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An abstract approach to convex and concave sets under refined neutrosophic set environment

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ABSTRACT. A refined neutrosophic set (RNS) is an extension of a neutrosophic set in which all the uncertain belonging-based entities like belonging-grade, non-belonging-grade, and indeterminate-grade are further categorized into their respective sub-belonging grades, sub-non-belonging-grades, and sub-indeterminate-grades, respectively. In other words, the RNS provides multi sub-grades for each uncertain component of theneutrosophic set. This study is aimed to integrate the classical concepts of convexity and concavity with RNS to make the RNS applicable to various optimization problems.Thus, convex RNS and concave RNS are developed. Some of their important aggregation operations and results are investigated and then modified.

Keywords: Sub-belonging grade; Sub non-belonging grade; Sub-indeterminacy grade; Infimum projection; Supremum projection; Ortho-convexity; Ortho-concavity.

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1. **Introduction**

To deal with uncertainty, Zadeh [1] proposed a fuzzy set (FS) in 1965. Each component of the universe under investigation is given a belonging grade from the range $[0, 1]$ in an FS. Zadeh [2] used his own idea of FSs as the foundation for a theory of possibility. The link between FSs and probability theories was studied by Dubois et al. [4, 5]. For algebraic operations carried out between random set-valued variables, they derived the monotonicity property. Dubois et al. [3] performed research on ranking fuzzy numbers in the context of possibility theory. Beg et al. computed similarities between FSs under specific implications [6–8]. The solution of nonlinear partial differential equations in a fuzzy environment was determined by Osman et al. [9]. Khan et al. [10] envisaged some semi-groups in the context of fuzzy interior intuitionistic ideals. With applications in both the first and second senses, Rahman et al. [11] and Ihsan et al. [26] proposed the conceptual framework of (m, n)-convexity-cum-concavity on fuzzy soft set and fuzzy soft expert set, respectively.

Only being a member is insufficient in some real-world situations. Atanassov conceptualized an intuitionistic fuzzy set (IFS) to make the FSs suitable for the non-belonging grade in 1986 [13, 14]. Each component of the universe of discourse receives an allocation of both belonging value and non-belonging value from a [0,1]. The generalization of the FS, the IFS, has shown to be a very useful tool for academics. With their study of operations, algebra, model operators, and normalization on IFSs, Ejegwa et al. [15] broadened the concept.

Since both Zadeh's FS and Atanassov's IFS are insufficient for the grade of indeterminacy, Smarandache [16] devised the neutrosophic set (NS) to overcome these drawbacks. Additionally, because the NS does not impose the dependency requirement on uncertain components, truthfulness, falseness, and indeterminacy grades are independent and can take on any value inside a closed unit interval.

The concept of a concave FS was presented by Chaudhuri [17, 18]. He also examined some of the sets' valuable qualities and defined some of their related concepts and computing methods. The development of fuzzy geometry and fuzzy structures can benefit from this idea. This idea was improved by Yu-Ru Syau [19] to include convex and concave fuzzy mappings. Concavo-convex FSs were introduced by Sarkar [20], who also established some of its intriguing characteristics. The discussion on convex IFSs given by Ban [21, 22] led to the development of convex temporal IFSs. The collection of convex IFSs was described and its generalized qualities were covered in depth by Dı́az et al. [23]. Sarkar [26] discusses convexity on the NS.

Smarandache [24] introduced refinements in FS-like structures including NS by developing their relevant models with refined settings which categorizes the uncertain grades of these models into their respective sub-grades. Rahman et al. [25] studied the fundamental properties, operations, and results of refined IFSs with examples. The researches [21, 22, 24, 26, 27] have many concepts which lead to the motivation of this study and thus convex and concave sets are generalized under refined NS (RNS). Additionally, few significant properties and results are investigated in this context.

The remaining portion of the paper has been divided into three sections: section 2, section 3, and section 4. Section 2 is about the recalling of some important definitions, section 3 is aimed to investigate the notions of classical convexity and concavity under the RNS environment along with modifications of various results, and the last section summarizes the paper accompanied by future scope.

2. Preliminaries

This portion is aimed to recall few definitions which assist the readers to understand the main concepts. The acronyms $\hat{\Delta}, \mathscr{G}, I, \hat{\zeta}, \hat{\vartheta}$ and $\hat{\xi}$ are meant for initial set of objects, $\mathscr{R}^n$, $[0,1]$, true-belonging, false-belonging and indeterminate-belonging functions respectively.

Definition 2.1. [1,2] A FS $\hat{\Lambda}$ is stated as $\hat{\Lambda} = \left\{\left(\hat{\wp}, \hat{\zeta}_{\hat{\Lambda}}(\hat{\wp})\right) : \hat{\wp} \in \hat{\Delta}\right\}$ such that $\hat{\zeta}_{\hat{\Lambda}} : \hat{\Delta} \to [0,1]$ with $\hat{\zeta}_{\hat{\Lambda}}(\hat{\wp}) \in [0,1]$ as belonging-grade of $\hat{\wp}$ in $\hat{\Delta}$. If $\hat{\Lambda}_1$ and $\hat{\Lambda}_2$ are FSs then

(1) $\hat{\Lambda}^c = \left\{\left(\hat{\wp}, 1 - \hat{\zeta}_{\hat{\Lambda}}(\hat{\wp})\right) : \hat{\wp} \in \hat{\Delta}\right\}$.
(2) $\hat{\Lambda}_3 = \hat{\Lambda}_1 \cup \hat{\Lambda}_2 = \left\{\left(\hat{\wp}, max\{\hat{\zeta}_{\hat{\Lambda}_1}(\hat{\wp}), \hat{\zeta}_{\hat{\Lambda}_2}(\hat{\wp})\}\right) : \hat{\wp} \in \hat{\Delta}\right\}$.
(3) $\hat{\Lambda}_4 = \hat{\Lambda}_1 \cap \hat{\Lambda}_2 = \left\{\left(\hat{\wp}, min\{\hat{\zeta}_{\hat{\Lambda}_1}(\hat{\wp}), \hat{\zeta}_{\hat{\Lambda}_2}(\hat{\wp})\}\right) : \hat{\wp} \in \hat{\Delta}\right\}$.

Definition 2.2. [1] A FS $\hat{\Lambda}$ is stated to be convex FS when its belonging function $\hat{\zeta}_{\hat{\Lambda}}$ satisfies the following inequality $\hat{\zeta}_{\hat{\Lambda}}\left(\hat{\varsigma}\hat{\wp}_1 + (1-\hat{\varsigma})\,\hat{\wp}_2\right) \geq min\left(\hat{\zeta}_{\hat{\Lambda}}\left(\hat{\wp}_1\right), \hat{\zeta}_{\hat{\Lambda}}\left(\hat{\wp}_2\right)\right)$ with $\hat{\varsigma} \in [0,1]$ and $\hat{\wp}_1, \hat{\wp}_2 \in \hat{\Delta}$.

Definition 2.3. [17] A FS $\hat{\Lambda}$ is stated to be concave FS when its belonging function $\hat{\zeta}_{\hat{\Lambda}}$ satisfies the following inequality $\hat{\zeta}_{\hat{\Lambda}}\left(\hat{\varsigma}\hat{\wp}_1 + (1-\hat{\varsigma})\,\hat{\wp}_2\right) \leq max\left(\hat{\zeta}_{\hat{\Lambda}}\left(\hat{\wp}_1\right), \hat{\zeta}_{\hat{\Lambda}}\left(\hat{\wp}_2\right)\right)$ with $\hat{\varsigma} \in [0,1]$ and $\hat{\wp}_1, \hat{\wp}_2 \in \hat{\Delta}$.

Definition 2.4. [13] A IFS $\hat{\Gamma}$ is stated as $\hat{\Gamma} = \left\{\left(\hat{\wp}, \langle\hat{\zeta}_{\hat{\Gamma}}(\hat{\wp}), \hat{\vartheta}_{\hat{\Gamma}}(\hat{\wp})\rangle\right) : \hat{\wp} \in \hat{\Delta}\right\}$ such that $\hat{\zeta}_{\hat{\Gamma}}, \hat{\vartheta}_{\hat{\Gamma}} : \hat{\Delta} \to [0,1]$ with $\hat{\zeta}_{\hat{\Gamma}}(\hat{\wp}), \hat{\vartheta}_{\hat{\Gamma}}(\hat{\wp}) \in [0,1]$ as belonging-grade and non belonging-grade of $\hat{\wp}$ in $\hat{\Delta}$ such that $0 \leq \hat{\zeta}_{\hat{\Gamma}}(\hat{\wp}) + \hat{\vartheta}_{\hat{\Gamma}}(\hat{\wp}) \leq 1$. If $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ are IFSs then

(1) $\hat{\Gamma}^c = \left\{\left(\hat{\wp}, \langle\hat{\vartheta}_{\hat{\Gamma}}(\hat{\wp}), \hat{\zeta}_{\hat{\Gamma}}(\hat{\wp})\rangle\right) : \hat{\wp} \in \hat{\Delta}\right\}$.
(2) $\hat{\Gamma}_3 = \hat{\Gamma}_1 \cup \hat{\Gamma}_2 = \left\{\left(\hat{\wp}, \langle max\{\hat{\zeta}_{\hat{\Gamma}_1}(\hat{\wp}), \hat{\zeta}_{\hat{\Gamma}_2}(\hat{\wp})\}, min\{\hat{\vartheta}_{\hat{\Gamma}_1}(\hat{\wp}), \hat{\vartheta}_{\hat{\Gamma}_2}(\hat{\wp})\}\rangle\right) : \hat{\wp} \in \hat{\Delta}\right\}$.
(3) $\hat{\Gamma}_4 = \hat{\Gamma}_1 \cap \hat{\Gamma}_2 = \left\{\left(\hat{\wp}, \langle min\{\hat{\zeta}_{\hat{\Gamma}_1}(\hat{\wp}), \hat{\zeta}_{\hat{\Gamma}_2}(\hat{\wp})\}, max\{\hat{\vartheta}_{\hat{\Gamma}_1}(\hat{\wp}), \hat{\vartheta}_{\hat{\Gamma}_2}(\hat{\wp})\}\rangle\right) : \hat{\wp} \in \hat{\Delta}\right\}$.

Definition 2.5. [21] A IFS $\hat{\Gamma}$ is stated to be concave IFS when its belonging function $\hat{\zeta}_{\hat{\Gamma}}$ and non belonging function $\hat{\vartheta}_{\hat{\Gamma}}$ satisfy the following inequalities

(1) $\hat{\zeta}_{\hat{\Gamma}}\left(\hat{\varsigma}\hat{\wp}_1 + (1-\hat{\varsigma})\,\hat{\wp}_2\right) \geq min\left(\hat{\zeta}_{\hat{\Gamma}}\left(\hat{\wp}_1\right), \hat{\zeta}_{\hat{\Gamma}}\left(\hat{\wp}_2\right)\right)$
(2) $\hat{\vartheta}_{\hat{\Gamma}}\left(\hat{\varsigma}\hat{\wp}_1 + (1-\hat{\varsigma})\,\hat{\wp}_2\right) \leq max\left(\hat{\vartheta}_{\hat{\Gamma}}\left(\hat{\wp}_1\right), \hat{\vartheta}_{\hat{\Gamma}}\left(\hat{\wp}_2\right)\right)$

with $\hat{\varsigma} \in [0,1]$ and $\hat{\wp}_1, \hat{\wp}_2 \in \hat{\Delta}$.

Definition 2.6. [16] A NS $\hat{\aleph}$ is stated as

$$\hat{\aleph} = \left\{\left(\hat{\wp}, \langle\hat{\zeta}_{\hat{\aleph}}(\hat{\wp}), \hat{\vartheta}_{\hat{\aleph}}(\hat{\wp}), \hat{\xi}_{\hat{\aleph}}(\hat{\wp})\rangle\right) : \hat{\wp} \in \hat{\Delta}, \hat{\zeta}_{\hat{\aleph}}, \hat{\vartheta}_{\hat{\aleph}}, \hat{\xi}_{\hat{\aleph}} \in \left]^{-}0, 1^{+}\right[\right\}$$

with $\hat{\zeta}_{\hat{\aleph}}$, $\hat{\vartheta}_{\hat{\aleph}}$ and $\hat{\xi}_{\hat{\aleph}}$ as belonging, non-belonging and indeterminate functions such that $^{-}0 \leq \hat{\zeta}_{\hat{\aleph}}(\hat{\wp}) + \hat{\vartheta}_{\hat{\aleph}}(\hat{\wp}) + \hat{\xi}_{\hat{\aleph}}(\hat{\wp}) \leq 3^{+}$.

Definition 2.7. [26] A NS $\hat{\aleph}$ is stated to be convex NS when its belonging function $\hat{\zeta}_{\hat{\aleph}}$, non belonging function $\hat{\vartheta}_{\hat{\aleph}}$ and indeterminate function $\hat{\xi}_{\hat{\aleph}}$ satisfy the following inequalities

(1) $\hat{\zeta}_{\hat{\aleph}}\left(\varsigma\hat{\wp}_1+(1-\varsigma)\,\hat{\wp}_2\right)\geq min\left(\hat{\zeta}_{\hat{\aleph}}\left(\hat{\wp}_1\right),\hat{\zeta}_{\hat{\aleph}}\left(\hat{\wp}_2\right)\right)$
(2) $\hat{\vartheta}_{\hat{\aleph}}\left(\varsigma\hat{\wp}_1+(1-\varsigma)\,\hat{\wp}_2\right)\leq max\left(\hat{\vartheta}_{\hat{\aleph}}\left(\hat{\wp}_1\right),\hat{\vartheta}_{\hat{\aleph}}\left(\hat{\wp}_2\right)\right)$
(3) $\hat{\xi}_{\hat{\aleph}}\left(\varsigma\hat{\wp}_1+(1-\varsigma)\,\hat{\wp}_2\right)\leq max\left(\hat{\xi}_{\hat{\aleph}}\left(\hat{\wp}_1\right),\hat{\xi}_{\hat{\aleph}}\left(\hat{\wp}_2\right)\right)$

with $\varsigma\in[0,1]$ and $\hat{\wp}_1,\hat{\wp}_2\in\hat{\Delta}$.

Definition 2.8. [24] A *refined FS* $\hat{\Omega}_{RFS}$ is stated as

$$\hat{\Omega}_{RFS}=\left\{\left(\hat{\wp},\left\langle\hat{\zeta}^1_{\hat{\Omega}_{RFS}}(\hat{\wp}),\hat{\zeta}^2_{\hat{\Omega}_{RFS}}(\hat{\wp}),...,\hat{\zeta}^p_{\hat{\Omega}_{RFS}}(\hat{\wp})\right\rangle\right):p\geq 2,\hat{\wp}\in\hat{\Omega}_{RFS}\right\}$$

with $\hat{\zeta}^k_{\hat{\Omega}_{RFS}}$ as sub-belonging grades of k^{th}-type entities of $\hat{\Delta}$ with respect to $\hat{\Omega}_{RFS}$, and for $k\in[1,p]$ and $\sum_{k=1}^{p}\sup\hat{\zeta}^k_{\hat{\wp}}\leq 1,\forall\hat{\wp}\in\hat{\Omega}_{RFS}$.

Definition 2.9. [24] A refined IFS $\hat{\Omega}_{RIFS}$ is stated as

$$\hat{\Omega}_{RIFS}=\left\{\left(\hat{\wp},\left\langle\begin{array}{c}\left(\hat{\zeta}^1_{\hat{\Omega}_{RIFS}}(\hat{\wp}),\hat{\zeta}^2_{\hat{\Omega}_{RIFS}}(\hat{\wp}),...,\hat{\zeta}^p_{\hat{\Omega}_{RIFS}}(\hat{\wp})\right);\\ \left(\hat{\vartheta}^1_{\hat{\Omega}_{RIFS}}(\hat{\wp}),\hat{\vartheta}^2_{\hat{\Omega}_{RIFS}}(\hat{\wp}),...,\hat{\vartheta}^s_{\hat{\Omega}_{RIFS}}(\hat{\wp})\right)\end{array}\right\rangle\right),p+s\geqslant 3,\hat{\wp}\in\hat{\Omega}_{RIFS}\right\}$$

with $\hat{\zeta}^k_{\hat{\Omega}_{RIFS}}$ as sub-belonging grades of k^{th}-type entities with respect to $\hat{\Omega}_{RIFS}$, and $\hat{\vartheta}^l_{\hat{\Omega}_{RIFS}}$ as sub non-belonging grades of l^{th}-type entities with respect to $\hat{\Omega}_{RIFS}$ and $\sum_{k=1}^{p}\sup\hat{\zeta}^k+\sum_{l=1}^{s}\sup\hat{\vartheta}^l\leq 1$, and $\hat{\zeta}^k_{\hat{\Omega}_{RIFS}},\hat{\vartheta}^l_{\hat{\Omega}_{RIFS}}\subseteq[0,1]$ for $k\in[1,p]$ and $l\in[1,s]$.

Definition 2.10. [24] A RNS $\hat{\Omega}_{RNS}$ is stated as

$$\hat{\Omega}_{RNS}=\left\{\left(\hat{\wp},\left\langle\begin{array}{c}\left(\hat{\zeta}^1_{\hat{\Omega}_{RNS}}(\hat{\wp}),\hat{\zeta}^2_{\hat{\Omega}_{RNS}}(\hat{\wp}),...,\hat{\zeta}^p_{\hat{\Omega}_{RNS}}(\hat{\wp})\right);\\ \left(\hat{\vartheta}^1_{\hat{\Omega}_{RNS}}(\hat{\wp}),\hat{\vartheta}^2_{\hat{\Omega}_{RNS}}(\hat{\wp}),...,\hat{\vartheta}^s_{\hat{\Omega}_{RNS}}(\hat{\wp})\right);\\ \left(\hat{\xi}^1_{\hat{\Omega}_{RNS}}(\hat{\wp}),\hat{\xi}^2_{\hat{\Omega}_{RNS}}(\hat{\wp}),...,\hat{\xi}^t_{\hat{\Omega}_{RNS}}(\hat{\wp})\right)\end{array}\right\rangle\right):p+s+t\geqslant 3,\hat{\wp}\in\hat{\Omega}_{RNS}\right\}$$

with $\hat{\zeta}^k_{\hat{\Omega}_{RNS}}$ as sub-belonging grades of k^{th}-type entities, $\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}$ as sub non-belonging grades of l^{th}-type entities and $\hat{\xi}^m_{\hat{\Omega}_{RNS}}$ as sub indeterminate grades of m^{th}-type entities with respect to $\hat{\Omega}_{RNS}$ and ${}^-0\leq\sum_{k=1}^{p}\sup\hat{\zeta}^k_{\hat{\Omega}_{RNS}}+\sum_{l=1}^{s}\sup\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}+\sum_{m=1}^{t}\sup\hat{\xi}^m_{\hat{\Omega}_{RNS}}\leq 3^+$, and $\hat{\zeta}^k_{\hat{\Omega}_{RNS}},\hat{\vartheta}^l_{\hat{\Omega}_{RNS}},\hat{\xi}^m_{\hat{\Omega}_{RNS}}\subseteq\,]^-0,1^+[$ for $k\in[1,p],l\in[1,s],m\in[1,t]$.

3. **Convexity and Concavity on RNSs**

This portion describes the notions of convexity and concavity for RNSs. Throughout the paper, the symbols "RNS" and "$\overline{\hat{z}_1\hat{z}_2}$" are meant for RNS and line-segment correspondingly.

Definition 3.1. In $\mathscr{G}$, a RNS $\hat{\Omega}_{RNS}$ is stated to be convex if the points $\hat{z}_1,\hat{z}_2,\hat{z}_3\in\mathscr{G}$ on $\overline{\hat{z}_1\hat{z}_2}$

$$\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right)\geq\min\left(\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right),\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right),k\in[1,p]$$

$$\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right)\leq\max\left(\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right),\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right),l\in[1,s]$$

$$\hat{\xi}^m_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\xi}^m_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\xi}^m_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), m \in [1,t]$$

where $\hat{\zeta}^k_{\hat{\Omega}_{RNS}}$ is k^{th}-type grade of sub-belonging of the entities with respect to $\hat{\Omega}_{RNS}$, and for $k \in [1,p]$, $\sum_{k=1}^{p} \sup \hat{\zeta}^k \leq 1$, $\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}$ is l^{th}-type grade of sub non-belonging of the entities with respect to $\hat{\Omega}_{RNS}$, and for $l \in [1,s]$ and $\sum_{l=1}^{s} \sup \hat{\vartheta}^l \leq 1$ and $\hat{\xi}^m_{\hat{\Omega}_{RNS}}$ is m^{th}-type grade of sub-indeterminacy of the entities with respect to $\hat{\Omega}_{RNS}$, and for $m \in [1,t]$, $\sum_{m=1}^{t} \sup \hat{\xi}^m \leq 1$ with condition $\sum_{k=1}^{p} \sup \hat{\zeta}^k + \sum_{l=1}^{s} \sup \hat{\vartheta}^l + \sum_{m=1}^{t} \sup \hat{\xi}^m \leq 3$. The symbol $\hat{\Xi}_{CxRNS}$ is meant for family of convex RNSs.

Definition 3.2. In $\mathscr{G}$, a RNS $\hat{\Omega}_{RNS}$ is stated to be ortho-convex if the points $\hat{z}_1, \hat{z}_2, \hat{z}_3 \in \mathscr{G}$ on $\overline{\hat{z}_1\hat{z}_2}$ which is lying on that line which is $\|$ axis

$$\hat{\zeta}^k_{\hat{\Omega}_{RNS}}(\hat{z}'_3) \geq \min\left(\hat{\zeta}^k_{\hat{\Omega}_{RNS}}(\hat{z}'_1), \hat{\zeta}^k_{\hat{\Omega}_{RNS}}(\hat{z}'_2)\right), k \in [1,p].$$
$$\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}(\hat{z}'_3) \leq \max\left(\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}(\hat{z}'_1), \hat{\vartheta}^l_{\hat{\Omega}_{RNS}}(\hat{z}'_2)\right), l \in [1,s].$$
$$\hat{\xi}^m_{\hat{\Omega}_{RNS}}(\hat{z}'_3) \leq \max\left(\hat{\xi}^m_{\hat{\Omega}_{RNS}}(\hat{z}'_1), \hat{\xi}^m_{\hat{\Omega}_{RNS}}(\hat{z}'_2)\right), m \in [1,t].$$

with same conditions as provided in Definition 3.1. The symbol $\hat{\Xi}^O_{CxRNS}$ is meant for family of ortho-convex RNSs.

Remark 3.3. If $\hat{\Omega}_{RNS} \in \hat{\Xi}^O_{CxRNS}$ then $\hat{\Omega}_{RNS} \in \hat{\Xi}_{CxRNS}$ but the converse is not true.

Definition 3.4. In $\mathscr{G}$, a RNS $\hat{\Omega}_{RNS}$ is stated to be concave if the points $\hat{z}_1, \hat{z}_2, \hat{z}_3 \in \mathscr{G}$ on $\overline{\hat{z}_1\hat{z}_2}$

$$\hat{\zeta}^k_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\zeta}^k_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\zeta}^k_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), k \in [1,p].$$
$$\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}(\hat{z}_3) \geq \min\left(\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\vartheta}^l_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), l \in [1,s].$$
$$\hat{\xi}^m_{\hat{\Omega}_{RNS}}(\hat{z}_3) \geq \min\left(\hat{\xi}^m_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\xi}^m_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), m \in [1,t].$$

where

$\hat{\zeta}^k_{\hat{\Omega}_{RNS}}$ is k^{th}-type grade of sub-belonging of the entities with respect to $\hat{\Omega}_{RNS}$, and is subset of I for $k \in [1,p]$ and $\sum_{k=1}^{p} \sup \hat{\zeta}^k \leq 1$, $\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}$ is l^{th}-type grade of sub non-belonging of the entities with respect to $\hat{\Omega}_{RNS}$, and is subset of I for $l \in [1,s]$ and $\sum_{l=1}^{s} \sup \hat{\vartheta}^l \leq 1$ with condition $\sum_{k=1}^{p} \sup \hat{\zeta}^k + \sum_{l=1}^{s} \sup \hat{\vartheta}^l \leq 1$ and $\hat{\xi}^m_{\hat{\Omega}_{RNS}}$ is m^{th}-type grade of sub indeterminacy of the entities with respect to $\hat{\Omega}_{RNS}$, and is subset of I for $m \in [1,t]$ and $\sum_{m=1}^{t} \sup \hat{\xi}^m \leq 1$ with condition $\sum_{k=1}^{p} \sup \hat{\zeta}^k + \sum_{l=1}^{s} \sup \hat{\vartheta}^l + \sum_{m=1}^{t} \sup \hat{\xi}^m \leq 3$.

The symbol $\hat{\Xi}_{CvRNS}$ is meant for family of concave RNSs.

Definition 3.5. In $\mathscr{G}$, a RNS $\hat{\Omega}_{RNS}$ is stated to be ortho-concave if the points $\hat{z}_1, \hat{z}_2, \hat{z}_3 \in \mathscr{G}$ on $\overline{\hat{z}_1\hat{z}_2}$ that is lying on line which is $\parallel$ axis

$$\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right) \leq \max\left(\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right), k \in [1, p].$$

$$\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right) \geq \min\left(\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right), l \in [1, s].$$

$$\hat{\zeta}^m_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right) \geq \min\left(\hat{\zeta}^m_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\vartheta}^m_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right), m \in [1, t].$$

with same conditions as provided in Definition 3.4.

The symbol $\hat{\Xi}^O_{CvRNS}$ is meant for family of ortho-concave RNSs.

Remark 3.6. If $\hat{\Omega}_{RNS} \in \hat{\Xi}^O_{CvRNS}$ then $\hat{\Omega}_{RNS} \in \hat{\Xi}_{CvRNS}$ but the converse is not true.

Theorem 3.7. *If $\hat{\Omega}_{RNS} \in \hat{\Xi}_{CxRNS}$ then $\hat{\Omega}^c_{RNS} \in \hat{\Xi}_{CvRNS}$.*

Proof. If $\hat{\Omega}_{RNS} \in \hat{\Xi}_{CxRNS}$ then for points $\hat{z}_1, \hat{z}_2, \hat{z}_3$ on $\overline{\hat{z}_1\hat{z}_2}$

$$\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \geq \min\left(\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), \hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right), k \in [1, p]$$

so

$$\overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \leq 1 - \min\left(1 - \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), 1 - \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right), k \in [1, p] \tag{1}$$

now if

$$1 - \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right) \leq 1 - \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)$$

then

$$\min\left(1 - \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), 1 - \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right) = 1 - \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right)$$

and there from (1)

$$\overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \leq \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right)$$

similarly if

$$1 - \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right) \leq 1 - \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right)$$

then

$$\min\left(1 - \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), 1 - \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right) = 1 - \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)$$

so from (1)

$$\overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \leq \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right).$$

Hence

$$\overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \leq \max\left(\overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right), k \in [1, p].$$

Again

$$\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \leq \max\left(\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), \hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right), l \in [1, s]$$

then

$$\overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \geq 1 - \max\left(1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right), l \in [1, s] \tag{2}$$

now if

$$1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right) \geq 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)$$

then

$$\max\left(1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right) = 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right)$$

and from (2)

$$\overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \geq \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right)$$

similarly if

$$1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right) \geq 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right)$$

then

$$\max\left(1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right) = 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)$$

so from (2)

$$\overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \geq \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right).$$

Hence

$$\overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \geq \min\left(\overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right), l \in [1, s].$$

Similarly

$$\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \leq \max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), \hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right), m \in [1, t]$$

so

$$\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \geq 1 - \max\left(1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), 1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right), m \in [1, t] \tag{3}$$

now if

$$1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right) \geq 1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)$$

then

$$\max\left(1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), 1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right) = 1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right)$$

and there from (3)

$$\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \geq \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right)$$

similarly if

$$1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right) \geq 1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right)$$

then

$$\max\left(1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), 1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right) = 1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)$$

so from (3)

$$\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \geq \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right).$$

Hence

$$\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \geq \min\left(\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), m \in [1, t]$$

consequently $\hat{\Omega}^c_{RNS} \in \hat{\Xi}_{CvRNS}$. □

Remark 3.8. If $\hat{\Omega}_{RNS} \in \hat{\Xi}^{O}_{CxRNS}$ then $\hat{\Omega}^c_{RNS} \in \hat{\Xi}^{O}_{CvRNS}$ and $\hat{\Omega}_{RNS} \in \hat{\Xi}_{CvRNS}$.

Theorem 3.9. *If $\hat{\Omega}_{RNS}, \hat{\Theta}_{RNS} \in \hat{\Xi}_{CxRNS}$ then $\hat{\Omega}_{RNS} \cup \hat{\Theta}_{RNS} \in \hat{\Xi}_{CxRNS}$.*

Proof. Let $\hat{\Omega}_{RNS}$ and $\hat{\Theta}_{RNS}$ be two convex RNSs and $\hat{\Psi}_{RNS} = \hat{\Omega}_{RNS} \cup \hat{\Theta}_{RNS}$ and the points $\hat{z}_1, \hat{z}_2, \hat{z}_3$ on $\overline{\hat{z}_1\hat{z}_2}$. Now

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_1) = \min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_1)\right), k \in [1, p]$$
$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_2) = \min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right), k \in [1, p]$$
$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_3), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_3)\right), k \in [1, p].$$

Now

$$\min\left(\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_2)\right)$$
$$= \min\left(\min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_1)\right), \min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)\right)$$
$$= \min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right) \tag{4}$$

let

$$\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_3)$$

in (4) so that

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_3)$$

as $\hat{\Omega}_{RNS}$ is convex RNS so

$$\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \geq \min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right)$$
$$\geq \min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)$$

i.e.

$$\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_3) = \hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_3) \geq \min\left(\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_1) \hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_2)\right)$$

similarly for $\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_3) \leq \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_3)$ in equation (4) so that

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_3)$$

as $\hat{\Theta}_{RNS}$ is convex RNS so (4) becomes

$$\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_3) \geq \min\left(\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)$$
$$\geq \min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)$$

i.e.

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_3) \geq \min\left(\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_1)\, \hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_2)\right)$$

Again

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_1) = \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_1)\right), l \in [1, s]$$
$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_2) = \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right), l \in [1, s]$$
$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_3), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_3)\right), l \in [1, s].$$

Now

$$\max\left(\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_1), \hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_2)\right)$$
$$= \max\left(\max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_1)\right), \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)\right)$$
$$= \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right) \tag{5}$$

let

$$\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_3)$$

in (5) so that

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_3)$$

as $\hat{\Omega}_{RNS}$ is convex RNS so

$$\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right)$$
$$\leq \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)$$

i.e.

$$\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_3) = \hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_1)\, \hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_2)\right)$$

similarly for $\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_3) \geq \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_3)$ in equation (5) so that

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_3)$$

as $\hat{\Theta}_{RNS}$ is convex RNS so (5) becomes

$$\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)$$
$$\leq \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)$$

i.e.

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_1)\, \hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_2)\right).$$

Similarly now

$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_1) = \max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_1)\right), m \in [1, t]$$
$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_2) = \max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right), m \in [1, t]$$
$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_3)\right), m \in [1, t].$$

Now

$$\max\left(\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}_{1}\right),\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}_{2}\right)\right)$$
$$=\max\left(\max\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_{1}\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}_{1}\right)\right),\max\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_{2}\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}_{2}\right)\right)\right)$$
$$=\max\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_{1}\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}_{1}\right),\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_{2}\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}_{2}\right)\right) \tag{6}$$

let

$$\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_{3}\right)\leq\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}_{3}\right)$$

in (6) so that

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}_{3}\right)=\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_{3}\right)$$

as $\hat{\Omega}_{RNS}$ is convex RNS so

$$\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_{3}\right)\leq\max\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_{1}\right),\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_{2}\right)\right)$$
$$\leq\max\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_{1}\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}_{1}\right),\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_{2}\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}_{2}\right)\right)$$

i.e.

$$\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_{3}\right)=\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}_{3}\right)\leq\max\left(\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}_{1}\right)\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}_{2}\right)\right)$$

similarly for $\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}_{3}\right)\geq\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_{3}\right)$ in equation (6) so that

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}_{3}\right)=\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}_{3}\right)$$

as $\hat{\Theta}_{RNS}$ is convex RNS so (6) becomes

$$\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}_{3}\right)\leq\max\left(\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}_{1}\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}_{2}\right)\right)$$
$$\leq\max\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_{1}\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}_{1}\right),\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_{2}\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}_{2}\right)\right)$$

i.e.

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}_{3}\right)\leq\max\left(\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}_{1}\right)\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}_{2}\right)\right).$$

□

Theorem 3.10. *If* $\hat{\Omega}_{RNS},\hat{\Theta}_{RNS}\in\hat{\Xi}^{O}_{CxRNS}$ *then* $\hat{\Omega}_{RNS}\cup\hat{\Theta}_{RNS}\in\hat{\Xi}^{O}_{CxRNS}$ *and* $\hat{\Omega}_{RNS}\cup\hat{\Theta}_{RNS}\in\hat{\Xi}_{CxRNS}$.

Proof. Let $\hat{\Omega}_{RNS}$ and $\hat{\Theta}_{RNS}$ be two convex RNSs and $\hat{\Psi}_{RNS}=\hat{\Omega}_{RNS}\cup\hat{\Theta}_{RNS}$ and the points $\hat{z}'_{1},\hat{z}'_{2},\hat{z}'_{3}$ on $\overline{\hat{z}'_{1}\hat{z}'_{2}}\|$ axis.
Now

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_{1}\right)=\min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_{1}\right),\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_{1}\right)\right),k\in[1,p]$$
$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_{2}\right)=\min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_{2}\right),\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_{2}\right)\right),k\in[1,p]$$
$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_{3}\right)=\min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_{3}\right),\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_{3}\right)\right),k\in[1,p].$$

Now

$$\min\left(\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right)$$

$$=\min\left(\min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right)\right),\min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)\right)$$

$$=\min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right) \tag{7}$$

let

$$\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)\leq\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)$$

in (7) so that

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)$$

as $\hat{\Omega}_{RNS}$ is ortho-convex RNS so

$$\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)\geq\min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right)$$

$$\geq\min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

i.e.

$$\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)=\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)\geq\min\left(\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right)\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right)$$

similarly for $\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)\leq\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)$ in equation (7) so that

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)$$

as $\hat{\Theta}_{RNS}$ is ortho-convex RNS so

$$\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)\geq\min\left(\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

$$\geq\min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

i.e.

$$\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)=\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)\geq\min\left(\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right)\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right).$$

Again

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right)=\max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right)\right),l\in[1,s]$$

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)=\max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right),l\in[1,s]$$

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right),\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)\right),l\in[1,s].$$

Now

$$\max\left(\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right)$$

$$=\max\left(\max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right)\right),\max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)\right)$$

$$=\max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right) \tag{8}$$

let

$$\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)\geq\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)$$

in (8) so that

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)$$

as $\hat{\Omega}_{RNS}$ is ortho-convex RNS so

$$\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)\leq\max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right)$$
$$\leq\max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

i.e.

$$\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)=\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)\leq\max\left(\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right)\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right)$$

similarly for $\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)\geq\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)$ in equation (8) so that

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)$$

as $\hat{\Theta}_{RNS}$ is ortho-convex RNS so

$$\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)\leq\max\left(\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$
$$\leq\max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

i.e.

$$\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)=\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)\leq\max\left(\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right)\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right).$$

Similarly

$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right)=\max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\xi}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right)\right),m\in[1,t]$$
$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)=\max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\xi}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right),m\in[1,t]$$
$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right),\hat{\xi}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)\right),m\in[1,t].$$

Now

$$\max\left(\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right),\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right)$$
$$=\max\left(\max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\xi}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right)\right),\max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\xi}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)\right)$$
$$=\max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\xi}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\xi}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right) \tag{9}$$

let

$$\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)\geq\hat{\xi}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)$$

in (9) so that

$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)$$

as $\hat{\Omega}_{RNS}$ is ortho-convex RNS so

$$\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)\leq\max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right)$$
$$\leq\max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\xi}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\xi}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

i.e.

$$\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)=\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)\leq\max\left(\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right)\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right)$$

similarly for $\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)\geq\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)$ in equation (9) so that

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)$$

as $\hat{\Theta}_{RNS}$ is ortho-convex RNS so

$$\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)\leq\max\left(\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

$$\leq\max\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

i.e.

$$\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)=\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)\leq\max\left(\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right)\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right)$$

since every ortho-convex RNS is also convex RNS. Hence the proof. □

Remark 3.11. If $\hat{\Omega}^{\alpha}_{RNS}\in\hat{\Xi}_{CxRNS}$ then $\bigcup\limits_{\alpha}\hat{\Omega}^{\alpha}_{RNS}\in\hat{\Xi}_{CxRNS}$.

Remark 3.12. If $\hat{\Omega}^{\alpha}_{RNS}\in\hat{\Xi}^{O}_{CxRNS}$ then $\bigcup\limits_{\alpha}\hat{\Omega}^{\alpha}_{RNS}\in\hat{\Xi}^{O}_{CxRNS}$ and $\bigcup\limits_{\alpha}\hat{\Omega}^{\alpha}_{RNS}\in\hat{\Xi}_{CxRNS}$.

Theorem 3.13. *If $\hat{\Omega}_{RNS}\in\hat{\Xi}_{CvRNS}$ then $\hat{\Omega}^{c}_{RNS}\in\hat{\Xi}_{CxRNS}$.*

Proof. Let $\hat{\Omega}_{RNS}\in\hat{\Xi}_{CvRNS}$ and the points $\hat{z}_1,\hat{z}_2,\hat{z}_3$ on $\overline{\hat{z}_1\hat{z}_2}$, then

$$\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right)\leq\max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right),\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right),k\in[1,p]$$

so

$$\overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right)\geq1-\max\left(1-\overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right),1-\overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right),k\in[1,p] \tag{10}$$

now if

$$1-\overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right)\leq1-\overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)$$

then

$$\max\left(1-\overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right),1-\overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right)=1-\overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)$$

and from (10)

$$\overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right)\geq\overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)$$

similarly if

$$1-\overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\leq1-\overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right)$$

then

$$\max\left(1-\overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right),1-\overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right)=1-\overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right)$$

so from (10)

$$\overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right)\geq\overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right).$$

Hence

$$\overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \geq \min\left(\overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), k \in [1,p]$$

consequently $\hat{\Omega}^{c}_{RNS} \in \hat{\Xi}_{CxRNS}$.

Again

$$\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \geq \min\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), l \in [1,s]$$

so we have

$$\overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq 1 - \min\left(1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1), 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), l \in [1,s] \tag{11}$$

now if

$$1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1) \geq 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)$$

then

$$\min\left(1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1), 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right) = 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)$$

and there from (11)

$$\overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)$$

similarly if

$$1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2) \geq 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1)$$

then

$$\min\left(1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1), 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right) = 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1)$$

so from (11)

$$\overline{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1).$$

Hence

$$\overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq \max\left(\overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), l \in [1,s].$$

Similarly

$$\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \geq \min\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), m \in [1,t]$$

so we have

$$\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq 1 - \min\left(1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), 1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), m \in [1,t] \tag{12}$$

now if

$$1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1) \geq 1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)$$

then

$$\min\left(1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), 1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right) = 1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)$$

and there from (12)

$$\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)$$

similarly if

$$1-\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\geq 1-\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1)$$

then

$$\min\left(1-\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1),1-\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right)=1-\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1)$$

so from (12)

$$\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3)\leq\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1).$$

Hence

$$\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3)\leq\max\left(\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1),\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right),m\in[1,t]$$

consequently $\hat{\Omega}^{c}_{RNS}\in\hat{\Xi}_{CxRNS}$. □

Remark 3.14. If $\hat{\Omega}_{RNS}\in\hat{\Xi}^{O}_{CvRNS}$ then $\hat{\Omega}^{c}_{RNS}\in\hat{\Xi}^{O}_{CxRNS}$ and $\hat{\Omega}_{RNS}\in\hat{\Xi}_{CxRNS}$.

Theorem 3.15. *If $\hat{\Omega}_{RNS},\hat{\Theta}_{RNS}\in\hat{\Xi}_{CvRNS}$ then $\hat{\Omega}_{RNS}\cup\hat{\Theta}_{RNS}\in\hat{\Xi}_{CvRNS}$.*

Proof. Let $\hat{\Omega}_{RNS},\hat{\Theta}_{RNS}\in\hat{\Xi}_{CvRNS}$, $\hat{\Psi}_{RNS}=\hat{\Omega}_{RNS}\cup\hat{\Theta}_{RNS}$ and the points $\hat{z}_1,\hat{z}_2,\hat{z}_3$ on $\overline{\hat{z}_1\hat{z}_2}$ now

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_1)=\max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_1),\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_1)\right),k\in[1,p]$$
$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_2)=\max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_2),\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right),k\in[1,p]$$
$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_3)=\max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_3),\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_3)\right),k\in[1,p]$$

now

$$\begin{aligned}&\max\left(\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_1),\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_2)\right)\\ =&\max\left(\max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_1),\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_1)\right),\max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_2),\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)\right)\\ =&\max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_1),\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_1),\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_2),\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)\end{aligned}\tag{13}$$

let

$$\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_3)\geq\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_3)$$

in equation (13) so that

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_3)=\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_3)$$

as $\hat{\Omega}_{RNS}$ is concave RNS so equation (13) becomes

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_3)=\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_3)\leq\max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_1),\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right)$$

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_3)\leq\max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_1),\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_1),\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\,\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)$$

i.e.

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_3)\leq\max\left(\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_1)\,\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_2)\right)$$

similarly for $\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}_3\right) \geq \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right)$,in equation (13) so that

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}_3\right) = \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}_3\right)$$

as $\hat{\Theta}_{RNS}$ is concave RNS so equation (13) becomes

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}_3\right) = \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}_3\right) \leq \max\left(\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}_1\right), \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right)$$

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}_3\right) \leq \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}_1\right), \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right) \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}_2\right)\right)$$

i.e.

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}_3\right) \leq \max\left(\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}_1\right) \hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}_2\right)\right).$$

Now

$$\begin{aligned}
\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_1\right) &= \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_1\right)\right), l \in [1, s] \\
\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_2\right) &= \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_2\right)\right), l \in [1, s] \\
\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_3\right) &= \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_3\right)\right), l \in [1, s]
\end{aligned} \tag{14}$$

now

$$\begin{aligned}
&\max\left(\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_1\right), \hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_2\right)\right) \\
&= \max\left(\max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_1\right)\right), \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_2\right)\right)\right) \\
&= \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_1\right), \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_2\right)\right)
\end{aligned}$$

let

$$\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \geq \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_3\right)$$

in equation (14) so that

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_3\right) = \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right)$$

as $\hat{\Omega}_{RNS}$ is concave RNS so equation (14) becomes

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_3\right) = \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \leq \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right)$$

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_3\right) \leq \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_1\right), \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right) \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_2\right)\right)$$

i.e.

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_3\right) \leq \max\left(\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_1\right) \hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_2\right)\right)$$

similarly for $\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_3\right) \geq \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right)$,in equation (14) so that

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_3\right) = \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_3\right)$$

as $\hat{\Theta}_{RNS}$ is concave RNS so equation (14) becomes

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_3\right) = \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_3\right) \leq \max\left(\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_1\right), \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right)$$

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_3\right) \leq \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_1\right), \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right) \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_2\right)\right)$$

i.e.

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_1)\,\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_2)\right).$$

Similarly

$$\begin{aligned}
\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_1) &= \max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_1)\right), m \in [1,t] \\
\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_2) &= \max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right), m \in [1,t] \\
\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_3) &= \max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_3)\right), m \in [1,t]
\end{aligned} \tag{15}$$

now

$$\begin{aligned}
&\max\left(\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_1), \hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_2)\right) \\
&= \max\left(\max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_1)\right), \max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)\right) \\
&= \max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)
\end{aligned}$$

let

$$\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \geq \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_3)$$

in equation (15) so that

$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3)$$

as $\hat{\Omega}_{RNS}$ is concave RNS so equation (15) becomes

$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right)$$

$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\,\hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)$$

i.e.

$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_1)\,\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_2)\right)$$

similarly for $\hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_3) \geq \hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3)$,in equation (15) so that

$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_3)$$

as $\hat{\Theta}_{RNS}$ is concave RNS so equation (15) becomes

$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right)$$

$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\,\hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)$$

i.e.

$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_1)\,\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_2)\right)$$

hence the proof. □

Theorem 3.16. *If* $\hat{\Omega}_{RNS}, \hat{\Theta}_{RNS} \in \hat{\Xi}^{O}_{CvRNS}$ *then* $\hat{\Omega}_{RNS} \cup \hat{\Theta}_{RNS} \in \hat{\Xi}^{O}_{CvRNS}$ *and* $\hat{\Omega}_{RNS} \cup \hat{\Theta}_{RNS} \in \hat{\Xi}_{CvRNS}$.

Proof. Let $\hat{\Omega}_{RNS}, \hat{\Theta}_{RNS} \in \hat{\Xi}^{O}_{CvRNS}$, $\hat{\Psi}_{RNS} = \hat{\Omega}_{RNS} \cup \hat{\Theta}_{RNS}$ and the points $\hat{z}'_1, \hat{z}'_2, \hat{z}'_3$ on $\overline{\hat{z}'_1\hat{z}'_2}$ so that $\overline{\hat{z}'_1\hat{z}'_2}\|$ axis.

Now

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right) = \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right)\right), k \in [1, p]$$
$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right) = \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right), k \in [1, p]$$
$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) = \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)\right), k \in [1, p]$$

now

$$\max\left(\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right)$$
$$= \max\left(\max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right)\right), \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)\right)$$
$$= \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right) \tag{16}$$

let

$$\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right) \geq \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)$$

in (16) so that

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) = \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)$$

as $\hat{\Omega}_{RNS}$ is ortho-concave RNS so

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) = \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right) \leq \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right)$$
$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) \leq \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right) \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

i.e.

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) \leq \max\left(\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right) \hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right)$$

similarly for $\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right) \geq \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)$,in equation (16) so that

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) = \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)$$

as $\hat{\Theta}_{RNS}$ is ortho-concave RNS so equation (16) becomes

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) = \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right) \leq \max\left(\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right)$$
$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) \leq \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right) \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

i.e.

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) \leq \max\left(\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right) \hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right).$$

Again

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right) = \min\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right)\right), l \in [1, s]$$
$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right) = \min\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right), l \in [1, s]$$
$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) = \min\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)\right), l \in [1, s]$$

now

$$\min\left(\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right)$$

$$=\min\left(\min\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right)\right),\min\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)\right)$$

$$=\min\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right) \tag{17}$$

let

$$\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)\le\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)$$

in (17) so that

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)$$

as $\hat{\Omega}_{RNS}$ is ortho-concave RNS so

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)\ge\min\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right)$$

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)\ge\min\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

i.e.

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)\ge\min\left(\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right)\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right)$$

similarly for $\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)\le\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)$,in equation (17) so that

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)$$

as $\hat{\Theta}_{RNS}$ is ortho-concave RNS so equation (17) becomes

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)\ge\min\left(\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right)$$

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)\ge\min\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

i.e.

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)\ge\min\left(\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right)\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right).$$

Similarly

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right)=\min\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right)\right),m\in[1,t]$$

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)=\min\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right),m\in[1,t]$$

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\min\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)\right),m\in[1,t]$$

now

$$\min\left(\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right)$$

$$=\min\left(\min\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right)\right),\min\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)\right)$$

$$=\min\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right) \tag{18}$$

let

$$\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)\le\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)$$

in (18) so that

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)$$

as $\hat{\Omega}_{RNS}$ is ortho-concave RNS so

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)\geq\min\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right)$$

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)\geq\min\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

i.e.

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)\geq\min\left(\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right)\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right)$$

similarly for $\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)\leq\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)$,in equation (18) so that

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)$$

as $\hat{\zeta}_{\hat{\Theta}_{RNS}}$ is ortho-concave RNS so equation (18) becomes

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)\geq\min\left(\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right)$$

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)\geq\min\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

i.e.

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)\geq\min\left(\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right)\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right).$$

Since every ortho-concave RNS is also concave RNS which leads to completion of proof. □

Remark 3.17. If $\hat{\Omega}^{\alpha}_{RNS}\in\hat{\Xi}_{CvRNS}$ then $\bigcup_{\alpha}\hat{\Omega}^{\alpha}_{RNS}\in\hat{\Xi}_{CvRNS}$.

Remark 3.18. If $\hat{\Omega}^{\alpha}_{RNS}\in\hat{\Xi}^{O}_{CvRNS}$ then $\bigcup_{\alpha}\hat{\Omega}^{\alpha}_{RNS}\in\hat{\Xi}^{O}_{CvRNS}$ and $\bigcup_{\alpha}\hat{\Omega}^{\alpha}_{RNS}\in\hat{\Xi}_{CvRNS}$.

Definition 3.19. If $\mathscr{L}$ be any line and p be any point on it with $\mathscr{L}_p\perp\mathscr{L}$ at $\hat{\Omega}_{RNS}$ then the inf projection of $\hat{\Omega}_{RNS}$, denoted by $\hat{\Omega}_{\mathscr{L}}$, is stated as a mapping $\hat{\psi}:\mathscr{L}\to\hat{X}$ such that for any $p\in\mathscr{L},\hat{\psi}(p)=\inf\left\{\hat{\Omega}_{RNS}(\hat{r}),\hat{r}\in\mathscr{L}_p\right\}$ where $\left\{\hat{\Omega}_{RNS}(\hat{r}),\hat{r}\in\mathscr{L}_p\right\}\subseteq\hat{X}$.

Definition 3.20. If $\mathscr{L}$ be any line and p be any point on it with $\mathscr{L}_p\perp\mathscr{L}$ at $\hat{\Omega}_{RNS}$ then the sup projection of $\hat{\Omega}_{RNS}$, denoted by $\hat{\Omega}_{\mathscr{L}}$, is stated as a mapping $\hat{\psi}:\mathscr{L}\to\hat{X}$ such that for any $p\in\mathscr{L},\hat{\psi}(p)=\sup\left\{\hat{\Omega}_{RNS}(\hat{r}),\hat{r}\in\mathscr{L}_p\right\}$ where $\left\{\hat{\Omega}_{RNS}(\hat{r}),\hat{r}\in\mathscr{L}_p\right\}\subseteq\hat{X}$.

Theorem 3.21. *If $\hat{\Omega}_{RNS}\in\hat{\Xi}_{CvRNS}$ then $\hat{\Omega}_{\mathscr{L}}\in\hat{\Xi}_{CvRNS}$.*

Proof. Let $\hat{z}_1,\hat{z}_2,\hat{z}_3$ are the points lying on $\mathscr{L}$ with $\hat{z}_3$ that is lying on $\overline{\hat{z}_1\hat{z}_2}$, for any $\hat{\varepsilon}>0$, let $\hat{z}'_1,\hat{z}'_2$ be the points lying on $\mathscr{L}_{\hat{z}_1}$ and $\mathscr{L}_{\hat{z}_2}$ with $\hat{\zeta}^{k}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_1\right)>\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right)-\hat{\varepsilon}$ and $\hat{\zeta}^{k}_{\hat{\Omega}_{K}}\left(\hat{z}_2\right)>\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)-\hat{\varepsilon}$. Let $\hat{z}'_3=\overline{\hat{z}'_1\hat{z}'_2}\cap\mathscr{L}_{\hat{z}_3}$. Since $\hat{\Omega}_{RNS}$ is concave and $\hat{z}'_3\in\overline{\hat{z}'_1\hat{z}'_2}$, then we have

$$\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)\leq\max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right),k\in[1,p],$$

$$< \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_1\right)+\hat{\varepsilon},\hat{\zeta}^{k}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_2\right)+\hat{\varepsilon}\right)$$
$$= \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_1\right),\hat{\zeta}^{k}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_2\right)\right)+\hat{\varepsilon}$$

but

$$\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right) \geq \hat{\zeta}^{k}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_3\right)$$

hence

$$\hat{\zeta}^{k}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_3\right) < \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_1\right),\hat{\zeta}^{k}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_2\right)\right)+\hat{\varepsilon}$$

as $\hat{\varepsilon} > 0$ is of arbitrary nature, therefore

$$\hat{\zeta}^{k}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_3\right) \leq \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_1\right),\hat{\zeta}^{k}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_2\right)\right).$$

Again

$$\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right) \geq \min\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right), l \in [1,s],$$
$$> \min\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_1\right)+\hat{\varepsilon},\hat{\vartheta}^{l}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_2\right)+\hat{\varepsilon}\right)$$
$$= \min\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_1\right),\hat{\vartheta}^{l}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_2\right)\right)+\hat{\varepsilon}$$

but

$$\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right) \leq \hat{\vartheta}^{l}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_3\right)$$

hence

$$\hat{\vartheta}^{l}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_3\right) > \min\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_1\right),\hat{\vartheta}^{l}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_2\right)\right)+\hat{\varepsilon}$$

as $\hat{\varepsilon} > 0$ is of arbitrary nature, therefore

$$\hat{\vartheta}^{l}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_3\right) \geq \min\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_1\right),\hat{\vartheta}^{l}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_2\right)\right).$$

Similarly

$$\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right) \geq \min\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right), m \in [1,t],$$
$$> \min\left(\hat{\xi}^{m}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_1\right)+\hat{\varepsilon},\hat{\xi}^{m}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_2\right)+\hat{\varepsilon}\right)$$
$$= \min\left(\hat{\xi}^{m}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_1\right),\hat{\xi}^{m}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_2\right)\right)+\hat{\varepsilon}$$

but

$$\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right) \leq \hat{\xi}^{m}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_3\right)$$

hence

$$\hat{\xi}^{m}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_3\right) > \min\left(\hat{\xi}^{m}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_1\right),\hat{\xi}^{m}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_2\right)\right)+\hat{\varepsilon}$$

as $\hat{\varepsilon} > 0$ is of arbitrary nature, therefore

$$\hat{\xi}^{m}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_3\right) \geq \min\left(\hat{\xi}^{m}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_1\right),\hat{\xi}^{m}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_2\right)\right)$$

so $\hat{\Omega}_{\mathscr{L}}$ is concave. □

Remark 3.22. If $\hat{\Omega}_{RNS} \in \hat{\Xi}_{CxRNS}$ then $\hat{\Omega}_{\mathscr{L}} \in \hat{\Xi}_{CxRNS}$.

4. Conclusion

Through this research, the existing idea of NS is refined by categorizing its uncertain components into their respective multi-sub-grades. This refined idea is then integrated with the classical theory of convexity and concavity to make it applicable to solving optimization-related problems. Several useful axiomatic results are generalized with convex and concave RNS settings. It is observed that all classical results that are discussed in the paper, are quite valid for such settings. By taking into consideration the various kinds of convexity, the proposed model may be extended to generalize the results for them. Additionally, these results can also be utilized successfully for establishing various types of mathematical inequalities.

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