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On Additive Analogues of Certain Arithmetic Functions

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1. The Smarandache, Pseudo-Smarandache, resp. Smarandache-simple functions are defined as ([7], [6])

\[ S(n) = \min \{ m \in \mathbb{N} : n \mid m! \} , \tag{1} \]
\[ Z(n) = \min \left\{ m \in \mathbb{N} : n \mid \frac{m(m+1)}{2} \right\} , \tag{2} \]
\[ S_p(n) = \min \{ m \in \mathbb{N} : p^n \mid m! \} \text{ for fixed primes } p. \tag{3} \]

The duals of \( S \) and \( Z \) have been studied e.g. in [2], [6], [6]:

\[ S_*(n) = \max \{ m \in \mathbb{N} : m! \mid n \} , \tag{4} \]
\[ Z_*(n) = \max \left\{ m \in \mathbb{N} : \frac{m(m+1)}{2} \mid n \right\} . \tag{5} \]

We note here that the dual of the Smarandache simple function can be defined in a similar manner, namely by

\[ S_{p*}(n) = \max \{ m \in \mathbb{N} : m! \mid p^n \} \tag{6} \]

This dual will be studied in a separate paper (in preparation).

2. The additive analogues of the functions \( S \) and \( S_* \) are real variable functions, and have been defined and studied in paper [3]. (See also our book [6], pp. 171-174). These functions have been recently further extended, by the use of Euler’s gamma function, in place of the factorial (see [1]). We note that in what follows, we could define also the additive analogues functions by the use of Euler’s gamma function. However, we shall apply the more transparent notation of a factorial of a positive integer.

The additive analogues of \( S \) and \( S_* \), from (1) and (4) have been introduced in [3] as follows:

\[ S(x) = \min \{ m \in \mathbb{N} : x \leq m! \}, \quad S : [1, \infty) \to \mathbb{R} , \tag{7} \]
\[ S_*(x) = \max \{ m \in \mathbb{N} : m! \leq x \}, \quad S_* : [1, \infty) \to \mathbb{R} . \tag{8} \]
Besides of properties relating to continuity, differentiability, or Riemann integrability of these functions, we have proved the following results:

**Theorem 1.**

\[ S_k(x) \sim \frac{\log x}{\log \log x} \quad (x \to \infty) \]  
(9)

(The same for \( S(x) \)).

**Theorem 2.** The series

\[ \sum_{n=1}^{\infty} \frac{1}{n(S(n))^\alpha} \]  
(10)

is convergent for \( \alpha > 1 \) and divergent for \( \alpha < 1 \) (the same for \( S(n) \) replaced by \( S(n) \)).

3. The additive analogues of \( Z \) and \( Z_1 \) from (2), resp. (4) will be defined as

\[ Z(x) = \min \left\{ m \in \mathbb{N} : x \leq \frac{m(m+1)}{2} \right\}, \]  
(11)

\[ Z_1(x) = \max \left\{ m \in \mathbb{N} : \frac{m(m+1)}{2} \leq x \right\}. \]  
(12)

In (11) we will assume \( x \in (0, \infty) \), while in (12) \( x \in [1, \infty) \).

The two additive variants of \( S_1(n) \) of (3) will be defined as

\[ P(x) = S_c(x) = \min \{ m \in \mathbb{N} : p^* \leq m \}; \]  
(13)

(\( \text{where in this case } p > 1 \) is an arbitrary fixed real number)

\[ P_c(x) = S_c(x) = \max \{ m \in \mathbb{N} : m \leq p^* \}. \]  
(14)

From the definitions follow at once that

\[ Z(x) = k \quad \Leftrightarrow \quad x \in \left( \frac{(k-1)k}{2}, \frac{k(k+1)}{2} \right] \quad \text{for } k \geq 1 \]  
(15)

\[ Z_1(x) = k \quad \Leftrightarrow \quad x \in \left[ \frac{k(k+1)}{2}, \frac{(k+1)(k+2)}{2} \right) \]  
(16)

For \( x \geq 1 \) it is immediate that

\[ Z_1(x) + 1 \geq Z(x) \geq Z_1(x) \]  
(17)

Therefore, it is sufficient to study the function \( Z_1(x) \).

The following theorems are easy consequences of the given definitions:

**Theorem 3.**

\[ Z_1(x) \sim \frac{1}{2} \sqrt{8x + 1} \quad (x \to \infty) \]  
(18)

**Theorem 4.**

\[ \sum_{n=1}^{\infty} \frac{1}{(Z_1(n))^\alpha} \]  
(19)

is convergent for \( \alpha > 2 \) and divergent for \( \alpha \leq 2 \). The series \( \sum_{n=1}^{\infty} \frac{1}{n(Z_1(n))^\alpha} \) is convergent for all \( \alpha > 0 \).

**Proof.** By (16) one can write \( \frac{k(k+1)}{2} \leq x < \frac{(k+1)(k+2)}{2} \), so \( k^2 + k - 2x \leq 0 \) and \( k^2 + 3k + 2 - 2x > 0 \). Since the solutions of these quadratic equations are \( k_{1,2} = \frac{-3 \pm \sqrt{8x + 1}}{2} \), resp. \( k_{3,4} = \frac{-1 \pm \sqrt{8x + 1}}{2} \), and remarking that \( \frac{\sqrt{8x + 1} - 3}{\sqrt{8x + 1} + 3} \geq 1 \) for \( x \geq 3 \), we obtain that the solution of the above system of inequalities is:

\[ \begin{cases} k \in \left[ 1, \frac{1 + \sqrt{8x - 1}}{2} \right] & \text{if } x \in [1, 3); \\ k \in \left( \frac{\sqrt{8x - 3} + 1}{2}, \frac{1 + \sqrt{8x - 1}}{2} \right] & \text{if } x \in [3, +\infty) \end{cases} \]  
(20)

So, for \( x \geq 3 \),

\[ \frac{\sqrt{8x - 3} + 3}{2} < Z_1(x) \leq \frac{1 + \sqrt{8x - 1}}{2} \]  
(21)

implying relation (18).

Theorem 4 now follows by (18) and the known fact that the generalized harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n^\theta} \) is convergent only for \( \theta > 1 \).

The things are slightly more complicated in the case of functions \( P \) and \( P_c \). Here it is sufficient to consider \( P_c \), too.

First remark that

\[ P_c(x) = m \Leftrightarrow x \in \left[ \frac{\log m!}{\log \log m}, \frac{\log (m+1)!}{\log \log m} \right]. \]  
(22)

The following asymptotic results have been proved in [31 (Lemma 2) (see also [61, p. 172)]

\[ \log m! \sim m \log m, \quad \frac{m \log m!}{\log m} \sim 1, \quad \log \left( \frac{m \log m!}{\log m} \right) \sim 1 \quad (m \to \infty) \]  
(23)

By (22) one can write

\[ \frac{m \log m!}{\log m} - \frac{m \log \log m}{\log \log m} \leq \frac{m \log x}{\log m} \leq \frac{m \log \log (m+1)!}{\log m} - \frac{m \log \log m}{\log m}, \]

giving \( \frac{m \log x}{\log m} \to 1 \) \((m \to \infty)\), and by (23) one gets \( \log x \sim \log m \). This means that:

**Theorem 5.**

\[ \log P_c(x) \sim \log x \quad (x \to \infty) \]  
(24)

The following theorem is a consequence of (24), and a convergence theorem established in [3]:

**Theorem 6.** The series \( \sum_{n=1}^{\infty} \frac{1}{n \log P_c(n)} \) is convergent for \( \alpha > 1 \) and divergent for \( \alpha \leq 1 \).

Indeed, by (24) it is sufficient to study the series \( \sum_{n=1}^{\infty} \frac{1}{n \log P_c(n)} \) (where \( n_0 \in \mathbb{N} \) is a fixed positive integer). This series has been proved to be convergent for \( \alpha > 1 \) and divergent for \( \alpha \leq 1 \) (see [61], p. 174).
ON SOME SMARANDACHE PROBLEMS

Edited by M. Perez

1. PROPOSED PROBLEM

Let \( n \geq 2 \). As a generalization of the integer part of a number one defines the Inferior Smarandache Prime Part as: \( ISPP(n) \) is the largest prime less than or equal to \( n \). For example: \( ISPP(9) = 7 \) because \( 7 < 9 < 11 \), also \( ISPP(13) = 13 \). Similarly the Superior Smarandache Prime Part is defined as: \( SSPP(n) \) is smallest prime greater than or equal to \( n \). For example: \( SSPP(9) = 11 \) because \( 7 < 9 < 11 \), also \( SSPP(13) = 13 \). Questions:

1) Show that a number \( p \) is prime if and only if \( ISPP(p) = SSPP(p) \).

2) Let \( k > 0 \) be a given integer. Solve the Diophantine equation:

\[ ISPP(x) + SSPP(x) = k. \]

Solution by Hans Gunter, Köln (Germany)

The Inferior Smarandache Prime Part, \( ISPP(n) \), does not exist for \( n < 2 \).

1) The first question is obvious (Carlos Rivera).

2) The second question:

a) If \( k = 2p \) and \( p \) = prime (i.e., \( k \) is the double of a prime), then the Smarandache diophantine equation

\[ ISPP(x) + SSPP(x) = 2p \]

has one solution only: \( x = p \) (Carlos Rivera).

b) If \( k \) is equal to the sum of two consecutive primes, \( k = p(n) + p(n+1) \), where \( p(n) \) is the \( n \)-th prime, then the above Smarandache diophantine equation has many solutions: all the integers between \( p(n) \) and \( p(n + 1) \) (of course, the extremes \( p(n) \) and \( p(n + 1) \) are excluded). Except the case \( k = 5 = 2 + 3 \), when this equation has no solution. The sub-cases when this equation has one solution only is when \( p(n) \) and \( p(n + 1) \) are twin primes, i.e. \( p(n+1) - p(n) = 2 \), and then the solution is \( p(n)+1 \). For example: \( ISPP(x) + SSPP(x) = 24 \) has the only solution \( x = 12 \) because \( 11 < 12 < 13 \) and \( 24 = 11 + 13 \) (Teresinha DaCosta).

Let's consider an example:

\[ ISPP(x) + SSPP(x) = 100. \]
because $100 = 47 + 53$ (two consecutive primes), then $x = 48, 19, 50, 51,$ and $52$ (all the integers between $47$ and $53$).

$$ISPP(48) + SSPP(48) = 47 + 53 = 100.$$  

Another example:  

$$ISPP(x) + SSPP(x) = 99$$  

has no solution, because if $x = 47$ then  

$$ISPP(47) + SSPP(47) = 47 + 47 < 99,$$  

and if $x = 48$ then  

$$ISPP(48) + SSPP(48) = 47 + 53 = 100 > 99.$$  

If $x \leq 47$ then  

$$ISPP(x) + SSPP(x) < 99,$$  

while if $x \geq 48$ then  

$$ISPP(x) + SSPP(x) > 99.$$  

c) If $k$ is not equal to the double of a prime, or $k$ is not equal to the sum of two consecutive primes, then the above Smarandache diophantine equation has no solution.

A remark: We can consider the equation more general: Find the real number $x$ (not necessarily integer number) such that  

$$ISPP(x) + SSPP(x) = k,$$  

where $k > 0$.

Example: Then if $k = 100$ then $x$ is any real number in the open interval $(47, 53)$, therefore infinitely many real solutions. While integer solutions are only five: $48, 49, 50, 51$, $52$.

A criterion of primality: The integers $p$ and $p + 2$ are twin primes if and only if the diophantine smarandacheian equation  

$$ISPP(x) + SSPP(x) = 2p + 2$$  

has only the solution $x = p + 1$.

References


where \( p(k) \) is the \( k \)-th prime; I mean any number is between two consecutive primes.

For another example:
\[ 27 = 29 \pm 2 \text{ and } 4 = 29 \pm 4, \] 4 is between 3 and 5 therefore \( 4 = 3 + 1 \), therefore \( 27 = 3 + 4 + 1 \) in the SPB (a unique representation).

Not allowed to say that \( 27 = 19 + 8 \) because 27 is not between 19 and 29 but between 23 and 29.

The proof that all digits are 0 or 1 relies on the Chebyshev's theorem that between a number \( n \) and \( 2n \) there is at least a prime. Thus, between a prime \( q \) and \( 2q \) there is at least a prime. Thus \( q(\pi(q)) > p(\pi(p)) \) where \( \pi(q) \) means the \( k \)-th prime.

References


3. PROPOSED PROBLEM

Let \( p \) be a positive prime, and \( S(n) \) the Smarandache Function, defined as the smallest integer such that \( S(n)! \) be divisible by \( n \). The factorial of \( m \) is the product of all integers from 1 to \( m \). Prove that
\[ S(p^2) = p^3. \]

Solution by Alecu Stuparu, 0945 Balcesti, Valcea, Romania

Because \( p \) is prime and \( S(p^2) \) must be divisible by \( p \), one gets that \( S(p^2) = p \), or \( 2p \), or \( 3p \), etc.

More, \( S(p^2) \) must be divisible by \( p^2 \), therefore
\[ S(p^2) = p \equiv p, \text{ or } p \equiv (p + 1), \text{ or } p \equiv (p + 2), \text{ etc.} \]

But the smallest one is \( p \equiv p \) [because \( p \equiv (p - 1)! \) is not divisible by \( p^2 \), but by \( p^2 - 1 \)]. Therefore
\[ S(p^2) = p^3. \]

4. PROPOSED PROBLEM

Let \( S3f(n) \) be the triple Smarandache function, i.e. the smallest integer \( m \) such that \( m!! = m(m - 3)(m - 6) \ldots \) the product of all such positive non-zero integers. For example \( 8!! = 8(8 - 3)(8 - 6) = (8)(5)(2) = 80 \).

\[ S3f(10) = 5 \text{ because } 5!! = 5(5 - 3)(5 - 2) = 10, \text{ which is divisible by } 10, \text{ and it is the smallest one with this property.} \]

Question: Prove that if \( n \) is divisible by 3 then \( S3f(n) \) is also divisible by 3.

Solution by K. L. Ramsharan, Madras, India

Let \( S3f(n) = m \).

\( S3f(n!!) = m!! \) has to be divisible by \( n \) according to the definition of this function, i.e. \( m \) has to be a multiple of \( 3 \), because \( n \) is a multiple of \( 3 \). In \( m \) is not a multiple of \( 3 \), then no factor of \( m!! = m(m - 3)(m - 6) \ldots \) will be a multiple of \( 3 \), therefore \( m!! \) would not be divisible by \( n \). Absurd.

5. PROPOSED PROBLEM

Let \( Sdf(n) \) represent the Smarandache double factorial function, i.e. the smallest positive integer such that \( Sdf(n)!! \) is divisible by \( n \), where double factorial \( m!! = 1 \times 3 \times 5 \times \ldots \times m \) if \( m \) is odd, and \( m!! = 2 \times 4 \times 6 \times \ldots \times m \) if \( m \) is even. Solve the diophantine equation \( Sdf(x) = p \), when \( p \) is prime. How many solutions are there?

Solution by Carlos Gustavo Moreira, Rio de Janeiro, Brazil

For the equation \( Sdf(x) = p = \text{prime}, \) the number of solutions is \( \geq 2^k \), where \( k = (p - 3)/2 \). The general solution of the equation \( Sdf(x) = p = \text{prime} \) is \( p = \text{prime} \), where \( m \) is any divisor of \( (p - 2)! \).

Let us consider the example for the Smarandache double factorial function \( Sdf(x) = 17 \).

The solutions are \( 17 \times m \), where \( m \) is any divisor of \( (17 - 2)! \) which is equal to \( 3 \times 5 \times 7 \times 11 \times 13 \times 15 = (3^3) \times (5^2) \times 7 \times 11 \times 13 \) which has \( (4 + 1) \times (2 + 1) \times (1 + 1) \times (1 + 1) = 120 \) divisor, therefore 120 solutions < \( 2^7 = 128 \).

The number of solutions is not \( 2^7 = 128 \) because some solutions were counted twice, for example: \( 17 \times 3 \times 5 \) is the same as \( 17 \times 15 \) or \( 17 \times 3 \times 15 \) is the same as \( 17 \times 5 \times 9 \).

Comment by Gilbert Johnson,
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How to determine the solutions and how to find a superior limit for the number of solutions.

Using the definition of Smarandache Function, we find that: $p!$ is divisible by $x$, and $p$ is the smallest positive integer with this property. Because $p$ is prime, $x$ should be a multiple of $p$ (otherwise $p$ would not be the smallest positive integer with that property). $p!$ is a multiple of $x$.

a) If $p = 2$, then $x = 2$.
b) If $p > 2$, then $p$ is odd and $p! = 1 \times 3 \times 5 \times \ldots \times p = Mx$ (multiple of $x$).

Solutions are formed by all combinations of $p$, times none, one, or more factors from 3, 5, ..., $p - 2$.

Let $(p - 3)/2 = k$ and $rC$s represent combinations of $s$ elements taken by $r$.

So:
- for one factor: $p$, we have 1 solution: $x = p$; i.e. $0Ck$ solution;
- for two factors: $p \times 3, p \times 5, \ldots, p \times (p - 2)$,
  we have $kCk$ solutions;
  i.e. $1Ck$ solutions;
- for three factors: $p \times 3 \times 5, p \times 3 \times 7, \ldots, p \times 3 \times (p - 2); p \times 5 \times 7, \ldots, p \times 5 \times (p - 2); \ldots, p \times (p - 4) \times (p - 2)$, we have $2Ck$ solutions; etc. and so on: for $k$ factors:
  $p \times 3 \times 5 \times \ldots \times (p - 2)$,
  we have $kCk$ solutions.

Thus, the general solution has the form:

$$x = p \times c_1 \times c_2 \times \ldots \times c_k,$$

with all $c_j$ distinct integers and belonging to $\{3, 5, \ldots, p - 2\}$, $0 \leq j \leq k$, and $k = (p - 3)/2$.

The smallest solution is $x = p$, the largest solution is $x = p!$.

The total number of solutions is less than or equal to $0Ck + 1Ck + 2Ck + \ldots + kCk = 2k$, where $k = (p - 3)/2$.

Therefore, the number of solutions of this equation is equal to the number of divisors of $(p - 2)!$.

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**ON SOME PROBLEMS RELATED TO SMARANDACHE NOTIONS**

Edited by M. Perez

1. **Problem of Number Theory by L. Seagull, Glendale Community College**

   Let $n$ be a composite integer $> 4$. Prove that in between $n$ and $S(n)$ there exists at least a prime number.

   **Solution:**

   T. Yau proved that the Smarandache Function has the following property: $S(n) \leq \frac{n}{2}$ for any composite number $n$, because: if $n = pq$, with $p < q$ and $(p, q) = 1$, then

   $$S(n) \leq \max(S(p), S(q)) = S(q) \leq \frac{n}{p} \leq \frac{n}{2}.$$

   Now, using Bertrand-Tchebichev's theorem, we get that in between $\frac{n}{2}$ and $n$ there exists at least a prime number.

2. **Proposed Problem by Antony Begay**

   Let $S(n)$ be the smallest integer number such that $S(n)!$ is divisible by $n$, where $m! = 1.2.3.\ldots.m$ (factoriel of $m$), and $S(1) = 1$ (Smarandache Function). Prove that if $p$ is prime then $S(p) = p$. Calculate $S(12)$.

   **Solution:**

   $S(p)$ cannot be less than $p$, because if $S(p) = n < p$ then $n! = 1.2.3.\ldots.n$ is not divisible by $p$ (assuming $p$ is prime). Thus $S(p) \geq p$. But $p! = 1.2.3.\ldots.p$ is divisible by $p$, and is the smallest one with this property. Therefore $S(p) = p$.

   $42 = 2 \times 3 \times 7 = 1.2.3.4.5.6.7$ which is divisible by 2, by 3, and by 7. Thus $S(42) \leq 7$. But $S(42)$ can not be less than 7, because for example 6! = 1.2.3.4.5.6 is not divisible by 7. Hence $S(42) = 7$.

3. **Proposed Problem by Leonardo Motta**

   Let $n$ be a square free integer, and $p$ the largest prime which divides $n$. Show that $S(n) = p$, where $S(n)$ is the Smarandache Function, i.e. the smallest integer such that $S(n)!$ is divisible by $n$. 

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Solution:

Because \( n \) is a square free number, there is no prime \( q \) such that \( q^2 \) divides \( n \). Thus \( n \) is a product of distinct prime numbers, each one to the first power only. For example 105 is square free because 105 = 3•5•7, i.e. 105 is a product of distinct prime numbers, each of them to the power 1 only. While 945 is not a square free number because 945 = 3^2•5•7, therefore 945 is divisible by \( 3^2 \) (which is 9, i.e. a square). Now, if we compute the Smarandache Function \( S(105) = 7 \) because \( 7! = 1•2•3•4•5•6•7 \) which is divisible by 3, and 7 in the same time, and 7 is smallest number with this property. But \( S(945) = 9 \), not 7. Therefore, if \( n = a•b•…•p \), where all \( a < b < \cdots < p \) are distinct two by two primes, then \( S(n) = \max(a, b, \ldots, p) = p \), because the factorial of \( p \), the largest prime which divides \( n \), includes the factors \( a, b, \ldots \) in its development: \( p! = 1, a, b, \ldots, p \).

4. Proposed Problem by Gilbert Johnson

Let \( Sdf(n) \) be the Smarandache Double Factorial Function, i.e. the smallest integer such that \( Sdf(n)!! \) is divisible by \( n \), where \( m!! = 1•3•5\ldots m \) if \( m \) is odd and \( m!! = 2•4•6\ldots m \) if \( m \) is even. If \( n \) is an even square free number and \( p \) the largest prime which divides \( n \), then \( Sdf(n) = 2p \).

Solution:

Because \( n \) is even and square free, then \( n = 2•a•b\ldots•p \) where all \( 2 < a < b < \cdots < p \) are distinct primes two by two, occurring to the power 1 only. \( Sdf(n) \) cannot be less that \( 2p \) because if it is \( 2p - k \), with \( 1 \leq k < 2p \), then \( (2p - k)!! \) would not be divisible by \( p \).

\[
(2p)!! = 2.4\ldots(2a)\ldots(2b)\ldots(2p)
\]

is divisible by \( n \) and it is the smallest number with this property.

GENERALIZED SMARANDACHE PALINDROME

Edited by George Gregory, New York, USA

A Generalized Smarandache Palindrome is a number of the form: \( a_1a_2\ldots a_na_{n-1}a_2a_1 \) or \( a_1a_2\ldots a_{n-1}a_na_{n-1} \ldots a_2a_1 \), where all \( a_1, a_2, \ldots, a_n \) are positive integers of various number of digits.

Examples:

a) 1235656312 is a GSP because we can group it as \( (12)(3)(56)(56)(12) \), i.e. ABCBAC.

b) Of course, any integer can be considered a GSP because we may consider the entire number as equal to \( a_1 \), which is Smarandache palindromic; say \( N = 176293 \) is GSP because we may take \( a_1 = 176293 \) and thus \( N = a_1 \). But one disregards this trivial case.

Very interesting GSP are formed from Smarandacheian sequences. Let us consider this one:

\[
11, 1221, 123321, 123456789087654321, 1234567891010987654321, 1234567891011110987654321, \ldots
\]

all of them are GSP.

It has been proven that 1234567891010987654321 is a prime (see http://www.kottke.org/notes/0103.html, and the Prime Curios site).

A question: How many other GSP are in the above sequence?
ON 15-TH SMARANDACHE’S PROBLEM
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Introduction

The 15-th Smarandache’s problem [1] is the following: “Smarandache’s simple numbers:

A number $n$ is called “Smarandache’s simple number” if the product of its proper divisors is less than or equal to $n$. Generally speaking, $n$ has the form $n = p$, or $n = p^2$, or $n = p^3$, or $n = pq$, where $p$ and $q$ are distinct primes”.

Let us denote by $S$ - the sequence of all Smarandache’s simple numbers and by $s_n$ - the $n$-th term of $S$; by $P$ - the sequence of all primes and by $p_n$ - the $n$-th term of $P$; by $P^2$ - the sequence $\{p_n^2\}_{n=1}^{\infty}$; by $P^3$ - the sequence $\{p_n^3\}_{n=1}^{\infty}$; by $PQ$ - the sequence $\{pq\}_{p,q \in P}$, where $p < q$.

For an arbitrary increasing sequence of natural numbers $C = \{c_n\}_{n=1}^{\infty}$ we denote by $\pi_C(n)$ the number of terms of $C$, which are not greater than $n$. When $n < c_1$ we must put $\pi(C,n) = 0$.

In the present paper we find $\pi_S(n)$ in an explicit form and using this, we find the $n$-th term of $S$ in explicit form, too.

1. $\pi_S(n)$-representation

First, we must note that instead of $\pi_P(n)$ we shall use the well known denotation $\pi(n)$. Hence

$$\pi_P(n) = \pi(\sqrt{n}), \pi_P(\sqrt{n}) = \pi(\sqrt{n}).$$

Thus, using the definition of $S$, we get

$$\pi_S(n) = \pi(n) + \pi(\sqrt{n}) + \pi(\sqrt{n}) + \pi_{PQ}(n). \quad (1)$$

Our first aim is to express $\pi_S(n)$ in an explicit form. For $\pi(n)$ some explicit formulae are proposed in [2]. Other explicit formulae for $\pi(n)$ are contained in [3]. One of them is known as Mináč’s formula. It is given below

$$\pi(n) = \sum_{k=2}^{\sqrt{n}} \left[ \left( \frac{k-1}{k} \right) + \frac{1}{k} - \left( \frac{k-1}{k} \right) \right]. \quad (2)$$

where $[.]$ denotes the function integer part. Therefore, the question about explicit formulae for functions $\pi(n), \pi(\sqrt{n}), \pi(\sqrt{n})$ is solved successfully. It remains only to express $\pi_{PQ}(n)$ in an explicit form.

Let $k \in \{1, 2, ..., \pi(\sqrt{n})\}$ be fixed. We consider all numbers of the kind $p_kq$, where $q \in P, q > p_k$ for which $p_kq \leq n$. The number of these numbers is $\pi(\frac{n}{p_k}) - \pi(p_k)$, or which is the same

$$\pi(n) = \pi(p_k) - k. \quad (3)$$

When $k = 1, 2, ..., \pi(\sqrt{n})$, numbers $p_kq$, that were defined above, describe all numbers of the kind $p_kq$, where $p, q \in P, p < q, p, q \leq n$. But the number of the last numbers is equal to $\pi_{PQ}(n)$. Hence

$$\pi_{PQ}(n) = \sum_{k=1}^{\pi(\sqrt{n})} \left( \pi(n) - \pi(p_k) \right), \quad (4)$$

because of (3). The equality (4), after a simple computation yields the formula

$$\pi_{PQ}(n) = \sum_{k=1}^{\pi(\sqrt{n})} \left( \pi(n) - \pi(\sqrt{n}) + \frac{\pi(\sqrt{n})}{2} \right). \quad (5)$$

In [4] the identity

$$\sum_{k=1}^{\pi(\sqrt{n})} \left( \pi(n) - \pi(p_k) \right) = \pi(\sqrt{n}) \sum_{k=1}^{\pi(\sqrt{n})} \pi(n) - \pi(\sqrt{n}) \sum_{k=1}^{\pi(\sqrt{n})} \pi(p_k) \quad (6)$$

is proved, under the condition $b \geq 2$ ($b$ is a real number). When $\pi(\sqrt{n}) = \pi(\sqrt{n})$, the right hand-side of (6) reduces to $\pi(\sqrt{n}) \pi(b)$. In the case $b = \sqrt{n}$ and $n \geq 4$ equality (6) yields

$$\sum_{k=1}^{\pi(\sqrt{n})} \left( \pi(n) - \pi(\sqrt{n}) \right) = \pi(\sqrt{n}) \frac{\pi(\sqrt{n})}{2} + \sum_{k=1}^{\pi(\sqrt{n})} \pi(p_k) \pi(\sqrt{n}). \quad (7)$$

If we compare (5) with (7) we obtain for $n \geq 4$

$$\pi_{PQ}(n) = \pi(\sqrt{n}) \left( \pi(n) - \pi(\sqrt{n}) \right) + \sum_{k=1}^{\pi(\sqrt{n})} \pi(\sqrt{n}) \pi(p_k) \pi(\sqrt{n}). \quad (8)$$

Thus, we have two different explicit representations for $\pi_{PQ}(n)$. These are formulae (5) and (8). We must note that the right hand-side of (8) reduces to $\pi(\sqrt{n}) \pi(\sqrt{n}) - \pi(\sqrt{n})$, when $\pi(\sqrt{n}) = \pi(\sqrt{n})$.

Finally, we observe that (1) gives an explicit representation for $\pi_S(n)$, since we may use formula (2) for $\pi(n)$ (or other explicit formulae for $\pi(n)$) and (5), or (8) for $\pi_{PQ}(n)$.
2. Explicit formulae for $s_n$

The following assertion decides the question about explicit representation of $s_n$.

Theorem: The $n$-th term $s_n$ of $S$ admits the following three different explicit representations:

\[
s_n = \sum_{k=0}^{n-1} \frac{1}{1 + \left[ \frac{\pi_S(k)}{n} \right]}, \tag{9}
\]

\[
s_n = -2 \sum_{k=0}^{n-1} \theta\left( -2 \frac{\pi_S(k)}{n} \right), \tag{10}
\]

\[
s_n = \sum_{k=0}^{n-1} \frac{1}{\Gamma\left( 1 - \left[ \frac{\pi_S(k)}{n} \right] \right)}, \tag{11}
\]

where

\[
\theta(n) = \left[ \frac{n^2 + 3n + 1}{4} \right], \quad n = 1, 2, \ldots, \tag{12}
\]

$\zeta$ is Riemann’s function zeta and $\Gamma$ is Euler’s function gamma.

Remark. We must note that in (9)-(11) $\pi_S(k)$ is given by (1), $\pi(k)$ is given by (2) (or by other formulae like (2)) and $\pi_{\mathcal{C}}(n)$ is given by (5), or by (8). Therefore, formulae (9)-(11) are explicit.

Proof of the Theorem. In [2] the following three universal formulae are proposed, using $\pi_{\mathcal{C}}(k)$ ($k = 0, 1, \ldots$), which one could apply to represent $c_n$. They are the following:

\[
c_n = \sum_{k=0}^{\infty} \frac{1}{1 + \left[ \frac{\pi_{\mathcal{C}}(k)}{n} \right]}, \tag{13}
\]

\[
c_n = -2 \sum_{k=0}^{\infty} \zeta\left( -2 \frac{\pi_{\mathcal{C}}(k)}{n} \right), \tag{14}
\]

\[
c_n = \sum_{k=0}^{\infty} \frac{1}{\Gamma\left( 1 - \left[ \frac{\pi_{\mathcal{C}}(k)}{n} \right] \right)}, \tag{15}
\]

In [5] is shown that the inequality

\[
p_n \leq \theta(n), \quad n = 1, 2, \ldots, \tag{16}
\]

holds. Hence

\[
s_n = \theta(n), \quad n = 1, 2, \ldots, \tag{17}
\]

since we have obviously

\[
s_n \leq p_n, \quad n = 1, 2, \ldots. \tag{18}
\]

Then to prove the Theorem it remains only to apply (13)-(15) in the case $C = S$, i.e., for $c_n = s_n$, putting there $\pi_S(k)$ instead of $\pi_{\mathcal{C}}(k)$ and $\theta(n)$ instead of $\infty$.

REFERENCES:


The second problem from [1] (see also 16-th problem from [2]) is the following:

\[ \text{Smarandache circular sequence:} \]

\[ \overline{1, 2, 1, 23, 231, 312, 2341, 3412, 4123, 23451, 23456, 45123, 51234, 345612, 456123, 561234, 612345, \ldots} \]

Let \([\lfloor \cdot \rfloor\) be the largest natural number strongly smaller than real (positive) number \(x\). For example, \([7.4]\) = 7, but \([7.9]\) = 6.

Let \(f(n)\) be the \(n\)-th member of the above sequence. We shall prove the following

**Theorem:** For every natural number \(n\):

\[ f(n) = s(s+1)\ldots k12\ldots(s-1), \tag{1} \]

where

\[ k \equiv k(n) = \frac{\sqrt{8n + 1} - 1}{2} \tag{2} \]

and

\[ s \equiv s(n) = n - \frac{k(k+1)}{2}. \tag{3} \]

**Proof:** When \(n = 1\), then from (1) and (2) it follows that \(k = 0\), \(s = 1\) and from (3) - that \(f(1) = 1\). Let us assume that the assertion is valid for some natural number \(n\). Then for \(n + 1\) we have the following two possibilities:

1. \(k(n + 1) = k(n)\), i.e., \(k\) is the same as above. Then

\[ s(n + 1) = n + 1 - \frac{k(n + 1)(k(n + 1) + 1)}{2} = n + 1 - \frac{k(n)(k(n) + 1)}{2} = s(n) + 1, \]

i.e.,

\[ f(n + 1) = (s + 1)\ldots k12\ldots s. \]

2. \(k(n + 1) = k(n) + 1\). Then

\[ s(n + 1) = n + 1 - \frac{k(n + 1)(k(n + 1) + 1)}{2} = n + 1 - \frac{k(n)(k(n) + 1)}{2} = s(n) + 1, \]

\[ f(n + 1) = (s + 1)\ldots k12\ldots(s-1). \]

On the other hand, it is seen directly, that in (2) number \(\sqrt{8n + 1} - 1\) is an integer if and only if \(n = \frac{m(m+1)}{2}\). Also, for every natural numbers \(n\) and \(m \geq 1\) such that

\[ \frac{(m - 1)m}{2} < n < \frac{m(m + 1)}{2} \]

it will be valid that

\[ \frac{\sqrt{8n + 1} - 1}{2} = \frac{\sqrt{n(n + 1) + 1} - 1}{2} = m. \]

Therefore, when \(k(n + 1) = k(n) + 1\), then

\[ n = \frac{m(m + 1)}{2} + 1 \]

and for it from (4) we obtain:

\[ s(n + 1) = 1, \]

i.e.,

\[ f(n + 1) = 12\ldots(n + 1). \]

Therefore, the assertion is valid.

Let

\[ S(n) = \sum_{i=1}^{n} f(i). \]

Then, we shall use again formulae (2) and (3). Therefore,

\[ S(n) = \sum_{i=1}^{p} f(i) + \sum_{i=p+1}^{n} f(i), \]

where

\[ p = \frac{m(m + 1)}{2}. \]

It can be seen directly, that

\[ \sum_{i=1}^{p} f(i) = \sum_{i=1}^{m} \frac{i(i + 1) + 1}{2} = \sum_{i=1}^{m} \frac{11i + 1}{2} = \frac{11m^2}{2}. \]

On the other hand, if \(s = n - p\), then

\[ \sum_{i=p+1}^{n} f(i) = \frac{11(m + 1)^2}{2} + \frac{33(m + 1)1}{2} + s(s + 1)\ldots m(m + 1)12\ldots(s - 1) \]

\[ = \frac{m(m + 1)}{2} \]
\[
= \sum_{i=0}^{n+1} \frac{(i+1)(i+2)}{2} - \frac{i(i+1)}{2}, 10^{-n-1}.
\]

REFERENCES:


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