Smarandache Multi-Space Theory(I)

-Algebraic multi-spaces

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Abstract. A Smarandache multi-space is a union of n different spaces equipped with some different structures for an integer $n \geq 2$, which can be both used for discrete or connected spaces, particularly for geometries and spacetimes in theoretical physics. This monograph concentrates on characterizing various multi-spaces including three parts altogether. The first part is on *algebraic multi-spaces with structures*, such as those of multi-groups, multi-rings, multi-vector spaces, multi-metric spaces, multi-operation systems and multi-manifolds, also multi-voltage graphs, multi-embedding of a graph in an n-manifold,$\cdots$, etc.. The second discusses *Smarandache geometries*, including those of map geometries, planar map geometries and pseudo-plane geometries, in which the *Finsler geometry*, particularly the *Riemann geometry* appears as a special case of these Smarandache geometries. The third part of this book considers the *applications of multi-spaces to theoretical physics*, including the relativity theory, the M-theory and the cosmology. Multi-space models for p-branes and cosmos are constructed and some questions in cosmology are clarified by multi-spaces. The first two parts are relative independence for reading and in each part open problems are included for further research of interested readers.

Key words: algebraic structure, multi-space, multi-group, multi-ring, multi-vector space, multi-metric space.

Classification: AMS(2000) 03C05,05C15,51D20,51H20,51P05,83C05, 83E50

Contents

1. **Algebraic multi-spaces**

The notion of multi-spaces was introduced by Smarandache in 1969, see his article uploaded to *arXiv* [86] under his idea of hybrid mathematics: *combining different fields into a unifying field*([85]), which is more closer to our real life world. Today, this idea is widely accepted by the world of sciences. For mathematics, a definite or an exact solution under a given condition is not the only object for mathematician. New creation power has emerged and new era for mathematics has come now. Applying the Smarandache's notion, this chapter concentrates on constructing various multi-spaces by algebraic structures, such as those of groups, rings, fields, vector spaces, $\cdots$,etc., also by metric spaces, which are more useful for constructing multi-voltage graphs, maps and map geometries in the following chapters.

§1.1 **Sets**

1.1.1. **Sets**

A *set* Ξ is a collection of objects with some common property P, denoted by

$$\Xi = \{x|x \textit{ has property } P\},$$

where, x is said an element of the set Ξ, denoted by $x \in \Xi$. For an element y not possessing the property P, i.e., not an element in the set Ξ, we denote it by $y \notin \Xi$.

The *cardinality* (or the number of elements if Ξ is finite) of a set Ξ is denoted by $|\Xi|$.

Some examples of sets are as follows.

$$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\};$$

$$B = \{p|\ p \textit{ is a prime number}\};$$

$$C = \{(x, y)|x^2 + y^2 = 1\};$$

$$D = \{\textit{the cities in the world}\}.$$

The sets A and D are finite with $|A| = 10$ and $|D| < +\infty$, but these sets B and C are infinite.

Two sets Ξ_1 and Ξ_2 are said to be *identical* if and only if for $\forall x \in \Xi_1$, we have $x \in \Xi_2$ and for $\forall x \in \Xi_2$, we also have $x \in \Xi_1$. For example, the following two sets

$$E = \{1, 2, -2\} \text{ and } F = \{\ x\ | x^3 - x^2 - 4x + 4 = 0\}$$

are identical since we can solve the equation $x^3 - x^2 - 4x + 4 = 0$ and get the solutions $x = 1, 2$ or -2. Similarly, for the cardinality of a set, we know the following result.

Theorem 1.1.1([6]) *For sets Ξ_1, Ξ_2, $|\Xi_1| = |\Xi_2|$ if and only if there is an $1-1$ mapping between Ξ_1 and Ξ_2.*

According to this theorem, we know that $|B| \neq |C|$ although they are infinite. Since B is countable, i.e., there is an $1-1$ mapping between B and the natural number set $N = \{1, 2, 3, \cdots, n, \cdots\}$, however C is not.

Let A_1, A_2 be two sets. If for $\forall a \in A_1 \Rightarrow a \in A_2$, then A_1 is said to be a *subset* of A_2, denoted by $A_1 \subseteq A_2$. If a set has no elements, we say it an empty set, denoted by $\emptyset$.

Definition 1.1.1 *For two sets Ξ_1, Ξ_2, two operations $\bigcup$and $\bigcap$ on Ξ_1, Ξ_2 are defined as follows:*

$$\Xi_1 \bigcup \Xi_2 = \{x | x \in \Xi_1 \ or \ x \in \Xi_2\},$$

$$\Xi_1 \bigcap \Xi_2 = \{x | x \in \Xi_1 \ and \ x \in \Xi_2\}$$

and Ξ_1 minus Ξ_2 is defined by

$$\Xi_1 \setminus \Xi_2 = \{x | x \in \Xi_1 \ but \ x \notin \Xi_2\}.$$

For the sets A and E, calculation shows that

$$A \bigcup E = \{1, 2, -2, 3, 4, 5, 6, 7, 8, 9, 10\},$$

$$A \bigcap E = \{1, 2\}$$

and

$$A \setminus E = \{3, 4, 5, 6, 7, 8, 9, 10\},$$

$$E \setminus A = \{-2\}.$$

For a set Ξ and $H \subseteq \Xi$, the set $\Xi \setminus H$ is said the *complement* of H in Ξ, denoted by $\overline{H}(\Xi)$. We also abbreviate it to $\overline{H}$ if each set considered in the situation is a subset of $\Xi = \Omega$, i.e., the *universal set*.

These operations defined in Definition 1.1.1 observe the following laws.

L1 Itempotent law. For $\forall S \subseteq \Omega$,

$$A\bigcup A = A\bigcap A = A.$$

L2 Commutative law. For $\forall U, V \subseteq \Omega$,

$$U\bigcup V = V\bigcup U;\ U\bigcap V = V\bigcap U.$$

L3 Associative law. For $\forall U, V, W \subseteq \Omega$,

$$U\bigcup(V\bigcup W) = (U\bigcup V)\bigcup W;\ U\bigcap(V\bigcap W) = (U\bigcap V)\bigcap W.$$

L4 Absorption law. For $\forall U, V \subseteq \Omega$,

$$U\bigcap(U\bigcup V) = U\bigcup(U\bigcap V) = U.$$

L5 Distributive law. For $\forall U, V, W \subseteq \Omega$,

$$U\bigcup(V\bigcap W) = (U\bigcup V)\bigcap(U\bigcup W);\ U\bigcap(V\bigcup W) = (U\bigcap V)\bigcup(U\bigcap W).$$

L6 Universal bound law. For $\forall U \subseteq \Omega$,

$$\emptyset\bigcap U = \emptyset, \emptyset\bigcup U = U;\ \Omega\bigcap U = U, \Omega\bigcup U = \Omega.$$

L7 Unary complement law. For $\forall U \subseteq \Omega$,

$$U\bigcap\overline{U} = \emptyset;\ U\bigcup\overline{U} = \Omega.$$

A set with two operations $\bigcap$ and $\bigcup$ satisfying the laws $L1 \sim L7$ is said to be a *Boolean algebra*. Whence, we get the following result.

Theorem 1.1.2 *For any set U, all its subsets form a Boolean algebra under the operations $\bigcap$ and $\bigcup$.*

1.1.2 **Partially order sets**

For a set Ξ, define its *Cartesian product* to be

$$\Xi \times \Xi = \{(x, y) | \forall x, y \in \Xi\}.$$

A subset $R \subseteq \Xi \times \Xi$ is called a *binary relation* on Ξ. If $(x, y) \in R$, we write xRy. A *partially order set* is a set Ξ with a binary relation $\preceq$ such that the following laws hold.

O1 Reflective law. For $x \in \Xi$, xRx.

O2 Antisymmetry law. For $x, y \in \Xi$, xRy and $yRx \Rightarrow x = y$.

O3 Transitive law. For $x, y, z \in \Xi$, xRy and $yRz \Rightarrow xRz$.

A partially order set Ξ with a binary relation $\preceq$ is denoted by $(\Xi, \preceq)$. Partially ordered sets with a finite number of elements can be conveniently represented by a diagram in such a way that each element in the set Ξ is represented by a point so placed on the plane that a point a is above another point b if and only if $b \prec a$. This kind of diagram is essentially a directed graph (see also Chapter 2 in this book). In fact, a directed graph is correspondent with a partially set and vice versa. Examples for the partially order sets are shown in Fig.1.1 where each diagram represents a finite partially order set.

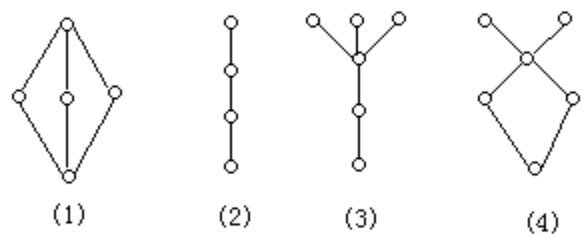

Fig.1.1

An element a in a partially order set $(\Xi, \preceq)$ is called *maximal* (or *minimal*) if for $\forall x \in \Xi$, $a \preceq x \Rightarrow x = a$ (or $x \preceq a \Rightarrow x = a$). The following result is obtained by the definition of partially order sets and the induction principle.

Theorem 1.1.3 *Any finite non-empty partially order set* $(\Xi, \preceq)$ *has maximal and minimal elements.*

A partially order set $(\Xi, \preceq)$ is an *order set* if for any $\forall x, y \in \Xi$, there must be $x \preceq y$ or $y \preceq x$. It is obvious that any partially order set contains an order subset, finding this fact in Fig.1.1.

An *equivalence relation* $R \subseteq \Xi \times \Xi$ on a set Ξ is defined by

R1 Reflective law. For $x \in \Xi$, xRx.

R2 Symmetry law. For $x, y \in \Xi$, $xRy \Rightarrow yRx$

R3 Transitive law. For $x, y, z \in \Xi$, xRy and $yRz \Rightarrow xRz$.

For a set Ξ with an equivalence relation R, we can classify elements in Ξ by R as follows:

$$R(x) = \{y|\ y \in \Xi \text{ and } yRx\ \}.$$

Then, we get the following useful result for the combinatorial enumeration.

Theorem 1.1.4 *For a finite set* Ξ *with an equivalence* R, $\forall x, y \in \Xi$, *if there is an bijection* ς *between* $R(x)$ *and* $R(y)$, *then the number of equivalence classes under* R

is

$$\frac{|\Xi|}{|R(x)|},$$

where x is a chosen element in Ξ.

Proof Notice that there is an bijection ς between $R(x)$ and $R(y)$ for $\forall x, y \in \Xi$. Whence, $|R(x)| = |R(y)|$. By definition, for $\forall x, y \in \Xi$, $R(x) \bigcap R(y) = \emptyset$ or $R(x) = R(y)$. Let T be a representation set of equivalence classes, i.e., choice one element in each class. Then we get that

$$\begin{aligned} |\Xi| &= \sum_{x \in T} |R(x)| \\ &= |T||R(x)|. \end{aligned}$$

Whence, we know that

$$|T| = \frac{|\Xi|}{|R(x)|}. \qquad \natural$$

1.1.3 Neutrosophic set

Let $[0,1]$ be a closed interval. For three subsets $T, I, F \subseteq [0,1]$ and $S \subseteq \Omega$, define a relation of an element $x \in \Omega$ with the subset S to be $x(T, I, F)$, i,e., the *confidence set* for $x \in S$ is T, the *indefinite set* is I and *fail set* is F. A set S with three subsets T, I, F is said to be a *neutrosophic set* ([85]). We clarify the conception of neutrosophic sets by abstract set theory as follows.

Let Ξ be a set and $A_1, A_2, \cdots, A_k \subseteq \Xi$. Define $3k$ functions $f_1^x, f_2^x, \cdots, f_k^x$ by $f_i^x : A_i \to [0,1]$, $1 \le i \le k$, where $x = T, I, F$. Denote by $(A_i; f_i^T, f_i^I, f_i^F)$ the subset A_i with three functions f_i^T, f_i^I, f_i^F, $1 \le i \le k$. Then

$$\bigcup_{i=1}^{k} (A_i; f_i^T, f_i^I, f_i^F)$$

is a union of neutrosophic sets. Some extremal cases for this union is in the following, which convince us that neutrosophic sets are a generalization of classical sets.

Case 1 $f_i^T = 1,\ f_i^I = f_i^F = 0$ for $i = 1, 2, \cdots, k$.

In this case,

$$\bigcup_{i=1}^{k} (A_i; f_i^T, f_i^I, f_i^F) = \bigcup_{i=1}^{k} A_i.$$

Case 2 $f_i^T = f_i^I = 0,\ f_i^F = 1$ for $i = 1, 2, \cdots, k$.

In this case,

$$\bigcup_{i=1}^{k}(A_i; f_i^T, f_i^I, f_i^F) = \overline{\bigcup_{i=1}^{k} A_i}.$$

Case 3 There is an integer s such that $f_i^T = 1$ $f_i^I = f_i^F = 0$, $1 \leq i \leq s$ but $f_j^T = f_j^I = 0, f_j^F = 1$ for $s+1 \leq j \leq k$.

In this case,

$$\bigcup_{i=1}^{k}(A_i, f_i) = \bigcup_{i=1}^{s} A_i \bigcup \overline{\bigcup_{i=s+1}^{k} A_i}.$$

Case 4 There is an integer l such that $f_l^T \neq 1$ or $f_l^F \neq 1$.

In this case, the union is a general neutrosophic set. It can not be represented by abstract sets.

If $A \bigcap B = \emptyset$, define the function value of a function f on the union set $A \bigcup B$ to be

$$f(A \bigcup B) = f(A) + f(B)$$

and

$$f(A \bigcap B) = f(A)f(B)$$

.
Then if $A \bigcap B \neq \emptyset$, we get that

$$f(A \bigcup B) = f(A) + f(B) - f(A)f(B).$$

Generally, by applying the Inclusion-Exclusion Principle to a union of sets, we get the following formulae.

$$f(\bigcap_{i=1}^{l} A_i) = \prod_{i=1}^{l} f(A_i),$$

$$f(\bigcup_{i=1}^{k} A_i) = \sum_{j=1}^{k}(-1)^{j-1} \prod_{s=1}^{j} f(A_s).$$

§1.2 Algebraic Structures

In this section, we recall some conceptions and results without proofs in algebra, such as, these groups, rings, fields, vectors $\cdots$, all of these can be viewed as a sole-space system.

1.2.1. **Groups**

A set G with a binary operation $\circ$, denoted by $(G;\circ)$, is called a *group* if $x \circ y \in G$ for $\forall x, y \in G$ such that the following conditions hold.

(i) $(x \circ y) \circ z = x \circ (y \circ z)$ for $\forall x, y, z \in G$;
(ii) There is an element $1_G, 1_G \in G$ such that $x \circ 1_G = x$;
(iii) For $\forall x \in G$, there is an element $y, y \in G$, such that $x \circ y = 1_G$.

A group G is *abelian* if the following additional condition holds.

(iv) For $\forall x, y \in G$, $x \circ y = y \circ x$.

A set G with a binary operation $\circ$ satisfying the condition (i) is called a *semigroup*. Similarly, if it satisfies the conditions (i) and (iv), then it is called a *abelian semigroup*.

Some examples of groups are as follows.

(1) $(R\ ;+)$ and $(R\ ;\cdot)$, where R is the set of real numbers.

(2) $(U_2;\cdot)$, where $U_2 = \{1, -1\}$ and generally, $(U_n;\cdot)$, where $U_n = \{e^{i\frac{2\pi k}{n}}, k = 1, 2, \cdots, n\}$.

(3) For a finite set X, the set $SymX$ of all permutations on X with respect to permutation composition.

The cases (1) and (2) are abelian group, but (3) is not in general.

A subset H of a group G is said to be *subgroup* if H is also a group under the same operation in G, denoted by $H \prec G$. The following results are well-known.

Theorem 1.2.1 *A non-empty subset H of a group $(G\ ;\circ)$ is a group if and only if for $\forall x, y \in H$, $x \circ y \in H$.*

Theorem 1.2.2(Lagrange theorem) *For any subgroup H of a finite group G, the order $|H|$ is a divisor of $|G|$.*

For $\forall x \in G$, denote the set $\{xh | \forall h \in H\}$ by xH and $\{hx | \forall h \in H\}$ by Hx. A subgroup H of a group $(G\ ;\circ)$ is *normal*, denoted by $H \lhd G$, if for $\forall x \in G$, $xH = Hx$.

For two subsets A, B of a group $(G\ ;\circ)$, define their product $A \circ B$ by

$$A \circ B = \{a \circ b |\ \forall a \in A,\ \forall b \in b\ \}.$$

For a subgroup $H, H \lhd G$, it can be shown that

$$(xH) \circ (yH) = (x \circ y)H \text{ and } (Hx) \circ (Hy) = H(x \circ y).$$

for $\forall x, y \in G$. Whence, the operation "$\circ$" is closed in the sets $\{xH | x \in G\} = \{Hx | x \in G\}$, denote this set by G/H. We know G/H is also a group by the facts

$$(xH \circ yH) \circ zH = xH \circ (yH \circ zH),\ \forall x, y, z \in G$$

and

$$(xH) \circ H = xH, \ (xH) \circ (x^{-1}H) = H.$$

For two groups G, G', let σ be a mapping from G to G'. If

$$\sigma(x \circ y) = \sigma(x) \circ \sigma(y),$$

for $\forall x, y \in G$, then call σ a *homomorphism* from G to G'. The *image* $Im\sigma$ and the *kernel* $Ker\sigma$ of a homomorphism $\sigma : G \to G'$ are defined as follows:

$$Im\sigma = G^{\sigma} = \{\sigma(x) | \ \forall x \in G \ \},$$

$$Ker\sigma = \{x | \ \forall x \in G, \ \sigma(x) = 1_{G'} \ \}.$$

A one to one homomorphism is called a *monomorphism* and an onto homomorphism an *epimorphism*. A homomorphism is called a *bijection* if it is one to one and onto. Two groups G, G' are said to be *isomorphic* if there exists a bijective homomorphism σ between them, denoted by $G \cong G'$.

Theorem 1.2.3 *Let $\sigma : G \to G'$ be a homomorphism of group. Then*

$$(G, \circ)/Ker\sigma \cong Im\sigma.$$

1.2.2. **Rings**

A set R with two binary operations $+$ and $\circ$, denoted by $(R\ ; +, \circ)$, is said to be a *ring* if $x + y \in R$, $x \circ y \in R$ for $\forall x, y \in R$ such that the following conditions hold.

(*i*) $(R\ ; +)$ is an abelian group;
(*ii*) $(R\ ; \circ)$ is a semigroup;
(*iii*) For $\forall x, y, z \in R$, $x \circ (y + z) = x \circ y + x \circ z$ and $(x + y) \circ z = x \circ z + y \circ z$.

Some examples of rings are as follows.

(1) $(Z\ ; +, \cdot)$, where Z is the set of integers.
(2) $(pZ\ ; +, \cdot)$, where p is a prime number and $pZ = \{pn | n \in Z\}$.
(3) $(\mathcal{M}_n(Z)\ ; +, \cdot)$, where $\mathcal{M}_n(Z)$ is a set of $n \times n$ matrices with each entry being an integer, $n \geq 2$.

For a ring $(R\ ; +, \circ)$, if $x \circ y = y \circ x$ for $\forall x, y \in R$, then it is called a *commutative ring*. The examples of (1) and (2) are commutative, but (3) is not.

If R contains an element 1_R such that for $\forall x \in R$, $x \circ 1_R = 1_R \circ x = x$, we call R a *ring with unit*. All of these examples of rings in the above are rings with unit. For (1), the unit is 1, (2) is Z and (3) is $I_{n \times n}$.

The unit of $(R\ ; +)$ in a ring $(R\ ; +, \circ)$ is called *zero*, denoted by 0. For $\forall a, b \in R$, if

$$a \circ b \ = \ 0,$$

then a and b are called *divisors of zero*. In some rings, such as the $(Z\ ;+,\cdot)$ and $(pZ\ ;+,\cdot)$, there must be a or b be 0. We call it only has a *trivial divisor of zero*. But in the ring $(pqZ\ ;+,\cdot)$ with p, q both being prime, since

$$pZ\ \cdot\ qZ\ =\ 0$$

and $pZ \neq 0$, $qZ \neq 0$, we get non-zero divisors of zero, which is called to have *non-trivial divisors of zero*. The ring $(\mathcal{M}_n(Z); +,\cdot)$ also has non-trivial divisors of zero, since

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} = O_{n\times n}.$$

A *division ring* is a ring which has no non-trivial divisors of zero and an *integral domain* is a commutative ring having no non-trivial divisors of zero.

A *body* is a ring $(R\ ;+,\circ)$ with a unit, $|R| \geq 2$ and $(R \setminus \{0\};\circ)$ is a group and a *field* is a commutative body. The examples (1) and (2) of rings are fields. The following result is well-known.

Theorem 1.2.4 *Any finite integral domain is a field.*

A non-empty subset R' of a ring $(R\ ;+,\circ)$ is called a *subring* if $(R'\ ;+,\circ)$ is also a ring. The following result for subrings can be obtained immediately by definition.

Theorem 1.2.5 *For a subset R' of a ring $(R\ ;+,\circ)$, if*
(i) $(R'\ ;+)$ is a subgroup of $(R\ ;+)$,
(ii) R' is closed under the operation $\circ$,
then $(R'\ ;+,\circ)$ is a subring of $(R\ ,+.\circ)$.

An *ideal* I of a ring $(R\ ;+,\circ)$ is a non-void subset of R with properties:
(*i*) $(I\ ;+)$ is a subgroup of $(R\ ;+)$;
(*ii*) $a \circ x \in I$ and $x \circ a \in I$ for $\forall a \in I, \forall x \in R$.
Let $(R\ ;+,\circ)$ be a ring. A chain

$$R \succ R_1 \succ \cdots \succ R_l = \{1_\circ\}$$

satisfying that R_{i+1} is an ideal of R_i for any integer $i, 1 \leq i \leq l$, is called an *ideal chain* of $(R\ ,+,\circ)$. A ring whose every ideal chain only has finite terms is called an *Artin ring*. Similar to normal subgroups, consider the set $x+I$ in the group $(R\ ;+)$. Calculation shows that $R/I = \{x + I|\ x \in R\}$ is also a ring under these operations $+$ and $\circ$. Call it a *quotient ring of R to I*.

For two rings $(R\ ;+,\circ)$, $(R'\ ;*,\bullet)$, let ι be a mapping from R to R'. If

$$\iota(x+y)=\iota(x)*\iota(y),$$

$$\iota(x\circ y)=\iota(x)\bullet\iota(y),$$

for $\forall x,y\in R$, then ι is called a *homomorphism* from $(R\ ;+,\circ)$ to $(R'\ ;*,\bullet)$. Similar to Theorem 2.3, we know that

Theorem 1.2.6 *Let $\iota : R\rightarrow R'$ be a homomorphism from $(R\ ;+,\circ)$ to $(R'\ ;*,\bullet)$. Then*

$$(R\ ;+,\circ)/Ker\iota\cong Im\iota.$$

1.2.3 **Vector spaces**

A *vector space* or *linear space* consists of the following:

(*i*) a field F of scalars;

(*ii*) a set V of objects, called vectors;

(*iii*) an operation, called vector addition, which associates with each pair of vectors $\mathbf{a},\mathbf{b}$ in V a vector $\mathbf{a}+\mathbf{b}$ in V, called the sum of $\mathbf{a}$ and $\mathbf{b}$, in such a way that

(1) addition is commutative, $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$;

(2) addition is associative, $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$;

(3) there is a unique vector $\mathbf{0}$ in V, called the zero vector, such that $\mathbf{a}+\mathbf{0}=\mathbf{a}$ for all $\mathbf{a}$ in V;

(4) for each vector $\mathbf{a}$ in Vthere is a unique vector $-\mathbf{a}$ in V such that $\mathbf{a}+(-\mathbf{a})=\mathbf{0}$;

(*iv*) an operation $\cdot$, called scalar multiplication, which associates with each scalar k in F and a vector $\mathbf{a}$ in V a vector $k\cdot\mathbf{a}$ in V, called the product of k with $\mathbf{a}$, in such a way that

(1) $1\cdot\mathbf{a}=\mathbf{a}$ for every $\mathbf{a}$ in V;

(2) $(k_1k_2)\cdot\mathbf{a}=k_1(k_2\cdot\mathbf{a})$;

(3) $k\cdot(\mathbf{a}+\mathbf{b})=k\cdot\mathbf{a}+k\cdot\mathbf{b}$;

(4) $(k_1+k_2)\cdot\mathbf{a}=k_1\cdot\mathbf{a}+k_2\cdot\mathbf{a}$.

We say that V is a *vector space over the field F, denoted by* $(V\ ;+,\cdot)$.

Some examples of vector spaces are as follows.

(1) *The n-tuple space R^n over the real number field R.* Let V be the set of all n-tuples $(x_1,x_2,\cdots,x_n)$ with $x_i\in R, 1\leq i\leq n$. If $\forall\mathbf{a}=(x_1,x_2,\cdots,x_n)$, $\mathbf{b}=(y_1,y_2,\cdots,y_n)\in V$, then the sum of $\mathbf{a}$ and $\mathbf{b}$ is defined by

$$\mathbf{a}+\mathbf{b}=(x_1+y_1,x_2+y_2,\cdots,x_n+y_n).$$

The product of a real number k with $\mathbf{a}$ is defined by

$$k\mathbf{a}=(kx_1,kx_2,\cdots,kx_n).$$

(2) *The space $Q^{m\times n}$ of $m\times n$ matrices over the rational number field Q.* Let $Q^{m\times n}$ be the set of all $m\times n$ matrices over the natural number field Q. The sum of two vectors A and B in $Q^{m\times n}$ is defined by

$$(A+B)_{ij} = A_{ij} + B_{ij},$$

and the product of a rational number p with a matrix A is defined by

$$(pA)_{ij} = pA_{ij}.$$

A *subspace* W of a vector space V is a subset W of V which is itself a vector space over F with the operations of vector addition and scalar multiplication on V. The following result for subspaces is known in references [6] and [33].

Theorem 1.2.7 *A non-empty subset W of a vector space $(V\ ;+,\cdot)$ over the field F is a subspace of $(V\ ;+,\cdot)$ if and only if for each pair of vectors $\mathbf{a},\mathbf{b}$ in W and each scalar k in F the vector $k\cdot\mathbf{a}+\mathbf{b}$ is also in W.*

Therefore, the intersection of two subspaces of a vector space V is still a subspace of V. Let U be a set of some vectors in a vector space V over F. The subspace spanned by U is defined by

$$\langle U\rangle\ = \{\ k_1\cdot\mathbf{a_1}+k_2\cdot\mathbf{a_2}+\cdots+k_l\cdot\mathbf{a}_l\ |\ l\geq 1, k_i\in F,\ \text{and}\ \mathbf{a_j}\in S, 1\leq i\leq l\ \}.$$

A subset W of V is said to be linearly dependent if there exist distinct vectors $\mathbf{a_1},\mathbf{a_2},\cdots,\mathbf{a_n}$ in W and scalars $k_1,k_2,\cdots,k_n$ in F, not all of which are 0, such that

$$k_1\cdot\mathbf{a_1}+k_2\cdot\mathbf{a_2}+\cdots+k_n\cdot\mathbf{a_n}=\mathbf{0}.$$

For a vector space V, its *basis* is a linearly independent set of vectors in V which spans the space V. Call a space V *finite-dimensional* if it has a finite basis. Denoted by $dimV$ the number of elements in a basis of V.

For two subspaces U,W of a space V, the sum of subspaces U,W is defined by

$$U+W=\{\ \mathbf{u}+\mathbf{w}\ |\ \mathbf{u}\in U,\ \mathbf{w}\in W\ \}.$$

Then, we have results in the following ([6][33]).

Theorem 1.2.8 *Any finite-dimensional vector space V over a field F is isomorphic to one and only one space F^n, where $n=dimV$.*

Theorem 1.2.9 *If W_1 and W_2 are finite-dimensional subspaces of a vector space V, then W_1+W_2 is finite-dimensional and*

$$dimW_1+dimW_2=dim(W_1\bigcap W_2)+dim(W_1+W_2).$$

§1.3 **Algebraic Multi-Spaces**

The notion of a multi-space was introduced by Smarandache in 1969 ([86]). Algebraic multi-spaces had be researched in references [58] – [61] and [103]. Vasantha Kandasamy researched various bispaces in [101], such as those of bigroups, bisemigroups, biquasigroups, biloops, bigroupoids, birings, bisemirings, bivectors, bisemivectors, bilnear-rings, $\cdots$, etc., considered two operation systems on two different sets.

1.3.1. **Algebraic multi-spaces**

Definition 1.3.1 *For any integers n, i, $n \geq 2$ and $1 \leq i \leq n$, let A_i be a set with ensemble of law L_i, and the intersection of k sets $A_{i_1}, A_{i_2}, \cdots, A_{i_k}$ of them constrains the law $I(A_{i_1}, A_{i_2}, \cdots, A_{i_k})$. Then the union $\widetilde{A}$*

$$\widetilde{A} = \bigcup_{i=1}^{n} A_i$$

is called a multi-space.

Notice that in this definition, each law may be contain more than one binary operation. For a binary operation $\times$, if there exists an element $1_\times^l$ (or $1_\times^r$) such that

$$1_\times^l \times a = a \quad \text{or} \quad a \times 1_\times^r = a$$

for $\forall a \in A_i, 1 \leq i \leq n$, then $1_\times^l$ ($1_\times^r$) is called a *left (right) unit*. If $1_\times^l$ and $1_\times^r$ exist simultaneously, then there must be

$$1_\times^l = 1_\times^l \times 1_\times^r = 1_\times^r = 1_\times.$$

Call $1_\times$ a *unit* of A_i.

Remark 1.3.1 In Definition 1.3.1, the following three cases are permitted:

(*i*) $A_1 = A_2 = \cdots = A_n$, i.e., n laws on one set.

(*ii*) $L_1 = L_2 = \cdots = L_n$, i.e., n set with one law

(*iii*) there exist integers $s_1, s_2, \cdots, s_l$ such that $I(s_j) = \emptyset, 1 \leq j \leq l$, i.e., some laws on the intersections may be not existed.

We give some examples for Definition 1.3.1.

Example 1.3.1 Take n disjoint two by two cyclic groups $C_1, C_2, \cdots, C_n$, $n \geq 2$ with

$$C_1 = (\langle a \rangle ; +_1), C_2 = (\langle b \rangle ; +_2), \cdots, C_n = (\langle c \rangle ; +_n).$$

Where $+_1, +_2, \cdots, +_n$ are n binary operations. Then their union

$$\widetilde{C} = \bigcup_{i=1}^{n} C_i$$

is a multi-space with the empty intersection laws. In this multi-space, for $\forall x, y \in \widetilde{C}$, if $x, y \in C_k$ for some integer k, then we know $x +_k y \in C_k$. But if $x \in C_s$, $y \in C_t$ and $s \neq t$, then we do not know which binary operation between them and what is the resulting element corresponds to them.

A general multi-space of this kind is constructed by choosing n algebraic systems $A_1, A_2, \cdots, A_n$ satisfying that

$$A_i \bigcap A_j = \emptyset \ \text{ and } \ O(A_i) \bigcap O(A_j) = \emptyset,$$

for any integers $i, j, i \neq j,\ 1 \leq i, j \leq n$, where $O(A_i)$ denotes the binary operation set in A_i. Then

$$\widetilde{A} = \bigcup_{i=1}^{n} A_i$$

with $O(\widetilde{A}) = \bigcup\limits_{i=1}^{n} O(A_i)$ is a multi-space. This kind of multi-spaces can be seen as a model of spaces with a empty intersection.

Example 1.3.2 Let $(G\ ; \circ)$ be a group with a binary operation $\circ$. Choose n different elements $h_1, h_2, \cdots, h_n$, $n \geq 2$ and make the extension of the group $(G\ ; \circ)$ by $h_1, h_2, \cdots, h_n$ respectively as follows:

$(G \bigcup \{h_1\}; \times_1)$, where the binary operation $\times_1 = \circ$ for elements in G, otherwise, new operation;

$(G \bigcup \{h_2\}; \times_2)$, where the binary operation $\times_2 = \circ$ for elements in G, otherwise, new operation;

$\cdots\cdots\cdots\cdots\cdots\cdots$;

$(G \bigcup \{h_n\}; \times_n)$, where the binary operation $\times_n = \circ$ for elements in G, otherwise, new operation.

Define

$$\widetilde{G} = \bigcup_{i=1}^{n} (G \bigcup \{h_i\}; \times_i).$$

Then $\widetilde{G}$ is a multi-space with binary operations $\times_1, \times_2, \cdots, \times_n$. In this multi-space, for $\forall x, y \in \widetilde{G}$, unless the exception cases $x = h_i, y = h_j$ and $i \neq j$, we know the binary operation between x and y and the resulting element by them.

For $n = 3$, this multi-space can be shown as in Fig.1.2, in where the central circle represents the group G and each angle field the extension of G. Whence, we call this kind of multi-space a *fan multi-space*.

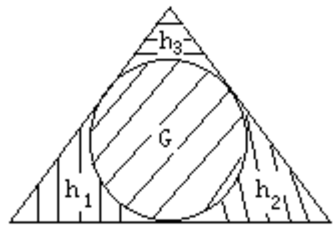

Fig.1.2

Similarly, we can also use a ring R to get fan multi-spaces. For example, let $(R\ ;+,\circ)$ be a ring and let $r_1, r_2, \cdots, r_s$ be two by two different elements. Make these extensions of $(R\ ;+,\circ)$ by $r_1, r_2, \cdots, r_s$ respectively as follows:

$(R\bigcup\{r_1\};+_1,\times_1)$, where binary operations $+_1 = +$, $\times_1 = \circ$ for elements in R, otherwise, new operation;

$(R\bigcup\{r_2\};+_2,\times_2)$, where binary operations $+_2 = +$, $\times_2 = \circ$ for elements in R, otherwise, new operation;

$\cdots\cdots\cdots\cdots\cdots\cdots$;

$(R\bigcup\{r_s\};+_s,\times_s)$, where binary operations $+_s = +$, $\times_s = \circ$ for elements in R, otherwise, new operation.

Define

$$\widetilde{R} = \bigcup_{j=1}^{s}(R\bigcup\{r_j\};+_j,\times_j).$$

Then $\widetilde{R}$ is a fan multi-space with ring-like structure. Also we can define a fan multi-space with field-like, vector-like, semigroup-like,$\cdots$, etc. structures.

These multi-spaces constructed in Examples 1.3.1 and 1.3.2 are not *completed*, i.e., there exist some elements in this space not have binary operation between them. In algebra, we wish to construct a *completed multi-space*, i.e., there is a binary operation between any two elements at least and their resulting is still in this space. The following example is a completed multi-space constructed by applying *Latin squares* in the combinatorial design.

Example 1.3.3 Let S be a finite set with $|S| = n \geq 2$. Constructing an $n\times n$ Latin square by elements in S, i.e., every element just appears one time on its each row and each column. Now choose k Latin squares $M_1, M_2, \cdots, M_k$, $k \leq \prod\limits_{s=1}^{n} s!$.

By a result in the reference [83], there are at least $\prod\limits_{s=1}^{n} s!$ distinct $n\times n$ Latin squares. Whence, we can always choose $M_1, M_2, \cdots, M_k$ distinct two by two. For a Latin square $M_i, 1 \leq i \leq k$, define an operation $\times_i$as follows:

$$\times_i : (s, f) \in S\times S \rightarrow (M_i)_{sf}.$$

The case of $n = 3$ is explained in the following. Here $S = \{1, 2, 3\}$ and there are 2 Latin squares L_1, L_2 as follows:

$$L_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \qquad L_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$$

.

Therefore, by the Latin square L_1, we get an operation $\times_1$as in table 1.3.1.

$\times_1$	1	2	3
1	1	2	3
2	2	3	1
3	3	1	2

table 1.3.1

and by the Latin square L_2, we also get an operation $\times_2$ as in table 1.3.2.

$\times_2$	1	2	3
1	1	2	3
2	3	1	2
3	2	3	1

table 1.3.2

For $\forall x, y, z \in S$ and two operations $\times_i$ and $\times_j$, $1 \leq i, j \leq k$, define

$$x \times_i y \times_j z = (x \times_i y) \times_j z.$$

For example, in the case $n = 3$, we know that

$$1 \times_1 2 \times_2 3 = (1 \times_2) \times_2 3 = 2 \times_2 3 = 2;$$

and

$$2 \times_1 3 \times_2 2 = (2 \times_1 3) \times_2 2 = 1 \times_2 3 = 3.$$

Whence S is a completed multi-space with k operations.

The following example is also a completed multi-space constructed by an algebraic system.

Example 1.3.4 For constructing a completed multi-space, let $(S\,; \circ)$ be an algebraic system, i.e., $a \circ b \in S$ for $\forall a, b \in S$. Whence, we can take $C, C \subseteq S$ being a cyclic group. Now consider a partition of S

$$S = \bigcup_{k=1}^{m} G_k$$

with $m \geq 2$ such that $G_i \bigcap G_j = C$ for $\forall i, j, 1 \leq i, j \leq m$.

For an integer $k, 1 \leq k \leq m$, assume $G_k = \{g_{k1}, g_{k2}, \cdots, g_{kl}\}$. We define an operation $\times_k$on G_k as follows, which enables $(G_k; \times_k)$ to be a cyclic group.

$$g_{k1} \times_k g_{k1} = g_{k2},$$

$$g_{k2} \times_k g_{k1} = g_{k3},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$g_{k(l-1)} \times_k g_{k1} = g_{kl},$$

and

$$g_{kl)} \times_k g_{k1} = g_{k1}.$$

Then $S = \bigcup_{k=1}^{m} G_k$ is a completed multi-space with $m+1$ operations.

The approach used in Example 1.3.4 enables us to construct a complete multi-spaces $\widetilde{A} = \bigcup_{i=1}^{n}$ with k operations for $k \geq n+1$, i.e., the intersection law $I(A_1, A_2, \cdots, A_n) \neq \emptyset$.

Definition 1.3.2 *A mapping f on a set X is called faithful if $f(x) = x$ for $\forall x \in X$, then $f = 1_X$, the unit mapping on X fixing each element in X.*

Notice that if f is faithful and $f_1(x) = f(x)$ for $\forall x \in X$, then $f_1^{-1} f = 1_X$, i.e., $f_1 = f$.

For each operation $\times$ and a chosen element g in a subspace $A_i, A_i \subset \widetilde{A}, 1 \leq i \leq n$, there is a *left-mapping* $f_g^l : A_i \to A_i$ defined by

$$f_g^l : a \to g \times a, \quad a \in A_i.$$

Similarly, we can also define the *right-mapping* f_g^r.

We adopt the following convention for multi-spaces in this book.

Convention 1.3.1 *Each operation $\times$in a subset $A_i, A_i \subset \widetilde{A}, 1 \leq i \leq n$ is faithful, i.e., for $\forall g \in A_i$, $\varsigma : g \to f_g^l$ (or $\tau : g \to f_g^r$) is faithful.*

Define the kernel $Ker\varsigma$ of a mapping ς by

$$\mathrm{Ker}\varsigma = \{g | g \in A_i \text{ and } \varsigma(g) = 1_{A_i}\}.$$

Then Convention 1.3.1 is equivalent to the next convention.

Convention 1.3.2 *For each $\varsigma : g \to f_g^l$ (or $\varsigma : g \to f_g^r$) induced by an operation $\times$ has kernel*

$$\mathrm{Ker}\varsigma = \{1_\times^l\}$$

if $1_\times^l$ *exists. Otherwise,* $\mathrm{Ker}\varsigma = \emptyset$.

We have the following results for multi-spaces $\widetilde{A}$.

Theorem 1.3.1 *For a multi-space* $\widetilde{A}$ *and an operation* $\times$*, the left unit* $1_\times^l$ *and right unit* $1_\times^r$ *are unique if they exist.*

Proof If there are two left units $1_\times^l, I_\times^l$ in a subset A_i of a multi-space $\widetilde{A}$, then for $\forall x \in A_i$, their induced left-mappings $f_{1_\times^l}^l$ and $f_{I_\times^l}^l$ satisfy

$$f_{1_\times^l}^l(x) = 1_\times^l \times x = x$$

and

$$f_{I_\times^l}^l(x) = I_\times^l \times x = x.$$

Therefore, we get that $f_{1_\times^l}^l = f_{I_\times^l}^l$. Since the mappings $\varsigma_1 : 1_\times^l \to f_{1_\times^l}^l$ and $\varsigma_2 : I_\times^l \to f_{I_\times^l}^l$ are faithful, we know that

$$1_\times^l = I_\times^l.$$

Similarly, we can also prove that the right unit $1_\times^r$ is also unique. ♮

For two elements a, b of a multi-space $\widetilde{A}$, if $a \times b = 1_\times^l$, then b is called a *left-inverse* of a. If $a \times b = 1_\times^r$, then a is called a *right-inverse* of b. Certainly, if $a \times b = 1_\times$, then a is called an *inverse* of b and b an *inverse* of a.

Theorem 1.3.2 *For a multi-space* $\widetilde{A}$*,* $a \in \widetilde{A}$*, the left-inverse and right-inverse of* a *are unique if they exist.*

Proof Notice that $\kappa_a : x \to ax$ is faithful, i.e., $\mathrm{Ker}\kappa = \{1_\times^l\}$ for $1_\times^l$ existing now.

If there exist two left-inverses b_1, b_2 in $\widetilde{A}$ such that $a \times b_1 = 1_\times^l$ and $a \times b_2 = 1_\times^l$, then we know that

$$b_1 = b_2 = 1_\times^l.$$

Similarly, we can also prove that the right-inverse of a is also unique. ♮

Corollary 1.3.1 *If* $\times$ *is an operation of a multi-space* $\widetilde{A}$ *with unit* $1_\times$*, then the equation*

$$a \times x = b$$

has at most one solution for the indeterminate x*.*

Proof According to Theorem 1.3.2, we know there is at most one left-inverse a_1 of a such that $a_1 \times a = 1_\times$. Whence, we know that

$$x = a_1 \times a \times x = a_1 \times b. \qquad \natural$$

We also get a consequence for solutions of an equation in a multi-space by this result.

Corollary 1.3.2 *Let $\widetilde{A}$ be a multi-space with a operation set $O(\widetilde{A})$. Then the equation*

$$a \circ x = b$$

has at most $o(\widetilde{A})$ solutions, where $\circ$is any binary operation of $\widetilde{A}$.

Two multi-spaces $\widetilde{A_1}, \widetilde{A_2}$ are said to be *isomorphic* if there is a one to one mapping $\zeta : \widetilde{A_1} \to \widetilde{A_2}$ such that for $\forall x, y \in \widetilde{A}_1$ with binary operation $\times$, $\zeta(x), \zeta(y)$ in $\widetilde{A_2}$ with binary operation $\circ$ satisfying the following condition

$$\zeta(x \times y) = \zeta(x) \circ \zeta(y).$$

If $\widetilde{A_1} = \widetilde{A_2} = \widetilde{A}$, then an isomorphism between $\widetilde{A_1}$ and $\widetilde{A_2}$ is called an *automorphism* of $\widetilde{A}$. All automorphisms of $\widetilde{A}$ form a group under the composition operation between mappings, denoted by Aut$\widetilde{A}$.

Notice that $\mathrm{Aut}Z_n \cong Z_n^*$, where Z_n^* is the group of reduced residue class mod$n$ under the multiply operation ([108]). It is known that $|\mathrm{Aut}Z_n| = \varphi(n)$, where $\varphi(n)$ is the Euler function. We know the automorphism group of the multi-space $\widetilde{C}$ in Example 1.3.1 is

$$\mathrm{Aut}\widetilde{C} = S_n[Z_n^*].$$

Whence, $|\mathrm{Aut}\widetilde{C}| = \varphi(n)^n n!$. For Example 1.3.3, determining its automorphism group is a more interesting problem for the combinatorial design (see also the final section in this chapter).

1.3.2 **Multi-Groups**

The conception of multi-groups is a generalization of classical algebraic structures, such as those of groups, fields, bodies, $\cdots$, etc., which is defined in the following definition.

Definition 1.3.3 *Let $\widetilde{G} = \bigcup\limits_{i=1}^{n} G_i$ be a complete multi-space with an operation set $O(\widetilde{G}) = \{\times_i, 1 \leq i \leq n\}$. If $(G_i; \times_i)$ is a group for any integer $i, 1 \leq i \leq n$ and for $\forall x, y, z \in \widetilde{G}$ and $\forall \times, \circ \in O(\widetilde{G})$, $\times \neq \circ$, there is one operation, for example the operation $\times$ satisfying the distribution law to the operation $\circ$provided all of these operating results exist , i.e.,*

$$x \times (y \circ z) = (x \times y) \circ (x \times z),$$

$$(y \circ z) \times x = (y \times x) \circ (z \times x),$$

then $\widetilde{G}$ is called a multi-group.

Remark 1.3.2 The following special cases for $n = 2$ convince us that multi-groups are a generalization of groups, fields and bodies, $\cdots$, etc..

(i) If $G_1 = G_2 = \widetilde{G}$, then $\widetilde{G}$ is a body.
(ii) If $(G_1; \times_1)$ and $(G_2; \times_2)$ are commutative groups, then $\widetilde{G}$ is a field.

For a multi-group $\widetilde{G}$ and a subset $\widetilde{G_1} \subset \widetilde{G}$, if $\widetilde{G_1}$ is also a multi-group under a subset $O(\widetilde{G_1}), O(\widetilde{G_1}) \subset O(\widetilde{G})$, then $\widetilde{G_1}$ is called a *sub-multi-group* of $\widetilde{G}$, denoted by $\widetilde{G_1} \preceq \widetilde{G}$. We get a criterion for sub-multi-groups in the following.

Theorem 1.3.3 *For a multi-group $\widetilde{G} = \bigcup\limits_{i=1}^{n} G_i$ with an operation set $O(\widetilde{G}) = \{\times_i | 1 \leq i \leq n\}$, a subset $\widetilde{G_1} \subset \widetilde{G}$ is a sub-multi-group of $\widetilde{G}$ if and only if $(\widetilde{G_1} \bigcap G_k; \times_k)$ is a subgroup of $(G_k; \times_k)$ or $\widetilde{G_1} \bigcap G_k = \emptyset$ for any integer $k, 1 \leq k \leq n$.*

Proof If $\widetilde{G_1}$ is a multi-group with an operation set $O(\widetilde{G_1}) = \{\times_{i_j} | 1 \leq j \leq s\} \subset O(\widetilde{G})$, then

$$\widetilde{G_1} = \bigcup_{i=1}^{n} (\widetilde{G_1} \bigcap G_i) = \bigcup_{j=1}^{s} G'_{i_j}$$

where $G'_{i_j} \preceq G_{i_j}$ and $(G_{i_j}; \times_{i_j})$ is a group. Whence, if $\widetilde{G_1} \bigcap G_k \neq \emptyset$, then there exist an integer $l, k = i_l$ such that $\widetilde{G_1} \bigcap G_k = G'_{i_l}$, i.e., $(\widetilde{G_1} \bigcap G_k; \times_k)$ is a subgroup of $(G_k; \times_k)$.

Now if $(\widetilde{G_1} \bigcap G_k; \times_k)$ is a subgroup of $(G_k; \times_k)$ or $\widetilde{G_1} \bigcap G_k = \emptyset$ for any integer k, let N denote the index set k with $\widetilde{G_1} \bigcap G_k \neq \emptyset$, then

$$\widetilde{G_1} = \bigcup_{j \in N} (\widetilde{G_1} \bigcap G_j)$$

and $(\widetilde{G_1} \bigcap G_j, \times_j)$ is a group. Since $\widetilde{G_1} \subset \widetilde{G}$, $O(\widetilde{G_1}) \subset O(\widetilde{G})$, the associative law and distribute law are true for the $\widetilde{G_1}$. Therefore, $\widetilde{G_1}$ is a sub-multi-group of $\widetilde{G}$. ♮

For finite sub-multi-groups, we get a criterion as in the following.

Theorem 1.3.4 *Let $\widetilde{G}$ be a finite multi-group with an operation set $O(\widetilde{G}) = \{\times_i | 1 \leq i \leq n\}$. A subset $\widetilde{G_1}$ of $\widetilde{G}$ is a sub-multi-group under an operation subset $O(\widetilde{G_1}) \subset O(\widetilde{G})$ if and only if $(\widetilde{G_1}; \times)$ is complete for each operation $\times$ in $O(\widetilde{G_1})$.*

Proof Notice that for a multi-group $\widetilde{G}$, its each sub-multi-group $\widetilde{G_1}$ is complete.

Now if $\widetilde{G_1}$ is a complete set under each operation $\times_i$ in $O(\widetilde{G_1})$, we know that $(\widetilde{G_1} \bigcap G_i; \times_i)$ is a group or an empty set. Whence, we get that

$$\widetilde{G_1} = \bigcup_{i=1}^{n} (\widetilde{G_1} \bigcap G_i).$$

Therefore, $\widetilde{G_1}$ is a sub-multi-group of $\widetilde{G}$ under the operation set $O(\widetilde{G_1})$. ♮

For a sub-multi-group $\widetilde{H}$ of a multi-group $\widetilde{G}$, $g \in \widetilde{G}$, define

$$g\widetilde{H} = \{g \times h | h \in \widetilde{H}, \times \in O(\widetilde{H})\}.$$

Then for $\forall x, y \in \widetilde{G}$,

$$x\widetilde{H} \bigcap y\widetilde{H} = \emptyset \quad \text{or} \quad x\widetilde{H} = y\widetilde{H}.$$

In fact, if $x\widetilde{H} \bigcap y\widetilde{H} \neq \emptyset$, let $z \in x\widetilde{H} \bigcap y\widetilde{H}$, then there exist elements $h_1, h_2 \in \widetilde{H}$ and operations $\times_i$ and $\times_j$ such that

$$z = x \times_i h_1 = y \times_j h_2.$$

Since $\widetilde{H}$ is a sub-multi-group, $(\widetilde{H} \bigcap G_i; \times_i)$ is a subgroup. Whence, there exists an inverse element h_1^{-1} in $(\widetilde{H} \bigcap G_i; \times_i)$. We get that

$$x \times_i h_1 \times_i h_1^{-1} = y \times_j h_2 \times_i h_1^{-1}.$$

i.e.,

$$x = y \times_j h_2 \times_i h_1^{-1}.$$

Whence,

$$x\widetilde{H} \subseteq y\widetilde{H}.$$

Similarly, we can also get that

$$x\widetilde{H} \supseteq y\widetilde{H}.$$

Thereafter, we get that

$$x\widetilde{H} = y\widetilde{H}.$$

Denote the union of two set A and B by $A \bigoplus B$ if $A \bigcap B = \emptyset$. Then the following result is implied in the previous proof.

Theorem 1.3.5 *For any sub-multi-group $\widetilde{H}$ of a multi-group $\widetilde{G}$, there is a representation set T, $T \subset \widetilde{G}$, such that*

$$\widetilde{G} = \bigoplus_{x \in T} x\widetilde{H}.$$

For the case of finite groups, since there is only one binary operation $\times$ and $|x\widetilde{H}| = |y\widetilde{H}|$ for any $x, y \in \widetilde{G}$, We get a consequence in the following, which is just the Lagrange theorem for finite groups.

Corollary 1.3.3(Lagrange theorem) *For any finite group G, if H is a subgroup of G, then $|H|$ is a divisor of $|G|$.*

For a multi-group $\widetilde{G}$ and $g \in \widetilde{G}$, denote all the binary operations associative with g by $\overrightarrow{O(g)}$ and the elements associative with the binary operation $\times$ by $\widetilde{G}(\times)$. For a sub-multi-group $\widetilde{H}$ of $\widetilde{G}$, $\times \in O(\widetilde{H})$, if

$$g \times h \times g^{-1} \in \widetilde{H},$$

for $\forall h \in \widetilde{H}$ and $\forall g \in \widetilde{G}(\times)$, then we call $\widetilde{H}$ a *normal sub-multi-group* of $\widetilde{G}$, denoted by $\widetilde{H} \lhd \widetilde{G}$. If $\widetilde{H}$ is a normal sub-multi-group of $\widetilde{G}$, similar to the normal subgroups of groups, it can be shown that $g \times \widetilde{H} = \widetilde{H} \times g$, where $g \in \widetilde{G}(\times)$. Thereby we get a result as in the following.

Theorem 1.3.6 *Let $\widetilde{G} = \bigcup\limits_{i=1}^{n} G_i$ be a multi-group with an operation set $O(\widetilde{G}) = \{\times_i | 1 \leq i \leq n\}$. Then a sub-multi-group $\widetilde{H}$ of $\widetilde{G}$ is normal if and only if $(\widetilde{H} \bigcap G_i; \times_i)$ is a normal subgroup of $(G_i; \times_i)$ or $\widetilde{H} \bigcap G_i = \emptyset$ for any integer $i, 1 \leq i \leq n$.*

Proof We have known that

$$\widetilde{H} = \bigcup_{i=1}^{n} (\widetilde{H} \bigcap G_i).$$

If $(\widetilde{H} \bigcap G_i; \times_i)$ is a normal subgroup of $(G_i; \times_i)$ for any integer $i, 1 \leq i \leq n$, then we know that

$$g \times_i (\widetilde{H} \bigcap G_i) \times_i g^{-1} = \widetilde{H} \bigcap G_i$$

for $\forall g \in G_i, 1 \leq i \leq n$. Whence,

$$g \circ \widetilde{H} \circ g^{-1} = \widetilde{H}$$

for $\forall \circ \in O(\widetilde{H})$ and $\forall g \in \overrightarrow{\widetilde{G}(\circ)}$. That is, $\widetilde{H}$ is a normal sub-multi-group of $\widetilde{G}$.

Now if $\widetilde{H}$ is a normal sub-multi-group of $\widetilde{G}$, by definition we know that

$$g \circ \widetilde{H} \circ g^{-1} = \widetilde{H}$$

for $\forall \circ \in O(\widetilde{H})$ and $\forall g \in \widetilde{G}(\circ)$. Not loss of generality, we assume that $\circ = \times_k$, then we get

$$g \times_k (\widetilde{H} \bigcap G_k) \times_k g^{-1} = \widetilde{H} \bigcap G_k.$$

Therefore, $(\widetilde{H}\bigcap G_k;\times_k)$ is a normal subgroup of $(G_k,\times_k)$. Since the operation $\circ$ is chosen arbitrarily, we know that $(\widetilde{H}\bigcap G_i;\times_i)$ is a normal subgroup of $(G_i;\times_i)$ or an empty set for any integer i, $1\leq i\leq n$. ♮

For a multi-group $\widetilde{G}$ with an operation set $O(\widetilde{G})=\{\times_i|\ 1\leq i\leq n\}$, an order of operations in $O(\widetilde{G})$ is said to be an *oriented operation sequence*, denoted by $\overrightarrow{O}(\widetilde{G})$. For example, if $O(\widetilde{G})=\{\times_1,\times_2\times_3\}$, then $\times_1\succ\times_2\succ\times_3$ is an oriented operation sequence and $\times_2\succ\times_1\succ\times_3$ is also an oriented operation sequence.

For a given oriented operation sequence $\overrightarrow{O}(\widetilde{G})$, we construct a series of normal sub-multi-group

$$\widetilde{G}\rhd\widetilde{G}_1\rhd\widetilde{G}_2\rhd\cdots\rhd\widetilde{G}_m=\{1_{\times_n}\}$$

by the following programming.

STEP 1: *Construct a series*

$$\widetilde{G}\rhd\widetilde{G}_{11}\rhd\widetilde{G}_{12}\rhd\cdots\rhd\widetilde{G}_{1l_1}$$

under the operation $\times_1$*.*

STEP 2: *If a series*

$$\widetilde{G}_{(k-1)l_1}\rhd\widetilde{G}_{k1}\rhd\widetilde{G}_{k2}\rhd\cdots\rhd\widetilde{G}_{kl_k}$$

has be constructed under the operation $\times_k$ *and* $\widetilde{G}_{kl_k}\neq\{1_{\times_n}\}$*, then construct a series*

$$\widetilde{G}_{kl_1}\rhd\widetilde{G}_{(k+1)1}\rhd\widetilde{G}_{(k+1)2}\rhd\cdots\rhd\widetilde{G}_{(k+1)l_{k+1}}$$

under the operation $\times_{k+1}$*.*

This programming is terminated until the series

$$\widetilde{G}_{(n-1)l_1}\rhd\widetilde{G}_{n1}\rhd\widetilde{G}_{n2}\rhd\cdots\rhd\widetilde{G}_{nl_n}=\{1_{\times_n}\}$$

has be constructed under the operation $\times_n$*.*

The number m is called the *length of the series of normal sub-multi-groups.* Call a series of normal sub-multi-group

$$\widetilde{G}\rhd\widetilde{G}_1\rhd\widetilde{G}_2\rhd\cdots\rhd\widetilde{G}_n=\{1_{\times_n}\}$$

maximal if there exists a normal sub-multi-group $\widetilde{H}$ for any integer $k,s,1\leq k\leq n,1\leq s\leq l_k$ such that

$$\widetilde{G}_{ks}\rhd\widetilde{H}\rhd\widetilde{G}_{k(s+1)},$$

then $\widetilde{H}=\widetilde{G}_{ks}$ or $\widetilde{H}=\widetilde{G}_{k(s+1)}$. For a maximal series of finite normal sub-multi-group, we get a result as in the following.

Theorem 1.3.7 *For a finite multi-group* $\widetilde{G} = \bigcup_{i=1}^{n} G_i$ *and an oriented operation sequence* $\overrightarrow{O}(\widetilde{G})$*, the length of the maximal series of normal sub-multi-group in* $\widetilde{G}$ *is a constant, only dependent on* $\widetilde{G}$ *itself.*

Proof The proof is by the induction principle on the integer n.

For $n = 1$, the maximal series of normal sub-multi-groups of $\widetilde{G}$ is just a composition series of a finite group. By the Jordan-Hölder theorem (see [73] or [107]), we know the length of a composition series is a constant, only dependent on $\widetilde{G}$. Whence, the assertion is true in the case of $n = 1$.

Assume that the assertion is true for all cases of $n \leq k$. We prove it is also true in the case of $n = k+1$. Not loss of generality, assume the order of those binary operations in $\overrightarrow{O}(\widetilde{G})$ being $\times_1 \succ \times_2 \succ \cdots \succ \times_n$ and the composition series of the group $(G_1, \times_1)$ being

$$G_1 \rhd G_2 \rhd \cdots \rhd G_s = \{1_{\times_1}\}.$$

By the Jordan-Hölder theorem, we know the length of this composition series is a constant, dependent only on $(G_1; \times_1)$. According to Theorem 3.6, we know a maximal series of normal sub-multi-groups of $\widetilde{G}$ gotten by STEP 1 under the operation $\times_1$ is

$$\widetilde{G} \rhd \widetilde{G} \setminus (G_1 \setminus G_2) \rhd \widetilde{G} \setminus (G_1 \setminus G_3) \rhd \cdots \rhd \widetilde{G} \setminus (G_1 \setminus \{1_{\times_1}\}).$$

Notice that $\widetilde{G} \setminus (G_1 \setminus \{1_{\times_1}\})$ is still a multi-group with less or equal to k operations. By the induction assumption, we know the length of the maximal series of normal sub-multi-groups in $\widetilde{G} \setminus (G_1 \setminus \{1_{\times_1}\})$ is a constant only dependent on $\widetilde{G} \setminus (G_1 \setminus \{1_{\times_1}\})$. Therefore, the length of a maximal series of normal sub-multi-groups is also a constant, only dependent on $\widetilde{G}$.

Applying the induction principle, we know that the length of a maximal series of normal sub-multi-groups of $\widetilde{G}$ is a constant under an oriented operations $\overrightarrow{O}(\widetilde{G})$, only dependent on $\widetilde{G}$ itself. ♮

As a special case of Theorem 1.3.7, we get a consequence in the following.

Corollary 1.3.4(Jordan-Hölder theorem) *For a finite group* G*, the length of its composition series is a constant, only dependent on* G*.*

Certainly, we can also find other characteristics for multi-groups similar to group theory, such as those to establish the decomposition theory for multi-groups similar to the decomposition theory of abelian groups, to characterize finite generated multi-groups, $\cdots$, etc.. More observations can be seen in the finial section of this chapter.

1.3.3 **Multi-Rings**

Definition 1.3.4 *Let* $\widetilde{R} = \bigcup_{i=1}^{m} R_i$ *be a complete multi-space with a double operation* *set* $O(\widetilde{R}) = \{(+_i, \times_i), 1 \leq i \leq m\}$. *If for any integers* $i, j,\ i \neq j, 1 \leq i, j \leq m$, $(R_i; +_i, \times_i)$ *is a ring and*

$$(x +_i y) +_j z = x +_i (y +_j z), \quad (x \times_i y) \times_j z = x \times_i (y \times_j z)$$

for $\forall x, y, z \in \widetilde{R}$ *and*

$$x \times_i (y +_j z) = x \times_i y +_j x \times_i z, \quad (y +_j z) \times_i x = y \times_i x +_j z \times_i x$$

if all of these operating results exist, then $\widetilde{R}$ *is called a multi-ring. If* $(R; +_i, \times_i)$ *is a field for any integer* $1 \leq i \leq m$*, then* $\widetilde{R}$ *is called a multi-field.*

For a multi-ring $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$, let $\widetilde{S} \subset \widetilde{R}$ and $O(\widetilde{S}) \subset O(\widetilde{R})$, if $\widetilde{S}$ is also a multi-ring with a double operation set $O(\widetilde{S})$, then we call $\widetilde{S}$ a *sub-multi-ring* of $\widetilde{R}$. We get a criterion for sub-multi-rings in the following.

Theorem 1.3.8 *For a multi-ring* $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$*, a subset* $\widetilde{S} \subset \widetilde{R}$ *with* $O(\widetilde{S}) \subset O(\widetilde{R})$ *is a sub-multi-ring of* $\widetilde{R}$ *if and only if* $(\widetilde{S} \bigcap R_k; +_k, \times_k)$ *is a subring of* $(R_k; +_k, \times_k)$ *or* $\widetilde{S} \bigcap R_k = \emptyset$ *for any integer* $k, 1 \leq k \leq m$.

Proof For any integer $k, 1 \leq k \leq m$, if $(\widetilde{S} \bigcap R_k; +_k, \times_k)$ is a subring of $(R_k; +_k, \times_k)$ or $\widetilde{S} \bigcap R_k = \emptyset$, then since $\widetilde{S} = \bigcup\limits_{i=1}^{m} (\widetilde{S} \bigcap R_i)$, we know that $\widetilde{S}$ is a sub-multi-ring by the definition of a sub-multi-ring.

Now if $\widetilde{S} = \bigcup\limits_{j=1}^{s} S_{i_j}$ is a sub-multi-ring of $\widetilde{R}$ with a double operation set $O(\widetilde{S}) = \{(+_{i_j}, \times_{i_j}), 1 \leq j \leq s\}$, then $(S_{i_j}; +_{i_j}, \times_{i_j})$ is a subring of $(R_{i_j}; +_{i_j}, \times_{i_j})$. Therefore, $S_{i_j} = R_{i_j} \bigcap \widetilde{S}$ for any integer $j, 1 \leq j \leq s$. But $\widetilde{S} \bigcap S_l = \emptyset$ for other integer $l \in \{i; 1 \leq i \leq m\} \setminus \{i_j; 1 \leq j \leq s\}$. $\natural$

Applying these criterions for subrings of a ring, we get a result in the following.

Theorem 1.3.9 *For a multi-ring* $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$*, a subset* $\widetilde{S} \subset \widetilde{R}$ *with* $O(\widetilde{S}) \subset O(\widetilde{R})$ *is a sub-multi-ring of* $\widetilde{R}$ *if and only if* $(\widetilde{S} \bigcap R_j; +_j) \prec (R_j; +_j)$ *and* $(\widetilde{S}; \times_j)$ *is complete for any double operation* $(+_j, \times_j) \in O(\widetilde{S})$.

Proof According to Theorem 1.3.8, we know that $\widetilde{S}$ is a sub-multi-ring if and only if $(\widetilde{S} \bigcap R_i; +_i, \times_i)$ is a subring of $(R_i; +_i, \times_i)$ or $\widetilde{S} \bigcap R_i = \emptyset$ for any integer $i, 1 \leq i \leq m$. By a well known criterion for subrings of a ring (see also [73]), we know that $(\widetilde{S} \bigcap R_i; +_i, \times_i)$ is a subring of $(R_i; +_i, \times_i)$ if and only if $(\widetilde{S} \bigcap R_j; +_j) \prec (R_j; +_j)$ and $(\widetilde{S}; \times_j)$ is a complete set for any double operation $(+_j, \times_j) \in O(\widetilde{S})$. This completes the proof. $\natural$

We use multi-ideal chains of a multi-ring to characteristic its structure properties. A *multi-ideal* $\widetilde{I}$ of a multi-ring $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$ with a double operation set $O(\widetilde{R})$ is a sub-multi-ring of $\widetilde{R}$ satisfying the following conditions:

(*i*) $\widetilde{I}$ is a sub-multi-group with an operation set $\{+|\ (+,\times)\in O(\widetilde{I})\}$;

(*ii*) for any $r\in\widetilde{R}, a\in\widetilde{I}$ and $(+,\times)\in O(\widetilde{I})$, $r\times a\in\widetilde{I}$ and $a\times r\in\widetilde{I}$ if all of these operating results exist.

Theorem 1.3.10 *A subset $\widetilde{I}$ with $O(\widetilde{I}), O(\widetilde{I})\subset O(\widetilde{R})$ of a multi-ring $\widetilde{R}=\bigcup\limits_{i=1}^{m}R_i$ with a double operation set $O(\widetilde{R})=\{(+_i,\times_i)|\ 1\leq i\leq m\}$ is a multi-ideal if and only if $(\widetilde{I}\bigcap R_i,+_i,\times_i)$ is an ideal of the ring $(R_i,+_i,\times_i)$ or $\widetilde{I}\bigcap R_i=\emptyset$ for any integer $i, 1\leq i\leq m$.*

Proof By the definition of a multi-ideal, the necessity of these conditions is obvious.

For the sufficiency, denote by $\widetilde{R}(+,\times)$ the set of elements in $\widetilde{R}$ with binary operations $+$ and $\times$. If there exists an integer i such that $\widetilde{I}\bigcap R_i\neq\emptyset$ and $(\widetilde{I}\bigcap R_i,+_i,\times_i)$ is an ideal of $(R_i,+_i,\times_i)$, then for $\forall a\in\widetilde{I}\bigcap R_i$, $\forall r_i\in R_i$, we know that

$$r_i\times_i a\in\widetilde{I}\bigcap R_i;\quad a\times_i r_i\in\widetilde{I}\bigcap R_i.$$

Notice that $\widetilde{R}(+_i,\times_i)=R_i$. Thereafter, we get that

$$r\times_i a\in\widetilde{I}\bigcap R_i\ \text{ and }\ a\times_i r\in\widetilde{I}\bigcap R_i,$$

for $\forall r\in\widetilde{R}$ if all of these operating results exist. Whence, $\widetilde{I}$ is a multi-ideal of $\widetilde{R}$. ♮

A multi-ideal $\widetilde{I}$ of a multi-ring $\widetilde{R}$ is said to be *maximal* if for any multi-ideal $\widetilde{I}'$, $\widetilde{R}\supseteq\widetilde{I}'\supseteq\widetilde{I}$ implies that $\widetilde{I}'=\widetilde{R}$ or $\widetilde{I}'=\widetilde{I}$. For an order of the double operations in the set $O(\widetilde{R})$ of a multi-ring $\widetilde{R}=\bigcup\limits_{i=1}^{m}R_i$, not loss of generality, let the order be $(+_1,\times_1)\succ(+_2,\times_2)\succ\cdots\succ(+_m,\times_m)$, we can define a *multi-ideal chain* of $\widetilde{R}$ by the following programming.

(*i*) Construct a multi-ideal chain

$$\widetilde{R}\supset\widetilde{R}_{11}\supset\widetilde{R}_{12}\supset\cdots\supset\widetilde{R}_{1s_1}$$

under the double operation $(+_1,\times_1)$, where $\widetilde{R}_{11}$ is a maximal multi-ideal of $\widetilde{R}$ and in general, $\widetilde{R}_{1(i+1)}$ is a maximal multi-ideal of $\widetilde{R}_{1i}$ for any integer $i,\ 1\leq i\leq m-1$.

(*ii*) If a multi-ideal chain

$$\widetilde{R}\supset\widetilde{R}_{11}\supset\widetilde{R}_{12}\supset\cdots\supset\widetilde{R}_{1s_1}\supset\cdots\supset\widetilde{R}_{i1}\supset\cdots\supset\widetilde{R}_{is_i}$$

has been constructed for $(+_1,\times_1)\succ(+_2,\times_2)\succ\cdots\succ(+_i,\times_i)$, $1\leq i\leq m-1$, then construct a multi-ideal chain of $\widetilde{R}_{is_i}$

$$\widetilde{R}_{is_i}\supset\widetilde{R}_{(i+1)1}\supset\widetilde{R}_{(i+1)2}\supset\cdots\supset\widetilde{R}_{(i+1)s_1}$$

under the double operation $(+_{i+1}, \times_{i+1})$, where $\widetilde{R}_{(i+1)1}$ is a maximal multi-ideal of $\widetilde{R}_{is_i}$ and in general, $\widetilde{R}_{(i+1)(i+1)}$ is a maximal multi-ideal of $\widetilde{R}_{(i+1)j}$ for any integer $j, 1 \leq j \leq s_i - 1$. Define a multi-ideal chain of $\widetilde{R}$ under $(+_1, \times_1) \succ (+_2, \times_2) \succ \cdots \succ (+_{i+1}, \times_{i+1})$ to be

$$\widetilde{R} \supset \widetilde{R}_{11} \supset \cdots \supset \widetilde{R}_{1s_1} \supset \cdots \supset \widetilde{R}_{i1} \supset \cdots \supset \widetilde{R}_{is_i} \supset \widetilde{R}_{(i+1)1} \supset \cdots \supset \widetilde{R}_{(i+1)s_{i+1}}.$$

Similar to multi-groups, we get a result for multi-ideal chains of a multi-ring in the following.

Theorem 1.3.11 *For a multi-ring $\widetilde{R} = \bigcup_{i=1}^{m} R_i$, its multi-ideal chain only has finite terms if and only if the ideal chain of the ring $(R_i; +_i, \times_i)$ has finite terms, i.e., each ring $(R_i; +_i, \times_i)$ is an Artin ring for any integer $i, 1 \leq i \leq m$.*

Proof Let the order of these double operations in $\overrightarrow{O}(\widetilde{R})$ be

$$(+_1, \times_1) \succ (+_2, \times_2) \succ \cdots \succ (+_m, \times_m)$$

and let a maximal ideal chain in the ring $(R_1; +_1, \times_1)$ be

$$R_1 \succ R_{11} \succ \cdots \succ R_{1t_1}.$$

Calculate

$$\widetilde{R}_{11} = \widetilde{R} \setminus \{R_1 \setminus R_{11}\} = R_{11} \bigcup (\bigcup_{i=2}^{m} R_i),$$

$$\widetilde{R}_{12} = \widetilde{R}_{11} \setminus \{R_{11} \setminus R_{12}\} = R_{12} \bigcup (\bigcup_{i=2}^{m} R_i),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\widetilde{R}_{1t_1} = \widetilde{R}_{1t_1} \setminus \{R_{1(t_1-1)} \setminus R_{1t_1}\} = R_{1t_1} \bigcup (\bigcup_{i=2}^{m} R_i).$$

According to Theorem 1.3.10, we know that

$$\widetilde{R} \supset \widetilde{R}_{11} \supset \widetilde{R}_{12} \supset \cdots \supset \widetilde{R}_{1t_1}$$

is a maximal multi-ideal chain of $\widetilde{R}$ under the double operation $(+_1, \times_1)$. In general, for any integer $i, 1 \leq i \leq m-1$, assume

$$R_i \succ R_{i1} \succ \cdots \succ R_{it_i}$$

is a maximal ideal chain in the ring $(R_{(i-1)t_{i-1}}; +_i, \times_i)$. Calculate

$$\widetilde{R}_{ik} = R_{ik} \bigcup (\bigcup_{j=i+1}^{m} \widetilde{R}_{ik} \bigcap R_i).$$

Then we know that

$$\widetilde{R}_{(i-1)t_{i-1}} \supset \widetilde{R}_{i1} \supset \widetilde{R}_{i2} \supset \cdots \supset \widetilde{R}_{it_i}$$

is a maximal multi-ideal chain of $\widetilde{R}_{(i-1)t_{i-1}}$ under the double operation $(+_i, \times_i)$ by Theorem 3.10. Whence, if the ideal chain of the ring $(R_i; +_i, \times_i)$ has finite terms for any integer $i, 1 \leq i \leq m$, then the multi-ideal chain of the multi-ring $\widetilde{R}$ only has finite terms. Now if there exists an integer i_0 such that the ideal chain of the ring $(R_{i_0}, +_{i_0}, \times_{i_0})$ has infinite terms, then there must also be infinite terms in a multi-ideal chain of the multi-ring $\widetilde{R}$. ♮.

A multi-ring is called an *Artin multi-ring* if its each multi-ideal chain only has finite terms. We get a consequence by Theorem 1.3.11.

Corollary 1.3.5 *A multi-ring* $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$ *with a double operation set* $O(\widetilde{R}) = \{(+_i, \times_i) | \ 1 \leq i \leq m\}$ *is an Artin multi-ring if and only if the ring* $(R_i; +_i, \times_i)$ *is an Artin ring for any integer* $i, 1 \leq i \leq m$.

For a multi-ring $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$ with a double operation set $O(\widetilde{R}) = \{(+_i, \times_i) | \ 1 \leq i \leq m\}$, an element e is an *idempotent* element if $e_{\times}^2 = e \times e = e$ for a double binary operation $(+, \times) \in O(\widetilde{R})$. We define the *directed sum* $\widetilde{I}$ of two multi-ideals $\widetilde{I}_1$ and $\widetilde{I}_2$ as follows:

(*i*) $\widetilde{I} = \widetilde{I}_1 \bigcup \widetilde{I}_2$;

(*ii*) $\widetilde{I}_1 \bigcap \widetilde{I}_2 = \{0_+\}$, or $\widetilde{I}_1 \bigcap \widetilde{I}_2 = \emptyset$, where 0_+ denotes an unit element under the operation $+$.

Denote the directed sum of $\widetilde{I}_1$ and $\widetilde{I}_2$ by

$$\widetilde{I} = \widetilde{I}_1 \bigoplus \widetilde{I}_2.$$

If $\widetilde{I} = \widetilde{I}_1 \bigoplus \widetilde{I}_2$ for any $\widetilde{I}_1, \widetilde{I}_2$ implies that $\widetilde{I}_1 = \widetilde{I}$ or $\widetilde{I}_2 = \widetilde{I}$, then $\widetilde{I}$ is called *non-reducible*. We get the following result for Artin multi-rings similar to a well-known result for Artin rings (see [107] for details).

Theorem 1.3.12 *Any Artin multi-ring* $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$ *with a double operation set* $O(\widetilde{R}) = \{(+_i, \times_i) | \ 1 \leq i \leq m\}$ *is a directed sum of finite non-reducible multi-ideals, and if* $(R_i; +_i, \times_i)$ *has unit* $1_{\times_i}$ *for any integer* $i, 1 \leq i \leq m$, *then*

$$\widetilde{R} = \bigoplus_{i=1}^{m} (\bigoplus_{j=1}^{s_i} (R_i \times_i e_{ij}) \bigcup (e_{ij} \times_i R_i)),$$

where $e_{ij}, 1 \leq j \leq s_i$ *are orthogonal idempotent elements of the ring* R_i*.*

Proof Denote by $\widetilde{M}$ the set of multi-ideals which can not be represented by a directed sum of finite multi-ideals in $\widetilde{R}$. According to Theorem 3.11, there is a minimal multi-ideal $\widetilde{I}_0$ in $\widetilde{M}$. It is obvious that $\widetilde{I}_0$ is reducible.

Assume that $\widetilde{I}_0 = \widetilde{I}_1 + \widetilde{I}_2$. Then $\widetilde{I}_1 \not\in \widetilde{M}$ and $\widetilde{I}_2 \not\in \widetilde{M}$. Therefore, $\widetilde{I}_1$ and $\widetilde{I}_2$ can be represented by a directed sum of finite multi-ideals. Thereby $\widetilde{I}_0$ can be also represented by a directed sum of finite multi-ideals. Contradicts that $\widetilde{I}_0 \in \widetilde{M}$.

Now let

$$\widetilde{R} = \bigoplus_{i=1}^{s} \widetilde{I}_i,$$

where each $\widetilde{I}_i, 1 \leq i \leq s$ is non-reducible. Notice that for a double operation $(+, \times)$, each non-reducible multi-ideal of $\widetilde{R}$ has the form

$$(e \times R(\times)) \bigcup (R(\times) \times e), \quad e \in R(\times).$$

Whence, we know that there is a set $T \subset \widetilde{R}$ such that

$$\widetilde{R} = \bigoplus_{e \in T, \ \times \in O(\widetilde{R})} (e \times R(\times)) \bigcup (R(\times) \times e).$$

For any operation $\times \in O(\widetilde{R})$ and the unit $1_\times$, assume that

$$1_\times = e_1 \oplus e_2 \oplus \cdots \oplus e_l, \ e_i \in T, \ 1 \leq i \leq s.$$

Then

$$e_i \times 1_\times = (e_i \times e_1) \oplus (e_i \times e_2) \oplus \cdots \oplus (e_i \times e_l).$$

Therefore, we get that

$$e_i = e_i \times e_i = e_i^2 \ \text{ and } \ e_i \times e_j = 0_i \ \text{ for } \ i \neq j.$$

That is, $e_i, 1 \leq i \leq l$ are orthogonal idempotent elements of $\widetilde{R}(\times)$. Notice that $\widetilde{R}(\times) = R_h$ for some integer h. We know that $e_i, 1 \leq i \leq l$ are orthogonal idempotent elements of the ring $(R_h, +_h, \times_h)$. Denote by e_{hi} for e_i, $1 \leq i \leq l$. Consider all units in $\widetilde{R}$, we get that

$$\widetilde{R} = \bigoplus_{i=1}^{m} (\bigoplus_{j=1}^{s_i} (R_i \times_i e_{ij}) \bigcup (e_{ij} \times_i R_i)).$$

This completes the proof. ♮

Corollary 1.3.6 *Any Artin ring* $(R\ ; +, \times)$ *is a directed sum of finite ideals, and if* $(R\ ; +, \times)$ *has unit* $1_\times$*, then*

$$R = \bigoplus_{i=1}^{s} R_i e_i,$$

where $e_i, 1 \leq i \leq s$ are orthogonal idempotent elements of the ring $(R; +, \times)$.

Similarly, we can also define *Noether multi-rings, simple multi-rings, half-simple multi-rings*, $\cdots$, etc. and find their algebraic structures.

1.3.4 Multi-Vector spaces

Definition 1.3.5 *Let $\widetilde{V} = \bigcup\limits_{i=1}^{k} V_i$ be a complete multi-space with an operation set $O(\widetilde{V}) = \{(\dot{+}_i, \cdot_i) \mid 1 \leq i \leq m\}$ and let $\widetilde{F} = \bigcup\limits_{i=1}^{k} F_i$ be a multi-filed with a double operation set $O(\widetilde{F}) = \{(+_i, \times_i) \mid 1 \leq i \leq k\}$. If for any integers $i, j,\ 1 \leq i, j \leq k$ and $\forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in \widetilde{V}$, $k_1, k_2 \in \widetilde{F}$,*

(i) $(V_i; \dot{+}_i, \cdot_i)$ is a vector space on F_i with vector additive $\dot{+}_i$ and scalar multiplication $\cdot_i$;

(ii) $(\mathbf{a}\dot{+}_i\mathbf{b})\dot{+}_j\mathbf{c} = \mathbf{a}\dot{+}_i(\mathbf{b}\dot{+}_j\mathbf{c})$;

(iii) $(k_1 +_i k_2) \cdot_j \mathbf{a} = k_1 +_i (k_2 \cdot_j \mathbf{a})$;

provided these operating results exist, then $\widetilde{V}$ is called a multi-vector space on the multi-filed space $\widetilde{F}$ with an double operation set $O(\widetilde{V})$, denoted by $(\widetilde{V}; \widetilde{F})$.

For subsets $\widetilde{V}_1 \subset \widetilde{V}$ and $\widetilde{F}_1 \subset \widetilde{F}$, if $(\widetilde{V}_1; \widetilde{F}_1)$ is also a multi-vector space, then we call $(\widetilde{V}_1; \widetilde{F}_1)$ a *multi-vector subspace* of $(\widetilde{V}; \widetilde{F})$. Similar to the linear space theory, we get the following criterion for multi-vector subspaces.

Theorem 1.3.13 *For a multi-vector space $(\widetilde{V}; \widetilde{F})$, $\widetilde{V}_1 \subset \widetilde{V}$ and $\widetilde{F}_1 \subset \widetilde{F}$, $(\widetilde{V}_1; \widetilde{F}_1)$ is a multi-vector subspace of $(\widetilde{V}; \widetilde{F})$ if and only if for any vector additive $\dot{+}$, scalar multiplication $\cdot$ in $(\widetilde{V}_1; \widetilde{F}_1)$ and $\forall \mathbf{a}, \mathbf{b} \in \widetilde{V}$, $\forall \alpha \in \widetilde{F}$,*

$$\alpha \cdot \mathbf{a}\dot{+}\mathbf{b} \in \widetilde{V}_1$$

provided these operating results exist.

Proof Denote by $\widetilde{V} = \bigcup\limits_{i=1}^{k} V_i, \widetilde{F} = \bigcup\limits_{i=1}^{k} F_i$. Notice that $\widetilde{V}_1 = \bigcup\limits_{i=1}^{k} (\widetilde{V}_1 \bigcap V_i)$. By definition, we know that $(\widetilde{V}_1; \widetilde{F}_1)$ is a multi-vector subspace of $(\widetilde{V}; \widetilde{F})$ if and only if for any integer $i, 1 \leq i \leq k$, $(\widetilde{V}_1 \bigcap V_i; \dot{+}_i, \cdot_i)$ is a vector subspace of $(V_i, \dot{+}_i, \cdot_i)$ and $\widetilde{F}_1$ is a multi-filed subspace of $\widetilde{F}$ or $\widetilde{V}_1 \bigcap V_i = \emptyset$.

According to a criterion for linear subspaces of a linear space ([33]), we know that $(\widetilde{V}_1 \bigcap V_i; \dot{+}_i, \cdot_i)$ is a vector subspace of $(V_i, \dot{+}_i, \cdot_i)$ for any integer $i, 1 \leq i \leq k$ if and only if for $\forall \mathbf{a}, \mathbf{b} \in \widetilde{V}_1 \bigcap V_i$, $\alpha \in F_i$,

$$\alpha \cdot_i \mathbf{a}\dot{+}_i\mathbf{b} \in \widetilde{V}_1 \bigcap V_i.$$

That is, for any vector additive $\dot{+}$, scalar multiplication$\cdot$in $(\widetilde{V}_1;\widetilde{F}_1)$ and $\forall \mathbf{a},\mathbf{b}\in\widetilde{V}$, $\forall\alpha\in\widetilde{F}$, if $\alpha\cdot\mathbf{a}\dot{+}\mathbf{b}$ exists, then $\alpha\cdot\mathbf{a}\dot{+}\mathbf{b}\in\widetilde{V}_1$. ♮

Corollary 1.3.7 *Let $(\widetilde{U};\widetilde{F}_1),(\widetilde{W};\widetilde{F}_2)$ be two multi-vector subspaces of a multi-vector space $(\widetilde{V};\widetilde{F})$. Then $(\widetilde{U}\bigcap\widetilde{W};\widetilde{F}_1\bigcap\widetilde{F}_2)$ is a multi-vector space.*

For a multi-vector space $(\widetilde{V};\widetilde{F})$, vectors $\mathbf{a}_1,\mathbf{a}_2,\cdots,\mathbf{a}_n\in\widetilde{V}$, if there are scalars $\alpha_1,\alpha_2,\cdots,\alpha_n\in\widetilde{F}$ such that

$$\alpha_1\cdot_1\mathbf{a}_1\dot{+}_1\alpha_2\cdot_2\mathbf{a}_2\dot{+}_2\cdots\dot{+}_{n-1}\alpha_n\cdot_n\mathbf{a}_n=\mathbf{0},$$

where $\mathbf{0}\in\widetilde{V}$ is a unit under an operation $+$ in $\widetilde{V}$ and $\dot{+}_i,\cdot_i\in O(\widetilde{V})$, then these vectors $\mathbf{a}_1,\mathbf{a}_2,\cdots,\mathbf{a}_n$ are said to be *linearly dependent*. Otherwise, $\mathbf{a}_1,\mathbf{a}_2,\cdots,\mathbf{a}_n$ are said to be *linearly independent*.

Notice that there are two cases for linearly independent vectors $\mathbf{a}_1,\mathbf{a}_2,\cdots,\mathbf{a}_n$ in a multi-vector space:

(i) for scalars $\alpha_1,\alpha_2,\cdots,\alpha_n\in\widetilde{F}$, if

$$\alpha_1\cdot_1\mathbf{a}_1\dot{+}_1\alpha_2\cdot_2\mathbf{a}_2\dot{+}_2\cdots\dot{+}_{n-1}\alpha_n\cdot_n\mathbf{a}_n=\mathbf{0},$$

where $\mathbf{0}$ is a unit of $\widetilde{V}$ under an operation $+$ in $O(\widetilde{V})$, then $\alpha_1=0_{+_1},\alpha_2=0_{+_2},\cdots,\alpha_n=0_{+_n}$, where 0_{+_i} is the unit under the operation $+_i$in $\widetilde{F}$ for integer $i, 1\leq i\leq n$.

(ii) the operating result of $\alpha_1\cdot_1\mathbf{a}_1\dot{+}_1\alpha_2\cdot_2\mathbf{a}_2\dot{+}_2\cdots\dot{+}_{n-1}\alpha_n\cdot_n\mathbf{a}_n$ does not exist.

Now for a subset $\widehat{S}\subset\widetilde{V}$, define its *linearly spanning set* $\left\langle\widehat{S}\right\rangle$ to be

$$\left\langle\widehat{S}\right\rangle=\{\ \mathbf{a}\mid\mathbf{a}=\alpha_1\cdot_1\mathbf{a}_1\dot{+}_1\alpha_2\cdot_2\mathbf{a}_2\dot{+}_2\cdots\in\widetilde{V},\mathbf{a}_i\in\widehat{S},\alpha_i\in\widetilde{F},i\geq 1\}.$$

For a multi-vector space $(\widetilde{V};\widetilde{F})$, if there exists a subset $\widehat{S},\widehat{S}\subset\widetilde{V}$ such that $\widetilde{V}=\left\langle\widehat{S}\right\rangle$, then we say $\widehat{S}$ is a *linearly spanning set* of the multi-vector space $\widetilde{V}$. If these vectors in a linearly spanning set $\widehat{S}$ of the multi-vector space $\widetilde{V}$ are linearly independent, then $\widehat{S}$ is said to be a *basis* of $\widetilde{V}$.

Theorem 1.3.14 Any multi-vector space $(\widetilde{V};\widetilde{F})$ has a basis.

Proof Assume $\widetilde{V}=\bigcup\limits_{i=1}^{k}V_i,\widetilde{F}=\bigcup\limits_{i=1}^{k}F_i$ and the basis of the vector space $(V_i;\dot{+}_i,\cdot_i)$ is $\Delta_i=\{\mathbf{a}_{i1},\mathbf{a}_{i2},\cdots,\mathbf{a}_{in_i}\}$, $1\leq i\leq k$. Define

$$\widehat{\Delta}\ =\ \bigcup_{i=1}^{k}\Delta_i.$$

Then $\widehat{\Delta}$ is a linearly spanning set for $\widetilde{V}$ by definition.

If these vectors in $\widehat{\Delta}$ are linearly independent, then $\widehat{\Delta}$ is a basis of $\widetilde{V}$. Otherwise, choose a vector $\mathbf{b}_1 \in \widehat{\Delta}$ and define $\widehat{\Delta}_1 = \widehat{\Delta} \setminus \{\mathbf{b}_1\}$.

If we have obtained a set $\widehat{\Delta}_s$, $s \geq 1$ and it is not a basis, choose a vector $\mathbf{b}_{s+1} \in \widehat{\Delta}_s$ and define $\widehat{\Delta}_{s+1} = \widehat{\Delta}_s \setminus \{\mathbf{b}_{s+1}\}$.

If these vectors in $\widehat{\Delta}_{s+1}$ are linearly independent, then $\widehat{\Delta}_{s+1}$ is a basis of $\widetilde{V}$. Otherwise, we can define a set $\widehat{\Delta}_{s+2}$ again. Continue this process. Notice that all vectors in Δ_i are linearly independent for any integer $i, 1 \leq i \leq k$. Therefore, we can finally get a basis of $\widetilde{V}$. ♮

Now we consider finite-dimensional multi-vector spaces. A multi-vector space $\widetilde{V}$ is *finite-dimensional* if it has a finite basis. By Theorem 1.2.14, if the vector space $(V_i; +_i, \cdot_i)$ is finite-dimensional for any integer $i, 1 \leq i \leq k$, then $(\widetilde{V}; \widetilde{F})$ is finite-dimensional. On the other hand, if there is an integer $i_0, 1 \leq i_0 \leq k$ such that the vector space $(V_{i_0}; +_{i_0}, \cdot_{i_0})$ is infinite-dimensional, then $(\widetilde{V}; \widetilde{F})$ is also infinite-dimensional. This enables us to get a consequence in the following.

Corollary 1.3.8 *Let $(\widetilde{V}; \widetilde{F})$ be a multi-vector space with $\widetilde{V} = \bigcup\limits_{i=1}^{k} V_i, \widetilde{F} = \bigcup\limits_{i=1}^{k} F_i$. Then $(\widetilde{V}; \widetilde{F})$ is finite-dimensional if and only if $(V_i; +_i, \cdot_i)$ is finite-dimensional for any integer $i, 1 \leq i \leq k$.*

Theorem 1.3.15 *For a finite-dimensional multi-vector space $(\widetilde{V}; \widetilde{F})$, any two bases have the same number of vectors.*

Proof Let $\widetilde{V} = \bigcup\limits_{i=1}^{k} V_i$ and $\widetilde{F} = \bigcup\limits_{i=1}^{k} F_i$. The proof is by the induction on k. For $k = 1$, the assertion is true by Theorem 4 of Chapter 2 in [33].

For the case of $k = 2$, notice that by a result in linearly vector spaces (see also [33]), for two subspaces W_1, W_2 of a finite-dimensional vector space, if the basis of $W_1 \bigcap W_2$ is $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_t\}$, then the basis of $W_1 \bigcup W_2$ is

$$\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_t, \mathbf{b}_{t+1}, \mathbf{b}_{t+2}, \cdots, \mathbf{b}_{dimW_1}, \mathbf{c}_{t+1}, \mathbf{c}_{t+2}, \cdots, \mathbf{c}_{dimW_2}\},$$

where, $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_t, \mathbf{b}_{t+1}, \mathbf{b}_{t+2}, \cdots, \mathbf{b}_{dimW_1}\}$ is a basis of W_1 and $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_t, \mathbf{c}_{t+1}, \mathbf{c}_{t+2}, \cdots, \mathbf{c}_{dimW_2}\}$ a basis of W_2.

Whence, if $\widetilde{V} = W_1 \bigcup W_2$ and $\widetilde{F} = F_1 \bigcup F_2$, then the basis of $\widetilde{V}$ is also

$$\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_t, \mathbf{b}_{t+1}, \mathbf{b}_{t+2}, \cdots, \mathbf{b}_{dimW_1}, \mathbf{c}_{t+1}, \mathbf{c}_{t+2}, \cdots, \mathbf{c}_{dimW_2}\}.$$

Assume the assertion is true for $k = l, l \geq 2$. Now we consider the case of $k = l + 1$. In this case, since

$$\widetilde{V} = (\bigcup_{i=1}^{l} V_i) \bigcup V_{l+1}, \ \ \widetilde{F} = (\bigcup_{i=1}^{l} F_i) \bigcup F_{l+1},$$

by the induction assumption, we know that any two bases of the multi-vector space

$(\bigcup\limits_{i=1}^{l} V_i; \bigcup\limits_{i=1}^{l} F_i)$ have the same number p of vectors. If the basis of $(\bigcup\limits_{i=1}^{l} V_i)\bigcap V_{l+1}$ is $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$, then the basis of $\widetilde{V}$ is

$$\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n, \mathbf{f}_{n+1}, \mathbf{f}_{n+2}, \cdots, \mathbf{f}_p, \mathbf{g}_{n+1}, \mathbf{g}_{n+2}, \cdots, \mathbf{g}_{dimV_{l+1}}\},$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n, \mathbf{f}_{n+1}, \mathbf{f}_{n+2}, \cdots, \mathbf{f}_p\}$ is a basis of $(\bigcup\limits_{i=1}^{l} V_i; \bigcup\limits_{i=1}^{l} F_i)$ and $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n, \mathbf{g}_{n+1}, \mathbf{g}_{n+2}, \cdots, \mathbf{g}_{dimV_{l+1}}\}$ is a basis of V_{l+1}. Whence, the number of vectors in a basis of $\widetilde{V}$ is $p + dimV_{l+1} - n$ for the case $n = l + 1$.

Therefore, we know the assertion is true for any integer k by the induction principle. ♮

The cardinal number of a basis of a finite dimensional multi-vector space $\widetilde{V}$ is called its *dimension*, denoted by $dim\widetilde{V}$.

Theorem 1.3.16(*dimensional formula*) *For a multi-vector space $(\widetilde{V}; \widetilde{F})$ with $\widetilde{V} = \bigcup\limits_{i=1}^{k} V_i$ and $\widetilde{F} = \bigcup\limits_{i=1}^{k} F_i$, the dimension $dim\widetilde{V}$ of $\widetilde{V}$ is*

$$dim\widetilde{V} = \sum_{i=1}^{k}(-1)^{i-1} \sum_{\{i1,i2,\cdots,ii\}\subset\{1,2,\cdots,k\}} dim(V_{i1}\bigcap V_{i2}\bigcap\cdots\bigcap V_{ii}).$$

Proof The proof is by the induction on k. For $k = 1$, the formula is turn to a trivial case of $dim\widetilde{V} = dimV_1$. for $k = 2$, the formula is

$$dim\widetilde{V} = dimV_1 + dimV_2 - dim(V_1 \bigcap dimV_2),$$

which is true by the proof of Theorem 1.3.15.

Now we assume the formula is true for $k = n$. Consider the case of $k = n + 1$. According to the proof of Theorem 1.3.15, we know that

$$\begin{aligned}
dim\widetilde{V} &= dim(\bigcup_{i=1}^{n} V_i) + dimV_{n+1} - dim((\bigcup_{i=1}^{n} V_i)\bigcap V_{n+1}) \\
&= dim(\bigcup_{i=1}^{n} V_i) + dimV_{n+1} - dim(\bigcup_{i=1}^{n}(V_i\bigcap V_{n+1})) \\
&= dimV_{n+1} + \sum_{i=1}^{n}(-1)^{i-1} \sum_{\{i1,i2,\cdots,ii\}\subset\{1,2,\cdots,n\}} dim(V_{i1}\bigcap V_{i2}\bigcap\cdots\bigcap V_{ii}) \\
&+ \sum_{i=1}^{n}(-1)^{i-1} \sum_{\{i1,i2,\cdots,ii\}\subset\{1,2,\cdots,n\}} dim(V_{i1}\bigcap V_{i2}\bigcap\cdots\bigcap V_{ii}\bigcap V_{n+1}) \\
&= \sum_{i=1}^{n}(-1)^{i-1} \sum_{\{i1,i2,\cdots,ii\}\subset\{1,2,\cdots,k\}} dim(V_{i1}\bigcap V_{i2}\bigcap\cdots\bigcap V_{ii}).
\end{aligned}$$

By the induction principle, we know the formula is true for any integer k. ♮

As a consequence, we get the following formula.

Corollary 1.3.9(*additive formula*) *For any two multi-vector spaces* $\widetilde{V}_1, \widetilde{V}_2$,

$$dim(\widetilde{V}_1 \bigcup \widetilde{V}_2) = dim\widetilde{V}_1 + dim\widetilde{V}_2 - dim(\widetilde{V}_1 \bigcap \widetilde{V}_2).$$

§1.4 **Multi-Metric Spaces**

1.4.1. **Metric spaces**

A set M associated with a metric function $\rho : M \times M \rightarrow R^+ = \{x \mid x \in R, x \geq 0\}$ is called a *metric space* if for $\forall x, y, z \in M$, the following conditions for ρ hold:

(1)(*definiteness*) $\rho(x, y) = 0$ if and only if $x = y$;
(*ii*)(*symmetry*) $\rho(x, y) = \rho(y, x)$;
(*iii*)(*triangle inequality*) $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$.

A metric space M with a metric function ρ is usually denoted by $(M\ ; \rho)$. Any $x, x \in M$ is called a point of $(M\ ; \rho)$. A sequence $\{x_n\}$ is said to be *convergent to* x if for any number $\epsilon > 0$ there is an integer N such that $n \geq N$ implies $\rho(x_n, x) < 0$, denoted by $\lim_n x_n = x$. We have known the following result in metric spaces.

Theorem 1.4.1 *Any sequence* $\{x_n\}$ *in a metric space has at most one limit point.*

For $x_0 \in M$ and $\epsilon > 0$, a ϵ-disk about x_0 is defined by

$$B(x_0, \epsilon) = \{\ x \mid x \in M, \rho(x, x_0) < \epsilon\}.$$

If $A \subset M$ and there is an ϵ-disk $B(x_0, \epsilon) \supset A$, we say A is a bounded point set of M.

Theorem 1.4.2 *Any convergent sequence* $\{x_n\}$ *in a metric space is a bounded point set.*

Now let (M, ρ) be a metric space and $\{x_n\}$ a sequence in M. If for any number $\epsilon > 0, \epsilon \in \mathbf{R}$, there is an integer N such that $n, m \geq N$ implies $\rho(x_n, x_m) < \epsilon$, we call $\{x_n\}$ a *Cauchy sequence.* A metric space (M, ρ) is called to be *completed* if its every Cauchy sequence converges.

Theorem 1.4.3 *For a completed metric space* (M, ρ), *if an* ϵ*-disk sequence* $\{B_n\}$ *satisfies*

(*i*) $B_1 \supset B_2 \supset \cdots \supset B_n \supset \cdots$;
(*ii*) $\lim_n \epsilon_n = 0$,

where $\epsilon_n > 0$ *and* $B_n = \{\ x \mid x \in M, \rho(x, x_n) \leq \epsilon_n\}$ *for any integer* $n, n = 1, 2, \cdots$, *then* $\bigcap_{n=1}^{\infty} B_n$ *only has one point.*

For a metric space (M, ρ) and $T : M \to M$ a mapping on (M, ρ), if there exists a point $x^* \in M$ such that

$$Tx^* = x^*,$$

then x^* is called a *fixed point* of T. If there exists a constant $\eta, 0 < \eta < 1$ such that

$$\rho(Tx, Ty) \leq \eta\rho(x, y)$$

for $\forall x, y \in M$, then T is called a *contraction*.

Theorem 1.4.4 (Banach) *Let (M, ρ) be a completed metric space and let $T : M \to M$ be a contraction. Then T has only one fixed point.*

1.4.2. **Multi-Metric spaces**

Definition 1.4.1 *A multi-metric space is a union $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ such that each M_i is a space with a metric ρ_i for $\forall i, 1 \leq i \leq m$.*

When we say a *multi-metric space* $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$, it means that a multi-metric space with metrics $\rho_1, \rho_2, \cdots, \rho_m$ such that (M_i, ρ_i) is a metric space for any integer $i, 1 \leq i \leq m$. For a multi-metric space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$, $x \in \widetilde{M}$ and a positive number R, a *R-disk* $B(x, R)$ in $\widetilde{M}$ is defined by

$$B(x, R) = \{\ y \mid \text{there exists an integer } k, 1 \leq k \leq m \text{ such that } \rho_k(y, x)\ < R, y \in \widetilde{M}\}$$

Remark 1.4.1 The following two extremal cases are permitted in Definition 1.4.1:

(i) there are integers $i_1, i_2, \cdots, i_s$ such that $M_{i_1} = M_{i_2} = \cdots = M_{i_s}$, where $i_j \in \{1, 2, \cdots, m\}$, $1 \leq j \leq s$;

(ii) there are integers $l_1, l_2, \cdots, l_s$ such that $\rho_{l_1} = \rho_{l_2} = \cdots = \rho_{l_s}$, where $l_j \in \{1, 2, \cdots, m\}$, $1 \leq j \leq s$.

For metrics on a space, we have the following result.

Theorem 1.4.5 *Let $\rho_1, \rho_2, \cdots, \rho_m$ be m metrics on a space M and let F be a function on $\mathbf{R}^m$ such that the following conditions hold:*

(i) $F(x_1, x_2, \cdots, x_m) \geq F(y_1, y_2, \cdots, y_m)$ for $\forall i, 1 \leq i \leq m$, $x_i \geq y_i$;

(ii) $F(x_1, x_2, \cdots, x_m) = 0$ only if $x_1 = x_2 = \cdots = x_m = 0$;

(iii) for two m-tuples $(x_1, x_2, \cdots, x_m)$ and $(y_1, y_2, \cdots, y_m)$,

$$F(x_1, x_2, \cdots, x_m) + F(y_1, y_2, \cdots, y_m) \geq F(x_1 + y_1, x_2 + y_2, \cdots, x_m + y_m).$$

Then $F(\rho_1, \rho_2, \cdots, \rho_m)$ is also a metric on M.

Proof We only need to prove that $F(\rho_1, \rho_2, \cdots, \rho_m)$ satisfies those of metric conditions for $\forall x, y, z \in M$.

By (ii), $F(\rho_1(x,y), \rho_2(x,y), \cdots, \rho_m(x,y)) = 0$ only if $\rho_i(x,y) = 0$ for any integer i. Since ρ_i is a metric on M, we know that $x = y$.

For any integer $i, 1 \le i \le m$, since ρ_i is a metric on M, we know that $\rho_i(x,y) = \rho_i(y,x)$. Whence,

$$F(\rho_1(x,y), \rho_2(x,y), \cdots, \rho_m(x,y)) = F(\rho_1(y,x), \rho_2(y,x), \cdots, \rho_m(y,x)).$$

Now by (i) and (iii), we get that

$$\begin{aligned}
&F(\rho_1(x,y), \rho_2(x,y), \cdots, \rho_m(x,y)) + F(\rho_1(y,z), \rho_2(y,z), \cdots, \rho_m(y,z)) \\
&\ge F(\rho_1(x,y) + \rho_1(y,z), \rho_2(x,y) + \rho_2(y,z), \cdots, \rho_m(x,y) + \rho_m(y,z)) \\
&\ge F(\rho_1(x,z), \rho_2(x,z), \cdots, \rho_m(x,z)).
\end{aligned}$$

Therefore, $F(\rho_1, \rho_2, \cdots, \rho_m)$ is a metric on M. ♮

Corollary 1.4.1 If $\rho_1, \rho_2, \cdots, \rho_m$ are m metrics on a space M, then $\rho_1 + \rho_2 + \cdots + \rho_m$ and $\frac{\rho_1}{1+\rho_1} + \frac{\rho_2}{1+\rho_2} + \cdots + \frac{\rho_m}{1+\rho_m}$ are also metrics on M.

A sequence $\{x_n\}$ in a multi-metric space $\widetilde{M} = \bigcup_{i=1}^{m} M_i$ is said to be *convergent to a point* $x, x \in \widetilde{M}$ if for any number $\epsilon > 0$, there exist numbers N and $i, 1 \le i \le m$ such that

$$\rho_i(x_n, x) \ < \ \epsilon$$

provided $n \ge N$. If $\{x_n\}$ is convergent to a point $x, x \in \widetilde{M}$, we denote it by $\lim_n x_n = x$.

We get a characteristic for convergent sequences in a multi-metric space as in the following.

Theorem 1.4.6 *A sequence $\{x_n\}$ in a multi-metric space $\widetilde{M} = \bigcup_{i=1}^{m} M_i$ is convergent if and only if there exist integers N and $k, 1 \le k \le m$ such that the subsequence $\{x_n | n \ge N\}$ is a convergent sequence in (M_k, ρ_k).*

Proof If there exist integers N and $k, 1 \le k \le m$ such that $\{x_n | n \ge N\}$ is a convergent sequence in (M_k, ρ_k), then for any number $\epsilon > 0$, by definition there exist an integer P and a point $x, x \in M_k$ such that

$$\rho_k(x_n, x) \ < \ \epsilon$$

if $n \ge max\{N, \ P\}$.

Now if $\{x_n\}$ is a convergent sequence in the multi-space $\widetilde{M}$, by definition for any positive number $\epsilon > 0$, there exist a point $x, x \in \widetilde{M}$, natural numbers $N(\epsilon)$ and integer $k, 1 \leq k \leq m$ such that if $n \geq N(\epsilon)$, then

$$\rho_k(x_n, x) \ < \ \epsilon.$$

That is, $\{x_n | n \geq N(\epsilon)\} \subset M_k$ and $\{x_n | n \geq N(\epsilon)\}$ is a convergent sequence in (M_k, ρ_k). $\natural$

Theorem 1.4.7 *Let $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ be a multi-metric space. For two sequences $\{x_n\}$, $\{y_n\}$ in $\widetilde{M}$, if $\lim\limits_{n} x_n = x_0$, $\lim\limits_{n} y_n = y_0$ and there is an integer p such that $x_0, y_0 \in M_p$, then $\lim\limits_{n} \rho_p(x_n, y_n) = \rho_p(x_0, y_0)$.*

Proof According to Theorem 1.4.6, there exist integers N_1 and N_2 such that if $n \geq max\{N_1, N_2\}$, then $x_n, y_n \in M_p$. Whence, we know that

$$\rho_p(x_n, y_n) \leq \rho_p(x_n, x_0) + \rho_p(x_0, y_0) + \rho_p(y_n, y_0)$$

and

$$\rho_p(x_0, y_0) \leq \rho_p(x_n, x_0) + \rho_p(x_n, y_n) + \rho_p(y_n, y_0).$$

Therefore,

$$|\rho_p(x_n, y_n) - \rho_p(x_0, y_0)| \ \leq \ \rho_p(x_n, x_0) + \rho_p(y_n, y_0).$$

Now for any number $\epsilon > 0$, since $\lim\limits_{n} x_n = x_0$ and $\lim\limits_{n} y_n = y_0$, there exist numbers $N_1(\epsilon), N_1(\epsilon) \geq N_1$ and $N_2(\epsilon), N_2(\epsilon) \geq N_2$ such that $\rho_p(x_n, x_0) \ \leq \ \frac{\epsilon}{2}$ if $n \geq N_1(\epsilon)$ and $\rho_p(y_n, y_0) \ \leq \ \frac{\epsilon}{2}$ if $n \geq N_2(\epsilon)$. Whence, if we choose $n \geq max\{N_1(\epsilon), N_2(\epsilon)\}$, then

$$|\rho_p(x_n, y_n) - \rho_p(x_0, y_0)| \ < \ \epsilon. \qquad \natural$$

Whether can a convergent sequence have more than one limiting points? The following result answers this question.

Theorem 1.4.8 *If $\{x_n\}$ is a convergent sequence in a multi-metric space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$, then $\{x_n\}$ has only one limit point.*

Proof According to Theorem 1.4.6, there exist integers N and $i, 1 \leq i \leq m$ such that $x_n \in M_i$ if $n \geq N$. Now if

$$\lim_{n} x_n = x_1 \text{ and } \lim_{n} x_n = x_2,$$

and $n \geq N$, by definition,

$$0 \le \rho_i(x_1, x_2) \le \rho_i(x_n, x_1) + \rho_i(x_n, x_2).$$

Whence, we get that $\rho_i(x_1, x_2) = 0$. Therefore, $x_1 = x_2$. ♮

Theorem 1.4.9 *Any convergent sequence in a multi-metric space is a bounded points set.*

Proof According to Theorem 1.4.8, we obtain this result immediately. ♮

A sequence $\{x_n\}$ in a multi-metric space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ is called a *Cauchy sequence* if for any number $\epsilon > 0$, there exist integers $N(\epsilon)$ and $s, 1 \le s \le m$ such that for any integers $m, n \ge N(\epsilon)$, $\rho_s(x_m, x_n) < \epsilon$.

Theorem 1.4.10 *A Cauchy sequence $\{x_n\}$ in a multi-metric space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ is convergent if and only if $|\{x_n\} \bigcap M_k|$ is finite or infinite but $\{x_n\} \bigcap M_k$ is convergent in (M_k, ρ_k) for $\forall k, 1 \le k \le m$.*

Proof The necessity of these conditions in this theorem is known by Theorem 1.4.6.

Now we prove the sufficiency. By definition, there exist integers $s, 1 \le s \le m$ and N_1 such that $x_n \in M_s$ if $n \ge N_1$. Whence, if $|\{x_n\} \bigcap M_k|$ is infinite and $\lim\limits_{n} \{x_n\} \bigcap M_k = x$, then there must be $k = s$. Denote by $\{x_n\} \bigcap M_k = \{x_{k1}, x_{k2}, \cdots, x_{kn}, \cdots\}$.

For any positive number $\epsilon > 0$, there exists an integer $N_2, N_2 \ge N_1$ such that $\rho_k(x_m, x_n) < \frac{\epsilon}{2}$ and $\rho_k(x_{kn}, x) < \frac{\epsilon}{2}$ if $m, n \ge N_2$. According to Theorem 1.4.7, we get that

$$\rho_k(x_n, x) \le \rho_k(x_n, x_{kn}) + \rho_k(x_{kn}, x) < \epsilon$$

if $n \ge N_2$. Whence, $\lim\limits_{n} x_n = x$. ♮

A multi-metric space $\widetilde{M}$ is said to be *completed* if its every Cauchy sequence is convergent. For a completed multi-metric space, we obtain two important results similar to Theorems 1.4.3 and 1.4.4 in metric spaces.

Theorem 1.4.11 *Let $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ be a completed multi-metric space. For an ϵ-disk sequence $\{B(\epsilon_n, x_n)\}$, where $\epsilon_n > 0$ for $n = 1, 2, 3, \cdots$, if the following conditions hold:*

(i) $B(\epsilon_1, x_1) \supset B(\epsilon_2, x_2) \supset B(\epsilon_3, x_3) \supset \cdots \supset B(\epsilon_n, x_n) \supset \cdots$;

(ii) $\lim\limits_{n \to +\infty} \epsilon_n = 0$,

then $\bigcap\limits_{n=1}^{+\infty} B(\epsilon_n, x_n)$ only has one point.

Proof First, we prove that the sequence $\{x_n\}$ is a Cauchy sequence in $\widetilde{M}$. By the condition (i), we know that if $m \geq n$, then $x_m \in B(\epsilon_m, x_m) \subset B(\epsilon_n, x_n)$. Whence $\rho_i(x_m, x_n) < \epsilon_n$ provided $x_m, x_n \in M_i$ for $\forall i, 1 \leq i \leq m$.

Now for any positive number ϵ, since $\lim\limits_{n \to +\infty} \epsilon_n = 0$, there exists an integer $N(\epsilon)$ such that if $n \geq N(\epsilon)$, then $\epsilon_n < \epsilon$. Therefore, if $x_n \in M_l$, then $\lim\limits_{m \to +\infty} x_m = x_n$. Thereby there exists an integer N such that if $m \geq N$, then $x_m \in M_l$ by Theorem 1.4.6. Choice integers $m, n \geq max\{N, N(\epsilon)\}$, we know that

$$\rho_l(x_m, x_n) \ < \ \epsilon_n \ < \ \epsilon.$$

So $\{x_n\}$ is a Cauchy sequence.

By the assumption that $\widetilde{M}$ is completed, we know that the sequence $\{x_n\}$ is convergent to a point $x_0, x_0 \in \widetilde{M}$. By conditions of (i) and (ii), we get that $\rho_l(x_0, x_n) < \epsilon_n$ if $m \to +\infty$. Whence, $x_0 \in \bigcap\limits_{n=1}^{+\infty} B(\epsilon_n, x_n)$.

Now if there is a point $y \in \bigcap\limits_{n=1}^{+\infty} B(\epsilon_n, x_n)$, then there must be $y \in M_l$. We get that

$$0 \leq \rho_l(y, x_0) = \lim_n \rho_l(y, x_n) \leq \lim_{n \to +\infty} \epsilon_n = 0$$

by Theorem 1.4.7. Therefore, $\rho_l(y, x_0) = 0$. By the definition of a metric function, we get that $y = x_0$. $\natural$

Let $\widetilde{M}_1$ and $\widetilde{M}_2$ be two multi-metric spaces and let $f : \widetilde{M}_1 \to \widetilde{M}_2$ be a mapping, $x_0 \in \widetilde{M}_1, f(x_0) = y_0$. For $\forall \epsilon > 0$, if there exists a number δ such that $f(x) = y \in B(\epsilon, y_0) \subset \widetilde{M}_2$ for $\forall x \in B(\delta, x_0)$, i.e.,

$$f(B(\delta, x_0)) \subset B(\epsilon, y_0),$$

then we say that f is *continuous at point* x_0. A mapping $f : \widetilde{M}_1 \to \widetilde{M}_2$ is called a *continuous mapping* from $\widetilde{M}_1$ to $\widetilde{M}_2$ if f is continuous at every point of $\widetilde{M}_1$.

For a continuous mapping f from $\widetilde{M}_1$ to $\widetilde{M}_2$ and a convergent sequence $\{x_n\}$ in $\widetilde{M}_1$, $\lim\limits_n x_n = x_0$, we can prove that

$$\lim_n f(x_n) = f(x_0).$$

For a multi-metric space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ and a mapping $T : \widetilde{M} \to \widetilde{M}$, if there is a point $x^* \in \widetilde{M}$ such that $Tx^* = x^*$, then x^* is called a *fixed point* of T. Denote the number of fixed points of a mapping T in $\widetilde{M}$ by $^{\#}\Phi(T)$. A mapping T is called a *contraction* on a multi-metric space $\widetilde{M}$ if there are a constant $\alpha, 0 < \alpha < 1$ and integers $i, j, 1 \leq i, j \leq m$ such that for $\forall x, y \in M_i$, $Tx, Ty \in M_j$ and

$$\rho_j(Tx, Ty) \leq \alpha \rho_i(x, y).$$

Theorem 1.4.12 *Let* $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ *be a completed multi-metric space and let* T *be a contraction on* $\widetilde{M}$*. Then*

$$1 \leq^{\#} \Phi(T) \leq m.$$

Proof Choose arbitrary points $x_0, y_0 \in M_1$ and define recursively

$$x_{n+1} = Tx_n, \quad y_{n+1} = Tx_n$$

for $n = 1, 2, 3, \cdots$. By definition, we know that for any integer $n, n \geq 1$, there exists an integer $i, 1 \leq i \leq m$ such that $x_n, y_n \in M_i$. Whence, we inductively get that

$$0 \leq \rho_i(x_n, y_n) \leq \alpha^n \rho_1(x_0, y_0).$$

Notice that $0 < \alpha < 1$, we know that $\lim\limits_{n\to+\infty} \alpha^n = 0$. Thereby there exists an integer i_0 such that

$$\rho_{i_0}(\lim_n x_n, \lim_n y_n) = 0.$$

Therefore, there exists an integer N_1 such that $x_n, y_n \in M_{i_0}$ if $n \geq N_1$. Now if $n \geq N_1$, we get that

$$\begin{aligned}
\rho_{i_0}(x_{n+1}, x_n) &= \rho_{i_0}(Tx_n, Tx_{n-1}) \\
&\leq \alpha\rho_{i_0}(x_n, x_{n-1}) = \alpha\rho_{i_0}(Tx_{n-1}, Tx_{n-2}) \\
&\leq \alpha^2\rho_{i_0}(x_{n-1}, x_{n-2}) \leq \cdots \leq \alpha^{n-N_1}\rho_{i_0}(x_{N_1+1}, x_{N_1}).
\end{aligned}$$

and generally, for $m \geq n \geq N_1$,

$$\begin{aligned}
\rho_{i_0}(x_m, x_n) &\leq \rho_{i_0}(x_n, x_{n+1}) + \rho_{i_0}(x_{n+1}, x_{n+2}) + \cdots + \rho_{i_0}(x_{n-1}, x_n) \\
&\leq (\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n)\rho_{i_0}(x_{N_1+1}, x_{N_1}) \\
&\leq \frac{\alpha^n}{1-\alpha}\rho_{i_0}(x_{N_1+1}, x_{N_1}) \to 0 (m, n \to +\infty).
\end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence in $\widetilde{M}$. Similarly, we can also prove $\{y_n\}$ is a Cauchy sequence.

Because $\widetilde{M}$ is a completed multi-metric space, we know that

$$\lim_n x_n = \lim_n y_n = z^*.$$

Now we prove z^* is a fixed point of T in $\widetilde{M}$. In fact, by $\rho_{i_0}(\lim\limits_n x_n, \lim\limits_n y_n) = 0$, there exists an integer N such that

$$x_n, y_n, Tx_n, Ty_n \in M_{i_0}$$

if $n \geq N+1$. Whence, we know that

$$\begin{aligned} 0 \leq \rho_{i_0}(z^*, Tz^*) &\leq \rho_{i_0}(z^*, x_n) + \rho_{i_0}(y_n, Tz^*) + \rho_{i_0}(x_n, y_n) \\ &\leq \rho_{i_0}(z^*, x_n) + \alpha\rho_{i_0}(y_{n-1}, z^*) + \rho_{i_0}(x_n, y_n). \end{aligned}$$

Notice that

$$\lim_{n\to+\infty} \rho_{i_0}(z^*, x_n) = \lim_{n\to+\infty} \rho_{i_0}(y_{n-1}, z^*) = \lim_{n\to+\infty} \rho_{i_0}(x_n, y_n) = 0.$$

We get $\rho_{i_0}(z^*, Tz^*) = 0$, i.e., $Tz^* = z^*$.

For other chosen points $u_0, v_0 \in M_1$, we can also define recursively

$$u_{n+1} = Tu_n, \quad v_{n+1} = Tv_n$$

and get a limiting point $\lim_n u_n = \lim_n v_n = u^* \in M_{i_0}, Tu^* \in M_{i_0}$. Since

$$\rho_{i_0}(z^*, u^*) = \rho_{i_0}(Tz^*, Tu^*) \leq \alpha\rho_{i_0}(z^*, u^*)$$

and $0 < \alpha < 1$, there must be $z^* = u^*$.

Similarly consider the points in $M_i, 2 \leq i \leq m$, we get that

$$1 \leq^{\#} \Phi(T) \leq m. \qquad \natural$$

As a consequence, we get the *Banach theorem* in metric spaces.

Corollary 1.4.2(Banach) *Let M be a metric space and let T be a contraction on M. Then T has just one fixed point.*

§1.5 **Remarks and Open Problems**

The central idea of Smarandache multi-spaces is to combine different fields (spaces, systems, objects, $\cdots$) into a unifying field and find its behaviors. Which is entirely new, also an application of combinatorial approaches to classical mathematics but more important than combinatorics itself. This idea arouses us to think why an assertion is true or not in classical mathematics. Then combine an assertion with its non-assertion and enlarge the filed of truths. A famous fable says that *each theorem in mathematics is an absolute truth*. But we do not think so. Our thinking is that *each theorem in mathematics is just a relative truth*. Thereby we can establish new theorems and present new problems boundless in mathematics. Results obtained in Section 1.3 and 1.4 are applications of this idea to these groups, rings, vector spaces or metric spaces. Certainly, more and more multi-spaces and their good behaviors can be found under this thinking. Here we present some remarks and open problems for multi-spaces.

1.5.1. **Algebraic Multi-Spaces** The algebraic multi-spaces are discrete representations for phenomena in the natural world. They maybe completed or not in cases. For a completed algebraic multi-space, it is a reflection of an equilibrium phenomenon. Otherwise, a reflection of a non-equilibrium phenomenon. Whence, more consideration should be done for algebraic multi-spaces, especially, by an analogous thinking as in classical algebra.

Problem 1.5.1 *Establish a decomposition theory for multi-groups.*

In group theory, we know the following decomposition result([107][82]) for groups.

Let G be a finite Ω-group. Then G can be uniquely decomposed as a direct product of finite non-decomposition Ω-subgroups.

Each finite abelian group is a direct product of its Sylow p-subgroups.

Then Problem 1.5.1 can be restated as follows.

Whether can we establish a decomposition theory for multi-groups similar to the above two results in group theory, especially, for finite multi-groups?

Problem 1.5.2 *Define the conception of simple multi-groups. For finite multi-groups, whether can we find all simple multi-groups?*

For finite groups, we know that there are four simple group classes ([108]):

Class 1: the cyclic groups of prime order;

Class 2: the alternating groups $A_n, n \geq 5$;

Class 3: the 16 groups of Lie types;

Class 4: the 26 sporadic simple groups.

Problem 1.5.3 *Determine the structure properties of multi-groups generated by finite elements.*

For a subset A of a multi-group $\widetilde{G}$, define its spanning set by

$$\langle A\rangle = \{a \circ b | a, b \in A \text{ and } \circ \in O(\widetilde{G})\}.$$

If there exists a subset $A \subset \widetilde{G}$ such that $\widetilde{G} = \langle A\rangle$, then call $\widetilde{G}$ is generated by A. Call $\widetilde{G}$ is *finitely generated* if there exist a finite set A such that $\widetilde{G} = \langle A\rangle$. Then Problem 5.3 can be restated by

Can we establish a finite generated multi-group theory similar to the finite generated group theory?

Problem 1.5.4 *Determine the structure of a Noether multi-ring.*

Let R be a ring. Call R a *Noether ring* if its every ideal chain only has finite terms. Similarly, for a multi-ring $\widetilde{R}$, if its every multi-ideal chain only has finite terms, it is called a *Noether multi-ring. Whether can we find its structures similar to Corollary* 1.3.5 *and Theorem* 1.3.12*?*

Problem 1.5.5 *Similar to ring theory, define a Jacobson or Brown-McCoy radical for multi-rings and determine their contribution to multi-rings.*

Notice that Theorem 1.3.14 has told us there is a similar linear theory for multi-vector spaces, but the situation is more complex.

Problem 1.5.6 *Similar to linear spaces, define linear transformations on multi-vector spaces. Can we establish a matrix theory for these linear transformations?*

Problem 1.5.7 *Whether a multi-vector space must be a linear space?*

Conjecture 1.5.1 *There are non-linear multi-vector spaces in multi-vector spaces.*

Based on Conjecture 1.5.1, there is a fundamental problem for multi-vector spaces.
Problem 1.5.8 *Can we apply multi-vector spaces to non-linear spaces?*

1.5.2. **Multi-Metric Spaces** On a tradition notion, only one metric maybe considered in a space to ensure the same on all the time and on all the situation. Essentially, this notion is based on an assumption that all spaces are homogeneous. In fact, it is not true in general.

Multi-metric spaces can be used to simplify or beautify geometrical figures and algebraic equations. For an explanation, an example is shown in Fig.1.3, in where the left elliptic curve is transformed to the right circle by changing the metric along x, y-axes and an elliptic equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

to equation

$$x^2 + y^2 = r^2$$

of a circle of radius r.

Fig.1.3

Generally, in a multi-metric space, we can simplify a polynomial similar to the approach used in the projective geometry. *Whether this approach can be contributed to mathematics with metrics?*

Problem 1.5.9 *Choose suitable metrics to simplify the equations of surfaces or curves in* $\mathbf{R}^3$.

Problem 1.5.10 *Choose suitable metrics to simplify the knot problem. Whether can it be used for classifying* 3*-dimensional manifolds?*

Problem 1.5.11 *Construct multi-metric spaces or non-linear spaces by Banach spaces. Simplify equations or problems to linear problems.*

1.5.3. **Multi-Operation Systems** By a complete Smarandache multi-space $\widetilde{A}$ with an operation set $O(\widetilde{A})$, we can get a *multi-operation system* $\widetilde{A}$. For example, if $\widetilde{A}$ is a multi-field $\widetilde{F} = \bigcup\limits_{i=1}^{n} F_i$ with an operation set $O(\widetilde{F}) = \{(+_i, \times_i)| \ 1 \leq i \leq n\}$, then $(\widetilde{F}; +_1, +_2, \cdots, +_n)$, $(\widetilde{F}; \times_1, \times_2, \cdots, \times_n)$ and $(\widetilde{F}; (+_1, \times_1), \ (+_2, \times_2), \cdots, (+_n, \times_n))$ are multi-operation systems. On this view, the classical operation system $(R\ ; +)$ and $(R\ ; \times)$ are only *sole operation systems*. For a multi-operation system $\widetilde{A}$, we can define these conceptions of equality and inequality, $\cdots$, etc.. For example, in the multi-operation system $(\widetilde{F}; +_1, +_2, \cdots, +_n)$, we define the equalities $=_1, =_2, \cdots, =_n$ such as those in sole operation systems $(\widetilde{F}; +_1), (\widetilde{F}; +_2), \cdots, (\widetilde{F}; +_n)$, for example, $2 =_1 2, 1.4 =_2 1.4, \cdots, \sqrt{3} =_n \sqrt{3}$ which is the same as the usual meaning and similarly, for the conceptions $\geq_1, \geq_2, \cdots, \geq_n$ and $\leq_1, \leq_2, \cdots, \leq_n$.

In a classical operation system $(R\ ; +)$, the equation system

$$\begin{aligned}
x + 2 + 4 + 6 &= 15 \\
x + 1 + 3 + 6 &= 12 \\
x + 1 + 4 + 7 &= 13
\end{aligned}$$

can not has a solution. But in the multi-operation system $(\widetilde{F}; +_1, +_2, \cdots, +_n)$, the equation system

$$\begin{aligned} x +_1 2 +_1 4 +_1 6 &=_1 & 15 \\ x +_2 1 +_2 3 +_2 6 &=_2 & 12 \\ x +_3 1 +_3 4 +_3 7 &=_3 & 13 \end{aligned}$$

may have a solution x if

$$\begin{aligned} 15 +_1 (-1) +_1 (-4) +_1 (-16) &= 12 +_2 (-1) +_2 (-3) +_2 (-6) \\ &= 13 +_3 (-1) +_3 (-4) +_3 (-7). \end{aligned}$$

in $(\widetilde{F}; +_1, +_2, \cdots, +_n)$. Whence, an element maybe have different disguises in a multi-operation system.

For the multi-operation systems, a number of open problems needs to research further.

Problem 1.5.12 *Find necessary and sufficient conditions for a multi-operation system with more than* 3 *operations to be the rational number field* Q*, the real number field* R *or the complex number field* C*.*

For a multi-operation system $(N\ ; (+_1, \times_1), (+_2, \times_2), \cdots, (+_n, \times_n))$ and integers $a, b, c \in N$, if $a = b \times_i c$ for an integer $i, 1 \leq i \leq n$, then b and c are called *factors* of a. An integer p is called a *prime* if there exist integers n_1, n_2 and $i, 1 \leq i \leq n$ such that $p = n_1 \times_i n_2$, then $p = n_1$ or $p = n_2$. Two problems for primes of a multi-operation system $(N\ ; (+_1, \times_1),\ (+_2, \times_2), \cdots, (+_n, \times_n))$ are presented in the following.

Problem 1.5.13 *For a positive real number* x*, denote by* $\pi_m(x)$ *the number of primes* $\leq x$ *in* $(N\ ; (+_1, \times_1), (+_2, \times_2), \cdots, (+_n, \times_n))$*. Determine or estimate* $\pi_m(x)$*.*

Notice that for the positive integer system, by a well-known theorem, i.e., *Gauss prime theorem*, we have known that([15])

$$\pi(x) \sim \frac{x}{\log x}.$$

Problem 1.5.14 *Find the additive number properties for* $(N\ ; (+_1, \times_1), (+_2, \times_2), \cdots, (+_n, \times_n))$*, for example, we have weakly forms for Goldbach's conjecture and Fermat's problem* ([34]) *as follows.*

Conjecture 1.5.2 *For any even integer* $n, n \geq 4$*, there exist odd primes* p_1, p_2 *and an integer* $i, 1 \leq i \leq n$ *such that* $n = p_1 +_i p_2$*.*

Conjecture 1.5.3 *For any positive integer q, the Diophantine equation* $x^q + y^q = z^q$ *has non-trivial integer solutions* (x, y, z) *at least for an operation* $+_i$ *with* $1 \leq i \leq n$.

A *Smarandache n-structure on a set S* means a weak structure $\{w(0)\}$ on S such that there exists a chain of proper subsets $P(n-1) \subset P(n-2) \subset \cdots \subset P(1) \subset S$ whose corresponding structures verify the inverse chain $\{w(n-1)\} \supset \{w(n-2)\} \supset \cdots \supset \{w(1)\} \supset \{w(0)\}$, i.e., structures satisfying more axioms.

Problem 1.5.15 *For Smarandache multi-structures, solves these Problems* 1.5.1 – 1.5.8.

1.5.4. **Multi-Manifolds** Manifolds are important objects in topology, Riemann geometry and modern mechanics. It can be seen as a local generalization of Euclid spaces. By the Smarandache's notion, we can also define multi-manifolds. To determine their behaviors or structure properties will useful for modern mathematics.

In an Euclid space $\mathbf{R}^n$, an *n-ball of radius r* is defined by

$$B^n(r) = \{(x_1, x_2, \cdots, x_n) | x_1^2 + x_2^2 + \cdots + x_n^2 \leq r\}.$$

Now we choose m n-balls $B_1^n(r_1), B_2^n(r_2), \cdots, B_m^n(r_m)$, where for any integers $i, j, 1 \leq i, j \leq m$, $B_i^n(r_i) \bigcap B_j^n(r_j) =$ or not and $r_i = r_j$ or not. An *n-multi-ball* is a union

$$\widetilde{B} = \bigcup_{k=1}^{m} B_k^n(r_k).$$

Then an *n-multi-manifold* is a Hausdorff space with each point in this space has a neighborhood homeomorphic to an n-multi-ball.

Problem 1.5.16 *For an integer* $n, n \geq 2$, *classifies n-multi-manifolds. Especially, classifies* 2*-multi-manifolds.*

For closed 2-manifolds, i.e., locally orientable surfaces, we have known a classification theorem for them.

Problem 1.5.17 *If we replace the word homeomorphic by points equivalent or isomorphic, what can we obtain for n-multi-manifolds? Can we classify them?*

Similarly, we can also define differential multi-manifolds and consider their contributions to modern differential geometry, Riemann geometry or modern mechanics, $\cdots$, etc..