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Algebraic Approach to Literal Neutrosophic Kumaraswamy Probability Distribution

Mohamed Bisher Zeina[1,*], Mohammad Abobala[2], Ahmad Hatip[3], Said Broumi[4], Sarah Jalal Mosa[5]

[1] Department of Mathematical Statistics, Faculty of Science, University of Aleppo, Aleppo, Syria; bisher.zeina@gmail.com
[2] Department of Mathematics, Faculty of Science, Tishreen University, Latakia, Syria; mohammadabobala777@gmail.com
[3] Department of Mathematics, Gaziantep University, Gaziantep, Turkey; Kollnaar5@gmail.com
[4] Laboratory of Information Processing, Faculty of Science Ben M'Sik, University Hassan II, B.P 7955, Sidi Othman, Casablanca, Morocco; broumisaid78@gmail.com
[5] College of Engineering Techniques, Al-Farahidi University, Baghdad, Iraq; s.mosa@uoalfarahidi.edu.iq
* Correspondence: bisher.zeina@gmail.com

Abstract: In this paper we have successfully constructed the literal neutrosophic Kumaraswamy probability distribution. We mean by literal neutrosophic probability distribution that parameters of the distribution and the values that the random variable describing the distribution all take literal neutrosophic numbers of the form $\theta_N = a + bI\,; I^2 = I$ which differs from interval-valued neutrosophic probability distributions in which parameters of theses distributions take the form $\theta_N \in [L, U]$. We have derived the neutrosophic form of the probability density function, cumulative distribution function, statistical properties and maximum likelihood estimations of the parameters. Finally, a simulation study is performed to show the efficiency of the estimators provided by the neutrosophic MLE method.

Keywords: Literal Neutrosophic Numbers; Probability Distributions Theory; Maximum Likelihood Estimation; Kumaraswamy Distribution; Simulation.

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1. Introduction

Neutrosophic probability distributions from one point of view are a generalization of the concept of crisp probability distributions and fuzzy probability distributions that allow for the modeling of indeterminacy and uncertainty. In traditional probability theory, probabilities are assigned to events and are represented as real numbers between 0 and 1. In neutrosophic probability theory, probabilities are assigned as a triplet of values (T, I, F) where T represents the degree of truth, I represents the degree of indeterminacy and F represents the degree of falsity. These values are used to model the degree to which an event is certain, uncertain, or false [1-6,32-38].

From another point of view according to the fact that neutrosophic field of reals $R(I)$ is a generalization of the field of reals R, literal neutrosophic probability theory is another way of generalizing crisp probability theory where each probability can be presented in the form $P = P_1 + P_2 I \,; P_1, P_2 \in [0,1] \,, I^2 = I$ [7-14].

Neutrosophic probability distributions can be used in a variety of fields such as decision making, artificial intelligence, and data analysis, where traditional probability distributions are inadequate to model the uncertainty and indeterminacy present in real-world systems. [15-29]

The Kumaraswamy distribution [30] is a two-parameter continuous probability distribution that is commonly used in Bayesian statistics, reliability theory and other fields. The probability density function (PDF) of the classical Kumaraswamy distribution is defined as:

$$f(x;\, a,b) = a\ b\ x^{a-1}\ (1-x^a)^{b-1}; x \in [0,1] \qquad (1)$$

Where a and b are the shape parameters of the distribution, and they are both positive real numbers. The cumulative distribution function (CDF) is given by:

$$F(x;\, a,b) = 1 - (1-x^a)^b \qquad (2)$$

The Kumaraswamy distribution is a generalization of the beta distribution, in the sense that the beta distribution is a special case of the Kumaraswamy distribution when $a = b$.

Many generalizations of the Kumaraswamy distribution were made to provide more flexibility in modeling various types of data, and they are widely used in various fields.

It's worth noting that the Kumaraswamy distribution has some desirable properties such as it is closed under convolution, it has increasing failure rate, and it has increasing hazard rate. These properties make it useful for modeling various types of data in different fields.

In this paper, we are going to construct the neutrosophic form of Kumaraswamy distribution and study some properties of it depending on the One-Dimensional AH-Isometry.

2. Preliminaries

Definition 2.1 [7]

Let $R(I) = \{a + bI\, ; a, b \in R\, , I^2 = I\}$ be the neutrosophic field of reals. One-dimensional AH-isometry presented by Abobala and Hatip and its inverse are given by:

$$T: R(I) \to R^2 : \ T(a + bI) = (a, a + b) \qquad (5)$$

$$T^{-1}: R^2 \to R(I) : \ T^{-1}(a, b) = a + (b - a)I \qquad (6)$$

Note:

Let $\boldsymbol{x_N}, \boldsymbol{y_N} \in \boldsymbol{R(I)}$ and $\boldsymbol{T}$ be the AH-Isometry, since $\boldsymbol{T}$ is an algebraic isomorphism then it has the following properties:

1. $\boldsymbol{T(x_N + y_N) = T(x_N) + T(y_N)}$
2. $\boldsymbol{T(x_N \cdot y_N) = T(x_N) \cdot T(y_N)}$
3. $\boldsymbol{T}$ is correspondence one-to-one.

Definition 2.2 [8]

Let $f: R(I) \to R(I); f = f(x_N)$ where $x_N = x + yI \in R(I)$ then f is called a neutrosophic real function with one neutrosophic variable.

Definition 2.3 [9]

Neutrosophic gamma function is a special function is defined by:

$$\Gamma(a_N) = \Gamma(a_1) + I\{\Gamma(a_1 + a_2) - \Gamma(a_1)\}\, ; a_N = a_1 + a_2 I\, , I^2 = I$$

Where:

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx \ ; a > 0$$

Definition 2.4 [9]

Neutrosophic beta function is a special function can be defined in one of the following forms:

$$\beta(a_N, b_N) = \int_0^1 x^{a_N-1}(1-x)^{b_N-1}dx = \beta(a_1, b_1) + \{\beta(a_1+a_2, b_1+b_2) - \beta(a_1, b_1)\}I$$

$$= \frac{\Gamma(a_N)\Gamma(b_N)}{\Gamma(a_N+b_N)} \; ; a_N = a_1 + a_2 I \, , b_1 + b_2 I \, , I^2 = I$$

Definition 2.5 [9,11]

A neutrosophic random variable is defined as follows:

$$X_N = X_1 + X_2 I \, ; I^2 = I \, , 0 \cdot I = 0 \qquad (7)$$

Where X, Y are crisp random variables taking values on R.

Definition 2.6 [8]

Neutrosophic power of neutrosophic numbers is defined as follows:

$$(a+bI)^{c+dI} = a^c + I[(a+b)^{c+d} - a^c] \qquad (8)$$

Definition 2.7 [10]

Let $\mathbb{X}_N = X_{1N}, X_{2N}, \dots, X_{nN}$ be a neutrosophic random sample of random variables, we call:

$$L_N = L(\mathbb{X}_N; \Theta_N) = f(\mathbb{X}_N; \Theta_N) = \prod_{i=1}^{n} f(X_{iN}; \Theta_N) = L(\mathbb{X}; \Theta_1) + [L(\mathbb{X}+\mathbb{Y}; \Theta_1+\Theta_2) - L(\mathbb{X}; \Theta_1)]I \qquad (9)$$

The neutrosophic likelihood function.

Definition 2.8 [10]

Let $\mathbb{X}_N = X_{1N}, X_{2N}, \dots, X_{nN}$ be a neutrosophic random sample of random variables, we call:

$$\mathcal{L}_N = ln\, L(\mathbb{X}_N; \Theta_N) \qquad (10)$$

The neutrosophic loglikelihood function and we have:

$$\mathcal{L}_N = \mathcal{L}(\mathbb{X}; \Theta_1) + [\mathcal{L}(\mathbb{X}+\mathbb{Y}; \Theta_1+\Theta_2) - \mathcal{L}(\mathbb{X}; \Theta_1)]I \qquad (11)$$

4. Neutrosophic Kumaraswamy probability distribution

In this section we are going to construct the neutrosophic form of Kumaraswamy probability distribution function, cumulative probability distribution function, statistical properties and MLE estimations. Building this probability distribution and its properties will be in an algebraic approach depending on the one-dimensional AH-Isometry.

3.1 Probability density function and cumulative distribution function

Definition 3.1

Neutrosophic Kumaraswamy probability density function is defined as follows:

$$f(x_N;\ a_N, b_N) = a_N\, b_N\ x_N^{a_N-1}\ \left(1 - x_N^{a_N}\right)^{b_N-1}; x_N \in [0,1] \qquad (12)$$

Where: $x_N = x_1 + x_2 I, a_N = a_1 + a_2 I, b_N = b_1 + b_2 I, I^2 = I$

Theorem 3.1

The neutrosophic formal form of (12) is:

$$f(x_N;\ a_N, b_N) = a_1\, b_1\ x_1^{a_1-1}\ \left(1 - x_1^{a_1}\right)^{b_1-1}$$
$$+ I\left[(a_1 + a_2)(b_1 + b_2)\ (x_1 + x_2)^{a_1+a_2-1}\ \left(1 - (x_1 + x_2)^{(a_1+a_2)}\right)^{b_1+b_2-1}\right.$$
$$\left. - a_1\, b_1\ x_1^{a_1-1}\ \left(1 - x_1^{a_1}\right)^{b_1-1}\right]; x_1 \in [0,1] \& x_1 + x_2 \in [0,1]$$

Proof

$$T[f(x_N;\ a_N, b_N)] = T\left[a_N\, b_N\ x_N^{a_N-1}\ \left(1 - x_N^{a_N}\right)^{b_N-1}\right]$$
$$= T[a_N]T[b_N\ ]T[\ x_N^{a_N-1}]T\left[\left(1 - x_N^{a_N}\right)^{b_N-1}\right]$$
$$= (a_1, a_1 + a_2)(b_1, b_1$$
$$+ b_2)\left(x_1^{a_1-1}, (x_1 + x_2)^{a_1+a_2-1}\right)\left(\left(1 - x_1^{a_1}\right)^{b_1-1}, (1 - (x_1 + x_2)^{a_1+a_2})^{b_1+b_2-1}\right)$$
$$= \left(a_1 b_1 x_1^{a_1-1}\left(1 - x_1^{a_1}\right)^{b_1-1}, (a_1 + a_2)(b_1\right.$$
$$\left. + b_2)(x_1 + x_2)^{a_1+a_2-1}(1 - (x_1 + x_2)^{a_1+a_2})^{b_1+b_2-1}\right)$$
$$= (f(x_1; a_1, b_1), f(x_1 + x_2; a_1 + a_2, b_1 + b_2)\ )$$

Taking T^{-1}:s

$$f(x_N;\ a_N, b_N) = a_1 b_1 x_1^{a_1-1}\left(1 - x_1^{a_1}\right)^{b_1-1}$$
$$+ \left[(a_1 + a_2)(b_1 + b_2)(x_1 + x_2)^{a_1+a_2-1}(1 - (x_1 + x_2)^{a_1+a_2})^{b_1+b_2-1}\right.$$
$$\left. - a_1 b_1 x_1^{a_1-1}\left(1 - x_1^{a_1}\right)^{b_1-1}\right] I$$
$$= f(x_1; a_1, b_1) + I[f(x_1 + x_2; a_1 + a_2, b_1 + b_2) - f(x_1; a_1, b_1)]$$

Theorem 3.2

Equation (12) represents probability density function in classical sense.

Proof

We have:

$$T\left[\int_0^1 f(x_N;\ a_N, b_N)\,dx_N\right] = \left(\int_0^1 f(x_1; a_1, b_1)dx_1\,, \int_0^1 f(x_1 + x_2; a_1 + a_2, b_1 + b_2)d(x_1 + x_2)\right)$$

$$= \left(\int_0^1 a_1 b_1 x_1^{a_1-1}\left(1 - x_1^{a_1}\right)^{b_1-1} dx_1\,, \int_0^1 (a_1 + a_2)(b_1 + b_2)(x_1 + x_2)^{a_1+a_2-1}(1 - (x_1 + x_2)^{a_1+a_2})^{b_1+b_2-1}\, d(x_1 + x_2)\right)$$

$$= \left(-\int_0^1 d\left(1 - x_1^{a_1}\right)^{b_1}, -\int_0^1 d(1 - (x_1 + x_2)^{a_1+a_2})^{b_1+b_2-1}\right) = (1,1)$$

So:

$$\int_0^1 f(x_N;\ a_N, b_N)\,dx_N = T^{-1}(1,1) = 1$$

Also, depending on [7] it is easy to see that $T[f(x_N;\ a_N, b_N)]$ are two continuous functions on $[0,1] \subseteq R$ so $f(x_N;\ a_N, b_N)$ is continuous on $[0,1]$.

Depending on previous results we can prove that given neutrosophic function is a neutrosophic probability density function in classical sense.

Theorem 3.3

Cumulative distribution function of neutrosophic Kumaraswamy distribution is:

$$F(x_N;\ a_N, b_N) = 1 - \left(1 - x_N^{a_N}\right)^{b_N} \qquad (13)$$

Proof

$$F(x_N;\ a_N, b_N) = \int_0^{x_N} f(t_N;\ a_N, b_N)\,dt_N$$

$$T[F(x_N;\ a_N, b_N)] = T\left[\int_0^{x_N} f(t_N;\ a_N, b_N)\, dt_N\right]$$

$$= \left(\int_0^{x_1} f(t_1; a_1, b_1)dt_1\,, \int_0^{x_1+x_2} f(t_1+t_2; a_1+a_2, b_1+b_2)d(t_1+t_2)\right)$$

$$= \left(\int_0^{x_1} a_1 b_1 t_1^{a_1-1}\left(1-t_1^{a_1}\right)^{b_1-1} dt_1\,, \int_0^{x_1+x_2} (a_1+a_2)(b_1\right.$$

$$\left. + b_2)(t_1+t_2)^{a_1+a_2}(1-(t_1+t_2)^{a_1+a_2-1})^{b_1+b_2-1}\, d(t_1+t_2)\right)$$

$$= \left(-\int_0^{x_1} d\left(1-t_1^{a_1}\right)^{b_1}, -\int_0^{x_1+x_2} d(1-(t_1+t_2)^{a_1+a_2-1})^{b_1+b_2-1}\right)$$

$$= \left(1-\left(1-x_1^{a_1}\right)^{b_1}, 1-(1-(x_1+x_2)^{a_1+a_2})^{b_1+b_2-1}\right)$$

So:

$$F(x_N;\ a_N, b_N) = T^{-1}\left[1-\left(1-x_1^{a_1}\right)^{b_1}, 1-(1-(x_1+x_2)^{a_1+a_2})^{b_1+b_2-1}\right]$$

$$= 1-\left(1-x_1^{a_1}\right)^{b_1} + \left[1-(1-(x_1+x_2)^{a_1+a_2})^{b_1+b_2-1} - 1 + \left(1-x_1^{a_1}\right)^{b_1}\right] I$$

Which is the neutrosophic formal form of the function:

$$F(x_N;\ a_N, b_N) \ = \ 1 \ - \ \left(1-x_N^{a_N}\right)^{b_N}$$

3.2 Statistical properties of Kumaraswamy distribution

Theorem 3.4

Let X_N be a neutrosophic random variable following Kumaraswamy distribution with parameters a_N, b_N then:

1) $E(X_N^r) = b_1\beta\left(\frac{r}{a_1}+1, b_1\right) + \left[(b_1+b_2)\beta\left(\frac{r}{a_1+a_2}+1, b_1+b_2\right) - b_1\beta\left(\frac{r}{a_1}+1, b_1\right)\right] I$
2) $E(X_N) = b_1\beta\left(\frac{1}{a_1}+1, b_1\right) + \left[(b_1+b_2)\beta\left(\frac{1}{a_1+a_2}+1, b_1+b_2\right) - b_1\beta\left(\frac{1}{a_1}+1, b_1\right)\right] I$
3) $V(X_N) = b_1\beta\left(\frac{2}{a_1}+1, b_1\right) + \left[(b_1+b_2)\beta\left(\frac{2}{a_1+a_2}+1, b_1+b_2\right) - b_1\beta\left(\frac{2}{a_1}+1, b_1\right)\right] I -$ $\left[b_1\beta\left(\frac{1}{a_1}+1, b_1\right) + \left[(b_1+b_2)\beta\left(\frac{1}{a_1+a_2}+1, b_1+b_2\right) - b_1\beta\left(\frac{1}{a_1}+1, b_1\right)\right] I\right]^2$
4) Median=$\left(1-2^{-\frac{1}{b_1}}\right)^{\frac{1}{a_1}} + \left[\left(1-2^{-\frac{1}{b_1+b_2}}\right)^{\frac{1}{a_1+a_2}} - \left(1-2^{-\frac{1}{b_1}}\right)^{\frac{1}{a_1}}\right] I$

Proof

1) We have:

$$x_N^r f(x_N;\ a_N, b_N) = a_N\ b_N\ x_N^{a_N+r-1}\ \left(1-x_N^{a_N}\right)^{b_N-1}$$

$$T[x_N^r f(x_N;\ a_N, b_N)] = T[x_N^r]T[f(x_N;\ a_N, b_N)]$$
$$= (x_1^r, (x_1+x_2)^r)\left(a_1 b_1 x_1^{a_1-1}\left(1-x_1^{a_1}\right)^{b_1-1}, (a_1+a_2)(b_1\right.$$
$$\left.+ b_2)(x_1+x_2)^{a_1+a_2-1}(1-(x_1+x_2)^{a_1+a_2})^{b_1+b_2-1}\right)$$
$$= \left(a_1 b_1 x_1^{a_1+r-1}\left(1-x_1^{a_1}\right)^{b_1-1}, (a_1+a_2)(b_1\right.$$
$$\left.+ b_2)(x_1+x_2)^{a_1+a_2+r-1}(1-(x_1+x_2)^{a_1+a_2})^{b_1+b_2-1}\right)$$

So:

$$T\left[\int_0^1 x_N^r f(x_N;\ a_N, b_N)\, dx_N\right]$$
$$= \left(\int_0^1 a_1 b_1 x_1^{a_1+r-1}\left(1-x_1^{a_1}\right)^{b_1-1} dx_1, \int_0^1 (a_1+a_2)(b_1\right.$$
$$\left.+ b_2)(x_1+x_2)^{a_1+a_2+r-1}(1-(x_1+x_2)^{a_1+a_2})^{b_1+b_2-1} d(x_1+x_2)\right) = (L, R)$$

In L let $x_1^{a_1} = t$ then $x_1^r = t^{\frac{r}{a_1}}$ and $a_1 x_1^{a_1-1} dx_1 = dt$ so:

$$L = \int_0^1 b_1 t^{\frac{r}{a}}(1-t)^{b_1-1} dt = b_1 \beta\left(\frac{r}{a_1}+1, b_1\right)$$

In R similarly we let $(x_1+x_2)^{a_1+a_2} = t$ so $(x_1+x_2)^r = t^{\frac{r}{a_1+a_2}}$ and $(a_1+a_2)(x_1+x_2)^{a_1+a_2-1} d(x_1+x_2) = dt$ that yields:

$$R = \int_0^1 (b_1+b_2) t^{\frac{r}{a_1+a_2}}(1-t)^{b_1+b_2-1} dt = (b_1+b_2)\beta\left(\frac{r}{a_1+a_2}+1, b_1+b_2\right)$$

Then we have:

$$T\left[\int_0^1 x_N^r f(x_N;\ a_N, b_N)\, dx_N\right] = \left(b_1\beta\left(\frac{r}{a_1}+1, b_1\right), (b_1+b_2)\beta\left(\frac{r}{a_1+a_2}+1, b_1+b_2\right)\right)$$

So:

$$E(X_N^r) = \int_0^1 x_N^r f(x_N;\ a_N, b_N)\, dx_N = T^{-1}\left(b_1\beta\left(\frac{r}{a_1}+1, b_1\right), (b_1+b_2)\beta\left(\frac{r}{a_1+a_2}+1, b_1+b_2\right)\right)$$
$$= b_1\beta\left(\frac{r}{a_1}+1, b_1\right) + \left[(b_1+b_2)\beta\left(\frac{r}{a_1+a_2}+1, b_1+b_2\right) - b_1\beta\left(\frac{r}{a_1}+1, b_1\right)\right] I$$

2) By substituting $r = 1$ we get the required formula directly.

3) Straightforward from definition of variance (see [9]).
4) Median is the point that 50% of the area under the density curve is preceded by it, so it satisfies the following:

$$\int_{0}^{Median} f(x_N;\ a_N, b_N)\, dx_N = 0.5$$

Or equivalently:

$$F(Median;\ a_N, b_N) = \ 1\ -\ (1 - Median^{a_N})^{b_N} = 0.5$$

By solving the previous equation with respect to the Median we get:

$$Median = \left(1 - 2^{-\frac{1}{b_N}}\right)^{\frac{1}{a_N}}$$

Following rules of calculating neutrosophic powers presented in equation (8) we get:

$$Median = \left(1 - 2^{-\frac{1}{b_N}}\right)^{\frac{1}{a_N}} = \left(1 - 2^{-\frac{1}{b_1}}\right)^{\frac{1}{a_1}} + \left[\left(1 - 2^{-\frac{1}{b_1+b_2}}\right)^{\frac{1}{a_1+a_2}} - \left(1 - 2^{-\frac{1}{b_1}}\right)^{\frac{1}{a_1}}\right] I$$

4.3 Parameters' estimation using neutrosophic MLE method

Let $\mathbb{X}_N = X_{1N}, X_{2N}, \dots, X_{nN}$ be a neutrosophic random sample drawn from neutrosophic Kumaraswamy distribution presented in equation (12) then the neutrosophic likelihood function will be:

$$L_N = L(\mathbb{X}_N; \Theta_N) = f(\mathbb{X}_N; \Theta_N) = \prod_{i=1}^{n} f(X_{iN}; a_N, b_N) = \prod_{i=1}^{n} a_N\, b_N\ X_{iN}^{a_N-1}\ \left(1 - X_{iN}^{a_N}\right)^{b_N-1}$$

$$= a_N^n\, b_N^n\ \prod_{i=1}^{n} X_{iN}^{a_N-1}\ \prod_{i=1}^{n}\left(1 - X_{iN}^{a_N}\right)^{b_N-1}$$

So, the loglikelihood function will be:

$\mathcal{L}_N = lnL(\mathbb{X}_N; \Theta_N) = n \ln a_N + n \ln\ b_N + (a_N - 1)\ \sum_{i=1}^{n} \ln X_{iN} + (b_N - 1)\ \sum_{i=1}^{n} \ln(1 - X_{iN}^{a_N})$ (14)

Taking partial derivatives of equation (14) with respect to a_N, b_N yields to:

$$\frac{\partial}{\partial a_N}\mathcal{L}_N = \frac{n}{a_N} + \sum_{i=1}^{n} \ln X_{iN} + (b_N - 1)\ \sum_{i=1}^{n} \frac{-X_{iN}^{a_N} \ln X_{iN}^{a_N}}{1 - X_{iN}^{a_N}} \qquad (15)$$

$$\frac{\partial}{\partial b_N}\mathcal{L}_N = \frac{n}{b_N} + \sum_{i=1}^{n} \ln(1 - X_{iN}^{a_N}) \qquad (16)$$

Equations (15-16) are equivalent to the following four equations in R^2 (using the AH-Isometry):

$$\begin{cases} \frac{\partial}{\partial a_1}\mathcal{L}_1 = \frac{n}{a_1} + \sum_{i=1}^{n} \ln X_{i1} + (b_1 - 1) \sum_{i=1}^{n} \frac{-X_{i1}^{a_1} \ln X_{i1}^{a_1}}{1-X_{i1}^{a_1}} \\ \frac{\partial(\mathcal{L}_1+\mathcal{L}_2)}{\partial(a_1+a_2)} = \frac{n}{a_1+a_2} + \sum_{i=1}^{n} \ln(X_{i1} + X_{i2}) + (b_1 + b_2 - 1) \sum_{i=1}^{n} \frac{-(X_{i1}+X_{i2})^{a_1+a_2} \ln(X_{i1}+X_{i2})^{a_1+a_2}}{1-(X_{i1}+X_{i2})^{a_1+a_2}} \end{cases}$$

(17)

$$\begin{cases} \frac{\partial}{\partial b_1}\mathcal{L}_1 = \frac{n}{b_1} + \sum_{i=1}^{n} \ln(1 - X_{i1}^{a_1}) \\ \frac{\partial(\mathcal{L}_1+\mathcal{L}_2)}{\partial(b_1+b_2)} = \frac{n}{b_1+b_2} + \sum_{i=1}^{n} \ln(1 - (X_{i1} + X_{i2})^{a_1+a_2}) \end{cases} \quad (18)$$

Solving these sets of equations is not easy analytically, we will provide simulation study to show the efficiency of these neutrosophic MLE estimation.

4.4 Simulation study and random numbers generating

To do a simulation study we first derive a formula for random numbers generating noticing that equation (13) can be written as follows:

$$F(x_N;\, a_N, b_N) = 1 - \left(1 - x_N^{a_N}\right)^{b_N} = p_1 + p_2 I = P_N$$

Where P_N is neutrosophically uniform distributed on $[0,1]$ So:

$$1 - x_N^{a_N} = (1 - P_N)^{\frac{1}{b_N}}$$

$$x_N = \left(1 - (1 - P_N)^{\frac{1}{b_N}}\right)^{\frac{1}{a_N}} \quad (19)$$

Taking AH-isometry to equation (19) yields to the following two equations:

$$x_1 = \left(1 - (1 - p_1)^{\frac{1}{b_1}}\right)^{\frac{1}{a_1}} \quad (20)$$

$$x_1 + x_2 = \left(1 - (1 - p_1 - p_2)^{\frac{1}{b_1+b_2}}\right)^{\frac{1}{a_1+a_2}} \quad (19)$$

We can use equations (20-21) to generate random numbers following classical Kumaraswamy distribution with selected parameters, and takin T^{-1} to the generated numbers yields to neutrosophic Kumaraswamy distribution.

Now, performance of MLE estimators will be evaluated based on Monte Carlo simulation to the Kumaraswamy neutrosophic probability distribution with total replication of $N = 10000$ times and with sample sizes of 5,15,30,50 and 100 and with fixed parameters $a_N = 3 + 2I, b_N = 2 + 4I$.

Goodness of estimation is assessed depending on average bias and root mean square error defined below: [31]

$$AB = \frac{\sum_{i=1}^{N}(\hat{\theta}_{Ni} - \theta_N)}{N}$$

$$RMSE = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\theta}_{Ni} - \theta_N)^2}{N}}$$

Where $\hat{\theta}_{Ni}$ is the i[th] estimator of θ_N.

Table 1. Simulation results of neutrosophic Kumaraswamy distribution parameters' estimation

N	Average $\hat{a}_N$	$RMSE(\hat{a}_N)$	$AB(\hat{a}_N)$	Average $\hat{b}_N$	$RMSE(\hat{b}_N)$	$AB(\hat{b}_N)$
5	$4.287 + 1.642I$	$2.380 - 0.237I$	$1.287 - 0.358I$	$3.311 + 2.109I$	$2.522 - 0.279I$	$1.311 - 1.891I$
15	$3.434 + 2.149I$	$1.109 + 0.390I$	$0.434 + 0.149I$	$2.586 + 3.365I$	$1.344 + 0.615I$	$0.586 - 0.635I$
30	$3.209 + 2.075I$	$0.714 + 0.270I$	$0.209 + 0.075I$	$2.297 + 3.953I$	$0.807 + 0.923I$	$0.297 - 0.047I$
50	$3.093 + 2.090I$	$0.503 + 0.233I$	$0.093 + 0.090I$	$2.139 + 4.123I$	$0.522 + 0.977I$	$0.139 + 0.123I$
100	$3.043 + 2.017I$	$0.339 + 0.169I$	$0.043 + 0.017I$	$2.063 + 4.140I$	$0.330 + 0.869I$	$0.063 + 0.140I$

Table (1) shows results of simulation analysis for neutrosophic Kumaraswamy distribution where we notice that average bias of estimators is when sample size increases, which proves by simulation that proposed estimators are asymptotically unbiased.

5. Conclusions and future research directions

We have derived the neutrosophic Kumaraswamy probability distribution function, cumulative distribution function and statistical properties of the distribution, such as the mean, median, variance, and general moments. Additionally, we have derived the maximum likelihood estimations of the distributions' parameters.

The simulation study demonstrated the efficiency of the derived estimators and have shown that the estimators are unbiased. These results indicate that the neutrosophic Kumaraswamy distribution and its associated estimators can be useful in a variety of applications, including those involving uncertain or incomplete information.

Overall, this work has contributed to the development of neutrosophic probability theory and has practical implications for data analysis in various fields. Further research can be done to explore the potential of the neutrosophic Kumaraswamy distribution in other statistical applications.

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