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Continuous and bounded operators on neutrosophic normed spaces

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Abstract. In this paper, we define the concept of continuous, sequentially continuous, and strongly continuous mappings neutrosophic normed spaces. Also, we have some important relationships between continuous, sequentially continuous, strongly continuous relationships mappings. Furthermore, the concept of neutrosophic Lipschitzian mapping is introduced and a neutrosophic version of Banach's contraction principle is achieved. Finally, the definition of neutrosophic bounded and weakly bounded linear operators are discussed and studied.

Keywords: Neutrosophic sets, neutrosophic normed spaces, continuous mappings, bounded linear operators, Banach spaces.

Mathematics Subject Classication: 46S40, 11B39, 03E72, 40G15.

1. **Introduction**

The concept of neutrosophic set, as a generalization of fuzzy set [18] and intuitionistic fuzzy set [5] was introduced by smarandache [16, 17]. Since 2005, the notion of the neutrosophic set received by attantion and have many applications [1–3]. The concept of neutrosophic normed space is a natural generalization of fuzzy normed space and intuitionistic fuzzy normed space. However, many different types of fuzzy normed spaces were introduced in [10,11,13]. In [6] Bag and Samanta introduced a new concept of fuzzy norm its more natural to the usual norm, they studied the properties of bounded sets and compact set in finite dimensional fuzzy normed linear spaces. Also, in [7] Bag and Samanta introduced types of continuous and bounded of linear operators. In [4] Abdulgawad et al present the notion of fuzzy strongly continuous, sequentially continuous, and continuous mappings. As well as they discussed the bounded and isometry of the fuzzy linear operator between fuzzy normed.

Recently, the concept of neutrosophic normed space, as a generalization of fuzzy normed spaces and the intuitionistic fuzzy normed space was introduced in [9], they studied the properties of convergence, completeness of such spaces.

In this paper, we extend the definitions of continuous and bounded operators in neutrosophic normed spaces. Moreover, we establish the main properties of bounded linear operators and continuous linear operators. We obtain a generalized version of boundedness and continuity of intuitionistic fuzzy norms, while will play an important role in study neutrosophic analysis. Furthermore, we introduce the notion of neutrosophic Lipsechitzian mapping and neutrosophic Banach space.

The paper is divided into the following sections:

Section 2 includes some basic results. In section 3, we introduce and study some types of continuous linear operators in neutrosophic normed spaces and neutrosophic Lipschitzian mapping. In section 4, we define and study some types of bounded and isometry linear operators in neutrosophic normed spaces. In section 5, we draw some conclusions.

2. Basic concepts

In this section, we remember the basic concepts and results that are required for the present work.

Definition 2.1. [12] A continuous t-norm is a binary operation $* : [0,1] \times [0,1] \to [0,1]$ with the following axioms:

(i) $*$ is commutative and associative.

(ii) $*$ is continuous.

(iii) $\ell * 1 = \ell, \,, \forall\, \ell \in [0,1]$.

(iv) $x * y \leq u * v$, $y \leq v$, $x \leq u$ and $x, y, u, v \in [0,1]$.

Definition 2.2. [14,15] A continuous t-co-norm is a binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ with the following axioms:

(i) $\diamond$ is commutative and associative.

(ii) $\diamond$ is continuous.

(iii) $\ell \diamond 0 = \ell, \forall\, \ell \in [0,1]$.

(iv) $x \diamond y \leq u \diamond v$, $y \leq v$, $x \leq u$ and $x, y, u, v \in [0,1]$.

Definition 2.3. [17] Let N be the universe set. A neutrosophic set $\mathcal{N}$ on N (NS $\mathcal{N}$) is defined as:

$$\mathcal{N} = \{< a, \rho(a), \xi(a), \eta(a) > | a \in N\}\,.$$

where $\rho, \xi, \eta : N \to [0,1]$.

Definition 2.4. [15] Let U be a linear space over $\mathbb{R}$ and $*, \diamond$ be a continuous t-norm, a continuous t-co-norm, respectively, then a neutrosophic subset $\mathcal{N} :< \rho, \xi, \eta >$ on $V \times \mathbb{R}$ be a neutrosophic norm on U if for $a, b \times U$ and $c,\ t,\ s \in \mathbb{R}$, if the following conditions hold.

(1) $0 \leq \rho(a,t), \xi(a,t), \eta(a,t) \leq 1$.
(2) $0 \leq \rho(a,t) + \xi(a,t) + \eta(a,t) \leq 3$.
(3) $\rho(a,t) = 0$ with $t \leq 0$.
(4) $\rho(a,t) = 1$ with $t > 0$ iff $x = 0$.
(5) $\rho(ca,t) = \rho(x, \frac{t}{\mid c \mid})\ \forall c \neq 0,\ t > 0$.
(6) $\rho(a,s) * \rho(b,t) \leq \rho(a+b, s+t)\ \forall s,t \in \mathbb{R}$.
(7) $\rho(a,.)$ is continuous non-decreasing function for $t > 0$, $\lim_{t\to\infty} \rho(a,t) = 1$.
(8) $\xi(a,t) = 1$ with $t \leq 0$.
(9) $\xi(a,t) = 0$ with $t > 0$ iff $x = 0$.
(10) $\xi(ca,t) = \xi(x, \frac{t}{\mid c \mid})\ \forall c \neq 0,\ t > 0$.
(11) $\xi(a,s) \diamond \xi(b,t) \geq \xi(a+b, s+t)$.
(12) $\xi(a,.)$ is continuous non-increasing function for $t > 0$, $\lim_{t\to\infty} \xi(a,t) = 0$.
(13) $\eta(a,t) = 1$ with $t \leq 0$.
(14) $\eta(a,t) = 0$ and $t > 0$ if and only if $x = 0$.
(15) $\eta(ca,t) = \xi(a, \frac{t}{\mid c \mid})\ \forall c \neq 0,\ t > 0$.
(16) $\eta(a,s) \diamond \eta(b,t) \geq \eta(a+b, s+t)$.
(17) $\eta(a,.)$ is continuous non-increasing function for $t > 0$, $\lim_{t\to\infty} \eta(a,t) = 0$.

Further $(V, \mathcal{N}, *, \diamond)$ is neutrosophic normed linear space (N$\mathcal{N}$LS).

Definition 2.5. [14,15] Let (a_n) be a sequence of points in an N$\mathcal{N}$LS $(U, \mathcal{N}, *, \diamond)$, then the sequence converges to a point $a \in U$ if and only if for given $0 < e < 1,\ t > 0\ \exists\ n_0 \in N$ such that,

$$\rho(a_n - a, t) > 1 - e,\ \xi(a_n - a, t) < e,\ \eta(a_n - a, t) < e\ \forall\ n \geq n_0.$$

$$\lim_{n\to\infty} \rho(a_n - a, t) = 1,\ \lim_{n\to\infty} \xi(a_n - a, t) = 0,\ \lim_{n\to\infty} \eta(a_n - a, t) = 0.$$

Then the sequence (a_n) is called a convergent sequence in the N$\mathcal{N}$LS $(U, \mathcal{N}, *, \diamond)$.

Definition 2.6. [15] Let (a_n) be a sequence in an N$\mathcal{N}$LS $(U, \mathcal{N}, *, \diamond)$, is said to be bounded for $0 < e < 1,\ t > 0$ if the following hold,

$$\rho(a_n, t) > 1 - e,\ \xi(a_n, t) < e,\ \eta(a_n, t) < e\ \forall n \in N.$$

Definition 2.7. [15] A sequence (a_n) of points in an N$\mathcal{N}$LS $(U, \mathcal{N}, *, \diamond)$, is called a Cauchy sequence if for given $0 < e < 1,\ t > 0\ \exists\ n_0 \in N$ such that,

$$\rho(a_n - a_m, t) > 1 - e,\ \xi(a_n - a_m, t) < e,\ \eta(a_n - a_m, t) < e\ \forall\ n, m \geq n_0.$$

$$\lim_{n,m\to\infty} \rho(a_n - a_m, t) = 1, \quad \lim_{n,m\to\infty} \xi(a_n - a_m, t) = 0, \quad \lim_{n,m\to\infty} \eta(a_n - a_m, t) = 0.$$

3. Continuous mappings

In this section, we introduce the concept of continuous, sequentially continuous, and strongly continuous mappings neutrosophic normed spaces. Also, we study the relationships between continuous, sequentially continuous, strongly continuous mappings. Moreover, this study is enhanced with an application

Definition 3.1. Let $(U, \mathcal{N}_U, *, \diamond)$ and $(V, \mathcal{N}_V, *, \diamond)$ be two neutrosophic normed spaces. The mapping $\mathcal{T} : (U, \mathcal{N}_U, *, \diamond) \to (V, \mathcal{N}_V, *, \diamond)$ is said to be continuous at $x_0 \in U$ if for all $x \in U$, for each $0 < \epsilon < 1$ and $t > 0$, there exists $0 < \delta < 1$ and $s > 0$, such that

$$\begin{aligned}
&\rho_V(\mathcal{T}(x) - \mathcal{T}(x_0), t) > (1 - \epsilon), \\
&\xi_V(\mathcal{T}(x) - \mathcal{T}(x_0), t) < \epsilon, \\
&\eta_V(\mathcal{T}(x) - \mathcal{T}(x_0), t) < \epsilon,
\end{aligned}$$

whenever

$$\begin{aligned}
&\rho_U(x - x_0, s) > (1 - \delta), \\
&\xi_U(x - x_0, s) < \delta, \\
&\eta_U(x - x_0, s) < \delta,
\end{aligned}$$

respectively. In other words:

$$\begin{aligned}
\rho_U(x - x_0, s) > (1 - \delta) &\Rightarrow \rho_V(\mathcal{T}(x) - \mathcal{T}(x_0), t) > (1 - \epsilon), \\
\xi_U(x - x_0, s) < \delta &\Rightarrow \xi_V(\mathcal{T}(x) - \mathcal{T}(x_0), t) < \epsilon, \qquad (1)\\
\eta_U(x - x_0, s) < \delta &\Rightarrow \eta_V(\mathcal{T}(x) - \mathcal{T}(x_0), t) < \epsilon,
\end{aligned}$$

$\mathcal{T}$ is continuous on U if it is continuous at every point in U.

Definition 3.2. Let $(U, \mathcal{N}_U, *, \diamond)$ and $(V, \mathcal{N}_V, *, \diamond)$ be two neutrosophic normed spaces. The mapping $\mathcal{T} : (U, \mathcal{N}_U, *, \diamond) \to (V, \mathcal{N}_V, *, \diamond)$ is called sequentially continuous at $x_0 \in U$, any sequence (x_n) in U satisfying $x_n \to x_0$ leads to $\mathcal{T}(x_n) \to \mathcal{T}(x_0)$. In other words:

$$\begin{aligned}
\lim_{n\to\infty} \rho_U(x_n - x_0, t) = 1 &\Rightarrow \lim_{n\to\infty} \rho_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) = 1, \\
\lim_{n\to\infty} \xi_U(x_n - x_0, t) = 0 &\Rightarrow \lim_{n\to\infty} \xi_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) = 0, \qquad (2)\\
\lim_{n\to\infty} \eta_U(x_n - x_0, t) = 0 &\Rightarrow \lim_{n\to\infty} \eta_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) = 0,
\end{aligned}$$

where $t > 0$. We call $\mathcal{T}$ is sequentially continuous on U when $\mathcal{T}$ is sequentially continuous at each point of U

Definition 3.3. Let $(U, \mathcal{N}_U, *, \diamond)$ and $(V, \mathcal{N}_V, *, \diamond)$ be two neutrosophic normed spaces. The mapping $\mathcal{T} : (U, \mathcal{N}_U, *, \diamond) \to (V, \mathcal{N}_V, *, \diamond)$ is called strongly continuous at $x_0 \in U$ if for each $t > 0$. $\exists\ s > 0$ such that $\forall\ x \in U$,

$$\begin{aligned} \rho_U(x - x_0, s) &\leq \rho_V(\mathcal{T}(x) - \mathcal{T}(x_0), t), \\ \xi_U(x - x_0, s) &\geq \xi_V(\mathcal{T}(x) - \mathcal{T}(x_0), t), \\ \eta_U(x - x_0, s) &\geq \eta_V(\mathcal{T}(x) - \mathcal{T}(x_0), t), \end{aligned} \tag{3}$$

we say $\mathcal{T}$ is strongly continuous on U when it is strongly continuous at every point in U.

Theorem 3.4. *Let $(U, \mathcal{N}_U, *, \diamond)$ and $(V, \mathcal{N}_V, *, \diamond)$ be two neutrosophic normed spaces. The mapping $\mathcal{T} : (U, \mathcal{N}_U, *, \diamond) \to (V, \mathcal{N}_V, *, \diamond)$ be continuous at $x_0 \in U$ if and only if $\mathcal{T}$ is sequentially continuous at $x_0 \in U$.*

Proof. Assume that $\mathcal{T}$ is continuous at $x_0 \in U$, $(x_n) \subset U$ if for all $x \in U$, for each $0 < \epsilon < 1$ and $t > 0\ \exists\ 0 < \delta < 1$ and $s > 0$, such that

$$\begin{aligned} \rho_U(x - x_0, s) > (1 - \delta) &\Rightarrow \rho_V(\mathcal{T}(x) - \mathcal{T}(x_0), t) > (1 - \epsilon), \\ \xi_U(x - x_0, s) < \delta &\Rightarrow \xi_V(\mathcal{T}(x) - \mathcal{T}(x_0), t) < \epsilon, \\ \eta_U(x - x_0, s) < \delta &\Rightarrow \eta_V(\mathcal{T}(x) - \mathcal{T}(x_0), t) < \epsilon, \end{aligned}$$

Since $x_n \to x_0$, then there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} &\rho_U(x_n - x_0, s) > (1 - \delta), \\ &\xi_U(x_n - x_0, s) < \delta, \\ &\eta_U(x_n - x_0, s) < \delta. \end{aligned}$$

□

Hence

$$\begin{aligned} &\rho_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) > (1 - \epsilon), \\ &\xi_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) < \epsilon, \\ &\eta_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) < \epsilon, \end{aligned}$$

as $0 < \epsilon < 1$ arbitrary; so $\mathcal{T}(x_n) \to \mathcal{T}(x_0)$. Thus, $\mathcal{T}$ is sequentially continuous at $x_0 \in U$.

Another direction, we suppose that $\mathcal{T}$ is sequentially continuous at $x_0 \in U$ and $\mathcal{T}$ is not continuous at x_0. Then there exists $0 < \epsilon < 1$ and $t > 0$, such that for any $0 < \delta < 1$ and $s > 0$, there exists $x \in U$, such that

$$\begin{aligned} \rho_U(x - x_0, s) > (1 - \delta) &\text{ but } \rho_V(\mathcal{T}(x) - \mathcal{T}(x_0), t) \leq (1 - \epsilon), \\ \xi_U(x - x_0, s) < \delta &\text{ but } \xi_V(\mathcal{T}(x) - \mathcal{T}(x_0), t) \geq \epsilon, \\ \eta_U(x - x_0, s) < \delta &\text{ but } \eta_V(\mathcal{T}(x) - \mathcal{T}(x_0), t) \geq \epsilon. \end{aligned} \tag{4}$$

So, for $\delta = 1 - \frac{1}{n+1}$, $s = \frac{1}{n+1}$, $n \in \mathbb{N}\ \exists\ x_n$ such that

$$\rho_U(x_n - x_0, \frac{1}{n+1}) > (\frac{1}{n+1}) \text{ but } \rho_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) \leq (1-\epsilon),$$
$$\xi_U(x_n - x_0, \frac{1}{n+1}) < 1 - \frac{1}{n+1} \text{ but } \xi_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) \geq \epsilon,$$
$$\eta_U(x_n - x_0, \frac{1}{n+1}) < 1 - \frac{1}{n+1} \text{ but } \eta_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) \geq \epsilon.$$

Taking $s > 0$, there exists n_0, such that $\frac{1}{n+1} < s$ for all $n \geq n_0$ then

$$\rho_U(x_n - x_0, s) > (\frac{1}{n+1}),$$
$$\xi_U(x_n - x_0, s) < 1 - \frac{1}{n+1},$$
$$\eta_U(x_n - x_0, s) < 1 - \frac{1}{n+1},$$

hence

$$\lim_{n\to\infty} \rho_U(x_n - x_0, s) = 1,$$
$$\lim_{n\to\infty} \xi_U(x_n - x_0, s) = 0,$$
$$\lim_{n\to\infty} \eta_U(x_n - x_0, s) = 0,$$

this lead to $x_n \to x_0$. However by (4),

$$\rho_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) \leq (1-\epsilon),$$
$$\xi_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) \geq \epsilon,$$
$$\eta_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) \geq \epsilon.$$

Thus, $\mathcal{T}(x_n)$ does not converges to $\mathcal{T}(x_0)$ but $x_n \to x_0$, which gives contradiction. Therefor, the mapping $\mathcal{T}$ is continuous at $x_0 \in U$.

Theorem 3.5. *Let $(U, \mathcal{N}_U, *, \diamond)$, $(V, \mathcal{N}_V, *, \diamond)$ be two neutrosophic normed spaces and $\mathcal{T} : (U, \mathcal{N}_U, *, \diamond) \to (V, \mathcal{N}_V, *, \diamond)$. If $\mathcal{T}$ is a strongly continuous, then $\mathcal{T}$ is sequentially continuous at $x_0 \in U$.*

Proof. Suppose that $\mathcal{T}$ is strongly continuous at x_0, then for each $t > 0$, there exists $s > 0$ such that for all $x \in U$ sequence (x_n) in U satisfying (3). Suppose that (x_n) is a sequence such that $x_n \to x_0$. If we put $x = x_n$ in (3), then we have

$$\rho_U(x_n - x_0, s) \leq \rho_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t),$$
$$\xi_U(x_n - x_0, s) \geq \xi_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t),$$
$$\eta_U(x_n - x_0, s) \geq \eta_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t).$$

This implies that

$$\lim_{n\to\infty} \rho_U(x_n - x_0, s) \leq \lim_{n\to\infty} \rho_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t),$$
$$\lim_{n\to\infty} \xi_U(x_n - x_0, s) \geq \lim_{n\to\infty} \xi_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t),$$
$$\lim_{n\to\infty} \eta_U(x_n - x_0, s) \geq \lim_{n\to\infty} \eta_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t).$$

Therefore,

$$\lim_{n\to\infty} \rho_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) = 1,$$
$$\lim_{n\to\infty} \xi_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) = 0,$$
$$\lim_{n\to\infty} \eta_V(\mathcal{T}(x_n) - \mathcal{T}(x_0), t) = 0.$$

Since $t > 0$ is arbitrary, we obtain that $\mathcal{T}(x_n) \to \mathcal{T}(x_0)$. Thus, $\mathcal{T}$ is sequentially continuous. □

Remark 3.6. The converse of the above Theorem 3.5 is not true, i.e., the sequentially continuity does not imply the strongly continuity.

Now, we give an example that illustrates the above remark.

Example 3.7. Let $(U = \mathbb{R}, \| x \|)$ be a normed linear space, where $\| x \| = | x | \ \forall x \in U$, and $a * b = min\{a, b\}$, $a \diamond b = max\{a, b\} \ \forall a, b \in [0, 1]$. Define $\rho_1, \rho_2, \xi_1, \xi_2, \eta_1, \eta_2 : U \times \mathbb{R}^+ \to [0, 1]$ by

$$\rho_1(x, t) = \frac{t}{t + | x |}, \qquad \rho_2(x, t) = \frac{t}{t + c | x |}, \ c > 0,$$

$$\xi_1(x, t) = \frac{| x |}{t + | x |}, \qquad \xi_2(x, t) = \frac{c | x |}{t + c | x |}, \ c > 0,$$

$$\eta_1(x, t) = \frac{| x |}{t}, \qquad \eta_2(x, t) = \frac{c | x |}{t}, \ c > 0.$$

It is easy to see that $(U, \mathcal{N}_1, *, \diamond)$ and $(U, \mathcal{N}_2, *, \diamond)$ are N$\mathcal{N}$LS. Let us now define, $f : (U, \mathcal{N}_1, *, \diamond) \to (U, \mathcal{N}_2, *, \diamond)$, $f(x) = \dfrac{x^4}{1 + x^2}$ for all $x \in U$. Let $x_0 \in U$ and (x_n) be a sequence in U such that $x_n \to x_0$ in $(U, \mathcal{N}_1, *, \diamond)$, that is, for all $t > 0$,

$$\lim_{n\to\infty} \rho_1(x_n - x_0, t) = \lim_{n\to\infty} \frac{t}{t + | x_n - x_0 |} = 1,$$
$$\lim_{n\to\infty} \xi_1(x_n - x_0, t) = \lim_{n\to\infty} \frac{c | x_n - x_0 |}{t + | x_n - x_0 |} = 0,$$
$$\lim_{n\to\infty} \eta_1(x_n - x_0, t) = \lim_{n\to\infty} \frac{| x_n - x_0 |}{t} = 0.$$

In other hand,

$$\begin{aligned}\rho_2(f(x_n)-f(x_0),t) &= \frac{t}{t+c\mid f(x_n)-f(x_0)\mid} \\ &= \frac{t}{t+c\mid \dfrac{x_n^4}{1+x_n^2}-\dfrac{x_0^4}{1+x_0^2}\mid} \\ &= \frac{t(1+x_n^2)(1+x_0^2)}{t(1+x_n^2)(1+x_0^2)+c\mid x_n^4(1+x_0^2)-x_0^4(1+x_n^2)\mid} \\ &= \frac{t(1+x_n^2)(1+x_0^2)}{t(1+x_n^2)(1+x_0^2)+c\mid (x_n^2+x_0^2)(x_n^2-x_0^2+x_n^2x_0^2(x_n^2-x_0^2))\mid} \\ &= \frac{t(1+x_n^2)(1+x_0^2)}{t(1+x_n^2)(1+x_0^2)+c\mid (x_n-x_0)(x_n+x_0)(x_n^2+x_0^2+x_n^2x_0^2)\mid}.\end{aligned}$$

So

$$\lim_{n\to\infty}\rho_2(x_n-x_0,t)=1.$$

$$\begin{aligned}\xi_2(f(x_n)-f(x_0),t) &= \frac{c\mid f(x_n)-f(x_0)\mid}{t+c\mid f(x_n)-f(x_0)\mid} \\ &= \frac{c\mid (x_n-x_0)(x_n+x_0)(x_n^2+x_0^2+x_n^2x_0^2)\mid}{t(1+x_n^2)(1+x_0^2)+c\mid (x_n-x_0)(x_n+x_0)(x_n^2+x_0^2+x_n^2x_0^2)\mid}.\end{aligned}$$

So

$$\lim_{n\to\infty}\xi_2(x_n-x_0,t)=0.$$

Finally,

$$\begin{aligned}\eta_2(f(x_n)-f(x_0),t) &= \frac{c\mid f(x_n)-f(x_0)\mid}{t} \\ &= \frac{c\mid (x_n-x_0)(x_n+x_0)(x_n^2+x_0^2+x_n^2x_0^2)\mid}{t\mid (1+x_n^2)(1+x_0^2)\mid},\end{aligned}$$

and this lead to

$$\lim_{n\to\infty}\eta_2(x_n-x_0,t)=0.$$

Thus, we see that f is sequentially continuous on U.

Now, we will explain that f is not strongly continuous by a contradiction. Let f be strongly continuous, then it holds that for all $x_0\in U$ and for each $t>0$ there exist $s>0$ such that for all $x_0\in U$,

$$\begin{aligned}&\rho_1(x-x_0,s)\le\rho_2(f(x)-f(x_0),t),\\ &\xi_1(x-x_0,s)\ge\xi_2(f(x)-f(x_0),t),\\ &\eta_1(x-x_0,s)\ge\eta_2(f(x)-f(x_0),t).\end{aligned}$$

Firstly, from the calculation of example [7, 8] and

$$\frac{c \mid (x-x_0)(x+x_0)(x^2+x_0^2+x^2x_0^2) \mid}{t \mid (1+x^2)(1+x_0^2) \mid} \leq \frac{\mid x-x_0 \mid}{s}$$

$$\frac{t \mid (1+x^2)(1+x_0^2) \mid}{\mid (x+x_0)(x^2+x_0^2+x^2x_0^2) \mid} \geq \frac{c}{t}s.$$

Then it holds that

$$Inf_{x\in U}\left\{\frac{t \mid (1+x^2)(1+x_0^2) \mid}{\mid (x+x_0)(x^2+x_0^2+x^2x_0^2) \mid}\right\} \geq \frac{c}{t}s.$$

Thus, $\frac{c}{t}s = 0$. Since $k, t > 0$ then it holds that $s = 0$. This gives a contradiction with the fact that $s > 0$. So f is not strongly continuous.

3.1. *Application*

Definition 3.8. A mapping $\mathcal{T} : (U, \mathcal{N}_U, *, \diamond) \rightarrow (V, \mathcal{N}_V, *, \diamond)$ is said to be neutrosophic Lipschitzian on U if $\exists\ c > 0$ such that

$$\rho_V(\mathcal{T}(x) - \mathcal{T}(y), t) \geq \rho_U(x-y, \frac{t}{c}),$$

$$\xi_V(\mathcal{T}(x) - \mathcal{T}(y), t) \leq \xi_U(x-y, \frac{t}{c}),$$

$$\eta_V(\mathcal{T}(x) - \mathcal{T}(y), t) \leq \eta_U(x-y, \frac{t}{c}),$$

$\forall t > 0,\ \forall x,\ y \in U$. If $c < 1$, we say that $\mathcal{T}$ is a neutrosophic contraction.

Remark 3.9. If $\mathcal{T}$ is a neutrosophic Lipschitzian mapping, then $\mathcal{T}$ is a neutrosophic continuous.

Definition 3.10. A neutrosophic Banach space is a complete neutrosophic normed linear space.

Theorem 3.11. *Let $(U, \mathcal{N}_U, *, \diamond)$ be a neutrosophic Banach space and $\mathcal{T} : (U, \mathcal{N}_U, *, \diamond) \rightarrow (U, \mathcal{N}_U, *, \diamond)$ be a neutrosophic contraction, then $\mathcal{T}$ has a unique fixed point.*

Proof. Let x be arbitary point in U, then $\{\mathcal{T}^n(x)\}$ is a Cauchy sequence. In fact, for $t > 0$ and $m \in \mathbb{N} - \{0\}$, we get

$$\rho(\mathcal{T}^{n+m}(x) - \mathcal{T}^n(x), t) \geq \rho(\mathcal{T}^{n+m-1}(x) - \mathcal{T}^{n-1}(x), \frac{t}{c}) \geq ... \geq \rho(\mathcal{T}^m(x) - x, \frac{t}{c^n}),$$

$$\xi(\mathcal{T}^{n+m}(x) - \mathcal{T}^n(x), t) \leq \xi(\mathcal{T}^{n+m-1}(x) - \mathcal{T}^{n-1}(x), \frac{t}{c}) \leq ... \leq \xi(\mathcal{T}^m(x) - x, \frac{t}{c^n}),$$

$$\eta(\mathcal{T}^{n+m}(x) - \mathcal{T}^n(x), t) \leq \eta(\mathcal{T}^{n+m-1}(x) - \mathcal{T}^{n-1}(x), \frac{t}{c}) \leq ... \leq \eta(\mathcal{T}^m(x) - x, \frac{t}{c^n}).$$

As $0 < c < 1$, we have that $\lim_{n\to\infty} \frac{t}{c^n} = \infty$. So

$$\lim_{n\to\infty} \rho(\mathcal{T}^m(x) - x, \frac{t}{c^n}) = 1,$$
$$\lim_{n\to\infty} \xi(\mathcal{T}^m(x) - x, \frac{t}{c^n}) = 0,$$
$$\lim_{n\to\infty} \eta(\mathcal{T}^m(x) - x, \frac{t}{c^n}) = 0.$$

Thus,

$$\lim_{n\to\infty} \rho(\mathcal{T}^{n+m}(x) - \mathcal{T}^n(x), t) = 1,$$
$$\lim_{n\to\infty} \xi(\mathcal{T}^{n+m}(x) - \mathcal{T}^n(x), t) = 0,$$
$$\lim_{n\to\infty} \eta(\mathcal{T}^{n+m}(x) - \mathcal{T}^n(x), t) = 0.$$

Since U is complete, we have that $\{\mathcal{T}^n(x)\}$ is a convergent sequence. So there exists $u \in U$ such that $\lim_{n\to\infty} \mathcal{T}^n(x) = u$. We find that

$$u = \lim_{n\to\infty} \mathcal{T}^{n+1}(x) = \lim_{n\to\infty} \mathcal{T}(\mathcal{T}^n(x)) = \mathcal{T}(u).$$

Now, we exhibit the uniqueness. Assume that $\exists u, v \in U$ with $u \neq v$ and $u = \mathcal{T}(u)$, $v = \mathcal{T}(v)$. As $u \neq v$, $\exists s > 0$ such that

$$\rho(u - v, s) = a < 1,$$
$$\xi(u - v, s) = b > 0,$$
$$\eta(u - v, s) = c > 0,$$

then, for all $n \in \mathbb{N}^*$ we obtain

$$a = \rho(v - u, s) = \rho(\mathcal{T}^n(v) - \mathcal{T}^n(u), s) \geq \rho(v - u, \frac{s}{c^n}) \to 1,$$
$$b = \xi(v - u, s) = \xi(\mathcal{T}^n(v) - \mathcal{T}^n(u), s) \leq \xi(v - u, \frac{s}{c^n}) \to 0,$$
$$c = \eta(v - u, s) = \eta(\mathcal{T}^n(v) - \mathcal{T}^n(u), s) \leq \eta(v - u, \frac{s}{c^n}) \to 0,$$

thus, $a = 1, b = 0, c = 0$, which gives contradiction, hence the claims of theorem. □

4. **Neutrosophic bounded**

In this section, we introduce the concept of boundedness and isometry of mappings neutrosophic linear operators between neutrosophic normed spaces. Also, we study the relationships between bounded and weakly bounded linear operators.

Definition 4.1. Let $(U,\mathcal{N}_U,*,\diamond)$ and $(V,\mathcal{N}_V,*,\diamond)$ be two neutrosophic normed spaces. A mapping $\mathcal{T}:U\to V$ is called neutrosophic isometry if for each $x\in U$, $t>0$ such that for all $x\in D$,

$$\begin{aligned}\rho_V(\mathcal{T}(x),t)&=\rho_U(x,t),\\ \xi_V(\mathcal{T}(x),t)&=\xi_U(x,t),\\ \eta_V(\mathcal{T}(x),t)&=\eta_U(x,t).\end{aligned}\tag{5}$$

Definition 4.2. Let $(U,\mathcal{N}_U,*,\diamond)$ and $(V,\mathcal{N}_V,*,\diamond)$ be two neutrosophic normed spaces and $\mathcal{T}:U\to V$ be a linear operator. The operator $\mathcal{T}$ is called neutrosophic bounded if there exist a constant $k\in\mathbb{R}-\{0\}$ such that for each $x\in U$ and $t>0$,

$$\begin{aligned}\rho_V(\mathcal{T}(x),t)&\geq\rho_U(kx,t),\\ \xi_V(\mathcal{T}(x),t)&\leq\xi_U(kx,t),\\ \eta_V(\mathcal{T}(x),t)&\leq\eta_U(kx,t).\end{aligned}\tag{6}$$

Definition 4.3. Let $(U,\mathcal{N}_U,*,\diamond)$ and $(V,\mathcal{N}_V,*,\diamond)$ be two neutrosophic normed spaces and $\mathcal{T}:U\to V$ be a linear operator. The operator $\mathcal{T}$ is called weakly neutrosophic bounded if for all $0<r<1$ there exist a constant $k\in\mathbb{R}-\{0\}$ such that for each $x\in U$ and $t>0$,

$$\begin{aligned}\rho_U(kx,t)\geq 1-r&\Rightarrow\rho_V(\mathcal{T}(x),t)\geq 1-r,\\ \xi_U(kx,t)\leq r&\Rightarrow\xi_V(\mathcal{T}(x),t)\leq r,\\ \eta_U(kx,t)\leq r&\Rightarrow\eta_V(\mathcal{T}(x),t)\leq r.\end{aligned}\tag{7}$$

Theorem 4.4. *Let $(U,\mathcal{N}_U,*,\diamond)$ and $(V,\mathcal{N}_V,*,\diamond)$ be two neutrosophic normed spaces. The linear operator $\mathcal{T}:(U,\mathcal{N}_U,*,\diamond)\to(V,\mathcal{N}_V,*,\diamond)$ be neutrosophic bounded if $\mathcal{T}$ is weakly neutrosophic bounded.*

Proof. Suppose that $\mathcal{T}$ is a neutrosophic bounded operator. Then there exist a constant $k\in\mathbb{R}-\{0\}$ such that for each $x\in U$, $t>0$, and satisfied (6). Using the fact that $\rho_U(kx,t)$, $\xi_U(kx,t)$, $\eta_U(kx,t)\in[0,1]$, we obtain that for any $0<r<1$ there exist a k_r depends on k such that

$$\begin{aligned}\rho_U(kx,t)\geq\rho_U(k_rx,t)\geq 1-r,\\ \rho_U(kx,t)\leq\rho_U(k_rx,t)\leq r,\\ \rho_U(kx,t)\leq\rho_U(k_rx,t)\leq r.\end{aligned}$$

Since (6) it holds that

$$\begin{aligned}\rho_V(\mathcal{T}(x),t)\geq 1-r,\\ \xi_V(\mathcal{T}(x),t)\leq r,\\ \eta_V(\mathcal{T}(x),t)\leq r.\end{aligned}$$

Thus, T is weakly neutrosophic bounded □

Theorem 4.5. *Let* $(U, \mathcal{N}_U, *, \diamond)$ *and* $(V, \mathcal{N}_V, *, \diamond)$ *be two neutrosophic normed spaces. The linear operator* $\mathcal{T} : (U, \mathcal{N}_U, *, \diamond) \to (V, \mathcal{N}_V, *, \diamond)$ *is continuous iff it is neutrosophic bounded.*

Proof. The first direction, let $\mathcal{T}$ be continuous on $(U, \mathcal{N}_U, *, \diamond)$, then it is continuous at $0 \in U$. Thus, for all $x \in U$, for each $0 < \epsilon < 1$ and $t > 0$, there exists $0 < \delta < 1$ and $s > 0$, such that if

$$\rho_U(x - 0, s) > (1 - \delta) \Rightarrow \rho_V(\mathcal{T}(x) - \mathcal{T}(0), t) > (1 - \epsilon),$$
$$\xi_U(x - 0, s) < \delta \Rightarrow \xi_V(\mathcal{T}(x) - \mathcal{T}(0), t) < \epsilon,$$
$$\eta_U(x - 0, s) < \delta \Rightarrow \eta_V(\mathcal{T}(x) - \mathcal{T}(0), t) < \epsilon.$$

Now, any way there exists $0 < \delta < 1$ such that

$$\rho_U(kx, t) > (1 - \delta),$$
$$\xi_U(kx, t) < \delta,$$
$$\eta_U(kx, t) < \delta.$$

So

$$\rho_U(x, \frac{t}{\mid k \mid}) = \rho_U(x, t) > (1 - \delta),$$
$$\xi_U(x, \frac{t}{\mid k \mid}) = \xi_U(kx, t) < \delta,$$
$$\eta_U(x, \frac{t}{\mid k \mid}) = \eta_U(kx, t) < \delta.$$

By putting $s = \frac{t}{\mid k \mid}$ we obtain that

$$\rho_U(x, s) > (1 - \delta) \Rightarrow \rho_V(\mathcal{T}(x), t) > (1 - \epsilon),$$
$$\xi_U(x, s) < \delta \Rightarrow \xi_V(\mathcal{T}(x), t) < \epsilon,$$
$$\eta_U(x, s) < \delta \Rightarrow \eta_V(\mathcal{T}(x), t) < \epsilon.$$

Hence

$$\rho_V(\mathcal{T}(x), t) \geq \rho_U(kx, t),$$
$$\xi_V(\mathcal{T}(x), t) \leq \xi_U(kx, t),$$
$$\eta_V(\mathcal{T}(x), t) \leq \eta_U(kx, t).$$

Therefore, $\mathcal{T}$ is neutrosophic bounded.

For the other direction, suppose that $\mathcal{T}$ is neutrosophic bounded, then there exist a constant

$k \in \mathbb{R} - \{0\}$ such that for each $x \in U$, $t > 0$, and satisfied (6). We have

$$\begin{aligned} \rho_V(\mathcal{T}(x), t) &\geq \rho_U(kx, t) = \rho_U(x, \frac{t}{\mid k \mid}) = \rho_U(x, s), \\ \xi_V(\mathcal{T}(x), t) &\leq \xi(kx, t) = \xi_U(x, \frac{t}{\mid k \mid}) = \xi_U(x, s), \\ \eta_V(\mathcal{T}(x), t) &\leq \eta_U(kx, t) = \eta_U(x, \frac{t}{\mid k \mid}) = \eta_U(x, s). \end{aligned} \tag{8}$$

Let $x_0 \in U$, $0 < \epsilon < 1,\ t > 0$, put $\delta = \epsilon$ and $s = \dfrac{t}{\mid k \mid} > 0$. Suppose that

$$\begin{aligned} &\rho_U(x - x_0) \geq (1 - \delta), \\ &\xi_U(x - x_0) \leq \delta, \\ &\eta_U(x - x_0) \leq \delta. \end{aligned}$$

Since (8) it holds that

$$\begin{aligned} &\rho_V(\mathcal{T}(x) - \mathcal{T}(x_0)) > (1 - \delta), \\ &\xi_V(\mathcal{T}(x) - \mathcal{T}(x_0)) < \delta, \\ &\eta_V(\mathcal{T}(x) - \mathcal{T}(x_0)) < \delta. \end{aligned}$$

Thus, $\mathcal{T}$ is continuous. □

5. **Conclusions**

In this paper, we have extended the definitions of continuous and bounded operators in neutrosophic normed spaces. Also, we have introduced a type of continuous and bounded operators in neutrosophic normed spaces. Moreover, we have studied some interesting relationships. These are illustrated by examples that are appropriate.

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