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Introduction to Neutrosophic Stochastic Processes

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Abstract: In this article, the definition of literal neutrosophic stochastic processes is presented for the first time in the form $\mathcal{N}_t = \xi_t + \eta_t I\,; I^2 = I$ where both $\{\xi(t), t \in T\}$ and $\{\eta(t), t \in T\}$ are classical real valued stochastic processes. Characteristics of the literal neutrosophic stochastic process are defined and its formulas are driven including neutrosophic ensemble mean, neutrosophic covariance function and neutrosophic autocorrelation function. Concept of literal neutrosophic stationary stochastic processes is well defined and many theorems are presented and proved using classical neutrosophic operations then using the one-dimensional AH-Isometry. Some solved examples are presented and solved successfully. We have proved that studying the literal neutrosophic stochastic process $\{\mathcal{N}(t), t \in T\}$ is equivalent to studying two classical stochastic processes which are $\{\xi(t), t \in T\}$ and $\{\xi_t + \eta_t, t \in T\}$.

Keywords: AH-Isometry; Neutrosophic Field of Reals; Neutrosophic Random Variables; Stationary Stochastic Processes; Characteristics of Stochastic Processes; Ensemble Mean; Covariance Function; Autocorrelation Function.

1. Introduction

In probability theory, a family of random variables is called a stochastic process usually noted by $\{\xi(t)\,, t \in T\}$. Stochastic processes have many applications in many fields of science like biology, physics, ecology, information theory, chemistry, telecommunications, finance, etc. [1]

Classical stochastic process depends on parameters which are determined and known with high precision and confidence, but sometimes those parameters may have some uncertainty and it may be imprecise which led to define what is known by fuzzy stochastic processes **[2]**, **[3]**, **[4]**.

In the recent years Prof. Smarandache introduced an extension of fuzzy and intuitionistic fuzzy sets called neutrosophic sets where elements are described using three independent functions; membership, indeterminacy and non-membership. Also, Smarandache extended the field of reals adding the indeterminacy component $\boldsymbol{I}$ which satisfies $\boldsymbol{I}^2 = \boldsymbol{I}$ and introduced the literal neutrosophic reals field $\boldsymbol{R}(\boldsymbol{I}) = \boldsymbol{R} \cup \{\boldsymbol{I}\}$.

These extensions have been applied in many fields of sciences like probability theory, statistics, game theory, geometry, decision making, artificial intelligence, machine learning, abstract algebra, linear algebra, operations research, etc.[5-34]

Zeina and Hatip defined literal neutrosophic random variable in the form $\xi_N = \xi + I$ and studied its properties including literal neutrosophic expected value, literal neutrosophic variance, literal neutrosophic moments, literal neutrosophic characteristic function, literal neutrosophic moments generating function, literal neutrosophic probability density function and literal neutrosophic cumulative distribution function, and this study has been extended by Carlos Granados et al in [5-8].

Abobala and Hatip defined an isometry mapping between $R(I)$ and $R \times R$ called One-Dimensional AH-Isometry [9]. Based on this isometry, strong theorems and definitions of Euclidian geometry was written. This isometry is a powerful tool to build mathematical concepts strongly and with logical steps.

In this paper we generalize the definition of literal neutrosophic random variables to literal neutrosophic stochastic processes which are families of literal neutrosophic random variables depending on the one-dimensional AH-isometry and depending on direct computation based on neutrosophic rules.

This paper opens new research fields in probability theory like queueing theory, dynamic systems, reliability theory, stochastic differential equations, etc.

2. Preliminaries

Definition 2.1

Literal neutrosophic real number $\boldsymbol{N}$ is defined by:

$$\boldsymbol{N} = \boldsymbol{n_1} + \boldsymbol{n_2 I} \,;\, \boldsymbol{I^2} = \boldsymbol{I} \,\&\, \boldsymbol{n_1}, \boldsymbol{n_2} \in \boldsymbol{R}$$

And we call $\boldsymbol{R(I)} = \{\boldsymbol{n_1} + \boldsymbol{n_2 I} \,;\, \boldsymbol{n_1}, \boldsymbol{n_2} \in \boldsymbol{R} \text{ } \boldsymbol{and} \text{ } \boldsymbol{I^2} = \boldsymbol{I}\}$ the literal neutrosophic real set.

Definition 2.2

Let $\boldsymbol{R(I)}$ be the literal neutrosophic real set, we say $\boldsymbol{n_1} + \boldsymbol{n_2 I} \leq \boldsymbol{n_3} + \boldsymbol{n_4 I}$ iff $\boldsymbol{n_1} \leq \boldsymbol{n_3}$ & $\boldsymbol{n_1} + \boldsymbol{n_2} \leq \boldsymbol{n_3} + \boldsymbol{n_4}$.

Definition 2.3

AH-Isometry is an isomorphism preserves distances between $\boldsymbol{R(I)}$ and $\boldsymbol{R} \times \boldsymbol{R}$ and defined as in the following equation:

$$\boldsymbol{g} : \boldsymbol{R(I)} \to \boldsymbol{R} \times \boldsymbol{R} \,;\, \boldsymbol{g}(\boldsymbol{n_1} + \boldsymbol{n_2 I}) = (\boldsymbol{n_1}, \boldsymbol{n_1} + \boldsymbol{n_2}) \quad (1)$$

and its inverse is defined as follows:

$$\boldsymbol{g^{-1}} : \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R(I)} \,;\, \boldsymbol{g}(\boldsymbol{n_1}, \boldsymbol{n_2}) = \boldsymbol{n_1} + (\boldsymbol{n_2} - \boldsymbol{n_1})\boldsymbol{I} \qquad (2)$$

Definition 2.4

Let $\vec{\boldsymbol{n}} = (\boldsymbol{n_1} + \boldsymbol{n_2 I}, \boldsymbol{n_3} + \boldsymbol{n_4 I})$ be a vector, then its norm is defined as:

$$||\vec{\boldsymbol{n}}|| = \sqrt{(\boldsymbol{n_1} + \boldsymbol{n_2 I})^2 + (\boldsymbol{n_3} + \boldsymbol{n_4 I})^2}$$

Remark 2.1

Since the one-dimensional AH-Isometry is an algebraic isomorphism and preserves distances then it has the following properties:

1. $g(n_1 + n_2 I + \mathrm{n}_3 + n_4 I) = g(n_1 + n_2 I) + g(n_3 + n_4 I)$
2. $g[(n_1 + n_2 I) \cdot (n_3 + n_4 I\,)] = g(n_1 + n_2 I) \cdot g(n_3 + n_4 I)$

3. g is correspondence one-to-one

4. $g(\|\overrightarrow{AB}\|) = \|g(\overrightarrow{AB})\|$

Definition 2.5 [11]

Let $\boldsymbol{\xi}, \boldsymbol{\eta}$ be two classical random variables, then literal neutrosophic random variable (LNRV) is defined by:

$$\xi_N = \xi + \eta I \,; I^2 = I$$

Remark 2.2

Let $\boldsymbol{\xi_N}$ be a LNRV then:

1. $E(\xi_N) = E(\xi) + I\,E(\eta)$

2. $V(\xi_N) = V(\xi) + I\,[V(\xi + \eta) - V(\xi)]$

3. Literal Neutrosophic Stochastic Processes

Definition 3.1

Let $\{\boldsymbol{\xi}\,(\boldsymbol{t}), \boldsymbol{t} \in \boldsymbol{T}\}$ and $\{\boldsymbol{\xi}(\boldsymbol{t}), \boldsymbol{t} \in \boldsymbol{T}\}$ be two crisp (classic) stochastic processes, we define the literal neutrosophic stochastic process $\{\boldsymbol{\mathcal{N}}(\boldsymbol{t}), \boldsymbol{t} \in \boldsymbol{T}\}$ as follows:

$$\boldsymbol{\mathcal{N}}: (\boldsymbol{\Omega} \times \boldsymbol{T}) \to \boldsymbol{R}(\boldsymbol{I})\,;\ \boldsymbol{\mathcal{N}}(\boldsymbol{t}) = \boldsymbol{\xi}(\boldsymbol{t}) + \boldsymbol{\eta}(\boldsymbol{t})\boldsymbol{I}; \boldsymbol{I}^{\boldsymbol{2}} = \boldsymbol{I}$$

We call $\boldsymbol{\xi}(\boldsymbol{t})$ the determinant part of $\boldsymbol{\mathcal{N}}(\boldsymbol{t})$ and we call $\boldsymbol{\eta}(\boldsymbol{t})$ the indeterminant part of $\boldsymbol{\mathcal{N}}(\boldsymbol{t})$**.**

Theorem 1

Let $\{\mathcal{N}(t), t \in T\}$ be a literal neutrosophic stochastic process then the ensemble average function of $\{\mathcal{N}(t), t \in T\}$ is:

$$\boldsymbol{\mu_{\mathcal{N}}}(\boldsymbol{t}) = \boldsymbol{\mu_{\xi}}(\boldsymbol{t}) + \boldsymbol{I}\boldsymbol{\mu_{\eta}}(\boldsymbol{t}) \quad (3)$$

Proof

For a fixed $t \in T$ both $\{\xi(t), t \in T\}$ and $\{\eta(t), t \in T\}$ become random variables (not stochastic processes), the $\{\mathcal{N}(t), t \in T\}$ becomes a literal neutrosophic random variable, so based on properties of literal neutrosophic random variables we can write:

$$\boldsymbol{\mu_{\mathcal{N}}(t) = E[\mathcal{N}(t)] = E[\xi(t) + I\,\eta(t)] = E[\xi(t)] + IE[\eta(t)] = \mu_{\xi}(t) + I\mu_{\eta}(t)}$$

Theorem 2

Let $\{\mathcal{N}(t), t \in T\}$ be a literal neutrosophic stochastic process then autocorrelation function is:

$$R_{\mathcal{N}}(s,t) = R_{\xi}(s,t) + I\{R_{\xi\eta}(s,t) + R_{\eta\xi}(s,t) + R_{\eta}(s,t)\} \quad (4)$$

Proof

$$R_{\mathcal{N}}(s,t) = E[\mathcal{N}(s) \cdot \mathcal{N}(t)] = E\{[\xi(s) + I\,\eta(s)] \cdot [\xi(t) + I\,\eta(t)]\}$$
$$= E\{\xi(s)\xi(t) + I\xi(s)\eta(t) + I\eta(s)\xi(t) + I^2\eta(s)\eta(t)\}$$
$$= R_{\xi}(s,t) + I\{R_{\xi\eta}(s,t) + R_{\eta\xi}(s,t) + R_{\eta}(s,t)\}$$

Remark 3.1

Notice that $R_{\mathcal{N}}(t,t) = R_{\xi}(t,t) + I\{2R_{\xi\eta}(t,t) + R_{\eta}(t,t)\} = E[\xi^2(t)] + I\{2R_{\xi\eta}(t,t) + E[\eta^2(t)]\}$

Theorem 3

Let $\{\mathcal{N}(t), t \in T\}$ be a literal neutrosophic stochastic process then its autocovariance function is:

$$C_{\mathcal{N}}(s,t) = R_{\mathcal{N}}(s,t) - \mu_{\mathcal{N}}(s)\mu_{\mathcal{N}}(t) \quad (5)$$

Proof

$$C_{\mathcal{N}}(s,t) = cov[\mathcal{N}(s), \mathcal{N}(t)] = E\{[\mathcal{N}(s) - \mu_{\mathcal{N}}(s)][\mathcal{N}(t) - \mu_{\mathcal{N}}(t)]\}$$
$$= E\{\mathcal{N}(s)\mathcal{N}(t) - \mu_{\mathcal{N}}(t)\mathcal{N}(s) - \mu_{\mathcal{N}}(s)\mathcal{N}(t) + \mu_{\mathcal{N}}(s)\mu_{\mathcal{N}}(t)\}$$
$$= R_{\mathcal{N}}(s,t) - \mu_{\mathcal{N}}(t)E[\mathcal{N}(s)] - \mu_{\mathcal{N}}(s)E[\mathcal{N}(t)] + \mu_{\mathcal{N}}(s)\mu_{\mathcal{N}}(t)$$
$$= R_{\mathcal{N}}(s,t) - \mu_{\mathcal{N}}(t)\mu_{\mathcal{N}}(s) - \mu_{\mathcal{N}}(s)\mu_{\mathcal{N}}(t) + \mu_{\mathcal{N}}(s)\mu_{\mathcal{N}}(t)$$
$$= R_{\mathcal{N}}(s,t) - \mu_{\mathcal{N}}(s)\mu_{\mathcal{N}}(t)$$

Remark 3.2

If $s = t$ then:

$$C_{\mathcal{N}}(s\,,t) = C_{\mathcal{N}}(t\,,t) = E\{[\mathcal{N}(t) - \mu_{\mathcal{N}}(t)][\mathcal{N}(t) - \mu_{\mathcal{N}}(t)]\} = Var[\mathcal{N}(t)]$$

Definition 3.2

Let $\{\mathcal{N}(t), t \in T\}$ be a literal neutrosophic stochastic process, we call $F(x_N, t) = P\{\mathcal{N}(t) \leq x_N\}$ the first order distribution of $\{\mathcal{N}(t), t \in T\}$ where $x_N = x + Iy$ and $x, y \in R$.

Definition 3.3

Let $\{\mathcal{N}(t), t \in T\}$ be a literal neutrosophic stochastic process, we call $\frac{\partial}{\partial x_N} F(x_N, t) = f(x_N, t)$ the first order density of $\{\mathcal{N}(t), t \in T\}$ where $x_N = x + Iy$ and $x, y \in R$.

Definition 3.4

A literal neutrosophic stochastic process is called strongly stationary if its distribution is invariant under neutrosophic transition of time, i.e., $f(x_N, t) = f(x_N, t + h_N); h_N = h_1 + Ih_2$

Definition 3.5

A literal neutrosophic stochastic process is called weakly stationary if it satisfies the following two conditions:

1. $\mu_{\mathcal{N}}(t) = \mu_N = \mu_1 + I\mu_2$
2. $E[\mathcal{N}(t) \cdot \mathcal{N}(t - \tau_N)] = R(\tau_N)$

4. Literal Neutrosophic Stochastic Processes Using AH-Isometry:

Consider the literal neutrosophic stochastic process $\{\mathcal{N}(t), t \in T\}$ then applying AH-isometry on it yields to:

$$g[\mathcal{N}(t)] = g[\xi(t) + \eta(t)I] = \big(\xi(t), \xi(t) + \eta(t)\big)$$

Notice that using the one-dimensional AH-Isometry we transfer the literal neutrosophic stochastic process $\{\mathcal{N}(t), t \in T\}$ into two classical stochastic processes $\{\xi(t), t \in T\}$ and $\{\xi(t) + \eta(t), t \in T\}$.

So, we can study the characteristics of $\{\mathcal{N}(t), t \in T\}$ by studying the characteristics of both $\{\xi(t), t \in T\}$ and $\{\xi(t) + \eta(t), t \in T\}$.

Example 4.1

In theorem 1 we show that $\mu_{\mathcal{N}}(t) = \mu_\xi(t) + \mu_\eta(t)I$, we can reach the same result by using the one-dimensional AH-Isometry as follows:

We have:

$$\mathcal{N}(t) = \xi(t) + \eta(t)I$$

So:

$$E[\mathcal{N}(t)] = E[\xi(t) + \eta(t)I]$$

$$g(E[\mathcal{N}(t)]) = g(E[\xi(t) + I\,\eta(t)]) = E[g(\xi(t) + \eta(t)I)] = E[\xi(t), \xi(t) + \eta(t)]$$

$$= \left(\mu_\xi(t), \mu_\xi(t) + \mu_\eta(t)\right)$$

Taking the inverse isometry:

$$g^{-1}g(E[\mathcal{N}(t)]) = E[\mathcal{N}(t)] = \mu_\xi(t) + \left[\mu_\xi(t) + \mu_\eta\,(t) - \mu_\xi(t)\right]I = \mu_\xi(t) + \mu_\eta(t)I$$

Which is the same result presented in theorem 1.

Example 4.2

Let's calculate the autocorrelation function $R_{\mathcal{N}}(s,t)$ using the AH-Isometry:

$$R_{\mathcal{N}}(s,t) = E[\mathcal{N}(s) \cdot \mathcal{N}(t)]$$

$$g\left(R_{\mathcal{N}}(s,t)\right) = E\{g[\mathcal{N}(s) \cdot \mathcal{N}(t)]\} = E\{g[\xi(s) + \eta(s)I][\xi(t) + \eta(t)I]\}$$

$$= E\{g[\xi(s) + \eta(s)I]g[\xi(t) + \eta(t)I]\}$$

$$= E\left\{\left(\xi(s), \xi(s) + \eta(s)\right)\left(\xi(t), \xi(t) + \eta(t)\right)\right\}$$

$$= \left\{E\left(\xi(s)\xi(t)\right), E\left(\xi(s) + \eta(s)\right)\left(\xi(t) + \eta(t)\right)\right\}$$

$$= \left(R_\xi(s,t), R_\xi(s,t) + R_{\xi\eta}(s,t) + R_{\eta\xi}(s,t) + R_\eta(s,t)\right)$$

Now taking g^{-1} yields:

$$R_{\mathcal{N}}(s,t) = R_\xi(s,t) + \left[R_\xi(s,t) + R_{\xi\eta}(s,t) + R_{\eta\xi}(s,t) + R_\eta(s,t) - R_\xi(s,t)\right]I$$

$$= R_\xi(s,t) + I\left\{R_{\xi\eta}(s,t) + R_{\eta\xi}(s,t) + R_\eta(s,t)\right\}$$

Which is the same result in theorem 2.

Theorem 4

A literal neutrosophic stochastic process $\mathcal{N}(t) = \xi(t) + \eta(t)I$ is weakly stationary if and only if $\{\xi(t), t \in T\}$ is weakly stationary and $\{\xi(t) + \eta(t), t \in T\}$ is weakly stationary.

Proof

We will first suppose that $\{\xi(t), t \in T\}$ and $\{\xi(t) + \eta(t), t \in T\}$ are weakly stationary and prove that $\mathcal{N}(t) = \xi(t) + \eta(t)I$ is also stationary:

Since $\{\xi(t), t \in T\}$ is weakly stationary then $\mu_\xi(t) = \mu_\xi = constant$ and $E[\xi(t) \cdot \xi(t - \tau)] = R_\xi(\tau)$

We also supposed that $\{\xi(t) + \eta(t), t \in T\}$ is weakly stationary so $\mu_{\xi+\eta}(t) = E[\xi(t) + \eta(t)] = \mu_{\xi+\eta} = costant,$ which means that $\mu_\eta(t) = \mu_\eta = constant.$

and $R_{\xi+\eta}(t,t-\tau) = E[\xi(t)+\eta(t)][\xi(t-\tau)+\eta(t-\tau)] = E[\xi(t)\xi(t-\tau)+\xi(t)\eta(t-\tau)+\eta(t)\xi(t-\tau)+\eta(t)\eta(t-\tau)] = R_{\xi}(t,t-\tau)+R_{\xi\eta}(t,t-\tau)+R_{\eta\xi}(t,t-\tau)+R_{\eta}(t,t-\tau)$

Since $\xi(t)+\eta(t)$ is weakly stationary then $R_{\xi+\eta}(t,t-\tau)$ must depend only on the difference τ, so the only possible form of it will be:

$$R_{\xi+\eta}(t,t-\tau) = R_{\xi}(\tau)+R_{\xi\eta}(\tau)+R_{\eta\xi}(\tau)+R_{\eta}(\tau) = R_{\xi+\eta}(\tau)$$

Which means that $R_{\xi\eta}(t,t-\tau) = R_{\xi\eta}(\tau), R_{\eta\xi}(t,t-\tau) = R_{\eta\xi}(\tau), R_{\eta}(t,t-\tau) = R_{\eta}(\tau)$

$$E\big(\mathcal{N}(t)\big) = E[\xi(t)+\eta(t)I] = \mu_{\xi}\,(t)+\mu_{\eta}(t)I = \mu_{\xi}+\mu_{\eta}I = \mu_{N} = constant$$

Using equation (4):

$$\begin{aligned} R_{\mathcal{N}}(t,t-\tau) &= E[\mathcal{N}(t)\cdot\mathcal{N}(t-\tau)] \\ &= R_{\xi}(t,t-\tau)+I\big\{R_{\xi\eta}(t,t-\tau)+R_{\eta\xi}(t,t-\tau)+R_{\eta}(t,t-\tau)\big\} \\ &= R_{\xi}(\tau)+I\big\{R_{\xi\eta}(\tau)+R_{\eta\xi}(\tau)+R_{\eta}(\tau)\big\} = R_{\mathcal{N}}(\tau) \end{aligned}$$

So, we conclude that $\{\mathcal{N}(t), t\in T\}$ is weakly stationary.

Now let's assume that $\{\mathcal{N}(t), t\in T\}$ is weakly stationary and prove that both $\{\xi(t), t\in T\}$ and $\{\xi(t)+\eta(t), t\in T\}$ are weakly stationary.

Since $\{\mathcal{N}(t), t\in T\}$ is weakly stationary then $E\big(\mathcal{N}(t)\big) = \mu_N(t) = \mu_N = constant$

but $E\big(\mathcal{N}(t)\big) = \mu_{\xi}(t)+I\mu_{\eta}(t)$ so both $\mu_{\xi}(t)$ and $\mu_{\eta}(t)$ must be dependent of time, then

$$\mu_{\xi}(t) = \mu_{\xi} \quad (6)$$

$$\mu_{\eta}(t) = \mu_{\eta} \quad (7)$$

which meant that:

$$\mu_{\xi+\eta}(t) = \mu_{\xi}+\mu_{\eta} = constant \; (8)$$

Also, we have: $R_{\mathcal{N}}(t,t-\tau) = R_{\xi}(t,t-\tau)+I\big\{R_{\xi\eta}(t,t-\tau)+R_{\eta\xi}(t,t-\tau)+R_{\eta}(t,t-\tau)\big\}$ and since $\{\mathcal{N}(t), t\in T\}$ is weakly stationary then $R_{\mathcal{N}}(t,t-\tau)$ must depend only on the difference τ so the following equations must hold:

$$R_{\xi}(t,t-\tau) = R_{\xi}(\tau) \quad (9)$$

$$R_{\xi\eta}(t,t-\tau) = R_{\xi\eta}(\tau) \; (10)$$

$$R_{\eta\xi}(t,t-\tau) = R_{\eta\xi}(\tau) \; (11)$$

$$R_{\eta}(t,t-\tau) = R_{\eta}(\tau) \quad (12)$$

From equations (6), (9) we conclude that $\{\xi(t), t\in T\}$ is weakly stationary.

And using equations (8), (9-12) we conclude that $\{\xi(t)+\eta(t), t \in T\}$ is weakly stationary.

Theorem 5

Suppose that $\{\mathcal{N}(t), t \in T\}$ is a weakly stationary literal neutrosophic stochastic process with autocorrelation function $R_{\mathcal{N}}(\tau)$, then the following holds:

1. $R_{\mathcal{N}}(\tau) = R_{\mathcal{N}}(-\tau)$
2. $|R_{\mathcal{N}}(\tau)| \leq R(0)$

Proof

1. we have:

$$R_{\mathcal{N}}(\tau) = R_{\xi}(\tau) + I\left\{R_{\xi\eta}(\tau) + R_{\eta\xi}(\tau) + R_{\eta}(\tau)\right\}$$

So:

$$R_{\mathcal{N}}(-\tau) = R_{\xi}(-\tau) + I\left\{R_{\xi\eta}(-\tau) + R_{\eta\xi}(-\tau) + R_{\eta}(-\tau)\right\}$$

And using properties of cross-correlation function in classical stationary processes we get:

$$R_{\mathcal{N}}(-\tau) = R_{\xi}(\tau) + I\left\{R_{\eta\xi}(\tau) + R_{\xi\eta}(\tau) + R_{\eta}(\tau)\right\} = R_{\mathcal{N}}(\tau)$$

2. Taking AH-Isometry:

$$g(|R_{\mathcal{N}}(\tau)|) = |E\{g[\mathcal{N}(t) \cdot \mathcal{N}(t-\tau)]\} = E\{g[\xi(t) + I\eta(t)][\xi(t-\tau) + I\eta(t-\tau)]\}|$$
$$= |E\{g[\xi(t) + I\eta(t)]g[\xi(t-\tau) + I\eta(t-\tau)]\}|$$
$$= \left|E\left\{\left(\xi(t), \xi(t) + \eta(t)\right)\left(\xi(t-\tau), \xi(t-\tau) + \eta(t-\tau)\right)\right\}\right|$$
$$= |E\{\xi(t)\xi(t-\tau), [\xi(t) + \eta(t)][\xi(t-\tau) + \eta(t-\tau)]\}|$$
$$= \left(\left|R_{\xi}(\tau)\right|, \left|R_{\xi+\eta}(\tau)\right|\right) \leq (0,0)$$

Now taking g^{-1}:

$$|R_{\mathcal{N}}(\tau)| = \left|R_{\xi}(\tau)\right| + \left(\left|R_{\xi+\eta}(\tau)\right| - \left|R_{\xi}(\tau)\right|\right)I \leq 0$$

5. Some Applications:

Example 5.1

Let $\{\mathcal{N}(t), t \in T\}$ be a literal neutrosophic stochastic process defined as follows:

$$\mathcal{N}(t) = A_N \cos(t) + \sin(t)\ I$$

Where distribution of the literal neutrosophic random variable A_N is:

A_N	0	1
$Prob$	$\frac{1}{3}I$	$1-\frac{1}{3}I$

Let's find $\mu_{\mathcal{N}}(t), R_{\mathcal{N}}(s,t)$ and show whether $\{\mathcal{N}(t), t \in T\}$ is stationary or not.

Solution

$$\boldsymbol{E(A_N) = 0 \cdot \frac{1}{3}I + 1 \cdot \left(1-\frac{1}{3}I\right) = 1-\frac{1}{3}I}$$

$$\boldsymbol{E(A_N^2) = 0^2 \cdot \frac{1}{3}I + 1^2 \cdot \left(1-\frac{1}{3}I\right) = 1-\frac{1}{3}I}$$

$$\boldsymbol{\mu_{\mathcal{N}}(t) = E(\mathcal{N}(t)) = E(A_N \cos(t) + \sin(t)\ I) = \left(1-\frac{1}{3}I\right) \cdot \cos(t) + \sin(t)\, I}$$

Since $\boldsymbol{\mu_{\mathcal{N}}(t)}$ is a function of $\boldsymbol{t}$ then $\{\boldsymbol{\mathcal{N}(t), t \in T}\}$ is not stationary stochastic process.

$$\begin{aligned} R_{\mathcal{N}}(s,t) &= E[\mathcal{N}(s) \cdot \mathcal{N}(t)] = E[(A_N \cos(t) + \sin(t)\ I)(A_N \cos(s) + \sin(s)\ I)] \\ &= E[A_N^2 \cos(t)\cos(s) + A_N \cos(t)\sin(s)\, I + \sin(t)\, I\, A_N \cos(s) \\ &+ \sin(t)\sin(s)\, I^2] \\ &= \left(1-\frac{1}{3}I\right)\cos(t)\cos(s) + \left(1-\frac{1}{3}I\right)\cos(t)\sin(s)\, I + \left(1-\frac{1}{3}I\right)\sin(t)\cos(s)\, I \\ &+ \sin(t)\sin(s)\, I \end{aligned}$$

Example 5.2

let $\{\mathcal{N}(t), t \in T\}$ be a neutrosophic stochastic process defined as follows:

$$\mathcal{N}(t) = \xi(t) + \xi(t)\, I$$

Where $\{\xi(t), t \in T\}$ is a classical stochastic process defined by:

$$\xi(t) = A\cos(t) + B\sin(t)$$

Where A, B are random variables both defined as by:

	-2	1
$Prob$	$\frac{1}{3}$	$\frac{2}{3}$

Let's find $\mu_{\mathcal{N}}(t), R_{\mathcal{N}}(s,t)$ and show whether $\{\mathcal{N}(t), t \in T\}$ is stationary or not.

solution

$$E(A) = E(B) = \frac{2}{3} - \frac{2}{3} = 0$$

$$E(A^2) = E(B^2) = \frac{2}{3} + \frac{4}{3} = 2$$

$$\mu_\xi(t) = \cos(t)\ E(A) + \sin(t)\ E(B) = 0$$

$$R_\xi(s,t) = E[\xi(s)\,\xi(t)] = E[(A\cos(s) + B\sin(s))(A\cos(t) + B\sin(t))]$$

$$= E(A^2\cos(s)\cos(t) + AB\cos(s)\sin(t) + BA\sin(s)\cos(t) + B^2\sin(s)\sin(t))$$

$$= 2(\cos(s)\cos(t) + \sin(s)\sin(t)) = 2\cos(t-s) = 2\cos\tau$$

So:

$$\mu_{\mathcal{N}}(t) = E[\mathcal{N}(t)] = E[\xi(t) + \xi(t)\ I] = \mu_\xi(t) + \mu_\xi(t)I = 0 = const$$

$$R_{\mathcal{N}}(s,t) = E[\mathcal{N}(s)\cdot\mathcal{N}(t)] = E[(\xi(s) + \xi(s)I)(\xi(t) + \xi(t)I)]$$

$$= E[\xi(s)\xi(t) + \xi(s)\xi(t)I + \xi(s)X(t)I + \xi(s)\xi(t)I^2]$$

$$= R_\xi(s,t) + R_\xi(s,t)I + R_\xi(s,t)I + R_\xi(s,t)I = 2\cos(\tau) + 6\cos(\tau)\,I = R_N(\tau)$$

We conclude that $\{\mathcal{N}(t), t \in T\}$ is weakly stationary process.

In fact, it is clear that $\{\mathcal{N}(t), t \in T\}$ is weakly stationary process since $\{\xi(t), t \in T\}$ and $\{2\xi(t), t \in T\}$ are both weakly stationary processes.

6. Conclusions and future research directions

Concept of literal neutrosophic stochastic process is well defined by $\boldsymbol{\mathcal{N}(t) = \xi(t) + \eta(t)I}$. We proved that a literal neutrosophic stochastic process can be presented in $\boldsymbol{R^2}$ as two classical stochastic processes, first is $\{\boldsymbol{\xi(t), t \in T}\}$ and second is the convolution $\{\boldsymbol{\xi(t) + \eta(t), t \in T}\}$. Many theorems were proved successfully especially the theorem of stationary stochastic process where we have seen that $\{\boldsymbol{\mathcal{N}(t), t \in T}\}$ is stationary if and only if $\{\boldsymbol{\xi(t), t \in T}\}$ is stationary and $\{\boldsymbol{\xi(t) + \eta(t), t \in T}\}$ is stationary. This paper can be applied in many fields related to probability theory including game theory, polling, statistical analysis, financial mathematics, etc. In future researches we are looking forward to study cross neutrosophic stochastic processes and define its characteristics and the theorems related to it. Also, we are looking forward to study applications of literal neutrosophic stochastic processes in related fields.

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