

The Algebraic Structure of Normal Groups Associated with Q-Neutrosophic Soft Sets

Majdoleen Abuqamar^{1,*}, Abd Ghafur Ahmad², Nasruddin Hassan³

^{1,2,3}School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi 43600, Selangor, Malaysia.

E-mail: ¹p90675@siswa.ukm.edu.my, ²mjabuqamar@gmail.com, ²ghafur@ukm.edu.my, ³nas@ukm.edu.my

*Correspondence: ¹mjabuqamar@gmail.com

Abstract: In this paper, we define the notion of Q-neutrosophic normal soft groups and discuss several related structural characteristics and properties. Additionally, we discuss the relation between Q-neutrosophic normal soft groups and normal soft groups. Furthermore, we define the concept of Q-neutrosophic soft cosets and discuss several relevant attributes.

Keywords: Neutrosophic soft group, Neutrosophic soft set, Q-neutrosophic soft group, Q-neutrosophic soft set.

1 Introduction

Fuzzy sets were established by Zadeh [1] as a tool to deal with uncertain data. The idea of neutrosophic fuzzy sets, an extension of fuzzy sets, was introduced by Smarandache [2, 3] to handle indeterminate and uncertain situations. As another way to deal with uncertain information, Molodtsov [4] introduced the concept of soft sets. Various researchers around the world have extended fuzzy sets and soft sets in different directions in order to make them more appropriate to handle different types of information. However, in some cases the description of objects by fuzzy soft sets in terms of one dimensional membership function only is not adequate. This motivates Adam and Hassan [5–8] to define the Q-fuzzy soft sets and matrix as a way to deal with situations with a set of parameters and two-dimensional data. Q-neutrosophic soft sets (Q-NSSs) [9] were introduced as a new model that deals with two-dimensional uncertain data. It is a model that generalizes neutrosophic and Q-fuzzy sets simultaneously. Q-NSSs were further investigated and their basic operations and relations were discussed in [9, 10].

Different hybrid models of fuzzy sets and soft sets were utilized in different branches of mathematics, including algebra [11–13]. Bera and Mahapatra [14, 15] introduced neutrosophic soft groups and neutrosophic normal soft groups. This motivates Solairaju and Nagarajan [16] to introduce the new structure of Q-fuzzy groups which combine the concepts of Q-fuzzy sets and groups. Recently, Q-fuzzy sets were utilized to different algebraic structures, for example, Q-fuzzy normal subgroups [17], anti-Q-fuzzy normal subgroups [18]. Furthermore, Sarala and Suganya [19] utilized Q-fuzzy soft sets to establish Q-fuzzy soft rings.

In a particular view on the utilization of Q-NSSs to algebraic structures, Abu Qamar and Hassan [20] applied Q-NSS to group theory by introducing Q-neutrosophic soft groups, they examined numerous properties

and basic attributes. Additionally, they characterized the thought of Q-level soft sets of a Q-neutrosophic soft set, which is a bridge between Q-neutrosophic soft groups and soft groups. Furthermore, rings and fields were studied under Q-neutrosophic soft settings in [21, 22].

In this paper, we provide a wider discussion on Q-NSGs, by defining the notions of Q-neutrosophic normal soft groups (Q-NNSGs) and Q-neutrosophic soft cosets. Also, we discuss the relation between Q-neutrosophic normal soft groups and normal soft groups. Further, we discuss several related structural characteristics and properties.

2 Preliminaries

In this section, we recall some basic definitions related to the work in this study.

Definition 2.1 ([9]). Let X be a universal set, Q be a nonempty set and $A \subseteq E$ be a set of parameters. Let $\mu^l QNS(X)$ be the set of all multi Q-NSs on X with dimension $l = 1$. A pair (Γ_Q, A) is called a Q-NSS over X , where $\Gamma_Q : A \rightarrow \mu^l QNS(X)$ is a mapping, such that $\Gamma_Q(e) = \phi$ if $e \notin A$.

Definition 2.2 ([10]). The union of two Q-neutrosophic soft sets (Γ_Q, A) and (Ψ_Q, B) is the Q-neutrosophic soft set (Λ_Q, C) written as $(\Gamma_Q, A) \cup (\Psi_Q, B) = (\Lambda_Q, C)$, where $C = A \cup B$ and for all $c \in C$, $(x, q) \in X \times Q$, the truth-membership, indeterminacy-membership and falsity-membership of (Λ_Q, C) are as follows:

$$T_{\Lambda_Q(c)}(x, q) = \begin{cases} T_{\Gamma_Q(c)}(x, q) & \text{if } c \in A - B, \\ T_{\Psi_Q(c)}(x, q) & \text{if } c \in B - A, \\ \max\{T_{\Lambda_Q(c)}(x, q), T_{\Psi_Q(c)}(x, q)\} & \text{if } c \in A \cap B, \end{cases}$$

$$I_{\Lambda_Q(c)}(x, q) = \begin{cases} I_{\Gamma_Q(c)}(x, q) & \text{if } c \in A - B, \\ I_{\Psi_Q(c)}(x, q) & \text{if } c \in B - A, \\ \min\{I_{\Gamma_Q(c)}(x, q), I_{\Psi_Q(c)}(x, q)\} & \text{if } c \in A \cap B, \end{cases}$$

$$F_{\Lambda_Q(c)}(x, q) = \begin{cases} F_{\Gamma_Q(c)}(x, q) & \text{if } c \in A - B, \\ F_{\Psi_Q(c)}(x, q) & \text{if } c \in B - A, \\ \min\{F_{\Gamma_Q(c)}(x, q), F_{\Psi_Q(c)}(x, q)\} & \text{if } c \in A \cap B. \end{cases}$$

Definition 2.3 ([10]). The intersection of two Q-neutrosophic soft sets (Γ_Q, A) and (Ψ_Q, B) is the Q-neutrosophic soft set (Λ_Q, C) written as $(\Gamma_Q, A) \cap (\Psi_Q, B) = (\Lambda_Q, C)$, where $C = A \cap B$ and for all $c \in C$ and $(x, q) \in X \times Q$ the truth-membership, indeterminacy-membership and falsity-membership of (Λ_Q, C) are as follows:

$$\begin{aligned} T_{\Lambda_Q(c)}(x, q) &= \min\{T_{\Gamma_Q(c)}(x, q), T_{\Psi_Q(c)}(x, q)\}, \\ I_{\Lambda_Q(c)}(x, q) &= \max\{I_{\Gamma_Q(c)}(x, q), I_{\Psi_Q(c)}(x, q)\}, \\ F_{\Lambda_Q(c)}(x, q) &= \max\{F_{\Gamma_Q(c)}(x, q), F_{\Psi_Q(c)}(x, q)\}. \end{aligned}$$

Definition 2.4. [22] Let G be a group and (Γ_Q, A) be a Q-NSS over a group G . Then (Γ_Q, A) is called a Q-neutrosophic soft group over G if for all $x, y \in G$ and $e \in A$ it satisfies:

1. $T_{\Gamma_Q(e)}(xy, q) \geq \min \{T_{\Gamma_Q(e)}(x, q), T_{\Gamma_Q(e)}(y, q)\}$, $I_{\Gamma_Q(e)}(xy, q) \leq \max \{I_{\Gamma_Q(e)}(x, q), I_{\Gamma_Q(e)}(y, q)\}$ and $F_{\Gamma_Q(e)}(xy, q) \leq \max \{F_{\Gamma_Q(e)}(x, q), F_{\Gamma_Q(e)}(y, q)\}$.
2. $T_{\Gamma_Q(e)}(x^{-1}, q) \geq T_{\Gamma_Q(e)}(x, q)$, $I_{\Gamma_Q(e)}(x^{-1}, q) \leq I_{\Gamma_Q(e)}(x, q)$ and $F_{\Gamma_Q(e)}(x^{-1}, q) \leq F_{\Gamma_Q(e)}(x, q)$.

3 Q-Neutrosophic Normal Soft Groups

In this section, we introduce the Q-NNSG and discuss several relevant structural properties.

Definition 3.1. A Q-NSG (Γ_Q, A) over the group G is called a Q-NNSG over G if $\Gamma_Q(e)$ is a Q-neutrosophic normal subgroup of G for each $e \in A$ i.e., for $x \in \Gamma_Q(e), y \in G, q \in Q$

$$\begin{aligned} T_{\Gamma_Q(e)}(yxy^{-1}, q) &\geq T_{\Gamma_Q(e)}(x, q), \\ I_{\Gamma_Q(e)}(yxy^{-1}, q) &\leq I_{\Gamma_Q(e)}(x, q), \\ F_{\Gamma_Q(e)}(yxy^{-1}, q) &\leq F_{\Gamma_Q(e)}(x, q). \end{aligned}$$

Definition 3.2. A Q-NSG (Γ_Q, A) over the group G is called abelian Q-NSG if $\forall x, y \in G, q \in Q, e \in A$ the following hold

$$\begin{aligned} T_{\Gamma_Q(e)}(xy, q) &= T_{\Gamma_Q(e)}(yx, q), \\ I_{\Gamma_Q(e)}(xy, q) &= I_{\Gamma_Q(e)}(yx, q), \\ F_{\Gamma_Q(e)}(xy, q) &= F_{\Gamma_Q(e)}(yx, q). \end{aligned}$$

Example 3.3. Let $G = (\mathbb{Z}, +)$ be a group and $A = \mathbb{N}$ be the parametric set. Define a Q-NSG (Γ_Q, A) as follows: For $q \in Q, x \in \mathbb{Z}, m \in \mathbb{N}$

$$\begin{aligned} T_{\Gamma_Q(m)}(x, q) &= \begin{cases} 0 & \text{if } x \text{ is odd} \\ \frac{2}{3} & \text{if } x \text{ is even,} \end{cases} \\ I_{\Gamma_Q(m)}(x, q) &= \begin{cases} \frac{1}{n} & \text{if } x \text{ is odd} \\ 0 & \text{if } x \text{ is even,} \end{cases} \\ F_{\Gamma_Q(m)}(x, q) &= \begin{cases} 1 - \frac{3}{n} & \text{if } x \text{ is odd} \\ 0 & \text{if } x \text{ is even.} \end{cases} \end{aligned}$$

It is clear that (Γ_Q, \mathbb{N}) is a Q-NNSG over G .

Proposition 3.4. Let (Γ_Q, A) be a Q-NNSG over a group G . Then, $\forall x, y \in G, q \in Q$ and $e \in A$,

1. $T_{\Gamma_Q(e)}(yxy^{-1}, q) = T_{\Gamma_Q(e)}(x, q)$, $I_{\Gamma_Q(e)}(yxy^{-1}, q) = I_{\Gamma_Q(e)}(x, q)$, $F_{\Gamma_Q(e)}(yxy^{-1}, q) = F_{\Gamma_Q(e)}(x, q)$.
2. (Γ_Q, A) is an abelian Q-NSG over G .

Proof. 1.

$$\begin{aligned} T_{\Gamma_Q(e)}(x, q) &= T_{\Gamma_Q(e)}\left((y^{-1}y)x(y^{-1}y), q\right) \\ &= T_{\Gamma_Q(e)}\left(y^{-1}(yxy^{-1})y, q\right) \\ &= T_{\Gamma_Q(e)}\left(y^{-1}(yxy^{-1})(y^{-1})^{-1}, q\right) \\ &\geq T_{\Gamma_Q(e)}(yxy^{-1}, q). \end{aligned}$$

Now, from Definition 3.1 $T_{\Gamma_Q(e)}(yxy^{-1}, q) = T_{\Gamma_Q(e)}(x, q)$.

In a similar manner we can show that $I_{\Gamma_Q(e)}(yxy^{-1}, q) = I_{\Gamma_Q(e)}(x, q)$ and $F_{\Gamma_Q(e)}(yxy^{-1}, q) = F_{\Gamma_Q(e)}(x, q)$.

2. $T_{\Gamma_Q(e)}(x, q) = T_{\Gamma_Q(e)}(yxy^{-1}, q)$, this implies $T_{\Gamma_Q(e)}(xy, q) = T_{\Gamma_Q(e)}(yxyy^{-1}, q) = T_{\Gamma_Q(e)}(yx, q)$. Similarly, we can show that $I_{\Gamma_Q(e)}(xy, q) = I_{\Gamma_Q(e)}(yx, q)$ and $F_{\Gamma_Q(e)}(xy, q) = F_{\Gamma_Q(e)}(yx, q)$. Hence, (Γ_Q, A) is an abelian Q-NSG over G . □

Theorem 3.5. Let (Γ_Q, A) and (Ψ_Q, B) be two Q-NNSG over a group G . Then, $(\Gamma_Q, A) \cap (\Psi_Q, B)$ is also a Q-NNSG over G .

Proof. Let $(\Lambda_Q, C) = (\Gamma_Q, A) \cap (\Psi_Q, B)$. Then, for $x, y \in G, q \in Q, e \in C$

$$\begin{aligned} T_{\Lambda_Q(e)}(yxy^{-1}, q) &= \min \left\{ T_{\Gamma_Q(e)}(yxy^{-1}, q), T_{\Psi_Q(e)}(yxy^{-1}, q) \right\} \\ &\geq \min \left\{ T_{\Gamma_Q(e)}(x, q), T_{\Psi_Q(e)}(x, q) \right\} \\ &= T_{\Lambda_Q(e)}(x, q), \end{aligned}$$

$$\begin{aligned} I_{\Lambda_Q(e)}(yxy^{-1}, q) &= \max \left\{ I_{\Gamma_Q(e)}(yxy^{-1}, q), I_{\Psi_Q(e)}(yxy^{-1}, q) \right\} \\ &\leq \max \left\{ I_{\Gamma_Q(e)}(x, q), I_{\Psi_Q(e)}(x, q) \right\} \\ &= I_{\Lambda_Q(e)}(x, q). \end{aligned}$$

Similarly, we can show that $F_{\Lambda_Q(e)}(yxy^{-1}, q) \leq F_{\Lambda_Q(e)}(x, q)$. This completes the proof. □

Remark 3.6. The union of two Q-NNSGs is not a Q-NNSG since the union is not a Q-NSG.

The next example illustrates the above remark.

Example 3.7. let $G = (\mathbb{Z}, +)$ and $E = 2\mathbb{Z}$. Define the two Q-neutrosophic soft groups (Γ_Q, E) and (Ψ_Q, E) over G as the following:

For $x, m \in \mathbb{Z}, q \in Q$

$$\begin{aligned}
 T_{\Gamma_Q(2m)}(x, q) &= \begin{cases} 0.50 & \text{if } x = 4tm, \exists t \in \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases} \\
 I_{\Gamma_Q(2m)}(x, q) &= \begin{cases} 0 & \text{if } x = 4tm, \exists t \in \mathbb{Z}, \\ 0.25 & \text{otherwise,} \end{cases} \\
 F_{\Gamma_Q(2m)}(x, q) &= \begin{cases} 0 & \text{if } x = 4tm, \exists t \in \mathbb{Z}, \\ 0.10 & \text{otherwise,} \end{cases}
 \end{aligned}$$

and

$$\begin{aligned}
 T_{\Psi_Q(2m)}(x, q) &= \begin{cases} 0.67 & \text{if } x = 6tm, \exists t \in \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases} \\
 I_{\Psi_Q(2m)}(x, q) &= \begin{cases} 0 & \text{if } x = 6tm, \exists t \in \mathbb{Z}, \\ 0.20 & \text{otherwise,} \end{cases} \\
 F_{\Psi_Q(2m)}(x, q) &= \begin{cases} 0 & \text{if } x = 6tm, \exists t \in \mathbb{Z}, \\ 0.17 & \text{otherwise.} \end{cases}
 \end{aligned}$$

Let $(\Gamma_Q, A) \cup (\Psi_Q, B) = (\Lambda_Q, E)$. For $m = 3, x = 12, y = 18$ we have

$$T_{\Lambda_Q(6)}(12.18^{-1}, q) = T_{\Lambda_Q(6)}(-6, q) = \max \{ T_{\Gamma_Q(6)}(-6, q), T_{\Psi_Q(6)}(-6, q) \} = \max \{ 0, 0 \} = 0$$

and

$$\begin{aligned}
 &\min \{ T_{\Lambda_Q(6)}(12, q), T_{\Lambda_Q(6)}(18, q) \} \\
 &= \min \{ \max \{ T_{\Gamma_Q(6)}(12, q), T_{\Psi_Q(6)}(12, q) \}, \max \{ T_{\Gamma_Q(6)}(18, q), T_{\Psi_Q(6)}(18, q) \} \} \\
 &= \min \{ \max \{ 0.50, 0 \}, \max \{ 0, 0.67 \} \} \\
 &= \min \{ 0.50, 0.67 \} = 0.50.
 \end{aligned}$$

Hence, $T_{\Lambda_Q(6)}(12.18^{-1}, q) = 0 < \min \{ T_{\Lambda_Q(6)}(12, q), T_{\Lambda_Q(6)}(18, q) \} = 0.50$; i.e. $(\Lambda_Q, E) = (\Gamma_Q, A) \cup (\Psi_Q, B)$ is not a Q-neutrosophic soft group.

Theorem 3.8. Let (Γ_Q, A) be a Q-NSS over G . Then, (Γ_Q, A) is a Q-NNSG over G if and only if for all $\alpha, \beta, \gamma \in [0, 1]$, the Q-level soft set $(\Gamma_Q, A)_{(\alpha, \beta, \gamma)} \neq \phi$ is a normal soft group over G .

Proof. We only need to prove the normality. For $x \in (\Gamma_Q, A)_{(\alpha, \beta, \gamma)}, y \in G$ and $q \in Q$, we have

$$\begin{aligned}
 T_{\Gamma_Q(e)}(yxy^{-1}, q) &= T_{\Gamma_Q(e)}(yy^{-1}x, q) = T_{\Gamma_Q(e)}(x, q) \geq \alpha, \\
 I_{\Gamma_Q(e)}(yxy^{-1}, q) &= I_{\Gamma_Q(e)}(yy^{-1}x, q) = I_{\Gamma_Q(e)}(x, q) \leq \beta, \\
 F_{\Gamma_Q(e)}(yxy^{-1}, q) &= F_{\Gamma_Q(e)}(yy^{-1}x, q) = F_{\Gamma_Q(e)}(x, q) \leq \gamma.
 \end{aligned}$$

It follows that $xyx^{-1} \in (\Gamma_Q, A)_{(\alpha, \beta, \gamma)}$, i.e. $(\Gamma_Q, A)_{(\alpha, \beta, \gamma)}$ is a Q-NNSG of G .

Conversely, assume that (Γ_Q, A) is not a Q-NNSG over G . Then, there exists $e \in A$ such that $\Gamma_Q(e)$ is not a Q-NN subgroup of G . Then, there exists $x_1, y_1 \in G$ and $q \in Q$ such that

$$\begin{aligned} &T_{\Gamma_Q(e)}(x_1y_1, q) < T_{\Gamma_Q(e)}(y_1x_1, q) \text{ or } T_{\Gamma_Q(e)}(x_1y_1, q) > T_{\Gamma_Q(e)}(y_1x_1, q) \text{ or} \\ &I_{\Gamma_Q(e)}(x_1y_1, q) < I_{\Gamma_Q(e)}(y_1x_1, q) \text{ or } I_{\Gamma_Q(e)}(x_1y_1, q) > I_{\Gamma_Q(e)}(y_1x_1, q) \text{ or} \\ &F_{\Gamma_Q(e)}(x_1y_1, q) < F_{\Gamma_Q(e)}(y_1x_1, q) \text{ or } F_{\Gamma_Q(e)}(x_1y_1, q) > F_{\Gamma_Q(e)}(y_1x_1, q). \end{aligned}$$

In case $T_{\Gamma_Q(e)}(x_1y_1, q) < T_{\Gamma_Q(e)}(y_1x_1, q)$, there exists $\alpha \in [0, 1]$ such that $T_{\Gamma_Q(e)}(x_1y_1, q) < \alpha < T_{\Gamma_Q(e)}(y_1x_1, q)$. It follows that $x_1y_1 \notin \Gamma_{Q(e)}(\alpha, \beta, \gamma)$, but for $I_{\Gamma_Q(e)}(x_1y_1, q) < \beta$ and $F_{\Gamma_Q(e)}(x_1y_1, q) < \gamma$, $x_1y_1 \notin \Gamma_{Q(e)}(\alpha, \beta, \gamma)$ this contradicts with the fact that $(\Gamma_Q, A)_{(\alpha, \beta, \gamma)}$ is a normal soft group over G . In the other cases the proof can be obtained in a similar way. □

Theorem 3.9. *Let (Γ_Q, A) be a Q-NNSG over G . Let*

$$\begin{aligned} (\Gamma_Q, A)|_{\acute{e}} = \left\{ x \in G : T_{\Gamma_Q(e)}(x, q) = T_{\Gamma_Q(e)}(\acute{e}, q), I_{\Gamma_Q(e)}(x, q) = I_{\Gamma_Q(e)}(\acute{e}, q), \right. \\ \left. F_{\Gamma_Q(e)}(x, q) = F_{\Gamma_Q(e)}(\acute{e}, q), e \in A \right\}, \end{aligned}$$

where \acute{e} is the unit element of G . Then, $(\Gamma_Q, A)|_{\acute{e}}$ is a normal soft group over G .

Proof. For each $e \in A$ and $x, y \in (\Gamma_Q, A)|_{\acute{e}}$, $q \in Q$, we have

$$\begin{aligned} T_{\Gamma_Q(e)}(xy^{-1}, q) &\geq \min \left\{ T_{\Gamma_Q(e)}(x, q), T_{\Gamma_Q(e)}(y, q) \right\} \\ &= \min \left\{ T_{\Gamma_Q(e)}(\acute{e}, q), T_{\Gamma_Q(e)}(\acute{e}, q) \right\} \\ &= T_{\Gamma_Q(e)}(\acute{e}, q), \end{aligned}$$

$$\begin{aligned} I_{\Gamma_Q(e)}(xy^{-1}, q) &\leq \max \left\{ I_{\Gamma_Q(e)}(x, q), I_{\Gamma_Q(e)}(y, q) \right\} \\ &= \max \left\{ I_{\Gamma_Q(e)}(\acute{e}, q), I_{\Gamma_Q(e)}(\acute{e}, q) \right\} \\ &= I_{\Gamma_Q(e)}(\acute{e}, q). \end{aligned}$$

Similarly, we can show $F_{\Gamma_Q(e)}(xy^{-1}, q) \leq F_{\Gamma_Q(e)}(\acute{e}, q)$. Always, $T_{\Gamma_Q(e)}(\acute{e}, q) \geq T_{\Gamma_Q(e)}(xy^{-1}, q)$, $I_{\Gamma_Q(e)}(\acute{e}, q) \leq I_{\Gamma_Q(e)}(xy^{-1}, q)$ and $F_{\Gamma_Q(e)}(\acute{e}, q) \leq F_{\Gamma_Q(e)}(xy^{-1}, q)$. Therefore, $T_{\Gamma_Q(e)}(xy^{-1}, q) = T_{\Gamma_Q(e)}(\acute{e}, q)$, $I_{\Gamma_Q(e)}(xy^{-1}, q) = I_{\Gamma_Q(e)}(\acute{e}, q)$, $F_{\Gamma_Q(e)}(xy^{-1}, q) = F_{\Gamma_Q(e)}(\acute{e}, q)$ and $xy^{-1} \in (\Gamma_Q, A)|_{\acute{e}}$.

Next, let $x \in (\Gamma_Q, A)|_{\acute{e}}$ and $y \in G$. Then,

$$\begin{aligned} T_{\Gamma_Q(e)}(yxy^{-1}, q) &= T_{\Gamma_Q(e)}(x, q) = T_{\Gamma_Q(e)}(\acute{e}, q), \\ I_{\Gamma_Q(e)}(yxy^{-1}, q) &= I_{\Gamma_Q(e)}(x, q) = I_{\Gamma_Q(e)}(\acute{e}, q), \\ F_{\Gamma_Q(e)}(yxy^{-1}, q) &= F_{\Gamma_Q(e)}(x, q) = F_{\Gamma_Q(e)}(\acute{e}, q). \end{aligned}$$

Therefore, $yxy^{-1} \in (\Gamma_Q, A)|_{\acute{e}}$. Hence, $(\Gamma_Q, A)|_{\acute{e}}$ is a normal soft group over G . □

4 Q-Neutrosophic Soft Cosets

In this section, we present the Q-neutrosophic soft cosets with some related properties.

Definition 4.1. Let (Γ_Q, A) be a Q-NSG over G and $g \in G$ be a fixed element. Then, the set $g(\Gamma_Q, A) = \{g\Gamma_Q(e) : e \in A\}$ is called a left Q-neutrosophic soft coset of (Γ_Q, A) , where

$$\begin{aligned} g\Gamma_Q(e) &= \left\{ \left\langle (x, q), T_{g\Gamma_Q(e)}(x, q), I_{g\Gamma_Q(e)}(x, q), F_{g\Gamma_Q(e)}(x, q) \right\rangle : x \in G, q \in Q \right\} \\ &= \left\{ \left\langle (x, q), T_{\Gamma_Q(e)}(g^{-1}x, q), I_{\Gamma_Q(e)}(g^{-1}x, q), F_{\Gamma_Q(e)}(g^{-1}x, q) \right\rangle : x \in G, q \in Q \right\}. \end{aligned}$$

The right Q-neutrosophic soft coset of (Γ_Q, A) in G is $(\Gamma_Q, A)g = \{\Gamma_Q(e)g : e \in A\}$, where

$$\Gamma_Q(e)g = \left\{ \left\langle (x, q), T_{\Gamma_Q(e)}(xg^{-1}, q), I_{\Gamma_Q(e)}(xg^{-1}, q), F_{\Gamma_Q(e)}(xg^{-1}, q) \right\rangle : x \in G, q \in Q \right\}.$$

Example 4.2. Let G be a classical group. Then, $(\Gamma_Q, A) = \{\Gamma_Q(e) : e \in A\}$, where

$$\Gamma_Q(e) = \left\{ \left\langle (x, q), T_{\Gamma_Q(e)}(x, q), I_{\Gamma_Q(e)}(x, q), F_{\Gamma_Q(e)}(x, q) \right\rangle : x \in G, q \in Q \right\}$$

with $T_{\Gamma_Q(e)}(x, q) = T_{\Gamma_Q(e)}(\acute{e}, q)$, $I_{\Gamma_Q(e)}(x, q) = I_{\Gamma_Q(e)}(\acute{e}, q)$ and $F_{\Gamma_Q(e)}(x, q) = F_{\Gamma_Q(e)}(\acute{e}, q)$; (\acute{e} being the identity element in G) is a Q-NNSG of G . In that case, we can get a neutrosophic soft coset.

Proposition 4.3. (Γ_Q, A) is called a Q-NNSG over G if and only if the left and right Q-neutrosophic soft cosets are equal.

Proof. Suppose that (Γ_Q, A) is a Q-NNSG over G . Then,

$$\begin{aligned} g\Gamma_Q(e) &= \left\{ \left\langle (x, q), T_{g\Gamma_Q(e)}(x, q), I_{g\Gamma_Q(e)}(x, q), F_{g\Gamma_Q(e)}(x, q) \right\rangle : x \in G, q \in Q \right\} \\ &= \left\{ \left\langle (x, q), T_{\Gamma_Q(e)}(g^{-1}x, q), I_{\Gamma_Q(e)}(g^{-1}x, q), F_{\Gamma_Q(e)}(g^{-1}x, q) \right\rangle : x \in G, q \in Q \right\} \\ &= \left\{ \left\langle (x, q), T_{\Gamma_Q(e)}(xg^{-1}, q), I_{\Gamma_Q(e)}(xg^{-1}, q), F_{\Gamma_Q(e)}(xg^{-1}, q) \right\rangle : x \in G, q \in Q \right\} \\ &= \left\{ \left\langle (x, q), T_{\Gamma_Q(e)g}(x, q), I_{\Gamma_Q(e)g}(x, q), F_{\Gamma_Q(e)g}(x, q) \right\rangle : x \in G, q \in Q \right\} \\ &= \Gamma_Q(e)g. \end{aligned}$$

Thus, $g(\Gamma_Q, A) = \{g\Gamma_Q(e) : e \in A\} = \{\Gamma_Q(e)g : e \in A\} = (\Gamma_Q, A)g$.

Next, suppose that $g(\Gamma_Q, A) = (\Gamma_Q, A)g$.

Then,

$$T_{g\Gamma_Q(e)}(x, q) = T_{\Gamma_Q(e)g}(x, q), I_{g\Gamma_Q(e)}(x, q) = I_{\Gamma_Q(e)g}(x, q) \text{ and } F_{g\Gamma_Q(e)}(x, q) = F_{\Gamma_Q(e)g}(x, q).$$

This implies,

$$T_{\Gamma_Q(e)}(g^{-1}x, q) = T_{\Gamma_Q(e)}(xg^{-1}, q), I_{\Gamma_Q(e)}(g^{-1}x, q) = I_{\Gamma_Q(e)}(xg^{-1}, q) \text{ and } F_{\Gamma_Q(e)}(g^{-1}x, q) = F_{\Gamma_Q(e)}(xg^{-1}, q).$$

Thus,

$$T_{\Gamma_Q(e)}(xg^{-1}, q) = T_{\Gamma_Q(e)}(g^{-1}x, q), I_{\Gamma_Q(e)}(xg^{-1}, q) = I_{\Gamma_Q(e)}(g^{-1}x, q) \text{ and } F_{\Gamma_Q(e)}(xg^{-1}, q) = F_{\Gamma_Q(e)}(g^{-1}x, q),$$

which implies

$$T_{\Gamma_Q(e)}(gxxg^{-1}, q) = T_{\Gamma_Q(e)}(x, q), I_{\Gamma_Q(e)}(gxxg^{-1}, q) = I_{\Gamma_Q(e)}(x, q), F_{\Gamma_Q(e)}(gxxg^{-1}, q) = F_{\Gamma_Q(e)}(x, q).$$

Thus, (Γ_Q, A) is a Q-NNSG over G .

Therefore, if (Γ_Q, A) is a Q-NNSG over G then the left and right Q-neutrosophic soft cosets coincide. In this case, we call it Q-neutrosophic soft cosets instead of left or right Q-neutrosophic soft cosets separately. \square

Theorem 4.4. Let (Γ_Q, A) be a Q-NNSG over the group G and the set ς be the collection of all distinct Q-neutrosophic soft cosets of (Γ_Q, A) in G . Then, ς is a group in classical sense under the operation of composition: $g_1(\Gamma_Q, A)y(\Gamma_Q, A) = (g_1y)(\Gamma_Q, A), \forall g_1, g_2 \in G$.

Proof. First we show that the operation is well defined in the sense that if $g_1(\Gamma_Q, A) = g'_1(\Gamma_Q, A)$ and $g_2(\Gamma_Q, A) = g'_2(\Gamma_Q, A)$, then $g_1(\Gamma_Q, A)g_2(\Gamma_Q, A) = (g'_1g'_2)(\Gamma_Q, A)$ for $g_1, g_2, g'_1, g'_2 \in G$.

Now, $g_1(\Gamma_Q, A) = g'_1(\Gamma_Q, A)$ implies $g_1^{-1}g'_1 = \Gamma_Q(e_1), e_1 \in A$ and $g_2(\Gamma_Q, A) = g'_2(\Gamma_Q, A)$ implies $g_2^{-1}g'_2 = \Gamma_Q(e_2), e_2 \in A$.

We show, $(g_1g_2)(\Gamma_Q, A) = (g'_1g'_2)(\Gamma_Q, A)$ i.e., $(g_1g_2)^{-1}(g'_1g'_2) \in G$. Now,

$$\begin{aligned} (g_1g_2)^{-1}(g'_1g'_2) &= g_2^{-1}g_1^{-1}g'_1g'_2 \\ &= g_2^{-1}\Gamma_Q(e_1)g'_2 \\ &= g_2^{-1}g_2\Gamma_Q(e_1) \\ &= \Gamma_Q(e_2)\Gamma_Q(e_1) \\ &= \Gamma_Q(e_3) \in (\Gamma_Q, A), e_3 \in A. \end{aligned}$$

Hence, the operation is well defined. Now,

1. the closure axiom is clearly satisfied.

2. $g_1(\Gamma_Q, A)[g_2(\Gamma_Q, A)g_3(\Gamma_Q, A)] = g_1(\Gamma_Q, A)(g_2g_3)(\Gamma_Q, A) = g_1(g_2g_3)(\Gamma_Q, A)$ and $[g_1(\Gamma_Q, A)g_2(\Gamma_Q, A)]g_3(\Gamma_Q, A) = (g_1g_2)g_3(\Gamma_Q, A) = (g_1g_2)g_3(\Gamma_Q, A)$ for $g_1, g_2, g_3 \in G$. Now, $g_1(g_2g_3) = (g_1g_2)g_3$, since G is a group.

3. $\acute{e}(\Gamma_Q, A)g_1(\Gamma_Q, A) = (\acute{e}g_1)(\Gamma_Q, A) = g_1(\Gamma_Q, A)$ and $g_1(\Gamma_Q, A)\acute{e}(\Gamma_Q, A) = (g_1\acute{e})(\Gamma_Q, A) = g_1(\Gamma_Q, A)$ for \acute{e} being the unity in G .

4. $g_1^{-1}(\Gamma_Q, A)g_1(\Gamma_Q, A) = (g_1^{-1}g_1)(\Gamma_Q, A) = \acute{e}(\Gamma_Q, A) = (\Gamma_Q, A)$ and $g_1(\Gamma_Q, A)g_1^{-1}(\Gamma_Q, A) = (g_1g_1^{-1})(\Gamma_Q, A) = \acute{e}(\Gamma_Q, A) = (\Gamma_Q, A)$.

Thus, ς is a group. This group is called the quotient group of G by (Γ_Q, A) and is denoted by $G/(\Gamma_Q, A)$. \square

Theorem 4.5. Let (Γ_Q, A) be a Q-NNSG over G . Then, there exists a natural homomorphism $\varphi : G \rightarrow G/(\Gamma_Q, A)$ defined by $\varphi(g) = g(\Gamma_Q, A), \forall g \in G$ in the classical sense.

Proof. Let $\varphi : G \rightarrow G/(\Gamma_Q, A)$ be given by $\varphi(g) = g(\Gamma_Q, A), \forall e \in A$. We show that φ is a homomorphism i.e.

$\varphi(g_1g_2) = \varphi(g_1)\varphi(g_2), \forall g_1, g_2 \in G$, i.e., $(g_1g_2)\Gamma_Q(e) = g_1\Gamma_Q(e)g_2\Gamma_Q(e)$. Now, for $x \in G, q \in Q$

$$\begin{aligned}(g_1\Gamma_Q(e))(x, q) &= \langle T_{g_1\Gamma_Q(e)}(x, q), I_{g_1\Gamma_Q(e)}(x, q), F_{g_1\Gamma_Q(e)}(x, q) \rangle \\ &= \langle T_{\Gamma_Q(e)}(g_1^{-1}x, q), I_{\Gamma_Q(e)}(g_1^{-1}x, q), F_{\Gamma_Q(e)}(g_1^{-1}x, q) \rangle,\end{aligned}$$

$$\begin{aligned}(g_2\Gamma_Q(e))(x, q) &= \langle T_{g_2\Gamma_Q(e)}(x, q), I_{g_2\Gamma_Q(e)}(x, q), F_{g_2\Gamma_Q(e)}(x, q) \rangle \\ &= \langle T_{\Gamma_Q(e)}(g_2^{-1}x, q), I_{\Gamma_Q(e)}(g_2^{-1}x, q), F_{\Gamma_Q(e)}(g_2^{-1}x, q) \rangle,\end{aligned}$$

$$(g_1g_2\Gamma_Q(e))(x, q) = \langle T_{\Gamma_Q(e)}((g_1g_2)^{-1}x, q), I_{\Gamma_Q(e)}((g_1g_2)^{-1}x, q), F_{\Gamma_Q(e)}((g_1g_2)^{-1}x, q) \rangle.$$

Then,

$$\begin{aligned}[(g_1\Gamma_Q(e))(g_2\Gamma_Q(e))](x, q) &= \langle \min \{T_{g_1\Gamma_Q(e)}(x, q), T_{g_2\Gamma_Q(e)}(x, q)\}, \\ &\quad \max \{I_{g_1\Gamma_Q(e)}(x, q), I_{g_2\Gamma_Q(e)}(x, q)\}, \\ &\quad \max \{F_{g_1\Gamma_Q(e)}(x, q), F_{g_2\Gamma_Q(e)}(x, q)\} \rangle \\ &= \langle \min \{T_{\Gamma_Q(e)}(g_1^{-1}x, q), T_{\Gamma_Q(e)}(g_2^{-1}x, q)\}, \\ &\quad \max \{I_{\Gamma_Q(e)}(g_1^{-1}x, q), I_{\Gamma_Q(e)}(g_2^{-1}x, q)\}, \\ &\quad \max \{F_{\Gamma_Q(e)}(g_1^{-1}x, q), F_{\Gamma_Q(e)}(g_2^{-1}x, q)\} \rangle\end{aligned}$$

Further,

$$\begin{aligned}T_{\Gamma_Q(e)}((g_1g_2)^{-1}x, q) &= T_{\Gamma_Q(e)}(g_2^{-1}g_1^{-1}x, q) \\ &= T_{\Gamma_Q(e)}(g_2^{-1}g_1^{-1}xg_2g_1, q) \\ &= T_{\Gamma_Q(e)}(g_1^{-1}xg_2, q) \\ &\geq \min \{T_{\Gamma_Q(e)}(g_1x, q), T_{\Gamma_Q(e)}(g_2x, q)\}.\end{aligned}$$

Hence, $T_{\Gamma_Q(e)}((g_1g_2)^{-1}x, q) = \min \{T_{\Gamma_Q(e)}(g_1^{-1}x, q), T_{\Gamma_Q(e)}(g_2^{-1}x, q)\}$, similarly, $I_{\Gamma_Q(e)}((g_1g_2)^{-1}x, q) = \max \{I_{\Gamma_Q(e)}(g_1^{-1}x, q), I_{\Gamma_Q(e)}(g_2^{-1}x, q)\}$ and $F_{\Gamma_Q(e)}((g_1g_2)^{-1}x, q) = \max \{F_{\Gamma_Q(e)}(g_1^{-1}x, q), F_{\Gamma_Q(e)}(g_2^{-1}x, q)\}$. This shows that, $[(g_1\Gamma_Q(e))(g_2\Gamma_Q(e))](x, q) = [(g_1g_2)\Gamma_Q(e)](x, q)$ which implies, $\varphi(g_1g_2) = \varphi(g_1)\varphi(g_2)$. \square

5 Conclusion

We have introduced the notions of Q-neutrosophic normal soft groups and Q-neutrosophic soft cosets. We have discussed several related structural characteristics and properties. For future research, we can extend these topics to hyperalgebra. Also, these topics may be discussed using t-norm and s-norm. We intend to further explore the applications of the algebraic structure to different extensions of fuzzy sets in order to provide a significant addition to existing theories for handling uncertainties, especially in the area of soft sets [23–25].

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Zadeh, L.A. Fuzzy sets. *Information and Control* 1965, 8, 338-353.
2. Smarandache, F. *Neutrosophy: Neutrosophic probability, set and logic*; American Research Press: Rehoboth, IL, USA, 1998.
3. Smarandache, F. Neutrosophic set, a generalisation of the intuitionistic fuzzy sets. *International Journal of Pure and Applied Mathematics* 2005, 24, 287-297.
4. Molodtsov, D. Soft set theory-first results. *Computers & Mathematics with Applications* 1999, 37, 19-31.
5. Adam, F.; Hassan, N. Operations on Q-fuzzy soft set. *Applied Mathematical Sciences* 2014, 8, 8697-8701.
6. Adam, F.; Hassan, N. Q-fuzzy soft set. *Applied Mathematical Sciences* 2014, 8, 8689-8695.
7. Adam, F.; Hassan, N. Properties on the multi Q-fuzzy soft matrix. *AIP Conference Proceedings* 2014, 1614, 834-839.
8. Adam, F.; Hassan, N. Q-fuzzy soft matrix and its application. *AIP Conference Proceedings* 2014, 1602, 772-778.
9. Abu Qamar, M.; Hassan, N. Q-neutrosophic soft relation and its application in decision making. *Entropy* 2018, 20(3), 172.
10. Abu Qamar, M.; Hassan, N. An approach toward a Q-neutrosophic soft set and its application in decision making. *Symmetry* 2019, 11(2), 139.
11. Yaqoob, N.; Akram, M.; Aslam, M. Intuitionistic fuzzy soft groups induced by (t, s) norm. *Indian Journal of Science and Technology* 2013, 6, 4282-4289.
12. Al-Masarwah, A.; Ahmad, A.G. m-polar fuzzy ideals of BCK/BCI-algebras. *Journal of King Saud University-Science* 2019, 31 (4), 1220-1226.
13. Al-Masarwah, A.; Ahmad, A.G. On some properties of doubt bipolar fuzzy H-ideals in BCK/BCI-algebras. *European Journal of Pure and Applied Mathematics* 2018, 11, 652-670.
14. Bera, T.; Mahapatra, N. K. Introduction to neutrosophic soft groups. *Neutrosophic Sets and Systems*, 2016, 13, 118-127.
15. Bera, T.; Mahapatra, N.K. On Neutrosophic normal soft groups. *International Journal of Applied and Computational Mathematics* 2017, 3, 3047-3066.
16. Solairaju, A.; Nagarajan, R. A new structure and construction of Q-fuzzy groups. *Advances in Fuzzy Mathematics* 2009, 4, 23-29.
17. Priya, T.; Ramachandran, T.; Nagalakshmi, K.T. On Q-fuzzy normal subgroups. *International Journal of Computer and Organization Trends* 2013, 3, 574-578.
18. Sithar, S.P.; Priya, T.; Nagalakshmi, K.; Ramachandran, T. On some properties of anti-Q-fuzzy normal subgroups. *General Mathematics Notes* 2014, 22, 1-10.
19. Sarala, N.; Suganya, B. Q- fuzzy soft ring. *International Journal of Engineering Research & Technology* 2015, 4, 879-882.
20. Abu Qamar, M.; Hassan, N. Characterizations of group theory under Q-neutrosophic soft environment. *Neutrosophic Sets and Systems*, 2019, 27, 114-130.

21. Abu Qamar, M.; Ahmad, A.G.; Hassan, N. An Approach to Q-Neutrosophic Soft Rings. *AIMS Mathematics*, 2019, 4(4), 1291-1306
22. Abu Qamar, M.; Ahmad, A.G.; Hassan, N. On Q-neutrosophic soft fields. *Neutrosophic Sets and Systems*, 2020, 32, 80-93.
23. Alhazaymeh, K.; Halim, S.A.; Salleh, A.R.; Hassan, N. Soft intuitionistic fuzzy sets. *Applied Mathematical Sciences* 2012, 6 (54), 2669-2680.
24. Salleh, A.R.; Alkhazaleh, S.; Hassan, N.; Ahmad, A.G. Multiparameterized soft set. *Journal of Mathematics and Statistics* 2012, 8(1), 92-97.
25. Adam, F.; Hassan, N. Multi Q-fuzzy soft set and its application, *Far East Journal of Mathematical Sciences* 2015, 97 (7), 871-881.

Received: Nov 30, 2021. Accepted: Feb 2, 2022