Smarandache Multi-Space Theory(II)

-Multi-spaces on graphs

Linfan Mao

Academy of Mathematics and System Sciences
Chinese Academy of Sciences, Beijing 100080
maolinfan@163.com

Abstract. A Smarandache multi-space is a union of n different spaces equipped with some different structures for an integer $n \geq 2$, which can be both used for discrete or connected spaces, particularly for geometries and spacetimes in theoretical physics. This monograph concentrates on characterizing various multi-spaces including three parts altogether. The first part is on *algebraic multi-spaces with structures*, such as those of multi-groups, multi-rings, multi-vector spaces, multi-metric spaces, multi-operation systems and multi-manifolds, also multi-voltage graphs, multi-embedding of a graph in an n-manifold,$\cdots$, etc.. The second discusses *Smarandache geometries*, including those of map geometries, planar map geometries and pseudo-plane geometries, in which the *Finsler geometry*, particularly the *Riemann geometry* appears as a special case of these Smarandache geometries. The third part of this book considers the *applications of multi-spaces to theoretical physics*, including the relativity theory, the M-theory and the cosmology. Multi-space models for p-branes and cosmos are constructed and some questions in cosmology are clarified by multi-spaces. The first two parts are relative independence for reading and in each part open problems are included for further research of interested readers.

Key words: graph, multi-voltage graph, Cayley graph of a multi-group, multi-embedding of a graph, map, graph model of a multi-space, graph phase.

Classification: AMS(2000) 03C05,05C15,51D20,51H20,51P05,83C05, 83E50

Contents

2. **Multi-spaces on graphs**

As a useful tool for dealing with relations of events, graph theory has rapidly grown in theoretical results as well as its applications to real-world problems, for example see [9], [11] and [80] for graph theory, [42] – [44] for topological graphs and combinatorial map theory, [7], [12] and [104] for its applications to probability, electrical network and real-life problems. By applying the Smarandache's notion, graphs are models of multi-spaces and matters in the natural world. For the later, graphs are a generalization of p-branes and seems to be useful for mechanics and quantum physics.

§2.1 **Graphs**

2.1.1. **What is a graph?**

A *graph* G is an ordered 3-tuple $(V, E; I)$, where V, E are finite sets, $V \neq \emptyset$ and $I : E \to V \times V$. Call V the *vertex set* and E the *edge set* of G, denoted by $V(G)$ and $E(G)$, respectively. Two elements $v \in V(G)$ and $e \in E(G)$ are said to be *incident* if $I(e) = (v, x)$ or (x, v), where $x \in V(G)$. If $(u, v) = (v, u)$ for $\forall u, v \in V$, the graph G is called a graph, otherwise, a directed graph with an orientation $u \to v$ on each edge (u, v). Unless Section 2.4, graphs considered in this chapter are non-directed.

The cardinal numbers of $|V(G)|$ and $|E(G)|$ are called the *order* and the *size* of a graph G, denoted by $|G|$ and $\varepsilon(G)$, respectively.

We can draw a graph G on a plane $\sum$ by representing each vertex u of G by a point $p(u)$, $p(u) \neq p(v)$ if $u \neq v$ and an edge (u, v) by a plane curve connecting points $p(u)$ and $p(v)$ on $\sum$, where $p : G \to P$ is a mapping from the graph G to P.

For example, a graph $G = (V, E; I)$ with $V = \{v_1, v_2, v_3, v_4\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$ and $I(e_i) = (v_i, v_i), 1 \leq i \leq 4; I(e_5) = (v_1, v_2) = (v_2, v_1), I(e_8) = (v_3, v_4) = (v_4, v_3), I(e_6) = I(e_7) = (v_2, v_3) = (v_3, v_2), I(e_8) = I(e_9) = (v_4, v_1) = (v_1, v_4)$ can be drawn on a plane as shown in Fig.2.1

Fig, 2.1

In a graph $G = (V, E; I)$, for $\forall e \in E$, if $I(e) = (u, u), u \in V$, then e is called a *loop*. For $\forall e_1, e_2 \in E$, if $I(e_1) = I(e_2)$ and they are not loops, then e_1 and e_2 are called *multiple edges* of G. A graph is *simple* if it is loopless and without multiple edges, i.e., $\forall e_1, e_2 \in E(\Gamma)$, $I(e_1) \neq I(e_2)$ if $e_1 \neq e_2$ and for $\forall e \in E$, if $I(e) = (u, v)$, then $u \neq v$. In a simple graph, an edge (u, v) can be abbreviated to uv.

An edge $e \in E(G)$ can be divided into two semi-arcs e_u, e_v if $I(e) = (u, v)$. Call u the *root vertex* of the semi-arc e_u. Two semi-arc e_u, f_v are said to be v-*incident* or $e-$*incident* if $u = v$ or $e = f$. The set of all semi-arcs of a graph G is denoted by $X_{\frac{1}{2}}(G)$.

A *walk* of a graph Γ is an alternating sequence of vertices and edges $u_1, e_1, u_2, e_2, \cdots, e_n, u_{n_1}$ with $e_i = (u_i, u_{i+1})$ for $1 \leq i \leq n$. The number n is the *length of the walk*. If $u_1 = u_{n+1}$, the walk is said to be *closed*, and *open* otherwise. For example, $v_1e_1v_1e_5v_2e_6v_3e_3v_3e_7v_2e_2v_2$ is a walk in Fig.2.1. A walk is called a *trail* if all its edges are distinct and a *path* if all the vertices are distinct. A closed path is said to be a *circuit*.

A graph $G = (V, E; I)$ is *connected* if there is a path connecting any two vertices in this graph. In a graph, a maximal connected subgraph is called a *component*. A graph G is k-*connected* if removing vertices less than k from G still remains a connected graph. Let G be a graph. For $\forall u \in V(G)$, the neighborhood $N_G(u)$ of the vertex u in G is defined by $N_G(u) = \{v|\forall(u, v) \in E(G)\}$. The cardinal number $|N_G(u)|$ is called the *valency of the vertex* u in the graph G and denoted by $\rho_G(u)$. A vertex v with $\rho_G(v) = 0$ is called an *isolated vertex* and $\rho_G(v) = 1$ a *pendent vertex*. Now we arrange all vertices valency of G as a sequence $\rho_G(u) \geq \rho_G(v) \geq \cdots \geq \rho_G(w)$. Call this sequence the *valency sequence* of G. By enumerating edges in $E(G)$, the following result holds.

$$\sum_{u \in V(G)} \rho_G(u) = 2|E(G)|.$$

Give a sequence $\rho_1, \rho_2, \cdots, \rho_p$ of non-negative integers. If there exists a graph whose valency sequence is $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_p$, then we say that $\rho_1, \rho_2, \cdots, \rho_p$ is a *graphical sequence*. We have known the following results (see [11] for details).

Theorem 2.1.1(Havel,1955 and Hakimi,1962) *A sequence $\rho_1, \rho_2, \cdots, \rho_p$ of non-negative integers with $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_p$, $p \geq 2, \rho_1 \geq 1$ is graphical if and only if the sequence $\rho_2 - 1, \rho_3 - 1, \cdots, \rho_{\rho_1+1} - 1, \rho_{\rho_1+2}, \cdots, \rho_p$ is graphical.*

Theorem 2.1.2(Erdös and Gallai,1960) *A sequence $\rho_1, \rho_2, \cdots, \rho_p$ of non-negative integers with $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_p$ is graphical if and only if $\sum\limits_{i=1}^{p} \rho_i$ is even and for each integer $n, 1 \leq n \leq p - 1$,*

$$\sum_{i=1}^{n} \rho_i \leq n(n-1) + \sum_{i=n+1}^{p} min\{n, \rho_i\}.$$

A graph G with a vertex set $V(G) = \{v_1, v_2, \cdots, v_p\}$ and an edge set $E(G) =$

$\{e_1, e_2, \cdots, e_q\}$ can be also described by means of matrix. One such matrix is a $p \times q$ *adjacency matrix* $A(G) = [a_{ij}]_{p \times q}$, where $a_{ij} = |I^{-1}(v_i, v_j)|$. Thus, the adjacency matrix of a graph G is symmetric and is a $0, 1$-matrix having 0 entries on its main diagonal if G is simple. For example, the adjacency matrix $A(G)$ of the graph in Fig.2.1 is

$$A(G) = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix}$$

Let $G_1 = (V_1, E_1; I_1)$ and $G_2 = (V_2, E_2; I_2)$ be two graphs. They are *identical*, denoted by $G_1 = G_2$ if $V_1 = V_2, E_1 = E_2$ and $I_1 = I_2$. If there exists a $1-1$ mapping $\phi : E_1 \to E_2$ and $\phi : V_1 \to V_2$ such that $\phi I_1(e) = I_2\phi(e)$ for $\forall e \in E_1$ with the convention that $\phi(u, v) = (\phi(u), \phi(v))$, then we say that G_1 is *isomorphic* to G_2, denoted by $G_1 \cong G_2$ and ϕ an *isomorphism* between G_1 and G_2. For simple graphs H_1, H_2, this definition can be simplified by $(u, v) \in I_1(E_1)$ if and only if $(\phi(u), \phi(v)) \in I_2(E_2)$ for $\forall u, v \in V_1$.

For example, let $G_1 = (V_1, E_1; I_1)$ and $G_2 = (V_2, E_2; I_2)$ be two graphs with

$$V_1 = \{v_1, v_2, v_3\},$$

$$E_1 = \{e_1, e_2, e_3, e_4\},$$

$$I_1(e_1) = (v_1, v_2), I_1(e_2) = (v_2, v_3), I_1(e_3) = (v_3, v_1), I_1(e_4) = (v_1, v_1)$$

and

$$V_2 = \{u_1, u_2, u_3\},$$

$$E_2 = \{f_1, f_2, f_3, f_4\},$$

$$I_2(f_1) = (u_1, u_2), I_2(f_2) = (u_2, u_3), I_2(f_3) = (u_3, u_1), I_2(f_4) = (u_2, u_2),$$

i.e., the graphs shown in Fig.2.2.

Fig. 2.2

Then they are isomorphic since we can define a mapping $\phi : E_1 \bigcup V_1 \rightarrow E_2 \bigcup V_2$ by

$$\phi(e_1) = f_2, \phi(e_2) = f_3, \phi(e_3) = f_1, \phi(e_4) = f_4$$

and $\phi(v_i) = u_i$ for $1 \leq i \leq 3$. It can be verified immediately that $\phi I_1(e) = I_2\phi(e)$ for $\forall e \in E_1$. Therefore, ϕ is an isomorphism between G_1 and G_2.

If $G_1 = G_2 = G$, an isomorphism between G_1 and G_2 is said to be an *automorphism* of G. All automorphisms of a graph G form a group under the composition operation, i.e., $\phi\theta(x) = \phi(\theta(x))$, where $x \in E(G) \bigcup V(G)$. We denote the automorphism group of a graph G by Aut$G$.

For a simple graph G of n vertices, it is easy to verify that Aut$G \leq S_n$, the symmetry group action on these n vertices of G. But for non-simple graph, the situation is more complex. The automorphism groups of graphs $K_m, m = |V(K_m)|$ and $B_n, n = |E(B_n)|$ in Fig.2.3 are Aut$K_m = S_m$ and Aut$B_n = S_n$.

Fig. 2.3

For generalizing the conception of automorphisms, the semi-arc automorphisms of a graph were introduced in [53], which is defined in the following definition.

Definition 2.1.1 *A one-to-one mapping ξ on $X_{\frac{1}{2}}(G)$ is called a semi-arc automorphism of a graph G if $\xi(e_u)$ and $\xi(f_v)$ are $v-$incident or $e-$incident if e_u and f_v are $v-$incident or $e-$incident for $\forall e_u, f_v \in X_{\frac{1}{2}}(G)$.*

All semi-arc automorphisms of a graph also form a group, denoted by $\mathrm{Aut}_{\frac{1}{2}}G$. For example, $\mathrm{Aut}_{\frac{1}{2}}B_n = S_n[S_2]$.

For $\forall g \in \mathrm{Aut}G$, there is an induced action $g|^{\frac{1}{2}} : X_{\frac{1}{2}}(G) \to X_{\frac{1}{2}}(G)$ on $X_{\frac{1}{2}}(G)$ defined by

$$\forall e_u \in X_{\frac{1}{2}}(G), g(e_u) = g(e)_{g(u)}.$$

All induced action of elements in $\mathrm{Aut}G$ is denoted by $\mathrm{Aut}G|^{\frac{1}{2}}$.

The graph B_n shows that $\mathrm{Aut}_{\frac{1}{2}}G$ may be not the same as $\mathrm{Aut}G|^{\frac{1}{2}}$. However, we get a result in the following.

Theorem 2.1.3([56]) *For a graph Γ without loops,*

$$\mathrm{Aut}_{\frac{1}{2}}\Gamma = \mathrm{Aut}\Gamma|^{\frac{1}{2}}.$$

Various applications of this theorem to graphs, especially, to combinatorial maps can be found in references [55] – [56] and [66] – [67].

2.1.2. **Subgraphs in a graph**

A graph $H = (V_1, E_1; I_1)$ is a *subgraph* of a graph $G = (V, E; I)$ if $V_1 \subseteq V$, $E_1 \subseteq E$ and $I_1 : E_1 \to V_1 \times V_1$. We denote that H is a subgraph of G by $H \subset G$. For example, graphs G_1, G_2, G_3 are subgraphs of the graph G in Fig.2.4.

Fig 2.4

For a nonempty subset U of the vertex set $V(G)$ of a graph G, the subgraph $\langle U\rangle$ of G *induced* by U is a graph having vertex set U and whose edge set consists of these edges of G incident with elements of U. A subgraph H of G is called *vertex-induced* if $H \cong \langle U\rangle$ for some subset U of $V(G)$. Similarly, for a nonempty subset F of $E(G)$, the subgraph $\langle F\rangle$ induced by F in G is a graph having edge set F and whose vertex set consists of vertices of G incident with at least one edge of F. A subgraph H of G is *edge-induced* if $H \cong \langle F\rangle$ for some subset F of $E(G)$. In Fig.2.4, subgraphs G_1

and G_2 are both vertex-induced subgraphs $\langle\{u_1, u_4\}\rangle$, $\langle\{u_2, u_3\}\rangle$ and edge-induced subgraphs $\langle\{(u_1, u_4)\}\rangle$, $\langle\{(u_2, u_3)\}\rangle$.

For a subgraph H of G, if $|V(H)| = |V(G)|$, then H is called a *spanning subgraph* of G. In Fig.2.4, the subgraph G_3 is a spanning subgraph of the graph G. Spanning subgraphs are useful for constructing multi-spaces on graphs, see also Section 2.4.

A spanning subgraph without circuits is called a *spanning forest*. It is called a *spanning tree* if it is connected. The following characteristic for spanning trees of a connected graph is well-known.

Theorem 2.1.4 *A subgraph T of a connected graph G is a spanning tree if and only if T is connected and $E(T) = |V(G)| - 1$.*

Proof The necessity is obvious. For its sufficiency, since T is connected and $E(T) = |V(G)| - 1$, there are no circuits in T. Whence, T is a spanning tree. ♮

A path is also a tree in which each vertex has valency 2 unless the two pendent vertices valency 1. We denote a path with n vertices by P_n and define the *length* of P_n to be $n - 1$. For a connected graph G, $x, y \in V(G)$, the distance $d(x, y)$ of x to y in G is defined by

$$d_G(x, y) = min\{\ |V(P(x, y))| - 1 \mid P(x, y)\ is\ a\ path\ connecting\ x\ and\ y\ \}.$$

For $\forall u \in V(G)$, the *eccentricity* $e_G(u)$ of u is defined by

$$e_G(u) = max\{\ d_G(u, x)\ |x \in V(G)\}.$$

A vertex u^+ is called an *ultimate vertex* of a vertex u if $d(u, u^+) = e_G(u)$. Not loss of generality, we arrange these eccentricities of vertices in G in an order $e_G(v_1), e_G(v_2), \cdots$, $e_G(v_n)$ with $e_G(v_1) \le e_G(v_2) \le \cdots \le e_G(v_n)$, where $\{v_1, v_2, \cdots, v_n\} = V(G)$. The sequence $\{e_G(v_i)\}_{1\le i\le s}$ is called an *eccentricity sequence* of G. If $\{e_1, e_2, \cdots,$ $e_s\} = \{e_G(v_1), e_G(v_2), \cdots, e_G(v_n)\}$ and $e_1 < e_2 < \cdots < e_s$, the sequence $\{e_i\}_{1\le i\le s}$ is called an *eccentricity value sequence* of G. For convenience, we abbreviate an integer sequence $\{r - 1 + i\}_{1\le i\le s+1}$ to $[r, r + s]$.

The *radius* $r(G)$ and the *diameter* $D(G)$ of G are defined by

$$r(G) = min\{e_G(u)|u \in V(G)\}\ \text{ and }\ D(G) = max\{e_G(u)|u \in V(G)\},$$

respectively. For a given graph G, if $r(G) = D(G)$, then G is called a *self-centered graph*, i.e., the eccentricity value sequence of G is $[r(G), r(G)]$. Some characteristics of self-centered graphs can be found in [47], [64] and [108].

For $\forall x \in V(G)$, we define a *distance decomposition* $\{V_i(x)\}_{1\le i\le e_G(x)}$ *of G with root x* by

$$G = V_1(x) \bigoplus V_2(x) \bigoplus \cdots \bigoplus V_{e_G(x)}(x)$$

where $V_i(x) = \{\ u\ |d(x, u) = i, u \in V(G)\}$ for any integer $i, 0 \leq i \leq e_G(x)$. We get a necessary and sufficient condition for the eccentricity value sequence of a simple graph in the following.

Theorem 2.1.5 *A non-decreasing integer sequence $\{r_i\}_{1\leq i\leq s}$ is a graphical eccentricity value sequence if and only if*

(i) $r_1 \leq r_s \leq 2r_1$;

(ii) $\triangle(r_{i+1}, r_i) = |r_{i+1} - r_i| = 1$ *for any integer* $i, 1 \leq i \leq s-1$.

Proof If there is a graph G whose eccentricity value sequence is $\{r_i\}_{1\leq i\leq s}$, then $r_1 \leq r_s$ is trivial. Now we choose three different vertices u_1, u_2, u_3 in G such that $e_G(u_1) = r_1$ and $d_G(u_2, u_3) = r_s$. By definition, we know that $d(u_1, u_2) \leq r_1$ and $d(u_1, u_3) \leq r_1$. According to the triangle inequality for distances, we get that $r_s = d(u_2, u_3) \leq d_G(u_2, u_1) + d_G(u_1, u_3) = d_G(u_1, u_2) + d_G(u_1, u_3) \leq 2r_1$. So $r_1 \leq r_s \leq 2r_1$.

Assume $\{e_i\}_{1\leq i\leq s}$ is the eccentricity value sequence of a graph G. Define $\triangle(i) = e_{i+1} - e_i$, $1 \leq i \leq n-1$. We assert that $0 \leq \triangle(i) \leq 1$. If this assertion is not true, then there must exists a positive integer $I, 1 \leq I \leq n-1$ such that $\triangle(I) = e_{I+1} - e_I \geq 2$. Choose a vertex $x \in V(G)$ such that $e_G(x) = e_I$ and consider the distance decomposition $\{V_i(x)\}_{0\leq i\leq e_G(x)}$ of G with root x.

Notice that it is obvious that $e_G(x) - 1 \leq e_G(u_1) \leq e_G(x) + 1$ for any vertex $u_1 \in V_1(G)$. Since $\triangle(I) \geq 2$, there does not exist a vertex with the eccentricity $e_G(x) + 1$. Whence, we get $e_G(u_1) \leq e_G(x)$ for $\forall u_1 \in V_1(x)$. If we have proved that $e_G(u_j) \leq e_G(x)$ for $\forall u_j \in V_j(x)$, $1 \leq j < e_G(x)$, we consider these eccentricity values of vertices in $V_{j+1}(x)$. Let $u_{j+1} \in V_{j+1}(x)$. According to the definition of $\{V_i(x)\}_{0\leq i\leq e_G(x)}$, there must exists a vertex $u_j \in V_j(x)$ such that $(u_j, u_{j+1}) \in E(G)$. Now consider the distance decomposition $\{V_i(u_j)\}_{0\leq j\leq e_G(u)}$ of G with root u_j. Notice that $u_{j+1} \in V_1(u_j)$. Thereby we get that

$$e_G(u_{j+1}) \leq e_G(u_j) + 1 \leq e_G(x) + 1.$$

Because we have assumed that there are no vertices with the eccentricity $e_G(x) + 1$, so $e_G(u_{j+1}) \leq e_G(x)$ for any vertex $u_{j+1} \in V_{j+1}(x)$. Continuing this process, we know that $e_G(y) \leq e_G(x) = e_I$ for any vertex $y \in V(G)$. But then there are no vertices with the eccentricity $e_I + 1$, which contradicts the assumption that $\triangle(I) \geq 2$. Therefore $0 \leq \triangle(i) \leq 1$ and $\triangle(r_{i+1}, r_i) = 1, 1 \leq i \leq s-1$.

For any integer sequence $\{r_i\}_{1\leq i\leq s}$ with conditions (i) and (ii) hold, it can be simply written as $\{r, r+1, \cdots, r+s-1\} = [r, r+s-1]$, where $s \leq r$. We construct a graph with the eccentricity value sequence $[r, r+s-1]$ in the following.

Case 1 $\quad s = 1$

In this case, $\{r_i\}_{1\leq i\leq s} = [r, r]$. We can choose any self-centered graph with $r(G) = r$, especially, the circuit C_{2r} of order $2r$. Then its eccentricity value sequence is $[r, r]$.

Case 2 $\quad s \geq 2$

Choose a self-centered graph H with $r(H) = r, x \in V(H)$ and a path $P_s = u_0u_1 \cdots u_{s-1}$. Define a new graph $G = P_s \bigodot H$ as follows:

$V(G) = V(P_s) \bigcup V(H) \setminus \{u_0\}$,

$E(G) = (E(P_s) \bigcup \{(x, u_1)\} \bigcup E(H) \setminus \{(u_1, u_0)\}$

such as the graph G shown in Fig.2.5.

Fig 2.5

Then we know that $e_G(x) = r$, $e_G(u_{s-1}) = r + s - 1$ and $r \leq e_G(x) \leq r + s - 1$ for all other vertices $x \in V(G)$. Therefore, the eccentricity value sequence of G is $[r, r + s - 1]$. This completes the proof. ♮

For a given eccentricity value l, the *multiplicity set* $N_G(l)$ is defined by $N_G(l) = \{ x \mid x \in V(G), e(x) = l \}$. Jordan proved that the $\langle N_G(r(G)) \rangle$ in a tree is a vertex or two adjacent vertices in 1869([11]). For a graph must not being a tree, we get the following result which generalizes Jordan's result for trees.

Theorem 2.1.6 *Let* $\{r_i\}_{1 \leq i \leq s}$ *be a graphical eccentricity value sequence. If* $|N_G(r_I)| = 1$, *then there must be* $I = 1$, *i.e.,* $|N_G(r_i)| \geq 2$ *for any integer* $i, 2 \leq i \leq s$.

Proof Let G be a graph with the eccentricity value sequence $\{r_i\}_{1 \leq i \leq s}$ and $N_G(r_I) = \{x_0\}, e_G(x_0) = r_I$. We prove that $e_G(x) > e_G(x_0)$ for any vertex $x \in V(G) \setminus \{x_0\}$. Consider the distance decomposition $\{V_i(x_0)\}_{0 \leq i \leq e_G(x_0)}$ of G with root x_0. First, we prove that $e_G(v_1) = e_G(x_0) + 1$ for any vertex $v_1 \in V_1(x_0)$. Since $e_G(x_0) - 1 \leq e_G(v_1) \leq e_G(x_0) + 1$ for any vertex $v_1 \in V_1(x_0)$, we only need to prove that $e_G(v_1) > e_G(x_0)$ for any vertex $v_1 \in V_1(x_0)$. In fact, since for any ultimate vertex x_0^+ of x_0, we have that $d_G(x_0, x_0^+) = e_G(x_0)$. So $e_G(x_0^+) \geq e_G(x_0)$. Since $N_G(e_G(x_0)) = \{x_0\}, x_0^+ \notin N_G(e_G(x_0))$. Therefore, $e_G(x_0^+) > e_G(x_0)$. Choose $v_1 \in V_1(x_0)$. Assume the shortest path from v_1 to x_0^+ is $P_1 = v_1v_2 \cdots v_s x_0^+$ and $x_0 \notin V(P_1)$. Otherwise, we already have $e_G(v_1) > e_G(x_0)$. Now consider the distance decomposition $\{V_i(x_0^+)\}_{0 \leq i \leq e_G(x_0^+)}$ of G with root x_0^+. We know that $v_s \in V_1(x_0^+)$. So we get that

$$e_G(x_0^+) - 1 \leq e_G(v_s) \leq e_G(x_0^+) + 1.$$

Thereafter we get that $e_G(v_s) \geq e_G(x_0^+) - 1 \geq e_G(x_0)$. Because $N_G(e_G(x_0)) = \{x_0\}$, so $v_s \notin N_G(e_G(x_0))$. We finally get that $e_G(v_s) > e_G(x_0)$.

Similarly, choose $v_s, v_{s-1}, \cdots, v_2$ to be root vertices respectively and consider these distance decompositions of G with roots $v_s, v_{s-1}, \cdots, v_2$, we get that

$e_G(v_s) > e_G(x_0)$,

$e_G(v_{s-1}) > e_G(x_0)$,

$\cdots\cdots\cdots\cdots\cdots$,

and

$e_G(v_1) > e_G(x_0)$.

Therefore, $e_G(v_1) = e_G(x_0) + 1$ for any vertex $v_1 \in V_1(x_0)$.

Now consider these vertices in $V_2(x_0)$. For $\forall v_2 \in V_2(x_0)$, assume that v_2 is adjacent to $u_1, u_1 \in V_1(x_0)$. We know that $e_G(v_2) \geq e_G(u_1) - 1 \geq e_G(x_0)$. Since $|N_G(e_G(x_0))| = |N_G(r_I)| = 1$, we get $e_G(v_2) \geq e_G(x_0) + 1$.

Now assume that we have proved $e_G(v_k) \geq e_G(x_0) + 1$ for any vertex $v_k \in V_1(x_0) \bigcup V_2(x_0) \bigcup \cdots \bigcup V_k(x_0)$ for $1 \leq k < e_G(x_0)$. Let $v_{k+1} \in V_{k+1}(x_0)$ and assume that v_{k+1} is adjacent to u_k in $V_k(x_0)$. Then we know that $e_G(v_{k+1}) \geq e_G(u_k) - 1 \geq e_G(x_0)$. Since $|N_G(e_G(x_0))| = 1$, we get that $e_G(v_{k+1}) \geq e_G(x_0) + 1$. Therefore, $e_G(x) > e_G(x_0)$ for any vertex $x, x \in V(G) \setminus \{x_0\}$. That is, if $|N_G(r_I)| = 1$, then there must be $I = 1$. $\natural$

Theorem 2.1.6 is the best possible in some cases of trees. For example, the eccentricity value sequence of a path P_{2r+1} is $[r, 2r]$ and we have that $|N_G(r)| = 1$ and $|N_G(k)| = 2$ for $r + 1 \leq k \leq 2r$. But for graphs not being trees, we only found some examples satisfying $|N_G(r_1)| = 1$ and $|N_G(r_i)| > 2$. A non-tree graph with the eccentricity value sequence $[2, 3]$ and $|NG(2)| = 1$ can be found in Fig.2 in the reference [64].

For a given graph G and $V_1, V_2 \in V(G)$, define an *edge cut* $E_G(V_1, V_2)$ by

$$E_G(V_1, V_2) = \{\ (u, v) \in E(G) \mid u \in V_1, v \in V_2\}.$$

A graph G is *hamiltonian* if it has a circuit containing all vertices of G. This circuit is called a *hamiltonian circuit*. A path containing all vertices of a graph G is called a *hamiltonian path*. For hamiltonian circuits, we have the following characteristic.

Theorem 2.1.7 *A circuit C of a graph G without isolated vertices is a hamiltonian circuit if and only if for any edge cut $\mathcal{C}$, $|E(C) \bigcap E(\mathcal{C})| \equiv 0(mod2)$ and $|E(C) \bigcap E(\mathcal{C})| \geq 2$.*

Proof For any circuit C and an edge cut $\mathcal{C}$, the times crossing $\mathcal{C}$ as we travel along C must be even. Otherwise, we can not come back to the initial vertex. if C is a hamiltonian circuit, then $|E(C) \bigcap E(\mathcal{C})| \neq 0$. Whence, $|E(C) \bigcap E(\mathcal{C})| \geq 2$ and $|E(C) \bigcap E(\mathcal{C})| \equiv 0(mod2)$ for any edge cut $\mathcal{C}$.

Now if a circuit C satisfies $|E(C) \bigcap E(\mathcal{C})| \geq 2$ and $|E(C) \bigcap E(\mathcal{C})| \equiv 0(mod2)$ for any edge cut $\mathcal{C}$, we prove that C is a hamiltonian circuit of G. In fact, if $V(G) \setminus V(C) \neq \emptyset$, choose $x \in V(G) \setminus V(C)$. Consider an edge cut $E_G(\{x\}, V(G) \setminus \{x\})$. Since $\rho_G(x) \neq 0$, we know that $|E_G(\{x\}, V(G) \setminus \{x\})| \geq 1$. But since $V(C) \bigcap (V(G) \setminus V(C)) = \emptyset$, we know that $|E_G(\{x\}, V(G) \setminus \{x\}) \bigcap E(C)| = 0$. Contradicts the fact that $|E(C)\bigcap E(\mathcal{C})| \geq 2$ for any edge cut $\mathcal{C}$. Therefore $V(C) = V(G)$ and C is a hamiltonian circuit of G. $\natural$

Let G be a simple graph. The *closure* of G, denoted by $C(G)$, is a graph obtained from G by recursively joining pairs of non-adjacent vertices whose valency sum is at least $|G|$ until no such pair remains. In 1976, Bondy and Chvátal proved a very useful theorem for hamiltonian graphs.

Theorem 2.1.8([5][8]) *A simple graph is hamiltonian if and only if its closure is hamiltonian.*

This theorem generalizes Dirac's and Ore's theorems simultaneously stated as follows:

Dirac (1952): *Every connected simple graph G of order $n \geq 3$ with the minimum valency$\geq \frac{n}{2}$ is hamiltonian.*

Ore (1960): *If G is a simple graph of order $n \geq 3$ such that $\rho_G(u) + \rho_G(v) \geq n$ for all distinct non-adjacent vertices u and v, then G is hamiltonian.*

In 1984, Fan generalized Dirac's theorem to a localized form ([41]). He proved that

Let G be a 2-connected simple graph of order n. If Fan's condition:

$max\{\rho_G(u), \rho_G(v)\} \geq \frac{n}{2}$

holds for $\forall u, v \in V(G)$ provided $d_G(u, v) = 2$, then G is hamiltonian.

After Fan's paper [17], many researches concentrated on weakening Fan's condition and found new localized conditions for hamiltonian graphs. For example, those results in references [4], [48] – [50], [52], [63] and [65] are this type. The next result on hamiltonian graphs is obtained by Shi in 1992 ([84]).

Theorem 2.1.9(Shi, 1992) *Let G be a 2-connected simple graph of order n. Then G contains a circuit passing through all vertices of valency$\geq \frac{n}{2}$.*

Proof Assume the assertion is false. Let $C = v_1v_2\cdots v_kv_1$ be a circuit containing as many vertices of valency$\geq \frac{n}{2}$ as possible and with an orientation on it. For $\forall v \in V(C)$, v^+ denotes the successor and v^- the predecessor of v on C. Set $R = V(G)\setminus V(C)$. Since G is 2-connected, there exists a path length than 2 connecting two vertices of C that is internally disjoint from C and containing one internal vertex x of valency$\geq \frac{n}{2}$ at least. Assume C and P are chosen in such a way that the length of P as small as possible. Let $N_R(x) = N_G(x)\bigcap R$, $N_C(x) = N_G(x)\bigcap C$, $N_C^+(x) = \{v|v^- \in N_C(x)\}$ and $N_C^-(x) = \{v|v^+ \in N_C(x)\}$.

Not loss of generality, we may assume $v_1 \in V(P)\bigcap V(C)$. Let v_t be the other vertex in $V(P)\bigcap V(C)$. By the way C was chosen, there exists a vertex v_s with $1 < s < t$ such that $\rho_G(v_s) \geq \frac{n}{2}$ and $\rho(v_i) < \frac{n}{2}$ for $1 < i < s$.

If $s \geq 3$, by the choice of C and P the sets

$$N_C^-(v_s)\setminus\{v_1\}, N_C(x), N_R(v_s), N_R(x), \{x, v_{s-1}\}$$

are pairwise disjoint, implying that

$$\begin{aligned} n &\geq |N_C^-(v_s)\setminus\{v_1\}| + |N_C(x)| + |N_R(v_s)| + |N_R(x)| + |\{x, v_{s-1}\}| \\ &= \rho_G(v_s) + \rho_G(x) + 1 \geq n+1, \end{aligned}$$

a contradiction. If $s = 2$, then the sets

$$N_C^-(v_s), N_C(x), N_R(v_s), N_R(x), \{x\}$$

are pairwise disjoint, which yields a similar contradiction. $\natural$

Three induced subgraphs used in the next result for hamiltonian graphs are shown in Fig.2.6.

Fig. 2.6

For an induced subgraph L of a simple graph G, a condition is called a *localized condition* $D_L(l)$ if $D_L(x,y) = l$ implies that $max\{\rho_G(x), \rho_G(y)\} \geq \frac{|G|}{2}$ for $\forall x, y \in V(L)$. Then we get the following result.

Theorem 2.1.10 *Let G be a 2-connected simple graph. If the localized condition $D_L(2)$ holds for induced subgraphs $L \cong K_{1.3}$ or Z_2 in G, then G is hamiltonian.*

Proof By Theorem 2.1.9, we denote by $c_{\frac{n}{2}}(G)$ the maximum length of circuits passing through all vertices$\geq \frac{n}{2}$. Similar to Theorem 2.1.8, we know that for $x, y \in V(G)$, if $\rho_G(x) \geq \frac{n}{2}$, $\rho_G(y) \geq \frac{n}{2}$ and $xy \notin E(G)$, then $c_{\frac{n}{2}}(G\bigcup\{xy\}) = c_{\frac{n}{2}}(G)$. Otherwise, if $c_{\frac{n}{2}}(G\bigcup\{xy\}) > c_{\frac{n}{2}}(G)$, there exists a circuit of length $c_{\frac{n}{2}}(G\bigcup\{xy\})$ and passing through all vertices$\geq \frac{n}{2}$. Let $C_{\frac{n}{2}}$ be such a circuit and $C_{\frac{n}{2}} = xx_1x_2\cdots x_syx$ with $s = c_{\frac{n}{2}}(G\bigcup\{xy\}) - 2$. Notice that

$$N_G(x)\bigcap(V(G)\setminus V(C_{\frac{n}{2}}(G\bigcup\{xy\}))) = \emptyset$$

and

$$N_G(y)\bigcap(V(G)\setminus V(C_{\frac{n}{2}}(G\bigcup\{xy\}))) = \emptyset.$$

If there exists an integer $i, 1 \le i \le s$, $xx_i \in E(G)$, then $x_{i-1}y \notin E(G)$. Otherwise, there is a circuit $C' = xx_ix_{i+1}\cdots x_syx_{i-1}x_{i-2}\cdots x$ in G passing through all vertices$\ge \frac{n}{2}$ with length $c_{\frac{n}{2}}(G\bigcup\{xy\})$. Contradicts the assumption that $c_{\frac{n}{2}}(G\bigcup\{xy\}) > c_{\frac{n}{2}}(G)$. Whence,

$$\rho_G(x) + \rho_G(y) \le |V(G) \setminus V(C(C_{\frac{n}{2}}))| + |V(C(C_{\frac{n}{2}}))| - 1 = n - 1,$$

also contradicts that $\rho_G(x) \ge \frac{n}{2}$ and $\rho_G(y) \ge \frac{n}{2}$. Therefore, $c_{\frac{n}{2}}(G\bigcup\{xy\}) = c_{\frac{n}{2}}(G)$ and generally, $c_{\frac{n}{2}}(C(G)) = c_{\frac{n}{2}}(G)$.

Now let C be a maximal circuit passing through all vertices$\ge \frac{n}{2}$ in the closure $C(G)$ of G with an orientation $\overrightarrow{C}$. According to Theorem 2.1.8, if $C(G)$ is non-hamiltonian, we can choose H be a component in $C(G) \setminus C$. Define $N_C(H) = (\bigcup\limits_{x \in H} N_{C(G)}(x))\bigcap V(C)$. Since $C(G)$ is 2-connected, we get that $|N_C(H)| \ge 2$. This enables us choose vertices $x_1, x_2 \in N_C(H)$, $x_1 \ne x_2$ and x_1 can arrive at x_2 along $\overrightarrow{C}$. Denote by $x_1\overrightarrow{C}x_2$ the path from x_1 to x_2 on $\overrightarrow{C}$ and $x_2\overleftarrow{C}x_1$ the reverse. Let P be a shortest path connecting x_1, x_2 in $C(G)$ and

$$u_1 \in N_{C(G)}(x_1)\bigcap V(H)\bigcap V(P), \quad u_2 \in N_{C(G)}(x_2)\bigcap V(H)\bigcap V(P).$$

Then

$$E(C(G))\bigcap(\{x_1^-x_2^-, x_1^+x_2^+\}\bigcup E_{C(G)}(\{u_1, u_2\}, \{x_1^-, x_1^+, x_2^-, x_2^+\})) = \emptyset$$

and

$$\left\langle\{x_1^-, x_1, x_1^+, u_1\}\right\rangle \ncong K_{1.3} \text{ or } \left\langle\{x_2^-, x_2, x_2^+, u_2\}\right\rangle \ncong K_{1.3}.$$

Otherwise, there exists a circuit longer than C, a contradiction. To prove this theorem, we consider two cases.

Case 1 $\left\langle\{x_1^-, x_1, x_1^+, u_1\}\right\rangle \ncong K_{1.3}$ and $\left\langle\{x_2^-, x_2, x_2^+, u_2\}\right\rangle \ncong K_{1.3}$

In this case, $x_1^-x_1^+ \in E(C(G))$ and $x_2^-x_2^+ \in E(C(G))$. By the maximality of C in $C(G)$, we have two claims.

Claim 1.1 $u_1 = u_2 = u$

Otherwise, let $P = x_1u_1y_1\cdots y_lu_2$. By the choice of P, there must be

$$\left\langle\{x_1^-, x_1, x_1^+, u_1, y_1\}\right\rangle \cong Z_2 \text{ and } \left\langle\{x_2^-, x_2, x_2^+, u_2, y_l\}\right\rangle \cong Z_2$$

Since $C(G)$ also has the $D_L(2)$ property, we get that

$$max\{\rho_{C(G)}(x_1^-), \rho_{C(G)}(u_1)\} \ge \frac{n}{2}, \quad max\{\rho_{C(G)}(x_12^-), \rho_{C(G)}(u_2)\} \ge \frac{n}{2}.$$

Whence, $\rho_{C(G)}(x_1^-) \geq \frac{n}{2}$, $\rho_{C(G)}(x_2^-) \geq \frac{n}{2}$ and $x_1^- x_2^- \in E(C(G))$, a contradiction.

Claim 1.2 $\quad x_1x_2 \in E(C(G))$

If $x_1x_2 \notin E(C(G))$, then $\left\langle \{x_1^-, x_1, x_1^+, u, x_2\} \right\rangle \cong Z_2$. Otherwise, we get $x_2x_1^- \in E(C(G))$ or $x_2x_1^+ \in E(C(G))$. But then there is a circuit

$$C_1 = x_2^+ \overrightarrow{C} x_1^- x_2 u x_1 \overrightarrow{C} x_2^- x_2^+ \text{ or } C_2 = x_2^+ \overrightarrow{C} x_1 u x_2 x_1^+ \overrightarrow{C} x_2^- x_2^+.$$

Contradicts the maximality of C. Therefore, we know that

$$\left\langle \{x_1^-, x_1, x_1^+, u, x_2\} \right\rangle \cong Z_2.$$

By the property $D_L(2)$, we get that $\rho_{C(G)}(x_1^-) \geq \frac{n}{2}$

Similarly, consider the induced subgraph $\left\langle \{x_2^-, x_2, x_2^+, u, x_2\} \right\rangle$, we get that $\rho_{C(G)}(x_2^-) \geq \frac{n}{2}$. Whence, $x_1^- x_2^- \in E(C(G))$, also a contradiction. Thereby we know the structure of G as shown in Fig.2.7.

Fig 2.7

By the maximality of C in $C(G)$, it is obvious that $x_1^{--} \neq x_2^+$. We construct an induced subgraph sequence $\{G_i\}_{1 \leq i \leq m}$, $m = |V(x_1^- \overleftarrow{C} x_2^+)| - 2$ and prove there exists an integer $r, 1 \leq r \leq m$ such that $G_r \cong Z_2$.

First, we consider the induced subgraph $G_1 = \left\langle \{x_1, u, x_2, x_1^-, x_1^{--}\} \right\rangle$. If $G_1 \cong Z_2$, take $r = 1$. Otherwise, there must be

$$\{x_1^- x_2, x_1^{--} x_2, x_1^{--} u, x_1^{--} x_1\} \bigcap E(C(G)) \neq \emptyset.$$

If $x_1^- x_2 \in E(C(G))$, or $x_1^{--} x_2 \in E(C(G))$, or $x_1^{--} u \in E(C(G))$, there is a circuit $C_3 = x_1^- \overleftarrow{C} x_2^+ x_2^- \overleftarrow{C} x_1 u x_2 x_1^-$, or $C_4 = x_1^{--} \overleftarrow{C} x_2^+ x_2^- \overleftarrow{C} x_1^+ x_1^- x_1 u x_2 x_1^{--}$, or $C_5 = x_1^{--} \overleftarrow{C} x_1^+ x_1^- x_1 u x_1^{--}$. Each of these circuits contradicts the maximality of C. Therefore, $x_1^{--} x_1 \in E(C(G))$.

Now let $x_1^- \overleftarrow{C} x_2^+ = x_1^- y_1 y_2 \cdots y_m x_2^+$, where $y_0 = x_1^-, y_1 = x_1^{--}$ and $y_m = x_2^{++}$. If we have defined an induced subgraph G_k for any integer k and have gotten $y_i x_1 \in E(C(G))$ for any integer $i, 1 \leq i \leq k$ and $y_{k+1} \neq x_2^{++}$, then we define

$$G_{k+1} = \langle \{y_{k+1}, y_k, x_1, x_2, u\} \rangle .$$

If $G_{k+1} \cong Z_2$, then $r = k+1$. Otherwise, there must be

$$\{y_k u, y_k x_2, y_{k+1} u, y_{k+1} x_2, y_{k+1} x_1\} \bigcap E(C(G)) \neq \emptyset.$$

If $y_k u \in E(C(G))$, or $y_k x_2 \in E(C(G))$, or $y_{k+1} u \in E(C(G))$, or $y_{k+1} x_2 \in E(C(G))$, there is a circuit $C_6 = y_k \overleftarrow{C} x_1^+ x_1^- \overleftarrow{C} y_{k-1} x_1 u y_k$, or $C_7 = y_k \overleftarrow{C} x_2^+ x_2^- \overleftarrow{C} x_1^+ x_1^- \overleftarrow{C} y_{k-1} x_1 u x_2 y_k$, or $C_8 = y_{k+1} \overleftarrow{C} x_1^+ x_1^- \overleftarrow{C} y_k x_1 u y_{k+1}$, or $C_9 = y_{k+1} \overleftarrow{C} x_2^+ x_2^- \overleftarrow{C} x_1^+ x_1^- \overleftarrow{C} y_k x_1 u x_2 y_{k+1}$. Each of these circuits contradicts the maximality of C. Thereby, $y_{k+1} x_1 \in E(C(G))$.

Continue this process. If there are no subgraphs in $\{G_i\}_{1 \leq i \leq m}$ isomorphic to Z_2, we finally get $x_1 x_2^{++} \in E(C(G))$. But then there is a circuit $C_{10} = x_1^- \overleftarrow{C} x_2^{++} x_1 u x_2 x_2^+ \overleftarrow{C} x_1^+ x_1^-$ in $C(G)$. Also contradicts the maximality of C in $C(G)$. Therefore, there must be an integer $r, 1 \leq r \leq m$ such that $G_r \cong Z_2$.

Similarly, let $x_2^- \overleftarrow{C} x_1^+ = x_2^- z_1 z_2 \cdots z_t x_1^-$, where $t = |V(x_2^- \overleftarrow{C} x_1^+)| - 2$, $z_0 = x_2^-, z_1^{++} = x_2, z_t = x_1^{++}$. We can also construct an induced subgraph sequence $\{G^i\}_{1 \leq i \leq t}$ and know that there exists an integer $h, 1 \leq h \leq t$ such that $G^h \cong Z_2$ and $x_2 z_i \in E(C(G))$ for $0 \leq i \leq h-1$.

Since the localized condition $D_L(2)$ holds for an induced subgraph Z_2 in $C(G)$, we get that $max\{\rho_{C(G)}(u), \rho_{C(G)}(y_{r-1})\} \geq \frac{n}{2}$ and $max\{\rho_{C(G)}(u), \rho_{C(G)}(z_{h-1})\} \geq \frac{n}{2}$. Whence $\rho_{C(G)}(y_{r-1}) \geq \frac{n}{2}$, $\rho_{C(G)}(z_{h-1}) \geq \frac{n}{2}$ and $y_{r-1} z_{h-1} \in E(C(G))$. But then there is a circuit

$$C_{11} = y_{r-1} \overleftarrow{C} x_2^+ x_2^- \overleftarrow{C} z_{h-2} x_2 u x_1 y_{r-2} \overrightarrow{C} x_1^- x_1^+ \overrightarrow{C} z_{h-1} y_{r-1}$$

in $C(G)$, where if $h = 1$, or $r = 1$, $x_2^- \overleftarrow{C} z_{h-2} = \emptyset$, or $y_{r-2} \overrightarrow{C} x_1^- = \emptyset$. Also contradicts the maximality of C in $C(G)$.

Case 2 $\left\langle \{x_1^-, x_1, x_1^+, u_1\} \right\rangle \ncong K_{1.3}$, $\left\langle \{x_2^-, x_2, x_2^+, u_2\} \right\rangle \cong K_{1.3}$ or $\left\langle \{x_1^-, x_1, x_1^+, u_1\} \right\rangle \cong K_{1.3}$, $\left\langle \{x_2^-, x_2, x_2^+, u_2\} \right\rangle \ncong K_{1.3}$

Not loss of generality, we assume that $\left\langle \{x_1^-, x_1, x_1^+, u_1\} \right\rangle \ncong K_{1.3}$, $\left\langle \{x_2^-, x_2, x_2^+, u_2\} \right\rangle \cong K_{1.3}$. Since each induced subgraph $K_{1.3}$ in $C(G)$ possesses $D_L(2)$, we get that $max\{\rho_{C(G)}(u), \rho_{C(G)}(x_2^-)\} \geq \frac{n}{2}$ and $max\{\rho_{C(G)}(u), \rho_{C(G)}(x_2^+)\} \geq \frac{n}{2}$. Whence $\rho_{C(G)}(x_2^-) \geq \frac{n}{2}$, $\rho_{C(G)}(x_2^+) \geq \frac{n}{2}$ and $x_2^- x_2^+ \in E(C(G))$. Therefore, the discussion of Case 1 also holds in this case and yields similar contradictions.

Combining Case 1 with Case 2, the proof is complete. ♮

Let G, $F_1, F_2, \cdots, F_k$ be $k+1$ graphs. If there are no induced subgraphs of G isomorphic to $F_i, 1 \leq i \leq k$, then G is called $\{F_1, F_2, \cdots, F_k\}$*-free.* we get a immediately consequence by Theorem 2.1.10.

Corollary 2.1.1 *Every 2-connected $\{K_{1.3}, Z_2\}$-free graph is hamiltonian.*

Let G be a graph. For $\forall u \in V(G)$, $\rho_G(u) = d$, let H be a graph with d pendent vertices $v_1, v_2, \cdots, v_d$. Define a splitting operator $\vartheta : G \to G^{\vartheta(u)}$ on u by

$$V(G^{\vartheta(u)}) = (V(G) \setminus \{u\}) \bigcup (V(H) \setminus \{v_1, v_2, \cdots, v_d\}),$$

$$\begin{aligned} E(G^{\vartheta(u)}) &= (E(G) \setminus \{ux_i \in E(G), 1 \leq i \leq d\}) \\ &\bigcup (E(H) \setminus \{v_i y_i \in E(H), 1 \leq i \leq d\}) \bigcup \{x_i y_i, 1 \leq i \leq d\}. \end{aligned}$$

We call d the *degree of the splitting operator* ϑ and $N(\vartheta(u)) = H \setminus \{x_i y_i, 1 \leq i \leq d\}$ the *nucleus of* ϑ . A splitting operator is shown in Fig.2.8.

Fig. 2.9

Erdös and Rényi raised a question in 1961 ([7]): *in what model of random graphs is it true that almost every graph is hamiltonian?* Pósa and Korshuuov proved independently that for some constant c almost every labelled graph with n vertices and at least $n \log n$ edges is hamiltonian in 1974. Contrasting this probabilistic result, there is another property for hamiltonian graphs, i.e., there is a splitting operator ϑ such that $G^{\vartheta(u)}$ is non-hamiltonian for $\forall u \in V(G)$ of a graph G.

Theorem 2.1.11 *Let G be a graph. For $\forall u \in V(G)$, $\rho_G(u) = d$, there exists a splitting operator ϑ of degree d on u such that $G^{\vartheta(u)}$ is non-hamiltonian.*

Proof For any positive integer i, define a simple graph Θ_i by $V(\Theta_i) = \{x_i, y_i, z_i, u_i\}$ and $E(\Theta_i) = \{x_i y_i, x_i z_i, y_i z_i, y_i u_i, z_i u_i\}$. For integers $\forall i, j \geq 1$, the point product $\Theta_i \odot \Theta_j$ of Θ_i and Θ_j is defined by

$$V(\Theta_i \odot \Theta_j) = V(\Theta_i) \bigcup V(\Theta_j) \setminus \{u_j\},$$

$$E(\Theta_i \odot \Theta_j) = E(\Theta_i) \bigcup E(\Theta_j) \bigcup \{x_i y_j, x_i z_j\} \setminus \{x_j y_j, x_j z_j\}.$$

Now let H_d be a simple graph with

$$V(H_d) = V(\Theta_1 \odot \Theta_2 \odot \cdots \Theta_{d+1}) \bigcup \{v_1, v_2, \cdots, v_d\},$$

$$E(H_d) = E(\Theta_1 \odot \Theta_2 \odot \cdots \Theta_{d+1}) \bigcup \{v_1 u_1, v_2 u_2, \cdots, v_d u_d\}.$$

Then $\vartheta : G \to G^{\vartheta(w)}$ is a splitting operator of degree d as shown in Fig.2.10.

Fig¸2.10

For any graph G and $w \in V(G), \rho_G(w) = d$, we prove that $G^{\vartheta(w)}$ is non-hamiltonian. In fact, If $G^{\vartheta(w)}$ is a hamiltonian graph, then there must be a hamiltonian path $P(u_i, u_j)$ connecting two vertices u_i, u_j for some integers $i, j, 1 \leq i, j \leq d$ in the graph $H_d \setminus \{v_1, v_2, \cdots, v_d\}$. However, there are no hamiltonian path connecting vertices u_i, u_j in the graph $H_d \setminus \{v_1, v_2, \cdots, v_d\}$ for any integer $i, j, 1 \leq i, j \leq d$. Therefore, $G^{\vartheta(w)}$ is non-hamiltonian. ♮

2.1.3. **Classes of graphs with decomposition**

(1) **Typical classes of graphs**

C1. Bouquets and Dipoles

In graphs, two simple cases is these graphs with one or two vertices, which are just bouquets or dipoles. A graph $B_n = (V_b, E_b; I_b)$ with $V_b = \{\ O\ \}$, $E_b = \{e_1, e_2, \cdots, e_n\}$ and $I_b(e_i) = (O, O)$ for any integer $i, 1 \leq i \leq n$ is called a *bouquet* of n edges. Similarly, a graph $D_{s.l.t} = (V_d, E_d; I_d)$ is called a *dipole* if $V_d = \{O_1, O_2\}$, $E_d = \{e_1, e_2, \cdots, e_s, e_{s+1}, \cdots, e_{s+l}, e_{s+l+1}, \cdots, e_{s+l+t}\}$ and

$$I_d(e_i) = \begin{cases} (O_1, O_1), & \text{if } 1 \leq i \leq s, \\ (O_1, O_2), & \text{if } s+1 \leq i \leq s+l, \\ (O_2, O_2), & \text{if } s+l+1 \leq i \leq s+l+t. \end{cases}$$

For example, B_3 and $D_{2,3,2}$ are shown in Fig.2.11.

Fig¸2.11

In the past two decades, the behavior of bouquets on surfaces fascinated many mathematicians. A typical example for its application to mathematics is the classification theorem of surfaces. By a combinatorial view, these connected sums of tori, or these connected sums of projective planes used in this theorem are just bouquets on surfaces. In Section 2.4, we will use them to construct completed multi-spaces.

C2. Complete graphs

A *complete graph* $K_n = (V_c, E_c; I_c)$ is a simple graph with $V_c = \{v_1, v_2, \cdots, v_n\}$, $E_c = \{e_{ij}, 1 \leq i, j \leq n, i \neq j\}$ and $I_c(e_{ij}) = (v_i, v_j)$. Since K_n is simple, it can be also defined by a pair (V, E) with $V = \{v_1, v_2, \cdots, v_n\}$ and $E = \{v_iv_j, 1 \leq i, j \leq n, i \neq j\}$. The one edge graph K_2 and the triangle graph K_3 are both complete graphs.

A complete subgraph in a graph is called a *clique*. Obviously, every graph is a union of its cliques.

C3. r-Partite graphs

A simple graph $G = (V, E; I)$ is *r-partite* for an integer $r \geq 1$ if it is possible to partition V into r subsets $V_1, V_2, \cdots, V_r$ such that for $\forall e \in E$, $I(e) = (v_i, v_j)$ for $v_i \in V_i$, $v_j \in V_j$ and $i \neq j, 1 \leq i, j \leq r$. Notice that by definition, there are no edges between vertices of V_i, $1 \leq i \leq r$. A vertex subset of this kind in a graph is called an *independent vertex subset*.

For $n = 2$, a 2-partite graph is also called a *bipartite graph*. It can be shown that *a graph is bipartite if and only if there are no odd circuits in this graph*. As a consequence, a tree or a forest is a bipartite graph since they are circuit-free.

Let $G = (V, E; I)$ be an r-partite graph and let $V_1, V_2, \cdots, V_r$ be its r-partite vertex subsets. If there is an edge $e_{ij} \in E$ for $\forall v_i \in V_i$ and $\forall v_j \in V_j$, where $1 \leq i, j \leq r, i \neq j$ such that $I(e) = (v_i, v_j)$, then we call G a *complete r-partite graph*, denoted by $G = K(|V_1|, |V_2|, \cdots, |V_r|)$. Whence, a complete graph is just a complete 1-partite graph. For an integer n, the complete bipartite graph $K(n, 1)$ is called a *star*. For a graph G, we have an obvious formula shown in the following, which corresponds to the neighborhood decomposition in topology.

$$E(G) = \bigcup_{x \in V(G)} E_G(x, N_G(x)).$$

C4. Regular graphs

A graph G is *regular of valency k* if $\rho_G(u) = k$ for $\forall u \in V(G)$. These graphs are also called *k-regular*. There 3-regular graphs are referred to as *cubic graphs*. A k-regular vertex-spanning subgraph of a graph G is also called a k-factor of G.

For a k-regular graph G, since $k|V(G)| = 2|E(G)|$, thereby one of k and $|V(G)|$ must be an even number, i.e., there are no k-regular graphs of odd order with $k \equiv 1(mod 2)$. A complete graph K_n is $(n-1)$-regular and a complete s-partite graph $K(p_1, p_2, \cdots, p_s)$ of order n with $p_1 = p_2 = \cdots = p_s = p$ is $(n-p)$-regular.

In regular graphs, those of simple graphs with high symmetry are particularly important to mathematics. They are related combinatorics with group theory and crystal geometry. We briefly introduce them in the following.

Let G be a simple graph and H a subgroup of Aut$G$. G is said to be *H-vertex transitive*, *H-edge transitive* or *H-symmetric* if H acts transitively on the vertex set $V(G)$, the edge set $E(G)$ or the set of ordered adjacent pairs of vertex of G. If $H = \mathrm{Aut}G$, an H-vertex transitive, an H-edge transitive or an H-symmetric graph is abbreviated to a *vertex-transitive*, an *edge-transitive* or a *symmetric* graph.

Now let Γ be a finite generated group and $S \subseteq \Gamma$ such that $1_\Gamma \notin S$ and $S^{-1} = \{x^{-1} | x \in S\} = S$. A *Cayley graph* $Cay(\Gamma : S)$ is a simple graph with vertex set $V(G) = \Gamma$ and edge set $E(G) = \{(g,h)|g^{-1}h \in S\}$. By the definition of Cayley graphs, we know that *a Cayley graph* $Cay(\Gamma : S)$ *is complete if and only if* $S = \Gamma \setminus \{1_\Gamma\}$ *and connected if and only if* $\Gamma = \langle S \rangle$.

Theorem 2.1.12 *A Cayley graph* $Cay(\Gamma : S)$ *is vertex-transitive.*

Proof For $\forall g \in \Gamma$, define a permutation ζ_g on $V(\mathrm{Cay}(\Gamma : S)) = \Gamma$ by $\zeta_g(h) = gh, h \in \Gamma$. Then ζ_g is an automorphism of Cay$(\Gamma : S)$ for $(h,k) \in E(\mathrm{Cay}(\Gamma : S)) \Rightarrow h^{-1}k \in S \Rightarrow (gh)^{-1}(gk) \in S \Rightarrow (\zeta_g(h), \zeta_g(k)) \in E(\mathrm{Cay}(\Gamma : S))$.

Now we know that $\zeta_{kh^{-1}}(h) = (kh^{-1})h = k$ for $\forall h, k \in \Gamma$. Whence, Cay$(\Gamma : S)$ is vertex-transitive. ♮

Not every vertex-transitive graph is a Cayley graph of a finite group. For example, the Petersen graph is vertex-transitive but not a Cayley graph(see [10], [21]] and [110] for details). However, every vertex-transitive graph can be constructed almost like a Cayley graph. This result is due to Sabidussi in 1964. The readers can see [110] for a complete proof of this result.

Theorem 2.1.13 *Let G be a vertex-transitive graph whose automorphism group is A. Let $H = A_b$ be the stabilizer of $b \in V(G)$. Then G is isomorphic with the group-coset graph $C(A/H, S)$, where S is the set of all automorphisms x of G such that $(b, x(b)) \in E(G)$, $V(C(A/H,S)) = A/H$ and $E(C(A/H,S)) = \{(xH, yH)|x^{-1}y \in HSH\}$.*

C5. Planar graphs

Every graph is drawn on the plane. A graph is *planar* if it can be drawn on the plane in such a way that edges are disjoint expect possibly for endpoints. When we remove vertices and edges of a planar graph G from the plane, each remained connected region is called a *face* of G. The length of the boundary of a face is called its *valency*. Two planar graphs are shown in Fig.2.12.

Fig¸2.12

For a planar graph G, its order, size and number of faces are related by a well-known formula discovered by Euler.

Theorem 2.1.14 *let G be a planar graph with $\phi(G)$ faces. Then*

$$|G| - \varepsilon(G) + \phi(G) = 2.$$

Proof We can prove this result by employing induction on $\varepsilon(G)$. See [42] or [23], [69] for a complete proof. ♮

For an integer $s, s \geq 3$, an s-regular planar graph with the same length r for all faces is often called an (s, r)-*polyhedron*, which are completely classified by the ancient Greeks.

Theorem 2.1.15 There are exactly five polyhedrons, two of them are shown in Fig.2.12, the others are shown in Fig.2.13.

Fig¸2.13

Proof Let G be a k-regular planar graph with l faces. By definition, we know that $|G|k = \phi(G)l = 2\varepsilon(G)$. Whence, we get that $|G| = \frac{2\varepsilon(G)}{k}$ and $\phi(G) = \frac{2\varepsilon(G)}{l}$. According to Theorem 2.1.14, we get that

$$\frac{2\varepsilon(G)}{k} - \varepsilon(G) + \frac{2\varepsilon(G)}{l} = 2.$$

i.e.,

$$\varepsilon(G) = \frac{2}{\frac{2}{k} - 1 + \frac{2}{l}}.$$

Whence, $\frac{2}{k} + \frac{2}{l} - 1 > 0$. Since k, l are both integers and $k \geq 3, l \geq 3$, if $k \geq 6$, we get

$$\frac{2}{k} + \frac{2}{l} - 1 \leq \frac{2}{3} + \frac{2}{6} - 1 = 0.$$

Contradicts that $\frac{2}{k} + \frac{2}{l} - 1 > 0$. Therefore, $k \leq 5$. Similarly, $l \leq 5$. So we have $3 \leq k \leq 5$ and $3 \leq l \leq 5$. Calculation shows that all possibilities for (k, l) are $(k, l) = (3, 3), (3, 4), (3, 5), (4, 3)$ and $(5, 3)$. The $(3, 3)$ and $(3, 4)$ polyhedrons have be shown in Fig.2.12 and the remainder $(3, 5), (4, 3)$ and $(5, 3)$ polyhedrons are shown in Fig.2.13. $\natural$

An *elementary subdivision* on a graph G is a graph obtained from G replacing an edge $e = uv$ by a path uwv, where, $w \notin V(G)$. A *subdivision* of G is a graph obtained from G by a succession of elementary subdivision. A graph H is defined to be a *homeomorphism of* G if either $H \cong G$ or H is isomorphic to a subdivision of G. Kuratowski found the following characterization for planar graphs in 1930. For its a complete proof, see [9], [11] for details.

Theorem 2.1.16 *A graph is planar if and only if it contains no subgraph homeomorphic with* K_5 *or* $K(3,3)$.

(2) **Decomposition of graphs**

A complete graph K_6 with vertex set $\{1, 2, 3, 4, 5, 6\}$ has two families of subgraphs $\{C_6, C_3^1, C_3^2, P_2^1, P_2^2, P_2^3\}$ and $\{S_{1.5}, S_{1.4}, S_{1.3}, S_{1.2}, S_{1.1}\}$, such as those shown in Fig.2.14 and Fig.2.15.

Fig 2.14

Fig 2.15

We know that

$$E(K_6) = E(C_6)\bigcup E(C_3^1)\bigcup E(C_3^2)\bigcup E(P_2^1)\bigcup E(P_2^2)\bigcup E(P_2^3);$$

$$E(K_6) = E(S_{1.5})\bigcup E(S_{1.4})\bigcup E(S_{1.3})\bigcup E(S_{1.2})\bigcup E(S_{1.1}).$$

These formulae imply the conception of decomposition of graphs. For a graph G, a *decomposition* of G is a collection $\{H_i\}_{1\leq i\leq s}$ of subgraphs of G such that for any integer $i, 1 \leq i \leq s$, $H_i = \langle E_i\rangle$ for some subsets E_i of $E(G)$ and $\{E_i\}_{1\leq i\leq s}$ is a partition of $E(G)$, denoted by $G = H_1 \bigoplus H_2 \bigoplus \cdots \bigoplus H_s$. The following result is obvious.

Theorem 2.1.17 *Any graph G can be decomposed to bouquets and dipoles, in where K_2 is seen as a dipole $D_{0.1.0}$.*

Theorem 2.1.18 *For every positive integer n, the complete graph K_{2n+1} can be decomposed to n hamiltonian circuits.*

Proof For $n = 1$, K_3 is just a hamiltonian circuit. Now let $n \geq 2$ and $V(K_{2n+1}) = \{v_0, v_1, v_2, \cdots, v_{2n}\}$. Arrange these vertices $v_1, v_2, \cdots, v_{2n}$ on vertices of a regular $2n$-gon and place v_0 in a convenient position not in the $2n$-gon. For $i = 1, 2, \cdots, n$, we define the edge set of H_i to be consisted of v_0v_i, v_0v_{n+i} and edges parallel to v_iv_{i+1} or edges parallel to $v_{i-1}v_{i+1}$, where the subscripts are expressed modulo $2n$. Then we get that

$$K_{2n+1} = H_1 \bigoplus H_2 \bigoplus \cdots \bigoplus H_n$$

with each $H_i, 1 \leq i \leq n$ being a hamiltonian circuit

$$v_0v_iv_{i+1}v_{i-1}v_{i+1}v_{i-2}\cdots v_{n+i-1}v_{n+i+1}v_{n+i}v_0. \qquad \natural$$

Every Cayley graph of a finite group Γ can be decomposed into 1-factors or 2-factors in a natural way as stated in the following theorems.

Theorem 2.1.19 *Let G be a vertex-transitive graph and let H be a regular subgroup of* Aut$G$*. Then for any chosen vertex $x, x \in V(G)$, there is a factorization*

$$G = (\bigoplus_{y \in N_G(x), |H_{(x,y)}|=1} (x,y)^H) \bigoplus (\bigoplus_{y \in N_G(x), |H_{(x,y)}|=2} (x,y)^H),$$

for G such that $(x,y)^H$ is a 2-factor if $|H_{(x,y)}| = 1$ and a 1-factor if $|H_{(x,y)}| = 2$.

Proof First, We prove the following claims.

Claim 1 $\forall x \in V(G), x^H = V(G)$ *and* $H_x = 1_H$.

Claim 2 *For* $\forall (x,y), (u,w) \in E(G)$, $(x,y)^H \bigcap (u,w)^H = \emptyset$ *or* $(x,y)^H = (u,w)^H$.

Claims 1 and 2 are holden by definition.

Claim 3 *For* $\forall (x,y) \in E(G), |H_{(x,y)}| = 1$ *or* 2.

Assume that $|H_{(x,y)}| \neq 1$. Since we know that $(x,y)^h = (x,y)$, i.e., $(x^h, y^h) = (x,y)$ for any element $h \in H_{(x,y)}$. Thereby we get that $x^h = x$ and $y^h = y$ or $x^h = y$ and $y^h = x$. For the first case we know $h = 1_H$ by Claim 1. For the second, we get that $x^{h^2} = x$. Therefore, $h^2 = 1_H$.

Now if there exists an element $g \in H_{(x,y)} \setminus \{1_H, h\}$, then we get $x^g = y = x^h$ and $y^g = x = y^h$. Thereby we get $g = h$ by Claim 1, a contradiction. So we get that $|H_{(x,y)}| = 2$.

Claim 4 *For any* $(x,y) \in E(G)$, *if* $|H_{(x,y)}| = 1$, *then* $(x,y)^H$ *is a* 2*-factor.*

Because $x^H = V(G) \subset V(\left\langle (x,y)^H \right\rangle) \subset V(G)$, so $V(\left\langle (x,y)^H \right\rangle) = V(G)$. Therefore, $(x,y)^H$ is a spanning subgraph of G.

Since H acting on $V(G)$ is transitive, there exists an element $h \in H$ such that $x^h = y$. It is obvious that $o(h)$ is finite and $o(h) \neq 2$. Otherwise, we have $|H_{(x,y)}| \geq 2$, a contradiction. Now $(x,y)^{\langle h \rangle} = x x^h x^{h^2} \cdots x^{h^{o(h)-1}} x$ is a circuit in the graph G. Consider the right coset decomposition of H on $\langle h \rangle$. Suppose $H = \bigcup_{i=1}^{s} \langle h \rangle a_i$, $\langle h \rangle a_i \bigcap \langle h \rangle a_j = \emptyset$, if $i \neq j$, and $a_1 = 1_H$.

Now let $X = \{a_1, a_2, ..., a_s\}$. We know that for any $a, b \in X$, $(\langle h \rangle a) \bigcap (\langle h \rangle b) = \emptyset$ if $a \neq b$. Since $(x,y)^{\langle h \rangle a} = ((x,y)^{\langle h \rangle})^a$ and $(x,y)^{\langle h \rangle b} = ((x,y)^{\langle h \rangle})^b$ are also circuits, if $V(\left\langle (x,y)^{\langle h \rangle a} \right\rangle) \bigcap V(\left\langle (x,y)^{\langle h \rangle b} \right\rangle) \neq \emptyset$ for some $a, b \in X, a \neq b$, then there must be two elements $f, g \in \langle h \rangle$ such that $x^{fa} = x^{gb}$. According to Claim 1, we get that $fa = gb$, that is $ab^{-1} \in \langle h \rangle$. So $\langle h \rangle a = \langle h \rangle b$ and $a = b$, contradicts to the assumption that $a \neq b$.

Thereafter we know that $(x,y)^H = \bigcup_{a \in X} (x,y)^{\langle h \rangle a}$ is a disjoint union of circuits. So $(x,y)^H$ is a 2-factor of the graph G.

Claim 5 *For any* $(x,y) \in E(G)$, $(x,y)^H$ *is an* 1*-factor if* $|H_{(x,y)}| = 2$.

Similar to the proof of Claim 4, we know that $V(\left\langle (x,y)^H \right\rangle) = V(G)$ and $(x,y)^H$ is a spanning subgraph of the graph G.

Let $H_{(x,y)} = \{1_H, h\}$, where $x^h = y$ and $y^h = x$. Notice that $(x,y)^a = (x,y)$ for $\forall a \in H_{(x,y)}$. Consider the coset decomposition of H on $H_{(x,y)}$, we know that $H = \bigcup\limits_{i=1}^{t} H_{(x,y)}b_i$, where $H_{(x,y)}b_i \bigcap H_{(x,y)}b_j = \emptyset$ if $i \neq j, 1 \leq i,j \leq t$. Now let $L = \{H_{(x,y)}b_i, 1 \leq i \leq t\}$. We get a decomposition

$$(x,y)^H = \bigcup_{b \in L} (x,y)^b$$

for $(x,y)^H$. Notice that if $b = H_{(x,y)}b_i \in L$, $(x,y)^b$ is an edge of G. Now if there exist two elements $c, d \in L, c = H_{(x,y)}f$ and $d = H_{(x,y)}g$, $f \neq g$ such that $V(\langle (x,y)^c \rangle) \bigcap V(\left\langle (x,y)^d \right\rangle) \neq \emptyset$, there must be $x^f = x^g$ or $x^f = y^g$. If $x^f = x^g$, we get $f = g$ by Claim 1, contradicts to the assumption that $f \neq g$. If $x^f = y^g = x^{hg}$, where $h \in H_{(x,y)}$, we get $f = hg$ and $fg^{-1} \in H_{(x,y)}$, so $H_{(x,y)}f = H_{(x,y)}g$. According to the definition of L, we get $f = g$, also contradicts to the assumption that $f \neq g$. Therefore, $(x,y)^H$ is an 1-factor of the graph G.

Now we can prove the assertion in this theorem. According to Claim 1- Claim 4, we get that

$$G = (\bigoplus_{y \in N_G(x), |H_{(x,y)}|=1} (x,y)^H) \bigoplus (\bigoplus_{y \in N_G(x), |H_{(x,y)}|=2} (x,y)^H).$$

for any chosen vertex $x, x \in V(G)$. By Claims 5 and 6, we know that $(x,y)^H$ is a 2-factor if $|H_{(x,y)}| = 1$ and is a 1-factor if $|H_{(x,y)}| = 2$. Whence, the desired factorization for G is obtained. ♮

Now for a Cayley graph $\mathrm{Cay}(\Gamma : S)$, by Theorem 2.1.13, we can always choose the vertex $x = 1_\Gamma$ and H the right regular transformation group on Γ. After then, Theorem 2.1.19 can be restated as follows.

Theorem 2.1.20 *Let Γ be a finite group with a subset $S, S^{-1} = S$, $1_\Gamma \notin S$ and H is the right transformation group on Γ. Then there is a factorization*

$$G = (\bigoplus_{s \in S, s^2 \neq 1_\Gamma} (1_\Gamma, s)^H) \bigoplus (\bigoplus_{s \in S, s^2 = 1_\Gamma} (1_\Gamma, s)^H)$$

for the Cayley graph $\mathrm{Cay}(\Gamma : S)$ *such that $(1_\Gamma, s)^H$ is a 2-factor if $s^2 \neq 1_\Gamma$ and 1-factor if $s^2 = 1_\Gamma$.*

Proof For any $h \in H_{(1_\Gamma, s)}$, if $h \neq 1_\Gamma$, then we get that $1_\Gamma h = s$ and $sh = 1_\Gamma$, that is $s^2 = 1_\Gamma$. According to Theorem 2.1.19, we get the factorization for the Cayley graph $\mathrm{Cay}(\Gamma : S)$. ♮

More factorial properties for Cayley graphs of a finite group can be found in the reference [51].

2.1.4. **Operations on graphs**

For two given graphs $G_1 = (V_1.E_1; I_1)$ and $G_2 = (V_2, E_2; I_2)$, there are a number of ways to produce new graphs from G_1 and G_2. Some of them are described in the following.

Operation 1. **Union**

The *union* $G_1 \bigcup G_2$ of graphs G_1 and G_2 is defined by

$$V(G_1 \bigcup G_2) = V_1 \bigcup V_2,\ E(G_1 \bigcup G_2) = E_1 \bigcup E_2 \text{ and } I(E_1 \bigcup E_2) = I_1(E_1) \bigcup I_2(E_2).$$

If a graph consists of k disjoint copies of a graph H, $k \geq 1$, then we write $G = kH$. Therefore, we get that $K_6 = C_6 \bigcup 3K_2 \bigcup 2K_3 = \bigcup\limits_{i=1}^{5} S_{1.i}$ for graphs in Fig.2.14 and Fig.2.15 and generally, $K_n = \bigcup\limits_{i=1}^{n-1} S_{1.i}$. For an integer $k, k \geq 2$ and a simple graph G, kG is a multigraph with edge multiple k by definition.

By the definition of a union of two graphs, we get decompositions for some well-known graphs such as

$$B_n = \bigcup_{i=1}^{n} B_1(O), \quad D_{k,m,n} = (\bigcup_{i=1}^{k} B_1(O_1)) \bigcup (\bigcup_{i=1}^{m} K_2) \bigcup (\bigcup_{i=1}^{n} B_1(O_2)),$$

where $V(B_1)(O_1) = \{O_1\}, V(B_1)(O_2) = \{O_2\}$ and $V(K_2) = \{O_1, O_2\}$. By Theorem 1.18, we get that

$$K_{2n+1} = \bigcup_{i=1}^{n} H_i$$

with $H_i = v_0 v_i v_{i+1} v_{i-1} v_{i+1} v_{i-2} \cdots v_{n+i-1} v_{n+i+1} v_{n+i} v_0$.

In Fig.2.16, we show two graphs C_6 and K_4 with a nonempty intersection and their union $C_6 \bigcup K_4$.

Fig ¸2.16

Operation 2. **Join**

The *complement* $\overline{G}$ of a graph G is a graph with the vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in G. The *join* $G_1 + G_2$ of G_1 and G_2 is defined by

$$V(G_1 + G_2) = V(G_1) \bigcup V(G_2),$$

$$E(G_1 + G_2) = E(G_1) \bigcup E(G_2) \bigcup \{(u, v) | u \in V(G_1), v \in V(G_2)\}$$

and

$$I(G_1 + G_2) = I(G_1) \bigcup I(G_2) \bigcup \{I(u, v) = (u, v) | u \in V(G_1), v \in V(G_2)\}.$$

Using this operation, we can represent $K(m, n) \cong \overline{K_m} + \overline{K_n}$. The join graph of circuits C_3 and C_4 is given in Fig.2.17.

Fig ¸2.17

Operation 3. **Cartesian product**

The *cartesian product* $G_1 \times G_2$ of graphs G_1 and G_2 is defined by $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 \times G_2$ are adjacent if and only if either $u_1 = v_1$ and $(u_2, v_2) \in E(G_2)$ or $u_2 = v_2$ and $(u_1, v_1) \in E(G_1)$.

For example, the cartesian product $C_3 \times C_3$ of circuits C_3 and C_3 is shown in Fig.2.18.

Fig ¸2.18

§2.2 Multi-Voltage Graphs

There is a convenient way for constructing a covering space of a graph G in topological graph theory, i.e., by a voltage graph (G,α) of G which was firstly introduced by Gustin in 1963 and then generalized by Gross in 1974. Youngs extensively used voltage graphs in proving Heawood map coloring theorem([23]). Today, it has become a convenient way for finding regular maps on surface. In this section, we generalize voltage graphs to two types of multi-voltage graphs by using finite multi-groups.

2.2.1. **Type** 1

Definition 2.2.1 *Let* $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ *be a finite multi-group with an operation set* $O(\widetilde{\Gamma}) = \{\circ_i | 1 \leq i \leq n\}$ *and* G *a graph. If there is a mapping* $\psi : X_{\frac{1}{2}}(G) \rightarrow \widetilde{\Gamma}$ *such that* $\psi(e^{-1}) = (\psi(e^{+}))^{-1}$ *for* $\forall e^{+} \in X_{\frac{1}{2}}(G)$, *then* (G,ψ) *is called a multi-voltage graph of type* 1.

Geometrically, a multi-voltage graph is nothing but a weighted graph with weights in a multi-group. Similar to voltage graphs, the importance of a multi-voltage graph is in its *lifting* defined in the next definition.

Definition 2.2.2 *For a multi-voltage graph* (G,ψ) *of type* 1, *the lifting graph* $G^{\psi} = (V(G^{\psi}),\ E(G^{\psi}); I(G^{\psi}))$ *of* (G,ψ) *is defined by*

$$V(G^{\psi}) = V(G) \times \widetilde{\Gamma},$$

$$E(G^{\psi}) = \{(u_a, v_{a\circ b}) | e^{+} = (u,v) \in X_{\frac{1}{2}}(G), \psi(e^{+}) = b, a \circ b \in \widetilde{\Gamma}\}$$

and

$$I(G^{\psi}) = \{(u_a, v_{a\circ b}) | I(e) = (u_a, v_{a\circ b})\ if\ e = (u_a, v_{a\circ b}) \in E(G^{\psi})\}.$$

For abbreviation, a vertex (x,g) in G^{ψ} is denoted by x_g. Now for $\forall v \in V(G)$, $v \times \widetilde{\Gamma} = \{v_g | g \in \widetilde{\Gamma}\}$ is called a *fiber over* v, denoted by F_v. Similarly, for $\forall e^{+} = (u,v) \in X_{\frac{1}{2}}(G)$ with $\psi(e^{+}) = b$, all edges $\{(u_g, v_{g\circ b}) | g, g \circ b \in \widetilde{\Gamma}\}$ is called the *fiber over* e, denoted by F_e.

For a multi-voltage graph (G,ψ) and its lifting G^{ψ}, there is a *natural projection* $p : G^{\psi} \rightarrow G$ defined by $p(F_v) = v$ for $\forall v \in V(G)$. It can be verfied that $p(F_e) = e$ for $\forall e \in E(G)$.

Choose $\widetilde{\Gamma} = \Gamma_1 \bigcup \Gamma_2$ with $\Gamma_1 = \{1, a, a^2\}$, $\Gamma_2 = \{1, b, b^2\}$ and $a \neq b$. A multi-voltage graph and its lifting are shown in Fig.2.19.

Fig ¸2.19

Let $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ be a finite multi-group with groups $(\Gamma_i; \circ_i), 1 \leq i \leq n$. Similar to the unique walk lifting theorem for voltage graphs, we know the following *walk multi-lifting theorem* for multi-voltage graphs of type 1.

Theorem 2.2.1 *Let $W = e^1e^2\cdots e^k$ be a walk in a multi-voltage graph (G, ψ) with initial vertex u. Then there exists a lifting W^ψ start at u_a in G^ψ if and only if there are integers $i_1, i_2, \cdots, i_k$ such that*

$$a \circ_{i_1} \psi(e_1^+) \circ_{i_2} \cdots \circ_{i_{j-1}} \psi(e_j^+) \in \Gamma_{i_{j+1}} \textit{ and } \psi(e_{j+1}^+) \in \Gamma_{i_{j+1}}$$

for any integer $j, 1 \leq j \leq k$

Proof Consider the first semi-arc in the walk W, i.e., e_1^+. Each lifting of e_1 must be $(u_a, u_{a\circ\psi(e_1^+)})$. Whence, there is a lifting of e_1 in G^ψ if and only if there exists an integer i_1 such that $\circ = \circ_{i_1}$ and $a, a \circ_{i_1} \psi(e_1^+) \in \Gamma_{i_1}$.

Now if we have proved there is a lifting of a sub-walk $W_l = e_1e_2\cdots e_l$ in G^ψ if and only if there are integers $i_1, i_2, \cdots, i_l$, $1 \leq l < k$ such that

$$a \circ_{i_1} \psi(e_1^+) \circ_{i_2} \cdots \circ_{i_{j-1}} \psi(e_j^+) \in \Gamma_{i_{j+1}}, \quad \psi(e_{j+1}^+) \in \Gamma_{i_{j+1}}$$

for any integer $j, 1 \leq j \leq l$, we consider the semi-arc e_{l+1}^+. By definition, there is a lifting of e_{l+1}^+ in G^ψ with initial vertex $u_{a\circ_{i_1}\psi(e_1^+)\circ_{i_2}\cdots\circ_{i_{j-1}}\psi(e_l^+)}$ if and only if there exists an integer i_{l+1} such that

$$a \circ_{i_1} \psi(e_1^+) \circ_{i_2} \cdots \circ_{i_{j-1}} \psi(e_l^+) \in \Gamma_{l+1} \text{ and } \psi(e_{l+1}^+) \in \Gamma_{l+1}.$$

According to the induction principle, we know that there exists a lifting W^ψ start at u_a in G^ψ if and only if there are integers $i_1, i_2, \cdots, i_k$ such that

$$a \circ_{i_1} \psi(e_1^+) \circ_{i_2} \cdots \circ_{i_{j-1}} \psi(e_j^+) \in \Gamma_{i_{j+1}}, \text{ and } \psi(e_{j+1}^+) \in \Gamma_{i_{j+1}}$$

for any integer $j, 1 \leq j \leq k$. $\natural$

For two elements $g, h \in \widetilde{\Gamma}$, if there exist integers $i_1, i_2, \cdots, i_k$ such that $g, h \in \bigcap\limits_{j=1}^{k} \Gamma_{i_j}$ but for $\forall i_{k+1} \in \{1, 2, \cdots, n\} \setminus \{i_1, i_2, \cdots, i_k\}$, $g, h \notin \bigcap\limits_{j=1}^{k+1} \Gamma_{i_j}$, we call $k = \Pi[g, h]$

the *joint number of* g *and* h. Denote $O(g,h)=\{\circ_{i_j};1\leq j\leq k\}$. Define $\widetilde{\Pi}[g,h]=\sum\limits_{\circ\in O(\widetilde{\Gamma})}\Pi[g,g\circ h]$, where $\Pi[g,g\circ h]=\Pi[g\circ h,h]=0$ if $g\circ h$ does not exist in $\widetilde{\Gamma}$. According to Theorem 2.2.1, we get an upper bound for the number of liftings in G^{ψ} for a walk W in (G,ψ).

Corollary 2.2.1 *If those conditions in Theorem* 2.2.1 *hold, the number of liftings of* W *with initial vertex* u_a *in* G^{ψ} *is not in excess of*

$$\widetilde{\Pi}[a,\psi(e_1^+)]\times$$
$$\prod_{i=1}^{k}\sum_{\circ_1\in O(a,\psi(e_1^+))}\cdots\sum_{\circ_i\in O(a;\circ_j,\psi(e_j^+),1\leq j\leq i-1)}\widetilde{\Pi}[a\circ_1\psi(e_1^+)\circ_2\cdots\circ_i\psi(e_i^+),\psi(e_{i+1}^+)],$$

where $O(a;\circ_j,\psi(e_j^+),1\leq j\leq i-1)=O(a\circ_1\psi(e_1^+)\circ_2\cdots\circ_{i-1}\psi(e_{i-1}^+),\psi(e_i^+))$.

The natural projection of a multi-voltage graph is not regular in general. For finding a regular covering of a graph, a typical class of multi-voltage graphs is the case of $\Gamma_i=\Gamma$ for any integer $i,1\leq i\leq n$ in these multi-groups $\widetilde{\Gamma}=\bigcup\limits_{i=1}^{n}\Gamma_i$. In this case, we can find the exact number of liftings in G^{ψ} for a walk in (G,ψ).

Theorem 2.2.2 *Let* $\widetilde{\Gamma}=\bigcup\limits_{i=1}^{n}\Gamma$ *be a finite multi-group with groups* $(\Gamma;\circ_i),1\leq i\leq n$ *and let* $W=e^1e^2\cdots e^k$ *be a walk in a multi-voltage graph* (G,ψ) , $\psi:X_{\frac{1}{2}}(G)\rightarrow\widetilde{\Gamma}$ *of type* 1 *with initial vertex* u. *Then there are* n^k *liftings of* W *in* G^{ψ} *with initial vertex* u_a *for* $\forall a\in\widetilde{\Gamma}$.

Proof The existence of lifting of W in G^{ψ} is obvious by Theorem 2.2.1. Consider the semi-arc e_1^+. Since $\Gamma_i=\Gamma$ for $1\leq i\leq n$, we know that there are n liftings of e_1 in G^{ψ} with initial vertex u_a for any $a\in\widetilde{\Gamma}$, each with a form $(u_a,u_{a\circ\psi(e_1^+)}),\circ\in O(\widetilde{\Gamma})$.

Now if we have gotten $n^s,1\leq s\leq k-1$ liftings in G^{ψ} for a sub-walk $W_s=e^1e^2\cdots e^s$. Consider the semi-arc e_{s+1}^+. By definition we know that there are also n liftings of e_{s+1} in G^{ψ} with initial vertex $u_{a\circ_{i_1}\psi(e_1^+)\circ_{i_2}\cdots\circ_s\psi(e_s^+)}$, where $\circ_i\in O(\widetilde{\Gamma}),1\leq i\leq s$. Whence, there are n^{s+1} liftings in G^{ψ} for a sub-walk $W_s=e^1e^2\cdots e^{s+1}$ in $(G;\psi)$.

By the induction principle, we know the assertion is true. ♮

Corollary 2.2.2([23]) *Let* W *be a walk in a voltage graph* $(G,\psi),\psi:X_{\frac{1}{2}}(G)\rightarrow\Gamma$ *with initial vertex* u. *Then there is an unique lifting of* W *in* G^{ψ} *with initial vertex* u_a *for* $\forall a\in\Gamma$.

If a lifting W^{ψ} of a multi-voltage graph (G,ψ) is the same as the lifting of a voltage graph $(G,\alpha),\alpha:X_{\frac{1}{2}}(G)\rightarrow\Gamma_i$, then this lifting is called a *homogeneous* *lifting of* Γ_i. For lifting a circuit in a multi-voltage graph, we get the following result.

Theorem 2.2.3 *Let* $\widetilde{\Gamma} = \bigcup_{i=1}^{n} \Gamma$ *be a finite multi-group with groups* $(\Gamma; \circ_i), 1 \leq i \leq n$, $C = u_1 u_2 \cdots u_m u_1$ *a circuit in a multi-voltage graph* (G, ψ) *and* $\psi : X_{\frac{1}{2}}(G) \to \widetilde{\Gamma}$. *Then there are* $\frac{|\Gamma|}{o(\psi(C,\circ_i))}$ *homogenous liftings of length* $o(\psi(C, \circ_i))m$ *in* G^{ψ} *of* C *for any integer* $i, 1 \leq i \leq n$, *where* $\psi(C, \circ_i) = \psi(u_1, u_2) \circ_i \psi(u_2, u_3) \circ_i \cdots \circ_i \psi(u_{m-1}, u_m) \circ_i \psi(u_m, u_1)$ *and there are*

$$\sum_{i=1}^{n} \frac{|\Gamma|}{o(\psi(C, \circ_i))}$$

homogenous liftings of C *in* G^{ψ} *altogether.*

Proof According to Theorem 2.2.2, there are liftings with initial vertex $(u_1)_a$ of C in G^{ψ} for $\forall a \in \widetilde{\Gamma}$. Whence, for any integer $i, 1 \leq i \leq n$, walks

$$W_a = (u_1)_a (u_2)_{a \circ_i \psi(u_1,u_2)} \cdots (u_m)_{a \circ_i \psi(u_1,u_2) \circ_i \cdots \circ_i \psi(u_{m-1},u_m)} (u_1)_{a \circ_i \psi(C, \circ_i)},$$

$$\begin{aligned} W_{a \circ_i \psi(C,\circ_i)} &= (u_1)_{a \circ_i \psi(C,\circ_i)} (u_2)_{a \circ_i \psi(C,\circ_i) \circ_i \psi(u_1,u_2)} \\ &\cdots (u_m)_{a \circ_i \psi(C,\circ_i) \circ_i \psi(u_1,u_2) \circ_i \cdots \circ_i \psi(u_{m-1},u_m)} (u_1)_{a \circ_i \psi^2(C,\circ_i)}, \end{aligned}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

and

$$\begin{aligned} W_{a \circ_i \psi^{o(\psi(C,\circ_i))-1}(C,\circ_i)} &= (u_1)_{a \circ_i \psi^{o(\psi(C,\circ_i))-1}(C,\circ_i)} (u_2)_{a \circ_i \psi^{o(\psi(C,\circ_i))-1}(C,\circ_i) \circ_i \psi(u_1,u_2)} \\ &\cdots (u_m)_{a \circ_i \psi^{o(\psi(C,\circ_i))-1}(C,\circ_i) \circ_i \psi(u_1,u_2) \circ_i \cdots \circ_i \psi(u_{m-1},u_m)} (u_1)_a \end{aligned}$$

are attached end-to-end to form a circuit of length $o(\psi(C, \circ_i))m$. Notice that there are $\frac{|\Gamma|}{o(\psi(C,\circ_i))}$ left cosets of the cyclic group generated by $\psi(C, \circ_i)$ in the group $(\Gamma, \circ_i)$ and each is correspondent with a homogenous lifting of C in G^{ψ}. Therefore, we get

$$\sum_{i=1}^{n} \frac{|\Gamma|}{o(\psi(C, \circ_i))}$$

homogenous liftings of C in G^{ψ}. ♮

Corollary 2.2.3([23]) *Let* C *be a* k*-circuit in a voltage graph* (G, ψ) *such that the order of* $\psi(C, \circ)$ *is* m *in the voltage group* $(\Gamma; \circ)$. *Then each component of the preimage* $p^{-1}(C)$ *is a* km*-circuit, and there are* $\frac{|\Gamma|}{m}$ *such components.*

The lifting G^ζ of a multi-voltage graph (G,ζ) of type 1 has a natural decomposition described in the next result.

Theorem 2.2.4 *Let $(G,\zeta),\zeta: X_{\frac{1}{2}}(G)\rightarrow\widetilde{\Gamma}=\bigcup\limits_{i=1}^{n}\Gamma_i$, be a multi-voltage graph of type 1. Then*

$$G^\zeta=\bigoplus_{i=1}^{n}H_i,$$

where H_i is an induced subgraph $\langle E_i\rangle$ of G^ζ for an integer $i, 1\leq i\leq n$ with

$$E_i=\{(u_a,v_{a\circ_i b})|a,b\in\Gamma_i \text{ and } (u,v)\in E(G),\zeta(u,v)=b\}.$$

For a finite multi-group $\widetilde{\Gamma}=\bigcup\limits_{i=1}^{n}\Gamma_i$ with an operation set $O(\widetilde{\Gamma})=\{\circ_i,1\leq i\leq n\}$ and a graph G, if there exists a decomposition $G=\bigoplus\limits_{j=1}^{n}H_i$ and we can associate each element $g_i\in\Gamma_i$ a homeomorphism φ_{g_i} on the vertex set $V(H_i)$ for any integer $i,1\leq i\leq n$ such that

(i) $\varphi_{g_i\circ_i h_i}=\varphi_{g_i}\times\varphi_{h_i}$ for all $g_i,h_i\in\Gamma_i$, where $\times$ is an operation between homeomorphisms;

(ii) φ_{g_i} is the identity homeomorphism if and only if g_i is the identity element of the group $(\Gamma_i;\circ_i)$,

then we say this association to be a *subaction of a multi-group $\widetilde{\Gamma}$ on the graph G*. If there exists a subaction of $\widetilde{\Gamma}$ on G such that $\varphi_{g_i}(u)=u$ only if $g_i=\mathbf{1}_{\Gamma_i}$ for any integer $i,1\leq i\leq n$, $g_i\in\Gamma_i$ and $u\in V_i$, then we call it a *fixed-free subaction*.

A *left subaction lA* of $\widetilde{\Gamma}$ on G^ψ is defined as follows:

For any integer $i,1\leq i\leq n$, let $V_i=\{u_a|u\in V(G),a\in\widetilde{\Gamma}\}$ and $g_i\in\Gamma_i$. Define $lA(g_i)(u_a)=u_{g_i\circ_i a}$ if $a\in V_i$. Otherwise, $g_i(u_a)=u_a$.

Then the following result holds.

Theorem 2.2.5 *Let (G,ψ) be a multi-voltage graph with $\psi: X_{\frac{1}{2}}(G)\rightarrow\widetilde{\Gamma}=\bigcup\limits_{i=1}^{n}\Gamma_i$ and $G=\bigoplus\limits_{j=1}^{n}H_i$ with $H_i=\langle E_i\rangle$, $1\leq i\leq n$, where $E_i=\{(u_a,v_{a\circ_i b})|a,b\in\Gamma_i$ and $(u,v)\in E(G),\zeta(u,v)=b\}$. Then for any integer $i,1\leq i\leq n$,*

(i) for $\forall g_i\in\Gamma_i$, the left subaction $lA(g_i)$ is a fixed-free subaction of an automorphism of H_i;

(ii) Γ_i is an automorphism group of H_i.

Proof Notice that $lA(g_i)$ is a one-to-one mapping on $V(H_i)$ for any integer $i,1\leq i\leq n$, $\forall g_i\in\Gamma_i$. By the definition of a lifting, an edge in H_i has the form $(u_a,v_{a\circ_i b})$ if $a,b\in\Gamma_i$. Whence,

$$(lA(g_i)(u_a),lA(g_i)(v_{a\circ_i b}))=(u_{g_i\circ_i a},v_{g_i\circ_i a\circ_i b})\in E(H_i).$$

As a result, $lA(g_i)$ is an automorphism of the graph H_i.

Notice that $lA : \Gamma_i \to \mathrm{Aut}H_i$ is an injection from Γ_i to $\mathrm{Aut}G^{\psi}$. Since $lA(g_i) \neq lA(h_i)$ for $\forall g_i, h_i \in \Gamma_i, g_i \neq h_i, 1 \leq i \leq n$. Otherwise, if $lA(g_i) = lA(h_i)$ for $\forall a \in \Gamma_i$, then $g_i \circ_i a = h_i \circ_i a$. Whence, $g_i = h_i$, a contradiction. Therefore, Γ_i is an automorphism group of H_i.

For any integer $i, 1 \leq i \leq n$, $g_i \in \Gamma_i$, it is implied by definition that $lA(g_i)$ is a fixed-free subaction on G^{ψ}. This completes the proof. ♮

Corollary 2.2.4([23]) *Let (G, α) be a voltage graph with $\alpha : X_{\frac{1}{2}}(G) \to \Gamma$. Then Γ is an automorphism group of G^{α}.*

For a finite multi-group $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ action on a graph $\widetilde{G}$, the vertex orbit $orb(v)$ of a vertex $v \in V(\widetilde{G})$ and the edge orbit $orb(e)$ of an edge $e \in E(\widetilde{G})$ are defined as follows:

$$orb(v) = \{g(v) | g \in \widetilde{\Gamma}\} \text{ and } orb(e) = \{g(e) | g \in \widetilde{\Gamma}\}.$$

The *quotient graph* $\widetilde{G}/\widetilde{\Gamma}$ of $\widetilde{G}$ under the action of $\widetilde{\Gamma}$ is defined by

$$V(\widetilde{G}/\widetilde{\Gamma}) = \{\ orb(v) \mid v \in V(\widetilde{G})\},\quad E(\widetilde{G}/\widetilde{\Gamma}) = \{orb(e) | e \in E(\widetilde{G})\}$$

and

$$I(orb(e)) = (orb(u), orb(v))\ if\ there\ exists\ (u, v) \in E(\widetilde{G})$$

.

For example, a quotient graph is shown in Fig.2.20, where, $\widetilde{\Gamma} = Z_5$.

Fig 2.20

Then we get a necessary and sufficient condition for the lifting of a multi-voltage graph in next result.

Theorem 2.2.6 *If the subaction $\mathcal{A}$ of a finite multi-group $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ on a graph $\widetilde{G} = \bigoplus\limits_{i=1}^{n} H_i$ is fixed-free, then there is a multi-voltage graph $(\widetilde{G}/\widetilde{\Gamma}, \zeta)$, $\zeta : X_{\frac{1}{2}}(\widetilde{G}/\widetilde{\Gamma}) \to \widetilde{\Gamma}$ of type 1 such that*

$$\widetilde{G} \cong (\widetilde{G}/\widetilde{\Gamma})^{\zeta}.$$

Proof First, we choose positive directions for edges of $\widetilde{G}/\widetilde{\Gamma}$ and $\widetilde{G}$ so that the quotient map $q_{\widetilde{\Gamma}} : \widetilde{G} \to \widetilde{G}/\widetilde{\Gamma}$ is direction-preserving and that the action $\mathcal{A}$ of $\widetilde{\Gamma}$ on $\widetilde{G}$ preserves directions. Next, for any integer $i, 1 \leq i \leq n$ and $\forall v \in V(\widetilde{G}/\widetilde{\Gamma})$, label one vertex of the orbit $q_{\widetilde{\Gamma}}^{-1}(v)$ in $\widetilde{G}$ as $v_{1_{\Gamma_i}}$ and for every group element $g_i \in \Gamma_i, g_i \neq 1_{\Gamma_i}$, label the vertex $\mathcal{A}(g_i)(v_{1_{\Gamma_i}})$ as v_{g_i}. Now if the edge e of $\widetilde{G}/\widetilde{\Gamma}$ runs from u to w, we assigns the label e_{g_i} to the edge of the orbit $q_{\Gamma_i}^{-1}(e)$ that originates at the vertex u_{g_i}. Since Γ_i acts freely on H_i, there are just $|\Gamma_i|$ edges in the orbit $q_{\Gamma_i}^{-1}(e)$ for each integer $i, 1 \leq i \leq n$, one originating at each of the vertices in the vertex orbit $q_{\Gamma_i}^{-1}(v)$. Thus the choice of an edge to be labelled e_{g_i} is unique for any integer $i, 1 \leq i \leq n$. Finally, if the terminal vertex of the edge $e_{1_{\Gamma_i}}$ is w_{h_i}, one assigns a voltage h_i to the edge e in the quotient $\widetilde{G}/\widetilde{\Gamma}$, which enables us to get a multi-voltage graph $(\widetilde{G}/\widetilde{\Gamma}, \zeta)$. To show that this labelling of edges in $q_{\Gamma_i}^{-1}(e)$ and the choice of voltages $h_i, 1 \leq i \leq n$ for the edge e really yields an isomorphism $\vartheta : \widetilde{G} \to (\widetilde{G}/\widetilde{\Gamma})^{\zeta}$, one needs to show that for $\forall g_i \in \Gamma_i, 1 \leq i \leq n$ that the edge e_{g_i} terminates at the vertex $w_{g_i \circ_i h_i}$. However, since $e_{g_i} = \mathcal{A}(g_i)(e_{1_{\Gamma_i}})$, the terminal vertex of the edge e_{g_i} must be the terminal vertex of the edge $\mathcal{A}(g_i)(e_{1_{\Gamma_i}})$, which is

$$\mathcal{A}(g_i)(w_{h_i}) = \mathcal{A}(g_i)\mathcal{A}(h_i)(w_{1_{\Gamma_i}}) = \mathcal{A}(g_i \circ_i h_i)(w_{1_{\Gamma_i}}) = w_{g_i \circ_i h_i}.$$

Under this labelling process, the isomorphism $\vartheta : \widetilde{G} \to (\widetilde{G}/\widetilde{\Gamma})^{\zeta}$ identifies orbits in $\widetilde{G}$ with fibers of G^{ζ}. Moreover, it is defined precisely so that the action of $\widetilde{\Gamma}$ on $\widetilde{G}$ is consistent with the left subaction lA on the lifting graph G^{ζ}. This completes the proof. $\natural$

Corollary 2.2.5([23]) *Let Γ be a group acting freely on a graph $\widetilde{G}$ and let G be the resulting quotient graph. Then there is an assignment α of voltages in Γ to the quotient graph G and a labelling of the vertices $\widetilde{G}$ by the elements of $V(G) \times \Gamma$ such that $\widetilde{G} = G^{\alpha}$ and that the given action of Γ on $\widetilde{G}$ is the natural left action of Γ on G^{α}.*

2.2.2. **Type** 2

Definition 2.2.3 *Let $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ be a finite multi-group and let G be a graph with vertices partition $V(G) = \bigcup\limits_{i=1}^{n} V_i$. For any integers $i, j, 1 \leq i, j \leq n$, if there is a mapping $\tau : X_{\frac{1}{2}}(\langle E_G(V_i, V_j)\rangle) \to \Gamma_i \bigcap \Gamma_j$ and $\varsigma : V_i \to \Gamma_i$ such that $\tau(e^{-1}) = (\tau(e^+))^{-1}$ for $\forall e^+ \in X_{\frac{1}{2}}(G)$ and the vertex subset V_i is associated with the group $(\Gamma_i, \circ_i)$ for any integer $i, 1 \leq i \leq n$, then (G, τ, ς) is called a multi-voltage graph of type* 2.

Similar to multi-voltage graphs of type 1, we construct a lifting from a multi-voltage graph of type 2.

Definition 2.2.4 *For a multi-voltage graph* (G,τ,ς) *of type 2, the lifting graph* $G^{(\tau,\varsigma)} = (V(G^{(\tau,\varsigma)}),\ E(G^{(\tau,\varsigma)}); I(G^{(\tau,\varsigma)}))$ *of* (G,τ,ς) *is defined by*

$$V(G^{(\tau,\varsigma)}) = \bigcup_{i=1}^{n}\{V_i \times \Gamma_i\},$$

$$E(G^{(\tau,\varsigma)}) = \{(u_a, v_{a\circ b}) | e^+ = (u,v) \in X_{\frac{1}{2}}(G), \psi(e^+) = b, a\circ b \in \widetilde{\Gamma}\}$$

and

$$I(G^{(\tau,\varsigma)}) = \{(u_a, v_{a\circ b}) | I(e) = (u_a, v_{a\circ b})\ if\ e = (u_a, v_{a\circ b}) \in E(G^{(\tau,\varsigma)})\}.$$

Two multi-voltage graphs of type 2 are shown on the left and their lifting on the right in (a) and (b) of Fig.21. In where, $\widetilde{\Gamma} = Z_2 \bigcup Z_3$, $V_1 = \{u\}$, $V_2 = \{v\}$ and $\varsigma : V_1 \to Z_2$, $\varsigma : V_2 \to Z_3$.

Fig 2.21

Theorem 2.2.7 *Let* (G,τ,ς) *be a multi-voltage graph of type* 2 *and let* $W_k = u_1u_2\cdots u_k$ *be a walk in* G. *Then there exists a lifting of* $W^{(\tau,\varsigma)}$ *with an initial vertex* $(u_1)_a, a \in \varsigma^{-1}(u_1)$ *in* $G^{(\tau,\varsigma)}$ *if and only if* $a \in \varsigma^{-1}(u_1)\bigcap\varsigma^{-1}(u_2)$ *and for any integer* $s, 1 \le s < k$, $a \circ_{i_1} \tau(u_1u_2) \circ_{i_2} \tau(u_2u_3) \circ_{i_3} \cdots \circ_{i_{s-1}} \tau(u_{s-2}u_{s-1}) \in \varsigma^{-1}(u_{s-1})\bigcap\varsigma^{-1}(u_s)$, *where* $\circ_{i_j}$ *is an operation in the group* $\varsigma^{-1}(u_{j+1})$ *for any integer* $j, 1 \le j \le s$.

Proof By the definition of the lifting of a multi-voltage graph of type 2, there exists a lifting of the edge u_1u_2 in $G^{(\tau,\varsigma)}$ if and only if $a\circ_{i_1}\tau(u_1u_2) \in \varsigma^{-1}(u_2)$, where $\circ_{i_j}$ is an operation in the group $\varsigma^{-1}(u_2)$. Since $\tau(u_1u_2) \in \varsigma^{-1}(u_1)\bigcap\varsigma^{-1}(u_2)$, we get that $a \in \varsigma^{-1}(u_1)\bigcap\varsigma^{-1}(u_2)$. Similarly, there exists a lifting of the subwalk $W_2 = u_1u_2u_3$ in $G^{(\tau,\varsigma)}$ if and only if $a \in \varsigma^{-1}(u_1)\bigcap\varsigma^{-1}(u_2)$ and $a \circ_{i_1} \tau(u_1u_2) \in \varsigma^{-1}(u_2)\bigcap\varsigma^{-1}(u_3)$.

Now assume there exists a lifting of the subwalk $W_l = u_1u_2u_3\cdots u_l$ in $G^{(\tau,\varsigma)}$ if and only if $a \circ_{i_1} \tau(u_1u_2) \circ_{i_2} \cdots \circ_{i_{t-2}} \tau(u_{t-2}u_{t-1}) \in \varsigma^{-1}(u_{t-1})\bigcap\varsigma^{-1}(u_t)$ for any integer $t, 1 \leq t \leq l$, where$\circ_{i_j}$is an operation in the group $\varsigma^{-1}(u_{j+1})$ for any integer $j, 1 \leq j \leq l$. We consider the lifting of the subwalk $W_{l+1} = u_1u_2u_3\cdots u_{l+1}$. Notice that if there exists a lifting of the subwalk W_l in $G^{(\tau,\varsigma)}$, then the terminal vertex of W_l in $G^{(\tau,\varsigma)}$ is $(u_l)_{a\circ_{i_1}\tau(u_1u_2)\circ_{i_2}\cdots\circ_{i_{l-1}}\tau(u_{l-1}u_l)}$. We only need to find a necessary and sufficient condition for existing a lifting of u_lu_{l+1} with an initial vertex $(u_l)_{a\circ_{i_1}\tau(u_1u_2)\circ_{i_2}\cdots\circ_{i_{l-1}}}\tau(u_{l-1}u_l)$. By definition, there exists such a lifting of the edge u_lu_{l+1} if and only if $(a\circ_{i_1}\tau(u_1u_2)\circ_{i_2}\cdots\circ_{i_{l-1}})\tau(u_{l-1}u_l))\circ_l\tau(u_lu_{l+1}) \in \varsigma^{-1}(u_{l+1})$. Since $\tau(u_lu_{l+1}) \in \varsigma^{-1}(u_{l+1})$ by the definition of multi-voltage graphs of type 2, we know that $a \circ_{i_1} \tau(u_1u_2) \circ_{i_2} \cdots \circ_{i_{l-1}} \tau(u_{l-1}u_l) \in \varsigma^{-1}(u_{l+1})$.

Continuing this process, we get the assertion of this theorem by the induction principle. ♮

Corollary 2.2.7 *Let G a graph with vertices partition $V(G) = \bigcup\limits_{i=1}^{n} V_i$ and let $(\Gamma;\circ)$ be a finite group, $\Gamma_i \prec \Gamma$ for any integer $i, 1 \leq i \leq n$. If (G,τ,ς) is a multi-voltage graph with $\tau : X_{\frac{1}{2}}(G) \to \Gamma$ and $\varsigma : V_i \to \Gamma_i$ for any integer $i, 1 \leq i \leq n$, then for a walk W in G with an initial vertex u, there exists a lifting $W^{(\tau,\varsigma)}$ in $G^{(\tau,\varsigma)}$ with the initial vertex $u_a, a \in \varsigma^{-1}(u)$ if and only if $a \in \bigcap_{v\in V(W)}\varsigma^{-1}(v)$.*

Similar to multi-voltage graphs of type 1, we can get the exact number of liftings of a walk in the case of $\Gamma_i = \Gamma$ and $V_i = V(G)$ for any integer $i, 1 \leq i \leq n$.

Theorem 2.2.8 *Let $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma$ be a finite multi-group with groups $(\Gamma;\circ_i), 1 \leq i \leq n$ and let $W = e^1e^2\cdots e^k$ be a walk with an initial vertex u in a multi-voltage graph (G,τ,ς) , $\tau : X_{\frac{1}{2}}(G) \to \bigcap\limits_{i=1}^{n} \Gamma$ and $\varsigma : V(G) \to \Gamma$, of type 2. Then there are n^k liftings of W in $G^{(\tau,\varsigma)}$ with an initial vertex u_a for $\forall a \in \widetilde{\Gamma}$.*

Proof The proof is similar to the proof of Theorem 2.2.2. ♮

Theorem 2.2.9 *Let $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma$ be a finite multi-group with groups $(\Gamma;\circ_i), 1 \leq i \leq n$, $C = u_1u_2\cdots u_mu_1$ a circuit in a multi-voltage graph (G,τ,ς) of type 2 where $\tau : X_{\frac{1}{2}}(G) \to \bigcap\limits_{i=1}^{n} \Gamma$ and $\varsigma : V(G) \to \Gamma$. Then there are $\frac{|\Gamma|}{o(\tau(C,\circ_i))}$ liftings of length $o(tau(C,\circ_i))m$ in $G^{(\tau,\varsigma)}$ of C for any integer $i, 1 \leq i \leq n$, where $\tau(C,\circ_i) = \tau(u_1,u_2)\circ_i \tau(u_2,u_3) \circ_i \cdots \circ_i \tau(u_{m-1},u_m) \circ_i \tau(u_m,u_1)$ and there are*

$$\sum_{i=1}^{n} \frac{|\Gamma|}{o(\tau(C,\circ_i))}$$

liftings of C in $G^{(\tau,\varsigma)}$ altogether.

Proof The proof is similar to the proof of Theorem 2.2.3. ♮

Definition 2.2.5 *Let G_1, G_2 be two graphs and H a subgraph of G_1 and G_2. A one-to-one mapping ξ between G_1 and G_2 is called an H-isomorphism if for any subgraph J isomorphic to H in G_1, $\xi(J)$ is also a subgraph isomorphic to H in G_2.*

If $G_1 = G_2 = G$, then an H-isomorphism between G_1 and G_2 is called an H-automorphism of G. Certainly, all H-automorphisms form a group under the composition operation, denoted by $\mathrm{Aut}_H G$ and $\mathrm{Aut}_H G = \mathrm{Aut} G$ if we take $H = K_2$.

For example, let $H = \langle E(x, N_G(x))\rangle$ for $\forall x \in V(G)$. Then the H-automorphism group of a complete bipartite graph $K(n,m)$ is $\mathrm{Aut}_H K(n,m) = S_n[S_m] = \mathrm{Aut} K(n,m)$. There H-automorphisms are called *star-automorphisms*.

Theorem 2.2.10 *Let G be a graph. If there is a decomposition $G = \bigoplus_{i=1}^{n} H_i$ with $H_i \cong H$ for $1 \leq i \leq n$ and $H = \bigoplus_{j=1}^{m} J_j$ with $J_j \cong J$ for $1 \leq j \leq m$, then*

(i) $\langle \iota_i, \iota_i : H_1 \to H_i,$ an isomorphism, $1 \leq i \leq n\rangle = S_n \preceq \mathrm{Aut}_H G$, and particularly, $S_n \preceq \mathrm{Aut}_H K_{2n+1}$ if $H = C$, a hamiltonian circuit in K_{2n+1}.

(ii) $\mathrm{Aut}_J G \preceq \mathrm{Aut}_H G$, and particularly, $\mathrm{Aut} G \preceq \mathrm{Aut}_H G$ for a simple graph G.

Proof (i) For any integer $i, 1 \leq i \leq n$, we prove there is a such H-automorphism ι on G that $\iota_i : H_1 \to H_i$. In fact, since $H_i \cong H$, $1 \leq i \leq n$, there is an isomorphism $\theta : H_1 \to H_i$. We define ι_i as follows:

$$\iota_i(e) = \begin{cases} \theta(e), & \text{if } e \in V(H_1) \bigcup E(H_1), \\ e, & \text{if} e \in (V(G) \setminus V(H_1)) \bigcup (E(G) \setminus E(H_1)). \end{cases}$$

Then ι_i is a one-to-one mapping on the graph G and is also an H-isomorphism by definition. Whence,

$$\langle \iota_i, \iota_i : H_1 \to H_i, \text{ an isomorphism}, \ 1 \leq i \leq n\rangle \preceq \mathrm{Aut}_H G.$$

Since $\langle \iota_i, 1 \leq i \leq n\rangle \cong \langle (1,i), 1 \leq i \leq n\rangle = S_n$, thereby we get that $S_n \preceq \mathrm{Aut}_H G$.

For a complete graph K_{2n+1}, we know a decomposition $K_{2n+1} = \bigoplus_{i=1}^{n} C_i$ with

$$C_i = v_0 v_i v_{i+1} v_{i-1} v_{i-2} \cdots v_{n+i-1} v_{n+i+1} v_{n+i} v_0$$

for any integer $i, 1 \leq i \leq n$ by Theorem 2.1.18. Therefore, we get that

$$S_n \preceq \mathrm{Aut}_H K_{2n+1}$$

if we choose a hamiltonian circuit H in K_{2n+1}.

(ii) Choose $\sigma \in \mathrm{Aut}_J G$. By definition, for any subgraph A of G, if $A \cong J$, then $\sigma(A) \cong J$. Notice that $H = \bigoplus_{j=1}^{m} J_j$ with $J_j \cong J$ for $1 \leq j \leq m$. Therefore, for any subgraph $B, B \cong H$ of G, $\sigma(B) \cong \bigoplus_{j=1}^{m} \sigma(J_j) \cong H$. This fact implies that $\sigma \in \mathrm{Aut}_H G$.

Notice that for a simple graph G, we have a decomposition $G = \bigoplus_{i=1}^{\varepsilon(G)} K_2$ and $\mathrm{Aut}_{K_2}G = \mathrm{Aut}G$. Whence, $\mathrm{Aut}G \preceq \mathrm{Aut}_H G$. $\natural$

The equality in Theorem 2.2.10(ii) does not always hold. For example, a one-to-one mapping σ on the lifting graph of Fig.2.21(a): $\sigma(u_0) = u_1$, $\sigma(u_1) = u_0$, $\sigma(v_0) = v_1$, $\sigma(v_1) = v_2$ and $\sigma(v_2) = v_0$ is not an automorphism, but it is an H-automorphism with H being a star $S_{1.2}$.

For automorphisms of the lifting $G^{(\tau,\varsigma)}$ of a multi-voltage graph (G, τ, ς) of type 2, we get a result in the following.

Theorem 2.2.11 *Let (G, τ, ς) be a multi-voltage graph of type 2 with $\tau : X_{\frac{1}{2}}(G) \rightarrow \bigcap_{i=1}^{n} \Gamma_i$ and $\varsigma : V_i \rightarrow \Gamma_i$. Then for any integers $i, j, 1 \leq i, j \leq n$,*

(i) for $\forall g_i \in \Gamma_i$, the left action $lA(g_i)$ on $\langle V_i \rangle^{(\tau,\varsigma)}$ is a fixed-free action of an automorphism of $\langle V_i \rangle^{(\tau,\varsigma)}$;

(ii) for $\forall g_{ij} \in \Gamma_i \bigcap \Gamma_j$, the left action $lA(g_{ij})$ on $\langle E_G(V_i, V_j) \rangle^{(\tau,\varsigma)}$ is a star-automorphism of $\langle E_G(V_i, V_j) \rangle^{(\tau,\varsigma)}$.

Proof The proof of (i) is similar to the proof of Theorem 2.2.4. We prove the assertion (ii). A star with a central vertex u_a, $u \in V_i, a \in \Gamma_i \bigcap \Gamma_j$ is the graph $S_{star} = \left\langle \{(u_a, v_{a \circ_j b}) \text{ if } (u, v) \in E_G(V_i, V_j), \tau(u, v) = b\} \right\rangle$. By definition, the left action $lA(g_{ij})$ is a one-to-one mapping on $\langle E_G(V_i, V_j) \rangle^{(\tau,\varsigma)}$. Now for any element $g_{ij}, g_{ij} \in \Gamma_i \bigcap \Gamma_j$, the left action $lA(g_{ij})$ of g_{ij} on a star S_{star} is

$$lA(g_{ij})(S_{star}) = \left\langle \{(u_{g_{ij} \circ_i a}, v_{(g_{ij} \circ_i a) \circ_j b}) \text{ if } (u, v) \in E_G(V_i, V_j), \tau(u, v) = b\} \right\rangle = S_{star}.$$

Whence, $lA(g_{ij})$ is a star-automorphism of $\langle E_G(V_i, V_j) \rangle^{(\tau,\varsigma)}$. $\natural$

Let $\widetilde{G}$ be a graph and let $\widetilde{\Gamma} = \bigcup_{i=1}^{n} \Gamma_i$ be a finite multi-group. If there is a partition for the vertex set $V(\widetilde{G}) = \bigcup_{i=1}^{n} V_i$ such that the action of $\widetilde{\Gamma}$ on $\widetilde{G}$ consists of Γ_i action on $\langle V_i \rangle$ and $\Gamma_i \bigcap \Gamma_j$ on $\langle E_G(V_i, v_j) \rangle$ for $1 \leq i, j \leq n$, then we say this action to be a *partially-action*. A partially-action is called *fixed-free* if Γ_i is fixed-free on $\langle V_i \rangle$ and the action of each element in $\Gamma_i \bigcap \Gamma_j$ is a star-automorphism and fixed-free on $\langle E_G(V_i, V_j) \rangle$ for any integers $i, j, 1 \leq i, j \leq n$. These orbits of a partially-action are defined to be

$$orb_i(v) = \{g(v) | g \in \Gamma_i, v \in V_i\}$$

for any integer $i, 1 \leq i \leq n$ and

$$orb(e) = \{g(e) | e \in E(\widetilde{G}), g \in \widetilde{\Gamma}\}.$$

A *partially-quotient graph* $\widetilde{G}/_p\widetilde{\Gamma}$ is defined by

$$V(\widetilde{G}/_p\widetilde{\Gamma}) = \bigcup_{i=1}^{n}\{\ orb_i(v) \mid v \in V_i\},\quad E(\widetilde{G}/_p\widetilde{\Gamma}) = \{orb(e)|e \in E(\widetilde{G})\}$$

and $I(\widetilde{G}/_p\widetilde{\Gamma}) = \{I(e) = (orb_i(u), orb_j(v))$ if $u \in V_i, v \in V_j$ and $(u,v) \in E(\widetilde{G}), 1 \leq i,j \leq n\}$. An example for partially-quotient graph is shown in Fig.2.22, where $V_1 = \{u_0, u_1, u_2, u_3\}$, $V_2 = \{v_0, v_1, v_2\}$ and $\Gamma_1 = Z_4$, $\Gamma_2 = Z_3$.

Fig.2.22

Then we have a necessary and sufficient condition for the lifting of a multi-voltage graph of type 2.

Theorem 2.2.12 *If the partially-action $\mathcal{P}_a$ of a finite multi-group $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n}\Gamma_i$ on a graph $\widetilde{G}$, $V(\widetilde{G}) = \bigcup\limits_{i=1}^{n} V_i$ is fixed-free, then there is a multi-voltage graph $(\widetilde{G}/_p\widetilde{\Gamma}, \tau, \varsigma)$, $\tau : X_{\frac{1}{2}}(\widetilde{G}/\widetilde{\Gamma}) \to \widetilde{\Gamma}$, $\varsigma : V_i \to \Gamma_i$ of type 2 such that*

$$\widetilde{G} \ \cong\ (\widetilde{G}/_p\widetilde{\Gamma})^{(\tau,\varsigma)}.$$

Proof Similar to the proof of Theorem 2.2.6, we also choose positive directions on these edges of $\widetilde{G}/_p\widetilde{\Gamma}$ and $\widetilde{G}$ so that the partially-quotient map $p_{\widetilde{\Gamma}} : \widetilde{G} \to \widetilde{G}/_p\widetilde{\Gamma}$ is direction-preserving and the partially-action of $\widetilde{\Gamma}$ on $\widetilde{G}$ preserves directions.

For any integer $i, 1 \leq i \leq n$ and $\forall v^i \in V_i$, we can label v^i as $v^i_{1_{\Gamma_i}}$ and for every group element $g_i \in \Gamma_i, g_i \neq 1_{\Gamma_i}$, label the vertex $\mathcal{P}_a(g_i)((v_i)_{1_{\Gamma_i}})$ as $v^i_{g_i}$. Now if the edge e of $\widetilde{G}/_p\widetilde{\Gamma}$ runs from u to w, we assign the label e_{g_i} to the edge of the orbit $p^{-1}(e)$ that originates at the vertex $u^i_{g_i}$ and terminates at $w^j_{h_j}$.

Since Γ_i acts freely on $\langle V_i\rangle$, there are just $|\Gamma_i|$ edges in the orbit $p^{-1}_{\Gamma_i}(e)$ for each integer $i, 1 \leq i \leq n$, one originating at each of the vertices in the vertex orbit $p^{-1}_{\Gamma_i}(v)$. Thus for any integer $i, 1 \leq i \leq n$, the choice of an edge in $p^{-1}(e)$ to be labelled e_{g_i} is unique. Finally, if the terminal vertex of the edge e_{g_i} is $w^j_{h_j}$, one assigns voltage $g_i^{-1} \circ_j h_j$ to the edge e in the partially-quotient graph $\widetilde{G}/_p\widetilde{\Gamma}$ if $g_i, h_j \in \Gamma_i \bigcap \Gamma_j$ for $1 \leq i, j \leq n$.

Under this labelling process, the isomorphism $\vartheta : \widetilde{G} \to (\widetilde{G}/_p\widetilde{\Gamma})^{(\tau,\varsigma)}$ identifies orbits in $\widetilde{G}$ with fibers of $G^{(\tau,\varsigma)}$. $\natural$

The multi-voltage graphs defined in this section enables us to enlarge the application field of voltage graphs. For example, a complete bipartite graph $K(n, m)$ is a lifting of a multi-voltage graph, but it is not a lifting of a voltage graph in general if $n \neq m$.

§2.3 Graphs in a Space

For two topological spaces $\mathcal{E}_1$ and $\mathcal{E}_2$, an embedding of $\mathcal{E}_1$ in $\mathcal{E}_2$ is a one-to-one continuous mapping $f : \mathcal{E}_1 \to \mathcal{E}_2$ (see [92] for details). Certainly, the same problem can be also considered for $\mathcal{E}_2$ being a metric space. By a topological view, a graph is nothing but a 1-complex, we consider the embedding problem for graphs in spaces or on surfaces in this section. The same problem had been considered by Grümbaum in [25]-[26] for graphs in spaces and in these references [6], [23],[42] – [44],[56], [69] and [106] for graphs on surfaces.

2.3.1. Graphs in an n-manifold

For a positive integer n, an n-manifold $\mathbf{M}^n$ is a Hausdorff space such that each point has an open neighborhood homeomorphic to an open n-dimensional ball $B^n = \{(x_1, x_2, \cdots, x_n)|x_1^2 + x_2^2 + \cdots + x_n^2 < 1\}$. For a given graph G and an n-manifold $\mathbf{M}^n$ with $n \geq 3$, the embeddability of G in $\mathbf{M}^n$ is trivial. We characterize an embedding of a graph in an n-dimensional manifold $\mathbf{M}^n$ for $n \geq 3$ similar to the rotation embedding scheme of a graph on a surface (see [23], [42] – [44], [69] for details) in this section.

For $\forall v \in V(G)$, a *space permutation* $P(v)$ of v is a permutation on $N_G(v) = \{u_1, u_2, \cdots, u_{\rho_G(v)}\}$ and all space permutation of a vertex v is denoted by $\mathcal{P}_s(v)$. We define a *space permutation* $P_s(G)$ *of a graph* G to be

$$P_s(G) = \{P(v)|\forall v \in V(G), P(v) \in \mathcal{P}_s(v)\}$$

and a *permutation system* $\mathcal{P}_s(G)$ *of* G to be all space permutation of G. Then we have the following characteristic for an embedded graph in an n-manifold $\mathbf{M^n}$ with $n \geq 3$.

Theorem 2.3.1 *For an integer $n \geq 3$, every space permutation $P_s(G)$ of a graph G defines a unique embedding of $G \to \mathbf{M}^n$. Conversely, every embedding of a graph $G \to \mathbf{M}^n$ defines a space permutation of G.*

Proof Assume G is embedded in an n-manifold $\mathbf{M}^n$. For $\forall v \in V(G)$, define an $(n-1)$-ball $B^{n-1}(v)$ to be $x_1^2 + x_2^2 + \cdots + x_n^2 = r^2$ with center at v and radius r as small as needed. Notice that all autohomeomorphisms $\mathrm{Aut}B^{n-1}(v)$ of $B^{n-1}(v)$ is a group under the composition operation and two points $A = (x_1, x_2, \cdots, x_n)$ and $B = (y_1, y_2, \cdots, y_n)$ in $B^{n-1}(v)$ are said to be combinatorially equivalent if there exists an autohomeomorphism $\varsigma \in \mathrm{Aut}B^{n-1}(v)$ such that $\varsigma(A) = B$. Consider intersection points of edges in $E_G(v, N_G(v))$ with $B^{n-1}(v)$. We get a permutation

$P(v)$ on these points, or equivalently on $N_G(v)$ by $(A, B, \cdots, C, D)$ being a cycle of $P(v)$ if and only if there exists $\varsigma \in \mathrm{Aut}B^{n-1}(v)$ such that $\varsigma^i(A) = B, \cdots, \varsigma^j(C) = D$ and $\varsigma^l(D) = A$, where $i, \cdots, j, l$ are integers. Thereby we get a space permutation $P_s(G)$ of G.

Conversely, for a space permutation $P_s(G)$, we can embed G in $\mathbf{M}^n$ by embedding each vertex $v \in V(G)$ to a point X of $\mathbf{M}^n$ and arranging vertices in one cycle of $P_s(G)$ of $N_G(v)$ as the same orbit of $\langle\sigma\rangle$ action on points of $N_G(v)$ for $\sigma \in \mathrm{Aut}B^{n-1}(X)$. Whence we get an embedding of G in the manifold $\mathbf{M}^n$. ♮

Theorem 2.3.1 establishes a relation for an embedded graph in an n-dimensional manifold with a permutation, which enables us to give a combinatorial definition for graphs embedded in n-dimensional manifolds, see Definition 2.3.6 in the finial part of this section.

Corollary 2.3.1 *For a graph G, the number of embeddings of G in $\mathbf{M}^n, n \geq 3$ is*

$$\prod_{v \in V(G)} \rho_G(v)!.$$

For applying graphs in spaces to theoretical physics, we consider an embedding of a graph in an manifold with some additional conditions which enables us to find good behavior of a graph in spaces. On the first, we consider rectilinear embeddings of a graph in an Euclid space.

Definition 2.3.1 *For a given graph G and an Euclid space $\mathbf{E}$, a rectilinear embedding of G in $\mathbf{E}$ is a one-to-one continuous mapping $\pi : G \to \mathbf{E}$ such that*

(i) for $\forall e \in E(G)$, $\pi(e)$ is a segment of a straight line in $\mathbf{E}$;

(ii) for any two edges $e_1 = (u, v), e_2 = (x, y)$ in $E(G)$, $(\pi(e_1) \setminus \{\pi(u), \pi(v)\}) \bigcap (\pi(e_2) \setminus \{\pi(x), \pi(y)\}) = \emptyset$.

In $\mathbf{R}^3$, a rectilinear embedding of K_4 and a cube Q_3 are shown in Fig.2.23.

Fig 2.23

In general, we know the following result for rectilinear embedding of G in an Euclid space $\mathbf{R}^n, n \geq 3$.

Theorem 2.3.2 *For any simple graph G of order n, there is a rectilinear embedding of G in $\mathbf{R}^n$ with $n \geq 3$.*

Proof We only need to prove this assertion for $n = 3$. In $\mathbf{R}^3$, choose n points $(t_1, t_1^2, t_1^3), (t_2, t_2^2, t_2^3), \cdots, (t_n, t_n^2, t_n^3)$, where $t_1, t_2, \cdots, t_n$ are n different real numbers. For integers $i, j, k, l, 1 \leq i, j, k, l \leq n$, if a straight line passing through vertices (t_i, t_i^2, t_i^3) and (t_j, t_j^2, t_j^3) intersects with a straight line passing through vertices (t_k, t_k^2, t_k^3) and (t_l, t_l^2, t_l^3), then there must be

$$\begin{vmatrix} t_k - t_i & t_j - t_i & t_l - t_k \\ t_k^2 - t_i^2 & t_j^2 - t_i^2 & t_l^2 - t_k^2 \\ t_k^3 - t_i^3 & t_j^3 - t_i^3 & t_l^3 - t_k^3 \end{vmatrix} = 0,$$

which implies that there exist integers $s, f \in \{k, l, i, j\}$, $s \neq f$ such that $t_s = t_f$, a contradiction.

Let $V(G) = \{v_1, v_2, \cdots, v_n\}$. We embed the graph G in $\mathbf{R}^3$ by a mapping $\pi : G \to \mathbf{R}^3$ with $\pi(v_i) = (t_i, t_i^2, t_i^3)$ for $1 \leq i \leq n$ and if $v_i v_j \in E(G)$, define $\pi(v_i v_j)$ being the segment between points (t_i, t_i^2, t_i^3) and (t_j, t_j^2, t_j^3) of a straight line passing through points (t_i, t_i^2, t_i^3) and (t_j, t_j^2, t_j^3). Then π is a rectilinear embedding of the graph G in $\mathbf{R}^3$. $\natural$

For a graph G and a surface S, an *immersion* ι of G on S is a one-to-one continuous mapping $\iota : G \to S$ such that for $\forall e \in E(G)$, if $e = (u, v)$, then $\iota(e)$ is a curve connecting $\iota(u)$ and $\iota(v)$ on S. The following two definitions are generalization of embedding of a graph on a surface.

Definition 2.3.2 *Let G be a graph and S a surface in a metric space $\mathcal{E}$. A pseudo-embedding of G on S is a one-to-one continuous mapping $\pi : G \to \mathcal{E}$ such that there exists vertices $V_1 \subset V(G)$, $\pi|_{\langle V_1 \rangle}$ is an immersion on S with each component of $S \setminus \pi(\langle V_1 \rangle)$ isomorphic to an open 2-disk.*

Definition 2.3.3 *Let G be a graph with a vertex set partition $V(G) = \bigcup\limits_{j=1}^{k} V_i$, $V_i \bigcap V_j = \emptyset$ for $1 \leq i, j \leq k$ and let $S_1, S_2, \cdots, S_k$ be surfaces in a metric space $\mathcal{E}$ with $k \geq 1$. A multi-embedding of G on $S_1, S_2, \cdots, S_k$ is a one-to-one continuous mapping $\pi : G \to \mathcal{E}$ such that for any integer $i, 1 \leq i \leq k$, $\pi|_{\langle V_i \rangle}$ is an immersion with each component of $S_i \setminus \pi(\langle V_i \rangle)$ isomorphic to an open 2-disk.*

Notice that if $\pi(G) \bigcap (S_1 \bigcup S_2 \cdots \bigcup S_k) = \pi(V(G))$, then every $\pi : G \to \mathbf{R}^3$ is a multi-embedding of G. We say it to be a *trivial multi-embedding* of G on $S_1, S_2, \cdots, S_k$. If $k = 1$, then every trivial multi-embedding is a trivial pseudo-embedding of G on S_1. The main object of this section is to find nontrivial multi-embedding of G on $S_1, S_2, \cdots, S_k$ with $k \geq 1$. The existence pseudo-embedding of a graph G is obvious by definition. We concentrate our attention on characteristics of multi-embeddings of a graph.

For a graph G, let $G_1, G_2, \cdots, G_k$ be k vertex-induced subgraphs of G. If $V(G_i)\bigcap V(G_j) = \emptyset$ for any integers $i, j, 1 \leq i, j \leq k$, it is called a *block decomposition* of G and denoted by

$$G = \biguplus_{i=1}^{k} G_i.$$

The *planar block number* $n_p(G)$ of G is defined by

$$n_p(G) = min\{k|G = \biguplus_{i=1}^{k} G_i, \text{For any integer } i, 1 \leq i \leq k, G_i \text{ is planar}\}.$$

Then we get a result for the planar black number of a graph G in the following.

Theorem 2.3.3 *A graph G has a nontrivial multi-embedding on s spheres $P_1, P_2, \cdots$, P_s with empty overlapping if and only if $n_p(G) \leq s \leq |G|$.*

Proof Assume G has a nontrivial multi-embedding on spheres $P_1, P_2, \cdots, P_s$. Since $|V(G)\bigcap P_i| \geq 1$ for any integer $i, 1 \leq i \leq s$, we know that

$$|G| = \sum_{i=1}^{s} |V(G)\bigcap P_i| \geq s.$$

By definition, if $\pi : G \to \mathbf{R}^3$ is a nontrivial multi-embedding of G on $P_1, P_2, \cdots, P_s$, then for any integer $i, 1 \leq i \leq s$, $\pi^{-1}(P_i)$ is a planar induced graph. Therefore,

$$G = \biguplus_{i=1}^{s} \pi^{-1}(P_i),$$

and we get that $s \geq n_p(G)$.

Now if $n_p(G) \leq s \leq |G|$, there is a block decomposition $G = \biguplus_{i=1}^{s} G_s$ of G such that G_i is a planar graph for any integer $i, 1 \leq i \leq s$. Whence we can take s spheres $P_1, P_2, \cdots, P_s$ and define an embedding $\pi_i : G_i \to P_i$ of G_i on sphere P_i for any integer $i, 1 \leq i \leq s$.

Now define an immersion $\pi : G \to \mathbf{R}^3$ of G on $\mathbf{R}^3$ by

$$\pi(G) = (\bigcup_{i=1}^{s} \pi(G_i))\bigcup\{(v_i, v_j)|v_i \in V(G_i), v_j \in V(G_j), (v_i, v_j) \in E(G), 1 \leq i, j \leq s\}.$$

Then $\pi : G \to \mathbf{R}^3$ is a multi-embedding of G on spheres $P_1, P_2, \cdots, P_s$. ♮

For example, a multi-embedding of K_6 on two spheres is shown in Fig.2.24, in where, $\langle\{x, y, z\}\rangle$ is on one sphere and $\langle\{u, v, w\}\rangle$ on another.

Fig 2.24

For a complete or a complete bipartite graph, we get the number $n_p(G)$ as follows.

Theorem 2.3.4 *For any integers $n, m, n, m \geq 1$, the numbers $n_p(K_n)$ and $n_p(K(m,n))$ are*

$$n_p(K_n) = \lceil \frac{n}{4} \rceil \quad and \quad n_p(K(m,n)) = 2,$$

if $m \geq 3, n \geq 3$, otherwise 1, respectively.

Proof Notice that every vertex-induced subgraph of a complete graph K_n is also a complete graph. By Theorem 2.1.16, we know that K_5 is non-planar. Thereby we get that

$$n_p(K_n) = \lceil \frac{n}{4} \rceil$$

by definition of $n_p(K_n)$. Now for a complete bipartite graph K(m,n), any vertex-induced subgraph by choosing s and l vertices from its two partite vertex sets is still a complete bipartite graph. According to Theorem 2.1.16, $K(3,3)$ is non-planar and $K(2,k)$ is planar. If $m \leq 2$ or $n \leq 2$, we get that $n_p(K(m,n)) = 1$. Otherwise, $K(m,n)$ is non-planar. Thereby we know that $n_p(K(m,n)) \geq 2$.

Let $V(K(m,n)) = V_1 \bigcup V_2$, where V_1, V_2 are its partite vertex sets. If $m \geq 3$ and $n \geq 3$, we choose vertices $u, v \in V_1$ and $x, y \in V_2$. Then the vertex-induced subgraphs $\langle \{u,v\} \bigcup V_2 \setminus \{x,y\} \rangle$ and $\langle \{x,y\} \bigcup V_2 \setminus \{u,v\} \rangle$ in $K(m,n)$ are planar graphs. Whence, $n_p(K(m,n)) = 2$ by definition. ♮

The position of surfaces $S_1, S_2, \cdots, S_k$ in a metric space $\mathcal{E}$ also influences the existence of multi-embeddings of a graph. Among these cases an interesting case is there exists an arrangement $S_{i_1}, S_{i_2}, \cdots, S_{i_k}$ for $S_1, S_2, \cdots, S_k$ such that in $\mathcal{E}$, S_{i_j} is a subspace of $S_{i_{j+1}}$ for any integer $j, 1 \leq j \leq k$. In this case, the multi-embedding is called an *including multi-embedding* of G on surfaces $S_1, S_2, \cdots, S_k$.

Theorem 2.3.5 *A graph G has a nontrivial including multi-embedding on spheres $P_1 \supset P_2 \supset \cdots \supset P_s$ if and only if there is a block decomposition $G = \biguplus_{i=1}^{s} G_i$ of G such that for any integer $i, 1 < i < s$,*

(*i*) *G_i is planar;*

(ii) for $\forall v \in V(G_i)$, $N_G(x) \subseteq (\bigcup_{j=i-1}^{i+1} V(G_j))$.

Proof Notice that in the case of spheres, if the radius of a sphere is tending to infinite, an embedding of a graph on this sphere is tending to a planar embedding. From this observation, we get the necessity of these conditions.

Now if there is a block decomposition $G = \biguplus_{i=1}^{s} G_i$ of G such that G_i is planar for any integer $i, 1 < i < s$ and $N_G(x) \subseteq (\bigcup_{j=i-1}^{i+1} V(G_j))$ for $\forall v \in V(G_i)$, we can so place s spheres $P_1, P_2, \cdots, P_s$ in $\mathbf{R}^3$ that $P_1 \supset P_2 \supset \cdots \supset P_s$. For any integer $i, 1 \le i \le s$, we define an embedding $\pi_i : G_i \to P_i$ of G_i on sphere P_i.

Since $N_G(x) \subseteq (\bigcup_{j=i-1}^{i+1} V(G_j))$ for $\forall v \in V(G_i)$, define an immersion $\pi : G \to \mathbf{R}^3$ of G on $\mathbf{R}^3$ by

$$\pi(G) = (\bigcup_{i=1}^{s} \pi(G_i)) \bigcup \{(v_i, v_j) | j = i-1, i, i+1 \ for \ 1 < i < s \ and \ (v_i, v_j) \in E(G)\}.$$

Then $\pi : G \to \mathbf{R}^3$ is a multi-embedding of G on spheres $P_1, P_2, \cdots, P_s$. ♮

Corollary 2.3.2 *If a graph* G *has a nontrivial including multi-embedding on spheres* $P_1 \supset P_2 \supset \cdots \supset P_s$, *then the diameter* $D(G) \ge s - 1$.

2.3.2. **Graphs on a surface**

In recent years, many books concern the embedding problem of graphs on surfaces, such as Biggs and White's [6], Gross and Tucker's [23], Mohar and Thomassen's [69] and White's [106] on embeddings of graphs on surfaces and Liu's [42]-[44], Mao's [56] and Tutte's [100] for combinatorial maps. Two disguises of graphs on surfaces, i.e., *graph embedding* and *combinatorial map* consist of two main streams in the development of topological graph theory in the past decades. For relations of these disguises with Klein surfaces, differential geometry and Riemman geometry, one can see in Mao's [55]-[56] for details.

(1) **The embedding of a graph**

For a graph $G = (V(G), E(G), I(G))$ and a surface S, an embedding of G on S is the case of $k = 1$ in Definition 2.3.3, which is also an embedding of a graph in a 2-manifold. It can be shown immediately that if there exists an embedding of G on S, then G is connected. Otherwise, we can get a component in $S \setminus \pi(G)$ not isomorphic to an open 2-disk. Thereafter all graphs considered in this subsection are connected.

Let G be a graph. For $v \in V(G)$, denote all of edges incident with the vertex v by $N_G^e(v) = \{e_1, e_2, \cdots, e_{\rho_G(v)}\}$. A permutation $C(v)$ on $e_1, e_2, \cdots, e_{\rho_G(v)}$ is said a *pure rotation* of v. All pure rotations incident with a vertex v is denoted by $\varrho(v)$. A *pure rotation system* of G is defined by

$$\rho(G) = \{C(v)|C(v) \in \varrho(v) \text{ for } \forall v \in V(G)\}$$

and all pure rotation systems of G is denoted by $\varrho(G)$.

Notice that in the case of embedded graphs on surfaces, a 1-dimensional ball is just a circle. By Theorem 2.3.1, we get a useful characteristic for embedding of graphs on orientable surfaces first found by Heffter in 1891 and then formulated by Edmonds in 1962. It can be restated as follows.

Theorem 2.3.6([23]) *Every pure rotation system for a graph G induces a unique embedding of G into an orientable surface. Conversely, every embedding of a graph G into an orientable surface induces a unique pure rotation system of G.*

According to this theorem, we know that the number of all embeddings of a graph G on orientable surfaces is $\prod_{v\in V(G)}(\rho_G(v) - 1)!$.

By a topological view, an embedded vertex or face can be viewed as a disk, and an embedded edge can be viewed as a 1-band which is defined as a topological space B together with a homeomorphism $h : I \times I \to B$, where $I = [0, 1]$, the unit interval. Whence, an edge in an embedded graph has two sides. One side is $h((0, x)), x \in I$. Another is $h((1, x)), x \in I$.

For an embedded graph G on a surface, the two sides of an edge $e \in E(G)$ may lie in two different faces f_1 and f_2, or in one face f without a twist ,or in one face f with a twist such as those cases (a), or (b), or (c) shown in Fig.25.

Fig¸2.25

Now we define a rotation system $\rho^L(G)$ to be a pair $(\mathcal{J}, \lambda)$, where $\mathcal{J}$ is a pure rotation system of G, and $\lambda : E(G) \to Z_2$. The edge with $\lambda(e) = 0$ or $\lambda(e) = 1$ is called *type* 0 or *type* 1 edge, respectively. The *rotation system* $\varrho^L(G)$ of a graph G are defined by

$$\varrho^L(G) = \{(\mathcal{J}, \lambda)|\mathcal{J} \in \varrho(G), \lambda : E(G) \to Z_2\}.$$

By Theorem 2.3.1 we know the following characteristic for embedding graphs on locally orientable surfaces.

Theorem 2.3.7([23],[91]) *Every rotation system on a graph G defines a unique locally orientable embedding of $G \to S$. Conversely, every embedding of a graph $G \to S$ defines a rotation system for G.*

Notice that in any embedding of a graph G, there exists a spanning tree T such that every edge on this tree is type 0 (see also [23],[91] for details). Whence, the number of all embeddings of a graph G on locally orientable surfaces is

$$2^{\beta(G)} \prod_{v \in V(G)} (\rho_G(v) - 1)!$$

and the number of all embedding of G on non-orientable surfaces is

$$(2^{\beta(G)} - 1) \prod_{v \in V(G)} (\rho(v) - 1)!.$$

The following result is the famous *Euler-Poincaré* formula for embedding a graph on a surface.

Theorem 2.3.8 *If a graph G can be embedded into a surface S, then*

$$\nu(G) - \varepsilon(G) + \phi(G) = \chi(S),$$

where $\nu(G), \varepsilon(G)$ and $\phi(G)$ are the order, size and the number of faces of G on S, and $\chi(S)$ is the Euler characteristic of S, i.e.,

$$\chi(S) = \begin{cases} 2 - 2p, & if\ S\ is\ orientable, \\ 2 - q, & if\ S\ is\ non-orientable. \end{cases}$$

For a given graph G and a surface S, whether G embeddable on S is uncertain. We use the notation $G \to S$ denoting that G can be embeddable on S. Define the *orientable genus range* $GR^O(G)$ and the *non-orientable genus range* $GR^N(G)$ of a graph G by

$$GR^O(G) = \{\frac{2 - \chi(S)}{2} | G \to S, S\ is\ an\ orientable\ surface\},$$

$$GR^N(G) = \{2 - \chi(S) | G \to S, S\ is\ a\ non-orientable\ surface\},$$

respectively and the orientable or non-orientable genus $\gamma(G)$, $\overline{\gamma}(G)$ by

$$\gamma(G) = min\{p | p \in GR^O(G)\}, \quad \gamma_M(G) = max\{p | p \in GR^O(G)\},$$

$$\widetilde{\gamma}(G) = min\{q | q \in GR^N(G)\}, \quad \widetilde{\gamma}_M(G) = max\{q | q \in GR^O(G)\}.$$

Theorem 2.3.9(Duke 1966) *Let G be a connected graph. Then*

$$GR^O(G) = [\gamma(G), \gamma_M(G)].$$

Proof Notice that if we delete an edge e and its adjacent faces from an embedded graph G on a surface S, we get two holes at most, see Fig.25 also. This implies that $|\phi(G) - \phi(G-e)| \leq 1$.

Now assume G has been embedded on a surface of genus $\gamma(G)$ and $V(G) = \{u, v, \cdots, w\}$. Consider those of edges adjacent with u. Not loss of generality, we assume the rotation of G at vertex v is $(e_1, e_2, \cdots, e_{\rho_G(u)})$. Construct an embedded graph sequence $G_1, G_2, \cdots, G_{\rho_G(u)!}$ by

$\varrho(G_1) = \varrho(G)$;
$\varrho(G_2) = (\varrho(G) \setminus \{\varrho(u)\}) \bigcup \{(e_2, e_1, e_3, \cdots, e_{\rho_G(u)})\}$;
$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$;
$\varrho(G_{\rho_G(u)-1}) = (\varrho(G) \setminus \{\varrho(u)\}) \bigcup \{(e_2, e_3, \cdots, e_{\rho_G(u)}, e_1)\}$;
$\varrho(G_{\rho_G(u)}) = (\varrho(G) \setminus \{\varrho(u)\}) \bigcup \{(e_3, e_2, \cdots, e_{\rho_G(u)}, e_1)\}$;
$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$;
$\varrho(G_{\rho_G(u)!}) = (\varrho(G) \setminus \{\varrho(u)\}) \bigcup \{(e_{\rho_G(u)}, \cdots, e_2, e_1,)\}$.

For any integer $i, 1 \leq i \leq \rho_G(u)!$, since $|\phi(G) - \phi(G-e)| \leq 1$ for $\forall e \in E(G)$, we know that $|\phi(G_{i+1}) - \phi(G_i)| \leq 1$. Whence, $|\chi(G_{i+1}) - \chi(G_i)| \leq 1$.

Continuing the above process for every vertex in G we finally get an embedding of G with the maximum genus $\gamma_M(G)$. Since in this sequence of embeddings of G, the genus of two successive surfaces differs by at most one, we get that

$$GR^O(G) = [\gamma(G), \gamma_M(G)]. \qquad \natural$$

The genus problem, i.e., *to determine the minimum orientable or non-orientable genus of a graph* is NP-complete (see [23] for details). Ringel and Youngs got the genus of K_n completely by *current graphs* (a dual form of voltage graphs) as follows.

Theorem 2.3.10 *For a complete graph K_n and a complete bipartite graph $K(m,n)$, $m, n \geq 3$,*

$$\gamma(K_n) = \lceil \frac{(n-3)(n-4)}{12} \rceil \text{ and } \gamma(K(m,n)) = \lceil \frac{(m-2)(n-2)}{4} \rceil.$$

Outline proofs for $\gamma(K_n)$ in Theorem 2.3.10 can be found in [42], [23],[69] and a complete proof is contained in [81]. For a proof of $\gamma(K(m,n))$ in Theorem 2.3.10 can be also found in [42], [23],[69].

For the maximum genus $\gamma_M(G)$ of a graph, the time needed for computation is bounded by a polynomial function on the number of $\nu(G)$ ([23]). In 1979, Xuong got the following result.

Theorem 2.3.11(Xuong 1979) *Let G be a connected graph with n vertices and q edges. Then*

$$\gamma_M(G) = \frac{1}{2}(q - n + 1) - \frac{1}{2}\min_T c_{odd}(G \setminus E(T)),$$

where the minimum is taken over all spanning trees T of G and $c_{odd}(G \setminus E(T))$ denotes the number of components of $G \setminus E(T)$ with an odd number of edges.

In 1981, Nebeský derived another important formula for the maximum genus of a graph. For a connected graph G and $A \subset E(G)$, let $c(A)$ be the number of connected component of $G \setminus A$ and let $b(A)$ be the number of connected components X of $G \setminus A$ such that $|E(X)| \equiv |V(X)|(mod 2)$. With these notations, his formula can be restated as in the next theorem.

Theorem 2.3.12(Nebeský 1981) *Let G be a connected graph with n vertices and q edges. Then*

$$\gamma_M(G) = \frac{1}{2}(q - n + 2) - \max_{A \subseteq E(G)} \{c(A) + b(A) - |A|\}.$$

Corollary 2.3.3 *The maximum genus of K_n and $K(m, n)$ are given by*

$$\gamma_M(K_n) = \lfloor \frac{(n-1)(n-2)}{4} \rfloor \text{ and } \gamma_M(K(m,n)) = \lfloor \frac{(m-1)(n-1)}{2} \rfloor,$$

respectively.

Now we turn to non-orientable embedding of a graph G. For $\forall e \in E(G)$, we define an *edge-twisting surgery* $\otimes(e)$ to be given the band of e an extra twist such as that shown in Fig.26.

Fig.2.26

Notice that for an embedded graph G on a surface S, $e \in E(G)$, if two sides of e are in two different faces, then $\otimes(e)$ will make these faces into one and if two sides of e are in one face, $\otimes(e)$ will divide the one face into two. This property of $\otimes(e)$ enables us to get the following result for the crosscap range of a graph.

Theorem 2.3.13(Edmonds 1965, Stahl 1978) *Let G be a connected graph. Then*

$$GR^N(G) = [\widetilde{\gamma}(G), \beta(G)],$$

where $\beta(G) = \varepsilon(G) - \nu(G) + 1$ *is called the Betti number of* G.

Proof It can be checked immediately that $\widetilde{\gamma}(G) = \widetilde{\gamma}_M(G) = 0$ for a tree G. If G is not a tree, we have known there exists a spanning tree T such that every edge on this tree is type 0 for any embedding of G.

Let $E(G) \setminus E(T) = \{e_1, e_2, \cdots, e_{\beta(G)}\}$. Adding the edge e_1 to T, we get a two faces embedding of $T + e_1$. Now make edge-twisting surgery on e_1. Then we get a one face embedding of $T + e_1$ on a surface. If we have get a one face embedding of $T + (e_1 + e_2 + \cdots + e_i)$, $1 \leq i < \beta(G)$, adding the edge e_{i+1} to $T + (e_1 + e_2 + \cdots + e_i)$ and make $\otimes(e_{i+1})$ on the edge e_{i+1}. We also get a one face embedding of $T + (e_1 + e_2 + \cdots + e_{i+1})$ on a surface again.

Continuing this process until all edges in $E(G) \setminus E(T)$ have a twist, we finally get a one face embedding of $T + (E(G) \setminus E(T)) = G$ on a surface. Since the number of twists in each circuit of this embedding of G is $1(mod2)$, this embedding is non-orientable with only one face. By the Euler-Poincaré formula, we know its genus $\widetilde{g}(G)$

$$\widetilde{g}(G) = 2 - (\nu(G) - \varepsilon(G) + 1) = \beta(G).$$

For a minimum non-orientable embedding $\mathcal{E}_G$ of G, i.e., $\widetilde{\gamma}(\mathcal{E}_G) = \widetilde{\gamma}(G)$, one can selects an edge e that lies in two faces of the embedding $\mathcal{E}_G$ and makes $\otimes(e)$. Thus in at most $\widetilde{\gamma}_M(G) - \widetilde{\gamma}(G)$ steps, one has obtained all of embeddings of G on every non-orientable surface N_s with $s \in [\widetilde{\gamma}(G), \widetilde{\gamma}_M(G)]$. Therefore,

$$GR^N(G) = [\widetilde{\gamma}(G), \beta(G)] \qquad \natural$$

Corollary 2.3.4 *Let* G *be a connected graph with* p *vertices and* q *edges. Then*

$$\widetilde{\gamma}_M(G) = q - p + 1.$$

Theorem 2.3.14 For a complete graph K_n and a complete bipartite graph $K(m,n)$, $m, n \geq 3$,

$$\widetilde{\gamma}(K_n) = \lceil \frac{(n-3)(n-4)}{6} \rceil$$

with an exception value $\widetilde{\gamma}(K_7) = 3$ *and*

$$\widetilde{\gamma}(K(m,n)) = \lceil \frac{(m-2)(n-2)}{2} \rceil.$$

A complete proof of this theorem is contained in [81], Outline proofs of Theorem 2.3.14 can be found in [42].

(2) **Combinatorial maps**

Geometrically, an embedded graph of G on a surface is called a combinatorial map M and say G underlying M. Tutte found an algebraic representation for an embedded graph on a locally orientable surface in 1973 ([98], which transfers a geometrical partition of a surface to a permutation in algebra.

According to the summaries in Liu's [43] $-$ [44], a *combinatorial map* $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ is defined to be a permutation $\mathcal{P}$ acting on $\mathcal{X}_{\alpha,\beta}$ of a disjoint union of quadricells Kx of $x \in X$, where X is a finite set and $K = \{1, \alpha, \beta, \alpha\beta\}$ is Klein 4-group with the following conditions hold.

(i) $\forall x \in \mathcal{X}_{\alpha,\beta}$, there does not exist an integer k such that $\mathcal{P}^k x = \alpha x$;

(ii) $\alpha\mathcal{P} = \mathcal{P}^{-1}\alpha$;

(iii) The group $\Psi_J = \langle \alpha, \beta, \mathcal{P} \rangle$ is transitive on $\mathcal{X}_{\alpha,\beta}$.

The *vertices* of a combinatorial map are defined to be pairs of conjugate orbits of $\mathcal{P}$ action on $\mathcal{X}_{\alpha,\beta}$, *edges* to be orbits of K on $\mathcal{X}_{\alpha,\beta}$ and *faces* to be pairs of conjugate orbits of $\mathcal{P}\alpha\beta$ action on $\mathcal{X}_{\alpha,\beta}$.

For determining a map $(\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ is orientable or not, the following condition is needed.

(iv) If the group $\Psi_I = \langle \alpha\beta, \mathcal{P} \rangle$ is transitive on $\mathcal{X}_{\alpha,\beta}$, then M is non-orientable. Otherwise, orientable.

For example, the graph $D_{0.4.0}$ (a dipole with 4 multiple edges) on Klein bottle shown in Fig.27,

Fig 2.27

can be algebraic represented by a combinatorial map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ with

$$\mathcal{X}_{\alpha,\beta} = \bigcup_{e \in \{x,y,z,w\}} \{e, \alpha e, \beta e, \alpha\beta e\},$$

$$\begin{aligned} \mathcal{P} &= (x, y, z, w)(\alpha\beta x, \alpha\beta y, \beta z, \beta w) \\ &\times (\alpha x, \alpha w, \alpha z, \alpha y)(\beta x, \alpha\beta w, \alpha\beta z, \beta y). \end{aligned}$$

This map has 2 vertices $v_1 = \{(x, y, z, w), (\alpha x, \alpha w, \alpha z, \alpha y)\}$, $v_2 = \{(\alpha\beta x, \alpha\beta y, \beta z, \beta w), (\beta x, \alpha\beta w, \alpha\beta z, \beta y)\}$, 4 edges $e_1 = \{x, \alpha x, \beta x, \alpha\beta x\}$, $e_2 = \{y, \alpha y, \beta y, \alpha\beta y\}$, $e_3 =$

$\{z, \alpha z, \beta z, \alpha\beta z\}$, $e_4 = \{w, \alpha w, \beta w, \alpha\beta w\}$ and 2 faces $f_2 = \{(x, \alpha\beta y, z, \beta y, \alpha x, \alpha\beta w), (\beta x, \alpha w, \alpha\beta x, y, \beta z, \alpha y)\}$, $f_2 = \{(\beta w, \alpha z), (w, \alpha\beta z)\}$. The Euler characteristic of this map is

$$\chi(M) = 2 - 4 + 2 = 0$$

and $\Psi_I = \langle \alpha\beta, \mathcal{P}\rangle$ is transitive on $\mathcal{X}_{\alpha,\beta}$. Thereby it is a map of $D_{0.4.0}$ on a Klein bottle with 2 faces accordance with its geometrical figure.

The following result was gotten by Tutte in [98], which establishes a relation for an embedded graph with a combinatorial map.

Theorem 2.3.15 *For an embedded graph G on a locally orientable surface S, there exists one combinatorial map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ with an underlying graph G and for a combinatorial map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, there is an embedded graph G underlying M on S.*

Similar to the definition of a multi-voltage graph (see [56] for details), we can define a multi-voltage map and its lifting by applying a multi-group $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ with $\Gamma_i = \Gamma_j$ for any integers $i, j, 1 \leq i, j \leq n$.

Definition 2.3.4 *Let $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma$ be a finite multi-group with $\Gamma = \{g_1, g_2, \cdots, g_m\}$ and an operation set $O(\widetilde{\Gamma}) = \{\circ_i | 1 \leq i \leq n\}$ and let $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ be a combinatorial map. If there is a mapping $\psi : \mathcal{X}_{\alpha,\beta} \to \widetilde{\Gamma}$ such that*

(i) for $\forall x \in \mathcal{X}_{\alpha,\beta}, \forall \sigma \in K = \{1, \alpha, \beta, \alpha\beta\}$, $\psi(\alpha x) = \psi(x)$, $\psi(\beta x) = \psi(\alpha\beta x) = \psi(x)^{-1}$;

(ii) for any face $f = (x, y, \cdots, z)(\beta z, \cdots, \beta y, \beta x)$, $\psi(f, i) = \psi(x) \circ_i \psi(y) \circ_i \cdots \circ_i \psi(z)$, where $\circ_i \in O(\widetilde{\Gamma})$, $1 \leq i \leq n$ and $\langle \psi(f, i) | f \in F(v)\rangle = G$ for $\forall v \in V(G)$, where $F(v)$ denotes all faces incident with v,
then (M, ψ) is called a multi-voltage map.

The *lifting of a multi-voltage map* is defined in the next definition.

Definition 2.3.5 *For a multi-voltage map (M, ψ), the lifting map $M^\psi = (\mathcal{X}^\psi_{\alpha^\psi,\beta^\psi}, \mathcal{P}^\psi)$ is defined by*

$$\mathcal{X}^\psi_{\alpha^\psi,\beta^\psi} = \{x_g | x \in \mathcal{X}_{\alpha,\beta}, g \in \widetilde{\Gamma}\},$$

$$\mathcal{P}^\psi = \prod_{g \in \widetilde{\Gamma}} \prod_{(x,y,\cdots,z)(\alpha z,\cdots,\alpha y,\alpha x) \in V(M)} (x_g, y_g, \cdots, z_g)(\alpha z_g, \cdots, \alpha y_g, \alpha x_g),$$

$$\alpha^\psi = \prod_{x \in \mathcal{X}_{\alpha,\beta}, g \in \widetilde{\Gamma}} (x_g, \alpha x_g),$$

$$\beta^{\psi} = \prod_{i=1}^{m} \prod_{x \in \mathcal{X}_{\alpha,\beta}} (x_{g_i}, (\beta x)_{g_i \circ_i \psi(x)})$$

with a convention that $(\beta x)_{g_i \circ_i \psi(x)} = y_{g_i}$ *for some quadricells* $y \in \mathcal{X}_{\alpha,\beta}$.

Notice that the lifting M^{ψ} is connected and $\Psi_I^{\psi} = \left\langle \alpha^{\psi}\beta^{\psi}, \mathcal{P}^{\psi} \right\rangle$ is transitive on $\mathcal{X}^{\psi}_{\alpha^{\psi},\beta^{\psi}}$ if and only if $\Psi_I = \langle \alpha\beta, \mathcal{P} \rangle$ is transitive on $\mathcal{X}_{\alpha,\beta}$. We get a result in the following.

Theorem 2.3.16 *The Euler characteristic* $\chi(M^{\psi})$ *of the lifting map* M^{ψ} *of a multi-voltage map* $(M, \widetilde{\Gamma})$ *is*

$$\chi(M^{\psi}) = |\Gamma|(\chi(M) + \sum_{i=1}^{n} \sum_{f \in F(M)} (\frac{1}{o(\psi(f, \circ_i))} - \frac{1}{n})),$$

where $F(M)$ *and* $o(\psi(f, \circ_i))$ *denote the set of faces in* M *and the order of* $\psi(f, \circ_i)$ *in* $(\Gamma; \circ_i)$, *respectively.*

Proof By definition the lifting map M^{ϑ} has $|\Gamma|\nu(M)$ vertices, $|\Gamma|\varepsilon(M)$ edges. Notice that each lifting of the boundary walk of a face is a homogenous lifting by definition of β^{ψ}. Similar to the proof of Theorem 2.2.3, we know that M^{ϑ} has $\sum\limits_{i=1}^{n} \sum\limits_{f \in F(M)} \frac{|\Gamma|}{o(\psi(f, \circ_i))}$ faces. By the Eular-Poincaré formula we get that

$$\begin{aligned}
\chi(M^{\psi}) &= \nu(M^{\psi}) - \varepsilon(M^{\psi}) + \phi(M^{\psi}) \\
&= |\Gamma|\nu(M) - |\Gamma|\varepsilon(M) + \sum_{i=1}^{n} \sum_{f \in F(M)} \frac{|\Gamma|}{o(\psi(f, \circ_i))} \\
&= |\Gamma|(\chi(M) - \phi(M) + \sum_{i=1}^{n} \sum_{f \in F(M)} \frac{1}{o(\psi(f, \circ_i))} \\
&= |G|(\chi(M) + \sum_{i=1}^{n} \sum_{f \in F(M)} \frac{1}{o(\psi(f, \circ_i))} - \frac{1}{n}). \qquad \natural
\end{aligned}$$

Recently, more and more papers concentrated on finding *regular maps* on surface, which are related with *discrete groups, discrete geometry* and *crystal physics*. For this object, an important way is by the voltage assignment on a map. In this field, general results for automorphisms of the lifting map are known, see [45] – [46] and [71] – [72] for details. It is also an interesting problem for applying these multi-voltage maps to finding non-regular or other maps with some constraint conditions.

Motivated by the Four Color Conjecture, Tait conjectured that *every simple 3-polytope is hamiltonian* in 1880. By Steinitz's a famous result (see [24]), this conjecture is equivalent to that *every 3-connected cubic planar graph is hamiltonian*. Tutte disproved this conjecture by giving a 3-connected non-hamiltonian cubic planar graph with 46 vertices in 1946 and proved that *every* 4*-connected planar graph is* *hamiltonian* in 1956([97],[99]). In [56], Grünbaum conjectured that *each* 4-*connected graph embeddable in the torus or in the projective plane is hamiltonian*. This conjecture had been solved for the projective plane case by Thomas and Yu in 1994 ([93]). Notice that the splitting operator ϑ constructed in the proof of Theorem 2.1.11 is a planar operator. Applying Theorem 2.1.11 on surfaces we know that *for every map* M *on a surface,* M^{ϑ} *is non-hamiltonian*. In fact, we can further get an interesting result related with Tait's conjecture.

Theorem 2.3.17 *There exist infinite* $3-$*connected non-hamiltonian cubic maps on each locally orientable surface.*

Proof Notice that there exist 3-connected triangulations on every locally orientable surface S. Each dual of them is a 3-connected cubic map on S. Now we define a splitting operator σ as shown in Fig.2.28.

Fig.2.28

For a 3-connected cubic map M, we prove that $M^{\sigma(v)}$ is non-hamiltonian for $\forall v \in V(M)$. According to Theorem 2.1.7, we only need to prove that there are no $y_1 - y_2$, or $y_1 - y_3$, or $y_2 - y_3$ hamiltonian path in the nucleus $N(\sigma(v))$ of operator σ.

Let $H(z_i)$ be a component of $N(\sigma(v))\backslash\{z_0z_i, y_{i-1}u_{i+1}, y_{i+1}v_{i-1}\}$ which contains the vertex $z_i, 1 \leq i \leq 3$(all these indices mod 3). If there exists a $y_1 - y_2$ hamiltonian path P in $N(\sigma(v))$, we prove that there must be a $u_i - v_i$ hamiltonian path in the subgraph $H(z_i)$ for an integer $i, 1 \leq i \leq 3$.

Since P is a hamiltonian path in $N(\sigma(v))$, there must be that $v_1y_3u_2$ or $u_2y_3v_1$ is a subpath of P. Now let $E_1 = \{y_1u_3, z_0z_3, y_2v_3\}$, we know that $|E(P)\bigcap E_1| = 2$. Since P is a $y_1 - y_2$ hamiltonian path in the graph $N(\sigma(v))$, we must have $y_1u_3 \notin E(P)$ or $y_2v_3 \notin E(P)$. Otherwise, by $|E(P)\bigcap S_1| = 2$ we get that $z_0z_3 \notin E(P)$. But in this case, P can not be a $y_1 - y_2$ hamiltonian path in $N(\sigma(v))$, a contradiction.

Assume $y_2v_3 \notin E(P)$. Then $y_2u_1 \in E(P)$. Let $E_2 = \{u_1y_2, z_1z_0, v_1y_3\}$. We also know that $|E(P)\bigcap E_2| = 2$ by the assumption that P is a hamiltonian path in $N(\sigma(v))$. Hence $z_0z_1 \notin E(P)$ and the $v_1 - u_1$ subpath in P is a $v_1 - u_1$ hamiltonian path in the subgraph $H(z_1)$.

Similarly, if $y_1u_3 \notin E(P)$, then $y_1v_2 \in E(P)$. Let $E_3 = \{y_1v_2, z_0z_2, y_3u_2\}$. We can also get that $|E(P)\bigcap E_3| = 2$ and a $v_2 - u_2$ hamiltonian path in the subgraph $H(z_2)$.

Now if there is a $v_1 - u_1$ hamiltonian path in the subgraph $H(z_1)$, then the graph $H(z_1) + u_1v_1$ must be hamiltonian. According to the Grinberg's criterion for planar hamiltonian graphs, we know that

$$\phi'_3 - \phi"_3 + 2(\phi'_4 - \phi"_4) + 3(\phi'_5 - \phi"_5) + 6(\phi'_8 - \phi"_8) = 0, \qquad (*)$$

where ϕ'_i or $\phi"_i$ is the number of i-gons in the interior or exterior of a chosen hamiltonian circuit C passing through u_1v_1 in the graph $H(z_1) + u_1v_1$. Since it is obvious that

$$\phi'_3 = \phi"_8 = 1, \quad \phi"_3 = \phi'_8 = 0,$$

we get that

$$2(\phi'_4 - \phi"_4) + 3(\phi'_5 - \phi"_5) = 5, \qquad (**)$$

by (*).

Because $\phi'_4 + \phi"_4 = 2$, so $\phi'_4 - \phi"_4 = 0, 2$ or -2. Now the valency of z_1 in $H(z_1)$ is 2, so the 4-gon containing the vertex z_1 must be in the interior of C, that is $\phi'_4 - \phi"_4 \neq -2$. If $\phi'_4 - \phi"_4 = 0$ or $\phi'_4 - \phi"_4 = 2$, we get $3(\phi'_5 - \phi"_5) = 5$ or $3(\phi'_5 - \phi"_5) = 1$, a contradiction.

Notice that $H(z_1) \cong H(z_2) \cong H(z_3)$. If there exists a $v_2 - u_2$ hamiltonian path in $H(z_2)$, a contradiction can be also gotten. So there does not exist a $y_1 - y_2$ hamiltonian path in the graph $N(\sigma(v))$. Similarly , there are no $y_1 - y_3$ or $y_2 - y_3$ hamiltonian paths in the graph $N(\sigma(v))$. Whence, $M^{\sigma(v)}$ is non-hamiltonian.

Now let n be an integer, $n \geq 1$. We get that

$$\begin{array}{lcl} M_1 & = & (M)^{\sigma(u)}, \quad u \in V(M); \\ M_2 & = & (M_1)^{N(\sigma(v))(v)}, \quad v \in V(M_1); \\ \cdots & \cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots; \\ M_n & = & (M_{n-1})^{N(\sigma(v))(w)}, \quad w \in V(M_{n-1}); \\ \cdots & \cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots. \end{array}$$

All of these maps are 3-connected non-hamiltonian cubic maps on the surface S. This completes the proof. $\natural$

Corollary 2.3.5 *There is not a locally orientable surface on which every 3-connected cubic map is hamiltonian.*

2.3.3. **Multi-Embeddings in an n-manifold**

We come back to determine multi-embeddings of graphs in this subsection. Let $S_1, S_2, \cdots, S_k$ be k locally orientable surfaces and G a connected graph. Define numbers

$$\gamma(G; S_1, S_2, \cdots, S_k) = \min\{\sum_{i=1}^{k} \gamma(G_i) | G = \biguplus_{i=1}^{k} G_i, G_i \to S_i, 1 \le i \le k\},$$

$$\gamma_M(G; S_1, S_2, \cdots, S_k) = \max\{\sum_{i=1}^{k} \gamma(G_i) | G = \biguplus_{i=1}^{k} G_i, G_i \to S_i, 1 \le i \le k\}.$$

and the *multi-genus range* $GR(G; S_1, S_2, \cdots, S_k)$ by

$$GR(G; S_1, S_2, \cdots, S_k) = \{\sum_{i=1}^{k} g(G_i) | G = \biguplus_{i=1}^{k} G_i, G_i \to S_i, 1 \le i \le k\},$$

where G_i is embeddable on a surface of genus $g(G_i)$. Then we get the following result.

Theorem 2.3.18 *Let G be a connected graph and let $S_1, S_2, \cdots, S_k$ be locally orientable surfaces with empty overlapping. Then*

$$GR(G; S_1, S_2, \cdots, S_k) = [\gamma(G; S_1, S_2, \cdots, S_k), \gamma_M(G; S_1, S_2, \cdots, S_k)].$$

Proof Let $G = \biguplus_{i=1}^{k} G_i, G_i \to S_i, 1 \le i \le k$. We prove that there are no gap in the multi-genus range from $\gamma(G_1) + \gamma(G_2) + \cdots + \gamma(G_k)$ to $\gamma_M(G_1) + \gamma_M(G_2) + \cdots + \gamma_M(G_k)$. According to Theorems 2.3.8 and 2.3.12, we know that the genus range $GR^O(G_i)$ or $GR^N(G)$ is $[\gamma(G_i), \gamma_M(G_i)]$ or $[\widetilde{\gamma}(G_i), \widetilde{\gamma}_M(G_i)]$ for any integer $i, 1 \le i \le k$. Whence, there exists a multi-embedding of G on k locally orientable surfaces $P_1, P_2, \cdots, P_k$ with $g(P_1) = \gamma(G_1)$, $g(P_2) = \gamma(G_2), \cdots$, $g(P_k) = \gamma(G_k)$. Consider the graph G_1, then G_2, and then G_3, $\cdots$ to get multi-embedding of G on k locally orientable surfaces step by step. We get a multi-embedding of G on k surfaces with genus sum at least being an unbroken interval $[\gamma(G_1) + \gamma(G_2) + \cdots + \gamma(G_k), \gamma_M(G_1) + \gamma_M(G_2) + \cdots + \gamma_M(G_k)]$ of integers.

By definitions of $\gamma(G; S_1, S_2, \cdots, S_k)$ and $\gamma_M(G; S_1, S_2, \cdots, S_k)$, we assume that $G = \biguplus_{i=1}^{k} G'_i, G'_i \to S_i, 1 \le i \le k$ and $G = \biguplus_{i=1}^{k} G''_i, G''_i \to S_i, 1 \le i \le k$ attain the extremal values $\gamma(G; S_1, S_2, \cdots, S_k)$ and $\gamma_M(G; S_1, S_2, \cdots, S_k)$, respectively. Then we know that the multi-embedding of G on k surfaces with genus sum is at least an unbroken intervals $[\sum\limits_{i=1}^{k}\gamma(G_i'), \sum\limits_{i=1}^{k}\gamma_M(G_i')]$ and $[\sum\limits_{i=1}^{k}\gamma(G_i''), \sum\limits_{i=1}^{k}\gamma_M(G_i'')]$ of integers.

Since

$$\sum_{i=1}^{k} g(S_i) \in [\sum_{i=1}^{k}\gamma(G_i'), \sum_{i=1}^{k}\gamma_M(G_i')] \bigcap [\sum_{i=1}^{k}\gamma(G_i''), \sum_{i=1}^{k}\gamma_M(G_i'')],$$

we get that

$$GR(G; S_1, S_2, \cdots, S_k) = [\gamma(G; S_1, S_2, \cdots, S_k), \gamma_M(G; S_1, S_2, \cdots, S_k)].$$

This completes the proof. ♮

For multi-embeddings of a complete graph, we get the following result.

Theorem 2.3.19 *Let $P_1, P_2, \cdots, P_k$ and $Q_1, Q_2, \cdots, Q_k$ be respective k orientable and non-orientable surfaces of genus≥ 1. A complete graph K_n is multi-embeddable in $P_1, P_2, \cdots, P_k$ with empty overlapping if and only if*

$$\sum_{i=1}^{k}\lceil\frac{3+\sqrt{16g(P_i)+1}}{2}\rceil \leq n \leq \sum_{i=1}^{k}\lfloor\frac{7+\sqrt{48g(P_i)+1}}{2}\rfloor$$

and is multi-embeddable in $Q_1, Q_2, \cdots, Q_k$ with empty overlapping if and only if

$$\sum_{i=1}^{k}\lceil 1+\sqrt{2g(Q_i)}\rceil \leq n \leq \sum_{i=1}^{k}\lfloor\frac{7+\sqrt{24g(Q_i)+1}}{2}\rfloor.$$

Proof According to Theorem 2.3.9 and Corollary 2.3.2, we know that the genus $g(P)$ of an orientable surface P on which a complete graph K_n is embeddable satisfies

$$\lceil\frac{(n-3)(n-4)}{12}\rceil \leq g(P) \leq \lfloor\frac{(n-1)(n-2)}{4}\rfloor,$$

i.e.,

$$\frac{(n-3)(n-4)}{12} \leq g(P) \leq \frac{(n-1)(n-2)}{4}.$$

If $g(P) \geq 1$, we get that

$$\lceil\frac{3+\sqrt{16g(P)+1}}{2}\rceil \leq n \leq \lfloor\frac{7+\sqrt{48g(P)+1}}{2}\rfloor.$$

Similarly, if K_n is embeddable on a non-orientable surface Q, then

$$\lceil\frac{(n-3)(n-4)}{6}\rceil \leq g(Q) \leq \lfloor\frac{(n-1)^2}{2}\rfloor,$$

i.e.,

$$\lceil 1+\sqrt{2g(Q)}\rceil \leq n \leq \lfloor \frac{7+\sqrt{24g(Q)+1}}{2}\rfloor.$$

Now if K_n is multi-embeddable in $P_1, P_2, \cdots, P_k$ with empty overlapping, then there must exists a partition $n = n_1 + n_2 + \cdots + n_k$, $n_i \geq 1, 1 \leq i \leq k$. Since each vertex-induced subgraph of a complete graph is still a complete graph, we know that for any integer $i, 1 \leq i \leq k$,

$$\lceil \frac{3+\sqrt{16g(P_i)+1}}{2}\rceil \leq n_i \leq \lfloor \frac{7+\sqrt{48g(P_i)+1}}{2}\rfloor.$$

Whence, we know that

$$\sum_{i=1}^{k}\lceil \frac{3+\sqrt{16g(P_i)+1}}{2}\rceil \leq n \leq \sum_{i=1}^{k}\lfloor \frac{7+\sqrt{48g(P_i)+1}}{2}\rfloor. \qquad (*)$$

On the other hand, if the inequality (*) holds, we can find positive integers $n_1, n_2, \cdots, n_k$ with $n = n_1 + n_2 + \cdots + n_k$ and

$$\lceil \frac{3+\sqrt{16g(P_i)+1}}{2}\rceil \leq n_i \leq \lfloor \frac{7+\sqrt{48g(P_i)+1}}{2}\rfloor.$$

for any integer $i, 1 \leq i \leq k$. This enables us to establish a partition $K_n = \biguplus_{i=1}^{k} K_{n_i}$ for K_n and embed each K_{n_i} on P_i for $1 \leq i \leq k$. Therefore, we get a multi-embedding of K_n in $P_1, P_2, \cdots, P_k$ with empty overlapping.

Similarly, if K_n is multi-embeddable in $Q_1, Q_2, \cdots Q_k$ with empty overlapping, there must exists a partition $n = m_1 + m_2 + \cdots + m_k$, $m_i \geq 1, 1 \leq i \leq k$ and

$$\lceil 1+\sqrt{2g(Q_i)}\rceil \leq m_i \leq \lfloor \frac{7+\sqrt{24g(Q_i)+1}}{2}\rfloor.$$

for any integer $i, 1 \leq i \leq k$. Whence, we get that

$$\sum_{i=1}^{k}\lceil 1+\sqrt{2g(Q_i)}\rceil \leq n \leq \sum_{i=1}^{k}\lfloor \frac{7+\sqrt{24g(Q_i)+1}}{2}\rfloor. \qquad (**)$$

Now if the inequality (**) holds, we can also find positive integers $m_1, m_2, \cdots, m_k$ with $n = m_1 + m_2 + \cdots + m_k$ and

$$\lceil 1+\sqrt{2g(Q_i)}\rceil \leq m_i \leq \lfloor \frac{7+\sqrt{24g(Q_i)+1}}{2}\rfloor.$$

for any integer $i, 1 \leq i \leq k$. Similar to those of orientable cases, we get a multi-embedding of K_n in $Q_1, Q_2, \cdots, Q_k$ with empty overlapping. ♮

Corollary 2.3.6 *A complete graph K_n is multi-embeddable in $k, k \geq 1$ orientable surfaces of genus $p, p \geq 1$ with empty overlapping if and only if*

$$\lceil\frac{3+\sqrt{16p+1}}{2} \leq \frac{n}{k} \leq \lfloor\frac{7+\sqrt{48p+1}}{2}\rfloor$$

and is multi-embeddable in $l, l \geq 1$ non-orientable surfaces of genus $q, q \geq 1$ with empty overlapping if and only if

$$\lceil 1+\sqrt{2q}\rceil \leq \frac{n}{k} \leq \lfloor\frac{7+\sqrt{24q+1}}{2}\rfloor.$$

Corollary 2.3.7 *A complete graph K_n is multi-embeddable in $s, s \geq 1$ tori with empty overlapping if and only if*

$$4s \leq n \leq 7s$$

and is multi-embeddable in $t, t \geq 1$ projective planes with empty overlapping if and only if

$$3t \leq n \leq 6t.$$

Similarly, the following result holds for a complete bipartite graph $K(n, n)$.

Theorem 2.3.20 *Let $P_1, P_2, \cdots, P_k$ and $Q_1, Q_2, \cdots, Q_k$ be respective k orientable and k non-orientable surfaces of genus≥ 1. A complete bipartite graph $K(n, n)$ is multi-embeddable in $P_1, P_2, \cdots, P_k$ with empty overlapping if and only if*

$$\sum_{i=1}^{k}\lceil 1+\sqrt{2g(P_i)}\rceil \leq n \leq \sum_{i=1}^{k}\lfloor 2+2\sqrt{g(P_i)}\rfloor$$

and is multi-embeddable in $Q_1, Q_2, \cdots, Q_k$ with empty overlapping if and only if

$$\sum_{i=1}^{k}\lceil 1+\sqrt{g(Q_i)}\rceil \leq n \leq \sum_{i=1}^{k}\lfloor 2+\sqrt{2g(Q_i)}\rfloor.$$

Proof Similar to the proof of Theorem 2.3.18, we get this result. ♮

2.3.4. **Classification of graphs in an n-manifold**

By Theorem 2.3.1 we can give a combinatorial definition for a graph embedded in an n-manifold, i.e., a *manifold graph* similar to the Tutte's definition for a map.

Definition 2.3.6 *For any integer $n, n \geq 2$, an n-dimensional manifold graph ${}^n\mathcal{G}$ is a pair ${}^n\mathcal{G} = (\mathcal{E}_\Gamma, \mathcal{L})$ in where a permutation $\mathcal{L}$ acting on $\mathcal{E}_\Gamma$ of a disjoint union $\Gamma x = \{\sigma x | \sigma \in \Gamma\}$ for $\forall x \in E$, where E is a finite set and $\Gamma = \{\mu, o | \mu^2 = o^n = 1, \mu o = o\mu\}$ is a commutative group of order $2n$ with the following conditions hold.*

(i) $\forall x \in \mathcal{E}_K$, *there does not exist an integer* k *such that* $\mathcal{L}^k x = o^i x$ *for* $\forall i, 1 \leq i \leq n-1$*;*

(ii) $\mu\mathcal{L} = \mathcal{L}^{-1}\mu$*;*

(iii) *The group* $\Psi_J = \langle \mu, o, \mathcal{L} \rangle$ *is transitive on* $\mathcal{E}_\Gamma$*.*

According to (i) and (ii), a *vertex* v *of an* n*-dimensional manifold graph* is defined to be an n-tuple $\{(o^i x_1, o^i x_2, \cdots, o^i x_{s_l(v)})(o^i y_1, o^i y_2, \cdots, o^i y_{s_2(v)}) \cdots (o^i z_1, o^i z_2, \cdots, o^i z_{s_{l(v)}(v)}); 1 \leq i \leq n\}$ of permutations of $\mathcal{L}$ action on $\mathcal{E}_\Gamma$, edges to be these orbits of Γ action on $\mathcal{E}_\Gamma$. The number $s_1(v) + s_2(v) + \cdots + s_{l(v)}(v)$ is called the *valency of* v, denoted by $\rho_G^{s_1, s_2, \cdots, s_{l(v)}}(v)$. The condition (iii) is used to ensure that an n-dimensional manifold graph is connected. Comparing definitions of a map with an n-dimensional manifold graph, the following result holds.

Theorem 2.3.21 *For any integer* $n, n \geq 2$, *every* n*-dimensional manifold graph* ${}^n\mathcal{G} = (\mathcal{E}_\Gamma, \mathcal{L})$ *is correspondent to a unique map* $M = (\mathcal{E}_{\alpha,\beta}, \mathcal{P})$ *in which each vertex* v *in* ${}^n\mathcal{G}$ *is converted to* $l(v)$ *vertices* $v_1, v_2, \cdots, v_{l(v)}$ *of* M*. Conversely, a map* $M = (\mathcal{E}_{\alpha,\beta}, \mathcal{P})$ *is also correspondent to an* n*-dimensional manifold graph* ${}^n\mathcal{G} = (\mathcal{E}_\Gamma, \mathcal{L})$ *in which* $l(v)$ *vertices* $u_1, u_2, \cdots, u_{l(v)}$ *of* M *are converted to one vertex* u *of* ${}^n\mathcal{G}$*.*

Two n-dimensional manifold graphs ${}^n\mathcal{G}_1 = (\mathcal{E}^1_{\Gamma_1}, \mathcal{L}_1)$ and ${}^n\mathcal{G}_2 = (\mathcal{E}^2_{\Gamma_2}, \mathcal{L}_2)$ are said to be *isomorphic* if there exists a one-to-one mapping $\kappa : \mathcal{E}^1_{\Gamma_1} \to \mathcal{E}^2_{\Gamma_2}$ such that $\kappa\mu = \mu\kappa, \kappa o = o\kappa$ and $\kappa\mathcal{L}_1 = \mathcal{L}_2\kappa$. If $\mathcal{E}^1_{\Gamma_1} = \mathcal{E}^2_{\Gamma_2} = \mathcal{E}_\Gamma$ and $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}$, an isomorphism between ${}^n\mathcal{G}_1$ and ${}^n\mathcal{G}_2$ is called an automorphism of ${}^n\mathcal{G} = (\mathcal{E}_\Gamma, \mathcal{L})$. It is immediately that all automorphisms of ${}^n\mathcal{G}$ form a group under the composition operation. We denote this group by $\mathrm{Aut}^n\mathcal{G}$.

It is obvious that for two isomorphic n-dimensional manifold graphs ${}^n\mathcal{G}_1$ and ${}^n\mathcal{G}_2$, their underlying graphs G_1 and G_2 are isomorphic. For an embedding ${}^n\mathcal{G} = (\mathcal{E}_\Gamma, \mathcal{L})$ in an n-dimensional manifold and $\forall \zeta \in \mathrm{Aut}_{\frac{1}{2}} G$, an induced action of ζ on $\mathcal{E}_\Gamma$ is defined by

$$\zeta(gx) = g\zeta(x)$$

for $\forall x \in \mathcal{E}_\Gamma$ and $\forall g \in \Gamma$. Then the following result holds.

Theorem 2.3.22 $\mathrm{Aut}^n\mathcal{G} \preceq \mathrm{Aut}_{\frac{1}{2}} G \times \langle \mu \rangle$.

Proof First we prove that two n-dimensional manifold graphs ${}^n\mathcal{G}_1 = (\mathcal{E}^1_{\Gamma_1}, \mathcal{L}_1)$ and ${}^n\mathcal{G}_2 = (\mathcal{E}^2_{\Gamma_2}, \mathcal{L}_2)$ are isomorphic if and only if there is an element $\zeta \in \mathrm{Aut}_{\frac{1}{2}}\Gamma$ such that $\mathcal{L}_1^\zeta = \mathcal{L}_2$ or $\mathcal{L}_2^{-1}$.

If there is an element $\zeta \in \mathrm{Aut}_{\frac{1}{2}}\Gamma$ such that $\mathcal{L}_1^\zeta = \mathcal{L}_2$, then the n-dimensional manifold graph ${}^n\mathcal{G}_1$ is isomorphic to ${}^n\mathcal{G}_2$ by definition. If $\mathcal{L}_1^\zeta = \mathcal{L}_2^{-1}$, then $\mathcal{L}_1^{\zeta\mu} = \mathcal{L}_2$. The n-dimensional manifold graph ${}^n\mathcal{G}_1$ is also isomorphic to ${}^n\mathcal{G}_2$.

By the definition of an isomorphism ξ between n-dimensional manifold graphs ${}^n\mathcal{G}_1$ and ${}^n\mathcal{G}_2$, we know that

$$\mu\xi(x)=\xi\mu(x),\ o\xi(x)=\xi o(x) \text{ and } \mathcal{L}_1^{\xi}(x)=\mathcal{L}_2(x).$$

$\forall x\in\mathcal{E}_\Gamma$. By definition these conditions

$$o\xi(x)=\xi o(x) \text{ and } \mathcal{L}_1^{\xi}(x)=\mathcal{L}_2(x).$$

are just the condition of an automorphism ξ or $\alpha\xi$ on $X_{\frac{1}{2}}(\Gamma)$. Whence, the assertion is true.

Now let $\mathcal{E}_{\Gamma_1}^1=\mathcal{E}_{\Gamma_2}^2=\mathcal{E}_\Gamma$ and $\mathcal{L}_1=\mathcal{L}_2=\mathcal{L}$. We know that

$$\mathrm{Aut}^n\mathcal{G}\preceq \mathrm{Aut}_{\frac{1}{2}}G\times\langle\mu\rangle\,. \qquad \natural$$

Similar to combinatorial maps, the action of an automorphism of a manifold graph on $\mathcal{E}_\Gamma$ is fixed-free.

Theorem 2.3.23 *Let ${}^n\mathcal{G}=(\mathcal{E}_\Gamma,\mathcal{L})$ be an n-dimensional manifold graph. Then $(\mathrm{Aut}^n\mathcal{G})_x$ is trivial for $\forall x\in\mathcal{E}_\Gamma$.*

Proof For $\forall g\in(\mathrm{Aut}^n\mathcal{G})_x$, we prove that $g(y)=y$ for $\forall y\in\mathcal{E}_\Gamma$. In fact, since the group $\Psi_J=\langle\mu,o,\mathcal{L}\rangle$ is transitive on $\mathcal{E}_\Gamma$, there exists an element $\tau\in\Psi_J$ such that $y=\tau(x)$. By definition we know that every element in Ψ_J is commutative with automorphisms of ${}^n\mathcal{G}$. Whence, we get that

$$g(y)=g(\tau(x))=\tau(g(x))=\tau(x)=y.$$

i.e., $(\mathrm{Aut}^n\mathcal{G})_x$ is trivial. $\natural$

Corollary 2.3.8 *Let $M=(\mathcal{X}_{\alpha,\beta},\mathcal{P})$ be a map. Then for $\forall x\in\mathcal{X}_{\alpha,\beta}$, $(\mathrm{Aut}M)_x$ is trivial.*

For an n-dimensional manifold graph ${}^n\mathcal{G}=(\mathcal{E}_\Gamma,\mathcal{L})$, an $x\in\mathcal{E}_\Gamma$ is said a *root* of ${}^n\mathcal{G}$. If we have chosen a root r on an n-dimensional manifold graph ${}^n\mathcal{G}$, then ${}^n\mathcal{G}$ is called a *rooted n-dimensional manifold graph*, denoted by ${}^n\mathcal{G}^r$. Two rooted n-dimensional manifold graphs ${}^n\mathcal{G}^{r_1}$ and ${}^n\mathcal{G}^{r_2}$ are said to be *isomorphic* if there is an isomorphism ς between them such that $\varsigma(r_1)=r_2$. Applying Theorem 2.3.23 and Corollary 2.3.1, we get an enumeration result for n-dimensional manifold graphs underlying a graph G in the following.

Theorem 2.3.24 *For any integer $n, n\geq 3$, the number $r_n^S(G)$ of rooted n-dimensional manifold graphs underlying a graph G is*

$$r_n^S(G)=\frac{n\varepsilon(G)\prod\limits_{v\in V(G)}\rho_G(v)!}{|Aut_{\frac{1}{2}}G|}.$$

Proof Denote the set of all non-isomorphic n-dimensional manifold graphs underlying a graph G by $\mathcal{G}^S(G)$. For an n-dimensional graph ${}^n\mathcal{G}=(\mathcal{E}_\Gamma,\mathcal{L})\in\mathcal{G}^S(G)$, denote the number of non-isomorphic rooted n-dimensional manifold graphs underlying ${}^n\mathcal{G}$ by $r({}^n\mathcal{G})$. By a result in permutation groups theory, for $\forall x \in \mathcal{E}_\Gamma$ we know that

$$|\mathrm{Aut}^n\mathcal{G}| = |(\mathrm{Aut}^n\mathcal{G})_x||x^{\mathrm{Aut}^n\mathcal{G}}|.$$

According to Theorem 2.3.23, $|(\mathrm{Aut}^n\mathcal{G})_x| = 1$. Whence, $|x^{\mathrm{Aut}^n\mathcal{G}}| = |\mathrm{Aut}^n\mathcal{G}|$. However there are $|\mathcal{E}_\Gamma| = 2n\varepsilon(G)$ roots in ${}^n\mathcal{G}$ by definition. Therefore, the number of non-isomorphic rooted n-dimensional manifold graphs underlying an n-dimensional graph ${}^n\mathcal{G}$ is

$$r({}^n\mathcal{G}) = \frac{|\mathcal{E}_\Gamma|}{|\mathrm{Aut}^n\mathcal{G}|} = \frac{2n\varepsilon(G)}{|\mathrm{Aut}^n\mathcal{G}|}.$$

Whence, the number of non-isomorphic rooted n-dimensional manifold graphs underlying a graph G is

$$r_n^S(G) = \sum_{{}^n\mathcal{G}\in\mathcal{G}^S(G)} \frac{2n\varepsilon(G)}{|\mathrm{Aut}^n\mathcal{G}|}.$$

According to Theorem 2.3.22, $\mathrm{Aut}^n\mathcal{G} \preceq \mathrm{Aut}_{\frac{1}{2}}G \times \langle\mu\rangle$. Whence $\tau \in \mathrm{Aut}^n\mathcal{G}$ for ${}^n\mathcal{G} \in \mathcal{G}^S(G)$ if and only if $\tau \in (\mathrm{Aut}_{\frac{1}{2}}G \times \langle\mu\rangle)_{{}^n\mathcal{G}}$. Therefore, we know that $\mathrm{Aut}^n\mathcal{G} = (\mathrm{Aut}_{\frac{1}{2}}G \times \langle\mu\rangle)_{{}^n\mathcal{G}}$. Because of $|\mathrm{Aut}_{\frac{1}{2}}G \times \langle\mu\rangle\,| = |(\mathrm{Aut}_{\frac{1}{2}}G \times \langle\mu\rangle)_{{}^n\mathcal{G}}||{}^n\mathcal{G}^{\mathrm{Aut}_{\frac{1}{2}}G\times\langle\mu\rangle}|$, we get that

$$|{}^n\mathcal{G}^{\mathrm{Aut}_{\frac{1}{2}}G\times\langle\mu\rangle}| = \frac{2|\mathrm{Aut}_{\frac{1}{2}}G|}{|\mathrm{Aut}^n\mathcal{G}|}.$$

Therefore,

$$\begin{aligned} r_n^S(G) &= \sum_{{}^n\mathcal{G}\in\mathcal{G}^S(G)} \frac{2n\varepsilon(G)}{|\mathrm{Aut}^n\mathcal{G}|} \\ &= \frac{2n\varepsilon(G)}{|\mathrm{Aut}_{\frac{1}{2}}G \times \langle\mu\rangle\,|} \sum_{{}^n\mathcal{G}\in\mathcal{G}^S(G)} \frac{|\mathrm{Aut}_{\frac{1}{2}}G \times \langle\mu\rangle\,|}{|\mathrm{Aut}^n\mathcal{G}|} \\ &= \frac{2n\varepsilon(G)}{|\mathrm{Aut}_{\frac{1}{2}}G \times \langle\mu\rangle\,|} \sum_{{}^n\mathcal{G}\in\mathcal{G}^S(G)} |{}^n\mathcal{G}^{\mathrm{Aut}_{\frac{1}{2}}G\times\langle\mu\rangle|} \\ &= \frac{n\varepsilon(G)\prod\limits_{v\in V(G)}\rho_G(v)!}{|\mathrm{Aut}_{\frac{1}{2}}G|} \end{aligned}$$

by applying Corollary 2.3.1. ♮

Notice the fact that an embedded graph in a 2-dimensional manifolds is just a map. Then Definition 3.6 is converted to Tutte's definition for combinatorial maps in this case. We can also get an enumeration result for rooted maps on surfaces underlying a graph G by applying Theorems 2.3.7 and 2.3.23 in the following.

Theorem 2.3.25([66],[67]) *The number $r^L(\Gamma)$ of rooted maps on locally orientable surfaces underlying a connected graph G is*

$$r^L(G) = \frac{2^{\beta(G)+1}\varepsilon(G)\prod\limits_{v\in V(G)}(\rho(v)-1)!}{|Aut_{\frac{1}{2}}G|},$$

where $\beta(G) = \varepsilon(G) - \nu(G) + 1$ is the Betti number of G.

Similarly, for a graph $G = \bigoplus\limits_{i=1}^{l} G_i$ and a multi-manifold $\widetilde{M} = \bigcup\limits_{i=1}^{l} \mathbf{M}^{l_i}$, choose l commutative groups $\Gamma_1, \Gamma_2, \cdots, \Gamma_l$, where $\Gamma_i = \left\langle \mu_i, o_i | \mu_i^2 = o^{h_i} = 1 \right\rangle$ for any integer $i, 1 \leq i \leq l$. Consider permutations acting on $\bigcup\limits_{i=1}^{l} \mathcal{E}_{\Gamma_i}$, where for any integer $i, 1 \leq i \leq l$, $\mathcal{E}_{\Gamma_i}$ is a disjoint union $\Gamma_i x = \{\sigma_i x | \sigma_i \in \Gamma\}$ for $\forall x \in E(G_i)$. Similar to Definition 2.3.6, we can also get a multi-embedding of G in $\widetilde{M} = \bigcup\limits_{i=1}^{l} \mathbf{M}^{h_i}$.

§2.4 Multi-Spaces on Graphs

A Smarandache multi-space is a union of k spaces $A_1, A_2, \cdots, A_k$ for an integer $k, k \geq 2$ with some additional constraint conditions. For describing a finite algebraic multi-space, graphs are a useful way. All graphs considered in this section are directed graphs.

2.4.1. A graph model for an operation system

A graph is called a *directed graph* if there is an orientation on its every edge. A directed graph $\overrightarrow{G}$ is called an *Euler graph* if we can travel all edges of $\overrightarrow{G}$ alone orientations on its edges with no repeat starting at any vertex $u \in V(\overrightarrow{G})$ and come back to u. For a directed graph $\overrightarrow{G}$, we use the convention that the orientation on the edge e is $u \to v$ for $\forall e = (u, v) \in E(\overrightarrow{G})$ and say that e is *incident from* u and *incident to* v. For $u \in V(\overrightarrow{G})$, the *outdegree* $\rho^+_{\overrightarrow{G}}(u)$ of u is the number of edges in $\overrightarrow{G}$ incident from u and the *indegree* $\rho^-_{\overrightarrow{G}}(u)$ of u is the number of edges in $\overrightarrow{G}$ incident to u. Whence, we know that

$$\rho^+_{\overrightarrow{G}}(u) + \rho^-_{\overrightarrow{G}}(u) = \rho_{\overrightarrow{G}}(u).$$

It is well-known that a graph $\overrightarrow{G}$ is Eulerian if and only if $\rho^+_{\overrightarrow{G}}(u) = \rho^-_{\overrightarrow{G}}(u)$ for $\forall u \in V(\overrightarrow{G})$, seeing examples in [11] for details. For a multiple 2-edge (a, b), if two orientations on edges are both to a or both to b, then we say it to be a *parallel* *multiple* 2*-edge*. If one orientation is to a and another is to b, then we say it to be an *opposite multiple* 2*-edge*.

Now let $(A;\circ)$ be an algebraic system with operation $\circ$. We associate a *weighted graph* $G[A]$ for $(A;\circ)$ defined as in the next definition.

Definition 2.4.1 *Let* $(A;\circ)$ *be an algebraic system. Define a weighted graph* $G[A]$ *associated with* $(A;\circ)$ *by*

$$V(G[A]) \ = \ A$$

and

$$E(G[A]) = \{(a,c) \ with \ weight \ \circ b \mid if \ a \circ b = c \ for \ \forall a,b,c \in A\}$$

as shown in Fig.2.29.

Fig.2.29

For example, the associated graph $G[Z_4]$ for the commutative group Z_4 is shown in Fig.2.30.

Fig.2.30

The advantage of Definition 2.4.1 is that for any edge in $G[A]$, if its vertices are a,c with a weight $\circ b$, then a$\circ b = c$ and vice versa, if a$\circ b = c$, then there is one and only one edge in $G[A]$ with vertices a, c and weight $\circ b$. This property enables us to find some structure properties of $G[A]$ for an algebraic system $(A;\circ)$.

*P*1. $G[A]$ *is connected if and only if there are no partition* $A = A_1 \bigcup A_2$ *such that for* $\forall a_1 \in A_1$, $\forall a_2 \in A_2$, *there are no definition for* $a_1 \circ a_2$ *in* $(A;\circ)$.

If $G[A]$ is disconnected, we choose one component C and let $A_1 = V(C)$. Define $A_2 = V(G[A]) \setminus V(C)$. Then we get a partition $A = A_1 \bigcup A_2$ and for $\forall a_1 \in A_1$,

$\forall a_2 \in A_2$, there are no definition for $a_1 \circ a_2$ in $(A;\circ)$, a contradiction and vice versa.

*P*2. *If there is a unit* $\mathbf{1}_A$ *in* $(A;\circ)$, *then there exists a vertex* $\mathbf{1}_A$ *in* $G[A]$ *such that the weight on the edge* $(\mathbf{1}_A, x)$ *is* $\circ x$ *if* $\mathbf{1}_A \circ x$ *is defined in* $(A;\circ)$ *and vice versa.*

*P*3. *For* $\forall a \in A$, *if* a^{-1} *exists, then there is an opposite multiple* 2*-edge* $(\mathbf{1}_A, a)$ *in* $G[A]$ *with weights* $\circ a$ *and* $\circ a^{-1}$, *respectively and vice versa.*

*P*4. *For* $\forall a, b \in A$ *if* $a \circ b = b \circ a$, *then there are edges* (a, x) *and* (b, x), $x \in A$ *in* $(A;\circ)$ *with weights* $w(a, x) = \circ b$ *and* $w(b, x) = \circ a$, *respectively and vice versa.*

*P*5. *If the cancellation law holds in* $(A;\circ)$, *i.e., for* $\forall a, b, c \in A$, *if* $a \circ b = a \circ c$ *then* $b = c$, *then there are no parallel multiple* 2*-edges in* $G[A]$ *and vice versa.*

The property $P2, P3, P4$ and $P5$ are gotten by definition. Each of these cases is shown in Fig.2.31(1), (2), (3) and (4), respectively.

Fig.2.31

Definition 2.4.2 *An algebraic system* $(A;\circ)$ *is called to be a one-way system if there exists a mapping* $\varpi : A \rightarrow A$ *such that if* $a \circ b \in A$, *then there exists a unique* $c \in A$, $c \circ \varpi(b) \in A$. ϖ *is called a one-way function on* $(A;\circ)$.

We have the following results for an algebraic system $(A;\circ)$ with its associated weighted graph $G[A]$.

Theorem 2.4.1 *Let* $(A;\circ)$ *be an algebraic system with a associated weighted graph* $G[A]$. *Then*

(*i*) *if there is a one-way function* ϖ *on* $(A;\circ)$, *then* $G[A]$ *is an Euler graph, and vice versa, if* $G[A]$ *is an Euler graph, then there exist a one-way function* ϖ *on* $(A;\circ)$.

(*ii*) *if* $(A;\circ)$ *is a complete algebraic system, then the outdegree of every vertex in* $G[A]$ *is* $|A|$; *in addition, if the cancellation law holds in* $(A;\circ)$, *then* $G[A]$ *is a complete multiple* 2*-graph with a loop attaching at each of its vertices such that each edge between two vertices in* $G[A]$ *is an opposite multiple* 2*-edge, and vice versa.*

Proof (*i*) Assume ϖ is a one-way function ϖ on $(A;\circ)$. By definition there exists $c \in A$, $c \circ \varpi(b) \in A$ for $\forall a \in A$, $a \circ b \in A$. Thereby there is a one-to-one correspondence between edges from a with edges to a. That is, $\rho^{+}_{G[A]}(a) = \rho^{-}_{G[A]}(a)$ for $\forall a \in V(G[A])$. Therefore, $G[A]$ is an Euler graph.

Now if $G[A]$ is an Euler graph, then there is a one-to-one correspondence between edges in $E^{-} = \{e_i^{-}; 1 \leq i \leq k\}$ from a vertex a with edges $E^{+} = \{e_i^{+}; 1 \leq i \leq k\}$ to the vertex a. For any integer $i, 1 \leq i \leq k$, define $\varpi : w(e_i^{-}) \rightarrow w(e_i^{+})$. Therefore, ϖ is a well-defined one-way function on $(A; \circ)$.

(ii) If $(A; \circ)$ is complete, then for $\forall a \in A$ and $\forall b \in A$, $a \circ b \in A$. Therefore, $\rho^{+}_{\overrightarrow{G}}(a) = |A|$ for any vertex $a \in V(G[A])$.

If the cancellation law holds in $(A; \circ)$, by $P5$ there are no parallel multiple 2-edges in $G[A]$. Whence, each edge between two vertices is an opposite 2-edge and weights on loops are $\circ \mathbf{1}_A$.

By definition, if $G[A]$ is a complete multiple 2-graph with a loop attaching at each of its vertices such that each edge between two vertices in $G[A]$ is an opposite multiple 2-edge, we know that $(A; \circ)$ is a complete algebraic system with the cancellation law holding by the definition of $G[A]$. ♮

Corollary 2.4.1 *Let Γ be a semigroup. Then $G[\Gamma]$ is a complete multiple 2-graph with a loop attaching at each of its vertices such that each edge between two vertices in $G[A]$ is an opposite multiple 2-edge.*

Notice that in a group Γ, $\forall g \in \Gamma$, if $g^2 \neq \mathbf{1}_\Gamma$, then $g^{-1} \neq g$. Whence, all elements of order> 2 in Γ can be classified into pairs. This fact enables us to know the following result.

Corollary 2.4.2 *Let Γ be a group of even order. Then there are opposite multiple 2-edges in $G[\Gamma]$ such that weights on its 2 directed edges are the same.*

2.4.2. **Multi-Spaces on graphs**

Let $\widetilde{\Gamma}$ be a Smarandache multi-space. Its associated weighted graph is defined in the following.

Definition 2.4.3 *Let $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ be an algebraic multi-space with $(\Gamma_i; \circ_i)$ being an algebraic system for any integer $i, 1 \leq i \leq n$. Define a weighted graph $G(\widetilde{\Gamma})$ associated with $\widetilde{\Gamma}$ by*

$$G(\widetilde{\Gamma}) = \bigcup_{i=1}^{n} G[\Gamma_i],$$

where $G[\Gamma_i]$ is the associated weighted graph of $(\Gamma_i; \circ_i)$ for $1 \leq i \leq n$.

For example, the weighted graph shown in Fig.2.32 is correspondent with a multi-space $\widetilde{\Gamma} = \Gamma_1 \bigcup \Gamma_2 \bigcup \Gamma_3$, where $(\Gamma_1; +) = (Z_3, +)$, $\Gamma_2 = \{e, a, b\}$, $\Gamma_3 = \{1, 2, a, b\}$ and these operations $\cdot$on Γ_2 and $\circ$ on Γ_3 are shown in tables 2.4.1 and 2.4.2.

Fig¸2.32

$\cdot$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

table¸ 2.4.1

$\circ$	1	2	a	b
1	*	a	b	*
2	b	*	*	a
a	*	*	*	1
b	*	*	2	*

table¸ 2.4.2

Notice that the correspondence between the multi-space $\widetilde{\Gamma}$ and the weighted graph $G[\widetilde{\Gamma}]$ is one-to-one. We immediately get the following result.

Theorem 2.4.2 *The mappings* $\pi : \widetilde{\Gamma} \to G[\widetilde{\Gamma}]$ *and* $\pi^{-1} : G[\widetilde{\Gamma}] \to \widetilde{\Gamma}$ *are all one-to-one.*

According to Theorems 2.4.1 and 2.4.2, we get some consequences in the following.

Corollary 2.4.3 *Let* $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ *be a multi-space with an algebraic system* $(\Gamma_i; \circ_i)$ *for any integer* $i, 1 \leq i \leq n$. *If for any integer* $i, 1 \leq i \leq n$, $G[\Gamma_i]$ *is a complete multiple* 2*-graph with a loop attaching at each of its vertices such that each edge between two vertices in* $G[\Gamma_i]$ *is an opposite multiple* 2*-edge, then* $\widetilde{\Gamma}$ *is a complete multi-space.*

Corollary 2.4.4 *Let* $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ *be a multi-group with an operation set* $O(\widetilde{\Gamma}) = \{\circ_i; 1 \leq i \leq n\}$. *Then there is a partition* $G[\widetilde{\Gamma}] = \bigcup\limits_{i=1}^{n} G_i$ *such that each* G_i *being a complete multiple* 2*-graph attaching with a loop at each of its vertices such that each* *edge between two vertices in $V(G_i)$ is an opposite multiple 2-edge for any integer $i, 1 \leq i \leq n$.*

Corollary 2.4.5 *Let F be a body. Then $G[F]$ is a union of two graphs $K^2(F)$ and $K^2(F^*)$, where $K^2(F)$ or $K^2(F^*)$ is a complete multiple 2-graph with vertex set F or $F^* = F \setminus \{0\}$ and with a loop attaching at each of its vertices such that each edge between two different vertices is an opposite multiple 2-edge.*

2.4.3. **Cayley graphs of a multi-group**

Similar to the definition of Cayley graphs of a finite generated group, we can also define *Cayley graphs of a finite generated multi-group*, where a multi-group $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ is said to be *finite generated* if the group Γ_i is finite generated for any integer $i, 1 \leq i \leq n$, i.e., $\Gamma_i = \langle x_i, y_i, \cdots, z_{s_i} \rangle$. We denote by $\widetilde{\Gamma} = \langle x_i, y_i, \cdots, z_{s_i}; 1 \leq i \leq n \rangle$ if $\widetilde{\Gamma}$ is finite generated by $\{x_i, y_i, \cdots, z_{s_i}; 1 \leq i \leq n\}$.

Definition 2.4.4 *Let $\widetilde{\Gamma} = \langle x_i, y_i, \cdots, z_{s_i}; 1 \leq i \leq n \rangle$ be a finite generated multi-group, $\widetilde{S} = \bigcup\limits_{i=1}^{n} S_i$, where $1_{\Gamma_i} \notin S_i$, $\widetilde{S}^{-1} = \{a^{-1} | a \in \widetilde{S}\} = \widetilde{S}$ and $\langle S_i \rangle = \Gamma_i$ for any integer $i, 1 \leq i \leq n$. A Cayley graph $Cay(\widetilde{\Gamma} : \widetilde{S})$ is defined by*

$$V(Cay(\widetilde{\Gamma} : \widetilde{S})) = \widetilde{\Gamma}$$

and

$$E(Cay(\widetilde{\Gamma} : \widetilde{S})) = \{(g, h) |\ if\ there\ exists\ an\ integer\ i, g^{-1} \circ_i h \in S_i, 1 \leq i \leq n\}.$$

By Definition 2.4.4, we immediately get the following result for Cayley graphs of a finite generated multi-group.

Theorem 2.4.3 *For a Cayley graph $Cay(\widetilde{\Gamma} : \widetilde{S})$ with $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ and $\widetilde{S} = \bigcup\limits_{i=1}^{n} S_i$,*

$$Cay(\widetilde{\Gamma} : \widetilde{S}) = \bigcup_{i=1}^{n} Cay(\Gamma_i : S_i).$$

It is well-known that *every Cayley graph of order≥ 3 is 2-connected*. But in general, a Cayley graph of a multi-group is not connected. For the connectedness of Cayley graphs of multi-groups, we get the following result.

Theorem 2.4.4 *A Cayley graph $Cay(\widetilde{\Gamma} : \widetilde{S})$ with $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ and $\widetilde{S} = \bigcup\limits_{i=1}^{n} S_i$ is connected if and only if for any integer $i, 1 \leq i \leq n$, there exists an integer $j, 1 \leq j \leq n$ and $j \neq i$ such that $\Gamma_i \bigcap \Gamma_j \neq \emptyset$.*

Proof According to Theorem 2.4.3, if there is an integer $i, 1 \leq i \leq n$ such that $\Gamma_i \bigcap \Gamma_j = \emptyset$ for any integer $j, 1 \leq j \leq n$, $j \neq i$, then there are no edges with the form (g_i, h), $g_i \in \Gamma_i$, $h \in \widetilde{\Gamma} \setminus \Gamma_i$. Thereby $Cay(\widetilde{\Gamma} : \widetilde{S})$ is not connected.

Notice that $Cay(\widetilde{\Gamma} : \widetilde{S}) = \bigcup\limits_{i=1}^{n} Cay(\Gamma_i : S_i)$. Not loss of generality, we assume that $g \in \Gamma_k$ and $h \in \Gamma_l$, where $1 \leq k, l \leq n$ for any two elements $g, h \in \widetilde{\Gamma}$. If $k = l$, then there must exists a path connecting g and h in $Cay(\widetilde{\Gamma} : \widetilde{S})$.

Now if $k \neq l$ and for any integer $i, 1 \leq i \leq n$, there is an integer $j, 1 \leq j \leq n$ and $j \neq i$ such that $\Gamma_i \bigcap \Gamma_j \neq \emptyset$, then we can find integers $i_1, i_2, \cdots, i_s$, $1 \leq i_1, i_2, \cdots, i_s \leq n$ such that

$$\Gamma_k \bigcap \Gamma_{i_1} \neq \emptyset,$$

$$\Gamma_{i_1} \bigcap \Gamma_{i_2} \neq \emptyset,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\Gamma_{i_s} \bigcap \Gamma_l \neq \emptyset.$$

Thereby we can find a path connecting g and h in $Cay(\widetilde{\Gamma} : \widetilde{S})$ passing through these vertices in $Cay(\Gamma_{i_1} : S_{i_1})$, $Cay(\Gamma_{i_2} : S_{i_2})$, $\cdots$, and $Cay(\Gamma_{i_s} : S_{i_s})$. Therefore, $Cay(\widetilde{\Gamma} : \widetilde{S})$ is connected. $\natural$

The following theorem is gotten by the definition of a Cayley graph and Theorem 2.4.4.

Theorem 2.4.5 *If* $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma$ *with* $|\Gamma| \geq 3$, *then a Cayley graph* $Cay(\widetilde{\Gamma} : \widetilde{S})$

(*i*) *is an* $|\widetilde{S}|$*-regular graph;*

(*ii*) *the edge connectivity* $\kappa(Cay(\widetilde{\Gamma} : \widetilde{S})) \geq 2n$.

Proof The assertion (*i*) is gotten by the definition of $Cay(\widetilde{\Gamma} : \widetilde{S})$. For (*ii*) since every Cayley graph of order≥ 3 is 2-connected, for any two vertices g, h in $Cay(\widetilde{\Gamma} : \widetilde{S})$, there are at least $2n$ edge disjoint paths connecting g and h. Whence, the edge connectivity $\kappa(Cay(\widetilde{\Gamma} : \widetilde{S})) \geq 2n$. $\natural$

Applying multi-voltage graphs, we get a structure result for Cayley graphs of a finite multi-group similar to that of Cayley graphs of a finite group.

Theorem 2.4.6 *For a Cayley graph* $Cay(\widetilde{\Gamma} : \widetilde{S})$ *of a finite multi-group* $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ *with* $\widetilde{S} = \bigcup\limits_{i=1}^{n} S_i$, *there is a multi-voltage bouquet* $\varsigma : B_{|\widetilde{S}|} \rightarrow \widetilde{S}$ *such that* $Cay(\widetilde{\Gamma} : \widetilde{S}) \cong (B_{|\widetilde{S}|})^{\varsigma}$.

Proof Let $\widetilde{S} = \{s_i; 1 \leq i \leq |\widetilde{S}|\}$ and $E(B_{|\widetilde{S}|}) = \{L_i; 1 \leq i \leq |\widetilde{S}|\}$. Define a multi-voltage graph on a bouquet $B_{|\widetilde{S}|}$ by

$$\varsigma : L_i \to s_i, \quad 1 \leq i \leq |\widetilde{S}|.$$

Then we know that there is an isomorphism τ between $(B_{|\widetilde{S}|})^{\varsigma}$ and $Cay(\widetilde{\Gamma} : \widetilde{S})$ by defining $\tau(O_g) = g$ for $\forall g \in \widetilde{\Gamma}$, where $V(B_{|\widetilde{S}|}) = \{O\}$. $\natural$

Corollary 2.4.6 *For a Cayley graph $Cay(\Gamma : S)$ of a finite group Γ, there exists a voltage bouquet $\alpha : B_{|S|} \to S$ such that $Cay(\Gamma : S) \cong (B_{|S|})^{\alpha}$.*

§2.5 Graph Phase Spaces

The behavior of a graph in an m-manifold is related with theoretical physics since it can be viewed as a model of p-branes in M-theory both for a microcosmic and macrocosmic world. For more details one can see in Chapter 6. This section concentrates on surveying some useful fundamental elements for graphs in n-manifolds.

2.5.1. Graph phase in a multi-space

For convenience, we introduce some notations used in this section in the following.

$\widetilde{\mathbf{M}}$ – a multi-manifold $\widetilde{\mathbf{M}} = \bigcup\limits_{i=1}^{n} \mathbf{M}^{n_i}$, where $\mathbf{M}^{n_i}$ is an n_i-manifold, $n_i \geq 2$. For multi-manifolds, see also those materials in Subsection 1.5.4.

$\overline{u} \in \widetilde{\mathbf{M}}$ – a point $\overline{u}$ of $\widetilde{\mathbf{M}}$.

$\mathcal{G}$ – a graph G embedded in $\widetilde{\mathbf{M}}$.

$C(\widetilde{\mathbf{M}})$ – the set of smooth mappings $\omega : \widetilde{\mathbf{M}} \to \widetilde{\mathbf{M}}$, differentiable at each point $\overline{u}$ in $\widetilde{\mathbf{M}}$.

Now we define the phase of a graph in a multi-space.

Definition 2.5.1 *Let $\mathcal{G}$ be a graph embedded in a multi-manifold $\widetilde{\mathbf{M}}$. A phase of $\mathcal{G}$ in $\widetilde{\mathbf{M}}$ is a triple $(\mathcal{G}; \omega, \Lambda)$ with an operation $\circ$ on $C(\widetilde{M})$, where $\omega : V(G) \to C(\widetilde{\mathbf{M}})$ and $\Lambda : E(\mathcal{G}) \to C(\widetilde{\mathbf{M}})$ such that $\Lambda(\overline{u}, \overline{v}) = \frac{\omega(\overline{u}) \circ \omega(\overline{v})}{\| \overline{u} - \overline{v} \|}$ for $\forall (\overline{u}, \overline{v}) \in E(\mathcal{G})$, where $\| \overline{u} \|$ denotes the norm of $\overline{u}$.*

For examples, the complete graph K_4 embedded in $\mathbf{R}^3$ has a phase as shown in Fig.2.33, where $g \in C(\mathbf{R}^3)$ and $h \in C(\mathbf{R}^3)$.

Fig.2.33

Similar to the definition of a adjacent matrix on a graph, we can also define matrixes on graph phases .

Definition 2.5.2 *Let* $(\mathcal{G};\omega,\Lambda)$ *be a phase and* $A[G]=[a_{ij}]_{p\times p}$ *the adjacent matrix of a graph* G *with* $V(G)=\{v_1,v_2,\cdots,v_p\}$*. Define matrixes* $V[\mathcal{G}]=[V_{ij}]_{p\times p}$ *and* $\Lambda[\mathcal{G}]=[\Lambda_{ij}]_{p\times p}$ *by*

$$V_{ij}=\frac{\omega(\overline{v}_i)}{\parallel \overline{v}_i-\overline{v}_j \parallel}\ if\ a_{ij}\neq 0;\ otherwise, V_{ij}=0$$

and

$$\Lambda_{ij}=\frac{\omega(\overline{v}_i)\circ\omega(\overline{v}_j)}{\parallel \overline{v}_i-\overline{v}_j \parallel^2}\ if\ a_{ij}\neq 0;\ otherwise, \Lambda_{ij}=0,$$

where $\circ$ *is an operation on* $C(\widetilde{M})$.

For example, for the phase of K_4 in Fig.2.33, if choice $g(u)=(x_1,x_2,x_3)$, $g(v)=(y_1,y_2,y_3)$, $g(w)=(z_1,z_2,z_3)$, $g(o)=(t_1,t_2,t_3)$ and $\circ=\times$, the multiplication of vectors in $\mathbf{R}^3$, then we get that

$$V(\mathcal{G})=\begin{bmatrix} 0 & \frac{g(u)}{\rho(u,v)} & \frac{g(u)}{\rho(u,w)} & \frac{g(u)}{\rho(u,o)} \\ \frac{g(v)}{\rho(v,u)} & 0 & \frac{g(v)}{\rho(v,w)} & \frac{g(v)}{\rho(v,t)} \\ \frac{g(w)}{\rho(w,u)} & \frac{g(w)}{\rho(w,v)} & 0 & \frac{g(w)}{\rho(w,o)} \\ \frac{g(o)}{\rho(o,u)} & \frac{g(o)}{\rho(o,v)} & \frac{g(o)}{\rho(o,w)} & 0 \end{bmatrix}$$

where

$$\rho(u,v)=\rho(v,u)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2},$$

$$\rho(u,w)=\rho(w,u)=\sqrt{(x_1-z_1)^2+(x_2-z_2)^2+(x_3-z_3)^2},$$

$$\rho(u,o)=\rho(o,u)=\sqrt{(x_1-t_1)^2+(x_2-t_2)^2+(x_3-t_3)^2},$$

$$\rho(v,w)=\rho(w,v)=\sqrt{(y_1-z_1)^2+(y_2-z_2)^2+(y_3-z_3)^2},$$

$$\rho(v,o)=\rho(o,v)=\sqrt{(y_1-t_1)^2+(y_2-t_2)^2+(y_3-t_3)^2},$$

$$\rho(w,o)=\rho(o,w)=\sqrt{(z_1-t_1)^2+(z_2-t_2)^2+(z_3-t_3)^2}$$

and

$$\Lambda(\mathcal{G}) = \begin{bmatrix} 0 & \frac{g(u)\times g(v)}{\rho^2(u,v)} & \frac{g(u)\times g(w)}{\rho^2(u,w)} & \frac{g(u)\times g(o)}{\rho^2(u,o)} \\ \frac{g(v)\times g(u)}{\rho^2(v,u)} & 0 & \frac{g(v)\times g(w)}{\rho^2(v,w)} & \frac{g(v\times g(o)}{\rho^2(v,o)} \\ \frac{g(w)\times g(u)}{\rho^2(w,u)} & \frac{g(w)\times g(v)}{\rho^2(w,v)} & 0 & \frac{g(w)\times g(o)}{\rho^2(w,o)} \\ \frac{g(o)\times g(u)}{\rho^2(o,u)} & \frac{g(o)\times g(v)}{\rho^2(o,v)} & \frac{g(o)\times g(w)}{\rho^2(o,w)} & 0 \end{bmatrix}.$$

where

$$g(u) \times g(v) = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1),$$

$$g(u) \times g(w) = (x_2z_3 - x_3z_2, x_3z_1 - x_1z_3, x_1z_2 - x_2z_1),$$

$$g(u) \times g(o) = (x_2t_3 - x_3t_2, x_3t_1 - x_1t_3, x_1t_2 - x_2t_1),$$

$$g(v) \times g(u) = (y_2x_3 - y_3x_2, y_3x_1 - y_1x_3, y_1x_2 - y_2x_1),$$

$$g(v) \times g(w) = (y_2z_3 - y_3z_2, y_3z_1 - y_1z_3, y_1z_2 - y_2z_1),$$

$$g(v) \times g(o) = (y_2t_3 - y_3t_2, y_3t_1 - y_1t_3, y_1t_2 - y_2t_1),$$

$$g(w) \times g(u) = (z_2x_3 - z_3x_2, z_3x_1 - z_1x_3, z_1x_2 - z_2x_1),$$

$$g(w) \times g(v) = (z_2y_3 - z_3y_2, z_3y_1 - z_1y_3, z_1y_2 - z_2y_1),$$

$$g(w) \times g(o) = (z_2t_3 - z_3t_2, z_3t_1 - z_1t_3, z_1t_2 - z_2t_1),$$

$$g(o) \times g(u) = (t_2x_3 - t_3x_2, t_3x_1 - t_1x_3, t_1x_2 - t_2x_1),$$

$$g(o) \times g(v) = (t_2y_3 - t_3y_2, t_3y_1 - t_1y_3, t_1y_2 - t_2y_1),$$

$$g(o) \times g(w) = (t_2z_3 - t_3z_2, t_3z_1 - t_1z_3, t_1z_2 - t_2z_1).$$

For two given matrixes $A = [a_{ij}]_{p\times p}$ and $B = [b_{ij}]_{p\times p}$, the *star product* $*$ on an operation $\circ$ is defined by $A * B = [a_{ij} \circ b_{ij}]_{p\times p}$. We get the following result for matrixes $V[\mathcal{G}]$ and $\Lambda[\mathcal{G}]$.

Theorem 2.5.1 $V[\mathcal{G}] * V^t[\mathcal{G}] = \Lambda[\mathcal{G}]$.

Proof Calculation shows that each (i,j) entry in $V[\mathcal{G}] * V^t[\mathcal{G}]$ is

$$\frac{\omega(\overline{v}_i)}{\| \overline{v}_i - \overline{v}_j \|} \circ \frac{\omega(\overline{v}_j)}{\| \overline{v}_j - \overline{v}_i \|} = \frac{\omega(\overline{v}_i) \circ \omega(\overline{v}_j)}{\| \overline{v}_i - \overline{v}_j \|^2} = \Lambda_{ij},$$

where $1 \le i, j \le p$. Therefore, we get that

$$V[\mathcal{G}] * V^t[\mathcal{G}] = \Lambda[\mathcal{G}]. \qquad \natural$$

An operation called *addition on graph phases* is defined in the next.

Definition 2.5.3 *For two phase spaces $(\mathcal{G}_1; \omega_1, \Lambda_1)$, $(\mathcal{G}_2; \omega_2, \Lambda_2)$ of graphs G_1, G_2 in $\widetilde{M}$ and two operations $\bullet$ and $\circ$ on $C(\widetilde{M})$, their addition is defined by*

$$(\mathcal{G}_1; \omega_1, \Lambda_1) \bigoplus (\mathcal{G}_2; \omega_2, \Lambda_2) = (\mathcal{G}_1 \bigoplus \mathcal{G}_2; \omega_1 \bullet \omega_2, \Lambda_1 \bullet \Lambda_2),$$

where $\omega_1 \bullet \omega_2 : V(\mathcal{G}_1 \bigcup \mathcal{G}_2) \to C(\widetilde{M})$ satisfying

$$\omega_1 \bullet \omega_2(\overline{u}) = \begin{cases} \omega_1(\overline{u}) \bullet \omega_2(\overline{u}), & if\ \overline{u} \in V(\mathcal{G}_1) \bigcap V(\mathcal{G}_2), \\ \omega_1(\overline{u}), & if\ \overline{u} \in V(\mathcal{G}_1) \setminus V(\mathcal{G}_2), \\ \omega_2(\overline{u}), & if\ \overline{u} \in V(\mathcal{G}_2) \setminus V(\mathcal{G}_1). \end{cases}$$

and

$$\Lambda_1 \bullet \Lambda_2(\overline{u}, \overline{v}) = \frac{\omega_1 \bullet \omega_2(\overline{u}) \circ \omega_1 \bullet \omega_2(\overline{v})}{\| \overline{u} - \overline{v} \|^2}$$

for $(\overline{u}, \overline{v}) \in E(\mathcal{G}_1) \bigcup E(\mathcal{G}_2)$

The following result is immediately gotten by Definition 2.5.3.

Theorem 2.5.2 *For two given operations $\bullet$ and $\circ$ on $C(\widetilde{M})$, all graph phases in $\widetilde{M}$ form a linear space on the field Z_2 with a phase $\bigoplus$ for any graph phases $(\mathcal{G}_1; \omega_1, \Lambda_1)$ and $(\mathcal{G}_2; \omega_2, \Lambda_2)$ in $\widetilde{M}$.*

2.5.2. **Transformation of a graph phase**

Definition 2.5.4 *Let $(\mathcal{G}_1; \omega_1, \Lambda_1)$ and $(\mathcal{G}_2; \omega_2, \Lambda_2)$ be graph phases of graphs G_1 and G_2 in a multi-space $\widetilde{M}$ with operations $\circ_1, \circ_2$, respectively. If there exists a smooth mapping $\tau \in C(\widetilde{M})$ such that*

$$\tau : (\mathcal{G}_1; \omega_1, \Lambda_1) \to (\mathcal{G}_2; \omega_2, \Lambda_2),$$

i.e., for $\forall \overline{u} \in V(\mathcal{G}_1)$, $\forall (\overline{u}, \overline{v}) \in E(\mathcal{G}_1)$, $\tau(\mathcal{G}_1) = \mathcal{G}_2$, $\tau(\omega_1(\overline{u})) = \omega_2(\tau(\overline{u}))$ and $\tau(\Lambda_1(\overline{u}, \overline{v})) = \Lambda_2(\tau(\overline{u}, \overline{v}))$, then we say $(\mathcal{G}_1; \omega_1, \Lambda_1)$ and $(\mathcal{G}_2; \omega_2, \Lambda_2)$ are transformable and τ a transform mapping.

For examples, a transform mapping t for embeddings of K_4 in $\mathbf{R}^3$ and on the plane is shown in Fig.2.34

Fig.2.34

Theorem 2.5.3 *Let* $(\mathcal{G}_1;\omega_1,\Lambda_1)$ *and* $(\mathcal{G}_2;\omega_2,\Lambda_2)$ *be transformable graph phases with transform mapping* τ*. If* τ *is one-to-one on* $\mathcal{G}_1$ *and* $\mathcal{G}_2$*, then* $\mathcal{G}_1$ *is isomorphic to* $\mathcal{G}_2$*.*

Proof By definitions, if τ is one-to-one on $\mathcal{G}_1$ and $\mathcal{G}_2$, then τ is an isomorphism between $\mathcal{G}_1$ and $\mathcal{G}_2$. ♮

A very useful case among transformable graph phases is that one can find parameters $t_1, t_2, \cdots, t_q, q \geq 1$, such that each vertex of a graph phase is a smooth mapping of $t_1, t_2, \cdots, t_q$, i.e., for $\forall \overline{u} \in \widetilde{M}$, we consider it as $\overline{u}(t_1, t_2, \cdots, t_q)$. In this case, we introduce two conceptions on graph phases.

Definition 2.5.5 *For a graph phase* $(\mathcal{G};\omega,\Lambda)$*, define its capacity* $Ca(\mathcal{G};\omega,\Lambda)$ *and entropy* $En(\mathcal{G};\omega,\Lambda)$ *by*

$$Ca(\mathcal{G};\omega,\Lambda) = \sum_{\overline{u}\in V(\mathcal{G})} \omega(\overline{u})$$

and

$$En(\mathcal{G};\omega,\Lambda) = \log(\prod_{\overline{u}\in V(\mathcal{G})} \parallel \omega(\overline{u}) \parallel).$$

Then we know the following result.

Theorem 2.5.4 *For a graph phase* $(\mathcal{G};\omega,\Lambda)$*, its capacity* $Ca(\mathcal{G};\omega,\Lambda)$ *and entropy* $En(\mathcal{G};\omega,\Lambda)$ *satisfy the following differential equations*

$$\mathrm{d}Ca(\mathcal{G};\omega,\Lambda) = \frac{\partial Ca(\mathcal{G};\omega,\Lambda)}{\partial u_i}\mathrm{d}u_i \ \ \textit{and} \ \ \mathrm{d}En(\mathcal{G};\omega,\Lambda) = \frac{\partial En(\mathcal{G};\omega,\Lambda)}{\partial u_i}\mathrm{d}u_i,$$

where we use the Einstein summation convention, i.e., a sum is over i *if it is appearing both in upper and lower indices.*

Proof Not loss of generality, we assume $\overline{u} = (u_1, u_2, \cdots, u_p)$ for $\forall \overline{u} \in \widetilde{M}$. According to the invariance of differential form, we know that

$$\mathrm{d}\omega = \frac{\partial \omega}{\partial u_i}\mathrm{d}u_i.$$

By the definition of the capacity $Ca(\mathcal{G};\omega,\Lambda)$ and entropy $En(\mathcal{G};\omega,\Lambda)$ of a graph phase, we get that

$$\begin{aligned}
\mathrm{d}Ca(\mathcal{G};\omega,\Lambda) &= \sum_{\overline{u}\in V(\mathcal{G})} \mathrm{d}(\omega(\overline{u})) \\
&= \sum_{\overline{u}\in V(\mathcal{G})} \frac{\partial \omega(\overline{u})}{\partial u_i}\mathrm{d}u_i = \frac{\partial(\sum\limits_{\overline{u}\in V(\mathcal{G})} \omega(\overline{u}))}{\partial u_i}\mathrm{d}u_i \\
&= \frac{\partial Ca(\mathcal{G};\omega,\Lambda)}{\partial u_i}\mathrm{d}u_i.
\end{aligned}$$

Similarly, we also obtain that

$$\begin{aligned}
\mathrm{d}En(\mathcal{G};\omega,\Lambda) &= \sum_{\overline{u}\in V(\mathcal{G})} \mathrm{d}(\log \parallel \omega(\overline{u}) \parallel) \\
&= \sum_{\overline{u}\in V(\mathcal{G})} \frac{\partial \log|\omega(\overline{u})|}{\partial u_i}\mathrm{d}u_i = \frac{\partial(\sum\limits_{\overline{u}\in V(\mathcal{G})} \log \parallel \omega(\overline{u}) \parallel)}{\partial u_i}\mathrm{d}u_i \\
&= \frac{\partial En(\mathcal{G};\omega,\Lambda)}{\partial u_i}\mathrm{d}u_i.
\end{aligned}$$

This completes the proof. ♮

In a 3-dimensional Euclid space we can get more concrete results for graph phases $(\mathcal{G};\omega,\Lambda)$. In this case, we get some formulae in the following by choice $\overline{u} = (x_1, x_2, x_3)$ and $\overline{v} = (y_1, y_2, y_3)$.

$$\omega(\overline{u}) = (x_1, x_2, x_3) \text{ for } \forall \overline{u} \in V(\mathcal{G}),$$

$$\Lambda(\overline{u},\overline{v}) = \frac{x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1}{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2} \text{ for } \forall(\overline{u},\overline{v}) \in E(\mathcal{G}),$$

$$Ca(\mathcal{G};\omega,\Lambda) = (\sum_{\overline{u}\in V(\mathcal{G})} x_1(\overline{u}), \sum_{\overline{u}\in V(\mathcal{G})} x_2(\overline{u}), \sum_{\overline{u}\in V(\mathcal{G})} x_3(\overline{u}))$$

and

$$En(\mathcal{G};\omega,\Lambda) = \sum_{\overline{u}\in V(\mathcal{G})} \log(x_1^2(\overline{u}) + x_2^2(\overline{u}) + x_3^2(\overline{u}).$$

§2.6 Remarks and Open Problems

2.6.1 A graphical property $P(G)$ is called to be *subgraph hereditary* if for any subgraph $H \subseteq G$, H posses $P(G)$ whenever G posses the property $P(G)$. For example, the properties: *G is complete* and *the vertex coloring number* $\chi(G) \leq k$ both are subgraph hereditary. The hereditary property of a graph can be generalized by the following way.

Let G and H be two graphs in a space $\widetilde{M}$. If there is a smooth mapping ς in $C(\widetilde{M})$ such that $\varsigma(G) = H$, then we say G and H are *equivalent in* $\widetilde{M}$. Many conceptions in graph theory can be included in this definition, such as *graph homomorphism, graph equivalent*, $\cdots$, etc.

Problem 2.6.1 *Applying different smooth mappings in a space such as smooth mappings in* $\mathbf{R}^3$ *or* $\mathbf{R}^4$ *to classify graphs and to find their invariants.*

Problem 2.6.2 *Find which parameters already known in graph theory for a graph is invariant or to find the smooth mapping in a space on which this parameter is invariant.*

2.6.2 As an efficient way for finding regular covering spaces of a graph, voltage graphs have been gotten more attentions in the past half-century by mathematicians. Works for regular covering spaces of a graph can seen in [23], [45] – [46] and [71] – [72]. But few works are found in publication for irregular covering spaces of a graph. The multi-voltage graph of type 1 or type 2 with multi-groups defined in Section 2.2 are candidate for further research on irregular covering spaces of graphs.

Problem 2.6.3 *Applying multi-voltage graphs to get the genus of a graph with less symmetries.*

Problem 2.6.4 *Find new actions of a multi-group on a graph, such as the left subaction and its contribution to topological graph theory. What can we say for automorphisms of the lifting of a multi-voltage graph?*

There is a famous conjecture for Cayley graphs of a finite group in algebraic graph theory, i.e., *every connected Cayley graph of order*≥ 3 *is hamiltonian*. Similarly, we can also present a conjecture for Cayley graphs of a multi-group.

Conjecture 2.6.1 *Every Cayley graph of a finite multi-group* $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ *with order*≥ 3 *and* $|\bigcap\limits_{i=1}^{n} \Gamma_i| \geq 2$ *is hamiltonian.*

2.6.3 As pointed out in [56], for applying combinatorics to other sciences, a good idea is pullback measures on combinatorial objects, initially ignored by the classical combinatorics and reconstructed or make a combinatorial generalization for the classical mathematics, such as, the algebra, the differential geometry, the Riemann geometry, $\cdots$ and the mechanics, the theoretical physics, $\cdots$. For this object, a more natural way is to put a graph in a metric space and find its good behaviors. The problem discussed in Sections 2.3 is just an elementary step for this target. More works should be done and more techniques should be designed. The following open problems are valuable to research for a researcher on combinatorics.

Problem 2.6.5 *Find which parameters for a graph can be used to a graph in a space. Determine combinatorial properties of a graph in a space.*

Consider a graph in an Euclid space of dimension 3. All of its edges are seen as a structural member, such as steel bars or rods and its vertices are hinged points. Then we raise the following problem.

Problem 2.6.6 *Applying structural mechanics to classify what kind of graph structures are stable or unstable. Whether can we discover structural mechanics of dimension*≥ 4 *by this idea?*

We have known the orbit of a point under an action of a group, for example, a torus is an orbit of $Z \times Z$ action on a point in $\mathbf{R}^3$. Similarly, we can also define an *orbit of a graph in a space* under an action on this space.

Let $\mathcal{G}$ *be a graph in a multi-space* $\widetilde{M}$ *and* Π *a family of actions on* $\widetilde{M}$. *Define an orbit* $Or(\mathcal{G})$ *by*

$$Or(\mathcal{G}) = \{\pi(\mathcal{G}) | \ \forall \pi \in \Pi\}.$$

Problem 2.6.7 *Given an action* π*, continuous or discontinuous on a space* $\widetilde{M}$*, for example* $\mathbf{R}^3$ *and a graph* $\mathcal{G}$ *in* $\widetilde{M}$*, find the orbit of* $\mathcal{G}$ *under the action of* π*. When can we get a closed geometrical object by this action?*

Problem 2.6.8 *Given a family* $\mathcal{A}$ *of actions, continuous or discontinuous on a space* $\widetilde{M}$ *and a graph* $\mathcal{G}$ *in* $\widetilde{M}$*, find the orbit of* $\mathcal{G}$ *under these actions in* $\mathcal{A}$*. Find the orbit of a vertex or an edge of* $\mathcal{G}$ *under the action of* $\mathcal{G}$*, and when are they closed?*

2.6.4 The central idea in Section 2.4 is that a graph is equivalent to Smarandache multi-spaces. This fact enables us to investigate Smarandache multi-spaces possible by a combinatorial approach. Applying infinite graph theory (see [94] for details), we can also define an infinite graph for an infinite Smarandache multi-space similar to Definition 2.4.3.

Problem 2.6.9 *Find its structural properties of an infinite graph of an infinite Smarandache multi-space.*

2.6.5 There is an alternative way for defining transformable graph phases, i.e., by homotopy groups in a topological space, which is stated as follows.

Let $(\mathcal{G}_1; \omega_1, \Lambda_1)$ and $(\mathcal{G}_2; \omega_2, \Lambda_2)$ be two graph phases. If there is a continuous mapping $H : C(\widetilde{M}) \times I \to C(\widetilde{M}) \times I$, $I = [0, 1]$ such that $H(C(\widetilde{M}), 0) = (\mathcal{G}_1; \omega_1, \Lambda_1)$ and $H(C(\widetilde{M}), 1) = (\mathcal{G}_2; \omega_2, \Lambda_2)$, then $(\mathcal{G}_1; \omega_1, \Lambda_1)$ and $(\mathcal{G}_2; \omega_2, \Lambda_2)$ are said two *transformable graph phases*.

Similar to topology, we can also introduce product on homotopy equivalence classes and prove that all homotopy equivalence classes form a group. This group is called a *fundamental group* and denote it by $\pi(\mathcal{G}; \omega, \Lambda)$. In topology there is a famous theorem, called the *Seifert and Van Kampen theorem* for characterizing fundamental groups $\pi_1(\mathcal{A})$ of topological spaces $\mathcal{A}$ restated as follows (see [92] for details).

Suppose $\mathcal{E}$ is a space which can be expressed as the union of path-connected open sets $\mathcal{A}$, $\mathcal{B}$ such that $\mathcal{A} \bigcap \mathcal{B}$ is path-connected and $\pi_1(\mathcal{A})$ and $\pi_1(\mathcal{B})$ have respective presentations

$$\langle a_1, \cdots, a_m; r_1, \cdots, r_n \rangle ,$$

$$\langle b_1, \cdots, b_m; s_1, \cdots, s_n \rangle$$

while $\pi_1(\mathcal{A} \bigcap \mathcal{B})$ is finitely generated. Then $\pi_1(\mathcal{E})$ has a presentation

$$\langle a_1, \cdots, a_m, b_1, \cdots, b_m; r_1, \cdots, r_n, s_1, \cdots, s_n, u_1 = v_1, \cdots, u_t = v_t \rangle ,$$

where $u_i, v_i, i = 1, \cdots, t$ are expressions for the generators of $\pi_1(\mathcal{A} \bigcap \mathcal{B})$ in terms of the generators of $\pi_1(\mathcal{A})$ and $\pi_1(\mathcal{B})$ respectively.

Then there is a problem for the fundamental group $\pi(\mathcal{G}; \omega, \Lambda)$ of a graph phase $(\mathcal{G}; \omega, \Lambda)$.

Problem 2.6.10 *Find a result similar to the Seifert and Van Kampen theorem for the fundamental group of a graph phase.*