Neutrosophic Sets and Systems, Vol. 36, 2020

University of New Mexico

Introduction to AntiRings

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⋄ In commemoration of the 60th birthday of the first author

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Abstract. The objective of this paper is to introduce the concept of AntiRings. Several examples of AntiRings are presented. Specifically, certain types of AntiRings and their substructures are studied. It is shown that nonempty subsets of an AntiRing can be AntiRings with algebraic properties different from the algebraic properties of the parent AntiRing under the same binary operations. AntiIdeals, AntiQuotientRings and AntiRingHomomorphisms are studied with several examples. It is shown that the quotient of an AntiRing factored by an AntiIdeal can exhibit algebraic properties different from the algebraic properties of the AntiRing.

Keywords: NeutroRing; AntiRing; AntiSubring; QuasiAntiSubring; AntiIdeal; QuasiAntiIdeal; PseudoAntiIdeal; AntiQuotientRing; AntiRingHomomorphism.

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1. Introduction

Mathematical modeling of the real life space X requires that all the possible laws that can be defined on X as well as all the possible axioms that can be defined on X should all be considered. In order to be very close to reality, the laws as well as axioms on X should not be rigidly defined. The laws on X should be so flexibly defined to make provisions for both totally inner-defined, totally outer-defined, partially-defined and indeterminately-defined cases. Also, the axioms on X should be such that provisions are made for both totally inner-defined, totally outer-defined, partially-defined and indeterminately-defined cases. When the laws and axioms on X are totally inner-defined, they are called and referred to as ClassicalLaws and ClassicalAxioms respectively. When the laws and axioms on X are partially-defined, they are called and referred to as NeutroLaws and NeutroAxioms respectively. When the laws and axioms on X are totally outer-defined, they are called and referred to as AntiLaws and AntiAxioms respectively. Naturally, we have the neutrosophic triplets (Law, NeutroLaw, AntiLaw) and (Axiom, NeutroAxiom, AntiAxiom) where NonLaw = NeutroLaw $\cup$ AntiLaw, NonAxiom = NeutroAxiom $\cup$ AntiAxiom, NeutroLaw $\cap$ AntiLaw $= \emptyset$ and NeutroAxiom $\cap$ AntiAxiom $= \emptyset$. These concepts have several applications in sciences, engineering, technology, soft computing, social sciences, psychology, politics, sociology and humanities in general. For details on NeutroSociology the readers should see [14] and [7–11, 19] for more details on neutrosophy and applications.

Smarandache in [15–18] introduced and studied extensively the concepts of Neutro-Algebraic Structures and Anti-Algebraic Structures. Rezaei and Smarandache in [12] presented and studied Neutro-BE Algebras and Anti-BE Algebras. Agboola et al. in [4] studied NeutroAlgebras and AntiAlgebras, in [5] and [6], Agboola studied NeutroGroups and NeutroRings respectively. In [3], Agboola further studied NeutroGroups, in [2], he studied AntiGroups and in [1], he further studied NeutroRings. In the present paper, the concept of AntiRings is introduced. Several examples of AntiRings are presented. Specifically, certain types of AntiRings and their substructures are studied. It is shown that nonempty subsets of an AntiRing can be AntiRings with algebraic properties different from the algebraic properties of the parent AntiRing under the same binary operations. AntiIdeals, AntiQuotientRings and AntiRingHomomorphisms are studied with several examples. It is shown that the quotient of an AntiRing factored by an AntiIdeal can exhibit algebraic properties different from the algebraic properties of the AntiRing.

2. **Preliminaries**

In this section, some definitions and results that will be used later in the paper are presented.

Definition 2.1. [15]

A classical operation is an operation well defined for all the set's elements. A NeutroOperation is an operation partially well defined, partially indeterminate, and partially outer defined on the given set while an AntiOperation is an operation that is outer defined for all set's elements.

A classical law/axiom defined on a nonempty set is a law/axiom that is totally true (i.e. true for all set's elements). A NeutroLaw/NeutroAxiom (or Neutrosophic Law/Neutrosophic Axiom) defined on a nonempty set is a law/axiom that is true for some set's elements [degree of truth (T)], indeterminate for other set's elements [degree of indeterminacy (I)], or false for the other set's elements [degree of falsehood (F)], where $T, I, F \in [0,1]$, with $(T, I, F) \neq (1, 0, 0)$ that represents the classical axiom, and $(T, I, F) \neq (0, 0, 1)$ that represents the AntiAxiom while an AntiLaw/AntiAxiom defined on a nonempty set is a law/axiom that is false for all set's elements.

A PartialOperation on a set is an operation that is well defined for some elements of the set and undefined for all the other elements of the set. A PartialAlgebra is an algebra that has at least one PartialOperation, and all its axioms are classical.

A NeutroAlgebra is an algebra that has at least one NeutroOperation or one NeutroAxiom (axiom that is true for some elements, indeterminate for other elements, and false for other elements), and no AntiOperation or AntiAxiom while an AntiAlgebra is an algebra endowed with at least one AntiOperation or at least one AntiAxiom. When a NeutroAlgebra has no NeutroAxiom, then it coincides with the PartialAlgebra.

Theorem 2.2. *[15] The NeutroAlgebra is a generalization of PartialAlgebra.*

Theorem 2.3. *[12] Let $\mathbb{U}$ be a nonempty finite or infinite universe of discourse and let S be a finite or infinite subset of $\mathbb{U}$. If n classical operations (laws and axioms) are defined on S where $n \geq 1$, then there will be $(2^n - 1)$ NeutroAlgebras and $(3^n - 2^n)$ AntiAlgebras.*

Definition 2.4. [Classical ring] [13]

Let R be a nonempty set and let $+, . : R \times R \to R$ be binary operations of the usual addition and multiplication respectively defined on R. The triple $(R, +, .)$ is called a classical ring if the following conditions $(R1 - R9)$ hold:

(R1) $x + y \in R \; \forall x, y \in R$ [closure law of addition].

(R2) $x + (y + z) = (x + y) + z \; \forall x, y, z \in R$ [axiom of associativity].

(R3) There exists $e \in R$ such that $x + e = e + x = x \; \forall x \in R$ [axiom of existence of neutral element].

(R4) There exists $-x \in R$ such that $x + (-x) = (-x) + x = e \; \forall x \in G$ [axiom of existence of inverse element]

(R5) $x + y = y + x \; \forall x, y \in R$ [axiom of commutativity].

(R6) $x.y \in R \; \forall x, y \in R$ [closure law of multiplication].

(R7) $x.(y.z) = (x.y).z \; \forall x, y, z \in R$ [axiom of associativity].

(R8) $x.(y + z) = (x.y) + (x.z) \; \forall x, y, z \in R$ [axiom of left distributivity].

(R9) $(y + z).x = (y.x) + (z.x) \; \forall x, y, z \in R$ [axiom of right distributivity].

If in addition we have,

(R10) $x.y = y.x \; \forall x, y \in R$ [axiom of commutativity],

then $(R, +, .)$ is called a commutative ring.

Definition 2.5. [1][NeutroSophication of the laws and axioms of the classical ring]

(NR1) There exist at least three duplets $(x, y), (u, v), (p, q) \in R$ such that $x+y \in R$ (inner-defined with degree of truth T) and [$u + v =$ indeterminate (with degree of indeterminacy I) or $p + q \notin R$ (outer-defined/falsehood with degree of falsehood F)] [NeutroClosure law of addition].

(NR2) There exist at least three triplets $(x, y, z), (p, q, r), (u, v, w) \in R$ such that $x+(y+z) = (x+y)+z$ (inner-defned with degree of truth T) and [[$p + (q + r)$]or[$(p + q) + r$] = indeterminate (with degree of indeterminacy I) or $u + (v + w) \neq (u + v) + w$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of associativity (NeutroAssociativity)].

(NR3) There exists an element $e \in R$ such that $x + e = x + e = x$ (inner-defined with degree of truth T) and [[$x + e$]or[$e + x$] = indeterminate (with degree of indeterminate I) or $x + e \neq x \neq e + x$ (outer-defined/falsehood with degree of falsehood F)] for at least one $x \in R$ [NeutroAxiom of existence of neutral element (NeutroNeutralElement)].

(NR4) There exists $-x \in R$ such that $x + (-x) = (-x) + x = e$ (inner-defined with degree of truth T) and [[$-x + x$]or[$x + (-x)$] = indeterminate (with the degree of indeterminate I) or

$-x + x \neq e \neq x + (-x)$ (outer-defined/falsehood with degree of falsehood F)] for at least one $x \in R$ [NeutroAxiom of existence of inverse element (NeutroInverseElement)].

(NR5) There exist at least three duplets $(x, y), (u, v), (p, q) \in R$ such that $x + y = y + x$ (inner-defined with degree of truth T) and $[[p + q]\text{or}[q + p]$ = indeterminate (with degree of indeterminacy I) or $u + v \neq v + u$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of commutativity (NeutroCommutativity)].

(NR6) There exist at least three duplets $(x, y), (p, q), (u, v) \in R$ such that $x.y \in R$ (inner-defined with degree of truth T) and $[u.v$ = indeterminate (with degree of indeterminacy I) or $p.q \notin R$ (outer-defined/falsehood with degree of falsehood F)] NeutroClosure law of multiplication].

(NR7) There exist at least three triplets $(x, y, z), (p, q, r), (u, v, w) \in R$ such that $x.(y.z) = (x.y).z$ (inner-defined with degree of truth T) and $[[p.(q.r)]\text{or}[(p.q).r]$ = indeterminate (with degree of indeterminacy I) or $u.(v.w) \neq (u.v).w$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of associativity (NeutroAssociativity)].

(NR8) There exist at least three triplets $(x, y, z), (p, q, r), (u, v, w) \in R$ such that $x.(y + z) = (x.y) + (x.z)$ (inner-defined with degree of truth T) and $[[p.(q + r)]\text{or}[(p.q) + (p.r)]$ = indeterminate (with degree of indeterminacy I) or $u.(v + w) \neq (u.v) + (u.w)$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of left distributivity (NeutroLeftDistributivity)].

(NR9) There exist at least three triplets $(x, y, z), (p, q, r), (u, v, w) \in R$ such that $(y + z).x = (y.x) + (z.x)$ (inner-defined with degree of truth T) and $[[(v + w).u]\text{or}[(v.u) + (w.u)]$ = indeterminate (with degree of indeterminacy I) or $(v + w).u \neq (v.u) + (w.u)$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of right distributivity (NeutroRightDistributivity)].

(NR10) There exist at least three duplets $(x, y), (p, q), (u, v) \in R$ such that $x.y = y.x$ (inner-defined with degree of truth T) and $[[p.q]\text{or}[q.p]$ = indeterminate (with degree of indeterminacy I) or $u.v \neq v.u$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of commutativity (NeutroCommutativity)].

Definition 2.6. [1][AntiSophication of the laws and axioms of the classical ring]

(AR1) For all the duplets $(x, y) \in R$, $x + y \notin R$ [AntiClosure law of addition].

(AR2) For all the triplets $(x, y, z) \in R$, $x + (y + z) \neq (x + y) + z$ [AntiAxiom of associativity (AntiAssociativity)].

(AR3) There doest not exist an element $e \in R$ such that $x + e = x + e = x \ \forall x \in R$ [AntiAxiom of existence of neutral element (AntiNeutralElement)].

(AR4) There does not exist $-x \in R$ such that $x + (-x) = (-x) + x = e \ \forall x \in R$ [AntiAxiom of existence of inverse element (AntiInverseElement)].

(AR5) For all the duplets $(x, y) \in R$, $x+y \neq y+x$ [AntiAxiom of commutativity (AntiCommutativity)].

(AR6) For all the duplets $(x, y) \in R$, $x.y \notin R$ [AntiClosure law of multiplication].

(AR7) For all the triplets $(x, y, z) \in R$, $x.(y.z) \neq (x.y).z$ [AntiAxiom of associativity (AntiAssociativity)].

(AR8) For all the triplets $(x, y, z) \in R$, $x.(y + z) \neq (x.y) + (x.z)$ [AntiAxiom of left distributivity (AntiLeftDistributivity)].

(AR9) For all the triplets $(x, y, z) \in R$, $(y + z).x \neq (y.x) + (z.x)$ [AntiAxiom of right distributivity (AntiRightDistributivity)].

(AR10) For all the duplets $(x, y) \in R$, $x.y \neq y.x$ [AntiAxiom of commutativity (AntiCommutativity)].

Definition 2.7. [1][NeutroRing]
A NeutroRing NR is an alternative to the classical ring R that has at least one NeutroLaw or at least one of $\{NR1, NR2, NR3, NR4, NR5, NR6, NR7, NR8, NR9\}$ with no AntiLaw or AntiAxiom.

Definition 2.8. [[1]][AntiRing]
An AntiRing AR is an alternative to the classical ring R that has at least one AntiLaw or at least one of $\{AR1, AR2, AR3, AR4, AR5, AR6, AR7, AR8, AR9\}$.

Definition 2.9. [1][NeutroCommutativeRing]
A NeutroCommutativeRing NR is an alternative to the classical commutative ring R that has at least one NeutroLaw or at least one of $\{NR1, NR2, NR3, NR4, NR5, NR6, NR7, NR8, NR9\}$ and $NR10$ with no AntiLaw or AntiAxiom.

Definition 2.10. [1][AntiCommutativeRing]
An AntiCommutativeRing AR is an alternative to the classical commutative ring R that has at least one AntiLaw or at least one of $\{AR1, AR2, AR3, AR4, AR5, AR6, AR7, AR8, AR9\}$ and $AR10$.

Theorem 2.11. *[1] Let $(R, +, .)$ be a finite or infinite classical ring. Then:*

(i) *There are 511 types of NeutroRings.*
(ii) *There are 19171 types of AntiRings.*

Theorem 2.12. *[1] Let $(R, +, .)$ be a finite or infinite classical commutative ring. Then:*

(i) *There are 1023 types of NeutroCommutativeRings.*
(ii) *There are 58025 types of AntiCommutativeRings.*

Example 2.13. [1] Let $NR = \mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ and let $\oplus$ and $\odot$ be two binary operations defined on NR by

$$x \oplus y = x + y - 1,\ x \odot y = x + xy\ \forall\ x, y \in NR.$$

Then, $(NR, \oplus, \odot)$ is a NeutroCommutativeRing.

Example 2.14. [1] Let $NR = \{a, b, c, d\}$ and let $''+''$ and $''.''$ be binary operations defined on NR as shown in the Cayley tables below:

$+$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

$.$	a	b	c	d
a	a	b	c	d
b	a	c	b	c
c	c	d	c	d
d	d	a	d	a

Then, $(NR, +, .)$ is a NeutroCommutativeRing.

Example 2.15. Let $AR = \mathbb{Z}$ and let $\oplus$ and $\odot$ be two binary operations defined on AR such that $\oplus$ is the usual addition of integers and $\forall x, y \in AR$, $\odot$ is defined by

$$x \odot y = x^2 + x^2y + 2.$$

Then $(AR, \oplus, \odot)$ is an AntiRing. To see this, we first note that $\oplus$ is well defined for all $x, y \in AR$ and that $R1 - R5$ are totally true. Hence, $(AR, \oplus)$ is an abelian group.

It is also noted that $\odot$ is well defined for all $x, y \in AR$ that is, $R6$ is totally true $\forall x, y \in AR$. Now let $x, y, z \in AR$. Then

$$\begin{aligned} x \odot (y \odot z) &= x^2 + x^2y^2 + x^2y^2z + 2x^2 + 2, \\ (x \odot y) \odot z &= x^4 + 2x^4y + x^4y^2 + 4x^2 + 4x^2y \\ &\quad +x^4z + 2x^4yz + x^4y^2z + 4x^2z + 4x^2yz + 4z + 6. \\ \therefore\ x \odot (y \odot z) &\neq (x \odot y) \odot z\ \forall x, y, z \in AR. \end{aligned}$$

It has just been shown that for all the elements of AR, $\odot$ is AntiAssociative over AR. Thus, $AR7$ is satisfied.

Also for all $x, y, z \in AR$, we have

$$\begin{aligned} x \odot (y \oplus z) &= x \odot (y + z) \\ &= x^2 + x^2y + x^2z + 2, \\ (x \odot y) \oplus (x \odot z) &= (x \odot y) + (x \odot z) \\ &= 2x^2 + x^2y + x^2z + 4. \\ \therefore\ x \odot (y \oplus z) &\neq (x \odot y) \oplus (x \odot y)\ \forall x, y, z \in AR. \end{aligned}$$

It has again been shown that over AR, $\odot$ is not left distributive over $\oplus$ for all $x, y, z \in AR$. Hence $AR8$ is satisfied.

Lastly for all $x, y, z \in AR$, we have

$$\begin{aligned} (y \oplus z) \odot x &= (y + z) \odot x \\ &= y^2 + z^2 + 2yz + xy^2 + 2xyz + xz^2 + 2, \\ (y \odot x) \oplus (z \odot x) &= (y \odot x) + (z \odot x) \\ &= y^2 + z^2 + y^2x + z^2x + 4. \\ \therefore\ (y \oplus z) \odot x &\neq (y \odot x) \oplus (y \odot x)\ \forall x, y, z \in AR. \end{aligned}$$

This also shows that over AR, $\odot$ is not right distributive over $\oplus$ for all $x, y, z \in AR$. Hence, $AR9$ is satisfied. It can easily be shown that $\odot$ is NeutroCommutative over AR. Accordingly by Definition 2.8, $(AR, \oplus, \odot)$ is an AntiRing.

Example 2.16. (i) Let $AR = M_{n\times n}[\mathbb{R}]$ be the set of all $n \times n$ matrices with real entries and let $\oplus$ and $\odot$ be two binary operations defined on AR such that $\oplus$ is the usual addition of matrices and $\forall X, Y \in AR$, $\odot$ is defined by

$$X \odot Y = X^2 + X^2Y + 2I$$

where I is the $n \times n$ unit matrix. Then, $(AR, \oplus, \odot)$ is an AntiRing.

(ii) Let M be an additive abelian group and let $AR = End(M)$ be the set of all endomorphisms of M into itself. Let $\oplus$ and $\odot$ be two binary operations defined on AR such that $\oplus$ is the usual addition of mappings and $\forall f, g \in AR$, $\odot$ is defined by

$$(f \odot g)(x) = f^2(x) + f^2(x)g(x) + 2i(x)$$

where i is the identity mapping. Then, $(AR, \oplus, \odot)$ is an AntiRing.

3. A Study of Certain Types of AntiRings

In this section, we are going to study certain types of AntiRings. Many examples and basic results will be presented. Since there are many types of AntiRings, then AntiRings in this section will be classified and named type-AR[,] according to which of $AR1 - AR10$ is(are) satisfied. If only $AR1$ is satisfied, the AntiRing will be called of type-AR[1], type-AR[3,4] if only $AR3$ and $AR4$ are satisfied and so on. AntiRings of type-AR[1,2,3,4-9] or of type-AR[1,2,3,4-10] will be called trivial AntiRings or trivial AntiCommutativeRings respectively.

Definition 3.1. Let $(AR, +, .)$ be an AntiRing.

(i) AR is called a finite AntiRing of order n if the cardinality of AR is n that is $o(AR) = n$. Otherwise, AR is called an infinite AntiRing and we write $o(AR) = \infty$.

(ii) AR is called an AntiRing with unity if there exists a multiplicative unit element $u \in AR$ such that $ux = xu = x$ for at least one $x \in R$.

(iii) If there exists a least positive integer n such that $nx = e$ for at least one $x \in AR$, then AR is called an AntiRing of characteristic n. If no such n exists, then AR is called an AntiRing of characteristic zero.

(iv) An element $x \in AR$ is called an idempotent element if $x^2 = x$.

(v) An element $x \in AR$ is called a nilpotent element if for the least positive integer n, we have $x^n = e$.

(vi) An element $e \neq x \in AR$ is called a zero divisor element if there exists an element $e \neq y \in AR$ such that $xy = e$ or $yx = e$.

(vii) An element $x \in AR$ is called a multiplicative inverse element if there exists at least one $y \in AR$ such that $xy = yx = u$ where u is the multiplicative unity element in AR.

Definition 3.2. Let $(AR, +, .)$ be an AntiCommutativeRing with unity. Then

(i) AR is called an AntiIntegralDomain if all the elements of AR are zero divisors.

(ii) AR is called an AntiField if all the elements of AR have no multiplicative inverse elements.

Definition 3.3. Let $(AR, +, .)$ be an AntiRing. A nonempty subset AS of AR is called an AntiSubring of AR if $(AS, +, .)$ is also an AntiRing of the same type as AR.

Definition 3.4. Let $(AR, +, .)$ be an AntiRing. A nonempty subset AS of AR is called a QuasiAntiSubring of AR if $(AS, +, .)$ is also an AntiRing not of the same type as AR.

Definition 3.5. Let $(AR, +, .)$ be an AntiRing. A nonempty subset AI of AR is called a left AntiIdeal of AR if the following conditions hold:

(i) AI is an AntiSubring of AR of the same type as AR.

(ii) $x \in AI$ and $r \in AR$ imply that $xr \notin AI$ for all $r \in AR$.

Definition 3.6. Let $(AR, +, .)$ be an AntiRing. A nonempty subset AI of AR is called a right AntiIdeal of AR if the following conditions hold:

(i) AI is an AntiSubring of AR of the same type as AR.

(ii) $x \in AI$ and $r \in AR$ imply that $rx \notin AI$ for all $r \in NR$.

Definition 3.7. Let $(AR, +, .)$ be an AntiRing. A nonempty subset AI of NR is called a two-sided AntiIdeal of AR if the following conditions hold:

(i) AI is an AntiSubring of AR of the same type as AR.

(ii) $x \in AI$ and $r \in AR$ imply that $xr \notin AI$ and $rx \notin AI$ for all $r \in AR$.

Definition 3.8. Let $(AR, +, .)$ be an AntiRing. A nonempty subset AI of AR is called a left QuasiAntiIdeal of AR if the following conditions hold:

(i) AI is a QuasiAntiSubring of AR.

(ii) $x \in AI$ and $r \in AR$ imply that $xr \notin AI$ for all $r \in AR$.

Definition 3.9. Let $(AR, +, .)$ be an AntiRing. A nonempty subset AI of AR is called a right QuasiAntiIdeal of AR if the following conditions hold:

(i) AI is a QuasiAntiSubring of AR.

(ii) $x \in AI$ and $r \in AR$ imply that $rx \notin AI$ for all $r \in NR$.

Definition 3.10. Let $(AR, +, .)$ be an AntiRing. A nonempty subset AI of NR is called a two-sided QuasiAntiIdeal of AR if the following conditions hold:

(i) AI is a QuasiAntiSubring of AR.

(ii) $x \in AI$ and $r \in AR$ imply that $xr \notin AI$ and $rx \notin AI$ for all $r \in AR$.

Definition 3.11. Let $(AR, +, .)$ be an AntiRing. A nonempty subset AI of AR is called a left PseudoAntiIdeal of AR if the following conditions hold:

(i) AI is an AntiSubring or a QuasiAntiSubring of AR.

(ii) For at least one $x \in AI$, $xr \notin AI$ for all $r \in AR$.

Definition 3.12. Let $(AR, +, .)$ be an AntiRing. A nonempty subset AI of AR is called a right PseudoAntiIdeal of AR if the following conditions hold:

(i) AI is an AntiSubring or a QuasiAntiSubring of AR.

(ii) For at least one $x \in AI$, $rx \notin AI$ for all $r \in AR$.

Definition 3.13. Let $(AR, +, .)$ be an AntiRing. A nonempty subset AI of NR is called a two-sided PseudoAntiIdeal of AR if the following conditions hold:

(i) AI is an AntiSubring or a QuasiAntiSubring of AR.

(ii) For at least one $x \in AI$, $xr \notin AI$ and $rx \notin AI$ for all $r \in AR$.

Example 3.14. Let $AR = \{a, b\}$ and let $''+''$ and $''.''$ be two binary operations defined on AR as shown in the Cayley tables below.

$+$	a	b
a	a	b
b	b	a

$.$	a	b
a	b	b
b	a	a

Since $x+y, xy \in AR$ $\forall x, y \in AR$ and $(AR, +)$ is an abelian group, it follows that $R1 - R6$ of Definition 2.6 are totally true for all the elements of AR. Now consider the following:

(AR7) $a(aa) = b, (aa)a = a \neq b$, $a(ab) = b, (aa)b = a \neq b$, $a(ba) = b, (ab)a = a \neq b$, $b(aa) = a, (ba)a = b \neq a$,$a(bb) = b, (ab)b = a \neq b$, $b(ab) = a, (ba)b = b \neq a$, $b(ba) = a, (bb)a = b \neq a$, $b(bb) = a, (bb)b = b \neq a$. These show that the binary operation $''.''$ is totally AntiAssociative in AR.

(AR8) $a(a + a) = b$ while $aa + aa = a \neq b$. Also, $b(b + b) = a$ while $bb + bb = a$. These show that the binary operation $''.''$ is NeutroLeftDistributive over the binary operation $''+''$.

(AR9) $(a+a)a = b, aa+aa = a \neq b$, $(a+a)b = b, ab+ab = a \neq b$, $(a+b)a = a, aa+ba = b \neq a$, $(b+a)a = a, ba+aa = b \neq a$, $(a+b)b = a, ab+bb = b \neq a$, $(b+a)b = a, bb+ab = b \neq a$, $(b+b)a = b, ba+ba = a \neq b$, $(b+b)b = b, bb+bb = a \neq b$. These show that the binary operation $''.''$ is AntiRightDistributive over the binary operation $''+''$.

(AR10) $aa = b, bb = a$ but $ab = b, ba = a \neq b$. These show that the binary operation $''.''$ is NeutroCommutative in AR.

Since $AR7$ and $AR9$ are totally true for all the elements of AR, it follows from Definition 2.8 that $(AR, +, .)$ is an AntiRing which we call an AntiRing of type-AR[7,9].

Example 3.15. Let $(AR, +, .)$ be the AntiRing of Example 3.14. It is clear that $e = a$ is the additive identity element. The element b is idempotent since $bb = a$. Since $''.''$ is totally AntiAssociative, it follows that AR has no nilpotent elements. AR has no unity and consequently, none of the elements of AR is invertible. Since AR is not an AntiCommutativeRing, it follows that AR is neither an AntiIntegralDomain nor an AntiField.

Example 3.16. Let $AR = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$. Let $*$ and $\circ$ be two binary operations defined such that $*$ is the usual addition modulo 6 and for all $x, y \in AR$, $\circ$ is defined by

$$x \circ y = x + xy + 2.$$

It is clear that $x * y, x \circ y \in AR\ \forall x, y \in AR$. This shows that $R1 - R6$ of Definition 2.6 are totally true for all the elements of AR. Now consider the following:

(AR7) $x \circ (y \circ z) = 3x + xy + xyz + 2$ and $(x \circ y) \circ z = x + xy + xz + 2z + xyz + 4$. Equating the two expressions we obtain $2x = xz + 2z + 2$ from which we have that the triplet $(2, y, 2) \in AR$ can verify the associativity of $\circ$ in AR. Thus, $\circ$ is NeutroAssociative in AR.

(AR8) $x \circ (y * z) = x + xy + xz + 2$ and $(x \circ y) * (x \circ z) = 2x + xy + xz + 4$. Equating the two expressions we have $x = -2 \equiv 4$ modulo 6. Hence, only the triplets $(4, y, z) \in AR$ can verify the left distributivity of $\circ$ over $*$ in AR. Thus, $\circ$ is NeutroLeftDistributive in AR.

(R9) $(y * z) \circ x = y + z + xy + xz + 2$ and $(y \circ x) * (z \circ x) = y + z + xy + xz + 4$. Since $2 \neq 4$ modulo 6, it follows that $\circ$ is not right distributive over $*$ for all the triplets $(x, y, z) \in AR$. Hence $\circ$ is totally AntiRightDistributive over $*$ in AR.

(R10) $x \circ y = x + xy + 2$ and $y \circ x = y + yx + 2$. Equating the two expressions we have $x = y$ showing that only the duplets $(x, x) \in AR$ can verify the commutativity of $\circ$. Hence, $\circ$ is NeutroCommutative in AR.

According to Definition 2.8, we have that $(AR, *, \circ)$ is an AntiRing of type-AR[9].

Example 3.17. Let $AS = \{0, 3\}$ be a subset of AR where $(AR, *, \circ)$ is the AntiRing of Example 3.16. Consider the compositions of the elements of AS as shown in the Cayley tables below.

$*$	0	3
0	0	3
3	3	0

$\circ$	0	3
0	2	2
3	5	2

$R1 - R5$ are totally true since for all $x, y \in AS$, $x * y \in AS$ and $(AS, *)$ is an abelian group. Also for all the elements of AS, $R6 - R10$ are totally false. Accordingly, $(AS, *, \circ)$ is an AntiRing of the type-AR[6,7,8,9,10] which is different from the class of the parent AntiRing. Hence, AS is a QuasiAntiSubring of AR.

Example 3.18. Let $AT = \{0, 2, 4\}$ be a subset of AR where $(AR, *, \circ)$ is the AntiRing of Example 3.16. Consider the compositions of the elements of AT as shown in the Cayley tables below.

$*$	0	2	4
0	0	2	4
2	2	4	0
4	4	0	2

$\circ$	0	2	4
0	2	2	2
2	4	2	0
4	0	2	4

$R1 - R6$ are totally true since for all $x, y \in AS$, $x * y, x \circ y \in AT$ and $(AT, *)$ is an abelian group. Now consider the following:

(AR7) $2 \circ (4 \circ 2) = (2 \circ 4) \circ 2 = 2$ but $2 \circ (0 \circ 4) = 2, (2 \circ 0) \circ 4 = 4 \neq 2$. These show that the binary operation $\circ$ is NeutroAssociative over AT.

(AR8) $4 \circ (2 * 4) = (4 \circ 2) * (4 \circ 4) = 0$ but $2 \circ (4 * 0) = 0, (2 \circ 4) * (2 \circ 0) = 4 \neq 0$. These show that the binary operation $\circ$ is NeutroLeftDistributive over $*$ in AT.

(AR9) For all the triplets $(x, y, z) \in AS$, we have $(y * z) \circ x \neq (y \circ x) * (z \circ x)$. This shows that the binary operation $\circ$ is AntiRightDistributive over $*$ in AT.

(AR10) Since $0 \circ 0 = 2, 2 \circ 2 = 2, 4 \circ 4 = 4$ but $0 \circ 2 = 2, 2 \circ 0 = 4 \neq 2, 2 \circ 4 = 0, 4 \circ 2 = 2 \neq 0, 4 \circ 0 = 0, 0 \circ 4 = 2 \neq 0$, it follows that the binary operation $\circ$ is NeutroCommutative over AT.

Accordingly, $(AT, *, \circ)$ is an AntiRing of the type-AR[9] which is the same as the class of the parent AntiRing. Hence, AT is an AntiSubring of AR.

Example 3.19. Let $AR = \mathbb{Z}^+ = \{1, 2, 3, 4, \cdots, \}$ and let $AS = 2\mathbb{Z}^+ = \{2, 4, 6, 8, \cdots, \}$, $AT = 3\mathbb{Z}^+ = \{3, 6, 9, 12, \cdots, \}$. Suppose that $*$ and $\circ$ are binary operations respectively of the usual addition and multiplication of integers defined on AR, AS and AT. It can easily be shown that $(AR, *, \circ)$, $(AS, *, \circ)$ and $(AT, *, \circ)$ are AntiRings of type-AR[3,4] since $R1, R2$ and $R5 - R10$ are totally true but $R3$ and $R4$ are totally false. Since AS and AT are subsets of AR, it follows that AS and AT are AntiSubrings of AR. In general, $(n\mathbb{Z}^+, *, \circ)$ are AntiSubrings of the AntiRing $(\mathbb{Z}^+, *, \circ)$ for $n = 2, 3, 4, 5, \cdots$.

Remark 3.20. It is evident from Example 3.17 that an AntiRing of a particular type can have nonempty subsets which are AntiRings of types different from the type of the parent AntiRing under the same binary operations.

Example 3.21. Let AR be an AntiRing of Example 3.16 and let AS and AT be its QuasiAntiSubring and AntiSubring of Examples 3.17 and 3.18 respectively. Then $AS \cup AT = \{0, 2, 3, 4\}$ and $AS \cap AT = \{0\}$. It is clear that $AS \cap AT$ is neither an AntiSubring nor a QuasiAntiSubring of AR. However, it can be shown that $(AS \cup AT, *, \circ)$ is an AntiRing of type-AR[9]. Hence, $AS \cup AT$ is an AntiSubring of AR.

Example 3.22. Let AR be the AntiRing of Example 3.19 and let AS and AT be its AntiSubrings. Then $AS \cup AT = \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, \cdots\}$ and $AS \cap AT = \{6, 12, 18, 24, \cdots\} = 6\mathbb{Z}^+$. It can be shown that $AS \cup AT$ is a QuasiAntiSubring of AR and $AS \cap AT$ is an AntiSubring of AR.

Remark 3.23. The union of two AntiSubrings of an AntiRing can produce a QuasiAntiSubring of the AntiRing.

Example 3.24. Let AR be an AntiRing of Example 3.16 and let AS be its QuasiAntiSubring of Example 3.17. Then $[0 \circ 0 = 0 \circ 1 = 0 \circ 2 = 0 \circ 3 = 0 \circ 4 = 0 \circ 5 = 2, 3 \circ 0 = 3 \circ 2 = 5, 3 \circ 1 = 3 \circ 5 = 2, 3 \circ 3 = 1, 3 \circ 4 = 4] \notin AS$. These show that AS is a left QuasiAntiIdeal of AR.

However, $[0 \circ 0 = 2, 2 \circ 0 = 4, 3 \circ 0 = 5, 5 \circ 0 = 1, 0 \circ 3 = 2, 2 \circ 3 = 4, 3 \circ 3 = 1, 5 \circ 3 = 4] \notin AS$ but $[1 \circ 0 = 3, 4 \circ 0 = 1 \circ 3 = 4 \circ 3 = 0] \in AS$. These show that AS is a right PseudoAntiIdeal of AR.

Example 3.25. Let AR be an AntiRing of Example 3.16 and let AT be its AntiSubring of Example 3.18. Then $[0 \circ 0 = 0 \circ 1 = 0 \circ 2 = 0 \circ 3 = 0 \circ 4 = 0 \circ 5 = 2, 2 \circ 0 = 2 \circ 3 = 4, 2 \circ 1 = 2 \circ 4 = 0, 2 \circ 2 = 2 \circ 5 = 2, 4 \circ 0 = 4 \circ 3 = 0, 4 \circ 1 = 4 \circ 4 = 4, 4 \circ 2 = 4 \circ 5 = 2] \in AT$. These show that AT is neither a left QuasiAntiIdeal nor a left PseudoAntiIdeal of AR.

Also, $[0 \circ 0 = 0 \circ 2 = 2 \circ 2 = 4 \circ 2 = 0 \circ 4 = 2, 2 \circ 0 = 1 \circ 2 = 3 \circ 4 = 4 \circ 4 = 4, 4 \circ 0 = 2 \circ 4 = 0] \in AT$ but $[1 \circ 0 = 5 \circ 4 = 3, 3 \circ 0 = 3 \circ 2 = 5 \circ 2 = 5, 5 \circ 0 = 1 \circ 4 = 1] \notin AT$. These show that AT is a right PseudoAntiIdeal of AR.

Example 3.26. Let AR be the AntiRing of Example 3.19 and let AS and AT be its AntiSubrings. Then AS and AT are two-sided PseudoAntiIdeals of AR. To see this, let $x \in AS$ and $r \in AR$. Then

$$x \circ r = r \circ x = \begin{cases} a \in AS \text{ if } r = 2, 4, 6, 8, \cdots \\ b \notin AS \text{ if } r = 1, 3, 5, 7, \cdots \end{cases}$$

Also,

$$x \circ r = r \circ x = \begin{cases} c \in AT \text{ if } r = 3, 6, 9, 12, \cdots \\ d \notin AT \text{ if } r = 1, 2, 4, 8, \cdots \end{cases}$$

Definition 3.27. Let $(AR, +, .)$ be an AntiRing and let AI be a left(right)(two-sided) AntiIdeal or a left(right)(two-sided) QuasiAntiIdeal or a left(right)(two-sided) PseudoAntiIdeal of AR. The set AR/AI is defined by

$$AR/AI = \{x + AI : x \in AR\}.$$

For all $x + AI, y + AI \in AR/AI$, let $\oplus$ and $\odot$ be two binary operations on AR/AI defined as follows:

$$(x + AI) \oplus (y + AI) = (x * y) + AI,$$
$$(x + AI) \odot (y + AI) = (x \circ y) + AI.$$

If $(AR/AI, \oplus, \odot)$ is an AntiRing, then AR/AI is called an AntiQuotientRing.

Example 3.28. Let AR be an AntiRing of Example 3.16 and let AS be its left QuasiAntiIdeal of Example 3.24. Then

$$AR/AS = \{AS, 1 + AS, 2 + AS\}.$$

Let $\oplus$ and $\odot$ be two binary operations defined on AR/AS as shown in the Cayley tables below.

$\oplus$	AS	$1+AS$	$2+AS$
AS	AS	$1+AS$	$2+AS$
$1+AS$	$1+AS$	$2+AS$	AS
$2+AS$	$2+AS$	AS	$1+AS$

$\odot$	AS	$1+AS$	$2+AS$
AS	$2+AS$	$2+AS$	$2+AS$
$1+AS$	AS	$1+AS$	$2+AS$
$2+AS$	$1+AS$	AS	$2+AS$

It can easily be shown that $R1 - R6$ are totally true, $R7, R8$ and $R10$ are partially true and partially false and $R9$ is totally false. Hence, $(AR/AS, \oplus, \odot)$ is an AntiRing of type-AR[9].

Example 3.29. Let AR be an AntiRing of Example 3.16 and let AT be its right PseudoAntiIdeal of Example 3.25. Then

$$AR/AT = \{AT, 1 + AT\}.$$

Let $\oplus$ and $\odot$ be two binary operations defined on AR/AT as shown in the Cayley tables below.

$\oplus$	AT	$1+AT$
AT	AT	$1+AT$
$1+AT$	$1+AT$	AT

$\odot$	AT	$1+AT$
AT	AT	AT
$1+AT$	$1+AT$	AT

It can easily be shown that $R1 - R6$ and $R9$ are totally true, $R7, R8$ and $R10$ are partially true and partially false. Hence, $(AR/AT, \oplus, \odot)$ is a NeutroRing.

Example 3.30. Let AR be the AntiRing of Example 3.19 and let AS be its PseudoAntiIdeal of Example 3.26. Then

$$AR/AS = \{1 + AS, 2 + AS, 3 + AS, 4 + AS, \cdots\}$$

If $\oplus$ and $\odot$ are two binary operations on AR/AS such that

$$(x + AS) \oplus (y + AS) = (x * y) + AS,$$
$$(x + AS) \odot (y + AS) = (x \circ y) + AS,$$

It can be shown that $(AR/AS, \oplus, \odot)$ is an AntiRing of type-AR[3,4].

Remark 3.31. If $(AR, +, .)$ is an AntiRing and AI is a left(right)(two-sided) AntiIdeal or a left(right)(two-sided) QuasiAntiIdeal or a left(right)(two-sided) PseudoAntiIdeal of AR, then an AntiQuotientRing AR/AI can have algebraic properties different from the algebraic properties of AR.

Definition 3.32. Let $(AR, +, .)$ and $(AS, +', .')$ be any two AntiRings of the same type/class. The mapping $\phi : AR \to AS$ is called an AntiRingHomomorphism if ϕ anti-preserves the binary operations of AR and AS that is if for all the duplets $(x, y) \in AR$, we have:

$$\begin{aligned} \phi(x+y) &\neq \phi(x) +' \phi(y), \\ \phi(x.y) &\neq \phi(x).'\phi(y). \end{aligned}$$

The kernel of ϕ denoted by $Ker\phi$ is defined as

$$Ker\phi = \{x : \phi(x) = e_{AR}\}.$$

The image of ϕ denoted by $Im\phi$ is defined as

$$Im\phi = \{y \in AS : y = \phi(x) \text{ for at least one } y \in AS\}.$$

If in addition ϕ is an AntiBijection, then ϕ is called an AntiRingIsomorphism. AntiRingEpimorphism, AntiRingMonomorphism, AntiRingEndomorphism and AntiRingAutomorphism are similarly defined.

Example 3.33. Let AR be the AntiRing of Example 3.19 and let AR/AS be the AntiQuotientRing of Example 3.30. Then $\phi : AR \to AR/AS$ defined by

$$\phi(x) = x + AS \ \forall x \in AR$$

is a classical homomorphism and not an AntiRingHomomorphism. To see this, for all $m, n \in AR$, we have $\phi(m) = m + AS$ and $\phi(n) = n + AS$ so that

$$\begin{aligned} \phi(m) + \phi(n) &= (m + AS) \oplus (n + AS) \\ &= (m + n) + AS \\ &= \phi(m + n). \\ \phi(m)\phi(n) &= (m + AS) \odot (n + AS) \\ &= (mn) + AS \\ &= \phi(mn). \\ Ker\phi &= \emptyset. \\ Im\phi &= \{1 + AS, 2 + AS, 3 + AS, 4 + AS, \cdots\} = AR/AS. \end{aligned}$$

Remark 3.34. It is evident from Example 3.33 that the fundamental theorem of homomorphisms of the classical rings cannot hold in the classes of AntiRings.

4. **Conclusions**

We have in this paper introduced the concept of AntiRings with several examples. Specifically, certain types of AntiRings and their substructures were studied. It was shown that nonempty subsets of an AntiRing can be AntiRings with algebraic properties different from the parent AntiRing under the same binary operations. Also, we studied with several examples the concepts of AntiIdeals, AntiQuotientRings and AntiRingHomomorphisms. It was shown that an AntiQuotientRing of an AntiRing factored by an AntiIdeal can exhibit algebraic properties different from the algebraic properties of the AntiRing. We hope to study morphisms and AntiMorphisms of AntiSubrings and QuasiAntiSubrings of AntiRings and present further properties of different types of AntiRings in our future papers.

Funding: This research received no external funding.

Acknowledgments: The private discussions and suggestions of Professor Florentin Smarandache towards the improvement of this paper are acknowledged. The valuable comments and suggestions of all the anonymous reviewers are equally acknowledged.

Conflicts of Interest: The author declares no conflict of interest.

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Received: July 22, 2020 Accepted: Sept 18, 2020