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Closed neutrosophic dominating set in neutrosophic graphs

Amir Majeed [1], Nabeel Arif [2]

[1]Department of Mechanic \ College of Technical Engineering\ Sulaimani Polytechnic University-Iraq

[1]Email:amir.majeed@spu.edu.iq

1,2 Department of mathematics\ college of Computer science and Mathematics \ Tikrit University-Iraq

[2]Email: nabarif@tu.edu.iq

Abstract: The aim of this article is to concentrate on the notion of closed neutrosophic domination (CND) number $\gamma cl\ (G)$ of a neutrosophic graph (NG) with using effective edge, furthermore we gain a few outcomes on this notion, the relation between $\gamma cl\ (G)$ and some other notions is acquired, eventually the notion of (CND) number of (join neutrosophic graphs) is came in.

Keywords: fuzzy graph, neutrosophic graph, domination set, domination number

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1. Introduction

A graph is a nonempty set whose elements are called vertices or points. It also contains a set of elements consisting of unordered pairs of vertices; these elements are called edges or lines [1]. There are many relations between graph theory and other branches of mathematics such as Topology, Algebra, Probability, Fuzzy and Numerical Analysis. In addition, there are relations with other sciences such as Engineering, Computer Science, Chemistry, Physics, and Biology[2]. The concept of graph domination is one of the topics in graph theory, in which it is used in all the above sciences. The first one who initiated this concept is Claude Berge in 1962[3] .

Ore [4] is the one who introduced the concepts of domination number and dominating sets. After that, this notion started to appear in different kinds and forms. In mathematics, this concept appeared in many fields including fuzzy graph, topological indices of graphs, etc. Additionally, many new definitions in this concept have been used, depending on putting some conditions on the dominating set. The concept of dominance which introduced by V.T.Chandrasekaran and Nagoorgani, and all the concepts of dominant sets, independent set, dominant number, The total dominant number in the fuzzy graph was developed by R.Parvathi and G.Thamizhenthi [5]. A. Somasundaram introduced dominance in fuzzy graph using effective edges, relying on fuzzy graph concept which introduced by Rosenfeld in 1975, which is consequently built on the basis of the fuzzy sets proposed by Zadeh[6] in 1965 as a new mathematical framework for the visualization of unreliability phenomena in a real-life situation[7].

The use of the intuitionistic fuzzy set also played an important role in the transition from mathematics to computer, information science, and communication systems. Use combinatorial optimization, physics, and statistical problem solving to see the graphs[8]. In 1998, Florentin Smarandache [9]introduced the concept of Neutrosophic set which is a powerful general formal framework that generalizes the concept of fuzzy set and intuitionistic fuzzy set by treating with indeterminate membership furthermore of truth and false memberships and then domination in neutrosophic graphs was introduced by M.Mullai [10] . The variety of applications for graphs and their domination sets are increased by appearing of SVNG since its domain is larger than that of FGs and IFGs. SVNGs model the relationships much like any other type of graph. Hence, it is used to address a variety of relationship-based issues. Where FGs and IFGs fail, it may mimic issues with fluctuating and ambiguous information in the actual world[11].

This work aims to introduce the concept of Closed neutrosophic dominating set in neutrosophic graphs which possess more properties than the traditional domination set concept and some other related concepts was provided.

2.preliminaries

Definition 2.1 [12]. Let V be a non-empty set, a fuzzy graph G = (σ, μ) is a couple of functions $\sigma : V \to [0,1]$ and $\mu: VxV \to [0,1]$ such that $\mu(x,y) \le \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$, where xy denotes the edge between the vertices x and y furthermore σ and μ represent the fuzzy vertices and fuzzy edges sets on V and E respectively. See figure (1A)

Definition 2.2 [11]. The form G =(V, E) is called an (IFG) where

i) $V = \{v_1, v_2, \dots, v_n\}$,where $\mu_1 : V \to [0,1]$ and $\gamma_1 : V \to [0,1]$ such that $\mu_1$is a membership grade and $\gamma_1$is a non-membership respectively of every $v_i \in V(\mathrm{i} = 1, \dots, \mathrm{n})$,

and $0 \le \mu_1(v_i) + \gamma_1(v_i) \le 1$

ii. $E \subseteq V \times V$ where $\mu_2 : VxV \to [0,1]$, and $\gamma_2 : VxV \to [0,1]$, are functions and

$\mu_2 : (v_i, v_j) \le \mu_1(v_i) \wedge \mu_1(v_j)$, $\gamma_2(v_i, v_j) \ge \gamma_1(v_i) \vee \gamma_1(v_j)$

and $0 \le \mu_2(v_i, v_j) + \gamma_2(v_i, v_j) \le 1, \forall (v_i, v_j) \in E, (i, j = 1,2,3, \dots, n)$ see figure (1B)

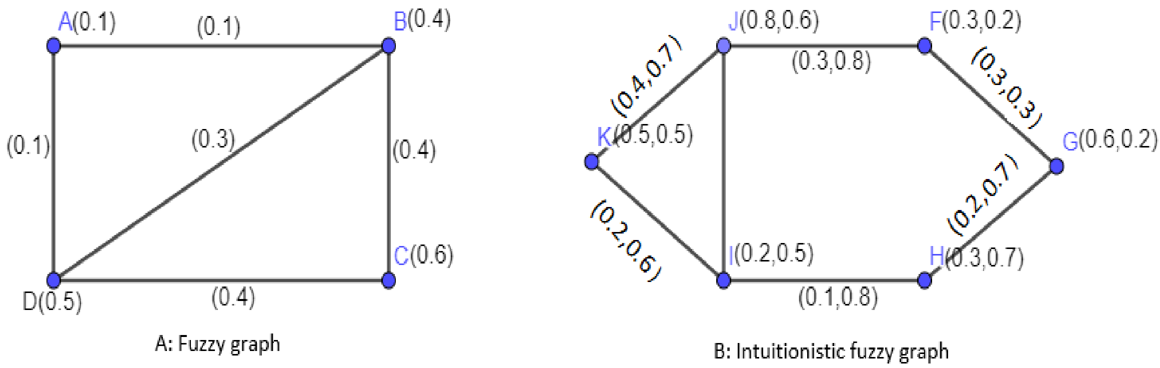


Figure :1

3. Single Valued Neutrosophic Graph (SVNG)[13].

Let $G^* = (V, E)$ refers to a traditional graph, and G = (A, B) to a (SVNG) On G^*

Definition 3.1. A single valued neutrosophic graph (SVNG) on vertices set V is a couple $G = (A, B)$ where $A = (T_A, I_A, F_A)$ and B=(T_B, I_B, F_B) as follows:

1.The T_A , I_A , F_A :V →[0, 1] are functions represent (truth, indeterminacy and falsity) membership degrees respectively, for all $v_i \in$V, (i=1,...,n), and $0 \le T_A(v_i) + I_A(v_i) + F_A(v_i) \le 3$

2. The functions T_B, I_B, F_B: E ⊆V x V → [0, 1], are satisfied the followings:
$T_B(v_i, v_j) \leq T_A(v_i) \wedge T_A(v_j)$
$I_B(v_i, v_j) \leq I_A(v_i) \wedge I_A(v_j)$ and

$F_B(v_i, v_j) \geq F_A(v_i) \vee F_A(v_j)$ for all v_j, v_j ∈V

where T_B, I_B, F_B *semilarly represent the three types of* membership degrees of each edge respectively, and

$0 \leq T_B(v_i, v_j) + I_B(v_i, v_j) + F_B(v_i, v_j) \leq 3$ for all $(v_i, v_J) \in E$, $(i, j = 1,2,3, \dots, n)$

Where A is denote the (SVN) vertex set of V, and B the (SVN) edge set of E, respectively. See figure 2.

Notes;

i) B is symmetric (SVN) relation on A.

ii) When $T_{Bij} = I_{Bij} = F_{Bij}$ =0 for some i and j, then V_i and V_j are not adjacent vertices otherwise there exists an edge $v_i v_j \in E$

iii) If at least one of the conditions in (1) and (2) is not satisfied, then G is not a (SVN) graph

Definition.3.2.[14]. Let $G = (A, B)$ be a (SVNG).

1) $\forall\, x \in V$ the neutrosophic degree $d(x)$ of x is

$$\sum T_B(xy), \sum I_B(xy), \sum F_B(xy), \ \forall\, y \in V \ adjacent\ to\ x$$

2) For every $v_i \in V$ $|v_i| = \left|\frac{1+T_A(v_i)+I_A(v_i)-F_A(v_i)}{3}\right|$ is called cardinality of the vertex v_i,

Then $|A| = \left|\sum_{x\in A} \frac{1+T_A(x)+I_A(x)-F_A(x)}{3}\right|$ is called vertex cardinality of G,

Similarly, $|e = xy| = \left|\sum_{x,y\in A} \frac{1+T_B(x,y)+I_B(x,y)-F_B(x,y)}{3}\right|$ is known as edge cardinality of G.

Definition. 3.3[15]. Let G = (A; B) be a (NG) on V. Then An edge $v_1 v_2 \in E$ in G is said to be an effective edge, if
$T_B(v_1, v_2) = T_A(v_1) \wedge T_A(v_2)$
$I_B(v_1, v_2) = I_A(v_1) \wedge I_A(v_1)$ and
$F_B(v_1, v_2) = F_A(v_1) \vee F_A(v_1)$ for v_1, v_2 ∈V

Definition. 3.4. [15]: take G = (A, B) as a (NG), then

1) G is renowned as strong neutrosophic graph if $\forall\, v_i v_j \in E$ is an effective edge.

2) G is renowned a complete neutrosophic graph if $\forall v_i, v_j \in V$, $\exists e = v_i v_j$ is an effective edge.

Definition. 3.5. [8]: A non-empty set $S \subseteq V(G)$ is called an independent neutrosophic set (INS) if

$T_B(xy) = I_B(xy) = F_B(xy) = 0$, *for all* $x, y \in S$

Definition. 3.6. $N(x)$ refers to open neighborhood of $x \in V$ (G) is define as
$N(x) = \{y \in V / (x, y)$ *is* an *effective edge*$\}$and
$N[x] = N(x) \cup \{x\}$ is closed neighborhood of x.
Definition .3.7. $A \subseteq V$(G) is called a neutrosophic vertex cover (NVC) of G if for each effective

Edge $e = (x, y)$, at least one of x *or* y *belong to* A .

The minimum neutrosophic cardinality (MNC) of all (MNVC) is called a **neutrosophic vertex covering number** of G which denoted by $\alpha_o(G)$.

Definition 3.8 [8]: Let G^* be underline graph of a neutrosophic graph G. The size m of G^* is a set of all edges in G^* and denoted by m=|E (G^*)|. Similarly, the order n=|V (G^*)| of $G^*$is the number of vertices in in G^* .
Definition 3.9. Let G = (A, B) be NG then the neutrosophic size $Ş_N$ and neutrosophic order O_N of G are define as
$Ş_N = (\sum T_B(u,v), \sum I_B(u,v), \sum F_B(u,v)), \forall\, uv \in E$ and O_N =
$(\sum T_A(u), \sum I_A(u), \sum F_A(u)), \forall\, u \in V$

Definition. 3.10. [16]. Let G = (A, B) be NG then the set $D \neq \emptyset, D \in V(G)$ is known as a neutrosophic dominating set (NDS) of G if $\forall\, y \in V - D$,$\exists$ *a vertex* $x \in D$ such that $T_B(x,y) = T_A(x) \wedge T_A(y), I_B(x,y) = I_A(x) \wedge I_A(y)$, and $F_B(x,y) = F_A(x) \vee F_A(y)$. The (MNC) for all minimum neutrosophic dominating set in G is called the neutrosophic domination number (NDN) of G which is denoted by γ_N.

4. Closed neutrosophic domination number (CNDN) in neutrosophic graph.

Definition. 4.1 Let G be (NG) with a vertex set V, and $D_k \subseteq V$ for some $k \in Z^+$, then Dk is called **closed neutrosophic dominating (*CND*)** set of G if the followings satisfied:

1) $\forall x \notin Dk$, $\exists$ $y \in D_k$ such that x dominate y and
2) If Dk contains more than two vertices then the two vertices have not been adjacent to the third one.
3) $N[Dk]$=$V(G)$

Algorithm for finding closed neutrosophic dominating set *Dk* can as follows:
Let V=\{$x1$,$x2$,…,xn\}, and $D_k = \{x_1, x_2, \dots, x_k\} \subseteq V$

1) Choose $x1 \in V(G)$, assume $D1$=\{$x1$\}, if $N[D_1] = V(G)$ then $D1$ is said to be closed neutrosophic dominating set, otherwise

2) Choose $x2 \in V-D1$(may $x1 and\ x2\ are\ adjacent$) put $D2=\{x1,x2\}$ if $N[D2]=V(G)$, then $D2$ is (CND) set, otherwise,
3) Choose $x_k \in V-N[Dk-1]$, $k \geq 3$, $k \in Z^+$ such that $N[DK] = V(G)$.

Definition. 4.2. A (CND) set Dk in a neutrosophic graph G is known as **minimum closed neutrosophic dominating (*MCND*) set** if the number of Dk elements less than or equal of number of vertices of each of other closed neutrosophic dominating set.

Definition. 4.3. Let $G = (A, B)$ be a NG The minimum neutrosophic cardinality of all (MCND) sets is known as **closed neutrosophic domination number (CNDN)** and denoted by $\boldsymbol{\gamma cl(G)}$ were

$$\gamma cl(G) = \{(min\ \{\sum_{x \in D_{ki}} |T_A(\mathrm{x}), I_A(\mathrm{x}), F_A(\mathrm{x})|\ ,\ ) x \in D_{\mathrm{ki}},\ Dki\ is\ MCND\ stes\}.$$

Definition. 4.4. The (MCND) set with **minimum neutrosophic cardinality** is said to be $\boldsymbol{\gamma cl}$**–*set*.**

Example. 4.5. Consider the given neutrosophic graph G, in figure 2.

We note two ($MCND$) sets: $Dk1$= {A, C, D, H} ={H}∪N(B) and $Dk2$= {I, J, B, D} = N(H)∪{B} such that $N[Dk1] = V(G)$ and $N[Dk2] = V(G)$.

$||Dk1||= |(1.7,2.4,1.7)| = 1.13333$, $||Dk2||= |(2.1,2.1,1.6)| = 1.2$

The closed neutrosophic domination number $\gamma cl = min\ \{||Dk1||, ||Dk2||\}$

=min $\{1.13333, 1.2\}$ = 1.1.333

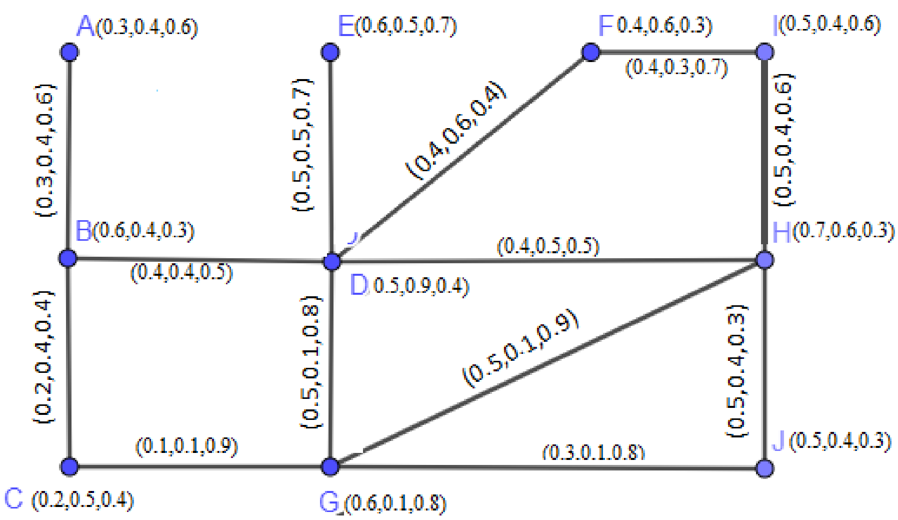


Figure 2: single value neutrosophic graph

Preposition. 4.6. Let G be a neutrosophic graph, then every closed neutrosophic dominating set of G is a neutrosophic dominating set of G.

Proof: It is clear a vertex set $D_k \subseteq V(G)$ in a neutrosophic graph G is closed neutrosophic dominating set if the following provisions satisfied:

1) $\forall x \notin Dk$, $\exists\ y \in D_k$ such that x dominate y and also if Dk contains more than two vertices then the two vertices have not been adjacent to the third one.

2) $N[Dk]=V(G)$, it is obviously $\forall x \notin Dk$ dominated by vertex $y \in D_k$,thus Dk is a neutrosophic dominating set of G.

Remark. 4.7.

i) The converse of proposition 4.6 is not always right, for instance in figure (2) where {B, D, H} is a neutrosophic dominating set but not closed neutrosophic dominating set.

ii) Let G be a neutrosophic graph with (CND) set, then $\gamma N \leq \gamma cl$ is not always true.

Preposition. 4.8. Let G=(A,B) be any (NG) , Where A=(T_A, I_A, F_A), B=(T_B, I_B, F_B) and H=(C,D) be any maximal spanning tree of G, then every closed neutrosophic dominating set of H, is closed neutrosophic dominating set of G and $\gamma_{cl}(G) = \gamma_{cl}(H)$ if $for\ each\ non-adjacent\ pair\ x, y \in D_K$ in H are non-adjacent in G

Proof: Let D_K be closed neutrosophic dominating set of H. Since H is a maximum spanning tree of G, we have A=C. thus, the vertices in $V - D_K$ are dominated by at least one vertex in D_K, then if D_K contains three or more vertices, then the third one has not be adjacent to other vertices and $N_H[D_K] = V(H) = V(G) = N[D_K]$.Therefore $\gamma_{cl}(G) = \gamma_{cl}(H)$.

Example: A graph H in a figure 3) below which is a maximal spanning tree of G, the sets $D1$= {A, C, D, H} and $D2$= {I, J, B, D} are closed neutrosophic domains in H and also in G

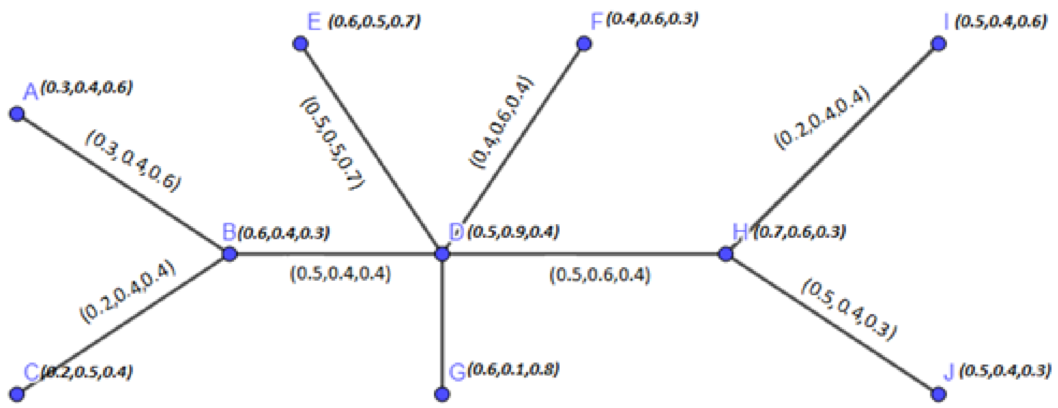


Figure3: spanning tree H of single value neutrosophic graph in figure 2

Note: If the two vertices A and D were adjacent in the figure 2, the theorem would not be true

Preposition. 4.9. Let $G \cong K_n^N$ be a complete neutrosophic graph and D_K is closed neutrosophic dominating set of Then, $V - D_K$ has a closed neutrosophic dominating set.

Proof: Given $G \cong K_n^N$ then every edge $e \in E(G)$ is an effective edge and each vertex $v \in V(G)$ is dominating all others. Thus, a closed neutrosophic dominating set is contains only one vertex then any singleton set of $V - D_K$ is closed neutrosophic dominating set. consequently $V - D_K$ has closed neutrosophic dominating set

Preposition. 4.10. For any neutrosophic graph G= (A, B), Where $A = (T_A, I_A, F_A)$, B= (T_B, I_B, F_B), $min|T_A(x), I_A(x), F_A(x)| \leq \gamma cl(G) \leq$more upper bound equality holds if

$T_B(x, y) < T_A(x) \wedge T_A(y)$, $I_B(x, y) < I_A(x) \wedge I_A(y)$, $F_B(x, y) < F_A(x) \vee F_A$, $\forall x, y \in V$

Proof: Straight forward from the definition of a closed neutrosophic dominating set

Preposition. 4.11. If G be a neutrosophic graph, every (IND) set of G is closed neutrosophic dominating set.

Proof: Assume that S_k be (IND) set of G then it has two probabilities:

Case1: If S_k be a singleton, then it is obviously SK is closed neutrosophic dominating set.

Case2: If the vertices in SK are more than two and since $T_B(\mathrm{x,y}) \le T_A(\mathrm{x}) \wedge T_A(\mathrm{Y}), I_B(\mathrm{x,y}) \le I_A(\mathrm{x}) \wedge I_A(\mathrm{Y}), F_B(\mathrm{x,y}) \ge F_A(\mathrm{x}) \wedge F_A(\mathrm{Y})$, for any $x, y \in S_K$,that's why $\forall\, z \in V - S_k$ has at least an effective edge e in S_k. Thus $N\,[S_k] = V(G)$ and for each $x \in S_k$ belongs to $V - N\,[S_k]$. consequentlyS_k is closed neutrosophic dominating set.

Preposition. 4.12. Let G= (A, B) be neutrosophic graph without isolated vertex and $(T_A(x) = I_A(x) = F_A(x)) = c\ \forall\ x\ \in V\ and\ c \in [0,1]$ then $\gamma cl\ (G) \le \mathrm{P} - \alpha_N$ where α_N is the neutrosophic covering number of G.

Proof: Let G= (A, B) be neutrosophic graph with no isolated vertex and V_C be a neutrosophic covering of G, then $V - V_C$ is independent neutrosophic set of G. Thus, $V - V_C$ is closed neutrosophic dominating set of G by proposition (4.5). Hence, $\gamma cl\ (G) \le \mid\mid V - V_C \mid\mid \le P - \alpha_N$.

Proposition. 4.13. For any neutrosophic graph G= (A, B), $\gamma_{cl}(G) + \gamma_{cl}\overline{(G)} \le 2(O_N)$.

$Further, equality\ hold\ if$

$0 < T_B(x,y) < T_A(x) \wedge T_A(y), 0 < I_B(x,y) < I_A(x) \wedge I_A(y), 0 < F_B(x,y) > F(x) \vee F_A(y),$

$\forall\ x, y \in E(G).$

Proof: *i)* Since, $\gamma_{cl}(G) = O_N$ it is trivial the inequality hold.

Since

$$0 < T_B(x,y) \ne T_A(x) \wedge T_A(y), 0 < I_B(x,y) \ne I_A(x) \wedge I_A(y), 0 < F_B(x,y) \ne F(x) \vee F_A(y), \forall\ x, y \in E(G).$$

i.e. $T_B(x,y) < T_A(x) \wedge T_A(y), I_B(x,y) < I_A(x) \wedge I_A(y), F_B(x,y) > F(x) \vee F_A(y), \forall\ x, y \in E(G)$

then $T_A(x) \wedge T_A(y) - T_B(x,y) < T_A(x) \wedge T_A(y),$

$I_A(x) \wedge I_A(y) - I_B(x,y) < I_A(x) \wedge I_A(y),$

$F_A(x) \wedge F_A(y) - F_B(x,y) < F(x) \vee F_A(y), \forall\ x, y \in \overline{E(G)}$

Then, $\gamma_{cl}\overline{(G)} = O_N$ .then , $\gamma_{cl}(G) + \gamma_{cl}\overline{(G)} = O_N + O_N = 2O_N$.

Preposition. 4.14. Let $G \cong K_n^N$ then $\gamma cl\ (K_n^N) = \min|(T_A(x_i), I_A(x_i), F_A(x_i))|, x_i \in V(G)$,i=1,2,…n

Proof: Let $G\cong K_n^N$ be complete (NG), then for each edge in G is an effective edge and each vertex in G dominates to all others of G. thus, the closed neutrosophic dominating set is contains a single vertex say DK={x}

such that $N[DK] = V\ (G)$, and x has minimum neutrosophic value. Hence, the outcome is gained.

Preposition 4.15. Let G = (A, B) be a strong neutrosophic star, then $\gamma cl\ (G) = |(T_A(\mathrm{x}),\ I_A(\mathrm{x}), F_A(\mathrm{x}))|$, where x is a root vertex.

Proof: Let G = (A, B) be strong neutrosophic star and $V\ (G) = \{x, x_1, x_2, \dots, x_n | x \text{ is a root of G}\}$, since all edges of G are effectives and x a dominating $x_i, i = 1,2,\dots,n.$ thus, a closed neutrosophic dominating set contains only one vertex such that $N[x] = V(G)$. Hence, $\gamma cl\ (G)$ $= |(T_A(\mathrm{x}),\ I_A(\mathrm{x}),\ F_A(\mathrm{x}))|$.

Preposition. 4.16. Let $G \cong K^N_{n,m}$ be complete bipartite (NG) with *n, m* vertices, then

$\gamma cl\ (K^N_{n,m}) =$

$$\begin{cases} |(T_A(x),\ I_A(x), F_A(x))| & \text{if either } n = 1 \text{ or } m = 1 \text{ where } x \in X \text{ or } x \in Y \\ \min|(T_A(x_i), I_A(x_i), F_A(x_i))| + \min|(T_A(y_i), I_A(y_i), F_A(y_i))| & \text{if } n, m \geq 2, \text{where } x_i \in X, i = 1,2,\dots n, yj \in Y, j = 1,2,\dots m \end{cases}$$

Proof: Assume $G \cong K_{n,m}$ $on\ V(K_{n,m}) = X \cup Y$,where $X = \{x1, x2, \dots, xn\}$ and $Y = \{y1, y2, \dots, ym\}$. Then there are couple of cases:

case 1: If either n=1 or m=1 then the graph is a star and the prove is hold by Preposition 4.9

case 2: If neither n=1 nor m=1. since for each $x \in X$ dominates to every $y \in Y$ and the contrariwise is true. Then, a $(MCND)$ set of $K_{n,m}$ contains pair of vertices.

Hence, $\gamma cl\ (Kn, m) = \min|(T_A(x_i), I_A(x_i), F_A(x_i))| + \min|(T_A(y_i), I_A(y_i), F_A(y_i))|$ where where $x_i \in X, i = 1,2, \dots n, yj \in Y, j = 1,2, \dots m$. The prove complete.

Preposition. 4.17. Let $G = (A, B)$ be a strong neutrosophic graph $G = C_n^N$,then

$\gamma cl(C_n^N) =$

$$\begin{cases} Min \sum_{i=0}^{\lceil \frac{n}{3} \rceil - 1} |T_A(x_{j+3i}), I_A(x_{j+3i}), F_A(x_{j+3i})| & j = 1,2, \dots, n \text{ and } n \equiv 0,2 (mod\ 3) \\ min \begin{cases} |(T_A(x), I(x), F_A(x)| + \sum_{i=0}^{\lceil \frac{n}{3} \rceil - 1} |(T_A(x_{j+3i+1}), I(x_{j+3i+1}), F_A(x_{j+3i+1}))| \\ |(T_A(x), I(x), F_A(x)| + \sum_{i=0}^{\lceil \frac{n}{3} \rceil - 1} |(T_A(x_{j+3i+2}), I(x_{j+3i+2}), F_A(x_{j+3i+2}))| \end{cases} & j = 1,2, \dots, n \text{ and } n \equiv 1 (mod\ 3) \end{cases}$$

Where taken $j + 3i\ , j + 3i\ + 1, j + 3i\ + 2\ modulo\ n.$

Proof: There are two cases depend on n as follows.

Case 1. If $n \equiv 0,2\ (mode3)$, then let $D_j = \left\{x_{j+3i}; i = 0,1, \dots, \left\lceil \frac{n}{3} \right\rceil - 1\right\}$,j=1, 2…, n, one can concluded that each one of the sets D_j is minimum dominating set and it independent. thus, according to proposition (4.5), it is closed neutrosophic dominating set.

Case 2. If $n \equiv 1(mode3)$, and let D_K be minimum closed neutrosophic dominating set of C_n^N,

Then there are two subcases:

i) If $x_i \in D_k$ then $(x_{i+1}\ or\ x_{i-1})$ vertex must be not belonged to D_k, then

$$D_j = \left\{x_{j+1+3i}; i = 0,1, \dots, \left\lceil \frac{n}{3} \right\rceil - 1\right\}, j=1, 2\dots, n,$$

ii) If any two vertices in D_k are not adjacent, then

$$D_j = \left\{x_{j+2+3i}; i = 0,1, \dots, \left\lceil \frac{n}{3} \right\rceil - 1\right\}, j=1, 2\dots, n,$$

From the above cases the prove is done.

Preposition. 4.18. Let W_{n+1}^N be strong neutrosophic wheel with x as a center, then

$\gamma cl\ (W_{n+1}^N) = (\ T_A(x),\ I_A(x),\ F_A(x)).$

Proof: Let W_{n+1}^N be a strong neutrosophic wheel, since all its edges are effective edge then *x* is dominating to *xi*, *i* = 1,2, …, *n*. Then, the (CND) set $DK = \{x\}$ such that $N[DK] = V\ (W_{n+1}^N)$.

Hence, $\gamma cl\ (W^{N}_{n+1})$ =$|(T_A(\mathrm{x}), I_A(\mathrm{x}), F_A(\mathrm{x}))|$

Preposition 4.19. For any strong neutrosophic graph G = (A, B), and $x \in v, d(x) = \Delta(g)$

$then\ \gamma cl \leq |O_N\ | - \sum_{y\in N(x)}|T_A\ (\mathrm{y}), I_A\ (\mathrm{y}).F_A\ (\mathrm{y})|$.

Proof: Let DK be a $\gamma cl - set$ of G and x be a vertex of G such that $d(x) = \Delta\ (G)$. Then, $V\ -\ N(x)$ is(CND) set, thus

$|DK| \leq |V\ -\ N(x)| = |n - \Delta\ (G)|$, take neutrosophic cardinality to both sides hence,

$\gamma cl \leq |O_N\ | - \sum_{y\in N(x)}|T_A\ (\mathrm{y}), I_A\ (\mathrm{y}).F_A\ (\mathrm{y})|$

5. Closed neutrosophic dominating set in some operation on neutrosophic graphs.

Definition. 5.1. [14]: Let $G_1\ =\ (A_1, B_1)$ and $G_2\ =\ (A_2, B_2)$ be any two neutrosophic graphs on $V1$and $V2$ respectively

then $G1 \cup G2$ is neutrosophic graph on $(V_1\ \cup\ V_2\ ).defined\ as$ $G = G1 \cup G2$ = ((A1 ∪ A2), (B1 ∪ B2)) where:

(A1∪A2)

(x)

$$= \begin{Bmatrix} (T_{A1}(x), I_{A1}(x), F_{A1}(x)) & If\ x \in V_1\ and\ x \notin\ V_2 \\ (T_{A2}(x), I_{A2}(x), F_{A2}(x)) & If\ x \in V_2\ and\ x \notin\ V_1 \\ (\max(T_{A1}(x), T_{A2}(x)), \max\ (I_{A1}(x), I_{A2}(x)), \min\ (F_{A1}(x), F_{A1}(x)) & If\ x \in V_1\ \cap\ V_2 \end{Bmatrix}$$

(B1 ∪ B2) (x, y) =

$$\begin{Bmatrix} (T_{B1}(xy), I_{B1}(xy), F_{B1}(xy)) & If\ xy \in E_1\ and\ xy \notin\ E_2 \\ (T_{B1}(xy), I_{B1}(xy), F_{B1}(xy)) & If\ xy \in E_2\ and\ xy \notin\ E_1 \\ \left(\max(T_{B1}(xy), T_{B2}(xy)), \max\ (I_{B1}(xy), I_{B2}(xy))\right), \min\ (F_{B1}(xy), F_{B1}(xy) & If\ xy \in E_1\ \cap\ E_2 \end{Bmatrix}$$

Example. 5.2. Consider $G_1\ =\ (A_1, B_1)$ and $G_2\ =\ (A_2, B_2)$ be any two neutrosophic graphs shown in 4 below

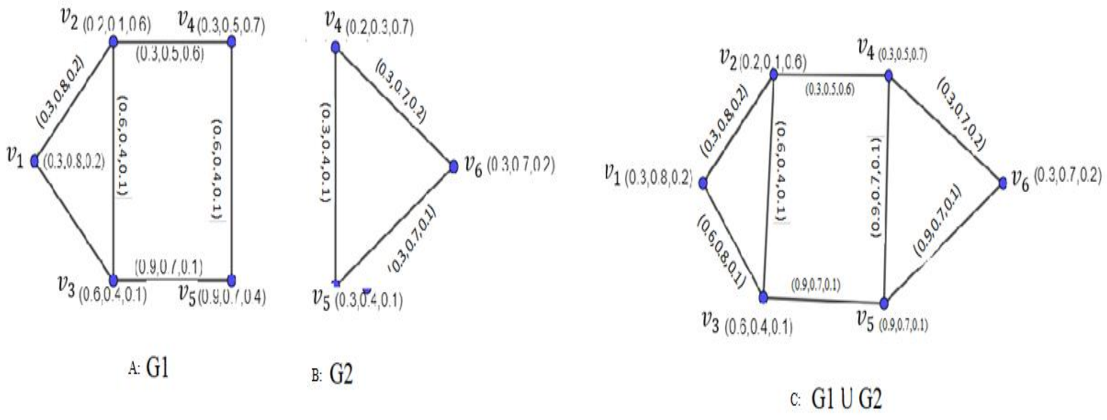


Figure4: A, B, C represent graph G1, G2, G1 U G2 respectively

Proposition. 5.3. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be any two strong neutrosophic graphs then

$\gamma cl\ (G1 \cup G2) =$

$$\begin{cases} \gamma cl\ (G1) + \gamma cl\ (G2) & \text{if } V1 \cap V2 = \emptyset \\ Min\{\gamma cl\ (G_1^U) + \gamma cl\ (G2 - (V1 \cap V2)), \gamma cl\ (G_2^U) + \gamma cl\ (G1 - (V1 \cap V2))\} & \text{if } V1 \cap V2 \neq \emptyset, \text{and } \forall v \in V1 \cap V2, v \notin D_k(G1 \cup G2) \end{cases}$$

Where G_1^U, G_2^U *the* G_1, G_2 after changing the (true, indeterminate, false) memberships of $V_1 \cap V_2$ under the union operation

Proof: Let $Dk1$ and $Dk2$ be a $\gamma cl - sets$ of $G1$ and $G2$ respectively.

Case 1. If $V1 \cap V2 = \emptyset$, then $Dk1 \cap Dk2 = \emptyset$. Therefore, $Dk = Dk1 \cup Dk2$ is (CND) set of $G = G1 \cup G2$. Hence, $\gamma cl\ (G) = \gamma cl\ (G1 \cup G2) = ||Dk1 \cup Dk2|| = ||Dk1 + Dk2|| = \gamma cl\ (G1) + \gamma cl\ (G2)$.

Case 2. If $V1 \cap V2 \neq \emptyset$, either $Dk = Dk1 \cup D\ (G_2 - (V_1 \cap V_2))$ or $Dk = Dk2 \cup D\ (G_1 - (V_1 \cap V_2))$

Then $\gamma cl\ (G1 \cup G2) Min\{\gamma cl\ (G_1) + \gamma cl\ (G_2 - (V_1 \cap V_2)), \gamma cl\ (G_2) + \gamma cl\ (G_1 - (V_1 \cap V_2))\}$

Definition. 5.4. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be any two (NG)s on $V1$ and $V2$ respectively, the join of $G1$ and $G2$ is a neutrosophic graph

$G = G1 + G2 = (A1 + A2, B1 + B2)$ where:

$(A_1 + A_2)(x, y) =$

$$\begin{cases} (T_{A1}(x), I_{A1}(x), F_{A1}(x)) & If\ x \in V_1\ and\ x \notin V_2 \\ (T_{A2}(x), I_{A2}(x), F_{A2}(x)) & If\ x \in V_2\ and\ x \notin V_1 \\ (\max(T_{A1}(x), T_{A2}(x)), \max(I_{A1}(x), I_{A2}(x)), \min(F_{A1}(x), F_{A1}(x)) & If\ x \in V_1 \cap V_2 \end{cases}$$

and

$(B1+B2)\ (x, y)$

=

$$\begin{cases} \left(T_{B1}(xy), I_{B1}(xy), F_{B1}(xy)\right) & If\ xy \in E_1\ and\ xy \notin E_2 \\ \left(T_{B1}(xy), I_{B1}(xy), F_{B1}(xy)\right) & If\ xy \in E_2\ and\ xy \notin E_1 \\ \left(\max\left(T_{B1}(xy), T_{B2}(xy)\right), \max\left(I_{B1}(xy), I_{B2}(xy)\right), \min\left(F_{B1}(xy), F_{B1}(xy)\right.\right. & If\ xy \in E_1 \cap E_2 \\ \left(\min\left(T_{A1}(x), T_{A2}(y)\right), \min\left(T_{A1}(x), T_{A2}(y)\right)\right), \max\left(F_{B1}(x), F_{B1}(y)\right. & If\ xy \in E' \end{cases}$$

where $E' = \{x_i y_j\ edges | x_i \in V1 \text{ and } y_i \in V2\}$

Example 5.5. In figure (5) below. Consider the graphs $G1$ = (A1, B1) and $G2$ = (A2, B2) be any two (NG)s on $V1$and $V2$ then $G1 + G2$ is given in figure 3.3

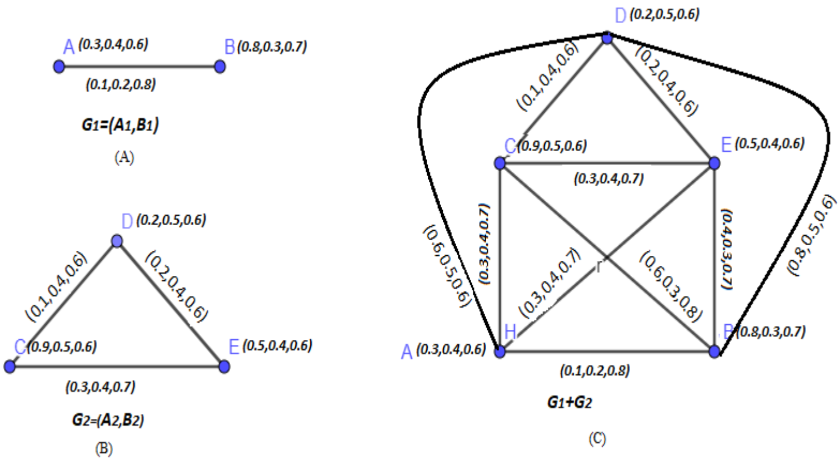


Figure5: A, B, C represent graph G1, G2, G1 + G2 respectively

Observation. 5.6. Consider $G1$ and $G2$ be two strong neutrosophic graphs, and $G = G1 + G2$. if $V1 \cap V2 = \emptyset$,For any $x \in V(G1)$ and $y \in V(G2)$ such that x and y have minimum neutrosophic cardinality values, the set $\{x, y\}$ is a closed neutrosophic dominating set in $G1 + G2$.Thus,

$\gamma cl\ (G) = \gamma cl\ (G1 + G2) = min|(T_A(x_i), I_A(x_i), I_A(x_i))\ | + min\left|\left(T_A(y_j), I_A(y_j), I_A(y_j)\right)\right|, x_i \in V(G1)$, i=1,2,..|V(G1)|, $and\ \ y_i \in V(G2)$,j=1,2,..|V(G2)|

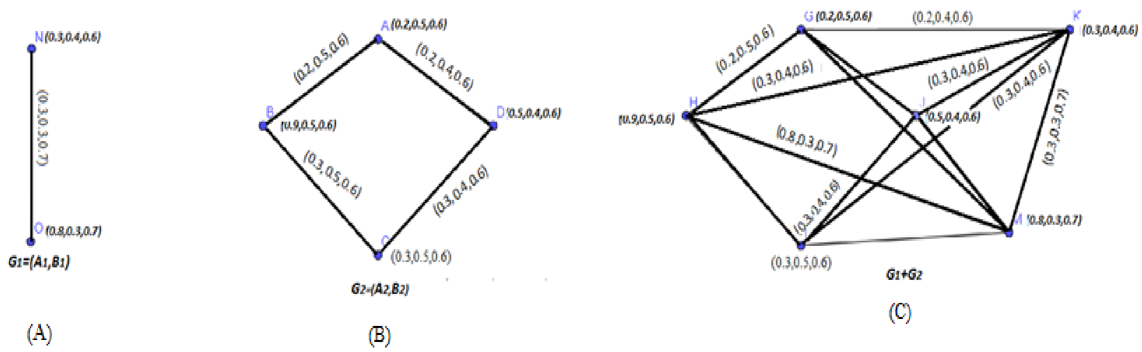


Figure6: A, B, C represent graph G1, G2, G1 + G2 respectively

Theorem. 5.7. Let $G1 = (A_1,\ B_1)$ and $G2 = (A_2,\ B_2)$ be any two strong neutrosophic graphs on $V1$and $V2$ respectively. Then, $\gamma cl\ (G) = \gamma cl\ (G1 + G2) \leq min\ \{\gamma cl(G1), \gamma cl\ (G2)\}$.

Proof: Let S1 and S2 be a γcl − sets of $G1$ and $G2$ respectively, by definition of join two neutrosophic graphs, we infer that S1 and S2 are (CND)sets of G. Hence $\gamma cl\ (G) = \gamma cl\ (G1 + G2) \leq min\ \{|S1|, |S2|\}$ = min $\{\gamma cl\ (G1), \gamma cl\ (G2)\}$.

Theorem. 5.8. Let G = (A, B) be a strong (NG) on V with $|V| = n$, then:

i) $\gamma cl\ (G) = |A(x)|$ if and only if $G = K_1^N$ or $G = K_1^N + \cup_{i=1}^{k} Hi$ for Some $k \geq 1$,and strong neutrosophic connected graph $H1, H2, \ldots, Hk$.

ii) $\gamma cl\ (G) = \min|A(x_i)|\ \ x_i \in V(G)$ if and only if $G = K_2^N$

iii) $\gamma cl\ (G) = |O_n^N|$ if and only if $G = \overline{K_n^N}$;

iv) $\gamma cl\ (G) = |O_n^N| - \min|A(x_i)|\ \ x_i \in V(K_2^N)$ if and only if $G = K_2^N \cup \overline{K_{n-2}^N}$.

Proof:

i) Suppose that $G = K_1^N + \cup_{i=1}^{k} Hi$ for some k ≥ 1, and strong neutrosophic connected graph $H1, H2, \ldots, Hk$, select $x \in V\ (K_1^N)$, since $V\ (G) = N[x]$. then $\gamma cl\ (G) = |A(x)|$.

Conversely, assume that $\gamma cl\ (G) = A(x)$ and let $x \in V\ (G)$ such that $\{x\}$ is a closed neutrosophic dominating set of G. If $G \neq K_1^N$,then $V\ (G) - \{x\} = N\ (x)$.

Consequently, $G = K_1^N + \cup_{i=1}^{k} Hi$ for some $k \geq 1$and strong connected neutrosophic graph $H1, H2, \ldots, Hk$. Hence, (i) is satisfied.

ii) When n = 2, the (CND)set is a singleton, thus by (i)

$\gamma cl\ (G) = A(x)$ If and only if $G = K_n^N$.

iii) If $G = \overline{K_n^N}$; it is obviously $D_K = V(G)$ $i.e.$ $\gamma cl\ (G) = O_n^N$. Suppose that $G \neq \overline{K_n^N}$; . If $G = K_n^N$,n=2, then $\gamma cl\ (G) = \min|A(x_i)| \neq O_n^N$. contradiction, suppose that $G \neq K_n^N$,n=2 and let the vertex x adjacent the vertex y in G construct a closed neutrosophic dominating set $\{x_1, x_2, x_3, \ldots, x_k\}$ in G such that $x_1 = x$ and $x_2 \neq y$. Then $K \leq n - 1$ vertices, thus $\gamma cl\ (G) < O_n^N$, a contradiction. then, (iii) is proved.

v) Now if $n \geq 3$. Suppose that $\gamma cl\ (G) = |O_n^N| - |A(x)|$ then $\Delta_E(G) \geq 1$.

assume that $\Delta_E\ (G) > 1$and let $x \in V\ (G)$ such that $d_E\ (x) = \Delta_E\ (G)$ construct closed neutrosophic dominating set $\{x1, x2, \ldots, xk\}$ in G such that $x_1 = x$ and $x_2 \in V(G) - N[x]$. Then $k \leq n - 2$, then $\gamma cl\ (G) \neq |O_n^N| - |A(x)|$, a contradiction. Thus, $\Delta_E\ (G) = 1$ therefore$G = K_2^N \cup K_{n-2}^N$ .The converse is directly.

6. Inverse Closed Neutrosophic Domination (ICND) in Neutrosophic Graphs

In this section, the notion of invers closed neutrosophic domination (ICND) γ_{cl}^{-1} in neutrosophic graph is introduced. some interesting relationships are known between closed neutrosophic domination and inverse closed neutrosophic domination.

in addition inverse closed neutrosophic domination in the join of new graphs discussed.

Definition. 6.1. Let D_K be a minimum closed neutrosophic dominating set in G. If $V - D_K$ contains a (CND) set ${D_K}^{-1}$ of G then ${D_K}^{-1}$ is said to be inverse closed neutrosophic dominating set according to D_K. An inverse closed neutrosophic domination number${\gamma_{cl}}^{-1}$ of G which is defined as ${\gamma_{cl}}^{-1} = (\min\{\sum_{x \in D_K^{-1}} |T_A(x), I_A(x), F_A(x)|\}, {D_K}^{-1}$ is minimum inverse closed neutrosophic dominating set of G. and minimum invers closed neutrosophic dominating set has minimum neutrosophic cardinality is called ${\gamma_{cl}}^{-1} -$ set of G

Example. 6.2. Let G=C_6^N as in the figure

Observation. 6.3. Let $\boldsymbol{G} = (\boldsymbol{A}, \boldsymbol{B})$ be neutrosophic graph of $n \geq 2$ vertices. if there is inverse closed neutrosophic dominating set in G. then

i) $\min|(T_A(x_i), I_A(x_i), F_A(x_i))| \leq {\gamma_{cl}}^{-1} < O_N,\ x_i \in V - D_K$ where D_K is minimum closed neutrosophic dominating set of G

ii) Not necessary $\gamma_{cl} \leq {\gamma_{cl}}^{-1}$

Example 6.4. Consider the following graph $G = (A, B)$

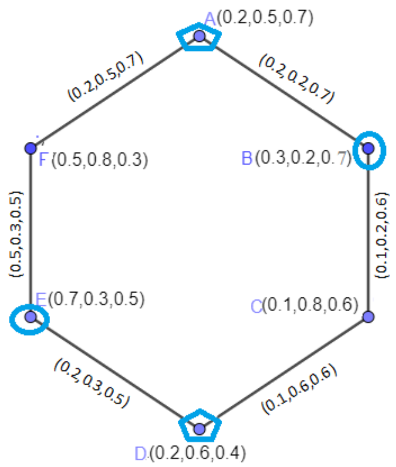


Figure7: Inverse closed neutrosophic dominating set

A minimum closed neutrosophic dominating sets are:

$D_K = \{A, D\}$ *and* $D_{K1}^{-1} = \{B, E\}$ *OR* $D_{K2}^{-1} = \{C, F\}$ *then*

$\gamma_{cl} D_K = |0.2,0.6,0.4| + |0.2,0.6,0.4| = 0.8$ *and* ${\gamma_{cl}}^{-1} D_{K1} = |(0.3,0.2,0.7)| + |(0.7,0.3,0.5)| = 0.76667$

${\gamma_{cl}}^{-1} D_{K2} = |(0.1,0.8,0.6)| + |(0.5,0.8,0.3)|$ =1.1 then ${\gamma_{cl}}^{-1} = \min(0.76667, 1.1) = 0.76667$

then $\gamma_{cl} > {\gamma_{cl}}^{-1}$

Proposition 6.5. Let G= (A, B) be a strong neutrosophic graph of $n \geq 2$ vertices. then,

${\gamma_{cl}}^{-1}(G) = \min|\ (T_A(x), (I_A(x), (F_A(x))|, x \in V - D_K$ where D_K is minimum closed neutrosophic dominating set of G, if and only if either G=K_2^N OR G=$K_2^N + H$ for some strong neutrosophic graph H.

Proof: Let D_K be a $\gamma_{cl} -$ set of G.

Case 1. If G= $G{=}K_2^N$ Then V(G) =2 then of the two vertices belong to D_K and the other belong to D_K^{-1} i.e., $\gamma_{cl}{}^{-1}(G) = \min|\,(T_A(x), (I_A(x), (F_A(x))|, x \in D_K^{-1}$

Case 2. If G=$K_2^N + H$, since G is strong neutrosophic graph then each of the vertices of K_2^N is adjacent with the all vertices of H, then obviously $\gamma_{cl}{}^{-1}(G) = \min|\,(T_A(x), (I_A(x), (F_A(x))|, x \in D_K^{-1}$

Conversely Suppose that $\gamma_{cl}{}^{-1}(\mathrm{G}) = \min|\,(T_A(x), (I_A(x), (F_A(x))|, x \in V - D_K$, i.e. a minimum inverse closed neutrosophic dominating set contains exactly one vertex say, $D_K{}^{-1} = \{x\}$, *then a* minimum closed neutrosophic dominating set D_K *of* G has only one vertex ,if $G \neq K_2^N$, then

$V - \{x\} = N(X)$.Hence, G=$K_2^N + H$, for some strong neutrosophic graph H. see figure 6

Theorem. 6.6. Let G= (A, B) be a strong neutrosophic graph of $n \geq 2$ vertices. then,

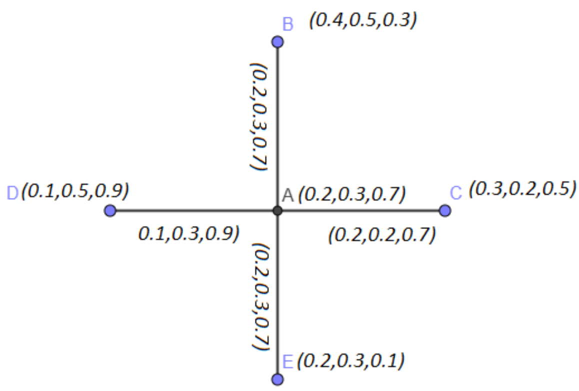
$\gamma_{cl}{}^{-1}(\mathrm{G}) = |O_n^N| - |A(x)|, x \in D_K$ if and only if G≅ strong neutrosophic star

Proof: Let $D_K{}^{-1}$ be a $\gamma_{cl}{}^{-1}$ – set of G and $\gamma_{cl}{}^{-1}(\mathrm{G}) = |O_n^N| - |A(x)|$. Let $x \in D_K \subseteq V(G)$. Then $N[x] = V(G)$,that is (x, y) effective edge for all $y \in V(G) - \{x\}$,

we claim that $(T_B(y,z), I_B(y,z), F_B(y,z)) = (0,0,0), \forall\, y, z \in V(G) - \{x\}$.suppose that $\exists\, y, z \in V(G) - \{x\}$ *such that*

$T_B(y,z) = (T_A(y) \wedge (T_A(z)$, $I_B(y,z) = (I_A(y) \wedge (I_A(z)$ and $(F_B(y,z) = (F_A(y) \vee (F_A(z)$ i.e. (y,z) is effective edge, thus $x, z \in N[y] \subseteq N[D_K{}^{-1} - \{z\}]$ then $D_K{}^{-1} - \{z\}$ *is* $\gamma_{cl}{}^{-1}$ – *set of G* , a contradiction. Hence G=$K_{1,n-1}^N$. Conversely, consider G=$K_{1,n-1}^N$it is clear that {x} is $\gamma_{cl}{}^{-1}$ – set of G then $D_K{}^{-1} = V(G) - \{x\}$.therefore, $\gamma_{cl}{}^{-1}(\mathrm{G}) = p^N - A(x). x \in D_K$.

Example. 6.7. Consider a strong neutrosophic graph G=$K_{1,4}^N$ in figure 6.2. a minimum closed neutrosophic dominating set $D_K = \{x\}$ and a minimum inverse closed neutrosophic dominating set $V - \{x\}$, then $\gamma_{cl}(K_{1,n-1}^N)$ =(1.2,1.8,2.5) -(0.2,0.3,0.7) = (1,1.5,1.8)



<u>**Figure8: Illustration theorem**</u> 6.2 $(K_{1,4}^N)$

Theorem. 6.8. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be any two strong neutrosophic graphs, then a minimum inverse closed neutrosophic dominating set $D_K{}^{-1}$ of $G_1 + G_2$ contains at most two vertices

Proof: Let D_{K1} and D_{K2} are minimum closed neutrosophic dominating set of G_1 and G_2 respectively, we know that a minimum closed neutrosophic dominating set D_K of join any two strong neutrosophic graphs $G_1 + G_2$ contains at most couple of vertices. Then, there exist two cases:

Case 1. If $D_K = \{x\}$ (contains a single vertex) is closed neutrosophic dominating set of $G_1 + G_2$. If $x \in V(G_1)$ then has n-1 neighborhood in G_1. Thus, assume that if there are S_1 and S_2 be the sets contains all vertices have $n-1$ neighborhood in G_1 and G_2 respectively $S_1 = \{x_i : \deg(x_i) = n-1, x_i \in G_1\}$ and $S_2 = \{y_i : \deg(y_i) = m-1, y_i \in G_2\}$, therefore $D_K{}^{-1} = \{x_i, x_i \in G_1 - \{x\}\ or\ D_K{}^{-1} = \{y_i, y_i \in G_2\}$. Hence , a minimum inverse closed neutrosophic dominating set $D_K{}^{-1}$ contains one vertex .Similarly if $\{x\} \in G_2$ if not, then it is clearly $D_K{}^{-1}$contains two vertices.

Case 2. If $D_K = \{x, y\}$(contains two vertices) is minimum closed neutrosophic dominating set of $G_1 + G_2$.If $x \in V(G_1)$ and $y \in V(G_2)$. Since $D_K = \{x, y\}$ is minimum closed neutrosophic dominating set of $G_1 + G_2$. Then, for any vertex $x_1 \in V(G_1) - D_K$ and $y_1 \in V(G_2) - D_K$, then the set $A = \{x_1, y_1\} \subseteq V(G_1 + G_2) - D_K$ is minimum closed neutrosophic dominating set of $G_1 + G_2$ which is inverse closed neutrosophic dominating set of $G_1 + G_2$. Hence ,from above cases the result is obtained.

Theorem. 6.9. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be any two strong neutrosophic graphs. If $\gamma_{cl}{}^{-1}(G_1 + G_2) = |(T_A(x), I_A(x), F_A(x))|$, then $\gamma_{cl}(G_1) = |(T_A(x_1), I_A(x_1), F_A(x_1)|$ or $\gamma_{cl}(G_2) = |(T_A(y_1), I_A(y_1), F_A(y_1)|$ where $x_1 \in V(G_1 + G_2),\ x_1 \in V(G_1)\ and\ y_1 \in V(G_2)$ with minimum neutrosophic value

proof: Given $G_1\ and\ G_2$ two strong neutrosophic graphs. Let $D_K{}^{-1} = \{x\}$ be minimum inverse closed neutrosophic dominating set of $G_1 + G_2$ then ,a minimum closed neutrosophic dominating set D_K of $G_1 + G_2$ also contained one vertex , therefore D_{K1} or D_{K2} contains one vertex, i.e. $\gamma_{cl}(G_1) = |(T_A(x_1), I_A(x_1), F_A(x_1)|$ or $\gamma_{cl}(G_2) = |(T_A(y_1), I_A(y_1), F_A(y_1)|$.hence ,the result obtain.

Remark 6.10. The propositions converse of theorem 6.4 is not true in general.

Example 6.11. Consider two strong neutrosophic graphs G_1=$K_{1,4}^N$ and G_2=p_7^N note that $\gamma_{cl}(G_1) = |(T_A(x), I_A(x), F_A(x)|, x$ is rote vertex but a minimum inverse closed neutrosophic dominating set of $G_1 + G_2$ contains two vertices, i.e. $\gamma_{cl}{}^{-1}(G_1 + G_2) \neq |(T_A(x), I_A(x), F_A(x)|$

Proposition 6.12. Let $G_1\ and\ G_2$ be two strong neutrosophic graphs such that $(T_{A1}(x), I_{A1}(x), F_{A1}(x) = (c, c, c)$, $c \in [0,1]$ and $(T_{A2}(y), I_{A2}(y), F_{A2}(y)) = (k, k, k)$, $k \in [0,1]$, then $\gamma_{cl}{}^{-1}(G_1 + G_2) = |(T_A(x), I_A(x), F_A(x)|$ if and only if one of the following is hold:

i) $\gamma_{cl}(G_1) = |T_{A1}(x), I_{A1}(x), F_{A1}(x)|$ or $\gamma_{cl}(G_2) = |(T_{A2}(y), I_{A2}(y), F_{A2}(y))|$
ii) $\gamma_{cl}(G_1) = |(T_A(x_1), I_A(x_1), F_A(x_1)|$ and G_1 has at least two minimums $\gamma_{cl} - sets$;
iii) $\gamma_{cl}(G_2) = |(T_A(y_1), I_A(y_1), F_A(y_1)|$and G_2 has at least two minimums $\gamma_{cl} - sets$;

Proof: Assume that (i) holds and $D_{K1} = \{x_1\} \subseteq V(G_1)$, $D_{K2} = \{y_1\} \subseteq V(G_2)$ are minimum closed neutrosophic dominating sets in $G_1 \, and \, G_2$ respectively ,then $D_{K1} \, and \, D_{K2}$ are minimum closed neutrosophic dominating set in $G_1 + G_2$, Since $D_{K1} \subseteq V(G_1 + G_2) - D_{K2}$, thus $D_{K1} \, is$

$\gamma cl^{-1} - set \, of(G_1 + G_2)$. Now suppose that (ii) hold. Let D_{K1} =$\{x_1\}$ $and \, D'_{K2} = \{x_2\}$ are minimum closed neutrosophic dominating set of G_1, then $D_{K1} \, and \, D'_{K1}$ are minimum closed neutrosophic dominating set of $G_1 + G_2$

Since $D_{K1} \subseteq V(G_1 + G_2) - D'_{K1}$ therefore, D_{K1} is minimum inverse closed neutrosophic dominating set of $G_1 + G_2$. Hence, $D_{K1} \, is \, \gamma cl^{-1} - set \, of(G_1 + G_2)$. Similarly, if (iii) holds.

Conversely, suppose that $D^{-1}_{K1} = \{x\}$ be a $\gamma cl^{-1} - set \, of(G_1 + G_2)$. i.e.,
$\gamma cl^{-1}(G_1 + G_2) = |(T_A(x), I_A(x), F_A(x)|$ then by proposition 6.4
$\gamma_{cl}(G_1) = |(T_A(x_1), I_A(x_1), F_A(x_1)|$ or
$\gamma_{cl}(G_2) = |(T_A(y_1), I_A(y_1), F_A(y_1)|$, if $\gamma_{cl}(G_1) = |(T_A(x_1), I_A(x_1), F_A(x_1)$ |then D_{K1} =$\{x_1\}$ is minimum closed neutrosophic dominating set of $G_1 + G_2$, since D^{-1}_{K1} has only one vertex thus a minimum closed neutrosophic dominating set (D_{K2}) of G_2contains one vertex then (i) is done.

Suppose that D_{K2} contains at least two vertices, then $\gamma_{cl}(G_1) = |(T_A(x_1), I_A(x_1), F_A(x_1)|$, let $D^{-1}_{K1} = \{x\}$ be a minimum inverse closed neutrosophic dominating set of $(G_1 + G_2)$, since V$(G_2) \subseteq N_{G_1+G_2}[x]$ and D_{K2} at least two vertices, $x \notin V(G_2)$.thus , $x \in V(G_1)$,necessarily $\{x\}$

is a $\gamma cl - set \, in \, G_1$, therefore G_1 has at least two a

$\gamma cl - sets \, and \, (ii) holds. by \, the \, same \, way \, we \, prove \, (iii)$

Corollary. 6.13. Let $G = (A, B)$ be any neutrosophic graphs. then
$\gamma_{cl}^{-1}(G_1 + H) = |(T_A(x), I_A(x), F_A(x))|$, x has minimum neutrosophic value if and only if $G \cong K^N_n, n \geq 2 \, or \, G = H_1 + H$ for some strong neutrosophic graphs $H_1 \, and \, H$ satisfying one of the following:

i) $\gamma_{cl}(H) = |(T_A(x_1), I_A(x_1), F_A(x_1)|$ and $\gamma_{cl}(H_1) = |(T_A(y_1), I_A(y_1), F_A(y_1)|$.
ii) $\gamma_{cl}(H) = |(T_A(x_1), I_A(x_1), F_A(x_1)|$ and H has at least two $\gamma cl - sets$.
iii) $\gamma_{cl}(H_1) = |(T_A(y_1), I_A(y_1), F_A(y_1)|$ and H_1 has at least two $\gamma cl - sets$.

Proof: Suppose that $\gamma_{cl}^{-1}(G) = |(T_A(x), I_A(x), F_A(x))|$, since

$\gamma_{cl}^{-1}(k^N_n) = (\min|(T_A(x), (I_A(x), (F_A(x))|)$, $x \in V(k^N_n) - D_{K1}$, where D_{K1} is minimum closed neutrosophic dominating set of k^N_n and $n \geq 2$.

Assume that $G \neq k^N_n$, suppose , $\gamma_{cl}^{-1}(k^N_n) = |(T_A(x), I_A(x), F_A(x))|$,.then, there exist two distinct vertices x_1 and x_2 of G such that $\{x_1\}$ and $\{x_2\}$ are $\gamma cl - sets$ of G.
Moreover, (x_1, x_2) *is effective edge, put* $H =< \{x_1, x_2\} >$ *and* $H_1 = G - \{x_1, x_2\}$. *Then* $G = H_1 + H$. *Furthermore,* $\{x_1\}$ and $\{x_2\}$ are distinct $\gamma cl - sets$ in H. Consequently, (ii)holds.

iv) Similarly, the converse follows immediately from theorem 6.9

Theorem. 6.1.4. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be any two strong neutrosophic graphs. Then, a minimum inverse closed neutrosophic dominating set of $G_1 + G_2$ contains two vertices. $|D_{K1}^{-1}| = 2$ if and only if any of the following is hold:

(i) $|D_{K1}| \geq 2$ and $|D_{K2}| \geq 2$ vertices, where D_{K1} and D_{K2} are minimum closed neutrosophic dominating sets of G_1 *and* G_2 respectively.

(ii) $|D_{K1}| = 1$ and $|D_{K2}| \geq 2$, $G \neq k_1^N + (k_1^N + \cup_j H_j)$ for some components H_j of G_j

Proof: Suppose that a minimum inverse closed neutrosophic dominating set of $G_1 + G_2$ contains two vertices, i.e., $|D_{K1}^{-1}| = 2.$ Then, a (MCND) set D_K of $G_1 + G_2$ either has one vertex or two vertices, then there are couple of cases:

Case1: If $|D_K| = 2$ then it is clear that $|D_{K1}| \geq 2$ and $|D_{K2}| \geq 2$, where D_{K1} and D_{K2} are minimum closed neutrosophic dominating set of G_1 *and* G_2 respectively

Case 2: If $|D_K| = 1$ *then* $|D_{K1}| = 1$ or $|D_{K2}| = 1$, i.e.

$\gamma_{cl}(G_1) = min|T_A(x_1), I_A(x_1), F_A(x_1)|$ or $\gamma_{cl}(G_2) = min|T_A(y_1), I_A(y_1), F_A(y_1)|$

Suppose that $\gamma_{cl}(G_1) = min|T_A(x_1), I_A(x_1), F_A(x_1)|$, then $G_1 = \{x_1\} + \cup_j H_j$, then for some component H_j of G_1. Thus, a minimum invers closed neutrosophic dominating set of $(\{x_1\} + \cup_j H_j + G_2)$ Contains two vertices with minimum neutrosophic value, i.e.,

$\gamma_{cl}{}^{-1}(G_1 + G_2) = \gamma_{cl}{}^{-1}(\{x_1\} + \cup_j H_j + G_2) = \sum_{j=1}^{2} x_j$, $x_j \in V(G_1 + G_2) - D_K$. Necessarily, D_{K2} and a minimum closed neutrosophic dominating of $[\cup_j H_j]$ contains two vertices. This, means that, in particular

$$G_1 \neq k_1 + (k_1 + \cup_j H_j).$$

Conversely, assume the first condition is true then a minimum closed neutrosophic dominating set of $G_1 + G_2$ contains two vertices say $D_K = \{x_1, y_1\}$, $x_1 \in V(G_1)$ *and* $y_1 \in V(G_2)$.

Let $x_2 \in V(G_1) - \{x_1\}$, $y_2 \in V(G_2) - \{y_1\}$. Then, $D_K^{-1} = \{x_2, y_2\}$ is minimum closed neutrosophic dominating set of $V(G_1 + G_2) - D_K$, *thus* $D_K^{-1}$is minimum inverse closed neutrosophic dominating set of $G_1 + G_2$ Contains a couple of vertices.

$|D_{K1}^{-1}| = 2.$ hence the result is done.

Now if (ii) hold, let $D_{K1} = \{x_1\} \subseteq V(G_1)$ be a closed neutrosophic dominating set of G_1.

then D_{K1} is closed neutrosophic dominating set in $G_1 + G_2$, consider

$(G_1 + G_2) - \{x_1\} = (G_1 - \{x_1\}) + G_2)$, by our imposition $(G_1 - \{x_1\}) \neq k_1^N + \cup_j H_j$

for some components H_j of G_1,thus a minimum closed neutrosophic dominating set of

$(G_1 - \{x_1\})$ contains at least a pair of vertices say $|D_K^*| \geq 2$. Now if $|D_K^{'}| \geq 2$ and $|D_{K2}| \geq 2$, then a minimum closed neutrosophic dominating set of $(G_1 - \{x_1\}) + G_2)$ contains two vertices, thus a minimum inverse closed neutrosophic dominating set of $G_1 + G_2$ also contain two vertices. Hence the prove is done.

Conclusion

Dominating sets can be used to model many other problems, including many relating to computer communication networks, social network theory, land surveying, and other similar issues. Determining the domination number for graphs and finding minimum dominating sets could thus prove very useful. Therefore, this study focused on the closed dominant sets, which are more in control of the network graphs, and theorems related to this concept presented and reinforced with necessary examples and graphics.

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