifies the extent of tolerance or a region that the parameter can possibly take. An extensive research and wide coverage on interval arithmetic and its applications can be found in [13, 19, 20]. More information about the interval numbers and its differences with other methods of representing uncertainty such as probability and fuzzy theory can be found in [18, 21, 22].According to these facts, when determining the exact values of the attributes is difficult or impossible, it is more appropriate to consider them as interval numbers. Therefore, in the present paper, we extend the VIKOR method to solve MADM problem with interval numbers.

3.1. Interval Arithmetic

If two intervals $Ix = [x^L, x^U]$ and $Iy = [y^L, y^U]$ are given, the sum, difference, product, quotient, and additive inverse of the intervals are calculated based on the following equations [23]:

$$I = k * Ix = [kx^L, kx^U];\ k \in \mathbb{R}^+. \tag{8}$$

$$I = -Iy = [-y^U, -y^L]. \tag{9}$$

$$I = Ix + Iy = [x^L + y^L, x^U + y^U]. \tag{10}$$

$$I = Ix - Iy = [x^L - y^U, x^U - y^L]. \tag{11}$$

$$I = Ix * Iy = [\min\{x^L y^L, x^L y^U, x^U y^L, x^U y^U\}, \max\{x^L y^L, x^L y^U, x^U y^L, x^U y^U\}]. \tag{12}$$

$$I = \frac{Ix}{Iy} = \left[\min\{\frac{x^L}{y^L}, \frac{x^L}{y^U}, \frac{x^U}{y^L}, \frac{x^U}{y^U}\}, \max\{\frac{x^L}{y^L}, \frac{x^L}{y^U}, \frac{x^U}{y^L}, \frac{x^U}{y^U}\}\right]; 0 \notin Iy\,. \tag{13}$$

3.2. Interval Ranking

For ranking intervals, the mean value of each of the intervals is first calculated, and the rankings are then specified based on the obtained values. The mean value of $Ix = [x^L, x^U]$ is represented by $\mathrm{me}(Ix)$, which is obtained from the following equation [23]:

$$\mathrm{me}(Ix) = \frac{x^L + x^U}{2}. \tag{14}$$

3.3. Presentation of an Extended VIKOR Method

Suppose that a decision matrix with interval numbers has the following form:

	$\mathbf{C_1}$	$\mathbf{C_2}$	...	$\mathbf{C_n}$
$\mathbf{A_1}$	$[f_{11}^L, f_{11}^U]$	$[f_{12}^L, f_{12}^U]$	...	$[f_{1n}^L, f_{1n}^U]$
$\mathbf{A_2}$	$[f_{21}^L, f_{21}^U]$	$[f_{22}^L, f_{22}^U]$	...	$[f_{2n}^L, f_{2n}^U]$
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$
$\mathbf{A_m}$	$[f_{m1}^L, f_{m1}^U]$	$[f_{m2}^L, f_{m2}^U]$	...	$[f_{mn}^L, f_{mn}^U]$
Iw	$[w_1^L, w_1^U]$	$[w_2^L, w_2^U]$	...	$[w_n^L, w_n^U]$

Where $A_1, A_2, \ldots, A_m$ are possible alternatives among which decision makers have to choose, $C_1, C_2, \ldots, C_n$ are criteria with which alternative performance are measured, If_{ij} is the rating of alternative A_i with respect to criterion C_j and is not known exactly and only we know $If_{ij} \in [f_{ij}^L, f_{ij}^U]$.

and $Iw_j = [w_j^L, w_j^U]$ is the weight of criterion C_j. The Interval VIKOR method consists of the following steps:

Step 1. Determine the PIS and NIS.

$$A^* = \{f_1^*, \dots, f_n^*\} = \left\{\left(\max_i f_{ij}^U \,\middle|\, j \in I\right) \; or \; \left(\min_i f_{ij}^L \,\middle|\, j \in J\right)\right\} \;\; ; \; j = 1,2,\dots,n. \tag{15}$$

$$A^- = \{f_1^-, \dots, f_n^-\} = \left\{\left(\min_i f_{ij}^L \,\middle|\, j \in I\right) \; or \; \left(\max_i f_{ij}^U \,\middle|\, j \in J\right)\right\} \;\; ; \; j = 1,2,\dots,n.$$

Where I is associated with benefit criteria, and J is associated with cost criteria. A^* and A^- are PIS and NIS.

Step 2. In this step compute $IS_i = [S_i^L, S_i^U]$ and $IR_i = [R_i^L, R_i^U]$ intervals below:

$$S_i^L = \sum_{j\in I} w_j^L \left(\frac{f_j^* - f_{ij}^U}{f_j^* - f_j^-}\right) + \sum_{j\in J} w_j^L \left(\frac{f_{ij}^L - f_j^*}{f_j^- - f_j^*}\right) \; ; i = 1,2, \dots m. \tag{16}$$

$$S_i^U = \sum_{j\in I} w_j^U \left(\frac{f_j^* - f_{ij}^L}{f_j^* - f_j^-}\right) + \sum_{j\in J} w_j^U \left(\frac{f_{ij}^U - f_j^*}{f_j^- - f_j^*}\right) \; ; i = 1,2, \dots m.$$

$$R_i^L = \max\left\{ w_j^L \left(\frac{f_j^* - f_{ij}^U}{f_j^* - f_j^-}\right) \middle| j \in I, w_j^L \left(\frac{f_{ij}^L - f_j^*}{f_j^- - f_j^*}\right) \middle| j \in J \right\}; i = 1,2,\dots,m\,. \tag{17}$$

$$R_i^U = \max\left\{ w_j^U \left(\frac{f_j^* - f_{ij}^L}{f_j^* - f_j^-}\right) \middle| j \in I, w_j^U \left(\frac{f_{ij}^U - f_j^*}{f_j^- - f_j^*}\right) \middle| j \in J \right\}; i = 1,2,\dots,m\,.$$

Step 3. Compute the interval $IQ_i = \left[Q_i^L, Q_i^U\right]; i = 1,2,\dots,m$, by these relations:

$$Q_i^L = v\left(\frac{S_i^L - S^*}{S^- - S^*}\right) + (1 - v)\left(\frac{R_i^L - R^*}{R^- - R^*}\right). \tag{18}$$

$$Q_i^U = v\left(\frac{S_i^U - S^*}{S^- - S^*}\right) + (1 - v)\left(\frac{R_i^U - R^*}{R^- - R^*}\right).$$

Where

$$S^* = \mathrm{Min}_i S_i^L \,,\; S^- = \mathrm{Max}_i S_i^U. \tag{19}$$

$$R^* = \mathrm{Min}_i R_i^L \,,\; R^- = \mathrm{Max}_i R_i^U. \tag{20}$$

Step 4. Based on the VIKOR method, the alternative that has minimum Q_i is the best alternative and it is chosen as compromise solution.

4. Numerical Example

In this section, we present a numerical example to illustrate how the proposed method can be used. Suppose that, there are three alternatives (A_1, A_2, A_3) and two criteria (C_1, C_2). The decision maker wants to choose an alternative that has minimum C_1 and maximumC_2. The values of decision matrix are not precise and interval numbers are used to describe and treat the uncertainty of the decision problem. The interval decision matrix is shown in *Table 1*. In this example, both criteria have similar relative importance, $Iw_1 = [0.45, 0.50], Iw_2 = [0.50, 0.55], v = 0.5$.

To solve this example using the Interval VIKOR (IVIKOR) method we go through the following steps.

Step 1. The PIS and NIS are computed by (15a) and (15b) and shown in Table 2.

Step 2. In this step, we compute $IS_i = [S_i^L, S_i^U]$ and $IR_i = [R_i^L, R_i^U]$ using *Eqs. (16)-(17)*. The result is presented in *Table 3*.

Step 3. We compute the interval $IQ_i = [Q_i^L, Q_i^U]; i = 1,2, \dots, m$, by (18a), (18b), (19) and (20). The results are shown in *Table 4*.

$$S^* = 0.3224,\ S^- = 0.6030.$$

$$R^* = 0.2172, R^- = 0.5500.$$

Final ranking is obtained as follows:

$$Q_2 = me(IQ_2) < Q_3 = me(IQ_3) < Q_1 = me(IQ_1) \quad \Rightarrow \text{Final ranking is:} A_2 > A_3 > A_1.$$

Table 1. *Interval decision matrix.*

	C_1	C_2
A_1	[0.75,1.24]	[2784,3192]
A_2	[1.83,2,11]	[3671,3857]
A_3	[4.90,5.37]	[4409,4681]

Table 2. *PIS and NIS.*

	C_1	C_2
f_j^*	0.8	4681
f_j^-	5.4	2784

Table 3. *IS and IR.*

	$IS = [S^L, S^U]$	$IR = [R^L, R^U]$
A_1	[0.3925,0.6030]	[0.3925,0.5500]
A_2	[0.3224,0.4400]	[0.2172,0.2928]
A_3	[0.4042,0.5789]	[0.4042,0.5000]

Table 4. *IQ and Q.*

	$\mathbf{IQ} = [\mathbf{Q^L}, \mathbf{Q^U}]$	$\mathbf{Q} = \mathbf{me}(\mathbf{IQ}) = \dfrac{\mathbf{Q^L} + \mathbf{Q^U}}{\mathbf{2}}$	**Rank**
$\mathbf{A_1}$	[0.3882,1]	0.6941	3
$\mathbf{A_2}$	[0,0.3232]	0.1616	1
$\mathbf{A_3}$	[0.4268,0.8818]	0.6543	2

The compromise solution of extended VIKOR method is A_2.

As mentioned in the introduction, Jahanshahloo et al. have extended TOPSIS method to solve decision making problems with interval data. This method uses different aggregation functions and different normalization methods. Here to make a comparison between these two methods, we solve this example using the extended TOPSIS method. Doing the introduced steps in the extended TOPSIS method, compromise solution is obtained as follows:

The ranking of extended TOPSIS is: $A_2 > A_3 > A_1$.

The compromise solution obtained by extended TOPSIS is different with the compromise solution of extended VIKOR.

These different solutions derive from differences in aggregation functions and normalization methods. Moreover, in extended TOPSIS, the interval numbers are reduced to exact values. These reductions lead to miss some information. In the extended VIKOR method by keeping interval numbers, considering the decision maker's optimism level and using the comparison of interval numbers, the compromise solution is obtained.

5. Conclusion

Because of the fact that determining the exact values of the attributes is difficult or impossible, it is more appropriate to consider them as interval numbers. In this paper, we extended the VIKOR (IVIKOR) method to MADM problem with interval numbers. This method introduced the ranking index based on particular measure of closeness to PIS. In the extended VIKOR method, we compute S, R and Q as interval numbers and to obtain the compromise solution, we need to compare interval numbers with each other. For that purpose, we utilized the interval means method.

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Fuzzy Logic in Accounting and Auditing

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PAPER INFO	ABSTRACT
Chronicle: *Received: 04 November 2019* *Revised: 01 February 2020* *Accepted: 31 February 2020* **Keywords:** *Fuzzy Logic.* *Accounting.* *Auditing.* *Ambiguity.*	Many areas of accounting have highly ambiguous due to undefined and inaccurate terms. Many ambiguities are generated by the human mind. In the field of accounting, these ambiguities lead to the creation of uncertain information. Many of the targets and concepts of accounting with binary classification are not consistent. Similarly, the discussion of the materiality or reliability of accounting is not a two-part concept. Because there are degrees of materiality or reliability. Therefore, these ambiguities lead to the presentation information that is not suitable for decision making. Lack of attention to the issue of ambiguity in management accounting techniques, auditing procedures, and financial reporting may lead to a reduced role of accounting information in decision-making processes. Because information plays an important role in economic decision-making, and no doubt, the quality of their, including accuracy in providing it to a wide range of users, can be useful for decision-making. One of the features of the fuzzy set is that it reduces the need for accurate data in decision making. Hence this information can be useful for users.

1. Introduction

Ambiguity and imprecision in human judgments are in many scientific disciplines. Accountants in dealing with the issue of ambiguity and imprecision, they behave there is no ambiguity or it is a random [1]. In recent years, fuzzy logic has gained wide acceptance in the field of accounting and business. This acceptance is due to the ability to management in situations of ambiguity and lack of consistency that does not exist within other approaches to dual value logic. In dual value logic, the proposition is true or false. Also, accounting has ambiguous in many important respects [2]. The problem of ambiguity and imprecision in accounting and auditing is related to the rules and accounting system [3]. Ro [4] argues that the concept of materiality is not essentially two-dimensional, such as black or white and good or bad, but that there are degrees of materiality that are overlooked in accounting. However, ambiguity and imprecision is different from being random. Randomness refers to the uncertainty about the occurrence or non-occurrence of an event and is expressed in the form of probabilities. While, ambiguity and imprecision are related to the inaccuracy and lack of clarity in the definition of words, the occurrence of events and judgments [2]. Zariffard [1] argues that neglect of ambiguity and imprecision in decision models can limit the usability of accounting models due to reduced usefulness and predictive power. Therefore, it is important attention to ambiguity. The purpose of this study is to

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introduce fuzzy logic and also is examine its major applications in accounting and auditing. The importance of this research is that considering that information is the main component of any decision making today, it has economic value.

2. Literature Review

2.1. Fuzzy Set Theory

In 1965, Zadeh discussed the existence of ambiguity and fuzziness in many human systems. According to Zadeh [5], the need to be very careful in decision analysis causes the analyst to ignore some related issues and consider only a part of this relationship with the real world. Fuzzy thinking followed the objection to Aristotelian logic about the distance between logic and reality. Aristotelian logic forms the basis of classical mathematics. This logic assumes that the world is black and white or two values one or 0. Zadeh [6] proposed the theory of fuzzy sets as a method for modeling in ambiguity and uncertainty. Sets can be divided into finite sets and fuzzy sets. In finite set, is there a member in a set or not? That is, it has no more than two values, one or 0. But not in the fuzzy set.

In fact, Aristotelian logic sacrifices accuracy for ease. But the real phenomena are not just black or white, they are somewhat gray. In other words, real phenomena are always fuzzy, that is, ambiguity and imprecision [7]. Fuzzy set theory reduces the possibility of making personal judgments by expressing qualitative and subjective information, and leads to more rational decisions [8].

2.2. Definition of Fuzzy Set

Let U be a classical (or ordinary) set of objects, called the universe, whose generic elements are denoted by x. That is, U={x}. A fuzzy set A in U is characterized by a membership function $\mu_A(X)$ which associates with each element in U a real number in the interval (0–1) [9]. The fuzzy set, A, is usually denoted by the set of pairs [10].

$$A=\{(x,\mu_A(X)),x\in U\}. \quad (1)$$

For an ordinary set, A

$$\mu_A(X)=\begin{cases}1, & x\in A\\ 0, & x\notin A\end{cases}. \quad (2)$$

When U is a finite set $\{x_1,\dots, x_n\}$, the fuzzy set on U may also be represented as

$$A=\sum_{i=1}^{n} x_i/\mu_A(x_i). \quad (3)$$

When U is an infinite set, the fuzzy set maybe represented as

$$A=\int (x/\mu_A(x))dx. \quad (4)$$

2.3. Basic Concepts of Fuzzy Set

The complement, support, a-cut, convexity, normality and cardinality of a fuzzy set are presented in the following sections [9].

Complement of a fuzzy set. The definition of the complement of fuzzy set A is defined as

$$\mu_A(X) = 1 - \mu_A(X) \quad x \in U. \tag{5}$$

Support of a fuzzy set. Those elements which have nonzero membership grades are considered as support of that fuzzy set

$$S(A) = \{x \in U | \mu_A(X) \geq 0\}. \tag{6}$$

a-Cut of a fuzzy set. a-Cut of a fuzzy set is an ordinary set whose elements belong to fuzzy set A, at least to the degree of a

$$A\alpha = \{x \in U | \mu_A(X) \geq \alpha\}. \tag{7}$$

It is a more general case of the support of a fuzzy set. If α=0 then Aα=S(A).

Convexity of a fuzzy set. A fuzzy set is convex if

$$\mu_A(\lambda X_1 + (1-\lambda)X_2) \geq \min(\mu_A(X_1), \quad \mu_A(X_2)). \tag{8}$$

X_1 and $X_2 \epsilon U$ also $\lambda \epsilon$ (0–1).

Normality of a fuzzy set. A fuzzy set A is normal only if there are one or more x′ values such that $\mu_A(X') = 1$.

Cardinality of a fuzzy set. The cardinality of fuzzy set A evaluates the proportion of elements of U having the property A. When U is finite, it is defined as

$$|A| = \sum \mu_A(x), \quad x \in U. \tag{9}$$

For infinite U, the cardinality is defined as

$$|A| = \int_x \mu_A(x)dx. \tag{10}$$

For more details, enormous materials can be found in the literature about fuzzy set theory.

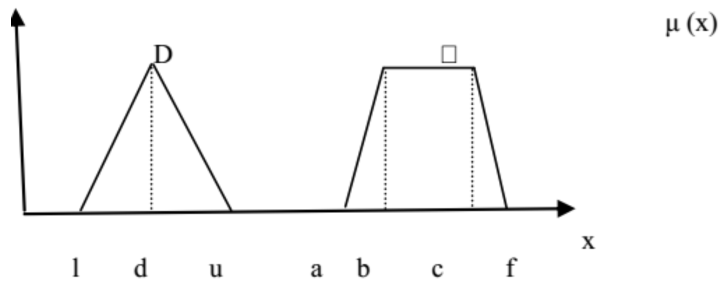

Fig. 1. *Triangular and trapezoidal fuzzy numbers.*

Fig. 1 defines a fuzzy and triangular number D as follows [8]:

D= {1, d, u} where 1, d, and u as the lower spread, the middle spread, and the upper spread. In this case the membership function $\mu_{(D)}$ X it is defined as follows:

$$\mu_D(x) = \begin{cases} 0; \quad x \le 1 \\ (x-1)/(d-1); \quad 1 < x \le d \\ (u-x)/(u-d); \quad d < x \le u \\ 0; \quad x > u \end{cases}. \tag{11}$$

If the climax of the triangular number D is not unique; the fuzzy number is known as a trapezoid.

3. Fuzzy Set Theory and Accounting and Auditing

For a variety of reasons, fuzzy set theory can be of great value to accountants in practice. First, fuzzy set theory provides a mathematical framework which fuzzy concepts of accounting can be examined on a regular basis, for instance, materiality errors. Therefore, using fuzzy set theory, accountants will be able to apply fuzzy set theory in accounting. As a result, they no longer have to ignore ambiguities in accounting matters. Also, they will be able to deal with it like random events using probability theory. Accountants with ignoring the ambiguity cause inaccuracies in accounting matters [1].

In addition, unlike ordinary set theory, fuzzy set theory abandons the rule of excluding the mean and logic of two values. As a result, there will be no need for a binary classification of accounting objectives that are generally unrealistic and artificial. Many of the targets and concepts of accounting with binary classification are not consistent. For example, neutrality is not a debate of being black and white. There are different degrees of neutrality, or in the discussion of deviation analysis, controllable deviations or uncontrollable deviations are kinds of unrealistic integration. Similarly, the discussion of the materiality or reliability of accounting is not a two-part concept. Because there are degrees of materiality or reliability. One of the features of the fuzzy set is that it reduces the need for accurate data in decision making.

Recently, this theory has been used to solve accounting problems. These studies can be divided into two groups. The first group deals with audit problems such as internal control, audit sampling, and judgment of materiality. The second group deals with management accounting issues and problems such as capital budgeting, cost deviations, and strategic planning. Some applications of fuzzy sets in the audit are summarized as:

Friedlob and Schleifer [11] argue that auditors usually express risk in the form of probabilities, examining different types of audit uncertainty. Finally, they introduced the fuzzy logic-based method as a new method of examining audit uncertainty.

Pathak et al. [12] indicates that in order to reduce the costs of detecting fraud in the claims made in their insurance companies, they designed fuzzy expert systems to evaluate and express the elements related to fraud in resolving insurance claims. This system is useful for deciding whether settled insurance claims are actual or whether there is evidence of fraud.

Comunale and Sexton [13] introduced a fuzzy logic approach to assess the importance of presenting financial statements. A fuzzy logic-based approach to significance assessment can provide an expert system for significance assessment compared to traditional approaches, which are based on binary valuation; In such a way that the importance of presenting financial statements correctly can be shown between zero and one, and on the other hand, quality criteria can be considered in evaluating the importance.

Dereli et al. [14] using a fuzzy mathematical programming model, they proposed a strategic algorithm to shape the quality audit team. In this study, the fuzzy ranking method has been used to determine the adequacy of the skills and expertise of each auditor in team auditing.

De Korvin et al. [15] examined the risk of internal controls in computer accounting information systems through a fuzzy set approach. The model presented in this research is used through a risk analysis matrix in a company active in the chemical industry. This model is useful in evaluating and applying new control procedures to increase the security of the company's information systems.

Also, some applications of fuzzy sets in the management accounting [16, 17] are summarized as:

Oderanti and De Wilde [18] used the concepts of fuzzy logic and game theory to model the strategic decision-making process by business organizations based on uncertain information. In this study, competition between business organizations is considered as a game and organizations are its actors. They model their decisions through strategic actions based on uncertain information.

Cassia et al. [19] examined the development of corporate management accounting systems in providing information to facilitate the strategic decision-making process and its relationship to the shape, development and size of companies through the general mode of fuzzy logic. The results of the study indicate that 511 Italian companies are always advances in the evolution of corporate management accounting system do not meet. In other words, you can find a large number of companies with a simple organizational structure but with an advanced management accounting system.

Rangone [20] according to strategic management accounting, strategic cost management and non-financial performance metrics are introduced as strategies to overcome the limitations of traditional management accounting systems. He provided an analytical framework using fuzzy logic to establish a relationship between the effectiveness of the organization, key indicators of success and performance measurement.

Nagasawa [21] using fuzzy set theory, a model for value engineering and cost management was designed. The existence of different tools and solutions for value engineering, their prioritization as well as the related ambiguities related to them, have been expressed as reasons for the need to address fuzzy set theory in value engineering.

Nachtmann and Needy [22] through the application of fuzzy logic concepts in costing, they developed an activity-based costing system. This study demonstrates the benefits of a fuzzy activity based costing system and the stages of development and implementation in a pharmaceutical company.

Nachtmann and Needy [23] have introduced and compared methods of overcoming ambiguity and uncertainty over the input data of the activity-based costing system from the perspective of cost-benefit analysis. According to the comparison, the use of fuzzy method in activity-based costing to consider the conditions of ambiguity and uncertainty is more appropriate than the methods based on each of the standard models, distance and Monte Carlo with normal input variables.

Yuan [24] Using a fuzzy expert system, designed a model to analyze costs, activity volume and profit in ambiguous conditions by management. In this new system, unlike the traditional mode, which uses the break-even point and assumes a state of confidence, the information of experts and the concepts of fuzzy sets are used to overcome inaccuracies and ambiguities.

4. Conclusion

The purpose of this paper is to introduce fuzzy set theory in accounting and examine its relationship as a way to solve accounting problems in conditions of ambiguity. Fuzzy theory, unlike traditional quantitative methods, provides a mathematical framework for inaccurate phenomena in human systems and decision making that can be applied on a regular basis. This theory does not require accurate measurements. As a result, fuzzy theory can be invaluable to accountants, especially in times of ambiguity and when care cannot be taken. Therefore, due to the ambiguities that exist in accounting and auditing issues; accountants and auditors should not hesitate to use fuzzy set theory. One of the features of the fuzzy set is that it reduces the need for accurate data in decision making. Because today, information plays an important role in economic decision-making, and no doubt [25], the quality of their, including accuracy in providing it to a wide range of users, can be useful for decision-making [26].

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Efficiency Study with Undesirable Inputs and Outputs in DEA

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Data Envelopment Analysis (DEA) is one of the well-known methods for calculating efficiency, determining efficient boundaries and evaluating efficiency that is used in specific input and output conditions. Traditional models of DEA do not try to reduce undesirable outputs and increase undesirable inputs. Therefore, in this study, in addition to determining the efficiency of Decision-Making Units (DMU) with the presence of some undesirable input and output components, its effect has also been investigated on the efficiency limit. To do this, we first defined the appropriate production possibility set according to the problem assumptions, and then we presented a new method to determine the unfavorable performance of some input and output components in decision-making units. And we determined the impact of unfavorable inputs and outputs on the efficient boundary. We also showed the model result by providing examples for both unfavorable input and output states and solving them and determining the efficiency score and driving them to the efficient boundary by plotting those boundaries.

1. Introduction

Data envelopment analysis is one of the most popular methods for determining efficiency, and the boundary of efficiency based on the concept of condition of defective units. In 1987, Charles et al. [2] identified efficient boundaries using linear programming and used them to determine productivity. In this way, they used both output-axis and input-axis models. Although these two models are not the only ones used, they are still the most popular DEA model. Many researchers use the DEA method to determine the boundary of performance and evaluate performance [4].

Over the past two decades, DEA has established itself as the strongest and most valuable methodology [3]. In many practical issues, some inputs of decision-making units may be such that increasing these inputs increases efficiency and decreasing it reduces efficiency, such as waste recycling operations, scrap metal and glass, etc., where it is necessary to increase the undesirable inputs to improve the level of efficiency, or some of the outputs of decision-making units may be such that increasing these outputs reduces efficiency and decreases it increases efficiency. Consider the waste of a factory or the deaths of patients in hospitals and the dismissals of doctors and nurses in

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training centers, which should be reduced as an undesirable output to increase efficiency. Undesirable outputs are generally desirable products and therefore can only be reduced by reducing them. There are methods for importing undesirable outputs into the DEA that can be divided into two categories:

- Direct methods.
- Indirect methods.

In indirect methods, undesirable inputs and outputs in each unit are converted into desired inputs and outputs by a uniform descending function and then the performance of the units is evaluated using DEA standard models. Direct ones are methods that use assumptions in production possibility set so that they are used in evaluating desirable input and output.

Conventional data envelopment analysis models in most studies treated the units under evaluation as a black box, producing only a series of primary inputs and using them to produce a series of final outputs; However, with the research of the last two decades, they came to the conclusion that the obtained efficiency is not accurate without considering their internal structure, and there are ambiguities in the analysis of its efficiency, and the role of undesirable factors should also be examined.

In the real world, it is not possible to match all inputs and outputs of inefficient units based on DEA results, which Chiang Kao showed with a new model design, which is possible. Unfavorable outputs are generally desirable products and can therefore be reduced only by a concomitant reduction in the second product. To understand this concept, the price of an undesirable output shadow must be negative and the opposite for a positive output. Based on these conditions, Kao et al. [6] in a paper presented a data envelopment analysis model that allows the production units under evaluation to determine the shadow price for both favorable and unfavorable outputs to maximize the measured performance score. The proposed model satisfies the assumption of poor usability of outputs. It is also shown that there is a directional function model in a group that has been widely used in modeling adverse outputs. However, unlike conventional directional distance measures, the proposed model is able to provide performance in the range of zero and one for easy comparison between inefficiently produced units.

Cross-productivity evaluation methods have long been proposed as an option for ranking decision-making units in data envelopment analysis. Neutral reciprocity performance evaluation methods are developed in a way that is only self-interested and indifferent to other DMUs. Accordingly, in 2019, Shi et al. [19] introduced a new cross-performance evaluation method in which each DMU has a neutral attitude towards its other peer units. This is done by introducing an Ideal Virtual Border (IVF) and a Non-Ideal Virtual Border (AVF). Unlike cross-performance evaluation methods, this cross-performance evaluation method determines the set of input and output weights for each DMU. The most important operation in this study is to introduce an ideal virtual boundary and a non-ideal virtual boundary improving DMU performance by considering IVF and AVF as evaluation criteria, minimizing deviation from IVF and maximizing deviation from AVF. In 2019, Wu et al. [5] studied the environmental efficiency measurement of thermoelectric power plants using an efficient frontier DEA approach with fixed-sum undesirable output.

In 2020, Song et al. [9] Studied accident deaths as undesirable output in the production and safety evaluation in Chinese coal mines. In 2020, Walheer [1] studied the output, input, and undesirable output interconnections in data envelopment analysis: Convexity and returns-to-scale. In 2020, Yu et al. [8] assessed environmental provincial eco-efficiency in China an improved network data envelopment analysis model with undesirable output. In 2020, Gómez-Calvet et al. [7] evaluated European energy efficiency evaluation based on the use of super-efficiency under undesirable outputs in SBM models. In this research, we have presented the possibility of production in accordance with the concept of undesirable inputs and outputs. Then, in the concept of inputs and outputs, we have examined the efficiency of decision-making units and the efficiency boundary diagram with the presence of undesirable inputs and outputs by providing an example.

2. Production Possibility Set

Suppose we have n observations on n DMUs with input and output vectors (x_j, y_j) for j = 1, 2,..., n. Let $x_j = (x_1,..., x_{mj})^T$ and $y_j = (y_{1j},..., y_{sj})$. All $x_j \in R^m$ and $y_j \in R^s$ and $x_j > 0$, $y_j > 0$ *for j = 1, 2,... n.* The input matrix X and output matrix Y can be represented as $X = [x_1,..., x_j,..., x_n]$, $Y = [y_1,..., y_j,..., y_n]$.

Where X is an $(m \times n)$ matrix and Y an $(s \times n)$ matrix.

The production possibility set T is generally defined as

$$T = \{(x,y)1 \text{ x can produce y}\}. \tag{1}$$

In DEA, the production possibility set under a Variable Return to Scale (VRS) technology is constructed form the observed data (x_j, y_j) *for j = 1, 2, ..., n* as follows:

$$T = \left\{ (x, y) \middle| x \geq \sum_{j=1}^{n} \lambda_j x_j \ , \ y \leq \sum_{j=1}^{n} \lambda_j y_j \ , \lambda_j \geq 0 \ , \ \sum_{j=1}^{n} \lambda_j = 1 \, , \, j = 1,..., n \right\}. \tag{2}$$

In the absence of undesirable factors when a DMU_O , $o \in \{1,2,..., n\}$, is under evaluation, we can use the following BCC model:

$$\begin{aligned} & \min \ \theta \\ & s.t \quad \theta x_o - X\lambda \geq 0 \\ & \qquad Y\lambda \geq y_o \ , \\ & \qquad 1^T \lambda = 1, \\ & \qquad \lambda \geq 0. \end{aligned} \tag{3}$$

Corresponding to each output *y, L(y)* is defined as the following:

$$L(y_j) = \{x | (x, y_j) \in T\}. \tag{4}$$

In fact, $L(y_j)$ is a function that y_j portrays to a subset of inputs so that inputs can produce y_j.

Now suppose that some inputs are undesirable so input matrix X can be represented

as $X=(X^g,X^b)^T$, where $X^g=(x^g_{1j},...,x^g_{m_1j}), j=1,...,n$ and $X^b=(x^b_{1j},...,x^b_{m_1j})$ $j=1,...,n$ *are* $(m_1\times n)$ *and* $(m_2\times n)$ matrixes that represent desirable (good) and undesirable (bad) inputs, respectively. And similarly, suppose that some outputs are undesirable so outputs. Matrix Y can be represented as $Y=(Y^g,Y^b)^T$, where $Y^g=(y^g_{1j},...,y^g_{s_1j}), j=1,...,n$ and $Y^b=(y^b_{1j},...,y^b_{s_2j})$, $j=1,...,n$ are $(s_1\times n)$ and $(s_2\times n)$ matrixes that represent. Desirable (good) and undesirable (bad) inputs, respectively.

Definition 1. Let DMU of $(x^g_1,x^b_1,y^g_1,y^b_1)$ is dominant to DMU of $(x^g_2,x^b_2,y^g_2,y^b_2)$ if $(x^g_1\le x^g_2,x^b_1\ge x^b_2,y^g_1\ge y^g_2)$ and $y^b_1\le y^b_2$ the unequal be strict at least in a component. So that,

$$\begin{pmatrix}-x^g_1\\ x^b_1\\ y^g_1\\ -y^b_1\end{pmatrix}\ge\begin{pmatrix}-x^g_2\\ x^b_2\\ y^g_2\\ -y^b_2\end{pmatrix}.$$

Definition 2. DMU_O is efficient if in T there is no DMU to be dominant over it.

We consider the properties of the Production Possibility Set as the following:

- T is convex.
- T is closed.
- The monotony property of desirable inputs and outputs. So that, $\forall u\in R^{m_1}_+, v\in R^{s_1}_+, (x^g,x^b,y^g,y^b)\in T\Rightarrow(x^g+u,x^b,y^g-v,y^b)\in T$

This is not necessarily established for undesirable factors, because in this case, T has no efficient DMU.

We can define the production possibility set T satisfying *Eq. (1)* through *Eq. (3)* by

$$T=\left\{(x^g,x^b,y^b,y^g)\left|\begin{array}{c}x^g\ge\sum_{j=1}^n\lambda_jx^g_j, x^b=\sum_{j=1}^n\lambda_jx^b_j, y^b=\sum_{j=}^n\lambda_jy^b_j, y^g\le\sum_{j=1}^n\lambda_jy^g_j\\ \sum_{j=1}^n\lambda_j=1, \lambda_j\ge0, j=1,...,n\end{array}\right.\right\} \quad (5)$$

3. Measures of Efficiency Using Undesirable Factors

In input oriented data, the efficiency of the DMU under evaluation is obtained by decreasing and increasing the desirable and undesirable input, respectively. And similarly, in output oriented data, we increase desirable output and decrease the undesirable output.

3.1. Nature of the Input

Suppose $DMU_o = (x_o^g, x_o^b, y_o^g, y_o^b)$ be unit under evaluation, corresponding to the output $y_o = (y_o^g, y_o^b)$ and using *Eq. (2)* $L(y_o^g, y_o^b)$ in defined as follows:

$$\mathrm{L}(\mathrm{y_o^g}, \mathrm{y_o^b}) = \left\{(\mathrm{x^g}, \mathrm{x^b}) \middle| (\mathrm{x^g}, \mathrm{x^b}, \mathrm{y_o^g}, \mathrm{y_o^b}) \in \mathrm{T}\right\}. \tag{6}$$

And we consider the subset of $L(y_o^g, y_o^b)$ as:

$$\partial^p \mathrm{L}(\mathrm{y_o^g}, \mathrm{y_o^b}) = \left\{(\mathrm{x^g}, \mathrm{x^b}) \middle| \forall(\mathrm{u}, \mathrm{v}) \geq 0, (\mathrm{u}, \mathrm{v}) \neq 0 \quad \Rightarrow \quad (\mathrm{x^g} - \mathrm{u}, \mathrm{x^b} + \mathrm{v}) \notin \mathrm{L}(\mathrm{y_o^g}, \mathrm{y_o^b})\right\}. \tag{7}$$

That $\partial^s L(y_o^g, y_o^b)$ includes all inputs of the efficient DMUs which can produce (y_o^g, y_o^b).

The model to evaluate the efficiency of DMUo with the most decrease of x_o^g and the most increase of x_o^b is as follows:

$$\mathrm{d_o}^{\mathrm{g}} = \mathrm{x_o}^{\mathrm{g}},$$

$$\mathrm{d_o}^{\mathrm{b}} = \mathrm{x_o}^{\mathrm{b}} - \mathrm{x_{max}^b}.$$

So that

$$(\mathrm{x_{max}^b})_i = \mathrm{Max}_j \left\{\mathrm{x_{ij}^b}\right\}.$$

Therefore, according to the definition of inefficiency we have:

$$\begin{aligned}
&\theta_o^* = \mathrm{Max} \quad \theta_o, \\
&\text{st.} \\
&\quad \sum_{j=1}^{n} \lambda_j x_j^g + s^- = x_o^g - \theta_o d_o^{\,g}. \\
&\quad \sum_{j=1}^{n} \lambda_J x_j^b = x_o^b - \theta d_o^{\,b}. \\
&\quad \sum_{j=1}^{n} \lambda_j y_j^g - s^+ = y_o^g. \\
&\quad \sum_{j=1}^{n} \lambda_j y_j^b = y_o^b. \\
&\quad \sum_{j=1}^{n} \lambda_j = 1. \\
&\quad \lambda_j \geq 0 \qquad \text{for} \quad \text{all} \quad j = 1, \ldots, n.
\end{aligned} \tag{8}$$

According to the definition of production possibility set, *model (1)* is possible in this set.

Theorem 1. The DMUo in *model (8)* is efficient if and only if

- $\theta_o^* = 1$.
- All slacks are zero for all optimal solutions.

Theorem 2. If all optimal solution of *model (8)* be (θ^*, s^{-*}), then

$$(x^g - \theta^* d^g - s^{-*}, x^b - \theta^* d^b) \in \partial^p L(y_o^b, y_o^g).$$

s^- is one of optimal answers.

4. Numerical Example 1

As an example, consider seven DMUs with one desirable input, one undesirable input to produce a desirable output normalized at level 1. These DMUs were explained in *Table 1*.

Regarding *Table 1* and *Fig. 1*, it can be seen that DMUs D, E, and F are efficient and they are on the $\partial^s L(y_G^g)$. On the other hand, efficiency of other DMUs have been examined through their image on $\partial^s L(y_G^g)$. (Efficient Frontiers).

Table 1. *The inputs and outputs data for 7 DMUs.*

DMU's	x^g	x^b	y^g	$1-\theta^*$
A	3	1	1	0.33
B	2	2	1	0.5
C	1	3	1	1
D	1	5	1	1
E	2	6	1	1
F	3	7	1	1
G	4	4	1	0.43

Similar discussion can be presented for the output oriented.

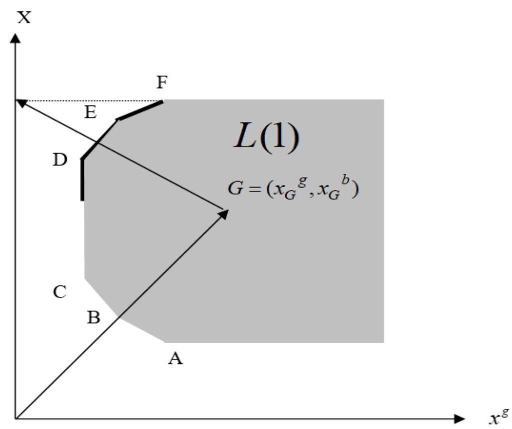

Fig. 1. *The graph of the* $L(y_G)$.

3.1. Nature of the Output

Suppose $DMU_o = (x_o^g, x_o^b, y_o^g, y_o^b)$ be unit under evaluation, corresponding to the output $x_o = (x_o^g, x_o^b)$ and using *Eq. (2)* $p(x_o^g, x_o^b)$ is defined as follows:

$$p(x_o^g, x_o^b) = \left\{(y^g, y^b) \middle| (x_o^g, x_o^b, y^g, y^b) \in T\right\}.$$

And we consider the subset of $p(x_o^g, x_o^b)$ as:

$$\partial^p p(x_o^g, x_o^b) = \left\{(y^g, y^b) \middle| \forall (u,v) \geq 0, (u,v) \neq 0 \quad \Rightarrow \quad (y^g + u, y^b - v) \notin p(x_o^g, x_o^b)\right\} \tag{9}$$

That $\partial^s L(y_o^g, y_o^b)$ includes all inputs of the efficient DMUs which can produce (y_o^g, y_o^b).

The model to evaluate the efficiency of DMUo with the most decrease of y_o^g and the most increase of y_o^b is as follows:

$$NE^d(x_o, y_o) = \sup\left\{\beta \middle| y_o + \beta d \in p(x_o)\right\}.$$

where $d = (d^g, d^b)$ indicate the direction of unit under evaluation such that $d^g \in R_+^{s_1}$ and $d \in R_-^{m_2}$ leads to increase the corresponding outputs and decreasing the unconfirmed outputs.

In this research, we direct the desired outputs to the efficient boundary in a radial direction. Thus: $d^g = y_o^g$.

We also reduce the undesirable outputs in the radial direction, i.e.

$$d^I = -y_o^b.$$

Therefore, according to the definition we have:

$$\begin{aligned}
&\beta_o^* = \text{Max} \quad \beta_o, \\
&\text{st.} \\
&\quad \sum_{j=1}^{n} \lambda_j x_j^g + s^- = x_o^g. \\
&\quad \sum_{j=1}^{n} \lambda_J x_j^b = x_o^b. \\
&\quad \sum_{j=1}^{n} \lambda_j y_j^g - s^+ = y_o^g + \beta_o d_o^g. \\
&\quad \sum_{j=1}^{n} \lambda_j y_j^b = y_o^b + \beta_o d_o^b. \\
&\quad \sum_{j=1}^{n} \lambda_j = 1. \\
&\quad \lambda_j \geq 0 \qquad \text{for} \quad \text{all} \quad j = 1, \ldots, n.
\end{aligned}$$

Theorem 3. The DMUo in *model (10)* is efficient if and only if

– $\beta_o^* = 1$.
– All slacks are zero for all optimal solutions.

Theorem 3. If be optimal solution of *model (10)* in, then

$$(y_o^* + \beta_o^* d_o^g + s^{+^*}, y_o^b + \beta_o^* d_o^b) \in \partial^p p(x_o^g, x_o^b).$$

5. Numerical Example 2

We consider five decision-making units with an optimal input to produce an undesirable output and a desirable output. These decision-making units are described in *Table 2*. *Fig. 2* shows that the decision-making units D, E and F are efficient. On the other hand, other decision-making units have been examined through their image on the (efficient border) of their efficiency.

Table 2. *The inputs and outputs data for 5 DMUs by model 10.*

$DMU's$	x^g	y^g	y^b	$1-\beta^*$
A	*1*	*4*	*1*	*1*
B	*1*	*5*	*2*	*1*
C	*1*	*5*	*4*	*1*
D	*1*	*4*	*5*	*0.25*
E	*1*	*3*	*3*	*0.5*

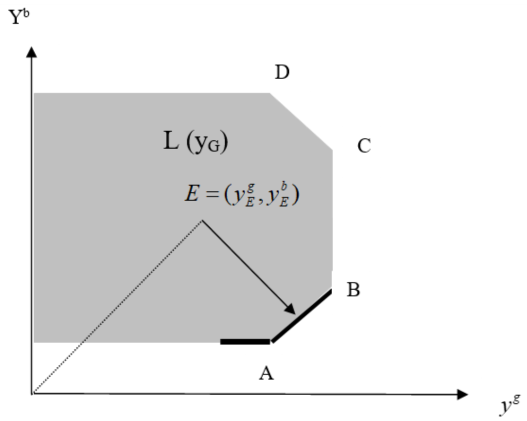

Fig. 2. *The graph of the L.*

6. Conclusion

Our proposed models in this study determine the efficiency of decision-making units, assuming that some of their input and output components may be undesirable. Numerical examples and model diagrams show that these models ensure that the presence of undesirable input and output factors is effective in determining the efficiency boundary of the decision-making units under evaluation and are compared with a unit corresponding to the efficient boundary set. By decreasing undesirable output and increasing undesirable input, the efficiency of decision-making units can improve and push them towards the efficient frontier.

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Generalizations and Alternatives of Classical Algebraic Structures to NeutroAlgebraic Structures and AntiAlgebraic Structures

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P A P E R I N F O	A B S T R A C T
Chronicle: *Received: 17 December 2019* *Revised: 20 February 2020* *Accepted: 20 April 2020* **Keywords:** *NeutroAlgebra, NeutroAxiom, NeutroOperation, NeutroTheorem. AntiAlgebra, AntiAxiom, AntiOperation.*	In this paper we present the development from paradoxism to neutrosophy, which gave birth to neutrosophic set and logic and especially to NeutroAlgebraic Structures (or NeutroAlgebras) and AntiAlgebraic Structures (or AntiAlgebras) that are generalizations and alternatives of the classical algebraic structures.

1. From Paradoxism to Neutrosophy

1.1. Paradoxism

Paradoxism is an international movement in science and culture, founded by Smarandache in 1980s, based on excessive use of antitheses, oxymoron, contradictions, and paradoxes in science, literature, and arts. During three decades (1980-2020) hundreds of authors from tens of countries around the globe contributed papers in various languages to 15 international paradoxist anthologies.

1.2. Neutrosophy

In 1995, the author extended the paradoxism (based on opposites) to a new branch of philosophy called neutrosophy (based on opposites and their neutral) that gave birth to many scientific branches, such as neutrosophic logic, neutrosophic set, neutrosophic probability and statistics, neutrosophic algebraic

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structures, and so on with multiple applications in engineering, computer science, administrative work, medical research, biology, psychology, social sciences etc.

1.3. Extensions

Neutrosophy is also an extension of Dialectics (characterized by the dynamics of opposites in philosophy), and of Yin-Yang Ancient Chinese philosophy (based also on opposites: male/female, good/bad, sky/earth, etc.) that was founded and studied two and half millennia ahead of Hegel's and Marx's Dialectics.

2. From Classical Algebras to NeutroAlgebras and AntiAlgebras

2.1. Operation, NeutroOperation, and AntiOperation

When we define an operation on a given set, it does not automatically mean that the operation is well-defined. There are three possibilities:

- The operation is well-defined (or inner-defined) for all set's elements (as in classical algebraic structures; this is classical *Operation*).
- The operation if well-defined for some elements, indeterminate for other elements, and outer-defined for others elements (this is *NeutroOperation*).
- The operation is outer-defined for all set's elements (this is *AntiOperation*).

2.2. Axiom, NeutroAxiom, and AntiAxiom

Similarly for an axiom defined on a given set endowed with some operation(s). When we define an axiom on a given set, it does not automatically mean that the axiom is true for all set's elements. We have three possibilities:

- The axiom is true for all set's elements [totally true] (as in classical algebraic structures; this is classical *Axiom*).
- The axiom if true for some elements, indeterminate for other elements, and false for other elements (this is *NeutroAxiom*).
- The axiom is false for all set's elements (this is *AntiAxiom*).

Similarly for any statement, theorem, lemma, algorithm, property, etc. For example: Classical *Theorem* (which is true for all space's elements), *NeutroTheorem* (which is partially true, partially indeterminate, and partially false), and *AntiTheorem* (which is false for all space's elements).

2.3. Algebra, NeutroAlgebra, and AntiAlgebra

An algebraic structure who's all operations are well-defined and all axioms are totally true is called Classical Algebraic Structure (or Algebra). An algebraic structure that has at least one NeutroOperation or one NeutroAxiom (and no AntiOperation and no AntiAxiom) is called NeutroAlgebraic Structure (or NeutroAlgebra).

An algebraic structure that has at least one AntiOperation or Anti Axiom is called AntiAlgebraic Structure (or AntiAlgebra). Therefore, a neutrosophic triplet structure is formed Algebra, NeutroAlgebra, and AntiAlgebra.

"Algebra" can be any classical algebraic structure, such as: groupoid, semigroup, monoid, group, commutative group, ring, field, vector space, BCK-Algebra, BCI-Algebra, etc.

3. Foundation of NeutroAlgebra and AntiAlgebra

The classical algebraic structures were generalized in 2019 and 2020 by Smarandache [1, 2, 3] to NeutroAlgebraic Structures (or NeutroAlgebras) whose operations and axioms are partially true, partially indeterminate, and partially false as extensions of partial algebra, and to AntiAlgebraic Structures (or AntiAlgebras) whose operations and axioms are totally false.

4. Foundation of NeutroStructures and AntiStructures

And in general, we extended any classical Structure, which is a space characterized by some properties, ideas, laws, shapes, hierarchy, etc., in no matter what field of knowledge, to a NeutroStructure and an AntiStructure.So, we formed a general neutrosophic triplet: Structure, NeutroStructure, and AntiStructure.

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Analyzing the Barriers of Organizational Transformation by Using Fuzzy SWARA

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ABSTRACT

The crucial role of bureaucracy in the economic, political, socio-cultural and political structures, and its impact in achieving the goals of organization is so important that in order to achieve the development, change directions consists of purifying and modernization of the administrative system in Iran also seems necessary. An important part of the transportation industry in each country, is the airports. So, dealing with the bureaucracy airports to implement better practices and removing unnecessary processes is the most issues. Hence, it can be stated that the aim of this study is to identify barriers of transformation in the organization administrative and then prioritizing these barriers in Mehrabad airport. For this purpose, the grounded theory and fuzzy SWARA methods was used to identifying the barriers and prioritizing them. Grounded theory results showed that cognitive barriers, structural barriers, participation barriers, economic and income barriers, legal barriers, strategic barriers, and management barriers are the barriers of the transformation in the Mehrabad airport administrative system. The fuzzy SWARA method used to prioritize these barriers, which according to the results, the structural barriers were the important barriers. Then cognitive and legal barriers were placed in the next rank. At the end, some solutions have been presented for overcoming these barriers in the Mehrabad airport.

1. Introduction

The present world is the world of changes and transformations [1]. In this regard, Drucker says: 'the first step for preparing ourselves in the present era is to forget yesterday' [2]. In the business world of today, organizations face with important conditions such as global competition, reduced cycle of technology innovation, universal and timely access to information, and also extensive changes in cultural, social and political environments, which have challenged stable competitive advantage, and more importantly, their survival [3]. So, in such turbulent conditions, organizations for their survival

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are compelled to coordinate themselves with this accelerating and unprecedented changes, and in parallel with hardware changes update their manpower and software, too [4].

The development of each country is closely connected to the administrative system and its effectiveness. The determinant role of administrative system in economic, political, social and cultural structures and its effect in realizing objectives of socio-political systems is very important. Therefore, administrative system because of association with other structures and affecting them has been of great importance. It is obvious that the efficiency of administrative system reform as a tool for managing and governing, a tool for providing sensitive and essential services to the community, a tool for dealing with necessary and special conditions, and finally a context for achieving economic and social growth and development is not only feasible by focusing on personnel issues, organization, paperwork, eliminating, merging, and dissolving departments, but also the traditional structure introverted and inflexible of administrative system with its hierarchical infrastructure in which task is completely separated of process requires profound changes and sometimes surface changes at different levels of the administrative system of each country which is possible by government’s effective and accurate planning [5]. Administrative reform follows several processes such as employees’ public participation, reforming administrative structures, appropriateness of duties and authorities, empowering employees and managers, authorities’ responsibility against citizens, monitoring administrative system, and development of information practices [6].

The transportation industry isn’t considered as the only factor in the development of a country and there are many other factors causing economic growth and development, but it should be noted that transportation is one of the key elements in economic development. The economic impacts of transportation can be observed in all economic activities such as agriculture industry, services, tourism, etc. One of the most important parts of transportation industry in each country is its airports. In the present century aviation industry has an important role in the relations among different countries around the world, culture exchange, showing economic and military powers and acceleration of critical affairs. Airports because of having various potentials in countries’ economic growth and contributing in creating stable development play important role [7]. The value and importance of airports in today's world is to the extent that some experts describe airports as economic locomotive of each country and believe that the existence of efficient and prosperous airports is a factor of economic growth and stable development. Therefore, attention to the administrative system of airports for better implementation of activities and elimination of unnecessary processes are of the most obvious issues and the most important mission of the International Airports Union is to integrate airports’ activities, determine policies and policy makings and improve its administrative system for faster workflow of works. Hence, paying attention to changes in the administrative system of airports is very essential to gain an appropriate and worthy position in the region.

Before starting change process, the organization must try to identify executive barriers for implementation of change and transformation and prepare itself to deal them properly and reasonably. Therefore, since there are many barriers in changing the administrative system of any organization, so the questions arising in this study are: “What are the main executive barriers in changing administrative system of country's airports company? Among these barriers, which has a greater priority than others and exacerbates other barriers and generally cause the failure of the project creation of change and transformation in administrative system of country's airports company?”

According to what was said, it can be stated that the goal of this research is to identify barriers of organizational transformation and then prioritize them. The highlights of this research as follows:

- This study proposes an integrated method to barriers of organizational transformation in aviation industry.
- Using the grounded theory as a powerful qualitative technique identifying the barriers.
- SWARA method extended with fuzzy numbers to obtain the weights of barriers.
- We apply Fuzzy Sets theory to handle the imprecise information in the real-world problems.

2. Research Background

2.1. Why Transformation?

Organizations are always subject to change and transformation and since these changes are caused by human, so it is necessary to assess change's contexts in him, both as the acceptor of change and as the creator of change. Most of rapid and accelerated changes cause unstable and transient behaviors and temporary improvement in organizations and this is due to this fact that human resources under subtle perception of organization's new situation and managers' expectations, act to cosmetic change in his behavior. Hence, we can conclude that organizational change and improvement is a function of the staff's behavioral changes and particularly managers. Therefore, as long as the senior management of an organization hasn't thought about the idea of change and improvement in the organization, it can't be expected to change and improve the organization [8].

The companies usually change to become global and people's thoughts and spirits, individually or as a group, are mobilized to reach the goals of all interest groups including: customers, employees and shareholders and anyway humans are the productive force and impetus for changing systems, structures and organizations. Usually it may that organizational change and transformation is created to transit from one stage of development to another one, organizations become mature by transition from different stages of development. Before making a decision about which of the aspects of the organization needs to be improved, the strengths and weaknesses of the organization must be analyzed. In this regard, needs assessment is of particular importance [9].

Organizational change firstly requires examining and diagnosing problem. Identifying problem and providing real problem is half of the change. If managers of changes scenario make mistake in defining problems, they will pay exorbitant costs, because it will direct organization and its resources towards the goals that haven't been designed towards the actual needs of the organization and can't be responsible for problems. Change management more than anything should examine and identify strategies for problem solving and its dimensions [10]. The message of most successful and unsuccessful administrative reforms in the world has been that transformation is rarely by chance. The success of administrative reform is guaranteed if people are responsible for its management and governance who have serious determination and clear and acceptable view about the future and reform path [11]. The responsible organization for management of administrative system reform, i.e. country's management and planning organization has summarized the main barriers to change country's administrative system as follows [3]:

- Resistance of groups affected by the reforms.
- Political basic Cost-Benefit measurement that concluded from Administrative revolution.
- Disagreement in goals, view and desirable future of country's administrative system.
- Disagreement in policies and strategies of administrative reform.
- Turbulent political environment.
- Little communication and connection with international environment.

Regardless of overcoming political attitude in identifying barriers to change country's administrative system and ignoring structural existing realities and quantity and quality of government employees among the barriers of administrative reform in this report had already been expressed as globalization and its requirements and expectations as one of the threats and weaknesses of administrative system [12], if globalization and its requirements and expectations are considered as a threat and weaknesses for administrative system, therefore how little communication and connection with international environment has been considered as an barrier to change administrative system [13]. *Table 1*, shows the background of some important studies:

Table 1. *The background of studies*.

N	Definition	Authors
1	*Organizational revolution is a respond to changes in the organizational beliefs, attitudes, values and structure, so that these factors can be adjusted aligned with technologies, markets, and challenges in the light of the speed of change.*	*[14]*
2	*Organizational revolution can be defined as a programmed and stabilized activity for applying behavioral sciences in improvement of systems via analytical and research methods.*	*[15]*
3	*Organizational revolution includes planned process of changes covering organizational culture which institutionalizes collective activities.*	*[10]*
4	*Organizational revolution aims at promoting compatibility of structures, processes, strategies, individuals, organization culture that tries to propose new and creative solutions and welcomes renovation.*	*[16]*
5	*Organizational revolution is meant changes in processes in all levels of an organization that fulfillment of goals are realized.*	*[7]*
6	*It has plans for changing the organization culture according to theories, research and behavioral sciences techniques.*	*[17]*

Literature review results about change in the administrative system and barriers showed that none of the previous researchers have provided a general framework for this purpose, showing all the barriers in the way of implementing the change in the administrative system. Also as yet, no studies have examined these barriers in country's airports company. For this purpose, in this study the researchers tried to provide an overall and comprehensive framework.

2.2. Why Grounded Theory?

The aim of this study is to develop a comprehensive model, which can identify the administrative barriers to change the administrative system of the Mehrabad airport. In order to develop this paradigmatic model, the grounded theory and Fuzzy SWARA were used. In fact, this study tries to investigate administrative barriers to create changes in the administrative system of country's airports company in actual mode and obtain an in-depth and comprehensive explanation of this phenomenon by making a model based on the experiences and attitudes of experts. In order to achieve this goal and based on the data-based theory approach, two secondary objectives were chosen for this study which include the following:

- Introducing the approach of data-based theory as efficient way to identify executive barriers.
- Assess and prioritize identified barriers and determine their importance.

Above objectives are proportional to identify executive barriers in creating changes in administrative system. The realization of these objectives requires the application of appropriate research design and methodology, which will be discussed in this section.

2.3. Fuzzy Sets Theory

Fuzzy set theory, which was introduced to deal with problems in which a source of vagueness is involved, has been utilized for incorporating imprecise data into the decision framework. A fuzzy set $\widetilde{M}$, can be defined mathematically by a membership function $\mu_{\widetilde{M}}(x)$, which assigns each element x in the universe of discourse X a real number in the interval [0, 1]. The higher the value of $\mu_{\widetilde{M}}(x)$, the higher the degree of membership of x in $\widetilde{M}$ [18].

Triangular and trapezoidal fuzzy numbers are the most common used fuzzy numbers both in theory and practice. Triangular fuzzy numbers are more practical in application because of their calculation easiness and features [19]. So, triangular fuzzy numbers are preferred for representing the linguistic variables in this study.

Let $\widetilde{M} = (l, m, u)$ is a triangular fuzzy number where l, m and u represent the smallest possible value, the most promising value, and the largest possible value, respectively and can be defined as *Eq. (1)*:

$$\mu_{\widetilde{M}}(x) = \begin{cases} 0, & x \langle l \text{ and } x \rangle u \\ (x-l)/(m-l) & l \leq x \leq m \\ (x-u)/(m-u) & m \leq x \leq u \end{cases}. \tag{1}$$

Some algebraic operations of the triangular fuzzy numbers ($\widetilde{M_1} = (l_1, m_1, u_1)$ *and* $\widetilde{M_2} = (l_2, m_2, u_2)$) can be expressed as follows [19, 20, 21]:

$$\widetilde{M}_1 \oplus \widetilde{M}_2 = (l_1 + l_2, m_1 + m_2, u_1 + u_2). \tag{2}$$

$$\widetilde{M}_1 \ominus \widetilde{M}_2 = (l_1 - u_2, m_1 - m_2, u_1 - l_2). \tag{3}$$

$$\widetilde{M}_1 \otimes \widetilde{M}_2 = (l_1 l_2, m_1 m_2, u_1 u_2). \tag{4}$$

$$\lambda \otimes \widetilde{M}_1 = (\lambda l_1, \lambda m_1, \lambda u_1)\ (\lambda > 0, \lambda \in R). \tag{5}$$

$$\widetilde{M}_1{}^{\lambda} = \left(l_1{}^{\lambda}, m_1{}^{\lambda}, u_1{}^{\lambda}\right)\ (\lambda > 0, \lambda \in R). \tag{6}$$

$$\widetilde{M}_1{}^{-1} = \left(\frac{1}{u_1}, \frac{1}{m_1}, \frac{1}{l_1}\right). \tag{7}$$

$$\widetilde{M}_1 \phi \widetilde{M}_2 = \left(\frac{l_1}{u_2}, \frac{m_1}{m_2}, \frac{u_1}{l_2}\right). \tag{8}$$

$$d(\widetilde{M}_1, \widetilde{M}_2) = \sqrt{\frac{1}{3}[(l_1 - l_2)^2 + (m_1 - m_2)^2 + (u_1 - u_2)^2]}. \tag{9}$$

2.4. Fuzzy SWARA

SWARA is a method where experts used their own knowledge. In addition, it is not considered to be complicated and time-consuming [22]. The main feature of the SWARA method is the possibility to estimate opinions of experts or stakeholder groups regarding the significance ratio of the attribute in the process of their weight determination [23]. The experts determine the most considerable attribute by the highest rank, the least considerable attribute by the lowest rank, and then estimate the overall ranks from the average value of ranks.

Crisp SWARA cannot effectively deal with problems with such imprecise information, hence, in this study fuzzy SWARA method has been applied to handle this issue. The process of evaluating the importance weights of attribute using the fuzzy SWARA method described in this section.

Step 1. Each of the Experts (DM=1,2,…,m) sort the evaluation attribute (j=1,2,…,n) in descending order of importance.

Step 2. According to *Table 2*, the relative importance of the attribute j in relation to the previous (j−1) attribute should be determined by each of the experts.

Table 2. *Linguistic comparison scale and fuzzy values [24].*

Linguistic Scale	Response Scale
Equally important	*(1, 1, 1)*
Moderately Less important	*(2/3, 1, 3/2)*
Less important	*(2/5, 1/2, 2/3)*
Very less important	*(2/7, 1/3, 2/5)*
Much less important	*(2/9, 1/4, 2/7)*

Step 3. Obtain the coefficient $\tilde{K}_j$:

$$\tilde{K}_j = \begin{cases} \tilde{1}, j = 1 \\ \tilde{S}_j + \tilde{1}, j > 1 \end{cases}. \tag{10}$$

Step 4. Calculate the fuzzy weight $\tilde{q}_j$:

$$\tilde{q}_j = \begin{cases} \tilde{1}, j = 1 \\ \frac{\tilde{q}_{j-1}}{\tilde{K}_j}, j > 1 \end{cases}. \tag{11}$$

Step 5. Calculate the relative weights of the evaluation Attribute:

$$\tilde{W}_j = \frac{\tilde{q}_j}{\sum \tilde{q}_j}. \tag{12}$$

Step 6. Calculate the defuzzied weights of the Attribute:

$$W = \frac{l + 2m + u}{4}. \tag{13}$$

Step 7. Calculate the normalized weights of the Attribute:

$$w_j' = \frac{w_j}{\sum w_j}. \tag{14}$$

Step 8. Calculate the average normalized weights of the Attribute:

$$w_j'' = \frac{1}{m} * \sum_{D=1}^{m} w_j'. \tag{15}$$

3. Research Methodology

This study also in terms of philosophical basis of research has interpretive paradigm. This study in terms of orientation, approach, background, goal and type of research is a practical, posteriori, combinatorial, descriptive and library and field research, respectively. In the first section of the study, qualitative method and in the second section semi-quantitative method has been used. The most important reasons for using qualitative method in the first section of this study include:

- Lack of paradigmatic and systematic view in studies conducted in this field.
- Multilateral approach towards executive barriers in creating changes in administrative system and try to identify them in a real manner.

Therefore, in this study, firstly, paradigmatic model to identify executive barriers in creating changes in administrative system of country's airports company using grounded theory approach and based on data collected with deep interviews, observation and reviewing documents, is provided. In grounded theory approach, data analysis is performed in two main levels: text level and conceptual level. Text level includes segmentation and organization of data files, data encryption and writing notes, while conceptual level emphasizes on making model, including sorting codes, and shaping networks. In the next step, in order to complete the model and assess the identified barriers, fuzzy SWARA method was used which has quantitative approach.

In this study, in order to choose sample size, the snowball sampling has been used. In this method, the process has been started of people who are experts in the field and have necessary criteria and in addition to the research questions, they were asked to introduce other experts in the field. Therefore, except for the first few people who were elected directly by the investigator based on specified criteria, other experts in addition to expertise criteria were chosen by other experts. Finally, 8 expert had choice that details of experts has been provided in the *Table 3*.

Table 3. *Expert's panel.*

N	Job position	Education	Work Experiences
1	*Adviser and expert of aviation industry*	*Master degree*	*Dean of the faculty of country's aviation industry, deputy of country's aviation industry, board member and deputy of management and resources of country's airports companies, director of Iran's aviation industries, and deputy of aviation industry research institute.*
2	*Deputy of management development and resources*	*Master degree*	*Deputy of management development and resources of country's airports companies, deputy of company, general manager of procurement, financial controller and director general of finance and income.*
3	*Chief of expert studies department of managing director's office*	*PhD*	*Chief of expert studies department of managing director's office, member of committee of administrative system reforms of country's airports company.*
4	*Veep (director general's assistant)*	*Master degree*	*Administrative and financial assistant of Islamic republic of Iran international exhibitions company, tax administration and income assistant of airports company and member of administrative health committee.*
5	*Veep (director general's assistant)*	*Master degree*	*Assistance of management development and resources of Mehrabad international airport, assistant of general department of finance and revenue.*
6	*Veep (director general's assistant)*	*Master degree*	*Assistant of general department of finance and revenue, member of policy development; administrative and budgetary development committee, inspector of airports company and Iran's air navigation, and deputy of management and resources development of technical and soil mechanics laboratory Co.*
7	*Assistant of general department of education and human resource development*	*Master degree*	*Assistant of general department of education and human resource development, legal general director' deputy.*
8	*Supervisor of reviewing operational manpower*	*Master degree*	*Chief of attracting and recruiting department, member of human resources committee.*

In this study, the researcher guided all interviews. The adoption of this procedure caused the researchers can use data from earlier interviews in subsequent interviews. Most of interviews were conducted in the interviewees' offices. Most of interview time was allocated to identify details of each executive barriers and related examples. Researcher in each organizational unit conducted interview, observed behavior and events, and in some cases studied some of documentations. In this study, data collection continued until theoretical saturation of categories and in other words, as far as access to new data was not possible. Form literature review, it can be realized that the subject of changing in administrative system is very diverse and there are great differences in the studied field, discussed issues, used tools, adopted policies and strategies among different researchers. The flowchart of the proposed MCDM model is shown in *Fig. 1*.

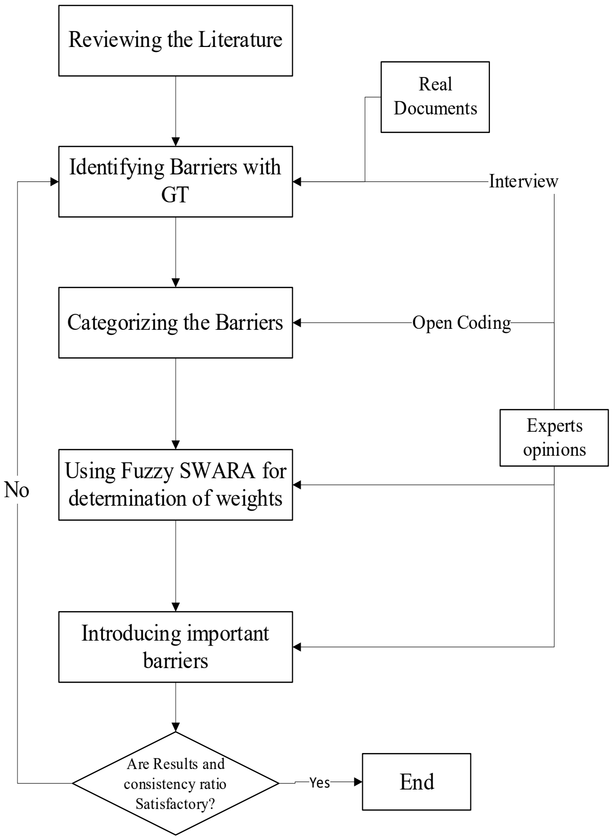

Fig. 1. *Flowchart of the proposed MCDM model.*

4. Data Analysis

4.1. Open Coding

Open coding is a part of the analysis performed by exact data analysis, naming and categorizing data. In order to classify concepts in categories accurately, each concept must be labeled after separation and raw data must be conceptualized by careful examination of interviews text and context notes. Data collected of Interviewees are encoded to identify their similarities and differences easier. Respondents in response to questions related to each dimension of the model described executive barriers to change and transform in the Mehrabad airport. The initial codes were extracted by analyzing their statements and views. In the next step, common codes and emphasized by all interviewees as well as important codes in researcher's view were identified as the final codes. Interviewees' descriptions in response to questions about the problem finding and causal conditions of executive barriers to create change and transformation in Mehrabad airport led to the identification of codes has been in *Table 4*.

Table 4. *Extracted codes related to barriers.*

N	Initial Extracted Code	N	Initial Extracted Code
1	*Lack of attention to research and development department as an independent unit.*	**20**	*Lack of participation by managers of different levels.*
2	*Lack of attention to successful patterns and benchmarks around the world.*	**21**	*Lack of knowledge management system.*
3	*Lack of strategic planning.*	**22**	*Lack of attention to enhance employees' knowledge and skill.*
4	*Lack of Action Plan (executive actions) and legal and executive projects.*	**23**	*Assigning task of changing and transforming only in personnel and administrative affairs unit.*
5	*Negative attitudes of managers to develop and implement strategic planning.*	**24**	*Indifference of other units to administrative change process.*
6	*Attitudes of managers and officials.*	**25**	*Superior rules.*
7	*Lack of manager's system attitude.*	**26**	*Managers' fear of losing position.*
8	*Macro policies governing the country.*	**27**	*Organizational Structure.*
9	*Political relations governing the organizations' communications.*	**28**	*Organization's Funds.*
10	*The lack of an integrated transformation system.*	**29**	*International and global standards and rules.*
11	*Economic conditions governing the society.*	**30**	*Mandatory costs.*
12	*International Relations and relationship with international organizations.*	**31**	*Poor coordination of government's economic and cultural policies.*
13	*Budget allocated to the change and transformation.*	**32**	*Lack of knowledge regarding the identification of weaknesses, power, opportunities and threats.*
14	*Lack of skill of organizational individual in changing and transforming.*	**33**	*Organization employees' low motivation.*
15	*Fear of losing company reputation.*	**34**	*Uncertainty of task aspects.*
16	*Low level of public participation.*	**35**	*Company's activities are professional.*
17	*Lack of proper operational and non-operational infrastructures.*	**36**	*Lack of full access to income sources.*
18	*Inadequate study of company's comprehensive plan.*	**37**	*Lack of coordination in company's macro decision makings.*
19	*Lack of understanding of the results of the project.*		

Investigation of extracted codes of interviewees' responses about the executive barriers to create change and transformation in Mehrabad airports has led to identification of the final. After applying experts' opinions and eliminating repetitious codes the final codes were obtained which showed in *Table 5*.

Table 5. *Final codes related to barriers.*

N	Final Code	N	Final Code
1	*Lack of attention to research and development department as an independent unit.*	15	*Macro policies governing the country.*
2	*Lack of attention to successful patterns and benchmarks around the world.*	16	*Political relations governing the organizations' communications.*
3	*Lack of strategic planning.*	17	*The lack of an integrated transformation system.*
4	*Lack of Action Plan (executive actions) and legal and executive projects.*	18	*Economic conditions governing the society.*
5	*Negative attitudes of managers to develop and implement strategic planning.*	19	*International Relations and relationship with international organizations.*
6	*Lack of manager's system attitude.*	20	*Budget allocated to the change and transformation.*
7	*Superior rules.*	21	*Lack of skill of organizational individual in changing and transforming.*
8	*Fear of losing company reputation.*	22	*Fear of losing company reputation.*
9	*Organization employees' low motivation.*	23	*Lack of proper operational and non-operational infrastructures.*
10	*Inadequate study of company's comprehensive plan.*	24	*Lack of understanding of the results of the project.*
11	*Lack of participation by managers of different levels.*	25	*Managers' fear of losing position.*
12	*Lack of knowledge management system.*	26	*International and global standards and rules.*
13	*Lack of attention to enhance employees' knowledge and skill.*	27	*Organization employees' low motivation.*
14	*Assigning task of changing and transforming only in personnel and administrative affairs unit.*	28	*Mandatory costs.*

4.2. Selective Coding

While open coding separates data into different categories, selected coding connects categories and their sub-categories given their characteristics and aspects. To discover how categories are connected to each other, the researcher uses paradigm. Paradigm is an analytic tool that Strauss and Corbin proposed for studying the data. The main components of the paradigm are: conditions, actions/ reactions and consequences. Strauss and Corbin proposed the paradigmatic model because in context theory sub-categories are related to categories in the form of a series of connections indicating casual conditions, phenomenon, context, intervention conditions, action/reaction strategies and results. During selective coding process, the researcher uses analytical tools such as asking question, and theoretical and permanent comparison of categories, sub-categories and their characteristics which have been appeared in open coding to develop relations between categories and sub-categories and form categories proportional to paradigmatic model. Simultaneous with doing open and selective coding a model was made indicating the relationship between the categories and sub-categories. When these relationships were developed, the selective coding procedure is used to facilitate integration of categories and sub-categories which have been identified in open and selective coding in the form of a new theory.

In selective coding the researcher by asking questions about the category which generally characterize a relationship refers to data and examines incidents confirming or rejecting questions. In selective coding process, the researcher is continuously moving between inductive and deductive thinking. It means that when he is working with data suggests their possible relations or properties deductively, and then tries to examine what he has expressed deductively against data. In Table 6, the identified concepts and categories associated with executive barriers to create change in Mehrabad airport has been presented.

Table 6. *Identified concepts and categories associated with executive barriers.*

Barriers	Sub-Barriers	Final Codes
Management barriers- B1	*Managers' poor planning- B11*	*Lack of coordination in company's macro decision makings.* *Lack of strategic planning.*
	Managers' traditional attitude- B12	*Negative attitudes of managers to develop and implement strategic planning.* *Attitudes of managers and officials.* *Lack of manager's system attitude.*
	Fear of change- B13	*Managers' fear of losing position.* *Lack of participation by managers of different levels.*
Strategic barriers- B2	*Lack of attention to the future- B21*	*Lack of attention to research and development department as an independent unit.* *Lack of attention to successful patterns and benchmarks around the world.* *Lack of strategic planning.*
	Poor knowledge for changing- B22	*Lack of knowledge management system.* *Lack of attention to enhance employees' knowledge and skill.* *Lack of knowledge regarding the identification of weaknesses, power, opportunities and threats.*
Legal barriers- B3	*Intra-organizational rules-B31*	*Lack of action plan (executive actions) and legal and executive projects.* *Macro policies governing the country.* *Superior rules.*
	extra-organizational rules- B32	*International relations and relationship with international organizations.* *Political relations governing the organizations' communications.* *International and global standards and rules.*
Economic and income barriers- B4	*Economic policies- B41*	*Poor coordination of government's economic and cultural policies.* *Economic conditions governing the society.* *Mandatory costs.*
	Company's financial constraints- B42	*Budget allocated to the change and transformation.* *Organization's funds.* *Lack of full access to income sources.*
Participatory barriers- B5	*Lack of inter-sectorial collaboration- B51*	*Indifference of other units to administrative change process.* *Low level of public participation.*
	Making processes professional- B52	*Assigning task of changing and transforming only in personnel and administrative affairs unit.* *Company's activities are professional.*
Structural barriers- B6	*Lack of transparency- B61*	*Uncertainty of task aspects.* *The lack of an integrated transformation system.*
	Poor infrastructure- B62	*Lack of proper operational and non-operational infrastructures.* *Organizational structure.*
Cognitive barriers- B7	*Insufficient knowledge of the change process- B71*	*Lack of understanding of the results of the project.* *Fear of losing company reputation.*
	Employees' poor attitude- B72	*Lack of skill of organizational individual in changing and transforming.* *Organization employees' low motivation.*

As seen in the above table, after conducting interviews with experts and open coding, the axial coding and selective coding were discussed. In axial coding 15 concepts were identified and in the next step according to the selective coding, the identified factors were classified in 7 main barrier groups. In the following, in order to determine the importance and priority level of each of the six main barrier groups, the fuzzy SWARA method was used.

4.3. Prioritizing the Barriers by Using Fuzzy SWARA

At this step, fuzzy which was explained in *Section 2.4* utilized to obtain importance weights of identified barriers. The barrier set is determined on the basis of the GT results as shown in the *Table 6*.

The results of fuzzy SWARA are presented in this section. Because of limitation of space, the method results presented for the first Decision Maker (DM1) and main barriers dimension of this research and showed in *Table 7* for instance. A similar procedure was followed for the other experts and sub-barriers.

Table 7. *Fuzzy SWARA results for DM1 and main barriers dimension.*

Barrier	$\tilde{S}_j$	$\tilde{K}_j$	$\tilde{q}_j$	$\tilde{w}_j$	w_j'
B6	–	(1, 1, 1)	(1, 1, 1)	(0.3969, 0.4848, 0.5828)	0.46979
B7	(0.6667, 1, 1.5)	(1.6667,2, 2.5)	(0.4, 0.5, 0.6)	(0.1588, 0.2424, 0.3497)	0.23937
B3	(0.6667, 1, 1.5)	(1.6667,2, 2.5)	(0.16, 0.25, 0.36)	(0.0635, 0.1212, 0.2098)	0.12429
B2	(0.4, 0.5, 0.6667)	(1.4, 1.5, 1.6667)	(0.096, 0.1667, 0.2571)	(0.0381, 0.0808, 0.1499)	0.08424
B5	(0.6667, 1, 1.5)	(1.6667,2, 2.5)	(0.0384, 0.08333, 0.1543)	(0.0152, 0.0404, 0.0899)	0.04481
B1	(0.6667, 1, 1.5)	(1.6667,2, 2.5)	(0.0154, 0.0417, 0.0926)	(0.0061, 0.0202, 0.0539)	0.02420
B4	(0.6667, 1, 1.5)	(1.6667,2, 2.5)	(0.0061, 0.0208, 0.0555)	(0.0024, 0.0101, 0.0324)	0.01325

At the final step, the average weights of barriers are calculated and showed in *Table 8*.

Table 8. *Transformation barriers weights.*

Barriers	Weight	Sub-Barriers	Local Weight	Global Weight	Rank
B1	0.030219283	B11	0.158874095	0.004801061	15
		B12	0.558670388	0.016882619	11
		B13	0.282455518	0.008535603	13
B2	0.089694687	B21	0.35444225	0.031791587	8
		B22	0.64555775	0.0579031	6
B3	0.132160409	B31	0.36443833	0.048164319	7
		B32	0.63556167	0.08399609	5
B4	0.01710169	B41	0.373803176	0.006392666	14
		B42	0.626196824	0.010709024	12
B5	0.050169669	B51	0.60455053	0.0303301	9
		B52	0.39544947	0.019839569	10
B6	0.445636176	B61	0.610242382	0.271946082	1
		B62	0.389757618	0.173690094	2
B7	0.235018085	B71	0.615934235	0.144755684	3
		B72	0.384065765	0.090262401	4

The prioritization of barriers (*Table 8*) explain that the 'structural barriers (B6)', 'cognitive barriers (B7)' and 'legal barriers (B3)' are the most important barriers and, 'economic and income barriers (B4)', 'management barriers (B1)' and 'participatory barriers (B5)' are the least important barriers respectively.

5. Conclusion and Recommendations

In general, any governmental administrative system depicts the government attitude toward administration and management of the country. The administrative system plays an important role in economic, political, socio-cultural structure and its effect in realization of the society macro-systems is of importance; so that the mentioned goals cannot be achieved without an efficient and effective administrative system. In recent years, improvement of administrative system is considered as a prerequisite of growth and its fundamental goal. The quality and efficacy of the administrative system is a determinant factor in implementation of developmental plans and providing welfare. Paying attention to the results, fulfillment of goals, continuous improvement of the quality of the public services and citizens' satisfaction, conducting organizational affairs in meaningful way and making significant changes in management knowledge in recent years, considering administrative system functions and evaluation of this system are essential. Nowadays, the role of management systems as an evaluation system and efficient supervision is obvious in improvement and perfection of organizations. While applying appropriate solutions, it is expected that the administrative system revolution will convert the government vision to service providing attitude which considers customer-oriented principle and customer satisfaction as one of the main indicators of efficacy and development of the system measurement. The components such as speed, accuracy and precision in providing the client services, the quality of performance, transparency and appropriate information dissemination are effective factors in customer satisfaction on services offered by the governmental system which provides the context for public trust as the biggest capital and support for the administrative system. It worthwhile to note that merely application of the revolutionary patterns proposed by different scholars will not be remedial in other organizations and communities that codify the rules and norms based on specific conditions. However, the Mehrabad airport needs for identification the barriers of changes and it should make effort to remove them. The results of fuzzy SWARA showed that the important barrier of changes in the administrative system in the Mehrabad airport is structural barriers. It is recommended this firm to try to remove these barriers. This firm also can identify jobs dimensions by clarification and establishing an integrated revolution system. Also, it should strengthen its infrastructural weak points in order to eliminate structural barriers before revolution.

The second case is cognitive barriers. The Mehrabad airports is recommended to recognize the change process in order to implement it without fear of losing its credibility. This firm also can take an action to change its staff vision on revolution. Legal barriers are considered as the third barrier of change in the administrative system. This firm should pay attention to codification of inter-organization rules; because it does not have authority to compile intra-organization regulations. Revolution is an essential, time-consuming, gradual and difficult process in this firm administrative system. Certainly, realization or non-realization of changes in this sector lies in the implementation of strategies and programs. In contrary to some extent, the firms are unable to approve laws and implement the rules in different contexts. Codification and implementation procedures are interwoven and providing the condition for implementation of a policy is superior to any other plans. Observance of this principle leads to compatibility in the nature of the implemented program and strategy which at the end it will pave the road for realization of the target. As before said, the findings of current research depict inconsistency of the strategy of change with the revolution program nature in the administrative system in the

Mehrabad airport, so that it has caused to violation from the goals. Thus, successful implementation coordinated with the revolution program nature will be possible merely when the dependency relations are minimized. In other words, this research emphasizes the decentralization process as the main element and fundamental solution for making changes. However, this firm should take an action to remove its strategic barriers. Following propositions are recommenced:

- It is recommended to conduct the organizational, technical and political decentralization process in short-term, middle-term and long-term.
- It is recommended to pay attention to planning and then implementation and control: If we consider any organizational change as a process, we will have three phases of planning, implementation and control. The governmental organizations put more energy on planning and then pay attention to implementation. Control is also the lost chain of organizational change process. In Iran it is emphasized documentation and strategy.

The cultural barriers encourages changes and rewards ideas and innovation. We can see this culture rarely in Iranian governmental organizations and what is tangible is habit and uniformity. The findings of this research reveal that slogans and politicized behaviors in the governmental organizations is the biggest barrier before organizational revolution. In such atmosphere seeking organizational revolution programs seems unlikely since some of the management of these organizations prioritize establishing a close relationship and strengthening of their position not development of organization.

The findings of this research propose various research opportunities for researchers. Some of these propositions include:

- Blocking sub-categories: In this research, blocking was done according to Strauss and Corbin categories paradigmatic model. The current research sub-categories paradigmatic model provides context for innovation and gaining knowledge.
- Examining variables and their relationships using survey research and multi-criteria decision making techniques is useful.

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Prediction of Pakistan Super League-2020 Using TOPSIS and Fuzzy TOPSIS Methods

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PAPER INFO	ABSTRACT
Chronicle: *Received: 21 January 2020* *Revised: 05 March 2020* *Accepted: 30 April 2020* **Keywords:** *PSL.* *TOPSIS.* *Prediction.* *Cricket.*	We lived in uncertain word and prediction in this uncertain word is a major issue. Prediction in cricket is very complex because there are many factors which are effecting on results. Weather, pitch, Conditions, Home grounds are some of these factors. In this article it is aimed to predict the results of PSL-2020 by using TOPSIS and Fuzzy TOPSIS methods. It will be interesting because first time in PSL history it is going to be held in Pakistan. But we use some serious factor to predict the winner of PSL-2020.

1. Introduction

Pakistan Super League (PSL) is a professional Twenty20 cricket league in Pakistan contested during February and March every year, with each team play matches in double round robin format. The league is founded by the Pakistan Cricket Board (PCB) in Lahore on 9 September 2015 with initial participation of 5 teams but now there are 6 teams represented by 6 major cities 0f Pakistan including Islamabad, Lahore, Multan, Karachi, Peshawar and Quetta.

The commercial rights to the initial franchises were sold for US$93 million for a ten-years period while the 6th franchise was originally formed in 2017 bought by Schone properties in US$5.2 million per annum for 8 years contract means the total contract for that team was worth US$41.6 million that makes it the most expensive team of PSL but unfortunately the contract was terminated by the PCB just after one year due to the non-payment of annual fee by the franchise. Ali Khan Tareen and Taimoor Malik are the new owners of the 6th team as they won the bid for US$6.2 million per annum for 7 years period.

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The 1st edition of the PSL was entirely played in UAE due to security reasons, Islamabad United was the 1st champions, Peshawar Zalmi were the 2017 PSL champions, Islamabad were again won the title in 2018 while Quetta Gladiators are the current champions of the HBL PSL.

The PSL 2020 season is now going to be held entirely in Pakistan for the 1st time as by the directions of Pakistani Prime Minister. The tournament will be played from February 20, 2020 to March 22, 2020. Four venues including Lahore, Karachi, Multan and Rawalpindi will host whole the tournament.

Cricket is said to be a game of uncertainties, if we talk about the T20 game then it's really a difficult task to predict about the Twenty20 game as it's the limited overs game and a single player can overcome on the situation [2] and changed the whole map of the game in just few minutes [3]. The combination of key players (Batsmen with good average and bowlers with good economy rates) also plays an important role in match winning [4] for this purpose these 3 factors (batting, bowling, key players) [1] are strongly recommended to add in the stats to help in prediction because more the key players will have the more chances to win the match.

The stats in this article are based on the upcoming edition of the PSL 2020. About 90% data in stats is taken from the website cicinfo.com that's considered to be the most authentic and most visiting website in all over the world for cricket's updates and stats. The 9 important attributes which are considered to be the most important in match wining or lost with names C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9,C_{10} and C_{11} are taken out with 6 teams named as T_1 , T_2 , T_3, T_4, T_5 and T_6.

2. Material and Methods

The 5th edition of the HBL PSL is going to start from February 20, 2020 in Pakistan. The best players from all over the world are taking part with their relative franchises with 6 of the following teams:

T_1= Quetta Gladiators.
T_2= Islamabad United.
T_3= Karachi Kings.
T_4= Peshawar Zalmi.
T_5= Multan Sultan.
T_6= Lahore Qalandars.

Attributes. Some important attributes that plays important role in match wining are as follows with explanation of each one separately.

C_1	C_2	C_3	C_4	C_5	C_6
Fifties	*Chased*	*Matches won*	*Matches lost*	*Win percentage*	*Lost percentage*
C_7	C_8	C_9	C_{10}	C_{11}	
2QA		*Batting average>24*	*Bowling S.R*	*Economy rate*	

C1–fifties. Fifties plays an important role in T20 cricket. It also helps to see the overall performance by whole the team, More the fifties maker in the team will have the more chances to adopt the pressure. The data shows the performance regarding fifties of all the teams in last 4 editions of the PSL.

C_2–chased. Chasing is a slightly a difficult task in T20 cricket if the opposite team makes a good total in 1st inning. So the team having good chasing skills will definitely have chances to win the match as well as to adopt the pressure in every situation. In stats the previous record is collected from last 4 editions which clearly shows the chasing abilities by a specific team.

C3–match won. It also an important factor to encourage the teams expectations and moral values. Winning matches shown the previous performance of the team and likely to have to perform well in future on the basis of previous record except of that the pitches, venues and some players are changed for upcoming PSL.

C_4–match lost. It shows the bad performance of the team .The team with most of the lost matches definitely will be considered as the weak team at all but restricted till the expectation as its T20 game and we can't predict with more confidently regarding the strongest and weakest team.

C_5–Win % AGE. Clearly shows the teams combined performance in last 4 editions. As the percentage mostly lies in decimal but in stats it's taken as exact numeric value i.e. instead of taking 58.69%. More the winning percentage shows the team's overall performance in last 4 editions of the PSL.

C_6–lost percentage. Lost percentage shows the weakness of the team. The team with more percentage in matches lost is likely to be consider as the weakest side of the tournament but except of that mostly players are changed or replaced every year so it's not the 100% surety to claim about its performance by just seeing previous record of the weakest team.

C_7–all out. It shows the team's confidence regarding to adopt the pressure in critical situations and also the consistency of the players in difficult situations. More the all outs shows the previous record about their performance in limited overs cricket.

C_8–key players. Key players also plays an important role in match winning .The team with more key players probably have the more chances to win the match or even title. Key players in stats are included with best batting, bowling averages and best economy rates.

C_9–best batting averages. Good batting line up is also a key factor for the team, batting averages taken in stats is >24 means no of best batsmen in a team. More the players with average >24 will increase the chances of winning the match or create a big total on board that helps to win the matches.

C_9 –bowling strike rate. Bowling strike rate is the average number of balls bowled per wicket taken, lower the strike rate means more effective a baller is taking wickets quickly. In stats just those players are taken out who have the best strike rates that is consider to be as <19.

C_{11}–economy rate. Economy rate is the average number of runs conceded for each over bowled. A lower economy rate is preferable. In stats just those bowlers are to be choose out to whom economy rates are less than 8.

Table 1 shows the statistical behavior of the teams performances. The data is collected up to 4 editions of the PSL.

Table 1. *Statically behavior of the collected data that is take up to 4 editions of the PSL.*

Teams/ Attributes	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}
T_1	31	20	26	16	62	38	6	6	5	2	2
T_2	31	18	25	18	58	42	5	6	6	4	4
T_3	27	8	17	23	43	57	4	6	6	3	3
T_4	36	15	27	19	59	41	4	4	6	2	2
T_5	12	5	7	12	37	63	6	4	4	1	4
T_6	24	5	10	24	31	69	7	4	4	2	3

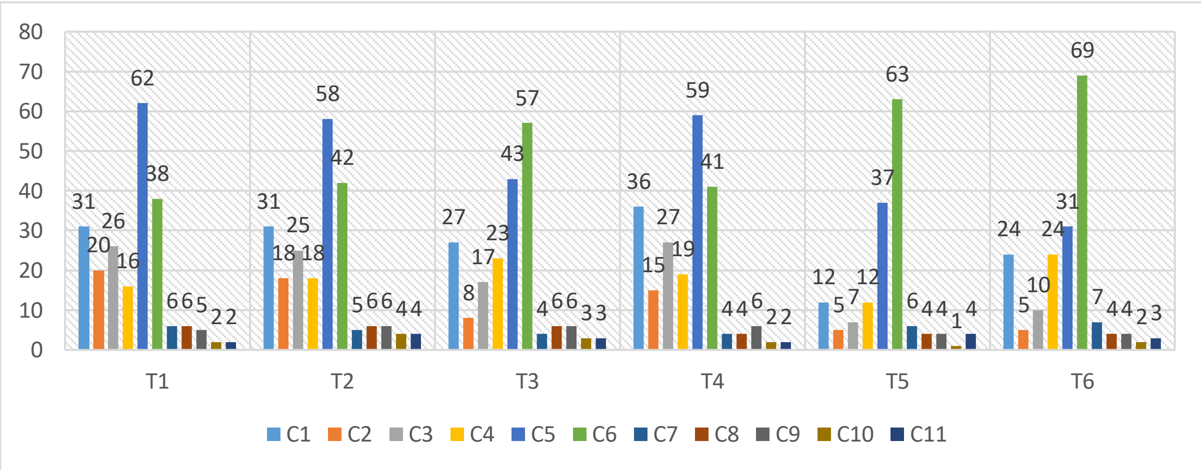

Fig. 1. *Graphical representation of the above mentioned data.*

3. Calculations

TOPSIS technique is used to calculate an MCDM problem. So the weights are assigned according to the importance of each of the attribute.

Weight	0.15	0.08	0.12	0.03	0.09	0.08	0.02	0.17	0.16	0.08	0.02
Attributes	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}

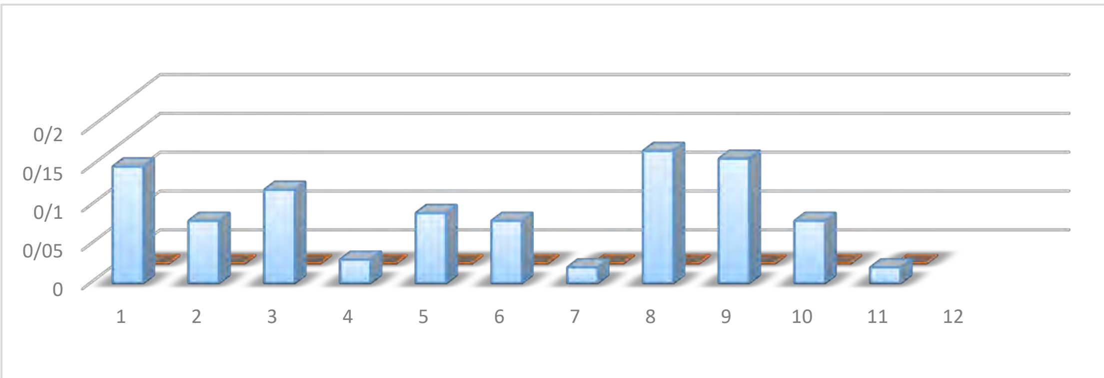

Fig. 2. *Graphical behavior of the above mentioned weights to the attributes.*

There is a great demand of game prediction for that many of the mathematical formulas are to be used for prediction by using MCDM techniques that is used to choose the best one amongst multiple criteria's.

3.1. TOPSIS Prediction

TOPSIS, Technique for order of preference by similarity to ideal solution, a tool for calculating the best one from the collected data. TOPSIS technique is applied to calculate the weights by given data that is collected up to 4 editions of the PSL which is easily available on "www.cricinfo.com" that is considered to be one of the best cricketing website in the world. To apply the technique some important attributes that are considered to play an important role in match or title winning including Fifties, chased, matches won, matches lost, win percentage, lost percentage, best batting average, best bowling strike rate, best economy rate with 6 teams T_1,T_2,T_3,T_4,T_5, and T_6 as decision makers are taken.

Step 1. Calculate normalized matrix.

$$X_{ij} = \frac{X_{ij}}{\sqrt{\sum_{j=1}^{n} X^2{}_{ij}}}.$$

Weight	**0.15**	**0.08**	**0.12**	**0.03**	**0.09**
T/C	**C_1**	**C_2**	**C_3**	**C_4**	**C_5**
T1	*0.453777*	*0.613428*	*0.523360293*	*0.341899075*	*0.508470092*
T2	*0.453777*	*0.552085*	*0.503231051*	*0.384636459*	*0.47566557*
T3	*0.395225*	*0.245371*	*0.342197115*	*0.49147992*	*0.352648612*
T4	*0.526967*	*0.460071*	*0.543489536*	*0.406005151*	*0.483866701*
T5	*0.175656*	*0.153357*	*0.140904694*	*0.256424306*	*0.303441829*
T6	*0.351311*	*0.153357*	*0.201292421*	*0.512848612*	*0.254235046*
0.08	**0.02**	**0.17**	**0.16**	**0.08**	**0.02**
C_6	**C_7**	**C_8**	**C_9**	**C_{10}**	**C_{11}**
0.292584827	*0.449719013*	*0.480384461*	*0.389249472*	*0.324442842*	*0.262612866*
0.32338323	*0.374765844*	*0.480384461*	*0.467099366*	*0.648885685*	*0.525225731*
0.43887724	*0.299812676*	*0.480384461*	*0.467099366*	*0.486664263*	*0.393919299*
0.315683629	*0.299812676*	*0.320256308*	*0.467099366*	*0.324442842*	*0.262612866*
0.485074845	*0.449719013*	*0.320256308*	*0.311399578*	*0.162221421*	*0.525225731*
0.531272449	*0.524672182*	*0.320256308*	*0.311399578*	*0.324442842*	*0.393919299*

Step 2. Calculate the weighted normalized matrix.

$$V_{ij} = X_{ij} \times W_j.$$

Weight	**0.15**	**0.08**	**0.12**	**0.03**	**0.09**
T/C	**C_1**	**C_2**	**C_3**	**C_4**	**C_5**
T1	*0.068067*	*0.049074*	*0.062803235*	*0.010256972*	*0.045762308*
T2	*0.068067*	*0.044167*	*0.060387726*	*0.011539094*	*0.042809901*
T3	*0.059284*	*0.01963*	*0.041063654*	*0.014744398*	*0.031738375*
T4	*0.079045*	*0.036806*	*0.065218744*	*0.012180155*	*0.043548003*
T5	*0.026348*	*0.012269*	*0.016908563*	*0.007692729*	*0.027309765*
T6	*0.052697*	*0.012269*	*0.02415509*	*0.015385458*	*0.022881154*

0.08	**0.02**	**0.17**	**0.16**	**0.08**	**0.02**
C_6	**C_7**	**C_8**	**C_9**	**C_{10}**	**C_{11}**
0.023406786	*0.00899438*	*0.081665358*	*0.062279916*	*0.025955427*	*0.005252257*
0.025870658	*0.007495317*	*0.081665358*	*0.074735899*	*0.051910855*	*0.010504515*
0.035110179	*0.005996254*	*0.081665358*	*0.074735899*	*0.038933141*	*0.007878386*
0.02525469	*0.005996254*	*0.054443572*	*0.074735899*	*0.025955427*	*0.005252257*
0.038805988	*0.00899438*	*0.054443572*	*0.049823932*	*0.012977714*	*0.010504515*
0.042501796	*0.010493444*	*0.054443572*	*0.049823932*	*0.025955427*	*0.007878386*

Step 3. Calculate the ideal best and ideal worst values.

	C_1	**C_2**	**C_3**	**C_4**	**C_5**	**C_6**	**C_7**	**C_8**	**C_9**	**C_{10}**	**C_{11}**
V+	*0.079045*	*0.049074*	*0.065218744*	*0.007692729*	*0.045762308*	*0.023406786*	*0.005996254*	*0.081665358*	*0.074735899*	*0.051910855*	*0.010504515*
V-	*0.026348*	*0.012269*	*0.016908563*	*0.015385458*	*0.022881154*	*0.042501796*	*0.010493444*	*0.054443572*	*0.049823932*	*0.012977714*	*0.005252257*

Step 4. Calculate the Euclidean distance from the idea best.

$$Si^{+} = \left[\left(\sum_{j=1}^{m} V_{ij} - V_j^{+}\right)^2\right]^{0.5}.$$

Step 5. Calculate the Euclidean distance from the ideal worst.

$$Si^{-} = \left[\left(\sum_{j=1}^{m} V_{ij} - V_j^{-}\right)^2\right]^{0.5}.$$

Step 6. Calculate the performance score.

$$P_i = \frac{Si^{-}}{Si^{+} + Si^{-}}.$$

Teams	**Si+**	**Si-**	**Pi**	**Rank**
Quetta Gladiators	*0.032*	*0.085*	*0.728429*	*2*
Islamabad United	*0.014*	*0.091*	*0.865348*	*1*
Karachi Kings	*0.049*	*0.063*	*0.560988*	*4*
Peshawar Zalmi	*0.04*	*0.085*	*0.679043*	*3*
Multan Sultan	*0.1*	*0.011*	*0.099888*	*6*
Lahore Qalandars	*0.082*	*0.03*	*0.269933*	*5*

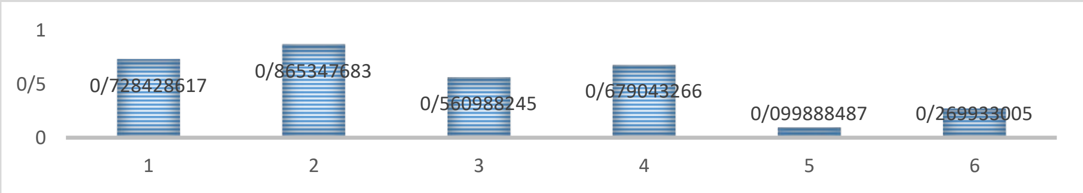

Fig. 3. *Graphical representation of the final results against each team.*

3.2. Calculations via Fuzzy TOPSIS

Step 1. Assign the fuzzy numbers to the group of decision makers on the basis of attributes regarding each of the criteria.

T/C	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}
T_1	*8,6,7*	*9,7,8*	*8,6,7*	*8,6,7*	*9,7,8*	*2,3,5*	*6,5,4*	*9,7,8*	*8,6,7*	*8,6,7*	*8,6,7*
T_2	*8,6,7*	*8,6,7*	*6,5,4*	*6,5,4*	*8,6,7*	*6,5,4*	*2,3,5*	*6,5,4*	*6,5,4*	*6,5,4*	*8,6,7*
T_3	*6,5,4*	*2,3,5*	*6,5,4*	*2,3,5*	*2,3,5*	*6,5,4*	*2,3,5*	*9,7,8*	*8,6,7*	*6,5,4*	*6,5,4*
T_4	*9,7,8*	*6,5,4*	*9,7,8*	*6,5,4*	*8,6,7*	*2,3,5*	*2,3,5*	*6,5,4*	*8,6,7*	*2,3,5*	*2,3,5*
T_5	*1,1,2*	*1,1,2*	*2,3,5*	*2,3,5*	*2,3,5*	*8,6,7*	*8,6,7*	*6,5,4*	*2,3,5*	*1,1,2*	*8,6,7*
T_6	*2,3,5*	*1,1,2*	*1,1,2*	*1,1,2*	*1,1,2*	*9,7,8*	*9,7,8*	*6,5,4*	*2,3,5*	*2,3,5*	*6,5,4*

Decision Maker 2.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}
T_1	*9,7,8*	*9,7,8*	*8,6,7*	*8,6,7*	*9,7,8*	*1,1,2*	*6,5,4*	*9,7,8*	*9,7,8*	*9,7,8*	*8,6,7*
T_2	*8,6,7*	*8,6,7*	*6,5,4*	*6,5,4*	*8,6,7*	*2,3,5*	*1,1,2*	*6,5,4*	*8,6,7*	*8,6,7*	*6,5,4*
T_3	*6,5,4*	*1,1,2*	*6,5,4*	*2,3,5*	*2,3,5*	*6,5,4*	*2,3,5*	*9,7,8*	*8,6,7*	*8,6,7*	*6,5,4*
T_4	*8,6,7*	*6,5,4*	*8,6,7*	*6,5,4*	*8,6,7*	*2,3,5*	*2,3,5*	*6,5,4*	*6,5,4*	*2,3,5*	*2,3,5*
T_5	*1,1,2*	*1,1,2*	*2,3,5*	*6,5,4*	*2,3,5*	*8,6,7*	*8,6,7*	*6,5,4*	*2,3,5*	*1,1,2*	*8,6,7*
T_6	*2,3,5*	*1,1,2*	*1,1,2*	*6,5,4*	*1,1,2*	*8,6,7*	*9,7,8*	*6,5,4*	*2,3,5*	*2,3,5*	*8,6,7*

Decision Maker 3.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}
T_1	*8,6,7*	*8,6,7*	*8,6,7*	*9,7,8*	*8,6,7*	*1,1,2*	*2,3,5*	*9,7,8*	*9,7,8*	*8,6,7*	*6,5,4*
T_2	*8,6,7*	*8,6,7*	*6,5,4*	*8,6,7*	*8,6,7*	*2,3,5*	*1,1,2*	*8,6,7*	*6,5,4*	*8,6,7*	*8,6,7*
T_3	*6,5,4*	*2,3,5*	*6,5,4*	*2,3,5*	*2,3,5*	*6,5,4*	*2,3,5*	*9,7,8*	*9,7,8*	*6,5,4*	*6,5,4*
T_4	*9,7,8*	*6,5,4*	*9,7,8*	*6,5,4*	*8,6,7*	*2,3,5*	*2,3,5*	*6,5,4*	*8,6,7*	*2,3,5*	*2,3,5*
T_5	*2,3,5*	*6.5.4*	*2,3,5*	*2,3,5*	*2,3,5*	*8,6,7*	*8,6,7*	*6,5,4*	*2,3,5*	*1,1,2*	*8,6,7*
T_6	*1,1,2*	*2,3,5*	*1,1,2*	*9,7,8*	*1,1,2*	*9,7,8*	*9,7,8*	*6,5,4*	*2,3,5*	*2,3,5*	*6,5,4*

Step 2. Construct the combined decision matrix by using the following formula.

$$\tilde{x}_{ij} = (a_{ij}, b_{ij}, c_{ij}) \text{ where } a_{ij} = \min_k\{a_{ij}{}^k\},\ b_{ij} = \frac{1}{k}\sum_{k=1}^{k} b_{ij}{}^k,\ c_{ij} = \max_k\{c_{ij}{}^k\}.$$

	C 1	C2	C3	C4	C5	C6	C 7	C8	C 9	C 10	C 11
T1	8,6.33,8	8,6.66,8	8,3,7	8,6.33,8	8,6.66,8	1,1.66,5	2,4.33,5	9,7,8	8,6.66,8	8,6.33,8	6,5.66,7
T2	8,6,7	8,6,7	6,5,4	6,5.33,7	8,6,7	2,3.66,5	1,1.66,5	6,5.33,7	6,5.33,7	6,5.66,7	6,5.66,7
T3	6,3,4	1,2.33,5	6,5,4	2,3,5	2,3,5	6,5,4	2,3,5	9,7,8	8,6.33,7	6,5.33,7	6,5,4
T4	8,6.66,8	6,5,4	8,6.66,8	6,5,4	8,6,7	2,3,5	2,3,5	6,5,4	6,5.66,7	2,3,5	2,3,5
T5	1,1.66,5	1,2.33,4	2,3,5	2,3.66,5	2,3,5	8,6,7	8,6,7	6,5,4	2,3,5	1,1,2	8,6,7
T6	1,2.33,5	1,1.66,5	1,1,2	1,4.33,8	1,1,2	8,7,8	9,7,8	6,5,4	2,3,5	2,3,5	6,5.33,7

Step 3. Compute the normalized fuzzy decision matrix by using formulae.

$$r_{ij} = \left(\frac{a_{ij}}{c_j^+}, \frac{b_{ij}}{c_j^+}, \frac{c_{ij}}{c_j^+}\right) \text{ and } c_j^+ = \max_i\{c_{ij}\} \text{ (Beneficial criteria).}$$

$$r_{ij} = \left(\frac{a_j^-}{c_{ij}}, \frac{a_j^-}{b_{ij}}, \frac{a_j^-}{a_{ij}}\right) \text{ and } a_j^- = \min_i\{a_{ij}\} \text{ (Non beneficial criteria).}$$

	Bnf	Bnf	Bnf	Non Bnf	Bnf
Wght	**8,6,9**	**6,5,4**	**7,5,8**	**6,5,4**	**6,3,5**
T/C	**C1**	**C2**	**C3**	**C4**	**C5**
T1	1,0.79125,1	1,0.8325,1	1,0.375,0.875	0.125,0.1579,0.125	1,0.8325,1
T2	1,0.75,1	1,0.75,0.875	0.75,0.625,0.5	0.1428,0.1876,0.166	1,0.75,0.875
T3	0.75,0.375,0.5	0.125,0.291,0.625	0.75,0.625,0.5	0.2,0.33,0.5	0.25,0.375,0.625
T4	1,0.8325,1	0.75,0.626,0.5	1,0.8325,1	0.25,0.2,0.166	1,0.75,0.875
T5	0.125,0.2075,0.625	0.125,0.291,0.5	0.25,0.375,0.625	0.2,0.27,0.166	0.25,0.375,0.625
T6	0.125,0.291,0.625	0.125,0.2075,0.625	0.125,0.125,0.25	0.125,0.230,1	0.125,0.125,0.25
Non Bnf	**Non Bnf**	**Bnf**	**Bnf**	**Bnf**	**Bnf**
3,4,7	**4,5,6**	**9,7,9**	**9,6,8**	**6,5,7**	**7,6,8**
C6	**C7**	**C8**	**C9**	**C10**	**C11**
0.2,0.602,1	0.2,0.230,0.5	1.125,0.875,1	1,0.8325,1	1,0.791,1	0.8571,0.8085,1
0.2,0.2732,0.5	0.2,0.6024,1	0.75,0.666,0.875	0.75,0.666,0.875	0.75,0.7075,0.875	0.8571,0.8085,1
0.25,0.22,0.166	0.2,0.33,0.5	1.125,0.875,1	0.75,0.666,0.875	0.75,0.666,0.875	0.8571,0.7142,0.5715
0.2,0.33,0.5	0.2,0.33,0.5	0.75,0.625,0.5	0.75,0.7075,0.875	0.25,0.375,0.625	0.285,0.4285,0.7142
0.1425,0.166,0.12	0.1425,0.166,0.125	0.75,0.625,0.5	0.25,0.375,0.625	0.125,0.125,0.25	1.1428,0.8571,1
0.125,0.1425,0.125	0.12,0.14,0.111	0.75,0.625,0.5	0.25,0.375,0.625	0.25,0.375,0.625	0.8571,0.7614,1

Step 4. Calculate weighted normalized fuzzy decision matrix.

$$v_{ij} = r_{ij} \times w_j.$$

Such that $A_1 \otimes A_2 = (a_1,b_1,c_1) \otimes (a_2,b_2,c_2) = (a_1 \times a_2, b_1 \times b_2, c_1 \times c_2)$.

	Bnf	**Bnf**	**Bnf**	**Non Bnf**	**Bnf**
Wght	**8,6,9**	**6,5,4**	**7,5,8**	**6,5,4**	**6,3,5**
T/C	**C_1**	**C_2**	**C_3**	**C_4**	**C_5**
T1	*8,4.7475,9*	*6,4.1625,4*	*7,1.87,6.7*	*0.75,0.7895,0.5*	*6,2.49755,5*
T2	*8,4.5,9*	*6,3.75,3.5*	*5.25,3.125,4*	*0.8568,0.938,0.664*	*6,2.25,4.375*
T3	*6,2.25,4.5*	*0.75,1.4575,2.5*	*5.25,3.125,4*	*1.2,1.65,2*	*1.5,1.125,3.1285*
T4	*8,4.995,9*	*4.5,3.125,2*	*7,4.1625,8*	*1.5,1,0.64*	*6,2.25,4.125*
T5	*1,1.245,5.626*	*0.75,1.4550,2*	*1.75,1.875,5*	*1.2,1.35,2*	*1.5,1.125,3.125*
T6	*1,1.7475,5.625*	*0.75,1.037,2.5*	*0.875,0.625,2*	*0.75,1.15,4*	*0.75,0.375,1.25*
Non Bnf	**Non Bnf**	**Bnf**	**Bnf**	**Bnf**	**Bnf**
3,4,7	**4,5,6**	**9,7,9**	**9,6,8**	**6,5,7**	**7,6,8**
C_6	**C_7**	**C_8**	**C_9**	**C_{10}**	**C_{11}**
0.6,2.408,7	*0.8,1.15,3*	*10.125,6.125,9*	*9,4.995,8*	*6,3.955,7*	*5.99,4.851,8*
0.6,1.092,3.5	*0.8,3.012,6*	*6.75,4.662,7.875*	*6.75,3.996,7*	*4.5,3.5375,6.125*	*5.99,4.851,8*
0.75,0.88,1.162	*0.8,1.65,3*	*9,5.39,7.92*	*9,4.74,7*	*4.5,3.3,6.125*	*5.25,3.75,4*
0.6,1.32,3.5	*0.8,1.65,3*	*5.94,3.85,3.96*	*5.58,4.62,7*	*1.5,1.875,4.37*	*1.75,2.25,5*
0.42,0.64,0.84	*0.56,0.80,0.72*	*5.94,3.85,3.96*	*2.25,2.25,5*	*0.75,0.625,1.75*	*7,4.5,7*
0.36,0.57,0.84	*0.48,0.70,0.66*	*5.94,3.85,3.96*	*2.25,2.25,5*	*1.5,1.875,4.375*	*5.25,3.96,7*

Step 5. Compute the Fuzzy Positive Ideal Solution (FPIS) and Fuzzy Negative Ideal Solution (FNIS) by given formula.

$$A^{+} = (v_1^{+}, v_2^{+} \ldots\ldots v_n^{+}) \text{ where } v_j^{+} = {}^{\max}_{i}\{v_{ij3}\}.$$
$$A^{-} = (v_1^{-}, v_2^{-} \ldots\ldots v_n^{-}) \text{ where } v_j^{-} = {}^{\min}_{i}\{v_{ij1}\}.$$

	C_1	**C_2**	**C_3**	**C_4**	**C_5**	**C_6**	**C_7**	**C_8**	**C_9**	**C_{10}**	**C_{11}**
FPIS = A+	*8,4.995,9*	*6,4.1625,4*	*7,4.1625,8*	*0.75, 1.15, 4*	*6,2.49755,5*	*0.6,2.408,7*	*0.8,3.012,6*	*10.125, 6.125,9*	*9,4.995,8*	*6,3.955,7*	*5.99,4.851,8*
FNIS = A-	*1,1.245,5.626*	*0.75,1.037,2.5*	*0.875, 0.625, 2*	*0.75, 1.15, 4*	*0.75,0.375,1.25*	*0.36,0.57,0.84*	*0.48,0.70,0.66*	*5.94,3.85,3.96*	*2.25, 2.25, 5*	*0.75,0.625,1.75*	*1.75,2.25,5*

Step 6. Calculate the distance from each alternative to the FPIS and then FNIS by using distance formula as mentioned below.

$$d(x, y) = \sqrt{\frac{1}{3}[(a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2]}.$$

Distance from FPIS.

T/C	C_1	C_2	C_3	C_4	C_5
T1	*0.142894192*	*0*	*1.521573336*	*2.0314166*	*0*
T2	*0.285788383*	*0.37423533*	*2.5909397*	*1.967769485*	*0.388107051*
T3	*3.25499744*	*3.830831348*	*2.5909397*	*1.2232095*	*2.916048*
T4	*0*	*1.562733*	*0*	*1.9887454*	*0.505181*
T5	*4.981511685*	*3.711749374*	*3.73252176*	*1.1913192*	*2.924003035*
T6	*4.862676946*	*3.632320207*	*5.355045323*	*0*	*3.912826*
C_6	C_7	C_8	C_9	C_{10}	C_{11}
0	*2.03854883*	*0*	*0*	*0*	*0*
2.158846606	*0*	*2.220857117*	*1.53411223*	*1.031169441*	*0*
3.485184261	*1.902195574*	*0.995364255*	*0.595825758*	*1.071548723*	*2.433091381*
2.116108378	*1.902195574*	*4.003808603*	*2.068576403*	*3.24002572*	*3.353744226*
3.70152329	*3.308017735*	*4.003808603*	*4.549634601*	*4.698010217*	*0.84524178*
3.714002513	*3.364686414*	*4.003808603*	*4.549634601*	*3.24002572*	*0.883455526*

Distance from FNIS.

T/C	C_1	C_2	C_3	C_4	C_5
T1	*4.921090064*	*3.63232*	*4.514999077*	*2.0314166*	*3.92131859*
T2	*4.864116946*	*3.70463*	*3.12999*	*1.967769485*	*3.664666*
T3	*3.015399*	*0.241188*	*3.12999*	*1.2232095*	*1.245493*
T4	*4.98048*	*2.49481*	*5.355045323*	*1.9887454*	*3.6214*
T5	*0*	*0.376364*	*1.942133*	*1.1913192*	*1.243734*
T6	*0.29011*	*0*	*0*	*0*	*0*
C_6	C_7	C_8	C_9	C_{10}	C_{11}
3.724002513	*1.388104223*	*4.003808603*	*4.549634601*	*4.698010217*	*3.353744226*
2.424574464	*3.364686414*	*2.355325101*	*3.016538634*	*3.727634695*	*3.494042*
0.34248309	*1.46975*	*3.023066875*	*4.311326169*	*3.667821788*	*2.273030283*
1.60163582	*1.46975*	*0*	*2.627153085*	*1.731030522*	*0*
0.053229	*0.081649658*	*0*	*0*	*0*	*3.494042549*
0	*0*	*0*	*0*	*1.733553052*	*2.52811524*

Step 7. Calculate the di^+ and di^- by using the formula mentioned below.

$$di^+ = \sum_{j=1}^{n} d(v_{ij}, v_j^+) \quad di^- = \sum_{j=1}^{n} d(v_{ij}, v_j^-).$$

Step 8. Calculate the Closeness Coefficient (CCi) for each alternative.

$$CCi = di^- / (di^+ + di^-).$$

Teams	di+	di-	Cci	RANK
Quetta Gladiators	*5.734432958*	*40.73844871*	*0.8766069*	*1*
Islamabad United	*12.55182534*	*35.71397374*	*0.739943695*	*2*
Karachi Kings	*24.29923594*	*23.9427577*	*0.496305312*	*4*
Peshawar Zalmi	*20.7411183*	*25.87005015*	*0.555018272*	*3*
Multan Sultan	*37.64734128*	*8.382471407*	*0.182109614*	*5*
Lahore Qalandars	*37.51848185*	*4.551778292*	*0.108194679*	*6*

So T_1 (Quetta Gladiator) has the most chances to win PSL-2020.

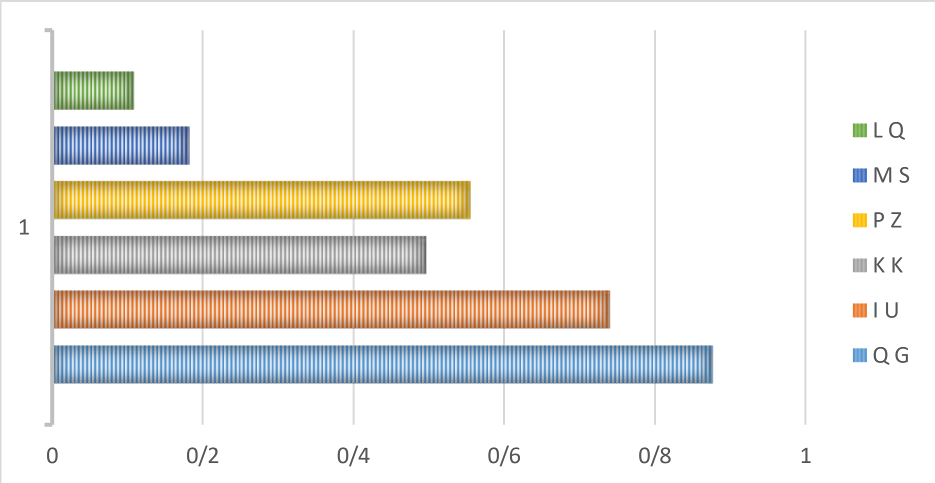

Fig. 4. *Graphical representation of the final results by FTOPSIS.*

4. Conclusion

The purpose of this research is to predict about the expected winner of PSL-5-2020.Although it's really a problematic task to predict about the game like this especially when we talk about limited overs cricket. Many of the factors including pitch ,weather conditions, availability of the key players directly impacts on the game, except of all that this research paper is based on the previous records of all the teams as well as availability of the key players for the upcoming edition of the PSL. So by collected records and mathematically point of view the results shows that team Islamabad United probably have the more chances to win the PSL-2020 by TOPSIS technique but fuzzy TOPSIS shows slightly different results with wining chances of another strongest and well deserved winning team Quetta Gladiators as well as the teams Multan Sultans and Lahore Qalandars have the lowest chances to win the title.

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Application of Fuzzy Logic in Portfolio Management: Evidence from Iranian Researches

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A B S T R A C T

Over the past decades, financial researchers have proposed different methods in portfolio selection, so that, Markwotiz [1] introduced risk and return criteria for a portfolio selection. Since it is difficult how to select an adequate stock portfolio, fuzzy models have been able to help researchers by considering uncertainty. In this research, we surveyed a portfolio management by reviewing the relevant literature of fuzzy model in financial management. The results showed that a fuzzy model can to determine an optimal portfolio.

1. Introduction

Portfolio is important because it is related to profitability and can lead to higher profits by providing a better model for portfolio selection. The first model for the portfolio problem was proposed by Markwotiz [1]. He stated that a rational investor not only focus on maximizing portfolio returns but also focuses on returns and risk [2]. Our decisions are made in conditions of uncertainty because investment environments are uncertain and financial markets are often accompanied by incomplete information. Fuzzy collections are one of the powerful tools to deal with uncertainly financial markets and predict investor's behavior. One of the most important features of these models, like humans, is the ability to intelligently design patterns to process qualitative information. These models, in fact, while creating flexibility in the model, take into account factors in the model such as knowledge, experience and human judgment and provide fully practical answers [3]. In recent years, with development of theories of using fuzzy logic in financial researches, we have seen the development of use of this model in domestic researches. Therefore, this article intends to refer to some of these researches in order to better understand the fuzzy models in investment decisions under portfolio management branch. Therefore, this article consists of three sections: Introduction to fuzzy logic, portfolio management, and the function of fuzzy model in portfolio management with research approach.

2. Fuzzy Logic

The theory of fuzzy sub-sets constitutes a very wide context in which to situate multivalent logic. Their origin can be found in the works, which in 1965 were developed by Zadeh [4], professor at the University of California, and today constitutes a mathematical theory constructed in all rigor that allows for the treatment of subjectivity and/or uncertainty. Its development has brought up an epistemological problem in the sense that is it better to use a certain model, which is unlikely to represent reality, or a fuzzy model that constitutes a valid reflection. In our understanding, it is necessary to observe economic and financial phenomena and determine their nature. It will be when they are presented in a fuzzy, vague manner, with limits that it will be necessary to use fuzzy mathematics. But we should not fall into the temptation of converting into fuzzy that which is not, but neither should we qualify as certain that which appears as fuzzy. Knowledge of the fact, persons and things is situated at different levels the specification of which is difficult. Between perfect knowledge of a phenomenon and total ignorance, knowledge that is more or less imprecise can be found.

3. Portfolio Management

Investment management includes two main topics: Securities analysis and portfolio management. Securities analysis includes to estimate each investment portfolio's benefits; while portfolio management includes to survey investments composition and investments management maintenance. Over the past decade, stock selection methods in investment topics have been shifted to portfolio management [5].

One of the most important issues in financial sciences is optimal portfolio selection, in which to achieve specific goals, are distributed a specific capital among the assets. In traditionally portfolio selection approach, when an investor invests in securities with the highest expected return that his aim to be obtain the highest expected return. But in 1952, this view was challenged by Markowitz [1]. In his view, it is irrational when an investor just pays attention to stock returns. Because in addition to maximizing stock return, the investor must be sure that it will be realize. On the other hand, if investors are only looking to maximize their returns, then they should only invest in a specific type of asset with the highest returns. According to Markowitz, investors should pay attention to both phenomena of risk and return at the same time. Accordingly, investors are faced two conflicting goals that they must balance them against each other [6]. In today's world, however, investment challenges are uncertainty in the future. What will be happened in the future about investment environment and what will be impacted these events? The traditional approach to uncertainty is to consider the return on assets as a stochastic factor. But this approach will impose unrealistic assumptions on optimal portfolio selection and will lead to many problems about finding the random distribution and related parameters. But we can done modeling more simply and efficiently this uncertainty by fuzzy logic. Experts' opinion is easily included in the model using fuzzy logic. Due to the fuzzy logic efficiency to take into account expert opinions and uncertainty in financial markets, one of the useful solutions for modeling return on assets in portfolio problem is using that approach. So, researchers have concluded that fuzzy logic can be useful, in which returns are considered as a fuzzy number. In next section, function of this model in portfolio management is discussed.

4. Fuzzy Logic and Portfolio Management

There is a lot of research on portfolio optimization. Given uncertainty in financial data, fuzzy logic tools help us to have a more accurate estimate in the future. Hence in our country as well, in this field various researches have been done that the results are remarkable. For example, Shams Lahroudi et al. [3] dealt with affecting factors to selection of optimal portfolio by integrated fuzzy MCDM techniques. Didehkhani [7] showed the multiobjective portfolio rebalancing model with fuzzy parameters is solved by fuzzy goal programming and a hybrid intelligent algorithm that combines fuzzy simulation with a genetic algorithm. The results in their paper demonstrated the effectiveness of the solution approach and efficiency of the model in practical applications of rebalancing an existing portfolio.

Khanjarpanah et al. [2] expressed portfolio selection always has been one of the interesting subjects in financial problems and markets. In this paper, the proposed model for evaluating, performance testing and logicality approved, is applied to some monthly return of company’s stock of Tehran Stock Exchange and the results is reported. The results showed that in lower values of confidence level in proposed portfolio problem, it's possible to obtain a higher profit with low risk.

Ghehi et al. [8] remarked the multi-period models running by MOPSO algorithm indicated for the models Mean-AVaR, Mean-Semi Entropy, and Mean-VaR, respectively, performed better, in terms of Sharpe and Treynor measures.

Jafaria and Dezfouli Khajehzadeh [6] explained investment portfolio selection as one of the most important issues raised in the area of financial engineering Mean-Variance model revolutionized portfolio selection problems. They showed a multi-objective portfolio selection model is considered including the uncertainty data. In particular, the aim of their paper presented a robust-fuzzy multi-objective model for portfolio selection. After presenting multi- objective optimization approach, robust optimization approach and fuzzy optimization approach, fuzzy-robust multi-objective model for portfolio selection is expressed. Finally, using real data to solve the proposed mode. Behnamian and Moshrefi [9] considered fuzzy concepts in the discussion of portfolio selection optimization in order to pursue this uncertainty. Then by using Bonison method, they determined priority and preference among the portfolios so that the investor can decide without confusion and finally through introducing combined metaheuristic algorithm for variable neighborhood search and genetics, can optimize the resulting model of previous process and comparing it with other solving algorithms. Fallahpoor et al. [5] showed that there is a significant difference between the mean of monthly sharpe ratio of 3, 5 and 50 shares portfolios obtained from the proposed model and Markowitz model. However, there is no a significant difference between the mean of monthly sharpe ratio of 10 shares portfolio obtained from the proposed model and Markowitz model. Nabizade and Behzadi [10] indicated that the proposed approach is well-suited, especially for portfolio models with higher moments. Conclusion: The findings showed that using entropy as a diversification index cannot cause any significant decrease in optimized values for other goals. Using Shannon entropy and Gini-Simpson entropy models can lead to an increase in return and Shannon entropy model can yield more diversification compared to Gini-Simpson entropy model Nabavi Chashmi and Yousefi Kerchangi [11] explained regarding to exerting the model in two unique investment companies during the years 2008 through 2009, research results represent that model application can provide a particular position for better and more precision adjustment recognition of investment companies portfolio in order for easier decision making of investors.

5. Conclusion

Portfolio selection is one of the fascinating issues in uncertainty planning. In recent years, we have seen studies in finance's field, in which extensive research has been done in portfolio selection and various methods have been presented for stock selection. One of these models has been using fuzzy approaches, which indicates its function in increasingly portfolio management efficiency. Therefore, it is expected that mutual funds managers will achieve to desirable return in the least risk and the shortest time by recognizing this application.

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A New Decision Making Approach for Winning Strategy Based on Muti Soft Set Logic

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ABSTRACT

We introduce a new concept of certainty and coverage of a parameter of the soft set and present a new decision making approach for the soft set over the universe using the certainty of a parameter. Also, we point out the drawbacks of the reduct definition by pointing out the delusion of Proposition 14 given by Herawan et al. [20] and provide the revised definition of the reduct of the multi soft set.

1. Introduction

Many practical problems involve data that contain uncertainties. These uncertainties may be dealt with existing theories such as fuzzy set theory [1] and rough set theory [2]. In 1999, Molodstov [3] pointed out the difficulties of these theories and he posited the concept of soft set theory. Maji et al. [4] made a theoretical study of soft set in 2003. Soft set theory has rich potential for applications. In [5], Maji et al. presented an application of soft sets in decision making problems and extend the concept into fuzzy soft sets in [6]. In the year 2010, Cagman et al. gave the $uni-int$ decision making algorithm using soft sets [7] and soft matrices [8]. Feng et al. [9] extended Cagman and Enginoglu's $uni-int$ decision making algorithm. They introduced several new soft decision making methods including $uni-int^k$, $uni-int_s^t$ and int^m-int^n decision making methods. Also, Han et al. [10] initiated the pruning method for solving int^m-int^n decision making method. Feng et al. [11] introduced the concept of discernibility matrix in 2014 and using this, they provided a decision making algorithm for soft sets. Also, in the same year, Dauda et al. [12] presented a decision making algorithm of soft sets using AND and OR operations. In 2020, Wang et al. [13] introduced a novel plausible reasoning based on intuitionistic fuzzy propositional logic and Meng et al. [14] proposed an inequality approach with the quasi-ordered set to evaluate the performances of decision making units. Recently many authors studied the concepts of decision making in terms of fuzzy set and intuitionistic fuzzy sets [15, 16, 17, 18]. In this paper, we define a new definition of certainty of a parameter of the soft set and with the help of this definition, we present a new decision making approach for the soft set over the universe which is a partition of objects. The standard soft set deals with a binary-valued information system. For a multi-valued information system, Herawan et al. [19] introduced a concept of multi soft sets in 2009. Also, they [20] introduced the reduct concept in multi soft sets using the value class of the multi soft matrix. In this paper, we point out the delusion of proposition 14 of [20] and provide the revised definition of the reduct of the multi soft set.

Definition 1. [20]. The idea of multi soft sets is based on a decomposition of a multi-valued information system $S=(U,A,V,f)$ into $|A|$ number of binary valued information systems $S=(U,A,V_{\{0,1\}},f)$ where $|A|$ denotes the cardinality of A. Consequently, the $|A|$ binary valued information systems, define multi soft sets $(F,A_m)=\{(F,a_i) \mid 1\le i\le |A|\}$.

Definition 2. [20]. Matrix $M_{a_i}, 1\le i\le |A|$ is called matrix representation of the soft set (F,a_i) over universe U. The dimension of matrices is defined by dim $(M_{a_i})=|U|\times|V_{a_i}|$. All entries of $M_{a_i}=[a_{ij}]$ is belong to a set $\{0,1\}$ where $a_{ij}=\begin{cases}0 & if\ |f(u,\alpha)|=0\\ 1 & if\ |f(u,\alpha)|=1\end{cases}$ where $1\le i\le |U|, 1\le j\le |V_{a_i}|, u\in U$ and $\alpha\in V_{a_i}$. The collection of all matrices representing (F,A_m) is denoted by M_A. That is, $M_A=\{M_{a_i} \mid 1\le i\le |A|\}$.

Definition 3. [20]. Let $M_{a_i}\in M_A$ be a matrix representation of a multi soft set (F,A_m) over U. The value class of M_{a_i}, that is, class of all value sets of M_{a_i}, denoted $C_{M_{a_i}}$ is defined by $C_{M_{a_i}}=\{\{u \mid |f(u,\alpha_1)|=1\},\ldots,\{u \mid |f(u,\alpha_{|V_{a_i}|})|=1\}\}, 1\le i\le |V_{a_i}|, u\in U$ and $\alpha\in V_{a_i}$. Clearly, $C_{M_{a_i}}\subseteq \wp(U)$.

Definition 4. [20]. Let $M_{a_i}=[a_{kl}], 1\le k\le |U|, 1\le l\le |V_{a_i}|$ and $M_{a_j}=[a_{mn}], 1\le m\le |U|, 1\le n\le |V_{a_j}|$ be two matrices in M_A. The AND operation between M_{a_i} and M_{a_j} is defined as M_{a_i} AND $M_{a_j}=M_{a_{ij}}=[a_{pq}]$ with dim($M_{a_{ij}}$)$=|U|\times(|V_{a_i}|\times|V_{a_j}|)$ where $a_{p1}=min\{a_{k1},a_{m1}\}, a_{p2}=min\{a_{k1},a_{m2}\},\ldots,a_{p(|V_{a_i}|\times|V_{a_j}|)}=min\{a_{k|V_{a_i}|},a_{m|V_{a_j}|}\}$.

Proposition 1. Let M_A be a multi soft matrix over U representing multi soft set (F,A_m). A set of attributes B of A is a reduct for A if only if $C_{\underset{b\in B}{AND} M_b}=C_{\underset{a\in A}{AND} M_a}$.

2. Soft Sets

In this section, we define a new definition for soft set over the universe.

Definition 5. Let (F,E) be a soft set over U. Suppose U is partitioned into t classes, namely, $U_1,U_2,\ldots,U_t$. If $U_1=\{u_1,u_2,\ldots,u_i\}, U_2=\{u_{i+1},u_{i+2},\ldots,u_j\},\ldots,U_t=\{u_{k+1},u_{k+2},\ldots,u_n\}$, then the soft set tabular representation of (F,E) as follows.

Table 1. *Tabular representation of soft set.*

U	e_1	e_2	…	e_m
u_1	$f(u_1,e_1)$	$f(u_1,e_2)$	…	$f(u_1,e_m)$
u_2	$f(u_2,e_1)$	$f(u_2,e_2)$	…	$f(u_2,e_m)$
…	…	…	…	…
u_i	$f(u_i,e_1)$	$f(u_i,e_2)$	…	$f(u_i,e_m)$
u_{i+1}	$f(u_{i+1},e_1)$	$f(u_{i+1},e_2)$	…	$f(u_{i+1},e_m)$
u_{i+2}	$f(u_{i+2},e_1)$	$f(u_{i+2},e_2)$	…	$f(u_{i+2},e_m)$
…	…	…	…	…
u_j	$f(u_j,e_1)$	$f(u_j,e_2)$	…	$f(u_j,e_m)$
…	…	…	…	…
u_{k+1}	$f(u_{k+1},e_1)$	$f(u_{k+1},e_2)$	…	$f(u_{k+1},e_m)$
u_{k+2}	$f(u_{k+2},e_1)$	$f(u_{k+2},e_2)$	…	$f(u_{k+2},e_m)$
…	…	…	…	…
u_n	$f(u_n,e_1)$	$f(u_n,e_2)$	…	$f(u_n,e_m)$

where $f(u_i,e_j)=1$ if $u_i \in F(e_j)$ and $f(u_i,e_j)=0$ otherwise.

Definition 6. Let (F,E) be a soft set over U with the partitions $U_1,U_2,\ldots,U_t$. Then we define the following.

- The support of $e \in E$ is defined as $supp(e)=\sum_{U_i \in U} supp_{U_i}(e)$ where $supp_{U_i}(e)=|\{u_j \in U_i \mid f(u_j,e)=1\}|$ and $supp(A_1,A_2)$ is the number of occurrences of A_2 with respect to the parameter A_1.
- The coverage of $A_1 \Rightarrow A_2$ is defined by $cov(A_1,A_2)=\dfrac{supp(A_1,A_2)}{|U|}$.
- The certainty of $A_1 \Rightarrow A_2$ is defined by $cer(A_1,A_2)=\sum_{U_i \in U} cer(U_i,A_1,A_2)$ where $cer(U_i,A_1,A_2)=\dfrac{supp_{U_i}(A_1,A_2)}{|U_i|}$.

Example 1. Consider the soft set (F,E) over the universe $U=\{U_1,U_2,U_3\}$ where $E=\{e_1,e_2,e_3,e_4,e_5\}, U_1=\{u_1,u_2,u_3\}, U_2=\{v_1,v_2,v_3,v_4\}$ and $U_3=\{w_1,w_2,w_3\}$ whose tabular representation is given below.

***Table 2.** Tabular representation of (F, E).*

U	e_1	e_2	e_3	e_4	e_5
u_1	1	0	1	0	0
u_2	0	1	1	0	1
u_3	0	0	1	1	0
v_1	1	0	1	0	1
v_2	0	1	0	1	1
v_3	1	0	0	0	0
v_4	0	1	1	1	0
w_1	1	0	1	0	1
w_2	0	0	1	1	1
w_3	1	0	0	0	1

Then $cer(U_1,e_1,1)=\frac{1}{3}=0.333$, $cer(U_1,e_1,0)=\frac{2}{3}=0.667$, $cer(U_2,e_1,1)=\frac{2}{4}=0.5$, $cer(U_2,e_1,0)=\frac{2}{4}$ $=0.5$, $cer(U_3,e_1,1)=\frac{2}{3}=0.667$, $cer(U_3,e_1,0)=\frac{1}{3}=0.333$. Hence $cer(e_1,1)=1.5$ and $cer(e_1,0)=1.5$. And $cer(U_1,e_2,1)=\frac{1}{3}=0.333$, $cer(U_1,e_2,0)=\frac{2}{3}=0.667$, $cer(U_2,e_2,1)=\frac{2}{4}=0.5$, $cer(U_2,e_2,0)=\frac{2}{4}=0.5$, $cer(U_3,e_2,1)=\frac{0}{3}=0$, $cer(U_3,e_2,0)=\frac{3}{3}=1$. Thus, we have $cer(e_2,1)=0.833$ and $cer(e_2,0)=2.167$. Also, $cer(U_1,e_3,1)=\frac{3}{3}=1$, $cer(U_1,e_3,0)=\frac{0}{3}=0$, $cer(U_2,e_3,1)=\frac{2}{4}=0.5$, $cer(U_2,e_3,0)=\frac{2}{4}=0.5$, $cer(U_3,e_3,1)=\frac{2}{3}=0.667$, $cer(U_3,e_3,0)=\frac{1}{3}=0.333$. Therefore, $cer(e_3,1)=2.167$ and $cer(e_3,0)=0.833$. Now, $cer(U_1,e_4,1)=\frac{1}{3}=0.333$, $cer(U_1,e_4,0)=\frac{2}{3}=0.667$, $cer(U_2,e_4,1)=\frac{2}{4}=0.5$, $cer(U_2,e_4,0)=\frac{2}{4}=0.5$, $cer(U_3,e_4,1)=\frac{1}{3}=0.333$, $cer(U_3,e_4,0)=\frac{2}{3}=0.667$. Hence $cer(e_4,1)=1.166$ and $cer(e_4,0)=1.834$. And $cer(U_1,e_5,1)=\frac{1}{3}=0.333$, $cer(U_1,e_5,0)=\frac{2}{3}=0.667$, $cer(U_2,e_5,1)=\frac{2}{4}=0.5$, $cer(U_2,e_5,0)=\frac{2}{4}=0.5$, $cer(U_3,e_5,1)=\frac{3}{3}=1$, $cer(U_3,e_5,0)=\frac{0}{3}=0$. Thus, $cer(e_5,1)=1.833$ and $cer(e_5,0)=1.167$.

Also, $cov(e_1,1)=\frac{5}{10}=0.5$ and $cov(e_1,0)=\frac{5}{10}=0.5$, $cov(e_2,1)=\frac{3}{10}=0.3$ and $cov(e_2,0)=\frac{7}{10}=0.7$, $cov(e_3,1)=\frac{7}{10}=0.7$ and $cov(e_3,0)=\frac{3}{10}=0.3$, $cov(e_4,1)=\frac{4}{10}=0.4$ and $cov(e_4,0)=\frac{6}{10}=0.6$, $cov(e_5,1)=\frac{6}{10}=0.6$ and $cov(e_5,0)=\frac{4}{10}=0.4$. Then the flow graph associated with certainty and coverage is given in the following *Fig. 1*.

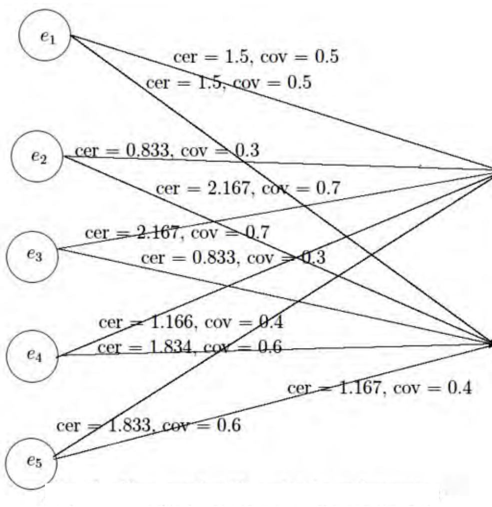

Fig. 1. *Flow graph with certainty and coverage.*

From the *Fig. 1*, the approximate graph for the flow graph is given as follows.

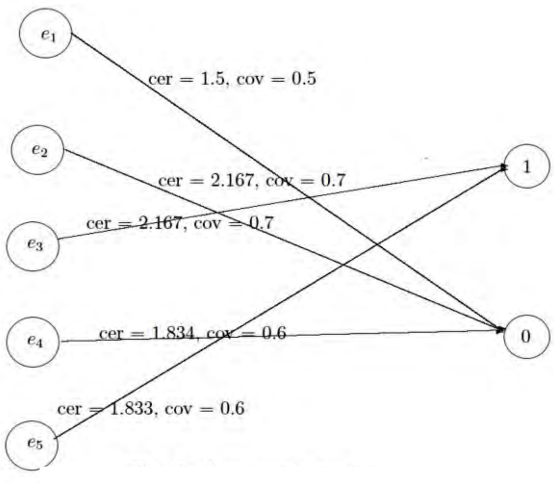

Fig. 2. *Approximate graph.*

3. Experimental Results

In this section, we illustrate the proposed approach through an example of a data set. Let (F,E) be a soft set over the universe $U=\{U_1,U_2,U_3,U_4\}$ and $E=\{e_1,e_2,e_3,e_4,e_5\}$ where e_1 stands for the Party 'A', e_2 stands for the Party 'B', e_3 stands for the Party 'C', e_4 stands for the Party 'D', e_5 stands for the Party 'E' and $U_1=\{u_1,u_2,u_3,u_4,u_5,u_6,u_7\}$ is the set of people who are politicians, $U_2=\{v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8,v_9,v_{10}\}$ is the set of formers, $U_3=\{w_1,w_2,w_3,w_4\}$ is the set of government employees and $U_4=\{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8\}$ is the set of students.

Table 3. *Tabular representation for the given soft set (F, E).*

U	e_1	e_2	e_3	e_4	e_5
u_1	0	1	1	0	0
u_2	1	1	0	0	0
u_3	0	1	0	1	0
u_4	1	0	1	1	0
u_5	0	1	0	1	1
u_6	1	1	1	0	0
u_7	0	1	0	1	0
v_1	1	0	1	1	0
v_2	0	1	1	0	0
v_3	1	0	1	0	1
v_4	1	1	0	1	0
v_5	0	0	1	0	1
v_6	1	0	1	1	1
v_7	1	1	0	0	0
v_8	0	1	0	1	0
v_9	1	0	0	1	0
v_{10}	0	0	1	0	0
w_1	0	1	1	0	0
w_2	1	0	1	0	0
w_3	1	1	1	0	0
w_4	0	1	0	1	0
x_1	1	0	0	1	0
x_2	1	1	0	1	0
x_3	1	0	1	0	0
x_4	1	0	0	1	0
x_5	1	1	0	0	0
x_6	1	1	1	0	0
x_7	1	0	1	1	0
x_8	1	1	1	1	0

Then the certainty and coverage of each party is given in the following *Table 4*.

Table 4. *Certainty and coverage of parties.*

E	Certainty	Coverage
$(e_1,1)$	2.52857	0.65518
$(e_1,0)$	1.47143	0.34483
$(e_2,1)$	2.50714	0.58621
$(e_2,0)$	1.49286	0.41379
$(e_3,1)$	2.27857	0.55173
$(e_3,0)$	1.72143	0.44827
$(e_4,1)$	1.94643	0.51723
$(e_4,0)$	2.05357	0.48276
$(e_5,1)$	0.44286	0.13793
$(e_5,0)$	3.55714	0.86207

From *Fig. 3*, branches of the flow graph represent the parties together with their certainty and coverage factors. For instance, the $(e_1,1)$ has the certainty factor 2.52857 and coverage factor 0.65518.

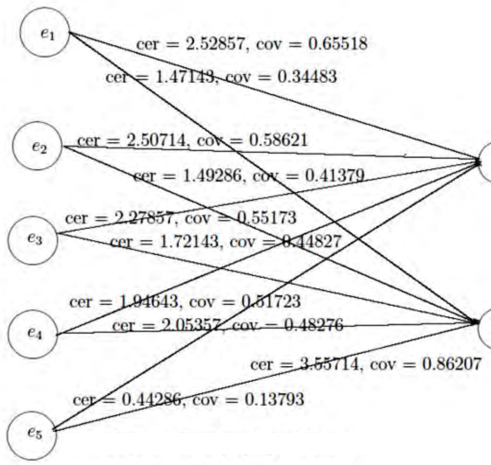

Fig. 3. *Flow graph for Table 4.*

The flow graph gives a clear insight into the winning strategy of all parties. We can replace flow graph shown in Figure by “approximate" flow graph shown in *Fig. 4*. From the *Fig. 4*, we can conclude that the Parties A,B and C are the winning parameters whose the coverage of 0.65518, 0.58621 and 0.55173, respectively.

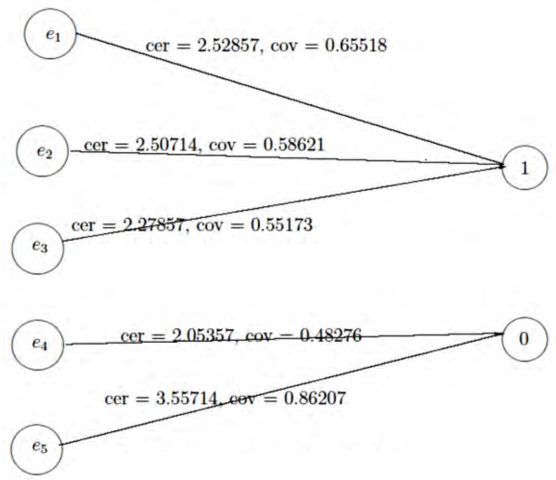

Fig. 4. *Approximate flow graph.*

4. Erratum

In [10], the authors defined the reduct of a multi soft set and gave a characterization for reduct of a multi soft set. The following Example shows that reduct of a multi soft set under the *Proposition 1* need not be unique.

Example 2. Consider the multi-valued information system given in Example 15 of [10]. Then the matrices representing the multi soft set (F,A_m) is $M_A=\{M_{a_1},M_{a_2},M_{a_3},M_{a_4}\}$ where

$$M_{a_1}=\begin{bmatrix}1&0\\1&0\\0&1\\0&1\\1&0\end{bmatrix},M_{a_2}=\begin{bmatrix}1&0\\0&1\\0&1\\0&1\\0&1\end{bmatrix},M_{a_3}=\begin{bmatrix}1&0\\1&0\\1&0\\1&0\\0&1\end{bmatrix},M_{a_4}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&1\\0&1&0\end{bmatrix}.$$

Here M_{a_3} AND $M_{a_4}=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&1&0&0&0\\0&0&0&0&1&0\end{bmatrix}$ and $C_{\underset{1\le j\le 4}{AND}\,M_{a_j}}=\{\{1\},\{2\},\{3,4\},\{5\}\}$.

Now, M_{a_2} AND $(M_{a_3}$ AND $M_{a_4})=\begin{bmatrix}1&0\\0&1\\0&1\\0&1\\0&1\end{bmatrix}\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&1&0&0&0\\0&0&0&0&1&0\end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Therefore, $C_{M_{a_2}\, AND\, M_{a_3}\, AND\, M_{a_4}} = \{\{1\},\{2\},\{3,4\},\{5\}\}$.

$$\text{Also, } M_{a_1} \text{ AND } (M_{a_3} \text{ AND } M_{a_4}) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, $C_{M_{a_1}\, AND\, M_{a_3}\, AND\, M_{a_4}} = \{\{1\},\{2\},\{3,4\},\{5\}\}$. Hence $\{a_1, a_3, a_4\}, \{a_2, a_3, a_4\}$ are reductions of (F, A_m). Also, in [10], the authors determined the reductions of (F, A_m) and gave the reductions as $\{a_1, a_2, a_3\}$ and $\{a_3, a_4\}$. That is, the reductions are not unique.

Based on the *Example 2*, we have the following Lemma.

Lemma 1. Suppose E_1 and E_2 are two members of the value class of $\underset{1 \le i \le n}{AND} M_{a_i}$. Then for any $e_1 \in E_1, e_2 \in E_2, \{e_1, e_2\}$ will not be a subset of any value class of $\underset{1 \le i \le n+1}{AND} M_{a_i}$.

Proof. Suppose $e_1 \in E_1$ and $e_2 \in E_2$. Since $E_1 \neq E_2, e_{1i} = 1$ and $e_{2j} = 1$ for some i and j. Suppose $\{e_1, e_2\} \subseteq F$ where F is a value class of $\underset{1 \le i \le n+1}{AND} M_{a_i}$. Then $e_{1k} = e_{2k} = 1$ for some k. By the definition of "AND" product, $e_{1l} = e_{2l} = 1$ in the matrix $\underset{1 \le i \le n}{AND} M_{a_i}$. That is, e_1 and e_2 belong to the same value class of $\underset{1 \le i \le n}{AND} M_{a_i}$ which is a contradiction.

The following Theorem shows that superset of a reduct set is again a reduct set.

Theorem 1. If B is a reduction of the multi soft set (F, A_m), then $B \cup C$ is also a reduction of (F, A_m) where $C \subseteq A - B$.

Proof. Suppose B is a reduction of (F, A_m). Then $C_{\underset{a_i \in B}{AND} M_{a_i}} = C_{\underset{a_i \in A}{AND} M_{a_i}}$. That is, the number of value classes of $\underset{a_i \in B}{AND} M_{a_i}$ and the number of value classes of $\underset{a_i \in A}{AND} M_{a_i}$ are equal. Also, by the definition of

"AND" product, the number of value classes of $\underset{1\le i\le n}{AND}\, M_{a_i}$ is less than or equal to the number of value classes of $\underset{1\le i\le n+1}{AND}\, M_{a_i}$. Therefore, by *Lemma 1*, for any $C\subseteq A-B$, the value classes of $\underset{a_i\in B}{AND}\, M_{a_i}$ and the value classes of $\underset{a_i\in B\cup C}{AND}\, M_{a_i}$ are equal. Hence $C_{\underset{a_i\in B}{AND} M_{a_i}} = C_{\underset{a_i\in B\cup C}{AND} M_{a_i}}$.

By the above Theorem, whenever B is a reduction of the multi soft set (F,A_m), $B\cup C$ is also a reduction of (F,A_m). In Example 15 of [10], the authors gave reduction of the multi soft set as $\{a_1,a_3,a_4\},\{a_2,a_3,a_4\},\{a_1,a_2,a_3\}$ and $\{a_3,a_4\}$. But by *Theorem 1*, any set containing a reduction set is a reduct set and hence the whole set A is a reduct set. Thus, we remove the redundancy, we modify the reduct definition for multi soft set as follows.

Definition 6. Let M_A be a multi soft set over U representing multi soft set (F,A_m). A set of attributes B of A is a reduct for A if B is a minimal subset of A such that $C_{\underset{b\in B}{AND} M_b} = C_{\underset{a\in A}{AND} M_a}$.

Example 3. Consider the multi soft set as in *Example 4.1*. Here $\{a_1,a_2,a_3\}$ and $\{a_3,a_4\}$ are reductions of (F,A_m) but $\{a_1,a_3,a_4\}$ and $\{a_2,a_3,a_4\}$ are not reductions of (F,A_m).

5. Conclusion

In this paper, we have presented a new decision making approach for the soft set over the universe with partition of objects using the certainty and coverage of a parameter. Also, we have pointed out the misconception of the reduct definition given by Herawan et al. [20].

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Medical Diagnostic Analysis on Some Selected Patients Based on Modified Thao et al.'s Correlation Coefficient of Intuitionistic Fuzzy Sets via an Algorithmic Approach

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A B S T R A C T

The concept of correlation coefficient of intuitionistic fuzzy sets is a reliable tool in information theory with numerous applications in diverse areas. Correlation coefficients of intuitionistic fuzzy sets have been studied through two-way approach by many researchers. This approach inappropriately discarded the hesitation margins of the concerned intuitionistic fuzzy sets, which makes the results of such experiments unreliable. In this paper, we modified the correlation coefficient of intuitionistic fuzzy sets of Thao et al. [36] in a three-way approach by including the hesitation margins in the computational process to enhance reliable output through an algorithmic method. We show that the modified correlation coefficient of intuitionistic fuzzy sets is more reasonable with precise outputs than correlation coefficient method. In terms of application, we demonstrate an analysis of medical diagnosis on some selected patients via an algorithm of the novel approach coded with JAVA programming language.

1. Introduction

Uncertainties are huge barrier to reckon with in decision-making processes because many real-life problems are enmeshed with indecisions. The invention of fuzzy sets technology by Zadeh [1] brought an amazing sight of relief to decision-makers, because of the ability of fuzzy model to curb the embedded uncertainties in decision-making. Some decision-making problems could not be properly resolved with fuzzy approach because fuzzy set only considered membership grade whereas, many real-life problems have the component of both membership grade and non-membership grade with the possibility of hesitation. However, with the invention of Intuitionistic Fuzzy Sets (IFSs) [2, 3], such cases can best be addressed. IFS consists of membership grade λ, non-membership grade υ and hesitation margin ϑ whereby their sum is one and $\lambda + \upsilon$ is less than or equal to one. IFS is a special case of fuzzy set with additional conditions and thus has more facility to curb uncertainties more appropriate

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with higher degree of precision. The concept of IFSs has found massive applications via measuring tools in myriad areas, namely; medical diagnosis as reported in [3-10], pattern recognition as found in [11, 12, 13], career determination [14, 15, 16], and group decision-making [17] to mention but a few.

Correlation coefficient proposed by Karl Pearson in 1895 is a vital tool for measuring similarity, interdependency and interrelationship between two variable or data. Statisticians found solace in the instrumentality of correlation analysis because of its vast application potentials. Also, some allied professions like engineering, sciences, among others have applied correlation analysis to resolve their peculiar problems. With the advent of fuzzy sets, some researchers have extended correlation analysis to fuzzy environment to handle fuzzy data [18, 19, 20]. In the same vein, correlation coefficient has been encapsulated in intuitionistic fuzzy domain and used to solve several Multi-Criteria Decision-Making (MCDM) issues [21-26].

The pioneer work on correlation coefficient between IFSs was carried out by Gerstenkorn and Manko [27] by using correlation and informational energies. Hung [28] used statistical approach to study correlation coefficient of IFSs by capturing only the membership and non-membership grades of IFSs. Correlation coefficient of IFSs was proposed based on centroid method in [29]. Park et al. [30] and Szmidt and Kacprzyk [31] improved the approach in [28] by including the hesitation margin of IFS. Liu et al. [32] introduced a new approach of computing correlation coefficient of IFSs with application. Garg and Kumar [33] proposed a novel method of correlation coefficient of IFSs based on set pair analysis and applied the method to solve some MCDM problems. The concept of correlation coefficient and its applications have been stretched to complex intuitionistic fuzzy and intuitionistic multiplicative environments [34, 35]. TOPSIS method based on correlation coefficient was proposed in [36] to solve decision-making problems with intuitionistic fuzzy soft set information. Thao et al. [36] proposed a new method of calculating correlation coefficient of IFSs using mean, variance and covariance with applications. The limitation of this approach is the omission of hesitation margin without minding its influence in the computational output. Also, this approach does not considered time factor in the computation since it was carried out manually with high possibility of errors. Although we cannot doubt the significant of similarity and distance measures as soft computing tools, but the penchant for correlation coefficient measure in information measure theory is because correlation coefficient measure considers both similarity (which is the dual of distance) and interrelationship indexes of IFSs.

The limitations of correlation coefficient measure of Thao et al. [36] motivated us to propose a new technique of estimating correlation coefficient between IFSs by incorporating hesitation margin to the approach in [36], to enhance accuracy and limiting information leakages. The new approach is studied from an algorithmic perspective to enable it to be coded with JAVA programming language and thus, reducing time of computation. The objectives of the work are to: Reiterate the correlation coefficient method in [36] to enable the introduction of a new correlation coefficient method with accuracy and reliability; mathematically justify the new method in corroborating to the axiomatic conditions for correlation coefficient methods, and shows its advantages over the correlation coefficient method in [36]; establish the application of the modified method in medical diagnostic analysis on some selected patients via an algorithmic approach coded with JAVA programming language. The rest of the article is delineated as follow; Section 2 discusses the fundamentals of IFSs and the correlation coefficient of IFSs according to Thao et al. [36]. Section 3 presents the modification of method of measuring correlation coefficient of Thao et al. [36] with some theoretical results and numerical verification. Section 4 demonstrates the application of the modified approach in medical diagnostic analysis on some selected patients via an algorithmic approach coded with JAVA programming language. Section 5 concludes the article with some possible research extensions.

2. Preliminaries

In this section, we present some basic concepts of IFSs and Thao et al.'s correlation coefficient measure of IFSs.

2.1. Concepts of Intuitionistic Fuzzy Sets

Take ℶ to be an intuitionistic fuzzy space defined in a non-empty set *X*.

Definition 1. [2]. Suppose we have an IFS $P \subseteq ℶ$. Then we define the construct *P* by

$$P = \{\langle \frac{\lambda_P(x), \upsilon_P(x)}{x} \rangle \mid x \in X\} \tag{1}$$

where the functions $\lambda_P(x), \upsilon_P(x) : X \rightarrow [0,1]$ define grades of membership and non-membership of $x \in X$ in which

$$0 \leq \lambda_P(x) + \upsilon_P(x) \leq 1. \tag{2}$$

For any IFS *P* in *X*, $\vartheta_P(x) = 1 - \lambda_P(x) - \upsilon_P(x)$ is the IFS index or hesitation margin of *P*.

Definition 2. [38]. Assume $P, Q \subseteq ℶ$, then

(i) $P = Q$ iff $\lambda P(x) = \lambda Q(x)$ and $\upsilon P(x) = \upsilon Q(x)\ \forall\ x \in X$.
(ii) $P \subseteq Q$ iff $\lambda_P(x) \leq \lambda_Q(x)$ and $\upsilon_P(x) \geq \upsilon_Q(x)\ \forall\ x \in X$.
(iii) $\overline{P} = \{\langle \frac{\upsilon_P(x), \lambda_P(x)}{x} \rangle | x \in X\}$
(iv) $P \cup Q = \{\langle \max(\frac{\lambda_P(x), \lambda_Q(x)}{x}), \min(\frac{\upsilon_P(x), \upsilon_Q(x)}{x}) \rangle | x \in X\}$
(v) $P \cap Q = \{\langle \min(\frac{\lambda_P(x), \lambda_Q(x)}{x}), \max(\frac{\upsilon_P(x), \upsilon_Q(x)}{x}) \rangle | x \in X\}$

Definition 3. [6]. Intuitionistic Fuzzy Values (IFVs) or Intuitionistic Fuzzy Pairs (IFPs) are characterized by the form $< x, y >$ such that $x + y \leq 1$ where $x,y \in [0,1]$. IFVs evaluate the IFS for which the components (*x* and *y*) are interpreted as grades of membership and non-membership.

2.2. Correlation Coefficient of Intuitionistic Fuzzy Sets

The concept of correlation coefficient measures the linear relationship between any two arbitrary IFSs. The correlation coefficient indicates positive sign when two intuitionistic fuzzy data sets are directly related, and a negative sign when two intuitionistic fuzzy data sets are inversely related. But whenever the correlation coefficient is zero, it indicates there is no linear relationship, neither positive nor negative. We recall the axiomatic definition of correlation measure of IFSs.

Definition 4. [27]. Suppose $P,Q \subseteq ℶ$ and $X = \{x_1,\ldots,x_n\}$ for $n \in]1,\infty[$. Then the correlation coefficient of *P* and *Q* denoted by $\sigma(P,Q)$ satisfies:

- $\sigma(P,Q) = \sigma(Q,P)$.
- $\sigma(P,Q) = 1$ implies $P = Q$.

– (iii) $-1 \leq \sigma(P,Q) \leq 1$.

2.2.1. Thao et al.'s correlation coefficient of intuitionistic fuzzy sets

Definition 5. [37]. The correlation coefficient $\sigma(P,Q)$ is given by

$$\sigma(P,Q) = \frac{\phi(P,Q)}{\sqrt{\psi(P)\psi(Q)}} \tag{3}$$

where $\psi(P)$, $\psi(Q)$ are the variances of P and Q defined by

$$\left.\begin{aligned} \psi(P) &= \frac{1}{n-1}\Sigma_{i=1}^{n}((\lambda_P(x_i) - \overline{\lambda_P})^2 + (\upsilon_P(x_i) - \overline{\upsilon_P})^2) \\ \psi(Q) &= \frac{1}{n-1}\Sigma_{i=1}^{n}((\lambda_Q(x_i) - \overline{\lambda_Q})^2 + (\upsilon_Q(x_i) - \overline{\upsilon_Q})^2) \end{aligned}\right\} \tag{4}$$

$\varphi(P,Q)$ is the covariance of (P,Q) defined by

$$\phi(P,Q) = \frac{1}{n-1}\Sigma_{i=1}^{n}((\lambda_P(x_i) - \overline{\lambda_P})(\lambda_Q(x_i) - \overline{\lambda_Q}) + (\upsilon_P(x_i) - \overline{\upsilon_P})(\upsilon_Q(x_i) - \overline{\upsilon_Q})), \tag{5}$$

for the means

$$\left.\begin{aligned} \overline{\lambda_P}, \overline{\lambda_Q} &= \frac{\Sigma_{i=1}^{n}\lambda_P(x_i)}{n}, \frac{\Sigma_{i=1}^{n}\lambda_Q(x_i)}{n} \\ \overline{\upsilon_P}, \overline{\upsilon_Q} &= \frac{\Sigma_{i=1}^{n}\upsilon_P(x_i)}{n}, \frac{\Sigma_{i=1}^{n}\upsilon_Q(x_i)}{n} \end{aligned}\right\} \tag{6}$$

3. Modified Thao et al.'s Correlation Coefficient of Intuitionistic Fuzzy Sets

In Thao et al.'s correlation coefficient of IFSs, the effect of hesitation margins in the computational procedure is not considered which will of necessity leads to an inaccurate results because hesitation margin is one of the three fundamental parameters of IFS. To remedy this setback, we modified Thao et al.'s correlation coefficient of IFSs by incorporating hesitation margins in the computational procedure.

Definition 6. With the same hypothesis in *Definition 4*, the variances of P and Q are defined by

$$\left.\begin{aligned} \hat{\psi}(P)&\frac{1}{n-1}\Sigma_{i=1}^{n}((\lambda_P(x_i) - \overline{\lambda_P})^2 + (\upsilon_P(x_i) - \overline{\upsilon_P})^2 + (\vartheta_P(x_i) - \overline{\vartheta_P})^2) \\ \hat{\psi}(Q) &= \frac{1}{n-1}\Sigma_{i=1}^{n}((\lambda_Q(x_i) - \overline{\lambda_Q})^2 + (\upsilon_Q(x_i) - \overline{\upsilon_Q})^2 + (\vartheta_Q(x_i) - \overline{\vartheta_Q})^2) \end{aligned}\right\}, \tag{7}$$

and the covariance of (P,Q) is defined by

$$\hat{\phi}(P,Q) = \frac{1}{n-1}\Sigma_{i=1}^{n}\Big(\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2\Big) \tag{8}$$

for

$$\lambda_P(x_i) - \overline{\lambda_P} = \alpha_1,\ \upsilon_P(x_i) - \overline{\upsilon_P} = \beta_1,\ \vartheta_P(x_i) - \overline{\vartheta_P} = \gamma_1$$

$$\lambda_Q(x_i) - \overline{\lambda_Q} = \alpha_2,\ \upsilon_Q(x_i) - \overline{\upsilon_Q} = \beta_2,\ \vartheta_Q(x_i) - \overline{\vartheta_Q} = \gamma_2$$

where the means of *P* and *Q* are defined by

$$\left.\begin{aligned}\overline{\lambda_P}, \overline{\lambda_Q} &= \frac{\Sigma_{i=1}^n \lambda_P(x_i)}{n}, \frac{\Sigma_{i=1}^n \lambda_Q(x_i)}{n}\\ \overline{\upsilon_P}, \overline{\upsilon_Q} &= \frac{\Sigma_{i=1}^n \upsilon_P(x_i)}{n}, \frac{\Sigma_{i=1}^n \upsilon_Q(x_i)}{n}\\ \overline{\vartheta_P}, \overline{\vartheta Q} &= \frac{\Sigma_{i=1}^n \vartheta_P(x_i)}{n}, \frac{\Sigma_{i=1}^n \vartheta_Q(x_i)}{n}\end{aligned}\right\} \quad (9)$$

Definition 7. The modified correlation coefficient $\sigma\hat{}(P,Q)$ is given by

$$\hat{\sigma}(P,Q) = \frac{\hat{\phi}(P,Q)}{\sqrt{\hat{\psi}(P)\hat{\psi}(Q)}} \quad (10)$$

where the components are defined in *Definition 6*.

Certainly, $\hat{\psi}(P) = \hat{\phi}(P,P)$ and $\hat{\psi}(Q) = \hat{\phi}(Q,Q)$. It worthy to note that *Eq. (10)* is more reliable than *Eq. (3)* because it considers grades of membership, non-membership and hesitation margin of the considered IFSs.

Theorem 1. The function $\sigma\hat{}(P,Q)$ is a correlation coefficient of IFSs *P* and *Q* contain in =.

Proof. We show that $\hat{\sigma}(P,Q) = \hat{\sigma}(Q,P)$, $\hat{\sigma}(P,Q) = 1$ implies $P = Q$ and $-1 \leq \hat{\sigma}(P,Q) \leq 1$. But,

$\hat{\sigma}(P,Q) = \hat{\sigma}(Q,P)$ because

$$\begin{aligned}\hat{\sigma}(P,Q) &= \frac{\hat{\phi}(P,Q)}{\sqrt{\hat{\phi}(P,P)\hat{\phi}(Q,Q)}} = \frac{\hat{\phi}(Q,P)}{\sqrt{\hat{\phi}(Q,Q)\hat{\phi}(P,P)}}\\ &= \hat{\sigma}(Q,P).\end{aligned}$$

Suppose $\hat{\sigma}(P,Q) = 1$, then we have

$$\hat{\sigma}(P,Q) = \frac{\hat{\phi}(P,Q)}{\sqrt{\hat{\phi}(P,P)\hat{\phi}(Q,Q)}} = \frac{\hat{\phi}(P,P)}{\hat{\phi}(P,P)} = 1$$

Hence, $P = Q$.

Again, it is certain that $\hat{\sigma}(P,Q) \geq -1$ because $\hat{\phi}(P,P)$ and $\hat{\phi}(Q,Q)$ are non-negative and $\hat{\phi}(P,Q) \geq -1$. Now, we prove that $\hat{\sigma}(P,Q) \leq 1$ as follows:

$$
\begin{aligned}
\hat{\sigma}(P,Q) &= \frac{\phi(\hat{P},Q)}{\sqrt{\phi(\hat{P},P)\phi(\hat{Q},Q)}} \\
&= \frac{\frac{1}{n-1}\Sigma_{i=1}^{n}\Big(\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2\Big)}{\sqrt{\frac{1}{n-1}\Sigma_{i=1}^{n}\Big(\alpha_1^2+\beta_1^2+\gamma_1^2\Big)\frac{1}{n-1}\Sigma_{i=1}^{n}\Big(\alpha_2^2+\beta_2^2+\gamma_2^2\Big)}} \\
&= \frac{\Sigma_{i=1}^{n}\Big(\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2\Big)}{\sqrt{\Sigma_{i=1}^{n}\Big(\alpha_1^2+\beta_1^2+\gamma_1^2\Big)\Sigma_{i=1}^{n}\Big(\alpha_2^2+\beta_2^2+\gamma_2^2\Big)}} \\
&= \frac{\Sigma_{i=1}^{n}\alpha_1\alpha_2+\Sigma_{i=1}^{n}\beta_1\beta_2+\Sigma_{i=1}^{n}\gamma_1\gamma_2}{\sqrt{\Sigma_{i=1}^{n}\Big(\alpha_1^2+\beta_1^2+\gamma_1^2\Big)\Sigma_{i=1}^{n}\Big(\alpha_2^2+\beta_2^2+\gamma_2^2\Big)}} \\
&\leq \frac{\sqrt{\Sigma_{i=1}^{n}\alpha_1^2\Sigma_{i=1}^{n}\alpha_2^2}+\sqrt{\Sigma_{i=1}^{n}\beta_1^2\Sigma_{i=1}^{n}\beta_2^2}+\sqrt{\Sigma_{i=1}^{n}\gamma_1^2\Sigma_{i=1}^{n}\gamma_2^2}}{\sqrt{\Sigma_{i=1}^{n}\Big(\alpha_1^2+\beta_1^2+\gamma_1^2\Big)\Sigma_{i=1}^{n}\Big(\alpha_2^2+\beta_2^2+\gamma_2^2\Big)}}
\end{aligned}
$$

Assume that

$$\Delta_1=\sum_{i=1}^{n}\alpha_1^2,\ \ \Delta_2=\sum_{i=1}^{n}\alpha_2^2,$$
$$\Pi_1=\sum_{i=1}^{n}\beta_1^2,\ \Pi_2=\sum_{i=1}^{n}\beta_2^2,$$

$$\Omega_1=\sum_{i=1}^{n}\gamma_1^2,\ \Omega_2=\sum_{i=1}^{n}\gamma_2^2.$$

Then

$$\hat{\sigma}(P,Q) \leq \frac{\sqrt{\Delta_1\Delta_2}+\sqrt{\Pi_1\Pi_2}+\sqrt{\Omega_1\Omega_2}}{\sqrt{\Big(\Delta_1+\Pi_1+\Omega_1\Big)\Big(\Delta_2+\Pi_2+\Omega_2\Big)}}$$

Consequently,

$$
\begin{aligned}
\hat{\sigma}^2(P,Q) &\leq \frac{\Big(\sqrt{\Delta_1\Delta_2}+\sqrt{\Pi_1\Pi_2}+\sqrt{\Omega_1\Omega_2}\Big)^2}{\Big(\Delta_1+\Pi_1+\Omega_1\Big)\Big(\Delta_2+\Pi_2+\Omega_2\Big)} \\
&= \frac{\Delta_1\Delta_2+\Pi_1\Pi_2+\Omega_1\Omega_2+2\Big(\sqrt{\Delta_1\Delta_2\Pi_1\Pi_2}+\sqrt{\Delta_1\Delta_2\Omega_1\Omega_2}+\sqrt{\Pi_1\Pi_2\Omega_1\Omega_2}\Big)}{\Big(\Delta_1+\Pi_1+\Omega_1\Big)\Big(\Delta_2+\Pi_2+\Omega_2\Big)}
\end{aligned}
$$

But

$$
\begin{aligned}
\hat{\sigma}^2(P,Q)-1 &= \frac{2\Big(\sqrt{\Delta_1\Delta_2\Pi_1\Pi_2}+\sqrt{\Delta_1\Delta_2\Omega_1\Omega_2}+\sqrt{\Pi_1\Pi_2\Omega_1\Omega_2}\Big)-\Big(\Delta_1(\Pi_2+\Omega_2)+\Pi_1(\Delta_2+\Omega_2)+\Omega_1(\Delta_2+\Pi_2)\Big)}{\Big(\Delta_1+\Pi_1+\Omega_1\Big)\Big(\Delta_2+\Pi_2+\Omega_2\Big)} \\
&= -\frac{\Big(\Delta_1(\Pi_2+\Omega_2)+\Pi_1(\Delta_2+\Omega_2)+\Omega_1(\Delta_2+\Pi_2)\Big)-2\Big(\sqrt{\Delta_1\Delta_2\Pi_1\Pi_2}+\sqrt{\Delta_1\Delta_2\Omega_1\Omega_2}+\sqrt{\Pi_1\Pi_2\Omega_1\Omega_2}\Big)}{\Big(\Delta_1+\Pi_1+\Omega_1\Big)\Big(\Delta_2+\Pi_2+\Omega_2\Big)} \\
&\leq 0.
\end{aligned}
$$

Thus, $\hat{\sigma}^2(P,Q) \leq 1$ implies $\hat{\sigma}(P,Q) \leq 1$. Hence, $-1 \leq \hat{\sigma}(P,Q) \leq 1$. Therefore, $\hat{\sigma}(P,Q)$ is a correlation coefficient of P and Q.

3.1. Numerical Verifications

We experiment the reliability of the Thao et al.'s approach and its modified version with some numerical examples.

3.1.1. Example I

Assume there are two IFSs defined in $X = \{x_1, x_2, x_3, x_4, x_5\}$ by

$$P_1 = \{\langle\frac{0.8, 0.1, 0.1}{x_1}\rangle, \langle\frac{0.6, 0.1, 0.3}{x_2}\rangle, \langle\frac{0.2, 0.8, 0.0}{x_3}\rangle, \langle\frac{0.6, 0.1, 0.3}{x_4}\rangle, \langle\frac{0.1, 0.6, 0.3}{x_5}\rangle\}$$
$$P_2 = \{\langle\frac{0.4, 0.0, 0.6}{x_1}\rangle, \langle\frac{0.3, 0.5, 0.2}{x_2}\rangle, \langle\frac{0.1, 0.7, 0.2}{x_3}\rangle, \langle\frac{0.4, 0.3, 0.3}{x_4}\rangle, \langle\frac{0.1, 0.7, 0.2}{x_5}\rangle\}$$

By using the Thao et al.'s approach, we have $\sigma\,(P1,P2) = 0.2095$. With the modified version,

$\hat{\sigma}\,(P1, P2) = 0.1631$. Thao et al.'s approach yields a better correlation index. Certainly, this "so called" advantage cannot be relied upon because Thao et al.'s approach do not take account of the hesitation margins.

3.1.2. Example II

Suppose we have two IFSs defined in $X = \{x_1, x_2, x_3, x_4, x_5\}$ by

$$Q_1 = \{\langle\frac{0.6, 0.1, 0.3}{x_1}\rangle, \langle\frac{0.5, 0.4, 0.1}{x_2}\rangle, \langle\frac{0.3, 0.4, 0.3}{x_3}\rangle, \langle\frac{0.7, 0.2, 0.1}{x_4}\rangle, \langle\frac{0.3, 0.4, 0.3}{x_5}\rangle\}$$
$$Q_2 = \{\langle\frac{0.1, 0.8, 0.1}{x_1}\rangle, \langle\frac{0.0, 0.8, 0.2}{x_2}\rangle, \langle\frac{0.2, 0.8, 0.0}{x_3}\rangle, \langle\frac{0.2, 0.8, 0.0}{x_4}\rangle, \langle\frac{0.8, 0.1, 0.1}{x_5}\rangle\}$$

Computing the correlation coefficient with Thao et al.'s approach, we have $\sigma\,(Q1,Q2) = -0.3794$. With the modified version, we obtain $\hat{\sigma}(Q1,Q2) = -0.3325$. Here, the modified version shows a better prospect of precision although both approaches indicate negative linear relationship.

It is worthy to note that, both approaches satisfied the conditions in *Definition 4*. In summary, the modified version of Thao et al.'s approach is more reliable, it losses no information due to omission and thus, has a precise output because it incorporates the orthodox parameters of IFSs unlike Thao et al.'s initiative.

4. Medical Diagnostic Analysis of Some Selected Patients

Medical diagnosis or diagnosis is the process of deciding which illness or disease describes a patient's signs and symptoms. The information necessary for diagnosis is usually collected from a history and frequently, physical examination of the patient seeking medical attention. Over and over again, one or more diagnosis processes, like medical tests, are also conducted during the procedure.

Diagnosis is time and again thought-provoking, because many signs and symptoms are uncertain. For example, headache by itself, is a sign of numerous diseases and thus does not show the physician what the patient is suffering from. Consequently differential diagnosis, in which some possible explanations are juxtaposed, must be performed, which could be best done by intuitionistic fuzzy approach. This involves correlation of many pieces of information followed by the recognition of patterns via correlation coefficient measures. In fact, the process of medical diagnosis is more challenging when a patient is showing symptoms of some closely related diseases, which also posed a problem to therapeutic process.

4.1. Hypothetical Experiment of Medical Diagnosis

In this section, we present an application of modified Thao et al.'s correlation coefficient to medical diagnostic analysis. In a given hypothetical diagnostic process, assume S is a set of symptoms, P is a set of patients, and D is a set of diseases. Now, we discuss the notion of intuitionistic fuzzy medical diagnosis via the following procedure viz; the determination of symptoms, the formulation of medical knowledge in the intuitionistic fuzzy domain, and the determination of diagnosis based on the greatest correlation coefficient value of the correlation coefficient of patients and diseases.

4.1.1. Example of medical diagnosis

Suppose we have four patients viz; Joe, Lil, Tony, and Tom who visit a medical facility for medical diagnosis. They are observed to possess the following symptoms; temperature, headache, stomach pain, cough, and chest pain. Mathematically, the set of the patients represented by P is $P = \{P_1, P_2, P_3, P_4\}$, where P_1= Joe, P_2 = Lil, P_3= Tony, P_4= Tom, and the set of symptoms S is $S = \{s_1, s_2, s_3, s_4, s_5\}$, in which s_1 = temperature, s_2= headache, s_3 = stomach pain, s_4 = cough, and s_5 = chest pain.

The patients P_i, $i = 1,2,3,4$ are observed to be showing symptoms of the diseases D_j, $j = 1,2,3,4,5$, given as $D = \{D_1,D_2,D_3,D_4,D_5\}$, where D_1 = viral fever, D_2 = malaria, D_3 = typhoid, D_4 = stomach problem, and D_5 = chest problem.

The intuitionistic fuzzy medical representations of the diseases based on medical knowledge of the diseases are given in *Table 1*. The intuitionistic fuzzy medical representations of the patients after medical examinations are presented hypothetically, in *Table 2*. Both the intuitionistic fuzzy medical representations of the diseases and the intuitionistic fuzzy medical representations of the patients are taken from [7].

Table 1. *Intuitionistic fuzzy medical representations I.*

Feature Space Diseases	s1	s2	s3	s4	s5
$\lambda D1$	0.4	0.3	0.1	0.4	0.1
$\upsilon D1$	0.0	0.5	0.7	0.3	0.7
$\vartheta D1$	0.6	0.2	0.2	0.3	0.2
$\lambda D2$	0.7	0.2	0.0	0.7	0.1
$\upsilon D2$	0.0	0.6	0.9	0.0	0.8
$\vartheta D2$	0.3	0.2	0.1	0.3	0.1
$\lambda D3$	0.3	0.6	0.2	0.2	0.1
$\upsilon D3$	0.3	0.1	0.7	0.6	0.9
$\vartheta D3$	0.4	0.3	0.1	0.2	0.0
$\lambda D4$	0.1	0.2	0.8	0.2	0.2
$\upsilon D4$	0.7	0.4	0.0	0.7	0.7
$\vartheta D4$	0.2	0.4	0.2	0.1	0.1
$\lambda D5$	0.1	0.0	0.2	0.2	0.8
$\upsilon D5$	0.8	0.8	0.8	0.8	0.1
$\vartheta D5$	0.1	0.2	0.0	0.0	0.1

Table 2. *Intuitionistic fuzzy medical representations II.*

Feature space Patients	s1	s2	s3	s4	s5
$\lambda P1$	0.8	0.6	0.2	0.6	0.1
$\upsilon P1$	0.1	0.1	0.8	0.1	0.6
$\vartheta P1$	0.1	0.3	0.0	0.3	0.3
$\lambda P2$	0.0	0.4	0.6	0.1	0.1
$\upsilon P2$	0.8	0.4	0.1	0.7	0.8
$\vartheta P2$	0.2	0.2	0.3	0.2	0.1
$\lambda P3$	0.8	0.8	0.0	0.2	0.0
$\upsilon P3$	0.1	0.1	0.6	0.7	0.5
$\vartheta P3$	0.1	0.1	0.4	0.1	0.5
$\lambda P4$	0.6	0.5	0.3	0.7	0.3
$\upsilon P4$	0.1	0.4	0.4	0.2	0.4
$\vartheta P4$	0.3	0.1	0.3	0.1	0.3

4.1.2. Algorithm of modified Thao et al.'s correlation coefficient

The algorithm for computing the correlation coefficient between the patients P_i and the diseases D_j using *Eq. (10)* is given as follows.

PRE. lambdaPi[si] is Membership Degrees (MDs) of Patients (Pi), upsilonPi[si] is Non-Membership Degrees (NMDs) of Pi, varthetaPi[si] is Hesitation Margins (HMs) of Pi where $i = 1,\ldots,4$; lambdaDi[si] is MDs of Diseases (Dj), upsilonDi[si] is NMDs of Dj, varthetaDj[si] is HMs of Dj where $j = 1,\ldots,5$; $S = \{s_1, s_2, s_3, s_4, s_5\}$, $n = 5$ is the number of feature space.

POST. This algorithm finds the correlation coefficients between *Pi* and *Dj*.

STEPS.

i: Set the value for n
ii: Initialize values for Pi and Dj
iii: Repeat for s = 1 to n
Set sumlambdaPi[si] = sumlambdaPi[si] + Pi[si]
Set sumupsilonPi[si] = sumupsilonPi[si] + Pi[si]
Set sumvarthetaPi[si] = sumvarthetaPi[si] + Pi[si]
End for
iv: Set lambdaPiBar = sumlambdaPi[si]/n; Set upsilonPiBar = sumupsilonPi[si]/n; Set varthetaPiBar = sumvarthetaPi[si]/n
v: Repeat for s = 1 to n
Set sumlambdaDj[si] = sumlambdaDj[si] + Dj[si]
Set sumupsilonDj[si] = sumupsilonDj[si] + Dj[si]
Set sumvarthetaDj[si] = sumvarthetaDj[si] + Dj[si]
End for
vi: Set lambdaDjBar = sumlambdaDj[si]/n; Set upsilonDjBar = sumupsilonDj[si]/n; Set varthetaDjBar = sumvarthetaDj[si]/n
vii: Repeat for s= 1 to n
Set templepi = ((lambdaPi[si]-lambdaPiBar)*(lambdaPi[si]-lambdaPiBar) + (upsilonPi[si]-upsilonPiBar)*(upsilonPi[si]-upsilonPiBar) + (varthetaPi[si]-varthetaPiBar)*(varthetaPi[si]-varthetaPiBar))
Set templedj = ((lambdaDj[si]-lambdaDjBar)*(lambdaDj[si]-lambdaDjBar) + (upsilonDj[si]-upsilonDjBar)*(upsilonDj[si]-upsilonDjBar) + (varthetaDj[si]-varthetaDjBar)*(varthetaDj[si]-varthetaDjBar))
Set templepidj = ((lambdaPi[si]-lambdaPiBar)*(lambdaDj[si]-lambdaDjBar) + (upsilonPi[si]-upsilonPiBar)*(upsilonDj[si]-upsilonDjBar) + (varthetaPi[si]varthetaPiBar)*(varthetaDj[si]-varthetaDjBar))
End for
viii: Set phiPiPi = (1/n-1) * templepi
ix: Set phiDjDj = (1/n-1) * templedj
x: Set phiPiDj = (1/n-1) * templepidj
xi: Set sigmaPiDj = phiPiDj/(sqrt(phiPiPi*phiDjDj))
xii: Exit.

4.1.3. Medical diagnostic results and discussions

After coding the algorithm for computing the correlation coefficient between patients P_i and diseases D_j via JAVA programming language, we obtain the result in *Table 3*.

Table 3. *Results for medical diagnosis.*

Diagnosis	Viral Fever	Malaria	Typhoid Fever	Stomach Problem	Chest Problem
Joe	*0.1631*	*0.1572*	*0.1886*	*0.0952*	*−0.3989*
Lil	*0.1573*	*0.1399*	*0.2099*	*0.1029*	*−0.2932*
Tony	*0.0717*	*0.0791*	*0.0586*	*0.0322*	*−0.2544*
Tom	*0.1232*	*0.1226*	*0.1327*	*0.0680*	*−0.3325*

From the results above, we obtain the following diagnoses: Joe is diagnosed of typhoid fever with some elements of viral fever and malaria. Lil is diagnosed of typhoid fever and should also be treated for viral fever. Tony has a very negligible symptoms of malaria and viral fever because of the values of the correlation coefficient. In fact, Tony is "near healthy". Finally, Tom has a mere symptoms of typhoid fever, viral fever and malaria; not a sever case at all.

We observe that none of the patients show positive for chest problem. The patients show positive for stomach problem in a very negligible stages. With these diagnoses, a physician can easily prescribe drugs for the patients because the diagnoses show degrees of severity and thus, minimize the possibility of wrong/unnecessary therapies.

5. Conclusion

In this paper, we have successfully modified the Thao et al.'s method of calculating correlation coefficient because of its limitation. The modified version of Thao et al.'s method remedied the limitation because it incorporated the impact of the hesitation margins of the intuitionistic fuzzy pairs in the computations. We showed that the new method satisfied the axiomatic description of correlation coefficient of IFSs. In addition, we integrated the new method in an algorithm for easy coding to enhance accuracy and ease of computations. We experimented the applicability of the new method with medical diagnosis conducted hypothetically on some patients and obtained their respective diagnoses with regard to the values of correlation coefficient between each patients and each diseases. Nonetheless, this approach could be extended to cluster algorithm with applications in future research.

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Computer and Fuzzy Theory Application: Review in Home Appliances

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PAPER INFO	ABSTRACT
Chronicle: *Received: 07 January 2020* *Revised: 14 March 2020* *Accepted: 04 June 2020* **Keywords:** *Fuzzy Theory.* *Fuzzy Control.* *Fuzzy Set.*	Clays have a tendency to this article first introduces the basic concepts of fuzzy theory, including comparisons between fuzzy sets and traditional explicit sets, fuzzy sets basic operations such as the membership function of the set and the colloquial variable, the intersection and union of the fuzzy set, and use the above concepts to guide into the four basic reasoning mechanisms of fuzzy mode and introduce several common types of fuzzy application examples such as fuzzy washing machine and fuzzy control of incinerator plant in China illustrate the application of fuzzy theory in real society.

1. Introduction

In 1965, Zadeh of the University of Califonia at Berkeley in "Information and Control". In this academic journal, published a "fuzzy set" (Fuzzy Sets) Thesis, the fuzzy theory was born. In this paper, professor Zadeh puts "high temperature", "giant man", "big sets that cannot be clearly defined, such as "number", are based on a new set theory. Representation, called fuzzy set. He specifically pointed out: The fuzzy collection pole suitable for abstract things, such as image recognition, information transmission, etc. These basic behaviors of human thinking that are difficult to express mathematically plus to quantify, and in the form of mathematical theory, to develop these situations. When Professor Zadeh published fuzzy set theory, the reaction of the academic community extremely indifferent, with many criticisms. He was once a "modern cybernetic a member of this rigorous theory has changed 180 degrees to advocate vague concepts, and this has aroused fierce criticism from everyone. Fuzzy theory has been despised since the beginning, but since 1974 British Mamdani announced the application of fuzzy logic to control small steam engines. In 1982, Denmark Ostergaard announced the successful operation of fuzzy logic after being built as a cement kiln factory, the practical potential of fuzzy theory was wide attention. In recent years, the blur has been like a whirlwind, in Europe and China. There are huge

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manpower in various parts of Japan and Japan. Research boom. Especially Japan, even more with its strong corporate economic to cooperate with the industrial R&D technology that no one can match, and develop vigorously blur product.

Today, as small as the camera' s autofocus device, washing the water flow controller of the clothes machine and the temperature adjustment of the air conditioner are as large as water treatment plant raw water treatment program, subway automatic driving system, the fuzzy trails can be seen everywhere, making the word fuzzy almost synonymous with smart technology. In this article, fuzzy logic develop basic theory, extend its theory to fuzzy control system, and take the fuzzy control system of automatic fuzzy washing machine and incinerator as the example is a simple and clear introduction. There are many applications, for example, see Faber and Stewart [1], Lugeri et al. [4], Markus et al. [5], and Pandey et al. [6]. Other studies have focused on disaster detection and early warning systems [2, 3, 7].

2. The Basic Concept

The so-called set is composed of some things with common properties. The quality of the organization can be used to summarize a group with the same characteristics tools to sign things. Generally speaking, collection is to express clear things mainly, in order to distinguish, it is customary to use a clear collection (crisp set). Explicit sets have the following common properties: The elements in the set are determined. The elements of the same set have certain identical properties. A whole composed of collective elements, the elements can be distinguished from each other do not. When using a collection to represent a concept, always consider the object restricted to a specific range, this range is called domain (universal of discourse, $U x$). Set A in the universe $U x$ has the following two basic representation methods: Enumeration method. If each element in the set can be listed one by one is enumerated, the set can be represented by enumeration. This method can only be used for a limited set of elements. For example Set A of elements, respectively, A1, A2, ..., An, can be expressed as: Descriptive method. If a set defines its elements according to specific properties at that time, this method is called descriptive method.

Common to this method the general formula is as follows: According to this symbol, A is $U x$ medium-oriented proposition (proposition) $P (x)$ is the set of all elements of truth. For an explicit set A in the universe of $U x$, its element x the relationship with the set can use the characteristic function (characteristic function) to illustrate, its definition is as follows: Definition 1. One of the explicit set A in the universe $U x$, its characteristic function [mu] A to the $U x$ of the element x is mapped to a set {0,1} being. Also which is or the above introduced the affiliation between elements and sets. In the set there is also the same definition of affiliation between combination and set, for example: Subset, equal set, etc. About collection the basic operation of, generally refers to intersection, union set, difference set, complement set. It exhibits many properties, such as commutativity, associativity, distributivity, transitivity, and exclusive neutrality law of excluded-middle, etc.

3. Fuzzy Set and Membership Function

Zadeh formally proposed the fuzzy set theory in 1965. Mold the biggest difference between fuzzy sets and explicit sets is that Zadeh proposed replace explicit sets with membership functions the characteristic function in the union. The membership function changes the original non-zero or 1 characteristic the eigen function value is expanded to a real number between 0 and 1. And the set defined by the membership function is called the fuzzy set. For a fuzzy set A in the universe of $U x$, generally as the following definition means: Definition 2. In a fuzzy set A in the universe U x , its membership function several [mu] A to the *U x* map elements to a range of [0,1] in the real number. That is μ A *(x): U x →[0, 1]* or *A (x): U x →[0, 1]* the above two expressions often appear in the literature, in this article the latter is used as the representation of fuzzy sets. In other words, mod the fuzzy set and its membership function are both represented by the same symbol. But use it is written as A (x) when representing the membership function, and there is a specific x value *(x *)* substitute the membership function value into the membership function, then write into *A (x *)*. In practical applications, the membership function is operated on site personnel, cooperating with experts in the application field, generally the membership function most commonly used in fuzzy logic is segment continuous.

4. The Application of Fuzzy Logic in Home Appliances

Take fuzzy washing machine as an example under the rapidly changing social structure, even the functions of home appliances the orientation has changed a lot. Take washing machines as an example, because small families the emergence of the structure, the proportion of professional women the work is therefore the exclusive work of the housewife. Work shared by the whole family and the flexibility of washing time is improved however, there may be quite extreme differences in the amount of laundry. In this way under the premise, the fuzzy control washing machine set the following development goals and technical issues: When the experienced housewife washes the clothes, computerization of the most appropriate washing method used for total quantity and material. In order to match the operation and use of the whole family, all operations it is best to use a single button, that is, the washer is responsible for pressing the start move the button, the rest are judged by the fuzzy system of the washing machine off.

Fuzzy fully automatic washing machine can be measured by enough sensors the amount of clothing and the quality of the clothing, and can be appropriately blurred control rule base to work out a good washing water volume and washing time, to make the most appropriate control. In order to allow all users to understand the current fuzzy laundry the operating status of the machine, for all expected and completed laundry the process must be clearly displayed to users. The fuzzy washing machine is based on the sophisticated family the housewife considers the problems when washing clothes, such as "do not hurt the cloth material", "strong cleaning power", "shorten washing time", etc. In addition to the theme of cleaning, it also achieves a balance with button operation the effect of control. Fuzzy control provides sensory information and develops sensors for the amount of clothes and fabrics. Based on this principle, developed the most moderate water flow intensity and washing time control technology.

5. Practical Method of Fuzzy Control

In order to make the product practical, the fuzzy control is put into the CPU in the 4 bit CPU, many functions including LCD display. It can realize fuzzy control of saving memory and shortening calculation time. For simplify the sequence and output results from input information to fuzzy control output single, so the possible output combination is fuzzy and real the test results are categorized to divide the range of output combinations and follow 19 practical ways to leave good fuzzy control. It is developed to control the most appropriate water flow, washing time, and dehydration time fuzzy fully automatic washing machine. The application examples of fuzzy automatic washing machines are described above. Needle for problems that are difficult to control by computers until today, fuzzy control can explain the scope of machinery equipment and process control in general industries in the domain, the realization, and home.

6. The Application of Fuzzy Logic in the Manufacturing Industry

Take incinerator control as an example in order to reduce the environmental pollution caused by the landfill method, disposal personnel have racked their brains to design appropriate treatment methods. In addition to the recycling of resources to reduce waste, the incineration method has become the main garbage disposal method adopted by many countries, because this incinerator has also become a very important equipment. But this is equipment that reduces environmental pollution, but it may be exhaust gas exceeds the standard value, which is harmful to the environment. Although discharged gas can use subsequent equipment to reduce pollution, but this is not only time-consuming and also increases the processing cost. So how to control the incinerator can fully burn the contents when burning waste. It has become an important subject of incinerator control. However, because every batch of waste incinerated in the incinerator has different physical and chemical properties, and there is no way before incineration effective screening, so a set of appropriate mathematical models cannot be established. The overall structure of the incinerator. To describe the incineration process. Therefore, the control encountered in the incineration process the control problem is not easy to solve with traditional control methods. And so, when the incinerator burns incompletely, the operator will often the previous combustion situation and the situation of the incinerator contents determine the future control action. In this case, it can simulate the control actions of human experts fuzzy controller may be more capable than traditional mode control strategy give full play to the effectiveness of its control. Collaboration between scholars in Korea and Samsung Heavy industries, we will try to achieve incinerator control with a fuzzy control architecture aims. The overall structure of the incinerator. The goal of its control is to achieve complete combustion while ensuring keep the evaporation rate and processing energy within the target area. This control system the system is based on the fuzzy controller architecture mentioned in the general literature, then make amendments to meet the needs of incinerator control.

As a whole said, this fuzzy control system is divided into three parts in total. First of all, in addition to the use of general meters to measure the incinerator. In addition to the numerical values of the combustion parameters, the more special part is the designer’ s adapt to the characteristics of the incinerator, and especially use the so-called fuzzy measuring device (fuzzy sensor) to capture what cannot be measured by the meter but is the data required for fuzzy control can be used as input variables of the fuzzy controller, the source of the information. Simply put, the so-called fuzzy sensor to collect some data that can be measured by the instrument through fuzzy collection, the calculation of another group of indicators cannot be measured with equipment. The average operator is judging when cutting off the combustion status of the incinerator, it may be the thickness of the waste, the calorific value of the incineration system, the nature of the waste, and the factors such as burning conditions are used as a reference for control and adjustment. But yes for fuzzy controllers, it is really difficult to describe the "burning condition" like this an abstract concept gives a clear definition. So in this system it is convenient to use the pressure drop of the incinerator, the switch state of the feed inlet, combustion bed length, evaporation rate, dry bed length, oxidation carbon concentration, oxygen concentration, etc. Can be measured by measuring equipment to estimate the size of the above indicators. The output of the entire fuzzy measure the relationship between input and output.

7. Fuzzy Decision Maker

Among the fuzzy control systems of the incinerator, the most special one is the so-called fuzzy decision maker (fuzzy decision maker). Actual mode the fuzzy decision maker is the set point of some parameters in the fuzzy controller decision mechanism. In simple terms, the fuzzy decision maker will the nature of furnaces, transmissions and other equipment and control target setting values take them into consideration in order to determine the setting value of the controller. For example in other words, the temperature setting of the combustion air is based on the current temperature and the nature of the waste measured by the fuzzy measurer decided. The decision process is based on the experience of operating experts so that the temperature of the incinerator can be maintained at a target area. Such decision rules generally have the following description: "When discarded, when the nature of the material is not good, if the temperature of the combustion air increases, the temperature of the chemical furnace will tend to stabilize." Another parameter determined by the fuzzy decision maker is the evaporation rate. Rate set point. In the past traditional control, in order to maintain evaporation rate in a certain target area, generally used to regulate waste feed the amount to achieve the goal. But because the ingredients of the feed are always there change, so it is difficult to get the desired effect. In this system, in order to obtain the appropriate evaporation rate set point, the waste and the difference between the current evaporation rate and the current set point, to calculate the next set point.

In this fuzzy controller, mainly use the parameter settings calculated by the aforementioned fuzzy decision maker point, compared with the data currently measured by the fuzzy measurer comparison, using fuzzy control rules constructed by expert knowledge reasoning, and finally get the control actions of each operation. In the output part of this fuzzy controller, you can subdivided into steam calorific value, feed port switch, fuel switch, feed rate, fuel rate, throttle angle, etc. And each output variable points have not different input parts. Overall, this multiple-input multiple-output fuzzy control can be regarded as the multiple input single output fuzzy controller combination. For example, for the waste and fuel inlet (feeder and stoker ON/OFF) of the switch control part, the input is the difference between the current evaporation rate and its set point, and the rate of change of the difference. The content of its control rules may be as follows narrative: "If the evaporation rate is high, at the same time the evaporation rate will increase the higher the trend, the waste inlet should be opened," or "if the evaporation rate is low, at the same time the evaporation rate tends to be lower and lower when the situation occurs, the waste inlet should be closed. The content of these rules it is accumulated by the experience of on-site control personnel.

8. Extraction of Control Rules

According to the operation of the aforementioned fuzzy controller components look, we can find that the most difficult part is probably the control rules the establishment. For a complex and variable system like an incinerator that said, the establishment of a rule base is particularly difficult. On-site control personnel are average only know the procedures of its on-site operation, but often cannot the accumulated experience is transformed into effective fuzzy control rules. The other party in the same situation, the on-site operators often there are different control strategies. These factors have increased the fuzzy control the complexity of establishing the rules of the device. In constructing this incinerator model when pasting the control system, the designer and multiple on-site operators staff have had many interviews and summed up their control experience reorganize, get rid of the chaos in addition, designers also use the program reaction of the chemical furnace program under normal feeding operation is used as the reference base point.

In addition, when engaging in computer simulations, they also refer to actual control status of the incinerator. Finally, the designers also incinerate the past traditional control strategy used by the furnace is integrated into the rules of the fuzzy controller in the library, to enrich the performance of the controller. In summary, the fuzzy control system of the incinerator is mainly three parts: Fuzzy measurer, fuzzy decision maker and fuzzy controller composition. This control system will be able to be measured by the actual measuring device data, through the fuzzy measurer, is converted to the operator to perform the operating state that is inferred but cannot be measured with instruments. Then profit use the fuzzy decision maker to estimate the set value of each variable, so you can let the control gas setting achieve the purpose of automatic adjustment. And these settings the fixed point is compared with the current value obtained by the measuring instrument, and the Paste the controller to infer the control action of each operating variable to complete into the entire control loop. The above control in the computer simulation, the strategy can be adapted to different incinerator conditions. Enough to get satisfactory control results in a short time.

9. Conclusions and Suggestions

This article is based on basic fuzzy logic theory, a series of discussed fuzzy sets, fuzzy patterns and fuzzy reasoning mechanisms, and use examples in the home appliance industry and factories for verification. But, whether applying fuzzy logic to fuzzy model modeling and fuzzy control above, the integration and implementation of expert knowledge in the entire application of fuzzy theory is very important, and the measurement values obtained by various types of sensors accuracy is the key to whether fuzzy theory can implement expert knowledge factor.

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The Picture Fuzzy Distance Measure in Controlling Network Power Consumption

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ABSTRACT

In order to solve the complex decision-making problems, there are many approaches and systems based on the fuzzy theory were proposed. In 1998, Smarandache [10] introduced the concept of single-valued neutrosophic set as a complete development of fuzzy theory. In this paper, we research on the distance measure between single-valued neutrosophic sets based on the H-max measure of Ngan et al. [8]. The proposed measure is also a distance measure between picture fuzzy sets which was introduced by Cuong in 2013 [15]. Based on the proposed measure, an Adaptive Neuro Picture Fuzzy Inference System (ANPFIS) is built and applied to the decision making for the link states in interconnection networks. In experimental evaluation on the real datasets taken from the UPV (Universitat Politècnica de València) university, the performance of the proposed model is better than that of the related fuzzy methods.

1. Introduction

The fuzzy theory was introduced the first time in 1965 by Zadeh [1]. A fuzzy set is determined by a membership function limited to [0, 1]. Until now, there is a giant research construction of fuzzy theory as well as its application. The fuzzy set is used in pattern recognition, artificial intelligent, decision making, or

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data mining [2, 3], and so on. Besides that, the expansion of fuzzy theory is also an interesting topic. The interval-valued fuzzy set [4], the type-2 fuzzy set [5], and the intuitionistic fuzzy set [6] are all developed from the fuzzy set. They replaced the value type or added the other evaluation to the fuzzy set in order to overcome the inadequate simple approach of this traditional fuzzy set. Such as in 1986, the intuitionistic fuzzy set of Atanassov [6] builds up the concept of the non-membership degree. This supplement gives more accurate results in pattern recognition, medical diagnosis and decision making [7-9], and so on. In 1998, Smarandache [10] introduced neutrosophic set to generalize intuitionistic fuzzy set by three independent components. Until today, many subclasses of neutrosophic sets were studied such as complex neutrosophic sets [11, 12]. As a particular case of standard neutrosophic sets [13, 14], the picture fuzzy set introduced in 2013 by Cuong [15], considered as a complete development of the fuzzy theory, allows an element to belong to it with three corresponding degrees where all of these degrees and their sum are limited to [0, 1]. Concerning extended fuzzy set, some recent publications may be mentioned here as in [16-20].

As one of the important pieces of set theory, distance measure between the sets is a tool for evaluating different or similar levels between them. Some literature on the application of intuitionistic fuzzy measure from 2012 to present can be found in [7, 21-23]. In 2018, Wei introduced the generalized Dice similarity measures for picture fuzzy sets [17]. However, the definition of Wei is without considering the condition related to order relation on picture fuzzy sets. In a decision-making model, a distance measure can be used to compare the similarities between the sets of attributes of the samples and that of the input, such as in predicting dental diseases from images [24]. In this paper, we define the concept of the single-valued neutrosophic distance measure, picture fuzzy distance measure, and represent the specific measure formula. We prove the characteristics of this formula as well as the relation among it and some of the other operators of picture fuzzy sets. The proposed distance measure is inspired by the H-max distance measure of intuitionistic fuzzy sets [8]. Hence, it inherits the advantage of the cross-evaluation in the H-max and moreover it has the completeness of picture fuzzy environment.

The decision-making problems appear in most areas aiming to provide the optimal solution. Saving interconnection network power is always interested, researched and becoming more and more urgent in the current technological era. In 2010, Alonso et al. introduced the power saving mechanism in regular interconnection network [25]. This decision-making model dynamically increases or reduces the number of links based on a thresholds policy. In 2015, they continue to study power consumption control in fat-tree interconnection networks based on the static and dynamic thresholds policies [26]. In general, these threshold policies are rough and hard because they are without any fuzzy approaches, parameter learning and optimizing processes. In 2017, Phan et al. [27] proposed a new method in power consumption estimation of network-on-chip based on fuzzy logic. However, this fuzzy logic system based on Sugeno model [27] is too rudimentary and the parameters here are chosen according to the authors' quantification.

In this paper, aiming to replace the above threshold policy, an Adaptive Neuro Picture Fuzzy Inference System (ANPFIS) based on picture fuzzy distance measure is proposed to make the decisions for the link states in interconnection networks. ANPFIS is a modification and combination between Adaptive Neuro Fuzzy Inference System (ANFIS) [28-30], picture fuzzy set, and picture fuzzy distance measure. Hence, ANPFIS operates based on the picture fuzzification and defuzzification processes, the picture fuzzy operators [18] and distance measure, and the learning capability for automatic picture fuzzy rule generation and parameter optimization. In order to evaluate performance, we tested the ANPFIS method on the real datasets of the network traffic history taken from the UPV (Universitat Politècnica de València) university with related methods. The result is that ANPFIS is the most effective algorithm.

The rest of the paper is organized as follows. Section 2 provides some fundamental concepts of the fuzzy, intuitionistic fuzzy, single-valued neutrosophic, and picture fuzzy theories. Section 3 proposes the distance measure of single-valued neutrosophic sets and points out its important properties. Section 4 shows the new decision-making method named Adaptive Neuro Picture Fuzzy Inference System (ANPFIS) and an application of ANPFIS to controlling network power consumption. Section 5 shows the experimental results of ANPFIS and the related methods on real-world datasets. Finally, conclusion is given in Section 6.

2. Preliminary

In this part, some concepts of the theories of fuzzy sets, intuitionistic fuzzy sets, single-valued neutrosophic sets, and picture fuzzy sets are showed.

Let X be a space of points.

Definition 1. [1]. A fuzzy set (FS) A in X,

$$A=\{(x:\mu_A(x))|x\in X\}, \tag{1}$$

is characterized by a membership function, μ_A, with a range in $[0,1]$.

Definition 2. [6]. A intuitionistic fuzzy set (IFS) A in X,

$$A=\{(x:\mu_A(x),\nu_A(x))|x\in X\}, \tag{2}$$

is characterized by a membership function μ_A and a non-membership function v_A with a range in $[0,1]$ such that $0\le\mu_A+v_A\le 1$.

Definition 3. [31]. A Single-Valued Neutrosophic Set (SVNS) A in X,

$$A=\{(x:T_A(x),I_A(x),F_A(x))|x\in X\}, \tag{3}$$

is characterized by a truth-membership function T_A, an indeterminacy-membership function I_A, and a false-nonmembership function T_A with a range in $[0,1]$ such that $0\le T_A+I_A+F_A\le 3$.

Definition 4. [15]. A Picture Fuzzy Set (PFS) A in X,

$$A=\{(x:\mu_A(x),\eta_A(x),\nu_A(x))|x\in X\}, \tag{4}$$

is characterized by a positive membership function μ_A, a neutral function η_A, and a negative membership function v_A with a range in $[0,1]$ such that $0 \le \mu_A + \eta_A + v_A \le 1$.

We denote that SVNS(X) is the set of all SVNSs in X and PFS(X) is the set of all PFSs in X. We consider the sets N^* and P^* defined by

$$N^* = \{x = (x_1, x_2, x_3) | 0 \le x_1, x_2, x_3 \le 1\}, \tag{5}$$

$$P^* = \{x = (x_1, x_2, x_3) | 0 \le x_1, x_2, x_3, x_1 + x_2 + x_3 \le 1\}. \tag{6}$$

Definition 5. The orders on N^* and P^* are defined as follows

- $x \le y \Leftrightarrow (x_1 < y_1, x_3 \ge y_3) \vee (x_1 = y_1, x_3 > y_3) \vee (x_1 = y_1, x_3 = y_3, x_2 \le y_2)$, $\forall x, y \in P^*$, [19].
- $x \ll y \Leftrightarrow x_1 \le y_1, x_2 \le y_2, x_3 \ge y_3, \forall x, y \in N^*$.

Clearly, on P^*, if $x \ll y$ then $x \le y$.

Remark 1. The lattice $(P^*, \le)$ is a complete lattice [19] but $(P^*, \ll)$ is not. For example, let $x = (0.2, 0.3, 0.5)$ and $y = (0.3, 0, 0.7)$, then there is not any supremum value of x and y on $(P^*, \ll)$.

Otherwise, we have $sup(x, y) = (0.3, 0, 0.5)$ on the lattice $(P^*, \le)$. We denote the units of $(P^*, \le)$ as follows $0_{P^*} = (0, 0, 1)$ and $1_{P^*} = (1, 0, 0)$ [19]. It is easy to see that 0_{P^*} and 1_{P^*} are also the units on $(P^*, \ll)$. Now, some logic operators on PFS(X) are presented.

Definition 6. [19]. A picture fuzzy negation N is a function satisfying

$N : P^* \to P^*$, $N(0_{P^*}) = 1_{P^*}$, $N(1_{P^*}) = 0_{P^*}$, and $N(x) \ge N(y) \Leftrightarrow x \le y$.

Example 1. For every $x \in P^*$, then $N_0(x) = (x_3, 0, x_1)$ and $N_S(x) = (x_3, x_4, x_1)$ are picture fuzzy negations, where $x_4 = 1 - x_1 - x_2 - x_3$.

Remark 2. The operator N_0 also satisfies $N_0(x) \gg N_0(y) \Leftrightarrow x \ll y, \forall x, y \in P^*$.

Now, let $x, y, z \in P^*$ and $I(x) = \{y \in P^* : y = (x_1, y_2, x_3), 0 \le y_2 \le x_2\}$.

Definition 7. [19] A picture fuzzy t-norm T is a function satisfying

$T : P^* \times P^* \to P^*$, $T(x, y) = T(y, x)$, $T(T(x, y), z) = T(x, T(y, z))$,

$T(1_{P^*}, x) \in I(x)$, and $T(x, y) \le T(x, z), \forall y \le z$.

Definition 8. [19] A picture fuzzy t-conorm S is a function satisfying

$S : P^* \times P^* \to P^*$, $S(x, y) = S(y, x)$, $S(S(x, y), z) = S(x, S(y, z))$,

$S(0_{P^*}, x) \in I(x)$, and $S(x, y) \leq S(x, z), \forall y \leq z$.

Example 2. For all $x, y \in P^*$, the following operators are the picture fuzzy t-norms:

- $T_0(x, y) = (min(x_1, y_1), min(x_2, y_2), max(x_3, y_3))$.
- $T_1(x, y) = (x_1 y_1, x_2 y_2, x_3 + y_3 - x_3 y_3)$.
- $T_2(x, y) = (max(0, x_1 + y_1 - 1), max(0, x_2 + y_2 - 1), min(1, x_3 + y_3))$.
- $T_3(x, y) = (max(0, x_1 + y_1 - 1), max(0, x_2 + y_2 - 1), x_3 + y_3 - x_3 y_3)$.
- $T_4(x, y) = (x_1 y_1, max(0, x_2 + y_2 - 1), x_3 + y_3 - x_3 y_3)$.
- $T_5(x, y) = (max(0, x_1 + y_1 - 1), x_2 y_2, x_3 + y_3 - x_3 y_3)$.

Example 3. For all $x, y \in P^*$, the following operators are the picture fuzzy t-conorms:

- $S_0(x, y) = (max(x_1, y_1), max(0, x_2 + y_2 - 1), min(x_3, y_3))$.
- $S_1(x, y) = (x_1 + y_1 - x_1 y_1, x_2 y_2, x_3 y_3)$.
- $S_2(x, y) = (min(1, x_1 + y_1), max(0, x_2 + y_2 - 1), max(0, x_3 + y_3 - 1))$.
- $S_3(x, y) = (x_1 + y_1 - x_1 y_1, max(0, x_2 + y_2 - 1), max(0, x_3 + y_3 - 1))$.
- $S_4(x, y) = (x_1 + y_1 - x_1 y_1, max(0, x_2 + y_2 - 1), x_3 y_3)$.
- $S_5(x, y) = (x_1 + y_1 - x_1 y_1, x_2 y_2, max(0, x_3 + y_3 - 1))$.

Remark 3. For all $x, y, z \in P^*$ and $y \ll z$, the operators $T_{i(i=0,...,5)}$ also satisfy the condition $T(x, y) \ll T(x, z)$. Similarly, $S_{i(i=0,...,5)}$ also satisfy $S(x, y) \ll S(x, z)$.

The logic operators N, T and S on P^* are corresponding to the basic set-theory operators on PFS(X) as follows.

Definition 9. Let N, T and S be the picture fuzzy negation, t-norm and t-conorm, respectively, and $A, B \in$ PFS(X). Then, the complement of A w.r.t N is defined as follows:

$$\overline{A}^N = \left\{ \left(x : N\left(\left(\mu_A(x), \eta_A(x), \nu_A(x) \right) \right) \right) \middle| x \in X \right\}, \tag{7}$$

the intersection of A and B w.r.t T is defined as follows:

$$A \cap_T B = \left\{ \left(x : T\left(\left(\mu_A(x), \eta_A(x), \nu_A(x) \right), \left(\mu_B(x), \eta_B(x), \nu_B(x) \right) \right) \right) \middle| x \in X \right\}, \tag{8}$$

and the union of A and B w.r.t T is defined as follows:

$$A \cup_S B = \left\{ \left(x : S\left(\left(\mu_A(x), \eta_A(x), \nu_A(x) \right), \left(\mu_B(x), \eta_B(x), \nu_B(x) \right) \right) \right) \middle| x \in X \right\}. \tag{9}$$

3. The Single-valued Neutrosophic Distance Measure and the Picture Fuzzy Distance Measure

Recently, Wei has introduced the generalized Dice similarity measures for picture fuzzy sets [17]. However, the definition of Wei is without considering the condition related to order relation on picture fuzzy sets. The new distance measure on picture fuzzy sets is proposed in this section. It is developed from intuitionistic distance measure of Wang et al. [32] and Ngan et al. [8].

Definition 10. A single-valued neutrosophic distance measure d is a function satisfying

- $d : N^* \times N^* \to [0,+\infty)$,
- $d(x,y) = d(y,x)$,
- $d(x,y) = 0 \Leftrightarrow x = y$,
- If $x \ll y \ll z$ then $d(x,y) \le d(x,z)$ and $d(y,z) \le d(x,z)$.

Definition 11. A picture fuzzy distance measure d is a single-valued neutrosophic distance measure and $d(x,y) \in [0,1], \forall x,y \in P^*$.

Definition 12. The measure D_0 is defined as follows

$$D_0(x,y) = \frac{1}{3}\left(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3| + \left| \max\{x_1, y_3\} - \max\{x_3, y_1\} \right| \right), \forall x,y \in P^*. \tag{10}$$

Proposition 1. The measure D_0 is a picture fuzzy distance measure.

Proof. Firstly, we have $\left| max\{x_1, y_3\} - max\{x_3, y_1\} \right| \in [0,1]$ and

$$|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3| \le (|x_1| + |y_1|) + (|x_2| + |y_2|) + (|x_3| + |y_3|) \le (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) \le 2.$$

Therefore, $0 \le \frac{1}{3}\left(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3| + \left| max\{x_1, y_3\} - max\{x_3, y_1\} \right| \right) \le 1.$

Secondly, we obtain that $D_0(x,y) = D_0(y,x)$ since D_0 has the symmetry property between the arguments.

Thirdly, $D_0(x,y) = 0 \Leftrightarrow |x_1 - y_1| = |x_2 - y_2| = |x_3 - y_3| = \left| max\{x_1, y_3\} - max\{x_3, y_1\} \right| = 0 \Leftrightarrow x = y.$

Finally, let $x \ll y \ll z$, then $x_1 \le y_1 \le z_1, x_2 \le y_2 \le z_2, x_3 \ge y_3 \ge z_3$. We obtain that

$|x_1 - y_1| \le |x_1 - z_1|, |x_2 - y_2| \le |x_2 - z_2|, |x_3 - y_3| \le |x_3 - z_3|.$

Moreover, $max\{z_1, x_3\} \ge max\{y_1, x_3\} \ge max\{x_1, y_3\} \ge max\{x_1, z_3\}$. Hence, $|max\{x_1, y_3\} - max\{x_3, y_1\}| \le |max\{x_1, z_3\} - max\{x_3, z_1\}|. \lim_{x\to\infty}$. Thus, $D_0(x,y) \le D_0(x,z)$. Similarly, we also have $D_0(y,z) \le D_0(x,z)$.

Remark 4. If d is a picture fuzzy distance measure, then d is a single-valued neutrosophic distance measure. The opposite is not necessarily true. Some picture fuzzy operations were introduced by the group of authors of this paper [18, 19]. Hence, this research is seen as a complete link to the authors' previous work on picture fuzzy inference systems. An inference system of neutrosophic theory will be developed in another paper as a future work.

Proposition 2. Let $x, y \in P^*$. The measure D_0 satisfies the following properties:

- $D_0(N_0(x), N_S(x)) = \frac{1}{3}x_4$.
- If $x_2 \ge x_4$, then $|D_0(x, N_0(x)) - D_0(x, N_S(x))| = \frac{1}{3}x_4$.
- $|D_0(x, N_0(y)) - D_0(N_0(x), y)| = \frac{1}{3}|x_2 - y_2|$.
- $|D_0(x,y) - D_0(N_0(x), N_0(y))| = \frac{1}{3}|x_2 - y_2|$.
- If $x_1 + x_3 = y_1 + y_3$, then $D_0(x,y) = D_0(N_S(x), N_S(y))$.
- If $x_1 + x_3 = y_1 + y_3$, then $D_0(x, N_S(y)) = D_0(N_S(x), y)$.
- $D_0(x, N_0(x)) = |x_1 - x_3| + \frac{1}{3}x_2$.
- $D_0(x, N_S(x)) = |x_1 - x_3| + \frac{1}{3}|x_2 - x_4|$.
- $D_0(x, 1_{P^*}) = \frac{1}{3}(2 - 2x_1 + x_2 + x_3)$.
- $D_0(x, 0_{P^*}) = \frac{1}{3}(2 - 2x_3 + x_1 + x_2)$.
- $D_0(0_{P^*}, 1_{P^*}) = 1.$
- $D_0(x, N_0(x)) = 1$ if and only if $x \in \{0_{P^*}, 1_{P^*}\}$.
- $D_0(x, N_S(x)) = 1$ if and only if $x \in \{0_{P^*}, 1_{P^*}\}$.
- $D_0(x, N_0(x)) = 0$ if and only if $x_1 = x_3, x_2 = 0.$
- $D_0(x, N_S(x)) = 0$ if and only if $x_1 = x_3, x_2 = x_4$.
- $D_0((0,0,0),(0,a,0)) < D_0((0,0,0),(a,0,0)) = D_0((0,0,0),(0,0,a))$

$$< D_0((a,0,0),(0,0,a)) = D_0((a,0,0),(0,a,0)) = D_0((0,0,a),(0,a,0)), \forall a \in (0,1].$$

Proof. These properties are proved as follows:

- We have $D_0\left(N_0(x), N_S(x)\right) = D_0\left((x_3, 0, x_1), (x_3, x_4, x_1)\right)$

$$= \frac{1}{3}\left(|x_3 - x_3| + |0 - x_4| + |x_1 - x_1| + |max\{x_3, x_1\} - max\{x_3, x_1\}|\right) = \frac{1}{3}x_4.$$

- We have

$$\left|D_0\left(x, N_0(x)\right) - D_0\left(x, N_S(x)\right)\right| = \left|D_0\left((x_1, x_2, x_3), (x_3, 0, x_1)\right) - D_0\left((x_1, x_2, x_3), (x_3, x_4, x_1)\right)\right|$$

$$= \left|\frac{1}{3}\left(|x_1 - x_3| + |x_2 - 0| + |x_3 - x_1| + |max\{x_1, x_1\} - max\{x_3, x_3\}|\right)\right.$$

$$\left. - \frac{1}{3}\left(|x_1 - x_3| + |x_2 - x_4| + |x_3 - x_1| + |max\{x_1, x_1\} - max\{x_3, x_3\}|\right)\right|$$

$$= \frac{1}{3}\left|x_2 - |x_2 - x_4|\right| = \frac{1}{3}x_4.$$

- We have

$$\left|D_0\left(x, N_0(y)\right) - D_0\left(N_0(x), y\right)\right| = \left|D_0\left((x_1, x_2, x_3), (y_3, 0, y_1)\right) - D_0\left((x_3, 0, x_1), (y_1, y_2, y_3)\right)\right|$$

$$= \left|\frac{1}{3}\left(|x_1 - y_3| + |x_2 - 0| + |x_3 - y_1| + |max\{x_1, y_1\} - max\{y_3, x_3\}|\right)\right.$$

$$\left. - \frac{1}{3}\left(|x_3 - y_1| + |0 - y_2| + |x_1 - y_3| + |max\{x_3, y_3\} - max\{y_1, x_1\}|\right)\right| = \frac{1}{3}|x_2 - y_2|.$$

- We have

$$\left|D_0(x, y) - D_0\left(N_0(x), N_0(y)\right)\right| = \left|D_0\left((x_1, x_2, x_3), (y_1, y_2, y_3)\right) - D_0\left((x_3, 0, x_1), (y_3, 0, y_1)\right)\right|$$

$$= \left|\frac{1}{3}\left(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3| + |max\{x_1, y_3\} - max\{x_3, y_1\}|\right)\right.$$

$$\left. - \frac{1}{3}\left(|x_3 - y_3| + |0 - 0| + |x_1 - y_1| + |max\{x_3, y_1\} - max\{y_3, x_1\}|\right)\right| = \frac{1}{3}|x_2 - y_2|.$$

- We have $D_0\left(N_S(x), N_S(y)\right) = D_0\left((x_3, x_4, x_1), (y_3, y_4, y_1)\right)$

$= \frac{1}{3}\left(|x_3 - y_3| + |x_4 - y_4| + |x_1 - y_1| + |max\{x_3, y_1\} - max\{y_3, x_1\}|\right)$. Further,
$|x_4 - y_4| = \left|(1 - x_1 - x_2 - x_3) - (1 - y_1 - y_2 - y_3)\right| = |x_2 - y_2|$. Thus, $D_0\left(N_S(x), N_S(y)\right) = D_0(x, y)$.

- We have $D_0(x, N_S(y)) = D_0((x_1, x_2, x_3), (y_3, y_4, y_1))$

  $= \frac{1}{3}(|x_1 - y_3| + |x_2 - y_4| + |x_3 - y_1| + |max\{x_1, y_1\} - max\{y_3, x_3\}|)$. In other hand,
  $D_0(N_S(x), y) = D_0((x_3, x_4, x_1), (y_1, y_2, y_3))$

  $= \frac{1}{3}(|x_3 - y_1| + |x_4 - y_2| + |x_1 - y_3| + |max\{x_3, y_3\} - max\{y_1, x_1\}|)$. Further,
  $|x_2 - y_4| = |x_2 - 1 + y_1 + y_2 + y_3| = |x_2 - 1 + x_1 + y_2 + x_3| = |y_2 - x_4|$. Thus,
  $D_0(x, N_S(y)) = D_0(N_S(x), y)$. $\lim_{x \to \infty}$

- We have $D_0(x, N_0(x)) = D_0((x_1, x_2, x_3), (x_3, 0, x_1))$

  $= \frac{1}{3}(|x_1 - x_3| + |x_2 - 0| + |x_3 - x_1| + |max\{x_1, x_1\} - max\{x_3, x_3\}|) = |x_1 - x_3| + \frac{1}{3}x_2$.

- We have
  $D_0(x, N_S(x)) = D_0((x_1, x_2, x_3), (x_3, x_4, x_1))$
  $= \frac{1}{3}(|x_1 - x_3| + |x_2 - x_4| + |x_3 - x_1| + |max\{x_1, x_1\} - max\{x_3, x_3\}|) = |x_1 - x_3| + \frac{1}{3}|x_2 - x_4|$.
- We have $D_0(x, 1_{P^*}) = D_0((x_1, x_2, x_3), (1, 0, 0))$

  $= \frac{1}{3}(|x_1 - 1| + |x_2 - 0| + |x_3 - 0| + |max\{x_1, 0\} - max\{x_3, 1\}|) = \frac{1}{3}(2 - 2x_1 + x_2 + x_3)$.

- We have $D_0(x, 0_{P^*}) = D_0((x_1, x_2, x_3), (0, 0, 1))$

  $= \frac{1}{3}(|x_1 - 0| + |x_2 - 0| + |x_3 - 1| + |max\{x_1, 1\} - max\{x_3, 0\}|) = \frac{1}{3}(2 - 2x_3 + x_1 + x_2)$.

- We have $D_0(0_{P^*}, 1_{P^*}) = D_0((1, 0, 0), (0, 0, 1)) = 1$.

Assume that $D_0(x, N_0(x)) = 1$, we have $|x_1 - x_3| + \frac{1}{3}x_2 = 1$. Since $|x_1 - x_3| + \frac{1}{3}x_2 \le (x_1 + x_3) + x_2 \le 1$.

Therefore, $|x_1 - x_3| + \frac{1}{3}x_2 = (x_1 + x_3) + x_2 = 1$. We obtain that $x_2 = 0$ and $|x_1 - x_3| = 1$. Thus, $x \in \{0_{P^*}, 1_{P^*}\}$. Assume that $D_0(x, N_S(x)) = 1$, we have $|x_1 - x_3| + \frac{1}{3}|x_2 - x_4| = 1$. Since $|x_1 - x_3| + \frac{1}{3}|x_2 - x_4| \le (x_1 + x_3) + (x_2 + x_4) = 1$. We obtain that $x_2 = x_4 = 0$ and $|x_1 - x_3| = 1$. Thus,

$x \in \{0_{P^*}, 1_{P^*}\}$. Assume that $D_0(x, N_0(x)) = 0$, we have $|x_1 - x_3| + \frac{1}{3}x_2 = 0$. Hence, $x_2 = 0$ and $x_1 = x_3$.

Assume that $D_0(x, N_S(x)) = 0$, we have $|x_1 - x_3| + \frac{1}{3}|x_2 - x_4| = 0$. Hence, $x_2 = x_4$ and $x_1 = x_3$.

– We have $D_0((0,0,0),(0,a,0)) = \frac{a}{3}$,

$D_0((0,0,0),(a,0,0)) = D_0((0,0,0),(0,0,a)) = \frac{2a}{3}$, and

$D_0((a,0,0),(0,0,a)) = D_0((a,0,0),(0,a,0)) = D_0((0,0,a),(0,a,0)) = a.$

Remark 5. The order "$\ll$" on P^* corresponds to the following order on $\mathrm{PFS}(X)$:

$$A \subseteq_* B \Leftrightarrow \mu_A(x) \le \mu_B(x), \eta_A(x) \le \eta_B(x), \nu_A(x) \ge \nu_B(x), \forall x \in X. \tag{11}$$

Remark 6. The picture fuzzy distance measure on P^* corresponds to the picture fuzzy distance measure on $\mathrm{PFS}(X)$, i.e., for all $A, B \in \mathrm{PFS}(X = \{x_1, x_2, \ldots, x_m\})$, we have the picture fuzzy distance measure D_0 between A and B as follows:

$$\begin{aligned} D_0(A,B) = \frac{1}{3m}\sum_{i=1}^{m}\Big(&|\mu_A(x_i) - \mu_B(x_i)| + |\eta_A(x_i) - \eta_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| \\ &+ \left|\max\{\mu_A(x_i), \nu_B(x_i)\} - \max\{\nu_A(x_i), \mu_B(x_i)\}\right|\Big). \end{aligned} \tag{12}$$

Proposition 3. Consider the picture fuzzy distance measure D_0 in Eq. (10), the picture fuzzy t-norms $T_{i(i=0,\ldots,5)}$ in Example 2, the picture fuzzy t-conorms $S_{i(i=0,\ldots,5)}$ in Example 3, and the picture fuzzy negation N_0 in Example 1. Let A and B be two picture fuzzy sets on the universe $X = \{x_1, x_2, \ldots, x_m\}$. Then, we have the following properties:

– $D_0(A \cap_{T_2} B, B) \ge \max\{D_0(A \cap_{T_i} B, A \cap_{T_j} B), D_0(A \cap_{T_k} B, B)\}$,

$D_0(A \cap_{T_2} B, A) \ge \max\{D_0(A \cap_{T_i} B, A \cap_{T_j} B), D_0(A \cap_{T_k} B, A)\}$,

$D_0\left(\overline{A \cap_{T_2} B}^{N_0}, \overline{A}^{N_0}\right) \ge \max\left\{D_0\left(\overline{A \cap_{T_i} B}^{N_0}, \overline{A \cap_{T_j} B}^{N_0}\right), D_0\left(\overline{A \cap_{T_k} B}^{N_0}, \overline{A}^{N_0}\right)\right\}$,

$D_0\left(\overline{A \cap_{T_2} B}^{N_0}, \overline{B}^{N_0}\right) \ge \max\left\{D_0\left(\overline{A \cap_{T_i} B}^{N_0}, \overline{A \cap_{T_j} B}^{N_0}\right), D_0\left(\overline{A \cap_{T_k} B}^{N_0}, \overline{B}^{N_0}\right)\right\}$,

$\forall (i,j) \in \{(x,y) \mid x, y = 0, \ldots, 5\} \setminus \{(4,5)\}$ and $k = 0,1,3,4,5.$

– $D_0\left(A\cup_{S_5} B,A\right)\geq max\left\{D_0\left(A\cup_{S_i} B,A\cup_{S_j} B\right),D_0\left(A\cup_{S_k} B,A\right)\right\}$,

$D_0\left(A\cup_{S_5} B,B\right)\geq max\left\{D_0\left(A\cup_{S_i} B,A\cup_{S_j} B\right),D_0\left(A\cup_{S_k} B,B\right)\right\}$,

$D_0\left(\overline{A\cup_{S_5} B}^{N_0},\overline{A}^{N_0}\right)\geq max\left\{D_0\left(\overline{A\cup_{S_i} B}^{N_0},\overline{A\cup_{S_j} B}^{N_0}\right),D_0\left(\overline{A\cup_{S_k} B}^{N_0},\overline{A}^{N_0}\right)\right\}$,

$D_0\left(\overline{A\cup_{S_5} B}^{N_0},\overline{B}^{N_0}\right)\geq max\left\{D_0\left(\overline{A\cup_{S_i} B}^{N_0},\overline{A\cup_{S_j} B}^{N_0}\right),D_0\left(\overline{A\cup_{S_k} B}^{N_0},\overline{B}^{N_0}\right)\right\}$,

$\forall\left(i,j\right)\in\left\{\left(x,y\right)\mid x,y=0,1,3,4,5\right\}\setminus\left\{\left(1,3\right)\right\}$ and $k=0,1,3,4.$

– $D_0\left(A\cup_{S_2} B,A\right)\geq max\left\{D_0\left(A\cup_{S_i} B,A\cup_{S_j} B\right),D_0\left(A\cup_{S_k} B,A\right)\right\}$,

$D_0\left(A\cup_{S_2} B,B\right)\geq max\left\{D_0\left(A\cup_{S_i} B,A\cup_{S_j} B\right),D_0\left(A\cup_{S_k} B,B\right)\right\}$,

$D_0\left(\overline{A\cup_{S_2} B}^{N_0},\overline{A}^{N_0}\right)\geq max\left\{D_0\left(\overline{A\cup_{S_i} B}^{N_0},\overline{A\cup_{S_j} B}^{N_0}\right),D_0\left(\overline{A\cup_{S_k} B}^{N_0},\overline{A}^{N_0}\right)\right\}$,

$D_0\left(\overline{A\cup_{S_2} B}^{N_0},\overline{B}^{N_0}\right)\geq max\left\{D_0\left(\overline{A\cup_{S_i} B}^{N_0},\overline{A\cup_{S_j} B}^{N_0}\right),D_0\left(\overline{A\cup_{S_k} B}^{N_0},\overline{B}^{N_0}\right)\right\}$,

$\forall i,j=0,2,3,4,$ and $k=0,3,4.$

– $D_0\left(A\cap_{T_i} B,A\cup_{S_j} B\right)\geq D_0\left(A\cap_{T_0} B,A\cup_{S_0} B\right)$ and

$D_0\left(\overline{A\cap_{T_i} B}^{N_0},\overline{A\cup_{S_j} B}^{N_0}\right)\geq D_0\left(\overline{A\cap_{T_0} B}^{N_0},\overline{A\cup_{S_0} B}^{N_0}\right),\forall i,j=0,...,5.$

Proof. These properties are proved as follows. Firstly, we see that for all $x,y\in[0,1]$,

$max\left(0,x+y-1\right)\leq xy\leq min\left(x,y\right)$ and $min\left(1,x+y\right)\geq x+y-xy\geq max\left(0,x+y-1\right)$. Hence, $\left(max\left(0,x_1+y_1-1\right),max\left(0,x_2+y_2-1\right),min\left(1,x_3+y_3\right)\right)\ll\left(x_1y_1,x_2y_2,x_3+y_3-x_3y_3\right)$. This means $T_2\ll T_1$. Similarly, we obtain that $T_2\ll T_3\ll T_4\ll T_1\ll T_0$ and $T_2\ll T_3\ll T_5\ll T_1\ll T_0$. Hence,

$A\cap_{T_2} B\subseteq_* A\cap_{T_3} B\subseteq_* A\cap_{T_4} B\subseteq_* A\cap_{T_1} B\subseteq_* A\cap_{T_0} B\subseteq_* A,$

$A\cap_{T_2} B\subseteq_* A\cap_{T_3} B\subseteq_* A\cap_{T_5} B\subseteq_* A\cap_{T_1} B\subseteq_* A\cap_{T_0} B\subseteq_* A,$

$A\cap_{T_2} B\subseteq_* A\cap_{T_3} B\subseteq_* A\cap_{T_4} B\subseteq_* A\cap_{T_1} B\subseteq_* A\cap_{T_0} B\subseteq_* B,$ and

$A\cap_{T_2} B\subseteq_* A\cap_{T_3} B\subseteq_* A\cap_{T_5} B\subseteq_* A\cap_{T_1} B\subseteq_* A\cap_{T_0} B\subseteq_* B.$

Since D_0 is the picture fuzzy distance measure, thus

$$D_0\left(A\cap_{T_2} B,A\right)\geq max\left\{D_0\left(A\cap_{T_i} B,A\cap_{T_j} B\right),D_0\left(A\cap_{T_k} B,A\right)\right\} \text{ and}$$

$$D_0\left(A\cap_{T_2} B,B\right)\geq max\left\{D_0\left(A\cap_{T_i} B,A\cap_{T_j} B\right),D_0\left(A\cap_{T_k} B,B\right)\right\},$$

$\forall(i,j)\in\{(x,y)\,|\,x,y=0,\ldots,5\}\setminus\{(4,5)\}$ and $k=0,1,3,4,5.$ Furthermore, we have

$$\overline{A}^{N_0}=\left\{\left(x:N_0\left(\left(\mu_A(x),\eta_A(x),v_A(x)\right)\right)\right)\middle|x\in X\right\}=\left\{\left(x:\left(v_A(x),0,\mu_A(x)\right)\right)\middle|x\in X\right\},\otimes.$$

It is easy to prove the following lemma: If $A\subseteq_* B,$ then $\overline{B}^{N_0}\subseteq_*\overline{A}^{N_0}.$ Thus,

$$D_0\left(\overline{A\cap_{T_2} B}^{N_0},\overline{A}^{N_0}\right)\geq max\left\{D_0\left(\overline{A\cap_{T_i} B}^{N_0},\overline{A\cap_{T_j} B}^{N_0}\right),D_0\left(\overline{A\cap_{T_k} B}^{N_0},\overline{A}^{N_0}\right)\right\} \text{ and}$$

$$D_0\left(\overline{A\cap_{T_2} B}^{N_0},\overline{B}^{N_0}\right)\geq max\left\{D_0\left(\overline{A\cap_{T_i} B}^{N_0},\overline{A\cap_{T_j} B}^{N_0}\right),D_0\left(\overline{A\cap_{T_k} B}^{N_0},\overline{B}^{N_0}\right)\right\},$$

$\forall(i,j)\in\{(x,y)\,|\,x,y=0,\ldots,5\}\setminus\{(4,5)\}$ and $k=0,1,3,4,5.$

Secondly, we have $S_0\ll S_4\ll S_3\ll S_5$, $S_0\ll S_4\ll S_3\ll S_2$, and $S_0\ll S_4\ll S_1\ll S_5$. Hence,

$A\subseteq_* A\cup_{S_0} B\subseteq_* A\cup_{S_4} B\subseteq_* A\cup_{S_3} B\subseteq_* A\cup_{S_5} B,$

$B\subseteq_* A\cup_{S_0} B\subseteq_* A\cup_{S_4} B\subseteq_* A\cup_{S_3} B\subseteq_* A\cup_{S_5} B,$

$A\subseteq_* A\cup_{S_0} B\subseteq_* A\cup_{S_4} B\subseteq_* A\cup_{S_1} B\subseteq_* A\cup_{S_5} B,$ and

$B\subseteq_* A\cup_{S_0} B\subseteq_* A\cup_{S_4} B\subseteq_* A\cup_{S_1} B\subseteq_* A\cup_{S_5} B.$

Therefore $D_0\left(A\cup_{S_5} B,A\right)\geq max\left\{D_0\left(A\cup_{S_i} B,A\cup_{S_j} B\right),D_0\left(A\cup_{S_k} B,A\right)\right\},$

$$D_0\left(A\cup_{S_5} B,B\right)\geq max\left\{D_0\left(A\cup_{S_i} B,A\cup_{S_j} B\right),D_0\left(A\cup_{S_k} B,B\right)\right\},$$

$$D_0\left(\overline{A\cup_{S_5} B}^{N_0},\overline{A}^{N_0}\right)\geq max\left\{D_0\left(\overline{A\cup_{S_i} B}^{N_0},\overline{A\cup_{S_j} B}^{N_0}\right),D_0\left(\overline{A\cup_{S_k} B}^{N_0},\overline{A}^{N_0}\right)\right\}, \text{ and}$$

$$D_0\left(\overline{A\cup_{S_5} B}^{N_0},\overline{B}^{N_0}\right)\geq max\left\{D_0\left(\overline{A\cup_{S_i} B}^{N_0},\overline{A\cup_{S_j} B}^{N_0}\right),D_0\left(\overline{A\cup_{S_k} B}^{N_0},\overline{B}^{N_0}\right)\right\},$$

$\forall(i,j)\in\{(x,y)\,|\,x,y=0,1,3,4,5\}\setminus\{(1,3)\}$ and $k=0,1,3,4.$

Now, we have $A\subseteq_* A\cup_{S_0} B\subseteq_* A\cup_{S_4} B\subseteq_* A\cup_{S_3} B\subseteq_* A\cup_{S_2} B$ and

$B \subseteq_* A \cup_{S_0} B \subseteq_* A \cup_{S_4} B \subseteq_* A \cup_{S_3} B \subseteq_* A \cup_{S_2} B$.

Thus, for all $i, j = 0,2,3,4,$ and $k = 0,3,4,$ we have

$$D_0\left(A \cup_{S_2} B, A\right) \geq max\left\{D_0\left(A \cup_{S_i} B, A \cup_{S_j} B\right), D_0\left(A \cup_{S_k} B, A\right)\right\},$$

$$D_0\left(A \cup_{S_2} B, B\right) \geq max\left\{D_0\left(A \cup_{S_i} B, A \cup_{S_j} B\right), D_0\left(A \cup_{S_k} B, B\right)\right\},$$

$$D_0\left(\overline{A \cup_{S_2} B}^{N_0}, \overline{A}^{N_0}\right) \geq max\left\{D_0\left(\overline{A \cup_{S_i} B}^{N_0}, \overline{A \cup_{S_j} B}^{N_0}\right), D_0\left(\overline{A \cup_{S_k} B}^{N_0}, \overline{A}^{N_0}\right)\right\}, \text{ and}$$

$$D_0\left(\overline{A \cup_{S_2} B}^{N_0}, \overline{B}^{N_0}\right) \geq max\left\{D_0\left(\overline{A \cup_{S_i} B}^{N_0}, \overline{A \cup_{S_j} B}^{N_0}\right), D_0\left(\overline{A \cup_{S_k} B}^{N_0}, \overline{B}^{N_0}\right)\right\}.$$

Finally, we see that $A \cap_{T_2} B \subseteq_* A \cap_{T_0} B \subseteq_* A \subseteq_* A \cup_{S_0} B \subseteq_* A \cup_{S_5} B.$

Thus, we obtain that $D_0\left(A \cap_{T_2} B, A \cup_{S_5} B\right) \geq D_0\left(A \cap_{T_0} B, A \cup_{S_0} B\right)$ and the remaining inequalities of ***Proposition 3***.

4. An Application of the Picture Fuzzy Distance Measure for Controlling Network Power Consumption

4.1. The Problem and the Solution

The interconnection network is important in the parallel computer systems. Saving interconnection network power is always interested, researched and becoming more and more urgent in the current technological era. In order to achieve high performance, the architectural design of the interconnection network requires an effective power saving mechanism. The aim of this mechanism is to reduce the network latency (the average latency of a message) and the percentages between the number of links that are kept switched on by the saving mechanism and the total number of links [25]. As a simplified way of understanding, this is a matter of optimizing the number of links opened in a networking system. This is a decision-making problem for the trunk link state.

In 2010, Alonso et al. introduced the power saving mechanism in regular interconnection network [25]. This model dynamically increases or reduces the number of links that compose a trunk link. This is done by measuring network traffic and dynamically turning these individual links on or off based on a $u_{\text{on}} / u_{\text{off}}$ threshold policy with keeping at least one operational link (see ***Fig. 1*** and ***Fig. 2***).

The two parameters u_{on} and u_{off} are designed based on different requirements of mechanism aggressiveness (controlled by the value $u_{\text{avg}} = (u_{\text{on}} + u_{\text{off}}) / 2$) and mechanism responsiveness (controlled by the difference $u_{\text{on}} - u_{\text{off}}$).

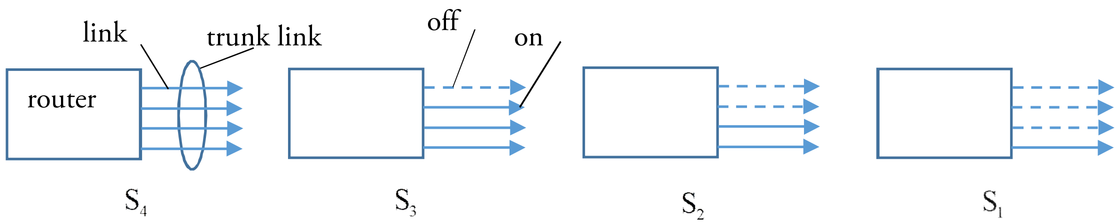

Fig. 1. Four trunk link states.

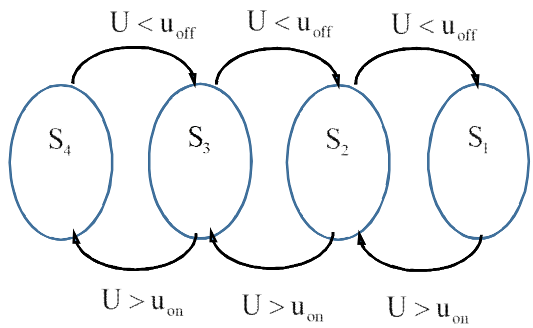

Fig. 2. The operational mechanism of switches.

In order to avoid the possibility of cyclic state transitions that makes the system become unstable, the following restrictions hold in the selection u_{on} and u_{off} :

$$0 < u_{\text{off}} \leq \frac{u_{\text{on}}}{2} \leq \frac{U_{\max}}{2}. \tag{13}$$

Thus, the different values of u_{off} and u_{on} that satisfy ***Eq. (13)*** are stiffly chosen in order to achieve different goals of responsiveness and aggressiveness for the power saving mechanism. In 2015, they continue to study and modify power consumption control in fat-tree interconnection networks based on the static and dynamic thresholds policies [26]. In general, this threshold policy is hard because it is without any fuzzy approaches, parameter learning and optimizing processes.

In 2017, Phan et al. [27] proposed a new method in power consumption estimation of network-on-chip based on fuzzy logic [27]. However, this fuzzy logic system based on Sugeno model is too rudimentary and the parameters here are chosen according to the authors' quantification. In this paper, aiming to replace the above threshold policy in decision making problem for the trunk link state, we propose a higher-level fuzzy system based on the proposed single-valued neutrosophic distance measure in Section 3.

4.2. The Adaptive Neuro Picture Fuzzy Inference System (ANPFIS)

In this subsection, an ANPFIS based on picture fuzzy distance measure is introduced to decision making problems. ANPFIS is a modification and combination between ANFIS [28], picture fuzzy set, and picture fuzzy distance measure. Hence, ANPFIS operates based on the picture fuzzification and defuzzification processes, the picture fuzzy operators and distance measure, and the learning capability for automatic picture fuzzy rule generation and parameter optimization. The model is showed as in the ***Fig. 3***.

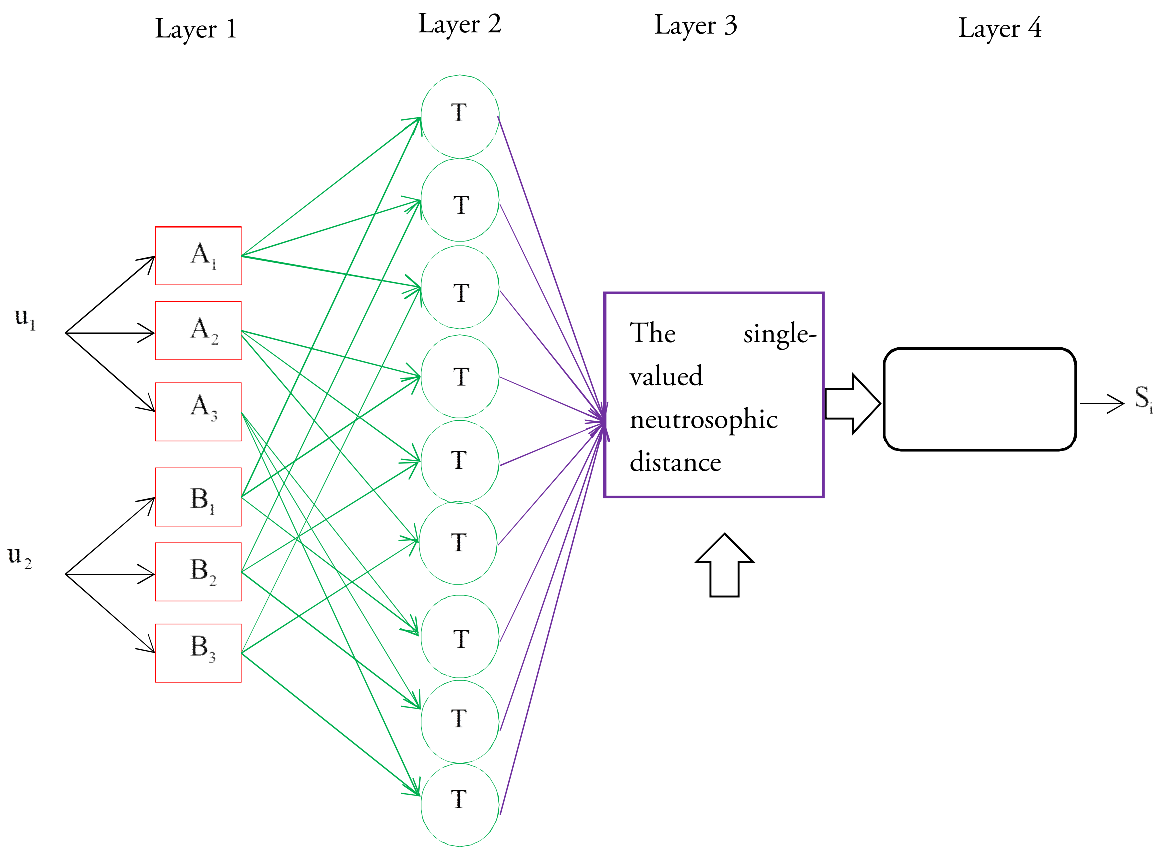

Fig. 3. The proposed ANPFIS decision making model.

The model has the inputs are number values and the output $S_i, i \in \{1,\ldots,n\}$ is the chosen solution. ANPFIS includes four layers as follows:

Layer 1-Picture Fuzzification. Each input value is connected to three neuros O_i^1 , in other words is fuzzified by three corresponding picture fuzzy sets named "High", "Medium", and "Low". We use the Picture Fuzzy Gaussian Function (PFGF): the PFGF is specified by two parameters. The Gaussian function is defined by a central value m and width $k > 0$. The smaller the k, the narrower the curve is. Picture fuzzy Gaussian positive membership, neutral, and negative membership functions are defined as follows

$\mu(x) = exp\left(-\frac{(x-m)^2}{2k^2}\right)$,

$v(x) = c_1(1-\mu(x))$, $(c_1 \in [0,1])$, and

$\eta(x) = c_2(1-\mu(x)-v(x))$, $(c_2 \in [0,1])$, where the parameters $m\otimes$ and $k \triangleright$ are trained.

Layer 2-Automatic Picture Fuzzy Rules. The picture fuzzy t-norm T (see. ***Definition 7*** and ***Example 2***) is used in this step in order to establish the IF-THEN picture fuzzy rules, i.e., the links between the neuros O_i^1 of Layer 1 and the neuros O_k^2 of Layer 2 as follows

"If O_i^1 is x and O_j^1 is y then O_k^2 is $T(x,y)$."

For examples $T(x,y)=T_1^{\lambda}(x,y)$, where [18]

$$T_1^{\lambda}(x,y)=\left(\frac{x_1y_1}{\lambda_1+(1-\lambda_1)(x_1+y_1-x_1y_1)},\frac{x_2y_2}{\lambda_2+(1-\lambda_2)(x_2+y_2-x_2y_2)},\left(x_3^{\lambda_3}+y_3^{\lambda_3}-x_3^{\lambda_3}y_3^{\lambda_3}\right)^{\frac{1}{\lambda_3}}\right),\text{ here } x,y\in P^*,$$

and the parameters $\lambda_1,\lambda_2,\lambda_3\in[1,+\infty)$ are trained.

Layer 3 – Calculate the difference to the samples. The difference between the input I and the sample K is calculated by the proposed picture fuzzy distance measure D_0 in ***Eq. (10)*** as follows

$$D_0(I,K)=\frac{1}{3m}\sum_{i=1}^{m}\omega_i.\{|\mu_1(x_i)-\mu_2(x_i)|+|\eta_1(x_i)-\eta_2(x_i)|+|v_1(x_i)-v_2(x_i)| \\ +|max\{\mu_1(x_i),v_2(x_i)\}-max\{\mu_2(x_i),v_1(x_i)\}|\},$$

where, m is the number of attribute neuro values and $\omega_{i(i=1,\ldots,m)}$ are the trained weights.

Layer 4-Picture Defuzzification. In this final step, we point out the minimum difference value in all values received from Layer 3,

$$Min\, D_0(I,K)=D_0(I,K_t).$$

Then, the output value of the ANPFIS is the solution S_t which is corresponding to the sample K_t.

4.3. Application of the ANPFIS algorithm in Controlling Network Power Consumption

In this part, we present the installation of ANPFIS algorithm in the trunk link state Controller of interconnection network.

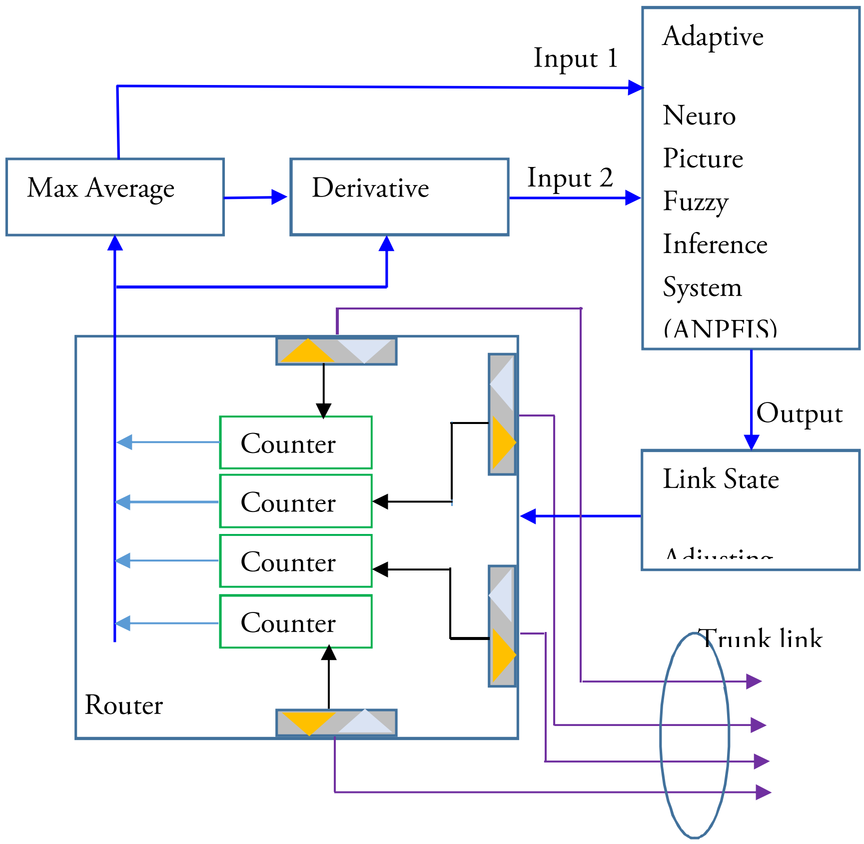

Fig. 4. The architecture of the trunk link state controller based on ANPFIS.

Fig. 4 describes the architecture of the trunk link state Controller based on ANPFIS. This Controller is developed from the previous architecture which is proposed by Phan et al. in 2017 for network-on-chip [27]. For details, each router input port will be equipped with a traffic counter. These counters count the data flits passing through the router in certain clock cycles based on the corresponding response signals from the router. The flits through the router is counted in a slot of time. When the counting finish, the traffic through the corresponding port will be calculated [27]. Each port of the router is connected with a Counter, then there are four average values of the traffic.

- The Max Average (MA) block receives the values of traffic from the counters which are connected with the routers ports. It compares these values and chose the maximum value for Input 1 of the ANPFIS.
- The Derivative (DER) block calculates the derivative of traffics obtained from the counters. This value is defined as an absolute value of the traffics change in a unit of time. This value is determined according to the maximum traffic value decided by MA block. After that, the DER gives it to the Input 2 of the ANPFIS for further processes.

  The value domain of Input 1 and Input 2 is from 0 to the maximum bandwidth value. They are normed into [0, 1] by Min Max normalization.

- Through the ANPFIS block, the received Output is the trunk link state $S_i, i \in \{1,2,3,4\}$. The received new state are adjusted by the Link State Adjusting block.

5. Experiments on Real-World Datasets

5.1. Experimental Environments

In order to evaluate performance, we test the ANPFIS method on the real datasets of the network traffic history taken from the UPV (Universitat Politècnica de València) university with related methods. The descriptions of the experimental dataset are presented in *Table 1.*

Table 1. The descriptions of the experimental dataset.

No. elements (checking-cycles)	16.571	
No. attributes	2 (MA, DER)	
The normalized value domain of attributes	MA [0,1]	DER [-1,1]
No. classes (No. link states)	4	

We compare the ANPFIS method against the methods of Hung (M2012) [21], Junjun et al. (M2013) [22], Maheshwari et al. (M2016) [23], Ngan et al. (H-max) [8], and ANFIS [28] in the Matlab 2015a programming language. The Mean Squared Error (MSE) degrees of these methods are given out to compare their performance.

5.2. The Quality

The MSE degree of the ANPFIS method are less than those of other methods. The specific values are expressed in **Table 2**.

Table 2. The performance of the methods.

Method	M2012 [21]	M2013 [22]	M2016 [23]	H-max [8]	ANFIS [28]	ANPFIS
MSE	0.2009	0.2606	0.2768	0.1259	0.2006	0.0089

Fig. 5 clearly show the difference between the performance values of six considered algorithms. In ***Fig. 5***, the blue columns illustrate the MSE values of the methods. It can be seen that the columns of the other methods are higher than that of the ANPFIS method. That means the accuracy of the proposed method is better than that of the related methods on the considered dataset.

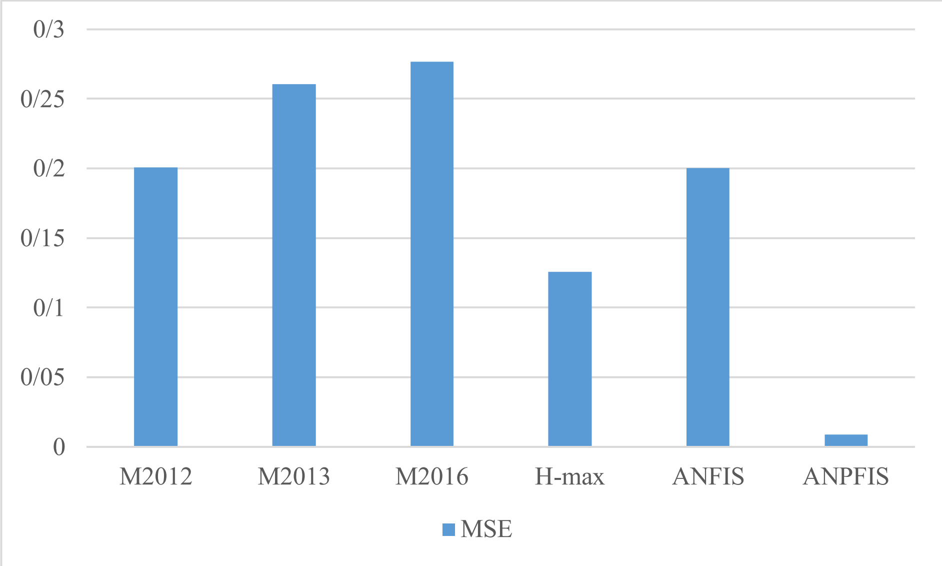

Fig. 5. The MSE values of 6 methods.

6. Conclusion

The neutrosophic theory increasingly attracts researchers and is applied in many fields. In this paper, a new single-valued neutrosophic distance measure is proposed. It is also a distance measure between picture fuzzy sets and is a development of the H-max measure which was introduced by Ngan et al. [8]. Further, an Adaptive NPFIS based on the proposed measure is shown and applied to the decision making for the link states in interconnection networks. The proposed model is tested on the real datasets taken from the UPV university. The MSE value of the proposed methods is less than that of other methods.

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Appendix

Source code and datasets of this paper can be found at this link, https://sourceforge.net/projects/pfdm-datasets-code/.

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Some Remarks on Neutro-Fine Topology

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Neutro-Fine Maximal Open Sets.

ABSTRACT

The neutro-fine topological space is a space that contains a combination of neutrosophic and fine sets. In this study, the various types of open sets such as generalized open and semi-open sets are defined in such space. The concept of interior and closure on semi-open sets are defined and some of their basic properties are stated. These definitions extend the concept to generalized semi-open sets. Moreover, the minimal and maximal open sets are defined and some of their properties are studied in this space. As well as, discussed the complement of all these sets as its closed sets. The basic properties of the union and intersection of these open sets are stated in some theorems. Only a few sets satisfy this postulates, and others are disproved as shown in the counterexamples. The converse of some theorems is proved in probable examples.

1. Introduction

The classical set theory developed by Zadeh [40] was termed as a Fuzzy Set (FS), whose elements amuse ambiguous features of true and false membership functions. The FS theory applied in the boundless area of a domain, while Atanassov [39] extended this theory as an Intuitionistic Fuzzy Set (IFS) theory. Later, Smarandache [21] explored a set that contains one more membership function called indeterminacy along with truth and falsity degrees as elements of the Neutrosophic Set (NS). Also, he generalized the NS on IFS [22] and recently proposed his work on attributes valued set, Plithogenic Set (PS) [23]. Nowadays, this set

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made an outstanding impact on many applications [1-4, 11-15, 16, 18, 19] and play a vital role in Decision Making (DM) problems [17, 20, 10] and Multi-Criteria DM (MCDM) problems [5, 9].

Topology is a study of flexible objects under frequent damages without splitting. In recent times, Topological Space (TS) is performing a lead character in the enormous branch of applied sciences and numerous categories of mathematics. The topological structure developed on NS as a generalization of IFTS which was originated by Salama & Alblowi [33, 34], named as Neutrosophic Topological Space (NTS). Few typical sets, open sets, and other TS explored [24, 7, 27, 28, 29, 31], and extended to bi-topological space [6] on such TS.

The most general class of sets which contains few open sets termed as Fine-Open Sets (FOSs), by Powar & Rajak [35], and investigated the special case of generalized TS, called Fine- Topological Space (FTS). Many researchers studied this concept on some sets like FS [26, 30], and others [25, 32]. Recently, this concept extends as Neutro-Fine Topological Space (NFTS) [8], which was introduced by Chinnadurai and Sindhu. The concept of minimal open (closed) and maximal open (closed) sets were exhibited by few researchers [36- 38].

The aspiration of this paper is to instigate the collection of open sets such as generalized open and semi-open sets defined on NFTS. The concept of interior and closure on neutro-fine-semi open sets are defined and some of their basic properties are stated. These definitions extend the concept to generalized semi-open sets. Moreover, the minimal and maximal open sets are defined and some of their properties are studied in this space. Simultaneously, discussed the complement of all these sets as its closed sets. The basic properties of the union and intersection of these sets are stated in some theorems. Only a few sets satisfy this postulates, and others are disproved as shown in the counterexamples. The converse of some theorems is proved in probable examples.

The layout of this proposal is as follows. In Portion 2, essential definitions of NFTS are recollected. In Portion 3, some type of generalized open sets are defined on NFTS and investigated its properties with illustrative examples. In Portion 4, some more open sets like neutro-fine minimal open sets and neutro-fine maximal open sets are explored via perfect examples. In the end, Portion 6 conveyed the conclusions with some future works.

2. Preliminaries

In this portion, we remind a few major descriptions connected to NFTS.

Definition 1. [8]. Let W be a set of universe and $w_i \in W$ where $i \in I$. Let R be a NS over W. Then the subset of NS R with respect to w_i (sub-NS R_{w_i}) and w_i, w_j (sub-NS R_{w_i,w_j}) are denoted as $\varsigma_R(w_i)$ and $\varsigma_R(w_i, w_j)$, and defined as

$$\varsigma_R(w_i)=\Big\{\big\langle w_i,T_R(w_i),I_R(w_i),F_R(w_i)\big\rangle,\big\langle w_{i,j},\max\big(T_R(w_i),T_R(w_j)\big),\max\big(I_R(w_i),I_R(w_j)\big),\min\big(F_R(w_i),F_R(w_j)\big)\big\rangle,$$
$$\big\langle w_k,T_R(0_n),I_R(0_n),F_R(0_n)\big\rangle,\big\langle w_{k,l},T_R(0_n),I_R(0_n),F_R(0_n)\big\rangle\Big\}$$

where $i\in I$, $j\in I-\{i\}$, $k,l\in I-\{i,j\}$ and $k\neq l$ and

$$\varsigma_R(w_i,w_j)=\Big\{\big\langle w_i,T_R(w_i),I_R(w_i),F_R(w_i)\big\rangle,\big\langle w_j,T_R(w_j),I_R(w_j),F_R(w_j)\big\rangle,\big\langle w_k,T_R(0_n),I_R(0_n),F_R(0_n)\big\rangle,$$
$$\big\langle w_{i,j},\max\big(T_R(w_i),T_R(w_j)\big),\max\big(I_R(w_i),I_R(w_j)\big),\min\big(F_R(w_i),F_R(w_j)\big)\big\rangle,$$
$$\big\langle w_{i,k},\max\big(T_R(w_i),T_R(w_k)\big),\max\big(I_R(w_i),I_R(w_k)\big),\min\big(F_R(w_i),F_R(w_k)\big)\big\rangle,$$
$$\big\langle w_{j,k},\max\big(T_R(w_j),T_R(w_k)\big),\max\big(I_R(w_j),I_R(w_k)\big),\min\big(F_R(w_j),F_R(w_k)\big)\big\rangle\Big\}$$

where $i,j,k\in I$ and $i\neq j\neq k$, respectively.

Definition 2. [8]. Let W be a set of universe and $w\in W$. Let R be a NS over W and V be any proper non-empty subset of W. Then $\varsigma_R(V)$ is said to be neutro-fine set (NFS) over W.

Definition 3. [8]. Let NFS(W) be the family of all NFSs over W. Then the fine collection of $\varsigma_R(V)$ is denoted as ${}^f\varsigma_W$ and defined over the NT (W, τ_n) as ${}^f\varsigma_W=\left\{0_{nf},1_{nf},\bigcup\varsigma_R(V)\right\}$.

Thus the triplet $\left(W,\tau_n,{}^f\varsigma_W\right)$ is said to be a NFTS over (W,τ_n). The elements belong to ${}^f\varsigma_W$ are said to be neutro-fine open sets (NFOSs) over (W,τ_n) and the complement of NFOSs are said to be neutro-fine closed sets (NFCSs) over (W,τ_n) and denote the collection by ${}^F\varsigma_W$.

Definition 4. [8]. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W, τ_n). Let $\varsigma_R(V)$ be a NFS over W. Then the neutro-fine interior of $\varsigma_R(V)$ is denoted as $Int_{nf}(\varsigma_R(V))$ and is defined as the union of all NFOSs contained in $\varsigma_R(V)$.

Clearly, $Int_{nf}(\varsigma_R(V))$ is the largest NFOS contained in $\varsigma_R(V)$.

Definition 5. [8]. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W, τ_n). Let $\varsigma_R(V)$ be a NFS over W. Then the neutro-fine closure of $\varsigma_R(V)$ is denoted as $Cl_{nf}(\varsigma_R(V))$ and is defined as the intersection of all NFCSs containing $\varsigma_R(V)$.

Clearly, $Cl_{nf}(\varsigma_R(V))$ is the smallest NFCS containing $\varsigma_R(V)$.

Definition 6. [8]. Let NF(W) be the family of all NFs over the universe W and $w\in W$. Then NFS $w^{\langle\alpha,\beta,\chi\rangle}$ is said to be a neutro-fine point (NFP), for $0\leq\alpha,\beta,\gamma\leq 1$ and is defined as follows:

$$\mathrm{w}^{\langle\alpha,\beta,\gamma\rangle}(\mathrm{v})=\begin{cases}(\alpha,\beta,\gamma), & \text{if } \mathrm{w}=\mathrm{v}\\(0,0,1), & \text{if } \mathrm{w}\neq\mathrm{v}\end{cases}.$$

Every NFS is the union of its NFPs.

Definition 7. [8]. Let $\left(W, \tau_n, {}^f\varsigma_W\right)$ be a NFTS over (W, τ_n). Let $\varsigma_R(V)$ be a NFS over W. Then $\varsigma_R(V)$ is said to be a neutro-fine neighborhood of the NFP $w^{\langle\alpha,\beta,\chi\rangle} \in \varsigma_R(V)$, if there exists a NFOS $\varsigma_R(U)$ such that $w^{\langle\alpha,\beta,\chi\rangle} \in \varsigma_R(U) \subseteq \varsigma_R(V)$.

Proposition 1. [8]. Let $\left(W, \tau_n, {}^f\varsigma_W\right)$ be a NFTS. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be two NFSs over W. Then,

$\mathrm{Int}_{nf}\left(0_{nf}\right) = 0_{nf}$ and $\mathrm{Int}_{nf}\left(1_{nf}\right) = 1_{nf}$:

$\varsigma_R(V_1)$ is NFOS $\Rightarrow \mathrm{Int}_{nf}\left(\varsigma_R(V_1)\right) = \varsigma_R(V_1)$;

$\mathrm{Int}_{nf}\left(\varsigma_R(V_1)\right) \subseteq \varsigma_R(V_1)$;

$\varsigma_R(V_1) \subseteq \varsigma_R(V_2) \Rightarrow \mathrm{Int}_{nf}\left(\varsigma_R(V_1)\right) \subseteq \mathrm{Int}_{nf}\left(\varsigma_R(V_2)\right)$;

$\mathrm{Int}_{nf}\left(\mathrm{Int}_{nf}\left(\varsigma_R(V_1)\right)\right) = \mathrm{Int}_{nf}\left(\varsigma_R(V_1)\right)$;

$\mathrm{Int}_{nf}\left(\varsigma_R(V_1) \cap \varsigma_R(V_2)\right) = \mathrm{Int}_{nf}\left(\varsigma_R(V_1)\right) \cap \mathrm{Int}_{nf}\left(\varsigma_R(V_2)\right)$;

$\mathrm{Int}_{nf}\left(\varsigma_R(V_1) \cup \varsigma_R(V_2)\right) \subseteq \mathrm{Int}_{nf}\left(\varsigma_R(V_1)\right) \cup \mathrm{Int}_{nf}\left(\varsigma_R(V_2)\right)$;

$\mathrm{Int}_{nf}\left(\varsigma_R(V_1)'\right) = \left[\mathrm{Cl}_{nf}\left(\varsigma_R(V_1)\right)\right]'$.

Proof. Straightforward.

Proposition 2. [8]. Let $\left(W, \tau_n, {}^f\varsigma_W\right)$ be a NFTS. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be two NFSs over W. Then,

$\mathrm{Cl}_{nf}\left(0_{nf}\right) = 0_{nf}$ and $\mathrm{Cl}_{nf}\left(1_{nf}\right) = 1_{nf}$;

$\varsigma_R(V_1)$ is NFCS $\Rightarrow \mathrm{Cl}_{nf}\left(\varsigma_R(V_1)\right) = \varsigma_R(V_1)$;

$\mathrm{Cl}_{nf}\left(\varsigma_R(V_1)\right) \supseteq \varsigma_R(V_1)$;

$\varsigma_R(V_1) \subseteq \varsigma_R(V_2) \Rightarrow \mathrm{Cl}_{nf}\left(\varsigma_R(V_1)\right) \subseteq \mathrm{Cl}_{nf}\left(\varsigma_R(V_2)\right)$;

$\mathrm{Cl}_{nf}\left(\mathrm{Cl}_{nf}\left(\varsigma_R(V_1)\right)\right) = \mathrm{Cl}_{nf}\left(\varsigma_R(V_1)\right)$;

$\mathrm{Cl}_{nf}\left(\varsigma_R(V_1) \cup \varsigma_R(V_2)\right) = \mathrm{Cl}_{nf}\left(\varsigma_R(V_1)\right) \cup \mathrm{Cl}_{nf}\left(\varsigma_R(V_2)\right)$;

$\mathrm{Cl}_{nf}\left(\varsigma_R(V_1) \cap \varsigma_R(V_2)\right) \subseteq \mathrm{Cl}_{nf}\left(\varsigma_R(V_1)\right) \cap \mathrm{Cl}_{nf}\left(\varsigma_R(V_2)\right)$;

$\mathrm{Cl}_{nf}(\varsigma_R(V_1)') = [\mathrm{Int}_{nf}(\varsigma_R(V_1))]'$.

Proof. Straightforward.

3. Some Form of Generalized Open Sets in NFTS

In this portion, some open types of generalized open sets on NFTS are defined and probable results are carried by some major expressive examples. This portion is splitted into 3 sub-portions which states neutro-fine-generalized, neutro-fine-semi, and neutro-fine-generalized semi-open sets on NFTS.

3.1. Neutro-Fine-Generalized Open Sets

Let $\varsigma_R(V)$ be a NFS over W of a NFTS $(W,\tau_n,{}^f\varsigma_W)$. Then $\varsigma_R(V_1)$ is said to be a neutro-fine-generalized closed set (nf-GCS) if $Cl_{nf}(\varsigma_R(V)) \subseteq \varsigma_R(U)$ whenever $\varsigma_R(V) \subseteq \varsigma_R(U)$ and $\varsigma_R(U)$ is NFOS. The complement of nf-GCS is said to be neutro-fine-generalized open set (nf-GOS).

Theorem 1. Every NFCS is a nf-GCS in NFTS $(W,\tau_n,{}^f\varsigma_W)$.

Proof. Let $\varsigma_R(V)$ be a NFCS on $(W,\tau_n,{}^f\varsigma_W)$. Let $\varsigma_R(V) \subseteq \varsigma_R(U)$, where $\varsigma_R(U)$ is NFOS in $(W,\tau_n,{}^f\varsigma_W)$. Since $\varsigma_R(V)$ is a NFCS, $\varsigma_R(V) = Cl_{nf}(\varsigma_R(V)) \Rightarrow Cl_{nf}(\varsigma_R(V)) \subseteq \varsigma_R(V)$. Thus $Cl_{nf}(\varsigma_R(V)) \subseteq \varsigma_R(V) \subseteq \varsigma_R(U)$. Hence $\varsigma_R(V)$ is a nf-GCS in NFTS $(W,\tau_n,{}^f\varsigma_W)$.

Remark 1. The converse of the above theorem is not true as shown in the following example.

Example 1. Let $W = \{w_1,w_2,w_3\}$ and $\tau_n = \{0_n,1_n,R,S,T,U\}$ where R, S, T and U are NSs over W and are defined as follows

$$R = \{\langle w_1,.2,.4,.7\rangle,\langle w_2,.6,.3,.1\rangle,\langle w_3,.4,.5,.6\rangle\},$$

$$S = \{\langle w_1,.9,.3,.6\rangle,\langle w_2,.6,.5,.4\rangle,\langle w_3,.7,.8,.1\rangle\},$$

$$T = \{\langle w_1,.9,.4,.6\rangle,\langle w_2,.6,.5,.1\rangle,\langle w_3,.7,.8,.1\rangle\} \text{ and}$$

$$U = \{\langle w_1,.2,.3,.7\rangle,\langle w_2,.6,.3,.4\rangle,\langle w_3,.4,.5,.6\rangle\}.$$

Thus (W,τ_n) is a NTS over W.

Then NFOSs over (W,τ_n) are ${}^f\varsigma_W = \{0_n,1_n,\varsigma_R(w_1),\varsigma_R(w_3),\varsigma_S(w_2,w_3)\}$, where

$$\varsigma_R(w_1) = \{\langle w_1,.2,.4,.7\rangle,\langle w_2,0,0,1\rangle,\langle w_3,0,0,1\rangle,\langle w_{1,2},.6,.4,.1\rangle,\langle w_{1,3},.4,.5,.6\rangle,\langle w_{2,3},0,0,1\rangle\},$$

$\varsigma_R(w_3)=\{\langle w_1,0,0,1\rangle,\langle w_2,0,0,1\rangle,\langle w_3,.4,.5,.6\rangle,\langle w_{1,2},0,0,1\rangle,\langle w_{1,3},.4,.5,.6\rangle,\langle w_{2,3},.6,.5,.1\rangle\}$,

$\varsigma_S(w_2,w_3)=\{\langle w_1,0,0,1\rangle,\langle w_2,.6,.5,.4\rangle,\langle w_3,.7,.8,.1\rangle,\langle w_{1,2},.9,.5,.4\rangle,\langle w_{1,3},.9,.8,.1\rangle,\langle w_{2,3},.7,.8,.1\rangle\}$

and NFCSs over (W,τ_n) are ${}^F\varsigma_W=\{0_n,1_n,\varsigma_R(w_1)',\varsigma_R(w_3)'\varsigma_S(w_2,w_3)'\}$, where

$\varsigma_R(w_1)'=\{\langle w_1,.7,.6,.2\rangle,\langle w_2,1,1,0\rangle,\langle w_3,1,1,0\rangle,\langle w_{1,2},.1,.6,.6\rangle,\langle w_{1,3},.6,.5,.4\rangle,\langle w_{2,3},1,1,0\rangle\}$,

$\varsigma_R(w_3)'=\{\langle w_1,1,1,0\rangle,\langle w_2,1,1,0\rangle,\langle w_3,.6,.5,.4\rangle,\langle w_{1,2},1,1,0\rangle,\langle w_{1,3},.6,.5,.4\rangle,\langle w_{2,3},.1,.5,.6\rangle\}$,

$\varsigma_S(w_2,w_3)'=\{\langle w_1,1,1,0\rangle,\langle w_2,.4,.5,.6\rangle,\langle w_3,.1,.2,.7\rangle,\langle w_{1,2},.4,.5,.9\rangle,\langle w_{1,3},.1,.2,.9\rangle,\langle w_{2,3},.1,.2,.7\rangle\}$.

Thus $(W,\tau_n,{}^f\varsigma_W)$ is a NFTS over (W,τ_n). Here nf-GCS $=\{\varsigma_S(w_2),\varsigma_S(w_3),\varsigma_S(w_{2,3})\}$. Thus $\varsigma_S(w_2)$ is nf-GCS but not NFCS.

Theorem 2. If $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ are nf-GCSs over $(W,\tau_n,{}^f\varsigma_W)$, then $\varsigma_R(V_1)\cup\varsigma_R(V_2)$ is also a nf-GCS over $(W,\tau_n,{}^f\varsigma_W)$.

Proof. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be nf-GCSs over $(W,\tau_n,{}^f\varsigma_W)$. Then $Cl_{nf}(\varsigma_R(V_1))\subseteq\varsigma_R(U)$ whenever $\varsigma_R(V_1)\subseteq\varsigma_R(U)$ and $\varsigma_R(U)$ is NFOS and $Cl_{nf}(\varsigma_R(V))\subseteq\varsigma_R(U)$ whenever $\varsigma_R(V)\subseteq\varsigma_R(U)$ and $\varsigma_R(U)$ is NFOS. Since $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ are subsets of $\varsigma_R(U)$, $\varsigma_R(V_1)\cup\varsigma_R(V_2)$ are subsets of $\varsigma_R(U)$ and $\varsigma_R(U)$ is NFOS. Then by ***Proposition 2***, $Cl_{nf}(\varsigma_R(V_1)\cup\varsigma_R(V_2))=Cl_{nf}(\varsigma_R(V_1))\cup Cl_{nf}(\varsigma_R(V_2))$. Thus $Cl_{nf}(\varsigma_R(V_1)\cup\varsigma_R(V_2))\subseteq\varsigma_R(U)$. Hence $\varsigma_R(V_1)\cup\varsigma_R(V_2)$ is a nf-GCSs over $(W,\tau_n,{}^f\varsigma_W)$.

Remark 2. The intersection of two nf-GCSs need not be a nf-GCS as shown in the following example.

Example 2. Consider the ***Example 1***. Here nf-GCS $=\{\varsigma_S(w_2),\varsigma_S(w_3),\varsigma_S(w_{2,3})\}$. Then

$\varsigma_S(w_2)\cap\varsigma_S(w_3)=\{\langle w_1,0,0,1\rangle,\langle w_2,0,0,1\rangle,\langle w_3,0,0,1\rangle,\langle w_{1,2},0,0,1\rangle,\langle w_{1,3},0,0,1\rangle,\langle w_{2,3},.7,.8,.1\rangle\}$, is not a nf-GCS.

Theorem 3. If $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ are nf-GCSs over $(W,\tau_n,{}^f\varsigma_W)$, then

$Cl_{nf}(\varsigma_R(V_1)\cap\varsigma_R(V_2))\subseteq Cl_{nf}(\varsigma_R(V_1))\cap Cl_{nf}(\varsigma_R(V_2))$.

Proof. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be nf-GCSs over $(W,\tau_n,{}^f\varsigma_W)$. Then $Cl_{nf}(\varsigma_R(V_1))\subseteq\varsigma_R(U)$ whenever $\varsigma_R(V_1)\subseteq\varsigma_R(U)$ and $\varsigma_R(U)$ is NFOS and $Cl_{nf}(\varsigma_R(V))\subseteq\varsigma_R(U)$ whenever $\varsigma_R(V)\subseteq\varsigma_R(U)$ and $\varsigma_R(U)$ is NFOS.

Since $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ are subsets of $\varsigma_R(U)$, $\varsigma_R(V_1)\cap\varsigma_R(V_2)$ are subsets of $\varsigma_R(U)$ and $\varsigma_R(U)$ is NFOS.

Since $\varsigma_R(V_1)\cap\varsigma_R(V_2)\subseteq\varsigma_R(V_1)$ and $\varsigma_R(V_1)\cap\varsigma_R(V_2)\subseteq\varsigma_R(V_1)$, $Cl_{nf}(\varsigma_R(V_1)\cap\varsigma_R(V_2))\subseteq Cl_{nf}(\varsigma_R(V_1))$ and $Cl_{nf}(\varsigma_R(V_1)\cap\varsigma_R(V_2))\subseteq Cl_{nf}(\varsigma_R(V_2))$, by ***Proposition 2***. Thus $Cl_{nf}(\varsigma_R(V_1)\cap\varsigma_R(V_2))\subseteq Cl_{nf}(\varsigma_R(V_1))\cap Cl_{nf}(\varsigma_R(V_2))$.

3.2. Neutro-Fine-Semi Open Sets

Definition 8. Let $\varsigma_R(V)$ be a NFS over W of a NFTS $(W,\tau_n,{}^f\varsigma_W)$. Then $\varsigma_R(V)$ is said to be a neutro-fine-semi closed set (nf-SCS) if $Int_{nf}(Cl_{nf}(\varsigma_R(V)))\subseteq\varsigma_R(V)$.

The complement of nf -SCS is said to be neutro-fine-semi open set (nf -SOS), i.e., $\varsigma_R(V)\subseteq Cl_{nf}(Int_{nf}(\varsigma_R(V)))$.

Theorem 4. Let $(W,\tau_n,{}^f\varsigma_W)$ be a NFTS and $\varsigma_R(V)$ be a NFS over W. Then $\varsigma_R(V)$ is nf-SCS if and only if $\varsigma_R(V)'$ is nf-SOS.

Proof. Let $\varsigma_R(V)$ be a nf-SCS. Then $Int_{nf}(Cl_{nf}(\varsigma_R(V)))\subseteq\varsigma_R(V)$.

Taking complement on both sides,

$$\varsigma_R(V)'\subseteq\left[Int_{nf}(Cl_{nf}(\varsigma_R(V)))\right]'=Cl_{nf}(Cl_{nf}(\varsigma_R(V)))'.$$

By using ***Proposition 1***, $\varsigma_R(V)'=Cl_{nf}(Int_{nf}(\varsigma_R(V)'))$. Thus $\varsigma_R(V)'$ is a nf-SOS. Conversely, assume that $\varsigma_R(V)'$ is a nf-SOS. Then $\varsigma_R(V)'=Cl_{nf}(Int_{nf}(\varsigma_R(V)'))$.

Taking complement on both sides,

$$\varsigma_R(V)\supseteq\left[Cl_{nf}(Int_{nf}(\varsigma_R(V)'))\right]'=Int_{nf}(Int_{nf}(\varsigma_R(V)'))',$$ by ***Proposition 1***.

By **Proposition 2**, $\varsigma_R(V)\supseteq Int_{nf}(Cl_{nf}(\varsigma_R(V)))$. Thus $\varsigma_R(V)$ is a nf-SCS.

Theorem 5. If $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ are nf-SCSs over NFTS $(W,\tau_n,{}^f\varsigma_W)$, then $\varsigma_R(V_1)\cap\varsigma_R(V_2)$ is also a nf -SCS in $(W,\tau_n,{}^f\varsigma_W)$.

Proof. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be nf-SCSs over $(W,\tau_n,{}^f\varsigma_W)$. Then $Int_{nf}(Cl_{nf}(\varsigma_R(V_1)))\subseteq\varsigma_R(V_1)$ and $Int_{nf}(Cl_{nf}(\varsigma_R(V_1)))\subseteq\varsigma_R(V_1)$. Thus $\varsigma_R(V_1)\cap\varsigma_R(V_2)\supseteq Int_{nf}(Cl_{nf}(\varsigma_R(V_1)))\cap Int_{nf}(Cl_{nf}(\varsigma_R(V_2)))$

$$=Int_{nf}(Cl_{nf}(\varsigma_R(V_1))\cap Cl_{nf}(\varsigma_R(V_2)))\supseteq Int_{nf}(Cl_{nf}(\varsigma_R(V_1)\cap\varsigma_R(V_2))),$$ by ***Propositions 1*** and ***2.***

Hence $\varsigma_R(V_1)\cap\varsigma_R(V_2)$ is a nf-SCSs in $(W,\tau_n,{}^f\varsigma_W)$.

Theorem 6. If $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ are nf-SOSs over NFTS $\left(W,\tau_n,{}^f\varsigma_W\right)$, then $\varsigma_R(V_1)\cup\varsigma_R(V_2)$ is also a nf-SOS in $\left(W,\tau_n,{}^f\varsigma_W\right)$.

Proof. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be nf-SCSs over $\left(W,\tau_n,{}^f\varsigma_W\right)$. Then $\varsigma_R(V_1)\subseteq Cl_{nf}\left(Int_{nf}\left(\varsigma_R(V_1)\right)\right)$ and $\varsigma_R(V_2)\subseteq Cl_{nf}\left(Int_{nf}\left(\varsigma_R(V_2)\right)\right)$. Thus $\varsigma_R(V_1)\cup\varsigma_R(V_2)\subseteq Cl_{nf}\left(Int_{nf}\left(\varsigma_R(V_1)\right)\right)\cup Cl_{nf}\left(Int_{nf}\left(\varsigma_R(V_2)\right)\right)$

$=Cl_{nf}\left(Int_{nf}\left(\varsigma_R(V_1)\right)\cup Int_{nf}\left(\varsigma_R(V_2)\right)\right)\supseteq Cl_{nf}\left(Int_{nf}\left(\varsigma_R(V_1)\cup\varsigma_R(V_2)\right)\right)$, by ***Propositions 1*** and ***2***.

Theorem 7. Every NFCS is a nf-SCS in NFTS $\left(W,\tau_n,{}^f\varsigma_W\right)$.

Proof. Let $\varsigma_R(V)$ be a NFCS on $\left(W,\tau_n,{}^f\varsigma_W\right)$. Then $\varsigma_R(V)=Cl_{nf}\left(\varsigma_R(V)\right)$. Thus

$Int_{nf}\left(Cl_{nf}\left(\varsigma_R(V)\right)\right)\subseteq Cl_{nf}\left(\varsigma_R(V)\right)\ \Rightarrow Int_{nf}\left(Cl_{nf}\left(\varsigma_R(V)\right)\right)\subseteq\varsigma_R(V)$. Hence $\varsigma_R(V)$ is a nf-SCS in $\left(W,\tau_n,{}^f\varsigma_W\right)$.

Definition 9. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n). Let $\varsigma_R(V)$ be a NFS over W. Then the neutro-fine-semi interior of $\varsigma_R(V)$ is denoted as $S^*Int_{nf}\left(\varsigma_R(V)\right)$ and is defined as the union of all nf-SOSs contained in $\varsigma_R(V)$. Clearly, $S^*Int_{nf}\left(\varsigma_R(V)\right)$ is the largest nf-SOS contained in $\varsigma_R(V)$.

Definition 10. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n). Let $\varsigma_R(V)$ be a NFS over W. Then the neutro-fine-semi closure of $\varsigma_R(V)$ is denoted as $S^*Cl_{nf}\left(\varsigma_R(V)\right)$ and is defined as the intersection of all nf-SCSs containing $\varsigma_R(V)$. Clearly, $S^*Cl_{nf}\left(\varsigma_R(V)\right)$ is the smallest nf-SCS containing $\varsigma_R(V)$.

Proposition 3. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be two NFSs over W. Then,

$S^*Int_{nf}\left(\varsigma_R(V_1)\right)\subseteq\varsigma_R(V_1)$;

$\varsigma_R(V_1)$ is nf-SOS $\Rightarrow S^*Int_{nf}\left(\varsigma_R(V_1)\right)=\varsigma_R(V_1)$;

$S^*Int_{nf}\left(S^*Int_{nf}\left(\varsigma_R(V_1)\right)\right)=S^*Int_{nf}\left(\varsigma_R(V_1)\right)$;

$\varsigma_R(V_1)\subseteq\varsigma_R(V_2)\Rightarrow S^*Int_{nf}\left(\varsigma_R(V_1)\right)\subseteq S^*Int_{nf}\left(\varsigma_R(V_2)\right)$.

Proof. Straightforward.

Proposition 4. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be two NFSs over W. Then,

$\varsigma_R(V_1)\subseteq S^*Cl_{nf}\left(\varsigma_R(V_1)\right)$;

$\varsigma_R(V_1)$ is nf-SCS $\Rightarrow S^*Cl_{nf}(\varsigma_R(V_1)) = \varsigma_R(V_1)$;

$S^*Cl_{nf}(S^*Cl_{nf}(\varsigma_R(V_1))) = S^*Cl_{nf}(\varsigma_R(V_1))$;

$\varsigma_R(V_1) \subseteq \varsigma_R(V_2) \Rightarrow S^*Cl_{nf}(\varsigma_R(V_1)) \subseteq S^*Cl_{nf}(\varsigma_R(V_2))$.

Proof. Straightforward.

Proposition 5. Let $(W, \tau_n, {}^f\varsigma_W)$ be a NFTS. Let $\varsigma_R(V)$ be any NFS over *W*. Then,

$Int_{nf}(\varsigma_R(V)) \subseteq S^*Int_{nf}(\varsigma_R(V)) \subseteq \varsigma_R(V)$;

$\varsigma_R(V) \subseteq S^*Cl_{nf}(\varsigma_R(V)) \subseteq Cl_{nf}(\varsigma_R(V))$;

$S^*Cl_{nf}(\varsigma_R(V)') = [S^*Int_{nf}(\varsigma_R(V))]'$;

$S^*Int_{nf}(\varsigma_R(V)') = [S^*Cl_{nf}(\varsigma_R(V))]'$.

Proof. Straightforward.

Proposition 6. Let $(W, \tau_n, {}^f\varsigma_W)$ be a NFTS. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be two NFSs over *W*. Then,

$S^*Int_{nf}(\varsigma_R(V_1) \cap \varsigma_R(V_2)) = S^*Int_{nf}(\varsigma_R(V_1)) \cap S^*Int_{nf}(\varsigma_R(V_2))$;

$S^*Int_{nf}(\varsigma_R(V_1) \cup \varsigma_R(V_2)) \supseteq S^*Int_{nf}(\varsigma_R(V_1)) \cup S^*Int_{nf}(\varsigma_R(V_2))$.

Proof. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be two NFSs over *W*.

Since $\varsigma_R(V_1) \cap \varsigma_R(V_2) \subseteq \varsigma_R(V_1)$ and $\varsigma_R(V_1) \cap \varsigma_R(V_2) \subseteq \varsigma_R(V_2)$, by using ***Proposition 3***, $S^*Int_{nf}(\varsigma_R(V_1) \cap \varsigma_R(V_2)) \subseteq S^*Int_{nf}(\varsigma_R(V_1))$ and $S^*Int_{nf}(\varsigma_R(V_1) \cap \varsigma_R(V_2)) \subseteq S^*Int_{nf}(\varsigma_R(V_2))$.

This implies that,

$$S^*Int_{nf}(\varsigma_R(V_1) \cap \varsigma_R(V_2)) \subseteq S^*Int_{nf}(\varsigma_R(V_1)) \cap S^*Int_{nf}(\varsigma_R(V_2)), \tag{1}$$

By using ***Proposition 3***,

$S^*Int_{nf}(\varsigma_R(V_1)) \subseteq \varsigma_R(V_1)$ and $S^*Int_{nf}(\varsigma_R(V_2)) \subseteq \varsigma_R(V_2)$
$\Rightarrow S^*Int_{nf}(\varsigma_R(V_1)) \cap S^*Int_{nf}(\varsigma_R(V_2)) \subseteq \varsigma_R(V_1) \cap \varsigma_R(V_2)$.

By using ***Proposition 3***,

$S^{*}Int_{nf}\left[S^{*}Int_{nf}\left(\varsigma_{R}(V_{1})\right)\cap S^{*}Int_{nf}\left(\varsigma_{R}(V_{2})\right)\right]\subseteq S^{*}Int_{nf}\left(\varsigma_{R}(V_{1})\cap\varsigma_{R}(V_{2})\right)$.

By ***Eq. (1)***,

$S^{*}Int_{nf}\left(S^{*}Int_{nf}\left(\varsigma_{R}(V_{1})\right)\right)\cap S^{*}Int_{nf}\left(S^{*}Int_{nf}\left(\varsigma_{R}(V_{2})\right)\right)\subseteq S^{*}Int_{nf}\left(\varsigma_{R}(V_{1})\cap\varsigma_{R}(V_{2})\right)$.

By using ***Proposition 3***,

$$\mathrm{S^{*}Int_{nf}}\left(\varsigma_{R}(\mathrm{V}_{1})\right)\cap \mathrm{S^{*}Int_{nf}}\left(\varsigma_{R}(\mathrm{V}_{2})\right)\subseteq \mathrm{S^{*}Int_{nf}}\left(\varsigma_{R}(\mathrm{V}_{1})\cap\varsigma_{R}(\mathrm{V}_{2})\right), \tag{2}$$

Hence from ***Eqs. (1)-(2)***,

$S^{*}Int_{nf}\left(\varsigma_{R}(V_{1})\cap\varsigma_{R}(V_{2})\right)=S^{*}Int_{nf}\left(\varsigma_{R}(V_{1})\right)\cap S^{*}Int_{nf}\left(\varsigma_{R}(V_{2})\right)$.
Since $\varsigma_{R}(V_1)\subseteq\varsigma_{R}(V_1)\cup\varsigma_{R}(V_2)$ and $\varsigma_{R}(V_1)\subseteq\varsigma_{R}(V_1)\cup\varsigma_{R}(V_2)$, by using ***Proposition 3***,

$S^{*}Int_{nf}\left(\varsigma_{R}(V_{1})\right)\subseteq S^{*}Int_{nf}\left(\varsigma_{R}(V_{1})\cup\varsigma_{R}(V_{2})\right)$ and $S^{*}Int_{nf}\left(\varsigma_{R}(V_{2})\right)\subseteq S^{*}Int_{nf}\left(\varsigma_{R}(V_{1})\cup\varsigma_{R}(V_{2})\right)$. Hence $S^{*}Int_{nf}\left(\varsigma_{R}(V_{1})\cup\varsigma_{R}(V_{2})\right)\supseteq S^{*}Int_{nf}\left(\varsigma_{R}(V_{1})\right)\cup S^{*}Int_{nf}\left(\varsigma_{R}(V_{2})\right)$.

Proposition 7. Let $\left(W,\tau_{n},{}^{f}\varsigma_{W}\right)$ be a NFTS. Let $\varsigma_{R}(V_1)$ and $\varsigma_{R}(V_2)$ be two NFSs over *W*. Then,

$\mathrm{S^{*}Cl_{nf}}\left(\varsigma_{R}(\mathrm{V}_{1})\cup\varsigma_{R}(\mathrm{V}_{2})\right)=\mathrm{S^{*}Cl_{nf}}\left(\varsigma_{R}(\mathrm{V}_{1})\right)\cup \mathrm{S^{*}Cl_{nf}}\left(\varsigma_{R}(\mathrm{V}_{2})\right)$;

$\mathrm{S^{*}Cl_{nf}}\left(\varsigma_{R}(\mathrm{V}_{1})\cap\varsigma_{R}(\mathrm{V}_{2})\right)\subseteq \mathrm{S^{*}Cl_{nf}}\left(\varsigma_{R}(\mathrm{V}_{1})\right)\cap \mathrm{S^{*}Cl_{nf}}\left(\varsigma_{R}(\mathrm{V}_{2})\right)$.

Proof. Let $\varsigma_{R}(V_1)$ and $\varsigma_{R}(V_2)$ be two NFSs over *W*.

Since $S^{*}Cl_{nf}\left(\varsigma_{R}(V_1)\cup\varsigma_{R}(V_2)\right)=S^{*}Cl_{nf}\left(\left(\varsigma_{R}(V_1)\cup\varsigma_{R}(V_2)\right)'\right)'$, by using ***Proposition 5***, $S^{*}Cl_{nf}\left(\varsigma_{R}(V_{1})\cup\varsigma_{R}(V_{2})\right)=\left[S^{*}Int_{nf}\left(\left(\varsigma_{R}(V_{1})\cup\varsigma_{R}(V_{2})\right)'\right)\right]'=\left[S^{*}Int_{nf}\left(\left(\varsigma_{R}(V_{1})'\right)\cap\left(\varsigma_{R}(V_{2})'\right)\right)\right]'$.

Again by using ***Proposition 5***, $S^{*}Cl_{nf}\left(\varsigma_{R}(V_{1})\cup\varsigma_{R}(V_{2})\right)=\left[S^{*}Int_{nf}\left(\varsigma_{R}(V_{1})'\right)\cap S^{*}Int_{nf}\left(\varsigma_{R}(V_{2})'\right)\right]'$

$=\left(S^{*}Int_{nf}\left(\varsigma_{R}(V_{1})'\right)\right)'\cup\left(S^{*}Int_{nf}\left(\varsigma_{R}(V_{2})'\right)\right)'$.

By using ***Proposition 5***, $S^{*}Cl_{nf}\left(\varsigma_{R}(V_{1})\cup\varsigma_{R}(V_{2})\right)=S^{*}Cl_{nf}\left(\left(\varsigma_{R}(V_{1})'\right)\right)'\cup S^{*}Cl_{nf}\left(\left(\varsigma_{R}(V_{2})'\right)\right)'$

$=S^{*}Cl_{nf}\left(\varsigma_{R}(V_{1})\right)\cup S^{*}Cl_{nf}\left(\varsigma_{R}(V_{2})\right)$.

Hence $S^{*}Cl_{nf}\left(\varsigma_{R}(V_{1})\cup\varsigma_{R}(V_{2})\right)=S^{*}Cl_{nf}\left(\varsigma_{R}(V_{1})\right)\cup S^{*}Cl_{nf}\left(\varsigma_{R}(V_{2})\right)$.

(ii) Since $\varsigma_R(V_1)\cap\varsigma_R(V_2)\subseteq\varsigma_R(V_1)$ and $\varsigma_R(V_1)\cap\varsigma_R(V_2)\subseteq\varsigma_R(V_2)$, by using ***Proposition 4***, $S^*Cl_{nf}(\varsigma_R(V_1)\cap\varsigma_R(V_2))\subseteq S^*Cl_{nf}(\varsigma_R(V_1))$ and $S^*Cl_{nf}(\varsigma_R(V_1)\cap\varsigma_R(V_2))\subseteq S^*Cl_{nf}(\varsigma_R(V_2))$.

Hence $S^*Cl_{nf}(\varsigma_R(V_1)\cap\varsigma_R(V_2))\subseteq S^*Cl_{nf}(\varsigma_R(V_1))\cap S^*Cl_{nf}(\varsigma_R(V_2))$.

3.3. Neutro-Fine-Generalized Semi Open Sets

Definition 11. Let $\varsigma_R(V)$ be a NFS over W of a NFTS $(W,\tau_n,{}^f\varsigma_W)$. Then $\varsigma_R(V)$ is said to be a neutro-fine-generalized semi closed set (nf -GSCS) if $S^*Cl_{nf}(\varsigma_R(V))\subseteq\varsigma_R(U)$ whenever $\varsigma_R(V)\subseteq\varsigma_R(U)$ and $\varsigma_R(U)$ is NFOS.

The complement of *nf* -GSCS is said to be neutro-fine-generalized semi open set (*nf* -GSOS), i.e., $\varsigma_R(U)\subseteq S^*Int_{nf}(\varsigma_R(V))$ whenever $\varsigma_R(U)\subseteq\varsigma_R(V)$ and $\varsigma_R(U)$ is NFCS.

Example 3. Consider ***Example 1***. Thus *nf* -SCS $=\{\varsigma_R(w_1,w_3),\varsigma_S(w_1)\}$, *nf* -SOS $=\{\varsigma_R(w_1,w_3)',\varsigma_S(w_1)'\}$ where $\varsigma_R(w_1,w_3)'=\{\langle w_1,.7,.6,.2\rangle,\langle w_2,1,1,0\rangle,\langle w_3,.6,.5,.4\rangle,\langle w_{1,2},.1,.6,.6\rangle,\langle w_{1,3},.6,.5,.4\rangle,\langle w_{2,3},.1,.5,.6\rangle\}$, $\varsigma_S(w_1)'=\{\langle w_1,.6,.7,.9\rangle,\langle w_2,.1,1,0\rangle,\langle w_3,1,1,0\rangle,\langle w_{1,2},.4,.5,.9\rangle,\langle w_{1,3},.1,.2,.9\rangle,\langle w_{2,3},1,1,0\rangle\}$ and *nf* -GSCS $=\{\varsigma_S(w_2)\}$.

Theorem 8. Every NFCS is a *nf*-GSCS in NFTS $(W,\tau_n,{}^f\varsigma_W)$.

Proof. Let $\varsigma_R(V)$ be a NFCS in NFTS $(W,\tau_n,{}^f\varsigma_W)$.

Let $\varsigma_R(V)\subseteq\varsigma_R(U)$, where $\varsigma_R(U)$ is NFOS in $(W,\tau_n,{}^f\varsigma_W)$. Since $\varsigma_R(V)$ is a NFCS, $\varsigma_R(V)=Cl_{nf}(\varsigma_R(V))$, by ***Proposition 2***. Also, by ***Proposition 5***, $S^*Cl_{nf}(\varsigma_R(V))\subseteq Cl_{nf}(\varsigma_R(V))$. Thus $S^*Cl_{nf}(\varsigma_R(V))\subseteq Cl_{nf}(\varsigma_R(V))=\varsigma_R(V)\subseteq\varsigma_R(U)$. Hence $\varsigma_R(V)$ is a *nf*-GSCS in NFTS $(W,\tau_n,{}^f\varsigma_W)$.

Theorem 9. If $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ are *nf*-GSCSs over NFTS $(W,\tau_n,{}^f\varsigma_W)$, then $\varsigma_R(V_1)\cap\varsigma_R(V_2)$ is also a *nf*-GSCS in $(W,\tau_n,{}^f\varsigma_W)$.

Proof. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be *nf*-GSCSs over $(W,\tau_n,{}^f\varsigma_W)$.

If $\varsigma_R(V_1)\cap\varsigma_R(V_2)\subseteq\varsigma_R(U)$ and $\varsigma_R(U)$ is a NFOS, then $\varsigma_R(V_1)\subseteq\varsigma_R(U)$ and $\varsigma_R(V_1)\subseteq\varsigma_R(U)$.

Since $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ are *nf*–GSCSs, $S^*Cl_{nf}(\varsigma_R(V_1))\subseteq\varsigma_R(U)$ and $S^*Cl_{nf}(\varsigma_R(V_2))\subseteq\varsigma_R(U)$.

Thus $S^*Cl_{nf}(\varsigma_R(V_1))\cap S^*Cl_{nf}(\varsigma_R(V_2))\subseteq\varsigma_R(U)$.

By ***Proposition 7***, $S^*Cl_{nf}(\varsigma_R(V_1)\cap\varsigma_R(V_2))\subseteq S^*Cl_{nf}(\varsigma_R(V_1))\cap S^*Cl_{nf}(\varsigma_R(V_2))\subseteq\varsigma_R(U)$. This implies that,

$S^*Cl_{nf}(\varsigma_R(V_1)\cap\varsigma_R(V_2))\subseteq\varsigma_R(U)$. Thus $S^*Cl_{nf}(\varsigma_R(V_1)\cap\varsigma_R(V_2))\subseteq\varsigma_R(U)$, $\varsigma_R(V_1)\cap\varsigma_R(V_2)\subseteq\varsigma_R(U)$ and $\varsigma_R(U)$ is a NFOS.

Hence $\varsigma_R(V_1)\cap\varsigma_R(V_2)$ is a nf-GSCS over $(W,\tau_n,{}^f\varsigma_W)$.

Theorem 10. Every NFOS is a nf-GSOS in NFTS $(W,\tau_n,{}^f\varsigma_W)$.

Proof. Let $\varsigma_R(V)$ be a NFOS in NFTS $(W,\tau_n,{}^f\varsigma_W)$. Let $\varsigma_R(U)\subseteq\varsigma_R(V)$, where $\varsigma_R(U)$ is NFCS in $(W,\tau_n,{}^f\varsigma_W)$.

Since $\varsigma_R(V)$ is a NFOS, $\varsigma_R(V)=Int_{nf}(\varsigma_R(V))$, by ***Proposition 1***.

Also, by ***Proposition 5,*** $Int_{nf}(\varsigma_R(V))\subseteq S^*Int_{nf}(\varsigma_R(V))\subseteq\varsigma_R(V)$. Thus $\varsigma_R(V)=S^*Int_{nf}(\varsigma_R(V))$ $\Rightarrow\varsigma_R(U)\subseteq\varsigma_R(V)=S^*Int_{nf}(\varsigma_R(V))$.

Hence $\varsigma_R(V)$ is a nf-GSCS in NFTS $(W,\tau_n,{}^f\varsigma_W)$.

Theorem 11. If $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ are nf-GSOSs over NFTS $(W,\tau_n,{}^f\varsigma_W)$, then $\varsigma_R(V_1)\cup\varsigma_R(V_2)$ is also a nf-GSOS in $(W,\tau_n,{}^f\varsigma_W)$.

Proof. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be nf-GSOSs over $(W,\tau_n,{}^f\varsigma_W)$.

If $\varsigma_R(U)\subseteq\varsigma_R(V_1)\cup\varsigma_R(V_2)$ and $\varsigma_R(U)$ is a NFCS, then $\varsigma_R(U)\subseteq\varsigma_R(V_1)$ and $\varsigma_R(U)\subseteq\varsigma_R(V_2)$.

Since $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ are nf–GSOSs, $\varsigma_R(U)\subseteq S^*Int_{nf}(\varsigma_R(V_1))$ and $\varsigma_R(U)\subseteq S^*Int_{nf}(\varsigma_R(V_2))$.

Thus $\varsigma_R(U)\subseteq S^*Int_{nf}(\varsigma_R(V_1))\cup S^*Int_{nf}(\varsigma_R(V_1))$.

By ***Proposition 6***, $\varsigma_R(U)\subseteq S^*Int_{nf}(\varsigma_R(V_1))\cup S^*Int_{nf}(\varsigma_R(V_1))\subseteq S^*Int_{nf}(\varsigma_R(V_1)\cup\varsigma_R(V_2))$.

This implies that, $\varsigma_R(U)\subseteq S^*Int_{nf}(\varsigma_R(V_1)\cup\varsigma_R(V_2))$.

Thus $\varsigma_R(U)\subseteq S^*Int_{nf}(\varsigma_R(V_1)\cup\varsigma_R(V_2))$, $\varsigma_R(U)\subseteq S^*Int_{nf}(\varsigma_R(V_1)\cup\varsigma_R(V_2))$ and $\varsigma_R(U)$ is a NFCS.

Hence $\varsigma_R(V_1)\cup\varsigma_R(V_2)$ is a nf-GSOS in $(W,\tau_n,{}^f\varsigma_W)$.

4. Neutro-Fine Minimal and Maximal Open Sets

In this portion, the minimal and maximal open sets on NFTS are defined and probable results are carried by some major expressive examples.

Definition 12. Let $\varsigma_R(V)$ be a proper non-empty NFOS of a NFTS $\left(W,\tau_n,{}^f\varsigma_W\right)$. Then $\varsigma_R(V)$ is said to be a neutro-fine minimal open set ($\min_{nf}$-OS) if any NFOS which is contained in $\varsigma_R(V)$ is 0_{nf} or $\varsigma_R(V)$. The complement of $\min_{nf}$-OS is said to be neutro-fine minimal closed set ($\min_{nf}$-CS).

Definition 13. Let $\varsigma_R(V)$ be a proper non-empty NFOS of a NFTS $\left(W,\tau_n,{}^f\varsigma_W\right)$. Then $\varsigma_R(V)$ is said to be a neutro-fine maximal open set ($\max_{nf}$-OS) if any NFOS which is contained in $\varsigma_R(V)$ is 1_{nf} or $\varsigma_R(V)$. The complement of $\max_{nf}$-OS is said to be neutro-fine maximal closed set ($\max_{nf}$-CS).

Example 4. Let $W=\{w_1,w_2,w_3\}$ and $\tau_n=\{0_n,1_n,R,S\}$ where R and S are NSs over W and are defined as follows

$R=\left\{\langle w_1,.1,.2,.8\rangle,\langle w_2,.4,.7,.3\rangle,\langle w_3,.6,.5,.2\rangle\right\}$ and $S=\left\{\langle w_1,.6,.5,.3\rangle,\langle w_2,.9,.8,.1\rangle,\langle w_3,.7,.6,.1\rangle\right\}$.

Thus (W,τ_n) is a NTS over W. Then ${}^f\varsigma_W=\left\{0_n,1_n,\varsigma_R(w_1),\varsigma_R(w_2,w_3),\varsigma_S(w_2)\right\}$, where

$\varsigma_R(w_1)=\left\{\langle w_1,.1,.2,.8\rangle,\langle w_2,0,0,1\rangle,\langle w_3,0,0,1\rangle,\langle w_{1,2},.4,.7,.3\rangle,\langle w_{1,3},.6,.5,.2\rangle,\langle w_{2,3},0,0,1\rangle\right\}$,

$\varsigma_R(w_2,w_3)=\left\{\langle w_1,0,0,1\rangle,\langle w_2,.4,.7,.3\rangle,\langle w_3,.6,.5,.2\rangle,\langle w_{1,2},.4,.7,.3\rangle,\langle w_{1,3},.6,.5,.2\rangle,\langle w_{2,3},.6,.7,.2\rangle\right\}$,

$\varsigma_S(w_2)=\left\{\langle w_1,0,0,1\rangle,\langle w_2,.9,.8,.1\rangle,\langle w_3,0,0,1\rangle,\langle w_{1,2},.9,.8,.1\rangle,\langle w_{1,3},0,0,1\rangle,\langle w_{2,3},.9,.8,.1\rangle\right\}$ are NFOSs over (W,τ_n).

Hence $\left(W,\tau_n,{}^f\varsigma_W\right)$ is a NFTS over (W,τ_n). Thus min_{nf}-OS $=\{0_n,\varsigma_R(w_1),\varsigma_S(w_2)\}$, min_{nf}-CS $=\{1_n,\varsigma_R(w_1)',\varsigma_S(w_2)'\}$, $\max_{nf}$-OS $=\{0_n,\varsigma_R(w_2,w_3)\}$ and max_{nf}-CS $=\{1_n,\varsigma_R(w_2,w_3)'\}$.

Example 5. Consider ***Example 1***. Here min_{nf}-OS $=\{0_n,\varsigma_R(w_1),\varsigma_R(w_3)\}$, min_{nf}-CS $=\{1_n,\varsigma_R(w_1)',\varsigma_R(w_3)'\}$,

$\max_{nf}$-OS $=\{0_n,\varsigma_S(w_2,w_3)\}$ and max_{nf}-CS $=\{1_n,\varsigma_S(w_2,w_3)'\}$.

Lemma 1. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n).

If $\varsigma_R(U)$ is a $\min_{nf}$-OS and $\varsigma_R(W)$ is NFOS, then $\varsigma_R(U)\cap\varsigma_R(W)=0_{nf}$ or $\varsigma_R(U)\subseteq\varsigma_R(W)$.

If $\varsigma_R(U)$ and $\varsigma_R(V)$ are $\min_{nf}$-OSs, then $\varsigma_R(U)\cap\varsigma_R(V)=0_{nf}$ or $\varsigma_R(U)=\varsigma_R(V)$.

Proof. Let $\varsigma_R(W)$ be a NFOS such that $\varsigma_R(U)\cap\varsigma_R(W)\neq 0_{nf}$.

Since $\varsigma_R(U)$ is a $\min_{nf}$-OS and $\varsigma_R(U)\cap\varsigma_R(W)\subseteq\varsigma_R(U)$, then $\varsigma_R(U)\cap\varsigma_R(W)=\varsigma_R(U)$. Hence $\varsigma_R(U)\subseteq\varsigma_R(W)$.

If $\varsigma_R(U)\cap\varsigma_R(W)\neq 0_{nf}$, then $\varsigma_R(U)\subseteq\varsigma_R(V)$ and $\varsigma_R(V)\subseteq\varsigma_R(U)$, by (i). Hence $\varsigma_R(U)=\varsigma_R(V)$.

Proposition 7. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n). Let $\varsigma_R(U)$ be a $\min_{nf}$-OS. If $w^{\langle\alpha,\beta,\gamma\rangle}$ is a NFP of $\varsigma_R(U)$, then $\varsigma_R(U)\subseteq\varsigma_R(W)$ for any neutro-fine neighborhood $\varsigma_R(W)$ of $w^{\langle\alpha,\beta,\gamma\rangle}$.

Proof. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n).

Let $\varsigma_R(W)$ be a neutro-fine neighborhood of $w^{\langle\alpha,\beta,\gamma\rangle}$ such that $\varsigma_R(U)\not\subset\varsigma_R(W)$. Then $\varsigma_R(U)\cap\varsigma_R(W)$ is a NFOS such that $\varsigma_R(U)\cap\varsigma_R(W)\not\subset\varsigma_R(U)$ and $\varsigma_R(U)\cap\varsigma_R(W)\neq 0_{nf}$.

This contradicts our assumption that $\varsigma_R(U)$ is a $\min_{nf}$-OS. Hence proved.

Proposition 8. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n). Let $\varsigma_R(U)$ be a $\min_{nf}$-OS. Then

$\varsigma_R(U)=\bigcap\left\{\varsigma_R(W):\varsigma_R(W)\text{ is a neutro-fine neighborhood of } w^{\langle\alpha,\beta,\gamma\rangle}\right\}$, for any NFP $w^{\langle\alpha,\beta,\gamma\rangle}$ of $\varsigma_R(U)$.

Proof. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n). Let $\varsigma_R(U)$ be a $\min_{nf}$-OS.

Since $\varsigma_R(U)$ is a neutro-fine neighborhood of $w^{\langle\alpha,\beta,\gamma\rangle}$, by ***Proposition 7***, then

$\varsigma_R(U)\subseteq\bigcap\left\{\varsigma_R(W):\varsigma_R(W)\text{ is a neutro-fine neighborhood of } w^{\langle\alpha,\beta,\gamma\rangle}\right\}\subseteq\varsigma_R(U)$. Thus
$\varsigma_R(U)=\bigcap\left\{\varsigma_R(W):\varsigma_R(W)\text{ is a neutro-fine neighborhood of } w^{\langle\alpha,\beta,\gamma\rangle}\right\}$.

Proposition 9. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n). Let $\varsigma_R(U)$ be a non-empty NFOS. Then the following conditions are equivalent:

- $\varsigma_R(U)$ is a min_{nf}-OS.
- $\varsigma_R(U)\subseteq Cl_{nf}\left(\varsigma_R(V)\right)$ for any NFS $\varsigma_R(V)$ of $\varsigma_R(U)$.
- $Cl_{nf}\left(\varsigma_R(U)\right)\subseteq Cl_{nf}\left(\varsigma_R(V)\right)$ for any NFS $\varsigma_R(V)$ of $\varsigma_R(U)$.

Proof. $(1)\Rightarrow(2)$. Let $\varsigma_R(V)$ be any NFS of $\varsigma_R(U)$.

By ***Proposition 7***, for any NFP $w^{\langle\alpha,\beta,\gamma\rangle}$ of $\varsigma_R(U)$ and any neutro-fine neighborhood $\varsigma_R(W)$ of $w^{\langle\alpha,\beta,\gamma\rangle}$, then $\varsigma_R(V)=\left(\varsigma_R(U)\cap\varsigma_R(V)\right)\subseteq\left(\varsigma_R(W)\cap\varsigma_R(V)\right)$. Thus $\varsigma_R(W)\cap\varsigma_R(V)\neq 0_{nf}$, and hence $\varsigma_R(U)\cap\varsigma_R(W)\neq 0_{nf}$ is a NFP of $Cl_{nf}\left(\varsigma_R(V)\right)$. Therefore $\varsigma_R(U)\subseteq Cl_{nf}\left(\varsigma_R(V)\right)$.

$(2)\Rightarrow(3)$. Since $\varsigma_R(V)$ is any NFS of $\varsigma_R(U)$, then $\varsigma_R(U)\subseteq Cl_{nf}(\varsigma_R(V))$.

Thus by (2), $Cl_{nf}(\varsigma_R(U))\subseteq Cl_{nf}(Cl_{nf}(\varsigma_R(V)))=Cl_{nf}(\varsigma_R(V))$. Hence $Cl_{nf}(\varsigma_R(U))\subseteq Cl_{nf}(\varsigma_R(V))$ for any NFS $\varsigma_R(V)$ of $\varsigma_R(U)$.

$(3)\Rightarrow(1)$. Suppose that $\varsigma_R(U)$ is not a $\min_{nf}$-OS.

Then there exists a NFS $\varsigma_R(V)$ such that $\varsigma_R(V)\not\subset\varsigma_R(U)$. Then there exists a NFP $w^{\langle\alpha,\beta,\gamma\rangle}\in\varsigma_R(U)$ such that $w^{\langle\alpha,\beta,\gamma\rangle}\notin\varsigma_R(V)$. This implies that, $w^{\langle\alpha,\beta,\gamma\rangle}$ is a NFS. Then it is clear that $Cl_{nf}\left(w^{\langle\alpha,\beta,\gamma\rangle}\right)\subseteq\varsigma_R(V)'$ $\Rightarrow Cl_{nf}\left(w^{\langle\alpha,\beta,\gamma\rangle}\right)\neq Cl_{nf}(\varsigma_R(U))$.

Hence the proof.

Lemma 2. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n).

If $\varsigma_R(U)$ is a $\max_{nf}$-OS and $\varsigma_R(W)$ is NFOS, then $\varsigma_R(U)\cup\varsigma_R(W)=1_{nf}$ or $\varsigma_R(W)\subseteq\varsigma_R(U)$.

If $\varsigma_R(U)$ and $\varsigma_R(V)$ are $\max_{nf}$-OSs, then $\varsigma_R(U)\cup\varsigma_R(V)=1_{nf}$ or $\varsigma_R(U)=\varsigma_R(V)$.

Proof. (i) Let $\varsigma_R(W)$ be a NFOS such that $\varsigma_R(U)\cup\varsigma_R(W)\neq 1_{nf}$.

Since $\varsigma_R(U)$ is a $\max_{nf}$-OS and $\varsigma_R(U)\subseteq\varsigma_R(U)\cup\varsigma_R(W)$, then $\varsigma_R(U)\cup\varsigma_R(W)=\varsigma_R(U)$. Hence $\varsigma_R(W)\subseteq\varsigma_R(U)$.

If $\varsigma_R(U)\cup\varsigma_R(W)\neq 1_{nf}$, then $\varsigma_R(U)\subseteq\varsigma_R(V)$ and $\varsigma_R(V)\subseteq\varsigma_R(U)$, by (i). Hence $\varsigma_R(U)=\varsigma_R(V)$.

Proposition 10. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n). Let $\varsigma_R(U)$ be a $\max_{nf}$-OS. If $w^{\langle\alpha,\beta,\gamma\rangle}$ is a NFP of $\varsigma_R(U)$, then for any neutro-fine neighborhood $\varsigma_R(W)$ of $w^{\langle\alpha,\beta,\gamma\rangle}$, $\varsigma_R(U)\cup\varsigma_R(W)=1_{nf}$ or $\varsigma_R(W)\subseteq\varsigma_R(U)$.

Proof. Follows from the ***Lemma 2***.

Proposition 11. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n). Let $\varsigma_R(U)$ be a $\max_{nf}$-OS. Then $\varsigma_R(U)=\bigcup\{\varsigma_R(W):\varsigma_R(W)\text{ is a neutro-fine neighborhood of } w^{\langle\alpha,\beta,\gamma\rangle} \text{ such that } \varsigma_R(U)\cup\varsigma_R(W)\neq 1_{nf}\}$.

Proof. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n). Let $\varsigma_R(U)$ be a $\max_{nf}$-OS.

Since $\varsigma_R(U)$ is a neutro-fine neighborhood of $w^{\langle\alpha,\beta,\gamma\rangle}$, by ***Proposition 10***, then $\varsigma_R(U)\subseteq\bigcup\{\varsigma_R(W):\varsigma_R(W)$ is a neutro-fine neighborhood of $w^{\langle\alpha,\beta,\gamma\rangle}$ such that $\varsigma_R(U)\bigcup\varsigma_R(W)\neq 1_{nf}\}\subseteq\varsigma_R(U)$. Hence the result.

Theorem 12. Let $(W,\tau_n,{}^f\varsigma_W)$ be a NFTS over (W,τ_n). Let $\varsigma_R(U_1)$, $\varsigma_R(U_2)$ and $\varsigma_R(U_3)$ be $\max_{nf}$-OSs such that $\varsigma_R(U_1)\neq\varsigma_R(U_2)$. If $(\varsigma_R(U_1)\bigcap\varsigma_R(U_2))\subseteq\varsigma_R(U_3)$, then $\varsigma_R(U_1)=\varsigma_R(U_3)$ or $\varsigma_R(U_2)=\varsigma_R(U_3)$.

Proof. Let $\varsigma_R(U_1)$, $\varsigma_R(U_2)$ and $\varsigma_R(U_3)$ be $\max_{nf}$-OSs such that $\varsigma_R(U_1)\neq\varsigma_R(U_2)$. Then

$(\varsigma_R(U_1)\bigcap\varsigma_R(U_3))=\varsigma_R(U_1)\bigcap(\varsigma_R(U_3)\bigcap 1_{nf})$

$=\varsigma_R(U_1)\bigcap(\varsigma_R(U_3)\bigcap(\varsigma_R(U_1)\bigcup\varsigma_R(U_2)))$ (by ***Lemma 2***)

$=\varsigma_R(U_1)\bigcap((\varsigma_R(U_3)\bigcap\varsigma_R(U_1))\bigcup(\varsigma_R(U_3)\bigcap\varsigma_R(U_2)))$

$=(\varsigma_R(U_1)\bigcap\varsigma_R(U_3))\bigcup(\varsigma_R(U_3)\bigcap\varsigma_R(U_1)\bigcap\varsigma_R(U_2))$

$=(\varsigma_R(U_1)\bigcap\varsigma_R(U_3))\bigcup(\varsigma_R(U_1)\bigcap\varsigma_R(U_2))$ (since $(\varsigma_R(U_1)\bigcap\varsigma_R(U_2))\subseteq\varsigma_R(U_3)$)

$=\varsigma_R(U_1)\bigcap(\varsigma_R(U_3)\bigcup\varsigma_R(U_2))$.

If $\varsigma_R(U_3)\neq\varsigma_R(U_2)$, then $(\varsigma_R(U_3)\bigcup\varsigma_R(U_2))=1_{nf}$.

Thus $(\varsigma_R(U_1)\bigcap\varsigma_R(U_3))=\varsigma_R(U_1)$ implies $\varsigma_R(U_1)\subseteq\varsigma_R(U_3)$. Since $\varsigma_R(U_1)$ and $\varsigma_R(U_3)$ are $\max_{nf}$-OSs, then hence $\varsigma_R(U_1)=\varsigma_R(U_3)$.

Theorem 13. Let $(W,\tau_n,{}^f\varsigma_W)$ be a NFTS over (W,τ_n). Let $\varsigma_R(U_1)$, $\varsigma_R(U_2)$ and $\varsigma_R(U_3)$ be , $\max_{nf}$-OSs, which are different from each other. Then $(\varsigma_R(U_1)\bigcap\varsigma_R(U_2))\not\subset(\varsigma_R(U_1)\bigcap\varsigma_R(U_3))$.

Proof. Let $\varsigma_R(U_1)$, $\varsigma_R(U_2)$ and $\varsigma_R(U_3)$ be , $\max_{nf}$-OSs.

Suppose assume that $(\varsigma_R(U_1)\bigcap\varsigma_R(U_2))\subseteq(\varsigma_R(U_1)\bigcap\varsigma_R(U_3))$. Then

$(\varsigma_R(U_1)\bigcap\varsigma_R(U_2))\bigcup(\varsigma_R(U_2)\bigcap\varsigma_R(U_3))\subseteq(\varsigma_R(U_1)\bigcap\varsigma_R(U_3))\bigcup(\varsigma_R(U_2)\bigcap\varsigma_R(U_3))$.

Thus $\varsigma_R(U_2)\bigcap(\varsigma_R(U_1)\bigcup\varsigma_R(U_3))\subseteq(\varsigma_R(U_1)\bigcup\varsigma_R(U_2))\bigcap\varsigma_R(U_3)$.

Since $\varsigma_R(U_1)\bigcup\varsigma_R(U_3)=1_{nf}=\varsigma_R(U_1)\bigcup\varsigma_R(U_2)$, then $\varsigma_R(U_2)\subseteq\varsigma_R(U_3)$.

This implies that $\varsigma_R(U_2)=\varsigma_R(U_3)$, which contradicts our assumption. Hence proved.

5. Conclusion

The main objective of this paper is to define some collection of open sets such as neutro-fine-generalized open and neutro-fine-semi open sets on NFTS and analyzed its basic properties with perfect examples. The notion of interior and closure on semi-open sets are described and specified certain properties. These definitions provide the idea of generalized semi-open sets on NFTS. Also, the neutro-fine-minimal and neutro-fine-maximal open sets are defined and some of their properties are studied in this space. Likewise, discussed the complement of all these sets as its closed sets. The basic properties of the union and intersection of these sets are stated in some theorems. Only a few sets satisfy this postulates, and others are disproved as shown in the counterexamples. The converse of some theorems is proved in probable examples. Consequently, the future researchers can extend this NFTS to some special types of sets, whereas soft sets, rough sets, crisp sets, cubic sets, etc., Also, the application part can widen on MCDM problems.

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An Overview of Portfolio Optimization Using Fuzzy Data Envelopment Analysis Models

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ABSTRACT

A combination of projects, assets, programs, and other components put together in a set is called a portfolio. Arranging these components helps to facilitate the efficient management of the set and subsequently leads to achieving the strategic goals. Generally, the components of the portfolio are quantifiable and measurable which makes it possible for management to manage, prioritize, and measure different portfolios. In recent years, the portfolio in various sectors of economics, management, industry, and especially project management has been widely applied and numerous researches have been done based on mathematical models to choose the best portfolio. Among the various mathematical models, the application of data envelopment analysis models due to the unique features as well as the capability of ranking and evaluating performances has been taken by some researchers into account. In this regard, several articles have been written on selecting the best portfolio in various fields, including selecting the best stocks portfolio, selecting the best projects, portfolio of manufactured products, portfolio of patents, selecting the portfolio of assets and liabilities, etc. After presenting the Markowitz mean-variance model for portfolio optimization, these pieces of research have witnessed significant changes. Moreover, after the presentation of the fuzzy set theory by Professor Lotfizadeh, despite the ambiguities in the selection of multiple portfolios, a wide range of applications in portfolio optimization was created by combining mathematical models of portfolio optimization.

1. Introduction

Nowadays, economic and financial issues of choosing the best investment portfolio for all individuals and legal entities are of special importance and have wide aspects.

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Investing means turning financial resources into one or more financial assets to achieve acceptable returns over a period of time and ultimately create wealth for the investor. Investing in monetary and capital markets is one of the most significant pillars of wealth creation and transfer in an economy [1].

Because of the prosperity of the stock market as well as the desire to invest and the influx of people's capital into this market, choosing the best investment portfolio by achieving the highest return along with the lowest risk, which means choosing the best stock portfolio, are some of the main concerns of the society. Since several factors are involved in choosing the best portfolio, including the financial structure of companies, products, sales and profitability, asset volume, political, commercial, psychological risks, etc., the selection of the right portfolio can be complicated. As a result, using traditional methods in today's complex situation will not much work. The use of mathematical models, due to their flexibility and the involvement of various quantitative and qualitative factors in the process of figuring out the best portfolio, by integrating facts such as ambiguities and doubts helps to make the answers obtained from the models closer to the facts and makes the range of multiple choices predictable. Among the various mathematical models, data envelopment analysis as a widely used technique can be effective in achieving the best portfolio. Especially when the ambiguities are applied in the model, the results of fuzzy data envelopment analysis in portfolio optimization will be much more accurate and consistent with the facts.

In the current work, first, the basic definitions of the concepts are concisely presented and then an overview of some research done in the field of portfolio optimization using fuzzy data envelopment analysis has been made.

2. Definitions

Portfolio. It is the combination of assets or plans the investor or management acquires to achieve the desired return in a given period of time, by accepting the corresponding risks, and by converting some other financial assets. This concept, especially in stock market investing, means choosing and buying a portfolio of various stocks or diverse securities aiming to increase wealth.

Data Envelopment Analysis. It is a non-parametric method in the field of operations research whose task is to evaluate the performance or efficiency of units/portfolios. This method, by considering various inputs and outputs, evaluates performances of multiple units/portfolios and identifies efficient and inefficient units/portfolios. This model was first proposed by Farrell in 1957 [10] and since then, the model has made extensive advances in various sciences and a lot of research has been done in evaluating the performance of different units as well as ranking them [2, 4].

Fuzzy. It means vague and indefinite in word. It is first introduced by Zadeh in 1965 leading to fundamental changes in the theory of classical mathematics [5]. This concept is based on logic and human decisions so that this logic can be considered as an extension of Aristotelian logic. Unlike the classical logic, which is in the state of right and wrong (zeros and ones), fuzzy logic is flexible, and the correctness or incorrectness of any logical statement is determined by the degree of membership (numerical between zero and one) [5].

In other words, instead of the characteristic function, the membership function is employed as follows:

$f_A: U \rightarrow [0,1]$.

In which $f_A(u) \in [0,1]$ refers to the level or degree of belonging of u to A.

Intuitionistic Fuzzy. Although fuzzy sets can plot and investigate the ambiguity, they cannot discuss and plot all the ambiguities that occur in real life. To this end, in the cases with insufficient information, a developed fuzzy theory called intuitive fuzzy theory is used.

An intuitive fuzzy subset of or an intuitive fuzzy subset in U is a set such as A in which to each member in $u \epsilon U$, two degrees are assigned including membership degree and non-membership degree. In other words, for each set of A, the membership function is defined as $f_A: U \rightarrow [0,1] * [0,1]$, so that

$$F_A(u) = (\mu_A(u), \nu_A(u)), \; 0 \leq \mu_A(u) \leq 1.$$

In the intuitive fuzzy sets, $\pi_A = 1 - \mu_A(u) - \nu_A(u)$ refers to the degree of doubt or ambiguity for each member $u \epsilon U$.

In all of the above-mentioned definitions, $\mu_A(u)$ and $\nu_A(u)$ are both fuzzy sets. On the other hand, the values of membership in intuitive fuzzy sets can be considered as $L = \{(x, y) \epsilon [0,1]^2 : 0 \leq x + y \leq 1\}$. Therefore, intuitive fuzzy sets are also called two-dimensional fuzzy sets [6, 7].

3. Markowitz Model

Markowitz was the first person who introduced a model capable of introducing a suitable criterion for portfolio selection by considering both risks and returns together. The Markowitz mean-variance model is the most popular selection approach for the stock portfolio. This model is based on the following statements:

- Investors are basically risk-averse and have the expected increased utility.
- Each investment option can be infinitely divisible.
- Investors select their portfolio based on the average and variance of expected returns.
- Investors have a one-period time horizon and this is the same for all investors.
- Investors prefer a higher return on a certain level of risk and, conversely, lower risk on a certain level of return.
- A commodity investment portfolio is a portfolio that has the highest return at a certain level of risk or has the lowest risk at a certain level of return [8].

4. Data Envelopment Analysis

Data envelopment analysis is a non-parametric method that is used to calculate the efficiency of homogeneous units, based on the inputs and outputs of the units, and finally the division of all units into efficient and inefficient units. This model was initially proposed by Farrell in 1957 [10] and then developed by Charnes et al. in 1978 [9] through which optimal solutions are sought by dividing a linear combination of outputs by a linear combination of inputs [9, 10].

The CCR model of data envelopment analysis is expressed as follows:

$$f_k = \underset{u,v}{\mathrm{Max}} \frac{\sum_{n=1}^{N}(xo)_{nk}v_{nk}}{\sum_{m=1}^{M}(xi)_{mk}u_{mk}},$$

s.t:

$$\frac{\sum_{n=1}^{N}(xo)_{nk}v_{nk}}{\sum_{m=1}^{M}(xi)_{mk}u_{mk}} \leq 1 \quad j = 1, \dots, J,$$

$$v_{nk}, u_{mk} \geq 0 \quad n = 1, \dots, N, m = 1, \dots, M.$$

Since data envelopment analysis models are considered as one of the multi-criteria decision-making methods, to regard multiple indicators and criteria in selecting the best options, various types of data envelopment analysis models can be utilized. One of the applications of data envelopment analysis models is for portfolio optimization. In this regard, several types of research have been done to evaluate the efficiency and ranking of stocks as well as to identify the effective variables [10].

Due to the increasing significance of investing in the stock market, choosing the optimal portfolio is one of the most important concerns of investors for which several methods of stock portfolio selection and investment have been presented so far.

Despite the income and profit that the investor can earn from the formation of her portfolio, the issue of reducing investment risk is of particular significance. Nowadays, risk overshadows all aspects of human life and always plays an important role in all matters and decisions of individuals.

Risk conventionally is a kind of danger that is said to happen due to uncertainty about the occurrence of an accident in the future, and the higher this uncertainty is, the higher the risk [1]. There are two viewpoints to risk definition:

First Viewpoint. The risk as any possible fluctuations of economic returns in the future.

Second Viewpoint. The risk as negative fluctuations of economic returns in the future.

In all financial markets, it is the principle that investors are always risk-averse. Therefore, risk will always be present along with return as the two main pillars in portfolio selection. In 1952, Markowit [8] for the first time, proposing the mean-variance model, proved that to form a stock portfolio, one could always minimize the risk by considering a certain level of return. Markowitz's model is known as the first mathematical model of stock portfolio optimization [2].

Before Markowitz introduced his model, it was traditionally believed that increasing diversity in the stock portfolio reduced portfolio risk, but they were unable to measure this risk. Markowitz considered the expected return per share as the average share return in previous periods and the risk per share as the variance of the return per share in previous periods. He showed that the average stock portfolio weight is equal to the stock returns, but the stock portfolio risk is not equal to the average stock weight risk. In order to calculate the expected return per share ($E\ (R)$), the shareholder must obtain the probable return on the securities (R) as well as the probability of the expected return (Pr) assuming that the sum of the probabilities is equal to one. In this case, the expected return is as follows:

$$E(R) = \sum_{i=1}^{m} R_i Pr_i.$$

In the above formula, m represents the number of potential returns per share.

Assuming the above, the stock portfolio returns $E(R_p)$ will be equal to the weighted return of each share.

$$E(R_p) = \sum_{i=1}^{m} w_i\, E(R_i),$$

$$\sum w_i = 1.$$

Where w_i is the weight of the stock portfolio for the ith share.

In the Markowitz model, the risk per share is considered equal to the return variance ($VAR\ (R)$) or its second root, the standard deviation ($SD\ (R)$) in previous periods [3].

$$VAR(R) = \sigma^2 = \sum_{r=1}^{m} (R_i - E(R))^2 Pr_i,$$

$$VAR(R) = \sigma^2 = \sum_{r=1}^{m} (R_i - E(R))^2 Pr_i.$$

Accordingly, stock portfolio risk based on Markowitz model is shown below [2]:

$$V_\Omega = \text{Min} \sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{\substack{l=1 \\ l \neq i}}^{n} w_i w_l \psi_{il}.$$

Where w_i and $w_l$are the weight of each share, σ_i^2 is the variance of stock returns, $\psi_{il}$is the covariance of double returns of shares and $V_\Omega$is the variance of stock returns.

Finally, the Markowitz model with two objective functions was defined as maximizing stock portfolio return and minimizing stock portfolio risk as follows:

$$\text{Max} \;\; f_1(x) = \sum_{i=1}^{m} w_i \bar{r}_i \,,$$

$$\text{Min} \;\; f_2(x) = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{\substack{l=1 \\ l \neq i}}^{n} w_i w_l \psi_{il},$$

s.t:

$$\sum w_i = 1,$$
$$w_i \geq 0.$$

In financial terms, the set of Pareto optimal solutions for the portfolio selection problem is called the efficient set or the efficient boundary. In other words, for a set of assets, a set of portfolios that have the least risk for a given return is called the efficient frontier. The efficient boundary is a non-descending function that shows the best interaction between risk and return. Markowitz used the concepts of mean returns, variance, and covariance to represent the efficient boundary. This model is usually called the EV model, in which E represents the mean and V represents the variance [11].

Tavana et al. [3], using the Markowitz model and using 7 macro criteria and 19 indicators (8 indicators as input and 11 indicators as output), examined various stock exchange industries and chose the best industries from among them to be in the stock portfolio. After calculating the variables, he entered them into the DEA model and calculated the Relative Financial Strength Indices (RFSI) of each company using the Anderson-Peterson model. After calculating the RFSI index, companies with financial strength greater than 0.9 in the BCC-O and CCR-I models as suitable investment options were selected. After selecting the companies, the weight of each share was determined using the Markowitz model (as shown below) and solving the model through genetic technique.

$$\text{Max} \quad \sum_{i=1}^{N} \mu_i x_i - \sum_{i=1}^{N} L(y_i) - \lambda \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij} x_i x_j \,,$$

S. t.:

$$\sum_{i=1}^{N} x_i \leq C^0,$$
$$y_i = \left| x_i - x_i^0 \right| \quad i = 1, \dots, N,$$
$$x_i \geq 0 \quad i = 1, \dots, N.$$

Sharifighazvini et al. [12] used the Beasley index in a research to obtain a suitable mathematical model with internal market realities to extract the weights of participation of companies' stocks or the fund manager in order to obtain the variance-covariance matrix between firm returns, using the Markowitz model. In the Beasley benchmark, the average and standard deviation of each company's weekly returns and correlation coefficients between 225 Nikkei-listed companies are collected. Therefore, in order to obtain the variance-covariance matrix between firm returns, the relationship between the standard deviation of the yield of each pair of firms and the correlation coefficient between them should be used.

$$\rho_{ab} = \frac{\mathrm{cov}(a, b)}{\sqrt{\mathrm{var}(a).\mathrm{var}(b)}} \quad \Rightarrow \quad \mathrm{cov}(a, b) = \rho_{ab} \,.\sqrt{\mathrm{var}(a).\mathrm{var}(b).}$$

Thus, the average weekly return and the variance-covariance matrix are obtained for 225 Nikkei-listed companies in the model. In addition, because the index provided by Beasley does not specify the name of each company, information about the price and number of shares of each company is generated as a random number from the corresponding ranges in 30 selected industries of the Iranian stock market. Also in the research, in addition to reducing risk and increasing returns, minimizing portfolio costs has been added to the Markowitz model as a goal function.

$$\begin{aligned}
&\mathrm{Max} \quad \sum_{i=1}^{N} w_i r_i \,, \\
&\mathrm{Min} \quad \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij}, \\
&\mathrm{Min} \quad \sum_{i=1}^{N} z_i \,, \\
&\mathrm{s.t:} \\
&w_i = \frac{c_i x_i}{\sum_{i=1}^{N} c_i x_i} \quad i = 1, \dots, N \,, \\
&\sum_{i=1}^{N} w_i \le 1 \,, \\
&\sum_{i=1}^{N} v_i \le 1 \,, \\
&v_i w_i \le 0.15 \qquad i = 1, \dots, N, \\
&(1 - v_i) w_i \le 0.1 \qquad i = 1, \dots, N, \\
&x_i \le 0.05 S_i \qquad i = 1, \dots, N, \\
&\sum_{i \in [j]} w_i \le 0.3 \quad j = 1, \dots, t, \\
&\sum_{i=1}^{N} Z_i \le k \,, \\
&x_i \epsilon \{\mathbb{Z}\} \,, v_i, w_i \epsilon \{0,1\}, w_i \ge 0.
\end{aligned}$$

Where x_i is the number of selected stocks of type i, and $v_i$is a zero-one variable that specifies which company's stocks are greater than 15% of the portfolio value.

w_i is the ratio of type i shares in the portfolio and $z_i$is a zero and one variable to indicate the participation or non-participation of type i shares in the portfolio. Model parameters are also:

r_i: Average stock return i.

N: Variety of stocks from which the portfolio is selected.

σ_{ij}: Covariance between stocks i and j.

c_i : Stock price i.

s_i: Number of shares of the company i.

t: Number of industry.

k: Maximum portfolio variety.

In order to optimize the stock portfolio, Ahmadi et al. [13] used a combination of data envelopment analysis and heuristic factor analysis methods. In the presented research, each company is considered as a decision-making unit and input and output indicators are defined for each company. Since each of the indicators shows different dimensions of the companies' performance, it divided the output indicators into the input indicators so that the target indicators become a single and comparable indicator called performance. Finally, in order to reduce the dimensions of the problem and eliminate the correlation between the data, exploratory factor analysis was used.

Factor analysis is a set of different mathematical and statistical techniques that aim to simplify complex data sets. The main question is the answer to the question of whether a set of variables can be described in terms of the number of indicators or fewer factors than the variables and what attribute or feature each of the indicators (factors) represents. Factor analysis is used for correlation between variables.

Due to the fact that the model obtained in this research is an integer programming type, it cannot be solved by mathematical methods. Therefore, to solve the obtained model, the method of genetic algorithm and simulated annealing has been used.

5. Fuzzy Markowitz Model

Mashayekhi and Omrani [14] presented a new multi-objective model for portfolio selection including cross-performance data envelopment analysis and Markowitz mean-variance model in addition to presenting the risk and performance of the portfolio. To take the uncertainty into account, they considered the return on assets as trapezoidal fuzzy numbers and finally solved the model by employing the second type of genetic algorithm –NSGAII. The basic model presented in the current research is the mean-variance cross-sectional performance model of fuzzy Markowitz data envelopment analysis.

Suppose $\tilde{A} = (a, b, \alpha, \beta)$ is a trapezoidal fuzzy number. The cross-sectional performance model of fuzzy Markowitz MV is expressed as:

$$\text{Max } E\left(\sum_{i=1}^{N} \widetilde{R}_i w_i\right) = \sum_{i=1}^{N} \frac{1}{2}\left[a_i + b_i + \frac{1}{3}(\beta_i - \alpha_i)\right] w_i ,$$

$$\text{Min } \sigma^2\left(\sum_{i=1}^{N} \widetilde{R}_i w_i\right) = (\sum_{i=1}^{N} \frac{1}{2}[b_i - a_i + \frac{1}{3}(\alpha_i + \beta_i)]w_i)^2 + \frac{1}{72}[\sum_{i=1}^{N}(\alpha_i + \beta_i)\, w_i]^2 ,$$

$$\text{Max} \ \sum_{i=1}^{N} w_i \bar{e}_i,$$

s.t:

$$\sum_{i=1}^{N} z_i \leq h,$$

$$l_i z_i \leq w_i \leq u_i z_i \quad i = 1, \dots, N,$$

$$\sum_{i=1}^{N} w_i = 1 \ ,$$

$$w_i \geq 0 \quad i = 1, \dots, N.$$

Chen et al. [16] illustrated a comprehensive model for selecting a multi-objective portfolio in a fuzzy environment by combining the semi-variance mean model and the cross-sectional data envelopment analysis model. Then, the proposed model was re-formulated with the Sharp ratio by considering resource constraints as well as other constraints. The sharp factor model is expressed as single-factor and multi-factor models. In the single-factor model, it is assumed that the returns of all securities are correlated with each other for only one reason as a common factor to which all securities react with varying degrees of intensity. This common factor is usually considered as the market basket. Due to the fact that in the return study, the positive deviation from the average as a profit is more than expected and this issue is considered as a positive criterion, so to achieve a more consistent result with reality, semi-variance models are used, where only the negative deviation from the expected return is minimized. These models are called the mean-half variance model of portfolio optimization.

Besides the half-variance deviation, there are some downside risks. To assess the Portfolio Performance (PE), Chen and Guy [15] considered three types of data envelopment analysis approaches based on fuzzy portfolio evaluation models and based on the size of different risk scales, namely the probable variance, probable semivariance, and probable semi-absolute deviation.

6. Conclusion and Suggestion

In today's world, due to the variety of choices in each field, reviewing and selecting the best options in each field is of particular importance and this issue is much more significant for everyone to choose the best investment portfolio. To select the best investment portfolio in the general sense and the best stock portfolio, in particular, there are various criteria and indicators. Concentrating on the financial structure of companies, sales, profitability, business environment, various business, political risks, etc. can be effective in choosing a portfolio. Nowadays, according to the development of mathematical models, the use of conceptual mathematical models helps investors to be better informed about the returns and risk of different stock portfolios based on the patterns and principles of mathematical models and let them choose the best type of investment according to their standards as well as their investment policies. Due to the unique features of data envelopment analysis, it has attracted the interest of many researchers among a wide variety of mathematical models to study and select the best portfolios. Despite the ambiguous nature of the data, combining this analysis and Fuzzy concepts leads to coming closer to reality in the models such as the Markowitz model. This paper provides an overview of some of the research conducted in optimizing investment portfolios using data envelopment analysis models, Markowitz model and the concept of fuzzy data. With the advancement of fuzzy concepts and finding new definitions of fuzzy concepts that bring issues and models closer to the realities of society, the presented models can be transformed into a variety of new fuzzy concepts and further studied. One of the new definitions and developments of the fuzzy concept is the description and concept of intuitive fuzzy in which the non-membership function is considered in addition to the membership function in the fuzzy concept and interfered in the relevant calculations. Given that based on field research, very little research has been done in the field of portfolio optimization using the Markowitz model and intuitive fuzzy concepts, more appropriate choices can be suggested to investors for future research by developing portfolio fuzzy optimization models to models with intuitive fuzzy data.

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Assessment and Linear Programming Under Fuzzy Conditions

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ABSTRACT

The present work focuses on two directions. First, a new fuzzy method using triangular/trapezoidal fuzzy numbers as tools is developed for evaluating a group's mean performance, when qualitative grades instead of numerical scores are used for assessing its members' individual performance. Second, a new technique is applied for solving Linear Programming problems with fuzzy coefficients. Examples are presented on student and basket-ball player assessment and on real life problems involving Linear Programming under fuzzy conditions to illustrate the applicability of our results in practice. A discussion follows on the perspectives of future research on the subject and the article closes with the general conclusions.

1. Introduction

Despite to the initial reserve of the classical mathematicians, fuzzy mathematics and Fuzzy Logic (FL) have found nowadays many and important applications to almost all sectors of human activity (e.g. [1], Chapter 6, [2] Chapters 4-8, [3], etc.). Due to its nature of characterizing the ambiguous real life situations with multiple values, FL offers among others rich resources for assessment purposes, which are more realistic than those of the classical logic [4-6].

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Fuzzy Numbers (FNs), which are a special form of Fuzzy Sets (FS) on the set of real numbers, play an important role in fuzzy mathematics analogous to the role played by the ordinary numbers in the traditional mathematics. The simplest forms of FNs are the Triangular FNs (TFNs) and the Trapezoidal FNs (TpFNs).

In the present work we study applications of TFNs and TpFNs to assessment processes and to Linear Programming (LP) under fuzzy conditions. The rest of the paper is formulated as follows: Section 2 contains all the information about FS, FNs and LP which is necessary for the understanding of its contents. Section 3 is divided in two parts. In the first part an assessment method is developed using TFNs/TpFNs as tools, which enables the calculation of the mean performance of a group of uniform objects (individuals, computer systems, etc.) with respect to a common activity performed under fuzzy conditions. In the second part a method is developed for solving LP problems with fuzzy data (Fuzzy LP). In Section 4 examples are presented illustrating the applicability of both methods to real world situations. The assessment outcomes are validated with the parallel use of the GPA index, while the solution of the FLP problems is reduced to the solution of ordinary LP problems by ranking the corresponding fuzzy coefficients. The article closes with a brief discussion for the perspectives of future research on those topics and the final conclusions that are presented in Section 5.

2. Background

2.1. Fuzzy Sets and Logic

For general facts on FS and FL we refer to [2] and for more details to [1]. The FL approach for a problem's solution involves the following steps:

- Fuzzification of the problem's data by representing them with properly defined FSs.
- Evaluation of the fuzzy data by applying principles and methods of FL in order to express the problem's solution in the form of a unique FS.
- Defuzzification of the problem's solution in order to "translate" it in the natural language for use with the original real-life problem.

One of the most popular defuzzification methods is the Center of Gravity (COG) technique. When using it, the fuzzy outcomes of the problem's solution are represented by the coordinates of the COG of the membership function graph of the FS involved in the solution [7].

2.2. Fuzzy Numbers

It is recalled that a FN is defined as follows:

Definition 1. A FN is a FS A on the set R of real numbers with membership function

$m_A: R \to [0, 1]$, such that:

- A is normal, i.e. there exists x in R such that $m_A(x) = 1$,
- A is convex, i.e. all its a-cuts $A^a = \{x \in U: m_A(x) \geq a\}$, a in [0, 1], are closed real intervals, and

- Its membership function $y = m_A(x)$ is a piecewise continuous function.

Remark 1. (Arithmetic operations on FNs): One can define the four basic arithmetic operations on FNS in the following two, equivalent to each other, ways:

- With the help of their a-cuts and the Representation-Decomposition Theorem of Ralesscou - Negoita ([8], Theorem 2.1, p.16) for FS. In this way the fuzzy arithmetic is turned to the well known arithmetic of the closed real intervals.
- By applying the Zadeh's extension principle ([1], Section 1.4, p.20), which provides the means for any function f mapping a crisp set X to a crisp set Y to be generalized so that to map fuzzy subsets of X to fuzzy subsets of Y.

In practice the above two general methods of the fuzzy arithmetic, requiring laborious calculations, are rarely used in applications, where the utilization of simpler forms of FNs is preferred. For general facts on FNs we refer to [9].

2.3. Triangular Fuzzy Numbers (TFNs)

A TFN (*a, b, c*), with *a, b, c* in *R* represents mathematically the fuzzy statement "the value of *b* lies in the interval [*a, c*]". The membership function of (*a, b, c*) is zero outside the interval [*a, c*], while its graph in [*a, c*] consists of two straight line segments forming a triangle with the OX axis (***Fig.1***).

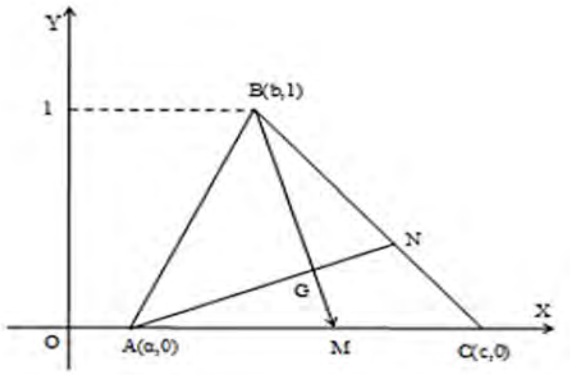

Fig. 1. Graph and COG of the TFN (a, b, c).

Therefore the analytic definition of a TFN is given as follows:

Definition 2. Let *a, b* and *c* be real numbers with *a < b < c*. Then the *TFN* (*a, b, c*) is a FN with membership function:

$$y = m(x) = \begin{cases} \frac{x-a}{b-a}, & x \in [a,b] \\ \frac{c-x}{c-b}, & x \in [b,c] \\ 0, & x < a \text{ or } x > c \end{cases}$$

Proposition 1. (Defuzzification of a TFN). The coordinates (*X*, *Y*) of the COG of the graph of the TFN (*a*, *b*, *c)* are calculated by the formulas $X = \frac{a+b+c}{3}$, $Y = \frac{1}{3}$.

Proof. The graph of the TFN (*a*, *b*, *c)* is the triangle ABC of ***Fig.1***, with A (*a*, 0), B(*b*, 1) and C (*c*, 0). Then, the COG, say G, of ABC is the intersection point of its medians AN and BM. The proof of the proposition is easily obtained by calculating the equations of AN and BM and by solving the linear system of those two equations.

Remark 2. (Arithmetic Operations on TFNs) It can be shown [9] that the two general methods of defining arithmetic operations on FNs mentioned in ***Remark 2*** lead to the following simple rules for the addition and subtraction of TFNs:

Let A = (a, b, c) and B = (a_1, b_1, c_1) be two TFNs. Then:

- The sum A + B = $(a+a_1, b+b_1, c+c_1)$.
- The difference A - B = A + (-B) = $(a-c_1, b-b_1, c-a_1)$, where –B = $(-c_1, -b_1, -a_1)$ is defined to be the opposite of.

In other words, the opposite of a TFN, as well as the sum and the difference of two TFNs are always TFNs. On the contrary, the product and the quotient of two TFNs, although they are FNs, they are not always TFNs, unless if a, b, c, a_1, b_1, c_1 are in R^+ [9].

The following two scalar operations can be also be defined:

- k + A= (k+*a*, k+*b*, k+*c*), k$\in R$.
- kA = (k*a*, k*b*, k*c*), if k>0 and kA = (k*c*, k*b*, k*a*), if k<0.

2.4. Trapezoidal Fuzzy Numbers (TpFNs)

A TpFN (*a, b, c, d*) with *a, b, c, d* in *R* represents the fuzzy statement approximately in the interval *[b, c]*. Its membership function *y=m(x)* is zero outside the interval [*a, d*], while its graph in this interval [*a, d*] is the union of three straight line segments forming a trapezoid with the X-axis (see ***Fig. 2***),

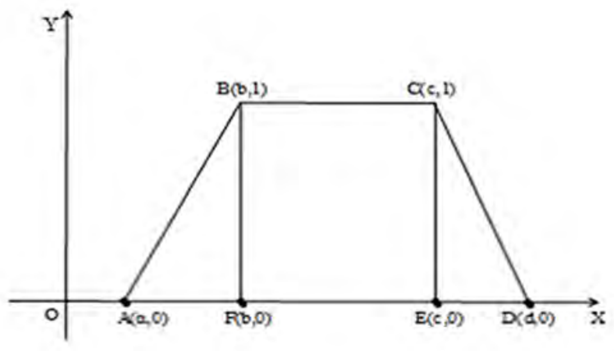

Fig. 2. Graph of the TpFN (a, b, c, d).

Therefore, the analytic definition of a TpFN is given as follows:

Definition 3. Let $a < b < c < d$ be given real numbers. Then the TpFN (*a, b, c, d*) is the FN with membership function:

$$y = m(x) = \begin{cases} \dfrac{x-a}{b-a}, & x \in [a,b] \\ x = 1, & x \in [b,c] \\ \dfrac{d-x}{d-c}, & x \in [c,d] \\ 0, & x < a \text{ and } x > d \end{cases}$$

Remark 3.

- It is easy to observe that the TpFNs are generalizations of TFNs. In fact, the TFN (*a, b, d*) can be considered as a special case of the TpFN (*a, b, c, d)* with $b=c$.
- The TFNs and the TpFNs are special cases of the *LR – FNs* of Dubois and Prade [10]. Generalizing the definitions of TFNs and TpFNs one can define n-agonal FNs of the form $(a_1, a_2, \dots, a_n)$ for any integer n, $n \geq 3$; e.g. see Section 2 of [11] for the definition of the hexagonal FNs.
- It can be shown [9] that the addition and subtraction of two TpFNs are performed in the same way that it was mentioned in ***Remark 2*** for TFNs. Also, the two scalar operations that have been defined in ***Remark 2*** for TFNs hold also for TpFNs.

The following two propositions provide two alternative ways for defuzzifying a given TpFN:

Proposition 2. (GOG of a TpFN): The coordinates (*X, Y*) of the COG of the graph of the TpFN (*a, b, c, d*) are calculated by the formulas $X = \dfrac{c^2+d^2-a^2-b^2+dc-ba}{3(c+d-a-b)}$, $Y = \dfrac{2c+d-a-2b}{3(c+d-a-b)}$.

Proof. We divide the trapezoid forming the graph of the TpFN (*a, b, c, d*) in three parts, two triangles and one rectangle (***Fig. 2***). The coordinates of the three vertices of the triangle ABE are (*a,* 0), (*b,* 1) and (*b,* 0) respectively, therefore by ***Proposition 4*** the COG of this triangle is the point $C_1\left(\dfrac{a+2b}{3}, \dfrac{1}{3}\right)$.

Similarly one finds that the COG of the triangle FCD is the point $C_2\left(\dfrac{d+2c}{3}, \dfrac{1}{3}\right)$. Also, it is easy to check that the COG of the rectangle BCFE, being the intersection point of its diagonals, is the point $C_3\left(\dfrac{b+c}{2}, \dfrac{1}{2}\right)$. Further, the areas of the two triangles are equal to $S_1 = \dfrac{b-a}{2}$ and $S_2 = \dfrac{d-c}{2}$ respectively, while the area of the rectangle is equal to $S_3 = c - b$.

Therefore, the coordinates of the COG of the trapezoid, being the resultant of the COGs C_i (x_i, yi), for i=1, 2, 3, are calculated by the formulas $X = \frac{1}{S}\sum_{i=1}^{3} S_i x_i$, $Y = \frac{1}{S}\sum_{i=1}^{3} S_i y_i$ (1), where $S = S_1 + S_2 + S_3 = \frac{c+d-b-a}{2}$ is the area of the trapezoid [12].

The proof is completed by replacing the above found values of S, S_i, x_i and y_i, i = 1, 2, 3, in formulas (1) and by performing the corresponding operations.

Proposition 3. (GOG of the GOGs of a TpFN): Consider the graph of the TpFN (α, b, c, d) (***Fig. 3***). Let G_1 and G_2 be the COGs of the rectangular triangles AEB and CFD and let G_3 be the COG of the rectangle BEFC respectively. Then $G_1G_2G_3$ is always a triangle, whose COG has coordinates

$$X = \frac{2(a+d)+7(b+c)}{18},\ Y = \frac{7}{18}.$$

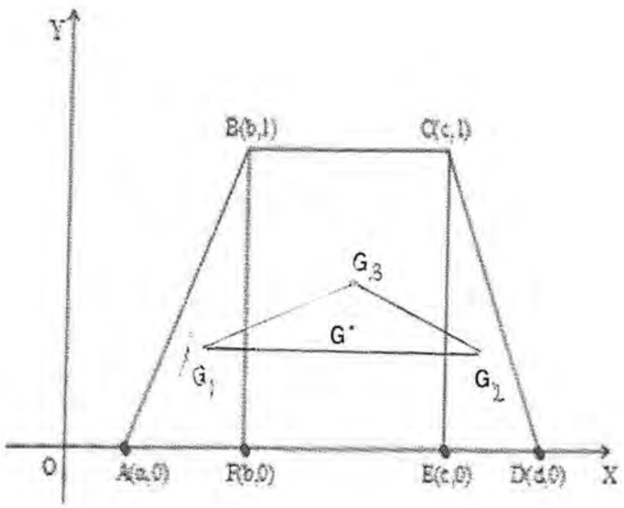

Fig. 3. The GOG of the GOGs of the TpFN (a, b, c, d).

Proof. By ***Proposition 4*** one finds that $G_1\ (\frac{a+2b}{3}, \frac{1}{3})$ and $G_2\ (\frac{d+2c}{3}, \frac{1}{3})$. Further, it is easy to check that the GOG G_3 of the rectangle BCFD, being the intersection of its diagonals, has coordinates $(\frac{b+c}{2}, \frac{1}{2})$.

The y – coordinates of all points of the straight line defined by the line segment G_1G_2 are equal to $\frac{1}{3}$, therefore the point G_3, having y – coordinate equal to $\frac{1}{2}$, does not belong to this line. Hence, by ***Proposition 6***, the COG G′ of the triangle $G_1G_2G_3$ has coordinates $X = (\frac{a+2b}{3} + \frac{d+2c}{3} + \frac{b+c}{2}) : 3 = \frac{2(a+d)+7(b+c)}{18}$ and $Y = (\frac{1}{3} + \frac{1}{3} + \frac{1}{2}) : 3 = \frac{7}{18}$.

Remark 4. Since the COGs G_1, G_2 and G_3 are the balancing points of the triangles AEB and CFD and of the rectangle BEFC respectively, the COG G′ of the triangle $G_1G_2G_3$, being the balancing point of the triangle formed by those COGs, can be considered instead of the COG G of the trapezoid ABCD for defuzzifying the TpFN (a, b, c, d). The advantage of the choice of G′ instead of G is that the formulas calculating the coordinates of G′ (***Proposition 3***) are simpler than those calculating the COG G of the trapezoid ABCD (***Proposition 2***).

An important problem of the fuzzy arithmetic is the ordering of FNs, i.e. the process of determining whether a given FN is larger or smaller than another one. This problem can be solved through the introduction of a ranking function, say R, which maps each FN on the real line, where a natural order exists. Several ranking methods have been proposed until today, like the lexicographic screening [13], the use of an area between the centroid and original points [14], the subinterval average method [11], etc.

Here, under the light of ***Propositions (1) & (3)*** and of ***Remark 4***, we define the ranking functions for TFNs and TpFNs as follows:

Definition 11. Let A be a FN. Then:

- If A is a TFN of the form A {α, b, c), we define $R(A) = \frac{a+b+c}{3}$.
- If A is a TpFN of the form A {α, b, c, d), we define $R(A) = \frac{2(a+d)+7(b+c)}{18}$.

2.5. Linear Programming

It is well known that Linear Programming (LP) is a technique for the optimization (maximization or minimization) of a linear objective function subject to linear equality and inequality constraints. The feasible region of a LP problem is a convex polytope, which is a generalization of the three-dimensional polyhedron in the n-dimensional real space R^n, where n is an integer, $n \geq 2$.

A LP algorithm determines a point of the LP polytope, where the objective function takes its optimal value, if such a point exists. In 1947, Dantzig invented the SIMPLEX algorithm [15] that has efficiently tackled the LP problem in most cases. Further, in 1948 Dantzig, adopting a conjecture of John von Neuman, who worked on an equivalent problem in Game Theory, provided a formal proof of the theory of Duality [16]. According to the above theory every LP problem has a dual problem the optimal solution of which, if there exists, provides an optimal solution of the original problem. For general facts about the SIMPLEX algorithm we refer to Chapters 3 and 4 of [17].

LP, apart from mathematics, is widely used nowadays in business and economics, in several engineering problems, etc. Many practical problems of operations research can be expressed as LP problems. However, in large and complex systems, like the socio-economic, the biological ones, etc. ., it is often very difficult to solve satisfactorily the LP problems with the standard theory, since the necessary data cannot be easily determined precisely and therefore estimates of them are used in practice. The reason for this is that such kind of systems usually involve many different and constantly changing factors the relationships among which are indeterminate, making their operation mechanisms to be not clear. In order to obtain good results in such cases one may apply techniques FLP, e.g. see [18, 19], etc.

3. Main Results

3.1. Assessment under Fuzzy Conditions

Assume that one wants to evaluate the mean performance of a group of uniform objects (individuals, computer systems, etc.) participating in a common activity. When the individual performance of the group's members is assessed by using numerical grades (scores), the traditional method for evaluating the group's mean performance is the calculation of the mean value of those scores. However, cases appear frequently in practice, where the individual performance is assessed by using linguistic instead of numerical grades. For example, this frequently happens for student assessment, usually for reasons of more elasticity that reduces the student pressure created by the existence of the strict numerical scores.

A standard method for such kind of assessment is the use of the linguistic expressions (labels) A = excellent, B = very good, C = good, D = fair and F = unsatisfactory (failed). In certain cases some insert the label E = less than satisfactory between D and F, while others use labels like B^+, B^-, etc., for a more strict assessment. It becomes evident that such kind of assessment involves a degree of fuzziness caused by the existence of the linguistic labels, which are less accurate than the numerical scores. Obviously, in the linguistic assessment the calculation of the mean value of the group's members' grades is not possible.

An alternative method for assessing a group's overall performance in such cases is the calculation of the Grade Point Average (GPA) index ([2], Chapter 6, p.125). The GPA index is a weighted mean calculated by the formula

$$GPA = \frac{0n_F + 1n_D + 2n_C + 3n_B + 4n_A}{n} . \tag{1}$$

In the above formula n denotes the total number of the group's members and n_A, n_B, n_C, n_D and n_F denote the numbers of the group's members that demonstrated excellent, very good, good, fair and unsatisfactory performance respectively. In case of the ideal performance ($n_A = n$) formula (1) gives that GPA = 4, whereas in case of the worst performance ($n_F = n$) it gives that $GPA = 4$. Therefore, we have in general that $0 \leq GPA \leq 4$, which means that values of GPA greater than 2 could be considered as indicating a more than satisfactory performance. However, since in ***Eq. (1)*** greater coefficients (weights) are assigned to the higher scores, it becomes evident that the GPA index reflects actually not the mean, but the group's quality performance.

Here a method will be developed for an approximate evaluation in such cases of the group's mean performance that uses TFNs or TpFNs as tools. For this, we need the following definition:

Definition 5. Let $A_i = (a_{1i}, a_{2i}, a_{3i}, a_{4i})$, $i = 1, 2,\ldots, n$ be TFNs/TpFNs (see ***Remark 3***), where n is a nonnegative integer, $n \geq 2$. Then the mean value of the A_i's is defined to be the TFN /TpFN,

$$A = \frac{1}{n}(A_1 + A_2 + \ldots + A_n). \qquad (2)$$

In case of utilizing TFNs as tools the steps of the new assessment method are the following:

- Assign a scale of numerical scores from 1 to 100 to the linguistic grades A, B, C, D and F as follows: A (85–100), B (75–84), C (60–74), D (50–59) and F (0–49)[1].
- For simplifying the notation use the same letters to represent the above grades by the TFNs.
- A = (85, 92.5, 100), B = (75, 79.5, 84), C (60, 67, 74), D (50, 54.5, 59) and F (0, 24.5, 49), respectively, where the middle entry of each of them is equal to the mean value of its other two entries.
- Evaluate the individual performance of all the group's members using the above qualitative grades. This enables one to assign a TFN A, B, C, D or F to each member. Then the mean value M of all those TFNs is equal to the TFN

$$M\,(a, b, c) = \frac{1}{n}\,(n_A A + n_B B + n_C C + n_D D + n_F F).$$

- Use the TFN M (a, b, c) for evaluating the group's mean performance. It is straightforward to check that the three entries of the TFN M are equal to

$$a = \frac{85n_A + 75n_B + 60n_C + 50n_D + 0n_F}{n} \quad b = \frac{92.5n_A + 79.5n_B + 67n_C + 54.5n_D + 24.5n_F}{n}$$ and

$$c - \frac{100n_A + 84n_B + 74n_C + 59n_D + 49n_F}{n}.$$ Then, by ***Proposition 6*** one gets that

$$X\,(M) = \frac{a+b+c}{3} = \frac{92.5n_A + 79.5n_B + 67n_C + 54.5n_D + 24.5n_F}{n} = \frac{a+c}{2} = b \;\; (3).$$

- Observe that, in the extreme (hypothetical) case where the lowest possible score has been assigned to each member of the group (i.e. the score 85 to nA members, the score 75 to nB members, etc.) the mean value of all those scores is equal to a. On the contrary, if the greatest score has been assigned to each member, then the mean value of all scores is equal to c. Therefore the value of X(M), being equal to the mean value of a and c, provides a reliable approximation of the group's mean performance.

In cases where multiple referees of the group's performance exist one could utilize TpFNs instead of TFNs for evaluating the group's mean performance. In that case a different TpFN is assigned to each member of the group representing its individual performance, while the other steps of the method remain unchanged (see ***Example 16***).

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[1]The scores assigned to the linguistic grades are not standard and may differ from case to case. For instance, in a more rigorous assessment one could take A(90-100), B (80-89), C(70-79), D (60-69), F(<60), etc.

3.2. Fuzzy Linear Programming

The general form of a FLP problem is the following: Maximize (or minimize) the linear expression

$F = A_1x_1 + A_2x_2 + \ldots + A_nx_n$ subject to constraints of the form $x_j \geq 0$,

$A_{i1}x_1 + A_{i2}x_2 + \ldots + A_{in}x_n \leq (\geq)$ B_i, where $i = 1, 2, \ldots, m$, $j = 1, 2,,,, n$ and A_j, A_{ij}, B_i are FNs. Here a new method will be proposed for solving FLP problems. We start with the following definition:

Definition 5. The *Degree of Fuzziness (DoF)* of a n-agonal FN $A = (a_1, \ldots, a_n)$ is defined to be the real number $D = a_n - a_1$. We write then $DoF(A) = D$.

The following two propositions are needed for developing the new method for solving FLP problems:

Proposition 4. Let A be a TFN with $DoF(A) = D$ and $R(A) = R$. Then A can be written in the form

A = (α, 3R-2α-D, α + D), where α is a real number such that $R - \frac{2D}{3} < \alpha < R - \frac{D}{3}$.

Proof. Let $A (\alpha, b, c)$ be the given TFN, with α, b, c real numbers such that $\alpha<b<c$. Then, since

$DoF(A) = c - \alpha = D$, is $c = \alpha + D$. Therefore, $R(A) = \frac{a+b+c}{3} = \frac{2a+b+D}{3} = R$, which gives that

$b = 3R-2\alpha-D$. Consequently we have that $\alpha < 3R-2\alpha-D < \alpha + D$. The left side of the last inequality implies that $3\alpha < 3R-D$, or $\alpha < R-\frac{D}{3}$. Also its right side implies that $-3\alpha < 2D-3R$, or $\alpha > R-\frac{2D}{3}$, which completes the proof.

Proposition 5. Let A be a TpFN with $DoF(A) = D$ and $R(A) = R$. Then A can be written in the form

$A = (\alpha, b, c, \alpha + D)$, where α, b and c are real numbers such that $\alpha < b \leq c < a + D$ and

$$b + c = \frac{18R - 4a - 2D}{7}.$$

Proof. Let $A (\alpha, b, c, d)$ be the given TFN, with α, b, c, d real numbers such that $a < b \leq c < d$. Since

$D(A) = d - \alpha = D$, it is $d = \alpha + D$. Also, by ***Definition 11*** we have that $R = \frac{2(2a+D)+7(b+c)}{18}$ wherefrom one gets the expression of $b + c$ in the required form.

The proposed in this work method for solving a Fuzzy LP problem involves the following steps:

- Ranking of the FNs A_j, A_{ij} and B_i.
- Solution of the obtained in the previous step ordinary LP problem with the standard theory.
- Conversion of the values of the decision variables in the optimal solution to FNs with the desired DoF.

The last step is not compulsory, but it is useful in problems of vague structure, where a fuzzy expression of their solution is often preferable than the crisp one (see ***Examples 17, 18***).

4. Applications

4.1. Examples of Assessment Problems

Example 1. ***Table 1*** depicts the performance of students of two Departments, say D_1 and D_2, of the School of Management and Economics of the Graduate Technological Educational Institute (T. E. I.) of Western Greece in their common progress exam for the course "Mathematics for Economists I" in terms of the linguistic grades A, B, C, D and F:

Table 1. Student performance in terms of the linguistic grades.

Grade	D_1	D_2
A	60	60
B	40	90
C	20	45
D	30	45
F	20	15
Total	170	255

It is asked to evaluate the two Departments overall quality and mean performance.

- Quality performance (GPA index): Replacing the data of ***Table 1*** to formula (1) and making the corresponding calculations one finds the same value $GPA = \frac{43}{17} \approx 2.53$ for the two Departments that indicates a more than satisfactory quality performance.
- Mean performance (using TFNs): According to the assessment method developed in the previous section it becomes clear that Table 1 gives rise to 170 TFNs representing the individual performance of the students of D_1 and 255 TFNs representing the individual performance of the students of D_2. Applying equation (2) it is straightforward to check that the mean values of the above TFNs are:

$D_1 = \frac{1}{170}.\ (60A+40B+20C+30D+20F) \approx (63.53,\ 73.5,\ 83.47)$, and $D_2 = \frac{1}{255}.\ (60A+90B+45C+45D+15F) \approx (65.88,\ 72.71,\ 79.53)$.

Therefore, ***Eq. (3)*** gives that $X(D_1) = 73.5$ and $X(D_2) = 72.71$. Consequently, both departments demonstrated a good (C) mean performance, with the performance of D_1 being slightly better.

Example 2. The individual performance of the five players of a basket-ball team who started a game was assessed by six different athletic journalists using a scale from 0 to 100 as follows: P_1 (player 1)**:** 43, 48, 49, 49, 50, 52, P_2**:** 81, 83. 85, 88, 91, 95, P_3**:** 76, 82, 89, 95, 95, 98, P_4**:** 86, 86, 87, 87, 87, 88 and P_5**:** 35, 40, 44, 52, 59, 62. It is asked to assess the mean performance of the five players and their overall quality performance by using the linguistic grades A, B, C, D and F. Also, for reasons of comparison, it is asked to approximate their mean performance in two ways, by using TFNs and TpFNs.

- Mean performance: Adding the $5 * 6 = 30$ in total scores assigned by the journalists to the five players and dividing the corresponding sum by 30 one finds that the mean value of those scores is approximately equal to 72.07. Therefore the mean performance of the five players can be characterized as good (C).
- Quality performance: A simple observation of the given data shows that 14 of the 30 in total scores correspond to the linguistic grade A, four to B, one to C, four to D and seven to F. Replacing those values to formula (1) one finds that the GPA index is approximately equal to 2.47. Therefore, the five players' overall quality performance can be characterized as more than satisfactory.

  Using TFNS: Forming the TFNs A, B, C, D and F and observing the $5*6 = 30$ in total player scores it becomes clear that 14 of them correspond to the TFN A, four to B, one to C, four to D and seven to F. The mean value of the above TFNs (***Definition 11***) is equal to $M = \frac{1}{30}(14A + 4B + C + 4D + 7F) \approx (58.33, 68.98, 79.63)$.

Therefore the mean performance of the five players is approximated by $X(M) = 68.98$ (good).

- Using TpFNs: A TpFN (denoted, for simplicity, by the same letter) is assigned to each basket-ball player as follows: $P_1 = (0, 43, 52, 59)$, $P_2 = (75, 81, 95, 100)$, $P_3 = (75, 76, 98, 100)$, $P_4 = (85, 86, 88, 100)$ and $P_5 = (0, 35, 62, 74)$. Each of the above TpFNs describes numerically the individual performance of the corresponding player in the form (*a, b, c, d*), where *a* and *d* are the lower and higher scores respectively corresponding to his performance, while *c* and *b* are the lower and higher scores respectively assigned to the corresponding player by the athletic journalists. For example, the performance of the player P_1 was characterized by the journalists from unsatisfactory (scores 43, 48, 49, 49) to fair (scores 50, 52), which means that $a = 0$ (lower score for F) and $d = 59$ (lower score for D), etc.

The mean value of the TpFNs P_i , i =1, 2, 3, 4, 5 (***Definition 13***), is equal to $P = \frac{1}{5}\sum_{i=1}^{5} P_i = (47, 64.2, 79, 86.6)$.

Therefore, under the light of ***Remark 10*** and ***Proposition 9*** one finds that $X(P) = \frac{2(47+86.6)+7(64.2+79)}{18} \approx 70.53$. This outcome shows that the five players demonstrated a good (C) mean performance.

The outcomes obtained from the application of the assessment methods used in ***Examples (15) & (16)*** are depicted in ***Tables (2)***-(***3)*** below.

Table 2. The outcomes of Example 1.

Method	D_1	D_2	Performance
GPA index	2.53	2.53	More than satisfactory
TFNs	73.5	72.68	Good (C)

Table 3. The outcomes of Example 2.

Method		Performance
Mean value	72.07	Good (C)
GPA index	2.47	More than satisfactory
TFNs	68.98	Good (C)
TpFNs	70.53	Good (C)

Observing those Tables one can see that the fuzzy outcomes (TFNs/TpFNs) are more or less compatible to the crisp ones (mean value/GPA index). This provides a strong indication that the fuzzy assessment method developed in this work "behaves" well. The appearing, relatively small, numerical differences are due to the different "philosophy" of the methods used (mean and quality performance, bi-valued and fuzzy logic).

The approximation of the player mean performance (70.53) obtained in ***Example 2*** using TpFNs is better (nearer to the accurate mean value 72.07 of the numerical scores) than that obtained by using TFNs (68.98). This is explained by the fact that the TpFNs, due to the way of their construction, describe more accurately than the TFNs each player's individual performance. However, it is not always easy in practice to use TpFNs instead of TFNs. Another advantage of using TpFNs as assessment tools is that, in contrast to TFNs, they make possible the comparison of the individual performance of any two members of the group, even among those whose performance has been characterized by the same qualitative grade. At any case, the fuzzy approximation of a group's mean performance is useful only when no numerical scores are given assessing the idividual performance of its members.

4.2. Examples of FLP Problems

Example 3. In a furniture factory it has been estimated that the construction of a set of tables needs 2 - 3 working hours (w. h.) for assembling, 2.5 - 3.5 w. h. for elaboration (plane, etc.) and 0.75 - 1.25 w. h. for polishing, while the construction of a set of desks needs 0.8 - 1.2, 2 - 4 and 1.5 - 2.5 w. h. respectively for each of the above procedures. According to the factory's existing number of workers, at most 20 w. h. per day can be spent for the assembling, at most 30 w. h. for the elaboration and at most 18 w. h. for the polishing of the tables and desks. If the profit from the sale of a set of tables is between 2.7 and 3.3 hundred euros and of a set of desks is between 3.8 and 4.2 hundred euros[2], find how many sets of tables and desks should be constructed daily to maximize the factory's total profit. Express the problem's optimal solution with TFNs of DoF equal to 1.

Solution. Let x_1 and x_2 be the sets of tables and desks to be constructed daily. Then, using TFNs, the problem can be mathematically formulated as follows[3]:

Maximize $F = (2.7, 3, 3.3)x_1 + (3.8, 4, 4.2)x_2$ *subject to* $x_1, x_2 \geq 0$ and

$(2, 2.5, 3)x_1 + (0.8, 1, 1.2]x_2 \leq (19, 20, 21)$,

$(2.5, 3, 3.5)x_1 + (2, 3, 4)x_2 \leq (29, 30, 31)$,

$(0.75, 1, 1.25)x_1 + (1.5, 2, 2.5)x_2 \leq (15, 16, 17)$.

The ranking of the TFNs involved leads to the following LP maximization problem of canonical form:

Maximize $f(x_1, x_2) = 3x_1 + 4x_2$ *subject to* $x_1, x_2 \geq 0$ and $2.5x_1 + x_2 \leq 20$, $3x_1 + 3x_2 \leq 30$, and $x_1 + 2x_2 \leq 16$.

Adding the slack variables s_1, s_2, s_3 for converting the last three inequalities to equations one forms the problem's first SIMPLEX matrix, which corresponds to the feasible solution $f(0, 0) = 0$, as follows:

$$\begin{bmatrix} x_1 & x_2 & s_1 & s_2 & s_3 & | & \text{Const.} \\ - & - & - & - & - & - & - \\ 2.5 & 1 & 1 & 0 & 0 & | & 20 = s_1 \\ 3 & 3 & 0 & 1 & 0 & | & 30 = s_2 \\ 1 & 2 & 0 & 0 & 1 & | & 16 = s_3 \\ - & - & - & - & - & | & - \\ -3 & -4 & 0 & 0 & 0 & | & 0 = f(0,0) \end{bmatrix}.$$

Denote by L_1, L_2, L_3, L_4 the rows of the above matrix, the fourth one being the *net evaluation row*. Since -4 is the smaller (negative) number of the net evaluation row and $\frac{16}{2} < \frac{30}{3} < \frac{20}{1}$, the *pivot element* 2 lies in the intersection of the third row and second column Therefore, applying the linear transformations $L_3 \rightarrow \frac{1}{2} L_3 = L'_3$ and $L_1 \rightarrow L_1 - L'_3$, $L_2 \rightarrow L_2 - 3L'_3$, $L_4 \rightarrow L_4 + 4L'_3$, one obtains the second SIMPLEX matrix, which corresponds to the feasible solution $f(0, 8) = 32$ and is the following:

---

[2]The profit is changing depending upon the price of the wood, the salaries of the workers, etc.

[3] The mathematical formulation of the problem using TFNs is not unique. Here we have taken $b = \frac{a+c}{2}$ for all the TFNs involved, but this is not compulsory. The change of the values of the above TFNs, changes of course the ordinary LP problem obtained by ranking them, but the change of its optimal solution is relatively small.

$$\begin{bmatrix} x_1 & x_2 & s_1 & s_2 & s_3 & | & \text{Const.} \\ - & - & - & - & - & - & - \\ 2 & 0 & 1 & 0 & -\frac{1}{2} & | & 12 = s_1 \\ \frac{3}{2} & 0 & 0 & 1 & -\frac{3}{2} & | & 6 = s_2 \\ \frac{1}{2} & 1 & 0 & 0 & \frac{1}{2} & | & 8 = x_2 \\ - & - & - & - & - & | & - \\ -1 & 0 & 0 & 0 & 0 & | & 32 = f(0,8) \end{bmatrix}.$$

In this matrix the pivot element $\frac{3}{2}$ lies in the intersection of the second row and first column, therefore working as above one obtains the third SIMPLEX matrix, which is:

$$\begin{bmatrix} x_1 & x_2 & s_1 & s_2 & s_3 & | & \text{Const.} \\ - & - & - & - & - & - & - \\ 0 & 0 & 1 & -\frac{4}{3} & -\frac{3}{2} & | & 4 = s_1 \\ 1 & 0 & 0 & \frac{2}{3} & -1 & | & 4 = x_1 \\ 0 & 1 & 0 & -\frac{1}{3} & 1 & | & 6 = x_2 \\ - & - & - & - & - & | & - \\ 0 & 0 & 0 & \frac{2}{3} & 1 & | & 36 = f(4,6) \end{bmatrix}.$$

Since there is no negative index in the net evaluation row, this is the last SIMPLEX matrix. Therefore $f(4, 6) = 36$ is the optimal solution maximizing the objective function. Further, since both the decision variables x_1 and x_2 are basic variables, i.e. they participate in the optimal solution, the above solution is unique.

Converting, with the help of ***Proposition 15***, the values of the decision variables in the above solution to TFNs with DoF equal to 1, one finds that $x_1 = (\alpha, 11\text{-}2\alpha, \alpha+1]$ with $\frac{10}{3} < a < \frac{11}{3}$ and $x_2 = (a, 17\text{-}2a, a+1)$ with. $\frac{16}{3} < a < \frac{17}{3}$. Therefore a fuzzy expression of the optimal solution states that the factory's maximal profit corresponds to a daily production between α and $\alpha+1$ groups of tables with $3.33 < a < 3.67$ and between a and $a+1$ groups of desks with $5.33 < a < 5.67$.

However, taking for example $\alpha = 3.5$ and $a = 5.5$ and considering the extreme in this case values of the daily construction of 4.5 groups of tables and 6.5 groups of desks, one finds that they are needed 33 in total w. h. for elaboration, whereas the maximum available w. h. are only 30. In other words, a fuzzy expression of the solution does not guarantee that all the values of the decision variables within the boundaries of the corresponding TFNs are feasible solutions.

Example 4. Three kinds of food, say F_1, F_2 and F_3, are used in a poultry farm for feeding the chickens, their cost varying between 38 - 42, 17 - 23 and 55 - 65 cents per kilo respectively. The food F_1 contains between 1.5 - 2.5 units of iron and 4 - 6 units of vitamins per kilo, F_2 contains 3.2 - 4.8, 0.6 – 1.4 and F_3 contains 1.7 – 2.3, 0.8 – 1.2 units per kilo respectively. It has been decided that the chickens must receive at least 24 units of iron and 8 units of vitamins per day. How must one mix the three foods so that to minimize the cost of the food? Express the problem's solution with TpFNs of DoF equal to 2.

Solution. Let x_1, x_2 and x_3 be the quantities of kilos to be mixed for each of the foods F_1, F_2 and F_3 respectively. Then, using TpFNs the problem's mathematical model could be formulated as follows[4]:

Minimize $F = (38, 39, 41, 42)\, x_1 + (17, 18, 22, 23)\, x_2 + (55, 56, 64, 65]\, x_3$ subject to $x_1, x_2, x_3 \geq 0$ and

$(1.5, 1.8, 2.2, 2.5)\, x_1 + (3.2, 3.5, 4.5, 4.8)\, x_2 + (1.7, 1.9, 2.1, 2.3]\, x_3 \geq [22, 23, 25, 26]$

$[4, 4.5, 5.5, 6]\, x_1 + [0.6, 0.8, 1.2, 1.4]\, x_2 + [0.8, 0.9, 1.1, 1.2] x_3 \geq (6, 7, 9, 1\ 0).$

The ranking of the TpFNs by ***Definition 13*** leads to the following LP minimization problem of canonical form:

Minimize $f(x_1, x_2, x_2) = 40x_1 + 20x_2 + 60x_3$ subject to $x_1, x_2, x_3 \geq 0$ and $2x_1 + 4x_2 + 2x_3 \geq 24$, $5x_1 + x_2 + x_3 \geq 8$.

The dual of the above problem is: the following:

Maximize $g(z_1, z_2) = 24z_1 + 8z_2$ subject to $t\ z_1, z_2 \geq 0$, $2z_1 + 5z_2 \leq 40$, $4z_1 + z_2 \leq 20$, $2z_1 + z_2 \leq 60$

Working similarly with ***Example 3*** it is straightforward to check that the last SIMPLEX matrix of the dual problem is the following:

$$\begin{bmatrix} z_1 & z_2 & s_1 & s_2 & s_3 & | & \text{Const.} \\ - & - & - & - & - & - & - \\ 0 & 1 & \frac{2}{9} & \frac{1}{9} & 0 & | & \frac{20}{3} = z_2 \\ 1 & 0 & -\frac{1}{18} & \frac{5}{18} & 0 & | & \frac{10}{3} = z_1 \\ 0 & 0 & -\frac{1}{9} & -\frac{4}{9} & 1 & | & \frac{140}{3} = s_3 \\ - & - & - & - & - & | & - \\ 0 & 0 & \frac{4}{9} & \frac{52}{9} & 0 & | & \frac{400}{3} = g(\frac{10}{3}, \frac{20}{3}) \end{bmatrix}.$$

[4]The problem's mathematical formulation using TpFNs is not unique, but the change of its optimal solution in each case is relatively small.

Therefore the solution of the original minimization problem is $f_{min} = f(\frac{4}{9}, \frac{52}{9}, 0) = \frac{400}{3}$.

In other words, the minimal cost of the chicken food is $\frac{400}{3} \approx 133$ cents and will be succeeded by mixing $\frac{4}{9} \approx 0.44$ kilos from food F_1 and $\frac{52}{9} \approx 5.77$ kilos from food F_2.

Converting the values of the decision variables in the above solution to TpFNs with DoF equal to 2 one finds with the help of ***Proposition 16*** that *x1, x2, x3* must be of the form $(\alpha, b, c, \alpha + 2)$ with

$\alpha < b \leq c < \alpha + 2,\ b + c = \frac{18R - 4a - 4}{7}$ and $R = \frac{4}{9}$ or $R = \frac{52}{9}$ or $R = 0$ respectively.

For R = $\frac{4}{9}$ one finds that b + c = $\frac{4-4a}{7}$. Therefore b < $\frac{4-4a}{7}$ - b or b < $\frac{2-2a}{7}$ which gives that

$\alpha < \frac{2-2a}{7}$ or $\alpha < \frac{2}{9}$. Taking for example $\alpha = \frac{1}{9}$, one finds that $b < \frac{2-\frac{2}{9}}{7} = \frac{16}{63}$. Therefore, taking for example $b = \frac{15}{63}$, we obtain that $c = \frac{4-\frac{4}{9}}{7} - \frac{15}{63} = \frac{17}{63}$. Therefore $x_1 = (\frac{7}{63}, \frac{15}{63}, \frac{17}{63}, \frac{133}{63})$.

Working similarly for $R = \frac{52}{9}$ and $R = 0$ one obtains that $x_2 = (\frac{196}{63}, \frac{340}{63}, \frac{362}{63}, \frac{488}{63})$ and $x_3 = (-\frac{21}{63}, -\frac{15}{63}, -\frac{9}{63}, \frac{60}{63})$, respectively.

Therefore, since a TpFN (a, b, c, d) expresses mathematically the fuzzy statement that the interval [b, c] lies within the interval [a, d], a fuzzy expression of the problem's optimal solution states that the minimal cost of the chickens' food will be succeeded by mixing between $\frac{15}{63} \approx 0.24$, $\frac{17}{63} \approx 0.27$, between $\frac{340}{63} \approx 5.4$, $\frac{362}{63} \approx 5.75$ and between $-\frac{15}{63} \approx -0.24$, $-\frac{17}{63} \approx -0.27$ kilos from each one of the foods F_1, F_2 and F_3 respectively. The values of x_3 are not feasible and must be replaced by 0, whereas the values of x_1 and x_2 must be checked as we did in ***Example 3***.

5. Discussion and Conclusions

The target of the present paper was two-folded. First, a combination of TFNs / TpFNs and of the COG defuzzification technique was used for assessment purposes. Examples were presented on student and basket-ball player assessment and the new fuzzy method was validated by comparing its outcomes with those of traditional assessment methods (calculation of the mean value of scores and of the GPA index). The advantage of this method is that it can be used for evaluating a group's mean performance when qualitative grades are used instead of numerical scores for assessing the individual performance of its members.

Second, a new technique was developed for solving Fuzzy LP problems by ranking the FNs involved and by solving the ordinary LP problem obtained in this way with the standard theory. Real-life examples were presented to illustrate the applicability of the new technique in practice. In LP problems with a vague structure a fuzzy expression of their solution is often preferable than a crisp one. This was attempted in the present work by converting the values of the decision variables in the optimal solution of the obtained ordinary LP problem to FNs with the desired DoF. The smaller the value of the chosen DoF, the more creditable is the fuzzy expression of the problem's optimal solution.

The new assessment method that has been developed in this work has a general character. This means that, apart for student and athlete assessment, it could be utilized for assessing a great variety of other human or machine (e.g. Case – Based Reasoning or Decision – Making systems) activities. This is an important direction for future research. Also a technique similar to that applied here for solving FLP problems could be used for solving systems of equations with fuzzy coefficients, as well as for solving LP problems and systems of equations with grey coefficients [20].

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Using Interval Type-2 Fuzzy Logic to Analyze Igbo Emotion Words

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ABSTRACT

Several attempts had been made to analyze emotion words in the fields of linguistics, psychology and sociology; with the advent of computers, the analyses of these words have taken a different dimension. Unfortunately, limited attempts have so far been made to using Interval Type-2 Fuzzy Logic (IT2FL) to analyze these words in native languages. This study used IT2FL to analyze Igbo emotion words. IT2F sets are computed using the interval approach method which is divided into two parts: the data part and the fuzzy set part. The data part preprocessed data and its statistics computed for the interval that survived the preprocessing stages while the fuzzy set part determined the nature of the footprint of uncertainty; the IT2F set mathematical models for each emotion characteristics of each emotion word is also computed. The data used in this work was collected from fifteen subjects who were asked to enter an interval for each of the emotion characteristics: Valence, Activation and Dominance on an interval survey of the thirty Igbo emotion words. With this, the words are being analyzed and can be used for the purposes of translation between vocabularies in consideration to context.

1. Introduction

Words are vital in our description and understanding of emotions and means different things to different people based on different instances and some are uncertain [1]. Hence, there is a need for a model that can capture the uncertainties of these words. Emotions are feelings or involuntary physiological response of a person to a situation, words or things. Emotion is defined using two approaches: the classical approach and the use of multidimensional space. The classical approach uses fixed number of emotion classes such as {positive, non-positive}, {negative, non-negative}, and {angry, non-angry} in describing emotion-related states. However, this approach lacks the ability to handle many types of real-life emotions while the second approach represents emotion using each point in the multidimensional space and the dimensions that are mostly used are valence, activation, and dominance [2, 3]. Valence represents negative

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to positive axis, activation represents calm to excited axis while dominance represents weak to strong axis in the three dimensional space [4]. However, the two approaches have failed to represent the uncertainty a person has about the emotion words as reflected in real-world and are somewhat vague and not precisely defined sets.

Therefore, application of fuzzy set model is well suitable because of its ability to cope with uncertainties [5, 6] and can represent each emotion dimension using intervals rather than fixed points [7].

The Type -1 Fuzzy Set (T1-FS) is a generalization of traditional classical sets in which a concept can possess a certain degree of truth, where the truth value may range between completely true and completely false. However, T1-FS lacks the ability to adequately represent or directly handle data uncertainty because its membership function is crisp in nature [8]. The optimal design of fuzzy systems enables making decisions based on a structure built from the knowledge of experts and guided by membership functions and fuzzy rules [28]. "The membership functions of type-2 fuzzy sets are (n+2)-dimensional, while membership functions of type-1 fuzzy sets are only (n+1)-dimensional (assuming that the universe of discourse has n dimensions). Thus, type-2 fuzzy sets allow more degrees of freedom in representing uncertainty" [32]. Type-2 Fuzzy Set (T2-FS) which is an extension of T1-FS was used in [9] to address the inabilities of T1-FLS. Interval Type-2 Fuzzy Logic (IT2FL) is a simplified version of the general T2FLS that uses intervals to handle uncertainty in the membership function. The structure of T2-FIS is similar to T1-FLS but with additional component called the type reducer modified to accommodate T2F set. Type Reducer is used to reduce the output of the T2 inference engine to type-1 before defuzzification. An IT2FL can better model intrapersonal and interpersonal uncertainties, which are intrinsic to natural language, because the membership grade of an IT2 Fuzzy set is an interval instead of a crisp number as in a T1 FS [31].

Though, many works have been done in the field of emotion words, [10], [8], [11], [12], [13], [14], [15], [16], [17], [18], however, attempts to analyze emotion words in Igbo non-English languages have not been impacted in any way. This study uses IT2FL to analyze Igbo Emotion Words. The term "Igbo" in this context is used to describe the language spoken primarily by Igbo ethnic group in Nigeria. The Igbo belong to the Sudanic linguistic group of the Kwa division according to [19, 20]. This wide presence of the Igbos is the basis for selecting the emotions words in Igbo language for analysis using IT2FL.

This study used IT2FL to analyze Igbo emotion words because of its ability to adequately handle emotion word uncertainties described by its Footprint Of Uncertainty (FOU), which is the uncertainty about the union of all the primary MFs. Uncertainty is a characteristic of information, which may be incomplete, inaccurate, undefined, inconsistent. Primary Membership Functions with where both the standard deviation and the uncertain are popular FOUs for a Gaussian because of their parsimony and differentiability [29]. IT2F sets are computed using the interval approach method which is divided into two parts: the data part and the fuzzy set part. The data part preprocessed data and its statistics computed for the interval that survived the preprocessing stages while the fuzzy set part determined the nature of the footprint of uncertainty; the IT2F set mathematical models for each emotion characteristics of each emotion word is also computed. The data used in this work was collected from 15 (fifteen) subjects who were asked to enter an interval for each of the emotion characteristics: Valence, Activation and Dominance on an interval survey of the 30 (thirty) Igbo emotion words.

IT2F sets are computed using the interval approach method which is divided into two parts: the data part and the fuzzy set part. The data part preprocessed data and its statistics computed for the interval that survived the preprocessing stages while the fuzzy set part determined the nature of the footprint of uncertainty; the IT2F set mathematical models for each emotion characteristics of each emotion word is also computed. The data used in this work was collected from 15 (fifteen) subjects who were asked to enter an interval for each of the emotion characteristics: Valence, Activation and Dominance on an interval survey of the 30 (thirty) Igbo emotion words. This paper is organized as follows: In the following section, the Emotion Space, Emotion Vocabularies and Variables, the Igbos and Emotions, T2FLS and Sets, and the IT2FLS are described. The research methodology is presented, the results and discussion are described and the conclusions are drawn.

1.1. Emotion Space, Emotion Vocabularies and Variables

The emotion space E can be considered as a set of all possible emotions and it is represented using variables on the Cartesian product space of valence, activation and dominance scales. An emotional variable $\mathcal{E}$ represents an arbitrary region in the emotion space i.e. $\varepsilon \subset E$. An emotional vocabulary in ***Eq. (1)***,

$$V = (W_V, eval_V) \tag{1}$$

is a set of words W_V and a function $eval_V$ that maps words of W_V to their corresponding region in the emotional space, $eval$: $W_v \rightarrow E$. Thus, an emotional vocabulary can be seen as a dictionary for looking up the meaning of an emotion word. Words in an emotional vocabulary can be seen as constant emotional variables.

1.2. Igbos and Emotions

Emotions affect every human including the Igbo's and the Igbo emotions have become even more complex over time hence when the Igbos speak of their emotions, the analysis, interpretation and translation to other languages is highly needed. For instance, an Igbo businessman may say "iwe ne enwe m" which means "I am angry". The emotion word in his statement is "iwe" translated as "anger" in English could either be with regards to a business situation or a person. Assuming it is a person that is an individual who annoyed him, it would not change his disposition about his business hence at almost the same time the same businessman could be heard saying "oba go" meaning "it has entered" which insinuates happiness of some sought.

1.3. Type-2 Fuzzy Logic System and Sets

T2FLS is the generalized standard type-1 in order to accommodate and handle more uncertainties in the MF [21]. According to [22] "words mean different things to different people". T2FLS is based on the T2F sets where the MF has multiple values for a crisp input of x, making the need for the creation of a 3-dimensional MF for all x Ɛ X. A characteristic feature of T2FS is the FOU, which is the union of all primary memberships and upper MFs and a lower MFs that are the bounds for the FOU of a T2Fs.

Uncertainty in relations and uncertainty in values of the variables are majorly the types of uncertainty considered in developing systems since the high overlapping of the MFS, defining precise values for linguistic variables is not possible [30].

Given a T2Fs $\tilde{A}$, the representation of $\tilde{A}$ is as shown ***Eq. (2)***.

$$\tilde{A} = \{((x, u), \mu_{\tilde{A}}(x, u)) | x \in X, u \in J_x \in [0,1]\}. \quad (2)$$

Where $\mu_{\tilde{A}}(x, u)$ is the type-2 fuzzy MF in which $0 \leq \mu_{\tilde{A}}(x, u) \leq 1$

1.4. Interval Type-2 Fuzzy Logic System (IT2FLS)

As a result of the computational complexity of using a general T2FS, IT2Fs is mostly used as a special case an express as, when all $\mu_{\tilde{A}}(x, u) = 1$, then $\tilde{A}$ can rightly be described as an IT2FL as seen in ***Eq. (3)*** and in ***Fig. 1***. The interval type-2 membership function is always equal to 1.

$$\tilde{A} = \int_{x \, \varepsilon \, X} \int_{u \, \varepsilon \, Jx} \frac{1}{(x,u)}, \int_x \varepsilon \, [0, 1]. \quad (3)$$

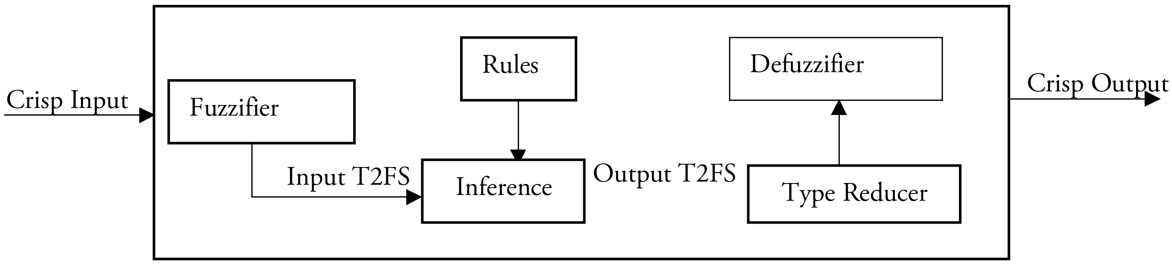

Fig. 1. Typical structure of interval type-2 fuzzy logic system [23].

IT2FLS consists of the fuzzifier, rule base, Inference Engine (IE), Type Reducer (TR). Fuzzification module maps the crisp input to a T2Fs using Gaussian MF. Inference Engine module evaluates the rules in the knowledge base against T2Fs from Fuzzification, to produce another T2Fs. TR uses Karnik-Mendel algorithm to reduce an IT2Fs to T1Fs. Defuzzification module maps the fuzzy set produced by TR to a crisp output using center of gravity defuzzification method. Fuzzy knowledge base stores rules generated from experts' knowledge used by the IE.

2. Research Methodology

The paper employs interval approach where data and fuzzy set parts are considered.

2.1. Interval Approach (IA)

The IA is a method for estimating MFs where the subject does not need to be knowledgeable about fuzzy sets and it has a simple and unambiguous mapping from data to FoU. Feilong and Mendel [24] used the

IA to capture the strong points of two previous methods: the Person-MF Approach and the End-point Approach. Using the IA, the data collected from different subjects is subjected to probability distribution. The mean and standard deviation of the distribution are then mapped into the parameters of a T1 MF which are then transformed into T2 MF from which the IT2 MF is derived. The IA is divided into two parts namely the Data Part and the Fuzzy Set Part (FSP) as seen in ***Figs. (2) & (3)***, respectively.

2.1.1. The Data Part (DP)

The DP of the IA consists of data collection, data preprocessing and probability distribution assignment parts as shown in ***Fig. 2***. The DP highlights the valence layer and this is repeated for each word in the vocabulary. In ***Fig. 2***, the following steps are carried out to achieve an input to the FSP.

- Data Collection. Here, interval survey is performed to collect human intuition about fuzzy predicate.
- Data Preprocessing. Pre-processing for n interval data $\left[a^{(i)}, b^{(i)}\right]$, i=1,…,n, are performed consisting of 3 stages:
- Bad Data Processing: At this stage, results from the survey that do not fall within the given range are removed. If interval end-points given by the respondents satisfy,

$$\left.\begin{array}{l} 0 \leq \mathrm{a}^{\mathrm{i}} \leq 10 \\ 0 \leq \mathrm{b}^{\mathrm{i}} \leq 10 \\ \mathrm{b}_{\mathrm{i}} \geq \mathrm{a}_{\mathrm{i}} \end{array}\right\} \forall\, \mathrm{i} = 1, \cdots, \mathrm{n}, \tag{4}$$

then accept the interval; else reject it. After this stage what remains is $n' \leq n$ intervals

- Outlier Processing. At this stage, a Box and Whisker test is used to eliminate outliers. Outliers are points which do not satisfy

$$\begin{array}{l} a^{(i)} \in [Q_a(0.25) - 1.5IQR_a, Q_a(0.75) + 1.5IQR_a] \\ b^{(i)} \in [Q_b(0.25) - 1.5IQR_b, Q_b(0.75) + 1.5IQR_b] \end{array}. \tag{5}$$

$Q_a(p)$ and $IQR_a = Q_a(0.75) - Q_a(0.25)$ are the p quartile and inter-quartile range for the left end-points and $Q_b(p)$ and $IQR_b = Q_b(0.75) - Q_b(0.25)$ are the p quartile and inter-quartile range for the right end-points, respectively.

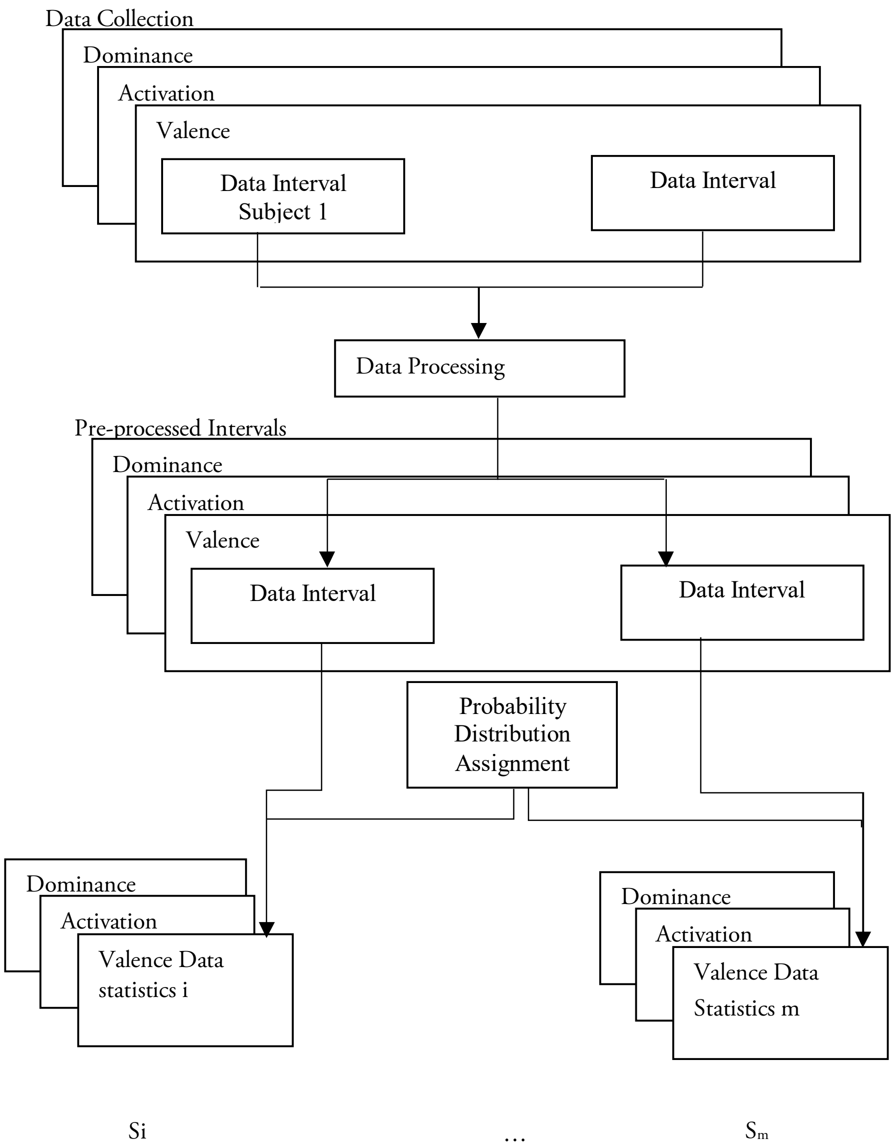

Fig. 2. The data part of the interval approach [25].

- Tolerance-Limit Processing. Here, tolerance-limit test is processed using ***Eqs. (6) & (7)*** and if the interval passes, it is accepted else, it is rejected. Then the data intervals are reduced to $m'' \leq m'$.

$$a^{(l)} \in [m_l - k\sigma_l, m_l + k\sigma_l,]. \tag{6}$$

$$b^{(r)} \in [m_r - k\sigma_r, m_r + k\sigma_r,]. \tag{7}$$

k is determined by confidence analysis, such that with a $100(1-\gamma)\%$ confidence the given limits contain at least the proportion$1-\alpha$ measurements.

After the data preprocessing, we are left with *m* data intervals such that $1 \leq m \leq n$.

- Assign Probability Distribution to the Interval Data. Here, a probability distribution is assigned to each subject's data interval. The distribution can be uniform, triangular, normal, etc. but for the purpose of this work, the uniform distribution is used.
- Compute Data Statistics for the Interval Data. The data statistics $m_l, \sigma_l, m_r and \sigma_r$, which are the sample means and standard deviation of the left- and right-end points respectively, are computed based on interval data that remains. The last two steps are often merged since they work hand in hand.

The probability distribution $S_i = (m_i, \sigma_i)$ is assigned to the remaining intervals, then the statistics are calculated using the formula for random variables with random distribution stated as

$$m = (a + b)/2 \text{ and} \tag{8}$$

$$\sigma = (b - a)/\sqrt{12}. \tag{9}$$

Then the probability distribution, S_i, is evaluated in ***Eq. (10)*** and forms the input to the fuzzy set part

$$S_i = (m_Y^i, \sigma_Y^i) \; \forall \; i = 1, \dots, m. \tag{10}$$

The S_i is then input to the next stage, the fuzzy set part.

2.1.2. The Fuzzy Set Part (FSP)

The fuzzy set part constructs the interval type-2 fuzzy sets as shown in ***Fig. 3.*** It takes the result of Data part, i.e. the S_i as input from which it creates the IT2Fs. The Layers denote individual fuzzy sets for valence, activation and dominance. This framework is repeated for each word in the vocabulary.

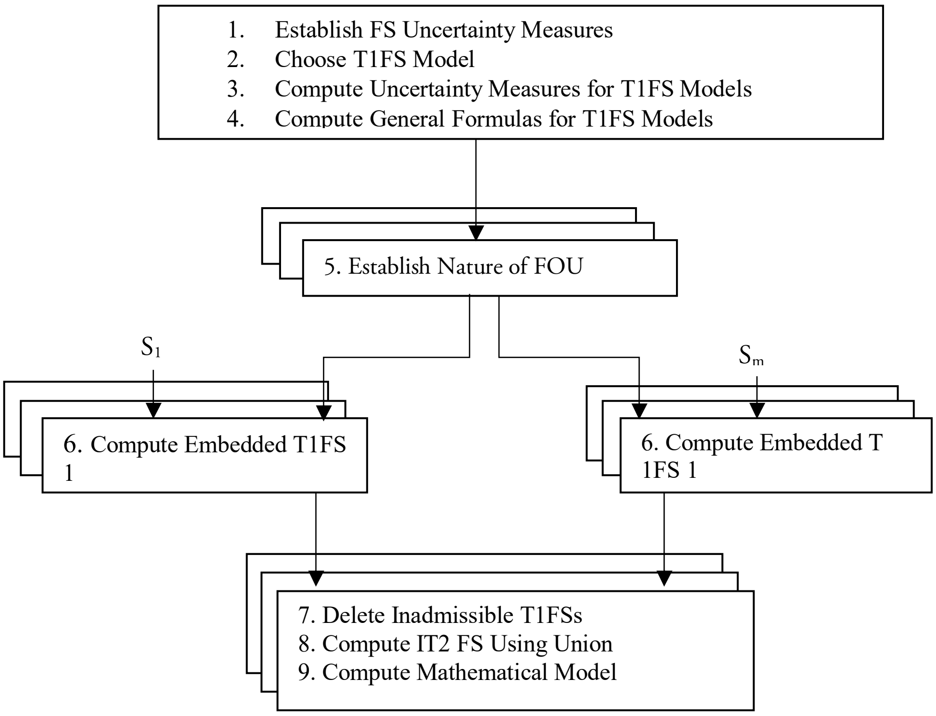

Fig. 3. Fuzzy set part of the interval approach [25].

In ***Fig. 3***, the fuzzy set part of IT2FL process is divided into nine (9) steps. Step 1, establishes FS uncertainty measures while Step 2 selects the T1FS model. In Step 3, the uncertainty measures for the selected models are computed. The uncertainty measures on the symmetrical interior triangle compute the mean and Standard Deviation (SD) derived from the mean and SD of the triangular and uniform distributions. In step 4, the S_i from the data part is recollected. The mean and standard SD from the interval model is equated with the corresponding parameters from the previous step. In Step 5, the nature of the FoU is established using the parameters of the models associated with each interval to classify whether an interval should be mapped to an interior MF, or a left or right shoulder MF. The input interval is mapped to a shoulder MF if the parameters show that the distribution is out of the range of the scales. In Step 6, the T1Fs is computed with the intervals and the decision is made in the previous step. Since the MFs derived in this step are based on statistics and not the raw intervals themselves, another preprocessing stage is carried out to delete inadmissible T1FS is with range outside of the limit of the scale of the variable of interest. In Step 7, the fuzzy sets derived are the subject-specific T1FSs. The aggregate is taken to compute the IT2FSs that contain all the subject-specific T1FS in its FOU. This aggregation can be said to be a type-2 union of T1FSs, where the embedded T1FSs describe the FOU of IT2FSs. This includes Steps 8 and 9. After these steps, the IT2FSs word model is derived. The study analyzes the 30 emotion words collected from a wide range of psychological domains based on their MF values of the three characteristics: Valence, Activation, and Dominance.

3. System Design

The architecture for analyzing emotion words from Igbo using IT2FL presented in ***Fig. 4***. The architecture is made up of the Knowledge Engine, the Knowledge Base and the User Interface. The Knowledge base further comprises the Database Model, the Interval Surveys and the IT2FL Model. The IT2FL model is composed of the Interval Approach Data Part and the Interval Approach Fuzzy Set Part. The knowledge engine stores all the variables required for the system, the knowledge base stores values for the variables defined in the knowledge engine. The Database stores an organized data to be used in the system as collected from the subjects. Interval survey is where the data are collected in a range of intervals which defines a subject's view of each emotion word given with regards to the emotion characteristics defined in the knowledge engine. The IT2FL model is used to represent words using IT2FSs. The IT2FL comprises Interval Approach which is made up of data and fuzzy set parts. The data part processes the data and makes it ready for use in estimating the MF of each emotion characteristics in each emotion word. The fuzzy set part evaluates the MF representing each emotion characteristics of each word.

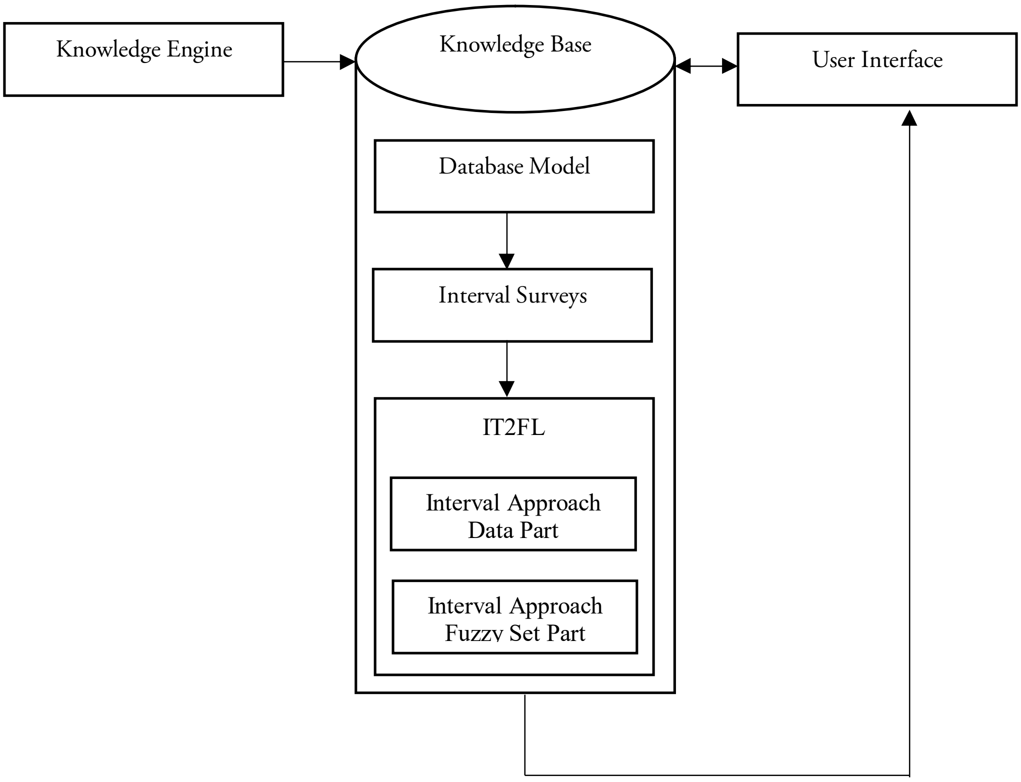

Fig. 4. Architecture for analyzing Igbo emotion words using interval IT2FL.

3.1. The Knowledge Engine for Analyzing Igbo Emotion Words

For the purpose of this paper, the variables used include the emotional characteristics (Valence, Activation and Dominance). Valence defines an emotion word on the basis of its positivity or negativity. Activation defines an emotion word on the basis of its calmness or excitement. Dominance defines an emotion word on the basis of its submissiveness or aggressiveness. The Igbo emotion vocabulary used in this work is contained in the knowledge engine. The emotion vocabulary used in this research work is the Igbo Emotions vocabulary of 30 words which include: Iwe, Obi Uto, Onuma, Ujo, Ntukwasi Obi, Obi Ojo, Onu, Anuri, Egwu, Ihunanya, Mgbagwoju Anya, Mwute, Ikpe Mara, Enyo, Ihere, Iwe Oku, Ekworo, Anyaufu, Anya Ukwu, Obi Ike, Nrugide, Akwa Uta, Kpebisiri Ike, Kenchekwube, Keechiche, Mkpako, Nwayoo, Obi Abuo, Obi Mgbawa, Ara.

Table 1. The Igbo emotion words and the english equivalence.

Igbo	**English**
Iwe	Anger, Annoyance
Obi Uto	Happiness, Delighted
Onuma	Wrath, Great Anger
Ujo	Fear, Shock
Ntukwasi Obi	Trust
Obi Ojo	Wickedness, Bitter
Onu	Joy
Anuri	Gladness
Egwu	Great Fear, Dread
Ihunaya	Love
Mgbagwoju Anya	Confused
Mwute	Regret
Ikpe Mara	Guilty
Enyo	Suspicious
Ihere	Shame
Iwe Oku	Hot-Temper
Ekworo	Jealousy
Anyaufu	Envy
Anya Ukwu	Greed, Discontentment
Obi Ike	Confidence, Courage
Nrugide	Persuasion
AkwaUta	Regret, Condemned
Kpebisiri Ike	Determination, Strong-willed
Kenchekwube	Hopeful
Keechiche	Worry
Mpako	Arrogance, Pride
Nwayoo	Gentle, Calm
Obi Abuo	Doubt, Unsteady
Obi Mgbawa	Heartbroken
Ara	Mad, Disturbed

3.2. The Database Model

The IT2F inference system uses the data as organized in the database. The database schema is created to hold the collected data, data statistics and the membership function values, respectively.

3.3. The Interval Survey

An interval survey is carried out using questionnaires which are given to 15 native speakers of Igbo. The questionnaire began by giving the users instructions then sequentially providing the words to the users. The reason for the interval survey is to collect human intuition about fuzzy predicate which is emotion in this case and provision is given for the subject to give an interval within the range 0-10 that defines the emotion characteristics.

3.4. The IT2FL Model to Analyze Igbo Emotion Words

The IT2FL model for analyzing Igbo emotion words using Interval Approach. The interval approach is used because it takes on the strength of the interval endpoints and the person membership function approaches. Based on ***Figs. (2)*** & (***3)***, the data and fuzzy set parts of Interval Approach for analyzing Igbo emotion words using IT2FL are evaluated.

3.4.1. Interval approach data part for analyzing Igbo emotion words using IT2FL

Using ***Fig. 2***, in the data collection part, data are collected using the interval surveys from 10 subjects who are native speakers of the Igbo Language. The 30 emotion words are randomly ordered and presented to the respondents. Each was asked to provide the end-points on an interval for each word on the scale of 0-10. Data processing is performed in estimating the MF of each emotion characteristic in each emotion word and then calculates the statistics of the data intervals. Validation of the data intervals is done starting at 'n' intervals at the data preprocessing part. Bad data processing is performed on the collected dataset and the intervals that do not satisfy the condition in ***Eq. (4)*** is removed. After bad processing, outlier processing is carried out on the remaining dataset and the intervals that do not satisfy the box and whisker test in (5) are eliminated. Tolerance –limit processing is performed on the remaining intervals and the data intervals are accepted if they satisfy ***Eqs. (6)*** & ***(7)*** otherwise, they are rejected.

The constant 'k' is used as estimated by Mendel and Liu [26, 27] as shown in ***Table 2*** for 10 intervals at 0.95 confidence limit since we have an average of ten intervals per characteristic.

Table 2. Tolerance factor k for a number of collected data (m'), a proportion of the data (1 - α), and a confidence level 1- γ.

m'	1-γ = 0.95		γ = 0.99	
	1 – α		1 – α	
	0.9	0.95	0.9	0.95
10	2.839	3.379	3.582	4.265
15	2.48	2.954	2.945	3.507
20	2.31	2.752	2.659	3.168
30	2.14	2.549	2.358	2.841
50	1.996	2.379	2.162	2.576
100	1.874	2.233	1.977	2.355
1000	1.709	2.036	1.736	2.718
∞	1.645	1.96	1.645	1.96

Reasonable data intervals are evaluated for the overlapped data intervals and only reasonable data intervals are kept. At this point, the intervals remain the same, i.e. $m = m''$. After the data has been validated, the mean and SD of the data intervals are computed in ***Eqs. (8)*** & ***(9)*** on the assumption that the data intervals are uniformly distributed and the probability distribution, $S_{i,}$ computed in ***Eq. (10)*** and then becomes an input to the fuzzy set part.

3.4.2. Interval approach fuzzy set part for analyzing Igbo emotion words using

The Interval Approach Fuzzy Set Part for Analyzing Igbo emotion words using IT2FL is used to evaluate the MF based on ***Fig. 3***. It consists of nine steps. Step 1 selects a T1FS and computes the mean and SD of the data intervals using symmetrical triangle interior T1FS, left-shoulder T1FS and the right-shoulder T1FS only. In Step 2, the mean and standard deviation are calculated to establish FS uncertainty measures using ***Eqs. (11)*** & ***(12)***.

$$m_A = \frac{\int_{aMF}^{bMF} x\mu_A(x)dx}{\int_{aMF}^{bMF} \mu_A(x)dx}. \tag{11}$$

$$\sigma_A = \left[\frac{\int_{aMF}^{bMF} (x - m_A)^2 \mu_A(x)dx}{\int_{aMF}^{bMF} \mu_A(x)dx}\right]^{1/2}. \tag{12}$$

Obviously, $\mu_A(x)/\int_{aMF}^{bMF}\mu_A(x)dx$ is the probability distribution of x, where$x \in \left[a_{MF}, b_{MF}\right]$, then m_A and σ_A are the same as the mean and standard deviation used in probability. In Step 3, the Uncertainty Measures are computed for T1FS by calculating the mean and SD for symmetric Triangle Interior (TI) and the Left-Shoulder (LS) and Right-Shoulder (RS) T1MFs using ***Eqs. (13) – (15)***, respectively.

$$TI{:}\ m_{MF} = (a_{MF} + b_{MF})/2;\ \sigma_{MF} = (b_{MF} - a_{MF})/2\sqrt{6}. \tag{13}$$

$$LS{:}\ m_{MF} = (2a_{MF} + b_{MF})/3; \sigma_{MF} = \left[\tfrac{1}{6}\left[(a_{MF} + b_{MF})^2 + 2a_{MF}^2\right] - m_{MF}^2\right]^{1/2}. \tag{14}$$

$$RS: m_{MF} = (2a_{MF} + b_{MF})/3; \sigma_{MF} = \left[\frac{1}{6}[(a'_{MF} + b'_{MF})^2 + 2a'^2_{MF}] - m'^2_{MF}\right]^{1/2}. \quad (15)$$

Where, $a'_{MF} = M - b_{MF}$; $b'_{MF} = M - a_{MF}$; $m'_{MF} = M - m_{MF}$. In Step 4, we compute for the parameters of T1FS models by equating the mean and SD of a T1FS to the mean and SD of the data intervals i.e. $m^i_{MF} = m^i_Y$ and $\sigma^i_{MF} = \sigma^i_Y$ to have ***Eqs. (16) – (18)***.

$$\text{IT: } a_{MF} = (a+b)/2 - \sqrt{2}(b-a)/2; b_{MF} = (a+b)/2 + \sqrt{2}(b-a)/2. \quad (16)$$

$$\text{LS: } a_{MF} = (a+b)/2 - (b-a)/\sqrt{6}; b_{MF} = (a+b)/2 + \sqrt{6}(b-a)/3. \quad (17)$$

$$\text{RS: } a_{MF} = M - (a'+b')/2 - (b'-a')/\sqrt{6}; \; b_{MF} = M - (a'+b')/2 + \sqrt{6}(b'-a')/3. \quad (18)$$

Where, $a' = M - b$ and $b' = M - a$. In Step 5, the nature of the FOU is established by mapping the 'm' data intervals into an IT, LS or a RS FOUs using a scale of [0, 10] if and only if (i), (ii) or (iii).

$$\left.\begin{matrix} a^i_{MF} \geq 0 \\ b^i_{MF} \leq 10 \\ b^i_{MF} \geq a^i_{MF} \end{matrix}\right\} \quad \left.\begin{matrix} a^i_{MF} \geq M \\ b^i_{MF} \leq 10 \\ b^i_{MF} \geq a^i_{MF} \end{matrix}\right\} \quad \left.\begin{matrix} a^i_{MF} \geq 0 \\ b^i_{MF} \leq M \\ b^i_{MF} \geq a^i_{MF} \end{matrix}\right\} \; \forall \; i = 1, \dots, m \quad (19)$$

(i) (ii) (iii)

From the mapping, we achieve, 12 TI FOUs, 35 LS FOUs and 23 RS FOUs respectively. In Step 6, embedded T1FSs is computed where the actual mapping to a T1FS is applied to the remaining 'm' data intervals for each word using ***Eq. (20)***.

$$A^i = (a^i, b^i) \to (a^i_{MF}, b^i_{MF}), i = 1, \dots, m. \quad (20)$$

In Step 7, we delete inadmissible T1FSs for all data intervals for any T1FS that do not satisfy (19(i)) – (19(iii)) and m reduces to m*. Step 8 computes an IT2FS using the representation theorem for an IT2FS $\tilde{A}$ in ***Eq. (21)***,

$$\tilde{A} = \bigcup_{i=1}^{m*} A^i. \quad (21)$$

Where, A^i is just the computed ith embedded T1FS. In Step 9, the mathematical model for FOU ($\tilde{A}$) is computed by first approximating the parameters of UMF ($\tilde{A}$) and the LMF($\tilde{A}$) for each of the FOU models as shown in ***Eqs. (22) – (27)***, respectively.

$$\left.\begin{matrix} \underline{a}_{MF} \equiv \min_{i=1,\dots m*} \{a^i_{MF}\} \\ \bar{a}_{MF} \equiv \max_{i=1,\dots m*} \{a^i_{MF}\} \end{matrix}\right\}. \quad (22)$$

$$\left.\begin{matrix} \underline{b}_{MF} \equiv \min_{i=1,\dots m*} \{b^i_{MF}\} \\ \bar{b}_{MF} \equiv \max_{i=1,\dots m*} \{b^i_{MF}\} \end{matrix}\right\}. \quad (23)$$

$$C^i_{MF} = \frac{a^i_{MF} + b^i_{MF}}{2}. \quad (24)$$

$$\left.\begin{matrix} \underline{C}_{MF} \equiv \min_{i=1,\dots m*} \{C^i_{MF}\} \\ \bar{C}_{MF} \equiv \max_{i=1,\dots m*} \{C^i_{MF}\} \end{matrix}\right\}. \quad (25)$$

$$p = \frac{\underline{b}_{MF}(\overline{c}_{MF}-\overline{a}_{MF})+\overline{a}_{MF}(\underline{b}_{MF}-\underline{c}_{MF})}{(\overline{c}_{MF}-\overline{a}_{MF})+(\underline{b}_{MF}-\underline{c}_{MF})}. \tag{26}$$

$$\mu_p = \frac{\underline{b}_{MF}-p}{(\underline{b}_{MF}-\underline{c}_{MF})}. \tag{27}$$

The mathematical model of the UMF and the LMF are shown in ***Table 3*** for TI, L-S and R-S FOU.

Table 3. The IT2FS UMF and the LMF of IT, L-S and R-S FOUs.

	Upper MF	Lower MF
Triangle (Interior) FOU	$(\underline{a}_{MF}, 0)$, $(\underline{c}_{MF}, 1)$, $(\overline{c}_{MF}, 1)$, $(\overline{b}_{MF}, 0)$	$(\underline{a}_{MF}, 0)$, $(\overline{a}_{MF}, 0)$, (p, μ_p), $(\underline{b}_{MF}, 0)$, $(\overline{b}_{MF}, 0)$
Left-Shoulder FOU	$(0, 1)$, $(\overline{a}_{MF}, 1)$, $(\overline{b}_{MF}, 0)$	$(0, 1)$, $(\underline{a}_{MF}, 1)$, $(\underline{b}_{MF}, 0)$, $(\overline{b}_{MF}, 0)$
Right-Shoulder FOU	$(\underline{a}_{MF}, 0)$, $(\underline{b}_{MF}, 1)$, $(M, 1)$	$(\underline{a}_{MF}, 0)$, $(\overline{a}_{MF}, 0)$, $(\overline{b}_{MF}, 1)$, $(M, 1)$

4. Model Experiment

This paper uses IT2FL to analyze Igbo emotion words. The IT2F sets are computed using the interval approach method which comprises data part and fuzzy set part. In order to illustrate the methodology proposed in this paper, we conduct some experiments for Igbo emotion words described in this work.

4.1. The Data Part

Table 4. shows the data intervals collected from the subjects. Bad data are processed and sample presented in ***Table 5.*** Outlier processing is performed in ***Table 5*** and the sample result is presented in ***Table 6.*** The tolerance limit processing in ***Table 6*** is shown in ***Table 7.*** Sample result of reasonable interval processing is seen in ***Table 8.*** The data statistics is computed for all the surviving *m* data intervals and sample are shown in ***Table 9.***

Table 4. Parts of the data intervals collected from the subjects.

		S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	S12	S13	S14	S15
	Val	8-10	0-8	3-6	7-9	5-8	5-7	0-5	7-10	6-8	6-9	5-8	8-10	7-10	4-5	4-8
	Act	3-5	1-6	4-7	5-8	7-9	6-7	7-10	0-4	6-10	7-10	7-10	8-10	0-5	9-10	5-9
Iwe	Dom	7-10	2-10	8-10	|||	7-10	8-10	2-9	6-8	8-10	8-10	8-10	8-10	5-8	8-10	6-9
	Val	9-10	5-6	4-7	7-10	5-8	6-8	5-10	1-4	0-5	0-4	3-5	1-3	5-8	4-5	3-8
	Act	9-10	7-9	5-8	5-8	7-10	8-10	1-10	9-10	8-10	5-9	8-10	8-10	6-9	7-10	5-9
Obi Uto	Dom	0-3	2-6	6-9	6-9	8-10	4-6	8-10	3-5	0-5	4-8	5-8	1-3	8-10	3-5	4-10
	Val	7-9	3-9	8-10	9-10	5-8	4-7	2-7	7-10	9-10	8-10	8-10	8-10	7-10	5-8	4-8
	Act	0-3	2-6	5-8	4-9	9-10	4-6	7-9	5-8	9-10	6-10	3-5	8-10	0-6	7-10	3-7
Onuma	Dom	8-10	3-9	7-10	8-9	5-8	6-9	2-6	10	9-10	7-10	4-6	8-10	5-8	7-10	5-9
	Val	0-3	4-10	2-4	1-3	5-8	6-9	8-10	1-3	8-10	5-8	7-9	8-10	8-10	4-6	2-10
	Act	0-4	2-6	3-5	5-8	7-10	3-6	4-5	0-2	0-5	2-7	0-5	1-3	5-7	7-8	3-9
Ujo	Dom	4-5	3-9	6-8	5-9	7-10	0-4	8-9	0-2	0-5	1-4	3-6	1-3	0-5	8-10	5-10
	Val	7-10	5-7	1-3	6-8	6-10	0-4	3-10	8-10	0-5	1-5	0-3	1-3	7-10	4-7	1-9
	Act	2-3	4-8	4-6	6-10	6-8	4-7	1-9	7-9	0-5	3-6	3-5	1-3	7-10	7-8	2-8
Ntukwasi Obi	Dom	1-3	5-6	8-10	4-8	6-9	4-7	4-9	6-7	0-5	5-8	3-5	1-3	8-10	8-10	5-7
	Val	5-8	0-10	2-5	6-10	5-10	6-10	2-4	5-8	8-10	8-10	8-10	8-10	6-10	4-5	2-8
Obi Ojo	Act	0-2	2-8	1-4	5-10	6-10	5-7	4-8	7-9	0-2	6-10	3-5	5-6	5-9	7-8	3-6

Table 5. Sample result of bad data processing.

		Valence	Activation	Dominance
1	IWE	15	15	14
2	OBI UTO	15	15	15
3	ONUMA	15	15	14
4	UJO	15	15	15
5	NTUKWASI OBI	15	15	15

Table 6. Sample result of outlier processing.

		Valence	Activation	Dominance
1	IWE	9	9	2
2	OBI UTO	12	5	11
3	ONUMA	5	10	5
4	UJO	15	13	12
5	NTUKWASI OBI	15	11	12

Table 7. Sample result of tolerance-limit processing.

		Valence	Activation	Dominance
1	IWE	9	9	2
2	OBI UTO	12	5	11
3	ONUMA	5	10	5
4	UJO	15	13	12
5	NTUKWASI OBI	15	11	12

Table 8. Sample result of reasonable interval processing.

		Valence	Activation	Dominance
1	IWE	9	9	2
2	OBI UTO	12	5	11
3	ONUMA	5	10	5
4	UJO	15	13	12
5	NTUKWASI OBI	15	11	12

4.2. The Fuzzy Set Part

In the fuzzy set part, only the symmetrical triangle interior T1FS, left-shoulder T1FS and the right-shoulder T1FS are used. The mean and SD are the same as in the data part as shown ***Table 9.*** The uncertainty measure for the chosen T1FS models are computed as is shown in parts in ***Table 10.*** The nature of FOU for each emotion characteristic for each word is determined as shown in ***Table 11.*** The embedded T1FSs, a_{MF} and b_{MF} of each interval are calculated as shown in ***Table 12*** where M=5. The LMF and the UMF of the FOU ($\tilde{A}$) for the five experimental intervals are calculated and summarized in ***Table 13.***

Table 10. Sample result of the uncertainty measures for T1FS models.

	m_{MF}	σ_{MF}
Interior	5	2.041241
Left	3.333333	3.535887
Right	3.333333	2.022657

Table 9. The sample mean and SD computed for all the surviving m data intervals.

			Mean						Standard Deviation					
			S1	S2	S3	S4	…	S15	S1	S2	S3	S4	…	S15
		Val	0	4	4.5	0	…	6	0	2.309401	0.866025	0	…	1.154701
1	Iwe	Act	4	3.5	5.5	6.5	…	7	0.57735	1.443376	0.866025	0.866025	…	1.154701
		Dom	0	0	0	0	…	0	0	0	0	0	…	0
							…						…	
							…						…	
		Val	0	5.5	5.5	0	…	5.5	0	0.288675	0.866025	0	…	1.443376
2	Obi Uto	Act	0	0	6.5	6.5	…	7	0	0	0.866025	0.866025	…	1.154701
		Dom	1.5	4	7.5	7.5	…	0	0.866025	1.154701	0.866025	0.866025	…	0
							…						…	
							…						…	
		Val	0	0	0	0	…	6	0	0	0	0	…	1.154701
3	Onuma	Act	1.5	4	6.5	6.5	…	5	0.866025	1.154701	0.866025	1.443376	…	1.154701
		Dom	0	0	0	0	…	7	0	0	0	0	…	1.154701
							…						…	
							…						…	
		Val	1.5	7	3	2	…	6	0.866025	1.732051	0.57735	0.57735	…	2.309401
4	Ujo	Act	2	4	4	6.5	…	0	1.154701	1.154701	0.57735	0.866025	…	0
		Dom	4.5	6	7	7	…	0	0.288675	1.732051	0.57735	1.154701	…	0
							…						…	
							…						…	
		Val	8.5	6	2	7	…	5	0.866025	0.57735	0.57735	0.57735	…	2.309401
5	Ntukwasi Obi	Act	2.5	6	5	0	…	5	0.288675	1.154701	0.57735	0	…	1.732051
		Dom	2	5.5	0	6	…	6	0.57735	0.288675	0	1.154701	…	0.57735

Table 11. Sample result of the nature of the FOU for each emotion characteristics.

			Interior	Left-Shoulder	Right-Shoulder
		Val		1	
1	Iwe	Act		1	
		Dom	1		
		Val		1	
2	Obi Uto	Act	1		
		Dom			1
		Val	1		
3	Onuma	Act		1	
		Dom	1		
		Val	1		
4	Ujo	Act	1		
		Dom	1		
		Val	1		
5	Ntukwasi Obi	Act	1		
		Dom	1		

Table 12. Embedded T1FS for each emotion characteristics.

			amf	Bmf
1	Iwe	Val	6.183503	8.632993
		Act	7.183503	9.632993
		Dom	5.585786	8.414214
2	Obi Uto	Val	6.183503	8.632993
		Act	5.37868	9.62132
		Dom	1.275255	3.724745
3	Onuma	Val	4.37868	8.62132
		Act	7.183503	9.632993
		Dom	4.171573	9.828427
4	Ujo	Val	7.585786	10.41421
		Act	6.792893	8.207107
		Dom	7.792893	9.207107
5	Ntukwasi Obi	Val	7.585786	10.41421
		Act	6.792893	8.207107
		Dom	5.37868	9.62132

Table 13. Computed LMF and the UMF of the FOU ($\tilde{A}$).

S/No	Word		UMF	LMF
1	Iwe	Val	(6,8)	(0,5,8)
		Act	(7,9)	(0,4,9)
		Dom	(5,6.5,7,8)	(5,6,6.8,8)
2	Obi Uto	Val	(6,8)	(0,3,8)
		Act	(5,6.5,7.5,9)	(5,6,7,8)
		Dom	(0,3,10)	(0,6,9,10)
3	Onma	Val	(2,4.5,6.5,8)	(2,5,5.7,7)
		Act	(7,9)	(0,3,9)
		Dom	(2,4,7,9)	(2,5,5.5,6)
4	Ujo	Val	(8,10)	(0,3,10)
		Act	(7,8)	(0,2,8)
		Dom	(8,9)	(0,2,9)
5	Ntukwasi Obi	Val	(8,10)	(0,3,10)
		Act	(7,8)	(0,3,8)
		Dom	(0,3,10)	(0,6,9,10)

5. Result and Discussion

5.1. The Fuzzy Set Part

The IT2FL model used to analyze the Igbo emotion words are simulated using Matlab, Microsoft excel and Netbeans. The data as collected are input into a spreadsheet, preprocessed as discussed in the paper.

Then, the model is run on the data yielding the MFs and the FOUs of each of the first 5 emotion words. Parts of the results of simulation for dimensions Valence, Activation and Dominance are shown in *Figs. (4)-(16)*.

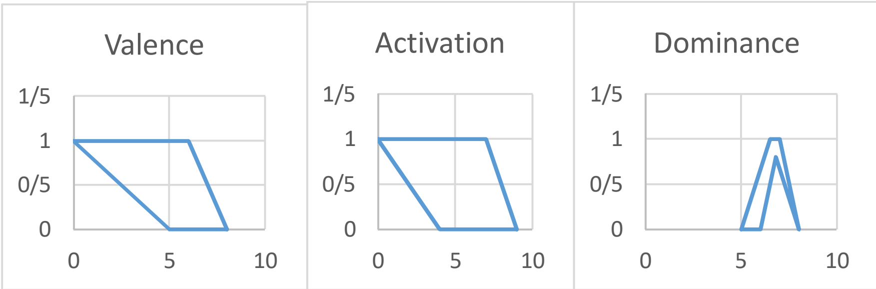

Fig. 4. The FOU of Iwe (Anger) for dimensions valence, activation and dominance.

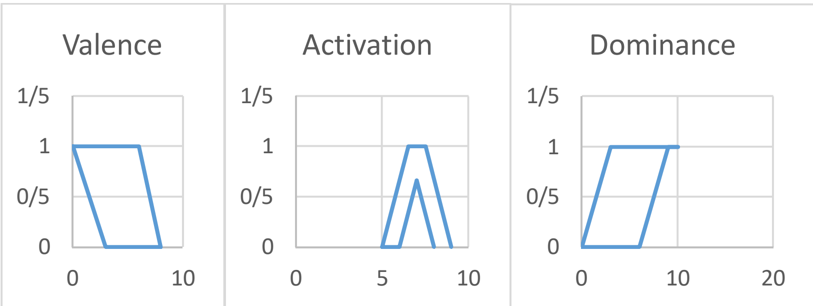

Fig. 5. The FOU of Obi Uto (Happiness) for dimensions valence, activation and dominance.

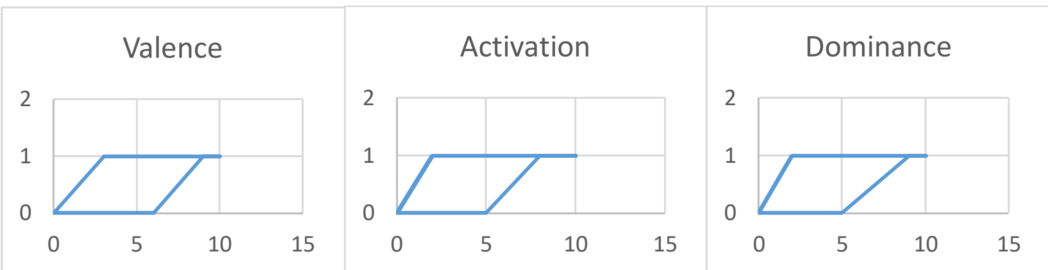

Fig. 6. The FOU of Ihere (Shame) for dimensions valence, activation and dominance.

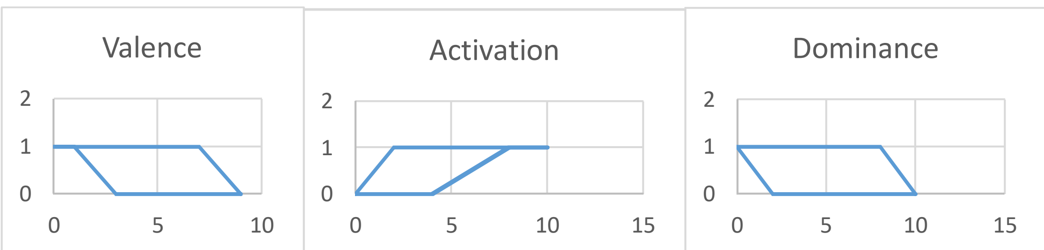

Fig. 7. The FOU of Akwa Uta (Remorse) for dimensions valence, activation and dominance.

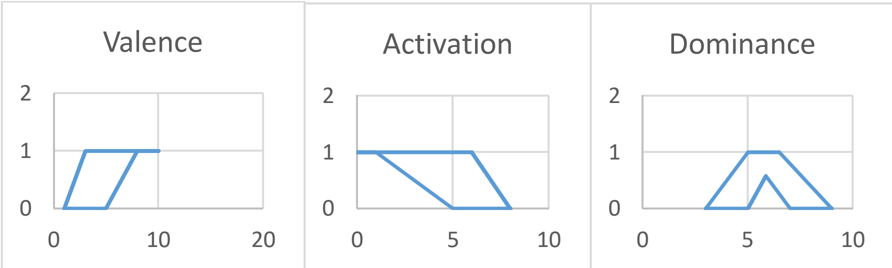

Fig. 8. The FOU of Ara (Mad) for dimensions valence, activation and dominance.

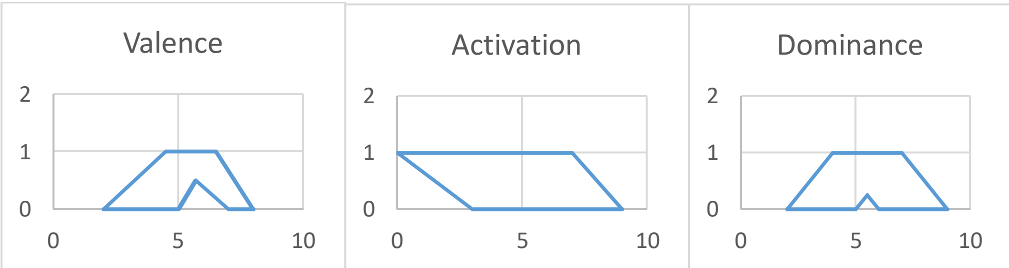

Fig. 9. The FOU of Onuma (Wrath) for dimensions valence, activation and dominance.

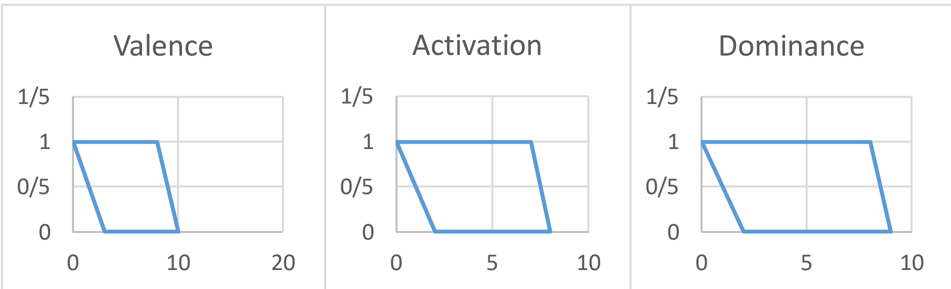

Fig. 10. The FOU of Ujo (Fear) for dimensions valence, activation and dominance.

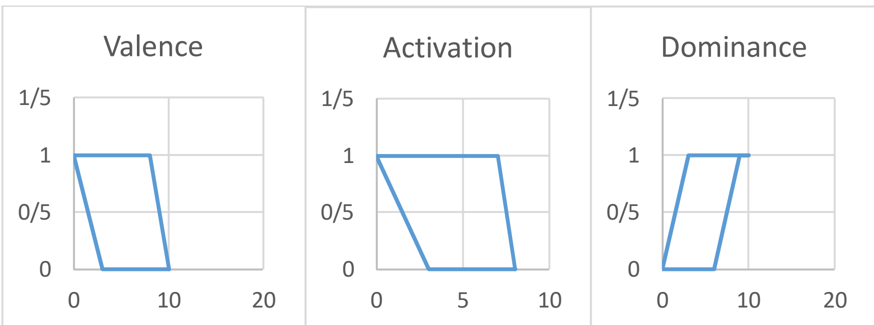

Fig. 11. The FOU of Ntukwasi (Trust) for dimensions valence, activation and dominance.

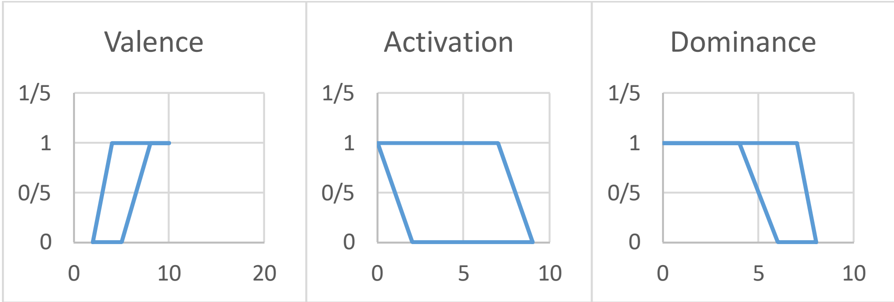

Fig. 12. The FOU of Ojo (Wicked) for dimensions valence, activation and dominance.

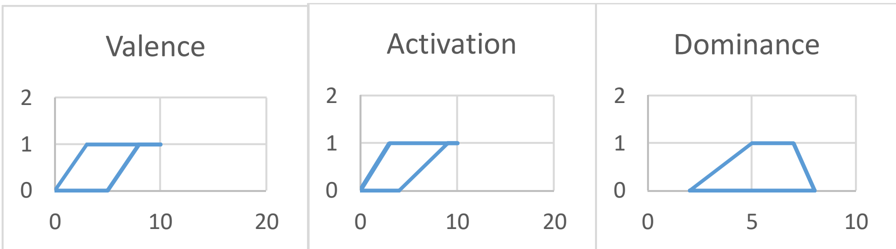

Fig. 13. The FOU of Onu (Joy) for dimensions valence, activation and dominance.

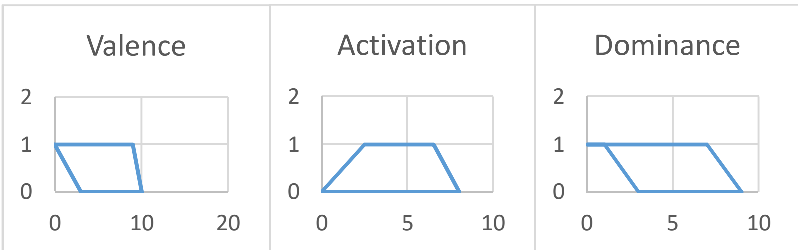

Fig. 14. The FOU of Anuri (Glad) for dimensions valence, activation and dominance.

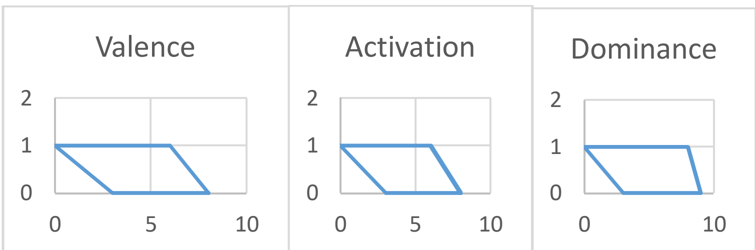

Fig. 15. The FOU of Egwu (Dread) for dimensions valence, activation and dominance.

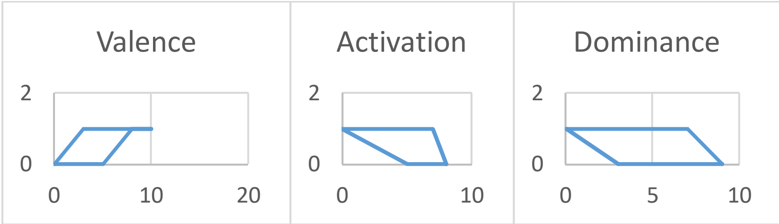

Fig. 16. The FOU of Ihunaya (Love) for dimensions valence, activation and dominance.

5.2. Discussion

Figs. (4) – (16) show parts of the MFs that were calculated from the survey data. These Figures indicate 3 general tendencies of graph: the MF could be either steeper and more peripheral if the emotion words are in accordance with their real meaning, or less steep and more central if the emotion words have ambiguous and undetermined meaning, or shifted to the opposite side of at least one of the scales if the emotion words do not conform with the real meaning. The valence dimension for the word "Iwe (Anger)" is not so broad, indicating a value a bit low. The activation dimension indicates a high value while the dominance dimension is narrow and has a small footprint of uncertainty. This means the word "Iwe (Anger)" carries a meaning that is well determined by the valence and dominance dimensions but is not well determined by the activation dimension. The word "Obi Uto (Happiness)" has a high value in its valence dimension, low value in its activation dimension and high value in its dominance dimension. This implies that the meaning of the word "Obi Uto (Happiness)" is well determined by its activation dimension. For the word "Ihere (Shame)", the valence dimension has a high value. The activation and dominance dimensions have low values. This indicates that the meaning of the word "Ihere (Shame)" is well determined by its activation and dominance dimensions and not the valence dimension. The word "AkwaUta (Remorse)" has a high value in its valence dimension, a low value in its activation dimension and a high value in its dominance dimension. This indicates that the meaning of the word "AkwaUta (Remorse)" is well determined by all the three dimensions. For the word "Ara (Mad)", the valence dimension has a low value while the activation dimension has a high value. Its dominance dimension though not so broad has a small footprint of uncertainty. This shows that the meaning of the word "Ara (Mad)" is well determined by all the three characteristics.

5. Conclusion

This paper involves the implementation of the IT2FL model for analyzing thirty (30) Igbo emotion words. Interval survey is conducted using Igbo native speakers to collect human intuition about fuzzy predicate which is emotion. Using an interval approach, user data are associated with each emotion word with intervals on the three scales of emotional characteristics-valence, activation, and dominance which are collected and used to estimate IT2F MFs for each scale. Results indicate that the study is able to demonstrate that the use of the proposed system will be of immense benefit to every aspect of Natural Language Processing (NLP) and affective computing and that the IT2FL model for words is more suitable for any purpose in which emotion words may be computed. This is because of the interval approach method used to analyze the words to yield IT2FSs which captures most uncertainty that are contained in an emotion word. Also, the study will help the users in selecting specific Igbo emotion words for easy communication and understanding.

In the future, more emotion words can be added to the system and IT2FL tool can be employed in the translation of Igbo emotion words in English language.

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Multi-Item Inventory Model Include Lead Time with Demand Dependent Production Cost and Set-Up-Cost in Fuzzy Environment

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A B S T R A C T

In this paper, we have developed the multi-item inventory model in the fuzzy environment. Here we considered the demand rate is constant and production cost is dependent on the demand rate. Set-up- cost is dependent on average inventory level as well as demand. Lead time crashing cost is considered the continuous function of leading time. Limitation is considered on storage of space. Due to uncertainty all cost parameters of the proposed model are taken as generalized trapezoidal fuzzy numbers. Therefore this model is very real. The formulated multi objective inventory problem has been solved by various techniques like as Geometric Programming (GP) approach, Fuzzy Programming Technique with Hyperbolic Membership Function (FPTHMF), Fuzzy Nonlinear Programming (FNLP) technique and Fuzzy Additive Goal Programming (FAGP) technique. An example is given to illustrate the model. Sensitivity analysis and graphical representation have been shown to test the parameters of the model.

1. Introduction

Inventory models deal with decisions that minimize the total average cost or maximize the total average profit. In that way to construct a real life mathematical inventory model on based on various assumptions and notations and approximations. Multi-item is also an important factor in the inventory control system. The basic well known Economic Order Quantity (EOQ) model was first introduced by Harris in 1913; Abou-el-ata and Kotb studied a multi-item EOQ inventory model with varying holding costs under two

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restrictions with a geometric programming approach [1]. Chen [7] presented an optimal determination of quality level, selling quantity and purchasing price for intermediate firms. Liang and Zhou [11] discussted two warehouse inventory model for deteriorating items and stock dependent demand under conditionally permissible delay in payment. Das et al. [12] developed a multi-item inventory model with quantity dependent inventory costs and demand-dependent unit cost under imprecise objectives and restrictions with a geometric programming approach. Das and Islam [13] considered a multi-objective two echelon supply chain inventory model with customer demand dependent purchase cost and production rate dependent production cost. Shaikh et al. [30] discussed an inventory model for deteriorating items with preservation facility of ramp type demand and trade credit.

The concept of fuzzy set theory was first introduced by Zadeh [27]. Afterward, Zimmermann [28] applied the fuzzy set theory concept with some useful membership functions to solve the linear programming problem with some objective functions. Bit [2] applied fuzzy programming with hyperbolic membership functions for multi objective capacitated transportation problems. Bortolan and Degani [4] discussed a review of some methods for ranking fuzzy subsets. Maiti [19] developed a fuzzy inventory model with two warehouses under possibility measure in fuzzy goals. Mandal et al. [22] presented a multi-objective fuzzy inventory model with three constraints with a geometric programming approach. Shaikh et al. [26] developed a fuzzy inventory model for a deteriorating Item with variable demand, permissible delay in payments and partial backlogging with Shortage Following Inventory (SFI) policy. Garai et al. [29] discussed multi-objective inventory model with both stock-dependent demand rate and holding cost rate under fuzzy random environment.

In the global market system lead time is an important matter. Ben-Daya and Rauf [3] considered an inventory model involving lead-time as a decision variable. Chuang et al. [8] presented a note on periodic review inventory model with controllable setup cost and lead time. Hariga and Ben-Daya [14] discussed some stochastic inventory models with deterministic variable lead time. Ouyang et al. [20] studied mixture inventory models with backorders and lost sales for variable lead time. Ouyang and Wu [21] established a min-max distribution free procedure for mixed inventory models with variable lead time. Sarkar et al. [24] developed an integrated inventory model with variable lead time and defective units and delay in payments. Sarkar et al. [25] studied quality improvement and backorder price discount under controllable lead time in an inventory model.

Geometric Programming (GP) is a powerful optimization technique developed to solve a class of non-linear optimization programming problems especially found in engineering design and manufacturing. Multi objective geometric programming techniques are also interesting in the EOQ model. GP was introduced by Duffin et al. in 1966 [10] and published a famous book in 1967 [9]. Beightler et al. [5] applied GP. Biswal [6] considered fuzzy programming techniques to solve multi-objective geometric programming problems. Islam [16] discussed multi-objective geometric-programming problem and its application. Mandal et al. [22] developed a multi-objective fuzzy inventory model with three constraints with a geometric programming approach. Mandal et al. [23] discussed an inventory model of deteriorating items with a constraint with a geometric Programming approach. Islam [17] studied a multi-objective marketing planning inventory model with a geometric programming approach. Kotb et al. [18] presented a multi-item EOQ model with both demand dependent on unit cost and varying lead time via geometric programming.

In this paper, we have developed an inventory model of multi-item with space constraint in a fuzzy environment. Here we considered the constant demand rate and production cost is dependent on the demand rate. Set-up- cost is dependent on average inventory level as well as demand. Lead time crashing cost is considered the continuous function of leading time. Due to uncertainty all cost parameters are taken as generalized trapezoidal fuzzy numbers. The proposal has been solved by various techniques like GP approach, FPTHMF, FNLP, and FAGP. Numerical example is given to illustrate the model. Finally sensitivity analysis and graphical representation have been shown to test the parameters of the model.

2. Mathematical Model

2.1. Notations

h_i: Holding cost per unit per unit time for i^{th} item.

T_i: The length of cycle time for i^{th}item, $T_i > 0$.

D_i : Demand rate per unit time for the i^{th} item.

L_i : Rate of leading time for the i^{th} item.

SS: Safety stock.

k: Safety factor.

$I_i(t)$: Inventory level of the i^{th} item at time t.

C_p^i : Unit production cost of i^{th} item.

${S_c}^i(Q_i, D_i)$: Set up cost for i^{th} item.

$R^i(L_i)$: Lead time crashing cost for the i^{th} item.

Q_i: The order quantity for the duration of a cycle of length $T_i$for i^{th} item.

$TAC_i(D_i, Q_i, L_i)$: Total average profit per unit for the i^{th} item.

w_i: Storage space per unit time for the i^{th} item.

W: Total area of space.

$\widetilde{w_i}$: Fuzzy storage space per unit time for the i^{th} item.

$\widetilde{h_i}$: Fuzzy holding cost per unit per unit time for the i^{th} item.

$\widetilde{TAC_i}(D_i, Q_i, L_i)$: Fuzzy total average cost per unit for the i^{th} item.

$\widehat{w_i}$: Defuzzyfication of the fuzzy number $\widetilde{w_i}$.

$\widehat{h_i}$: Defuzzyfication of the fuzzy number $\widetilde{h_i}$.

$\widehat{TAC_i}(D_i, Q_i, L_i)$: Defuzzyfication of the fuzzy number $\widetilde{TAC_i}(D_i, Q_i, L_i)$.

2.2. Assumptions

- Multi-item is considered.
- The replenishment occurs instantaneously at infinite rate.
- The lead time is considered.
- Shortages are not allowed.
- Production cost is inversely related to the demand. Here considered $C_p^i(D_i) = \alpha_i {D_i}^{-\beta_i}$, where $\alpha_i > 0$ and $\beta_i > 1$ are constant real numbers.
- The set up cost is dependent on the demand as well as average inventory level. Here considered ${S_c}^i(Q_i, D_i) = \gamma_i \left(\frac{Q_i}{2}\right)^{\delta_i} D_i^{\sigma_i}$ where $0 < \gamma_i$, $0 < \delta_i \ll 1$ and $0 < \sigma_i \ll 1$ are constant real numbers.
- Lead time crashing cost is dependent on the lead time by a function of the form $R^i(L_i) = \rho_i {L_i}^{-\tau_i}$, where $\rho_i > 0$ and $0 < \tau_i \leq 0.5$ are constant real numbers.
- *SS*= $k\omega\sqrt{L_i}$.
- Deterioration is not allowed.

2.3. Formulation of the Model

The inventory level for i^th item is illustrated in ***Fig. 1***. During the period $[0, T_i]$ the inventory level reduces due to demand rate. In this time period, the governing differential equation is

$$\frac{dI_i(t)}{dt} = -D_i, 0 \leq t \leq T_i. \tag{1}$$

With boundary condition, $I_i(0) = Q_i$, $I_i(T_i) = 0$.

Solving (1) we have,

$$I_i(t) = Q_i - D_i t, \; 0 \leq t \leq T_i \; . \tag{2}$$

$$T_i = \frac{Q_i}{D_i} . \tag{3}$$

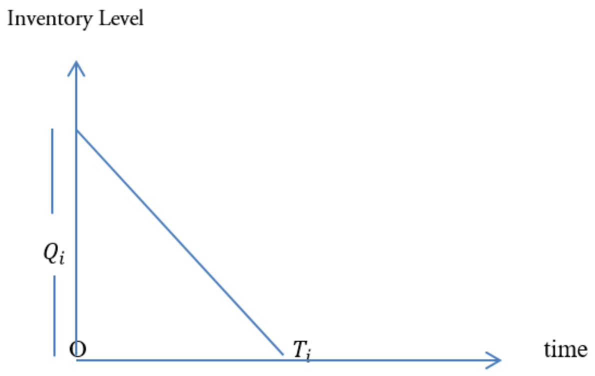

Fig. 1. Inventory level for the i[th] item.

Now calculating various averages cost for i[th] item,

Average production cost $(PC_i) = \frac{Q_i C_P^i(D_i)}{T_i} = \alpha_i D_i^{(1-\beta_i)}$;

Average holding cost$(HC_i) = \frac{1}{T_i}\int_0^{T_i} h_i I_i(t)dt + h_i k\omega\sqrt{L_i} \; = h_i\left(\frac{Q_i}{2} + k\omega\sqrt{L_i}\right)$;

Average set-up-cost $(SC_i) = \frac{1}{T_i}\left[\gamma_i\left(\frac{Q_i}{2}\right)^{\delta_i} D_i^{\sigma_i}\right] = \frac{\gamma_i {Q_i}^{\delta_i - 1} D_i^{\sigma_i + 1}}{2^{\delta_i}}$;

Average lead time crashing cost $(CC_i) = \frac{\rho_i {L_i}^{-\tau_i}}{T_i} = \frac{D_i \rho_i {L_i}^{-\tau_i}}{Q_i}$.

Total average cost for i[th] item is

$$\mathrm{TAC_i}(\mathrm{D_i}, \mathrm{Q_i}, \mathrm{L_i}) = (\mathrm{PC_i} + \mathrm{HC_i} + \mathrm{SC_i} + \mathrm{CC_i}) = \alpha_i \mathrm{D_i}^{(1-\beta_i)} + \mathrm{h_i}\left(\frac{\mathrm{Q_i}}{2} + \mathrm{k}\omega\sqrt{\mathrm{L_i}}\right) + \frac{\gamma_i {\mathrm{Q_i}}^{\delta_i - 1} \mathrm{D_i}^{\sigma_i + 1}}{2^{\delta_i}} + \frac{\mathrm{D_i} \rho_i {\mathrm{L_i}}^{-\tau_i}}{\mathrm{Q_i}}. \tag{4}$$

A Multi-Objective Inventory Model (MOIM) can be written as:

$$\mathrm{Min}\ \{\mathrm{TAC_1}, \mathrm{TAC_2}, \mathrm{TAC_3}, \dots\dots\dots\dots\dots, \mathrm{TAC_n}\},$$

$$\mathrm{TAC_i}(\mathrm{D_i}, \mathrm{Q_i}, \mathrm{L_i}) = \alpha_i \mathrm{D_i}^{(1-\beta_i)} + \mathrm{h_i}\left(\frac{\mathrm{Q_i}}{2} + \mathrm{k}\omega\sqrt{\mathrm{L_i}}\right) + \frac{\gamma_i {\mathrm{Q_i}}^{\delta_i - 1} \mathrm{D_i}^{\sigma_i + 1}}{2^{\delta_i}} + \frac{\mathrm{D_i} \rho_i {\mathrm{L_i}}^{-\tau_i}}{\mathrm{Q_i}}, \tag{5}$$

Subject to

$$\sum_{i=1}^{n} \mathrm{w_i Q_i} \le \mathrm{W}, \mathrm{D_i} > 0, \mathrm{Q_i} > 0, \mathrm{L_i} > 0, \text{for } i = 1,2, \dots\dots\dots..\mathrm{n}.$$

2.4. Fuzzy Model

Due to uncertainty, we consider all the parameters $(\alpha_i, \beta_i, h_i, \rho_i, \gamma_i, \delta_i, \sigma_i, \tau_i)$of the model and storage space w_i as Generalized Trapezoidal Fuzzy Number (GTrFN)$(\widetilde{\alpha_i}, \widetilde{\beta_i}, \widetilde{h_i}, \widetilde{\rho_i}, \widetilde{\gamma_i}, \widetilde{\delta_i}, \widetilde{\sigma_i}, \widetilde{w_i}, \widetilde{\tau_i})$. Here

$\widetilde{\alpha_\iota} = (\alpha_i{}^1, \alpha_i{}^2, \alpha_i{}^3, \alpha_i{}^4; \varphi_{\alpha_i}), 0 < \varphi_{\alpha_i} \le 1; \widetilde{h_\iota} = (h_i{}^1, h_i{}^2, h_i{}^3, h_i{}^4; \varphi_{h_i}), 0 < \varphi_{h_i} \le 1;$

$\widetilde{\beta_\iota} = (\beta_i{}^1, \beta_i{}^2, \beta_i{}^3, \beta_i{}^4; \varphi_{\beta_i}), 0 < \varphi_{\beta_i} \le 1; \widetilde{\rho_\iota} = (\rho_i{}^1, \rho_i{}^1, \rho_i{}^1, \rho_i{}^1; \varphi_{\rho_i}), 0 < \varphi_{\rho_i} \le 1;$

$\widetilde{\gamma_\iota} = (\gamma_i{}^1, \gamma_i{}^2, \gamma_i{}^3, \gamma_i{}^4; \varphi_{\gamma_i}), 0 < \varphi_{\gamma_i} \le 1; \widetilde{w_\iota} = (w_i{}^1, w_i{}^2, w_i{}^3, w_i{}^4; \varphi_{w_i}), 0 < \varphi_{w_i} \le 1;$

$\widetilde{\delta_\iota} = (\delta_i{}^1, \delta_i{}^2, \delta_i{}^3, \delta_i{}^4; \varphi_{\delta_i}), 0 < \varphi_{\delta_i} \le 1; \widetilde{\sigma_\iota} = (\sigma_i{}^1, \sigma_i{}^2, \sigma_i{}^3, \sigma_i{}^4; \varphi_{\sigma_i}), 0 < \varphi_{\sigma_i} \le 1;$

$\widetilde{\tau_\iota} = (\tau_i{}^1, \tau_i{}^2, \tau_i{}^3, \tau_i{}^4; \varphi_{\tau_i}), 0 < \varphi_{\tau_i} \le 1; (i = 1,2, \dots\dots\dots, n).$

Then the above inventory model (5) becomes the fuzzy inventory model as

$$\text{Min} \quad \{\widetilde{\text{TAC}_1}, \widetilde{\text{TAC}_2}, \widetilde{\text{TAC}_3}, \dots\dots\dots\dots\dots\dots\dots., \widetilde{\text{TAC}_\text{n}}\},$$

Subject to

$$\textstyle\sum_{i=1}^{n} \widetilde{w_\iota} Q_i \le W, \text{ for } i = 1,2, \dots\dots\dots..n. \tag{6}$$

Where

$$\text{TAC}_\iota(\widetilde{D_\iota, Q_\iota}, L_\iota) = \widetilde{\alpha_\iota} D_i{}^{(1-\widetilde{\beta_\iota})} + \widetilde{h_\iota}\left(\frac{Q_i}{2} + k\omega\sqrt{L_i}\right) + \frac{\widetilde{\gamma_\iota} Q_i{}^{\widetilde{\delta_\iota}-1} D_i^{\widetilde{\sigma_\iota}+1}}{2^{\widetilde{\delta_\iota}}} + \frac{D_i \widetilde{\rho_\iota} L_i{}^{-\widetilde{\tau_\iota}}}{Q_i}.$$

λ –Integer method is used to defuzzify the fuzzy number. In this method the defuzzify value of the fuzzy number $\tilde{A} = (a, b, c, d; \varphi)$ is $\varphi\left(\frac{a+b+c+d}{4}\right)$. So using the defuzzified values $(\widehat{\alpha_\iota}, \widehat{\beta_\iota}, \widehat{h_\iota}, \widehat{\rho_\iota}, \widehat{\gamma_\iota}, \widehat{\delta_\iota}, \widehat{\sigma_\iota}, \widehat{w_\iota}, \widehat{\tau_\iota})$ of the GTrFN $(\widetilde{\alpha_\iota}, \widetilde{\beta_\iota}, \widetilde{h_\iota}, \widetilde{\rho_\iota}, \widetilde{\gamma_\iota}, \widetilde{\delta_\iota}, \widetilde{\sigma_\iota}, \widetilde{w_\iota}, \widetilde{\tau_\iota})$, the above fuzzy inventory model (6) reduces to

$$\text{Min} \{\widehat{\text{TAC}_1}, \widehat{\text{TAC}_2}, \widehat{\text{TAC}_3}, \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots., \widehat{\text{TAC}_\text{n}}\},$$

Subject to

$$\textstyle\sum_{i=1}^{n} \widehat{w_\iota} Q_i \le W, \tag{7}$$

Where

$$\text{TAC}_\iota(\widehat{D_\iota, Q_\iota}, L_\iota) = \widehat{\alpha_\iota} D_i{}^{(1-\widehat{\beta_\iota})} + \widehat{h_\iota}\left(\frac{Q_i}{2} + k\omega\sqrt{L_i}\right) + \frac{\widehat{\gamma_\iota} Q_i{}^{\widehat{\delta_\iota}-1} D_i^{\widehat{\sigma_\iota}+1}}{2^{\widehat{\delta_\iota}}} + \frac{D_i \widehat{\rho_\iota} L_i{}^{-\widehat{\tau_\iota}}}{Q_i},$$

$D_i > 0, Q_i > 0, L_i > 0$, for $i = 1,2, \dots\dots\dots..n$.

3. Fuzzy Programming Techniques to Solve MOIM

Solve the MOIM as a single objective NLP using only one objective at a time and ignoring the others. So we get the ideal solutions. Using the ideal solutions the pay-off matrix as follows:

$$\begin{array}{c} \\ (D_1^1,Q_1^1,L_1^1) \\ (D_2^2,Q_2^2,L_2^2) \\ \\ \\ (D_n^n,Q_n^n,L_n^n) \end{array} \begin{array}{cccc} TAC_1(D_1,Q_1,L_1) & TAC_2(D_2,Q_2,L_2)\ldots & \ldots & TAC_n(D_n,Q_n,L_n) \\ \left(\begin{array}{c} TAC_1{}^*(D_1^1,Q_1^1,L_1^1) \\ TAC_1(D_2^2,Q_2^2,L_2^2) \\ \ldots\ldots \\ \ldots\ldots \\ TAC_1(D_n^n,Q_n^n,L_n^n) \end{array}\right. & \begin{array}{c} TAC_2(D_1^1,Q_1^1,L_1^1) \\ TAC_2{}^*(D_2^2,Q_2^2,L_2^2) \\ \ldots\ldots \\ \ldots\ldots \\ TAC_2(D_n^n,Q_n^n,L_n^n) \end{array} & \begin{array}{c} \ldots\ldots \\ \ldots\ldots \\ \ldots\ldots \\ \ldots\ldots \\ \ldots\ldots \end{array} & \left.\begin{array}{c} TAC_n(D_1^1,Q_1^1,L_1^1) \\ TAC_n(D_2^2,Q_2^2,L_2^2) \\ \ldots\ldots \\ \ldots\ldots \\ TAC_n{}^*(D_n^n,Q_n^n,L_n^n) \end{array}\right) \end{array} \tag{8}$$

Let $U^k = \max\{TAC_k(D_i^i,Q_i^i,L_i^i), i = 1,2,\ldots,n\}$ for $k = 1,2,\ldots,n$ and

$L^k = TAC_k{}^*(D_k^k,Q_k^k,L_k^k)$ for $k = 1,2,\ldots,n$.

Hence U^k, L^k are identified, $L^k \le TAP_k(D_i^i,Q_i^i,L_i^i) \le U^k$, for $i = 1,2,\ldots,n$; $k = 1,2,\ldots,n$.

3.1. Fuzzy Programming Technique Using Hyperbolic Membership Function (FPTHMF)

Now fuzzy non-linear hyperbolic membership functions $\mu^H_{TAC_k}(TAC_k(D_k,Q_k,L_k))$ for the k[th] objective functions $TAC_k(D_k,Q_k,L_k)$ respectively for $k = 1,2,\ldots,n$ are defined as follows:

$$\mu^H_{TAC_k}(TAC_k(D_k,Q_k,L_k)) = \frac{1}{2}\tanh\left(\left(\frac{U^k+L^k}{2} - TAC_k(D_k,Q_k,L_k)\right)\sigma_k\right) + \frac{1}{2}.$$

Where α_k is a parameter, $\sigma_k = \frac{3}{(U^k-L^k)/2} = \frac{6}{U^k-L^k}$.

In this technique the problem is defined as follows:

Max λ,

Subject to

$\frac{1}{2}\tanh\left(\left(\frac{U^k+L^k}{2} - TAC_k(D_k,Q_k,L_k)\right)\sigma_k\right) + \frac{1}{2} \ge \lambda$,

$\sum_{i=1}^{n} \widehat{w_i} Q_i \le W, \lambda \ge 0,\ D_k >, Q_k > 0, L_k > 0$, for $k = 1,2,\ldots\ldots n$.

After simplification the above problem can be written as

Max y ,

Subject to

$y + \sigma_k TAC_k(D_k,Q_k,L_k) \le \frac{U^k+L^k}{2}\sigma_k$,

$\sum_{i=1}^{n} \widehat{w_i} Q_i \leq W$, $y \geq 0$, $D_k >, Q_k > 0, L_k > 0$ for $k = 1,2, \ldots \ldots \ldots .. n$.

Now the above problem can be freely solved by suitable mathematical programming algorithm and then we shall get the appropiet solution of the MOIM.

3.2. Fuzzy Non-Linear Programming (FNLP) Technique based on Max-Min

In this technique fuzzy membership function $\mu_{TAC_k}(TAC_k(Q_k, D_k))$ for the k[th] objective function $TAC_k(D_k, Q_k, L_k)$ respectively for $k = 1,2, \ldots, n$ are defined as follows:

$$\mu_{TAC_k}(TAC_k(D_k, Q_k, L_k)) = \begin{cases} 1 & \text{for } TAC_k(D_k, Q_k, L_k) < L^k \\ \dfrac{U^k - TAC_k(D_k, Q_k, L_k)}{U^k - L^k} & \text{for } L^k \leq TAC_k(D_k, Q_k, L_k) \leq U^k \\ 0 & \text{for } TAC_k(D_k, Q_k, L_k) > U^k \end{cases}$$

for $k = 1,2, \ldots, n$.

In this technique the problem is defined as follows:

Max α',

Subject to

$TAC_k(D_k, Q_k, L_k) + \alpha'(U^k - L^k) \leq U^k$, for $k = 1,2, \ldots, n$,

$\sum_{i=1}^{n} \widehat{w_i} Q_i \leq W, 0 \leq \alpha' \leq 1$, $D_k >, Q_k > 0, L_k > 0$.

Now the above problem can be freely solved by suitable mathematical programming algorithm and then we shall get the required solution of the MOIM.

3.3. Fuzzy Additive Goal Programming (FAGP) Technique Based on Additive Operator

Using the above membership function, fuzzy non-linear programming problem is formulated as

Max $\sum_{k=1}^{n} \frac{U^k - TAC_k(D_k, Q_k, L_k)}{U^k - L^k}$,

Subject to

$U^k - TAC_k(D_k, Q_k, L_k) \leq U^k - L^k$,

$\sum_{i=1}^{n} \widehat{w_i} Q_i \leq W$, $D_k >, Q_k > 0, L_k > 0$ for $k = 1,2, \ldots, n$.

Now the above problem can be solved by suitable mathematical programming algorithm and then we shall get the solution of the MOIM.

4. Geometric Programming Technique

Let us consider a Multi Objective Geometric Programming (MOGP) problem is as follows

$$\text{Minimize } g_{s0}(t)=\sum_{k=1}^{T_{s0}} c_{s0k}\prod_{j=1}^{m} t_j^{\alpha_{s0kj}},\ s=1,2,3,\dots\dots\dots,n,$$

Subject to

$$g_r(t)=\sum_{k=1}^{l_r} c_{rk}\prod_{j=1}^{m} t_j^{\alpha_{rkj}}\le 1,\ r=1,2,3,\dots\dots\dots,p,$$
$$t_j>0,\ j=1,2,\dots,m.$$

Where $c_{rk}, c_{s0k}(>0)$, α_{rkj} and α_{s0kj} $(j=1,2,\dots,m; r=0,1,2,\dots,p;\ k=1,2,\dots\dots l_r;\ s=1,2,3,\dots\dots\dots,n)$are all real numbers. T_{s0} is the number of terms in the s^{th} objective function and l_r is the number of terms in the r^{th} constraint.

Now introducing the weights w_i $(i=1,2,3,\dots\dots\dots,n)$, the above MOGP converted into the single objective geometric programming problem as following

Primal Problem.

$$\begin{aligned}
&\text{Minimize } g(t)=\sum_{s=1}^{n} w_s\sum_{k=1}^{T_{s0}} c_{s0k}\prod_{j=1}^{m} t_j^{\alpha_{s0kj}},\ s=1,2,3,\dots\dots\dots,n,\\
&\text{i.e.} =\sum_{s=1}^{n}\sum_{k=1}^{T_{s0}} w_s c_{s0k}\prod_{j=1}^{m} t_j^{\alpha_{s0kj}},\\
&\text{Subject to}\\
&g_r(t)=\sum_{k=1}^{l_r} c_{rk}\prod_{j=1}^{m} t_j^{\alpha_{rkj}}\le 1,\ r=1,2,3,\dots\dots\dots,p,\\
&t_j>0,\ j=1,2,\dots,m,\\
&\sum_{i=1}^{n} w_i=1,\ w_i>0,\ i=1,2,3,\dots\dots\dots,n.
\end{aligned}\tag{9}$$

Let T be the total numbers of terms (including constraints), number of variables is m. Then the degree of the difficulty (DD) is $T-(m+1)$.

Dual Program.

The dual programming of ***Eq. (9)*** is given as follows:

$$\text{Maximize } v(\theta)=\prod_{s=1}^{n}\prod_{k=1}^{T_{s0}}\left(\frac{w_s c_{s0k}}{\theta_{0sk}}\right)^{\theta_{0sk}}\prod_{r=1}^{p}\prod_{k=1}^{l_r}\left(\frac{c_{rk}}{\theta_{rk}}\right)^{\theta_{rk}}\left(\sum_{k=1}^{l_r}\theta_{rk}\right)^{\sum_{k=1}^{l_r}\theta_{rk}},$$

Subject to

$$\sum_{s=1}^{n}\sum_{k=1}^{T_{s0}}\theta_{0sk}=1,\ \text{(Normality condition)}$$
$$\sum_{r=1}^{p}\sum_{k=1}^{l_r}\alpha_{rkj}\theta_{rk}+\sum_{s=1}^{n}\sum_{k=1}^{T_{s0}}\alpha_{s0kj}\theta_{0sk}=0,\ (j=1,2,\dots,m)\ \text{(Orthogonality conditions)}$$
$$\theta_{0sk},\theta_{rk}>0,\ (r=0,1,2,\dots,p;\ k=1,2,\dots\dots l_r;\ s=1,2,3,\dots\dots\dots,n).\ \text{(Positivity conditions)}$$

Now here three cases may arises

Case I. $T_0 = m+1$, (i.e. DD=0). So DP presents a system of linear equations for the dual variables. So we have a unique solution vector of dual variables.

Case II. $T_0 > m+1$, So a system of linear equations is presented for the dual variables, where the number of linear equations is less than the number of dual variables. So it is concluded that dual variables vector have many solutions.

Case III. $T_0 < m+1$, so a system of linear equations is presented for the dual variables, where the number of linear equations is greater than the number of dual variables. It is seen that generally no solution vector exists for the dual variables here.

4.1. Solution Procedure of My Proposed Problem

Primal Problem.

$$\text{Minimize } \mathrm{TAC}(D,Q,L) = \sum_{i=1}^{n} w_i' \left(\widehat{\alpha_\imath} D_i^{(1-\widehat{\beta_\imath})} + \widehat{h_\imath}\left(\frac{Q_i}{2} + k\omega\sqrt{L_i}\right) + \frac{\widehat{\gamma_\imath} {Q_i}^{\widehat{\delta_\imath}-1} D_i^{\widehat{\sigma_\imath}+1}}{2^{\widehat{\delta_\imath}}} + D_i \widehat{\rho_\imath} {L_i}^{-\widehat{\tau_\imath}} {Q_i}^{-1} \right),$$

Subject to (10)

$\sum_{i=1}^{n} \frac{\widehat{w_\imath}}{W} Q_i \le 1$,

$\sum_{i=1}^{n} w_i' = 1, w_i' > 0, i = 1,2,3, \ldots\ldots\ldots.., n$.

Dual Program.

The dual programming of ***Eq. (10)*** is given as follows:

$$\text{Maximize } v(\theta) = \prod_{i=1}^{n} \left(\frac{w_i'\widehat{\alpha_\imath}}{\theta_{i1}}\right)^{\theta_{i1}} \left(\frac{w_i'\widehat{h_\imath}}{2\theta_{i2}}\right)^{\theta_{i2}} \left(\frac{w_i' k\omega}{\theta_{i3}}\right)^{\theta_{i3}} \left(\frac{w_i'\widehat{\gamma_\imath}}{2^{\widehat{\delta_\imath}}\theta_{i4}}\right)^{\theta_{i4}} \left(\frac{w_i'\widehat{\rho_\imath}}{\theta_{i5}}\right)^{\theta_{i5}} \left(\frac{\widehat{w_\imath}}{W\theta_{i1}'}\right)^{\theta_{i1}'} \left(\sum_{i=1}^{n}\theta_{i1}'\right)^{\sum_{i=1}^{n}\theta_{i1}'},$$

Subject to

$$\theta_{i1} + \theta_{i2} + \theta_{i3} + \theta_{i4} + \theta_{i5} = 1, \qquad (11)$$

$$\left(1-\widehat{\beta_\imath}\right)\theta_{i1} + (\widehat{\sigma_\imath}+1)\theta_{i4} + \theta_{i5} = 0,$$

$$\theta_{i2} + \left(\widehat{\delta_\imath}-1\right)\theta_{i4} - \theta_{i5} + \theta_{i1}' = 0,$$

$$\frac{\theta_{i3}}{2} - \widehat{\tau_\imath}\theta_{i5} = 0\,,$$

$$\sum_{i=1}^{n} w_i' = 1, w_i' > 0,$$

$\theta_{i1}, \theta_{i2}, \theta_{i3}, \theta_{i4}, \theta_{i5}, \theta'_{i1} \geq 0$ for $i = 1,2,3, \dots\dots\dots, n$.

Solving the above linear equations we have

$$\theta_{i1} = \frac{\widehat{\sigma_\iota}+1}{\widehat{\beta_\iota}-1} y_i + \frac{1}{\widehat{\beta_\iota}-1} x_i,\ \theta_{i2} = 1 - \left\{(1 + 2\widehat{\tau_\iota}) + \frac{1}{\widehat{\beta_\iota}-1} x_i\right\} - \frac{\widehat{\sigma_\iota}+\widehat{\beta_\iota}}{\widehat{\beta_\iota}-1} y_i,\ \theta_{i3} = 2\widehat{\tau_\iota} x_i,\ \theta_{i4} = y_i,\ \theta_{i5} = x_i,$$

$$\theta'_{i1} = 1 - \left(2\widehat{\tau_\iota} + \frac{1}{\widehat{\beta_\iota} - 1}\right) x_i - \left(\frac{\widehat{\sigma_\iota} + \widehat{\beta_\iota}}{\widehat{\beta_\iota} - 1} + \widehat{\delta_\iota} - 1\right) y_i .$$

Putting the above values in ***Eq. (11)*** we have

Maximize $v(x, y)$

$$= \prod_{i=1}^{n} \left(\frac{w'_i(\widehat{\beta_\iota} - 1)\widehat{\alpha_\iota}}{(\widehat{\sigma_\iota} + 1)y_i + x_i}\right)^{\frac{\widehat{\sigma_\iota}+1}{\widehat{\beta_\iota}-1}y_i + \frac{1}{\widehat{\beta_\iota}-1}x_i} \left(\frac{w'_i \widehat{h_\iota}}{2\left\{1 - \left\{(1 + 2\widehat{\tau_\iota}) + \frac{1}{\widehat{\beta_\iota} - 1} x_i\right\} - \frac{\widehat{\sigma_\iota} + \widehat{\beta_\iota}}{\widehat{\beta_\iota} - 1} y_i\right\}}\right)^{1-\left\{(1+2\widehat{\tau_\iota}) + \frac{1}{\widehat{\beta_\iota}-1}x_i\right\} - \frac{\widehat{\sigma_\iota}+\widehat{\beta_\iota}}{\widehat{\beta_\iota}-1}y_i}$$

$$\left(\frac{w'_i k\omega}{2\widehat{\tau_\iota} x_i}\right)^{2\widehat{\tau_\iota} x_i} \left(\frac{w'_i \widehat{\gamma_\iota}}{2^{\widehat{\delta_\iota}} y_i}\right)^{y_i} \left(\frac{w'_i \widehat{\rho_\iota}}{x_i}\right)^{x_i} \left(\frac{\widehat{w_\iota}}{W z_i}\right)^{z_i} \left(\sum_{i=1}^{n} z_i\right)^{\sum_{i=1}^{n} z_i},$$

$$\sum_{i=1}^{n} w'_i = 1,$$

Where $z_i = 1 - \left(2\widehat{\tau_\iota} + \frac{1}{\widehat{\beta_\iota}-1}\right) x_i - \left(\frac{\widehat{\sigma_\iota}+\widehat{\beta_\iota}}{\widehat{\beta_\iota}-1} + \widehat{\delta_\iota} - 1\right) y_i$,

$x_i, y_i > 0, w'_i > 0$ and $x = (x_1, x_2, x_3, \dots\dots, x_n)$, $y = (y_1, y_2, y_3, \dots\dots\dots, y_n)$.

Now using the primal-dual relation we have

$$TAC^*(D, Q, L) = n\left(v^*(x, y)\right)^{1/n};$$

$$w'_i \widehat{\alpha_\iota} D_i^{*(1-\widehat{\beta_\iota})} = \theta_{i1}^*\left(v^*(x, y)\right)^{1/n};$$

$$\frac{w'_i \widehat{h_\iota} Q_i^*}{2} = \theta_{i2}^*\left(v^*(x, y)\right)^{1/n};$$

$$w'_i \widehat{h_\iota} k\omega\sqrt{L_i^*} = \theta_{i3}^*\left(v^*(x, y)\right)^{1/n},\ \text{for } i = 1,2,3, \dots\dots\dots, n.$$

5. Numerical Example

Here we consider an inventory system which consists of two items with following parameter values in proper units. Total storage area $W = 500\ Sq.ft.$ and $k = 3, \omega = 5, w'_1 = 0.5, w'_1 = 0.5$.

Table 1. Input imprecise data for shape parameters.

Parameters	Items I	Items II
$\widetilde{\alpha}_1$	$(200,205,210,215; 0.9)$	$(215,220,225,230; 0.8)$
$\widetilde{\beta}_1$	$(4,5,6,7; 0.8)$	$(5,6,7,8; 0.8)$
$\widetilde{h}_1$	$(2,4,5,6; 0.9)$	$(2,2.5,3,3.5; 0.8)$
$\widetilde{\rho}_1$	$(2,2.3,2.4,2.5; 0.9)$	$(3,3.1,3.2,3.3; 0.9)$
$\widetilde{\gamma}_1$	$(90,95,100,105; 0.7)$	$(92,95,98,102; 0.8)$
$\widetilde{\delta}_1$	$(0.02,0.03,0.04,0.05; 0.8)$	$(0.04,0.05,0.06,0.07; 0.8)$
$\widetilde{\sigma}_1$	$(0.2,0.3,0.4,0.5; 0.8)$	$(0.2,0.3,0.4,0.5; 0.9)$
$\widetilde{w}_1$	$(1.5,1.6,1.7,1.8; 0.7)$	$(1.7,1.8,1.9,2.0; 0.9)$
$\widetilde{\tau}_1$	$\left(\frac{1}{6},\frac{1}{5},\frac{1}{4},\frac{1}{3}; 0.9\right)$	$\left(\frac{1}{7},\frac{1}{6},\frac{1}{5},\frac{1}{4}; 0.9\right)$

Approximate value of the above parameter is

Table 2. Defuzzification of the fuzzy numbers.

Defuzzification of the Fuzzy Numbers	Items I	Items II
$\widehat{\alpha}_1$	186.75	178
$\widehat{\beta}_1$	4.4	5.2
$\widehat{h}_1$	3.825	2.2
$\widehat{\rho}_1$	2.07	2.835
$\widehat{\gamma}_1$	68.25	77.4
$\widehat{\delta}_1$	0.028	0.044
$\widehat{\sigma}_1$	0.28	0.315
$\widehat{w}_1$	1.155	2.115
$\widehat{\tau}_1$	0.21375	0.17089

Table 3. Optimal solutions of MOIM using different methods.

Methods	D_1^*	Q_1^*	L_1^*	TAC_1^*	D_2^*	Q_2^*	L_2^*	TAC_2^*
FPTHMF	2.50	11.51	0.34×10^{-3}	53.95	2.28	15.54	0.30×10^{-3}	41.03
FNLP	2.50	11.51	0.34×10^{-3}	53.95	2.30	15.57	0.33×10^{-3}	41.03
FAGP	2.49	11.35	0.35×10^{-3}	53.96	2.30	15.63	0.37×10^{-3}	41.03
GP	2.58	10.07	0.27×10^{-3}	54.58	2.20	17.58	0.42×10^{-3}	41.52

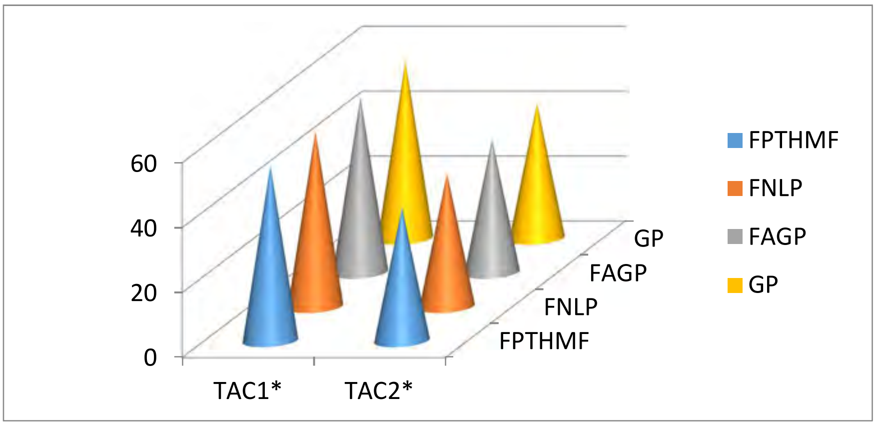

Fig. 2. Minimizing cost of both items using different methods.

From the above figure shows that GP, FPTHMF, FNLP, and FAGP methods almost provide the same results.

6. Sensitivity Analysis

In the sensitivity analysis the optimal solutions have been found buy using FNLP method.

Table 4. Optimal solution of MOIM for different values of α_1, α_2.

Method	α_1, α_2	D_1^*	Q_1^*	L_1^*	TAC_1^*	D_2^*	Q_2^*	L_2^*	TAC_2^*
FNLP	-20%	2.36	11.12	0.33×10^{-3}	52.14	2.19	15.12	0.29×10^{-3}	39.82
	-10%	2.43	11.33	0.34×10^{-3}	53.09	2.24	15.36	0.33×10^{-3}	40.45
	10%	2.56	11.68	0.35×10^{-3}	54.75	2.34	15.78	0.37×10^{-3}	41.55
	20%	2.61	10.84	0.35×10^{-3}	55.49	2.38	17.97	0.42×10^{-3}	42.03

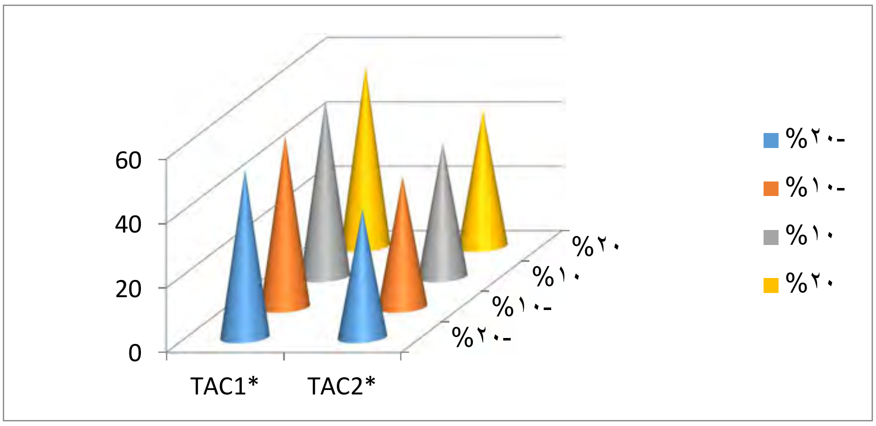

Fig. 3. Minimizing cost of both items for different values of α_1, α_2.

From the ***Fig. 3*** suggests that the minimum cost of both items is increased when values of α_1, α_2 are increased.

Method	β_1, β_2	D_1^*	Q_1^*	L_1^*	TAC_1^*	D_2^*	Q_2^*	L_2^*	TAC_2^*
	-20%	2.93	12.73	0.37×10^{-3}	62.82	2.67	17.19	0.32×10^{-3}	47.24
	-10%	2.69	12.05	0.35×10^{-3}	57.77	2.46	16.30	0.31×10^{-3}	43.71
FNLP	10%	2.35	11.07	0.33×10^{-3}	50.98	2.16	15.00	0.29×10^{-3}	38.92
	20%	2.22	10.70	0.32×10^{-3}	48.59	2.06	14.52	0.28×10^{-3}	37.23

Table 5. Optimal solution of MOIM for different values of β_1, β_2.

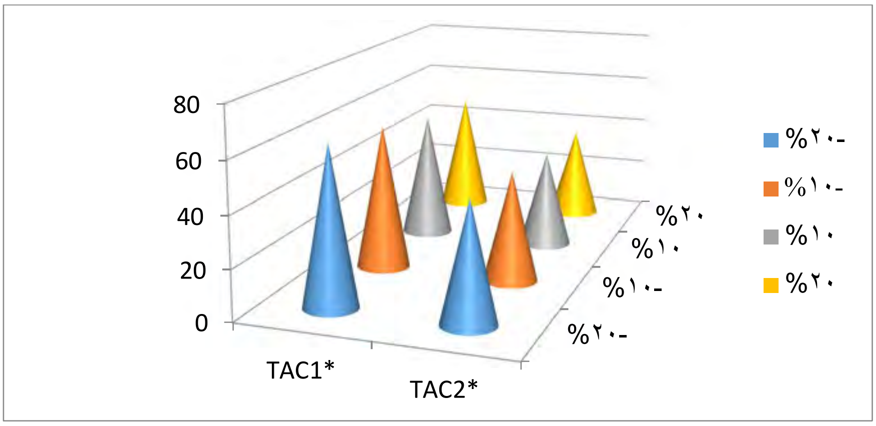

Fig. 4. Minimizing cost of 1[st] and 2[nd] items for different values of β_1, β_2.

From the ***Fig. 4*** suggests that the optimal cost of both items is decreased when values of β_1, β_2 are increased.

Table 6. Optimal solution of MOIM for different values of γ_1, γ_2.

Method	γ_1, γ_2	D_1^*	Q_1^*	L_1^*	TAC_1^*	D_2^*	Q_2^*	L_2^*	TAC_2^*
FNLP	-20%	2.57	10.58	0.40×10^{-3}	49.70	2.34	14.26	0.35×10^{-3}	37.60
	10%-	2.53	11.06	0.37×10^{-3}	51.89	2.32	14.94	0.32×10^{-3}	39.36
	10%	2.47	11.94	0.32×10^{-3}	55.92	2.27	16.19	0.28×10^{-3}	42.60
	20%	2.45	12.35	0.30×10^{-3}	57.78	2.25	16.77	0.26×10^{-3}	44.11

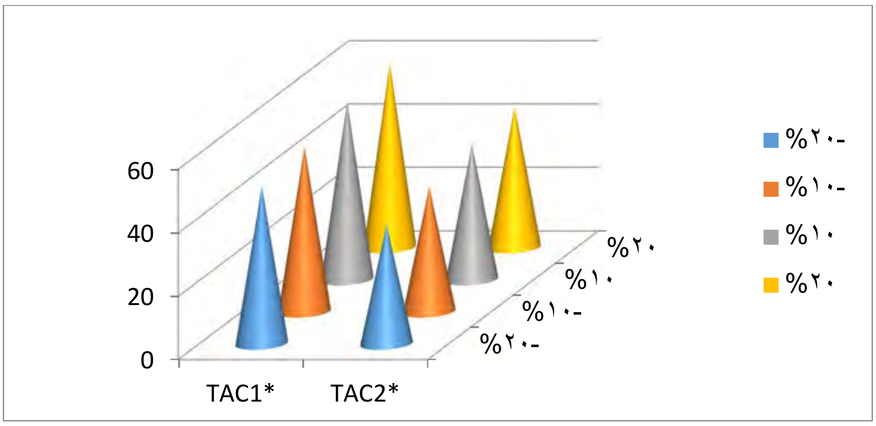

Fig. 5. Minimizing cost of both items for different values of γ_1, γ_2.

From the above *Fig. 5* suggests that the optimal cost of both items is increased when values of γ_1, γ_2 are increased.

Table 7. Optimal solutions of MOIM for different values of σ_1, σ_2.

Method	σ_1, σ_2	D_1^*	Q_1^*	L_1^*	TAC_1^*	D_2^*	Q_2^*	L_2^*	TAC_2^*
	-20%	2.54	11.36	0.35×10^{-3}	52.92	2.33	15.35	0.31×10^{-3}	40.18
	-10%	2.52	11.44	0.35×10^{-3}	53.44	2.31	15.46	0.31×10^{-3}	40.60
FNLP	10%	2.48	11.59	0.33×10^{-3}	54.47	2.28	15.70	0.29×10^{-3}	41.45
	20%	2.46	11.67	0.33×10^{-3}	54.99	2.26	15.81	0.29×10^{-3}	41.87

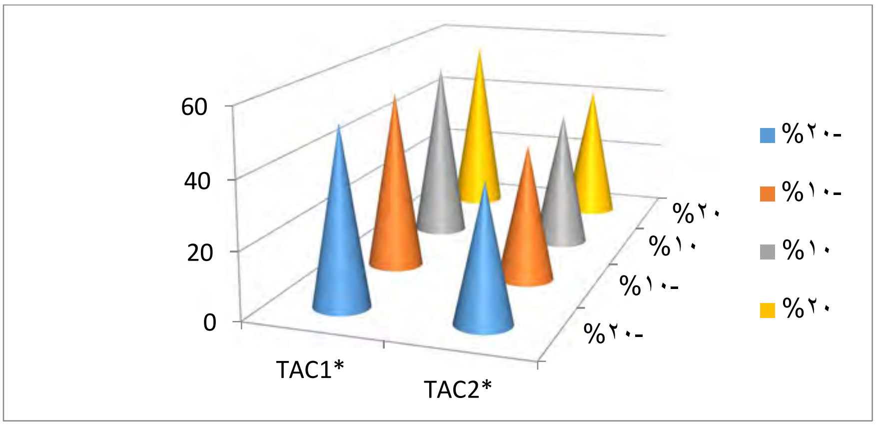

Fig. 6. Minimizing cost of 1st and 2nd items for different values of σ_1, σ_2.

From the above *Fig. 6* suggests that the minimum cost of both items is increased when values of σ_1, σ_2 are increased.

Table 8. Optimal solutions of MOIM for different values of ρ_1, ρ_2.

Method	ρ_1, ρ_2	D_1^*	Q_1^*	L_1^*	TAC_1^*	D_2^*	Q_2^*	L_2^*	TAC_2^*
	-20%	2.50	11.42	0.25×10^{-3}	53.44	2.29	15.47	0.23×10^{-3}	40.68
	-10%	2.50	11.47	0.30×10^{-3}	53.70	2.29	15.53	0.26×10^{-3}	40.86
FNLP	10%	2.50	11.55	0.39×10^{-3}	54.20	2.29	15.64	0.34×10^{-3}	41.19
	20%	2.50	11.60	0.43×10^{-3}	54.45	2.29	15.69	0.39×10^{-3}	41.35

Table 9. Optimal solutions of MOIM for different values of τ_1, τ_2.

Method	τ_1, τ_2	D_1^*	Q_1^*	L_1^*	TAC_1^*	D_2^*	Q_2^*	L_2^*	TAC_2^*
	-20%	2.50	11.40	0.15×10^{-3}	53.15	2.29	15.47	0.14×10^{-3}	40.58
	-10%	2.50	11.46	0.23×10^{-3}	53.54	2.29	15.52	0.21×10^{-3}	40.80
FNLP	10%	2.50	11.57	0.48×10^{-3}	54.39	2.29	15.64	0.42×10^{-3}	41.26
	20%	2.50	11.62	0.67×10^{-3}	54.83	2.29	15.69	0.57×10^{-3}	41.50

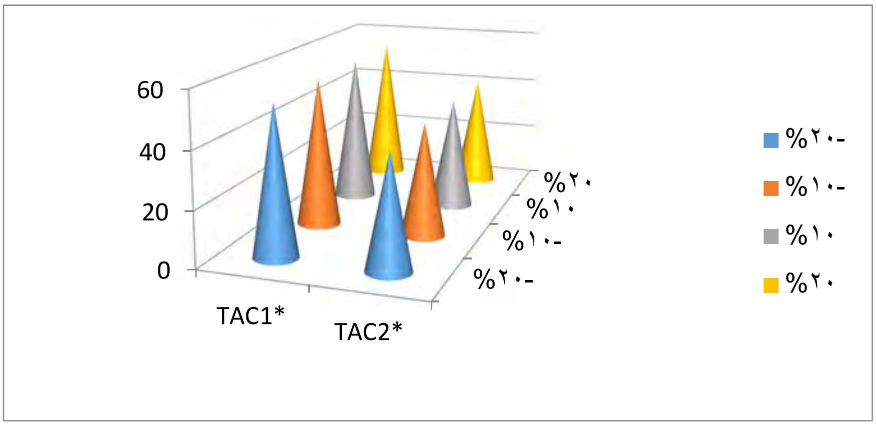

Fig. 7. Minimizing cost of both items for different values of ρ_1, ρ_2.

From the above ***Fig.*** *7* suggests that the minimum cost of both items is increased when values of ρ_1, ρ_2 are increased.

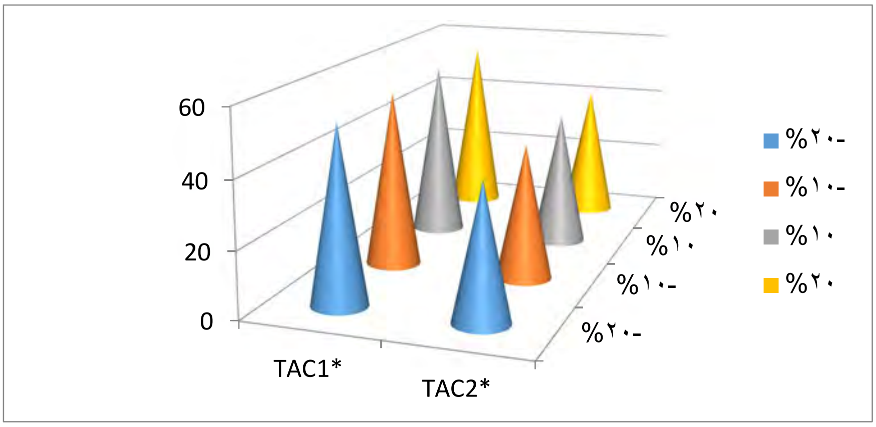

Fig. 8. Minimizing cost of 1st and 2nd items for different values of τ_1, τ_2.

From the above ***Fig.*** *8* suggests that the minimum cost of both items is increased when values of τ_1, τ_2 are increased.

7. Conclusion

In this article, we have developed an inventory model of multi-item with limitations on storage space in a fuzzy environment. Here we considered the constant demand rate and production cost is dependent on the demand rate. Set-up- cost is dependent on average inventory level as well as demand. Lead time crashing cost is considered the continuous function of leading time. Due to uncertainty all cost parameters are taken as a generalized trapezoidal fuzzy number. The formulated problem has been solved by various techniques like GP approach, FPTHMF, FNLP, and FAGP. Numerical example is given under considering two items to illustrate the model. A numerical problem is solved by using LINGO13 software.

This paper will be extended by using linear, quadratic demand, ramp type demand, power demand, and stochastic demand etc., introduce shortages, generalize the model under two-level credit period strategy etc. Inflation plays a crucial position in Inventory Management (IM) but here it is not considered. So inflation can be used in this model for practical. Also other types of fuzzy numbers like triangular fuzzy numbers; PfFN, pFN, etc. may be used for all cost parameters of the model.

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Some Similarity Measures of Spherical Fuzzy Sets Based on the Euclidean Distance and Their Application in Medical Diagnosis

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ABSTRACT

Similarity measure is an important tool in multiple criteria decision-making problems, which can be used to measure the difference between the alternatives. In this paper, some new similarity measures of Spherical Fuzzy Sets (SFS) are defined based on the Euclidean distance measure and the proposed similarity measures satisfy the axiom of the similarity measure. Furthermore, we apply the proposed similarity measures to medical diagnosis decision making problem; the numerical example is used to illustrate the feasibility and effectiveness of the proposed similarity measures of SFS, which are then compared to other existing similarity measures.

1. Introduction

The concept of Fuzzy Set (FS) $A = \{< x_i, \mu_{A_{x_i}} > | x_i \in X\}$ in $X = \{x_1, x_2, \dots, x_n\}$ was proposed by Zadeh [1], where the membership degree $\mu_{A_{x_i}}$ is a single value between zero and one. The FS has been widely applied in many fields, such as medical diagnosis, image processing, supply decision-making [2-4], and so on. In some uncertain decision-making problems, the degree of membership is not exactly as a numerical value but as an interval. Hence, Zadeh [5] proposed the Interval-Valued Fuzzy Sets (IVFS). However, the FS and the IVFS only have the membership degree, and they cannot describe the non-membership degree of the element belonging to the set. Then, Atanassov [6] proposed the Intuitionistic Fuzzy Set (IFS) $E =$

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$\{< x_i, \mu_E(x_i), \vartheta_E(x_i) > | x_i \in X\}$, where $\mu_E(x_i)$ $(0 \leq \mu_E(x_i) \leq 1)$ and $\vartheta_E(x_i)$ $(0 \leq \vartheta_E(x_i) \leq 1)$ represent the membership and the non-membership degree, respectively, and the indeterminacy- membership degree $\pi_E(x_i) = 1 - \mu_E(x_i) - \vartheta_E(x_i)$. The IFS is more effective to deal with the vague information more than the FS and IVFS.

Yang and Chiclana [7] proposed a spherical representation, which allowed us to define a distance function between intuitionistic fuzzy sets. In the spherical representation, hesitancy can be calculated based on the given membership and non-membership values since they only consider the surface of the sphere. Besides, they measure the spherical arc distance between two IFSs. Furthermore, Gong et al. [8] introduced an approach generalizing Yang and Chiclana's work.

The Spherical Fuzzy Sets (SFSs) are based on the fact that the hesitancy of a decision maker can be defined independently from membership and non-membership degrees, satisfying the following condition:

$$0 \leq \mu_{\tilde{A}}{}^2(u) + \vartheta_{\tilde{A}}{}^2(u) + \pi_{\tilde{A}}{}^2(u) \leq 1. \qquad \forall\, u \in U. \tag{1}$$

On the surface of the sphere, ***Eq. (1)*** becomes

$$\mu_{\tilde{A}}{}^2(u) + \vartheta_{\tilde{A}}{}^2(u) + \pi_{\tilde{A}}{}^2(u) = 1. \qquad \forall\, u \in U. \tag{2}$$

On the other hand, similarity measure is an important tool in multiple-criteria decision making problems, which can be used to measure the difference between the alternatives. Many studies about the similarity measures have been obtained. For example, Beg and Ashraf [9] proposed a similarity measure of fuzzy sets based on the concept of $\epsilon -$ fuzzy transitivity and discussed the degree of transitivity of different similarity measures. Song et al. [4] considered the similarity measure and proposed corresponding distance measure between intuitionistic fuzzy belief functions. In addition, cosine similarity measure is also an important similarity measure, and it can be defined as the inner product of two vectors divided by the product of their lengths. There are some scholars who studied the cosine similarity measures [10-15]. Various forms of Spherical fuzzy sets which are applied in Multi–attribute decision making problems are developed in [16-18].

In this paper, we propose a new method to construct the similarity measure of SFSs. They play an important role in practical application, especially in pattern recognition, medical diagnosis, and so on. Furthermore, the proposed similarity measure can be applied more widely in the field of decision–making problems.

2. Preliminaries

Definition 1. [19]. A SFS $\tilde{A}_s$ of the universe of discourse U is given by,

$\tilde{A}_s = \{\langle \mu_{\tilde{A}_s}(u), \vartheta_{\tilde{A}_s}(u), \pi_{\tilde{A}_s}(u) | u \in U \rangle\}$, where $\mu_{\tilde{A}_s}: U \rightarrow [0,1]$, $\vartheta_{\tilde{A}_s}: U \rightarrow [0,1]$, $\pi_{\tilde{A}_s}: U \rightarrow [0,1]$ and $0 \leq \mu_{\tilde{A}_s}{}^2(u) + \vartheta_{\tilde{A}_s}{}^2(u) + \pi_{\tilde{A}_s}{}^2(u) \leq 1.\quad \forall\, u \in U$.

For each u, the numbers $\mu_{\tilde{A}_s}(u), \vartheta_{\tilde{A}_s}(u)$ *and* $\pi_{\tilde{A}_s}(u)$ are the degree of membership, non-membership and hesitancy of u to $\tilde{A}_s$, respectively.

Definition 2. [9]. Basic operators of spherical fuzzy sets:

Union. $\tilde{A}_s \cup \tilde{B}_s = \{max\{\mu_{\tilde{A}_s}, \mu_{\tilde{B}_s}\}, min\{\vartheta_{\tilde{A}_s}, \vartheta_{\tilde{B}_s}\}, min\{\pi_{\tilde{A}_s}, \pi_{\tilde{B}_s}\}\}$.

Intersection. $\tilde{A}_s \cap \tilde{B}_s = \{min\{\mu_{\tilde{A}_s}, \mu_{\tilde{B}_s}\}, max\{\vartheta_{\tilde{A}_s}, \vartheta_{\tilde{B}_s}\}, max\{\pi_{\tilde{A}_s}, \pi_{\tilde{B}_s}\}\}$.

Addition. $\tilde{A}_s \oplus \tilde{B}_s = \left\{\left({\mu_{\tilde{A}_s}}^2 + {\mu_{\tilde{B}_s}}^2 - {\mu_{\tilde{A}_s}}^2 {\mu_{\tilde{B}_s}}^2\right)^{1/2}, \vartheta_{\tilde{A}_s}\vartheta_{\tilde{B}_s},\ \pi_{\tilde{A}_s}\pi_{\tilde{B}_s}\right\}$.

Multiplication. $\tilde{A}_s \otimes \tilde{B}_s = \left\{\mu_{\tilde{A}_s}\mu_{\tilde{B}_s},\ \left({\vartheta_{\tilde{A}_s}}^2 + {\vartheta_{\tilde{B}_s}}^2 - {\vartheta_{\tilde{A}_s}}^2 {\vartheta_{\tilde{B}_s}}^2\right)^{1/2}, \pi_{\tilde{A}_s}\pi_{\tilde{B}_s}\right\}$.

Multiplication by a scalar, $\lambda > 0$. $\lambda.\tilde{A}_s = \left\{\left(1 - \left(1 - {\mu_{\tilde{A}_s}}^2\right)^{\lambda}\right)^{1/2},\ {\vartheta_{\tilde{A}_s}}^{\lambda},\ {\pi_{\tilde{A}_s}}^{\lambda}\right\}$.

Power of $\tilde{A}_s$, $\lambda > 0$. ${\tilde{A}_s}^{\lambda} = \left\{{\mu_{\tilde{A}_s}}^{\lambda}, \left(1 - \left(1 - {\vartheta_{\tilde{A}_s}}^2\right)^{\lambda}\right)^{1/2},\ {\pi_{\tilde{A}_s}}^{\lambda}\right\}$.

3. Several New Similarity Measures

The similarity measure is a most widely used tool to evaluate the relationship between two sets. The following axiom about the similarity measure of IVSFSs should be satisfied:

Lemma 1. Let $X = \{x_1, x_2, \dots, x_n\}$ be the universal set [12] if the similarity measure S (A, B) between SFSs A and B satisfies the following properties:

- $0 \le S(A, B) \le 1$;
- $S(A, B) = 1$ if and only if $A = B$;
- $S(A, B) = S(B, A)$.

Then, the similarity measure $S(A, B)$ is a genuine similarity measure.

3.1. The New Similarity Measures between SFSs

Definition 3: Let $X = \{x_1, x_2, \dots, x_n\}$ be the universal set for any two SFSs $A = \{< x_i, \mu_{A_{x_i}}, \vartheta_{A_{x_i}}, \pi_{A_{x_i}} > | x_i \in X\}$ and $B = \{< x_i, \mu_{B_{x_i}}, \vartheta_{B_{x_i}}, \pi_{B_{x_i}} > | x_i \in X\}$; then the Euclidean distance between SFSs A and B is defined as follows:

$$D_{SFSs}(A, B) = \sqrt{\frac{\sum_{i=1}^{n}\left[\left(\mu_A{}^2(x_i) - \mu_B{}^2(x_i)\right)^2 + \left(\vartheta_A{}^2(x_i) - \vartheta_B{}^2(x_i)\right)^2 + \left(\pi_A{}^2(x_i) - \pi_B{}^2(x_i)\right)^2\right]}{3n}}. \tag{3}$$

Now, we construct new similarity measures of SFSs based on the Euclidean distance measures.

Definition 4. Let $X = \{x_1, x_2, \dots, x_n\}$ be the universal set for any two SFSs $A = \{< x_i, \mu_{A_{x_i}}, \vartheta_{A_{x_i}}, \pi_{A_{x_i}} > | x_i \in X\}$ and $B = \{< x_i, \mu_{B_{x_i}}, \vartheta_{B_{x_i}}, \pi_{B_{x_i}} > | x_i \in X\}$; the similarity measure of SFSs between A and B is defined as follows:

$$S_{1SFSs}(A,B)=\frac{\sum_{i=1}^{n}\left(\min\left(\mu_A{}^2(x_i),\mu_B{}^2(x_i)\right)+\min\left(\vartheta_A{}^2(x_i),\vartheta_B{}^2(x_i)\right)+\min\left(\pi_A{}^2(x_i),\pi_B{}^2(x_i)\right)\right)}{\sum_{i=1}^{n}\left(\max\left(\mu_A{}^2(x_i),\mu_B{}^2(x_i)\right)+\max\left(\vartheta_A{}^2(x_i),\vartheta_B{}^2(x_i)\right)+\max\left(\pi_A{}^2(x_i),\pi_B{}^2(x_i)\right)\right)}. \quad (4)$$

The similarity measure S_{1SFSs} satisfies the properties in ***Lemma 1***.

Next, we propose a new method to construct a new similarity measure of SFSs, and the Euclidean distance, it can be defined as follows:

Definition 5. Let $X=\{x_1,x_2,\dots,x_n\}$ be the universal set for any two SFSs $A=\{<x_i,\mu_{A_{x_i}},\vartheta_{A_{x_i}},\pi_{A_{x_i}}>|x_i\in X\}$ and$B=\{<x_i,\mu_{B_{x_i}},\vartheta_{B_{x_i}},\pi_{B_{x_i}}>|x_i\in X\}$; a new similarity measure $S^*{}_{1SFSs}(A,B)$ is defined as follows:

$$S^*{}_{1SFSs}(A,B)=\frac{1}{2}\left(S_{1SFSs}(A,B)+1-D_{SFSs}(A,B)\right). \quad (5)$$

The proposed similarity measure of SFSs satisfies the ***Theorem 1***.

Theorem 1. The similarity measure $S^*{}_{1SFSs}(A,B)$ between $A=\{<x_i,\mu_{A_{x_i}},\vartheta_{A_{x_i}},\pi_{A_{x_i}}>|x_i\in X\}$ and$B=\{<x_i,\mu_{B_{x_i}},\vartheta_{B_{x_i}},\pi_{B_{x_i}}>|x_i\in X\}$ satisfies the following properties:

- $0\le S^*{}_{1SFSs}(A,B)\le 1$
- $S^*{}_{1SFSs}(A,B)=1$ if and only if $A=B$
- $S^*{}_{1SFSs}(A,B)=S^*{}_{1SFSs}(B,A)$.

Proof. Because $D_{SFSs}(A,B)$ is an Euclidean distance measure, obviously, $0\le D_{SFSs}(A,B)\le 1$. Furthermore, according to lemma 1, we know that $0\le S_{1SFSs}(A,B)\le 1$. Then, $0\le\frac{1}{2}\left(S_{1SFSs}(A,B)+1-D_{SFSs}(A,B)\right)\le 1$, i.e., $0\le S^*{}_{1SFSs}(A,B)\le 1$.

If $S^*{}_{1SFSs}(A,B)=1$, we have $S_{1SFSs}(A,B)+1-D_{SFSs}(A,B)=2$, that is $S_{1SFSs}(A,B)=1+D_{SFSs}(A,B)$. Because $D_{SFSs}(A,B)$ is the Euclidean distance measure $0\le D_{SFSs}(A,B)\le 1$. Furthermore, $0\le S_{1SFSs}(A,B)\le 1$, then $S_{1SFSs}(A,B)=1$ and $D_{SFSs}(A,B)=0$ should be established at the same time. If the Euclidean distance measure $D_{SFSs}(A,B)=0, A= B$ is obvious. According to lemma 1, when $S_{1SFSs}(A,B)=1, A= B$; so if $S^*{}_{1SFSs}(A,B)=1, A= B$ is obtained.

On the other hand, when $A=B$, according to ***Eqs. (3)*** and *(4)* $D_{SFSs}(A,B)=0$ and $S_{1SFSs}(A,B)=1$ are obtained respectively. Furthermore, we can get $S^*{}_{1SFSs}(A,B)=1$. $S^*{}_{1SFSs}(A,B)=S^*{}_{1SFSs}(B,A)$ is straightforward.

From ***Theorem 1***, we know that the proposed new similarity measure $S^*{}_{1SFSs}(A,B)$ is a genuine similarity measure. On the other hand, cosine similarity measure is also an important similarity measure. The cosine similarity measure between SFSs is as follows:

Definition 6. Let $X=\{x_1,x_2,\dots,x_n\}$ be the universal set for any two SFSs $A=\{<x_i,\mu_{A_{x_i}},\vartheta_{A_{x_i}},\pi_{A_{x_i}}>|x_i\in X\}$ and $B=\{<x_i,\mu_{B_{x_i}},\vartheta_{B_{x_i}},\pi_{B_{x_i}}>|x_i\in X\}$; the cosine similarity measure of SFSs between A and B is defined as follows:

$$S_{2SFSs}(A,B)=\frac{1}{n}\sum_{i=1}^{n}\frac{\left(\left(\mu_A{}^2(x_i)\mu_B{}^2(x_i)\right)+\left(\vartheta_A{}^2(x_i)\,\vartheta_B{}^2(x_i)\right)+\left(\pi_A{}^2(x_i)\,\pi_B{}^2(x_i)\right)\right)}{\sqrt{(\mu_A{}^2(x_i))^2+(\vartheta_A{}^2(x_i))^2+(\pi_A{}^2(x_i))^2}\sqrt{(\mu_B{}^2(x_i))^2+(\vartheta_B{}^2(x_i))^2+(\pi_B{}^2(x_i))^2}}\ . \quad (6)$$

Now, we are going to propose another similarity measure of SFSs based on the cosine similarity measure and the Euclidean distance D_{SFSs}. It considers the similarity measure not only from the point of view of algebra but also from the point of view of geometry, which can be defined as:

Definition 7. Let $X=\{x_1,x_2,\dots,x_n\}$ be the universal set for any two SFSs

$A=\{< x_i,\mu_{A_{x_i}},\vartheta_{A_{x_i}},\pi_{A_{x_i}} > |x_i\in X\}$ and $B=\{< x_i,\mu_{B_{x_i}},\vartheta_{B_{x_i}},\pi_{B_{x_i}} > |x_i\in X\}$; a new similarity measure $S^*{}_{2SFSs}(A,B)$ is defined as follows:

$$S^*{}_{2SFSs}(A,B)=\frac{1}{2}\left(S_{2SFSs}(A,B)+1-D_{SFSs}(A,B)\right). \quad (7)$$

Theorem 2. The similarity measure $S^*{}_{2SFSs}(A,B)$ between $A=\{< x_i,\mu_{A_{x_i}},\vartheta_{A_{x_i}},\pi_{A_{x_i}} > |x_i\in X\}$ and

$B=\{< x_i,\mu_{B_{x_i}},\vartheta_{B_{x_i}},\pi_{B_{x_i}} > |x_i\in X\}$ satisfies the following properties:

- $0\le S^*{}_{2SFSs}(A,B)\le 1$;
- $S^*{}_{2SFSs}(A,B)=1$ if and only if $A=B$;
- $S^*{}_{2SFSs}(A,B)=S^*{}_{2SFSs}(B,A)$.

Proof. Because $D_{SFSs}(A,B)$ is an Euclidean distance measure, obviously, $0\le D_{SFSs}(A,B)\le 1$. Furthermore, according to lemma 1, we know that $0\le S_{SFSs}(A,B)\le 1$. Then, $0\le\frac{1}{2}\left(S_{2SFSs}(A,B)+1-D_{SFSs}(A,B)\right)\le 1$, i.e., $0\le S^*{}_{1SFSs}(A,B)\le 1$.

If $S^*{}_{2SFSs}(A,B)=1$, we have $S_{2SFSs}(A,B)+1-D_{SFSs}(A,B)=2$, that is $S_{2SFSs}(A,B)=1+D_{SFSs}(A,B)$. Because $D_{SFSs}(A,B)$ is the Euclidean distance measure $0\le D_{SFSs}(A,B)\le 1$. Furthermore, $0\le S_{2SFSs}(A,B)\le 1$, then $S_{2SFSs}(A,B)=1$ and $D_{SFSs}(A,B)=0$ should be established at the same time. When $S_{2SFSs}(A,B)=1$, we have $\mu_A(x_i)=k\mu_B(x_i)$, $\vartheta_A(x_i)=k\vartheta_B(x_i), and\ \pi_A(x_i)=k\pi_B(x_i)$ (k is a constant). When the Euclidean distance measure $D_{SFSs}(A,B)=0, A=B$. Then $A=B$ is obtained.

On the other hand, when $A=B$, according to ***Eqs. (3)*** and ***(6)*** if $A=B$, $D_{SFSs}(A,B)=0$ and $S_{2SFSs}(A,B)=1$ are obtained respectively. Furthermore, we can get $S^*{}_{2SFSs}(A,B)=1$. $S^*{}_{2SFSs}(A,B)=S^*{}_{2SFSs}(B,A)$ is straightforward.

Thus $S^*{}_{2SFSs}(A,B)$ satisfies all the properties of the ***Theorem 2***.

In the next section, we will apply the proposed new similarity measures to medical diagnosis decision problem; numerical examples are also given to illustrate the application and effectiveness of the proposed new similarity measures.

4. Applications of the Proposed Similarity Measures

4.1. The Proposed Similarity Measures between SFSs for Medical Diagnosis

We first give a numerical example medical diagnosis to illustrate the feasibility of the proposed new similarity measur $S^{*}_{1SFSs}(A,B)$e and $S^{*}_{2SFSs}(A,B)$ between SFSs.

Example 1. Consider a medical diagnosis decision problem; Suppose a set of diagnosis $Q = \{Q_1$(viral fever), Q_2(malaria), Q_3(typhoid), Q_4(Gastritis), Q_5(stenocardia)} and a set of symptoms S ={S_1(fever), S_2(headache), S_3(stomach), S_4(cough), S_5(chestpain)}. Assume a patient P_1 has all the symptoms in the process of diagnosis, the SFS evaluate information about P_1 is

P_1(Patient= $\{< S_1, 0.8, 0.2, 0.1 >, < S_2, 0.6, 0.3, 0.1 >, < S_3, 0.2, 0.1, 0.8 >) < S_4, 0.6, 0.5, 0.1 >,$
$<S_5, 0.1, 0.4, 0.6 >\}$.

The diagnosis information $Q_i (i = 1,2, \dots ,5)$ with respect to symptoms $S_i (i = 1,2, \dots ,5)$ also can be represented by the SFSs, which is shown in ***Table 1***.

Table 1. Diagnosis information.

	S_1	S_2	S_3	S_4	S_5
Q_1	[0.4, 0.6,0.0]	[0.3, 0.2, 0.5]	[0.1, 0.3, 0.7]	[0.4, 0.3, 0.3]	[0.1, 0.2, 0.7]
Q_2	[0.7, 0.3, 0.0]	[0.2, 0.2, 0.6]	[0.0, 0.1, 0.9]	[0.7, 0.3, 0.0]	[0.1, 0.1, 0.8]
Q_3	[0.3, 0.4, 0.3]	[0.6, 0.3, 0.1]	[0.2, 0.1, 0.7]	[0.2, 0.2, 0.6]	[0.1, 0.0, 0.9]
Q_4	[0.1, 0.2, 0.7]	[0.2, 0.2, 0.4]	[0.8, 0.2, 0.0]	[0.2, 0.1, 0.7]	[0.2, 0.1, 0.7]
Q_5	[0.1, 0.1, 0.8]	[0.0, 0.2,0.8]	[0.2, 0.0, 0.8]	[0.3, 0.1, 0.8]	[0.8, 0.1, 0.1]

By applying ***Eqs. (5)*** and ***(7)*** we can obtain the similarity measure values$S^{*}_{1SFSs}(P_1, Q_i)$ and $S^{*}_{2SFSs}(P_1, Q_i)$; the results are shown in ***Table 2***.

Table 2. Similarity measures.

	Q_1	Q_2	Q_3	Q_4	Q_5
$S^{*}_{1SFSs}(P_1, Q_i)$	0.5980	0.6801	0.5729	0.3919	0.3820
$S^{*}_{2SFSs}(P_1, Q_i)$	0.4277	0.4581	0.4024	0.3514	0.3155

From the above two similarity measures S^{*}_{1SFSs} and S^{*}_{2SFSs} , we can conclude that the diagnoses of the patient P_1 are all malaria (Q_2). The proposed two similarity measures are feasible and effective.

4.2. Comparative Analysis of Existing Similarity Measures

To illustrative the effectiveness of the proposed similarity measures for medical diagnosis, we change the existing similarity measures for SFS and thus will apply the existing similarity measures for comparative analyses.

At first, we introduce the existing similarity measures between SFSs as follows:

Let $A = \{< x_i, \mu_{A_{x_i}}, \vartheta_{A_{x_i}}, \pi_{A_{x_i}} > | x_i \in X\}$ an $B = \{< x_i, \mu_{B_{x_i}}, \vartheta_{B_{x_i}}, \pi_{B_{x_i}} > | x_i \in X\}$d be two SFSs in $X = \{x_1, x_2, \dots, x_n\}$, the existing measures between A and B are defined as follows:

Broumi et al. [20] proposed the similarity measure SM_{SFS}:

$$\mathrm{SM_{SFS}(A, B) = 1 - D_{SFS}(A, B)}. \tag{8}$$

Sahin and Küçük [21] proposed the similarity measure SD_{SFS}:

$$\mathrm{SD_{SFS}} = \frac{1}{1 + \mathrm{D_{SFS}(A, B)}}. \tag{9}$$

Ye [22] proposed the improved cosine similarity measure SC_{1SFS} and SC_{2SFS}:

$$\mathrm{SC_{1SFS}(A, B)} = \frac{1}{n}\sum_{i=1}^{n} \cos\left[\frac{\pi. \max\left(|\mu_A{}^2(x_i) - \mu_B{}^2(x_i)|, |\vartheta_A{}^2(x_i) - \vartheta_B{}^2(x_i)|, |\pi_A{}^2(x_i) - \pi_B{}^2(x_i)|\right)}{2}\right]. \tag{10}$$

$$\mathrm{SC_{2SFS}(A, B)} = \frac{1}{n}\sum_{i=1}^{n} \cos\left[\frac{\pi. \left(|\mu_A{}^2(x_i) - \mu_B{}^2(x_i)| + |\vartheta_A{}^2(x_i) - \vartheta_B{}^2(x_i)| + |\pi_A{}^2(x_i) - \pi_B{}^2(x_i)|\right)}{6}\right]. \tag{11}$$

Yong-Wei et al. [23] proposed the similarity measure $SY_{SFS}(A, B)$:

$$\mathrm{SY_{SFS}(A, B)} = \frac{\mathrm{SC_{SFS}(A, B)}}{\mathrm{SC_{SFS}(A, B) + D_{SFS}(A, B)}}. \tag{12}$$

Example 2. We apply ***Eqs. (4)***, ***(6)*** and ***(8) – (12)*** to calculate ***Example 1*** again; the similarity measure values between P_1 and Q_i $(i = 1,2,..,5)$ are shown on ***Table 3***.

As we can see from ***Table 3***, the patient P_1 is still assigned to malaria (Q_2), and the results are same as the proposed similarity measures in this paper, which means the proposed similarity measures are feasible and effective.

Table 3. Similarity values between patient and symptoms.

	Q_1	Q_2	Q_3	Q_4	Q_5
$\mathbf{SM_{SFS}}$	0.8003	0.8314	0.7449	0.6388	0.6007
$\mathbf{SD_{SFS}}$	0.8335	0.8557	0.7967	0.7346	0.7146
$\mathbf{SC_{1SFS}}$	0.8555	0.9325	0.6469	0.7324	0.6391
$\mathbf{SC_{2SFS}}$	0.9648	0.9759	0.7531	0.885	0.8585
$\mathbf{SY_{SFS}}$	0.8107	0.8468	0.7171	0.6697	0.6154
$\mathbf{S_{1SFS}}$	0.3958	0.5289	0.4010	0.1451	0.1633
$\mathbf{S_{2SFS}}$	0.0551	0.0849	0.0600	0.0191	0.0304

The proposed similarity measures in the paper have some advantages in solving multiple criteria decision making problems. They are constructed based on the existing similarity measures and Euclidean distance, which not only satisfy the axiom of the similarity measure but also consider the similarity measure from the points of view of algebra and geometry. Furthermore, they can be applied more widely in the field of decision making problems.

5. Conclusion

The similarity measure is widely used in multiple criteria decision making problems. This paper proposed a new method to construct the similarity measures combining the existing cosine similarity measure and the Euclidean distance measure. And, the similarity measures are proposed not only from the points of view of algebra and geometry but also satisfy the axiom of the similarity measure. Furthermore, we apply the proposed similarities measures to the medical diagnosis decision problems, and the numerical example is used to illustrate the feasibility and effectiveness of the proposed similarity measure, which are then compared to other existing similarity measures.

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Fuzzy Programming Approach to Bi-level Linear Programming Problems

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ABSTRACT

In this study, we discussed a fuzzy programming approach to bi-level linear programming problems and their application. Bi-level linear programming is characterized as mathematical programming to solve decentralized problems with two decision-makers in the hierarchal organization. They become more important for the contemporary decentralized organization where each unit seeks to optimize its own objective. In addition to this, we have considered Bi-Level Linear Programming (BLPP) and applied the Fuzzy Mathematical Programming (FMP) approach to get the solution of the system. We have suggested the FMP method for the minimization of the objectives in terms of the linear membership functions. FMP is a supervised search procedure (supervised by the upper Decision Maker (DM)). The upper-level decision-maker provides the preferred values of decision variables under his control (to enable the lower level DM to search for his optimum in a wider feasible space) and the bounds of his objective function (to direct the lower level DM to search for his solutions in the right direction).

1. Introduction

Decision making problems in decentralized organizations are often modeled as stackelberg games, and they are formulated as bi-level mathematical programming problems. A bi-level problem with a single decision maker at the upper level and two or more decision makers at the lower level is referred to as a decentralized bi-level programming problem. Real-world applications under non cooperative situations are formulated by bi-level mathematical programming problems and their effectiveness is demonstrated.

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The use of fuzzy set theory for decision problems with several conflicting objectives was first introduced by Zimmermann. Thereafter, various versions of Fuzzy Programming (FP) have been investigated and widely circulated in literature. The use of the concept of tolerance membership function of fuzzy set theory to Bi-Linear Programming Problems(BLPPs) for satisfactory decisions was first introduced by Lai in 1996 [1]. Shih and Lee further extended Lai's concept by introducing the compensatory fuzzy operator for solving BLPPs [2]. Sinha studied alternative BLP techniques based on Fuzzy Mathematical Programming (FMP).

The basic concept of these FMP approaches is the same as Fuzzy Goal Programming (FGP) approach which implies that the lower level DMs optimizes, his/her objective function, taking a goal or preference of the higher level DMs in to consideration. In the decision process, considering the membership functions of the fuzzy goals for the decision variables of the higher level DM, the lower level DM solves a FMP problem with a constraint on an overall satisfactory degree of the higher level DMs. If the proposed solution is not satisfactory, to the higher level DMs, the solution search is continued by redefining the elicited membership functions until a satisfactory solution is reached [2]. The main difficulty that arises with the FMP approach of Sinha is that there is possibility of rejecting the solution again and again by the higher level DMs and re-evaluation of the problem is repeatedly needed to reach the satisfactory decision, where the objectives of the DMs are over conflicting [2].

Taking in to account vagueness of judgments of the decision makers, we will present interactive fuzzy programming for bi-level linear programming problems. In the interactive method, after determining the fuzzy goals of the decision makers at both levels, a satisfactory solution is derived by updating some reference points with respect to the satisfactory level. In the real world, we often encounter situations where there are two or more decision makers in an organization with a hierarchical structure, and they make decisions in turn or at the same time so as to optimize their objective functions. In particular, consider a case where there are two decision makers; one of the decision makers first makes a decision. Such a situation is formulated as a bi-level programming problem. Although a large number of algorithms for obtaining stackelberg solutions have been developed, it is also known that solving the mathematical programming problems for obtaining stackelberg solution is NP-hard [3]. From such difficulties, a new solution concept which is easy to compute and reflects structure of bi-level programming problems had been expected [4] proposed a solution method, which is different from the concept of stackelberg solutions, for bi-level linear programming problems with cooperative relationship between decision makers. Sakawa and Nishizaki [5] present interactive fuzzy programming for bi-level linear programming problems. In order to overcome the problem in the methods of [4], after eliminating the fuzzy goals for decision variables, they formulate the bi-level linear programming problem.

In their interactive method, after determining the fuzzy goals of the decision makers at all the levels, a satisfactory solution is derived efficiently by updating the satisfactory degree of the decision maker at the upper level with considerations of overall satisfactory balance among all the levels. By eliminating the fuzzy goals for the decision variables to avoid such problems in the method of [4-6] develop interactive fuzzy programming for bi-level linear programming problems. Moreover, from the viewpoint of experts' imprecise or fuzzy understanding of the nature of parameters in a problem-formulation process, they extend it to interactive fuzzy programming for bi-level linear programming problems with fuzzy parameters [5]. Interactive fuzzy programming can also be extended so as to manage decentralized bi-level linear programming problems by taking in to consideration individual satisfactory balance between the upper level DM and each of the lower level DMs as well as overall satisfactory balance between the two levels [7]. Moreover, by using some decomposition methods which take advantage of the structural features of the decentralized bi-level problems, efficient methods for computing satisfactory solutions are also developed [7, 8].

Recently, [9-11] considered the L-R fuzzy numbers and the lexicography method in conjunction with crisp linear programming and designed a new model for solving FFLP. The proposed scheme presented promising results from the aspects of performance and computing efficiency. Moreover, comparison between the new model and two mentioned methods for the studied problem shows a remarkable agreement and reveals that the new model is more reliable in the point of view of optimality. Also, an author in [12-15] has been proposed a new efficient method for FFLP, in order to obtain the fuzzy optimal solution with unrestricted variables and parameters. This proposed method is based on crisp nonlinear programming and has a simple structure.

Furthermore, several authors deal with the modeling and optimization of a bi-level multi-objective production planning problem, where some of the coefficients of objective functions and parameters of constraints are multi-choice. They has been used a general transformation technique based on a binary variable to transform the multi-choices parameters of the problem into their equivalent deterministic form [16-21].

In this study, we discuss a procedure for solving bi-level linear programming problems through linear FMP approach. In order to reach the optimal solution of bi-level linear programming problems, using fuzzy programming approach, the report contains section three chapters. In section two we describe the basic concept of fuzzy set, and linear programming using fuzzy approach. In section three the basic concept of bi level linear programming characteristics and general model of mathematical formulation of bi -level linear programming problems are presented. In section, four the procedure for solving bi-level linear programming problems and FMP solution approach are discussed.

2. Preliminary

2.1. Fuzzy Set Theory

Fuzzy set theory has been developed to solve problems where the descriptions of activities and observations are imprecise, vague, or uncertain. The term "fuzzy" refers to a situation where there are no well-defined boundaries of the set of activities or observations to which the descriptions apply. For example, one can easily assign a person 180 cm tall to the class of tall men". But it would be difficult to justify the inclusion or exclusion of a 173 cm tall person to that class, because the term "tall" does not constitute a well- defined boundary. This notion of fuzziness exists almost everywhere in our daily life, such as a "class of red flowers," a "class of good shooters," a "class of comfortable speeds for travelling," a "number close to 10,"etc.These classes of objects cannot be well represented by classical set theory. In classical set theory, an object is either in a set or not in a set. An object cannot partially belong to a set .In fuzzy set theory, we extend the image set of the characteristic function from the binary set $B = \{0,1\}$ which contains only two alternatives, to the unit interval $U = [0,1]$ which has an infinite number of alternatives. We even give the characteristic function a new name, the membership function, and a new symbol μ, instead of χ. The vagueness of language, and its mathematical representation and processing, is one of the major areas of study in fuzzy set theory.

2.2. Definition of Fuzzy and Crisp Sets

Definition 1. Let X be a space of points (objects) called universal or referential set .An ordinary (a crisp) subset A in X is characterized by its characteristic function X_A as mapping from the elements of X to the elements of the set $\{0,1\}$ defined by;

$$\mathrm{X_A}\ (\mathrm{x}) = \begin{cases} 1, & \text{if } \mathrm{x} \in \mathrm{A} \\ 0, & \text{if } \mathrm{x} \notin \mathrm{A} \end{cases}.$$

Where $\{0, 1\}$ is called a valuation set. However, in the fuzzy set t, the membership function will have not only 0 and 1 but also any number in between. This implies that if the valuation set is allowed to be the real interval $[0, 1]$, A is called a fuzzy set.

Definition 2. If X is a collection of objects denoted by x, then a fuzzy set A is a set of ordered pairs denoted by $A = \{(x, \mu_A(x)) \mid x \in X\}$. Where $\mu_A(x): X \to [0,1]$ is called membership function or degree of membership (degree of compatibility or degree of truth).

Definition 3. A fuzzy set A in a non empty set X is categorized by its membership function $\mu_A(x): X \to [0,1]$ and $\mu_A(x)$ is called the degree of membership of element x in fuzzy set A for each x is an element of X that makes values in the interval $[0, 1]$.

Definition 4. Let X be a universal set and A is a subset of X. A fuzzy set of A in x is a set of ordered pairs $A = \{(x, \mu_A(x)) \mid x \in X\}$ where, $\mu_A(x) \to [0,1]$ is called the membership function at x in membership, the value one is used to represent complete membership and value zero is used to represent intermediate degree of membership.

Example 1. let $X = \{a , b , c\}$ and define the fuzzy set A as follows:

$$\mu_A(a) = 1.0, \qquad \mu_A(b) = 0.7, \qquad \mu_A(c) = 0.4\,,$$

$$A = \{(a, 1.0), (b, 0.7), (c, 0.4)\}.$$

Note. The statement, $\mu_A(b) = 0.7$ is interpreted as saying that the membership grade of 'b' in the fuzzy set A is seven-tenths. i.e. the degree or grade to which b belongs to A is 0.7.

Definition 5. A fuzzy set $A = \emptyset$ if and only if it is identically zero on X.

Definition 6. If two fuzzy sets A andfuzzy set B are equal then $A = B$, if and only if $A(x) = B(x)$, $\forall x \in X$.

2.3. Fuzzy Linear Programming

Crisp linear programming is one of the most important operational research techniques. It is a problem of maximizing or minimizing a crisp objective function subject to crisp constraints (crisp linear-inequalities and/or equations). It has been applied to solve many real world problems but it fails to deal with imprecise data, that is, in many practical situations it may not be possible for the decision maker to specify the objective and/or the constraint in crisp manner rather he/she may have put them in ‘’fuzzy sense’’. So many researchers succeeded in capturing such vague and imprecise information by fuzzy programming problem. In this case, the type of the problem he/she put in the fuzziness should be specified, that means, there is no general or unique definition of fuzzy linear problems. The fuzziness may appear in a linear programming problem in several ways such as the inequality may be fuzzy (p1–FLP), the objective function may be fuzzy (P2-FLP) or the parameters c, A, b may be fuzzy (P3-FLP) and so on.

Definition 7. If an imprecise aspiration level is assigned to the objective function, then this fuzzy objective is termed as fuzzy goal. It is characterized by its associated membership function by defining the tolerance limits for achievement of its aspired level.

We consider the general model of a linear programming

$$\begin{aligned} & \max \ \ C^T x, \\ & \text{s.t.} \\ & A_i x \leq b_i \quad (i = 1,2,3, \dots m), \\ & x \geq 0, \end{aligned} \tag{1}$$

Where A_i is an n-vector C is an n-column vector and $x \in \mathbb{R}^n$.

To a standard linear programming ***problem (1)*** above, taking in to account the imprecision or fuzziness of a decision maker’s judgment, Zimmermann considers the following linear programming problem with a fuzzy goal (objective function) and fuzzy constraints.

$$\begin{aligned} & C^T x \lesssim Z_0, && \text{(1a)} \\ & A_i x \lesssim b_i \quad (i = 1,2,3, \dots m), && \text{(1b)} \\ & x \geq 0. \end{aligned} \tag{2}$$

Where the symbol $\lesssim$ denotes a relaxed or fuzzy version of the ordinary inequality $<$. From the decision maker’s preference, the fuzzy goal (1a) and the fuzzy constraints (1b) mean that the objective function $C^T x$ should be “essentially smaller than or equal to” a certain level Z_0, and that the values of the constraints AX should be “essentially smaller than or equal to” b, respectively. Assuming that the fuzzy goal and the fuzzy constraints are equally important, he employed the following unified formulation.

$Bx \lesssim b'$,

$x \geq 0$.

Where $B = \begin{bmatrix} C \\ A_i \end{bmatrix}$ and $b' = \begin{bmatrix} Z_0 \\ b_i \end{bmatrix}$.

Definition 8. Fuzzy decision is the fuzzy set of alternatives resulting from the intersection of the fuzzy constraints and fuzzy objective functions. Fuzzy objective functions and fuzzy constraints are characterized by their membership functions.

2.4. Solution Techniques of Solving Some Fuzzy Linear Programming Problems

The solution techniques for fuzzy linear programming problems follow the following procedure. We consider the following linear programming problem with fuzzy goal and fuzzy constraints (the coefficients of the constraints are fuzzy numbers).

Where $\widetilde{a_{ij}}$ and $\widetilde{b_i}$ are fuzzy numbers with the following linear membership functions:

$$\mu_{ij}(x) = \begin{cases} 1, & \text{if } x \leq a_{ij}, \\ \dfrac{a_{ij} + d_{ij} - x}{d_{ij}}, & \text{if } a_{ij} < x < a_{ij} + d_{ij}, \\ 0, & \text{if } x \geq a_{ij} + d_{ij}. \end{cases}$$

$$\mu_{\widetilde{b_i}}(x) = \begin{cases} 1, & \text{if } x \leq b_i, \\ \dfrac{b_i + p_i - x}{p}, & \text{if } b_i < x < b_i + p_i, \\ 0, & \text{if } x \geq b_i + p_i. \end{cases}$$

and $x \in R, dij > 0$ is the maximum tolerance for the corresponding constraint coefficients and p_i is the maximum tolerance for the i^{th} constraint. For defuzzification of the problem, we first fuzzify the objective function. This is done by calculating the lower and upper bounds of the optimal values. These optimal values z_l and z_u can be defined by solving the following standard linear programming problems, for which we assume that both of them have finite optimal values.

$$z_1 = \max \sum_{j=1}^{n} c_j x_j,$$

$$s.t.$$

$$\sum_{j=1}^{n} (a_{ij} + d_{ij}) x_j \leq b_i, \; 1 \leq i \leq m) \; x_j \geq 0, \tag{3}$$

and

$$z_2 = \max \sum_{j=1}^{n} c_j x_j \,,$$

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i + p_i, \ \ 1 \leq i \leq m)x_j \geq 0.$$

Let $z_l = min(z_1, z_2)$ and $z_u = max(z_1, z_2)$. The objective function takes values between z_l and z_u while the constraint coefficients take values between a_{ij} and $a_{ij} + d_{ij}$and the right-hand side numbers take values between b_i and $b_i + p_i$ .Then, the fuzzy set optimal values, G, which is a subset of R^n is defined by:

$$\mu_G(x) = \begin{cases} 0, & \text{if} \sum_{j=1}^{n} c_jx_j \leq z_l, \\ \dfrac{\sum_{j=1}^{n} c_jx_j - z_l}{z_u - z_l}, & \text{if}\, z_l < \sum_{j=1}^{n} c_jx_j \leq z_u. \\ 1, & \text{if} \sum_{j=1}^{n} c_jx_j \geq z_u. \end{cases}$$

The fuzzy set of the $i^{\ th}$ constraint, C_i , which is a subset of R^n is defined by:

$$\mu_{ci}(x) = \begin{cases} 0, & \text{if}\, b_i \leq \sum_{j=1}^{n} a_{ij}x_j \,, \\ \dfrac{b_i - \sum_{j=1}^{n} a_{ij}x_j}{\sum_{j=1}^{n} d_{ij}x_j + p_i}, & \text{if} \sum_{j=1}^{n} a_{ij}x_j < b_i < \sum_{j=1}^{n} \left(a_{ij}x_j + d_{ij}\right)x_j + p_i. \\ 1, & \text{if}\ b \geq \sum_{j=1}^{n} \left(a_{ij}x_j + d_{ij}\right)x_j + p_i. \end{cases}$$

Using the above membership functions $\mu_{ci}(x)$ and $\mu_G(x)$and following Bellmann and Zadeh approach, we construct the membership function $\mu_D(x)$as follows.

$$\mu_D(x) \ = \ \text{mini}(\mu_G(x), \mu_{ci}(x)).$$

Where $\mu_D(x)$ is the membership function of the fuzzy decision set. The min. section is selected as the aggregation operator. Then the optimal decision x^* is the solution of

$$x^* = \arg(\max \text{mini}\{\mu_G(x), \mu_{ci}(x)\}.$$

Then, ***problem (1)*** is reduced to the following crisp problem by introducing the auxiliary variable λ which indicates the common degree of satisfaction of both the fuzzy constraints and objective function.

$$\max \lambda,$$

$$\text{s.t.}$$

$$\mu_G(x) \geq \lambda,$$

$$\mu_{ci}(x) \geq \lambda,$$

$$x \geq 0, 0 \leq \lambda \leq 1,\ 1 \leq i \leq m.$$

This problem is equivalent to the following non-convex optimization problem

$$\max \lambda,$$

$$\lambda(z_1 - z_2) - \sum_{j=1}^{n} c_j x_j - z_1 \leq 0,$$

$$\sum_{j=1}^{n} \left(a_{ij} + \lambda d_{ij}\right) x_j + \lambda p_i - b_i \leq 0\,,$$

$$x \geq 0, 0 \leq \lambda \leq 1, \qquad 1 \leq i \leq m\,.$$

Which contains the cross product terms λx_j that makes non- convex. Therefore, the solution of this problem requires the special approach such as fuzzy decisive method adopted for solving general non-convex optimization problems. Here solving the above linear programming problem gives us an optimum $\lambda^* \in [0,1]$. Then the solution of the problem is any $x \geq 0$ satisfying the problem constraint with $\lambda = \lambda^*$.

3. Bi-Level Programming

3.1. Basic Definitions

3.1.1. Decision making

Decision making is a process of choosing an action (solution) from a set of possible actions to optimize a given objective.

3.1.2. Decision making under multi objectives

In most real situation a decision maker needs to choose an action to optimize more than one objective simultaneously. Most of these objectives are usually conflicting. For example, a manufacturer wants to increase his profit and at the same time want to produce a product with better quality. Mathematically a multi objective optimization with k objectives, for a natural number $K > 1$, can be given as:

$$\max F(x) = (f_1(x), \qquad f_2(x), \ldots, f_k(x))\,,$$

$$\text{s.}t.$$

$$x \in S \subseteq \mathbb{R}^n.$$

3.1.3. Hierarchical decision making

An optimization problem which has other optimization problems in the constraint set and has a decision maker for each objective function controlling part of the variables is called multi-level optimization problem. If there are only two nested objective functions then it is called a bi-level optimization problem. The decision maker at the first level, with objective function f_1, is called the leader and the other decision makers are called the followers. A solution is supposed to fulfill all the feasibility conditions and optimize each objectives it is uncommon to find a solution which makes all the decision makers happy. Hence to choose an action the preference of the decision makers for all the levels or objectives play a big role.

3.1.4. Bi-level programming (BLP)

is a mathematical programming problem that solves decentralized planning problems with two DMs in a two level or hierarchical organization. It has been studied extensively since the 1980s. It often represents an adequate tool for modeling non-cooperative hierarchical decision process, where one player optimizes over a subset of decision variables, while taking in to account the independent reaction of the other player to his or course of action. In the real world, we often encounter situations where there are two or more decision makers in an organization with a hierarchical structure, and they make decisions in turn or at the same time so as to optimize their objective functions. In particular, consider a case where there are two decision makers; one of the decision makers first makes a decision, and then the other who knows the decision of the opponent makes a decision. Such a situation is formulated as a bi-level programming problem. We call the decision maker who first makes a decision the leader, and the other decision maker the follower. For bi-level programming problems, the leader first specifies (decides) a decision and then the follower determines a decision so as to optimize the objective function of the follower with full knowledge of the decision of the leader. According to this rule, the leader also makes a decision so as to optimize the objective function of self. This decision making process is extremely practical to such decentralized systems as agriculture, government policy, economic systems, finance, warfare, transportation, network designs, and is especially for conflict resolution.

Bi-level programming is particularly appropriate for problems with the following characteristics:

- **Interaction:** Interactive decision-making units within a predominantly hierarchical structure.
- **Hierarchy:** Execution of decision is sequential, from upper to lower level.
- **Full information:** Each DM is fully informed about all prior choices when it is his turn to move.
- **Nonzero sum:** The loss for the cost of one level is unequal to the gain for the cost of the other level. External effect on a DM's problem can be reflected in both the objective function and the set of feasible decision space.
- Each DM controls only a subset of the decision variables in an organization.

3.2. Mathematical Formulation of a Bi-Level Linear Programming Problem (BLPP)

For the bi-level programming problems, the leader first specifies a decision and then the follower determines a decision so as to optimize the objective function of self with full knowledge of the decision of the leader.

According to this rule, the leader also makes a decision so as to optimize the objective function of self. The solution defined as the above mentioned procedure is a stackelberg solution.

A bi-level LPP for obtaining the stackelberg solution is formulated as:

$$\max z_1(x_1, x_2) = c_1x_1 + d_1x_2,$$

$$x_1. \tag{4}$$

Where x_2 solves

$$\max z_2(x_1, x_2) = c_2x_1 + d_2x_2,$$

$$x_2,$$

$$\text{s.t.}$$

$$Ax_1 + Bx_2 \leqq b.$$

Where $c_i, i = 1,2$ are n_1-dimensional row coefficient vector$d_i, i = 1,2,$ are n_2- dimensional row coefficient vector, A is an mxn1 coefficient matrix, B is a mxn_2 coefficient matrix, b-is an m-dimensional column constant vector. In the bi-level linear programming problem above$z_1(x_1, x_2)$, and $z_2(x_1, x_2)$represent the objective functions of the leader and the follower, respectively, and x_1 and x_2 represent the decision variables of the leader and the follower respectively. Each decision maker knows the objective function of self and the constraints. The leader first makes a decision, and then the follower makes a decision so as to maximize the objective function with full knowledge of the decision of the leader. Namely, after the leader chooses x_1, he solves the following linear programming problem:

$$\max z_2(x_1, x_2) = c_2x_1 + d_2x_2,$$

$$x_2, \tag{5}$$

$$\text{s.t.}$$

$$Bx_2 \leq b - Ax_1,$$

$$x_2 \geqq 0.$$

And chooses an optimal solution $x_2(x_1)$ to the problem above as a rational response. Assuming that the follower chooses the rational response, the leader also makes a decision such that the objective function $z_1(x_1, x_2(x_1))$ is maximized.

3.3. BLP Problem Description

The linear bi-level programming problem is similar to standard linear programming, except that the constraint region is modified to include a linear objective function constrained to be optimal with respect to one set of variables. The linear BLPP characterized by two planners at different hierarchical levels each independently controlling only a set of decision variables, and with different conflicting objectives. The lower- level executes its policies after and in view of, the decision of the higher level , and the higher level optimizes its objective independently which is usually affected by the reactions of the lower level. Let the control over all real-valued decision variables in the vector $x = (x_1^1, x_1^2, \dots, x_1^{N(1)}, x_2^1, x_2^2, \dots, x_2^{N(2)})$ be partitioned between two planners ,hereafter known as level-one(the superior or top planner) and level-two(the inferior or bottom planner).Assume that the level-one has control over the vector $x = (x_1^1, x_1^2, \dots, x_1^{N(1)})$, the first $N(1)$ components of the vector x, and that the level-two has control over the vector $x = (x_2^1, x_2^2, \dots, x_2^{N(2)})$ the remaining $N(2)$ components .Further, assume that $f_1, f_2 : R^{N(1)} x\, R^{N(2)} \to R^1$ linear. Then, the linear BLPP can be formulated as:

$$\max_{x_1} f_1(x_1, x_2) = c_1x_1 + d_1x_2, \tag{6}$$

Where x_2 solves

$$\max_{x_2} f_2(x_1, x_2) = c_2x_1 + d_2x_2, \quad \text{s.t.}\ (x_1, x_2) \in S. \tag{7}$$

Where $S \subseteq R^{N(1)+N(2)}$ is the feasible choices of (x_1, x_2), and is closed and bounded. For any fixed choice of x_1, level-two will choose a value of x_2 to maximize the objective function $f_1(x_1, x_2)$. Hence, for each feasible value of x_1, level-two will react with a corresponding value of x_2. This induces a functional reaction ship between the decisions of level-one and the reactions of level-two. Say, $x_2 = W(x_1)$ .We will assume that the reaction function, $W(.)$, is completely known by level one.

Definition 9. The set $Wf_2(S)$ given by $Wf_2(S) = \{(x_1^*, x_2^*) \in S : f_2(x_1^*, x_2^*) = max\ f_2(x_1^*, x_2^*)$ is the set of rational reactions of $f_2$over S. Hence level-one is really restricted to choosing a point in the set of rational reactions of f_2 over S. So, if level-one wishes to maximize its objective function, $f_1(x_1, x_2)$,by controlling only the vector x_1, it must solve the following mathematical programming problem:

$$\max f_1(x_1, x_2), \quad \text{s.t.}\ (x_1, x_2) \in Wf_2(S). \tag{8}$$

For convenience of notation and terminology, we will refer to $S^1 = Wf_2(S)$ as the level-one feasible region or in general, the feasible region, and $S^1 = S$ as the level two feasible regions.

The following are the basic concepts of the bi-level linear programming problem of ***Eq. 3***:

The feasible region of the bi-level linear programming problem:

$$S = \{(x_1, x_2) : Ax_1 + Bx_2 \leqq b\}.$$

The decision space (feasible set) of the follower after x_1 is specified by the leader:

$$S(x_1) = \{x_2 \geqq 0 :\ Bx_2 < b - Ax_1, x_1 \geqq 0\}.$$

The decision space of the leader:

$$S_x = \{x_1 \geqq 0 \text{ there is an } x_2 \text{ such that } Ax_1 + Bx_2 \leqq b, x_2 \geqq 0\}.$$

The set of rational responses of the follower for x_1 specified by the leader

$$R(x_1) = \begin{cases} x_2 \geqq 0: x_2 \in \arg\max z_1(x_1, x_2) \\ \qquad x_2 \in S(x_1) \end{cases}.$$

Inducible region:

$$IR = \{(x_1, x_2): (x_1, x_2) \in S, x_2 \in R(x_1)\}.$$

Stackleberg solution:

$$\{(x_1, x_2): (x_1, x_2) \in \arg\max z_1(x_1, x_2), (x_1, x_2) \in R(x_1)\}.$$

Computational methods for obtaining stackelberg solution to bi-level linear programming problems are classified roughly in to three categories. These are;

The vertex enumeration approach [2]. This takes advantage of the property that there exists a stackelberg solution in a set of extreme points of the feasible region. The solution search procedure of the method starts from the first best point namely an optimal solution to the upper level problem which is the first best solution, is computed, and then it is verified whether the first best solution is also an optimal solution to the lower level problem. If the first best point is not the stackelberg solution, the procedure continues to examine the second best solution to the problem of the upper level, and so forth.

The Kuhn-Tucker approach. In this approach, the leader's problem with constraints involving the optimality conditions of the follower's problem is solved.

The penalty function approach. In this approach, a penalty term is appended to the objective function of the leader so as to satisfy the optimality of the follower's problem.

Fuzzy approach:-that will be discussed in detail under the next chapter

4. Fuzzy Approach to Bi-Level Linear Programming Problems

4.1. Fuzzy Bi-Level Linear Programming

As discussed under chapter two, a bi-level linear programming problem is formulated as:

$$\begin{aligned} &\max f_1(x_1, x_2) = c_{11}x_1 + c_{12}x_2, \\ &x_1. \\ &\text{Where } x_2 \text{ solves} \\ &\max f_2(x_1, x_2) = c_{21}x_1 + c_{22}x_2, \\ &\text{s.t.} \\ &A_1x_1 + A_2x_2 \le b, \\ &(x_1, x_2) \ge 0. \end{aligned} \tag{9}$$

Where $x_i, i = 1,2$ is an n_i-dimensional decision variable column vector ;

$C_{i1}, i = 1,2$ is an n_1-dimensional constant column vector;

$C_{i2}, i = 1,2$ is an n_2-dimensional constant column vector;

b-is an m-dimensional constant column vector, and

$A_i, i = 1,2$ is an mxni coefficient matrix.

For the sake of simplicity, we use the following notations:

$X = (x_1, x_2) \in R^{n_1+n_2}, C_i = (C_{i1}, C_{i2}), i = 1,2$ and $A = [A_1, A_2]$ and Let DM_1 denotes the decision maker at the upper level and DM_2 denotes the decision maker at the lower level. In the bi-level linear programming problem (7) above, $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$represent the objective functions of DM_1 and DM_2 respectively; and x_1 and x_2 represent the decision variables of DM_1 and DM_2 respectively.

Instead of searching through vertices as the k^{th} best algorithm, or the transformation approach based on Kuhn-Tucker conditions, we here introduce a supervised search procedure (supervised by DM_1) which will generate (non dominated) satisfactory solution for a bi-level programming problem. In this solution search, DM_1 specifies(decides) a fuzzy goal and a minimal satisfactory level for his objective function and decision vector and evaluates a solution proposed by DM_2, and DM_2 solves an optimization problem, referring to the fuzzy goal and the minimal satisfactory level of DM_1. The DM_2 then presents his/her solution to the DM_1. If the DM_1 agrees to the proposed solution, a solution is reached and it is called a satisfactory solution here. If he/she rejects this proposal, then DM_1 will need to re-evaluate and change former goals and decisions as well as their corresponding leeway or tolerances until a satisfactory solution is reached. It is natural that decision makers have fuzzy goals for their objective functions and their decision variables when they take fuzziness of human judgments in to consideration .For each of the objective functions $f_i(x)$, $i = 1,2$, assume that the decision makers have fuzzy goals such as "the objective function $f_i(x)$ should be substantially less than or equal to some value q_i " and the range of the decision on x_i , $i = 1,2$,should be " around $x_i^*$with its negative and positive –side tolerances $p_i^-$and p_i^+,respectively.

We obtain optimal solution of each DM_1 and DM_2 calculated in isolation. If the individual optimal solution x_i^0 , $i =$ 1.2; are the same then a satisfactory solution of the system has been attained. But this rarely happens due to conflicting objective functions of the two DMs. The decision-making process then begins at the first level. Thus, the first-level DM provides his preferred ranges for $f_1$and decision vector x_1 to the second level DM. This information can be modeled by fuzzy set theory using membership functions.

4.2. Fuzzy Programming Formulation of BLPPs

To formulate the fuzzy programming model of a BLPP, the objective functions f_i, $(i = 1,2)$ and the decision vectors x_i, $(i = 1,2)$ would be transformed in to fuzz goals by means of assigning an aspiration level (the optimal solutions of both of the DMs calculated in isolation can be taken as the aspiration levels of their associated fuzzy goals) to each of them. Then, they are to be characterized by the associated membership functions by defining tolerance limits for achievement of the aspired levels of the corresponding fuzzy goals.

4.3. Fuzzy Programming Approach for Bi-Level LPPs

In the decision making context, each DM is interested in maximizing his or her own objective function, the optimal solution of each DM when calculated in isolation would be considered as the best solution and the associated objective value can be considered as the aspiration level of the corresponding fuzzy goal because both the DMs are interested of maximizing their own objective functions over the same feasible region defined by the system of constraints. Let x_i^B be the best (optimal) solution of the i^{th} level DM. It is quite natural that objective values which are equal to or larger than $f_i^B = f_i(x_i^B) = max\ f_i(x), i = 1,2., x \in S$ should be absolutely satisfactory to thei^{th} level DM. If the individual best (optimal) solution $x_i^B, i = 1,2$ are the same, then a satisfactory optimal solution of the system is reached. However, this rarely happens due to the conflicting nature of the objectives. To obtain a satisfactory solution, higher level DM should give some tolerance (relaxation) and the relaxation of decision of the higher level DM depends on the needs, desires and practical situations in the decision making situation .Then the fuzzy goals take the form $f_i(x) \lesssim f_i(x_i^B),\ i = 1,2, x_i \cong x_i^B$.

To build membership functions, goals and tolerance should be determined first. However, they could hardly be determined without meaningful supporting data. Using the individual best solutions, we find the values of all the objective functions at each best solution and construct a payoff matrix

$$\begin{bmatrix} & f_1(x) & f_2(x) \\ x_1^0 & f_1(x_1^0) & f_2(x_1^0) \\ x_2^0 & f_1(x_2^0) & f_2(x_2^0) \end{bmatrix}.$$

The maximum value of each column $\left(f_i(x_i^0)\right)$ gives upper tolerance limit or aspired level of achievement for the ith objective function where $f_i^u = f_i(x_i^0) = \max f_i(x_i^0), i = 1,2$.

The minimum value of each column gives lower tolerance limit or lowest acceptable level of achievement for the i[th] objective function where $f_i^L = min\ f_i(x_i^0), i = 1.2$. For the maximization type objective function, the upper tolerance limit $f_t^u, t = 1,2$, are kept constant at their respective optimal values calculated in isolation but the lower tolerance limit f_i^L are changed. The idea being that $f_i(x) \to f_t^u$, then the fuzzy objective goals take the form $f_i(x) \lesssim f_i(x_i^u), i = 1,2$. And the fuzzy goal for the control vector x_i is obtained a $x_i \cong x_i^u$. Now, in the decision situation, it is assumed that all DMs that are up to i^{th} motivation to cooperate each other to make a balance of decision powers, and they agree to give a possible relaxation of their individual optimal decision. The i^{th} level DM must adjust his/her goal by assuming the lowest acceptable level of achievement f_i^L based on indefiniteness of the decentralized organization. Thus, all values of $f_i(x) \geq f_t^u$are absolutely acceptable (desired) to objective function $f_i(x)$ satisfactory to the ith level DM. All values o$f_i(x)$f with $f_i(x) \leq f_t^L$ are absolutely unacceptable (undesired) to the objective function $f_i(x)$ for $i = 1,2$. Based on this interval of tolerance, we can establish the following linear membership functions for the defined fuzzy goals as ***Fig.1*** below.

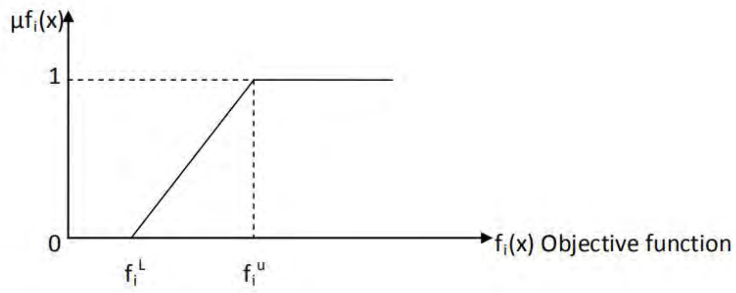

Fig. 1. Membership function of maximization-type objective function.

$$\mu_i(f_i(x)) = \begin{cases} 1, & \text{if } f_i(x) \geq f_i^u, \\ \dfrac{f_i(x) - f_i^L}{f_i^u \geq f_i^L}, & \text{if } f_i^L \leq f_i(x) \leq f_i^u, i = 1,2 \\ 0, & \text{if } f_i(x) \leq f_i^L. \end{cases} \tag{10}$$

By identifying the membership functions $\mu_1(f_1(x))$and $\mu_2(f_2(x))$for the objective functions $f_1(x)$ and $f_2(x)$, and following the principle of the fuzzy decision by Bellman and Zadeh, the original bi-level linear programming ***problem (9)*** can be interpreted as the membership function maxmin problem defined by:

$$\max \ \min\{\mu_i(f_i(x)), \ i = 1,2\},$$

$$\text{s.t.} \tag{11}$$

$$A_1x_1 + A_2x_2 \leq b, \ x_1, x_2 \geq 0.$$

Then the linear membership functions for decision vector x_1 can be formulated as:

$$\mu_{x1}(f_1(x)) = \begin{cases} \dfrac{x_1 - (x_1^0 - e_1^-)}{e_1^-}, & \text{if } x_1^0 - e_1^- \leq x_1 \leq x_1^0 \\ \dfrac{(x_1^0 + e_1^+) - x_1}{e_1^+}, & \text{if } x_1^0 \leq x_1 \leq (x_1^0 + e_1^+) \\ 0, & \text{if otherwise.} \end{cases} \tag{12}$$

Where x_1^0 is the optimal solution of first level DM;

e_1^- the negative tolerance value on x_1;

e_1^+ the positive tolerance value on x_1.

To derive an overall satisfactory solution to the membership function maximization ***problem (11)***, we first find the maximizing decision of the fuzzy decision proposed by Bellman and Zadeh [22]. Namely, the following problem is solved for obtaining a solution which maximizes the smaller degree of satisfaction between those of the two decision makers:

$$\max \min\{\mu_1(f_1(x)),\ \mu_2(f_2(x)),\ \mu_{x1}(x_1)\},$$
$$\text{s.t.} \tag{13}$$
$$A_1x_1 + A_2x_2 \le b,\ \ x_1, x_2 \ge 0.$$

By introducing an auxiliary variable λ, this problem can be transformed into the following equivalent problem:

$$\max \lambda,$$
$$\text{s.t. } \mu_1(f_1(x)) \ge \lambda,$$
$$\mu_2(f_2(x)) \ge \lambda, \tag{14}$$
$$\mu_{x1}(x_1) \ge \lambda,$$
$$A_1x_1 + A_2x_2 \le b,\ \ x_1, x_2 \ge 0.$$

Solving ***problem (14)***, we can obtain a solution which maximizes the smaller satisfactory degree between those of both decision makers. It should be noted that if the membership function$\mu_i(f_i(x)), i = 1.2$ are linear membership functions such as ***Eq. (10)***, ***problem (14)*** becomes a linear programming problem. Let x^*denotes an optimal solution to ***problem (14)***. Then we define the satisfactory degree of both decision makers under the constraints as

$$\lambda^* = \min\{\mu_1(f_1(x^*)),\qquad \mu_2(f_2(x^*))\} \tag{15}$$

$\mu_1(f_1(x))$If DM1 is satisfied with the optimal solution x^*, it follows that the optimal solution x^*becomes a satisfactory solution; however DM1 is not always satisfied with the solution x^*. It is quite natural to assume that DM1 specifies (decides) the minimal satisfactory level $\delta \in [0,1]$ for his membership function subjectively. Consequently, DM2 optimizes his objective under the new constraints as the following problem:

$$\max \mu_2(f_2(x)),$$
$$\text{s.t.} \tag{16}$$
$$\mu_1(f_1(x)) \le \delta$$
$$A_1x_1 + A_2x_2 \le b,\ \ x_1, x_2 \ge 0.$$

If an optimal solution to ***problem (16)*** exists, it follows that DM1 obtains a satisfactory solution having a satisfactory degree larger than or equal to the minimal satisfactory level specified (decided) by DM1's own self. However, the larger the minimal satisfactory level is assessed, the smaller DM2's satisfactory degree becomes. Consequently, a relative difference between the satisfactory degrees of DM1 and DM2 becomes larger than it is feared that overall satisfactory balance between both levels cannot be maintained. To take account of overall satisfactory balance between both levels, DM1 needs to compromise (agree) with DM2 on DM1' s own minimal satisfactory level. To do so, the following ratio of the satisfactory degree of DM2 to that of DM1 is defined as:

$$\Delta = \frac{\mu_2(f_2(x^*))}{\mu_1(f_1(x^*))}. \tag{17}$$

This is originally introduced by Lai [6].

Let $\Delta > \Delta^L$ denote the lower bound and the upper bound of Δ specified by DM1. If $\Delta > \Delta^U, i.e\ \mu_2(f_2(x^*)) > \Delta^U \mu_1(f_1(x^*))$, then DM1 updates (improves) the minimal satisfactory level δ by increasing δ . Then DM1 obtains a larger satisfactory degree and DM2 accepts a smaller satisfactory degree. Conversely, if $\Delta > \Delta^L, i.e\ \mu_2(f_2(x^*)) < \Delta^i \mu_1(f_1(x^*))$, then DM1 updates the minimal satisfactory level δ by decreasing δ, and DM1 accepts a smaller satisfactory degree and DM2 obtains a larger satisfactory degree.

At an iteration l , let $\mu_1(f_1(x^l))$, $\mu_2(f_2(x^l)), \lambda^l$ and $\Delta^l = \frac{\mu_2(f_2(x_1^l))}{\mu_1(f_1(x^l))}$ denote DM1's and DM2's satisfactory degrees, a satisfactory degree of both levels and the ratio of satisfactory degrees between both DMs, respectively, and let a corresponding solution be l^x at the iteration. The iterated interactive process terminates if the following two conditions are satisfied and DM1 concludes the solution as a satisfactory solution.

4.3.1. Termination conditions of the interactive processes for bi-level linear programming problems

DM1's satisfactory degree is larger than or equal to the minimal satisfactory level δ specified by DM1, i.e. $\mu_1(f_1(x^l)) \geq \delta$.

The ratio Δ^l of satisfactory degrees lies in the closed interval between the lower and upper bounds specified by DM1, i.e. $\Delta^l \in [\Delta\, min, \Delta max]$.

Condition (i) is DM1's required condition for solutions, and Condition (ii) is provided in order to keep overall satisfactory balance between both levels. Unless the conditions are satisfied simultaneously, DM1 needs to update the minimal satisfactory level δ.

Procedure for updating the minimal satisfactory level δ.

If Condition (i) is not satisfied, then DM1 decreases the minimal satisfactory level by δ.

If the ratio Δ^l exceeds its upper bound, then DM1 increases the minimal satisfactory level δ. Conversely, if the ratio Δ^l is below its lower bound, then DM1 decreases the minimal satisfactory level δ.

4.4. Algorithm of Interactive Fuzzy Programming for BLPPs.

Step 1. Find the solution of the first level and second level independently with the same feasible set given.

Step 2. Do these solutions coincide?

-If yes, an optimal solution is reached.

-If No, go to Step 3.

Step 3. Define a fuzzy goal, construct a payoff matrix, and then find upper tolerance limit f_t^u and lower tolerance limit f_t^L.

Step 4. Build member ship functions for maximization objective functions $\mu f_i(f_i(x))$ and decision vector x_1 using ***Eq. (8)*** and ***(10)***, respectively.

Step 5. set $\ell = 1$ and solve the auxiliary ***problems (14)***. If DM1 is satisfied with the optimal solution, the solution becomes a satisfactory solution x^*. Otherwise, ask DM1 to specify (decide) the minimal satisfactory level δ together with the lower and the upper bounds $[\Delta\, min, \Delta\, max]$ of the ratio of satisfactory degrees Δ^l with the satisfactory degree $\lambda^*$of both decision makers and the related information about the solution in mind.

Step 6. Solve ***problem (16)***, in which the satisfactory degree of DM1 is maximized under the condition that the satisfactory degree of DM1 is larger than or equal to the minimal satisfactory level δ, and then an optimal solution x^l to ***problem (16)*** is proposed to DM1 together with λ^l, $\mu_1(f_1(x^l))$, $\mu_2(f_2(x^l))$ and Δ^l.

Step 7. If the solution x^l satisfies the termination conditions and DM1 accepts it, then the procedure stops, and the solution x^l is determined to be a satisfactory solution.

Step 8. Ask DM1 to revise the minimal satisfactory level δ in accordance with the procedure for updating minimal satisfactory level. Return to Step 7.

Example 2. Solve (Linear BLPP)

$$\max f_1(x) = 5x_1 + 6x_2 + 4x_3 + 2x_4,$$

$$x_1, x_2.$$

$$\text{Where } x_3, x_4 \text{ solves} \tag{18}$$

$$\max f_2(x) = 8x_1 + 9x_2 + 2x_3 + 4x_4,$$

$x_3, x_4,$

$$
\begin{aligned}
&\text{s.t.}\\
&3x_1 + 2x_2 + x_3 + 3x_4 \le 40,\\
&x_1 + 2x_2 + x_3 + 2x_4 \le 30,\\
&2x_1 + 4x_2 + x_3 + 2x_4 \le 35,\\
&x_1, x_2, x_3, x_4 \ge 0.
\end{aligned}
$$

Solution.

Step 1. Find the solution of the top-level and lower-level independently with the same feasible set. i.e.

$$
\begin{aligned}
&\max f_1(x) = 5x_1 + 6x_2 + 4x_3 + 2x_4,\\
&\text{s.t.}\\
&3x_1 + 2x_2 + x_3 + 3x_4 \le 40,\\
&x_1 + 2x_2 + x_3 + 2x_4 \le 30,\\
&2x_1 + 4x_2 + x_3 + 2x_4 \le 35,\\
&x_1, x_2, x_3, x_4 \ge 0.
\end{aligned} \tag{19}
$$

Then we find the optimal solution

$f_1 = 125$ at $x_1^0 = (5, 0, 25,0)$;

$f_2 = 118.125$ at $x_2^0 = (11.25, 3.125, 0,0)$;

But this is not a satisfactory solution (since $x_1^0 \neq x_2^0$)

Step 2. Define fuzzy goals, construct the payoff matrix and we need to find the upper and lower tolerance limit.

Objective function as: $f_1 \lesssim 125, f_2 \lesssim 118.125$;

Decision variables as: $x_1 \cong 5, x_2 \cong 0$;

$$
\text{Payoff matrix}=\begin{bmatrix} & f_1(x_1^0) & f_2(x_2^0) \\ x_1^0 & 125 & 90 \\ x_2^0 & 75 & 118.125 \end{bmatrix};
$$

Upper tolerance limits are $f_1^u = 125, f_2^u \lesssim 118.125$;

Lower tolerance limits are $f_1^L = 75, f_2^L \lesssim 90$.

Step 3. Build membership functions for:

Objective functions as:

$$\mu f_1\big(f_1(x)\big) = \begin{cases} 1, & \text{if } f_1(x) \geq 125 \\ \dfrac{f_1(x) - 75}{125 - 75}, & \text{if } 75 \leq f_1(x) \leq 125 \\ 0, & \text{if } f_1(x) \leq 75 \end{cases}.$$

Decision variable function as

$$\mu f_2\big(f_2(x)\big) = \begin{cases} 1, & \text{if } f_2(x) \geq 118.125 \\ \dfrac{f_2(x) - 90}{118.125 - 90}, & \text{if } 90 \leq f_2(x) \leq 119.125. \\ 0, & \text{if } f_2(x) \leq 90 \end{cases}$$

Let the upper level DM specifies (decides) $x_1 = 5$ with 2.5 (negative) and 2.5 (positive) tolerances and $x_2 = 0$ with 0 (negative) and 3 (positive) tolerance values.

$$\mu x_1(x_1) = \begin{cases} \dfrac{x_1 - (5 - 2.5)}{2.5}, & \text{if } 2.5 \leq x_1 \leq 5 \\ \dfrac{(5 + 2.5) - x_1}{2.5}, & \text{if } 5 \leq x_1 \leq 7.5' \\ 0, & \text{otherwise} \end{cases}$$

$$\mu x_2(x_2) = \begin{cases} x_2, & \text{if } x_2 \leq 3 \\ \dfrac{3 - x_2}{3}, & \text{if } 0 \leq x_2 \leq 3 \\ 0, & \text{otherwise} \end{cases}.$$

Step 4. Solve the auxiliary problem

$$\begin{aligned} &\max \lambda, \\ &\text{s.t.} \\ &\mu f_1\big(f_1(x)\big) \geq \lambda, \\ &\mu f_2\big(f_2(x)\big) \geq \lambda, \\ &\mu x_1(x_1) \geq \lambda, \\ &3x_1 + 2x_2 + x_3 + 3x_4 \leq 40, \\ &x_1 + 2x_2 + x_3 + 2x_4 \leq 30, \\ &2x_1 + 4x_2 + x_3 + 2x_4 \leq 35, \\ &x_1, x_2, x_3, x_4 \geq 0. \end{aligned} \tag{20}$$

The result of the first iteration including an optimal solution to the problem is:

$x_1^1 = 6.41, x_2^1 = 1.95, x_3^1 = 10.52, x_4^1 = 1.42,$
and

$\lambda^1 = 0.316,$

$f_1^1(x) = 88.67, f_2^1(x) = 95.55, \mu_1\big(f_1(x)\big) = 0.2734.$

Suppose that DM1 is not satisfied with the solution obtained in iteration 1, and then let him specify (decide) the minimal satisfactory level at $\delta = 0.3$ and the bounds of the ratio at the interval $[\Delta min, \Delta\, max] = [0.3, 0.4]$, taking account of the result of the first iteration. Then, the problem with the minimal satisfactory level is written as:

$$\begin{aligned} &\max \mu f_2(f_2(x)),\\ &\text{s.t.}\\ &\mu f_1(f_1(x)) \geq 0.3,\\ &x \in S. \end{aligned} \tag{21}$$

Applying simplex algorithm, the result of the second iteration including an optimal solution to ***problem (21)*** is

$$\begin{aligned} &x_1^2 = 6.71, x_2^2 = 2.05, x_3^2 = 10.52, x_4^2 = 1.42,\\ &\text{and}\\ &\lambda^2 = 0.316,\\ &f_1^2(x) = 90.77, f_2^2(x) = 98.85, \mu_{f1}\big(f_1(x)\big) = 0.3154,\\ &\text{and}\\ &\Delta^2 = 0.3165. \end{aligned} \tag{22}$$

Therefore, this solution satisfies the termination conditions.

5. Conclusion

The fuzzy mathematically programming approach is simple to implement, interactive and applicable to BLPP. The satisfactory solution obtained is realistic. We can take any membership function other than linear. The results will hold good, however, the problem will become a non linear programming problem. We observe that even though the decision making process is from higher to lower level, the lower level becomes the most important. This is because the decision vector under the control of the lower level DM is not given any tolerance limits. Hence this decision vector either remains unchanged or closest to its valued obtained in isolation. But at higher level, the decision vectors are given some tolerance and hence they are free to move within the tolerance limits. The tolerance levels can also be considered as variables and if the DMs cooperate then the entire system as a whole can be optimized. We can easily apply the same approach to non linear BLPPs.

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An Overview of Data Envelopment Analysis Models in Fuzzy Stochastic Environments

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ABSTRACT

One of the appropriate and efficient tools in the field of productivity measurement and evaluation is data envelopment analysis, which is used as a non-parametric method to calculate the efficiency of decision-making units. Today, the use of data envelopment analysis technique is expanding rapidly and is used in the evaluation of various organizations and industries such as banks, postal service, hospitals, training centers, power plants, refineries, etc. In real-world problems, the values observed from input and output data are often ambiguous and random. To solve this problem, data envelopment analysis in stochastic fuzzy environment was proposed. Although the DEA has many advantages, one of the disadvantages of this method is that the classic DEA does not actually give us a definitive conclusion and does not allow random changes in input and output. In this paper, we review some of the proposed models in data envelopment analysis with fuzzy and random inputs and outputs.

1. Introduction

Data envelopment analysis is a linear programming method whose basic purpose is to compare and evaluate the performance of a number of identical decision-making units that have different amounts of inputs used and outputs produced. Data Envelopment Analysis (DEA) models used in evaluating the performance of the unit under study can use two separate approaches: reducing the amount of inputs without decreasing the amount of outputs, increasing the outputs without increasing the amount of inputs.

In real world problems, inputs and outputs are considered vague and random. In fact, decision makers may face a specific hybrid environment where there is fuzziness and randomness in the problem. Hatami-Marbini et al. classified the fuzzy DEA methods in the literature into five general groups [1], the tolerance approach [2, 3], the α-level based approach, the fuzzy ranking approach [4, 5], the possibility approach [6], and the fuzzy arithmetic approach [7]. Among these approaches, the α-level based approach is probably the

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most popular fuzzy DEA model in the literature. This approach generally tries to transform the FDEA model into a pair of parametric programs for each α-level. Kao and Liu, one of the most cited studies in the α-level approach's category, used Chen and Klein [8] method for ranking fuzzy numbers to convert the FDEA model to a pair of parametric mathematical programs for the given level of α [9]. Saati et al. proposed a fuzzy CCR model as a possibilistic programming problem and changed it into an interval programming problem by means of the α-level based approach [10]. Parameshwaran et al. proposed an integrated fuzzy analytic hierarchy process and DEA approach for the service performance measurement [11]. Puri and Yadav [12] applied the suggested methodology by Saati et al. [10] to solve fuzzy DEA model with undesirable outputs. Khanjani et al. [13] proposed fuzzy free disposal hull models under possibility and credibility measures. Momeni et al. used fuzzy DEA models to address the impreciseness and ambiguity associated with input and output data in supply chain performance evaluation problems [14]. Payan evaluated the performance of DMUs with fuzzy data by using the common set of weights based on a linear program [15]. Aghayi et al. formulated a model to measure the efficiency of DMUs with interval inputs and outputs based on common sets weights [16].

In recent years, several scholars work on DEA with fuzzy set extension. For example, Edalatpanah et al. [17] for the first time established triangular single-valued neutrosophic data envelopment analysis with application to hospital performance. He also presented data envelopment analysis based on triangular neutrosophic numbers [18]; see also [19-22].

In this research, some models of data envelopment analysis with fuzzy and random data will be mentioned.

2. Existing Models

In this section, we review the proposed models in a random fuzzy environment.

Tavana et al. [23] developed an imprecise DEA-based formulation for dealing with the randomness of fuzzy parameters on a possibility space $(\theta, P(\theta), Pos)$ through efficiency measurement. They considered n DMUs, indexed by $j=1,\ldots,n$, where each of them consumes m different random fuzzy inputs, indexed by $\tilde{\tilde{x}}_{ij}\,(i=1,\ldots,m)$, to secure s different random fuzzy outputs indexed by $\tilde{\tilde{y}}_{rj}(r=1,\ldots,s)$. Finally, the final model is as follows:

For $\delta > 0.5$:

$$
\begin{aligned}
&\max \ \varphi \\
&\text{s.t.} \quad \varphi + \theta_o^O \phi^{-1}(\delta) \le \sum_{r=1}^{s} u_r (y_{ro}^{m_2} + R^{-1}(\gamma) y_{ro}^{\beta}), \\
&\sum_{i=1}^{m} v_i (x_{io}^{m_2} + R^{-1}(\gamma) x_{io}^{\beta}) - \theta_o^I \phi^{-1}(\delta) \ge 1, \\
&\sum_{i=1}^{m} v_i (x_{io}^{m_1} - L^{-1}(\gamma) x_{io}^{\alpha}) + \theta_o^I \phi^{-1}(\delta) \le 1, \\
&\sum_{r=1}^{s} u_r (y_{rj}^{m_1} - L^{-1}(\gamma) y_{rj}^{\alpha}) - \sum_{i=1}^{m} v_i (x_{ij}^{m_2} + R^{-1}(\gamma) x_{ij}^{\beta}) + \phi^{-1}(\delta) \lambda_j \le 0, \quad j = 1,\dots,n, \\
&(\theta_o^O)^2 = \sum_{r=1}^{s} u_r^2 (\hat{y}_{ro}^{m_1} - L^{-1}(\gamma) \hat{y}_{ro}^{\alpha}), \\
&(\theta_o^I)^2 = \sum_{i=1}^{m} v_i^2 (\hat{x}_{io}^{m_1} - L^{-1}(\gamma) \hat{x}_{io}^{\alpha}), \\
&(\lambda_j)^2 = \sum_{r=1}^{s} u_r^2 \hat{y}_{rj}^{m_2} + \sum_{i=1}^{m} v_i^2 \hat{x}_{ij}^{m_2} - L^{-1}(\gamma) (\sum_{r=1}^{s} u_r^2 \hat{y}_{rj}^{\beta} + \sum_{i=1}^{m} v_i^2 \hat{x}_{ij}^{\beta}), \quad j = 1,\dots,n, \\
&u_r, v_i, \theta_p^O, \theta_p^I, \bar{\theta}_p^I, \lambda_j \ge 0, r = 1,\dots,s, i = 1,\dots,m, j = 1,\dots,n.
\end{aligned} \tag{1}
$$

And for $\delta \le 0.5$:

$$
\begin{aligned}
&\max \ \varphi \\
&\text{s.t.} \quad \varphi - \sum_{r=1}^{s} u_r (y_{ro}^{m_2} + R^{-1}(\gamma) \tilde{\tilde{y}}_{ro}^{\beta}) + \bar{\theta}_o^O \phi^{-1}(\delta) \le 0, \\
&\sum_{i=1}^{m} v_i (x_{io}^{m_2} + R^{-1}(\gamma) x_{io}^{\beta}) - \bar{\theta}_o^I \phi^{-1}(\delta) \ge 1, \\
&\sum_{i=1}^{m} v_i (x_{io}^{m_1} - L^{-1}(\gamma) x_{io}^{\alpha}) + \bar{\theta}_o^I \phi^{-1}(\delta) \le 1, \\
&\sum_{r=1}^{s} u_r (y_{rj}^{m_1} - L^{-1}(\gamma) \sum_{r=1}^{s} u_r y_{rj}^{\alpha}) - \sum_{i=1}^{m} v_i (\tilde{\tilde{x}}_{ip}^{m_2} + R^{-1}(\gamma) \sum_{i=1}^{m} v_i \tilde{\tilde{x}}_{ip}^{\beta}) + \phi^{-1}(\delta) \bar{\lambda}_j \le 0, \quad j = 1,\dots,n, \\
&(\bar{\theta}_o^O)^2 = \sum_{r=1}^{s} u_r^2 (\hat{y}_{ro}^{m_2} + R^{-1}(\gamma) \hat{y}_{ro}^{\beta}), \\
&(\bar{\theta}_o^I)^2 = \sum_{i=1}^{m} v_i^2 (\hat{x}_{io}^{m_2} + R^{-1}(\gamma) \hat{x}_{io}^{\beta}), \\
&(\bar{\lambda}_j)^2 = \sum_{r=1}^{s} u_r^2 \hat{y}_{rj}^{m_2} + \sum_{i=1}^{m} v_i^2 \hat{x}_{ij}^{m_2} + R^{-1}(\gamma) (\sum_{r=1}^{s} u_r^2 \hat{y}_{rj}^{\beta} + \sum_{i=1}^{m} v_i^2 \hat{x}_{ij}^{\beta}), \quad j = 1,\dots,n, \\
&u_r, v_i, \bar{\theta}_p^O, \bar{\theta}_p^I, \lambda_j \ge 0, r = 1,\dots,s, i = 1,\dots,m, j = 1,\dots,n.
\end{aligned} \tag{2}
$$

Furthermore, they presented a Necessity-Probability constrained programming model under fuzzy probability necessity constraints as follow:

For $\delta > 0.5$:

$$
\begin{aligned}
&\max \quad \bar{\varphi} \\
&\text{s.t.} \quad \bar{\varphi}-\sum_{r=1}^{s} u_r y_{ro}^{m_1} + L^{-1}(1-\gamma)\sum_{r=1}^{s} u_r y_{ro}^{\alpha} + \tilde{\theta}_o^{O}\phi^{-1}(\delta) \le 0, \\
&\qquad \sum_{i=1}^{m} v_i (x_{io}^{m_1} - L^{-1}(1-\gamma)x_{io}^{\alpha}) - \tilde{\theta}_o^{I}\phi^{-1}(\delta) \ge 1, \\
&\qquad \sum_{i=1}^{m} v_i x_{io}^{m_2} + R^{-1}(\gamma)\sum_{i=1}^{m} v_i x_{ip}^{\beta} + \tilde{\theta}_o^{I}\phi^{-1}(\delta) \le 1, \\
&\qquad \sum_{r=1}^{s} u_r (y_{rj}^{m_2} + R^{-1}(\gamma)y_{rj}^{\beta}) - \sum_{i=1}^{m} v_i (x_{ij}^{m_1} - L^{-1}(1-\gamma)x_{ij}^{\alpha}) + \tilde{\lambda}_j\phi^{-1}(\delta) \le 0, \quad j=1,\dots,n, \\
&\qquad (\tilde{\theta}_o^{O})^2 = \sum_{r=1}^{s} u_r^2 (\hat{y}_{ro}^{m_1} + R^{-1}(\gamma)\hat{y}_{ro}^{\alpha}), \\
&\qquad (\tilde{\theta}_o^{I})^2 = \sum_{i=1}^{m} v_i^2 (\hat{x}_{io}^{m_2} + R^{-1}(\gamma)\hat{x}_{io}^{\beta}), \\
&\qquad (\tilde{\lambda}_j)^2 = \sum_{r=1}^{s} u_r^2 \hat{y}_{rj}^{m_1} + \sum_{i=1}^{m} v_i^2 \hat{x}_{ij}^{m_1}, \quad j=1,\dots,n, \\
&\qquad u_r, v_i, \tilde{\theta}_o^{O}, \tilde{\theta}_o^{I}, \tilde{\lambda}_j \ge 0, r=1,\dots,s, i=1,\dots,m, j=1,\dots,n.
\end{aligned} \tag{3}
$$

For $\delta \le 0/5$:

$$
\begin{aligned}
&\max \quad \bar{\varphi} \\
&\text{s.t.} \quad \bar{\varphi}-\sum_{r=1}^{s} u_r y_{ro}^{m_1} + L^{-1}(1-\gamma)\sum_{r=1}^{s} u_r y_{ro}^{\alpha} + \hat{\theta}_o^{O}\phi^{-1}(\delta) \le 0, \\
&\qquad \sum_{i=1}^{m} v_i x_{io}^{m_1} - L^{-1}(1-\gamma)\sum_{i=1}^{m} v_i x_{io}^{\alpha} - \hat{\theta}_o^{I}\phi^{-1}(\delta) \ge 1, \\
&\qquad \sum_{i=1}^{m} v_i x_{io}^{m_2} + R^{-1}(\gamma)\sum_{i=1}^{m} v_i x_{ip}^{\beta} + \hat{\theta}_o^{I}\phi^{-1}(\delta) \le 1, \\
&\qquad \sum_{r=1}^{s} u_r (y_{rj}^{m_2} + R^{-1}(\gamma)y_{rj}^{\beta}) - \sum_{i=1}^{m} v_i (x_{ij}^{m_1} - L^{-1}(1-\gamma)x_{ij}^{\alpha}) + \hat{\lambda}_j\phi^{-1}(\delta) \le 0, \quad j=1,\dots,n, \\
&\qquad (\hat{\theta}_o^{O})^2 = \sum_{r=1}^{s} u_r^2 (\hat{y}_{ro}^{m_1} - L^{-1}(1-\gamma)\hat{y}_{ro}^{\alpha}), \\
&\qquad (\hat{\theta}_o^{I})^2 = \sum_{i=1}^{m} v_i^2 (\hat{x}_{io}^{m_1} - L^{-1}(\gamma)\hat{x}_{io}^{\beta}), \\
&\qquad (\hat{\lambda}_j)^2 = \sum_{r=1}^{s} u_r^2 \hat{y}_{rj}^{m_1} + \sum_{i=1}^{m} v_i^2 \hat{x}_{ij}^{m_1} - L^{-1}(1-\gamma)(\sum_{i=1}^{m} v_i^2 \hat{x}_{ij}^{\alpha} + \sum_{r=1}^{s} u_r^2 \hat{y}_{rj}^{\alpha}), \quad j=1,\dots,n, \\
&\qquad u_r, v_i, \hat{\theta}_o^{O}, \hat{\theta}_o^{I}, \hat{\lambda}_j \ge 0, r=1,\dots,s, i=1,\dots,m, j=1,\dots,n.
\end{aligned} \tag{4}
$$

In 2016, Nasseri et al. [24] proposed a new model of fuzzy stochastic DEA with input-oriented primal data. In this model, the properties and characteristics of the extended normal distribution are used. They considered n DMUs, each unit consumes m fuzzy stochastic inputs, denoted by $\tilde{\tilde{x}}_{ij} = (x_{ij}^{m}, x_{ij}^{\alpha}, x_{ij}^{\beta})_{LR}$, $i=1,\dots,m$, $j=1,\dots,n$, and produces s fuzzy stochastic outputs, denoted by $\tilde{\tilde{y}}_{rj} = (y_{rj}^{m}, y_{rj}^{\alpha}, y_{rj}^{\beta})_{LR}$, $r=1,\dots,s$, $j=1,\dots,n$. Also, they considered x_{ij}^{m} and y_{rj}^{m}, denoted by $x_{ij}^{m} \sim N(x_{ij}, \sigma_{ij}^2)$ and $y_{rj}^{m} \sim N(y_{rj}, \sigma_{rj}^2)$ be normally distributed. Therefore, $x_{ij}(y_{rj})$ and $\sigma_{ij}^2(\sigma_{rj}^2)$ are the mean and the variance of $x_{ij}^{m}(y_{rj}^{m})$ for DMU_j, respectively. Each unit has an extended normal distribution as $\tilde{\bar{x}}_{ij} \sim \overline{N}(\bar{x}_{ij}, \sigma_{ij})$ with $\bar{x}_{ij} = (x_{ij}, x_{ij}^{\alpha}, x_{ij}^{\beta})$ and $\tilde{\bar{y}}_{rj} \sim \overline{N}(\bar{y}_{rj}, \sigma_{rj})$ with $\bar{y}_{rj} = (y_{rj}, y_{rj}^{\alpha}, y_{rj}^{\beta})$. Finally, the final model is as follows:

$$
\begin{aligned}
& E_K^T(\delta,\gamma) = \max \ \varphi \\
& \text{s.t.} \\
& \quad \varphi \le \sum_{r=1}^{s} \hat{y}_{rk}, \\
& \quad \sum_{i=1}^{m} \hat{x}_{ik} = 1, \\
& \quad \sum_{r=1}^{s} \hat{y}_{rj} - \sum_{i=1}^{m} \hat{x}_{ij} \le 0 \quad \forall j, \\
& \quad u_r(y_{rj} - L^{-1}(\delta) y_{rj}^{\alpha} - \sigma_{rj}\phi^{-1}_{1-\frac{\gamma}{2}}) \le \hat{y}_{rj} \le u_r(y_{rj} + R^{-1}(\delta) y_{rj}^{\beta} + \sigma_{rj}\phi^{-1}_{1-\frac{\gamma}{2}}), \forall r, j \\
& \quad v_i(x_{rj} - L^{-1}(\delta) x_{rj}^{\alpha} - \sigma_{ij}\phi^{-1}_{1-\frac{\gamma}{2}}) \le \hat{x}_{ij} \le v_i(x_{ij} + R^{-1}(\delta) x_{ij}^{\beta} + \sigma_{ij}\phi^{-1}_{1-\frac{\gamma}{2}}), \forall i, j \\
& \quad u_r, v_i \ge 0.
\end{aligned} \tag{5}
$$

In 2017, Nasseri et al. [25] introduced a new model of fuzzy stochastic data envelopment analysis with undesirable outputs. The application of this model to the banking industry was demonstrated. They solved the proposed model by using the probability-possibility, probability-necessity and probability-credibility measures in CCP approach. The final models will be as follows:

Probability-possibility approach:

$$
\begin{aligned}
& E_k^{Pos}(\gamma,\delta) = \max \ \varphi \\
& \text{s.t.} \\
& \quad \varphi - \sum_{r=1}^{s_1} u_r^g(\tilde{y}_{rk}^g + R^{-1}(\delta) y_{rk}^{g,\beta}) + \sum_{p=1}^{s_2} u_p^b(\tilde{y}_{pk}^b - L^{-1}(\delta) y_{pk}^{b,\alpha}) \le \sigma_k^y \phi^{-1}_{1-\gamma}, \\
& \quad \sum_{i=1}^{m} v_i(\tilde{x}_{ik} + R^{-1}(\delta) x_{ik}^{\beta}) + \sigma_k^x \phi^{-1}_{1-\gamma} \ge 1, \\
& \quad \sum_{i=1}^{m} v_i(\tilde{x}_{ik} - L^{-1}(\delta) x_{ik}^{\alpha}) - \sigma_k^x \phi^{-1}_{1-\gamma} \le 1, \\
& \quad \sum_{r=1}^{s_1} u_r^g(\tilde{y}_{rj}^g + R^{-1}(\delta_j) y_{rj}^{g,\beta}) - \sum_{p=1}^{s_2} u_p^b(\tilde{y}_{pj}^b - L^{-1}(\delta_j) y_{pj}^{b,\alpha}) - \sigma_j^y \phi^{-1}_{1-\gamma} \ge 0, \\
& \quad \sum_{r=1}^{s_1} u_r^g(\tilde{y}_{rj}^g - L^{-1}(\delta) y_{rj}^{g,\alpha}) - \sum_{p=1}^{s_2} u_p^b(\tilde{y}_{pj}^b + R^{-1}(\delta) y_{pj}^{b,\beta}) - \sum_{i=1}^{m} v_i(\tilde{x}_{ik} + R^{-1}(\delta) x_{ik}^{\beta}) - \sigma_j^A \phi^{-1}_{1-\gamma} \le 0, \\
& \quad u_r^g, u_p^b, v_i \ge 0.
\end{aligned} \tag{6}
$$

Probability-Necessity model:

$$
\begin{aligned}
&E_k^{Nec}(\gamma,\delta)=\max\ \varphi\\
&\text{s.t.}\\
&\quad \varphi\le 1,\\
&\quad \varphi\le \sum_{r=1}^{s_1}u_r^g(\tilde{y}_{rk}^g - L^{-1}(1-\delta)y_{rk}^{g,\alpha})-\sum_{p=1}^{s_2}u_p^b(\tilde{y}_{pk}^b + R^{-1}(1-\delta)y_{pk}^{b,\beta})+\sigma_k^y\phi_{1-\gamma}^{-1},\\
&\quad \sum_{i=1}^{m}v_i(\tilde{x}_{ik}-L^{-1}(1-\delta)x_{ik}^{\alpha})+\sigma_k^x\phi_{1-\gamma}^{-1}\ge 1,\\
&\quad \sum_{J=1}^{n}v_i(\tilde{x}_{ik}+R^{-1}(1-\delta)x_{ik}^{\beta})-\sigma_k^x\phi_{1-\gamma}^{-1}\le 1,\\
&\quad \sum_{r=1}^{s_1}u_r^g(\tilde{y}_{rj}^g - L^{-1}(1-\delta)y_{rj}^{g,\alpha})-\sum_{p=1}^{s_2}u_p^b(\tilde{y}_{pj}^b + R^{-1}(1-\delta)y_{pj}^{b,\beta})+\sigma_j^y\phi_{1-\gamma}^{-1}\ge 0,\\
&\quad \sum_{r=1}^{s_1}u_r^g(\tilde{y}_{rj}^g + R^{-1}(1-\delta)y_{rj}^{g,\beta})-\sum_{p=1}^{s_2}u_p^b(\tilde{y}_{pj}^b - L^{-1}(1-\delta)y_{pj}^{b,\alpha})-\sum_{i=1}^{m}v_i(\tilde{x}_{ij}-L^{-1}(1-\delta)x_{ij}^{\alpha})-\sigma_j^A\phi_{1-\gamma}^{-1}\le 0,\\
&\quad \sigma_k^x=(\sum_{i=1}^{m}v_i^2\sigma_{ik}^2)^{\frac{1}{2}},\\
&\quad \sigma_j^y=(\sum_{r=1}^{s_1}(u_r^g)^2(\sigma_{rj}^g)^2+\sum_{p=1}^{s_2}(u_p^b)^2(\sigma_{pj}^b)^2)^{\frac{1}{2}}),\\
&\quad \sigma_j^A=(\sum_{r=1}^{s_1}(u_r^g)^2(\sigma_{rj}^g)^2+\sum_{p=1}^{s_2}(u_p^b)^2(\sigma_{pj}^b)^2)+\sum_{i=1}^{m}v_i^2\sigma_{ij}^2)^{\frac{1}{2}},\\
&\quad u_r^g,u_p^b,v_i\ge 0.
\end{aligned}
\tag{7}
$$

Probability-Credibility model:

For $\delta \le 0/5$:

$$
\begin{aligned}
&E_k^{Cr}(\gamma,\delta)=\max\ \varphi\\
&\text{s.t.}\\
&\quad \varphi\le 1,\\
&\quad \varphi\le \sum_{r=1}^{s_1}u_r^g(\tilde{y}_{rk}^g + R^{-1}(2\delta)y_{rk}^{g,\beta})-\sum_{p=1}^{s_2}u_p^b(\tilde{y}_{pk}^b - L^{-1}(2\delta)y_{pk}^{b,\alpha})+\sigma_k^y\phi_{1-\gamma}^{-1},\\
&\quad \sum_{i=1}^{m}v_i(\tilde{x}_{ik}+R^{-1}(2\delta)x_{ik}^{\beta})+\sigma_k^x\phi_{1-\gamma}^{-1}\ge 1,\\
&\quad \sum_{j=1}^{n}v_i(\tilde{x}_{ik}-L^{-1}(2\delta)x_{ik}^{\alpha})-\sigma_k^x\phi_{1-\gamma}^{-1}\le 1,\\
&\quad \sum_{r=1}^{s_1}u_r^g(\tilde{y}_{rj}^g + R^{-1}(2\delta)y_{rj}^{g,\beta})-\sum_{p=1}^{s_2}u_p^b(\tilde{y}_{pj}^b - L^{-1}(2\delta)y_{pj}^{b,\alpha})+\sigma_j^y\phi_{1-\gamma}^{-1}\ge 0,\\
&\quad \sum_{r=1}^{s_1}u_r^g(\tilde{y}_{rj}^g - L^{-1}(2\delta)y_{rj}^{g,\alpha})-\sum_{p=1}^{s_2}u_p^b(\tilde{y}_{pj}^b + R^{-1}(2\delta)y_{pj}^{b,\beta})-\sum_{i=1}^{m}v_i(\tilde{x}_{ij}+R^{-1}(2\delta)x_{ij}^{\beta})-\sigma_j^A\phi_{1-\gamma}^{-1}\le 0,\\
&\quad \sigma_k^x=(\sum_{i=1}^{m}v_i^2\sigma_{ik}^2)^{\frac{1}{2}},\\
&\quad \sigma_j^y=(\sum_{r=1}^{s_1}(u_r^g)^2(\sigma_{rk}^g)^2+\sum_{p=1}^{s_2}(u_p^b)^2(\sigma_{pk}^b)^2)^{\frac{1}{2}}),\\
&\quad \sigma_j^A=(\sum_{r=1}^{s_1}(u_r^g)^2(\sigma_{rj}^g)^2+\sum_{p=1}^{s_2}(u_p^b)^2(\sigma_{pj}^b)^2)+\sum_{i=1}^{m}v_i^2\sigma_{ij}^2)^{\frac{1}{2}},\\
&\quad u_r^g,u_p^b,v_i\ge 0.
\end{aligned}
\tag{8}
$$

For $\delta > 0.5$:

$$
\begin{aligned}
& E_k^{Nec}(\gamma,\delta) = \max\ \varphi \\
& s.t. \\
& \quad \varphi \le 1, \\
& \quad \varphi \le \sum_{r=1}^{s_1} u_r^g(\tilde{y}_{rk}^g - L^{-1}(2(1-\delta))y_{rk}^{g,\alpha}) - \sum_{p=1}^{s_2} u_p^b(\tilde{y}_{pk}^b + R^{-1}(2(1-\delta))y_{pk}^{b,\beta}) + \sigma_k^y \phi_{1-\gamma}^{-1}, \\
& \quad \sum_{i=1}^{m} v_i(\tilde{x}_{ik} - L^{-1}(2(1-\delta))x_{ik}^{\alpha}) + \sigma_k^x \phi_{1-\gamma}^{-1} \ge 1, \\
& \quad \sum_{J=1}^{n} v_i(\tilde{x}_{ik} + R^{-1}(2(1-\delta))x_{ik}^{\beta}) - \sigma_k^x \phi_{1-\gamma}^{-1} \le 1, \\
& \quad \sum_{r=1}^{s_1} u_r^g(\tilde{y}_{rj}^g - L^{-1}(2(1-\delta))y_{rj}^{g,\alpha}) - \sum_{p=1}^{s_2} u_p^b(\tilde{y}_{pj}^b + R^{-1}(2(1-\delta))y_{pj}^{b,\beta}) + \sigma_j^y \phi_{1-\gamma}^{-1} \ge 0, \\
& \quad \sum_{r=1}^{s_1} u_r^g(\tilde{y}_{rj}^g + R^{-1}(2(1-\delta))y_{rj}^{g,\beta}) - \sum_{p=1}^{s_2} u_p^b(\tilde{y}_{pj}^b - L^{-1}(2(1-\delta))y_{pj}^{b,\alpha}) - \\
& \quad \sum_{i=1}^{m} v_i(\tilde{x}_{ij} - L^{-1}(2(1-\delta))x_{ij}^{\alpha}) - \sigma_j^A \phi_{1-\gamma}^{-1} \le 0, \\
& \quad \sigma_k^x = (\sum_{i=1}^{m} v_i^2 \sigma_{ik}^2)^{\frac{1}{2}}, \\
& \quad \sigma_j^y = (\sum_{r=1}^{s_1} (u_r^g)^2 (\sigma_{rj}^g)^2 + \sum_{p=1}^{s_2} (u_p^b)^2 (\sigma_{pj}^b)^2)^{\frac{1}{2}}), \\
& \quad \sigma_j^A = (\sum_{r=1}^{s_1} (u_r^g)^2 (\sigma_{rj}^g)^2 + \sum_{p=1}^{s_2} (u_p^b)^2 (\sigma_{pj}^b)^2) + \sum_{i=1}^{m} v_i^2 \sigma_{ij}^2)^{\frac{1}{2}}, \\
& \quad u_r^g, u_p^b, v_i \ge 0.
\end{aligned}
\tag{9}
$$

3. Conclusion

A DEA model basically draws three critical elements: the model specification, the reference set itself, and the definition of the production possibility set. Starting from the latter, the production possibility set can either be defined as complete and known (like in conventional DEA) or as potentially extending beyond or excluding the reference set (like in stochastic DEA). The reference set, the very observations that form the engine of the non-parametric approach, can be either precise (as in conventional DEA), outcomes of stochastic processes (as in stochastic frontier analysis), or imprecise (as in the fuzzy DEA models).

Classic DEA models were originally formulated for optimal inputs and outputs, although undesirable outputs may also appear during production, which should be minimized. In addition, in the real world, there are dimensions and uncertainties in the data. Although DEA has many advantages, one of the disadvantages of this method is that in fact the classic DEA does not lead us to a definite conclusion and does not allow random changes in input and output.

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A Study on Fundamentals of Refined Intuitionistic Fuzzy Set with Some Properties

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ABSTRACT

Zadeh conceptualized the theory of fuzzy set to provide a tool for the basis of the theory of possibility. Atanassov extended this theory with the introduction of intuitionistic fuzzy set. Smarandache introduced the concept of refined intuitionistic fuzzy set by further subdivision of membership and non-membership value. The meagerness regarding the allocation of a single membership and non-membership value to any object under consideration is addressed with this novel refinement. In this study, this novel idea is utilized to characterize the essential elements e.g. subset, equal set, null set, and complement set, for refined intuitionistic fuzzy set. Moreover, their basic set theoretic operations like union, intersection, extended intersection, restricted union, restricted intersection, and restricted difference, are conceptualized. Furthermore, some basic laws are also discussed with the help of an illustrative example in each case for vivid understanding.

1. Introduction

Zadeh [1] introduced the concept of fuzzy set for the first time in 1965 which covers all weak aspects of the classical set theory. In fuzzy set, the membership value is allocated from the interval [0, 1] to all the elements of the universe under consideration. Zadeh [2] used his own concept as a basis for a theory of possibility. Dubois and Prade [3, 4] established relationship between fuzzy sets and probability theories and also derive monotonicity property for algebraic operations performed between random set-valued variables. Ranking fuzzy numbers in the setting of possibility theory was done by Dubois and Prade [5]. This concept was used by Liang et al. in data analysis, similarities measures in fuzzy sets were discussed by Beg and Ashraf [6-8]. Set difference and symmetric difference of fuzzy sets were established by Vemuri et al., after that, Neog and

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Sut [9] extended the work to complement of an extended fuzzy set. A lot of work is done by researchers in fuzzy mathematics and its hybrids [10-16].

In some real life situations, the values are in the form of intervals due to which it is hard to allocate a membership value to the element of the universe of discourse. Therefore, the concept of interval-valued is introduced which proved a very powerful tool in this area.

In 1986, Atanassov [17, 18] introduced the concept of intuitionistic fuzzy set in which the membership value and non-membership value is allocated from the interval [0,1] to all the elements of the universe under consideration. It is the generalization of the fuzzy set. The invention of intuitionistic fuzzy set proved very important tool for researchers. Ejegwa et al. [19] discussed about operations, algebra, model operators and normalization on intuitionistic fuzzy sets. Szmidt and Kacprzyk [20] gave geometrical representation of an intuitionistic fuzzy set is a point of departure for our proposal of distances between intuitionistic fuzzy sets and also discussed properties. Szmidt and Kacprzyk [21] also discussed about non-probabilistic-type entropy measure for these sets and used it for geometrical interpretation of intuitionistic fuzzy sets. Proposed measure in terms of the ratio of intuitionistic fuzzy cardinalities was also defined and discussed. Ersoy and Davvaz [22] discussed the basic definitions and properties of intuitionistic fuzzy Γ- hyperideals of a Γ- semi-hyperring with examples are introduced and described some characterizations of Artinian and Noetherian Γ- semi hyper ring. Bustince and Burillo [23] proved that vague sets are intuitionistic fuzzy sets. A lot of work is done by researchers in intuitionistic fuzzy environment and its hybrids [24-33].

In 2019, Smarandache defined the concept of refined intuitionistic fuzzy set [34]. In this paper, we extend the concept to refined intuitionistic fuzzy set and defined some fundamental concepts and aggregation operations of refined intuitionistic fuzzy set.

Imprecision is a critical viewpoint in any decision making procedure. Various tools have been invented to deal with the uncertain environment. Perhaps the important tool in managing with imprecision is intuitionistic fuzzy sets. Besides, the most significant thing is that in real life scenario, it is not sufficient to allocate a single membership and non-membership value to any object under consideration. This inadequacy is addressed with the introduction of refined intuitionistic fuzzy set. Having motivation from this novel concept, essential elements, set theoretic operations and basic laws are characterized for refined intuitionistic fuzzy set in this work.

The remaining article is outlined in such a way that the Section 2 recalls some basic definitions along with illustrative example. Section 3 explains basic notions of Refined Intuitionistic Fuzzy Set (RIFS) including subset, equal set, null set and complement set along with their examples for the clear understanding. Section 4 explains the aggregation operations of RIFS with the help of example, Section 5 gives some basic laws of RIFS and in the last, Section 6 concludes the work and gives the future directions.

2. Preliminaries

In this section, some basic concepts of Fuzzy Set (FS), Intuitionistic Fuzzy Set (IFS) and RIFS are discussed.

Let us consider $\breve{U}$ be a universal set, N be a set of natural numbers, $\breve{I}$ represent the interval $[0,1]$, T_{η}^{ω} denotes the degree of sub-truth of type $\omega = 1,2,3,\ldots,\alpha$ and F_{η}^{λ} denotes the degree of sub-falsity of type $\lambda = 1,2,3,\ldots,\beta$ such that$\alpha$and β are natural numbers. An illustrative example is considered to understand these entire basic concepts throughout the paper.

Definition 1. [1, 2] The fuzzy set $\breve{\eta}_f = \{< \breve{\delta}, \alpha_{\breve{\eta}_f}(\breve{\delta}) > | \breve{\delta} \in \breve{U}\}$ on $\breve{U}$ such that $\alpha_{\breve{\eta}_f}(\breve{\delta}) : \breve{U} \to \breve{I}$ where $\alpha_{\breve{\eta}_f}(\breve{\delta})$ describes the membership of $\breve{\delta} \in \breve{U}$.

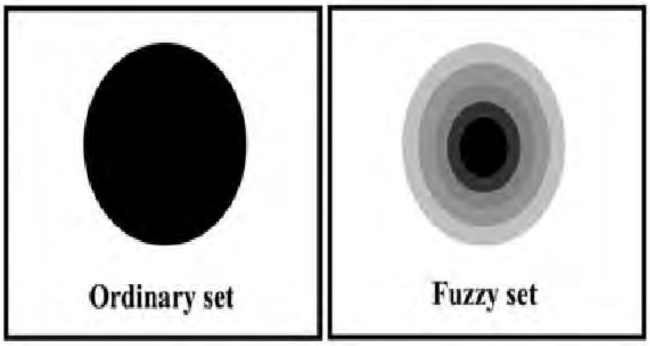

Fig. 1. Representation of fuzzy set.

Example 1. Hamna wants to purchase a dress for farewell party event of her university. She expected to purchase such dress which meets her desired requirements according to the event. Let $\breve{U} = \{\breve{B}_1, \breve{B}_2, \breve{B}_3, \breve{B}_4\}$, be different well-known brands of clothes in Pakistan such that

$\breve{B}_1 =$ Ideas Gul Ahmad;
$\breve{B}_2 =$ Khaadi;
$\breve{B}_3 =$ Nishat Linen;
$\breve{B}_4 =$ Junaid jamshaid.

Then fuzzy set $\breve{\eta}_f$ on the universe $\breve{U}$ is written in such a way that $\breve{\eta}_f = \left\{ \begin{matrix} < \breve{B}_1, 0.45 >, < \breve{B}_2, 0.57 >, \\ < \breve{B}_3, 0.6 >, < \breve{B}_4, 0.64 > \end{matrix} \right\}$.

Definition 2. [18]. An IFS $\breve{\eta}_{IFS}$ on $\breve{U}$ is given by $\breve{\eta}_{IFS} = \{< \breve{\delta}, T_{\breve{\eta}}(\breve{\delta}), F_{\breve{\eta}}(\breve{\delta}) > | \breve{\delta} \in \breve{U}\}$,

where $T_{\breve{\eta}}(\breve{\delta}), F_{\breve{\eta}}(\breve{\delta}) : \breve{U} \to P([0,1])$, respectively, with the condition $sup\, T_{\breve{\eta}}(\breve{\delta}) + sup\, F_{\breve{\eta}}(\breve{\delta}) \le 1$.

Example 2. Consider the illustrative example, and then the intuitionistic fuzzy set $\breve{\eta}_{IFS}$ on the universe $\breve{U}$ is given as $\breve{\eta}_{IFS} = \{< \breve{B}_1, 0.75, 0.14 >, < \breve{B}_2, 0.57, 0.2 >, < \breve{B}_3, 0.6, 0.3 >, < \breve{B}_4, 0.64, 0.16 >\}$.

Definition 3. [34] A RIFS $\breve{\eta}_{RIFS}$ on $\breve{U}$ is given by $\breve{\eta}_{RIFS} = \left\{< \breve{\delta}, T_{\breve{\eta}}^{\omega}(\breve{\delta}), F_{\breve{\eta}}^{\lambda}(\breve{\delta}) > : \ \omega \in N_1^{\alpha},\ \lambda \in N_1^{\beta},\ \alpha + \beta \ge 3,\ \ \breve{\delta} \in \breve{U}\right\}$, where $\alpha, \beta \in \breve{I}$ such that $T_{\breve{\eta}}^{\omega}, F_{\breve{\eta}}^{\lambda} \subseteq \breve{I}$, respectively, with the condition

$$\sum_{\omega=1}^{\alpha} sup\ T_{\breve{\eta}}^{\omega}(\breve{\delta}) + \sum_{\lambda=1}^{\beta} sup\ F_{\breve{\eta}}^{\lambda}(\breve{\delta}) \le 1.$$

It is denoted by$(\breve{\delta}, \breve{G})$, where $\breve{G} = (T_{\breve{\eta}}^{\omega}, F_{\breve{\eta}}^{\lambda})$.

Example 3. Consider the illustrative example, then the RIFS $\breve{\eta}_{RIFS}$ can be written in such a way that

$\breve{\eta}_{RIFS} = \{ < \breve{B}_1, (0.5, 0.4), (0.3, 0.25) >, < \breve{B}_2, (0.35, 0.3), (0.15, 0.1) >,$
$< \breve{B}_3, (0.35, 0.25), (0.3, 0.2) >, < \breve{B}_4, (0.6, 0.1), (0.12, 0.2) > \}$.

3. Basic Notions of RIFS

In this section, some basic notions of subset, equal sets, null set and complement set for RIFS are introduced.

Definition 4. Refined intuitionistic fuzzy subset

Let $\breve{\eta}_{1RIFS} = (\breve{\delta}, \breve{G}_1)$ and $\breve{\eta}_{2RIFS} = (\breve{\delta}, \breve{G}_2)$ be two RIFS, then $\breve{\eta}_{1RIFS} \subseteq \breve{\eta}_{2RIFS}$, if

$$\sum_{\omega=1}^{\alpha} sup\ T_{\breve{\eta}_1}^{\omega}(\breve{\delta}) \leq \sum_{\omega=1}^{\alpha} sup\ T_{\breve{\eta}_2}^{\omega}(\breve{\delta}),\ \sum_{\lambda=1}^{\beta} sup\, F_{\breve{\eta}_1}^{\lambda}(\breve{\delta}) \geq \sum_{\lambda=1}^{\beta} sup\, F_{\breve{\eta}_2}^{\lambda}(\breve{\delta})\ \forall\ \breve{\delta} \in \breve{U}.$$

Remark 1. If

$$\sum_{\omega=1}^{\alpha} sup\ T_{\breve{\eta}_1}^{\omega}(\breve{\delta}) < \sum_{\omega=1}^{\alpha} sup\ T_{\breve{\eta}_2}^{\omega}(\breve{\delta}),\ \sum_{\lambda=1}^{\beta} sup\, F_{\breve{\eta}_1}^{\lambda}(\breve{\delta}) > \sum_{\lambda=1}^{\beta} sup\, F_{\breve{\eta}_2}^{\lambda}(\breve{\delta})\ \forall\ \breve{\delta} \in \breve{U}.$$

Then it is denoted by $(\breve{\delta}, \breve{G}_1) \subset (\breve{\delta}, \breve{G}_2)$.

Suppose $(\breve{\delta}, \breve{G}_1^i) \subset (\breve{\delta}, \breve{G}_2^i)$ be two families of RIFS, then $(\breve{\delta}, \breve{G}_1^i)$ is called family of refined intuitionistic fuzzy subset of $(\breve{\delta}, \breve{G}_2^i)$, if $\breve{G}_1^i \subset \breve{G}_2^i$ and

$$\sum_{\omega=1}^{\alpha} sup\ T_{\breve{\eta}_1}^{\omega}(\breve{\delta}) < \sum_{\omega=1}^{\alpha} sup\ T_{\breve{\eta}_2}^{\omega}(\breve{\delta}), \sum_{\lambda=1}^{\beta} sup\, F_{\breve{\eta}_1}^{\lambda}(\breve{\delta}) > \sum_{\lambda=1}^{\beta} sup\, F_{\breve{\eta}_2}^{\lambda}(\breve{\delta}), \forall\, \breve{\delta} \in \breve{U}.$$

We denote it by $(\breve{\delta}, \breve{G}_1^i) \subset (\breve{\delta}, \breve{G}_2^i) \forall\ \ i = 1, 2, 3, \ldots, n$.

Example 4. Consider the illustrative example, let $\breve{\eta}_{1RIFS}$ and $\breve{\eta}_{2RIFS}$ be two RIFS such that

$\breve{\eta}_{1RIFS} = \{ < \breve{B}_1, (0.35, 0.1), (0.22, 0.19) >, < \breve{B}_2, (0.25, 0.03), (0.15, 0.19) >,$
$< \breve{B}_3, (0.2, 0.1), (0.2, 0.24) >, < \breve{B}_4, (0.3, 0.4), (0.06, 0.04) > \}$,

and
$\breve{\eta}_{2RIFS} = \{ < \breve{B}_1, (0.38, 0.11), (0.2, 0.14) >, < \breve{B}_2, (0.45, 0.04), (0.1, 0.14) >,$
$< \breve{B}_3, (0.3, 0.2), (0.01, 0.06) >, < \breve{B}_4, (0.31, 0.41), (0.01, 0.011) > \}$.

Then from above equations, it is clear that $\breve{\eta}_{1RIFS} \subseteq \breve{\eta}_{2RIFS}$.

Definition 5. Equal refined intuitionistic fuzzy sets

Let $\breve{\eta}_{1_{RIFS}} = (\breve{\delta},\ \breve{G}_1)$ and $\breve{\eta}_{2_{RIFS}} = (\breve{\delta},\ \breve{G}_2)$ be two RIFS, then $\breve{\eta}_{1_{RIFS}} = \breve{\eta}_{2_{RIFS}}$, if $\breve{\eta}_{1_{RIFS}} \subseteq \breve{\eta}_{2_{RIFS}}$ and $\breve{\eta}_{2_{RIFS}} \subseteq \breve{\eta}_{1_{RIFS}}$.

Example 5. Consider the illustrative example, let $\breve{\eta}_{1_{RIFS}}$ and $\breve{\eta}_{2_{RIFS}}$ be two RIFS such that

$\breve{\eta}_{1_{RIFS}} = (\breve{\delta},\breve{G}_1) = \{< \breve{B}_1, (0.4, 0.5), (0.03, 0.04) >, < \breve{B}_2, (0.5,0.4), (0.05,0.04) >,$
$< \breve{B}_3, (0.5,0.2), (0.01,0.06) >, < \breve{B}_4, (0.3,0.4), (0.06,0.04) > \},$

and

$\breve{\eta}_{2_{RIFS}} = (\breve{\delta},\breve{G}_2) = \{< \breve{B}_1, (0.4, 0.5), (0.03, 0.04) >, < \breve{B}_2, (0.5,0.4), (0.05,0.04) >,$
$< \breve{B}_3, (0.5,0.2), (0.01,0.06) >, < \breve{B}_4, (0.3,0.4), (0.06,0.04) > \}.$

Then from above equations, it is clear that $\breve{\eta}_{1_{RIFS}} = \breve{\eta}_{2_{RIFS}}$.

Definition 6. Null refined intuitionistic fuzzy set

Let RIFS $(\breve{\delta},\ \breve{G})$ is said to be null RIFS if

$$\sum_{\omega=1}^{\alpha} sup\ T_{\breve{\eta}}^{\omega}(\breve{\delta}) = 0,\ \sum_{\lambda=1}^{\beta} sup\ F_{\breve{\eta}}^{\lambda}(\breve{\delta}) = 0, \quad \forall\ \ \breve{\delta}\ \in \breve{U}.$$

It is denoted by $(\breve{\delta},\ \breve{G})_{null}$.

Example 6. Consider the illustrative example, the null RIFS is given as

$(\breve{\delta},\breve{G}) = \{< \breve{B}_1, (0, 0), (0, 0) >, < \breve{B}_2, (0,0), (0,0) >,$

$< \breve{B}_3, (0,0), (0,0) >, < \breve{B}_4, (0,0), (0,0) > \}.$

Definition 7. Complement of refined intuitionistic fuzzy set

The complement of RIFS$(\breve{\delta},\breve{G})$ is denoted by $(\breve{\delta},\breve{G}^c)$ and is defined that if

$$\sum_{\omega=1}^{\alpha} sup\ T_{\breve{\eta}^c}^{\omega}(\breve{\delta}) = \sum_{\lambda=1}^{\beta} sup\ F_{\breve{\eta}}^{\lambda}(\breve{\delta}), \quad \sum_{\lambda=1}^{\beta} sup\ F_{\breve{\eta}^c}^{\lambda}(\breve{\delta}) = \sum_{\omega=1}^{\alpha} sup\ T_{\breve{\eta}}^{\omega}(\breve{\delta}), \quad \forall\ \ \breve{\delta}\ \in \breve{U}.$$

Remark 2. The complement of family of RIFS$(\breve{\delta},\breve{G}^c)$ is denoted by $(\breve{\delta},\breve{G}^c)$ and is defined in a way that if

$$\sum_{\omega=1}^{\alpha} sup\ T_{\breve{\eta}_i^c}^{\omega}(\breve{\delta}) = \sum_{\lambda=1}^{\beta} sup\ F_{\breve{\eta}}^{\lambda}(\breve{\delta}), \sum_{\lambda=1}^{\beta} sup\ F_{\breve{\eta}_i^c}^{\lambda}(\breve{\delta}) = \sum_{\omega=1}^{\alpha} sup\ T_{\breve{\eta}}^{\omega}(\breve{\delta}), \forall\ i = 1,2,3,\ldots,n.$$

Example 7. Consider the illustrative example, if there is a RIFS $\breve{\eta}_{RIFS}$ given as

$\breve{\eta}_{RIFS} = \{ < \breve{B}_1, (0.2, 0.1), (0.3, 0.35) >, < \breve{B}_2, (0.05,0.34), (0.45,0.04) >,$
$< \breve{B}_3, (0.01,0.6), (0.1,0.02) >, < \breve{B}_4, (0.3,0.04), (0.12,0.2) > \}.$

Then the complement of RIFS$\breve{\eta}_{RIFS}$ given as

$\breve{\eta}_{RIFS} = \{ < \breve{B}_1 (0.3, 0.35), (0.2, 0.1) >, < \breve{B}_2, (0.45,0.04), (0.05,0.34) >,$
$< \breve{B}_3, (0.1,0.02), (0.01,0.6) >, < \breve{B}_4, (0.12,0.2), (0.3,0.04) > \}.$

4. Aggregation Operators of RIFS

In this section, union, intersection, extended intersection, restricted union, restricted intersection and restricted difference of RIFS is defined with the help of illustrative example.

Definition 8. Union of two RIFS

The union of two RIFS $(\breve{\delta},\ \breve{G}_1)$ and $(\breve{\delta},\ \breve{G}_2)$is denoted by $(\breve{\delta},\ \breve{G}_1) \cup (\breve{\delta},\ \breve{G}_2)$ and it isdefined as $(\breve{\delta},\ \breve{G}_1) \cup (\breve{\delta},\ \breve{G}_2) = (\breve{\delta},\ \breve{\Upsilon})$, where $\breve{\Upsilon} = \breve{G}_1 \cup \breve{G}_2$, and truth and falsemembership of $(\breve{\delta},\ \breve{\Upsilon})$ is defined in such a way that

$$\mathrm{T}_{\breve{\Upsilon}}(\breve{\delta}) = \max\left(\sum_{\omega=1}^{\alpha} \sup\ \mathrm{T}^{\omega}_{\breve{\eta}_1}(\breve{\delta}), \sum_{\omega=1}^{\alpha} \sup\ \mathrm{T}^{\omega}_{\breve{\eta}_2}(\breve{\delta})\right),$$

$$F_{\breve{\Upsilon}}(\breve{\delta}) = min\left(\sum_{\lambda=1}^{\beta} sup\, F^{\lambda}_{\breve{\eta}_1}(\breve{\delta}), \sum_{\lambda=1}^{\beta} sup\, F^{\lambda}_{\breve{\eta}_2}(\breve{\delta})\right).$$

Remark 3. The union of two families of RIFS $(\breve{\delta},\ \breve{G}^i_1)$ and $(\breve{\delta},\ \breve{G}^i_2)$is denoted by $(\breve{\delta},\ \breve{G}^i_1) \cup (\breve{\delta},\ \breve{G}^i_2)$ and it is defined as $(\breve{\delta},\ \breve{G}^i_1) \cup (\breve{\delta},\ \breve{G}^i_2) = (\breve{\delta},\ \breve{\Upsilon}^i)$, where$\breve{\Upsilon}^i = \breve{G}^i_1 \cup \breve{G}^i_2$, $i = 1,2,3, \ldots, n$, and truth and false membership of $(\breve{\delta},\ \breve{\Upsilon}^i)$ is defined in such a way that

$$T_{\breve{\Upsilon}^i}(\breve{\delta}) = max\left(\sum_{\omega=1}^{\alpha} sup\ T^{\omega}_{\breve{\eta}_1}(\breve{\delta}), \sum_{\omega=1}^{\alpha} sup\ T^{\omega}_{\breve{\eta}_2}(\breve{\delta})\right),$$

$$F_{\breve{\Upsilon}^i}(\breve{\delta}) = min\left(\sum_{\lambda=1}^{\beta} sup\, F^{\lambda}_{\breve{\eta}_1}(\breve{\delta}), \sum_{\lambda=1}^{\beta} sup\, F^{\lambda}_{\breve{\eta}_2}(\breve{\delta})\right).$$

Example 8. Consider the illustrative example, suppose that

$(\breve{\delta},\ \breve{G}_1) = \{ < \breve{B}_1,$ *(0.13,0.19),(0.24, 0.1)*$>, < \breve{B}_2, (0.2,0.25), (0.15,0.24) >,$

$< \breve{B}_3, (0.1,0.36), (0.34,0.12) >, < \breve{B}_4, (0.16,0.14), (0.23,0.37) > \},$

and

$(\breve{\delta},\ \breve{G}_2) = \{ < \breve{B}_1,$ *(0.2,0.3),(0.3, 0.15)*$>,$

$< \breve{B}_2, (0.32,0.38), (0.1,0.04) >$,
$< \breve{B}_3, (0.01,0.16), (0.5,0.2) >, < \breve{B}_4, (0.26,0.15), (0.12,0.2) > \}$,

be two RIFS. Then the union of RIFS $(\breve{\delta},\ \breve{G}_1)$ and $(\breve{\delta},\ \breve{G}_2)$is given as

$(\breve{\delta},\ \breve{Y}), = \{ < \breve{B}_1, (0.2, 0.3),(0.24,0.1)>, < \breve{B}_2, (0.32,0.38), (0.1,0.04) >$,
$< \breve{B}_3, (0.1,0.36), (0.34,0.12) >, < \breve{B}_4, (0.26,0.15), (0.12,0.2) > \}$.

Definition 9. Intersection of two RIFS

The intersection of two RIFS $(\breve{\delta},\ \breve{G}_1)$ and $(\breve{\delta},\ \breve{G}_2)$is denoted by $(\breve{\delta},\ \breve{G}_1) \cap (\breve{\delta},\ \breve{G}_2)$ and it is defined as $(\breve{\delta},\ \breve{G}_1) \cap (\breve{\delta},\ \breve{G}_2) = (\breve{\delta},\ \breve{Y})$, where $\breve{Y} = \breve{G}_1 \cap \breve{G}_2$, and truth and falsemembership of $(\breve{\delta},\ \breve{Y})$ is defined in such a way that

$$T_{\breve{Y}}(\breve{\delta}) = min\left(\sum_{\omega=1}^{\alpha} sup\ T^{\omega}_{\breve{\eta}_1}(\breve{\delta}), \sum_{\omega=1}^{\alpha} sup\ T^{\omega}_{\breve{\eta}_2}(\breve{\delta})\right),$$

$$F_{\breve{Y}}(\breve{\delta}) = max\left(\sum_{\lambda=1}^{\beta} sup\, F^{\lambda}_{\breve{\eta}_1}(\breve{\delta}), \sum_{\lambda=1}^{\beta} sup\, F^{\lambda}_{\breve{\eta}_2}(\breve{\delta})\right).$$

Remark 4. The intersection of two families of RIFS $(\breve{\delta},\ \breve{G}^i_1)$ and $(\breve{\delta},\ \breve{G}^i_2)$is denoted by $(\breve{\delta},\ \breve{G}^i_1) \cap (\breve{\delta},\ \breve{G}^i_2)$ and it is defined as$(\breve{\delta},\ \breve{G}^i_1) \cap (\breve{\delta},\ \breve{G}^i_2) = (\breve{\delta},\ \breve{Y}^i)$, where $\breve{Y}^i = \breve{G}^i_1 \cap \breve{G}^i_2,\ i = 1,2,3, \dots, n$, and truth and false membership of $(\breve{\delta},\ \breve{Y}^i)$ is defined in such a way that

$$T_{\breve{Y}^i}(\breve{\delta}) = min\left(\sum_{\omega=1}^{\alpha} sup\ T^{\omega}_{\breve{\eta}_1}(\breve{\delta}), \sum_{\omega=1}^{\alpha} sup\ T^{\omega}_{\breve{\eta}_2}(\breve{\delta})\right),$$

$$F_{\breve{Y}^i}(\breve{\delta}) = max\left(\sum_{\lambda=1}^{\beta} sup\, F^{\lambda}_{\breve{\eta}_1}(\breve{\delta}), \sum_{\lambda=1}^{\beta} sup\, F^{\lambda}_{\breve{\eta}_2}(\breve{\delta})\right).$$

Example 9. Consider the illustrative example, suppose that

$(\breve{\delta},\ \breve{G}_1) = \{ < \breve{B}_1, (0.13, 0.19),(0.24,0.1)>$,

$< \breve{B}_2, (0.2,0.25), (0.15,0.24) >$,
$< \breve{B}_3, (0.1,0.36), (0.34,0.12) >, < \breve{B}_4, (0.16,0.14), (0.23,0.37) > \}$,

and

$(\breve{\delta},\ \breve{G}_2) = \{ < \breve{B}_1, (0.2,0.3),(0.3, 0.15)>$,

$< \breve{B}_2, (0.32,0.38), (0.1,0.04) >$,
$< \breve{B}_3, (0.01,0.16), (0.5,0.2) >, < \breve{B}_4, (0.26,0.15), (0.12,0.2) > \}$,

be two RIFS. Then the intersection of RIFS $(\breve{\delta},\ \breve{G}_1)$ and $(\breve{\delta},\ \breve{G}_2)$is given as

$(\breve{\delta},\ \breve{Y}) = \{<\breve{B}_1, (0.13,\ 0.19),(0.24,\ 0.1)>,$

$<\breve{B}_2,(0.2,0.25),(0.15,0.24)>,$
$<\breve{B}_3,(0.01,0.16),(0.5,0.2)>,<\breve{B}_4,(0.16,0.14),(0.23,0.37)>\}.$

Definition 10. Extended intersection of two RIFS

The intersection of two RIFS $(\breve{\delta},\ \breve{G}_1)$ and $(\breve{\delta},\ \breve{G}_2)$is denoted by $(\breve{\delta},\ \breve{G}_1) \cap_\varepsilon (\breve{\delta},\ \breve{G}_2)$ and it is defined as $(\breve{\delta},\ \breve{G}_1) \cap_\varepsilon (\breve{\delta},\ \breve{G}_2) = (\breve{\delta},\ \breve{\Upsilon})$, where $\breve{\Upsilon} = \breve{G}_1 \cup \breve{G}_2$, and truth and falsemembership of $(\breve{\delta},\ \breve{\Upsilon})$ is defined in such a way that

$$T_{\breve{Y}}(\breve{\delta}) = min\left(\sum_{\omega=1}^{\alpha} sup\ T_{\breve{\eta}_1}^{\omega}(\breve{\delta}), \sum_{\omega=1}^{\alpha} sup\ T_{\breve{\eta}_2}^{\omega}(\breve{\delta})\right),$$

$$F_{\breve{Y}}(\breve{\delta}) = max\left(\sum_{\lambda=1}^{\beta} sup\, F_{\breve{\eta}_1}^{\lambda}(\breve{\delta}), \sum_{\lambda=1}^{\beta} sup\, F_{\breve{\eta}_2}^{\lambda}(\breve{\delta})\right).$$

Remark 5. The extended intersection of two families of RIFS $(\breve{\delta},\ \breve{G}_1^i)$ and $(\breve{\delta},\ \breve{G}_2^i)$is denoted by $(\breve{\delta},\ \breve{G}_1^i) \cap_\varepsilon (\breve{\delta},\ \breve{G}_2^i)$ and it isdefined as$(\breve{\delta},\ \breve{G}_1^i) \cap_\varepsilon (\breve{\delta},\ \breve{G}_2^i) = (\breve{\delta},\ \breve{Y}^i)$, where $\breve{Y}^i = \breve{G}_1^i \cup \breve{G}_2^i,\ i = 1,2,3,\dots,n$, and truth and false membership of $(\breve{\delta},\ \breve{Y}^i)$ is defined in such a way that

$$T_{\breve{Y}^i}(\breve{\delta}) = min\left(\sum_{\omega=1}^{\alpha} sup\ T_{\breve{\eta}_1}^{\omega}(\breve{\delta}), \sum_{\omega=1}^{\alpha} sup\ T_{\breve{\eta}_2}^{\omega}(\breve{\delta})\right),$$

$$F_{\breve{Y}^i}(\breve{\delta}) = max\left(\sum_{\lambda=1}^{\beta} sup\, F_{\breve{\eta}_1}^{\lambda}(\breve{\delta}), \sum_{\lambda=1}^{\beta} sup\, F_{\breve{\eta}_2}^{\lambda}(\breve{\delta})\right).$$

Example 10. Consider the illustrative example, suppose that

$(\breve{\delta},\ \breve{G}_1) = \{<\breve{B}_1, (0.13,\ 0.19),(0.24,0.1)>,$

$<\breve{B}_2,(0.2,0.25),(0.15,0.24)>,$
$<\breve{B}_3,(0.1,0.36),(0.34,0.12)>\},$

and

$(\breve{\delta},\breve{G}_2) = \{<\breve{B}_3,(0.01,0.16),(0.5,0.2)>,<\breve{B}_4,(0.26,0.15),(0.12,0.2)>\},$

be two RIFS. Then the extended intersection of RIFS $(\breve{\delta},\ \breve{G}_1)$ and $(\breve{\delta},\ \breve{G}_2)$is given as

$(\breve{\delta},\ \breve{Y}) = \{<\breve{B}_1, (0.13,0.19),(0.24,0.1)>,<\breve{B}_2,(0.2,0.25),(0.15,0.24)>,$
$<\breve{B}_3,(0.01,0.16),(0.5,0.2)>,<\breve{B}_4,(0.26,0.15),(0.12,0.2)>\}.$

Definition 11. Restricted union of two RIFS

The restricted union of two RIFS $(\breve{\delta},\ \breve{G}_1)$ and $(\breve{\delta},\ \breve{G}_2)$is denoted by $(\breve{\delta},\ \breve{G}_1) \cup_R (\breve{\delta},\ \breve{G}_2)$ and it isdefined as $(\breve{\delta},\ \breve{G}_1) \cup_R (\breve{\delta},\ \breve{G}_2) = (\breve{\delta},\ \breve{Y})$, where $\breve{Y} = \breve{G}_1 \cap_R \breve{G}_2$, and truth and falsemembership of $(\breve{\delta},\ \breve{Y})$ is defined in such a way that

$$T_{\breve{Y}}(\breve{\delta}) = max\left(\sum_{\omega=1}^{\alpha} sup\ T^{\omega}_{\breve{\eta}_1}(\breve{\delta}), \sum_{\omega=1}^{\alpha} sup\ T^{\omega}_{\breve{\eta}_2}(\breve{\delta})\right).$$

$$F_{\breve{Y}}(\breve{\delta}) = min\left(\sum_{\lambda=1}^{\beta} sup\, F^{\lambda}_{\breve{\eta}_1}(\breve{\delta}), \sum_{\lambda=1}^{\beta} sup\, F^{\lambda}_{\breve{\eta}_2}(\breve{\delta})\right).$$

Remark 6. The restricted union of two families of RIFS $(\breve{\delta},\ \breve{G}_1^i)$ and $(\breve{\delta},\ \breve{G}_2^i)$is denoted by

$(\breve{\delta},\ \breve{G}_1^i) \cup_R (\breve{\delta},\ \breve{G}_2^i)$ and it is defined as $(\breve{\delta},\ \breve{G}_1^i) \cup_R (\breve{\delta},\ \breve{G}_2^i) = (\breve{\delta},\ \breve{Y}^i)$, where$\breve{Y}^i = \breve{G}_1^i \cap_R \breve{G}_2^i,\ i = 1,2,3,\dots,n$, and truth and false membership of $(\breve{\delta},\ \breve{Y}^i)$ is defined in such a way that

$$T_{\breve{Y}^i}(\breve{\delta}) = max\left(\sum_{\omega=1}^{\alpha} sup\ T^{\omega}_{\breve{\eta}_1}(\breve{\delta}), \sum_{\omega=1}^{\alpha} sup\ T^{\omega}_{\breve{\eta}_2}(\breve{\delta})\right),$$

$$F_{\breve{Y}^i}(\breve{\delta}) = min\left(\sum_{\lambda=1}^{\beta} sup\, F^{\lambda}_{\breve{\eta}_1}(\breve{\delta}), \sum_{\lambda=1}^{\beta} sup\, F^{\lambda}_{\breve{\eta}_2}(\breve{\delta})\right).$$

Example 11. Consider the illustrative example, suppose that

$(\breve{\delta},\ \breve{G}_1) = \{< \breve{B}_1, (0.13, 0.19), (0.24, 0.1)>,$

$< \breve{B}_2, (0.2,0.25), (0.15,0.24) >,$
$< \breve{B}_3, (0.1,0.36), (0.34,0.12) > \},$

and

$(\breve{\delta}, \breve{G}_2) = \{< \breve{B}_3, (0.01,0.16), (0.5,0.2) >, < \breve{B}_4, (0.26,0.15), (0.12,0.2) > \},$

be two RIFS. Then the restricted union of RIFS $(\breve{\delta},\ \breve{G}_1)$ and $(\breve{\delta},\ \breve{G}_2)$is given as

$(\breve{\delta}, \breve{Y}) = \{< \breve{B}_3, (0.1,0.36), (0.34,0.12) > \}.$

Definition 12. Restricted intersection of two RIFS

The restricted intersection of two RIFS $(\breve{\delta},\ \breve{G}_1)$ and $(\breve{\delta},\ \breve{G}_2)$is denoted by $(\breve{\delta},\ \breve{G}_1) \cap_R (\breve{\delta},\ \breve{G}_2)$ and it isdefined as $(\breve{\delta},\ \breve{G}_1) \cap_R (\breve{\delta},\ \breve{G}_2) = (\breve{\delta},\ \breve{Y})$, where$\breve{Y} = \breve{G}_1 \cap_R \breve{G}_2$, and truth and false membership of $(\breve{\delta},\ \breve{Y})$ is defined in such a way that

$$T_{\breve{Y}}(\breve{\delta}) = min\left(\sum_{\omega=1}^{\alpha} sup\ T_{\breve{\eta}_1}^{\omega}(\breve{\delta}), \sum_{\omega=1}^{\alpha} sup\ T_{\breve{\eta}_2}^{\omega}(\breve{\delta})\right),$$

$$F_{\breve{Y}}(\breve{\delta}) = max\left(\sum_{\lambda=1}^{\beta} sup\ F_{\breve{\eta}_1}^{\lambda}(\breve{\delta}), \sum_{\lambda=1}^{\beta} sup\ F_{\breve{\eta}_2}^{\lambda}(\breve{\delta})\right).$$

Remark 7. The restricted intersection of two families of RIFS $(\breve{\delta},\ \breve{G}_1^i)$ and $(\breve{\delta},\ \breve{G}_2^i)$is denoted by

$(\breve{\delta},\ \breve{G}_1^i) \cap_R (\breve{\delta},\ \breve{G}_2^i)$ and it isdefined as $(\breve{\delta},\ \breve{G}_1^i) \cap_R (\breve{\delta},\ \breve{G}_2^i) = (\breve{\delta},\ \breve{Y}^i)$, where $\breve{Y}^i = \breve{G}_1^i \cap_R \breve{G}_2^i$, $i = 1,2,3,\dots,n$, and truth and false membership of $(\breve{\delta},\ \breve{Y}^i)$ is defined in such a way that

$$T_{\breve{Y}^i}(\breve{\delta}) = min\left(\sum_{\omega=1}^{\alpha} sup\ T_{\breve{\eta}_1}^{\omega}(\breve{\delta}), \sum_{\omega=1}^{\alpha} sup\ T_{\breve{\eta}_2}^{\omega}(\breve{\delta})\right),$$

$$F_{\breve{Y}^i}(\breve{\delta}) = max\left(\sum_{\lambda=1}^{\beta} sup\ F_{\breve{\eta}_1}^{\lambda}(\breve{\delta}), \sum_{\lambda=1}^{\beta} sup\ F_{\breve{\eta}_2}^{\lambda}(\breve{\delta})\right).$$

Example 12. Consider the illustrative example, suppose that

$(\breve{\delta},\ \breve{G}_1) = \{ < \breve{B}_1, (0.13,\ 0.19),(0.24,0.1)>,$

$< \breve{B}_2, (0.2,0.25), (0.15,0.24) >,$
$< \breve{B}_3, (0.1,0.36), (0.34,0.12) > \},$

and

$(\breve{\delta}, \breve{G}_2) = \{< \breve{B}_2, (0.32,0.38), (0.1,0.04) >, < \breve{B}_4, (0.26,0.15), (0.12,0.2) > \},$

be two RIFS. Then the restricted intersection of RIFS $(\breve{\delta},\ \breve{G}_1)$ and $(\breve{\delta},\ \breve{G}_2)$is given as

$(\breve{\delta}, \breve{Y}) = \{< \breve{B}_2, (0.2,0.25), (0.15,0.24) > \}.$

Definition 13. Restricted difference of two RIFS

The restricted difference of two RIFS $(\breve{\delta},\ \breve{G}_1)$ and $(\breve{\delta},\ \breve{G}_2)$is denoted by $(\breve{\delta},\ \breve{G}_1) -_R (\breve{\delta},\ \breve{G}_2)$ and it is defined as $(\breve{\delta},\ \breve{G}_1) -_R (\breve{\delta},\ \breve{G}_2) = (\breve{\delta},\ \breve{Y})$, where $\breve{Y} = \breve{G}_1 -_R \breve{G}_2$.

Example 13. Consider the illustrative example, suppose that

$(\breve{\delta},\ \breve{G}_1) = \{ < \breve{B}_1, (0.13,\ 0.19),(0.24,0.1)>,$

$< \breve{B}_2, (0.2,0.25), (0.15,0.24) >,$
$< \breve{B}_3, (0.1,0.36), (0.34,0.12) >, < \breve{B}_4, (0.16,0.14), (0.23,0.37) > \},$

and

$(\breve{\delta},\ \breve{G}_2) = \{ < \breve{B}_1, (0.2,0.3),(0.3,\ 0.15) >, < \breve{B}_2, (0.32,0.38), (0.1,0.04) >,$

$< \breve{B}_3, (0.01,0.16), (0.5,0.2) > \},$

be two RIFS. Then the restricted difference of RIFS $(\breve{\delta},\ \breve{G}_1)$ and $(\breve{\delta},\ \breve{G}_2)$ is given as

$(\breve{\delta}, \breve{Y}) = \{< \breve{B}_4, (0.16,0.14), (0.23,0.37) > \}.$

5. Some Basic Laws of RIFS

In this section, we prove some basic fundamental laws including idempotent law, identity law, domination law, De-Morgan law and commutative law with the help of illustrative example.

5.1. Idempotent Law

$(\breve{\delta},\breve{G}) \cup (\breve{\delta},\breve{G}) = (\breve{\delta},\breve{G}) = (\breve{\delta},\breve{G}) \cup_R (\breve{\delta},\breve{G}).$

$(\breve{\delta},\breve{G}) \cap (\breve{\delta},\breve{G}) = (\breve{\delta},\breve{G}) = (\breve{\delta},\breve{G}) \cap_\varepsilon (\breve{\delta},\breve{G}).$

Example 14. To prove (1) law, we consider illustrative example. For this, suppose that

$(\breve{\delta},\ \breve{G}) = \{ < \breve{B}_1, (0.13,\ 0.19),(0.24,0.1) >,$

$< \breve{B}_2, (0.2,0.25), (0.15,0.24) >,$
$< \breve{B}_3, (0.1,0.36), (0.34,0.12) >, < \breve{B}_4, (0.16,0.14), (0.23,0.37) > \}.$

One can observe

$(\breve{\delta},\ \breve{G}) \cup (\breve{\delta},\breve{G}) = \{ < \breve{B}_1, (0.13,\ 0.19),(0.24,0.1) >,$

$< \breve{B}_2, (0.2,0.25), (0.15,0.24) >,$
$< \breve{B}_3, (0.1,0.36), (0.34,0.12) >, < \breve{B}_4, (0.16,0.14), (0.23,0.37) > \}$

$= (\breve{\delta},\breve{G}) = (\breve{\delta},\breve{G}) \cup_R (\breve{\delta},\breve{G}).$

Similarly, we can prove (2).

5.2. Identity Law

$(\breve{\delta},\breve{G}) \cup \breve{\emptyset} = (\breve{\delta},\breve{G}) = (\breve{\delta},\breve{G}) \cup_R \breve{\emptyset}.$

$(\breve{\delta},\breve{G}) \cap \breve{U} = (\breve{\delta},\breve{G}) = (\breve{\delta},\breve{G}) \cap_\varepsilon \breve{U}.$

Example 15. To prove (1) law, we consider illustrative example. For this, suppose that

$(\breve{\delta},\ \breve{G}) = \{ < \breve{B}_1, (0.13,\ 0.19),(0.24,0.1) >,$

$< \breve{B}_2, (0.2, 0.25), (0.15, 0.24) >$,
$< \breve{B}_3, (0.1, 0.36), (0.34, 0.12) >, < \breve{B}_4, (0.16, 0.14), (0.23, 0.37) > \}$.

One can observe

$(\breve{\delta}, \breve{G}) \cup \breve{\emptyset} = \{ < \breve{B}_1, (0.13, 0.19), (0.24, 0.1) >$,

$< \breve{B}_2, (0.2, 0.25), (0.15, 0.24) >$,
$< \breve{B}_3, (0.1, 0.36), (0.34, 0.12) >, < \breve{B}_4, (0.16, 0.14), (0.23, 0.37) > \}$

$= (\breve{\delta}, \breve{G}) = (\breve{\delta}, \breve{G}) \cup_R \breve{\emptyset}$.

Similarly, we can prove (2).

5.3. Domination Law

$(\breve{\delta}, \breve{G}) \cup \breve{U} = \breve{U} = (\breve{\delta}, \breve{G}) \cup_R \breve{U}$.

$(\breve{\delta}, \breve{G}) \cap \breve{\emptyset} = \breve{\emptyset} = (\breve{\delta}, \breve{G}) \cap_\varepsilon \breve{\emptyset}$.

Example 16. To prove (1) law, we consider illustrative example. For this, suppose that

$(\breve{\delta}, \breve{G}) = \{ < \breve{B}_1, (0.13, 0.19), (0.24, 0.1) >$,

$< \breve{B}_2, (0.2, 0.25), (0.15, 0.24) >$,
$< \breve{B}_3, (0.1, 0.36), (0.34, 0.12) >, < \breve{B}_4, (0.16, 0.14), (0.23, 0.37) > \}$.

One can observe

$(\breve{\delta}, \breve{G}) \cup \breve{U} = \{ < \breve{B}_1, (0.13, 0.19), (0.24, 0.1) >$,

$< \breve{B}_2, (0.2, 0.25), (0.15, 0.24) >$,
$< \breve{B}_3, (0.1, 0.36), (0.34, 0.12) >, < \breve{B}_4, (0.16, 0.14), (0.23, 0.37) > \} \cup \breve{U}$

$= \breve{U} = (\breve{\delta}, \breve{G}) \cup_R \breve{U}$.

Similarly, we can prove (2).

5.4. De-Morgan Law

$\left((\breve{\delta}, \breve{G}_1) \cup (\breve{\delta}, \breve{G}_2)\right)^c = (\breve{\delta}, \breve{G}_1)^c \cap_\varepsilon (\breve{\delta}, \breve{G}_2)^c$.

$\left((\breve{\delta}, \breve{G}_1) \cap_\varepsilon (\breve{\delta}, \breve{G}_2)\right)^c = (\breve{\delta}, \breve{G}_1)^c \cup (\breve{\delta}, \breve{G}_2)^c$.

Example 17. To prove (1) law, we consider illustrative example. For this, suppose that L.H.S is

$\left(\breve{\delta},\breve{G}_1\right) \cup \left(\breve{\delta},\breve{G}_2\right) = \{<\breve{B}_1, (0.2, 0.3),(0.24,0.1)>,$

$<\breve{B}_2, (0.32,0.38), (0.1,0.04)>,$
$<\breve{B}_3, (0.1,0.36), (0.34,0.12)>, <\breve{B}_4, (0.26,0.15), (0.12,0.2)>\}.$

Then

$\left(\left(\breve{\delta},\breve{G}_1\right) \cup \left(\breve{\delta},\breve{G}_2\right)\right)^c = \{<\breve{B}_1, (0.24,0.1), (0.2,0.3)>,$

$<\breve{B}_2, (0.1,0.04), (0.32,0.38)>,$
$<\breve{B}_3, (0.34,0.12), (0.1,0.36)>, <\breve{B}_4, (0.12,0.2), (0.26,0.15)>\}.$

Now consider R.H.S.

$\left(\breve{\delta},\breve{G}_1\right)^c \cap_\varepsilon \left(\breve{\delta},\breve{G}_2\right)^c = \{<\breve{B}_1, (0.24,0.1), (0.2,0.3)>,$

$<\breve{B}_2, (0.1,0.04), (0.32,0.38)>,$
$<\breve{B}_3, (0.34,0.12), (0.1,0.36)>, <\breve{B}_4, (0.12,0.2), (0.26,0.15)>\}.$

From this, it is clear that L.H.S.=R.H.S. Similarly, we can prove (2).

5.5. Commutative Law

$\left(\breve{\delta},\breve{G}_1\right) \cup \left(\breve{\delta},\breve{G}_2\right) = \left(\breve{\delta},\breve{G}_2\right) \cup \left(\breve{\delta},\breve{G}_1\right).$

$\left(\breve{\delta},\breve{G}_1\right) \cup_R \left(\breve{\delta},\breve{G}_2\right) = \left(\breve{\delta},\breve{G}_2\right) \cup_R \left(\breve{\delta},\breve{G}_1\right).$

$\left(\breve{\delta},\breve{G}_1\right) \cap \left(\breve{\delta},\breve{G}_2\right) = \left(\breve{\delta},\breve{G}_2\right) \cap \left(\breve{\delta},\breve{G}_1\right).$

$\left(\breve{\delta},\breve{G}_1\right) \cap_\varepsilon \left(\breve{\delta},\breve{G}_2\right) = \left(\breve{\delta},\breve{G}_2\right) \cap_\varepsilon \left(\breve{\delta},\breve{G}_1\right).$

Example 18. To prove (1) law, we consider illustrative example. For this, suppose that

L.H.S:

$\left(\breve{\delta},\breve{G}_1\right) \cup \left(\breve{\delta},\breve{G}_2\right) = \{<\breve{B}_1, (0.2, 0.3),(0.24,0.1)>,$

$<\breve{B}_2, (0.32,0.38), (0.1,0.04)>,$
$<\breve{B}_3, (0.1,0.36), (0.34,0.12)>, <\breve{B}_4, (0.26,0.15), (0.12,0.2)>\}.$

R.H.S:

$\left(\breve{\delta},\breve{G}_2\right) \cup \left(\breve{\delta},\breve{G}_1\right) = \{<\breve{B}_1, (0.2, 0.3), (0.24,0.1)>,$

$<\breve{B}_2, (0.32,0.38), (0.1,0.04)>,$
$<\breve{B}_3, (0.1,0.36), (0.34,0.12)>, <\breve{B}_4, (0.26,0.15), (0.12,0.2)>\}.$

From above equation, we meet the required result. Similarly, we can prove the remaining.

6. Conclusion

In this article, the basic fundamentals of refined intuitionistic fuzzy Set (RIFS) i.e. RIF subset, Equal RIFS, Complement of RIFS, Null RIFS and aggregation operators i.e. union, intersection, restricted intersection, extended union, extended intersection and restricted difference of two RIFS is defined. All these fundamentals are explained using an illustrative example. Further extension can be sought through developing similarity measures for comparison purposes.

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Interval Valued Pythagorean Fuzzy Ideals in Semigroups

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ABSTRACT

In this paper, we define the new notion of interval-valued Pythagorean fuzzy ideals in semigroups and established the properties of its with suitable examples. Also, we introduce the concept of interval valued Pythagorean fuzzy sub-semigroup, interval valued Pythagorean fuzzy left (resp. right) ideal, interval valued Pythagorean fuzzy bi-ideal, interval valued Pythagorean fuzzy interior ideal and homomorphism of an interval valued Pythagorean fuzzy ideal in semigroups with suitable illustration. We show that every interval valued Pythagorean fuzzy left (resp. right) ideal is an interval valued Pythagorean fuzzy bi-ideal.

1. Introduction

In 1965, Zadeh [18, 19] introduced the concept of a fuzzy set. He also developed the notion of interval-valued fuzzy set in 1975, which extends the fuzzy set. A semigroup is an algebraic structure comprising a non-empty set together with an associative binary operation. Atanassov [2] introduced the intuitionistic fuzzy set with some properties. Atanassov [3] developed the concept of interval-valued intuitionistic fuzzy set. Thillaigovindan and Chinnadurai [15, 16] discussed interval-valued fuzzy ideals in algebraic structures. In 2018, Chen [4, 5] introduced the concept of interval-valued Pythagorean fuzzy outranking of various methods in the application. Garg [8, 9] presented the notion of interval-valued Pythagorean fuzzy sets of multi-criteria decision-making methods. In 2013, Yager [17] started the notion of Pythagorean fuzzy set, the sum of the squares of membership and non-membership belongs to the unit interval [0, 1]. Peng [13]

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developed the new operations for an interval-valued Pythagorean fuzzy set. Peng and Yang [14] presented the notion of interval-valued Pythagorean fuzzy set. In 2019, Hussain et al. [10] started the notions of rough Pythagorean fuzzy ideals in the semigroups. Akram[1] established the properties of fuzzy lie algebras. Kumar et al. [11] approached transportation decision making problems using Pythagorean fuzzy set. Das and Edalatpanah [6] studied the concept of fuzzy linear fractional progress with trapezoidal fuzzy numbers. Edalatpanah [7] used triangular intuitionistic fuzzy numbers to deal with data envelopment analysis model. Najafi and Edalatpanah [12] used iterative methods to study linear complementarily problems. In this paper, we discuss some of the properties of interval-valued Pythagorean fuzzy ideals in the semigroups.

2. Preliminaries

Definition 1. [12]. Let X be a universe of discourse, A Pythagorean fuzzy set (PFS) $P = \{w, \phi_p(w), \psi_p(w)/w \in X\}$ where $\phi: X \to [0,1]$ and $\psi: X \to [0,1]$ represent the degree of membership and non-membership of the object $w \in X$ to the set P subset to the condition $0 \leq (\phi_p(w))^2 + (\psi_p(w))^2 \leq 1$ for all $w \in X$. For the sake of simplicity a PFS is denoted as $P = (\phi_p(w), \psi_p(w))$.

3. Interval-Valued Pythagorean Fuzzy Ideals in Semigroups

Definition 2. An Interval-Valued Pythagorean Fuzzy Set (IVPFS) $\tilde{P} = [\widetilde{\phi_p}, \widetilde{\psi_p}]$ on S is known to be an interval-valued Pythagorean fuzzy sub-semigroup of S. If for all $w_1, w_2 \in S$, it holds.

$$\widetilde{\phi_p}(w_1 w_2) \geq \min\{\widetilde{\phi_p}(w_1), \widetilde{\phi_p}(w_2)\},$$

$$\widetilde{\psi_p}(w_1 w_2) \leq \max\{\widetilde{\psi_p}(w_1), \widetilde{\psi_p}(w_2)\}.$$

Example 1. Consider a semigroup $S = \{u, v, w, x, y\}$ with the Cayley Table.

Table 1. Cayley table.

•	u	v	w	x	y
u	u	u	u	u	u
v	u	v	u	x	u
w	u	y	w	w	y
x	u	v	x	x	v
y	u	y	u	w	u

Define an interval-valued Pythagorean fuzzy set(IVPFS) $\tilde{P} = [\widetilde{\phi_p}, \widetilde{\psi_p}]$ in S as follows.

S	$[\widetilde{\phi_p}(w_1), \widetilde{\psi_p}(w_1)]$
u	[0.7,0.8], [0.1,0.2]
v	[0.4,0.6], [0.4,0.5]
w	[0.3,0.5], [0.5,0.6]
x	[0.1,0.2], [0.3,0.5]
y	[0.3,0.5], [0.5,0.6]

$\widetilde{\phi_p}(uv) \geq \min\{\widetilde{\phi_p}(u), \widetilde{\phi_p}(v)\}$
([0.7,0.8],[0.1,0.2]) ≥[0.4,0.6],[0.1,0.2].
$\widetilde{\psi_p}(uv) \leq \max\{\widetilde{\psi_p}(u), \widetilde{\psi_p}(v)\}$.
([0.7,0.8],[0.1,0.2])≤[0.7,0.8],[0.4,0.5].

Thus $\tilde{P} = [\widetilde{\phi_p}, \widetilde{\psi_p}]$ is an Interval-Valued Pythagorean Fuzzy Sub-Semigroup (IVPFSS) of S.

Definition 3. An IVPFS $\tilde{P} = (\widetilde{\phi_p}, \widetilde{\psi_p})$ on semigroup S, is said to be an interval-valued Pythagorean fuzzy left ($\tilde{P}_{LI}$)(resp.right($\tilde{P}_{RI}$)) ideal of S. If for all $w_1, w_2 \in S$, it holds.

$\widetilde{\phi_p}(w_1 w_2) \geq \widetilde{\phi_p}(w_2)$;
$\widetilde{\psi_p}(w_1 w_2) \leq \widetilde{\psi_p}(w_2)$ (resp.right($\tilde{P}_{RI}$));
$\widetilde{\phi_p}(w_1 w_2) \geq \widetilde{\phi_p}(w_1)$;
$\widetilde{\psi_p}(w_1 w_2) \leq \widetilde{\psi_p}(w_1)$.

Definition 4. An IVPFS $\tilde{P} = [\widetilde{\phi_p}, \widetilde{\psi_p}]$ on S is called IVPFI ($\tilde{P}_I$) of S. If for all $w_1, w_2 \in S$, it $\tilde{P}$ is both a left and right IVPFI of S.

$(w_1 w_2) \geq \max\{\widetilde{\phi_p}(w_1), \widetilde{\phi_p}(w_2)\}$;

$\widetilde{\psi_p}(w_1 w_2) \leq \min\{\widetilde{\psi_p}(w_1), \widetilde{\psi_p}(w_2)\}$.

Definition 5. An IVPFS $\tilde{P} = [\widetilde{\phi_p}, \widetilde{\psi_p}]$ on S is known to be an interval-Valued Pythagorean Fuzzy Bi-Ideal (IVPFBI) ($\tilde{P}_{BI}$) of S. If for all $a, w_1, w_2 \in S$ and satisfy.

$\widetilde{\phi_p}(w_1 a w_2) \geq \min\{\widetilde{\phi_p}(w_1), \widetilde{\phi_p}(w_2)\}$;

$\widetilde{\psi_p}(w_1 a w_2) \leq max\{\widetilde{\psi_p}(w_1), \widetilde{\psi_p}(w_2)\}$.

Example 2. Consider a semigroup $S = \{u, v, w, x, y\}$ with the Cayley Table.

Define an interval-valued Pythagorean fuzzy set $\tilde{P} = [\widetilde{\phi_p}, \widetilde{\psi_p}]$ in S as follows.

S	$[\widetilde{\phi_p}(w_1), \widetilde{\psi_p}(w_1)]$
u	[0.8,0.9], [0.1,0.3]
v	[0.3,0.5], [0.7,0.9]
w	[0.4,0.6], [0.6,0.7]
x	[0.3,0.5], [0.7,0.9]
y	[0.7,0.8], [0.4,0.5]

Thus $\tilde{P} = [\widetilde{\phi_p}, \widetilde{\psi_p}]$ is an interval valued Pythagorean fuzzy bi-ideal of S.

Definition 7. An IVPFS $\tilde{P} = \langle[\widetilde{\phi_p}, \widetilde{\psi_p}]\rangle$ on S is known to be an interval-valued Pythagorean fuzzy interior ideal (IVPFII) ($\tilde{P}_{II}$) of S. If for all $a, w_1, w_2 \in S$ and satisfy.

$\widetilde{\phi_p}(w_1 a w_2) \geq \widetilde{\phi_p}(a)$;

$\widetilde{\psi_p}(w_1aw_2) \leq \widetilde{\psi_p}(a)$.

Definition 8. For any non-empty subset N of a semigroup S is defined to be a structure $\chi_N = \{w_1, [\tilde{\phi}_{\chi_N}(w_1), \tilde{\psi}_{\chi_N}(w_1)] | w_1 \in S\}$ which is briefly denoted by $\chi_N = [\tilde{\phi}_{\chi_N}, \tilde{\psi}_{\chi_N}]$

where, $\tilde{\phi}_{\chi_N}(w_1) = \begin{cases} \tilde{1} \; if \; x \in N \\ \tilde{0} \; otherwise \end{cases}$ $\widetilde{\psi}_{\chi_N}(w_1) = \begin{cases} \tilde{0} \; if \; x \in N \\ \tilde{1} \; otherwise \end{cases}$.

Theorem 1. Let S be a semigroup. Then the following are equivalent.

- The intersection of two interval-valued Pythagorean fuzzy sub-semigroup of S, is an interval-valued Pythagorean fuzzy sub-semigroup of S.
- The intersection of two interval-valued Pythagorean fuzzy left (resp. right) ideal of S, is IVPFLI (resp. IVPFRI) of S.

Proof. Let $\widetilde{P_1} = [\tilde{\phi}_{p_1}, \tilde{\psi}_{p_1}]$ and $\widetilde{P_2} = [\tilde{\phi}_{p_2}, \tilde{\psi}_{p_2}]$ be two interval-valued Pythagorean fuzzy sub-semigroup of S. Let $w_1, w_2 \in S$.

Then,

$$\begin{aligned}
&(\widetilde{\phi}_{p_1} \cap \widetilde{\phi}_{p_2})(w_1, w_2) = \min\{\widetilde{\phi}_{p_1}(w_1, w_2), \widetilde{\phi}_{p_2}(w_1, w_2)\} \\
&\geq \min\left\{\min\{\widetilde{\phi}_{p_1}(w_1), \widetilde{\phi}_{p_1}(w_2)\}, \min\{\widetilde{\phi}_{p_2}(w_1), \widetilde{\phi}_{p_2}(w_2)\}\right\} \\
&= \min\left\{\min\{\widetilde{\phi}_{p_1}(w_1), \widetilde{\phi}_{p_2}(w_1)\}, \min\{\widetilde{\phi}_{p_1}(w_2), \widetilde{\phi}_{p_2}(w_2)\}\right\} \\
&= \min\{\widetilde{\phi}_{p_1} \cap \widetilde{\phi}_{p_2}(w_1), \widetilde{\phi}_{p_1} \cap \widetilde{\phi}_{p_2}(w_2)\};
\end{aligned}$$

$$\begin{aligned}
&(\widetilde{\psi}_{p_1} \cup \widetilde{\psi}_{p_2})(w_1, w_2) = \max\{\widetilde{\psi}_{p_1}(w_1, w_2), \widetilde{\psi}_{p_2}(w_1, w_2)\} \\
&\leq \max\left\{\max\{\widetilde{\psi}_{p_1}(w_1), \widetilde{\psi}_{p_1}(w_2)\}, \max\{\widetilde{\psi}_{p_2}(w_1), \widetilde{\psi}_{p_2}(w_2)\}\right\} \\
&= \max\left\{\max\{\widetilde{\psi}_{p_1}(w_1), \widetilde{\psi}_{p_2}(w_1)\}, \max\{\widetilde{\psi}_{p_1}(w_2), \widetilde{\psi}_{p_2}(w_2)\}\right\} \\
&= \max\{\widetilde{\psi}_{p_1} \cup \widetilde{\psi}_{p_2}(w_1), \widetilde{\psi}_{p_1} \cup \widetilde{\psi}_{p_2}(w_2)\}.
\end{aligned}$$

Therefore, $\tilde{P}_1 \cap \tilde{P}_2 = \{\langle(\widetilde{\phi}_{p_1} \cap \widetilde{\phi}_{p_2}), (\widetilde{\psi}_{p_1} \cup \widetilde{\psi}_{p_2})\rangle\}$.

Interval-valued Pythagorean fuzzy sub-semigroup of S.

$$\begin{aligned}
&(\widetilde{\phi}_{p_1} \cap \widetilde{\phi}_{p_2})(w_1, w_2) = \min\{\widetilde{\phi}_{p_1}(w_1, w_2), \widetilde{\phi}_{p_2}(w_1, w_2)\} \\
&\geq \min\{\widetilde{\phi}_{p_1}(w_2), \widetilde{\phi}_{p_2}(w_2)\} \\
&= (\widetilde{\phi}_{p_1} \cap \widetilde{\phi}_{p_2})(w_2);
\end{aligned}$$

$$\begin{aligned}
&(\widetilde{\psi}_{p_1} \cup \widetilde{\psi}_{p_2})(w_1, w_2) = \max\{\widetilde{\psi}_{p_1}(w_1, w_2), \widetilde{\psi}_{p_2}(w_1, w_2)\} \\
&\leq \max\{\widetilde{\psi}_{p_1}(w_2), \widetilde{\psi}_{p_2}(w_2)\} \\
&= (\widetilde{\psi}_{p_1} \cup \widetilde{\psi}_{p_2})(w_2).
\end{aligned}$$

Therefore, $\tilde{P}_1 \cap \tilde{P}_2 = \{\langle(\tilde{\phi}_{p_1} \cap \tilde{\phi}_{p_2}), (\tilde{\psi}_{p_1} \cup \tilde{\psi}_{p_2})\rangle\}$ is an interval-valued Pythagorean fuzzy left (resp. right) ideal of S.

Theorem 2. An IVPFS $\tilde{P} = [\tilde{\phi}_p, \tilde{\psi}_p]$ of a semigroup S is an IVPFBI of S, if and only if $\langle(\phi_p^L, \phi_p^U), (\psi_p^L, \psi_p^U)\rangle$ of S.

Proof. Let $\tilde{P} = [\tilde{\phi}_p, \tilde{\psi}_p]$ be an interval-valued Pythagorean fuzzy bi-ideal of S, for any $w_1, w_2 \in S$. Then, we have membership

$$[\phi_p^L(w_1w_2), \phi_p^U(w_1w_2)] = \widetilde{\phi}_p(w_1w_2)$$
$$\geq \min\{\widetilde{\phi}_p(w_1), \widetilde{\phi}_p(w_2)\}$$
$$= \min\{[\phi_p^L(w_1), \phi_p^U(w_1)], [\phi_p^L(w_2), \phi_p^U(w_2)]\}$$
$$= \min\{[\phi_p^L(w_1), \phi_p^L(w_2)], [\phi_p^U(w_1), \phi_p^U(w_2)]\}.$$

It follows that $\phi_p^L(w_1w_2) \geq min\{\phi_p^L(w_1), \phi_p^L(w_2)\}$ and $\phi_p^U(w_1w_2) \geq min\{\phi_p^U(w_1), \phi_p^U(w_2)\}$and non-membership

$$[\psi_p^L(w_1w_2), \psi_p^U(w_1w_2)] = \widetilde{\psi}_p(w_1w_2)$$
$$\leq \max\{\widetilde{\psi}_p(w_1), \widetilde{\psi}_p(w_2)\}$$
$$= \max\{[\psi_p^L(w_1), \psi_p^U(w_1)], [\psi_p^L(w_2), \psi_p^U(w_2)]\}$$
$$= \max\{[\psi_p^L(w_1), \psi_p^L(w_2)], [\psi_p^U(w_1), \psi_p^U(w_2)]\}.$$

It follows that $\psi_p^L(w_1w_2) \leq max\{\psi_p^L(w_1), \psi_p^L(w_2)\}$ and $\psi_p^U(w_1w_2) \leq max\{\psi_p^U(w_1), \phi_p^U(w_2)\}$

Therefore, $\tilde{P} = \langle(\phi_p^L, \phi_p^U), (\psi_p^L, \psi_p^U)\rangle$ are Pythagorean fuzzy ideal of S.

Conversely, suppose that $([\phi_p^L, \phi_p^U], [\psi_p^L, \psi_p^U])$ are Pythagorean fuzzy ideal of S, le$w_1, w_2 \in S$ t.

$$\widetilde{\phi}_p(w_1w_2) = [\phi_p^L(w_1w_2), \phi_p^U(w_1w_2)]$$
$$\geq [\min\{\phi_p^L(w_1), \phi_p^L(w_2)\}, \min\{\phi_p^U(w_1), \phi_p^U(w_2)\}]$$
$$= \min\{[\phi_p^L(w_1), \phi_p^U(w_1)], [\phi_p^L(w_2), \phi_p^U(w_2)]\}$$
$$= \min\{\widetilde{\phi}_p(w_1), \widetilde{\phi}_p(w_2)\};$$

$$\widetilde{\psi}_p(w_1w_2) = [\psi_p^L(w_1w_2), \psi_p^U(w_1w_2)]$$
$$\leq [\max\{\psi_p^L(w_1), \psi_p^L(w_2)\}, \max\{\psi_p^U(w_1), \psi_p^U(w_2)\}]$$
$$= \max\{[\psi_p^L(w_1), \psi_p^U(w_1)], [\psi_p^L(w_2), \psi_p^U(w_2)]\}$$
$$= \max\{\widetilde{\psi}_p(w_1), \widetilde{\psi}_p(w_2)\}.$$

$\tilde{P} = [\tilde{\phi}_p, \tilde{\psi}_p]$ is an interval-valued Pythagorean fuzzy sub-semigroup of S.

$$\widetilde{\phi}_p(w_1aw_2) = [\phi_p^L(w_1aw_2), \phi_p^U(w_1aw_2)]$$
$$\geq [\min\{\phi_p^L(w_1), \phi_p^L(w_2)\}, \min\{\phi_p^U(w_1), \phi_p^U(w_2)\}]$$
$$= \min\{[\phi_p^L(w_1), \phi_p^U(w_1)], [\phi_p^L(w_2), \phi_p^U(w_2)]\}$$
$$= \min\{\widetilde{\phi}_p(w_1), \widetilde{\phi}_p(w_2)\};$$

$$\widetilde{\psi}_p(w_1aw_2) = [\psi_p^L(w_1aw_2), \psi_p^U(w_1aw_2)]$$
$$\leq [\max\{\psi_p^L(w_1), \psi_p^L(w_2)\}, \max\{\psi_p^U(w_1), \psi_p^U(w_2)\}]$$
$$= \max\{[\psi_p^L(w_1), \psi_p^U(w_1)], [\psi_p^L(w_2), \psi_p^U(w_2)]\}$$

$= \max\{\widetilde{\psi}_p(w_1), \widetilde{\psi}_p(w_2)\}$.

$\tilde{P} = [\tilde{\phi}_p, \tilde{\psi}_p]$ is an interval-valued Pythagorean fuzzy bi-ideal of S.

Theorem 3. If $\{P_i\}_{i\in I}$ is a family of interval-valued Pythagorean fuzzy bi-ideal of a semigroup S. Then $\cap P_i$ is an interval-valued Pythagorean fuzzy bi-ideal of S. Where $\cap P_i = (\cap \tilde{\phi}_{p_i}, \cup \tilde{\psi}_{p_i})$.

$\cap (\tilde{\phi}_{p_i}) = inf\{(\tilde{\phi}_{p_i})(w_1)/i \in I, w_1 \in S\}$, $\cup (\tilde{\psi}_{p_i}) = sup\{(\tilde{\psi}_{p_i})(w_1)/i \in I, w_1 \in S\}$ and $i \in I$ is any index set.

Proof. Since $\tilde{P}_i = \langle [\tilde{\phi}_{p_i}, \tilde{\psi}_{p_i}] | i \in I\rangle$ is a family of interval-valued Pythagorean fuzzy bi-ideal of S.

Let $a, w_1, w_2 \in S$.

$$\begin{aligned}
&\cap \widetilde{\phi}_{p_i}(w_1, w_2) = \inf\{\widetilde{\phi}_{p_i}(w_1, w_2)/i \in I, w_1, w_2 \in S\}\\
&\geq \inf\left\{\min\{\widetilde{\phi}_{p_i}(w_1), \widetilde{\phi}_{p_i}(w_2)\}\right\}\\
&= \min\left\{\inf\left(\widetilde{\phi}_{p_i}(w_1)\right), \inf\left(\widetilde{\phi}_{p_i}(w_2)\right)\right\}\\
&= \min\{\cap \widetilde{\phi}_{p_i}(w_1), \cap \widetilde{\phi}_{p_i}(w_2)\};
\end{aligned}$$

$$\begin{aligned}
&\cup \widetilde{\psi}_{p_i}(w_1 w_2) = \sup\{\widetilde{\psi}_{p_i}(w_1 w_2)/i \in I, w_1, w_2 \in S\}\\
&\leq \sup\left\{\max\{\widetilde{\psi}_{p_i}(w_1), \widetilde{\psi}_{p_i}(w_2)\}\right\}\\
&= \max\left\{\sup\left(\widetilde{\psi}_{p_i}(w_1)\right), \sup\left(\widetilde{\psi}_{p_i}(w_2)\right)\right\}\\
&= \max\{\cup \widetilde{\psi}_{p_i}(w_1), \cup \widetilde{\psi}_{p_i}(w_2)\}.
\end{aligned}$$

Hence, $\cap \tilde{P}_i = (\cap \tilde{\phi}_{p_i}, \cup \tilde{\psi}_{p_i})$ is an interval-valued Pythagorean fuzzy sub-semigorup of S.

$$\begin{aligned}
&\cap \widetilde{\phi}_{p_i}(w_1 a w_2) = \inf\{\widetilde{\phi}_{p_i}(w_1 a w_2)/i \in I, a, w_1, w_2 \in S\}\\
&\geq \inf\left\{\min\{\widetilde{\phi}_{p_i}(w_1), \widetilde{\phi}_{p_i}(w_2)\}\right\}\\
&= \min\left\{\inf\left(\widetilde{\phi}_{p_i}(w_1)\right), \inf\left(\widetilde{\phi}_{p_i}(w_2)\right)\right\}\\
&= \min\{\cap \widetilde{\phi}_{p_i}(w_1), \cap \widetilde{\phi}_{p_i}(w_2)\}.
\end{aligned}$$

$$\begin{aligned}
&\cup \widetilde{\psi}_{p_i}(w_1 a w_2) = \sup\{\widetilde{\psi}_{p_i}(w_1 a w_2)/i \in I, a, w_1, w_2 \in S\}\\
&\leq \sup\left\{\max\{\widetilde{\psi}_{p_i}(w_1), \widetilde{\psi}_{p_i}(w_2)\}\right\}\\
&= \max\left\{\sup\left(\widetilde{\psi}_{p_i}(w_1)\right), \sup\left(\widetilde{\psi}_{p_i}(w_2)\right)\right\}\\
&= \max\{\cup \widetilde{\psi}_{p_i}(w_1), \cup \widetilde{\psi}_{p_i}(w_2)\}.
\end{aligned}$$

Hence, $\cap P_i = (\cap \tilde{\phi}_{p_i}, \cup \tilde{\psi}_{p_i})$ is an interval-valued Pythagorean fuzzy bi-ideals of S.

Theorem 4. Let N be any non-empty subset of a semigroup S. Then N is a bi-ideal of S, if and only if the characteristic interval-valued Pythagorean fuzzy set $\chi_N = [\tilde{\phi}_{p\chi_N}, \tilde{\psi}_{p\chi_N}]$ is IVPFBI of S.

Proof. Assume that N is a bi-ideal of S. Let $a, w_1, w_2 \in S$.

Suppose that $\tilde{\phi}_{p\chi_N}(w_1w_2) < min\{\tilde{\phi}_{p\chi_N}(w_1), \tilde{\phi}_{p\chi_N}(w_2)\}$ and $\tilde{\psi}_{p\chi_N}(w_1w_2) > max\{\tilde{\psi}_{p\chi_N}(w_1), \tilde{\psi}_{p\chi_N}(w_2)\}$ it follows that $\tilde{\phi}_{p\chi_N}(w_1w_2) = 0,\ min\{\tilde{\phi}_{p\chi_N}(w_1), \tilde{\phi}_{p\chi_N}(w_2)\} = 1$

,$\tilde{\psi}_{p\chi_N}(w_1w_2) = 1,\ max\{\tilde{\psi}_{p\chi_N}(w_1), \tilde{\psi}_{p\chi_N}(w_2)\} = 0$.

This implies that $w_1, w_2 \in N$ by $w_1, w_2 \notin N$ a contradiction to N.

So $\tilde{\phi}_{p\chi_N}(w_1w_2) \geq min\{\tilde{\phi}_{p\chi_N}(w_1), \tilde{\phi}_{p\chi_N}(w_2)\}$, $\tilde{\psi}_{p\chi_N}(w_1w_2) \leq max\{\tilde{\psi}_{p\chi_N}(w_1), \tilde{\psi}_{p\chi_N}(w_2)\}$.

Suppose that $\tilde{\phi}_{p\chi_N}(w_1aw_2) < min\{\tilde{\phi}_{p\chi_N}(w_1), \tilde{\phi}_{p\chi_N}(w_2)\}$ and $\tilde{\psi}_{p\chi_N}(w_1aw_2) > max\{\tilde{\psi}_{p\chi_N}(w_1), \tilde{\psi}_{p\chi_N}(w_2)\}$ it follows that $\tilde{\phi}_{p\chi_N}(w_1aw_2) = 0,\ min\{\tilde{\phi}_{p\chi_N}(w_1), \tilde{\phi}_{p\chi_N}(w_2)\} = 1$, $\tilde{\psi}_{p\chi_N}(w_1w_2) = 1$, $max\{\tilde{\psi}_{p\chi_N}(w_1), \tilde{\psi}_{p\chi_N}(w_2)\} = 0$.

This implies that $a, w_1, w_2 \in N$ by $a, w_1, w_2 \notin N$ a contradiction to N.

So $\tilde{\phi}_{p\chi_N}(w_1aw_2) \geq min\{\tilde{\phi}_{p\chi_N}(w_1), \tilde{\phi}_{p\chi_N}(w_2)\}$, $\tilde{\psi}_{p\chi_N}(w_1aw_2) \leq max\{\tilde{\psi}_{p\chi_N}(w_1), \tilde{\psi}_{p\chi_N}(w_2)\}$.

This shows that χ_N is an interval-valued Pythagorean fuzzy bi-ideal of S.

Conversely, $\chi_N = [\tilde{\phi}_{p\chi_N}, \tilde{\psi}_{p\chi_N}]$ is an IVPFBI of S for any subset N of S.

Let $w_1, w_2 \in N$ then $\tilde{\phi}_{p\chi_N}(w_1) = \tilde{\phi}_{p\chi_N}(w_2) = \tilde{1}$, $\tilde{\psi}_{p\chi_N}(w_1) = \tilde{\psi}_{p\chi_N}(w_2) = \tilde{0}$, since χ_N is an IVPFBI of S.

$\tilde{\phi}_{p\chi_N}(w_1w_2) \geq min\{\tilde{\phi}_{p\chi_N}(w_1), \tilde{\phi}_{p\chi_N}(w_2)\} \geq min\{\tilde{1}, \tilde{1}\} = \tilde{1}$, $\tilde{\psi}_{p\chi_N}(w_1w_2) \leq max\{\tilde{\psi}_{p\chi_N}(w_1), \tilde{\psi}_{p\chi_N}(w_2)\} \leq max\{\tilde{0}, \tilde{0}\} = \tilde{0}$.

This implies that $w_{1,}w_2 \in N$.

Let $a, w_1, w_2 \in N$ then $\tilde{\phi}_{p\chi_N}(w_1) = \tilde{\phi}_{p\chi_N}(a) = \tilde{\phi}_{p\chi_N}(w_2) = \tilde{1}$,

$\tilde{\psi}_{p\chi_N}(w_1) = \tilde{\psi}_{p\chi_N}(a) = \tilde{\psi}_{p\chi_N}(w_2) = \tilde{0}$, since χ_N is an IVPFBI of S.

$\tilde{\phi}_{p\chi_N}(w_1aw_2) \geq min\{\tilde{\phi}_{p\chi_N}(w_1), \tilde{\phi}_{p\chi_N}(w_2)\} \geq min\{\tilde{1}, \tilde{1}\} = \tilde{1}$, $\tilde{\psi}_{p\chi_N}(w_1aw_2) \leq max\{\tilde{\psi}_{p\chi_N}(w_1), \tilde{\psi}_{p\chi_N}(w_2)\} \leq max\{\tilde{0}, \tilde{0}\} = \tilde{0}$.

Which implies that $w_1, w_2 \in N$. Hence N is a bi- ideal of S.

Theorem 5. If $\{\tilde{P}_i\}_{i\in I}$ is a family of interval-valued Pythagorean fuzzy interior ideal of a semigroup S. Then $\cap\ \tilde{P}_i$ is an interval-valued Pythagorean fuzzy interior ideal (IVPFII) of S.

Where $\cap\ \tilde{P}_i = (\cap\, \tilde{\phi}_{p_i}, \cup\, \tilde{\psi}_{p_i})$;

$\cap\,(\tilde{\phi}_{p_i}) = inf\{(\tilde{\phi}_{p_i})(w_1)/i \in I, w_1 \in S\}$, $\cup\,(\tilde{\psi}_{p_i}) = sup\{(\tilde{\psi}_{p_i})(w_1)/i \in I, w_1 \in S\}$ and $i \in I$ is any index set.

Theorem 6. Let N be any non-empty subset of a semigroup S. Then N is a interior ideal of S, if and only if the characteristic interval-valued Pythagorean fuzzy set $\chi_N = [\tilde{\phi}_{p\chi_N}, \tilde{\psi}_{p\chi_N}]$ is IVPFII of S.

4. Homomorphism of Interval-Valued Pythagorean Fuzzy Ideals in Semigroups

Let R and T be two non-empty sets of semigroup S. A mapping $f: R \to T$ is called a homomorphism if $(rt) = f(r)f(t)\ \forall r, t \in R$.

Definition 9. Let f be a mapping from a set R to a set T and $\tilde{P} = [\tilde{\phi}_p, \tilde{\psi}_p]$ be an interval-valued Pythagorean fuzzy set R the image of R (i.e.) $f(\tilde{P}) = (f(\tilde{\phi}_p), f(\tilde{\psi}_p))$ is an interval-valued Pythagorean fuzzy set of T is defined by

$$f(\tilde{P})(r) = \begin{cases} f(\widetilde{\phi_P})(r) = \begin{cases} \sup\limits_{t \in f'(r)} (\widetilde{\phi_P})(t), & \text{if} f^{-1}(r) = 0 \\ [0,0] & \text{otherwise} \end{cases} \\ f(\widetilde{\psi_P})(r) = \begin{cases} \inf\limits_{t \in f'(r)} (\widetilde{\psi_P})(t), & \text{if} f^{-1}(r) = 0 \\ [1,1] & \text{otherwise} \end{cases} \end{cases}$$

Let f be a mapping from a set R to T and $\tilde{P} = [\tilde{\phi}_p, \tilde{\psi}_p]$ be an interval-valued Pythagorean fuzzy set of T then the preimage of T (i.e.) $f^{-1}(\tilde{P}) = \{(f^{-1}(\tilde{\phi}_p), f^{-1}(\tilde{\psi}_p))\}$ is an interval-valued Pythagorean fuzzy set of R is defined as

$$f^{-1}(\tilde{P})(r) = \begin{cases} f^{-1}(\widetilde{\phi_p})(r) = \widetilde{\phi_p}(f(r)) \\ f^{-1}(\widetilde{\psi_p})(r) = \widetilde{\psi_p}(f(r)) \end{cases}.$$

Theorem 7. Let R, T be a semigroups, $f: R \to T$ be a homomorphism of semigroups.

If $\tilde{P} = [\tilde{\phi}_p, \tilde{\psi}_p]$ is an interval-valued Pythagorean fuzzy sub-semigroup of T the the preimage $f^{-1}(\tilde{P}) = (f^{-1}(\tilde{\phi}_p), f^{-1}(\tilde{\psi}_p))$ is an interval-valued Pythagorean fuzzy sub-semigroup of R.

If $\tilde{P} = [\tilde{\phi}_p, \tilde{\psi}_p]$ is an interval-valued Pythagorean fuzzy left (resp.right) ideal of T the the preimage $f^{-1}(\tilde{P}) = (f^{-1}(\tilde{\phi}_p), f^{-1}(\tilde{\psi}_p))$ is an interval-valued Pythagorean fuzzy left ideal (resp. right ideal) of R.

Proof. Assume that $\tilde{P} = [\tilde{\phi}_p, \tilde{\psi}_p]$ is an interval-valued Pythagorean fuzzy sub-semigroup of T and $r, t \in R$. Then

$$\begin{aligned} f^{-1}(\widetilde{\phi}_p)(rt) &= \widetilde{\phi}_p(f(rt)) \\ &= \widetilde{\phi}_p(f(r)f(t)) \\ &\geq \min\{\widetilde{\phi}_p(f(r)), \widetilde{\phi}_p(f(t))\} \\ &= \min\{f^{-1}(\widetilde{\phi}_p)(r), f^{-1}(\widetilde{\phi}_p)(f(t))\}; \end{aligned}$$

$$f^{-1}(\widetilde{\psi}_p)(rt) = \widetilde{\psi}_p(f(rt))$$

$= \widetilde{\psi}_p(f(r)f(t))$
$\leq \max\{\widetilde{\psi}_p(f(r)), \widetilde{\psi}_p(f(t))\}$
$= \max\{f^{-1}(\widetilde{\psi}_p)(r), f^{-1}(\widetilde{\psi}_p)(f(t))\}.$

Hence, $f^{-1}(\tilde{P}) = (f^{-1}(\tilde{\phi}_p), f^{-1}(\widetilde{\psi}_p))$ is an interval-valued Pythagorean fuzzy sub-semigroup of R.

$f^{-1}(\widetilde{\phi}_p)(rt) = \widetilde{\phi}_p(f(rt))$
$= \widetilde{\phi}_p(f(r)f(t))$
$\geq \widetilde{\phi}_p(f(t))$
$= f^{-1}(\widetilde{\phi}_p)(f(t));$

$f^{-1}(\widetilde{\psi}_p)(rt) = \widetilde{\psi}_p(f(rt))$
$= \widetilde{\psi}_p(f(r)f(t))$
$\leq \widetilde{\psi}_p(f(t))$
$= f^{-1}(\widetilde{\psi}_p)(f(t)).$

Hence, $f^{-1}(\tilde{P}) = (f^{-1}(\tilde{\phi}_p), f^{-1}(\tilde{\psi}_p))$ is an interval-valued Pythagorean fuzzy left (resp.right) ideal of R.

Theorem 8. Let R, T be a semigroups, $f: R \to T$ be a homomorphism of semigroups. If $\tilde{P} = [\tilde{\phi}_p, \tilde{\psi}_p]$ is an interval-valued Pythagorean fuzzy bi-ideal of T the the preimage $f^{-1}(\tilde{P}) = (f^{-1}(\tilde{\phi}_p), f^{-1}(\tilde{\psi}_p))$ is an interval-valued Pythagorean fuzzy bi-ideal of R.

Proof. Assume that $\tilde{P} = [\tilde{\phi}_p, \tilde{\psi}_p]$ is an interval-valued Pythagorean fuzzy sub-semigroup of T and $a, r, t \in R$. Then

$f^{-1}(\widetilde{\phi}_p)(rat) = \widetilde{\phi}_p(f(rat))$
$= \widetilde{\phi}_p(f(r)f(a)f(t))$
$\geq \min\{\widetilde{\phi}_p(f(r)), \widetilde{\phi}_p(f(t))\}$
$= \min\{f^{-1}(\widetilde{\phi}_p)(r), f^{-1}(\widetilde{\phi}_p)(f(t))\};$

$f^{-1}(\widetilde{\psi}_p)(rat) = \widetilde{\psi}_p(f(rat))$
$= \widetilde{\psi}_p(f(r)f(a)f(t))$
$\leq \max\{\widetilde{\psi}_p(f(r)), \widetilde{\psi}_p(f(t))\}$
$= \max\{f^{-1}(\widetilde{\psi}_p)(r), f^{-1}(\widetilde{\psi}_p)(f(t))\}.$

Hence $f^{-1}(\tilde{P}) = (f^{-1}(\tilde{\phi}_p), f^{-1}(\tilde{\psi}_p))$ is an interval-valued Pythagorean fuzzy bi-ideal of R.

Theorem 9. Let R, T be a semigroups, $f: R \to T$ be a homomorphism of semigroups. If $\tilde{P} = [\tilde{\phi}_p, \tilde{\psi}_p]$ is an interval-valued Pythagorean fuzzy interior ideal of T the preimage $f^{-1}(P) = (f^{-1}(\tilde{\phi}_p), f^{-1}(\tilde{\psi}_p))$ is an interval-valued Pythagorean fuzzy interior ideal of R.

5. Conclusion

In this paper interval valued Pythagorean fuzzy sub-semigroup, interval valued Pythagorean fuzzy left (resp. right) ideal, interval valued Pythagorean fuzzy ideal, interval valued Pythagorean fuzzy bi-ideal, interval valued Pythagorean fuzzy interior ideal and Homomorphism of interval valued Pythagorean fuzzy ideal in semigroups are studied and investigated some properties with suitable examples.

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Some Similarity Measures of Rough Interval Pythagorean Fuzzy Sets

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ABSTRACT

The purpose of this study is to propose new similarity measures namely cosine, jaccard and dice similarity measures. The weighted cosine, weighted jaccard and weighted dice similarity measures has been also defined. Some of the important properties of the defined similarity measures and weighted similarity measures have been established. We develop a new multi attribute decision making problem based on the proposed similarity measures. To demonstrate the applicability, a numerical example is solved.

1. Introduction

The concept of fuzzy set was introduced by Zadeh [15] in his classic paper in 1965 and has been applied to many branches in mathematics. Later Zadeh [14] also introduced the concept of interval valued fuzzy set by considering the values of membership functions as the intervals of numbers instead of the numbers alone. The notion of rough set theory was proposed by Pawlak [7]. The concept of rough set theory is an extension

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of crisp set theory for the study of intelligent systems characterized by inexact, uncertain or insufficient information. Dubois and Prade [3] were introduced the concept of rough fuzzy set. This theory was found to be more useful in decision making and medical diagnosis problems. A similarity measure is an important tool for determining the degree of similarity between two objects. Similarity measures between fuzzy sets is an important content in fuzzy mathematics. Yager [13] examined Pythagorean fuzzy set characterized by a membership degree and a non-membership degree that satisfies the case in which the square sum of its membership degree and non-membership degree is less than or equal to one. Peng and Yang [9] introduced the concept of interval Pythagorean fuzzy sets which is a generalization of Pythagorean fuzzy sets and interval valued fuzzy sets. Hussain et al. [2] introduced the concept of rough Pythagorean fuzzy sets. The Pythagorean fuzzy set has been investigated from different perspectives, including decision-making technologies [8], medical diagnosis [10], and transportation problem [6]. In particular, an extension of Pythagorean fuzzy set, named Interval-Valued Pythagorean Fuzzy Sets in decision making [8], complex Pythagoren fuzzy set in pattern recognition [12].

To facilitate our discussion, the remainder of this paper is organized as follows. In Section 2 we review some fundamental conceptions rough sets, interval valued fuzzy sets, Pythgorean fuzzy sets. In Section 3 we propose cosine similarity measure of rough interval Pythagorean fuzzy sets and some properties of this similarity measure discussed. Sections 4 and 5 deals with jaccard, dice similarity measures. In Section 6 we present algorithm for proposed measures. Section 7 deals with numerical example of proposed measures.

2.Basic Concepts

In this section we list some basic concepts.

Definition 1. Let x be a nonempty set. A mapping $\widetilde{\Omega}: x \to D[0,1]$ is called an interval valued fuzzy subset of x, where $\widetilde{\Omega}(x) = [\Omega^-(x), \Omega^+(x)], x \in X,$ and Ω^- and Ω^+ are the fuzzy subets in X such that $\Omega^-(x) \leq \Omega^+(x)\ x \in X$. $D[0,1]$ denotes the set of closed subsets of [0,1].

Definition 2. [5]. Let ϑ be a congruence relation on X. Le Λ t be any nonempty subset of X. The sets $\underline{\vartheta}(\Lambda) = \{x \in X/[x]_\vartheta \subseteq \Lambda\}$ and $\overline{\vartheta}(\Lambda) = \{x \in X/[x]_\vartheta \cap \vartheta \neq \emptyset\}$ are called the lower and upper approximations of Λ. Then $\vartheta(\Lambda) = (\underline{\vartheta}(\Lambda), \overline{\vartheta}(\Lambda))$ is called rough set in $(X, \vartheta) \Leftrightarrow \underline{\vartheta}(\Lambda) \neq \overline{\vartheta}(\Lambda)$.

Definition 3. [3]. Let ϑ be an congruence relation on X. Let Λ fuzzy subset of X. The upper and lower approximations of Λ defined by $\overline{\vartheta}(\Lambda)(x) = \bigvee_{a \in [x]_\vartheta} \Lambda(a)$ and $\underline{\vartheta}(\Lambda)(x) = \bigwedge_{a \in [x]_\vartheta} \Lambda(a)$. $\vartheta(\Lambda) = (\underline{\vartheta}(\Lambda), \overline{\vartheta}(\Lambda))$ is called a rough fuzzy set of Λ with respect to ϑ if $\underline{\vartheta}(\Lambda) \neq \overline{\vartheta}(\Lambda)$.

Definition 4. [4]. Let $\widetilde{\Omega}$ be an interval-valued fuzzy subset of X and let ϑ be the complete congruence relation on X. Let $\underline{\vartheta}(\widetilde{\Omega})$ and $\overline{\vartheta}(\widetilde{\Omega})$ be the interval-valued fuzzy subset of X defined by, $\underline{\vartheta}(\widetilde{\Omega})(n) = \wedge_{n \in [y]_\vartheta} \widetilde{\Omega}(n)$ and $\overline{\vartheta}(\widetilde{\Omega})(n) = \vee_{n \in [y]_\vartheta} \widetilde{\Omega}(n)$. Then $\vartheta(\widetilde{\Omega}) = \left(\underline{\vartheta}(\widetilde{\Omega}), \overline{\vartheta}(\widetilde{\Omega})\right)$ is called an interval-valued rough fuzzy subset of X if $\underline{\vartheta}(\widetilde{\Omega}) \neq \overline{\vartheta}(\widetilde{\Omega})$.

Definition 5. [1]. Let X be a nonempty set then an Intutionistic fuzzy set can be defined as $\Lambda\Omega = \left\{\left(x, \mu_\Omega(x), \gamma_{\Omega_\Lambda}(x)\right)/x \in X\right\}$ where $\mu_{\Omega_\Lambda}(x)$ *and* $\gamma_\Lambda(x)$ are mapping from X to [0,1] also $0 \leq$

$\mu_{\Omega_\Lambda}(x) \le 1, 0 \le \gamma_{\Omega_{\Lambda_\Lambda}}(x) \le 1, 0 \le \mu_{\Omega_\Lambda}(x) + \gamma_{\Omega_\Lambda}(x) \le 1$ for all $x \in X$ and represent the degrees of membership and non-membership of element $x \in X$ to set X.

Definition 6. [11]. Let X be a nonempty set then an Pythagorean fuzzy set can be defined as $\Omega = \left\{\left(x, \mu_\Omega(x), \gamma_{\Omega_\Lambda}(x)\right)/x \in X\right\}$ where $\mu_\Omega(x)$ *and* $\gamma_{\Omega_\Lambda}(x)$ are mapping from X to [0,1] also $0 \le \mu_{\Omega_\Lambda}(x) \le 1, 0 \le \gamma_\Omega(x) \le 1, 0 \le {\mu_\Omega}^2{}_\Lambda(x) + {\gamma_\Omega}^2{}_\Lambda(x) \le 1$ for all $x \in X$, and represent the degrees of membership and non membership of element $x \in X$ to set X.

Definition 7. [7]. Let X be a non-empty set then an Interval Pythagorean fuzzy set can be defined as follows $\widetilde{\Omega} = \{(x, \mu_{\widetilde{\Omega}}(x), \gamma_{\widetilde{\Omega}}(x))/x \in X\}$ where $\mu_{\widetilde{\Omega}}(x) = [{\mu_{\widetilde{\Omega}}}^-(x), {\mu_{\widetilde{\Omega}}}^+(x)]$ and $\gamma_{\widetilde{\Omega}}(x) = [{\gamma_{\widetilde{\Omega}}}^-(x), {\gamma_{\widetilde{\Omega}}}^+(x)]$ are the intervals in [0,1] also $0 \le (\mu^+{}_{\widetilde{\Omega}}(x))^2 + (\gamma^+{}_{\widetilde{\Omega}}(x))^2 \le 1$.

3. Cosine Similarity Measures (CSM) of Rough Interval Pythagorean

Fuzzy (RIPF) Sets.

In this section we introduce the notion of *CSM* of *RIPF* sets also discuss some properties of *RIPF* sets. Also weighted *CSM* of *RIPF* sets are discussed.

Definition 8. Let X be a nonempty set. Let $\widetilde{\Omega} = \{(n, \mu_{\widetilde{\Omega}}(n), \gamma_{\widetilde{\Omega}}(n))/n \in X\}$ be a pythagorean fuzzy set of X. Then rough interval Pythagorean fuzzy set is defined as $\vartheta(\widetilde{\Omega}) = (\underline{\vartheta}(\widetilde{\Omega}), \overline{\vartheta}(\widetilde{\Omega}))$ where

$\underline{\vartheta}(\widetilde{\Omega}) = \{\langle n, \underline{\vartheta}(\mu_{\widetilde{\Omega}}), \underline{\vartheta}(\gamma_{\widetilde{\Omega}})\rangle, n \in X\}$ and $\overline{\vartheta}(\widetilde{\Omega}) = \{\langle n, \overline{\vartheta}(\mu_{\widetilde{\Omega}}), \overline{\vartheta}(\gamma_{\widetilde{\Omega}})\rangle, n \in X\}$,

with the condition that $0 \le \left(\underline{\vartheta}(\mu_{\widetilde{\Omega}})\right)^2 + \left(\underline{\vartheta}(\gamma_{\widetilde{\Omega}})\right)^2 \le 1$, $0 \le \left(\overline{\vartheta}(\mu_{\widetilde{\Omega}})\right)^2 + \left(\overline{\vartheta}(\gamma_{\widetilde{\Omega}})\right)^2 \le 1$.

Here, $\underline{\vartheta}(\mu_{\widetilde{\Omega}})(n) = \wedge_{n \in [y]_\vartheta} \mu_{\widetilde{\Omega}}(y)$ and $\underline{\vartheta}(\gamma_{\widetilde{\Omega}})(n) = \vee_{n \in [y]_\vartheta} \gamma_{\widetilde{\Omega}}(y)$ also,

$\overline{\vartheta}(\mu_{\widetilde{\Omega}})(n) = \vee_{n \in [y]_\vartheta} \mu_{\widetilde{\Omega}}(y)$ and $\overline{\vartheta}(\gamma_{\widetilde{\Omega}})(n) = \wedge_{n \in [y]_\vartheta} \gamma_{\widetilde{\Omega}}(y)$.

Definition 9. Let ϑ be an congruence relation on X. Consider two *RIPF* sets $\vartheta(\widetilde{\Omega_1})$, $\vartheta(\widetilde{\Omega_2})$ in $X = \{x_1, x_2 \dots\dots x_n\}$. A *CSM* between $\vartheta(\widetilde{\Omega_1})$ and $\vartheta(\widetilde{\Omega_2})$ is defined as follows:

$$C_{RIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = \frac{1}{n}\sum_{i=1}^{n} \frac{\left(\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i) + \delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)\right)}{\sqrt{(\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i))^2 + (\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i))^2}\sqrt{(\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i))^2 + (\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i))^2}}. \quad (1)$$

Where

$$\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i) = \frac{(\underline{\vartheta}(\mu^-(x_i)) + \underline{\vartheta}(\mu^+(x_i)) + \overline{\vartheta}(\mu^-(x_i)) + \overline{\vartheta}(\mu^+(x_i)))}{4};$$

$$\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i) = \frac{(\underline{\vartheta}(\gamma^-(x_i)) + \underline{\vartheta}(\gamma^+(x_i)) + \overline{\vartheta}(\gamma^-(x_i)) + \overline{\vartheta}(\gamma^+(x_i)))}{4};$$

$$\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i) = \frac{\left(\underline{\vartheta}(\mu^-(x_i)) + \underline{\vartheta}(\mu^+(x_i)) + \overline{\vartheta}(\mu^-(x_i)) + \overline{\vartheta}(\mu^+(x_i))\right)}{4};$$

$$\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i) = \frac{\left(\underline{\vartheta}(\gamma^-(x_i)) + \underline{\vartheta}(\gamma^+(x_i)) + \overline{\vartheta}(\gamma^-(x_i)) + \overline{\vartheta}(\gamma^+(x_i))\right)}{4}.$$

Proposition 1. A *RIPCSM* between $\vartheta(\widetilde{\Omega_1})$ and $\vartheta(\widetilde{\Omega_2})$ satisfies the following properties:

$$0 \leq C_{RIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) \leq 1;$$

$$C_{RIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = 1 \Leftrightarrow \vartheta(\widetilde{\Omega_1}) = \vartheta(\widetilde{\Omega_2});$$

$$C_{RIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = C_{RIPF}\left(\vartheta(\widetilde{\Omega_2}), \vartheta(\widetilde{\Omega_1})\right).$$

Proof. It is obvious because all positive values of cosine function are within 0 and 1; it is obvious; for any two *RIPF* sets $\vartheta(\widetilde{\Omega_1})$ and $\vartheta(\widetilde{\Omega_2})$, if $\vartheta(\widetilde{\Omega_1}) = \vartheta(\widetilde{\Omega_2})$ then,

$\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i) = \delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i)$ and $\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i) = \delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)$. Hence $\cos(0) = 1$. Conversely, if $C_{RIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = 1$, then $\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i) = \delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i)$ and $\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i) = \delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i)$. Hence $\vartheta(\widetilde{\Omega_1}) = \vartheta(\widetilde{\Omega_2})$.

If we consider weight ω_i of each elementx_i, a weighted *RICSM* between *RIPF* sets $\vartheta(\widetilde{\Omega_1})$ and $\vartheta(\widetilde{\Omega_2})$ is defined as follows:

$$C_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = \frac{1}{n}\sum_{i=1}^{n} \frac{\left(\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i) + \delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)\right)}{\sqrt{(\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i))^2 + (\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i))^2}\sqrt{(\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i))^2 + (\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i))^2}}. \quad (2)$$

$\omega_i \in [0,1], i = 1,2,3 \dots n$ and $\sum_{i=1}^{n} \omega_i = 1$. If we take $\omega_i = \frac{1}{n}, i = 1,2, \dots . n$ then

$$C_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = C_{RIPF}\left(\vartheta(\widetilde{\Omega_2}), \vartheta(\widetilde{\Omega_1})\right).$$

The weighted *RICSM* between two *RIPF* sets $\vartheta(\widetilde{\Omega_1})$ and $\vartheta(\widetilde{\Omega_2})$ also satisfies the following properties.

Proposition 2.

$$0 \leq C_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) \leq 1;$$

$$C_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = 1 \Leftrightarrow \vartheta(\widetilde{\Omega_1}) = \vartheta(\widetilde{\Omega_2});$$

$$C_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = C_{WRIPF}\left(\vartheta(\widetilde{\Omega_2}), \vartheta(\widetilde{\Omega_1})\right).$$

4. Jaccard Similarity Measure (JSM) of Rough Interval Pythagorean Fuzzy (RIPF) Set

In this section we introduce the concept of *JSM* of *RIPF* sets. Weighted *JSM* of *RIPF* also derived.

Definition 10. Let ϑ be an congruence relation on X. Consider two *RIPF* sets $\vartheta(\widetilde{\Omega_1})$, $\vartheta(\widetilde{\Omega_2})$ in $X = \{x_1, x_2 \dots \dots x_n\}$. A *JSM* between $\vartheta(\widetilde{\Omega_1})$and $\vartheta(\widetilde{\Omega_2})$ is defined as follows:

$$\mathrm{J_{IRPF}}(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})) = \frac{1}{n}\sum_{i=1}^{n} \frac{\left(\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i) + \delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)\right)}{\begin{array}{c}[(\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i))^2 + \left(\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i)\right)^2 + \left(\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i)\right)^2 + \left(\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)\right)^2 + \\ \delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i) + \delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)]\end{array}}, \tag{3}$$

where

$$\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i) = \frac{(\underline{\vartheta}(\mu^-(x_i)) + \underline{\vartheta}(\mu^+(x_i)) + \overline{\vartheta}(\mu^-(x_i)) + \overline{\vartheta}(\mu^+(x_i)))}{4},$$

$$\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i) = \frac{(\underline{\vartheta}(\gamma^-(x_i)) + \underline{\vartheta}(\gamma^+(x_i)) + \overline{\vartheta}(\gamma^-(x_i)) + \overline{\vartheta}(\gamma^+(x_i)))}{4} \text{ and}$$

$$\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i) = \frac{(\underline{\vartheta}(\mu^-(x_i)) + \underline{\vartheta}(\mu^+(x_i)) + \overline{\vartheta}(\mu^-(x_i)) + \overline{\vartheta}(\mu^+(x_i)))}{4},$$

$$\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i) = \frac{(\underline{\vartheta}(\gamma^-(x_i)) + \underline{\vartheta}(\gamma^+(x_i)) + \overline{\vartheta}(\gamma^-(x_i)) + \overline{\vartheta}(\gamma^+(x_i)))}{4}.$$

Proposition 3. A RIPJSM between $\vartheta(\widetilde{\Omega_1})$ and $\vartheta(\widetilde{\Omega_2})$ satisfies the following properties:

$$0 \le \mathrm{J_{RIPF}}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) \le 1,$$

$$\mathrm{J_{RIPF}}(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})) = 1 \Leftrightarrow \vartheta(\widetilde{\Omega_1}) = \vartheta(\widetilde{\Omega_2}),$$

$$\mathrm{J_{RIPF}}(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})) = \mathrm{J_{IRPF}}(\vartheta(\widetilde{\Omega_2}), \vartheta(\widetilde{\Omega_1})).$$

Proof. It is obvious because all positive values of cosine function are within 0 and 1; it is obvious; for any two RIPF sets $\vartheta(\widetilde{\Omega_1})$ and $\vartheta(\widetilde{\Omega_2})$, if $\vartheta(\widetilde{\Omega_1}) = \vartheta(\widetilde{\Omega_2})$ then, $\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i) = \delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i)$ and $\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i) = \delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)$. Hence $\cos(0) = 1$. Conversely, if $J_{RIPF}(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})) = 1$, then $\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i) = \delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i)$ and $\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i) = \delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)$. Hence $\vartheta(\widetilde{\Omega_1}) = \vartheta(\widetilde{\Omega_2})$.

If we consider weight ω_i of each element x_i, a weighted RIPJSM between RIPF sets $\vartheta(\widetilde{\Omega_1})$ and $\vartheta(\widetilde{\Omega_2})$ is defined as follows:

$$J_{IRPF}(\vartheta(\widetilde{\Omega_1}),\vartheta(\widetilde{\Omega_2}))$$
$$=\frac{1}{n}\sum_{i=1}^{n}\omega_i\frac{\left(\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i)+\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)\right)}{\begin{gathered}[(\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i))^2+\left(\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i)\right)^2+\left(\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i)\right)^2+\left(\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)\right)^2+\\ \delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i)+\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)]\end{gathered}}. \quad (4)$$

$\omega_i \in [0,1], i = 1,2,3 \ldots n$ and $\sum_{i=1}^{n}\omega_i = 1$. If we take $\omega_i = \frac{1}{n}, i = 1,2, \ldots . n$ then $J_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}),\vartheta(\widetilde{\Omega_2})\right) = J_{RIPF}\left(\vartheta(\widetilde{\Omega_2}),\vartheta(\widetilde{\Omega_1})\right)$.

The weighted RIPJSM between two RIPF sets $\vartheta(\widetilde{\Omega_1})$ and $\vartheta(\widetilde{\Omega_2})$ also satisfies the following properties.

Proposition 4.

$$0 \le J_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}),\vartheta(\widetilde{\Omega_2})\right) \le 1;$$

$$J_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}),\vartheta(\widetilde{\Omega_2})\right) = 1 \Leftrightarrow \vartheta(\widetilde{\Omega_1}) = \vartheta(\widetilde{\Omega_2});$$

$$J_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}),\vartheta(\widetilde{\Omega_2})\right) = J_{WRIPF}\left(\vartheta(\widetilde{\Omega_2}),\vartheta(\widetilde{\Omega_1})\right).$$

5. Dice Similarity Measure (DSM) of Rough Interval Pythagorean Fuzzy (RIPF) Set

This section deals with *DSM* of *RIPF* sets. Some properties of this similarity measure are discussed.

Definition 11. Let $\vartheta(\widetilde{\Omega_1})$ and $\vartheta(\widetilde{\Omega_2})$ be two *RIPF* set in$X = \{x_1, x_2 \ldots \ldots x_n\}$. A *DSM* between $\vartheta(\widetilde{\Omega_1})$and $\vartheta(\widetilde{\Omega_2})$ is defined as follows:

$$D_{RIPF}(\vartheta(\widetilde{\Omega_1}),\vartheta(\widetilde{\Omega_2}))$$
$$=\frac{1}{n}\sum_{i=1}^{n}\frac{2\left(\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i)+\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)\right)}{\sqrt{(\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i))^2+(\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i))^2}\sqrt{(\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i))^2+(\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i))^2}}. \quad (6)$$

Where

$$\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i) = \frac{\left(\underline{\vartheta}(\mu^-(x_i)) + \underline{\vartheta}(\mu^+(x_i)) + \overline{\vartheta}(\mu^-(x_i)) + \overline{\vartheta}(\mu^+(x_i))\right)}{4},$$

$$\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i) = \frac{\left(\underline{\vartheta}(\gamma^-(x_i)) + \underline{\vartheta}(\gamma^+(x_i)) + \overline{\vartheta}(\gamma^-(x_i)) + \overline{\vartheta}(\gamma^+(x_i))\right)}{4} \text{ and}$$

$$\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i) = \frac{\left(\underline{\vartheta}(\mu^-(x_i)) + \underline{\vartheta}(\mu^+(x_i)) + \overline{\vartheta}(\mu^-(x_i)) + \overline{\vartheta}(\mu^+(x_i))\right)}{4},$$

$$\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i) = \frac{\left(\underline{\vartheta}(\gamma^-(x_i)) + \underline{\vartheta}(\gamma^+(x_i)) + \overline{\vartheta}(\gamma^-(x_i)) + \overline{\vartheta}(\gamma^+(x_i))\right)}{4}.$$

Proposition 5. A *RIPJSM* between $\vartheta(\widetilde{\Omega_1})$ and $\vartheta(\widetilde{\Omega_2})$ satisfies the following properties:

$$0 \leq \mathrm{D}_{\mathrm{RIPF}}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) \leq 1;$$

$$\mathrm{D}_{\mathrm{RIPF}}(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})) = 1 \Leftrightarrow \vartheta(\widetilde{\Omega_1}) = \vartheta(\widetilde{\Omega_2});$$

$$\mathrm{D}_{\mathrm{RIPF}}(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})) = \mathrm{D}_{\mathrm{IRPF}}(\vartheta(\widetilde{\Omega_2}), \vartheta(\widetilde{\Omega_1})).$$

Proof. Proof is similar to ***Proposition 3***.

If we consider weight ω_i of each elementx_i, a weighted *RIPDSM* between *RIPF* sets $\vartheta(\widetilde{\Omega_1})$ and $\vartheta(\widetilde{\Omega_2})$ is defined $\sum_{i=1}^{n} \omega_i = 1$.as follows:

$$D_{RIPF}(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})) = \frac{1}{n}\sum_{i=1}^{n} \omega_i \frac{2\left(\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i)+\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i)\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i)\right)}{\sqrt{(\delta\mu_{\vartheta(\widetilde{\Omega_1})}(x_i))^2+(\delta\gamma_{\vartheta(\widetilde{\Omega_1})}(x_i))^2}\sqrt{(\delta\mu_{\vartheta(\widetilde{\Omega_2})}(x_i))^2+(\delta\gamma_{\vartheta(\widetilde{\Omega_2})}(x_i))^2}}. \tag{7}$$

$\omega_i \in [0,1], i = 1,2,3 \ldots n$ and If we take $\omega_i = \frac{1}{n}, i = 1,2, \ldots. n$ then $D_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = D_{RIPF}\left(\vartheta(\widetilde{\Omega_2}), \vartheta(\widetilde{\Omega_1})\right)$.

The weighted *RIPDSM* between two *RIPF* sets $\vartheta(\widetilde{\Omega_1})$ and $\vartheta(\widetilde{\Omega_2})$ also satisfies the following properties.

Proposition 6.

$$0 \leq \mathrm{D}_{\mathrm{WRIPF}}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) \leq 1;$$

$$\mathrm{D}_{\mathrm{WRIPF}}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = 1 \Leftrightarrow \vartheta(\widetilde{\Omega_1}) = \vartheta(\widetilde{\Omega_2});$$

$$\mathrm{D}_{\mathrm{WRIPF}}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = \mathrm{D}_{\mathrm{WRIPF}}\left(\vartheta(\widetilde{\Omega_2}), \vartheta(\widetilde{\Omega_1})\right).$$

6. Decision Making Based on CSM, JSM and DSM under RIPF Environment

This section deals with *RIPSM* between *RIPF* sets to the multi-criteria decision making problem. Assume that $K = \{K_1, K_2, \ldots. K_m\}$ be the set of attributes and $Q = \{Q_1, Q_2, \ldots. Q_n\}$ be the set of alternatives. The proposed decision making approach is described by the following steps.

Algorithm 1. (See Fig 1).

Step 1. Construct the Decision Matrix with *R*IPF Number. The decision maker forms a decision matrix with respect to n alternatives and m attributes in terms of *RIPF* numbers.

Step 2. Determine RIP Mean Operator.

$$\langle \delta\mu(x_i), \delta\gamma(x_i) \rangle = \left(\frac{\left(\underline{\vartheta}(\mu^-(x_i)) + \underline{\vartheta}\left(\mu^+(x_i)\right) + \overline{\vartheta}(\mu^-(x_i)) + \overline{\vartheta}\left(\mu^+(x_i)\right)\right)}{4}, \frac{(\underline{\vartheta}(\gamma^-(x_i)) + \underline{\vartheta}(\gamma^+(x_i)) + \overline{\vartheta}(\gamma^-(x_i)) + \overline{\vartheta}(\gamma^+(x_i)))}{4} \right)$$

for $i = 1,2, \ldots . n$.

Step 3. Determine the Weights of the Attributes. Assume that the weight of the attributes K_j(j=1,2,…m) considered by the decision maker is ω_j (j=1,2,…m) where all $\omega_j \in [0,1], j = 1,2,3 \ldots m$ and $\sum_{j=1}^{m} \omega_j = 1$.

Step 4. Determine the Benefit Type Attributes and Cost Type Attributes. Generally, the evaluation attribute can be categorized into two types: benefit type attribute and cost type attribute.

For benefit type attribute: $Z^* = \{\max(\mu_{Q_i}), min(\gamma_{Q_i})\}$.

For cost type attribute: $Z^* = \{\min(\mu_{Q_i}), max(\gamma_{Q_i})\}$.

Step 5. Determine the Weighted RIPSM of the Alternatives.

$$C_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = \sum_{i=1}^{n} \omega_i \, C_{RIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right);$$

$$J_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = \sum_{i=1}^{n} \omega_i \, J_{RIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right);$$

$$D_{WRIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right) = \sum_{i=1}^{n} \omega_i \, D_{RIPF}\left(\vartheta(\widetilde{\Omega_1}), \vartheta(\widetilde{\Omega_2})\right).$$

Step 6. Ranking the Alternatives. The ranking order of all alternatives can be determined based on the descending order of similarity measures.

Step 7. End.

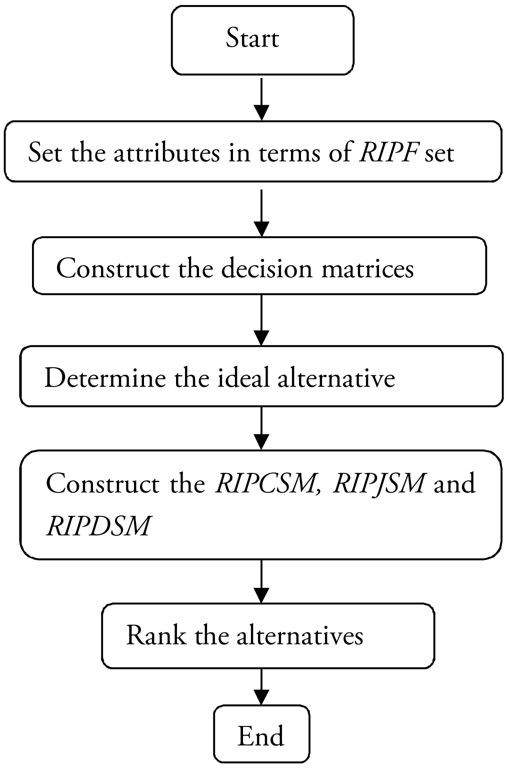

Fig 1. A flowchart of the proposed decision making.

7. Numerical Example for RIPCSM, RIPJSM and RIPDSM

Let us consider a decision maker wants to select the house from $Q = \{Q_1, Q_2, Q_3\}$ by considering four attributes, namely expensive (K_1), reasonable price (K_2), low price (K_3) and the risk factor (K_4). By proposed approach discussed above, the considered problem solved by the following steps:

Step 1. The decision maker forms a decision matrix with respect to the three alternatives and four attributes in terms of *RIP* number as follows.

Table 1. Decision matrix.

	$\mathbf{K_1}$	$\mathbf{K_2}$	$\mathbf{K_3}$	$\mathbf{K_4}$
$\mathbf{Q_1}$	([.3,.4],[.5,.7]), ([.3,.4],[.5,.7])	([.5,.6],[.8,.9]), ([.5,.6],[.8,.9])	([.1,.2],[.7,.8]), ([.5,.8],[.4,.6])	([.1,.2],[.7,.8]), ([.5,.8],[.4,.6])
$\mathbf{Q_2}$	([.7,.8],[.6,.7]), ([.7,.8],[.6,.7])	([.7,.8],[.6,.7]), ([.8,.9],[.4,.5])	([.5,.6],[.4,.5]), ([.5,.6],[.4,.5])	([.7,.8],[.6,.7]), ([.8,.9],[.4,.5])
$\mathbf{Q_3}$	([.5,.7],[.3,.4]), ([.8,.9],[.1,.2])	([.5,.7],[.3,.4]), ([.8,.9],[.1,.2])	([.5,.7],[.3,.4]), ([.8,.9],[.1,.2])	([.8,.9],[.1,.2]), ([.8,.9],[.1,.2])

Step 2. Determine the *RIP* mean operator.

Table 2. Transformed decision matrix.

	K_1	K_2	K_3	K_4
Q_1	[.35,.6]	[.55,.85]	[.4,.625]	[.4,.625]
Q_2	[.75,.65]	[.8,.55]	[.55,.45]	[.8,.55]
Q_3	[.725,.25]	[.725,.25]	[.725,.25]	[.825,.15]

Step 3. The weight vectors considered by the decision maker are 0.35, 0.25, 0.25 and 0.15 respectively.

Step 4. Determine the benefit type attribute and cost type attribute. Here three benefit types attributes K_1,K_2,K_3 and one cost type attribute K_4.

$$Z^* = \{[0.75,0.25],[.8,.25],[.725,.25],[.825,.15]\}.$$

Step 5. Calculate the weighted *RIP* similarity measures of the alternatives. Calculated values of weighted *RIP* similarity values are

$$C_{WIRPF}(Q_1, Z^*) = .7582;$$

$$C_{WIRPF}(Q_2, Z^*) = .9336;$$

$$C_{WIRPF}(Q_3, Z^*) = .9999;$$

$$J_{WIRPF}(Q_1, Z^*) = .6046;$$

$$J_{WIRPF}(Q_2, Z^*) = .8538;$$

$$J_{WIRPF}(Q_3, Z^*) = .9975;$$

$$D_{WIRPF}(Q_1, Z^*) = .7018;$$

$$D_{WIRPF}(Q_2, Z^*) = .9208;$$

$$D_{WIRPF}(Q_3, Z^*) = .9988.$$

Step 6. Ranking the alternatives is prepared based on the descending order of similarity measures. Highest value reflects the best alternative. HencQ_3e is the best alternative.

8. Conclusion

In this paper, we have defined Cosine, Jaccard, Dice similarity measure, Weighted Cosine, Jaccard and Dice similarity measures. We have also proved their basic properties. We have developed MADM strategies based on the proposed measures respectively. We have presented an example for select a best house for live. The thrust of the concept presented in this article will be in pattern recognition, medical diagnosis etc. in rough interval Pythagorean fuzzy sets.

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Spherical Interval-Valued Fuzzy Bi-Ideals of Gamma Near-Rings

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ABSTRACT

In this paper we introduce the concept of the spherical interval-valued fuzzy bi-ideal of gamma near-ring $\mathcal{R}$ and its some results. The union and intersection of the spherical interval-valued fuzzy bi-ideal of gamma near-ring $\mathcal{R}$ is also a spherical interval-valued fuzzy bi-ideal of gamma near-ring $\mathcal{R}$. Further we discuss about the relationship between bi-ideal and spherical interval-valued fuzzy bi-ideal of gamma near-ring $\mathcal{R}$.

1. Introduction

The Fuzzy Set (FS) was introduced by Zadeh [14] in 1965. It is identified as better tool for the scientific study of uncertainty, and came as a boost to the researchers working in the field of uncertainty. Many extensions and generalizations of FS was conceived by a number of researchers and a large number of real-life applications were developed in a variety of areas. In addition to this, parallel analysis of the classical results of many branches of Mathematics was also carried out in the fuzzy settings. Properties of fuzzy ideals in near-rings was studied by Hong et al. [3]. The monograph by Chinnadurai [1] gives a detailed discussion on fuzzy ideals in algebraic structures. Fuzzy ideals in Gamma near-ring $\mathcal{R}$ was discussed by Jun et al. [6 ,7] and Satyanarayana [8]. Thillaigovindan et al. [13] studied the interval valued fuzzy quasi-ideals of semigroups. Meenakumari and Tamizh chelvam [9] have defined fuzzy bi-ideal in $\mathcal{R}$ and

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established some properties of this structure. Srinivas and Nagaiah [11] have proved some results on T-fuzzy ideals of Γ-near-rings. Thillaigovindan et al. [12] worked on interval valued fuzzy ideals of near-rings. Chinnadurai and Kadalarasi [2] have defined the direct product of fuzzy ideals in near-rings. Kutlu Gündoğdu and Kahraman [8] introduced spherical fuzzy sets as an extension of picture fuzzy sets. Chinnadurai and Shakila [3, 4] discussed T-fuzzy bi-ideal of gamma near-ring and spherical fuzzy bi-ideals of gamma near-rings.

In this research work, we introduce the notion of Spherical Interval-Valued Fuzzy Bi-Ideal (SIVFBI) of gamma near-ring $\mathcal{R}$ as a generalization of spherical fuzzy bi-ideals of gamma near-rings $\mathcal{R}$. We will discuss some of the properties of spherical interval-valued fuzzy bi-ideal of gamma near-ring $\mathcal{R}$.

2. Preliminaries

In this section we present some definitions which are used for this research. Let $\mathcal{R}$ be a near-ring and Γ be a non-empty set such tha $\mathcal{R}$ t is a Gamma near-ring. A subgroup H of $(\mathcal{R},+)$ is a Bi-Ideal (BI) if and only if $H\Gamma\mathcal{R}\Gamma H \subseteq H$. A Spherical Fuzzy Set (SFS) $\tilde{A}_s$ of the universe of discourse U is given by, $\tilde{A}_s = \left\{u, \left(\tilde{\mu}(u), \tilde{\nu}(u), \tilde{\xi}(u)\right) \middle| u \in U\right\}$ where $\tilde{\mu}(u): U \longrightarrow [0,1]$, $\tilde{\nu}(u): U \longrightarrow [0,1]$ and $\tilde{\xi}(u): U \longrightarrow [0,1]$ and $0 \leq \tilde{\mu}^2(u) + \tilde{\nu}^2(u) + \tilde{\xi}^2(u) \leq 1$, $u \in U$.

For each u, the numbers $\tilde{\mu}(u)$, $\tilde{\nu}(u)$ and $\tilde{\xi}(u)$ are the degree of membership, non-membership and hesitancy of u to $\tilde{A}_s$, respectively.

A SFS $A_s = (\mu, \nu, \xi)$, where $\mu: \mathcal{R} \longrightarrow [0,1]$, $\nu: \mathcal{R} \longrightarrow [0,1]$ and $\xi: \mathcal{R} \longrightarrow [0,1]$ of $\mathcal{R}$ is said to be a Spherical Fuzzy Bi-Ideal (SFBI) of $\mathcal{R}$ if the following conditions are satisfied

$$\mu(u-v) \geq \min\{\mu(u), \mu(v)\},$$
$$\nu(u-v) \geq \min\{\nu(u), \nu(v)\},$$
$$\xi(u-v) \leq \max\{\xi(u), \xi(v)\},$$
$$\mu(u\alpha v\beta w) \geq \min\{\mu(u), \mu(w)\},$$
$$\nu(u\alpha v\beta w) \geq \min\{\nu(u), \nu(w)\},$$
$$\xi(u\alpha v\beta w) \leq \max\{\xi(u), \xi(w)\},$$

for all $u, v, w \in \mathcal{R}$ and $\alpha, \beta \in \Gamma$.

3. Spherical Interval-Valued Fuzzy Bi-Ideals of Gamma Near-Rings

In this section we define SIVFBI of $\mathcal{R}$ and study some of it properties. We obtain the condition for an arbitrary fuzzy subset of $\mathcal{R}$ is said to be SIVFBI.

Definition 1. A spherical fuzzy *set* $\tilde{A}_s = (\tilde{\mu}, \tilde{\nu}, \tilde{\xi})$ *of* $\mathcal{R}$ is to be SIVFBI of $\mathcal{R}$ if the following conditions are satisfied

$$\tilde{\mu}(u-v) \geq min^i\{\tilde{\mu}(u), \tilde{\mu}(v)\},$$
$$\tilde{\nu}(u-v) \geq min^i\{\tilde{\nu}(u), \tilde{\nu}(v)\},$$
$$\tilde{\xi}(u-v) \leq max^i\{\tilde{\xi}(u), \tilde{\xi}(v)\},$$
$$\tilde{\mu}(u\alpha v\beta w) \geq min^i\{\tilde{\mu}(u), \tilde{\mu}(w)\},$$
$$\tilde{\nu}(u\alpha v\beta w) \geq min^i\{\tilde{\nu}(u), \tilde{\nu}(w)\},$$
$$\tilde{\xi}(u\alpha v\beta w) \leq max^i\{\tilde{\xi}(u), \tilde{\xi}(w)\},$$

for all $u, v, w \in \mathcal{R}$ and $\alpha, \beta \in \Gamma$, where $\tilde{\mu}: \mathcal{R} \longrightarrow D[0,1]$, $\tilde{v}: \mathcal{R} \longrightarrow D[0,1]$ and $\tilde{\xi}: \mathcal{R} \longrightarrow D[0,1]$. Here $D[0,1]$ denotes the family of closed subintervals of $[0,1]$.

Example 1. Let $\mathcal{R} = \{0,1,2,3\}$ with binary operation $+$" on $\mathcal{R}$, $\Gamma = \{0,1\}$ and $\mathcal{R} \times \Gamma \times \mathcal{R} \longrightarrow \mathcal{R}$ be a mapping. From the cayley table,

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	1	0
3	3	2	0	1

0	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	2	2	2
3	0	3	3	3

1	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0

Define SFS $\tilde{\mu}: \mathcal{R} \longrightarrow D[0,1]$ by $\tilde{\mu}(0) = [0.2,0.3], \tilde{\mu}(1) = [0.3,0.6], \tilde{\mu}(2) = [0.7,0.9], \tilde{\mu}(3) = [0.5,0.9]$; $\tilde{v}: \mathcal{R} \longrightarrow D[0,1]$ by $\tilde{v}(0) = [0.2,0.4], \tilde{v}(1) = [0.5,0.6], \tilde{v}(2) = [0.6,0.7], \tilde{v}(3) = [0.7,0.9]$; $\tilde{\xi}: \mathcal{R} \longrightarrow D[0,1]$ by $\tilde{\xi}(0) = [0.1,0.3], \tilde{\xi}(1) = [0.4,0.6], \tilde{\xi}(2) = [0.8,0.9], \tilde{\xi}(3) = [0.5,0.7]$. Then $\tilde{A}_s$ is SIVFBI of $\mathcal{R}$.

Theorem 1. Let $\tilde{A}_s = [A_s^-; A_s^+]$ be a Spherical interval-valued fuzzy subset of a gamma near-ring $\mathcal{R}$, then $\tilde{A}_s$ is a SIVFBI of $\mathcal{R}$ if and only if A_s^-, A_s^+ are SFBI of $\mathcal{R}$.

Proof. If $\tilde{A}_s$ is a SIVFBI of $\mathcal{R}$. For any $u, v, w \in \mathcal{R}$. Now,

$$
\begin{aligned}
&[\mu^-(u-v), \mu^+(u-v)] = \tilde{\mu}(u-v)\\
&\geq \min^i\{\tilde{\mu}(u), \tilde{\mu}(v)\}\\
&= \min^i\{[\mu^-(u), \mu^+(u)], [\mu^-(v), \mu^+(v)]\}\\
&= \min^i\{[\mu^-(u), \mu^-(v)]\}, \min^i\{[\mu^+(u), \mu^+(v)]\},
\end{aligned}
$$

$$
\begin{aligned}
&[v^-(u-v), v^+(u-v)] = \tilde{v}(x-y)\\
&\geq \min^i\{\tilde{v}(u), \tilde{v}(v)\}\\
&= \min^i\{[v^-(u), v^+(u)], [v^-(v), v^+(v)]\}\\
&= \min^i\{[v^-(u), v^-(v)]\}, \min^i\{[v^+(u), v^+(v)]\}, \text{ and}
\end{aligned}
$$

$$[\boldsymbol{\xi}^-(\mathbf{u}-\mathbf{v}), \boldsymbol{\xi}^+(\mathbf{u}-\mathbf{v})] = \tilde{\boldsymbol{\xi}}(\mathbf{u}-\mathbf{v})$$

$$
\begin{aligned}
&\leq \max^i\{\tilde{\xi}(u), \tilde{\xi}(v)\}\\
&= \max^i\{[\xi^-(u), \xi^+(u)], [\xi^-(v), \xi^+(v)]\}\\
&= \max^i\{[\xi^-(u), \xi^-(v)]\}, \max^i\{[\xi^+(u), \xi^+(v)]\};
\end{aligned}
$$

$$[\mu^-(u\alpha v\beta w), \mu^+(u\alpha v\beta w)] = \tilde{\mu}(u\alpha v\beta w)$$
$$\geq \min^i\{\tilde{\mu}(u), \tilde{\mu}(w)\}$$
$$= \min^i\{[\mu^-(u), \mu^+(u)], [\mu^-(w), \mu^+(w)]\}$$
$$= \min^i\{[\mu^-(u), \mu^-(w)]\}, \min^i\{[\mu^+(u), \mu^+(w)]\};$$

$$[\nu^-(u\alpha v\beta w), \nu^+(u\alpha v\beta w)] = \tilde{\nu}(u\alpha v\beta w)$$
$$\geq \min^i\{\tilde{\nu}(u), \tilde{\nu}(w)\}$$
$$= \min^i\{[\nu^-(u), \nu^+(u)], [\nu^-(w), \nu^+(w)]\}$$
$$= \min^i\{[\nu^-(u), \nu^-(w)]\}, \min^i\{[\nu^+(u), \nu^+(w)]\}, \text{ and}$$

$$[\xi^-(u\alpha v\beta w), \xi^+(u\alpha v\beta w)] = \tilde{\xi}(u\alpha v\beta w)$$
$$\leq \max^i\{\tilde{\xi}(u), \tilde{\xi}(w)\}$$
$$= \max^i\{[\xi^-(u), \xi^+(u)], [\xi^-(w), \xi^+(w)]\}$$
$$= \max^i\{[\xi^-(u), \xi^-(w)]\}, \max^i\{[\xi^+(u), \xi^+(w)]\}.$$

Therefore A_s^-, A_s^+ are SFBI o $\mathcal{R}$ f.
Conversely let A_s^-, A_s^+ are SFBI of $\mathcal{R}$. Let $u, v, w \in \mathcal{R}$. Now,

$$\tilde{\mu}(u - v) = [\mu^-(u - v), \mu^+(u - v)]$$
$$\geq [\min^i\{\mu^-(u), \mu^-(v)\}, \min^i\{\mu^+(u), \mu^+(v)\}]$$
$$= \min^i\{\mu^-(u), \mu^+(u)\}, \min^i\{\mu^-(v), \mu^+(v)\}$$
$$= \min^i\{\tilde{\mu}(u), \tilde{\mu}(v)\};$$

$$\tilde{\nu}(u - v) = [\nu^-(u - v), \nu^+(u - v)]$$
$$\geq [\min^i\{\nu^-(u), \nu^-(v)\}, \min^i\{\nu^+(u), \nu^+(v)\}]$$
$$= \min^i\{\nu^-(u), \nu^+(u)\}, \min^i\{\nu^-(v), \nu^+(v)\}$$
$$= \min^i\{\tilde{\nu}(u), \tilde{\nu}(v)\}, \text{ and}$$

$$\tilde{\xi}(u - v) = [\xi^-(u - v), \xi^+(u - v)]$$
$$\leq [\max^i\{\xi^-(u), \xi^-(v)\}, \max^i\{\xi^+(u), \xi^+(v)\}]$$
$$= \max^i\{\xi^-(u), \xi^+(u)\}, \max^i\{\xi^-(v), \xi^+(v)\}$$
$$= \max^i\{\tilde{\xi}(u), \tilde{\xi}(v)\};$$

$$\tilde{\mu}(u\alpha v\beta w) = [\mu^-(u\alpha v\beta w), \mu^+(u\alpha v\beta w)]$$
$$\geq [\min^i\{\mu^-(u), \mu^-(w)\}, \min^i\{\mu^+(u), \mu^+(w)\}]$$
$$= \min^i\{\mu^-(u), \mu^+(u)\}, \min^i\{\mu^-(w), \mu^+(w)\}$$
$$= \min^i\{\tilde{\mu}(u), \tilde{\mu}(w)\};$$

$$\tilde{\nu}(u\alpha v\beta w) = [\nu^-(u\alpha v\beta w), \nu^+(u\alpha v\beta w)]$$

$$\geq [\min^i\{\nu^-(u), \nu^-(w)\}, \min^i\{\nu^+(u), \nu^+(w)\}]$$
$$= \min^i\{\nu^-(u), \nu^+(u)\}, \min^i\{\nu^-(w), \nu^+(w)\}$$
$$= \min^i\{\tilde{\nu}(u), \tilde{\nu}(w)\}, \text{ and}$$

$$\tilde{\xi}(u\alpha v\beta w) = [\xi^-(u\alpha v\beta w), \xi^+(u\alpha v\beta w)]$$
$$\leq [\max^i\{\xi^-(u), \xi^-(w)\}, \max^i\{\xi^+(u), \xi^+(w)\}]$$
$$= \max^i\{\xi^-(u), \xi^+(u)\}, \max^i\{\xi^-(w), \xi^+(w)\}$$

$= \max^{i}\{\tilde{\xi}(u), \tilde{\xi}(w)\}$.

So $\tilde{A}_s$ is a SIVFBI of $\mathcal{R}$.
Hence the proof.

Theorem 2. If $\{\tilde{A}_{s_i}; i \in I\}$ be a family of SIVFBI of a gamma near-ring $\mathcal{R}$, then $\bigcap_{i\in I} \tilde{A}_{s_i}$ is also SIVFBI of $\mathcal{R}$, where I is an index set.

Proof. Let $\{\tilde{A}_{s_i}; i \in I\}$ be a family of SIVFBI of a gamma near-ring $\mathcal{R}$. For any $u, v, w \in \mathcal{R}$ and $\alpha, \beta \in \Gamma$.

$$\bigcap_{i\in I} \tilde{\mu}_i(u-v) = \inf^{i}_{i\in I}\tilde{\mu}_i(u-v)$$
$$\geq \inf^{i}_{i\in I}\min^{i}\{\tilde{\mu}_i(u), \tilde{\mu}_i(v)\}$$
$$= \min^{i}\{\inf^{i}_{i\in I}\tilde{\mu}_i(u), \inf^{i}_{i\in I}\tilde{\mu}_i(v)\}$$
$$= \min^{i}\{\bigcap_{i\in I} \tilde{\mu}_i(u), \bigcap_{i\in I} \tilde{\mu}_i(v)\};$$

$$\bigcap_{i\in I} \tilde{\nu}_i(u-v) = \inf^{i}_{i\in I}\tilde{\nu}_i(u-v)$$
$$\geq \inf^{i}_{i\in I}\min^{i}\{\tilde{\nu}_i(u), \tilde{\nu}_i(v)\}$$
$$= \min^{i}\{\inf^{i}_{i\in I}\tilde{\nu}_i(u), \inf^{i}_{i\in I}\tilde{\nu}_i(v)\}$$
$$= \min^{i}\{\bigcap_{i\in I} \tilde{\nu}_i(u), \bigcap_{i\in I} \tilde{\nu}_i(v)\};$$

$$\bigcap_{i\in I} \tilde{\xi}_i(u-v) = \inf^{i}_{i\in I}\tilde{\xi}_i(u-v)$$
$$\leq \inf^{i}_{i\in I}\max^{i}\{\tilde{\xi}_i(u), \tilde{\xi}_i(v)\}$$
$$= \max^{i}\{\inf^{i}_{i\in I}\tilde{\xi}_i(u), \inf^{i}_{i\in I}\tilde{\xi}_i(v)\}$$
$$= \max^{i}\{\bigcap_{i\in I} \tilde{\xi}_i(u), \bigcap_{i\in I} \tilde{\xi}_i(v)\};$$

$$\bigcap_{i\in I} \tilde{\mu}_i(u\alpha v\beta w) = \inf^{i}_{i\in I}\tilde{\mu}_i(u\alpha v\beta w)$$
$$\geq \inf^{i}_{i\in I}\min^{i}\{\tilde{\mu}_i(u), \tilde{\mu}_i(w)\}$$
$$= \min^{i}\{\inf^{i}_{i\in I}\tilde{\mu}_i(u), \inf^{i}_{i\in I}\tilde{\mu}_i(w)\}$$
$$= \min^{i}\{\bigcap_{i\in I} \tilde{\mu}_i(u), \bigcap_{i\in I} \tilde{\mu}_i(w)\};$$

$$\bigcap_{i\in I} \tilde{\nu}_i(u\alpha v\beta w) = \inf^{i}_{i\in I}\tilde{\nu}_i(u\alpha v\beta w)$$
$$\geq \inf^{i}_{i\in I}\min^{i}\{\tilde{\nu}_i(u), \tilde{\nu}_i(w)\}$$
$$= \min^{i}\{\inf^{i}_{i\in I}\tilde{\nu}_i(u), \inf^{i}_{i\in I}\tilde{\nu}_i(w)\}$$
$$= \min^{i}\{\bigcap_{i\in I} \tilde{\nu}_i(u), \bigcap_{i\in I} \tilde{\nu}_i(w)\}, \text{ and}$$

$$\bigcap_{i\in I} \tilde{\xi}_i(u\alpha v\beta w) = \inf^{i}_{i\in I}\tilde{\xi}_i(u\alpha v\beta w)$$
$$\leq \inf^{i}_{i\in I}\max^{i}\{\tilde{\xi}_i(u), \tilde{\xi}_i(w)\}$$
$$= \max^{i}\{\inf^{i}_{i\in I}\tilde{\xi}_i(u), \inf^{i}_{i\in I}\tilde{\xi}_i(w)\}$$
$$= \max^{i}\{\bigcap_{i\in I} \tilde{\xi}_i(u), \bigcap_{i\in I} \tilde{\xi}_i(w)\}.$$

Hence the proof.

Theorem 3. If $\{\tilde{A}_{s_i}; i \in I\}$ be a family of SIVFBI of a gamma near-ring $\mathcal{R}$, then $\bigcup_{i \in I} \tilde{A}_{s_i}$ is also SIVFBI of $\mathcal{R}$, where I is an index set.

Proof. Let $\{\tilde{A}_{s_i}; i \in I\}$ be a family of SIVFBI of a gamma near-ring $\mathcal{R}$. For any $u, v, w \in \mathcal{R}$ and $\alpha, \beta \in \Gamma$.

$$\begin{aligned}\bigcup_{i\in I} \tilde{\mu}_i(u-v) &= \sup^i_{i\in I}\tilde{\mu}_i(u-v)\\ &\geq \sup^i_{i\in I}\min^i\{\tilde{\mu}_i(u), \tilde{\mu}_i(v)\}\\ &= \min^i\{\sup^i_{i\in I}\tilde{\mu}_i(u), \sup^i_{i\in I}\tilde{\mu}_i(v)\}\\ &= \min^i\{\bigcup_{i\in I}\tilde{\mu}_i(u), \bigcup_{i\in I}\tilde{\mu}_i(v)\};\end{aligned}$$

$$\begin{aligned}\bigcup_{i\in I} \tilde{\nu}_i(u-v) &= \sup^i_{i\in I}\tilde{\nu}_i(u-v)\\ &\geq \sup^i_{i\in I}\min^i\{\tilde{\nu}_i(u), \tilde{\nu}_i(v)\}\\ &= \min^i\{\sup^i_{i\in I}\tilde{\nu}_i(u), \sup^i_{i\in I}\tilde{\nu}_i(v)\}\\ &= \min^i\{\bigcup_{i\in I}\tilde{\nu}_i(u), \bigcup_{i\in I}\tilde{\nu}_i(v)\};\end{aligned}$$

$$\begin{aligned}\bigcup_{i\in I} \tilde{\xi}_i(u-v) &= \inf^i_{i\in I}\tilde{\xi}_i(u-v)\\ &\leq \sup^i_{i\in I}\max^i\{\tilde{\xi}_i(u), \tilde{\xi}_i(v)\}\\ &= \max^i\{\sup^i_{i\in I}\tilde{\xi}_i(u), \sup^i_{i\in I}\tilde{\xi}_i(v)\}\\ &= \max^i\{\bigcup_{i\in I}\tilde{\xi}_i(u), \bigcup_{i\in I}\tilde{\xi}_i(v)\};\end{aligned}$$

$$\begin{aligned}\bigcup_{i\in I} \tilde{\mu}_i(u\alpha v\beta w) &= \sup^i_{i\in I}\tilde{\mu}_i(u\alpha v\beta w)\\ &\geq \sup^i_{i\in I}\min^i\{\tilde{\mu}_i(u), \tilde{\mu}_i(w)\}\\ &= \min^i\{\sup^i_{i\in I}\tilde{\mu}_i(u), \sup^i_{i\in I}\tilde{\mu}_i(w)\}\\ &= \min^i\{\bigcap_{i\in I}\tilde{\mu}_i(u), \bigcup_{i\in I}\tilde{\mu}_i(w)\};\end{aligned}$$

$$\begin{aligned}\bigcup_{i\in I} \tilde{\nu}_i(u\alpha v\beta w) &= \sup^i_{i\in I}\tilde{\nu}_i(u\alpha v\beta w)\\ &\geq \sup^i_{i\in I}\min^i\{\tilde{\nu}_i(u), \tilde{\nu}_i(w)\}\\ &= \min^i\{\sup^i_{i\in I}\tilde{\nu}_i(u), \sup^i_{i\in I}\tilde{\nu}_i(w)\}\\ &= \min^i\{\bigcup_{i\in I}\tilde{\nu}_i(u), \bigcup_{i\in I}\tilde{\nu}_i(w)\}, \text{ and}\end{aligned}$$

$$\begin{aligned}\bigcup_{i\in I} \tilde{\xi}_i(u\alpha v\beta w) &= \sup^i_{i\in I}\tilde{\xi}_i(u\alpha v\beta w)\\ &\leq \inf^i_{i\in I}\max^i\{\tilde{\xi}_i(u), \tilde{\xi}_i(w)\}\\ &= \max^i\{\sup^i_{i\in I}\tilde{\xi}_i(u), \sup^i_{i\in I}\tilde{\xi}_i(w)\}\\ &= \max^i\{\bigcup_{i\in I}\tilde{\xi}_i(u), \bigcup_{i\in I}\tilde{\xi}_i(w)\}.\end{aligned}$$

Hence the proof.

Theorem 4. If $\tilde{A}_s$ and $\tilde{\sigma}_s$ are SIVFBIs of $\mathcal{R}$, then $\tilde{A}_s \wedge \tilde{\sigma}_s$ is SIVFBI of $\mathcal{R}$.

Proof. Let and $\tilde{\sigma}_s$ are SFBIs of $\mathcal{R}$. Let $u, v, w \in \mathcal{R}$ and $\alpha, \beta \in \Gamma$. Then,

$(\tilde{\mu} \wedge \tilde{\sigma}_s)(u-v) = \min^i\{\tilde{\mu}(u-v), \tilde{\sigma}_s(u-v)\}$, since by $(\tilde{\mu} \wedge \tilde{\sigma})(u) = \min^i\{\tilde{\mu}(u), \tilde{\sigma}(u)\}$

$\geq \min^i\{\min^i\{\tilde{\mu}(u), \tilde{\mu}(v)\}, \min^i\{\tilde{\sigma}_s(u), \tilde{\sigma}_s(v)\}\}$
$= \min^i\{\min^i\{\min^i\{\tilde{\mu}(u), \tilde{\mu}(v), \tilde{\sigma}_s(u)\}, \tilde{\sigma}_s(v)\}\}$
$= \min^i\{\min^i\{\min^i\{\tilde{\mu}(u), \tilde{\sigma}_s(u)\}, \tilde{\mu}(v)\}, \tilde{\sigma}_s(v)\}$
$= \min^i\{\min^i\{\tilde{\mu}(u), \tilde{\sigma}_s(u)\}, \min^i\{\tilde{\mu}(v), \tilde{\sigma}_s(v)\}$
$= \min^i\{\tilde{\mu} \wedge \tilde{\sigma}(u)), (\tilde{\mu} \wedge \tilde{\sigma}(v))\}.$

Also $(\tilde{\nu} \wedge \tilde{\sigma}_s)(u-v) \geq min^i\{\tilde{\nu} \wedge \tilde{\sigma}(u)), (\tilde{\nu} \wedge \tilde{\sigma}(v))\}$ and $(\tilde{\xi} \wedge \tilde{\sigma}_s)(u-v) \leq max^i\{\tilde{\xi} \wedge \tilde{\sigma}(u)), (\tilde{\xi} \wedge \tilde{\sigma}(v))\}$.
Since $(\tilde{\mu}(u\alpha v\beta w) \geq min^i\{\tilde{\mu}(u), \tilde{\mu}(v))$.

$(\tilde{\mu} \wedge \tilde{\sigma}_s)(u\alpha v\beta w) = \min^i\{\tilde{\mu}(u\alpha v\beta w), \tilde{\sigma}_s(u\alpha v\beta w))$
$\geq \min^i\{\min^i\{\tilde{\mu}(u), \tilde{A}_s(w)\}, \min^i\{\tilde{\sigma}_s(u), \tilde{\sigma}_s(w)\}\}$
$= \min^i\{\min^i\{\tilde{\mu}(u), \tilde{\sigma}_s(u)\}, \min^i\{\tilde{\mu}(w)\}, \tilde{\sigma}_s(w)\}\}$
$= \min^i\{(\tilde{\mu} \wedge \tilde{\sigma}_s)(u), (\tilde{\mu} \wedge \tilde{\sigma}_s)(w)\}.$

Also $(\tilde{\nu} \wedge \tilde{\sigma}_s)(u\alpha v\beta w) \geq min^i\{(\tilde{\nu} \wedge \tilde{\sigma}_s)(u), (\tilde{\nu} \wedge \tilde{\sigma}_s)(w)$ and $(\tilde{\xi} \wedge \tilde{\sigma}_s)(u\alpha v\beta w) \leq max^i\{(\tilde{\xi} \wedge \tilde{\sigma}_s)(u), (\tilde{\xi} \wedge \tilde{\sigma}_s)(w))$.

Hence $(\tilde{A}_s \wedge \tilde{\sigma}_s)$ is a SIVFBI of $\mathcal{R}$.

Lemma 1. Let A be BI of $\mathcal{R}$. For any $0 < m < 1$, there exists a SIVFBI A_s of $\mathcal{R}$ such that $\tilde{A}_{s_m} = A$.

Proof. Let A be BI of $\mathcal{R}$. Define $\tilde{A}_s: \mathcal{R} \longrightarrow [0,1]$ by

$$\tilde{A}_s(u) = \begin{cases} m, & \text{if } u \in A \\ 0, & \text{if } u \notin A. \end{cases}$$

where m be a constant in $(0,1)$. Clearly $\tilde{A}_{s_m} = A$. Let $u, v \in \mathcal{R}$. If $u, v \in A$, then $\tilde{\mu}(u-v) = m \geq min^i\{\tilde{\mu}(u), \tilde{\mu}(v)\}$, $\tilde{\nu}(u-v) = m \geq min^i\{\tilde{\nu}(u), \tilde{\nu}(v)\}$ and $\tilde{\xi}(u-v) = m \leq max^i\{\tilde{\xi}(u), \tilde{\xi}(v)\}$.

If at least one of u and v is not in A, then $u - v \notin A$ and so $\tilde{\mu}(u-v) = 0 = min^i\{\tilde{\mu}(u), \tilde{\mu}(v)\}$, $\tilde{\nu}(u-v) = 0 = min^i\{\tilde{\nu}(u), \tilde{\nu}(v)\}$ and $\tilde{\xi}(u-v) = 0 = max^i\{\tilde{\xi}(u), \tilde{\xi}(v)\}$.

Let $u, v, w \in \mathcal{R}$ and $\alpha, \beta \in \Gamma$. If $u, w \in A$, then $\tilde{\mu}(u), \tilde{\nu}(u), \tilde{\xi}(u) = m; \tilde{\mu}(w), \tilde{\nu}(w), \tilde{\xi}(w) = m$. Also $\tilde{\mu}(u\alpha v\beta w) = m \geq min^i\{\tilde{\mu}(u), \tilde{\mu}(w)\}$, $\tilde{\nu}(u\alpha v\beta w) = m \geq min^i\{\tilde{\nu}(u), \tilde{\nu}(w)\}$ and $\tilde{\xi}(u\alpha v\beta w) = m \leq max^i\{\tilde{\xi}(u), \tilde{\xi}(w)\}$.

If at least one of u and w is not in A, then $\tilde{\mu}(u\alpha v\beta w) \geq 0 = min^i\{\tilde{\mu}(u), \tilde{\mu}(w)\}$, $\tilde{\nu}(u\alpha v\beta w) \geq 0 = min^i\{\tilde{\nu}(u), \tilde{\nu}(w)\}$ and $\tilde{\xi}(u\alpha v\beta w) \leq 0 = max^i\{\tilde{\xi}(u), \tilde{\xi}(w)\}$.

Thus $\tilde{A}_s$ is SIVFBI of $\mathcal{R}$.

Theorem 5. If $\tilde{A}_s$ be SIVFBI of $\mathcal{R}$, then the complement $\tilde{A}_s$ is also SIVFBI o $\mathcal{R}$ f.

Proof. For $u, v, w \in \mathcal{R}$ and $\alpha, \beta \in \Gamma$, we have

$\tilde{\mu}(u-v)=1-\tilde{\mu}(u-v)\geq 1-min^i\{\tilde{\mu}(u),\tilde{\mu}(v)\}$=$min^i\{1-\tilde{\mu}(u),1-\tilde{\mu}(v)\}$= $min^i\{\tilde{\mu}(u),\tilde{\mu}(v)\}$, and also $\tilde{v}(u-v)\geq min^i\{\tilde{v}(u),\tilde{v}(v)\}$, $\tilde{\xi}(u-v)\leq max^i\{\tilde{\xi}(u),\tilde{\xi}(v)\}$.

$\tilde{\mu}(u\alpha v\beta w)=1-\tilde{\mu}(u\alpha v\beta w)\geq 1-min^i\{\tilde{\mu}(u),\tilde{\mu}(w)\}$=$min^i\{1-\tilde{\mu}(u),1-\tilde{\mu}(w)\}$= $min^i\{\tilde{\mu}(u),\tilde{\mu}(w))$, and also $\tilde{v}(u\alpha v\beta w)\geq min^i\{\tilde{v}(u),\tilde{v}(w)\}$, $\tilde{\xi}(u\alpha v\beta w)\leq max^i\{\tilde{\xi}(u),\tilde{\xi}(w)\}$.
Hence $\tilde{A}_s$ is also SIVFBI of $\mathcal{R}$.

Lemma 6. Let U is fuzzy subset of $\mathcal{R}$. Then U is BI of $\mathcal{R}$ if and only if $\tilde{A}_{s_U}$ is SIVFBI of $\mathcal{R}$.

Proof. Let U be BI of $\mathcal{R}$. For $u,v\in U$, $u-v\in U$.

Let $u,v\in\mathcal{R}$.
case(a): If $u,v\in U$, then $\tilde{\mu}_U(u)=1$ and $\tilde{\mu}_U(v)=1$. Thus $\tilde{\mu}_U(u-v)=1\geq min^i\{\tilde{\mu}(u),\tilde{\mu}(v)\}$.
case(b): If $u\in U$ and $v\notin U$, then $\tilde{\mu}_U(u)=1$ and $\tilde{\mu}_U(v)=0$. Thus $\tilde{\mu}_U(u-v)=0\geq min^i\{\tilde{\mu}(u),\tilde{\mu}(v))$.
case(c): If $u\notin U$ and $v\in U$, then $\tilde{\mu}_U(u)=0$ and $\tilde{\mu}_U(v)=1$. Thus $\tilde{\mu}_U(u-v)=0\geq min^i\{\tilde{\mu}(u),\tilde{\mu}(v)\}$.
case(d): If $u\notin U$ and $v\notin U$, then $\tilde{\mu}_U(u)=0$ and $\tilde{\mu}_U(v\tilde{\xi}_U(u-v)\leq max^i\{\tilde{\xi}(u),\tilde{\xi}(v)\})=0$. Thu$\tilde{\mu}_U(u-v)=$ 0s $\geq min^i\{\tilde{\mu}(u),\tilde{\mu}(v)\}$.

In the above four cases$\tilde{v}_U(u-v)\geq min^i\{\tilde{v}(u),\tilde{v}(v)\}$ and.

Let $u,v,w\in\mathcal{R}$.
case(a): If $u\in U$ and $w\in U$, then $\tilde{\mu}_U(u)=1$ and $\tilde{\mu}_U(w)=1$. Thus $\tilde{\mu}_U(u\alpha v\beta w)=1\geq min^i\{\tilde{A}_{s_U}\tilde{\mu}(u),\tilde{\mu}(w)\}$.
case(b): If $u\in U$ an $w\notin U$ d, then$\tilde{\mu}_U(u)=1$ and $\tilde{\mu}_U(w)=0$. Thus$\tilde{\mu}_U(u\alpha v\beta w)=0\geq min^i\{\tilde{\mu}(u),\tilde{\mu}(w)\}$.
case(c): If $u\notin U$ and $w\in U$, then $\tilde{\mu}_U(u)=0$ and $\tilde{\mu}_U(w)=1$. Thus $\mu_U(u\alpha v\beta w)=0\geq min^i\{\tilde{\mu}(u),\tilde{\mu}(w)\}$.
case(d): If $u\notin U$ and $w\notin U$, then $\tilde{\mu}_U(u)=0$ and $\tilde{\mu}_U(w)=0$. Thus $\tilde{\mu}_U(u\alpha v\beta w)=0\geq min^i\{\tilde{\mu}(u),\tilde{\mu}(w)\}$.

Also $\tilde{v}_U(u\alpha v\beta w)\geq min^i\{\tilde{v}(u),\tilde{v}(w)\}$ and$\tilde{\xi}_U(u\alpha v\beta w)\leq max^i\{\tilde{\xi}(u),\tilde{\xi}(w)\}$ Thus $\tilde{A}_{s_U}$ is a SIVFBI of $\mathcal{R}$.
Conversely, suppose is a SIVFBI of $\mathcal{R}$. Then by ***lemma 4*** $\tilde{A}_{s_U}$ has only two elements.
Hence U is BI of $\mathcal{R}$.

Theorem 7. If $\mathcal{R}$ be a gamma near-ring and $\tilde{A}_s$ be SIVFBI of $\mathcal{R}$, then the set $\mathcal{R}_{\tilde{A}_s}=\{u\in\mathcal{R}|\tilde{A}_s(u)=\tilde{A}_s(0)\}$ is BI of $\mathcal{R}$. $\mathcal{R}$

Proof. Let $\tilde{A}_s$ be SIVFBI of and le $u,v,w\in\mathcal{R}$ t. Then

$\tilde{\mu}(u-v)\geq min^i\{\tilde{\mu}(u),\tilde{\mu}(v)\}=min^i\{\tilde{\mu}(0),\tilde{\mu}(0)\}=\tilde{\mu}(0)$.
So $\tilde{\mu}(u-v)=\tilde{\mu}(0)$, then $u-v\in\mathcal{R}_{\tilde{A}_s}$.

$\tilde{v}(u-v)\geq min^i\{\tilde{v}(u),\tilde{v}(v)\}=min^i\{\tilde{v}(0),\tilde{v}(0)\}=\tilde{v}(0)$.
So $\tilde{v}(u-v)=\tilde{v}(0)$, then $u-v\in\mathcal{R}_{\tilde{A}_s}$.

$\tilde{\xi}(u-v)\leq max^i\{\tilde{\xi}(u),\tilde{\xi}(v)\}=max^i\{\tilde{\xi}(0),\tilde{\xi}(0)\}=\tilde{\xi}(0)$.
So $\tilde{\xi}(u-v)=\tilde{\xi}(0)$, then $u-v\in\mathcal{R}_{\tilde{A}_s}$.

$\tilde{\mu}(u\alpha v\beta w) \geq \min^i\{\tilde{\mu}(u), \tilde{\mu}(w)\} = \min^i\{\tilde{\mu}(0), \tilde{\mu}(0)\} = \tilde{\mu}(0)$.
So $\tilde{\mu}(u\alpha v\beta w) = \tilde{\mu}(0)$, then $u\alpha v\beta w \in \mathcal{R}_{\tilde{A}_s}$.

$\tilde{\nu}(u\alpha v\beta w) \geq \min^i\{\tilde{\nu}(u), \tilde{\nu}(w)\} = \min^i\{\tilde{\nu}(0), \tilde{\nu}(0)\} = \tilde{\nu}(0)$.
So $\tilde{\nu}(u\alpha v\beta w) = \tilde{\nu}(0)$, then $u\alpha v\beta w \in \mathcal{R}_{\tilde{A}_s}$.

$\tilde{\xi}(u\alpha v\beta w) \leq \max^i\{\tilde{\xi}(u), \tilde{\xi}(w)\} = \max^i\{\tilde{\xi}(0), \tilde{\xi}(0)\} = \tilde{\xi}(0)$.
So $\tilde{\xi}(u\alpha v\beta w) = \tilde{\xi}(0)$, then $u\alpha v\beta w \in \mathcal{R}_{\tilde{A}_s}$.

Then $\mathcal{R}_{\tilde{A}_s}$ is BI of $\mathcal{R}$.

Theorem 8. If B be a non-empty subset of $\mathcal{R}$ and$\tilde{A}_{s_B}$ be a Spherical Interval-Valued Fuzzy Set (SIVFS) $\mathcal{R}$ defined by

$$\tilde{A}_{s_B}(u) = \begin{cases} \tilde{p}, & \text{if } u \in B \\ \tilde{q}, & \text{otherw}\tilde{p} \geq \tilde{q}\text{ise.} \end{cases}$$

for $u \in \mathcal{R}$, $\tilde{p}, \tilde{q} \in D[0,1]$ and. Then $\tilde{A}_{s_B}(u)$ is a SIVFBI of $\mathcal{R}$ if and only if B is a BI of $\mathcal{R}$. Als$\mathcal{R}_{\tilde{A}_{s_B}} = B$ o.

Proof. Le$\tilde{A}_{s_B}$t be a SIVFS $\mathcal{R}$ and le $u, v, w \in B$ t. Then $\tilde{A}_{s_B}(u) = \tilde{p} = \tilde{A}_{s_B}(v) = \tilde{A}_{s_B}(w)$. Now,

$\tilde{A}_{s_B}(u - v) \geq \min^i\{\tilde{A}_{s_B}(u), \tilde{A}_{s_B}(v)\}$
$= \min^i\{\tilde{p}, \tilde{p}\}$
$= \tilde{p}$.
$\tilde{A}_{s_B}(u - v) = \tilde{p}$, so $u - v \in B$.
$\tilde{A}_{s_B}(u\alpha v\beta w) \geq \min^i\{\tilde{A}_{s_B}(u), \tilde{A}_{s_B}(w)\}$
$= \min^i\{\tilde{p}, \tilde{p}\}$
$= \tilde{p}$.
$\tilde{A}_{s_B}B(u\alpha v\beta w) = \tilde{p}$, so $u\alpha v\beta w \in B$.

Then is a BI of $\mathcal{R}$.
Conversely let B be a BI of $\mathcal{R}$ and le $u, v, w \in \mathcal{R}$ t.
If at $\tilde{A}_{s_B}(u)$least one u, v is not in B, then $u - v \notin B$ and so$\tilde{A}_{s_B}(u - v) \geq min^i\{\tilde{A}_{s_B}(u), \tilde{A}_{s_B}(v)\} = \tilde{q}$.
If at least one u, w is not in B, then $u\alpha v\beta w \notin B$ and so $\tilde{A}_{s_B}(u\alpha v\beta w) \geq min^i\{\tilde{A}_{s_B}(u), \tilde{A}_{s_B}(w)\} = \tilde{q}$.

Thus is a SIVFBI of $\mathcal{R}$.

4. Conclusion

We obtained the union and intersection of the spherical interval-valued fuzzy bi-ideal of gamma near-ring $\mathcal{R}$ is also a spherical interval-valued fuzzy bi-ideal of gamma near-ring. And for that condition, $\tilde{A}_{s_m} = A$, for any $0 < \mathrm{m} < 1$, bi-ideal of gamma near-ring $\mathcal{R}$ becomes spherical interval-valued fuzzy bi-ideal of gamma near-ring $\mathcal{R}$. In future we will discuss the spherical fuzzy sets in some other algebraic structures.

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K-algebras on Quadripartitioned Single Valued Neutrosophic Sets

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ABSTRACT

Quadripartitioned Single Valued Neutrosophic (QSVN) set is a powerful structure where we have four components: Truth-T, Falsity-F, Unknown-U and Contradiction-C. And also it generalizes the concept of fuzzy, initutionstic and single valued neutrosophic set. In this paper we have proposed the concept of K-algebras on QSVN, level subset of QSVN and studied some of the results. In addition to this we have also investigated the characteristics of QSVN K-subalgebras under homomorphism.

1. Introduction

Dar and Akram [12] proposed a novel logical algebra known as K-algebra. The algebraic structure of a group G which K-algebra was built on should have a right identity element and satisfy the properties of non-commutative and non-associative. Furthermore this group G is of the type where each non-identity element is not of order 2 and K-algebra was built by adjoining the induced binary operation on G [11, 12, 13]. Zadeh's fuzzy set theory [22] was a powerful framework which deals the concept of uncertainty, imprecision and also it represented by membership function which lies in a unit interval of [0,1]. Fuzzy K-algebra was introduced by Akram et al. [2, 3, 5] and also they established this in a wide-reaching way

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through other researchers. Later Atanassov [9] introduced the concept of intuitionistic fuzzy set in 1983. It has an additional degree called the degree of nonmembership. Intuitionistic fuzzy K-subalgebras was proposed by Akram et al. [4, 6]. Intuitionistic fuzzy Ideals of BCK-Algebras was proposed by Jun and Kim [14].

Neutrosophic set which is a generalization of fuzzy set and intuitionistic fuzzy set was introduced by Smarandache [20] in 1998. Along with membership and non-membership function neutrosophic set has one more extra component called indeterminacy membership function. Also all the values of these three components lie in the real standard or non-standard subset of unit interval]−0,1+[where −0 = 0 − ϵ, 1+ = 1 + ϵ, ϵ is an infinitesimal number. In neutrosophic set theory algebraic structures were studied in soft topological K-algebras [7]. Agboola and Davvaz [1] presented the introduction to neutrosophic BCI/BCK algebras. Smarandache and Wang et al. [21] introduced single-valued neutrosophic set which plays a vital place in many real life problems and it takes the values from the subset of [0,1]. Akram et al. [8] studied K-algebras on single valued neutrosophic sets and also discussed homomorphisms between the single valued neutrosophic K-subalgebras. Belnap [10] introduced the concept of four valued logic that is the information are represented by four components T, F, None, Both which denote true, false, neither true nor false, both true and false, respectively. Based on this concept, Smarandache proposed four numerical valued neutrosophic logic where indeterminacy is splitted into two terms known as Contradiction (C) and Unknown (U). Chatterjee et al. [19] introduced Quadripartitioned Single Valued Neutrosophic (QSVN) set in which we have four components T, C, U, and F, respectively, and also it lies in the real unit interval of [0,1]. K. Mohana and M. Mohanasundari [15, 17] studied the concept of Quadripartitioned Single Valued Neutrosophic Relations (QSVNR) as well as some properties of quadripartitioned single valued neutrosophic rough sets and its axiomatic characterizations. Under QSVN environment multicriteria decision making problems has been discussed in [16, 18].

In this paper Section 2 deals with the basic definitions of QSVN set and the concept of K-algebras on single valued neutrosophic set. Section 3 discusses about K-algebras on QSVN, level subset of QSVN and also studies some of the results. Section 4 defines the homomorphism of quadripartitioned single valued neutrosophic K-algebras, characteristic and fully invariant K-subalgebras. Section 5 concludes the paper.

2. Preliminaries

This section deals with the basic definitions of QSVNS and K-algebra of single valued neutrosophic set that helps us to study the rest of the paper.

Definition 1. [19]. Let X be a non-empty set. A quadripartitioned neutrosophic set A over X characterizes each element x in X by a truth-membership function T_A, a contradiction membership function C_A, an ignorance – membership function U_A, and a falsity membership function F_A such that for each $x \in X, T_A, C_A, U_A, F_A \in [0,1]$ and $0 \le T_A(x)+C_A(x)+U_A(x)+F_A(x) \le 4$. When X is discrete A is represented as,

$$A = \sum_{i=1}^{n} \left\langle T_A(x_i), C_A(x_i), U_A(x_i), F_A(x_i) \right\rangle / x_i, x_i \in X.$$

However, when the universe of discourse is continuous A is represented as $A=\int_X \langle T_A(x),C_A(x),U_A(x),F_A(x)\rangle / x, x\in X$.

Definition 2. [19]. Consider two QSVNS A and B over X. A is said to be contained in B, denoted by $A\subseteq B$ *iff* $T_A(x)\le T_B(x), C_A(x)\le C_B(x), U_A(x)\ge U_B(x)$ *and* $F_A(x)\ge F_B(x)$.

Definition 3. [19]. The complement of a QSVNS A is denoted by A^C and is defined as, $A^C=\sum_{i=1}^{n}\langle F_A(x_i),U_A(x_i),C_A(x_i),T_A(x_i)\rangle / x_i, x_i\in X$,
$T_{A^C}(x_i)=F_A(x_i), C_{A^C}(x_i)=U_A(x_i), U_{A^C}(x_i)=C_A(x_i)$ *and* $F_{A^C}(x_i)=T_A(x_i), x_i\in X$.

Definition 4. [19]. The union of two QSVNS A and B is denoted by $A\cup B$ and is defined as, $A\cup B=\sum_{i=1}^{n}\langle T_A(x_i)\vee T_B(x_i), C_A(x_i)\vee C_B(x_i), U_A(x_i)\wedge U_B(x_i), F_A(x_i)\wedge F_B(x_i)\rangle / x_i, x_i\in X$.

Definition 5. [19]. The intersection of two QSVNS A and B is denoted by $A\cap B$ and is defined as, $A\cap B=\sum_{i=1}^{n}\langle T_A(x_i)\wedge T_B(x_i), C_A(x_i)\wedge C_B(x_i), U_A(x_i)\vee U_B(x_i), F_A(x_i)\vee F_B(x_i)\rangle / x_i, x_i\in X$.

Definition 6. [12]. Let $(G,\cdot,\odot,e)$ be a group in which each non-identity element is not of order 2. Then a K-algebra is a structure Ƙ= $(G,\cdot,\odot,e)$ on a group G in which induced binary operation $\odot: G\times G\rightarrow G$ is defined b $\odot(x,y)=x\odot y=xy^{-1}$y and satisfies the following axioms:

$$(x\odot y)\odot(x\odot z)=\big(x\odot((e\odot z)\odot(e\odot y))\big)\odot x,$$

$$x\odot(x\odot y)=\big(x\odot(e\odot y)\big)\odot x,$$

$$x\odot x=e,$$

$$x\odot e=x,$$

$$e\odot x=x^{-1}, \text{ for all } x,y,z\in G.$$

Definition 7. [8]. A single-valued neutrosophic set $A=(T_A,I_A,F_A)$ in a K-algebra Ƙ is called a single-valued neutrosophic K-subalgebra of Ƙ if it satisfies the following conditions:

$$T_A(s\odot t)\ge \min\{T_A(s),T_A(t)\},$$

$$I_A(s\odot t)\ge \min\{I_A(s),I_A(t)\},$$

$$F_A(s\odot t)\le \max\{F_A(s),F_A(t)\}, \text{ for all } s,t\in G.$$

Note that $T_A(e)\ge T_A(s), I_A(e)\ge I_A(s), F_A(e)\le F_A(s), for\ all\ s\in G$.

3. Quadripartitioned Single Valued Neutrosophic K-Algebras

Definition 8. A quadripartitioned single valued neutrosophic set $X=(T_X, C_X, U_X, F_X)$ in a K-algebra Ꝁ is called a quadripartitioned single valued neutrosophic K-subalgebra of Ꝁ if it satisfies the following conditions:

$T_X(e) \geq T_X(u), C_X(e) \geq C_X(u), U_X(e) \leq U_X(u), F_X(e) \leq F_X(u)$ for all $u \in G$.

$T_X(u \odot v) \geq \min\{T_X(u), T_X(v)\}$,

$C_X(u \odot v) \geq \min\{C_X(u), C_X(v)\}$,

$U_X(u \odot v) \leq \max\{U_X(u), U_X(v)\}$,

$F_X(u \odot v) \leq \max\{F_X(u), F_X(v)\}$ for all $u, v \in G$.

Example 1. Let $G=\{e, g, g^2, g^3, g^4\}$ is the cyclic group of order 5 in a K-algebra Ꝁ= $(G, \cdot, \odot, e)$. The Cayley's table for ʘ is given as follows.

$\odot$	e	g	g^2	g^3	g^4
e	e	g^4	g^3	g^2	g
g	g	e	g^4	g^3	g^2
g^2	g^2	g	e	g^4	g^3
g^3	g^3	g^2	g	e	g^4
g^4	g^4	g^4	g^3	g	e

We define a quadripartitioned single valued neutrosophic set $X=(T_X, C_X, U_X, F_X)$ in K-algebra as follows:

$T_X(e)=0.5,\quad C_X(e)=0.7,\quad U_X(e)=0.3,\quad F_X(e)=0.5$
$T_X(u)=0.2,\quad C_X(u)=0.4,\quad U_X(u)=0.5,\quad F_X(u)=0.8$

for all $u \neq e \in G$. Clearly, it shows that $X=(T_X, C_X, U_X, F_X)$ is a quadripartitioned single valued neutrosophic K-algebras of Ꝁ.

Proposition 1. If $X=(T_X, C_X, U_X, F_X)$ denotes a quadripartitioned single valued neutrosophic K-algebras of Ꝁ then,

$a)$ $(\forall\, u, v \in G), (T_X(u \odot v) = T_X(v) \Rightarrow T_X(u) = T_X(e))$

$(\forall\, u, v \in G), \big(T_X(u) = T_X(e) \Rightarrow T_X(u \odot v) \geq T_X(v)\big)$;

b) $(\forall \mathrm{u}, \mathrm{v} \in \mathrm{G}), (\mathrm{C_X}(\mathrm{u}\odot\mathrm{v}) = \mathrm{C_X}(\mathrm{v}) \Rightarrow \mathrm{C_X}(\mathrm{u}) = \mathrm{C_X}(\mathrm{e}))$

$(\forall \mathrm{u}, \mathrm{v} \in \mathrm{G}), \big(\mathrm{C_X}(\mathrm{u}) = \mathrm{C_X}(\mathrm{e}) \Rightarrow \mathrm{C_X}(\mathrm{u}\odot\mathrm{v}) \geq \mathrm{C_X}(\mathrm{v})\big);$

c) $(\forall \mathrm{u}, \mathrm{v} \in \mathrm{G}), (\mathrm{U_X}(\mathrm{u}\odot\mathrm{v}) = \mathrm{U_X}(\mathrm{v}) \Rightarrow \mathrm{U_X}(\mathrm{u}) = \mathrm{U_X}(\mathrm{e}))$

$(\forall \mathrm{u}, \mathrm{v} \in \mathrm{G}), \big(\mathrm{U_X}(\mathrm{u}) = \mathrm{U_X}(\mathrm{e}) \Rightarrow \mathrm{U_X}(\mathrm{u}\odot\mathrm{v}) \leq \mathrm{U_X}(\mathrm{v})\big);$

d) $(\forall \mathrm{u}, \mathrm{v} \in \mathrm{G}), (\mathrm{F_X}(\mathrm{u}\odot\mathrm{v}) = \mathrm{F_X}(\mathrm{v}) \Rightarrow \mathrm{F_X}(\mathrm{u}) = \mathrm{F_X}(\mathrm{e}))$

$(\forall \mathrm{u}, \mathrm{v} \in \mathrm{G}), \big(\mathrm{F_X}(\mathrm{u}) = \mathrm{F_X}(\mathrm{e}) \Rightarrow \mathrm{F_X}(\mathrm{u}\odot\mathrm{v}) \geq \mathrm{F_X}(\mathrm{v})\big);$

Proof. We only prove (a) and (c). (b) and (d) proved in a similar way.

(a) First we assume that $T_X(u\odot v) = T_X(v)\ \forall\, u, v \in G$. Put $v = e$ and use (iii) of ***Definition 6*** we get $T_X(u) = T_X(u\odot e) = T_X(e)$. Let for $u, v\ \in G$ be such that $T_X(u) = T_X(e)$ then$T_X(u\odot v) \geq min\{T_X(u), T_X(v)\} = min\{T_X(e), T_X(v)\} = T_X(v)$.

Now to prove (c) consider that $U_X(u\odot v) = U_X(v)\ \forall\, u, v \in \mathrm{G}$. Put $v = e$ and use (iii) of ***Definition 6***, we have $U_X(u) = U_X(u\odot e) = U_X(e)$. Let for $u, v\ \in G$ be such that $U_X(u) = U_X(e)$ then, $U_X(u\odot v) \leq max\{U_X(u), U_X(v)\} = max\{U_X(e), U_X(v)\} = U_X(v)$. Hence the proof.

Definition 9. Let $X = (T_X, C_{X,} U_X, F_X)$ be a quadripartitioned single valued neutrosophic set in a K-algebra of Ҡ and let $(\lambda, \mu, \vartheta, \xi) \in [0,1] \times [0,1] \times [0,1] \times [0,1]$ with $\lambda + \mu + \vartheta + \xi \leq 4$. Then the sets,

$$\mathrm{X}_{(\lambda,\mu,\vartheta,\xi)} = \{\mathrm{u} \in \mathrm{G} |\ \mathrm{T_X}(\mathrm{u}) \geq \lambda,\ \mathrm{C_X}(\mathrm{u}) \geq \mu,\ \mathrm{U_X}(\mathrm{u}) \leq \vartheta,\ \mathrm{F_X}(\mathrm{u}) \leq \xi\},$$

$$(\lambda, \mu, \vartheta, \xi)\mathrm{X}_{(\lambda,\mu,\vartheta,\xi)} = \mathrm{U}(\mathrm{T_X}, \lambda) \cap \mathrm{U}^{'}(\mathrm{C_X}, \mu) \cap \mathrm{L}(\mathrm{U_X}, \vartheta) \cap \mathrm{L}^{'}(\mathrm{F_X}, \xi)$$

are called $(\lambda, \mu, \vartheta, \xi)$ level subsets of quadripartitioned single valued neutrosophic set X.

And also the set $X_{(\lambda,\mu,\vartheta,\xi)} = \{u \in G |\ T_X(u) > \lambda,\ C_X(u) > \mu,\ U_X(u) < \vartheta,\ F_X(u) < \xi\}$ is known as strong level subset of X.

Note. The set of all $(\lambda, \mu, \vartheta, \xi) \in Im(T_X) \times Im(C_X) \times Im(U_X) \times Im(F_X)$ is known as image of $X = \big(T_X, C_{X,} U_X, F_X\big)$.

Proposition 2. If $X = (T_X, C_{X,} U_X, F_X)$ is a quadripartitioned single valued neutrosophic K-algebra of Ҡ then the level subsets,

$$\mathrm{U}(\mathrm{T_X}, \lambda) = \{\mathrm{u} \in \mathrm{G} |\ \mathrm{T_X}(\mathrm{u}) \geq \lambda\}, \mathrm{U}'(\mathrm{C_X}, \mu) = \{\mathrm{u} \in \mathrm{G} |\ \mathrm{C_X}(\mathrm{u}) \geq \mu\},$$

$$\mathrm{L}(\mathrm{U_X}, \vartheta) = \{\mathrm{u} \in \mathrm{G} |\ \mathrm{U_X}(\mathrm{u}) \leq \vartheta\}, \mathrm{L}'(\mathrm{F_X}, \xi) = \{\mathrm{u} \in \mathrm{G} |\ \mathrm{F_X}(\mathrm{u}) \leq \xi\}$$

are K-subalgebras of Ҡ for every $(\lambda, \mu, \vartheta, \xi) \in Im(T_X) \times Im(C_X) \times Im(U_X) \times Im(F_X) \subseteq [0,1]$

where $Im(T_X), Im(C_X), Im(U_X)$ *and* $Im(F_X)$ are sets of values $T(X), C(X), U(X)$ *and* $F(X)$, respectively.

Proof. Let $X = (T_X, C_X, U_X, F_X)$ be a quadripartitioned single valued neutrosophic set in a K-algebra of K and $(\lambda, \mu, \vartheta, \xi) \in Im(T_X) \times Im(C_X) \times Im(U_X) \times Im(F_X)$ be such that $U(T_X, \lambda) \neq \emptyset$, $U'(C_X, \mu) \neq \emptyset$, $L(U_X, \vartheta) \neq \emptyset$ *and* $L'(F_X, \xi) \neq \emptyset$. We have to show that U, U', L *and* L' are level K-subalgebras. Let for $u, v \in U(T_X, \lambda)$, $T_X(u) \geq \lambda$ *and* $T_X(v) \geq \lambda$. Then from ***Definition 8*** we get $T_X(u \odot v) \geq min\{T_X(u), T_X(v)\} \geq \lambda$. It shows that $u \odot v \in U(T_X, \lambda)$. Hence $U(T_X, \lambda)$ is a level K-subalgebra of K. Similarly, we can prove for $U'(C_X, \mu), L(U_X, \vartheta)$ *and* $L'(F_X, \xi)$.

Theorem 1. Let $X = (T_X, C_X, U_X, F_X)$ be a quadripartitioned single valued neutrosophic set in a K-algebra of K. Then $X = (T_X, C_X, U_X, F_X)$ is a quadripartitioned single valued neutrosophic K-subalgebra of K if and only if $X_{(\lambda,\mu,\vartheta,\xi)}$ is a K-subalgebra of K for every $(\lambda, \mu, \vartheta, \xi) \in Im(T_X) \times Im(C_X) \times Im(U_X) \times Im(F_X)$ with $\lambda + \mu + \vartheta + \xi \leq 4$.

Proof. First assume tha$X_{(\lambda,\mu,\vartheta,\xi)}$t is a K-subalgebra of K. If the conditions in ***Definition 8*** fail, then there exist $s, t \in G$ such that,

$$T_X(s \odot t) < \min\{T_X(s), T_X(t)\},$$

$$C_X(s \odot t) < \min\{C_X(s), C_X(t)\},$$

$$U_X(s \odot t) > \max\{U_X(s), U_X(t)\},$$

$$F_X(s \odot t) > \max\{F_X(s), F_X(t)\}.$$

Now let $\lambda_1 = \frac{1}{2}(T_X(s \odot t) + min\{T_X(s), T_X(t)\})$, $\mu_1 = \frac{1}{2}(C_X(s \odot t) + min\{C_X(s), C_X(t)\})$,

$\vartheta_1 = \frac{1}{2}(U_X(s \odot t) + max\{U_X(s), U_X(t)\})$, $\xi_1 = \frac{1}{2}(F_X(s \odot t) + max\{F_X(s), F_X(t)\})$.
Now we have,

$$T_X(s \odot t) < \lambda_1 < \min\{T_X(s), T_X(t)\},$$

$$C_X(s \odot t) < \mu_1 < \min\{C_X(s), C_X(t)\},$$

$$U_X(s \odot t) > \vartheta_1 > \max\{U_X(s), U_X(t)\},$$

$$F_X(s \odot t) > \xi_1 > \max\{F_X(s), F_X(t)\}.$$

This implies that $s, t \in X_{(\lambda,\mu,\vartheta,\xi)}$ *and* $s \odot t \notin X_{(\lambda,\mu,\vartheta,\xi)}$ which is a contradiction. This proves that the conditions of ***Definition 8*** is true. Hence $X = (T_X, C_X, U_X, F_X)$ is a quadripartitioned single valued neutrosophic K-subalgebra of K.

Now assume that $X = (T_X, C_X, U_X, F_X)$ be a quadripartitioned single valued neutrosophic K-subalgebra of K. Let for $(\lambda, \mu, \vartheta, \xi) \in Im(T_X) \times Im(C_X) \times Im(U_X) \times Im(F_X)$ with $\lambda + \mu + \vartheta + \xi \leq 4$ such that $X_{(\lambda,\mu,\vartheta,\xi)} \neq \emptyset$. Let $u, v \in X_{(\lambda,\mu,\vartheta,\xi)}$ be such that,

$T_X(u) \geq \lambda, T_X(v) \geq \lambda'$,

$C_X(u) \geq \mu, C_X(v) \geq \mu'$,

$U_X(u) \leq \vartheta, U_X(v) \leq \vartheta'$,

$F_X(u) \leq \xi, F_X(v) \leq \xi'$.

Now assume that $\lambda \leq \lambda', \mu \leq \mu', \vartheta \geq \vartheta'$ *and* $\xi \geq \xi'$. It follows from ***Definition 8*** that,

$T_X(u \odot v) \geq \lambda = \min\{T_X(u), T_X(v)\}$,

$C_X(u \odot v) \geq \mu = \min\{C_X(u), C_X(v)\}$,

$U_X(u \odot v) \leq \vartheta = \max\{U_X(u), U_X(v)\}$,

$F_X(u \odot v) \leq \xi = \max\{F_X(u), F_X(v)\}$.

This shows that $u \odot v \in X_{(\lambda,\mu,\vartheta,\xi)}$. Hence $X_{(\lambda,\mu,\vartheta,\xi)}$ is a K-subalgebra of Ҝ.

Theorem 2. Let $X = (T_X, C_X, U_X, F_X)$ be a quadripartitioned single valued neutrosophic K-subalgebra and $(\lambda_1, \mu_1, \vartheta_1, \xi_1), (\lambda_2, \mu_2, \vartheta_2, \xi_2) \in Im(T_X) \times Im(C_X) \times Im(U_X) \times Im(F_X)$ with $\lambda_i + \mu_i + \vartheta_i + \xi_i \leq 4$ for $i = 1,2$. Then $X_{(\lambda_1,\mu_1,\vartheta_1,\xi_1)} = X_{(\lambda_2,\mu_2,\vartheta_2,\xi_2)}$ if $(\lambda_1, \mu_1, \vartheta_1, \xi_1) = (\lambda_2, \mu_2, \vartheta_2, \xi_2)$.

Proof. When $(\lambda_1, \mu_1, \vartheta_1, \xi_1) = (\lambda_2, \mu_2, \vartheta_2, \xi_2)$ then the result is obvious for $X_{(\lambda_1,\mu_1,\vartheta_1,\xi_1)} = X_{(\lambda_2,\mu_2,\vartheta_2,\xi_2)}$. Conversely assume that $X_{(\lambda_1,\mu_1,\vartheta_1,\xi_1)} = X_{(\lambda_2,\mu_2,\vartheta_2,\xi_2)}$. Since $(\lambda_1, \mu_1, \vartheta_1, \xi_1) \in Im(T_X) \times Im(C_X) \times Im(U_X) \times Im(F_X)$ there exists $u \in G$ such that $T_X(u) = \lambda_1$, $C_X(u) = \mu_1, U_X(u) = \vartheta_1$ *and* $F_X(u) = \xi_1$. This implies that $u \in X_{(\lambda_1,\mu_1,\vartheta_1,\xi_1)} = X_{(\lambda_2,\mu_2,\vartheta_2,\xi_2)}$. Hence $\lambda_1 = T_X(u) \geq \lambda_2, \mu_1 = C_X(u) \geq \mu_2, \vartheta_1 = U_X(u) \leq \vartheta_2$ *and* $\xi_1 = F_X(u) \leq \xi_2$. Also $(\lambda_2, \mu_2, \vartheta_2, \xi_2) \in Im(T_X) \times Im(C_X) \times Im(U_X) \times Im(F_X)$ there exists $v \in G$ such that $T_X(v) = \lambda_2$, $C_X(v) = \mu_2, U_X(v) = \vartheta_2$ *and* $F_X(v) = \xi_2$. This implies that $v \in X_{(\lambda_2,\mu_2,\vartheta_2,\xi_2)} = X_{(\lambda_1,\mu_1,\vartheta_1,\xi_1)}$. Hence $\lambda_2 = T_X(v) \geq \lambda_1, \mu_2 = C_X(v) \geq \mu_1, \vartheta_2 = U_X(v) \leq \vartheta_1$ *and* $\xi_2 = F_X(v) \leq \xi_1$. Hence $(\lambda_1, \mu_1, \vartheta_1, \xi_1) = (\lambda_2, \mu_2, \vartheta_2, \xi_2)$.

Theorem 3. Let I be a K-subalgebra of K-algebra Ҝ. Then there exists a quadripartitioned single valued neutrosophic K-subalgebra $X = (T_X, C_X, U_X, F_X)$ of K-algebra Ҝ such that $X = (T_X, C_X, U_X, F_X) = I$ for some $\lambda, \mu \in (0,1]$ *and* $\vartheta, \xi \in [0,1)$.

Proof. Let $X = (T_X, C_X, U_X, F_X)$ be a quadripartitioned single valued neutrosophic set in K-algebra Ҝ given by,

$$T_X(u) = \begin{cases} \lambda \in (0,1], & \text{if } u \in I \\ 0, & \text{otherwise} \end{cases}$$

$$C_X(u) = \begin{cases} \mu \in (0,1], & \text{if } u \in I \\ 0, & \text{otherwise} \end{cases}$$

$$\mathrm{U_X(u)} = \begin{cases} \vartheta \in [0,1), & \text{if } u \in I \\ 0, & \text{otherwise} \end{cases}$$

$$\mathrm{F_X(u)} = \begin{cases} \xi \in [0,1), & \text{if } u \in I \\ 0, & \text{otherwise} \end{cases}.$$

Let $u, v \in G$. If $u, v \in I$, then $u \odot v \in I$ and so,

$$T_X(u \odot v) \geq \min\{T_X(u), T_X(v)\},$$

$$C_X(u \odot v) \geq \min\{C_X(u), C_X(v)\},$$

$$U_X(u \odot v) \leq \max\{U_X(u), U_X(v)\},$$

$$F_X(u \odot v) \leq \max\{F_X(u), F_X(v)\}.$$

Suppose $u \notin I \text{ or } v \notin I$ then,

$$T_X(u) = 0 \text{ or } T_X(v), C_X(u) = 0 \text{ or } C_X(v), U_X(u) = 0 \text{ or } U_X(v) \text{ and } F_X(u) = 0 \text{ or } F_X(v).$$

It implies that,

$$T_X(u \odot v) \geq \min\{T_X(u), T_X(v)\},$$

$$C_X(u \odot v) \geq \min\{C_X(u), C_X(v)\},$$

$$U_X(u \odot v) \leq \max\{U_X(u), U_X(v)\},$$

$$F_X(u \odot v) \leq \max\{F_X(u), F_X(v)\}.$$

Hence $X = (T_X, C_{X,} U_X, F_X)$ is a quadripartitioned single valued neutrosophic K-subalgebra of Ꝁ.

Consequently $X_{(\lambda,\mu,\vartheta,\xi)} = I$

Theorem 4. Let Ꝁ be a K-algebra. Let a chain of K-subalgebras: $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = G$. Then the level K-subalgebras of the quadripartitioned single valued neutrosophic K-subalgebra remains same as the K-subalgebras of this chain.

Proof. Let $\{\lambda_i | i = 0,1, \dots, n\}$, $\{\mu_i | i = 0,1, \dots, n\}$ be finite decreasing sequences and $\{\vartheta_i | i = 0,1, \dots, n\}$, $\{\xi_i | i = 0,1, \dots, n\}$ be finite increasing sequences in $[0,1]$ such that $\lambda_k + \mu_k + \vartheta_k + \xi_k \leq 4$ for $k = 0,1,2, \dots, n$. Let $X = (T_X, C_{X,} U_X, F_X)$ be a quadripartitioned single valued neutrosophic set in Ꝁ defined by $T_X(X_0) = \lambda_0$, $C_X(X_0) = \mu_0, U_X(X_0) = \vartheta_0 \text{ and } F_X(X_0) = \xi_0$,

$$T_X(X_i \backslash X_{i-1}) = \lambda_i, C_X(X_i \backslash X_{i-1}) = \mu_i, U_X(X_i \backslash X_{i-1}) = \vartheta_i \text{ and } F_X(X_i \backslash X_{i-1}) = \xi_i \text{ for } 0 < i \leq n.$$

We have t $u \odot v \in X_{i-1}$o prove that $X = (T_X, C_{X,} U_X, F_X)$ is a quadripartitioned single valued neutrosophic K-subalgebra of Ꝁ. Let $u, v \in G$. If $u, v \in X_i \backslash X_{i-1}$ then it implies that $T_X(u) = \lambda_i = T_X(v)$, $C_X(u) = \mu_i =$

$C_X(v)$, $U_X(u) = \vartheta_i = U_X(v)$ and$F_X(u) = \xi_i = F_X(v)$. Since each X_i is a K-subalgebra, we get $u \odot v \in X_i$. So that either $u \odot v \in X_i \backslash X_{i-1}$ or. In any of the above case it follows that,

$$T_X(u \odot v) \geq \lambda_i = \min\{T_X(u), T_X(v)\},$$

$$C_X(u \odot v) \geq \mu_i = \min\{C_X(u), C_X(v)\},$$

$$U_X(u \odot v) \leq \vartheta_i = \max\{U_X(u), U_X(v)\},$$

$$F_X(u \odot v) \leq \xi_i = \max\{F_X(u), F_X(v)\}.$$

For $k > l$ if $u \in X_k \backslash X_{k-1}$ and $v \in X_l \backslash X_{l-1}$ then,

$$T_X(u) = \lambda_k, T_X(v) = \lambda_l,$$

$$C_X(u) = \mu_k, C_X(v) = \mu_{l,}$$

$$U_X(u) = \vartheta_k, U_X(v) = \vartheta_l,$$

$$F_X(u) = \xi_k, F_X(v) = \xi_l,$$

and $u \odot v \in X_k$ because X_k is a K-subalgebra and $X_l \subset X_k$. It follows that,

$$T_X(u \odot v) \geq \lambda_k = \min\{T_X(u), T_X(v)\},$$

$$C_X(u \odot v) \geq \mu_k = \min\{C_X(u), C_X(v)\},$$

$$U_X(u \odot v) \leq \vartheta_k = \max\{U_X(u), U_X(v)\},$$

$$F_X(u \odot v) \leq \xi_k = \max\{F_X(u), F_X(v)\}.$$

Hence $X = (T_X, C_{X,} U_X, F_X)$ is a quadripartitioned single valued neutrosophic K-subalgebra of Ꝁ and all its non-empty level subsets are level K-subalgebras of Ꝁ. Since $Im(T_X) = \{\lambda_0, \lambda_1, \dots, \lambda_n\}$, $Im(C_X) = \{\mu_0, \mu_1, \dots, \mu_n\}$, $Im(U_X) = \{\vartheta_0, \vartheta_1, \dots, \vartheta_n\}\ and\ Im(F_X) = \{\xi_0, \xi_1, \dots, \xi_n\}$. Therefore, the level K-subalgebras of $X = (T_X, C_{X,} U_X, F_X)$ are given by the chain of K-subalgebras:

$$U(T_X, \lambda_0) \subset U(T_X, \lambda_1) \subset \cdots \subset U(T_X, \lambda_n) = G,$$

$$U'(C_X, \mu_0) \subset U'(C_X, \mu_1) \subset \cdots \subset U'(C_X, \mu_n) = G,$$

$$L(U_X, \vartheta_0) \subset L(U_X, \vartheta_1) \subset \cdots \subset L(U_X, \vartheta_n) = G,$$

$$L'(F_X, \xi_0) \subset L'(F_X, \xi_1) \subset \cdots \subset L'(F_X, \xi_n) = G,$$

respectively. Indeed,

$$U(T_X, \lambda_0) = \{u \in G |\ T_X(u) \geq \lambda_0\} = X_0,$$

$$U'(C_X, \mu_0) = \{u \in G|\ C_X(u) \geq \mu_0\} = X_0,$$

$$L(U_X, \vartheta_0) = \{u \in G|\ U_X(u) \leq \vartheta_0\} = X_0,$$

$$L'(F_X, \xi_0) = \{u \in G|\ F_X(u) \leq \xi_0\} = X_0.$$

Now we have to prove that,

$U(T_X, \lambda_i) = X_i, U'(C_X, \mu_i) = X_i, L(U_X, \vartheta_i) = X_i\ and\ L'(F_X, \xi_i) = X_i\ for\ 0 < i \leq n.$ Clearly $X_i \subseteq U(T_X, \lambda_i), X_i \subseteq U'(C_X, \mu_i), X_i \subseteq L(U_X, \vartheta_i)\ and\ X_i \subseteq L'(F_X, \xi_i)$. If $u \in U(T_X, \lambda_i)$ then $T_X(u) \geq \lambda_i$ and so $u \notin A_k\ for\ k > i$. Hence $T_X(u) \in \{\lambda_0, \lambda_1, \ldots, \lambda_i\}$ which shows that $u \in X_k\ for\ k \leq i$, since $X_k \subseteq X_i$. It follows that $u \in X_i$. Consequently $U(T_X, \lambda_i) = X_i$ for some $0 < i \leq n$. Similarly, it is proved for $U'(C_X, \mu_i) = X_i$. Now if $v \in L(U_X, \vartheta_i)$ then $U_X(v) \leq \vartheta_i$ and so $v \notin X_k\ for\ some\ i \leq k$. Thus $U_X(u) \in \{\vartheta_0, \vartheta_1, \ldots, \vartheta_i\}$ which shows that $u \in X_l\ for\ some\ l \leq i$, since $X_l \subseteq X_i$. It follows that $v \in X_i$. Consequently, $L(U_X, \vartheta_i) = X_i$ for some $0 < i \leq n$. Similarly, it is proved for $L'(F_X, \xi_i) = X_i$. Hence the proof.

4. Homomorphism of Quadripartitioned Single Valued Neutrosophic K-Algebras

Definition 10. Consider two K-algebras $\mathbb{K}_1 = (G_1, \cdot, \odot, e_1)\ and\ \mathbb{K}_2 = (G_2, \cdot, \odot, e_2)$ and f be a function from $\mathbb{K}_1$ into $\mathbb{K}_2$. If $Y = (T_Y, C_Y, U_Y, F_Y)$ is a quadripartitioned single valued neutrosophic K-subalgebra of $\mathbb{K}_2$, then the preimage of $Y = (T_Y, C_Y, U_Y, F_Y)$ under f is a quadripartitioned single valued neutrosophic K-subalgebra of $\mathbb{K}_1$ defined by,

$$f^{-1}(T_Y)(u) = T_Y(f(u)), f^{-1}(C_Y)(u) = C_Y(f(u)),$$

$$f^{-1}(U_Y)(u) = U_Y(f(u)), f^{-1}(F_Y)(u) = F_Y(f(u)),$$

for all $u \in G$.

Definition 11. A quadripartitioned single valued neutrosophic K-subalgebra $X = (T_X, C_X, U_X, F_X)$ of a K-algebra $\mathbb{K}$ is called characteristic if $T_X(f(u)) = T_X(u), C_X(f(u)) = C_X(u), U_X(f(u)) = U_X(u)\ and\ F_X(f(u)) = F_X(u)$ for all $u \in G$ and $f \in Aut(K)$.

Definition 12. A K-subalgebra U of a K-algebra $\mathbb{K}$ is said to be fully invariant if $f(U) \subseteq U$ for all $f \in End(K)$ where $End(K)$ is the set of all endomorphisms of a K-algebra $\mathbb{K}$. A quadripartitioned single valued neutrosophic K-subalgebra $X = (T_X, C_X, U_X, F_X)$ of a K-algebra $\mathbb{K}$ is called fully invariant if $T_X(f(u)) \leq T_X(u), C_X(f(u)) \leq C_X(u), U_X(f(u)) \geq U_X(u)\ and\ F_X(f(u)) \geq F_X(u)$ for all $u \in G$ and $f \in End(K)$.

Definition 13. Let $X_1 = (T_{X_1}, C_{X_1}, U_{X_1}, F_{X_1})$ and $X_2 = (T_{X_2}, C_{X_2}, U_{X_2}, F_{X_2})$ be two quadripartitioned single valued neutrosophic K-subalgebras of $\mathbb{K}$. Then $X_1 = (T_{X_1}, C_{X_1}, U_{X_1}, F_{X_1})$ is said to be the same type of $X_2 = (T_{X_2}, C_{X_2}, U_{X_2}, F_{X_2})$ if there exists $f \in Aut(K)$ such that $X_1 = X_2 \circ f$ i.e.,$T_{X_1}(u) = T_{X_2}(f(u)), C_{X_1}(u) = C_{X_2}(f(u)), U_{X_1}(u) = U_{X_2}(f(u))\ and\ F_{X_1}(u) = F_{X_2}(f(u))$ for all $u \in G$.

Theorem 5. Let $f: K_1 \rightarrow K_2$ be an epimorphism of K-algebras. If $Y = (T_Y, C_Y, U_Y, F_Y)$ is a quadripartitioned single valued neutrosophic K-subalgebra of K_2, then $f^{-1}(Y)$ is a quadripartitioned single valued neutrosophic K-subalgebra of K_1.

Proof. It is obvious that,

$$f^{-1}(T_Y)(e) \geq f^{-1}(T_Y)(u), f^{-1}(C_Y)(e) \geq f^{-1}(C_Y)(u),$$

$$f^{-1}(U_Y)(e) \leq f^{-1}(U_Y)(u), f^{-1}(F_Y)(e) \leq f^{-1}(F_Y)(u),$$

for all $u \in G_1. Let\ u, v \in G_1$ then,

$$f^{-1}(T_Y)(u \odot v) = T_Y(f(u \odot v)),$$

$$f^{-1}(T_Y)(u \odot v) = T_Y(f(u) \odot f(v)),$$

$$f^{-1}(T_Y)(u \odot v) \geq \min\{T_Y(f(u)), T_Y(f(v))\},$$

$$f^{-1}(T_Y)(u \odot v) \geq \min\{f^{-1}(T_Y)(u), f^{-1}(T_Y)(v)\};$$

$$f^{-1}(C_Y)(u \odot v) = C_Y(f(u \odot v)),$$

$$f^{-1}(C_Y)(u \odot v) = C_Y(f(u) \odot f(v)),$$

$$f^{-1}(C_Y)(u \odot v) \geq \min\{C_Y(f(u)), C_Y(f(v))\},$$

$$f^{-1}(C_Y)(u \odot v) \geq \min\{f^{-1}(C_Y)(u), f^{-1}(C_Y)(v)\};$$

$$f^{-1}(U_Y)(u \odot v) = U_Y(f(u \odot v)),$$

$$f^{-1}(U_Y)(u \odot v) = U_Y(f(u) \odot f(v)),$$

$$f^{-1}(U_Y)(u \odot v) \leq \max\{U_Y(f(u)), U_Y(f(v))\},$$

$$f^{-1}(U_Y)(u \odot v) \leq \max\{f^{-1}(U_Y)(u), f^{-1}(U_Y)(v)\};$$

$$f^{-1}(F_Y)(u \odot v) = F_Y(f(u \odot v)),$$

$$f^{-1}(F_Y)(u \odot v) = F_Y(f(u) \odot f(v)),$$

$$f^{-1}(F_Y)(u \odot v) \leq \max\{F_Y(f(u)), F_Y(f(v))\},$$

$$f^{-1}(F_Y)(u \odot v) \le \max\{f^{-1}(F_Y)(u), f^{-1}(F_Y)(v)\}.$$

Hence $f^{-1}(Y)$ is a quadripartitioned single valued neutrosophic K-subalgebra of K_1.

Theorem 6. Let $f: \mathrm{K}_1 \to \mathrm{K}_2$ be an epimorphism of K-algebras. If $Y = (T_Y, C_Y, U_Y, F_Y)$ is a quadripartitioned single valued neutrosophic K-subalgebra of K_2 and $X = (T_X, C_X, U_X, F_X)$ is the preimage of Y under f. Then X is a quadripartitioned single valued neutrosophic K-subalgebra of K_1.

Proof. It is obvious that $T_X(e) \ge T_X(u), C_X(e) \ge C_X(u), U_X(e) \le U_X(u)$ and $F_X(e) \le F_X(u)$

for all $u \in G_1$. Now for any $u, v \in G_1$,

$$T_X(u \odot v) = T_Y(f(u \odot v)),$$

$$T_X(u \odot v) = T_Y(f(u) \odot f(v)),$$

$$T_X(u \odot v) \ge \min\{T_Y(f(u)), T_Y(f(v))\},$$

$$T_X(u \odot v) \ge \min\{T_X(u), T_X(v)\};$$

$$C_X(u \odot v) = C_Y(f(u \odot v)),$$

$$C_X(u \odot v) = C_Y(f(u) \odot f(v)),$$

$$C_X(u \odot v) \ge \min\{C_Y(f(u)), C_Y(f(v))\},$$

$$C_X(u \odot v) \ge \min\{C_X(u), C_X(v)\};$$

$$U_X(u \odot v) = U_Y(f(u \odot v)),$$

$$U_X(u \odot v) = U_Y(f(u) \odot f(v)),$$

$$U_X(u \odot v) \le \max\{U_Y(f(u)), U_Y(f(v))\},$$

$$U_X(u \odot v) \le \max\{U_X(u), U_X(v)\};$$

$$F_X(u \odot v) = F_Y(f(u \odot v)),$$

$$F_X(u \odot v) = F_Y(f(u) \odot f(v)),$$

$$F_X(u \odot v) \le \max\{F_Y(f(u)), F_Y(f(v))\},$$

$F_X(u \odot v) \le \max\{F_X(u), F_X(v)\}$.

Hence X is a quadripartitioned single valued neutrosophic K-subalgebra of K_1.

Definition 14. Let f be a mapping from K_1 into K_2 i.e., $f: K_1 \to K_2$ of K-algebras and let $X = (T_X, C_X, U_X, F_X)$ be a quadripartitioned single valued neutrosophic set of K_2. The map $X = (T_X, C_X, U_X, F_X)$ is called the preimage of X under f if $T_X^f(u) = T_X(f(u)), C_X^f(u) = C_X(f(u)), U_X^f(u) = U_X(f(u))$ and $F_X^f(u) = F_X(f(u))$ for all $u \in G_1$.

Theorem 7. Let $f: K_1 \to K_2$ be an epimorphism of K-algebras. Then $X^f = (T_X^f, C_X^f, U_X^f, F_X^f)$ is a quadripartitioned single valued neutrosophic K-subalgebra of K_1 if and only if $X = (T_X, C_X, U_X, F_X)$ is a quadripartitioned single valued neutrosophic K-subalgebra of K_2.

Proof. Let $f: K_1 \to K_2$ be an epimorphism of K-algebras. First assume that $X^f = (T_X^f, C_X^f, U_X^f, F_X^f)$ is a quadripartitioned single valued neutrosophic K-subalgebra of K_1. Then we have to prove that $X = (T_X, C_X, U_X, F_X)$ is a quadripartitioned single valued neutrosophic K-subalgebra of K_2. Since there exists $u \in G_1$ such that $v = f(u)$ for any $v \in G_2$:

$$T_X(v) = T_X(f(u)) = T_X^{f(u)} \le T_X^{f(e_1)} = T_X(f(e_1)) = T_X(e_2),$$

$$C_X(v) = C_X(f(u)) = C_X^{f(u)} \le C_X^{f(e_1)} = C_X(f(e_1)) = C_X(e_2),$$

$$U_X(v) = U_X(f(u)) = U_X^{f(u)} \ge U_X^{f(e_1)} = U_X(f(e_1)) = U_X(e_2),$$

$$F_X(v) = F_X(f(u)) = F_X^{f(u)} \ge F_X^{f(e_1)} = F_X(f(e_1)) = F_X(e_2).$$

For any $u, v \in G_2, s, t \in G_1$ such that $u = f(s)$ and $v = f(t)$. It follows that:

$$T_X(u \odot v) = T_X(f(s \odot t)),$$

$$T_X(u \odot v) = T_X^f(s \odot t),$$

$$T_X(u \odot v) \ge \min\{T_X^f(s), T_X^f(t)\},$$

$$T_X(u \odot v) \ge \min\{T_X(f(s)), T_X(f(t))\},$$

$$T_X(u \odot v) \ge \min\{T_X(u), T_X(v)\};$$

$$C_X(u \odot v) = C_X(f(s \odot t)),$$

$$C_X(u \odot v) = C_X^f(s \odot t),$$

$$C_X(u \odot v) \ge \min\{C_X^f(s), C_X^f(t)\},$$

$$C_X(u \odot v) \geq \min\{C_X(f(s)), C_X(f(t))\},$$

$$C_X(u \odot v) \geq \min\{C_X(u), C_X(v)\};$$

$$U_X(u \odot v) = U_X(f(s \odot t)),$$

$$U_X(u \odot v) = U_X^f(s \odot t),$$

$$U_X(u \odot v) \leq \max\{U_X^f(s), U_X^f(t)\},$$

$$U_X(u \odot v) \leq \max\{U_X(f(s)), U_X(f(t))\},$$

$$U_X(u \odot v) \leq \max\{U_X(u), U_X(v)\};$$

$$F_X(u \odot v) = F_X(f(s \odot t)),$$

$$F_X(u \odot v) = F_X^f(s \odot t),$$

$$F_X(u \odot v) \leq \max\{F_X^f(s), F_X^f(t)\},$$

$$F_X(u \odot v) \leq \max\{F_X(f(s)), F_X(f(t))\},$$

$$F_X(u \odot v) \leq \max\{F_X(u), F_X(v)\}.$$

Hence $X = (T_X, C_X, U_X, F_X)$ is a quadripartitioned single valued neutrosophic K-subalgebra of K_2. Conversely, assume that $X = (T_X, C_X, U_X, F_X)$ is a quadripartitioned single valued neutrosophic K-subalgebra of K_2. Then we have to prove that $X^f = (T_X^f, C_X^f, U_X^f, F_X^f)$ is a quadripartitioned single valued neutrosophic K-subalgebra of K_1. For any $u \in G_1$ we have:

$$T_X^f(e_1) = T_X(f(e_1)) = T_X(e_2) \geq T_X(f(u)) = T_X^f(u),$$

$$C_X^f(e_1) = C_X(f(e_1)) = C_X(e_2) \geq C_X(f(u)) = C_X^f(u),$$

$$U_X^f(e_1) = U_X(f(e_1)) = U_X(e_2) \leq U_X(f(u)) = U_X^f(u),$$

$$F_X^f(e_1) = F_X(f(e_1)) = F_X(e_2) \leq F_X(f(u)) = F_X^f(u).$$

Since X is a quadripartitioned single valued neutrosophic K-subalgebra of K_2 and for any $u, v \in G_1$,

$$T_X^f(u \odot v) = T_X(f(u \odot v)),$$

$$T_X^f(u \odot v) = T_X(f(u) \odot f(v)),$$

$T_X^f(u \odot v) \geq \min\{T_X(f(u)), T_X(f(v))\}$,

$T_X^f(u \odot v) \geq \min\{T_X^f(u), T_X^f(v)\}$;

$C_X^f(u \odot v) = C_X(f(u \odot v))$,

$C_X^f(u \odot v) = C_X(f(u) \odot f(v))$,

$C_X^f(u \odot v) \geq \min\{C_X(f(u)), C_X(f(v))\}$,

$C_X^f(u \odot v) \geq \min\{C_X^f(u), C_X^f(v)\}$;

$U_X^f(u \odot v) = U_X(f(u \odot v))$,

$U_X^f(u \odot v) = U_X(f(u) \odot f(v))$,

$U_X^f(u \odot v) \leq \max\{U_X(f(u)), U_X(f(v))\}$,

$U_X^f(u \odot v) \leq \max\{U_X^f(u), U_X^f(v)\}$;

$F_X^f(u \odot v) = F_X(f(u \odot v))$,

$F_X^f(u \odot v) = F_X(f(u) \odot f(v))$,

$F_X^f(u \odot v) \leq \max\{F_X(f(u)), F_X(f(v))\}$,

$F_X^f(u \odot v) \leq \max\{F_X^f(u), F_X^f(v)\}$.

Hence $X^f = (T_X^f, C_X^f, U_X^f, F_X^f)$ is a quadripartitioned single valued neutrosophic K-subalgebra of $\breve{K}_1$.

Theorem 8. Let $X_1 = (T_{X_1}, C_{X_1}, U_{X_1}, F_{X_1})$ and $X_2 = (T_{X_2}, C_{X_2}, U_{X_2}, F_{X_2})$ be two quadripartitioned single valued neutrosophic K-subalgebras of $\breve{K}$. Then a quadripartitioned single valued neutrosophic K-subalgebra $X_1 = (T_{X_1}, C_{X_1}, U_{X_1}, F_{X_1})$ is of the same type of quadripartitioned single valued neutrosophic K-subalgebra $X_2 = (T_{X_2}, C_{X_2}, U_{X_2}, F_{X_2})$ if and only if X_1 is isomorphic to X_2.

Proof. It is enough to prove only the necessary condition since sufficient condition holds trivially. Let $X_1 = (T_{X_1}, C_{X_1}, U_{X_1}, F_{X_1})$ be quadripartitioned single valued neutrosophic K-subalgebra having same type of $X_2 = (T_{X_2}, C_{X_2}, U_{X_2}, F_{X_2})$. Then there exists $f \in Aut(K)$ such that $T_{X_1}(u) = T_{X_2}(f(u)), C_{X_1}(u) = C_{X_2}(f(u)), U_{X_1}(u) = U_{X_2}(f(u))$ *and* $F_{X_1}(u) = F_{X_2}(f(u))$ for all $u \in G$.

Let $g: X_1(K) \rightarrow X_2(K)$ be a mapping defined by $g\big(X_1(s)\big) = X_2\big(f(u)\big)$ for all $u \in G$ i.e., $g\left(T_{X_1}(u)\right) = T_{X_2}(f(u)), g\left(C_{X_1}(u)\right) = C_{X_2}(f(u)), g\left(U_{X_1}(u)\right) = U_{X_2}(f(u))$ *and* $g\left(F_{X_1}(u)\right) = F_{X_2}(f(u))$ for all $u \in G$. g is surjective obviously. And if $g\left(T_{X_1}(u)\right) = g\left(T_{X_1}(v)\right)$ for all $u, v \in G$ then $T_{X_2}(f(u)) = T_{X_2}(f(v))$ and we get $T_{X_1}(u) = T_{X_1}(v)$. Similarly we can prove for $C_{X_1}(u) = C_{X_1}(v), U_{X_1}(u) = U_{X_1}(v)$ *and* $F_{X_1}(u) = F_{X_1}(v)$.

Hence g is injective. Therefore g is a homomorphism such that for $u, v \in G$ we have:

$$g\big(T_{X_1}(u \odot v)\big) = T_{X_2}(f(u \odot v)) = T_{X_2}(f(u) \odot f(v)),$$

$$g\big(C_{X_1}(u \odot v)\big) = C_{X_2}(f(u \odot v)) = C_{X_2}(f(u) \odot f(v)),$$

$$g\big(U_{X_1}(u \odot v)\big) = U_{X_2}(f(u \odot v)) = U_{X_2}(f(u) \odot f(v)),$$

$$g\big(F_{X_1}(u \odot v)\big) = F_{X_2}(f(u \odot v)) = F_{X_2}(f(u) \odot f(v)).$$

Hence $X_1 = (T_{X_1}, C_{X_1}, U_{X_1}, F_{X_1})$ is isomorphic to $X_2 = \big(T_{X_2}, C_{X_2}, U_{X_2}, F_{X_2}\big)$.

5. Conclusion

In recent years, a new branch of logical algebra known as K-algebra applied in fuzzy set, intuitionistic fuzzy set and single valued neutrosophic set which helps us to extend the concept to K-algebra on quadripartitioned single valued neutrosophic sets. Quadripartitioned single valued neutrosophic set has four components truth,contradiction,unknown,false which helps to deal the concept of indeterminacy effectively. In this paper we defined K-algebras on quadripartitioned single valued neutrosophic sets and studied some of the results. Further the homomorphism of quadripartitioned single valued neutrosophic K-algebras, characteristic and fully invariant K-subalgebras also discussed in detail.

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