Neutrosophic Sets and Systems, Vol. 36, 2020
University of New Mexico

Introduction to Interval-valued Neutrosophic Subring

Sudipta Gayen[1], Florentin Smarandache[2], Sripati Jha[1] and Ranjan Kumar[3,*]

[1] National Institute of Technology Jamshedpur, India.

[2] University of New Mexico, USA.

[3] Jain Deemed to be University, Jayanagar, Bengaluru, India; ranjank.nit52@gmail.com

[*] Correspondence: ranjank.nit52@gmail.com

Abstract. The main purpose of this article is to develop and study the notion of interval-valued neutrosophic subring. Also, we have studied some homomorphic characteristics of interval-valued neutrosophic subring. Again, we have defined the concept of product of two interval-valued neutrosophic subrings and analyzed some of its important properties. Furthermore, we have developed the notion of interval-valued neutrosophic normal subring and studied some of its basic characteristics and homomorphic properties.

Keywords: Neutrosophic set; Interval-valued neutrosophic set; Interval-valued neutrosophic subring; Interval-valued neutrosophic normal subring

---

ABBREVIATIONS

TN signifies "T-norm".

SN signifies "S-norm".

IVTN signifies "interval-valued T-norm".

IVSN signifies "interval-valued S-norm".

CS signifies "crisp set".

FS signifies "fuzzy set".

IFS signifies "intuitionistic fuzzy set".

NS signifies "neutrosophic set".

PS signifies "plithogenic set".

FSG signifies "fuzzy subgroup".

IFSG signifies "intuitionistic fuzzy subgroup".

NSG signifies "neutrosophic subgroup".

CR signifies "crisp ring".

FSR signifies "fuzzy subring".

IFSR signifies "intuitionistic fuzzy subring".

NSR signifies "neutrosophic subring".

IVFSR signifies "interval-valued fuzzy subring".

IVIFSR signifies "interval-valued intuitionistic fuzzy subring".

IVNSR signifies "interval-valued neutrosophic subring".

IVNNSR signifies "interval-valued neutrosophic normal subring".

DMP signifies "decision making problem".

$\boldsymbol{\psi(P)}$ signifies "power set of P".

$\boldsymbol{L}$ signifies "the set $[0,1]$".

1. Introduction

Zadeh's vision behind introducing the revolutionary concept of FS [1] theory was to tackle uncertainty in a better way than CS theory, which has certain drawbacks. Later on, following his vision Atanassov introduced a more general version of it, which is known as IFS [2] theory. These IFSs are a little step ahead in managing ambiguities and hence are welcomed by numerous researchers. Furthermore, following their footsteps Smarandache introduced NS [3] theory, which is more capable of handling vague situations. It is a significant generalization over CS, FS, and IFS theories. Smarandache has also initiated the concept of PS [4] theory which has broader aspects than those previously discussed concepts. In NS and PS theory, he has also developed the notions of neutrosophic calculus [5], neutrosophic probability [6], neutrosophic statistics [7], integral, measure [8], neutrosophic psychology [9], neutrosophic robotics [10], neutrosophic triplet group [11], plithogenic hypersoft set [12], plithogenic fuzzy whole hypersoft set [13], plithogenic logic, probability [14], plithogenic subgroup [15], plithogenic hypersoft subgroup [16], etc. Again, NS theory has various other contributions in different scientific researches, like in linear programming [17–20], decision making [21–27], healthcare [28,29], shortest path problem [30–37], neutrosophic forecasting [38], resource leveling [39], transportation problem [40, 41], project scheduling [42], brain processing [43], etc.

Gradually, interval-valued versions of FS [44], IFS [45], and NS [46] were introduced, which are further generalizations of their CS, FS, IFS, and NS counterparts. Presently, these set theories are extensively used in different scientific domains. From the very start, various researchers have carried out this concepts and explored them in different dimensions. In the subsequent Table 1 we have referred some significant aspects of these notions.

Table 1. Importance of interval-valued notions in different domains.

Author & references	Year	Contributions in various fields
Biswas [47]	1994	Introduced interval-valued FSG.

continued ...

Author & references	Year	Contributions in various fields
Atanassov [45]	1999	Studied basic definition and some properties of IVFS.
Mondal & Samanta [48]	2001	Defined and studied topology of IVIFSs.
Wang et al. [46]	2005	Proposed and studied IVNS and interval-valued neutrosophic logic.
Ye [49]	2009	Worked on multi-criteria DMP under IVIFSs.
Kang & Hur [50]	2010	Introduced and studied the notion of IVFSR.
Akram & Dudek [51]	2011	Defined some basic operations on interval-valued fuzzy graphs and studied some of their properties.
Aygünoğlu et al. [52]	2012	Introduced interval-valued IFSG and studied some homomorphic properties of it.
Moorthy & Arjunan [53]	2014	Introduced and studied some properties of IVIFSR.
Aiwu et al. [54]	2015	Worked on multi-attribute DMP under IVNSs.
Broumi et al. [56]	2016	Worked on interval-valued neutrosophic graph theory.
Deli [55]	2017	Applied soft version of IVNS in DMP.
Broumi et al. [56]	2019	Studied some properties of interval-valued neutrosophic graphs.

Group theory and ring theory are fundamental building blocks of abstract algebra, which are utilized in different scientific domains. But, initially, these concepts were introduced upon crisp environment. Gradually, from 1971 on-wards researchers started introducing these concepts under various uncertain environments. Some significant developments of these notions under uncertainty are the concepts of FSG [57], IFSG [58], NSG [59], FSR [60, 61], IFSR [62], NSR [63], etc. Again some researchers have introduced these concepts under interval-valued environments and initiated the notions of interval-valued FSG [47], interval-valued IFSG [52], interval-valued NSG [64], interval-valued FSR [50], interval-valued IFSR [53], etc. Some more articles which can be helpful to different researchers are [65–71], etc. But, still, the notion of interval-valued NSR is undefined. Hence, by mixing interval-valued environment with neutrosophic environment, we can introduce a more general version of NSR, which will be called IVNSR. Also, their homomorphic properties can be studied. Again, their product and normal forms can be developed and analyzed. Based on these observations, the followings are some of our main objectives for this article:

- Introducing the notion of IVNSR and a analyzing its homomorphic properties.
- Introducing the product of IVNSRs.
- Introducing subring of a IVNSR.

- Introducing the notion of IVNNSR and a analyzing its homomorphic attributes.

The subsequent arrangement of this article is: in Section 2, some desk researches of FS, IFS, NS, IVFS, IVIFS, IVNS, FSR, IFSR, NSR, IVFSR, IVIFSR, etc., are discussed. In Section 3, the idea of IVNSR has been introduced and some basic theories are provided. Also, their product and normal versions are defined. Also, some theories are given to understand their algebraic attributes. Lastly, in Section 4, the concluding segment is given and also some opportunities for further studies are mentioned.

2. **Literature Review**

Definition 2.1. [1] A FS of a CS P is defined as the function $\nu : P \to L$.

Definition 2.2. [2] An IFS ρ of a CS P is defined as $\rho = \{(r, t_\rho(r), f_\gamma(r)) : r \in P\}$, where $\forall r \in P$, $t_\rho(r)$ and $f_\rho(r)$ known as the degree of membership and non-membership which satisfy the inequality $0 \le t_\rho(r) + f_\rho(r) \le 1$.

Definition 2.3. [3] A NS κ of a CS P is defined as $\kappa = \{(r, t_\kappa(r), i_\kappa(r), f_\kappa(r)) : r \in P\}$, where $\forall r \in P$, $t_\kappa(r), i_\kappa(r),$ and $f_\kappa(r)$ are known as degree of truth, indeterminacy, and falsity which satisfy the inequality $^-0 \le t_\kappa(r) + i_\kappa(r) + f_\kappa(r) \le 3^+$.

Definition 2.4. [52] An interval number of $L = [0, 1]$ is denoted as $\bar{k} = [k^-, k^+]$, where $1 \ge k^+ \ge k^- \ge 0$.

Definition 2.5. [44] An IVFS of P is defined as the mapping $\nu : P \to \psi(L)$.

Definition 2.6. [45] An IVIFS of P is defined as the mapping $\bar{\rho} : P \to \psi(L) \times \psi(L)$, It is denoted as $\bar{\rho} = \{(r, \bar{t}_{\bar{\rho}}(r), \bar{f}_{\bar{\rho}}(r)) : r \in P\}$, where $\bar{t}_{\bar{\rho}}(r),\ \bar{f}_{\bar{\rho}}(r) \subseteq [0, 1]$.

Definition 2.7. [46] An IVNS of P is defined as the mapping $\bar{\kappa} : P \to \psi(L) \times \psi(L) \times \psi(L)$, It is denoted as $\bar{\kappa} = \{(r, \bar{t}_{\bar{\kappa}}(r), \bar{i}_{\bar{\kappa}}(r), \bar{f}_{\bar{\kappa}}(r)) : r \in P\}$ where $\forall r \in P$, $\bar{t}_{\bar{\kappa}}(r),\ \bar{i}_{\bar{\kappa}}(r),$ and $\bar{f}_{\bar{\kappa}}(r) \subseteq L$.

Definition 2.8. [46] Let $\bar{\kappa_1} = \{(r, \bar{t}_{\bar{\kappa_1}}(r), \bar{i}_{\bar{\kappa_1}}(r), \bar{f}_{\bar{\kappa_1}}(r)) : r \in P\}$ and $\bar{\kappa_2} = \{(r, \bar{t}_{\bar{\kappa_2}}(r), \bar{i}_{\bar{\kappa_2}}(r), \bar{f}_{\bar{\kappa_2}}(r)) : r \in P\}$ be two IVNSs of P. Then union of $\bar{\kappa_1}$ and $\bar{\kappa_2}$ is defined as

$$\bar{t}_{\bar{\kappa_1} \cup \bar{\kappa_2}} = \left[\max\{\bar{t}^-_{\bar{\kappa_1}}, \bar{t}^-_{\bar{\kappa_2}}\}, \max\{\bar{t}^+_{\bar{\kappa_1}}, \bar{t}^+_{\bar{\kappa_2}}\}\right]$$
$$\bar{t}_{\bar{\kappa_1} \cup \bar{\kappa_2}} = \left[\min\{\bar{i}^-_{\bar{\kappa_1}}, \bar{i}^-_{\bar{\kappa_2}}\}, \min\{\bar{i}^+_{\bar{\kappa_1}}, \bar{i}^+_{\bar{\kappa_2}}\}\right]$$
$$\bar{t}_{\bar{\kappa_1} \cup \bar{\kappa_2}} = \left[\min\{\bar{f}^-_{\bar{\kappa_1}}, \bar{f}^-_{\bar{\kappa_2}}\}, \min\{\bar{f}^+_{\bar{\kappa_1}}, \bar{f}^+_{\bar{\kappa_2}}\}\right]$$

Then intersection of $\bar{\kappa_1}$ and $\bar{\kappa_2}$ is defined as

$$\bar{t}_{\bar{\kappa_1} \cap \bar{\kappa_2}} = \left[\min\{\bar{t}^-_{\bar{\kappa_1}}, \bar{t}^-_{\bar{\kappa_2}}\}, \min\{\bar{t}^+_{\bar{\kappa_1}}, \bar{t}^+_{\bar{\kappa_2}}\}\right]$$
$$\bar{t}_{\bar{\kappa_1} \cap \bar{\kappa_2}} = \left[\max\{\bar{i}^-_{\bar{\kappa_1}}, \bar{i}^-_{\bar{\kappa_2}}\}, \max\{\bar{i}^+_{\bar{\kappa_1}}, \bar{i}^+_{\bar{\kappa_2}}\}\right]$$
$$\bar{t}_{\bar{\kappa_1} \cap \bar{\kappa_2}} = \left[\max\{\bar{f}^-_{\bar{\kappa_1}}, \bar{f}^-_{\bar{\kappa_2}}\}, \max\{\bar{f}^+_{\bar{\kappa_1}}, \bar{f}^+_{\bar{\kappa_2}}\}\right]$$

Definition 2.9. [72] A function $T : L \to L$ is called a TN iff $\forall r, v, z \in L$, the followings can be concluded

(i) $T(r, 1) = r$
(ii) $T(r, v) = T(v, r)$
(iii) $T(r, v) \leq T(z, v)$ if $r \leq z$
(iv) $T(r, T(v, z)) = T(T(r, v), z)$

Definition 2.10. [73] A function $\bar{T} : \psi(L) \times \psi(L) \to \psi(L)$ defined as $\bar{T}(\bar{k}, \bar{w}) = [T(k^-, w^-), T(k^+, w^+)]$, where T is a TN is known as an IVTN.

Definition 2.11. [72] A function $S : L \to L$ is called a SN iff $\forall r, v, z \in L$, the followings can be concluded

(i) $S(r, 0) = r$
(ii) $S(r, v) = S(v, r)$
(iii) $S(r, v) \leq S(z, v)$ if $r \leq z$
(iv) $S(r, S(v, z)) = S(S(r, v), z)$

Definition 2.12. [73] The function $\bar{S} : \psi(L) \times \psi(L) \to \psi(L)$ defined as $\bar{S}(\bar{k}, \bar{w}) = [S(k^-, w^-), S(k^+, w^+)]$, where S is a SN is called an IVSN.

2.1. *Fuzzy, Intuitionistic fuzzy & Neutrosophic subrings*

Definition 2.13. [60] Let $(P, +, \cdot)$ be a crisp ring. A FS λ is called a FSR of P, iff $\forall r, v \in P$,

(i) $\lambda(r - v) \geq \min\{\lambda(r), \lambda(v)\}$,
(ii) $\lambda(r \cdot v) \geq \min\{\lambda(r), \lambda(v)\}$

The set of all FSR of a crisp ring $(P, +, \cdot)$ will be denoted as FSR(P).

Theorem 2.1. *[61] Any FS λ of a ring $(P, +, \cdot)$ is a FSR of P iff the level sets λ_s $(\lambda(\theta_P) \geq s \geq 0)$ are crisp subrings of P, where θ_P is the zero element of P.*

Definition 2.14. [61] Let λ be a FSR of $(P, +, \cdot)$ and $\lambda(\theta_P) \geq s \geq 0$, where θ_P is the zero element of P. Then λ_s is called a level subring of λ.

Proposition 2.2. *[61] Let $\lambda_1, \lambda_2 \in FSR(P)$. Then $\lambda_1 \cap \lambda_2 \in FSR(P)$.*

Theorem 2.3. *[61] Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings. Also, let $l : P \to R$ be a homomorphism. If λ is a FSR of P then $l(\lambda)$ is a FSR of R.*

Theorem 2.4. *[61] Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings. Also, let $l : P \to R$ be a homomorphism. If λ' is a FSR of R then $l^{-1}(\lambda')$ is a FSR of P.*

Definition 2.15. [62] Let $(P,+,\cdot)$ be a crisp ring. An IFS $\gamma = \big\{\big(r, t_\gamma(r), f_\gamma(r)\big) : r \in P\big\}$ is called an IFSR of P, iff $\forall r, v \in P$,

(i) $t_\gamma(r+v) \geq T\big(t_\gamma(r), t_\gamma(v)\big),\ f_\gamma(r+v) \leq S\big(f_\gamma(r), i_\gamma(v)\big)$

(ii) $t_\gamma(-r) \geq t_\gamma(r),\ f_\gamma(-r) \leq f_\gamma(r)$

(iii) $t_\gamma(r\cdot v) \geq T\big(t_\gamma(r), t_\gamma(v)\big),\ f_\gamma(r\cdot v) \leq S\big(f_\gamma(r), i_\gamma(v)\big)$.

Here, T is a TN and S is a SN.

The set of all IFSR of a crisp ring $(P,+,\cdot)$ will be denoted as IFSR(P).

Proposition 2.5. *[62] Let $\gamma \in$ IFSR(P). Then the followings will hold*

(i) $t_\gamma(-r) = t_\gamma(r),\ f_\gamma(-r) = f_\gamma(r)$ *and*

(ii) $t_\gamma(\theta_P) \geq t_\gamma(r),\ f_\gamma(\theta_P) \leq f_\gamma(r)$, *where θ_P is the zero element of P.*

Proposition 2.6. *[62] An IFS $\gamma = \big\{\big(r, t_\gamma(r), f_\gamma(r)\big) : r \in P\big\}$ is called an IFSR of P, iff $\forall r, v \in P$,*

(i) $t_\gamma(r-v) \geq T\big(t_\gamma(r), t_\gamma(v)\big),\ f_\gamma(r-v) \leq S\big(f_\gamma(r), f_\gamma(v)\big)$

(ii) $t_\gamma(r\cdot v) \geq T\big(t_\gamma(r), t_\gamma(v)\big),\ f_\gamma(r\cdot v) \leq S\big(f_\gamma(r), f_\gamma(v)\big)$

Proposition 2.7. *[62] Let $\gamma_1, \gamma_2 \in$ IFSR(P). Then $\gamma_1 \cap \gamma_2 \in$ IFSR(P).*

Theorem 2.8. *[62] Let $(P,+,\cdot)$ and $(R,+,\cdot)$ be two crisp rings. Also, let $l : P \to R$ be a homomorphism. If γ is an IFSR of P then $l(\gamma)$ is an IFSR of R.*

Theorem 2.9. *[62] Let $(P,+,\cdot)$ and $(R,+,\cdot)$ be two crisp rings. Also, let $l : P \to R$ be a homomorphism. If γ' is an IFSR of R then $l^{-1}(\gamma')$ is an IFSR of P.*

Definition 2.16. [63] Let $(P,+,\cdot)$ be a crisp ring. A NS $\omega = \big\{\big(r, t_\omega(r), i_\omega(r), f_\omega(r)\big) : r \in P\big\}$ is called a NSR of P, iff $\forall r, v \in P$,

(i) $t_\omega(r+v) \geq T\big(t_\omega(r), t_\omega(v)\big),\ i_\omega(r+v) \geq I\big(i_\omega(r), i_\omega(v)\big),\ f_\omega(r+v) \leq F\big(f_\omega(r), f_\omega(v)\big)$

(ii) $t_\omega(-r) \geq t_\omega(r),\ i_\omega(-r) \geq i_\omega(r),\ f_\omega(-r) \leq f_\omega(r)$

(iii) $t_\omega(r\cdot v) \geq T\big(t_\omega(r), t_\omega(v)\big),\ i_\omega(r\cdot v) \geq I\big(i_\omega(r), i_\omega(v)\big),\ f_\omega(r\cdot v) \leq S\big(f_\omega(r), f_\omega(v)\big)$.

Here, T and I are two TNs and S is a SN.

The set of all NSR of a crisp ring $(P,+,\cdot)$ will be denoted as NSR(P).

Proposition 2.10. *[63] A NS $\omega = \big\{\big(r, t_\omega(r), i_\omega(r), f_\omega(r)\big) : r \in P\big\}$ is called a NSR of P, iff $\forall r, v \in P$,*

(i) $t_\omega(r-v) \geq T\big(t_\omega(r), t_\omega(v)\big),\ i_\omega(r-v) \geq I\big(i_\omega(r), i_\omega(v)\big),\ f_\omega(r-v) \leq F\big(f_\omega(r), f_\omega(v)\big)$

(ii) $t_\omega(r\cdot v) \geq T\big(t_\omega(r), t_\omega(v)\big),\ i_\omega(r\cdot v) \geq I\big(i_\omega(r), i_\omega(v)\big),\ f_\omega(r\cdot v) \leq S\big(f_\omega(r), f_\omega(v)\big)$.

Here, T and I are two TNs and S is a SN.

Proposition 2.11. *[63] Let $\omega_1, \omega_2 \in NSR(P)$. Then $\omega_1 \cap \omega_2 \in NSR(P)$.*

Theorem 2.12. *[63] Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings. Also, let $l : P \to R$ be a homomorphism. If ω is a NSR of P then $l(\omega)$ is a NSR of R.*

Theorem 2.13. *[63] Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings. Also, let $l : P \to R$ be a homomorphism. If ω' is a NSR of R then $l^{-1}(\omega')$ is a NSR of P.*

Definition 2.17. [63] Let $\omega = \{(r, t_\omega(r), i_\omega(r), f_\omega(r)) : r \in P\}$ be a NSR of P. Then $\forall s \in [0, 1]$ the s-level sets of P are defined as

(i) $(t_\omega)_s = \{r \in P : t_\omega(r) \geq s\}$,
(ii) $(i_\omega)_s = \{r \in P : i_\omega(r) \geq s\}$, and
(iii) $(f_\omega)^s = \{r \in P : f_\omega(r) \leq s\}$.

Proposition 2.14. *[63] A NS $\omega = \{(r, t_\omega(r), i_\omega(r), f_\omega(r)) : r \in P\}$ of a crisp ring $(P, +, \cdot)$ is a NSR of P iff $\forall s \in [0, 1]$ the s-level sets of P, i.e. $(t_\omega)_s$, $(i_\omega)_s$, and $(f_\omega)^s$ are crisp rings of P.*

2.2. *Interval-valued Fuzzy and intuitionistic fuzzy subrings*

Definition 2.18. [50] Let $(P, +, \cdot)$ be a crisp ring. An IVFS $\Lambda = \{(r, \bar{t}_\Lambda(r)) : r \in P\}$ is called an IVFSR of $(P, +, \cdot)$ with respect to IVTN $\bar{T}$ if $\forall r, v \in P$, the followings can be concluded:

(i) $\bar{t}_\Lambda(r + v) \geq \bar{T}(\bar{t}_\Lambda(r), \bar{t}_\Lambda(v))$,
(ii) $\bar{t}_\Lambda(-r) \geq \Lambda(r)$, and
(iii) $\bar{t}_\Lambda(r \cdot v) \geq \bar{T}(\bar{t}_\Lambda(r), \bar{t}_\Lambda(v))$,

The set of all IVFSR of a crisp ring $(P, +, \cdot)$ with respect to an IVTN $\bar{T}$ will be denoted as IVFSR$(P, \bar{T})$.

Proposition 2.15. *[50] Let $\lambda = \{(r, t_\lambda(r)) : r \in P\}$ be a FSR of $(P, +, \cdot)$. Then $\Lambda = [t_\lambda, t_\lambda]$ is an IVFSR of P.*

Proposition 2.16. *[50] Let $\Lambda = \{(r, \bar{t}_\Lambda(r)) : r \in P\}$ be an IVFSR of $(P, +, \cdot)$. Then $\Lambda^- = \{(r, \bar{t}^-_\Lambda(r)) : r \in P\}$ and $\Lambda^+ = \{(r, \bar{t}^+_\Lambda(r)) : r \in P\}$ are FSRs of P.*

Definition 2.19. [53] Let $(P, +, \cdot)$ be a crisp ring. An IVIFS $\Gamma = \{(r, \bar{t}_\Gamma(r), \bar{f}_\Gamma(r)) : r \in P\}$ is called an IVIFSR of $(P, +, \cdot)$ if $\forall r, v \in P$, the followings can be concluded:

(i) $\bar{t}_\Gamma(r + v) \geq \bar{T}(\bar{t}_\Gamma(r), \bar{t}_\Gamma(v))$, $\bar{f}_\Gamma(r + v) \leq \bar{F}(\bar{f}_\Gamma(r), \bar{f}_\Gamma(v))$,
(ii) $\bar{t}_\Gamma(-r) \geq \bar{t}_\Gamma(r)$, $\bar{f}_\Gamma(-r) \leq \bar{f}_\Gamma(r)$, and
(iii) $\bar{t}_\Gamma(r \cdot v) \geq \bar{T}(\bar{t}_\Gamma(r), \bar{t}_\Gamma(v))$, $\bar{f}_\Gamma(r \cdot v) \leq \bar{F}(\bar{f}_\Gamma(r), \bar{f}_\Gamma(v))$.

The set of all IVIFSR of a crisp ring $(P, +, \cdot)$ will be denoted as IVIFSR(P).

Theorem 2.17. *[53] If* $\Gamma = \{(r, \bar{t}_\Gamma(r), \bar{f}_\Gamma(r)) : r \in P\} \in IVIFSR(P)$*, then* $\bar{t}_\Gamma(r) \leq \bar{t}_\Gamma(\theta_P)$ *and* $\bar{f}_\Gamma(r) \geq \bar{f}_\Gamma(\theta_P)$.

Theorem 2.18. *[53] If* Γ_1 *and* $\Gamma_2 \in IVIFSR(P)$*, then* $\Gamma_1 \cap \Gamma_2 \in IVIFSR(P)$.

Theorem 2.19. *[53] Let* $\Gamma = \{(r, \bar{t}_\Gamma(r), \bar{f}_\Gamma(r)) : r \in P\} \in IVIFSR(P)$*, then* $\forall r, v \in P$

(i) $\bar{t}_\Gamma(r - v) = \bar{t}_\Gamma(\theta_P)$ *implies that* $\bar{t}_\Gamma(r) = \bar{t}_\Gamma(v)$.
(ii) $\bar{f}_\Gamma(r - v) = \bar{f}_\Gamma(\theta_P)$ *implies that* $\bar{f}_\Gamma(r) = \bar{f}_\Gamma(v)$.

3. Proposed notion of interval-valued neutrosophic subring

Definition 3.1. Let $(P, +, \cdot)$ be a crisp ring. An IVNS $\Omega = \{(r, \bar{t}_\Omega(r), \bar{i}_\Omega(r), \bar{f}_\Omega(r)) : r \in P\}$ is called an IVNSR of $(P, +, \cdot)$ if $\forall r, v \in P$, the followings can be concluded:

$$\text{(i)} \begin{cases} \bar{t}_\Omega(r+v) \geq \bar{T}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(v)\big), \\ \bar{i}_\Omega(r+v) \leq \bar{I}\big(\bar{i}_\Omega(r), \bar{i}_\Omega(v)\big), \\ \bar{f}_\Omega(r+v) \leq \bar{F}\big(\bar{f}_\Omega(r), \bar{f}_\Omega(v)\big) \end{cases}$$
$$\text{(ii)} \begin{cases} \bar{t}_\Omega(-r) \geq t_\Omega(r), \\ \bar{i}_\Omega(-r) \leq i_\Omega(r), \\ \bar{f}_\Omega(-r) \leq f_\Omega(r) \end{cases}$$
$$\text{(iii)} \begin{cases} \bar{t}_\Omega(r \cdot v) \geq \bar{T}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(v)\big), \\ \bar{i}_\Omega(r \cdot v) \leq \bar{I}\big(\bar{i}_\Omega(r), \bar{i}_\Omega(v)\big), \\ \bar{f}_\Omega(r \cdot v) \leq \bar{F}\big(\bar{f}_\Omega(r), \bar{f}_\Omega(v)\big), \end{cases}$$

where $\bar{T}$ is an IVTN, $\bar{I}$ and $\bar{F}$ are two IVSNs.

The set of all IVNSR of a crisp ring $(P, +, \cdot)$ will be denoted as IVNSR(P).

Example 3.2. Let $(\mathbb{Z}, +, \cdot)$ be the ring of integers with respect to usual addition and multiplication. Let $\Omega = \{(r, \bar{t}_\Omega(r), \bar{i}_\Omega(r), \bar{f}_\Omega(r)) : r \in \mathbb{Z}\}$ be an IVNS of $\mathbb{Z}$, where $\forall r \in \mathbb{Z}$

$$\bar{t}_\Omega(r) = \begin{cases} [0.2, 0.25] & \text{if } r \in 2\mathbb{Z} \\ [0, 0] & \text{if } r \in 2\mathbb{Z}+1 \end{cases},$$
$$\bar{i}_\Omega(r) = \begin{cases} [0, 0] & \text{if } r \in 2\mathbb{Z} \\ [0.1, 0.12] & \text{if } r \in 2\mathbb{Z}+1 \end{cases}, \text{ and}$$
$$\bar{f}_\Omega(r) = \begin{cases} [0, 0] & \text{if } r \in 2\mathbb{Z} \\ [0.75, 0.8] & \text{if } r \in 2\mathbb{Z}+1 \end{cases}.$$

Now, if we consider minimum TN and maximum SNs, then $\Omega \in$ IVNSR$(\mathbb{Z})$.

Example 3.3. Let $(\mathbb{Z}_4, +, \cdot)$ be the ring of integers modulo 4 with usual addition and multiplication. Let $\Omega = \{(r, \bar{t}_\Omega(r), \bar{i}_\Omega(r), \bar{f}_\Omega(r)) : r \in \mathbb{Z}_4\}$ be an IVNS of $\mathbb{Z}_4$, where interval-valued memberships of elements belonging to Ω are mentioned in Table 2.

Table 2. Membership values of elements belonging to Ω

Ω	$\bar{t}_\Omega$	$\bar{i}_\Omega$	$\bar{f}_\Omega$
$\bar{0}$	$[0.6, 0.7]$	$[0.33, 0.35]$	$[0.2, 0.3]$
$\bar{1}$	$[0.7, 0.8]$	$[0.21, 0.23]$	$[0.5, 0.6]$
$\bar{2}$	$[0.75, 0.85]$	$[0.24, 0.26]$	$[0.3, 0.7]$
$\bar{3}$	$[0.75, 0.9]$	$[0.31, 0.33]$	$[0.5, 0.7]$

Now, if we consider the Łukasiewicz T-norm $(T(r,v) = \max\{0, r+v-1\})$ and bounded sum S-norms $(S(r,v) = \min\{r+v, 1\})$, then $\Omega \in \text{IVNSR}(\mathbb{Z}_4)$.

Proposition 3.1. *An IVNS $\Omega = \{(r, \bar{t}_\Omega(r), \bar{i}_\Omega(r), \bar{f}_\Omega(r)) : r \in P\}$ of a crisp ring $(P, +, \cdot)$ is an IVNSR iff the followings can be concluded (assuming that all the IVTN and IVSNs are idempotent):*

$$\text{(i)} \begin{cases} \bar{t}_\Omega(r-v) \geq \bar{T}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(v)\big), \\ \bar{i}_\Omega(r-v) \leq \bar{I}\big(\bar{i}_\Omega(r), \bar{i}_\Omega(v)\big), \\ \bar{f}_\Omega(r-v) \leq \bar{F}\big(\bar{f}_\Omega(r), \bar{f}_\Omega(v)\big) \end{cases}$$

$$\text{(ii)} \begin{cases} \bar{t}_\Omega(r \cdot v) \geq \bar{T}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(v)\big), \\ \bar{i}_\Omega(r \cdot v) \leq \bar{I}\big(\bar{i}_\Omega(r), \bar{i}_\Omega(v)\big), \\ \bar{f}_\Omega(r \cdot v) \leq \bar{F}\big(\bar{f}_\Omega(r), \bar{f}_\Omega(v)\big). \end{cases}$$

Proof. Let $\Omega \in$ IVNSR(P). Then we have

$$\begin{aligned} \bar{t}_\Omega(r-v) &\geq \bar{T}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(-v)\big) \text{ [by condition (i) of Definition 3.1]} \\ &\geq \bar{T}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(v)\big) \text{ [by condition (ii) of Definition 3.1]} \end{aligned}$$

Similary, we will have

$$\begin{aligned} \bar{i}_\Omega(r-v) &\leq \bar{I}\big(\bar{i}_\Omega(r), \bar{i}_\Omega(v)\big), \text{ and} \\ \bar{f}_\Omega(r-v) &\leq \bar{F}\big(\bar{f}_\Omega(r), \bar{f}_\Omega(v)\big), \end{aligned}$$

which proves (i).

Again, (ii) follows immediately from condition (iii) of Definition 3.1.

Conversely, let (i) and (ii) of Proposition 3.1 hold. Also, let θ_P be the additive neutral element in $(P, +, \cdot)$. Then

$$\begin{aligned} \bar{t}_\Omega(\theta_P) &= \bar{t}_\Omega(r - r) \\ &\geq \bar{T}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(r)\big) \\ &= \bar{t}_\Omega(r) \end{aligned} \tag{3.1}$$

Similaly, we can show that

$$\bar{i}_\Omega(\theta_P) \leq \bar{i}_\Omega(r) \tag{3.2}$$

$$\bar{f}_\Omega(\theta_P) \leq \bar{f}_\Omega(r) \tag{3.3}$$

Now,

$$\begin{aligned} \bar{t}_\Omega(-r) &= \bar{t}_\Omega(\theta_P - r) \\ &\geq \bar{T}\big(\bar{t}_\Omega(\theta_P), \bar{t}_\Omega(r)\big) \\ &\geq \bar{T}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(r)\big) \text{ [by 3.1]} \\ &= \bar{t}_\Omega(r) \text{ [since } \bar{T} \text{ is idempotent]} \end{aligned} \tag{3.4}$$

Similarly, we can prove

$$\bar{i}_\Omega(-r) \leq \bar{i}_\Omega(r) \text{ [since } \bar{I} \text{ is idempotent]} \tag{3.5}$$

$$\bar{f}_\Omega(-r) \leq \bar{f}_\Omega(r) \text{ [since } \bar{F} \text{ is idempotent]} \tag{3.6}$$

Hence,

$$\begin{aligned} \bar{t}_\Omega(r + v) &= \bar{t}_\Omega(r - (-v)) \\ &\geq \bar{T}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(-v)\big) \\ &\geq \bar{T}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(v)\big) \text{ [by 3.4]} \end{aligned} \tag{3.7}$$

Similarly,

$$\bar{i}_\Omega(r + v) \leq \bar{I}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(v)\big) \text{ [by 3.5]} \tag{3.8}$$

$$\bar{f}_\Omega(r + v) \leq \bar{F}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(v)\big) \text{ [by 3.6]} \tag{3.9}$$

So, by Equations 3.7, 3.8, and 3.9 condition (i) of Proposition 3.1 has been proved. Also, condition (ii) of Proposition 3.1 is same as condition (iii) of Definition 3.1. Hence, $\Omega \in$ IVNSR(P). □

Theorem 3.2. *Let $(P, +, \cdot)$ be a crisp ring. If $\Omega_1, \Omega_2 \in IVNSR(P)$, then $\Omega_1 \cap \Omega_2 \in IVNSR(P)$ (assuming all the IVTN and IVSNs are idempotent).*

Proof. Let $\Omega = \Omega_1 \cap \Omega_2$. Now, $\forall r, v \in P$

$$\begin{aligned}\bar{t}_\Omega(r+v) &= \bar{T}\big(\bar{t}_{\Omega_1}(r+v), \bar{t}_{\Omega_2}(r+v)\big)\\ &\geq \bar{T}\Big(\bar{T}\big(\bar{t}_{\Omega_1}(r), \bar{t}_{\Omega_1}(v)\big), \bar{T}\big(\bar{t}_{\Omega_2}(r), \bar{t}_{\Omega_2}(v)\big)\Big)\\ &= \bar{T}\Big(\bar{T}\big(\bar{t}_{\Omega_1}(r), \bar{t}_{\Omega_1}(v)\big), \bar{T}\big(\bar{t}_{\Omega_2}(v), \bar{t}_{\Omega_2}(r)\big)\Big) \text{ [as } \bar{T} \text{ is commutative]}\\ &= \bar{T}\Big(\bar{T}\big(\bar{t}_{\Omega_1}(r), \bar{t}_{\Omega_2}(r)\big), \bar{T}\big(\bar{t}_{\Omega_1}(v), \bar{t}_{\Omega_2}(v)\big)\Big) \text{ [as } \bar{T} \text{ is associative]}\\ &= \bar{T}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(v)\big) \qquad (3.10)\end{aligned}$$

Similarly, as both $\bar{I}$ and $\bar{S}$ are commutative as well as associative, we will have

$$\bar{i}_\Omega(r+v) \leq \bar{I}\big(\bar{i}_\Omega(r), \bar{i}_\Omega(v)\big) \qquad (3.11)$$

$$\bar{f}_\Omega(r+v) \leq \bar{F}\big(\bar{f}_\Omega(r), \bar{f}_\Omega(v)\big) \qquad (3.12)$$

Again,

$$\begin{aligned}\bar{t}_\Omega(-r) &= \bar{T}\big(\bar{t}_{\Omega_1}(-r), \bar{t}_{\Omega_2}(-r)\big)\\ &\geq \bar{T}\big(\bar{t}_{\Omega_1}(r), \bar{t}_{\Omega_2}(r)\big) \text{ [by Definition 3.1]}\\ &= \bar{t}_\Omega(r) \qquad (3.13)\end{aligned}$$

Also,

$$\bar{i}_\Omega(-r) \leq \bar{i}_\Omega(r) \qquad (3.14)$$

$$\bar{f}_\Omega(-r) \leq \bar{f}_\Omega(r) \qquad (3.15)$$

Similarly, we can show that

$$\bar{t}_\Omega(r \cdot v) \geq \bar{T}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(v)\big), \qquad (3.16)$$

$$\bar{i}_\Omega(r \cdot v) \leq \bar{I}\big(\bar{i}_\Omega(r), \bar{i}_\Omega(v)\big), \text{ and} \qquad (3.17)$$

$$\bar{f}_\Omega(r \cdot v) \leq \bar{F}\big(\bar{f}_\Omega(r), \bar{f}_\Omega(v)\big) \qquad (3.18)$$

Hence, by Equations 3.10–3.18 $\Omega = \Omega_1 \cap \Omega_2 \in \text{IVNSR}(P)$. □

Remark 3.3. *In general, if $\Omega_1, \Omega_2 \in IVNSR(P)$, then $\Omega_1 \cup \Omega_2$ may not always be an IVNSR of $(P, +, \cdot)$.*

The following Example 3.4 will prove our claim.

Example 3.4. Let $(\mathbb{Z}, +, \cdot)$ be the ring of integers with respect to usual addition and multiplication. Let $\Omega_1 = \big\{\big(r, \bar{t}_{\Omega_1}(r), \bar{i}_{\Omega_1}(r), \bar{f}_{\Omega_1}(r)\big) : r \in \mathbb{Z}\big\}$ and $\Omega_2 = \big\{\big(r, \bar{t}_{\Omega_2}(r), \bar{i}_{\Omega_2}(r), \bar{f}_{\Omega_2}(r)\big) :$

$r \in \mathbb{Z}\big\}$ be two IVNSs of $\mathbb{Z}$, where $\forall r \in \mathbb{Z}$

$$\bar{t}_{\Omega_1}(r) = \begin{cases} [0.25, 0.4] & \text{if } r \in 2\mathbb{Z} \\ [0,0] & \text{if } r \in 2\mathbb{Z}+1 \end{cases},$$

$$\bar{i}_{\Omega_1}(r) = \begin{cases} [0,0] & \text{if } r \in 2\mathbb{Z} \\ [0.17, 0.2] & \text{if } r \in 2\mathbb{Z}+1 \end{cases}, \text{ and}$$

$$\bar{f}_{\Omega_1}(r) = \begin{cases} [0,0] & \text{if } r \in 2\mathbb{Z} \\ [0.33, 0.4] & \text{if } r \in 2\mathbb{Z}+1 \end{cases}.$$

and

$$\bar{t}_{\Omega_2}(r) = \begin{cases} [0.5, 0.67] & \text{if } r \in 3\mathbb{Z} \\ [0,0] & \text{if } r \in 3\mathbb{Z}+1 \end{cases},$$

$$\bar{i}_{\Omega_2}(r) = \begin{cases} [0,0] & \text{if } r \in 3\mathbb{Z} \\ [0.2, 0.25] & \text{if } r \in 3\mathbb{Z}+1 \end{cases}, \text{ and}$$

$$\bar{f}_{\Omega_2}(r) = \begin{cases} [0,0] & \text{if } r \in 3\mathbb{Z} \\ [0.33, 0.5] & \text{if } r \in 3\mathbb{Z}+1 \end{cases}.$$

Now, if we consider minimum TN and maximum SNs, then $\Omega_1, \Omega_2 \in$ IVNSR($\mathbb{Z}$).
Now let $\Omega = \Omega_1 \cup \Omega_2$. Then for $r = 4$ and $v = 9$

$$\begin{aligned} \bar{t}_{\Omega}(r+v) &= \bar{t}_{\Omega}(4+9) \\ &= \bar{t}_{\Omega}(13) \\ &= \max\{\bar{t}_{\Omega_1}(13), \bar{t}_{\Omega_2}(13)\} \\ &= \max\{[0,0],[0,0]\} \\ &= [0,0] \end{aligned}$$

Again, if $\Omega \in$ IVNSR(P), then $\forall r, v \in P$, $\bar{t}_{\Omega}(r+v) \geq \min\{\bar{t}_{\Omega}(r), \bar{t}_{\Omega}(v)\}$. But, here for $r = 4$ and $v = 9$, $\min\{\bar{t}_{\Omega}(4), \bar{t}_{\Omega}(9)\} = \min\{[0.25, 0.4], [0.5, 0.67]\} = [0.25, 0.4] \nleq [0,0] = \bar{t}_{\Omega}(4+9)$. Hence, $\Omega \notin$ IVNSR(P).

Corollary 3.4. *If $\Omega_1, \Omega_2 \in$ IVNSR(P), then $\Omega_1 \cup \Omega_2 \in$ IVNSR(P) iff one is contained in other.*

Definition 3.5. Let $\Omega = \big\{\big(r, \bar{t}_{\Omega}(r), \bar{i}_{\Omega}(r), \bar{f}_{\Omega}(r)\big) : r \in P\big\}$ be an IVNS of a crisp ring $(P, +, \cdot)$. Also, let $[k_1, s_1]$, $[k_2, s_2]$ and $[k_3, s_3] \in \Psi(L)$. Then the crisp set $\Omega_{\big([k_1,s_1],[k_2,s_2],[k_3,s_3]\big)}$ is called a level set of IVNSR Ω, where for any $r \in \Omega_{\big([k_1,s_1],[k_2,s_2],[k_3,s_3]\big)}$ the following inequalities will hold: $\bar{t}_{\Omega}(r) \geq [k_1, s_1]$, $\bar{i}_{\Omega}(r) \leq [k_2, s_2]$, and $\bar{f}_{\Omega}(r) \leq [k_3, s_3]$.

Theorem 3.5. *Let $(P,+,\cdot)$ be a crisp ring. Then $\Omega \in IVNSR(P)$ iff $\forall [k_1,s_1],[k_2,s_2],[k_3,s_3] \in \Psi(L)$ with $\bar{t}_\Omega(\theta_P) \geq [k_1,s_1]$, $\bar{i}_\Omega(\theta_P) \leq [k_2,s_2]$, and $\bar{f}_\Omega(\theta_P) \leq [k_3,s_3]$, $\Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}$ is a crisp subring of $(P,+,\cdot)$ (assuming all the IVTN and IVSNs are idempotent).*

Proof. Since, $\bar{t}_\Omega(\theta_P) \geq [k_1,s_1]$, $\bar{i}_\Omega(\theta_P) \leq [k_2,s_2]$, and $\bar{f}_\Omega(\theta_P) \leq [k_3,s_3]$, $\theta_P \in \Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}$, i.e., $\Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}$ is non-empty.
Now, let $\Omega \in$ IVNSR(P) and $r,v \in \Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}$. To show that, $(r-v)$ and $r\cdot v \in \Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}$. Here,

$$\begin{aligned}\bar{t}_\Omega(r-v) &\geq \bar{T}\big(\bar{t}_\Omega(r),\bar{t}_\Omega(v)\big) \text{ [by Proposition 3.1]}\\ &\geq \bar{T}\big([k_1,s_1],[k_1,s_1]\big) \left[\text{as } r,v \in \Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}\right]\\ &\geq [k_1,s_1] \text{ [as } \bar{T} \text{ is idempotent]} \end{aligned} \tag{3.19}$$

Again,

$$\begin{aligned}\bar{t}_\Omega(r\cdot v) &\geq \bar{T}\big(\bar{t}_\Omega(r),\bar{t}_\Omega(v)\big) \text{ [by Proposition 3.1]}\\ &\geq \bar{T}\big([k_1,s_1],[k_1,s_1]\big) \left[\text{as } r,v \in \Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}\right]\\ &\geq [k_1,s_1] \text{ [as } \bar{T} \text{ is idempotent]} \end{aligned} \tag{3.20}$$

Similarly, we can show that

$$\bar{i}_\Omega(r-v) \leq [k_2,s_2], \tag{3.21}$$

$$\bar{i}_\Omega(r\cdot v) \leq [k_2,s_2], \tag{3.22}$$

$$\bar{f}_\Omega(r-v) \leq [k_3,s_3], \text{ and} \tag{3.23}$$

$$\bar{f}_\Omega(r\cdot v) \leq [k_3,s_3] \tag{3.24}$$

Hence, by Equations 3.19–3.24 $(r-v)$ and $r\cdot v \in \Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}$, i.e., $\Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}$ is a crisp subring of $(P,+,\cdot)$.
Conversely, let $\Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}$ is a crisp subgroup of $(P,+,\cdot)$. To show that, $\Omega \in$ IVNSR(P).
Let $r,v \in P$, then there exists $[k_1,s_1] \in \Psi(L)$ such that $\bar{T}\big(\bar{t}_\Omega(r),\bar{t}_\Omega(v)\big) = [k_1,s_1]$. So, $\bar{t}_\Omega(r) \geq [k_1,s_1]$ and $\bar{t}_\Omega(v) \geq [k_1,s_1]$. Also, let there exist $[k_2,s_2],[k_3,s_3] \in \Psi(L)$ such that $\bar{I}\big(\bar{i}_\Omega(r),\bar{i}_\Omega(v)\big) = [k_2,s_2]$ and $\bar{F}\big(\bar{f}_\Omega(r),\bar{f}_\Omega(v)\big) = [k_3,s_3]$. Then $r,v \in \Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}$.
Again, as $\Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}$ is a crisp subring, $r-v \in \Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}$ and $r\cdot v \in \Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}$.

Hence,

$$\begin{aligned}\bar{t}_\Omega(r-v) &\geq [k_1, s_1] \\ &= \bar{T}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(v)\big) \text{ and} \qquad (3.25)\\ \bar{t}_\Omega(r\cdot v) &\geq [k_1, s_1] \\ &= \bar{T}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(v)\big) \qquad (3.26)\end{aligned}$$

Similarly, we can prove that

$$\begin{aligned}\bar{i}_\Omega(r-v) &\leq [k_2, s_2] \\ &= \bar{I}\big(\bar{i}_\Omega(r), \bar{i}_\Omega(v)\big), \qquad (3.27)\\ \bar{i}_\Omega(r\cdot v) &\leq [k_2, s_2] \\ &= \bar{I}\big(\bar{i}_\Omega(r), \bar{i}_\Omega(v)\big), \qquad (3.28)\\ \bar{f}_\Omega(r-v) &\leq [k_3, s_3] \\ &= \bar{F}\big(\bar{f}_\Omega(r), \bar{f}_\Omega(v)\big), \text{ and} \qquad (3.29)\\ \bar{f}_\Omega(r\cdot v) &\leq [k_3, s_3] \\ &= \bar{F}\big(\bar{f}_\Omega(r), \bar{f}_\Omega(v)\big) \qquad (3.30)\end{aligned}$$

So, Equations 3.25–3.30 imply that Ω follows Proposition 3.1, i.e., $\Omega \in \text{IVNSR}(P)$. □

Definition 3.6. Let Ω and Ω' be two IVNSs of two CSs P and R, respectively. Also, let $l : P \to R$ be a function. Then

(i) image of Ω under l will be $l(\Omega) = \big\{\big(v, \bar{t}_{l(\Omega)}(v), \bar{i}_{l(\Omega)}(v), \bar{f}_{l(\Omega)}(v)\big) : v \in R\big\}$, where $\bar{t}_{l(\Omega)}(v) = \underset{s\in l^{-1}(v)}{\vee} \bar{t}_\Omega(s)$, $\bar{i}_{l(\Omega)}(v) = \underset{s\in l^{-1}(v)}{\wedge} \bar{i}_\Omega(s)$, $\bar{f}_{l(\Omega)}(v) = \underset{s\in l^{-1}(v)}{\wedge} \bar{f}_\Omega(s)$. Wherefrom, if l is injective then $\bar{t}_{l(\Omega)}(v) = \bar{t}_\Omega\big(l^{-1}(v)\big)$, $\bar{i}_{l(\Omega)}(v) = \bar{i}_\Omega\big(l^{-1}(v)\big)$, $\bar{f}_{l(\Omega)}(v) = \bar{f}_\Omega\big(l^{-1}(v)\big)$, and

(ii) preimage of Ω' under l will be $l^{-1}(\Omega') = \big\{\big(r, \bar{t}_{l^{-1}(\Omega')}(r), \bar{i}_{l^{-1}(\Omega')}(r), \bar{f}_{l^{-1}(\Omega')}(r)\big) : r \in R\big\}$, where $\bar{t}_{l^{-1}(\Omega')}(r) = \bar{t}_{\Omega'}\big(l(r)\big)$, $\bar{i}_{l^{-1}(\Omega')}(r) = \bar{i}_{\Omega'}\big(l(r)\big)$, $\bar{f}_{l^{-1}(\Omega')}(r) = \bar{f}_{\Omega'}\big(l(r)\big)$.

Theorem 3.6. *Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings. Also, let $l : P \to R$ be a ring isomorphism. If Ω is an IVNSR of P then $l(\Omega)$ is an IVNSR of R.*

Proof. Let $v_1 = l(r_1)$ and $v_2 = l(r_2)$, where $r_1, r_2 \in P$ and $v_1, v_2 \in R$. Now,

$$\begin{aligned}
\bar{t}_{l(\Omega)}(v_1 - v_2) &= \bar{t}_{\Omega}\big(l^{-1}(v_1 - v_2)\big) \text{ [as } l \text{ is injective]} \\
&= \bar{t}_{\Omega}\big(l^{-1}(v_1) - l^{-1}(v_2)\big) \text{ [as } l^{-1} \text{ is a homomorphism]} \\
&= \bar{t}_{\Omega}(r_1 - r_2) \\
&\geq \bar{T}\big(\bar{t}_{\Omega}(r_1), \bar{t}_{\Omega}(r_2)\big) \\
&= \bar{T}\Big(\bar{t}_{\Omega}\big(l^{-1}(v_1)\big), \bar{t}_{\Omega}\big(l^{-1}(v_2)\big)\Big) \\
&= \bar{T}\big(\bar{t}_{l(\Omega)}(v_1), \bar{t}_{l(\Omega)}(v_2)\big) \qquad (3.31)
\end{aligned}$$

Again,

$$\begin{aligned}
\bar{t}_{l(\Omega)}(v_1 \cdot v_2) &= \bar{t}_{\Omega}\big(l^{-1}(v_1 \cdot v_2)\big) \text{ [as } l \text{ is injective]} \\
&= \bar{t}_{\Omega}\big(l^{-1}(v_1) \cdot l^{-1}(v_2)\big) \text{ [as } l^{-1} \text{ is a homomorphism]} \\
&= \bar{t}_{\Omega}(r_1 \cdot r_2) \\
&\geq \bar{T}\big(\bar{t}_{\Omega}(r_1), \bar{t}_{\Omega}(r_2)\big) \\
&= \bar{T}\Big(\bar{t}_{\Omega}\big(l^{-1}(v_1)\big), \bar{t}_{\Omega}\big(l^{-1}(v_2)\big)\Big) \\
&= \bar{T}\big(\bar{t}_{l(\Omega)}(v_1), \bar{t}_{l(\Omega)}(v_2)\big) \qquad (3.32)
\end{aligned}$$

Similarly,

$$\bar{i}_{l(\Omega)}(v_1 - v_2) \leq \bar{I}\big(\bar{i}_{l(\Omega)}(v_1), \bar{i}_{l(\Omega)}(v_2)\big), \qquad (3.33)$$

$$\bar{i}_{l(\Omega)}(v_1 \cdot v_2) \leq \bar{I}\big(\bar{i}_{l(\Omega)}(v_1), \bar{i}_{l(\Omega)}(v_2)\big), \qquad (3.34)$$

$$\bar{f}_{l(\Omega)}(v_1 - v_2) \leq \bar{F}\big(\bar{f}_{l(\Omega)}(v_1), \bar{f}_{l(\Omega)}(v_2)\big), \text{ and} \qquad (3.35)$$

$$\bar{f}_{l(\Omega)}(v_1 \cdot v_2) \leq \bar{F}\big(\bar{f}_{l(\Omega)}(v_1), \bar{f}_{l(\Omega)}(v_2)\big) \qquad (3.36)$$

Hence, Equations 3.31–3.36 imply that $l(\Omega)$ follows Proposition 3.1, i.e., $l(\Omega)$ is an IVNSR of R. □

Theorem 3.7. *Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings. Also, let $l : P \rightarrow R$ be a ring homomorphism. If Ω' is an IVNSR of R then $l^{-1}(\Omega')$ is an IVNSR of P (Note that, l^{-1} may not be an inverse mapping but $l^{-1}(\Omega')$ is an inverse image of Ω').*

Proof. Let $v_1 = l(r_1)$ and $v_2 = l(r_2)$, where $r_1, r_2 \in P$ and $v_1, v_2 \in R$. Now,

$$\begin{aligned}
\bar{t}_{l^{-1}(\Omega')}(r_1 - r_2) &= \bar{t}_{\Omega'}\big(l(r_1 - r_2)\big) \\
&= \bar{t}_{\Omega'}\big(l(r_1) - l(r_2)\big) \text{ [as } l \text{ is a homomorphism]} \\
&= \bar{t}_{\Omega'}(v_1 - v_2) \\
&\geq \bar{T}\big(\bar{t}_{\Omega'}(v_1), \bar{t}_{\Omega'}(v_2)\big) \\
&= \bar{T}\Big(\bar{t}_{\Omega'}\big(l(r_1)\big), \bar{t}_{\Omega'}\big(l(r_2)\big)\Big) \\
&= \bar{T}\big(\bar{t}_{l^{-1}(\Omega')}(r_1), \bar{t}_{l^{-1}(\Omega')}(r_2)\big) \qquad (3.37)
\end{aligned}$$

Again,

$$\begin{aligned}
\bar{t}_{l^{-1}(\Omega')}(r_1 \cdot r_2) &= \bar{t}_{\Omega'}\big(l(r_1 \cdot r_2)\big) \\
&= \bar{t}_{\Omega'}\big(l(r_1) \cdot l(r_2)\big) \text{ [as } l \text{ is a homomorphism]} \\
&= \bar{t}_{\Omega'}(v_1 \cdot v_2) \\
&\geq \bar{T}\big(\bar{t}_{\Omega'}(v_1), \bar{t}_{\Omega'}(v_2)\big) \\
&= \bar{T}\Big(\bar{t}_{\Omega'}\big(l(r_1)\big), \bar{t}_{\Omega'}\big(l(r_2)\big)\Big) \\
&= \bar{T}\big(\bar{t}_{l^{-1}(\Omega')}(r_1), \bar{t}_{l^{-1}(\Omega')}(r_2)\big) \qquad (3.38)
\end{aligned}$$

Similarly,

$$\bar{i}_{l^{-1}(\Omega')}(r_1 - r_2) \leq \bar{I}\big(\bar{i}_{l^{-1}(\Omega')}(r_1), \bar{i}_{l^{-1}(\Omega')}(r_2)\big) \qquad (3.39)$$

$$\bar{i}_{l^{-1}(\Omega')}(r_1 \cdot r_2) \leq \bar{I}\big(\bar{i}_{l^{-1}(\Omega')}(r_1), \bar{i}_{l^{-1}(\Omega')}(r_2)\big) \qquad (3.40)$$

$$\bar{f}_{l^{-1}(\Omega')}(r_1 - r_2) \leq \bar{F}\big(\bar{f}_{l^{-1}(\Omega')}(r_1), \bar{f}_{l^{-1}(\Omega')}(r_2)\big) \qquad (3.41)$$

$$\bar{f}_{l^{-1}(\Omega')}(r_1 \cdot r_2) \leq \bar{F}\big(\bar{f}_{l^{-1}(\Omega')}(r_1), \bar{f}_{l^{-1}(\Omega')}(r_2)\big) \qquad (3.42)$$

Hence, Equations 3.37–3.42 imply that $l^{-1}(\Omega')$ follows Proposition 3.1, i.e., $l^{-1}(\Omega')$ is an IVNSR of P. □

Definition 3.7. Let $(P, +, \cdot)$ be a crisp ring and $\Omega \in \text{IVNSR}(P)$. Again, let $\bar{\sigma} = [\sigma_1, \sigma_2], \bar{\tau} = [\tau_1, \tau_2], \bar{\delta} = [\delta_1, \delta_2] \in \Psi(L)$. Then

(i) Ω is called a $(\bar{\sigma},\bar{\tau},\bar{\delta})-$identity IVNSR over P, if $\forall r \in P$

$$\bar{t}_\Omega(r) = \begin{cases} \bar{\sigma} & \text{if } r = \theta_P \\ [0,0] & \text{if } r \neq \theta_P \end{cases},$$

$$\bar{i}_\Omega(r) = \begin{cases} \bar{\tau} & \text{if } r = \theta_P \\ [1,1] & \text{if } r \neq \theta_P \end{cases}, \text{ and}$$

$$\bar{f}_\Omega(r) = \begin{cases} \bar{\delta} & \text{if } r = \theta_P \\ [1,1] & \text{if } r \neq \theta_P \end{cases},$$

where θ_P is the zero element of P.

(ii) Ω is called a $(\bar{\sigma},\bar{\tau},\bar{\delta})-$absolute IVNSR over P, if $\forall r \in P$, $\bar{t}_\Omega(r) = \bar{\sigma}$, $\bar{i}_\Omega(r) = \bar{\tau}$, and $\bar{f}_\Omega(r) = \bar{\delta}$.

Theorem 3.8. *Let $(P,+,\cdot)$ and $(R,+,\cdot)$ be two crisp rings and $\Omega \in$IVNSR (P). Again, let $l : P \to R$ be a ring homomorphism. Then*

(i) *$l(\Omega)$ will be a $(\bar{\sigma},\bar{\tau},\bar{\delta})-$identity IVNSR over R, if $\forall r \in P$*

$$\bar{t}_\Omega(r) = \begin{cases} \bar{\sigma} & \text{if } r \in Ker(l) \\ [0,0] & \text{otherwise} \end{cases},$$

$$\bar{i}_\Omega(r) = \begin{cases} \bar{\tau} & \text{if } r \in Ker(l) \\ [1,1] & \text{otherwise} \end{cases}, \text{ and}$$

$$\bar{f}_\Omega(r) = \begin{cases} \bar{\delta} & \text{if } r \in Ker(l) \\ [1,1] & \text{otherwise} \end{cases},$$

(ii) *$l(\Omega)$ will be a $(\bar{\sigma},\bar{\tau},\bar{\delta})-$absolute IVNSR over R, if Ω is a $(\bar{\sigma},\bar{\tau},\bar{\delta})-$absolute IVNSR over P.*

Proof. (i) Clearly, by Theorem 3.6 $l(\Omega) \in$ IVNSR(R). Let $r \in Ker(l)$, then $l(r) = \theta_R$.
So,

$$\begin{aligned} \bar{t}_{l(\Omega)}(\theta_R) &= \bar{t}_\Omega\big(l^{-1}(\theta_R)\big) \\ &= \bar{t}_\Omega(r) \\ &= \bar{\sigma} \end{aligned} \tag{3.43}$$

Similarly, we can show that

$$\bar{i}_{l(\Omega)}(\theta_R) = \bar{\tau}, \text{ and} \tag{3.44}$$

$$\bar{f}_{l(\Omega)}(\theta_R) = \bar{\delta} \tag{3.45}$$

Again, let $r \in P \setminus Ker(l)$ and $l(r) = v$. Then

$$\begin{aligned}\bar{t}_{l(\Omega)}(v) &= \bar{t}_{\Omega}\big(l^{-1}(v)\big)\\ &= \bar{t}_{\Omega}(r)\\ &= [0,0] \qquad (3.46)\end{aligned}$$

Similarly, we can show that

$$\bar{i}_{l(\Omega)}(v) = [1,1] \text{ and} \qquad (3.47)$$

$$\bar{f}_{l(\Omega)}(v) = [1,1] \qquad (3.48)$$

Hence, by the Equations 3.43–3.48 $l(\Omega)$ is a $(\bar{\sigma}, \bar{\tau}, \bar{\delta})-$identity IVNSR over R.

(ii) Let $l(r) = v$, for $r \in P$ and $v \in R$. Then

$$\begin{aligned}\bar{t}_{l(\Omega)}(v) &= \bar{t}_{\Omega}\big(l^{-1}(v)\big)\\ &= \bar{t}_{\Omega}(r)\\ &= \bar{\sigma} \qquad (3.49)\end{aligned}$$

Similarly, we can show that

$$\bar{i}_{l(\Omega)}(v) = \bar{\tau} \text{ and} \qquad (3.50)$$

$$\bar{f}_{l(\Omega)}(v) = \bar{\delta} \qquad (3.51)$$

Hence, by the Equations 3.48–3.51 $l(\Omega)$ is a $(\bar{\sigma}, \bar{\tau}, \bar{\delta})-$absolute IVNSR over R. □

3.1. *Product of interval-valued neutrosophic subrings*

Definition 3.8. Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings. Again, let $\Omega_1 = \big\{\big(r, \bar{t}_{\Omega_1}(r), \bar{i}_{\Omega_1}(r), \bar{f}_{\Omega_1}(r)\big) : r \in P\big\}$ and $\Omega_2 = \big\{\big(v, \bar{t}_{\Omega_2}(v), \bar{i}_{\Omega_2}(v), \bar{f}_{\Omega_2}(v)\big) : v \in R\big\}$ are IVNSRs of P and R respectively. Then Cartesian product of Ω_1 and Ω_2 will be

$$\begin{aligned}\Omega &= \Omega_1 \times \Omega_2\\ &= \Big\{\Big((r,v), \bar{T}\big(\bar{t}_{\Omega_1}(r), \bar{t}_{\Omega_2}(v)\big), \bar{I}\big(\bar{i}_{\Omega_1}(r), \bar{i}_{\Omega_2}(v)\big), \bar{F}\big(\bar{f}_{\Omega_1}(r), \bar{f}_{\Omega_2}(v)\big)\Big) : (r,v) \in P \times R\Big\}\end{aligned}$$

Similarly, product of 3 or more IVNSRs can be defined.

Theorem 3.9. *Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings with $\Omega_1 \in$ IVNSR(P) and $\Omega_2 \in$ IVNSR(R). Then $\Omega_1 \times \Omega_2$ is a IVNSR of $P \times R$.*

Proof. Let $\Omega = \Omega_1 \times \Omega_2$ and $(r_1, v_1), (r_2, v_2) \in P \times R$. Then

$$\begin{aligned}
\bar{t}_\Omega\big((r_1, v_1) - (r_2, v_2)\big) &= \bar{t}_{\Omega_1 \times \Omega_2}\big((r_1 - r_2, v_1 - v_2)\big) \\
&= \bar{T}\big(\bar{t}_{\Omega_1}(r_1 - r_2), \bar{t}_{\Omega_2}(v_1 - v_2)\big) \text{ [by Definition 3.8]} \\
&\geq \bar{T}\Big(\bar{T}\big(\bar{t}_{\Omega_1}(r_1), \bar{t}_{\Omega_1}(r_2)\big), \bar{T}\big(\bar{t}_{\Omega_2}(v_1), \bar{t}_{\Omega_2}(v_2)\big)\Big) \text{ [by Proposition 3.1]} \\
&= \bar{T}\Big(\bar{T}\big(\bar{t}_{\Omega_1}(r_1), \bar{t}_{\Omega_2}(v_1)\big), \bar{T}\big(\bar{t}_{\Omega_1}(r_2), \bar{t}_{\Omega_2}(v_2)\big)\Big) \text{ [as } \bar{T} \text{ is associative]} \\
&= \bar{T}\big(\bar{t}_\Omega(r_1, v_1), \bar{t}_\Omega(r_2, v_2)\big) \qquad (3.52)
\end{aligned}$$

Again,

$$\begin{aligned}
\bar{t}_\Omega\big((r_1, v_1) \cdot (r_2, v_2)\big) &= \bar{t}_{\Omega_1 \times \Omega_2}\big((r_1 \cdot r_2, v_1 \cdot v_2)\big) \\
&= \bar{T}\big(\bar{t}_{\Omega_1}(r_1 \cdot r_2), \bar{t}_{\Omega_2}(v_1 \cdot v_2)\big) \text{ [by Definition 3.8]} \\
&\geq \bar{T}\Big(\bar{T}\big(\bar{t}_{\Omega_1}(r_1), \bar{t}_{\Omega_1}(r_2)\big), \bar{T}\big(\bar{t}_{\Omega_2}(v_1), \bar{t}_{\Omega_2}(v_2)\big)\Big) \text{ [by Proposition 3.1]} \\
&= \bar{T}\Big(\bar{T}\big(\bar{t}_{\Omega_1}(r_1), \bar{t}_{\Omega_2}(v_1)\big), \bar{T}\big(\bar{t}_{\Omega_1}(r_2), \bar{t}_{\Omega_2}(v_2)\big)\Big) \text{ [as } \bar{T} \text{ is associative]} \\
&= \bar{T}\big(\bar{t}_\Omega(r_1, v_1), \bar{t}_\Omega(r_2, v_2)\big) \qquad (3.53)
\end{aligned}$$

Similary, the followings can be shown

$$\bar{i}_\Omega\big((r_1, v_1) - (r_2, v_2)\big) \leq \bar{I}\big(\bar{i}_\Omega(r_1, v_1), \bar{i}_\Omega(r_2, v_2)\big), \qquad (3.54)$$

$$\bar{i}_\Omega\big((r_1, v_1) \cdot (r_2, v_2)\big) \leq \bar{I}\big(\bar{i}_\Omega(r_1, v_1), \bar{i}_\Omega(r_2, v_2)\big), \qquad (3.55)$$

$$\bar{f}_\Omega\big((r_1, v_1) - (r_2, v_2)\big) \leq \bar{F}\big(\bar{f}_\Omega(r_1, v_1), \bar{f}_\Omega(r_2, v_2)\big), \text{ and} \qquad (3.56)$$

$$\bar{f}_\Omega\big((r_1, v_1) \cdot (r_2, v_2)\big) \leq \bar{F}\big(\bar{f}_\Omega(r_1, v_1), \bar{f}_\Omega(r_2, v_2)\big) \qquad (3.57)$$

Hence, using Proposition 3.1 and by Equations 3.52–3.57 $\Omega_1 \times \Omega_2 \in$ IVNSR$(P \times R)$. □

Corollary 3.10. *Let $\forall i \in \{1, 2, ..., n\}$, $(P_i, +, \cdot)$ are crisp rings and $\Omega_i \in$ IVNSR(P_i). Then $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$ is a IVNSR of $P_1 \times P_2 \times \cdots \times P_n$, where $n \in \mathbb{N}$.*

3.2. *Subring of a interval-valued neutrosophic subgring*

Definition 3.9. Let $(P, +, \cdot)$ be a crisp ring and $\Omega_1, \Omega_2 \in$ IVNSR(P), where $\Omega_1 = \big\{\big(r, \bar{t}_{\Omega_1}(r), \bar{i}_{\Omega_1}(r), \bar{f}_{\Omega_1}(r)\big) : r \in P\big\}$ and $\Omega_2 = \big\{\big(r, \bar{t}_{\Omega_2}(r), \bar{i}_{\Omega_2}(r), \bar{f}_{\Omega_2}(r)\big) : r \in P\big\}$. Then Ω_1 is called a subring of Ω_2 if $\forall r \in P$, $\bar{t}_{\Omega_1}(r) \leq \bar{t}_{\Omega_2}(r)$, $\bar{i}_{\Omega_1}(r) \geq \bar{i}_{\Omega_2}(r)$, and $\bar{f}_{\Omega_1}(r) \geq \bar{f}_{\Omega_2}(r)$.

Theorem 3.11. *Let $(P, +, \cdot)$ be a crisp ring and $\Omega \in$ IVNSR(P). Again, let Ω_1 and Ω_2 be two subrings of Ω. Then $\Omega_1 \cap \Omega_2$ is also a subring of Ω, assuming that all the IVTN and IVSNs are idempotent.*

Proof. Here, $\forall r \in P$

$$\begin{aligned}\bar{t}_{\Omega_1 \cap \Omega_2}(r) &= \bar{T}\big(\bar{t}_{\Omega_1}(r), \bar{t}_{\Omega_2}(r)\big) \\ &\leq \bar{T}\big(\bar{t}_{\Omega}(r), \bar{t}_{\Omega}(r)\big) \\ &= \bar{t}_{\Omega}(r) \text{ [as } \bar{T} \text{ is idempotent]} \end{aligned} \tag{3.58}$$

Similarly, as $\bar{I}$ and $\bar{F}$ are idempotent we can show that,

$$\bar{i}_{\Omega_1 \cap \Omega_2}(r) \geq \bar{i}_{\Omega}(r) \text{ and} \tag{3.59}$$

$$\bar{f}_{\Omega_1 \cap \Omega_2}(r) \geq \bar{f}_{\Omega}(r) \tag{3.60}$$

Hence, by Equations 3.58–3.60 $\Omega_1 \cap \Omega_2$ is a subring of Ω. □

Theorem 3.12. *Let $(P, +, \cdot)$ be a crisp ring and $\Omega_1, \Omega_2 \in IVNSR(P)$ such that Ω_1 is a subring of Ω_2. Let $(R, +, \cdot)$ is another crisp ring and $l : P \to R$ be a ring isomorphism. Then*

(i) *$l(\Omega_1)$ and $l(\Omega_2)$ are two IVNSRs over R and*

(ii) *$l(\Omega_1)$ is a subring of $l(\Omega_2)$.*

Proof. (i) can be proved by using Theorem 3.6.
(ii) Let $v = l(r)$, where $r \in P$ and $v \in R$. Then

$$\begin{aligned}&\bar{t}_{\Omega_1}(r) \leq \bar{t}_{\Omega_2}(r) \text{ [as } \Omega_1 \text{ is a subring of } \Omega_2] \\ \Rightarrow &\bar{t}_{\Omega_1}\big(l^{-1}(v)\big) \leq \bar{t}_{\Omega_2}\big(l^{-1}(v)\big) \\ \Rightarrow &\bar{t}_{l(\Omega_1)}(v) \leq \bar{t}_{l(\Omega_2)}(v)\end{aligned} \tag{3.61}$$

Similarly, we can prove that

$$\bar{i}_{l(\Omega_1)}(v) \geq \bar{i}_{l(\Omega_2)}(v) \text{ and} \tag{3.62}$$

$$\bar{f}_{l(\Omega_1)}(v) \geq \bar{f}_{l(\Omega_2)}(v) \tag{3.63}$$

Hence, by Equations 3.61–3.63 $l(\Omega_1)$ is a subring of $l(\Omega_2)$. □

3.3. *Interval-valued neutrosophic normal subrings*

Definition 3.10. Let $(P, +, \cdot)$ be a crisp ring and Ω is an IVNS of P, where $\Omega = \big\{\big(r, \bar{t}_{\Omega}(r), \bar{i}_{\Omega}(r), \bar{f}_{\Omega}(r)\big) : r \in P\big\}$. Then Ω is called an IVNNSR over P if

(i) Ω is an IVNSR of P and

(ii) $\forall r, v \in P$, $\bar{t}_{\Omega}(r \cdot v) = \bar{t}_{\Omega}(v \cdot r)$, $\bar{i}_{\Omega}(r \cdot v) = \bar{i}_{\Omega}(v \cdot r)$, and $\bar{f}_{\Omega}(r \cdot v) = \bar{f}_{\Omega}(v \cdot r)$.

The set of all IVNNSR of a crisp ring $(P, +, \cdot)$ will be denoted as IVNNSR(P).

Example 3.11. Let $(\mathbb{Z}, +, \cdot)$ be the ring of integers with respect to usual addition and multiplication. Let $\Omega = \{(r, \bar{t}_\Omega(r), \bar{i}_\Omega(r), \bar{f}_\Omega(r)) : r \in \mathbb{Z}\}$ be an IVNS of $\mathbb{Z}$, where $\forall r \in \mathbb{Z}$

$$\bar{t}_\Omega(r) = \begin{cases} [0.67, 1] & \text{if } r \in 2\mathbb{Z} \\ [0, 0] & \text{if } r \in 2\mathbb{Z} + 1 \end{cases},$$

$$\bar{i}_\Omega(r) = \begin{cases} [0, 0] & \text{if } r \in 2\mathbb{Z} \\ [0.33, 0.5] & \text{if } r \in 2\mathbb{Z} + 1 \end{cases}, \text{ and}$$

$$\bar{f}_\Omega(r) = \begin{cases} [0, 0] & \text{if } r \in 2\mathbb{Z} \\ [0, 0.33] & \text{if } r \in 2\mathbb{Z} + 1 \end{cases}.$$

Now, if we consider minimum TN and maximum SNs, then $\Omega \in \text{IVNNSR}(\mathbb{Z})$.

Theorem 3.13. *Let $(P, +, \cdot)$ be a crisp ring. If $\Omega_1, \Omega_2 \in IVNNSR(P)$, then $\Omega_1 \cap \Omega_2 \in IVNNSR(P)$.*

Proof. As $\Omega_1, \Omega_2 \in \text{IVNSR}(P)$ by Theorem 3.2 $\Omega_1 \cap \Omega_2 \in \text{IVNSR}(P)$. Again,

$$\begin{aligned} \bar{t}_{\Omega_1 \cap \Omega_2}(r \cdot v) &= \bar{T}(\bar{t}_{\Omega_1}(r \cdot v), \bar{t}_{\Omega_2}(r \cdot v)) \\ &= \bar{T}(\bar{t}_{\Omega_1}(v \cdot r), \bar{t}_{\Omega_2}(v \cdot r)) \text{ [as } \Omega_1, \Omega_2 \in \text{IVNNSR}(P)] \\ &= \bar{t}_{\Omega_1 \cap \Omega_2}(v \cdot r) \end{aligned} \tag{3.64}$$

Similarly,

$$\bar{i}_{\Omega_1 \cap \Omega_2}(r \cdot v) = \bar{i}_{\Omega_1 \cap \Omega_2}(v \cdot r) \tag{3.65}$$

$$\bar{f}_{\Omega_1 \cap \Omega_2}(r \cdot v) = \bar{f}_{\Omega_1 \cap \Omega_2}(v \cdot r) \tag{3.66}$$

Hence, $\Omega_1 \cap \Omega_2 \in \text{IVNNSR}(P)$. □

Remark 3.14. *In general, if $\Omega_1, \Omega_2 \in IVNNSR(P)$, then $\Omega_1 \cup \Omega_2$ may not always be an IVNNSR of $(P, +, \cdot)$.*

Remark 3.14 can be proved by Example 3.4.

Theorem 3.15. *Let $(P, +, \cdot)$ be a crisp ring. Then $\Omega \in IVNNSR(P)$ iff $\forall [k_1, s_1], [k_2, s_2], [k_3, s_3] \in \Psi(L)$ with $\bar{t}_\Omega(\theta_P) \geq [k_1, s_1]$, $\bar{i}_\Omega(\theta_P) \leq [k_2, s_2]$, and $\bar{f}_\Omega(\theta_P) \leq [k_3, s_3]$, $\Omega_{([k_1,s_1],[k_2,s_2],[k_3,s_3])}$ is a crisp normal subring of $(P, +, \cdot)$ (assuming all the IVTN and IVSNs are idempotent).*

Proof. This can be proved using Theorem 3.5. □

Theorem 3.16. *Let $(P,+,\cdot)$ and $(R,+,\cdot)$ be two crisp rings. Also, let $l : P \to R$ be a ring isomorphism. If Ω is an IVNNSR of P then $l(\Omega)$ is an IVNNSR of R.*

Proof. As Ω is an IVNSR of P by Theorem 3.6 $l(\Omega)$ is an IVNSR of R. Let $l(r_1) = v_1$ and $l(r_2) = v_2$, where $r_1, r_2 \in P$ and $v_1, v_2 \in R$. Then

$$\begin{aligned}
\bar{t}_{l(\Omega)}(v_1 \cdot v_2) &= \bar{t}_{\Omega}\big(l^{-1}(v_1 \cdot v_2)\big) \text{ [as } l \text{ is injective]} \\
&= \bar{t}_{\Omega}\big(l^{-1}(v_1) \cdot l^{-1}(v_2)\big) \text{ [as } l^{-1} \text{ is a homomorphism]} \\
&= \bar{t}_{\Omega}(r_1 \cdot r_2) \\
&= \bar{t}_{\Omega}(r_2 \cdot r_1) \text{ [as } \Omega \text{ is an IVNNSR of } P] \\
&= \bar{t}_{\Omega}\big(l^{-1}(v_2) \cdot l^{-1}(v_1)\big) \\
&= \bar{t}_{\Omega}\big(l^{-1}(v_2 \cdot v_1)\big) \\
&= \bar{t}_{l(\Omega)}(v_2 \cdot v_1) \qquad (3.67)
\end{aligned}$$

Similarly,

$$\bar{i}_{l(\Omega)}(v_1 \cdot v_2) = \bar{i}_{l(\Omega)}(v_2 \cdot v_1) \text{ and} \qquad (3.68)$$

$$\bar{f}_{l(\Omega)}(v_1 \cdot v_2) = \bar{f}_{l(\Omega)}(v_2 \cdot v_1) \qquad (3.69)$$

Hence, by Equations 3.67–3.69 $l(\Omega)$ is an IVNNSR of R. □

4. **Conclusions**

As interval-valued neutrosophic environment is more general than regular one, we have adopted it and defined the notions of interval-valued neutrosophic subring and its normal version. Also, we have analyzed some homomorphic properties of these newly defined notions. Again, we have studied product of two interval-valued neutrosophic subrings. Furthermore, we have provided some essential theories to study some of their algebraic structures. These newly introduced notions have potentials to become fruitful research areas. For instance, soft set theory can be implemented and the notion of interval-valued neutrosophic soft subring can be defined.

References

1. L. A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–358, 1965.
2. K. T. Atanassov. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20(1):87–96, 1986.
3. F. Smarandache. A unifying field in logics: neutrosophic logic. In *Philosophy*, 1–141, American Research Press, 1999.
4. F. Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite Study, 2018.
5. F. Smarandache and H. E. Khalid. *Neutrosophic precalculus and neutrosophic calculus*. Infinite Study, 2015.

6. F. Smarandache. *An introduction to the neutrosophic probability applied in quantum physics*. Infinite Study, 2000.
7. F. Smarandache. *Introduction to neutrosophic statistics*. Infinite Study, 2014.
8. F. Smarandache. *Introduction to neutrosophic measure, neutrosophic integral, and neutrosophic probability*. Infinite Study, 2013.
9. F. Smarandache. *Neutropsychic personality: A mathematical approach to psychology*. Infinite Study, 2018.
10. F. Smarandache and L. Vlădăreanu. Applications of neutrosophic logic to robotics: an introduction. In *2011 IEEE International Conference on Granular Computing*, 607–612, 2011.
11. F. Smarandache and M. Ali. Neutrosophic triplet group. *Neural Computing and Applications*, 29(7):595–601, 2018.
12. F. Smarandache. *Extension of soft set to hypersoft set, and then to plithogenic hypersoft set*. Infinite Study, 2018.
13. S. Rana, M. Qayyum, M. Saeed, F. Smarandache, and B. Khan. Plithogenic fuzzy whole hypersoft set, construction of operators and their application in frequency matrix multi attribute decision making technique. *Neutrosophic Sets and Systems*, 28:34–50, 2019.
14. F. Smarandache. *Plithogeny, plithogenic set, logic, probability, and statistics*. Infinite Study, 2017.
15. S. Gayen, F. Smarandache, S. Jha, M. K. Singh, S. Broumi, and R. Kumar. Introduction to plithogenic subgroup. In *Neutrosophic Graph Theory and Algorithm*, 209–233, IGI-Global, 2019.
16. S. Gayen, F. Smarandache, S. Jha, M. K. Singh, S. Broumi, and R. Kumar. Introduction to plithogenic hypersoft subgroup. *Neutrosophic Sets and Systems*, 33:208–233, 2020.
17. M. Abdel-Basset, G. Manogaran, M. Mohamed, and F. Smarandache. A novel method for solving the fully neutrosophic linear programming problems. *Neural Computing and Applications*, 31(5):1595–1605, 2019.
18. M. Abdel-Basset, M. Mohamed, and F. Smarandache. Comment on" a novel method for solving the fully neutrosophic linear programming problems: suggested modifications". *Neutrosophic Sets and Systems*, 31:305–309, 2020.
19. T. Bera and N. K. Mahapatra. Generalised single valued neutrosophic number and its application to neutrosophic linear programming. *Neutrosophic Sets and Systems*, 25:85–103, 2019.
20. S. Pramanik and P. P. Dey. Bi-level linear programming problem with neutrosophic numbers. *Neutrosophic Sets and Systems*, 21:110–121, 2018.
21. M. Abdel-Basset, M. Saleh, A. Gamal, and F. Smarandache. An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number. *Applied Soft Computing*, 77:438–452, 2019.
22. A. Guleria and R. K. Bajaj. Technique for reducing dimensionality of data in decision-making utilizing neutrosophic soft matrices. *Neutrosophic Sets and Systems*, 29:129–141, 2019.
23. M. Abdel-Basset, G. Manogaran, A. Gamal, and F. Smarandache. A group decision making framework based on neutrosophic TOPSIS approach for smart medical device selection. *Journal of medical systems*, 43(2), 2019.
24. M. Mullai, S. Broumi, R. Surya, and G. M. Kumar. Neutrosophic intelligent energy efficient routing for wireless ad-hoc network based on multi-criteria decision making. *Neutrosophic Sets and Systems*, 30:113–121, 2019.
25. V. Uluçay, A. Kılıç, İ. Yıldız, and M. Şahin. An outranking approach for MCDM-problems with neutrosophic multi-sets. *Neutrosophic Sets and Systems*, 30:213–224, 2019.
26. N. A. Nabeeh, M. Abdel-Basset, H. A. El-Ghareeb, and A. Aboelfetouh. Neutrosophic multi-criteria decision making approach for IOT-based enterprises. *IEEE Access*, 7:59559–59574, 2019.
27. Irfan Deli. Some operators with IVGSTrn-numbers and their applications to multiple criteria group decision making. *Neutrosophic Sets and Systems*, 25:33–53, 2019.

28. M. Abdel-Basset and M. Mohamed. A novel and powerful framework based on neutrosophic sets to aid patients with cancer. *Future Generation Computer Systems*, 98:144–153, 2019.
29. M. Abdel-Basset, A. Gamal, G. Manogaran, L. H. Son, and H. V. Long. A novel group decision making model based on neutrosophic sets for heart disease diagnosis. *Multimedia Tools and Applications*, 79, 9977–10002, 2020.
30. R. Kumar, SA Edalatpanah, H Mohapatra. Note on "Optimal path selection approach for fuzzy reliable shortest path problem" *Journal of Intelligent & Fuzzy Systems*, 1–4, 2020, doi: 10.3233/JIFS-200923.
31. R. Kumar, S. A. Edaltpanah, S. Jha, S. Broumi, and A. Dey. Neutrosophic shortest path problem. *Neutrosophic Sets and Systems*, 23:5–15, 2018.
32. S. S. Biswas. Neutrosophic shortest path problem (NSPP) in a directed multigraph. *Neutrosophic Sets and Systems*, 29:174–185, 2019.
33. R. Kumar, A. Dey, F. Smarandache, and S. Broumi. Introduction to plithogenic subgroup. In *Neutrosophic Graph Theory and Algorithm*, 144–175, IGI-Global, 2019.
34. R. Kumar, S. A. Edalatpanah, S. Jha, S. Broumi, R. Singh, and A. Dey. A multi objective programming approach to solve integer valued neutrosophic shortest path problems. *Neutrosophic Sets and Systems*, 24:134–149, 2019.
35. R. Kumar, S. A. Edalatpanah, S. Jha, S. Gayen, and R. Singh. Shortest path problems using fuzzy weighted arc length. *International Journal of Innovative Technology and Exploring Engineering*, 8(6):724–731, 2019.
36. R. Kumar, S. A. Edalatpanah, S. Jha, and R. Singh. A novel approach to solve gaussian valued neutrosophic shortest path problems. *International Journal of Engineering and Advanced Technology*, 8(3):347–353, 2019.
37. R. Kumar, S. Jha, and R. Singh. A different approach for solving the shortest path problem under mixed fuzzy environment. *International Journal of Fuzzy System Applications*, 9(2):132–161, 2020.
38. M. Abdel-Basset, V. Chang, M. Mohamed, and F. Smarandache. A refined approach for forecasting based on neutrosophic time series. *Symmetry*, 11(4):1–23, 2019.
39. M. Abdel-Basset, M. Ali, and A. Atef. Resource leveling problem in construction projects under neutrosophic environment. *The Journal of Supercomputing*, 76(2):964–988, 2020.
40. J. Pratihar, R. Kumar, A. Dey, and S. Broumi. Modified Vogel's approximation method for transportation problem under uncertain environment. *Complex & Intelligent Systems*, 2020, https://doi.org/10.1007/s40747-020-00153-4
41. R. Kumar, S. A. Edalatpanah, S. Jha, and R. Singh. A pythagorean fuzzy approach to the transportation problem. *Complex & Intelligent Systems*, 5(2):255–263, 2019.
42. M. Abdel-Basset, M. Ali, and A. Atef. Uncertainty assessments of linear time-cost tradeoffs using neutrosophic set. *Computers & Industrial Engineering*, 141:1–9, 2020.
43. N. A. Nabeeh, F. Smarandache, M. Abdel-Basset, H. A. El-Ghareeb, and A. Aboelfetouh. An integrated neutrosophic-TOPSIS approach and its application to personnel selection: a new trend in brain processing and analysis. *IEEE Access*, 7:29734–29744, 2019.
44. L. A. Zadeh. The concept of a linguistic variable and its application to approximate reasoning—I. *Information Sciences*, 8(3):199–249, 1975.
45. K. T. Atanassov. Interval valued intuitionistic fuzzy sets. In *Intuitionistic Fuzzy Sets*, 139–177, Springer, 1999.
46. H. Wang, F. Smarandache, Y. Q. Zhang, and R. Sunderraman. *Interval neutrosophic sets and logic: theory and applications in computing*. Infinite Study, 2005.
47. R. Biswas. Rosenfeld's fuzzy subgroups with interval-valued membership functions. *Fuzzy Sets and Systems*, 63(1):87–90, 1994.
48. T. K. Mondal and S. K. Samanta. Topology of interval-valued intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 119(3):483–494, 2001.

49. J. Ye. Multicriteria fuzzy decision-making method based on a novel accuracy function under interval-valued intuitionistic fuzzy environment. *Expert Systems with Applications*, 36(3):6899–6902, 2009.
50. H. W. Kang and K. Hur. Interval-valued fuzzy subgroups and rings. *Honam Mathematical Journal*, 32(4):593–617, 2010.
51. M. Akram and W. A. Dudek. Interval-valued fuzzy graphs. *Computers & Mathematics with Applications*, 61(2):289–299, 2011.
52. A. Aygünoğlu, B. P. Varol, V. Çetkin, and H. Aygün. Interval-valued intuitionistic fuzzy subgroups based on interval-valued double t-norm. *Neural Computing and Applications*, 21(1):207–214, 2012.
53. M. G. S. Moorthy and K. Arjunan. Interval valued intuitionistic fuzzy subrings of a ring. *International Journal of Engineering Research and development*, 10(7):61–66, 2014.
54. Z. Aiwu, D. Jianguo, and G. Hongjun. Interval valued neutrosophic sets and multi-attribute decision-making based on generalized weighted aggregation operator. *Journal of Intelligent & Fuzzy Systems*, 29(6):2697–2706, 2015.
55. I. Deli. Interval-valued neutrosophic soft sets and its decision making. *International Journal of Machine Learning and Cybernetics*, 8(2):665–676, 2017.
56. S. Broumi, A. Bakali, M. Talea, F. Smarandache, and P. K. Singh. Properties of interval-valued neutrosophic graphs. In *Fuzzy Multi-criteria Decision-Making Using Neutrosophic Sets*, 173–202. Springer, 2019.
57. A. Rosenfeld. Fuzzy groups. *Journal of Mathematical Analysis and Applications*, 35(3):512–517, 1971.
58. R. Biswas. Intuitionistic fuzzy subgroups. *Notes on Intuitionistic Fuzzy Sets*, 3(2):53–60, 1997.
59. V. Çetkin and H. Aygün. An approach to neutrosophic subgroup and its fundamental properties. *Journal of Intelligent & Fuzzy Systems*, 29(5):1941–1947, 2015.
60. W. Liu. Fuzzy invariant subgroups and fuzzy ideals. *Fuzzy sets and systems*, 8(2):133–139, 1982.
61. V. N. Dixit, R. Kumar, and N. Ajmal. On fuzzy rings. *Fuzzy sets and systems*, 49(2):205–213, 1992.
62. L. Yan. Intuitionistic fuzzy ring and its homomorphism image. In *2008 International Seminar on Future Bio-Medical Information Engineering*, 75–77, IEEE, 2008.
63. V. Çetkin and H. Aygün. *An approach to neutrosophic subrings*. Infinite Study, 2019.
64. S. Gayen, F. Smarandache, S. Jha, and R. Kumar. Interval-valued neutrosophic subgroup based on interval-valued triple t-norm. In *Neutrosophic Sets in Decision Analysis and Operations Research*, 215–243, IGI-Global, 2019.
65. H. Hashim, L. Abdullah, and A. Al-Quran. Interval neutrosophic vague sets. *Neutrosophic Sets and Systems*, 25:66–75, 2019.
66. P. Muralikrishna and D. S. Kumar. Neutrosophic approach on normed linear space. *Neutrosophic Sets and Systems*, 30:225–240, 2019.
67. S. Gayen, S. Jha, M. Singh, and R. Kumar. On a generalized notion of anti-fuzzy subgroup and some characterizations. *International Journal of Engineering and Advanced Technology*, 8(3):385–390, 2019.
68. D. Preethi, S. Rajareega, J. Vimala, G. Selvachandran, and F. Smarandache. Single-valued neutrosophic hyperrings and single-valued neutrosophic hyperideals. *Neutrosophic Sets and Systems*, 29:121–128, 2019.
69. S. Gayen, S. Jha, and M. Singh. On direct product of a fuzzy subgroup with an anti-fuzzy subgroup. *International Journal of Recent Technology and Engineering*, 8(2):1105–1111, 2019.
70. D. Nagarajan, M. Lathamaheswari, S. Broumi, and J. Kavikumar. Blockchain single and interval valued neutrosophic graphs. *Neutrosophic Sets and Systems*, 24:23–35, 2019.
71. T. Bera and N. Mahapatra. Behaviour of ring ideal in neutrosophic and soft sense. *Neutrosophic Sets and Systems*, 25:1–24, 2019.
72. M. M. Gupta and J. Qi. Theory of t-norms and fuzzy inference methods. *Fuzzy Sets and Systems*, 40(3):431–450, 1991.
73. E. P. Klement, R. Mesiar, and E. Pap. *Triangular norms*. Springer Science & Business Media, 2013.

Received: June 12, 2020 / Accepted: October 1, 2020