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Partial Foundation of Neutrosophic Number Theory

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Abstract: The aim of this paper is to establish a partial foundation of number theoretical concepts in the neutrosophic ring of integers $Z(I)$ because it is based on a partial order relationship. This work partially generalizes and deals with necessary and sufficient conditions for division, Euler's function, congruencies, and some other classical concepts in $Z(I)$. The main result of this work is to show that Euler's famous theorem is still true in the case of neutrosophic integers for our partial ordering relationship. Also, this work introduces an algorithm to solve Pell's equation in the neutrosophic ring of integers Z(I).

Keywords: Partial order, **n**eutrosophic Euler's theorem, neutrosophic integers, neutrosophic congruence, neutrosophic Pell's equation

1. Introduction

Neutrosophy is a new branch of philosophy founded by Smarandache to deal with indeterminacy in nature and science [12]. Neutrosophy has many important applications in many fields of knowledge such as computing [21], decision making [20], medical research [15], and applied science [22]. Then, it plays an important role in algebra, where many neutrosophic algebraic structures were defined and studied widely such as neutrosophic rings [1,8], neutrosophic vector spaces [4,14], neutrosophic modules [5,18], and refined neutrosophic rings [2,3,6,7,19]. Also, neutrosophy has many applications and effects on the progression of optimization [16], intelligent systems [13], and medical researches [15].

In the literature, number theory was a mathematical way to deal with the properties of integers such as Diophantine equations, primes, Euclidean division, and congruencies [10].

Neutrosophic number theory began in [9], where some properties of neutrosophic integers were introduced such as the form of primes in Z(I). Also, neutrosophic linear Diophantine equation was solved for the first time in [11].

This work is devoted to establish the theoretical partial foundations of neutrosophic number theory to deal with properties of neutrosophic integers. We aim to close an important research gap by determining algorithms and conditions for division, congruencies, neutrosophic Pell's equation, and Euler's function and theorem in Z(I).

Preliminaries

Definition 2.1: [1]

Let R be any ring, I be an indeterminacy with the property $I^2 = I$. Then $R(I) = \{a + bI; a, b \in R\}$ is called a neutrosophic ring.

If $R = Z$ is the ring of integers, then $Z(I) = \{a + bI; a, b \in Z\}$ is called the neutrosophic ring of integers. Elements of $Z(I)$ are called neutrosophic integers.

Remark: The notion of indeterminacy I was proposed by Smarandache and Kandasamy in [8] as an algebraic element instead of logical meaning. We deal with it by using its multiplicative property $I^2 = I$, which helps in the building of neutrosophic algebraic structures.

Definition 2.2: [10]

Pell's equation is the Diophantine equation with form $X^2 - DY^2 = N$; where $D, N \in Z$.

Theorem 2.3: [10]

If the equation $X^2 - DY^2 = 1$ has a solution, then $D > 0$ and D is square free.

Theorem 2.4: [10]

$Z[\sqrt{d_1}]$ is an integral domain, where d_1 is a square free integer.

Theorem 2.5: [9]

Let Z(I)= $\{a + bI; a, b \in Z\}$ the neutrosophic ring of integers. Then primes in Z(I) have one of the following forms:

$x = \pm p + (\pm 1 \pm p)I \text{ or } x = \pm 1 + (\pm p \pm 1)I; p \text{ is any prime in } Z.$

Definition 2.6: [19]

Let $R(I) = \{a + bI;\ a, b \in R\}$ be the real neutrosophic field, we say that $a + bI \le c + dI$ if and only if $a \le c \text{ and } a + b \le c + d$.

Theorem 2.7: [19]

The relation defined in Definition 2.6 is a partial order relation.

Remark 2.8: [19]

According to Theorem 2.7, we are able to define positive neutrosophic real numbers as follows:

$a + bI \geq 0 = 0 + 0.I$ implies that $a \geq 0, a + b \geq 0$.

Absolute value on R(I) can be defined as follows:

$|a + bI| = |a| + I[|a + b| - |a|]$, we can see that $|a + bI| \geq 0$.

Example 2.9: [19]

$x = 2 - I$ is a neutrosophic positive real number, since $2 \geq 0$ *and* $(2 - 1) = 1 \geq 0$.

$2 + I \geq 2$, that is because $2 \geq 2$ *and* $(2 + 1) = 3 \geq (2 + 0) = 2$.

3. Number Theory in $\boldsymbol{Z(I)}$

Definition 3.1: (Division)

Let $Z(I) = \{a + bI; a, b \in Z\}$ the neutrosophic ring of integers. For any $x, y \in Z(I)$, we say that $x|y$ if there is $r \in Z(I)$; $r.x = y$.

Theorem 3.2: (Form of division in *Z(I))*

Let Z(I)= $\{a + bI; a, b \in Z\}$ the neutrosophic ring of integers, $x = x_1 + x_2 I, y = y_1 + y_2 I$ be two arbitrary elements in Z(I). Then $x|y$ if and only if $x_1|y_1$ *and* $x_1 + x_2|y_1 + y_2$.

Proof:

Suppose that $x|y$, hence there is $r = r_1 + r_2 I \in Z(I)$; $r.x = y$. This implies

(I) $r_1 x_1 = y_1$, i.e. $x_1|y_1$.

(II) $r_1 x_2 + r_2 x_1 + r_2 x_2 = y_2$. By adding (I) to (II) we get

$r_1 x_1 + r_1 x_2 + r_2 x_1 + r_2 x_2 = y_1 + y_2$, this means that $(r_1 + r_2)(x_1 + x_2) = y_1 + y_2$.

Thus $x_1 + x_2|y_1 + y_2$.

Conversely, assume that $x_1|y_1$ *and* $x_1 + x_2|y_1 + y_2$, hence there is $a, b \in Z$ such that $ax_1 = y_1$ *and* $b(x_1 + x_2) = y_1 + y_2$. We put $r = a + (b - a)I$.

It is easy to see that $r.x = y$ and $x|y$.

Definition 3.3: (primes)

Let Z(I)= $\{a + bI; a, b \in Z\}$ the neutrosophic ring of integers. An arbitrary element $x \in Z(I)$ is called prime if $x|y.z$ implies $x|y$ *or* $x|z$.

Theorem 3.4: (Form of primes in Z(I))

This result was proved in [9].

Let Z(I)= $\{a + bI; a, b \in Z\}$ the neutrosophic ring of integers**.** Then primes in Z(I) have one of the following forms:

$x = \pm p + (\pm 1 \pm p)I \text{ or } x = \pm 1 + (\pm p \pm 1)I; p \text{ is any prime in } Z.$

Definition 3.5: (Congruence)

(a) Let $x = a + bI, y = c + dI, z = m + nI$ be three elements in Z(I). We say that $x \equiv y(mod z)$ if and only if $z|x - y$.

(b) We say that $z = \gcd(x, y)$ if and only if $z|x \text{ and } z|y$ and for each divisor $c|x \text{ and } c|y$, then $c|z$.

x, y are called relatively prime in Z(I) if and only if $\gcd(x, y) = 1$.

Theorem 3.6: (Form of congruencies in Z(I))

Let $x = a + bI, y = c + dI, z = m + nI$ be three elements in Z(I). Then $x \equiv y(mod z)$ if and only if $a \equiv c(mod\ m), a + b \equiv c + d(mod\ m + n)$.

Proof:

We suppose that $x \equiv y(mod z)$, hence $z|x - y$, i.e. $m + nI|(a - c) + (b - d)I$. This implies $m|a - c \text{ and } m + n|(a + b) - (c + d)$, thus $a \equiv c(mod\ m), a + b \equiv c + d(mod\ m + n)$.

Conversely, we suppose that $a \equiv c(mod\ m), a + b \equiv c + d(mod\ m + n)$, hence $m|a - c \text{ and } m + n|(a + b) - (c + d)$, this implies that $m + nI|(a - c) + (b - d)I$, i.e. $z|x - y$, which means that $x \equiv y(mod z)$.

Theorem 3.7:

Let $x = a + bI, y = c + dI, z = m + nI$ be three elements in *Z(I)*. Then $z = \gcd(x, y)$ if $m = \gcd(a, c) \text{ and } m + n = \gcd(a + b, c + d)$.

Proof:

Consider $z = m + nI$, where $m = \gcd(a, c) \text{ and } m + n = \gcd(a + b, c + d)$.

It is easy to check that $z|x \text{ and } z|y$, that is because $m = \gcd(a, c)\,|a, m = \gcd(a, c)\,|c$, and $m + n = \gcd(a + b, c + d)\,|a + b, m + n = \gcd(a + b, c + d)|c + d$. On the other hand, we assume that $l = f + gI$ is a common divisor of x and y. We shall prove that $l|z$.

Since l is a common divisor, then we have $f|a \text{ and } f|c$, hence $f|\gcd(a, c) = m$. Also, we have

$f+g|a+b$ and $f+g|c+d$, hence $f+g|\gcd(a+b,c+d)=m+n$. This implies that

$l|z$, and $z=\gcd(x,y)$.

Example 3.8:

(a) $3+5I\equiv(1+3I)(mod\ 2+2I)$. This is because $3\equiv1(mod\ 2), 3+5=8\equiv1+3=4(mod 4)$.

(b) $\gcd(3+5I,1+3I)=1+3I$, that is because $\gcd(3,1)=1=m, \gcd(3+5,1+3)=\gcd(8,4)=4=$ $m+n$, thus $m+nI=1+3I=\gcd(3+5I,1+3I)$.

Theorem 3.9: (Euclidian division theorem in Z(I))

Let Z(I) be the neutrosophic ring of integers, $x=a+bI, y=c+dI$ be two arbitrary elements in Z(I). There are two elements $q=s+tI, r=m+nI$ such that $x=q.y+r$.

Proof:

This proof is different from the proof which was introduced in [9].

By the division theorem in Z, we can find the following integers:

q_1,q_2,r_1,r_2: $a=q_1c+r_1$, and $a+b=(c+d)q_2+r_2$. By putting $s=q_1, t=(q_2-q_1), m=r_1, n=$ (r_2-r_1), we find that $x=q.y+r$.

Example 3.10:

Consider the following neutrosophic integers $x=5+4I, y=3+I$. There are $q=1+I, r=2-I$ such that $x=q.y+r$.

Remark 3.11: (Solvability of a linear congruence in Z(I))

To solve a linear congruence $x+yI\equiv a+bI(mod\ m+nI)$. We should take its equivalent congruencies according to Theorem 3.6:

$x\equiv a(mod\ m)$, and $x+y\equiv(a+b)(mod\ m+n)$. We solve the equivalent system, and compute x,y.

Example 3.12:

Consider the following neutrosophic linear congruence (*) $x+yI\equiv1+7I(mod\ 4+I)$. Its equivalent system is:

(a) $x\equiv1(mod\ 4)$. (It has a solution $x=1$).

$(b) x+y\equiv8(mod\ 5)$. (It has a solution $x+y=3$, $hence$ $y=2$. This means that $1+2I$ is a solution of the neutrosophic congruence (*).

We can see that $4+I|(1+2I)-(1+7I)$, that is because $(4+I)(-I)=-5I$.

Definition 3.14: (Euler's function in Z(I))

We define the neutrosophic Euler's function on Z(I) as follows:

$\varphi(a+bI) = |\{x = c+dI; \gcd(c+dI, a+bI) = 1\}|, where\ c+dI \leq a+bI.$

Theorem 3.15: (Euler's Theorem in *Z(I))*

(a) Let $x = a+bI$ be any element in *Z(I)*, then $\varphi(x) = \varphi(a) \times \varphi(b+a)$.

(b) If $y = c+dI$ is a neutrosophic integer with $\gcd(x,y) = 1$, hence $y^{\varphi(x)} \equiv 1(mod\ x)$.

(neutrosophic Euler's Theorem).

Proof:

(a) Let $y = c+dI$ be any neutrosophic integer with, $c+dI \leq a+bI, and$ $\gcd(x,y) = 1$. We can see by Theorem 3.7 that

$\gcd(a,c) = 1, \gcd(a+b, c+d) = 1$, i.e. (a,c) are relatively prime and $(a+b, c+d)$ are relatively prime, hence we get that $\varphi(x) = \varphi(a) \times \varphi(b+a)$.

(b) By classical Euler's Theorem, we have $c^{\varphi(a)} \equiv 1(mod\ a), and\ (c+d)^{\varphi(a+b)} \equiv 1(mod\ a+b)$, that is because $\gcd(a,c) = \gcd(a+b, c+d) = 1$ under the assumption of $\gcd(x,y) = 1$. Now, we can write $c^{\varphi(a)\times\varphi(b+a)} = c^{\varphi(x)} \equiv 1(mod\ a)$, $(c+d)^{\varphi(a)\times\varphi(b+a)} = (c+d)^{\varphi(x)} \equiv 1(mod\ a+b)$.

Now, we compute

$y^{\varphi(x)} = (c+dI)^{\varphi(x)} = c^{\varphi(x)} + I[\sum_{i=1}^{\varphi(x)} \binom{\varphi(x)}{i} c^{\varphi(x)-i} d^i] = c^{\varphi(x)} + I\left[(c+d)^{\varphi(x)} - c^{\varphi(x)}\right] = m + nI.$

We remark that $m = c^{\varphi(x)} \equiv 1(mod\ a), m+n = (c+d)^{\varphi(x)} \equiv 1(mod\ a+b)$, this implies that $y^{\varphi(x)} = m + nI \equiv 1(mod\ a+bI)$, according to Theorem 3.6.

The previous theorem will open a new door in the study of neutrosophic number theory, since it clarifies that Euler's famous theorem is still true in the case of neutrosophic integers.

Remark 3.16: (Solving a congruence linear system in Z(I))

To solve a linear system of congruencies in Z(I), we can solve the corresponding equivalent system in Z.

Example 3.17:

Consider the following linear system of congruencies in Z(I).

$2x + (3y-2x)I \equiv 3 + I(mod\ 7+4I), 4x + (y-4x)I \equiv 7 - 5I(mod\ 13-10I)$, we aim to find x, y.

The corresponding linear system in Z according to Theorem 3.6 is

$2x \equiv 3(mod7), 3y \equiv 4(mod\ 11), 4x \equiv 7(mod\ 13), y \equiv 2(mod3)$, it has a solution $x = y = 5$.

Thus the neutrosophic congruence in Z(I) has a solution $10 + 5I, 20 - 15I$.

4. Neutrosophic Pell's equation

Definition 4.1:

Let $Z(I) = \{a + bI;\ a, b \in Z\}$ be the neutrosophic ring of integers. The neutrosophic Pell's Equation in Z(I) is defined as follows:

$X^2 - DY^2 = C;\ X, Y, D, C \in Z(I)$.

We show the sufficient condition for solvability of neutrosophic Pell's equation.

Theorem 4.2:

Let $Z(I) = \{a + bI;\ a, b \in Z\}$ be the neutrosophic ring of integers,(*) $X^2 - DY^2 = C;\ X, Y, D, C \in Z(I)$ be a neutrosophic Pell's equation with $X = x_1 + x_2I, Y = y_1 + y_2I, D = d_1 + d_2I, C = c_1 + c_2I$. This equation is equivalent to the following two classical Pell's equations:

(a) ${x_1}^2 - d_1{y_1}^2 = c_1$.
(b) $(x_1 + x_2)^2 - (d_1 + d_2)(y_1 + y_2)^2 = c_1 + c_2$.
Proof:

It is sufficient to prove that equation (*) implies (a), (b).

By computing (*), we get

$(x_1 + x_2I)^2 - (d_1 + d_2I)(y_1 + y_2I)^2 = c_1 + c_2I$, this implies

$[{x_1}^2 - d_1{y_1}^2] + I[2x_1x_2 + {x_2}^2 - d_1{y_2}^2 - d_2{y_1}^2 - 2d_1y_1y_2 - 2d_2y_1y_2 - d_2{y_1}^2 - d_2{y_2}^2 - 2d_1d_2{y_1}^2 - 2d_1d_2y_1y_2 - 2d_1d_2{y_2}^2] = c_1 + c_2I$, thus

${x_1}^2 - d_1{y_1}^2 = c_1$. (Equation (a)), and

(**) $2x_1x_2 + {x_2}^2 - d_1{y_2}^2 - d_2{y_1}^2 - 2d_1y_1y_2 - 2d_2y_1y_2 - d_2{y_1}^2 - d_2{y_2}^2 - 2d_1d_2{y_1}^2 - 2d_1d_2y_1y_2 - 2d_1d_2{y_2}^2 = c_2$, by adding equation (a) to (**), we get

${x_1}^2 - d_1{y_1}^2 + 2x_1x_2 + {x_2}^2 - d_1{y_2}^2 - d_2{y_1}^2 - 2d_1y_1y_2 - 2d_2y_1y_2 - d_2{y_1}^2 - d_2{y_2}^2 - 2d_1d_2{y_1}^2 - 2d_1d_2y_1y_2 - 2d_1d_2{y_2}^2 = c_1 + c_2$, hence

$(x_1 + x_2)^2 - (d_1 + d_2)(y_1 + y_2)^2 = c_1 + c_2$. (Equation (b)).

Remark 4.3:

To solve the neutrosophic Pell's equation $X^2 - DY^2 = C$, follow these steps

1) Solve ${x_1}^2 - d_1{y_1}^2 = c_1$, if it is possible.

2) Solve $(x_1+x_2)^2-(d_1+d_2)(y_1+y_2)^2=c_1+c_2$, if it is possible.

3) Compute x_2, y_2.

We study some special neutrosophic Pell's equations.

Theorem 4.4:

If the neutrosophic Pell's equation $X^2-DY^2=1$ has non trivial solutions, then

$d_1>0, d_1+d_2>0$, and d_1, d_1+d_2 are square free.

Proof:

According to Theorem 4.2, the equation $X^2-DY^2=1$ is equivalent to
(a) $x_1{}^2-d_1y_1{}^2=1$.

(b) $(x_1+x_2)^2-(d_1+d_2)(y_1+y_2)^2=1$.

By Theorem , thus (a), (b) have non trivial solutions. By Theorem 2.3 , we find that $d_1>0, d_1+d_2>0$, and d_1, d_1+d_2 are square free.

Example 4.5:

The equation $X^2-(2+3I)Y^2=1$ has non trivial solution, that is because:

The equivalent system is: (a) $x_1{}^2-2y_1{}^2=1$, (b) $(x_1+x_2)^2-5(y_1+y_2)^2=1$.

Equation (a) has a solution $x_1=3, y_1=2$. Equation (b) has a solution $x_1+x_2=9, y_1+y_2=4$, thus

$x_2=9-x_1=6, y_2=4-y_1=2$. So $X=3+6I, Y=2+2I$. We can see easily that $2>0, 2+3=5>0$, and $2, 2+3=5$ are square free.

Example 4.6:

Let $X^2-(3-I)Y^2=-3+I$ be a neutrosophic Pell's equation. Its equivalent system is

$x_1{}^2-3y_1{}^2=-3, (x_1+x_2)^2-(2)(y_1+y_2)^2=-2$. The first equation has the solution

$x_1=3, y_1=2$, the second one has the solution

$x_1+x_2=4, y_1+y_2=3$, thus $x_2=1, y_2=1$. We find that $X=3+I, Y=2+I$ is a solution of $X^2-(3-I)Y^2=-3+I$.

Theorem 4.7:

If the Pell's equation $x_1{}^2-d_1y_1{}^2=c_1;\ d_1, c_1\in Z$ has m solutions exactly. Then the neutrosophic Pell's equation

$X^2 - d_1Y^2 = c_1; X = x_1 + x_2I, Y = y_1 + y_2I$ has exactly m^2 solutions.

Proof:

$X^2 - d_1Y^2 = c_1$ is equivalent to the system:

(a) ${x_1}^2 - d_1{y_1}^2 = c_1$.

(b) $(x_1 + x_2)^2 - d_1(y_1 + y_2)^2 = c_1$.

We can see that (a), (b) are the same Pell's equation, thus each one has m solutions. Hence we have for each value of x_1, *(m)* corresponding values of x_2, and we get the same thing for y_1, y_2. Thus we have exactly m^2 solutions for equation $X^2 - d_1Y^2 = c_1$.

Theorem 4.8:

If the neutrosophic Pell's equation $X^2 - Dy^2 = C;\ D = a - aI\ ; a \in Z$ is solvable, then $c_1 + c_2$ is a square.

Proof:

Suppose that $X^2 - Dy^2 = C$ has a solution $X = x_1 + x_2I, Y = y_1 + y_2I$, then

${x_1}^2 - a{y_1}^2 = c_1,\ (x_1 + x_2)^2 - (a - a)(y_1 + y_2)^2 = c_1 + c_2$ are solvable equations, thus

$(x_1 + x_2)^2 = c_1 + c_2$, and $c_1 + c_2$ is a square.

Theorem 4.9:

If the neutrosophic Pell's equation $X^2 - Dy^2 = C; D = aI; a \in Z$ is solvable, then c_1 is a square.

Proof:

Suppose that $X^2 - Dy^2 = C$ has a solution $X = x_1 + x_2I, Y = y_1 + y_2I$, then

${x_1}^2 - 0.{y_1}^2 = c_1,\ (x_1 + x_2)^2 - (a)(y_1 + y_2)^2 = c_1 + c_2$ are solvable equations, thus

$\mathrm{x_1}^2 = \mathrm{c_1}$, and c_1 is a square.

Remark 4.10:

If the neutrosophic Pell's equation $X^2 - Dy^2 = C; D = aI; a \in Z$ is solvable, then it has an infinite number of solutions. This is because $x_1 = \pm\sqrt{c_1}$ and $(y_1 + y_2)^2$ is constant, i.e there is an infinite number of possible solutions. For every value of y_1, there is a single related value of y_2.

Example 4.11:

Consider the following neutrosophic Pell's equation $X^2 - IY^2 = 1 + 4I$, the equivalent system is

${x_1}^2 = 1,\ (x_1 + x_2)^2 - (y_1 + y_2)^2 = 5$. It has a solution $x_1 = 1, x_2 = 2, y_1 + y_2 = 2$.

We can see that the solutions of $X^2 - IY^2 = 1 + 4I$ are:

$X = 1 + 2I\ or\ X = -1 + 4I, Y = y_1 + (2 - y_1)I$.

Theorem 4.12:

Let ${x_1}^2 - d_1{y_1}^2 = c_1, {x_2}^2 - d_2{y_2}^2 = c_2$ be two classical Pell's equations. They can be transformed into one corresponding neutrosophic Pell's equation (*) $X^2 - DY^2 = C;\ X = x_1 + (x_2 - x_1)I, Y = y_1 + (y_2 - y_1)I$,

$D = d_1 + (d_2 - d_1)I, C = c_1 + (c_2 - c_1)I$.

Proof:

The proof holds directly by easy computing of equation (*).

Example 4.13:

Let ${x_1}^2 - 2{y_1}^2 = 1, {x_2}^2 - 3{y_2}^2 = 5$ be two Pell's equations. The corresponding neutrosophic Pell's equation is $[x_1 + (x_2 - x_1)I]^2 - (2 + I)[y_1 + (y_2 - y_1)I]^2 = 1 + 4I$.

Theorem 4.14:

The neutrosophic Pell's equation (*) $X^2 - DY^2 = aI$; (d_1 is a positive integer and square free) has solutions if and only if the equation

${x_2}^2 - (d_1 + d_2){y_2}^2 = a$ has solutions. Its solution has the form $X = x_2I, Y = y_2I$.

Proof:

The equivalent system of (*) is:

(a) ${x_1}^2 - d_1{y_1}^2 = 0$.

(b) $(x_1 + x_2)^2 - (d_1 + d_2)(y_1 + y_2)^2 = a$.

Equation (a) has only the zero solution, that is because $Z[\sqrt{d_1}]$ is an integral domain, thus $x_1 = y_1 = 0$.

Equation (b) becomes ${x_2}^2 - (d_1 + d_2){y_2}^2 = a$. Hence (*) has solutions if and only if (b) has solutions.

The solutions of (*) have the property $x_1 = y_1 = 0$, so they have the form $X = x_2I, Y = y_2I$.

4. Conclusions

In this article, we have established the partial basic theory of neutrosophic numbers. Concepts such as division, relatively primes, congruencies, and Pell's equation were discussed and handled in the case of neutrosophic integers. Also, we have proved that Euler's famous theorem is still true in *Z(I).*

This work can be considered as a primary step in the study of neutrosophic number theory, we aim that it will be very effective in the study of neutrosophic integers.

We want to refer that we obtained our results about Euler's theorem under a partial order relation on neutrosophic integers.

As a future research direction, we aim to find a total order relation and to check Euler's theorem under it.

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