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On The Algebraic Properties of 2-Cyclic Refined Neutrosophic Matrices and The Diagonalization Problem

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Abstract:

The n-cyclic refined neutrosophic algebraic structures are very diverse and rich materials. In this paper, we study the elementary algebraic properties of 2-cyclic refined neutrosophic square matrices, where we find formulas for computing determinants, eigen values, and inverses. On the other hand, we solve the diagonalization problem of these matrices, where a complete algorithm to diagonlaize every diagonalizable 2-cyclic refined neutrosophic square matrix is obtained and illustrated by many related examples.

Key Words: n-cyclic refined neutrosophic ring, n –cyclic refined neutrosophic matrix, the diagonalization problem.

1.Introduction

Neutrosophic algebraic structures were defined firstly in [1], by adding an algebraic indeterminacy element I to classical algebraic structures to obtain n novel extensions. For example, we can find neutrosophic geometry, neutrosophic functions, neutrosophic rings, and neutrosophic spaces [2-7].

The concept of n-cyclic neutrosophic algebraic structure was supposed in [8], and then it has been studied widely, see [9-12].

As an important algebraic structure, neutrosophic matrices with many types were handled and studied, where we can see many results about inverses, eigen vectors, diagonalizations, and determinants were proven and established [13-24]. In the literature, we have many types of neutrosophic matrices, refined neutrosophic matrices, and n-refined neutrosophic matrices, and n-cyclic refined neutrosophic matrices [17].

The diagonalization algorithm for n-cyclic refined neutrosophic matrix has been asked as an open problem in [12], and it is still open for all values of n.

This motivates us to study the diagonalization problem for n =2, and to present an effective algorithm to diagonlaize a 2-cyclic refined neutrosophic square matrix, as well as many related concepts, especially eigen values computing.

2. Preliminaries

Definition [8]

Let (R,+,×) be a ring and $I_k; 1 \leq k \leq n$ be n sub-indeterminacies. We define R_n(I)={$a_0 + a_1 I + \cdots + a_n I_n$; $a_i \in R$} to be n-cyclic refined neutrosophic ring.

Operations on R_n(I) are defined as:

$\sum_{i=0}^{n} x_i I_i + \sum_{i=0}^{n} y_i I_i = \sum_{i=0}^{n}(x_i + y_i) I_i$,$\sum_{i=0}^{n} x_i I_i \times \sum_{i=0}^{n} y_i I_i = \sum_{i,j=0}^{n}(x_i \times y_j) I_i I_j = \sum_{i,j=0}^{n}(x_i \times y_j) I_{(i+j\ mod n)}$.

× is the multiplication on the ring R.

In this paper, we study open problem 3, open problem 4, and open problem 5 in [12].

3. Main discussion :

Definition.

Let $M = M_0 + M_1 I_1 + M_2 I_2$ be a 2-cyclic refined neutrosophic matrix, then M is diagonalizable if and only if there exists a 2-cyclic refined neutrosophic diagonal matrix K and invertible matrix U such that $M = UKU^{-1}$.

Theorem.

Let $M = M_0 + M_1 I_1 + M_2 I_2$ be a 2-cyclic refined neutrosophic matrix, then M is diagonalizable if and only if: $M_0, M_0 + M_1 + M_2, M_0 - M_1 + M_2$ are diagonalizable.

Proof.

Assume that M is diagonalizable, then there exists a diagonal matrix $K = K_0 + K_1 I_1 + K_2 I_2$ and an invertible matrix $U = U_0 + U_1 I_1 + U_2 I_2$ such that $M = UKU^{-1}$.

The matrix equation $UKU^{-1} = M$ is equivalent to:

$$\begin{aligned} U_0 K_0 {U_0}^{-1} + \frac{1}{2} I_1 [& (U_0 + U_1 + U_2)(K_0 + K_1 + K_2)(U_0 + U_1 + U_2)^{-1} \\ & - (U_0 - U_1 + U_2)(K_0 - K_1 + K_2)(U_0 - U_1 + U_2)^{-1}] \\ & + \frac{1}{2} I_2 \big[(U_0 + U_1 + U_2)(K_0 + K_1 + K_2)(U_0 + U_1 + U_2)^{-1} \\ & - (U_0 - U_1 + U_2)(K_0 - K_1 + K_2)(U_0 - U_1 + U_2)^{-1} - 2U_0 K_0 {U_0}^{-1}\big] \\ & = M_0 + M_1 I_1 + M_2 I_2 \end{aligned}$$

Thus:

$$\begin{cases} U_0 K_0 {U_0}^{-1} = M_0 \\ (U_0 + U_1 + U_2)(K_0 + K_1 + K_2)(U_0 + U_1 + U_2)^{-1} = M_0 + M_1 + M_2 \\ (U_0 - U_1 + U_2)(K_0 - K_1 + K_2)(U_0 - U_1 + U_2)^{-1} = M_0 - M_1 + M_2 \end{cases}$$

This implies $M_0, M_0 + M_1 + M_2, M_0 - M_1 + M_2$ are diagonalizable.

Conversely, assume that $M_0, M_0 + M_1 + M_2, M_0 - M_1 + M_2$ are diagonalizable, then there exists diagonal matrices D_0, D_1, D_2 and invertible matrices P_0, P_1, P_2 such that $P_0 D_0 {P_0}^{-1} = M_0, P_1 D_1 {P_1}^{-1} = M_0 + M_1 + M_2,\ P_2 D_2 {P_2}^{-1} = M_0 - M_1 + M_2$.

This implies that $M_1 = \frac{1}{2}\left(P_1 D_1 {P_1}^{-1} - P_2 D_2 {P_2}^{-1}\right), M_2 = \frac{1}{2}\left(P_1 D_1 {P_1}^{-1} + P_2 D_2 {P_2}^{-1} - 2P_0 D_0 {P_0}^{-1}\right)$

We put $D = D_0 + \frac{1}{2} I_1 (D_1 - D_2) + \frac{1}{2} I_2 (D_1 + D_2 - 2D_0) = L_0 + \frac{1}{2} I_1 L_1 + \frac{1}{2} I_2 L_2; \begin{cases} L_0 = D_0 \\ L_1 = D_1 - D_2 \\ L_2 = D_1 + D_2 - 2D_0 \end{cases}$.

$$P = P_0 + \frac{1}{2} I_1 (P_1 - P_2) + \frac{1}{2} I_2 (P_1 + P_2 - 2P_0) = N_0 + \frac{1}{2} I_1 N_1 + \frac{1}{2} I_2 N_2; \begin{cases} N_0 = P_0 \\ N_1 = P_1 - P_2 \\ N_2 = P_1 + P_2 - 2P_0 \end{cases}.$$

We have:

$P^{-1} = {N_0}^{-1} + \frac{1}{2} I_1 [(N_0 + N_1 + N_2)^{-1} - (N_0 - N_1 + N_2)^{-1}] + \frac{1}{2} I_2 \big[(N_0 + N_1 + N_2)^{-1} - (N_0 - N_1 + N_2)^{-1} - 2{N_0}^{-1}\big] = {P_0}^{-1} + \frac{1}{2} I_1 \big[{P_1}^{-1} - {P_2}^{-1}\big] + \frac{1}{2} I_2 \big[{P_1}^{-1} + {P_2}^{-1} - 2{P_0}^{-1}\big]$

It is easy to check that $PDP^{-1} = M_0 + M_1 I_1 + M_2 I_2 = M$, thus M is diagonalizable.

Example.

Consider the following 2×2 2-cyclic refined matrix:

$$X=\begin{pmatrix} 3+\frac{1}{2}I_1-\frac{3}{2}I_2 & \frac{1}{2}I_1+\frac{1}{2}I_2 \\ \frac{1}{2}I_1-\frac{1}{2}I_2 & 2-\frac{3}{2}I_1+\frac{1}{2}I_2 \end{pmatrix}=\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}+\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-3}{2} \end{pmatrix}I_1+\begin{pmatrix} \frac{-3}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{pmatrix}I_2$$

$$=X_0+X_1I_1+X_2I_2$$

We have:

$$X_0=\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, X_0+X_1+X_2=\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, X_0-X_1+X_2=\begin{pmatrix} 1 & 0 \\ -1 & 4 \end{pmatrix}$$

X_0 is diagonalizable with $X_0={P_0}^{-1}A_0P_0$, where $P_0=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_0=\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

$X_0+X_1+X_2$ is diagonalizable with $X_0+X_1+X_2={P_1}^{-1}A_1P_1$, where $P_1=\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, $A_1=\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

$X_0-X_1+X_2$ is diagonalizable with $X_0-X_1+X_2={P_2}^{-1}A_2P_2$, where $P_2=\begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$, $A_2=\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

According the previous theorem, we have.

$X=P^{-1}YP$, where:

$$Y=A_0+\frac{1}{2}I_1(A_1-A_2)+\frac{1}{2}I_2(A_1+A_2-2A_0)=\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}+\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-3}{2} \end{pmatrix}I_1+\begin{pmatrix} \frac{-3}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}I_2$$

$$=\begin{pmatrix} 3+\frac{1}{2}I_1-\frac{3}{2}I_2 & 0 \\ 0 & 2-\frac{3}{2}I_1+\frac{1}{2}I_2 \end{pmatrix}$$

$$P={P_0}^{-1}+\frac{1}{2}I_1\left[{P_1}^{-1}-{P_2}^{-1}\right]+\frac{1}{2}I_2\left[{P_1}^{-1}+{P_2}^{-1}-2{P_0}^{-1}\right]$$

$$=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+\begin{pmatrix} -1 & \frac{1}{2} \\ \frac{-1}{2} & -1 \end{pmatrix}I_1+\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & -1 \end{pmatrix}I_2=\begin{pmatrix} 1-I_1+I_2 & \frac{1}{2}I_1+\frac{1}{2}I_2 \\ -\frac{1}{2}I_1+\frac{1}{2}I_2 & 1-I_1-I_2 \end{pmatrix}$$

The Eigen Values.

Definition.

Let $A=A_0+A_1I_1+A_2I_2$ be an n-cyclic refined neutrosophic matrix, we say that $T=t_0+t_1I_1+t_2I_2\in R_2(I)$ is an eigen value if and only if $AX=tX$; $X=X_0+X_1I_1+x_2I_2$ is an n-cyclic refined neutrosophic vector, where $X_i\in R^n$.

X is called n-cyclic refined neutrosophic vector.

Theorem.

Let $A = A_0 + A_1 I_1 + A_2 I_2$ be an n-cyclic refined neutrosophic matrix, then $T = t_0 + t_1 I_1 + t_2 I_2 \in R_2(I)$ is an eigen value with $X = X_0 + X_1 I_1 + X_2 I_2$ as eigen vector if and only if:

t_0 is an eigen value of A_0 with X_0 as eigen vector, $t_0 + t_1 + t_2$ is an eigen value of $A_0 + A_1 + A_2$ with $X_0 + X_1 + X_2$ as eigen vector, $t_0 - t_1 + t_2$ is an eigen value of $A_0 - A_1 + A_2$ with $X_0 - X_1 + X_2$ as eigen vector.

Proof.

The equation $AX = tX$ is equivalent to:

$$A_0X_0 + \frac{1}{2}I_1[(A_0 + A_1 + A_2)(X_0 + X_1 + X_2) - (A_0 - A_1 + A_2)(X_0 - X_1 + X_2)]$$
$$+ \frac{1}{2}I_2[(A_0 + A_1 + A_2)(X_0 + X_1 + X_2) + (A_0 - A_1 + A_2)(X_0 - X_1 + X_2) - 2A_0X_0]$$
$$= t_0X_0 + \frac{1}{2}I_1[(t_0 + t_1 + t_2)(X_0 + X_1 + X_2) - (t_0 - t_1 + t_2)(X_0 - X_1 + X_2)]$$
$$+ \frac{1}{2}I_2[(t_0 + t_1 + t_2)(X_0 + X_1 + X_2) + (t_0 - t_1 + t_2)(X_0 - X_1 + X_2) - 2t_0X_0]$$

So that:

$$\begin{cases} t_0X_0 = A_0X_0 \ldots (1) \\ (t_0 + t_1 + t_2)(X_0 + X_1 + X_2) - (t_0 - t_1 + t_2)(X_0 - X_1 + X_2) = (A_0 + A_1 + A_2)(X_0 + X_1 + X_2) - (A_0 - A_1 + A \\ (t_0 + t_1 + t_2)(X_0 + X_1 + X_2) - (t_0 - t_1 + t_2)(X_0 - X_1 + X_2) = (A_0 + A_1 + A_2)(X_0 + X_1 + X_2) - (A_0 - A_1 + A \end{cases}$$

This equivalents:

$$\begin{cases} A_0X_0 = t_0X_0 \\ (A_0 + A_1 + A_2)(X_0 + X_1 + X_2) = (t_0 + t_1 + t_2)(X_0 + X_1 + X_2) \\ (A_0 - A_1 + A_2)(X_0 - X_1 + X_2) = (t_0 - t_1 + t_2)(X_0 - X_1 + X_2) \end{cases}$$

Thus, the proof is complete.

Example.

Consider the matrix:

$$A = \begin{pmatrix} 3 + \frac{1}{2}I_1 - \frac{3}{2}I_2 & \frac{1}{2}I_1 + \frac{1}{2}I_2 \\ \frac{1}{2}I_1 - \frac{1}{2}I_2 & 2 - \frac{3}{2}I_1 + \frac{1}{2}I_2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-3}{2} \end{pmatrix} I_1 + \begin{pmatrix} \frac{-3}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{pmatrix} I_2$$
$$= A_0 + A_1I_1 + A_2I_2$$

The eigen values of A_0 are $\{3,2\}$.

The eigen values of $A_0 + A_1 + A_2$ are $\{2,1\}$.

The eigen values of $A_0 - A_1 + A_2$ are $\{1,4\}$.

To find the corresponding 2×2 2-cyclic refined neutrosophic matrix A, we discuss the following cases:

Case(1). If $t_0 = 3,\ t_0 + t_1 + t_2 = 2,\ t_0 - t_1 + t_2 = 1$, then:

$t_1 = \frac{1}{2}\ ,\ t_2 = \frac{-3}{2}$, thus $T_1 = 3 + \frac{1}{2}I_1 - \frac{3}{2}I_2$.

Case(2). If $t_0 = 3,\ t_0 + t_1 + t_2 = 2,\ t_0 - t_1 + t_2 = 4$, then:

$t_1 = -1\ ,\ t_2 = 0$, thus $T_2 = 3 - I_1$.

Case(3). If $t_0 = 3,\ t_0 + t_1 + t_2 = 1,\ t_0 - t_1 + t_2 = 1$, then:

$t_1 = 0\ ,\ t_2 = -2$, thus $T_3 = 3 - 2I_2$.

Case(4). If $t_0 = 3,\ t_0 + t_1 + t_2 = 1,\ t_0 - t_1 + t_2 = 4$, then:

$t_1 = \frac{-3}{2}\ ,\ t_2 = \frac{-1}{2}$, thus $T_4 = 3 - \frac{3}{2}I_1 - \frac{1}{2}I_2$.

Case(5). If $t_0 = 2,\ t_0 + t_1 + t_2 = 2,\ t_0 - t_1 + t_2 = 1$, then:

$t_1 = \frac{1}{2}\ ,\ t_2 = \frac{-1}{4}$, thus $T_5 = 3 + \frac{1}{2}I_1 - \frac{1}{4}I_2$.

Case(6). If $t_0 = 2,\ t_0 + t_1 + t_2 = 2,\ t_0 - t_1 + t_2 = 4$, then:

$t_1 = -1\ ,\ t_2 = 1$, thus $T_6 = 3 - I_1 + I_2$.

Case(7). If $t_0 = 2,\ t_0 + t_1 + t_2 = 1,\ t_0 - t_1 + t_2 = 1$, then:

$t_1 = 0\ ,\ t_2 = -1$, thus $T_7 = 3 - I_2$.

Case(8). If $t_0 = 2,\ t_0 + t_1 + t_2 = 1,\ t_0 - t_1 + t_2 = 4$, then:

$t_1 = \frac{-3}{2}\ ,\ t_2 = \frac{1}{2}$, thus $T_8 = 3 - \frac{3}{2}I_1 + \frac{1}{2}I_2$.

This implies that A has 8 eigen values.

The determinant of an n-cyclic refined neutrosophic matrix.

According to the previous discussion, we have found an algorithm to compute n-cyclic refined neutrosophic matrix.

From the point of view, we are forced to study the computing of eigen values by determinants.

Definition.

Let $A = A_0 + A_1 I_1 + A_2 I_2$ be an n-cyclic refined neutrosophic matrix, we define its determinant as follows:

$\det A = \det A_0 + \frac{1}{2}I_1[\det(A_0 + A_1 + A_2) - \det(A_0 - A_1 + A_2)] + \frac{1}{2}I_2[\det(A_0 + A_1 + A_2) + \det(A_0 - A_1 + A_2) - 2\det A_0]$.

Theorem.

Let $A = A_0 + A_1 I_1 + A_2 I_2, B = B_0 + B_1 I_1 + B_2 I_2$ be two $n \times n$ n-cyclic refined neutrosophic matrices, then:

1). A is invertible if and only if $\det A$ is invertible.

2). $\det A^T = \det A$.

3). $\det(A.B) = \det A . \det B$.

4). $T = t_0 + t_1 I_1 + t_2 I_2$ is an eigen of A if and only if $\det(A - TU_{n\times n}) = 0$.

Proof.

1). It is clear and easy.

2). $A^T = {A_0}^T + {A_1}^T I_1 + {A_2}^T I_2$, thus:

$$\det A^T = \det {A_0}^T + \frac{1}{2} I_1 [\det(A_0 + A_1 + A_2)^T - \det(A_0 - A_1 + A_2)^T]$$
$$+ \frac{1}{2} I_2 \left[\det(A_0 + A_1 + A_2)^T = \det(A_0 - A_1 + A_2)^T - 2\det {A_0}^T\right] = \det A$$

3). $A.B = A_0 B_0 + \frac{1}{2} I_1 [(A_0 + A_1 + A_2)(B_0 + B_1 + B_2) - (A_0 - A_1 + A_2)(B_0 - B_1 + B_2)] + \frac{1}{2} I_2 [(A_0 + A_1 + A_2)(B_0 + B_1 + B_2) + (A_0 - A_1 + A_2)(B_0 - B_1 + B_2) - 2A_0 B_0] = A_0 B_0 + \frac{1}{2} I_1 (T_1 - T_2) + \frac{1}{2} I_2 (T_1 + T_2 - 2A_0 B_0)$, where:

$T_1 = (A_0 + A_1 + A_2)(B_0 + B_1 + B_2), T_2 = (A_0 - A_1 + A_2)(B_0 - B_1 + B_2)$

$$\det(A.B) = \det A_0 B_0$$
$$+ \frac{1}{2} I_1 \left[\det\left(\frac{1}{2}T_1 - \frac{1}{2}T_2 + \frac{1}{2}T_1 + \frac{1}{2}T_2 - A_0 B_0 + A_0 B_0\right)\right.$$
$$\left. - \det\left(A_0 B_0 - \frac{1}{2}T_1 + \frac{1}{2}T_2 + \frac{1}{2}T_1 + \frac{1}{2}T_2 - A_0 B_0\right)\right]$$
$$+ \frac{1}{2} I_2 \left[\det\left(\frac{1}{2}T_1 - \frac{1}{2}T_2 + \frac{1}{2}T_1 + \frac{1}{2}T_2 - A_0 B_0 + A_0 B_0\right)\right.$$
$$\left. - \det\left(A_0 B_0 - \frac{1}{2}T_1 + \frac{1}{2}T_2 + \frac{1}{2}T_1 + \frac{1}{2}T_2 - A_0 B_0\right) - 2\det A_0 B_0\right]$$
$$= \det A_0 \det B_0 + \frac{1}{2} I_1 [\det T_1 - \det T_2] + \frac{1}{2} I_2 [\det T_1 + \det T_2 - 2\det A_0 \det B_0]$$
$$= \det A_0 \det B_0$$
$$+ \frac{1}{2} I_1 [\det(A_0 + A_1 + A_2) . \det(B_0 + B_1 + B_2)$$
$$+ \det(A_0 - A_1 + A_2) . \det(B_0 - B_1 + B_2)]$$
$$+ \frac{1}{2} I_2 [\det(A_0 + A_1 + A_2) . \det(B_0 + B_1 + B_2)$$
$$- \det(A_0 - A_1 + A_2) . \det(B_0 - B_1 + B_2) - 2\det A_0 \det B_0] = \det A \det B$$

4). We have $A - TU_{n\times n} = (A_0 + A_1 I_1 + A_2 I_2) - (t_0 + t_1 I_1 + t_2 I_2)U_{n\times n} = (A_0 - t_0 U_{n\times n}) + (A_1 - t_1 U_{n\times n})I_1 + (A_2 - t_2 U_{n\times n})I_2$.

$$\begin{aligned}\det(A - TU_{n\times n}) &\\ &= \det(A_0 - t_0 U_{n\times n})\\ &+ \frac{1}{2} I_1[\det(A_0 + A_1 + A_2 - (t_0 + t_1 + t_2)U_{n\times n})\\ &- \det(A_0 - A_1 + A_2 - (t_0 - t_1 + t_2)U_{n\times n})]\\ &+ \frac{1}{2} I_2[\det(A_0 + A_1 + A_2 - (t_0 + t_1 + t_2)U_{n\times n})\\ &+ \det(A_0 - A_1 + A_2 - (t_0 - t_1 + t_2)U_{n\times n}) - 2\det(A_0 - t_0 U_{n\times n})]\end{aligned}$$

The equation $\det(A - TU_{n\times n}) = 0$ is equivalent to:

$$\begin{cases} \det(A_0 - t_0 U_{n\times n}) = 0 \\ \det(A_0 + A_1 + A_2 - (t_0 + t_1 + t_2)U_{n\times n}) = 0 \\ \det(A_0 - A_1 + A_2 - (t_0 - t_1 + t_2)U_{n\times n}) = 0 \end{cases}$$

This is equivalent to:

To is eigen value of A_0, $t_0 + t_1 + t_2$ is eigen value of $A_0 + A_1 + A_2$, $t_0 - t_1 + t_2$ is eigen value of $A_0 - A_1 + A_2$, thus T is an eigen value of A.

Theorem.

Let $A = A_0 + A_1 I_1 + A_2 I_2, B = B_0 + B_1 I_1 + B_2 I_2$ be two $n \times n$ n-cyclic refined neutrosophic matrices, then:

$$\begin{aligned}A.B = A_0 B_0 &+ \frac{1}{2} I_1[(A_0 + A_1 + A_2)(B_0 + B_1 + B_2) - (A_0 - A_1 + A_2)(B_0 - B_1 + B_2)]\\ &+ \frac{1}{2} I_2[(A_0 + A_1 + A_2)(B_0 + B_1 + B_2) + (A_0 - A_1 + A_2)(B_0 - B_1 + B_2)\\ &- 2A_0 B_0]\end{aligned}$$

The proof is easy and clear.

Example.

Consider the following 2×2 2-cyclic refined neutrosophic matrix:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} I_1 + \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} I_2 = \begin{pmatrix} 1 + 2I_1 + I_2 & 1 + I_1 \\ I_1 + 3I_2 & 2 + I_1 + I_2 \end{pmatrix} = A_0 + A_1 I_1 + A_2 I_2$$

${A_0}^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix}$, $(A_0 + A_1 + A_2) = \begin{pmatrix} 4 & 3 \\ 4 & 4 \end{pmatrix}$, $(A_0 + A_1 + A_2)^{-1} = \frac{1}{4}\begin{pmatrix} 4 & -3 \\ -4 & 4 \end{pmatrix}$, $(A_0 - A_1 + A_2) = \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}$,$(A_0 - A_1 + A_2)^{-1} = -\frac{1}{2}\begin{pmatrix} 2 & -1 \\ -2 & 0 \end{pmatrix}$

$$A^{-1}=\frac{1}{2}\begin{pmatrix}2 & -2\\ 0 & 1\end{pmatrix}+\frac{1}{2}I_1\left[\frac{1}{4}\begin{pmatrix}4 & -3\\ -4 & 4\end{pmatrix}-\frac{-1}{2}\begin{pmatrix}2 & -1\\ -2 & 0\end{pmatrix}\right]$$
$$+\frac{1}{2}I_2\left[\frac{1}{4}\begin{pmatrix}4 & -3\\ -4 & 4\end{pmatrix}+\frac{-1}{2}\begin{pmatrix}2 & -1\\ -2 & 0\end{pmatrix}-2\left(\frac{1}{2}\right)\begin{pmatrix}2 & -2\\ 0 & 1\end{pmatrix}\right]$$
$$=\begin{pmatrix}1 & -1\\ 0 & \frac{1}{2}\end{pmatrix}+\frac{1}{2}I_1\left[\begin{pmatrix}1 & -\frac{3}{4}\\ -1 & 1\end{pmatrix}+\begin{pmatrix}1 & -\frac{1}{2}\\ -1 & 0\end{pmatrix}\right]$$
$$+\frac{1}{2}I_2\left[\begin{pmatrix}1 & -\frac{3}{4}\\ -1 & 1\end{pmatrix}+\begin{pmatrix}-1 & \frac{1}{2}\\ 1 & 0\end{pmatrix}-\begin{pmatrix}2 & -2\\ 0 & 1\end{pmatrix}\right]$$
$$=\begin{pmatrix}1 & -1\\ 0 & \frac{1}{2}\end{pmatrix}+\frac{1}{2}I_1\begin{pmatrix}2 & -\frac{5}{4}\\ -2 & 1\end{pmatrix}+\frac{1}{2}I_2\begin{pmatrix}-2 & \frac{7}{4}\\ 0 & 0\end{pmatrix}$$
$$=\begin{pmatrix}1+I_1-I_2 & -1-\frac{5}{8}I_1+\frac{7}{8}I_2\\ -I_1 & \frac{1}{2}+\frac{1}{2}I_1\end{pmatrix}$$

Theorem.

Let $X=X_0+X_1I_1+X_2I_2$ be a 2-cyclic refined neutrosophic matrix, then X is invertible if and only if $X_0, X_0+X_1+X_2, X_0-X_1+X_2$ are invertible, also:

$$X^{-1}={X_0}^{-1}+\frac{1}{2}I_1[(X_0+X_1+X_2)^{-1}-(X_0-X_1+X_2)^{-1}]$$
$$+\frac{1}{2}I_2[(X_0+X_1+X_2)^{-1}+(X_0-X_1+X_2)^{-1}-2X_0]$$

Proof.

Assume that X is invertible, the exists $Y=Y_0+Y_1I_1+Y_2I_2$ such that $X.Y=U_{n\times n}$.

$$X.Y=X_0Y_0+I_1[X_0Y_1+X_1Y_0+X_2Y_1+X_1Y_2]+I_2[X_0Y_2+X_2Y_0+X_1Y_1+X_2Y_2]$$
$$=X_0Y_0+\frac{1}{2}I_1[(X_0+X_1+X_2)(Y_0+Y_1+Y_2)-(X_0-X_1+X_2)(Y_0-Y_1+Y_2)]$$
$$+\frac{1}{2}I_2[(X_0+X_1+X_2)(Y_0+Y_1+Y_2)+(X_0-X_1+X_2)(Y_0-Y_1+Y_2)-2X_0Y_0]$$
$$=U_{n\times n}$$

This implies that:

$$\begin{cases} X_0Y_0=U_{n\times n} \\ (X_0+X_1+X_2)(Y_0+Y_1+Y_2)=(X_0-X_1+X_2)(Y_0-Y_1+Y_2)=U_{n\times n}\end{cases}$$

Hence $X_0, X_0+X_1+X_2, X_0-X_1+X_2$ are invertible.

On the other hand, we get $Y_0={X_0}^{-1}$, $Y_0-Y_1+Y_2=(X_0-X_1+X_2)^{-1}$, $Y_0+Y_1+Y_2=(X_0+X_1+X_2)^{-1}$, thus:

$$Y_1=\frac{1}{2}[(X_0+X_1+X_2)^{-1}-(X_0-X_1+X_2)^{-1}]$$

$Y_2 = \frac{1}{2}\left[(X_0 + X_1 + X_2)^{-1} + (X_0 - X_1 + X_2)^{-1} - 2{X_0}^{-1}\right].$

Conclusion

In this paper, we have presented a full solution of the diagonalization problem of 2-cyclic refined neutrosophic matrices, where we have presented a novel algorithm to compute the eigen values and vectors of 2-cyclic refined neutrosophic matrices that helps in representing them as a product $A^{-1}DA$, where A is an invertible matrix, and D is diagonal matrix.

In the future, we suggest researchers to continue our efforts, and to study the possibility of diagonalization problem of 3-cyclic refined neutrosophic matrices.

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