BMBJ-neutrosophic ideals in $BCK/BCI$-algebras

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Abstract: The concepts of a BMBJ-neutrosophic $\circ$-subalgebra and a (closed) BMBJ-neutrosophic ideal are introduced, and several properties are investigated. Conditions for an MBJ-neutrosophic set to be a BMBJ-neutrosophic ideal in $BCK/BCI$-algebras are provided. Characterizations of BMBJ-neutrosophic ideal are discussed. Relations between a BMBJ-neutrosophic subalgebra, a BMBJ-neutrosophic $\circ$-subalgebra and a (closed) BMBJ-neutrosophic ideal are considered.

Keywords: MBJ-neutrosophic set; BMBJ-neutrosophic subalgebra; BMBJ-neutrosophic ideal; BMBJ-neutrosophic $\circ$-subalgebra.

1 Introduction

Smarandache introduced the notion of neutrosophic set which is a more general platform that extends the notions of classic set, (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set (see [11, 12]). Neutrosophic set theory is applied to various part which is referred to the site http://fs.gallup.unm.edu/neutrosophy.htm. Jun and his colleagues applied the notion of neutrosophic set theory to $BCK/BCI$-algebras (see [4, 5, 6, 7, 10, 13, 14]). Borzooei et al. [2] studied commutative generalized neutrosophic ideals in $BCK$-algebras. Mohseni et al. [9] introduced the notion of MBJ-neutrosophic sets which is another generalization of neutrosophic set. They introduced the concept of MBJ-neutrosophic subalgebras in $BCK/BCI$-algebras, and investigated related properties. They gave a characterization of MBJ-neutrosophic subalgebra, and established a new MBJ-neutrosophic subalgebra by using an MBJ-neutrosophic subalgebra of a $BCI$-algebra. They considered the homomorphic inverse image of MBJ-neutrosophic subalgebra, and discussed translation of MBJ-neutrosophic subalgebra. Bordbar et al. [1] introduced the notion of BMBJ-neutrosophic subalgebras, and investigated related properties.

In this paper, we apply the notion of MBJ-neutrosophic sets to ideals of $BCK/BCI$-algebras. We introduce the concepts of a BMBJ-neutrosophic $\circ$-subalgebra and a (closed) BMBJ-neutrosophic ideal, and investigate several properties. We provide conditions for an MBJ-neutrosophic set to be a BMBJ-neutrosophic ideal in $BCK/BCI$-algebras, and discuss characterizations of BMBJ-neutrosophic ideal. We consider relations between a BMBJ-neutrosophic subalgebra, a BMBJ-neutrosophic $\circ$-subalgebra and a (closed) BMBJ-neutrosophic ideal.
2 Preliminaries

By a *BCI*-algebra, we mean a set $X$ with a binary operation $*$ and a special element 0 that satisfies the following conditions:

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,

(IV) $x * y = 0$, $y * x = 0 \Rightarrow x = y$

for all $x, y, z \in X$. If a *BCI*-algebra $X$ satisfies the following identity:

(V) $(\forall x \in X) (0 * x = 0)$,

then $X$ is called a *BCK*-algebra.

By a weakly *BCK*-algebra (see [3]), we mean a *BCI*-algebra $X$ satisfying $0 * x \leq x$ for all $x \in X$.

Every *BCK*-BCI-algebra $X$ satisfies the following conditions (see [3]):

\begin{align*}
(\forall x \in X) (x * 0 &= x), \quad (2.1) \\
(\forall x, y, z \in X) (x \leq y &\Rightarrow x * z \leq y * z, z * y \leq z * x), \quad (2.2) \\
(\forall x, y, z \in X) ((x * y) * z = (x * z) * y), \quad (2.3) \\
(\forall x, y, z \in X) ((x * z) * (y * z) \leq x * y) \quad (2.4)
\end{align*}

where $x \leq y$ if and only if $x * y = 0$. Any *BCI*-algebra $X$ satisfies the following conditions (see [3]):

\begin{align*}
(\forall x, y \in X) (x * (x * (x * y))) &= x * y), \quad (2.5) \\
(\forall x, y \in X) (0 * (x * y) = (0 * x) * (0 * y)). \quad (2.6)
\end{align*}

A *BCI*-algebra $X$ is said to be *p*-semisimple (see [3]) if

\begin{equation*}
(\forall x \in X) (0 * (0 * x) = x). \quad (2.7)
\end{equation*}

In a *p*-semisimple *BCI*-algebra $X$, the following holds:

\begin{equation*}
(\forall x, y \in X) (0 * (x * y) = y * x, x * (x * y) = y). \quad (2.8)
\end{equation*}

A *BCI*-algebra $X$ is said to be associative (see [3]) if

\begin{equation*}
(\forall x, y, z \in X) ((x * y) * z = x * (y * z)). \quad (2.9)
\end{equation*}

By an *$(S)$*-BCK-algebra, we mean a BCK-algebra $X$ such that, for any $x, y \in X$, the set

\begin{equation*}
\{ z \in X \mid z * x \leq y \}
\end{equation*}

has the greatest element, written by $x \circ y$ (see [8]).
A nonempty subset S of a BCK/BCI-algebra X is called a subalgebra of X if \(x \ast y \in S\) for all \(x, y \in S\). A subset I of a BCK/BCI-algebra X is called an ideal of X if it satisfies:

\[
0 \in I, \\
(\forall x \in X) (\forall y \in I) (x \ast y \in I \Rightarrow x \in I).
\]

A subset I of a BCI-algebra X is called a closed ideal of X (see [3]) if it is an ideal of X which satisfies:

\[
(\forall x \in X)(x \in I \Rightarrow 0 \ast x \in I).
\]

By an interval number we mean a closed subinterval \(\bar{a} = [a^-, a^+]\) of I, where \(0 \leq a^- \leq a^+ \leq 1\). Denote by \([I]\) the set of all interval numbers.

Let X be a nonempty set. A function \(A : X \to [I]\) is called an interval-valued fuzzy set (briefly, an IVF set) in X. Let \([I]^X\) stand for the set of all IVF sets in X. For every \(A \in [I]^X\) and \(x \in X\), \(A(x) = [A^-(x), A^+(x)]\) is called the degree of membership of an element \(x\) to \(A\), where \(A^- : X \to I\) and \(A^+ : X \to I\) are fuzzy sets in X which are called a lower fuzzy set and an upper fuzzy set in X, respectively. For simplicity, we denote \(A = [A^-, A^+]\).

Let X be a non-empty set. A neutrosophic set (NS) in X (see [11]) is a structure of the form:

\[
A := \{ \langle x; A_T(x), A_I(x), A_F(x) \rangle \mid x \in X \}
\]

where \(A_T : X \to [0, 1]\) is a truth membership function, \(A_I : X \to [0, 1]\) is an indeterminate membership function, and \(A_F : X \to [0, 1]\) is a false membership function. For the sake of simplicity, we shall use the symbol \(A = (A_T, A_I, A_F)\) for the neutrosophic set

\[
A := \{ \langle x; A_T(x), A_I(x), A_F(x) \rangle \mid x \in X \}.
\]

We refer the reader to the books [3, 8] for further information regarding BCK/BCI-algebras, and to the site “http://fs.gallup.unm.edu/neutrosophy.htm” for further information regarding neutrosophic set theory.

Let X be a non-empty set. By an MBJ-neutrosophic set in X (see [9]), we mean a structure of the form:

\[
\mathcal{A} := \{ \langle x; M_A(x), \tilde{B}_A(x), J_A(x) \rangle \mid x \in X \}
\]

where \(M_A\) and \(J_A\) are fuzzy sets in X, which are called a truth membership function and a false membership function, respectively, and \(\tilde{B}_A\) is an IVF set in X which is called an indeterminate interval-valued membership function.

For the sake of simplicity, we shall use the symbol \(\mathcal{A} = (M_A, \tilde{B}_A, J_A)\) for the MBJ-neutrosophic set

\[
\mathcal{A} := \{ \langle x; M_A(x), \tilde{B}_A(x), J_A(x) \rangle \mid x \in X \}.
\]

Let X be a BCK/BCI-algebra. An MBJ-neutrosophic set \(\mathcal{A} = (M_A, \tilde{B}_A, J_A)\) in X is called a BMBJ-neutrosophic subalgebra of X (see [1]) if it satisfies:

\[
(\forall x \in X)(M_A(x) + \tilde{B}_A^-(x) \leq 1, \tilde{B}_A^+(x) + J_A(x) \leq 1)
\]
and

\[
(\forall x, y \in X) \begin{cases}
M_A(x \ast y) \geq \min\{M_A(x), M_A(y)\} \\
B_A^{-}(x \ast y) \leq \max\{B_A^{-}(x), B_A^{-}(y)\} \\
B_A^{+}(x \ast y) \geq \min\{B_A^{+}(x), B_A^{+}(y)\} \\
J_A(x \ast y) \leq \max\{J_A(x), J_A(y)\}
\end{cases}
\]  \quad (2.14)

3 BMBJ-neutrosophic ideals

**Definition 3.1.** Let $X$ be a $BCK/BCI$-algebra. An MBJ-neutrosophic set $A = (M_A, B_A, J_A)$ in $X$ is called a $BMBJ$-neutrosophic ideal of $X$ if it satisfies (2.13) and

\[
(\forall x \in X) \begin{cases}
M_A(0) \geq M_A(x) \\
B_A^{-}(0) \leq B_A^{-}(x) \\
B_A^{+}(0) \geq B_A^{+}(x) \\
J_A(0) \leq J_A(x)
\end{cases}
\]  \quad (3.1)

\[
(\forall x, y \in X) \begin{cases}
M_A(x \ast y) \geq \min\{M_A(x \ast y), M_A(y)\} \\
B_A^{-}(x \ast y) \leq \max\{B_A^{-}(x \ast y), B_A^{-}(y)\} \\
B_A^{+}(x \ast y) \geq \min\{B_A^{+}(x \ast y), B_A^{+}(y)\} \\
J_A(x \ast y) \leq \max\{J_A(x \ast y), J_A(y)\}
\end{cases}
\]  \quad (3.2)

A BMBJ-neutrosophic ideal $A = (M_A, B_A, J_A)$ of a $BCI$-algebra $X$ is said to be closed if

\[
(\forall x \in X) \begin{cases}
M_A(0 \ast x) \geq M_A(x) \\
B_A^{-}(0 \ast x) \leq B_A^{-}(x) \\
B_A^{+}(0 \ast x) \geq B_A^{+}(x) \\
J_A(0 \ast x) \leq J_A(x)
\end{cases}
\]  \quad (3.3)

**Example 3.2.** Consider a set $X = \{0, 1, 2, a\}$ with the binary operation $\ast$ which is given in Table 1. Then

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>a</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

$(X; \ast, 0)$ is a $BCI$-algebra (see [3]). Let $A = (M_A, B_A, J_A)$ be an MBJ-neutrosophic set in $X$ defined by Table 2. It is routine to verify that $A = (M_A, B_A, J_A)$ is a closed MBJ-neutrosophic ideal of $X$. 

\[\text{M. Mohseni Takallo, Hashem Bordbar, R.A. Borzooei, Y.B. Jun, BMBJ-neutrosophic ideals in } \text{BCK/BCI-algebras.}\]
Table 2: MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$

<table>
<thead>
<tr>
<th>$X$</th>
<th>$M_A(x)$</th>
<th>$\tilde{B}_A(x)$</th>
<th>$J_A(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.7</td>
<td>[0.02, 0.08]</td>
<td>0.2</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>[0.02, 0.06]</td>
<td>0.2</td>
</tr>
<tr>
<td>2</td>
<td>0.4</td>
<td>[0.02, 0.06]</td>
<td>0.7</td>
</tr>
<tr>
<td>$a$</td>
<td>0.3</td>
<td>[0.02, 0.06]</td>
<td>0.7</td>
</tr>
</tbody>
</table>

**Proposition 3.3.** Let $X$ be a $BCK/BCI$-algebra. Then every BMBJ-neutrosophic ideal $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ of $X$ satisfies the following assertion.

$$x * y \leq z \Rightarrow \begin{cases} M_A(x) \geq \min\{M_A(y), M_A(z)\}, \\ \tilde{B}_A(x) \leq \max\{\tilde{B}_A(y), \tilde{B}_A(z)\}, \\ B_A^+(x) \geq \min\{B_A^+(y), B_A^+(z)\}, \\ J_A(x) \leq \max\{J_A(y), J_A(z)\} \end{cases}$$

(3.4)

for all $x, y, z \in X$.

**Proof.** Let $x, y, z \in X$ be such that $x * y \leq z$. Then

$$M_A(x * y) \geq \min\{M_A((x * y) * z), M_A(z)\} = \min\{M_A(0), M_A(z)\} = M_A(z),$$

$$\tilde{B}_A(x * y) \leq \max\{\tilde{B}_A((x * y) * z), \tilde{B}_A(z)\} = \max\{\tilde{B}_A(0), \tilde{B}_A(z)\} = \tilde{B}_A(z),$$

$$B_A^+(x * y) \geq \min\{B_A^+(((x * y) * z), B_A^+(z))\} = \min\{B_A^+(0), B_A^+(z)\} = B_A^+(z),$$

and

$$J_A(x * y) \leq \max\{J_A(((x * y) * z), J_A(z))\} = \max\{J_A(0), J_A(z)\} = J_A(z).$$

It follows that

$$M_A(x) \geq \min\{M_A(x * y), M_A(y)\} = \min\{M_A(y), M_A(z)\},$$

$$\tilde{B}_A(x) \leq \max\{\tilde{B}_A(x * y), \tilde{B}_A(y)\} = \max\{\tilde{B}_A(y), \tilde{B}_A(z)\},$$

$$B_A^+(x) \geq \min\{B_A^+(x * y), B_A^+(y)\} = \min\{B_A^+(y), B_A^+(z)\},$$

and

$$J_A(x) \leq \max\{J_A(x * y), J_A(y)\} = \max\{J_A(y), J_A(z)\}.$$
Therefore, \( b \in y \bigcup_U \) that \( U, t, s, \alpha \) for all \( \text{Theorem 3.5.} \) An MBJ-neutrosophic set \( \text{where} \)

\[ \text{Proof.} \] Let \( A = (M_A, \tilde{B}_A, J_A) \) be an MBJ-neutrosophic set in \( X \) satisfying (3.1) and (3.4). Note that \( x \ast (x \ast y) \leq y \) for all \( x, y \in X \). It follows from (3.4) that

\[ M_A(x) \geq \min \{M_A(x \ast y), M_A(y)\}, \]

\[ B^-_A(x) \leq \max \{B^-_A(x \ast y), B^-_A(y)\}, \]

\[ B^+_A(x) \geq \min \{B^+_A(x \ast y), B^+_A(y)\}, \]

and

\[ J_A(x) \leq \max \{J_A(x \ast y), J_A(y)\}. \]

Therefore \( A = (M_A, \tilde{B}_A, J_A) \) is a BMBJ-neutrosophic ideal of \( X \).

Given an MBJ-neutrosophic set \( A = (M_A, \tilde{B}_A, J_A) \) in a \( BCK/BCI \)-algebra \( X \), we consider the following sets.

\[ U(M_A; t) := \{x \in X \mid M_A(x) \geq t\}, \]

\[ L(B^-_A; \alpha^-) := \{x \in X \mid B^-_A(x) \leq \alpha^-\}, \]

\[ U(B^+_A; \alpha^+) := \{x \in X \mid B^+_A(x) \geq \alpha^+\}, \]

\[ L(J_A; s) := \{x \in X \mid J_A(x) \leq s\} \]

where \( t, s, \alpha^-, \alpha^+ \in [0, 1] \).

**Theorem 3.5.** An MBJ-neutrosophic set \( A = (M_A, \tilde{B}_A, J_A) \) in a \( BCK/BCI \)-algebra \( X \) is an MBJ-neutrosophic ideal of \( X \) if and only if the non-empty sets \( U(M_A; t), L(B^-_A; \alpha^-), U(B^+_A; \alpha^+)\) and \( L(J_A; s) \) are ideals of \( X \) for all \( t, s, \alpha^-, \alpha^+ \in [0, 1] \).

**Proof.** Suppose that \( A = (M_A, \tilde{B}_A, J_A) \) is an MBJ-neutrosophic ideal of \( X \). Let \( t, s, \alpha^-, \alpha^+ \in [0, 1] \) be such that \( U(M_A; t), L(B^-_A; \alpha^-), U(B^+_A; \alpha^+)\) and \( L(J_A; s) \) are non-empty. Obviously, \( 0 \in U(M_A; t) \cap L(B^-_A; \alpha^-) \cap U(B^+_A; \alpha^+) \cap L(J_A; s) \). For any \( x, y, a, b, p, q, u, v \in X \), if \( x \ast y \in U(M_A; t), y \in U(M_A; t), a \ast b \in L(B^-_A; \alpha^-), b \in L(B^-_A; \alpha^-), p \ast q \in U(B^+_A; \alpha^+), q \in U(B^+_A; \alpha^+), u \ast v \in L(J_A; s) \) and \( v \in L(J_A; s) \), then

\[ M_A(x) \geq \min \{M_A(x \ast y), M_A(y)\} \geq \min \{t, t\} = t, \]

\[ B^-_A(a) \leq \max \{B^-_A(a \ast b), B^-_A(b)\} \leq \max \{\alpha^-, \alpha^-\} = \alpha^- , \]

\[ B^+_A(p) \geq \min \{B^+_A(p \ast q), B^+_A(q)\} \geq \min \{\alpha^+, \alpha^+\} = \alpha^+ , \]

\[ J_A(u) \leq \max \{J_A(u \ast v), J_A(v)\} \leq \min \{s, s\} = s , \]

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and so \( x \in U(M_A; t) \), \( a \in L(B_A^-; \alpha^-) \), \( p \in U(B_A^+; \alpha^+) \) and \( u \in L(J_A; s) \). Therefore \( U(M_A; t), L(B_A^-; \alpha^-), U(B_A^+; \alpha^+) \) and \( L(J_A; s) \) are ideals of \( X \).

Conversely, assume that the non-empty sets \( U(M_A; t), L(B_A^-; \alpha^-), U(B_A^+; \alpha^+) \) and \( L(J_A; s) \) are ideals of \( X \) for all \( t, s, \alpha^- \), \( \alpha^+ \in [0, 1] \). Assume that \( M_A(0) < M_A(a), B_A^- (0) > B_A^- (a), B_A^+ (0) < B_A^+ (a) \) and \( J_A(0) > J_A(a) \) for some \( a \in X \). Then \( 0 \notin U(M_A; M_A(a)) \cap L(B_A^-; B_A^- (a)) \) \cap \( U(B_A^+; B_A^+ (a)) \) \cap \( L(J_A; J_A(a)) \), which is a contradiction. Hence \( M_A(0) \leq M_A(x), B_A^- (0) \leq B_A^- (x), B_A^+ (0) \geq B_A^+ (x) \) and \( J_A(0) \leq J_A(x) \) for all \( x \in X \). If \( M_A(a_0) < \min\{ M_A(a_0 * b_0), M_A(b_0) \} \) for some \( a_0, b_0 \in X \), then \( a_0 * b_0 \in U(M_A; t_0) \) and \( b_0 \in U(M_A; t_0) \) but \( a_0 \notin U(M_A; t_0) \) for \( t_0 := \min\{ M_A(a_0 * b_0), M_A(b_0) \} \). This is a contradiction, and thus \( M_A(a) \geq \min\{ M_A(a * b), M_A(b) \} \) for all \( a, b \in X \). Similarly, we can show that \( J_A(a) \leq \max\{ J_A(a * b), J_A(b) \} \) for all \( a, b \in X \). Suppose that \( B_A^- (a_0) > \max\{ B_A^- (a_0 * b_0), B_A^- (b_0) \} \) for some \( a_0, b_0 \in X \). Taking \( \alpha^- = \max\{ B_A^- (a_0 * b_0), B_A^- (b_0) \} \) implies that \( a_0 * b_0 \in L(B_A^-; \alpha^-) \) and \( b_0 \in L(B_A^-; \alpha^-) \) but \( a_0 \notin L(B_A^-; \alpha^-) \). This is a contradiction. Thus \( B_A^- (x) \leq \max\{ B_A^- (x * y), B_A^- (y) \} \) for all \( x, y \in X \). Similarly, we obtain \( B_A^+ (x) \geq \min\{ B_A^+ (x * y), B_A^+ (y) \} \) for all \( x, y \in X \). Consequently \( A = (M_A, B_A, J_A) \) is a BMBJ-neutrosophic ideal of \( X \).

\[\square\]

**Theorem 3.6.** An MBJ-neutrosophic set \( A = (M_A, B_A, J_A) \) in a BCK/BCI-algebra \( X \) is a BMBJ-neutrosophic ideal of \( X \) if and only if \( (M_A, B_A^-) \) and \( (B_A^+, J_A) \) are intuitionistic fuzzy ideals of \( X \).

**Proof.** Straightforward. \[\square\]

**Theorem 3.7.** Given an ideal \( I \) of a BCK/BCI-algebra \( X \), let \( A = (M_A, B_A, J_A) \) be an MBJ-neutrosophic set in \( X \) defined by

\[
M_A(x) = \begin{cases} 
  t & \text{if } x \in I, \\
  0 & \text{otherwise,}
\end{cases} \quad B_A^- (x) = \begin{cases} 
  \alpha^- & \text{if } x \in I, \\
  1 & \text{otherwise,}
\end{cases} \quad J_A(x) = \begin{cases} 
  s & \text{if } x \in I, \\
  1 & \text{otherwise,}
\end{cases}
\]

where \( t, \alpha^+ \in (0, 1], s, \alpha^- \in [0, 1) \). Then \( A = (M_A, B_A, J_A) \) is a BMBJ-neutrosophic ideal of \( X \) such that \( U(M_A; t) = L(B_A^-; \alpha^-) = U(B_A^+; \alpha^+) = L(J_A; s) = I \).

**Proof.** It is clear that \( U(M_A; t) = L(B_A^-; \alpha^-) = U(B_A^+; \alpha^+) = L(J_A; s) = I \). Let \( x, y \in X \). If \( x * y \in I \) and \( y \in I \), then \( x \in I \) and so

\[
M_A(x) = t = \min\{ M_A(x * y), M_A(y) \} \\
B_A^- (x) = \alpha^- = \max\{ B_A^- (x * y), B_A^- (y) \}, \\
B_A^+ (x) = \alpha^+ = \min\{ B_A^+ (x * y), B_A^+ (y) \}, \\
J_A(x) = s = \max\{ J_A(x * y), J_A(y) \}.
\]

If any one of \( x * y \) and \( y \) is contained in \( I \), say \( x * y \in I \), then \( M_A(x * y) = t, B_A^- (x * y) = \alpha^-, J_A(x * y) = s, M_A(y) = 0, B_A^- (y) = 1, B_A^+ (y) = 0 \) and \( J_A(y) = 1 \). Hence

\[
M_A(x) \geq 0 = \min\{ t, 0 \} = \min\{ M_A(x * y), M_A(y) \} \\
B_A^- (x) \leq 1 = \max\{ B_A^- (x * y), B_A^- (y) \}, \\
B_A^+ (x) \geq 0 = \min\{ B_A^+ (x * y), B_A^+ (y) \}, \\
J_A(x) \leq 1 = \max\{ s, 1 \} = \max\{ J_A(x * y), J_A(y) \}.
\]

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M. Mohseni Takallo, Hashem Bordbar, R.A. Borzooei, Y.B. Jun, BMBJ-neutrosophic ideals in BCK/BCI-algebras.
If \( x \ast y \notin I \), then \( M_A(x \ast y) = 0 = M_A(y), B_A^-(x \ast y) = 1 = B_A^-(y), B_A^+(x \ast y) = 0 = B_A^+(y) \) and \( J_A(x \ast y) = 1 = J_A(y) \). It follows that
\[
M_A(x) \geq 0 = \min\{M_A(x \ast y), M_A(y)\}, \\
B_A^-(x) \leq 1 = \max\{B_A^-(x \ast y), B_A^-(y)\}, \\
B_A^+(x) \geq 0 = \min\{B_A^+(x \ast y), B_A^+(y)\}, \\
J_A(x) \leq 1 = \max\{J_A(x \ast y), J_A(y)\}.
\]

It is obvious that \( M_A(0) \geq M_A(x), B_A^-(0) \leq B_A^-(x), B_A^+(0) \geq B_A^+(x) \) and \( J_A(0) \leq J_A(x) \) for all \( x \in X \). Therefore \( A = (M_A, \tilde{B}_A, J_A) \) is a BMBJ-neutrosophic ideal of \( X \).

**Theorem 3.8.** For any non-empty subset \( I \) of \( X \), let \( A = (M_A, \tilde{B}_A, J_A) \) be an MBJ-neutrosophic set in \( X \) which is given in Theorem 3.7. If \( A = (M_A, \tilde{B}_A, J_A) \) is a BMBJ-neutrosophic ideal of \( X \), then \( I \) is an ideal of \( X \).

**Proof.** Obviously, \( 0 \in I \). Let \( x, y \in X \) be such that \( x \ast y \in I \) and \( y \in I \). Then \( M_A(x \ast y) = t = M_A(y), B_A^-(x \ast y) = \alpha^- = B_A^-(y), B_A^+(x \ast y) = \alpha^+ = B_A^+(y) \) and \( J_A(x \ast y) = s = J_A(y) \). Thus
\[
M_A(x) \geq \min\{M_A(x \ast y), M_A(y)\} = t, \\
B_A^-(x) \leq \max\{B_A^-(x \ast y), B_A^-(y)\} = \alpha^-, \\
B_A^+(x) \geq \min\{B_A^+(x \ast y), B_A^+(y)\} = \alpha^+, \\
J_A(x) \leq \max\{J_A(x \ast y), J_A(y)\} = s,
\]
and hence \( x \in I \). Therefore \( I \) is an ideal of \( X \).

**Theorem 3.9.** In a BCK-algebra, every BMBJ-neutrosophic ideal is a BMBJ-neutrosophic subalgebra.

**Proof.** Let \( A = (M_A, \tilde{B}_A, J_A) \) be a BMBJ-neutrosophic ideal of a BCK-algebra \( X \). Since \( (x \ast y) \ast x \leq y \) for all \( x, y \in X \), it follows from Proposition 3.3 that
\[
M_A(x \ast y) \geq \min\{M_A(x), M_A(y)\}, \\
B_A^-(x \ast y) \leq \max\{B_A^-(x), B_A^-(y)\}, \\
B_A^+(x \ast y) \geq \min\{B_A^+(x), B_A^+(y)\}, \\
J_A(x \ast y) \leq \max\{J_A(x), J_A(y)\}
\]
for all \( x, y \in X \). Hence \( A = (M_A, \tilde{B}_A, J_A) \) is a BMBJ-neutrosophic subalgebra of a BCK-algebra \( X \).

The converse of Theorem 3.9 may not be true as seen in the following example.

**Example 3.10.** Consider a BCK-algebra \( X = \{0, 1, 2, 3\} \) with the binary operation \( \ast \) which is given in Table 3. Let \( A = (M_A, \tilde{B}_A, J_A) \) be an MBJ-neutrosophic set in \( X \) defined by Table 4. Then \( A = (M_A, \tilde{B}_A, J_A) \) is a BMBJ-neutrosophic subalgebra of \( X \), but it is not a BMBJ-neutrosophic ideal of \( X \) since
\[
B_A^+(1) \nless \min\{B_A^+(1 \ast 2), B_A^+(2)\}.
\]

We provide a condition for a BMBJ-neutrosophic subalgebra to be a BMBJ-neutrosophic ideal in a BCK-algebra.
Table 3: Cayley table for the binary operation “∗”

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4: MBJ-neutrosophic set \( A = (M_A, \tilde{B}_A, J_A) \)

<table>
<thead>
<tr>
<th>( X )</th>
<th>( M_A(x) )</th>
<th>( \tilde{B}_A(x) )</th>
<th>( J_A(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.7</td>
<td>[0.03, 0.08]</td>
<td>0.2</td>
</tr>
<tr>
<td>1</td>
<td>0.4</td>
<td>[0.02, 0.06]</td>
<td>0.3</td>
</tr>
<tr>
<td>2</td>
<td>0.4</td>
<td>[0.03, 0.08]</td>
<td>0.4</td>
</tr>
<tr>
<td>3</td>
<td>0.6</td>
<td>[0.02, 0.06]</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Theorem 3.11. Let \( A = (M_A, \tilde{B}_A, J_A) \) be a BMBJ-neutrosophic subalgebra of a BCK-algebra \( X \) satisfying the condition (3.4). Then \( A = (M_A, \tilde{B}_A, J_A) \) is a BMBJ-neutrosophic ideal of \( X \).

Proof. For any \( x \in X \), we get

\[
M_A(0) = M_A(x \ast x) \geq \min\{M_A(x), M_A(x)\} = M_A(x),
\]

\[
B_A^- (0) = B_A^- (x \ast x) \leq \max\{B_A^- (x), B_A^- (x)\} = B_A^- (x),
\]

\[
B_A^+ (0) = B_A^+ (x \ast x) \geq \min\{B_A^+ (x), B_A^+ (x)\} = B_A^+ (x),
\]

and

\[
J_A(0) = J_A(x \ast x) \leq \max\{J_A(x), J_A(x)\} = J_A(x).
\]

Since \( x \ast (x \ast y) \leq y \) for all \( x, y \in X \), it follows from (3.4) that

\[
M_A(x) \geq \min\{M_A(x \ast y), M_A(y)\},
\]

\[
B_A^- (x) \leq \max\{B_A^- (x \ast y), B_A^- (y)\},
\]

\[
B_A^+ (x) \geq \min\{B_A^+ (x \ast y), B_A^+ (y)\},
\]

\[
J_A(x) \leq \max\{J_A(x \ast y), J_A(y)\}
\]

for all \( x, y \in X \). Therefore \( A = (M_A, \tilde{B}_A, J_A) \) is a BMBJ-neutrosophic ideal of \( X \). 

M. Mohseni Takallo, Hashem Bordbar, R.A. Borzooei, Y.B. Jun, BMBJ-neutrosophic ideals in BCK/BCI-algebras.
Theorem 3.9 is not true in a $BCI$-algebra as seen in the following example.

**Example 3.12.** Let $(Y, *, 0)$ be a $BCI$-algebra and let $(Z, -, 0)$ be an adjoint $BCI$-algebra of the additive group $(Z, +, 0)$ of integers. Then $X = Y \times Z$ is a $BCI$-algebra and $I = Y \times \mathbb{N}$ is an ideal of $X$ where $\mathbb{N}$ is the set of all non-negative integers (see [3]). Let $A = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in $X$ which is given in Theorem 3.7. Then $A = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic ideal of $X$ by Theorem 3.7. But it is not a BMBJ-neutrosophic subalgebra of $X$ since

\[
M_A((0, 0) \ast (0, 1)) = M_A((0, -1)) = 0 < t = \min \{M_A((0, 0)), M_A(0, 1)\},
\]

\[
B_A^-((0, 0) \ast (0, 2)) = B_A^-((0, -2)) = 1 > \alpha^- = \max \{B_A^-((0, 0)), B_A^-((0, 2))\},
\]

\[
B_A^+((0, 0) \ast (0, 2)) = B_A^+((0, -2)) = 0 < \alpha^+ = \min \{B_A^+((0, 0)), B_A^+((0, 2))\},
\]

and/or

\[
J_A((0, 0) \ast (0, 3)) = J_A((0, -3)) = 1 > s = \max \{J_A((0, 0)), J_A(0, 3)\}.
\]

**Definition 3.13.** A BMBJ-neutrosophic ideal $A = (M_A, \tilde{B}_A, J_A)$ of a $BCI$-algebra $X$ is said to be **closed** if

\[
(\forall x \in X)(M_A(0 \ast x) \geq M_A(x), B_A^-((0 \ast x)) \leq B_A^-(x), B_A^+(0 \ast x) \geq B_A^+(x), J_A((0 \ast x)) \leq J_A(x)).
\]

**(3.5)**

**Theorem 3.14.** In a $BCI$-algebra, every closed BMBJ-neutrosophic ideal is a BMBJ-neutrosophic subalgebra.

**Proof.** Let $A = (M_A, \tilde{B}_A, J_A)$ be a closed BMBJ-neutrosophic ideal of a $BCI$-algebra $X$. Using (3.2), (2.3), (III) and (3.3), we have

\[
M_A(x \ast y) \geq \min \{M_A((x \ast y) \ast x), M_A(x)\} = \min \{M_A(0 \ast y), M_A(x)\} \geq \min \{M_A(y), M_A(x)\},
\]

\[
B_A^-((x \ast y) \ast x) \leq \max \{B_A^-(x \ast y) \ast x), B_A^-(x)\} = \max \{B_A^-(0 \ast y), B_A^-(x)\} \leq \max \{B_A^-(y), B_A^-(x)\},
\]

\[
B_A^+(x \ast y) \geq \min \{B_A^+(x \ast y) \ast x), B_A^+(x)\} = \min \{B_A^+(0 \ast y), B_A^+(x)\} \geq \min \{B_A^+(y), B_A^+(x)\},
\]

and

\[
J_A((x \ast y) \ast x) \leq \max \{J_A((x \ast y) \ast x), J_A(x)\} = \max \{J_A(0 \ast y), J_A(x)\} \leq \max \{J_A(y), J_A(x)\}
\]

for all $x, y \in X$. Hence $A = (M_A, \tilde{B}_A, J_A)$ is a BMBJ-neutrosophic subalgebra of $X$. \hfill \Box

**Theorem 3.15.** In a weakly $BCK$-algebra, every BMBJ-neutrosophic ideal is closed.

**Proof.** Let $A = (M_A, \tilde{B}_A, J_A)$ be a BMBJ-neutrosophic ideal of a weakly $BCK$-algebra $X$. For any $x \in X$, we obtain

\[
M_A(0 \ast x) \geq \min \{M_A((0 \ast x) \ast x), M_A(x)\} = \min \{M_A(0), M_A(x)\} = M_A(x),
\]
\[ B^+_A(0 \ast x) \geq \min\{B^+_A((0 \ast x) \ast x), B^+_A(x)\} = \min\{B^+_A(0), B^+_A(x)\} = B^+_A(x), \]

\[ B^-_A(0 \ast x) \leq \max\{B^-_A((0 \ast x) \ast x), B^-_A(x)\} = \max\{B^-_A(0), B^-_A(x)\} = B^-_A(x), \]

and

\[ J_A(0 \ast x) \leq \max\{J_A((0 \ast x) \ast x), J_A(x)\} = \max\{J_A(0), J_A(x)\} = J_A(x). \]

Therefore \( A = (M_A, \tilde{B}_A, J_A) \) is a closed BMBJ-neutrosophic ideal of \( X \).

\[ \square \]

**Corollary 3.16.** In a weakly BCK-algebra, every BMBJ-neutrosophic ideal is a BMBJ-neutrosophic subalgebra.

The following example shows that any BMBJ-neutrosophic subalgebra is not a BMBJ-neutrosophic ideal in a BCI-algebra.

**Example 3.17.** Consider a BCI-algebra \( X = \{0, a, b, c, d, e\} \) with the \( \ast \)-operation in Table 5.

<table>
<thead>
<tr>
<th>( \ast )</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>c</td>
<td>b</td>
<td>c</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>c</td>
<td>b</td>
<td>c</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td>c</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>b</td>
<td>0</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>d</td>
<td>d</td>
<td>b</td>
<td>a</td>
<td>c</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>e</td>
<td>e</td>
<td>b</td>
<td>a</td>
<td>c</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

Let \( A = (M_A, \tilde{B}_A, J_A) \) be an MBJ-neutrosophic set in \( X \) defined by Table 6.

<table>
<thead>
<tr>
<th>( X )</th>
<th>( M_A(x) )</th>
<th>( \tilde{B}_A(x) )</th>
<th>( J_A(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.7</td>
<td>[0.14, 0.19]</td>
<td>0.3</td>
</tr>
<tr>
<td>a</td>
<td>0.4</td>
<td>[0.04, 0.45]</td>
<td>0.6</td>
</tr>
<tr>
<td>b</td>
<td>0.7</td>
<td>[0.14, 0.19]</td>
<td>0.3</td>
</tr>
<tr>
<td>c</td>
<td>0.7</td>
<td>[0.14, 0.19]</td>
<td>0.3</td>
</tr>
<tr>
<td>d</td>
<td>0.4</td>
<td>[0.04, 0.45]</td>
<td>0.6</td>
</tr>
<tr>
<td>e</td>
<td>0.4</td>
<td>[0.04, 0.45]</td>
<td>0.6</td>
</tr>
</tbody>
</table>
It is routine to verify that $A = (M_A, \bar{B}_A, J_A)$ is a BMBJ-neutrosophic subalgebra of $X$. But it is not a BMBJ-neutrosophic ideal of $X$ since

$$M_A(d) < \min\{M_A(d * c), M_A(c)\},$$

$$B_A^- (d) > \max\{B_A^- (d * c), B_A^-(c)\},$$

$$B_A^+ (d) < \min\{B_A^+ (d * c), B_A^+(c)\},$$

and/or

$$J_A(d) > \max\{J_A(d * c), J_A(c)\}.$$

**Theorem 3.18.** *In a p-semisimple BCI-algebra $X$, the following are equivalent.

1. $A = (M_A, \bar{B}_A, J_A)$ is a closed BMBJ-neutrosophic ideal of $X$.

2. $A = (M_A, \bar{B}_A, J_A)$ is a BMBJ-neutrosophic subalgebra of $X$.

**Proof.** (1) $\Rightarrow$ (2). See Theorem 3.14.

(2) $\Rightarrow$ (1). Suppose that $A = (M_A, \bar{B}_A, J_A)$ is a BMBJ-neutrosophic subalgebra of $X$. For any $x \in X$, we get

$$M_A(0) = M_A(x * x) \geq \min\{M_A(x), M_A(x)\} = M_A(x),$$

$$B_A^- (0) = B_A^-(x * x) \leq \max\{B_A^- (x), B_A^-(x)\} = B_A^-(x),$$

$$B_A^+ (0) = B_A^+(x * x) \geq \min\{B_A^+(x), B_A^+(x)\} = B_A^+(x),$$

and

$$J_A(0) = J_A(x * x) \leq \max\{J_A(x), J_A(x)\} = J_A(x).$$

Hence $M_A(0 * x) \geq \min\{M_A(0), M_A(x)\} = M_A(x), B_A^- (0 * x) \leq \max\{B_A^- (0), B_A^-(x)\} = B_A^-(x) B_A^+(0 * x) \geq \min\{B_A^+(0), B_A^+(x)\} = B_A^+(x)$ and $J_A(0 * x) \leq \max\{J_A(0), J_A(x)\} = J_A(x)$ for all $x \in X$. Let $x, y \in X$. Then

$$M_A(x) = M_A(y * (y * x)) \geq \min\{M_A(y), M_A(y * x)\}$$

$$= \min\{M_A(y), M_A(0 * (x * y))\}$$

$$\geq \min\{M_A(x * y), M_A(y)\},$$

$$B_A^-(x) = B_A^-(y * (y * x)) \leq \max\{B_A^-(y), B_A^-(y * x)\}$$

$$= \max\{B_A^-(y), B_A^-(0 * (x * y))\}$$

$$\leq \max\{B_A^-(x * y), B_A^-(y)\}$$

---

*M. Mohseni Takallo, Hashem Bordbar, R.A. Borzooei, Y.B. Jun, BMBJ-neutrosophic ideals in BCK/BCI-algebras.*
\[ B^+_A(x) = B^+_A(y \ast (y \ast x)) \geq \min\{B^+_A(y), B^+_A(y \ast x)\} \]
\[ = \min\{B^+_A(y), B^+_A(0 \ast (x \ast y))\} \]
\[ \geq \min\{B^+_A(x \ast y), B^+_A(y)\} \]

and

\[ J_A(x) = J_A(y \ast (y \ast x)) \leq \max\{J_A(y), J_A(y \ast x)\} \]
\[ = \max\{J_A(y), J_A(0 \ast (x \ast y))\} \]
\[ \leq \max\{J_A(x \ast y), J_A(y)\} \]

Therefore \( A = (M_A, \tilde{B}_A, J_A) \) is a closed BMBJ-neutrosophic ideal of \( X \).

Since every associative \( BCI \)-algebra is \( p \)-semisimple, we have the following corollary.

**Corollary 3.19.** In an associative \( BCI \)-algebra \( X \), the following are equivalent.

1. \( A = (M_A, \tilde{B}_A, J_A) \) is a closed BMBJ-neutrosophic ideal of \( X \).
2. \( A = (M_A, \tilde{B}_A, J_A) \) is a BMBJ-neutrosophic subalgebra of \( X \).

**Definition 3.20.** Let \( X \) be an \((S)\)-\( BCK \)-algebra. An MBJ-neutrosophic set \( A = (M_A, \tilde{B}_A, J_A) \) in \( X \) is called a BMBJ-neutrosophic \( o \)-subalgebra of \( X \) if the following assertions are valid.

\[ M_A(x \circ y) \geq \min\{M_A(x), M_A(y)\}, \]
\[ B^-_A(x \circ y) \leq \max\{B^-_A(x), B^-_A(y)\}, \]
\[ B^+_A(x \circ y) \geq \min\{B^+_A(x), B^+_A(y)\}, \]
\[ J_A(x \circ y) \leq \max\{J_A(x), J_A(y)\} \]  \( (3.6) \)

for all \( x, y \in X \).

**Lemma 3.21.** Every BMBJ-neutrosophic ideal of a \( BCK \)/\( BCI \)-algebra \( X \) satisfies the following assertion.

\[ (\forall x, y \in X) \ (x \leq y \Rightarrow M_A(x) \geq M_A(y), B^-_A(x) \leq B^-_A(y), B^+_A(x) \geq B^+_A(y), J_A(x) \leq J_A(y)) \]  \( (3.7) \)

**Proof.** Assume that \( x \leq y \) for all \( x, y \in X \). Then \( x \ast y = 0 \), and so

\[ M_A(x) \geq \min\{M_A(x \ast y), M_A(y)\} = \min\{M_A(0), M_A(y)\} = M_A(y), \]
\[ B^-_A(x) \leq \max\{B^-_A(x \ast y), B^-_A(y)\} = \max\{B^-_A(0), B^-_A(y)\} = B^-_A(y), \]
\[ B^+_A(x) \geq \min\{B^+_A(x \ast y), B^+_A(y)\} = \min\{B^+_A(0), B^+_A(y)\} = B^+_A(y), \]

and

\[ J_A(x) \leq \max\{J_A(x \ast y), J_A(y)\} = \max\{J_A(0), J_A(y)\} = J_A(y). \]

This completes the proof.
\textbf{Theorem 3.22.} In an \((S)\)-BC\(K\)-algebra, every BMBJ-neutrosophic ideal is a BMBJ-neutrosophic \(\circ\)-subalgebra.

\textbf{Proof.} Let \(\mathcal{A} = (M_A, \tilde{B}_A, J_A)\) be a BMBJ-neutrosophic ideal of an \((S)\)-BC\(K\)-algebra \(X\). Note that \((x \circ y) \ast x \leq y\) for all \(x, y \in X\). Using Lemma 3.21 and (3.2) implies that

\[
M_A(x \circ y) \geq \min \{M_A((x \circ y) \ast x), M_A(x)\} \geq \min \{M_A(y), M_A(x)\},
\]

\[
B_A^-(x \circ y) \leq \max \{B_A^-(y \circ x), B_A^-(x)\} \leq \max \{B_A^- (y), B_A^-(x)\},
\]

\[
B_A^+(x \circ y) \geq \min \{B_A^+(y \circ x), B_A^+(x)\} \geq \min \{B_A^+(y), B_A^+(x)\},
\]

and

\[
J_A(x \circ y) \leq \max \{J_A((x \circ y) \ast x), J_A(x)\} \leq \max \{J_A(y), J_A(x)\}.
\]

Therefore \(\mathcal{A} = (M_A, \tilde{B}_A, J_A)\) is a BMBJ-neutrosophic \(\circ\)-subalgebra of \(X\). \(\Box\)

We provide a characterization of a BMBJ-neutrosophic ideal in an \((S)\)-BC\(K\)-algebra.

\textbf{Theorem 3.23.} Let \(\mathcal{A} = (M_A, \tilde{B}_A, J_A)\) be an MBJ-neutrosophic set in an \((S)\)-BC\(K\)-algebra \(X\). Then \(\mathcal{A} = (M_A, \tilde{B}_A, J_A)\) is a BMBJ-neutrosophic ideal of \(X\) if and only if the following assertions are valid.

\[
\begin{align*}
M_A(x) & \geq \min \{M_A(y), M_A(z)\}, \quad B_A^-(x) \leq \max \{B_A^-(y), B_A^-(z)\}, \\
B_A^+(x) & \geq \min \{B_A^+(y), B_A^+(z)\}, \quad J_A(x) \leq \max \{J_A(y), J_A(z)\}
\end{align*}
\]

(3.8)

for all \(x, y, z \in X\) with \(x \leq y \circ z\).

\textbf{Proof.} Assume that \(\mathcal{A} = (M_A, \tilde{B}_A, J_A)\) is a BMBJ-neutrosophic ideal of \(X\) and let \(x, y, z \in X\) be such that \(x \leq y \circ z\). Using (3.1), (3.2) and Theorem 3.22, we have

\[
M_A(x) \geq \min \{M_A(x \ast (y \circ z)), M_A(y \circ z)\} = \min \{M_A(0), M_A(y \circ z)\} = M_A(y \circ z) \geq \min \{M_A(y), M_A(z)\},
\]

\[
B_A^-(x) \leq \max \{B_A^-(x \ast (y \circ z)), B_A^-(y \circ z)\} = \max \{B_A^-(0), B_A^-(y \circ z)\} = B_A^-(y \circ z) \leq \max \{B_A^- (y), B_A^-(z)\},
\]

\[
B_A^+(x) \geq \min \{B_A^+(x \ast (y \circ z)), B_A^+(y \circ z)\} = \min \{B_A^+(0), B_A^+(y \circ z)\} = B_A^+(y \circ z) \geq \min \{B_A^+(y), B_A^+(z)\},
\]

\[
J_A(x) \leq \max \{J_A((x \circ y) \ast x), J_A(x)\} \leq \max \{J_A(y), J_A(x)\}.
\]
and

\[ J_A(x) \leq \max\{J_A(x \ast (y \circ z)), J_A(y \circ z)\} = \max\{J_A(0), J_A(y \circ z)\} = J_A(y \circ z) \leq \max\{J_A(y), J_A(z)\}. \]

Conversely, let \( A = (M_A, \tilde{B}_A, J_A) \) be an MBJ-neutrosophic set in an \((S)\)-\(BCK\)-algebra \( X \) satisfying the condition (3.8) for all \( x, y, z \in X \) with \( x \leq y \circ z \). Since \( 0 \leq x \circ x \) for all \( x \in X \), it follows from (3.8) that

\[ M_A(0) \geq \min\{M_A(x), M_A(x)\} = M_A(x), \]

\[ B_A^-(0) \leq \max\{B_A^-(x), B_A^-(x)\} = B_A^-(x), \]

\[ B_A^+(0) \geq \min\{B_A^+(x), B_A^+(x)\} = B_A^+(x), \]

and

\[ J_A(0) \leq \max\{J_A(x), J_A(x)\} = J_A(x). \]

Note that \( x \leq (x \ast y) \circ y \) for all \( x, y \in X \). Hence we have

\[ M_A(x) \geq \min\{M_A(x \ast y), M_A(y)\}, B_A^-(x) \leq \max\{B_A^-(x \ast y), B_A^-(y)\}, \]

\[ B_A^+(x) \geq \min\{B_A^+(x \ast y), B_A^+(y)\} \text{ and } J_A(x) \leq \max\{J_A(x \ast y), J_A(y)\}. \]

Therefore \( A = (M_A, \tilde{B}_A, J_A) \) is a BMBJ-neutrosophic ideal of \( X \).

4 Conclusions

As a generalization of neutrosophic set, Mohseni et al. [9] have introduced the notion of MBJ-neutrosophic sets, and have applied it to \( BCK/BCI \)-algebras. BMBJ-neutrosophic set has been introduced in [1] with an application in \( BCK/BCI \)-algebras. In this article, we have applied the notion of MBJ-neutrosophic sets to ideals of \( BCK/BI \)-algebras. We have introduced the concepts of a BMBJ-neutrosophic \( \circ \)-subalgebra and a (closed) BMBJ-neutrosophic ideal, and have investigated several properties. We have provided conditions for an MBJ-neutrosophic set to be a BMBJ-neutrosophic ideal in \( BCK/BCI \)-algebras, and have discussed characterizations of BMBJ-neutrosophic ideal. We have considered relations between a BMBJ-neutrosophic subalgebra, a BMBJ-neutrosophic \( \circ \)-subalgebra and a (closed) BMBJ-neutrosophic ideal. Using the results and ideas in this paper, our future work will focus on the study of several algebraic structures and substructures.

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References


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