Neutrosophic Sets and Systems, Vol. xx, 20xx
University of New Mexico

A Development of Pythagorean fuzzy hypersoft set with basic operations and decision-making approach based on the correlation coefficient

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Abstract: The idea of the Pythagorean fuzzy hypersoft set is a generalization of the intuitionistic fuzzy hypersoft set, which is used to express insufficient evaluation, uncertainty, and anxiety in decision-making. Compared with the intuitionistic fuzzy hypersoft set, the Pythagorean fuzzy hypersoft set can accommodate more uncertainty, which is the most important strategy for analyzing fuzzy information in the decision-making process. The most important determination of the present research is to perform basic operations under the Pythagorean fuzzy hypersoft set (PFHSS) with their mandatory properties. In it, we establish logical operators and propose the idea of necessity and possibility operations under PFHSS. In the following research under PFHSS, some desirable properties are proposed by using the proposed operations. We also introduce the correlation coefficient under the PFHSS structure and develop an algorithm for decision-making by using the developed correlation coefficient. Furthermore, a case study on decision-making difficulties proves the application of the proposed algorithm. Finally, a comparative analysis with the advantages, effectiveness, flexibility, and numerous existing studies demonstrates this method's effectiveness.

Keywords: Hypersoft set, intuitionistic fuzzy set, Pythagorean fuzzy soft set, Pythagorean fuzzy hypersoft set, correlation coefficient.

1. Introduction

Imprecision performs a dynamic part in many facets of life (such as modeling, medicine, engineering, etc.). However, people have raised a general question, that is, how can we express and use the concept of uncertainty in mathematical modeling. Many researchers in the world have proposed and recommended different methods of using uncertainty theory. First, Zadeh stepped forward the theory of fuzzy set (FS) [1] to resolve the problem of uncertainty and ambiguity. In some cases, we need to investigate membership as a non-membership value to properly interpret objects that FS cannot handle. To overcome the above-mentioned issues, Atanasov proposed the idea of intuitionistic fuzzy sets (IFS) [2]. Researchers have also used several other theories, such as cubic intuitionistic fuzzy sets [3], interval value IFS [4], linguistic interval-valued IFS [5], etc. After carefully considering the above theories, the experts considered the essence, and the sum of its two membership values and non-membership values cannot exceed one.

Atanassov's intuitionistic fuzzy sets only deal with insufficient data due to membership and non-membership values, but IFS cannot deal with incompatible and imprecise information. Molodtsov [6] proposed a general mathematical tool to deal with uncertain, ambiguous, and uncertain matters, called soft set (SS). Maji et al. [7] extended the concept of SS and developed some operations with properties and used the established concepts for decision-making [8]. Maji et al. [9] proposed the concept of a fuzzy soft set (FSS) by combining FS and SS. They also proposed an Intuitionistic Fuzzy Soft Set (IFSS) with basic operations and properties [10]. Yang et al. [11] proposed the concept of interval-valued fuzzy soft sets with operations (IVFSS) and proved some important results by combining IVFS and SS, and they also used the developed concepts for decision-making. Jiang et al. [12] proposed the concept of interval-valued intuitionistic fuzzy soft sets (IVIFSS) by extending IVIFS. They also introduced the necessity and possibility operators of IVIFSS and their properties.

Garg and Arora [13] progressed the generalized version of the IFSS with weighted averaging and geometric aggregation operators and built a decision-making technique to resolve complications beneath an intuitionistic fuzzy environment. Garg [14] developed some improved score functions to analyze the ranking of the normal intuitionistic and interval-valued intuitionistic sets and established the new methodologies to solve multi-attribute decision making (MADM) problems. The idea of entropy measure and TOPSIS under the correlation coefficient (CC) has been developed by using complex q-rung orthopair fuzzy information and used the established strategies for decision making [15]. The authors [16] developed the aggregate operators by using dual hesitant fuzzy soft numbers and utilized the proposed operators to solve multi-criteria decision making (MCDM) problems. To measure the relationship among dual hesitant fuzzy soft set Arora and Garg [17] introduced the CC and developed a decision-making approach under the presented environment to solve the MCDM approach, they also used the proposed methodology for decision making, medical diagnoses, and pattern recognition. They also developed the operational laws and presented some prioritized aggregation operators under linguistic IFS environment [18] and extended the Maclaurin symmetric mean (MSM) operators to IFSS based on Archimedean T-conorm and T-norm [19].

As the above work is considered an environment where linear inequalities have been examined between membership degree (MD) as well as non-membership degree (NMD). However, if the decision-maker goes steady with object MD = 0.7 and NDM = 0.6, then $\mathbf{0.7+0.6 \nleq 1}$. We can see that, it cannot be handled by the above studied IFS theories. To overcome the above-mentioned limitations, Yager [20, 21] prolonged the IFS to Pythagorean fuzzy sets (PFSs) by modifying the condition $\boldsymbol{\mathcal{T}+\mathcal{J} \leq 1}$ to $\boldsymbol{\mathcal{T}^2+\mathcal{J}^2 \leq 1}$. Zhang and Xu [22] defined some operational laws and extended the TOPSIS technique to solve MCDM problems under PFSs environment. Many researchers used the TOPSIS method for medical diagnoses, pattern recognition, and decision-making, etc. to find the positive ideal alternative in different structures [23-31]. Wei and Lu [32] presented several Pythagorean fuzzy power aggregation operators with their properties and proposed the decision-making approaches to solving MADM problems based on developed operators. Wang and Li [33] proposed the Pythagorean fuzzy interaction operational laws and power Bonferroni mean operators. Then, they discussed some specific cases of established operators and considered their properties. Zhang [34] established a novel decision-making technique based on Pythagorean fuzzy numbers (PFNs) to solve multiple criteria group decision making (MCGDM) problems. He also developed the accuracy function for the ranking of PFNs and similarity measures under a PFSs environment with some desirable properties. Guleria and Bajaj [35] introduced a Pythagorean fuzzy soft matrix and its various possible types and binary operations with their properties. Further, they used the proposed Pythagorean fuzzy soft matrices for decision making by developing a new algorithm by using a choice matrix and weighted choice matrix. They also presented some noel information measures to solve MCDM problems [36]. Bajaj and Guleria [37] proposed the notion of object-oriented Pythagorean fuzzy soft matrix and the parameter-oriented Pythagorean fuzzy soft matrix has been utilized to outline an algorithm for the dimensionality reduction in the process of decision making. The authors developed the new (R, S)-norm discriminant measure of PFSs has been proposed along with its various properties and proposed a decision-making approach to solving MCDM problems [38].

Recently, Smarandache [39] extended the concept of soft sets to hypersoft sets (HSS) by replacing the one-parameter function F with a multi-parameter (sub-attribute) function defined on the Cartesian product of n different attributes. The established HSS is more flexible than soft sets and is more suitable for the decision-making environment. He also introduced the further extension of HSS, such as crisp HSS, fuzzy HSS, intuitionistic fuzzy HSS, neutrosophic HSS, and plithogenic HSS. Nowadays, HSS theory and its extensions are developing rapidly. Many researchers have developed different operators and properties based on HSS and its extensions [40-44]. Abdel-Basset [45] uses plithogenic set theory to resolve uncertain information and evaluate the financial performance of manufacturing. Then, they use VIKOR and TOPSIS methods to find the weight vector of financial ratios using the AHP method to achieve this goal. Abdel basset, etc. [46] proposed an effective combination of plithogenic aggregation operations and quality feature deployment methods. The advantage of this combination is that it can improve accuracy and thus evaluate decision-makers.

The following research is organized as follows: In Section 2, we review some basic definitions used in the following sequels, such as SS, FSS, IFS, IFSS, and IFHSS, etc. In Section 3, we propose some operations with their necessary properties such as union, intersection, restricted union, and extended intersection, etc. under PFHSS. We develop the AND operator, OR operator, necessity operation, and possibility operation with their several desirable properties in section 4. In section 5, the idea of correlation coefficient in PFHSS structure is introduced, and develop the decision-making technique based on the presented CC. We also used the developed approach to solve decision making problems in an uncertain environment. Furthermore, we use some existing techniques to present comparative studies between our proposed methods. Likewise, present the advantages, naivety, flexibility as well as effectiveness of the planned algorithms. We organized a brief discussion and a comparative analysis of the recommended approach and the existing techniques in section 6.

2. Preliminaries

In this section, we recollect some basic definitions which are helpful to build the structure of the following manuscript such as soft set, hypersoft set, fuzzy hypersoft set, and intuitionistic fuzzy hypersoft set.

Definition 2.1 [6]

Let $\mathcal{U}$ be the universal set and $\mathcal{E}$ be the set of attributes concerning $\mathcal{U}$. Let $\mathcal{P}(\mathcal{U})$ be the power set of $\mathcal{U}$ and $\mathcal{A} \subseteq \mathcal{E}$. A pair $(\mathcal{F}, \mathcal{A})$ is called a soft set over $\mathcal{U}$ and its mapping is given as

$$\mathcal{F}: \mathcal{A} \rightarrow \mathcal{P}(\mathcal{U})$$

It is also defined as:

$$(\mathcal{F}, \mathcal{A}) = \{\mathcal{F}(e) \in \mathcal{P}(\mathcal{U}): e \in \mathcal{E}, \mathcal{F}(e) = \emptyset \; if \; e \notin \mathcal{A}\}$$

Definition 2.2 [9]

$\mathcal{F}(\mathcal{U})$ be a collection of all fuzzy subsets over $\mathcal{U}$ and $\mathcal{E}$ be a set of attributes. Let $\mathcal{A} \subseteq \mathcal{E}$, then a pair $(\mathcal{F}, \mathcal{A})$ is called FSS over $\mathcal{U}$, where $\mathcal{F}$ is a mapping such as $\mathcal{F}: \mathcal{A} \rightarrow F(\mathcal{U})$.

Definition 2.3 [39]

Let $\mathcal{U}$ be a universe of discourse and $\mathcal{P}(\mathcal{U})$ be a power set of $\mathcal{U}$ and $k = \{k_1, k_2, k_3, \ldots, k_n\}$, $(n \geq 1)$ be a set of attributes and set K_i a set of corresponding sub-attributes of k_i respectively with $K_i \cap K_j = \varphi$ for $n \geq 1$ for each $i, j \in \{1,2,3 \ldots n\}$ and $i \neq j$. Assume $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathcal{A}} = \{a_{1h} \times a_{2k} \times \cdots \times a_{nl}\}$ be a collection of multi-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and α, β, and $\gamma \in \mathbb{N}$. Then the pair $(\mathcal{F}, K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathcal{A}})$ is said to be HSS over $\mathcal{U}$ and its mapping is defined as

$\mathcal{F}: K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathcal{A}} \rightarrow \mathcal{P}(\mathcal{U})$.

It is also defined as

$(\mathcal{F},\ \ddot{\mathcal{A}}) = \{\breve{a}, \mathcal{F}_{\ddot{\mathcal{A}}}(\breve{a}) : \breve{a} \in \ddot{\mathcal{A}},\ \mathcal{F}_{\ddot{\mathcal{A}}}(\breve{a}) \in \mathcal{P}(\mathcal{U})\}$

Definition 2.4 [2]

An IFS is an object of the form $\mathcal{A} = \{\langle(\delta_i, \sigma_{\mathcal{A}}(\delta_i), \tau_{\mathcal{A}}(\delta_i)) \mid \delta_i \in \mathcal{U}\rangle\}$ on a universe $\mathcal{U}$, where $\sigma_{\mathcal{A}}$ and $\tau_{\mathcal{A}}$: $\mathcal{U} \rightarrow [0,1]$ represents the degree of membership and non-membership respectively of any element $\delta_i \in \mathcal{U}$, to set $\mathcal{A}$ with the following condition $0 \leq \sigma_{\mathcal{A}}(\delta_i) + \tau_{\mathcal{A}}(\delta_i) \leq 1$.

Definition 2.5 [10]

A mapping $\mathcal{F}$: $\mathcal{A} \rightarrow F(\mathcal{U})$ is known as an IFSS and defined as $\mathcal{F}_{\delta_i}(e) = \{(\delta_i, \sigma_{\mathcal{A}}(\delta_i), \tau_{\mathcal{A}}(\delta_i)) \mid \delta_i \in \mathcal{U}\}$, where $\sigma_{\mathcal{A}}(\delta_i)$ and $\tau_{\mathcal{A}}(\delta_i)$ are the degree of acceptance and rejection respectively for all $\delta_i \in \mathcal{U}$ and $0 \leq \sigma_{\mathcal{A}}(\delta_i), \tau_{\mathcal{A}}(\delta_i),\ \sigma_{\mathcal{A}}(\delta_i) + \tau_{\mathcal{A}}(\delta_i) \leq 1$.

Definition 2.6 [39]

Let $\mathcal{U}$ be a universe of discourse and $\mathcal{P}(\mathcal{U})$ be a power set of $\mathcal{U}$ and $k = \{k_1,\ k_2,\ k_3, \ldots, k_n\}$, $(n \geq 1)$ be a set of attributes and set K_i a set of corresponding sub-attributes of k_i respectively with $K_i \cap K_j = \varphi$ for $n \geq 1$ for each $i,\ j \in \{1,2,3 \ldots n\}$ and $i \neq j$. "Assume $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathcal{A}} = \{a_{1h} \times a_{2k} \times \cdots \times a_{nl}\}$ be a collection of sub-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and α, β, and $\gamma \in \mathbb{N}$ and $\mathbb{F}^{\mathcal{U}}$ be a collection of all fuzzy subsets over $\mathcal{U}$. Then the pair $(\mathcal{F},\ K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathcal{A}})$ is said to be FHSS over $\mathcal{U}$ and its mapping is defined as

$\mathcal{F}$: $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathcal{A}} \rightarrow \mathbb{F}^{\mathcal{U}}$.

It is also defined as

$(\mathcal{F},\ \ddot{\mathcal{A}}) = \{(\breve{a}, \mathcal{F}_{\ddot{\mathcal{A}}}(\breve{a})) : \breve{a} \in \ddot{\mathcal{A}},\ \mathcal{F}_{\ddot{\mathcal{A}}}(\breve{a}) \in \mathbb{F}^{\mathcal{U}} \in [0,1]\}$

Example 2.7

Consider the universe of discourse $\mathcal{U} = \{\delta_1, \delta_2\}$ and $\mathfrak{L} = \{\ell_1 = Teaching\ methdology, \ell_2 = Subjects, \ell_3 = Classes\}$ be a collection of attributes with following their corresponding attribute values are given as teaching methodology = $L_1 = \{a_{11} = project\ base, a_{12} = class\ discussion\}$, Subjects = $L_2 = \{a_{21} = Mathematics, a_{22} = Computer\ Science, a_{23} = Statistics\}$, and Classes = $L_3 = \{a_{31} = Masters, a_{32} = Doctorol\}$. Let $\ddot{\mathcal{A}} = L_1 \times L_2 \times L_3$ be a set of attributes

$\ddot{\mathcal{A}} = L_1 \times L_2 \times L_3 = \{a_{11}, a_{12}\} \times \{a_{21}, a_{22}, a_{23}\} \times \{a_{31}, a_{32}\}$

$$= \begin{Bmatrix} (a_{11}, a_{21}, a_{31}), (a_{11}, a_{21}, a_{32}), (a_{11}, a_{22}, a_{31}), (a_{11}, a_{22}, a_{32}), (a_{11}, a_{23}, a_{31}), (a_{11}, a_{23}, a_{32}), \\ (a_{12}, a_{21}, a_{31}), (a_{12}, a_{21}, a_{32}), (a_{12}, a_{22}, a_{31}), (a_{12}, a_{22}, a_{32}), (a_{12}, a_{23}, a_{31}), (a_{12}, a_{23}, a_{32}), \end{Bmatrix}$$

$\ddot{\mathcal{A}} = \{\breve{a}_1, \breve{a}_2, \breve{a}_3, \breve{a}_4, \breve{a}_5, \breve{a}_6, \breve{a}_7, \breve{a}_8, \breve{a}_9, \breve{a}_{10}, \breve{a}_{11}, \breve{a}_{12}\}$

Then the FHSS over $\mathcal{U}$ is given as follows

$$(\mathcal{F},\ \ddot{\mathcal{A}}) = \begin{Bmatrix} (\breve{a}_1, (\delta_1, .6), (\delta_2, .3)), (\breve{a}_2, (\delta_1, .7), (\delta_2, .5)), (\breve{a}_3, (\delta_1, .8), (\delta_2, .3)), (\breve{a}_4, (\delta_1, .2), (\delta_2, .8)), \\ (\breve{a}_5, (\delta_1, .4), (\delta_2, .3)), (\breve{a}_6, (\delta_1, .2), (\delta_2, .5)), (\breve{a}_7, (\delta_1, .6), (\delta_2, .9)), (\breve{a}_8, (\delta_1, .2), (\delta_2, .3)), \\ (\breve{a}_9, (\delta_1, .4), (\delta_2, .7)), (\breve{a}_{10}, (\delta_1, .1), (\delta_2, .7)), (\breve{a}_{11}, (\delta_1, .4), (\delta_2, .6)), (\breve{a}_5, (\delta_1, .2), (\delta_2, .7)) \end{Bmatrix}$$

Definition 2.8 [44]

Let $\mathcal{U}$ be a universe of discourse and $\mathcal{P}(\mathcal{U})$ be a power set of $\mathcal{U}$ and $k = \{k_1,\ k_2,\ k_3, \ldots, k_n\}$, $(n \geq 1)$ be a set of attributes and set K_i a set of corresponding sub-attributes of k_i respectively with $K_i \cap K_j = \varphi$ for $n \geq 1$ for each $i,\ j \in \{1,2,3 \ldots n\}$ and $i \neq j$. Assume $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathcal{A}} = \{a_{1h} \times a_{2k} \times \cdots \times a_{nl}\}$ be a collection of sub-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and α, β, and $\gamma \in \mathbb{N}$ and $IFS^{\mathcal{U}}$ be a collection of all intuitionistic fuzzy subsets over $\mathcal{U}$. Then the pair $(\mathcal{F},\ K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathcal{A}})$ is said to be IFHSS over $\mathcal{U}$ and its mapping is defined as

$\mathcal{F}$: $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathcal{A}} \rightarrow IFS^{\mathcal{U}}$.

It is also defined as

$(\mathcal{F},\ \ddot{\mathcal{A}}) = \{(\breve{a}, \mathcal{F}_{\ddot{\mathcal{A}}}(\breve{a})) : \breve{a} \in \ddot{\mathcal{A}},\ \mathcal{F}_{\ddot{\mathcal{A}}}(\breve{a}) \in IFS^{\mathcal{U}} \in [0,1]\}$, where $\mathcal{F}_{\ddot{\mathcal{A}}}(\breve{a}) = \{\langle\delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\rangle : \delta \in \mathcal{U}\}$, where $\sigma_{\mathcal{F}(\breve{a})}(\delta)$ and $\tau_{\mathcal{F}(\breve{a})}(\delta)$ represents the membership and non-membership values of the attributes such as $\sigma_{\mathcal{F}(\breve{a})}(\delta),\ \tau_{\mathcal{F}(\breve{a})}(\delta) \in [0,1]$, and $0 \leq \sigma_{\mathcal{F}(\breve{a})}(\delta) + \tau_{\mathcal{F}(\breve{a})}(\delta) \leq 1$.

Simply an intuitionistic fuzzy hypersoft number (IFHSN) can be expressed as $\mathcal{F} = \left\{\left(\sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\right)\right\}$, where $0 \leq \sigma_{\mathcal{F}(\breve{a})}(\delta) + \tau_{\mathcal{F}(\breve{a})}(\delta) \leq 1$.

3. Basic Operations and properties on a Pythagorean fuzzy hypersoft set

In this section, we introduce PFHSS and some basic operations with their properties under the Pythagorean fuzzy hypersoft environment.

Definition 3.1

Let $\mathcal{U}$ be a universe of discourse and $\mathcal{P}(\mathcal{U})$ be a power set of $\mathcal{U}$ and $k = \{k_1, k_2, k_3, ..., k_n\}$, $(n \geq 1)$ be a set of attributes and set K_i a set of corresponding sub-attributes of k_i respectively with $K_i \cap K_j = \varphi$ for $n \geq 1$ for each $i, j \in \{1,2,3 \ldots n\}$ and $i \neq j$. Assume $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathcal{A}} = \{a_{1h} \times a_{2k} \times \cdots \times a_{nl}\}$ be a collection of sub-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and α, β, and $\gamma \in \mathbb{N}$ and $PFS^{\mathcal{U}}$ be a collection of all Pythagorean fuzzy subsets over $\mathcal{U}$. Then the pair $(\mathcal{F}, K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathcal{A}})$ is said to be PFHSS over $\mathcal{U}$ and its mapping is defined as
$\mathcal{F}: K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathcal{A}} \rightarrow PFS^{\mathcal{U}}$.

It is also defined as
$(\mathcal{F}, \ddot{\mathcal{A}}) = \left\{\left(\breve{a}, \mathcal{F}_{\ddot{\mathcal{A}}}(\breve{a})\right): \breve{a} \in \ddot{\mathcal{A}}, \mathcal{F}_{\ddot{\mathcal{A}}}(\breve{a}) \in PFS^{\mathcal{U}} \in [0,1]\right\}$, where $\mathcal{F}_{\ddot{\mathcal{A}}}(\breve{a}) = \left\{\langle\delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\rangle: \delta \in \mathcal{U}\right\}$, where $\sigma_{\mathcal{F}(\breve{a})}(\delta)$ and $\tau_{\mathcal{F}(\breve{a})}(\delta)$ represents the membership and non-membership values of the attributes such as $\sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta) \in [0,1]$, and $0 \leq \left(\sigma_{\mathcal{F}(\breve{a})}(\delta)\right)^2 + \left(\tau_{\mathcal{F}(\breve{a})}(\delta)\right)^2 \leq 1$.

Simply a Pythagorean fuzzy hypersoft number (PFHSN) can be expressed as $\mathcal{F} = \left\{\left(\sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\right)\right\}$, where $0 \leq \left(\sigma_{\mathcal{F}(\breve{a})}(\delta)\right)^2 + \left(\tau_{\mathcal{F}(\breve{a})}(\delta)\right)^2 \leq 1$.

Example 3.2

Consider the universe of discourse $\mathcal{U} = \{\delta_1, \delta_2\}$ and $\mathfrak{L} = \{\ell_1 = Teaching\ methdology, \ell_2 = Subjects, \ell_3 = Classes\}$ be a collection of attributes with following their corresponding attribute values are given as teaching methodology = $L_1 = \{a_{11} = project\ base, a_{12} = class\ discussion\}$, Subjects = $L_2 = \{a_{21} = Mathematics, a_{22} = Computer\ Science, a_{23} = Statistics\}$", and Classes = $L_3 = \{a_{31} = Masters, a_{32} = Doctorol\}$. Let $\ddot{\mathcal{A}} = L_1 \times L_2 \times L_3$ be a set of attributes

$\ddot{\mathcal{A}} = L_1 \times L_2 \times L_3 = \{a_{11}, a_{12}\} \times \{a_{21}, a_{22}, a_{23}\} \times \{a_{31}, a_{32}\}$

$$= \left\{\begin{matrix} (a_{11}, a_{21}, a_{31}), (a_{11}, a_{21}, a_{32}), (a_{11}, a_{22}, a_{31}), (a_{11}, a_{22}, a_{32}), (a_{11}, a_{23}, a_{31}), (a_{11}, a_{23}, a_{32}), \\ (a_{12}, a_{21}, a_{31}), (a_{12}, a_{21}, a_{32}), (a_{12}, a_{22}, a_{31}), (a_{12}, a_{22}, a_{32}), (a_{12}, a_{23}, a_{31}), (a_{12}, a_{23}, a_{32}), \end{matrix}\right\}$$

$\ddot{\mathcal{A}} = \{\breve{a}_1, \breve{a}_2, \breve{a}_3, \breve{a}_4, \breve{a}_5, \breve{a}_6, \breve{a}_7, \breve{a}_8, \breve{a}_9, \breve{a}_{10}, \breve{a}_{11}, \breve{a}_{12}\}$

Then the PFHSS over $\mathcal{U}$ is given as follows

$(\mathcal{F}, \ddot{\mathcal{A}}) =$

$$\left\{\begin{matrix} \left(\breve{a}_1, (\delta_1, (.6, .3)), (\delta_2, (.5, .7))\right), \left(\breve{a}_2, (\delta_1, (.6, .7)), (\delta_2, (.7, .5))\right), \left(\breve{a}_3, (\delta_1, (.4, .8)), (\delta_2, (.3, .7))\right), \\ \left(\breve{a}_4, \left(\delta_1, (.6, .5)\right), \left(\delta_2, (.5, .6)\right)\right), \left(\breve{a}_5, \left(\delta_1, (.7, .3)\right), \left(\delta_2, (.4, .8)\right)\right), \left(\breve{a}_6, \left(\delta_1, (.5, .4)\right), \left(\delta_2, (.6, .5)\right)\right), \\ \left(\breve{a}_7, (\delta_1, (.5, .6)), (\delta_2, (.4, .5))\right), \left(\breve{a}_8, (\delta_1, (.2, .5)), (\delta_2, (.3, .9))\right), \left(\breve{a}_9, (\delta_1, (.4, .6)), (\delta_2, (.8, .5))\right), \\ \left(\breve{a}_{10}, (\delta_1, (.7, .4)), (\delta_2, (.7, .2))\right), \left(\breve{a}_{11}, (\delta_1, (.4, .5)), (\delta_2, (.5, .3))\right), \left(\breve{a}_{12}, (\delta_1, (.5, .7)), (\delta_2, (.4, .7))\right) \end{matrix}\right\}$$

Definition 3.3

Let $(\mathcal{F}, \ddot{\mathcal{A}})$ and $(\mathcal{G}, \ddot{\mathfrak{B}})$ be two PFHSS over $\mathcal{U}$, then $(\mathcal{F}, \ddot{\mathcal{A}})$ is said to be a Pythagorean fuzzy hypersoft subset of $(\mathcal{G}, \ddot{\mathfrak{B}})$, if

1. $\ddot{\mathcal{A}} \subseteq \ddot{\mathfrak{B}}$
2. $\mathcal{F}_{\ddot{\mathcal{A}}(\breve{a})}(\delta) \subseteq \mathcal{G}_{\ddot{\mathfrak{B}}(\breve{a})}(\delta)$ for all $\delta \in \mathcal{U}$.

Where $\sigma_{\mathcal{F}(\breve{a})}(\delta) \le \sigma_{\mathcal{G}(\breve{a})}(\delta)$, and $\tau_{\mathcal{F}(\breve{a})}(\delta) \ge \tau_{\mathcal{G}(\breve{a})}(\delta)$: $\delta \in \mathcal{U}$.

Definition 3.4

Let $(\mathcal{F}, \ddot{\mathcal{A}})$ and $(\mathcal{G}, \ddot{\mathfrak{B}})$ be two PFHSS over $\mathcal{U}$, then $(\mathcal{F}, \ddot{\mathcal{A}}) = (\mathcal{G}, \ddot{\mathfrak{B}})$, if $(\mathcal{F}, \ddot{\mathcal{A}}) \subseteq (\mathcal{G}, \ddot{\mathfrak{B}})$ and $(\mathcal{G}, \ddot{\mathfrak{B}}) \subseteq (\mathcal{F}, \ddot{\mathcal{A}})$.

Definition 3.5

Let $\mathcal{U}$ be a universe of discourse and $\ddot{\mathcal{A}}$ be a set of attributes, then a pair $(\emptyset, \ddot{\mathcal{A}})$ is said to be empty PFHSS, if $\sigma_{\mathcal{F}(\breve{a})}(\delta) = 0$, and $\tau_{\mathcal{F}(\breve{a})}(\delta) = 1$ for all $\breve{a} \in \ddot{\mathcal{A}}$ and $\delta \in \mathcal{U}$. It can be represented by $\emptyset_{\mathcal{F}(\breve{a})}(\delta)$ and defined as follows

$(\emptyset, \ddot{\mathcal{A}}) = \{\breve{a}, (\delta, (0,1)) : \delta \in \mathcal{U}, \breve{a} \in \ddot{\mathcal{A}}\}$.

Definition 3.6

Let $\mathcal{U}$ be a universe of discourse and $\ddot{\mathcal{A}}$ be a set of attributes, then a pair $(\emptyset, \ddot{\mathcal{A}})$ is said to be universal PFHSS, if $\sigma_{\mathcal{F}(\breve{a})}(\delta) = 1$, and $\tau_{\mathcal{F}(\breve{a})}(\delta) = 0$ for all $\breve{a} \in \ddot{\mathcal{A}}$ and $\delta \in \mathcal{U}$. It can be represented by $\mathbb{E}_{\mathcal{F}(\breve{a})}(\delta)$ and defined as follows

$(\mathbb{E}, \ddot{\mathcal{A}}) = \{\breve{a}, (\delta, (1,0)) : \delta \in \mathcal{U}, \breve{a} \in \ddot{\mathcal{A}}\}$.

Definition 3.7

Let $(\mathcal{F}, \ddot{\mathcal{A}}) = \{(\breve{a}, \langle \delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\rangle : \delta \in \mathcal{U}) : \breve{a} \in \ddot{\mathcal{A}}\}$ be a PFHSS over $\mathcal{U}$, then its complement is denoted by $(\mathcal{F}, \ddot{\mathcal{A}})^c$, and is defined as follows $(\mathcal{F}, \ddot{\mathcal{A}})^c = \{(\breve{a}, \langle \delta, \tau_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{F}(\breve{a})}(\delta)\rangle : \delta \in \mathcal{U}) : \breve{a} \in \ddot{\mathcal{A}}\}$.

Proposition 3.8

If $(\mathcal{F}, \ddot{\mathcal{A}})$ be a PFHSS, then

1. $(\mathcal{F}^c, \ddot{\mathcal{A}})^c = (\mathcal{F}, \ddot{\mathcal{A}})$
2. $(\emptyset^c, \ddot{\mathcal{A}}) = (\mathbb{E}, \ddot{\mathcal{A}})$
3. $(\mathbb{E}^c, \ddot{\mathcal{A}}) = (\emptyset, \ddot{\mathcal{A}})$

Proof 1

Let $(\mathcal{F}, \ddot{\mathcal{A}}) = \{(\breve{a}, \langle \delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\rangle : \delta \in \mathcal{U}) : \breve{a} \in \ddot{\mathcal{A}}\}$ be a PFHSS over $\mathcal{U}$, then by using definition 3.7. we have

$(\mathcal{F}^c, \ddot{\mathcal{A}}) = \{(\breve{a}, \langle \delta, \tau_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{F}(\breve{a})}(\delta)\rangle : \delta \in \mathcal{U}) : \breve{a} \in \ddot{\mathcal{A}}\}$. Again, "by using definition 3.7

$(\mathcal{F}^c, \ddot{\mathcal{A}})^c = \{(\breve{a}, \langle \delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\rangle : \delta \in \mathcal{U}) : \breve{a} \in \ddot{\mathcal{A}}\}$

Hence,

$(\mathcal{F}^c, \ddot{\mathcal{A}})^c = (\mathcal{F}, \ddot{\mathcal{A}})$.

Similarly, we can prove 2 and 3.

Definition 3.9

Let $(\mathcal{F}, \ddot{\mathcal{A}})$ and $(\mathcal{G}, \ddot{\mathfrak{B}})$ be two PFHSS over $\mathcal{U}$, then their union is defined as

$(\mathcal{F}, \ddot{\mathcal{A}}) \cup (\mathcal{G}, \ddot{\mathfrak{B}}) = \{\delta, (max\{\sigma_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{G}(\breve{a})}(\delta)\}, min\{\tau_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\}) : \delta \in \mathcal{U}, \breve{a} \in \ddot{\mathcal{A}}\}$.

Proposition 3.10

Let $(\mathcal{F}, \ddot{\mathcal{A}})$, $(\mathcal{G}, \ddot{\mathfrak{B}})$, and $(\mathcal{H}, \ddot{\mathrm{C}})$ be three PFHSS over $\mathcal{U}$. Then

1. $(\mathcal{F}, \ddot{\mathcal{A}}) \cup (\mathcal{F}, \ddot{\mathcal{A}}) = (\mathcal{F}, \ddot{\mathcal{A}})$
2. $(\mathcal{F}, \ddot{\mathcal{A}}) \cup (\emptyset, \ddot{\mathcal{A}}) = (\mathcal{F}, \ddot{\mathcal{A}})$
3. $(\mathcal{F}, \ddot{\mathcal{A}}) \cup (\mathbb{E}, \ddot{\mathcal{A}}) = (\mathbb{E}, \ddot{\mathcal{A}})$
4. $(\mathcal{F}, \ddot{\mathcal{A}}) \cup (\mathcal{G}, \ddot{\mathfrak{B}}) = (\mathcal{G}, \ddot{\mathfrak{B}}) \cup (\mathcal{F}, \ddot{\mathcal{A}})$
5. $((\mathcal{F}, \ddot{\mathcal{A}}) \cup (\mathcal{G}, \ddot{\mathfrak{B}})) \cup (\mathcal{H}, \ddot{\mathrm{C}}) = (\mathcal{F}, \ddot{\mathcal{A}}) \cup ((\mathcal{G}, \ddot{\mathfrak{B}}) \cup (\mathcal{H}, \ddot{\mathrm{C}}))$

Proof 1 As we know that

$(\mathcal{F}, \ddot{\mathcal{A}}) = \left\{\left(\delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\right\}$ be an PFHSS, then

$(\mathcal{F}, \ddot{\mathcal{A}}) \cup (\mathcal{F}, \ddot{\mathcal{A}}) = \{\delta, (max\{\sigma_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{F}(\breve{a})}(\delta)\}, min\{\tau_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\}): \delta \in \mathcal{U}, \breve{a} \in \ddot{\mathcal{A}}\}$

$(\mathcal{F}, \ddot{\mathcal{A}}) \cup (\mathcal{F}, \ddot{\mathcal{A}}) = \left\{\left(\delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\right\}$

Hence

$(\mathcal{F}, \ddot{\mathcal{A}}) \cup (\mathcal{F}, \ddot{\mathcal{A}}) = (\mathcal{F}, \ddot{\mathcal{A}})$.

Proof 2 As we know that

$(\mathcal{F}, \ddot{\mathcal{A}}) = \left\{\left(\delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\right\}$ be a PFHSS, and $(\emptyset, \ddot{\mathcal{A}}) = \{\breve{a}, (\delta, (0, 1)): \delta \in \mathcal{U}, \breve{a} \in \ddot{\mathcal{A}}\}$ be an empty PFHSS. Then,

$(\mathcal{F}, \ddot{\mathcal{A}}) \cup (\emptyset, \ddot{\mathcal{A}}) = \{\delta, (max\{\sigma_{\mathcal{F}(\breve{a})}(\delta), 0\}, min\{\tau_{\mathcal{F}(\breve{a})}(\delta), 1\}): \delta \in \mathcal{U}, \breve{a} \in \ddot{\mathcal{A}}\}$

$(\mathcal{F}, \ddot{\mathcal{A}}) \cup (\emptyset, \ddot{\mathcal{A}}) = \left\{\left(\delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\right\}$.

$(\mathcal{F}, \ddot{\mathcal{A}}) \cup (\emptyset, \ddot{\mathcal{A}}) = (\mathcal{F}, \ddot{\mathcal{A}})$.

Proof 3 As we know that

$(\mathcal{F}, \ddot{\mathcal{A}}) = \left\{\left(\delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\right\}$ be a PFHSS, and $(\mathbb{E}, \ddot{\mathcal{A}}) = \{\breve{a}, (\delta, (1, 0)): \delta \in \mathcal{U}, \breve{a} \in \ddot{\mathcal{A}}\}$ be an empty PFHSS. Then,

$(\mathcal{F}, \ddot{\mathcal{A}}) \cup (\mathbb{E}, \ddot{\mathcal{A}}) = \{\delta, (max\{\sigma_{\mathcal{F}(\breve{a})}(\delta), 1\}, min\{\tau_{\mathcal{F}(\breve{a})}(\delta), 0\}): \delta \in \mathcal{U}, \breve{a} \in \ddot{\mathcal{A}}\}$

$(\mathcal{F}, \ddot{\mathcal{A}}) \cup (\mathbb{E}, \ddot{\mathcal{A}}) = \{\breve{a}, (\delta, (1, 0)): \delta \in \mathcal{U}, \breve{a} \in \ddot{\mathcal{A}}\}$.

$(\mathcal{F}, \ddot{\mathcal{A}}) \cup (\mathbb{E}, \ddot{\mathcal{A}}) = (\mathbb{E}, \ddot{\mathcal{A}})$.

Proof 4 As

$(\mathcal{F}, \ddot{\mathcal{A}}) = \left\{\left(\delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\right\}$ and $(\mathcal{G}, \ddot{\mathfrak{B}}) = \left\{\left(\delta, \sigma_{\mathcal{G}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\right\}$ be two PFHSS, then

$(\mathcal{F}, \ddot{\mathcal{A}}) \cup (\mathcal{G}, \ddot{\mathfrak{B}}) = \{\delta, (max\{\sigma_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{G}(\breve{a})}(\delta)\}, min\{\tau_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\}): \delta \in \mathcal{U}, \breve{a} \in \ddot{\mathcal{A}} \cup \ddot{\mathfrak{B}}\}$

$(\mathcal{F}, \ddot{\mathcal{A}}) \cup (\mathcal{G}, \ddot{\mathfrak{B}}) = \{\delta, (max\{\sigma_{\mathcal{G}(\breve{a})}(\delta), \sigma_{\mathcal{F}(\breve{a})}(\delta)\}, min\{\tau_{\mathcal{G}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\}): \delta \in \mathcal{U}, \breve{a} \in \ddot{\mathcal{A}} \cup \ddot{\mathfrak{B}}\}$

$(\mathcal{F}, \ddot{\mathcal{A}}) \cup (\mathcal{G}, \ddot{\mathfrak{B}}) = (\mathcal{G}, \ddot{\mathfrak{B}}) \cup (\mathcal{F}, \ddot{\mathcal{A}})$.

Similarly, we can prove 5.

Definition 3.11

Let $(\mathcal{F}, \ddot{\mathcal{A}})$ and $(\mathcal{G}, \ddot{\mathfrak{B}})$ be two PFHSS over $\mathcal{U}$, then their intersection is defined as follows:

$(\mathcal{F}, \ddot{\mathcal{A}}) \cap (\mathcal{G}, \ddot{\mathfrak{B}}) = \{\delta, (min\{\sigma_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{G}(\breve{a})}(\delta)\}, max\{\tau_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\}): \delta \in \mathcal{U}, \breve{a} \in \ddot{\mathcal{A}}\}$.

Proposition 3.12

Let $(\mathcal{F}, \ddot{\mathcal{A}})$, $(\mathcal{G}, \ddot{\mathfrak{B}})$, and $(\mathcal{H}, \ddot{\mathrm{C}})$ be three PFHSS over $\mathcal{U}$. Then

1. $(\mathcal{F}, \ddot{\mathcal{A}}) \cap (\mathcal{F}, \ddot{\mathcal{A}}) = (\mathcal{F}, \ddot{\mathcal{A}})$
2. $(\mathcal{F}, \ddot{\mathcal{A}}) \cap (\emptyset, \ddot{\mathcal{A}}) = (\mathcal{F}, \ddot{\mathcal{A}})$
3. $(\mathcal{F}, \ddot{\mathcal{A}}) \cap (\mathbb{E}, \ddot{\mathcal{A}}) = (\mathbb{E}, \ddot{\mathcal{A}})$
4. $(\mathcal{F}, \ddot{\mathcal{A}}) \cap (\mathcal{G}, \ddot{\mathfrak{B}}) = (\mathcal{G}, \ddot{\mathfrak{B}}) \cap (\mathcal{F}, \ddot{\mathcal{A}})$
5. $((\mathcal{F}, \ddot{\mathcal{A}}) \cap (\mathcal{G}, \ddot{\mathfrak{B}})) \cap (\mathcal{H}, \ddot{\mathrm{C}}) = (\mathcal{F}, \ddot{\mathcal{A}}) \cap ((\mathcal{G}, \ddot{\mathfrak{B}}) \cap (\mathcal{H}, \ddot{\mathrm{C}}))$

Proof By using Definition 3.11 we can prove easily.

Proposition 3.13

Let $(\mathcal{F}, \ddot{\mathcal{A}})$ and $(\mathcal{G}, \ddot{\mathfrak{B}})$ be three PFHSS, then

1. $\left((\mathcal{F}, \ddot{\mathcal{A}}) \cup (\mathcal{G}, \ddot{\mathfrak{B}})\right)^c = \left(\mathcal{F}, \ddot{\mathcal{A}}\right)^c \cap \left(\mathcal{G}, \ddot{\mathfrak{B}}\right)^c$
2. $\left((\mathcal{F}, \ddot{\mathcal{A}}) \cap (\mathcal{G}, \ddot{\mathfrak{B}})\right)^c = \left(\mathcal{F}, \ddot{\mathcal{A}}\right)^c \cup \left(\mathcal{G}, \ddot{\mathfrak{B}}\right)^c$

Proof 1

As we know that

$(\mathcal{F}, \ddot{\mathcal{A}}) = \left\{\left(\delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\right\}$ and $(\mathcal{G}, \ddot{\mathfrak{B}}) = \left\{\left(\delta, \sigma_{\mathcal{G}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\right\}$ be two PFHSS, then by using Definition 3.9

$(\mathcal{F}, \ddot{\mathcal{A}}) \cup (\mathcal{G}, \ddot{\mathfrak{B}}) = \{\delta, (max\{\sigma_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{G}(\breve{a})}(\delta)\}", min\{\tau_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\}): \delta \in \mathcal{U}, \breve{a} \in \ddot{\mathcal{A}}\}$.

By using definition 3.7, we have

$\left((\mathcal{F}, \ddot{\mathcal{A}}) \cup (\mathcal{G}, \ddot{\mathfrak{B}})\right)^c = \{\delta, (min\{\tau_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\}, max\{\sigma_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{G}(\breve{a})}(\delta)\}): \delta \in \mathcal{U}, \breve{a} \in \ddot{\mathcal{A}}\}$

Now

$\left(\mathcal{F}, \ddot{\mathcal{A}}\right)^c = \left\{\langle\left(\delta, \tau_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{F}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\rangle\right\}$ and $\left(\mathcal{G}, \ddot{\mathfrak{B}}\right)^c = \left\{\langle\left(\delta, \tau_{\mathcal{G}(\breve{a})}(\delta), \sigma_{\mathcal{G}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\rangle\right\}$.

By using Definition 3.11

$\left(\mathcal{F}, \ddot{\mathcal{A}}\right)^c \cap \left(\mathcal{G}, \ddot{\mathfrak{B}}\right)^c = \{\delta, (min\{\tau_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\}, max\{\sigma_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{G}(\breve{a})}(\delta)\}): \delta \in \mathcal{U}, \breve{a} \in \ddot{\mathcal{A}}\}$

So

$\left((\mathcal{F}, \ddot{\mathcal{A}}) \cup (\mathcal{G}, \ddot{\mathfrak{B}})\right)^c = \left(\mathcal{F}, \ddot{\mathcal{A}}\right)^c \cap \left(\mathcal{G}, \ddot{\mathfrak{B}}\right)^c$

Proof 2

As we know that

$(\mathcal{F}, \ddot{\mathcal{A}}) = \left\{\langle\left(\delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\rangle\right\}$ and $(\mathcal{G}, \ddot{\mathfrak{B}}) = \left\{\langle\left(\delta, \sigma_{\mathcal{G}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\rangle\right\}$ be two PFHSS, then by using Definition 3.11

$(\mathcal{F}, \ddot{\mathcal{A}}) \cap (\mathcal{G}, \ddot{\mathfrak{B}}) = \{\delta, (min\{\sigma_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{G}(\breve{a})}(\delta)\}, max\{\tau_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\}): \delta \in \mathcal{U}, \breve{a} \in \ddot{\mathcal{A}}\}$.

By using Definition 3.7, we have

$\left((\mathcal{F}, \ddot{\mathcal{A}}) \cap (\mathcal{G}, \ddot{\mathfrak{B}})\right)^c = \{\delta, (max\{\tau_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\}, min\{\sigma_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{G}(\breve{a})}(\delta)\}): \delta \in \mathcal{U}, \breve{a} \in \ddot{\mathcal{A}}\}$

Now

$\left(\mathcal{F}, \ddot{\mathcal{A}}\right)^c = \left\{\left(\delta, \tau_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{F}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\right\}$ and $\left(\mathcal{G}, \ddot{\mathfrak{B}}\right)^c = \left\{\left(\delta, \tau_{\mathcal{G}(\breve{a})}(\delta), \sigma_{\mathcal{G}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\right\}$.

By using Definition 3.9

$\left(\mathcal{F}, \ddot{\mathcal{A}}\right)^c \cup \left(\mathcal{G}, \ddot{\mathfrak{B}}\right)^c = \{\delta, (max\{\tau_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\}, min\{\sigma_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{G}(\breve{a})}(\delta)\}): \delta \in \mathcal{U}, \breve{a} \in \ddot{\mathcal{A}}\}$

So

$\left((\mathcal{F}, \ddot{\mathcal{A}}) \cap (\mathcal{G}, \ddot{\mathfrak{B}})\right)^c = \left(\mathcal{F}, \ddot{\mathcal{A}}\right)^c \cup \left(\mathcal{G}, \ddot{\mathfrak{B}}\right)^c$

Definition 3.14

Let $(\mathcal{F}, \ddot{\mathcal{A}})$ and $(\mathcal{G}, \ddot{\mathfrak{B}})$ be two PFHSS over $\mathcal{U}$, then their restricted union is defined as

$$\sigma(\mathcal{F}, \ddot{\mathcal{A}}) \cup_R (\mathcal{G}, \ddot{\mathfrak{B}}) = \begin{cases} \sigma_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ \sigma_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ max\{\sigma_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

$$\tau(\mathcal{F}, \ddot{\mathcal{A}}) \cup_R (\mathcal{G}, \ddot{\mathfrak{B}}) = \begin{cases} \tau_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ \tau_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ min\{\tau_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

Definition 3.15

Let $(\mathcal{F}, \ddot{\mathcal{A}})$ and $(\mathcal{G}, \ddot{\mathfrak{B}})$ be two PFHSS over $\mathcal{U}$, then their extended intersection is defined as

$$\sigma(\mathcal{F}, \ddot{\mathcal{A}}) \cap_{\varepsilon} (\mathcal{G}, \ddot{\mathfrak{B}}) = \begin{cases} \sigma_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ \sigma_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ min\{\sigma_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

$$\tau(\mathcal{F}, \ddot{\mathcal{A}}) \cap_{\varepsilon} (\mathcal{G}, \ddot{\mathfrak{B}}) = \begin{cases} \tau_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ \tau_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ max\{\tau_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

Proposition 3.16

Let $(\mathcal{F}, \ddot{\mathcal{A}})$, and $(\mathcal{G}, \ddot{\mathfrak{B}})$ be two PFHSS over $\mathcal{U}$, then

1. $(\mathcal{F}, \ddot{\mathcal{A}}) \cup \big((\mathcal{F}, \ddot{\mathcal{A}}) \cap (\mathcal{G}, \ddot{\mathfrak{B}})\big) = (\mathcal{F}, \ddot{\mathcal{A}})$
2. $(\mathcal{F}, \ddot{\mathcal{A}}) \cap \big((\mathcal{F}, \ddot{\mathcal{A}}) \cup (\mathcal{G}, \ddot{\mathfrak{B}})\big) = (\mathcal{F}, \ddot{\mathcal{A}})$
3. $(\mathcal{F}, \ddot{\mathcal{A}}) \cup_R \big((\mathcal{F}, \ddot{\mathcal{A}}) \cap_{\varepsilon} (\mathcal{G}, \ddot{\mathfrak{B}})\big) = (\mathcal{F}, \ddot{\mathcal{A}})$
4. $(\mathcal{F}, \ddot{\mathcal{A}}) \cap_{\varepsilon} \big((\mathcal{F}, \ddot{\mathcal{A}}) \cup_R (\mathcal{G}, \ddot{\mathfrak{B}})\big) = (\mathcal{F}, \ddot{\mathcal{A}})$

Proof 1 Consider

$(\mathcal{F}, \ddot{\mathcal{A}}) = \{(\breve{a}, \langle\delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\rangle : \delta \in \mathcal{U}) : \breve{a} \in \ddot{\mathcal{A}},\}$, and $(\mathcal{G}, \ddot{\mathfrak{B}}) = \{(\breve{a}, \langle\delta, \sigma_{\mathcal{G}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\rangle : \delta \in \mathcal{U}) : \breve{a} \in \ddot{\mathfrak{B}},\}$ are two PFHSS over the universe of discourse $\mathcal{U}$

$(\mathcal{F}, \ddot{\mathcal{A}}) \cap (\mathcal{G}, \ddot{\mathfrak{B}}) = \{\delta, (min\{\sigma_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{G}(\breve{a})}(\delta)\}, max\{\tau_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\}) : \delta \in \mathcal{U}\}$

$(\mathcal{F}, \ddot{\mathcal{A}}) \cup \big((\mathcal{F}, \ddot{\mathcal{A}}) \cap (\mathcal{G}, \ddot{\mathfrak{B}})\big) =$

$$\left\{\delta, \left(max\left\{\sigma_{\mathcal{F}(\breve{a})}(\delta), min\{\sigma_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{G}(\breve{a})}(\delta)\}\right\}, min\left\{\tau_{\mathcal{F}(\breve{a})}(\delta), max\{\tau_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\}\right\}\right) : \delta \in \mathcal{U}\right\}$$

$(\mathcal{F}, \ddot{\mathcal{A}}) \cup \big((\mathcal{F}, \ddot{\mathcal{A}}) \cap (\mathcal{G}, \ddot{\mathfrak{B}})\big) = \{(\breve{a}, \langle\delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\rangle : \delta \in \mathcal{U}) : \breve{a} \in \ddot{\mathcal{A}}\}$.

Therefore,

$(\mathcal{F}, \ddot{\mathcal{A}}) \cup \big((\mathcal{F}, \ddot{\mathcal{A}}) \cap (\mathcal{G}, \ddot{\mathfrak{B}})\big) = (\mathcal{F}, \ddot{\mathcal{A}})$.

Proof 2 Consider

$(\mathcal{F}, \ddot{\mathcal{A}}) = \{(\breve{a}, \langle\delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\rangle : \delta \in \mathcal{U}) : \breve{a} \in \ddot{\mathcal{A}}\}$, and $(\mathcal{G}, \ddot{\mathfrak{B}}) = \{(\breve{a}, \langle\delta, \sigma_{\mathcal{G}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\rangle : \delta \in \mathcal{U}) : \breve{a} \in \ddot{\mathfrak{B}}\}$ are two PFHSS over the universe of discourse $\mathcal{U}$

$(\mathcal{F}, \ddot{\mathcal{A}}) \cup (\mathcal{G}, \ddot{\mathfrak{B}}) = \{\delta, (max\{\sigma_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{G}(\breve{a})}(\delta)\}, min\{\tau_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\}) : \delta \in \mathcal{U}\}$

$(\mathcal{F}, \ddot{\mathcal{A}}) \cap \big((\mathcal{F}, \ddot{\mathcal{A}}) \cup (\mathcal{G}, \ddot{\mathfrak{B}})\big) =$

$$\left\{\delta, \left(min\left\{\sigma_{\mathcal{F}(\breve{a})}(\delta), max\{\sigma_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{G}(\breve{a})}(\delta)\}\right\}, max\left\{\tau_{\mathcal{F}(\breve{a})}(\delta), min\{\tau_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\}\right\}\right) : \delta \in \mathcal{U}\right\}$$

$(\mathcal{F}, \ddot{\mathcal{A}}) \cap \big((\mathcal{F}, \ddot{\mathcal{A}}) \cup (\mathcal{G}, \ddot{\mathfrak{B}})\big) = \{(\breve{a}, \langle\delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\rangle : \delta \in \mathcal{U}) : \breve{a} \in \ddot{\mathcal{A}}\}$.

Therefore,

$(\mathcal{F}, \ddot{\mathcal{A}}) \cap \big((\mathcal{F}, \ddot{\mathcal{A}}) \cup (\mathcal{G}, \ddot{\mathfrak{B}})\big) = (\mathcal{F}, \ddot{\mathcal{A}})$.

Similarly, we can prove 3 and 4.

4. Logical operators and Necessity and Possibility operators under the Pythagorean fuzzy hypersoft set.

In this section, we propose the idea of AND-operator, OR-operator, Necessity operator, and possibility operator under the PFHSS with their several desirable properties. We also introduce the correlation coefficient under the PFHSS environment.

Definition 4.1

Let $(\mathcal{F},\ \ddot{\mathcal{A}})$ and $(\mathcal{G},\ \ddot{\mathfrak{B}})$ be two PFHSS over $\mathcal{U}$, then their OR-operator is represented by $(\mathcal{F},\ \ddot{\mathcal{A}}) \vee (\mathcal{G},\ \ddot{\mathfrak{B}})$ and defined as follows

$(\mathcal{F},\ \ddot{\mathcal{A}}) \vee (\mathcal{G},\ \ddot{\mathfrak{B}}) = (\lambda, \ddot{\mathcal{A}} \times \ddot{\mathfrak{B}})$, where $\lambda(\breve{a}_1 \times \breve{a}_2) = \mathcal{F}_{\ddot{\mathcal{A}}}(\breve{a}_1) \cup \mathcal{G}_{\ddot{\mathfrak{B}}}(\breve{a}_2)$ for all $(\breve{a}_1 \times \breve{a}_2) \in \ddot{\mathcal{A}} \times \ddot{\mathfrak{B}}$.

$\lambda(\breve{a}_1 \times \breve{a}_2) = \{max\{\sigma_{\mathcal{F}(\breve{a}_1)}(\delta), \sigma_{\mathcal{G}(\breve{a}_2)}(\delta)\}, min\{\tau_{\mathcal{F}\breve{a}_1}(\delta), \tau_{\mathcal{G}(\breve{a}_2)}(\delta)\} : \delta \in \mathcal{U}, \breve{a} \in \ddot{\mathcal{A}}\}$

Definition 4.2

Let $(\mathcal{F},\ \ddot{\mathcal{A}})$ and $(\mathcal{G},\ \ddot{\mathfrak{B}})$ be two PFHSS over $\mathcal{U}$, then their AND-operator is represented by $(\mathcal{F},\ \ddot{\mathcal{A}}) \wedge (\mathcal{G},\ \ddot{\mathfrak{B}})$ and defined as follows

$(\mathcal{F},\ \ddot{\mathcal{A}}) \wedge (\mathcal{G},\ \ddot{\mathfrak{B}}) = (\lambda, \ddot{\mathcal{A}} \times \ddot{\mathfrak{B}})$, where $\lambda(\breve{a}_1 \times \breve{a}_2) = \mathcal{F}_{\ddot{\mathcal{A}}}(\breve{a}_1) \cap \mathcal{G}_{\ddot{\mathfrak{B}}}(\breve{a}_2)$ for all $(\breve{a}_1 \times \breve{a}_2) \in \ddot{\mathcal{A}} \times \ddot{\mathfrak{B}}$.

$\lambda(\breve{a}_1 \times \breve{a}_2) = \{min\{\sigma_{\mathcal{F}(\breve{a}_1)}(\delta), \sigma_{\mathcal{G}(\breve{a}_2)}(\delta)\}, max\{\tau_{\mathcal{F}(\breve{a}_1)}(\delta), \tau_{\mathcal{G}(\breve{a}_2)}(\delta)\} : \delta \in \mathcal{U}\}$

Proposition 4.3

Let $(\mathcal{F},\ \ddot{\mathcal{A}})$, $(\mathcal{G},\ \ddot{\mathfrak{B}})$, and $(\mathcal{H},\ \ddot{C})$ be three PFHSS over $\mathcal{U}$. Then

1. $(\mathcal{F}, \ddot{\mathcal{A}}) \vee (\mathcal{G}, \ddot{\mathfrak{B}}) = (\mathcal{G}, \ddot{\mathfrak{B}}) \vee (\mathcal{F},\ \ddot{\mathcal{A}})$
2. $(\mathcal{F}, \ddot{\mathcal{A}}) \wedge (\mathcal{G}, \ddot{\mathfrak{B}}) = (\mathcal{G}, \ddot{\mathfrak{B}}) \wedge (\mathcal{F},\ \ddot{\mathcal{A}})$
3. $(\mathcal{F}, \ddot{\mathcal{A}}) \vee ((\mathcal{G}, \ddot{\mathfrak{B}}) \vee (\mathcal{H},\ \ddot{C})) = ((\mathcal{F}, \ddot{\mathcal{A}}) \vee (\mathcal{G}, \ddot{\mathfrak{B}})) \vee (\mathcal{H},\ \ddot{C})$
4. $(\mathcal{F}, \ddot{\mathcal{A}}) \wedge ((\mathcal{G}, \ddot{\mathfrak{B}}) \wedge (\mathcal{H},\ \ddot{C})) = ((\mathcal{F}, \ddot{\mathcal{A}}) \wedge (\mathcal{G}, \ddot{\mathfrak{B}})) \wedge (\mathcal{H},\ \ddot{C})$
5. $((\mathcal{F}, \ddot{\mathcal{A}}) \vee (\mathcal{G}, \ddot{\mathfrak{B}}))^c = \mathcal{F}^c(\ddot{\mathcal{A}}) \wedge \wp^c(\ddot{\mathfrak{B}})$
6. $((\mathcal{F}, \ddot{\mathcal{A}}) \wedge (\mathcal{G}, \ddot{\mathfrak{B}}))^c = \mathcal{F}^c(\ddot{\mathcal{A}}) \vee \wp^c(\ddot{\mathfrak{B}})$

Proof 1 By using Definitions 4.1, 4.2 we can prove easily.

Definition 4.4

Let $(\mathcal{F},\ \ddot{\mathcal{A}})$ be a PFHSS, then necessity operation on PFHSS represented by $\oplus\ (\mathcal{F},\ \ddot{\mathcal{A}})$ and defined as follows

$\oplus\ (\mathcal{F},\ \ddot{\mathcal{A}}) = \{(\breve{a}, \langle \delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), 1 - \sigma_{\mathcal{F}(\breve{a})}(\delta) \rangle : \delta \in \mathcal{U}) : \breve{a} \in \ddot{\mathcal{A}},\}$

Definition 4.5

Let $(\mathcal{F},\ \ddot{\mathcal{A}})$ be a PFHSS, then possibility operation on PFHSS represented by $(\mathcal{F},\ \ddot{\mathcal{A}})$ and defined as follows

$\otimes\ (\mathcal{F},\ \ddot{\mathcal{A}}) = \{(\breve{a}, \langle \delta, 1 - \tau_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta) \rangle : \delta \in \mathcal{U}) : \breve{a} \in \ddot{\mathcal{A}},\}$

Proposition 4.6

Let $(\mathcal{F},\ \ddot{\mathcal{A}})$ and $(\mathcal{G},\ \ddot{\mathfrak{B}})$ be two PFHSS, then

1. $\oplus\left((\mathcal{F}, \ddot{\mathcal{A}}) \cup_R (\mathcal{G}, \ddot{\mathfrak{B}})\right) = \oplus\ (\mathcal{G},\ \ddot{\mathfrak{B}}) \cup_R \oplus\ (\mathcal{F},\ \ddot{\mathcal{A}})$
2. $\oplus\left((\mathcal{F}, \ddot{\mathcal{A}}) \cap_\varepsilon (\mathcal{G}, \ddot{\mathfrak{B}})\right) = \oplus\ (\mathcal{G},\ \ddot{\mathfrak{B}}) \cap_\varepsilon \oplus\ (\mathcal{F},\ \ddot{\mathcal{A}})$

Proof 1

As we know that

$(\mathcal{F}, \ddot{\mathcal{A}}) = \left\{\left(\delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\right\}$ and $(\mathcal{G}, \ddot{\mathfrak{B}}) = \left\{\left(\delta, \sigma_{\mathcal{G}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\right\}$ are two PFHSS.

Let $\left((\mathcal{F}, \ddot{\mathcal{A}}) \cup_R (\mathcal{G}, \ddot{\mathfrak{B}})\right) = (\mathcal{H},\ \ddot{C})$

$$\sigma(\mathcal{H},\ \ddot{C}) = \begin{cases} \sigma_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ \sigma_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ max\{\sigma_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

$$\tau(\mathcal{H},\ \ddot{C}) = \begin{cases} \tau_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ \tau_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ min\{\tau_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

By using Definition 4.4

$$\oplus\, \sigma(\mathcal{H},\ \ddot{C}) = \begin{cases} \sigma_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ \sigma_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ max\{\sigma_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

$$\oplus\, \tau(\mathcal{H},\ \ddot{C}) = \begin{cases} 1 - \sigma_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ 1 - \sigma_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ min\{1 - \sigma_{\mathcal{F}(\breve{a})}(\delta), 1 - \sigma_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

Assume $\oplus(\mathcal{F}, \ddot{\mathcal{A}}) \cup_R \oplus(\mathcal{G}, \ddot{\mathfrak{B}}) = \aleph$, where $\oplus(\mathcal{F}, \ddot{\mathcal{A}})$ and $\oplus(\mathcal{G}, \ddot{\mathfrak{B}})$ are given as follows by using the definition of necessity operation.

$\oplus\,(\mathcal{F}, \ddot{\mathcal{A}}) = \left\{\left(\delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), 1 - \sigma_{\mathcal{F}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\right\}$ and $\oplus\,(\mathcal{G}, \ddot{\mathfrak{B}}) = \left\{\left(\delta, \sigma_{\mathcal{G}(\breve{a})}(\delta), 1 - \sigma_{\mathcal{G}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\right\}$. By using Definition 3.14

$$\sigma\aleph = \begin{cases} \sigma_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ \sigma_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ max\{\sigma_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

$$\tau\aleph = \begin{cases} 1 - \sigma_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ 1 - \sigma_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ min\{1 - \sigma_{\mathcal{F}(\breve{a})}(\delta), 1 - \sigma_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

Consequently $\oplus\,(\mathcal{H},\ \ddot{C})$ and $\aleph$ are the same, so

$\oplus\left((\mathcal{F}, \ddot{\mathcal{A}}) \cup_R (\mathcal{G}, \ddot{\mathfrak{B}})\right) = \oplus\,(\mathcal{G},\ \ddot{\mathfrak{B}}) \cup_R \oplus\,(\mathcal{F},\ \ddot{\mathcal{A}})$.

Proof 2

Let $\left((\mathcal{F}, \ddot{\mathcal{A}}) \cap_\varepsilon (\mathcal{G}, \ddot{\mathfrak{B}})\right) = (\mathcal{H},\ \ddot{C})$

$$\sigma(\mathcal{H},\ \ddot{C}) = \begin{cases} \sigma_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ \sigma_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ min\{\sigma_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

$$\tau(\mathcal{H},\ \ddot{C}) = \begin{cases} \tau_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ \tau_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ max\{\tau_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

By using Definition 4.4

$$\oplus\, \sigma(\mathcal{H},\ \ddot{C}) = \begin{cases} \sigma_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ \sigma_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ min\{\sigma_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

$$\oplus\, \tau(\mathcal{H},\ \ddot{C}) = \begin{cases} 1 - \sigma_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ 1 - \sigma_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ max\{1 - \sigma_{\mathcal{F}(\breve{a})}(\delta), 1 - \sigma_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

Assume $\oplus\,(\mathcal{G},\ \ddot{\mathfrak{B}}) \cap_\varepsilon \oplus\,(\mathcal{F},\ \ddot{\mathcal{A}}) = \aleph$, where $\oplus(\mathcal{F}, \ddot{\mathcal{A}})$ and $\oplus(\mathcal{G}, \ddot{\mathfrak{B}})$ are given as follows by using the definition of necessity operation.

$\oplus\,(\mathcal{F}, \ddot{\mathcal{A}}) = \left\{\left(\delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), 1 - \sigma_{\mathcal{F}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\right\}$ and $\oplus\,(\mathcal{G}, \ddot{\mathfrak{B}}) = \left\{\left(\delta, \sigma_{\mathcal{G}(\breve{a})}(\delta), 1 - \sigma_{\mathcal{G}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\right\}$. By using Definition 3.15

$$\sigma\aleph = \begin{cases} \sigma_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ \sigma_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ min\{\sigma_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

$$\tau\aleph = \begin{cases} 1 - \sigma_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ 1 - \sigma_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ max\{1 - \sigma_{\mathcal{F}(\breve{a})}(\delta), 1 - \sigma_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

Consequently $\oplus\,(\mathcal{H},\ \ddot{C})$ and $\aleph$ are the same, so

$\oplus\left((\mathcal{F}, \ddot{\mathcal{A}}) \cap_\varepsilon (\mathcal{G}, \ddot{\mathfrak{B}})\right) = \oplus\,(\mathcal{G},\ \ddot{\mathfrak{B}}) \cap_\varepsilon \oplus\,(\mathcal{F},\ \ddot{\mathcal{A}})$.

Proposition 4.7

Let $(\mathcal{F}, \ddot{\mathcal{A}})$ and $(\mathcal{G}, \ddot{\mathfrak{B}})$ be two PFHSS, then

1. $\otimes\left((\mathcal{F},\ddot{\mathcal{A}}) \cup_R (\mathcal{G},\ddot{\mathfrak{B}})\right) = \otimes(\mathcal{G}, \ddot{\mathfrak{B}}) \cup_R \otimes(\mathcal{F}, \ddot{\mathcal{A}})$
2. $\otimes\left((\mathcal{F},\ddot{\mathcal{A}}) \cap_\varepsilon (\mathcal{G},\ddot{\mathfrak{B}})\right) = \otimes(\mathcal{G}, \ddot{\mathfrak{B}}) \cap_\varepsilon \otimes(\mathcal{F}, \ddot{\mathcal{A}})$

Proof 1

As we know that

$(\mathcal{F},\ddot{\mathcal{A}}) = \left\{\left(\delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\right\}$ and $(\mathcal{G},\ddot{\mathfrak{B}}) = \left\{\left(\delta, \sigma_{\mathcal{G}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\right\}$ are two PFHSS.

Let $\left((\mathcal{F},\ddot{\mathcal{A}}) \cup_R (\mathcal{G},\ddot{\mathfrak{B}})\right) = (\mathcal{H}, \ddot{\text{C}})$

$$\sigma(\mathcal{H}, \ddot{\text{C}}) = \begin{cases} \sigma_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ \sigma_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ max\{\sigma_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

$$\tau(\mathcal{H}, \ddot{\text{C}}) = \begin{cases} \tau_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ \tau_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ min\{\tau_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

By using Definition 4.5

$$\otimes\,\sigma(\mathcal{H}, \ddot{\text{C}}) = \begin{cases} 1 - \tau_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ 1 - \tau_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ max\{1 - \tau_{\mathcal{F}(\breve{a})}(\delta), 1 - \tau_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

$$\otimes\,\tau(\mathcal{H}, \ddot{\text{C}}) = \begin{cases} \tau_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ \tau_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ min\{\tau_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

Assume $\otimes(\mathcal{F},\ddot{\mathcal{A}}) \cup_R \otimes(\mathcal{G},\ddot{\mathfrak{B}}) = \aleph$, where $\otimes(\mathcal{F},\ddot{\mathcal{A}})$ and $\otimes(\mathcal{G},\ddot{\mathfrak{B}})$ are given as follows by using the definition of necessity operation.

$\otimes(\mathcal{F},\ddot{\mathcal{A}}) = \left\{\left(\delta, 1 - \tau_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\right\}$ and $\otimes(\mathcal{G},\ddot{\mathfrak{B}}) = \left\{\left(\delta, 1 - \tau_{\mathcal{G}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\right\}$. By using Definition 3.14

$$\sigma\aleph = \begin{cases} 1 - \tau_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ 1 - \tau_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ max\{1 - \tau_{\mathcal{F}(\breve{a})}(\delta), 1 - \tau_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

$$\tau\aleph = \begin{cases} \tau_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ \tau_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ min\{\tau_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

Consequently $\otimes(\mathcal{H}, \ddot{\text{C}})$ and $\aleph$ are the same, so

$\otimes\left((\mathcal{F},\ddot{\mathcal{A}}) \cup_R (\mathcal{G},\ddot{\mathfrak{B}})\right) = \otimes(\mathcal{G}, \ddot{\mathfrak{B}}) \cup_R \otimes(\mathcal{F}, \ddot{\mathcal{A}})$.

Proof 2

As we know that

$(\mathcal{F},\ddot{\mathcal{A}}) = \left\{\left(\delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\right\}$ and $(\mathcal{G},\ddot{\mathfrak{B}}) = \left\{\left(\delta, \sigma_{\mathcal{G}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\right\}$ are two PFHSS.

Let $\left((\mathcal{F},\ddot{\mathcal{A}}) \cap_\varepsilon (\mathcal{G},\ddot{\mathfrak{B}})\right) = (\mathcal{H}, \ddot{\text{C}})$

$$\sigma(\mathcal{H}, \ddot{\text{C}}) = \begin{cases} \sigma_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ \sigma_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ min\{\sigma_{\mathcal{F}(\breve{a})}(\delta), \sigma_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

$$\tau(\mathcal{H}, \ddot{\text{C}}) = \begin{cases} \tau_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ \tau_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ max\{\tau_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

By using Definition 4.5

$$\otimes\ \sigma(\mathcal{H},\ \ddot{C}) = \begin{cases} 1-\tau_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ 1-\tau_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ min\{1-\tau_{\mathcal{F}(\breve{a})}(\delta), 1-\tau_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

$$\otimes\ \tau(\mathcal{H},\ \ddot{C}) = \begin{cases} \tau_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ \tau_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ max\{\tau_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

Assume $\otimes\ (\mathcal{F}, \ddot{\mathcal{A}})\ \cap_\varepsilon\ \otimes\ (\mathcal{G}, \ddot{\mathfrak{B}}) = \aleph$, where $\otimes\ (\mathcal{F}, \ddot{\mathcal{A}})$ and $\otimes\ (\mathcal{G}, \ddot{\mathfrak{B}})$ are given as follows by using the definition of necessity operation.

$\otimes\ (\mathcal{F}, \ddot{\mathcal{A}}) = \left\{\left(\delta, 1-\tau_{\mathcal{F}(\breve{a})}(\delta),\ \tau_{\mathcal{F}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\right\}$ and $\otimes\ (\mathcal{G}, \ddot{\mathfrak{B}}) = \left\{\left(\delta, 1-\tau_{\mathcal{G}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\right) \mid \delta \in \mathcal{U}\right\}$. By using Definition 3.14

$$\sigma\aleph = \begin{cases} 1-\tau_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ 1-\tau_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ min\{1-\tau_{\mathcal{F}(\breve{a})}(\delta), 1-\tau_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

$$\tau\aleph = \begin{cases} \tau_{\mathcal{F}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathcal{A}} - \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \\ \tau_{\mathcal{G}(\breve{a})}(\delta) & if\ \breve{a} \in \ddot{\mathfrak{B}} - \ddot{\mathcal{A}}, \delta \in \mathcal{U} \\ max\{\tau_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{G}(\breve{a})}(\delta)\} & if\ \breve{a} \in \ddot{\mathcal{A}} \cap \ddot{\mathfrak{B}}, \delta \in \mathcal{U} \end{cases}$$

Consequently $\otimes\ (\mathcal{H},\ \ddot{C})$ and $\aleph$ are the same, so

$\otimes\ \left((\mathcal{F}, \ddot{\mathcal{A}})\ \cap_\varepsilon\ (\mathcal{G}, \ddot{\mathfrak{B}})\right) = \otimes\ (\mathcal{G},\ \ddot{\mathfrak{B}})\ \cap_\varepsilon\ \otimes\ (\mathcal{F},\ \ddot{\mathcal{A}})$.

Proposition 4.8

Let $(\mathcal{F},\ \ddot{\mathcal{A}})$ and $(\mathcal{G},\ \ddot{\mathfrak{B}})$ be two PFHSS, then

1. $\oplus\left((\mathcal{F}, \ddot{\mathcal{A}}) \wedge (\mathcal{G}, \ddot{\mathfrak{B}})\right) = \oplus\ (\mathcal{F},\ \ddot{\mathcal{A}})\ \wedge\ \oplus\ (\mathcal{G},\ \ddot{\mathfrak{B}})$
2. $\oplus\ \left((\mathcal{F}, \ddot{\mathcal{A}}) \vee (\mathcal{G}, \ddot{\mathfrak{B}})\right) = \oplus\ (\mathcal{F},\ \ddot{\mathcal{A}})\ \vee\ \oplus\ (\mathcal{G},\ \ddot{\mathfrak{B}})$
3. $\otimes\ \left((\mathcal{F}, \ddot{\mathcal{A}}) \wedge (\mathcal{G}, \ddot{\mathfrak{B}})\right) = \otimes\ (\mathcal{F}, \ddot{\mathcal{A}})\ \wedge\ \otimes\ (\mathcal{G}, \ddot{\mathfrak{B}})$
4. $\otimes\ \left((\mathcal{F}, \ddot{\mathcal{A}}) \vee (\mathcal{G}, \ddot{\mathfrak{B}})\right) = \otimes\ (\mathcal{F}, \ddot{\mathcal{A}})\ \vee\ \otimes\ (\mathcal{G}, \ddot{\mathfrak{B}})$

Proof 1 Proof is straight forward.

5. Application of Correlation Coefficient for Decision Making Under PFHSS Environment

In this section, we present the correlation coefficient under the PFHSS environment and establish an algorithm based on the proposed CC under PFHSS and utilize the proposed approach for decision making in real-life problems.

Definition 5.1

Let $(\mathcal{F}, \ddot{\mathcal{A}}) = \left\{\left(\delta_i, \sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G}, \ddot{\mathfrak{B}}) = \left\{\left(\delta_i, \sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ be two PFHSSs defined over a universe of discourse $\mathcal{U}$. Then, their informational energies of $(\mathcal{F}, \ddot{\mathcal{A}})$ and $(\mathcal{G}, \ddot{\mathfrak{B}})$ can be described as follows:

$$\varsigma_{PFHSS}(\mathcal{F}, \ddot{\mathcal{A}}) = \sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^4 + \left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^4\right)$$

$$\boldsymbol{\varsigma_{PFHSS}(\mathcal{G}, \ddot{\mathfrak{B}}) = \sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^4 + \left(\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^4\right)}.$$

Definition 5.2

Let $(\mathcal{F}, \ddot{\mathcal{A}}) = \left\{\left(\delta_i, \sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G}, \ddot{\mathfrak{B}}) = \left\{\left(\delta_i, \sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ be two PFHSSs defined over a universe of discourse $\mathcal{U}$. Then, their correlation measure between $(\mathcal{F}, \ddot{\mathcal{A}})$ and $(\mathcal{G}, \ddot{\mathfrak{B}})$ can be described as follows:

$$\mathcal{C}_{PFHSS}((\mathcal{F}, \ddot{\mathcal{A}}), (\mathcal{G}, \ddot{\mathfrak{B}})) = \sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2 * \left(\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2 * \left(\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right).$$

Definition 5.3

Let $(\mathcal{F},\ddot{\mathcal{A}}) = \left\{\left(\delta_i, \sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G},\ddot{\mathfrak{B}}) = \left\{\left(\delta_i, \sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i), \tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right) \mid \delta_i \in \mathcal{U}\right\}$ be two PFHSSs, then correlation coefficient between them given as $\delta_{IFHSS}((\mathcal{F},\ddot{\mathcal{A}}),(\mathcal{G},\ddot{\mathfrak{B}}))$ and expressed as follows:

$$\delta_{PFHSS}((\mathcal{F},\ddot{\mathcal{A}}),(\mathcal{G},\ddot{\mathfrak{B}})) = \frac{\mathcal{C}_{PFHSS}((\mathcal{F},\ddot{\mathcal{A}}),(\mathcal{G},\ddot{\mathfrak{B}}))}{\sqrt{\varsigma_{PFHSS}(\mathcal{F},\ddot{\mathcal{A}})} * \sqrt{\varsigma_{PFHSS}(\mathcal{G},\ddot{\mathfrak{B}})}}$$

$$\boldsymbol{\delta_{IFHSS}((\mathcal{F},\ddot{\mathcal{A}}),(\mathcal{G},\ddot{\mathfrak{B}}))} = \frac{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2 * \left(\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2 * \left(\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)}{\sqrt{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^4 + \left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^4\right)}\sqrt{\sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^4 + \left(\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^4\right)}}$$

5.1 Algorithm for Correlation Coefficient under PFHSS

Step 1. Pick out the set containing sub-attributes of parameters.

Step 2. Construct the PFHSS according to experts in form of PFHSNs.

Step 3. Find the informational energies of PFHSS.

Step 4. Calculate the correlation between PFHSSs by using the following formula

$$\mathcal{C}_{PFHSS}((\mathcal{F},\ddot{\mathcal{A}}),(\mathcal{G},\ddot{\mathfrak{B}})) = \sum_{k=1}^{m}\sum_{i=1}^{n}\left(\left(\sigma_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2 * \left(\sigma_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2 + \left(\tau_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2 * \left(\tau_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)$$

Step 5. Calculate the CC between PFHSSs by using the following formula

$$\delta_{PFHSS}((\mathcal{F},\ddot{\mathcal{A}}),(\mathcal{G},\ddot{\mathfrak{B}})) = \frac{\mathcal{C}_{PFHSS}((\mathcal{F},\ddot{\mathcal{A}}),(\mathcal{G},\ddot{\mathfrak{B}}))}{\sqrt{\varsigma_{PFHSS}(\mathcal{F},\ddot{\mathcal{A}})} * \sqrt{\varsigma_{PFHSS}(\mathcal{G},\ddot{\mathfrak{B}})}}$$

Step 6. Choose the alternative with a maximum value of CC.

Step 7. Analyze the ranking of the alternatives.

A flowchart of the above-presented algorithm can be seen in Figure 1.

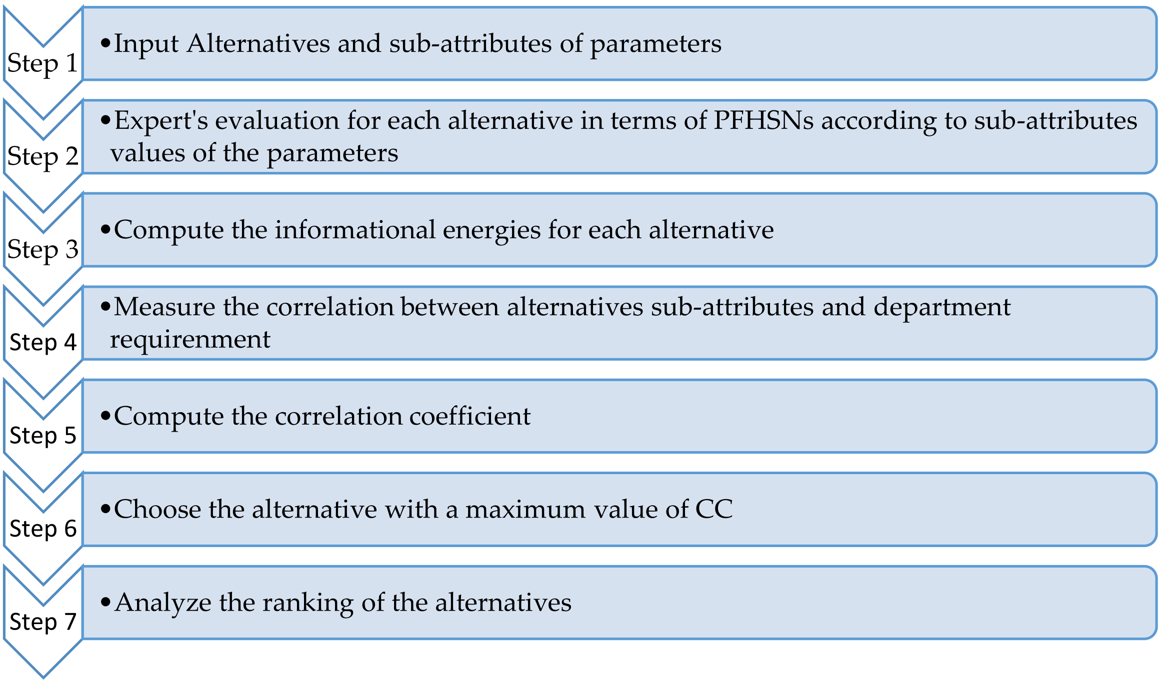

Figure 1: Flowchart for correlation coefficient under PFHSS

5.2 Problem Formulation and Application of PFHSS For Decision Making

Department of the scientific discipline of some university $\mathcal{U}$ will have one scholarship for the position of the doctoral degree. Several students apply to get a scholarship but referable probabilistic along with CGPA (cumulative grade points average), simply four students call for enrolled for undervaluation such as $\mathcal{T} = \{\mathcal{T}^1, \mathcal{T}^2, \mathcal{T}^3, \mathcal{T}^4\}$ be a set of selected students call for the interview. The president of the university hires a committee of four decision-makers (DM) $\mathcal{U} = \{\partial_1, \partial_2, \partial_3, \partial_4\}$ for the selection of doctoral degree student. The team of DM decides the criteria (attributes) for the selection of doctoral degree such as $\mathfrak{L} = \{\ell_1 = \text{Publications}, \ell_2 = \text{Subjects}, \ell_3 = \text{IF}\}$ be a collection of attributes and their corresponding sub-attribute are given as Publications $= \ell_1 = \{a_{11} = \mathit{more\ than}\ 10, a_{12} = \mathit{less\ than}\ 10\}$, Subjects $= \ell_2 = \{a_{21} = \mathit{Mathematics}, a_{22} = \mathit{Computer\ Science}\}$, and IF $= \ell_3 = \{a_{31} = 45, a_{32} = 47\}$. Let $\mathfrak{L}' = \ell_1 \times \ell_2 \times \ell_3$ be a set of sub-attributes

$\mathfrak{L}' = \ell_1 \times \ell_2 \times \ell_3 = \{a_{11}, a_{12}\} \times \{a_{21}, a_{22}\} \times \{a_{31}, a_{32}\}$

$= \begin{Bmatrix} (a_{11}, a_{21}, a_{31}), (a_{11}, a_{21}, a_{32}), (a_{11}, a_{22}, a_{31}), (a_{11}, a_{22}, a_{32}), \\ (a_{12}, a_{21}, a_{31}), (a_{12}, a_{21}, a_{32}), (a_{12}, a_{22}, a_{31}), (a_{12}, a_{22}, a_{32}) \end{Bmatrix}$, $\mathfrak{L}' = \{\breve{a}_1, \breve{a}_2, \breve{a}_3, \breve{a}_4, \breve{a}_5, \breve{a}_6, \breve{a}_7, \breve{a}_8\}$ be a set of all multi sub-attributes. Each DM will evaluate the ratings of each alternative in the form of PFHSNs under the considered multi sub-attributes. The developed method to find the best alternative is as follows.

5.3 Application of PFHSS For Decision Making

Assume $\mathcal{T} = \{\mathcal{T}^1, \mathcal{T}^2, \mathcal{T}^3, \mathcal{T}^4\}$ be a set of alternatives who are shortlisted for interview and $\mathfrak{L} = \{\ell_1 = \text{Publications}, \ell_2 = \text{Subjects}, \ell_3 = \text{Qualification}\}$ be a set of parameters for the selection of scholarship positions. Let the corresponding sub-attribute are given as Publications $= \ell_1 = \{a_{11} = \mathit{more\ than}\ 10, a_{12} = \mathit{less\ than}\ 10\}$, Subjects $= \ell_2 = \{a_{21} = \mathit{Mathematics}, a_{22} = \mathit{Computer\ Science}\}$, and IF $= \ell_3 = \{a_{31} = 45, a_{32} = 47\}$. Let $\mathfrak{L}' = \ell_1 \times \ell_2 \times \ell_3$ be a set of sub-attributes. The development of the decision matrix according to the requirement of the scientific discipline department in terms of PFHSNs is given in Table 1.

Table 1. Decision Matrix for Concerning Department

$\wp$	$\breve{a}_1$	$\breve{a}_2$	$\breve{a}_3$	$\breve{a}_4$	$\breve{a}_5$	$\breve{a}_6$	$\breve{a}_7$	$\breve{a}_8$
∂_1	(.3,.8)	(.7 .3)	(.6,.7)	(.5,.4)	(.2,.4)	(.4,.6)	(.5,.8)	(.9,.3)
∂_2	(.6,.7)	(.4,.6)	(.3,.4)	(.9,.2)	(.3,.8)	(.2,.4)	(.7,.5)	(.4,.5)
∂_3	(.7,.3)	(.2,.5)	(.1,.6)	(.3,.4)	(.4 .6)	(.8,.4)	(.6,.7)	(.2,.5)
∂_4	(.8,.4)	(.2,.9)	(.2,.4)	(.4,.6)	(.6,.5)	(.5,.6)	(.4,.5)	(.8,.3)

Develop the decision matrices for each alternative in terms of PFHSNs by considering their sub-attributes values of given attributes can be seen in Table 2- Table 5.

Table 2. Decision Matrix for Alternative $\mathcal{T}^{(1)}$

$\mathcal{T}^{(1)}$	$\breve{a}_1$	$\breve{a}_2$	$\breve{a}_3$	$\breve{a}_4$	$\breve{a}_5$	$\breve{a}_6$	$\breve{a}_7$	$\breve{a}_8$
∂_1	(.7,.6)	(.3,.4)	(.6,.5)	(.3,.9)	(.5,.4)	(.4,.6)	(.7,.5)	(.4,.8)
∂_2	(.8,.5)	(.7,.4)	(.9,.2)	(.7,.4)	(.4,.5)	(.9,.3)	(.2,.7)	(.3,.8)
∂_3	(.3,.7)	(.4,.5)	(.4,.8)	(.3,.4)	(.6,.7)	(.3,.4)	(.9,.2)	(.7,.2)
∂_4	(.5,.4)	(.7,.6)	(.9,.3)	(.8,.5)	(.9,.2)	(.2,.4)	(.4,.6)	(.6,.5)

Table 3. Decision Matrix for Alternative $\mathcal{T}^{(2)}$

$\mathcal{T}^{(2)}$	$\breve{\alpha}_1$	$\breve{\alpha}_2$	$\breve{\alpha}_3$	$\breve{\alpha}_4$	$\breve{\alpha}_5$	$\breve{\alpha}_6$	$\breve{\alpha}_7$	$\breve{\alpha}_8$
∂_1	(.6,.5)	(.3,.8)	(.4,.5)	(.7,.4)	(.6,.4)	(.4,.5)	(.3,.4)	(.7,.5)
∂_2	(.8,.4)	(.9,.3)	(.1,.8)	(.1,.2)	(.4,.6)	(.3,.7)	(.6,.8)	(.8,.4)
∂_3	(.6,.7)	(.7,.4)	(.7,.5)	(.3,.4)	(.9,.2)	(.6,.5)	(.3,.5)	(.6,.7)
∂_4	(.5,.4)	(.4,.8)	(.5,.6)	(.3,.4)	(.7,.6)	(.7,.5)	(.4,.9)	(.5,.2)

Table 4. Decision Matrix for Alternative $\mathcal{T}^{(3)}$

$\mathcal{T}^{(3)}$	$\breve{\alpha}_1$	$\breve{\alpha}_2$	$\breve{\alpha}_3$	$\breve{\alpha}_4$	$\breve{\alpha}_5$	$\breve{\alpha}_6$	$\breve{\alpha}_7$	$\breve{\alpha}_8$
∂_1	(.5,.7)	(.8,.5)	(.7,.4)	(.4,.3)	(.4,.9)	(.2,.4)	(.8,.4)	(.7,.5)
∂_2	(.8,.5)	(.7,.4)	(.8,.5)	(.5,.2)	(.5,.7)	(.7,.5)	(.7,.6)	(.6,.4)
∂_3	(.6,.8)	(.4,.5)	(.6,.5)	(.6,.4)	(.7,.5)	(.8,.4)	(.5,.8)	(.4 .5)
∂_4	(.5,.7)	(.9,.3)	(.3,.5)	(.5,.7)	(.3,.5)	(.8,.5)	(.7,.5)	(.2,.5)

Table 5. Decision Matrix for Alternative $\mathcal{T}^{(5)}$

$\mathcal{T}^{(4)}$	$\breve{\alpha}_1$	$\breve{\alpha}_2$	$\breve{\alpha}_3$	$\breve{\alpha}_4$	$\breve{\alpha}_5$	$\breve{\alpha}_6$	$\breve{\alpha}_7$	$\breve{\alpha}_8$
∂_1	(.6,.5)	(.3,.8)	(.4,.5)	(.7,.4)	(.6,.4)	(.4,.5)	(.3,.4)	(.7,.5)
∂_2	(.8,.4)	(.9,.3)	(.1,.8)	(.1,.2)	(.4,.6)	(.3,.7)	(.6,.8)	(.8,.4)
∂_3	(.6,.7)	(.7,.4)	(.7,.5)	(.3,.4)	(.9,.2)	(.6,.5)	(.3,.5)	(.6,.7)
∂_4	(.5,.4)	(.4,.8)	(.5,.6)	(.3,.4)	(.7,.6)	(.7,.5)	(.4,.9)	(.5,.2)

By using Tables 1-5, compute the correlation coefficient between $\delta_{PFHSS}(\wp,\mathcal{T}^{(1)})$, $\delta_{PFHSS}(\wp,\mathcal{T}^{(2)})$, $\delta_{PFHSS}(\wp,\mathcal{T}^{(3)})$, $\delta_{PFHSS}(\wp,\mathcal{T}^{(4)})$ by using Definition 5.3 given as follows:
$\delta_{PFHSS}(\wp,\mathcal{T}^{(1)})$ = .99658, $\delta_{PFHSS}(\wp,\mathcal{T}^{(2)})$ = .99732, $\delta_{PFHSS}(\wp,\mathcal{T}^{(3)})$ = .99894, and $\delta_{PFHSS}(\wp,\mathcal{T}^{(4)})$ = .99669. This shows that $\delta_{PFHSS}(\wp,\mathcal{T}^{(3)}) > \delta_{PFHSS}(\wp,\mathcal{T}^{(2)}) > \delta_{PFHSS}(\wp,\mathcal{T}^{(4)}) > \delta_{PFHSS}(\wp,\mathcal{T}^{(1)})$. It can be seen from this ranking alternative $\mathcal{T}^{(3)}$ is the most suitable alternative. Therefore $\mathcal{T}^{(3)}$ is the best alternative, the ranking of other alternatives given as $\mathcal{T}^{(3)} > \mathcal{T}^{(2)} > \mathcal{T}^{(4)} > \mathcal{T}^{(1)}$. Graphical results of alternatives ratings can be seen in Figure 2.

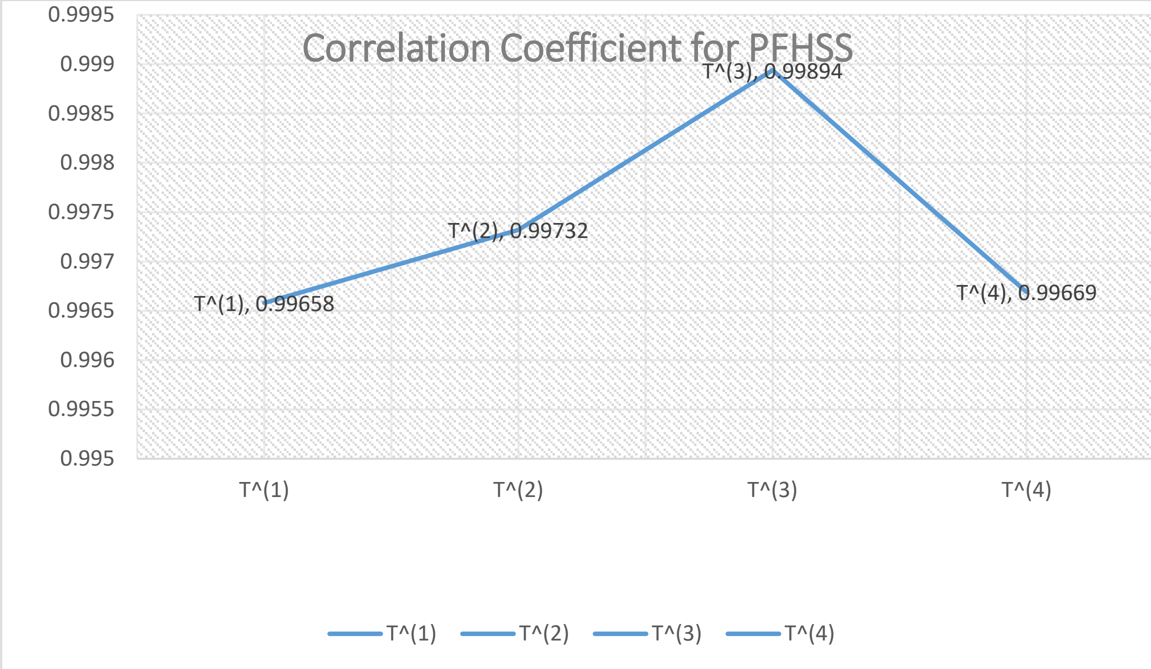

Figure 2: Alternatives rating based on correlation coefficient under PFHSS

6. Discussion and Comparative Analysis

In the following section, we are going to debate the effectivity, naivety, flexibility as well as favorable position of the suggested method along with the algorithm. We also organized a brief comparative analysis of the following: The suggested method along with the prevailing approaches.

6.1 Superiority of the Proposed Method

Through this research and comparative analysis, we have concluded that the proposed methods' results are more general than prevailing techniques. However, in the decision-making process, compared with the existing decision-making methods, it contains more information to deal with the data's uncertainty. Moreover, many of FS's mixed structure has become a special case of PFHSS, add some suitable conditions. In it, the information related to the object can be expressed more accurately and empirically, so it is a convenient tool for combining inaccurate and uncertain information in the decision-making process. Therefore, our proposed method is effective, flexible, simple, and superior to other hybrid structures of fuzzy sets.

Table 6: Comparative analysis between some existing techniques and the proposed approach

	Set	Truthiness	Falsity	Attributes	Multi sub-attributes	**Advantages**
Zadeh [1]	FS	✓	×	✓	×	Deals uncertainty by using fuzzy interval
Atanassov [2]	IFS	✓	✓	✓	×	Deals uncertainty by using MD and NMD
Yager [21]	PFS	✓	✓	✓	×	Deals more uncertainty by using MD and NMD comparative to IFS
Zulqarnain et al. [44]	IFHSS	✓	✓	✓	✓	Deals uncertainty of multi sub-attributes

Proposed approach	PFHSS	✓	✓	✓	✓	Deals more uncertainty comparative to IFHSS

6.2 Comparative Analysis

By using the technique of Zadeh [1], we deal with the true information of the alternatives, but this method cannot deal with falsity objects and multi sub-attributes of the alternatives. By utilizing the Atanassov [2], and Yager [21] methodologies, we cannot deal with the alternatives' multi-sub-attribute information. But our proposed method can easily solve these obstacles and provide more effective results for the MCDM problem. Zulqarnain et al. [44] presented IFHSS deals with the uncertainty by using the MD and NMD of the sub-attributes of a set of parameters, but the sum of MD and NMD of sub-attributes cannot exceed 1. In some cases the sum of MD and NMD exceeds 1, then existing IFHSS fails to deal with such situations. Instead of this, our developed method is an advanced technique that can handle alternatives with multi-sub-attributes information when the sum of MD and NMD exceeds 1. A comparison can be seen in the above-listed Table 6. However, on the other hand, the methodology we have established deals with the truthiness and falsity of alternatives with multi sub-attributes. Therefore, the technique we developed is more efficient and can provide better results for decision-makers through a variety of information comparative to existing techniques.

7. Conclusion

The Pythagorean fuzzy hypersoft set is a novel concept that is an extension of the intuitionistic fuzzy soft set and generalization of the intuitionistic fuzzy hypersoft set. In this work, we studied some basic concepts and developed some basic operations for PFHSS with their properties. We proposed the AND-operation and OR-operation under the PFHSS environment with their desirable properties in the following research. The idea of necessity and possibility operations with numerous properties under the Pythagorean fuzzy hypersoft set is also presented in it. Furthermore, the concept of CC is also established in this research with its decision-making methodology. We used the developed methodology to solve decision-making problems to ensure the validity and applicability of the developed decision-making methodology. Furthermore, A comparative analysis is presented to verify the validity and demonstration of the proposed method. Finally, based on the results obtained, it is concluded that the suggested techniques showed higher stability and practicality for decision-makers in the decision-making process. In the future, anyone can extend the PFHSS to interval-valued PFHSS, aggregation operators, and TOPSIS technique under PFHSS.

6. Acknowledgment

This research is partially supported by a grant of National Natural Science Foundation of China (11971384).

References

1. L. A. Zadeh, Fuzzy Sets, *Information and Control*, 8(1965) 338–353.
2. K. Atanassov, Intuitionistic Fuzzy Sets, *Fuzzy Sets and Systems*, 20(1986) 87–96.
3. H. Garg, G. Kaur, Cubic intuitionistic fuzzy sets and its fundamental properties, *J. Mult-valued Log. S*, 33 (2019) 507–537.
4. K. Atanassov, G. Gargov, Interval valued intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 31 (1989) 343-349.
5. H. Garg, K. Kumar, Linguistic interval-valued Atanassov intuitionistic fuzzy sets and their applications to group decision-making problems, *IEEE T. Fuzzy Syst.*, 27(2019) 2302–2311.
6. D. Molodtsov, Soft Set Theory First Results, *Computers & Mathematics with Applications*, 37(1999) 19–31.
7. P. K. Maji, R. Biswas, A. R. Roy, Soft set theory, *Computers and Mathematics with Applications*, 45(4–5) (2003) 555–562.
8. P. K. Maji, A. R. Roy, R. Biswas, An Application of Soft Sets in A Decision Making Problem, *Computers and Mathematics with Applications*, 44(2002) 1077–1083.

9. P. K. Maji, R. Biswas, A. R. Roy, Fuzzy soft sets, *Journal of Fuzzy Mathematics*, 9 (2001) 589–602.
10. P. K. Maji, R. Biswas, A. Roy, Intuitionistic fuzzy soft sets, *Journal of Fuzzy Mathematics*, 9 (2001) 677–692.
11. X. Yang, T. Y. Lin, J. Yang, Y. Li, D. Yu, Combination of interval-valued fuzzy set and soft set, *Computers and Mathematics with Applications*, 58 (2009) 521–527.
12. Y. Jiang, Y. Tang, Q. Chen, H. Liu, J. Tang, Interval-valued intuitionistic fuzzy soft sets and their properties, *Computers and Mathematics with Applications*, 60 (2010) 906–918.
13. H. Garg, R. Arora, Generalized and group-based generalized intuitionistic fuzzy soft sets with applications in decision-making, *Applied intelligence*, 48 (2018) 343–356.
14. H. Garg, New ranking method for normal intuitionistic sets under crisp, interval environments and its applications to multiple attribute decision making process, *Complex & Intelligent Systems*, 6(3) (2020) 559-571.
15. T. Mahmood, Z. Ali, Entropy measure and TOPSIS method based on correlation coefficient using complex q-rung orthopair fuzzy information and its application to multi-attribute decision making. Soft Computing, 11 (2020) 1-27.
16. H. Garg and R Arora, Maclaurin symmetric mean aggregation operators based on t-norm operations for the dual hesitant fuzzy soft set, *Journal of Ambient Intelligence and Humanized Computing*, 11 (2020) 375–410.
17. R. Arora, H. Garg, A robust correlation coefficient measure of dual hesitant fuzzy soft sets and their application in decision making, *Eng Appl Artif Intell*, 72 (2018) 80–92.
18. R. Arora, H. Garg, Group decision-making method based on prioritized linguistic intuitionistic fuzzy aggregation operators and its fundamental properties, *Comput Appl Math*, 38(2) (2019)1–36.
19. H. Garg, R. Arora, Generalized Maclaurin symmetric mean aggregation operators based on Archimedean t-norm of the intuitionistic fuzzy soft set information, *Artifcial Intelligence Review*, (2020) 1-41.
20. R. R. Yager, Pythagorean fuzzy subsets. Proceedings joint IFSA world congress and NAFIPS annual meeting. Edmonton, Canada, (2013) 57–61.
21. R. R. Yager, Pythagorean membership grades in multicriteria decision making, *IEEE Trans Fuzzy Syst*, 22(4) (2014) 958–965.
22. X. Zhang, Z. Xu, Extension of topsis to multiple criteria decision making with Pythagorean fuzzy sets, *International Journal of Intelligent Systems* 29(12) (2014) 1061–1078.
23. R. M. Zulqarnain, S. Abdal, A. Maalik, B. Ali, Z. Zafar, M. I. Ahamad, S. Younas, A. Mariam, F. Dayan, Application of TOPSIS Method in Decision Making Via Soft Set, *BIOMEDICAL Journal of Scientific & Technical Research*, 24(3) (2020) 18208-18215.
24. M. Zulqarnain, F. Dayan, Choose Best Criteria for Decision Making Via Fuzzy TOPSIS Method, Mathematics and Computer Science, 2(6) (2017) 113-119.
25. R. M. Zulqarnain, S. Abdal, B. Ali, L. Ali, F. Dayan, M. I. Ahamad, Z. Zafar, Selection of Medical Clinic for Disease Diagnosis by Using TOPSIS Method, *International Journal of Pharmaceutical Sciences Review and Research*, 61(1) (2020) 22-27.
26. M. Zulqarnain, F. Dayan, Selection of Best Alternative For An Automotive Company By Intuitionistic Fuzzy TOPSIS Method, *International journal of scientific & technology research*, 6(10) (2017) 126-132.
27. M. Zulqarnain, F. Dayan, M. Saeed, TOPSIS analysis for the prediction of diabetes based on general characteristics of humans, *International Journal of Pharmaceutical Sciences and Research*, 9(7) (2018) 2932-2939.
28. R. M. Zulqarnain, X. L. Xin, M. Saeed, N. Ahmad, F. Dayan, B. Ahmad, Recruitment of Medical Staff in Health Department by Using TOPSIS Method, *International Journal of Pharmaceutical Sciences Review and Research*, 62(1) (2020) 1-7.
29. R. M. Zulqarnain, M. Saeed, N. Ahmad, F. Dayan, B. Ahmad, Application of TOPSIS Method for Decision Making, *International Journal of Scientific Research in Mathematical and Statistical Sciences*, 7(2) (2020) 76-81.
30. R. M. Zulqarnain, M. Saeed, B. Ali, S. Abdal, M. Saqlain, M. I. Ahamad, Z. Zafar, Generalized Fuzzy TOPSIS to Solve Multi-Criteria Decision-Making Problems, *Journal of New Theory,* 32 (2020) 40-50.
31. R. M. Zulqarnain, X. L. Xin, M. Saeed, F. Smarandache, N. Ahmad, Generalized Neutrosophic TOPSIS to Solve Multi-Criteria Decision-Making Problems, *Neutrosophic Sets and Systems*, 38 (2020) 276-292.
32. G. Wei, M. Lu, Pythagorean fuzzy power aggregation operators in multiple attribute decision making, *International Journal of Intelligent Systems, 33(1), 169-186, 2018.*
33. L. Wang, N. Li, Pythagorean fuzzy interaction power Bonferroni mean aggregation operators in multiple attribute decision making, *International Journal of Intelligent Systems, 35(1) (2020) 150-183.*

34. X. L. Zhang, A novel approach based on similarity measure for Pythagorean fuzzy multiple criteria group decision making, *Int J Intell Syst*, 31(2016) 593–611.
35. A. Guleria, R. K. Bajaj, On Pythagorean fuzzy soft matrices, operations and their applications in decision making and medical diagnosis, *Soft Computing*, 23(17) (2019) 7889-7900.
36. A. Guleria, R. K. Bajaj, Pythagorean Fuzzy-Norm Information Measure for Multicriteria Decision-Making Problem, *Advances in Fuzzy Systems*, 2018 Jan 1;2018.
37. R. K. Bajaj, A. Guleria, Dimensionality reduction technique in decision making using Pythagorean fuzzy soft matrices, *Recent Advances in Computer Science and Communications (Formerly: Recent Patents on Computer Science)*, 13(3) (2020) 406-413.
38. A. Guleria, R. K. Bajaj, Pythagorean fuzzy (R, S)-norm discriminant measure in various decision making processes, *Journal of Intelligent & Fuzzy Systems*, 38(1) (2020) 761-777.
39. F. Smarandache, Extension of Soft Set to Hypersoft Set, and then to Plithogenic Hypersoft Set. *Neutrosophic Sets and Systems*, 22 (2018) 168-170.
40. S. Gayen, F. Smarandache, S. Jha, . K. Singh, S. Broumi, R. Kumar, Introduction to Plithogenic Subgroup. In Neutrosophic Graph Theory and Algorithms, *IGI Global*, (2020) 213-259.
41. R. M. Zulqarnain, X. L. Xin, M. Saqlain, F. Smarandache, Generalized Aggregate Operators on Neutrosophic Hypersoft Set, *Neutrosophic Sets and Systems*, 36 (2020) 271-281.
42. M. Saqlain, N. Jafar, S. Moin, M. Saeed, S. Broumi, Single and Multi-valued Neutrosophic Hypersoft set and Tangent Similarity Measure of Single valued Neutrosophic Hypersoft Sets, *Neutrosophic Sets and Systems*, *32*(1) (2020) 20.
43. M. Saqlain, S. Moin, M. N. Jafar, M. Saeed, F. Smarandache, Aggregate Operators of Neutrosophic Hypersoft Set. *Neutrosophic Sets and Systems*, *32*(1) (2020) 18.
44. R. M. Zulqarnain, X. L. Xin, M. Saeed, Extension of TOPSIS method under intuitionistic fuzzy hypersoft environment based on correlation coefficient and aggregation operators to solve decision making problem, *AIMS Mathematics*, 6 (3) (2021) 2732-2755.
45. M. Abdel-Basset, W. Ding, R. Mohamed, N. Metawa, An integrated plithogenic MCDM approach for financial performance evaluation of manufacturing industries, *Risk Manag*, 22(2020) 192–218.
46. M. Abdel-Basset, R. Mohamed, A. E. N. H. Zaied, F. Smarandache, A Hybrid Plithogenic Decision-Making Approach with Quality Function Deployment for Selecting Supply Chain Sustainability Metrics, *Symmetry*, 11(7) (2019) 90.

Received: Sep 15, 2020. Accepted: Feb 10, 2021