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Matrix Games with Single-Valued Triangular Neutrosophic Numbers as Pay-offs

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Abstract. Game theory is commonly used in competitive situations because of its significance in decision-making. Different types of fuzzy sets can handle uncertainty in matrix games. Neutrosophic set theory plays a vital role in analyzing complexity, ambiguity, incompleteness, and inconsistency in real-world problems. This study develops a novel approach to solve neutrosophic matrix games using linear programming problems with single-valued triangular neutrosophic numbers as pay-offs. This paper establishes some theoretical aspects of game theory in a neutrosophic environment. A numerical example verifies the theoretical results using the traditional simplex approach to achieve the strategy and value of the game. The proposed work is useful to model and solve conflict situations in decision-making problems with partial knowledge as data in a simple manner.

Keywords: Matrix game; Neutrosophic set; Single valued triangular neutrosophic number; Neutrosophic matrix game

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1. Introduction

Real-world conflict scenarios are often investigated using game theory. It is difficult to collect the right data from decision-makers in today's situations. Fuzzy set theory is based on unreliable information and vagueness due to a lack of some pieces of information and accurate data. Previous research investigated complexity in game theory using fuzzy sets, intuitionistic fuzzy sets, and rough fuzzy sets. The concept of neutrosophic set theory in games is new at the moment, and it is a common research subject all over the world for dealing with competitive situations.

Neumann and Morgenstern [1] established the notion of game theory. Although, the classical game theory has exact data and factual information about the players. In uncertain situations, the notion of the fuzzy set theory proposed by Zadeh [2] is applied to many fields. Campus [5] introduced a model based on a linear programming approach to interpreting fuzzy matrix games. Sakawa and Nishizaki [6, 10] investigated max-min solution methods for multi-objective conflict resolution problems. Bector et al. [11, 12] determined the matrix games with fuzzy goals and fuzzy payoffs. The concept of dual linear programming approach employed by Vijay et al. [13]. Several researchers [14, 15, 25, 27, 37] developed fuzzy matrix games. To determine the uncertainty about non-membership degrees Atanassov [4, 8] inducted intuitionistic fuzzy set theory. Further, intuitionistic fuzzy concept applied by [16–19, 21–24, 26, 38] to study game-theoretic models using linear programming approach. After that, Intuitionistic fuzzy sets were extended to interval-valued intuitionistic fuzzy sets and hesitant fuzzy sets. Kumar and Garg [28] suggested the TOPSIS method under interval-valued intuitionistic fuzzy environment. Xue et al. [45] applied the Ambika method to determine the matrix games with hesitant fuzzy knowledge and investigated the counter-terrorism problem. A methodology based on the linear programming approach was applied to solve the matrix games with triangular dual hesitant fuzzy numbers as payoffs by Yang and Song [39].

The intuitionistic fuzzy sets can not successfully deal in the circumstances of good, unacceptable, and uncertain decision-making problems. Therefore a novel theory was necessary. Smarandache [7, 9] filled the gap and introduced the concept of neutrosophic set theory, which deals with incomplete, inconsistent, and indeterminate situations. Single valued neutrosophic sets as an extension of neutrosophic sets were presented by Wang et al. [20]. A de-neutrosophication idea for linear and non-linear generalized triangular neutrosophic numbers was performed by Chakraborty et al. [30]. The concept of neutrosophic set and number has been successfully applied by Abdel-basset et al. [31–33], and developed methods for sustainable supplier selection problems. [34–36] investigated decision making models based on neutrosophic sets. A similar study of neutrosophic sets and numbers was provided by Broumi et al. [29]. Khalil et al. [40] suggested a new idea for the single-valued neutrosophic fuzzy soft set. Neutrosophic soft, rough topology and its applications to multicriteria decision-making problems were proposed by Riaz et al. [41]. Based on the neutrosophic fuzzy approach, an economical production quantity model was suggested by De et al. [42] for imperfect production processes under game. Du et al. [44] in neutrosophic Z-numbers conditions investigated a multicriteria decision-making approach. In contemporary situations to handle the conflicting political circumstances, a neutrosophic model for non-cooperative games was inducted by Arias et al. [43] using single-valued triangular neutrosophic numbers. Bhaumik et al. [46] introduced a new ranking approach to solve bi-matrix games based on (α, β, γ) -cut set of a single-valued triangular neutrosophic number.

Game theory is widely used in competitive scenarios due to its importance in decision-making. In real-world problems, the concept of neutrosophic set theory is useful for analyzing complexity, uncertainty, incompleteness, and inconsistency. In matrix games with single-valued triangular neutrosophic numbers as pay-offs, we developed a novel approach focused on linear programming using de-neutrosophication as values and ambiguities. The standard simplex approach is used to accomplish the strategy and value of the game for the individual player by providing a numerical representation. The proposed work is capable of quickly resolving conflict situations in decision-making problems using partial information as data.

The main novelties of this work are pointed as:

- A new class of matrix game, namely neutrosophic matrix game, is defined under partial informative situations.
- A mathematical model of neutrosophic matrix game is developed.
- Values and ambiguities are derived for single-valued triangular neutrosophic numbers, and some new theorems are provided.
- The theoretical results are verified by a numerical example arising in conflict situations in decision-making problems with partial knowledge as data.

The research paper is designed as: Section 2 contains preliminaries and definitions. Values and ambiguities are determined in Section 3. Section 4 deals with value index and ambiguity index. Section 5 describes a mathematical model of a matrix game. A numerical example is demonstrated in Section 6. Section 7 concludes the results of the paper.

2. Preliminaries and definitions

In this section we recall some basic definitions and notations which are useful throughout the paper.

Definition 2.1. Let $X = \{X_1, X_2, X_3, \ldots, X_n\}$ be the universal set. A neutrosophic set $\tilde{A}$ in the universal set X, is characterized by its truth membership function $\mu_{\tilde{A}}$, indeterminacy membership function $\pi_{\tilde{A}}$ and falsity membership function $\nu_{\tilde{A}}$ which associates with $X_i \in X$ to a real number in the interval $[0,1]$ and defined as

$$\tilde{A} = \{\left\langle X_i, \mu_{\tilde{A}}(X_i), \pi_{\tilde{A}}(X_i), \nu_{\tilde{A}}(X_i)\right\rangle | X_i \in X\}. \tag{1}$$

Definition 2.2. A single valued triangular neutrosophic number defined on the set of real numbers is a neutrosophic set, denoted by $\tilde{A}^{TNN} = \langle(\xi, \eta, \zeta)\,; \sigma, \rho, \tau\rangle$ whose truth membership, indeterminacy membership and falsity membership functions respectively are given as follows:

$$\mu_{\tilde{A}^{TNN}}(x) = \begin{cases} \frac{(x-\xi)\sigma}{(\eta-\xi)} & ; \xi \leq x \leq \eta \\ \sigma & ; x = \eta \\ \frac{(\zeta-x)\sigma}{(\zeta-\eta)} & ; \eta \leq x \leq \zeta \\ 0 & ; otherwise \end{cases} \tag{2}$$

$$\pi_{\tilde{A}^{TNN}}(x) = \begin{cases} \frac{(\eta-x)+\rho(x-\xi)}{(\eta-\xi)} & ; \xi \leq x \leq \eta \\ \rho & ; x = \eta \\ \frac{(x-\eta)+\rho(\zeta-x)}{(\zeta-\eta)} & ; \eta \leq x \leq \zeta \\ 1 & ; otherwise \end{cases} \tag{3}$$

$$\nu_{\tilde{A}^{TNN}}(x) = \begin{cases} \frac{(\eta-x)+\tau(x-\xi)}{(\eta-\xi)} & ; \xi \leq x \leq \eta \\ \tau & ; x = \eta \\ \frac{(x-\eta)+\tau(\zeta-x)}{(\zeta-\eta)} & ; \eta \leq x \leq \zeta \\ 1 & ; otherwise \end{cases} \tag{4}$$

where $0 \leq \sigma \leq 1$, $0 \leq \rho \leq 1$, $0 \leq \tau \leq 1$ and $0 \leq \sigma + \rho + \tau \leq 3$. σ represents the maximum degree of truth membership, ρ represents the minimum degree of indeterminacy membership and τ represents the minimum degree of falsity membership.

Definition 2.3. Let $\tilde{A}^{TNN} = \langle (\xi_1, \eta_1, \zeta_1); \sigma_1, \rho_1, \tau_1 \rangle$ and $\tilde{B}^{TNN} = \langle (\xi_2, \eta_2, \zeta_2); \sigma_2, \rho_2, \tau_2 \rangle$ be two single valued triangular neutrosophic numbers and λ be a real number, then some arithmetical operations are stipulated as follows:

- Addition

$$\tilde{A}^{TNN} + \tilde{B}^{TNN} = \langle (\xi_1 + \xi_2, \eta_1 + \eta_2, \zeta_1 + \zeta_2); \min(\sigma_1, \sigma_2), \max(\rho_1, \rho_2), \max(\tau_1, \tau_2) \rangle . \tag{5}$$

- Symmetric Image

$$-\tilde{A}^{TNN} = \langle (-\zeta_1, -\eta_1, -\xi_1); \sigma_1, \rho_1, \tau_1 \rangle . \tag{6}$$

- Subtraction

$$\tilde{A}^{TNN} - \tilde{B}^{TNN} = \langle (\xi_1 - \zeta_2, \eta_1 - \eta_2, \zeta_1 - \xi_2); \min(\sigma_1, \sigma_2), \max(\rho_1, \rho_2), \max(\tau_1, \tau_2) \rangle . \tag{7}$$

- Multiplication

$$\tilde{A}^{TNN} \times \tilde{B}^{TNN} = \langle (\xi_1 \xi_2, \eta_1 \eta_2, \zeta_1 \zeta_2); \min(\sigma_1, \sigma_2), \max(\rho_1, \rho_2), \max(\tau_1, \tau_2) \rangle . \tag{8}$$

- Scalar Product

$$\lambda \tilde{A}^{TNN} = \begin{cases} \langle (\lambda\xi_1, \lambda\eta_1, \lambda\zeta_1); \sigma_1, \rho_1, \tau_1 \rangle & ; \lambda > 0 \\ \langle (\lambda\zeta_1, \lambda\eta_1, \lambda\xi_1); \sigma_1, \rho_1, \tau_1 \rangle & ; \lambda < 0 \end{cases} \tag{9}$$

Definition 2.4. The (α,β,γ)- cut of a single valued triangular neutrosophic number $\tilde{A}^{TNN} = \langle(\xi,\eta,\zeta);\sigma,\rho,\tau\rangle$ is a closed crisp interval of real numbers denoted by $\tilde{A}^{TNN}_{(\alpha,\beta,\gamma)}$ and defined as

$$\tilde{A}^{TNN}_{(\alpha,\beta,\gamma)} = \{x|\mu_{\tilde{A}^{TNN}}(x) \geq \alpha, \pi_{\tilde{A}^{TNN}}(x) \leq \beta, \nu_{\tilde{A}^{TNN}}(x) \leq \gamma\}. \tag{10}$$

where $0 \leq \alpha \leq \sigma$, $\rho \leq \beta \leq 1$, $\tau \leq \gamma \leq 1$ and $0 \leq \alpha+\beta+\gamma \leq 3$.

Definition 2.5. The α- cut of a single valued triangular neutrosophic number $\tilde{A}^{TNN} = \langle(\xi,\eta,\zeta);\sigma,\rho,\tau\rangle$ is a closed crisp interval of real numbers denoted by $\tilde{A}^{TNN}_{\alpha} = \left[L_{\tilde{A}^{TNN}_{\alpha}}, R_{\tilde{A}^{TNN}_{\alpha}}\right]$ and defined as

$$\begin{aligned}\tilde{A}^{TNN}_{\alpha} &= \{x|\mu_{\tilde{A}^{TNN}}(x) \geq \alpha\}\\ &= \left[L_{\tilde{A}^{TNN}_{\alpha}}, R_{\tilde{A}^{TNN}_{\alpha}}\right]\\ &= \left[\xi + \frac{\alpha(\eta-\xi)}{\sigma}, \zeta - \frac{\alpha(\zeta-\eta)}{\sigma}\right].\end{aligned} \tag{11}$$

Definition 2.6. The β- cut of a single valued triangular neutrosophic number $\tilde{A}^{TNN} = \langle(\xi,\eta,\zeta);\sigma,\rho,\tau\rangle$ is a closed crisp interval of real numbers denoted by $\tilde{A}^{TNN}_{\beta} = \left[L_{\tilde{A}^{TNN}_{\beta}}, R_{\tilde{A}^{TNN}_{\beta}}\right]$ and defined as

$$\begin{aligned}\tilde{A}^{TNN}_{\beta} &= \{x|\pi_{\tilde{A}^{TNN}}(x) \leq \beta\}\\ &= \left[L_{\tilde{A}^{TNN}_{\beta}}, R_{\tilde{A}^{TNN}_{\beta}}\right]\\ &= \left[\frac{(1-\beta)\eta + (\beta-\rho)\xi}{(1-\rho)}, \frac{(1-\beta)\eta + (\beta-\rho)\zeta}{(1-\rho)}\right].\end{aligned} \tag{12}$$

Definition 2.7. The γ- cut of a single valued triangular neutrosophic number $\tilde{A}^{TNN} = \langle(\xi,\eta,\zeta);\sigma,\rho,\tau\rangle$ is a closed crisp interval of real numbers denoted by $\tilde{A}^{TNN}_{\gamma} = \left[L_{\tilde{A}^{TNN}_{\gamma}}, R_{\tilde{A}^{TNN}_{\gamma}}\right]$ and defined as

$$\begin{aligned}\tilde{A}^{TNN}_{\gamma} &= \{x|\nu_{\tilde{A}^{TNN}}(x) \leq \gamma\}\\ &= \left[L_{\tilde{A}^{TNN}_{\gamma}}, R_{\tilde{A}^{TNN}_{\gamma}}\right]\\ &= \left[\frac{(1-\gamma)\eta + (\gamma-\tau)\xi}{(1-\tau)}, \frac{(1-\gamma)\eta + (\gamma-\tau)\zeta}{(1-\tau)}\right].\end{aligned} \tag{13}$$

Theorem 2.8. *Let* $\tilde{A}^{TNN} = \langle(\xi,\eta,\zeta);\sigma,\rho,\tau\rangle$ *be any single valued triangular neutrosophic number then for any* $\alpha \in [0,\sigma]$, $\beta \in [\rho,1]$ *and* $\gamma \in [\tau,1]$ *the following equality hold*

$$\tilde{A}^{TNN}_{(\alpha,\beta,\gamma)} = \tilde{A}^{TNN}_{\alpha} \cap \tilde{A}^{TNN}_{\beta} \cap \tilde{A}^{TNN}_{\gamma}.$$

3. Values and ambiguities for the membership functions of $\tilde{A}^{TNN}$

Let $\tilde{A}^{TNN}_{\alpha}$, $\tilde{A}^{TNN}_{\beta}$ and $\tilde{A}^{TNN}_{\gamma}$ be the α-cut, β-cut and γ-cut of a single valued triangular neutrosophic number $\tilde{A}^{TNN}$ respectively, then the values and ambiguities for different membership functions of $\tilde{A}^{TNN}$ are defined as follows:

Value of the true membership function:

$$V_{\mu}(\tilde{A}^{TNN}) = \int_{0}^{\sigma} \frac{\left(L_{\tilde{A}_{\alpha}^{TNN}} + R_{\tilde{A}_{\alpha}^{TNN}}\right)}{2} \phi\left(\alpha\right) d\alpha. \tag{14}$$

Value of the indeterminacy membership function:

$$V_{\pi}(\tilde{A}^{TNN}) = \int_{\rho}^{1} \frac{\left(L_{\tilde{A}_{\beta}^{TNN}} + R_{\tilde{A}_{\beta}^{TNN}}\right)}{2} \psi\left(\beta\right) d\beta. \tag{15}$$

Value of the falsity membership function:

$$V_{\nu}(\tilde{A}^{TNN}) = \int_{\tau}^{1} \frac{\left(L_{\tilde{A}_{\gamma}^{TNN}} + R_{\tilde{A}_{\gamma}^{TNN}}\right)}{2} \chi\left(\gamma\right) d\gamma. \tag{16}$$

Ambiguity of the true membership function:

$$Amb_{\mu}(\tilde{A}^{TNN}) = \int_{0}^{\sigma} \left(R_{\tilde{A}_{\alpha}^{TNN}} - L_{\tilde{A}_{\alpha}^{TNN}}\right) \phi\left(\alpha\right) d\alpha. \tag{17}$$

Ambiguity of the indeterminacy membership function:

$$Amb_{\pi}(\tilde{A}^{TNN}) = \int_{\rho}^{1} \left(R_{\tilde{A}_{\beta}^{TNN}} - L_{\tilde{A}_{\beta}^{TNN}}\right) \psi\left(\beta\right) d\beta. \tag{18}$$

Ambiguity of the falsity membership function:

$$Amb_{\nu}(\tilde{A}^{TNN}) = \int_{\tau}^{1} \left(R_{\tilde{A}_{\gamma}^{TNN}} - L_{\tilde{A}_{\gamma}^{TNN}}\right) \chi\left(\gamma\right) d\gamma. \tag{19}$$

Here $\phi\left(\alpha\right)$ is a nonnegative increasing function defined on $[0,\sigma]$ with $\phi\left(0\right) = 0$ and $\int_{0}^{\sigma} \phi\left(\alpha\right) d\alpha = \sigma$. $\psi\left(\beta\right)$ is a nonnegative decreasing function defined on $[\rho,1]$ with $\psi\left(1\right) = 0$ and $\int_{\rho}^{1} \psi\left(\beta\right) d\beta = 1-\rho$ and $\chi\left(\gamma\right)$ is a nonnegative decreasing function defined on $[\tau,1]$ with $\chi\left(1\right) = 0$ and $\int_{\tau}^{1} \chi\left(\gamma\right) d\gamma = 1-\tau$.

According to the equations (11), (14) and suitable nonnegative functions $\phi\left(\alpha\right)$, $\psi\left(\beta\right)$ and $\chi\left(\gamma\right)$ as $\phi\left(\alpha\right) = \frac{2\alpha}{\sigma}$, $\psi\left(\beta\right) = \frac{2(1-\beta)}{(1-\rho)}$ and $\chi\left(\gamma\right) = \frac{2(1-\gamma)}{(1-\tau)}$. Then the value of true membership function of $\tilde{A}^{TNN}$ is

$$\begin{aligned} V_{\mu}(\tilde{A}^{TNN}) = &\int_{0}^{\sigma} \frac{\alpha\left(\xi + \frac{\alpha(\eta-\xi)}{\sigma} + \zeta - \frac{\alpha(\zeta-\eta)}{\sigma}\right)}{\sigma} d\alpha \\ =&\frac{1}{\sigma^2} \int_{0}^{\sigma} \left[\sigma\left(\xi+\zeta\right) + \alpha\left(2\eta-\xi-\zeta\right)\right] \alpha d\alpha \\ =&\frac{1}{\sigma^2}\left[\frac{\sigma^3\left(\xi+\zeta\right)}{2} + \frac{\sigma^3\left(2\eta-\xi-\zeta\right)}{3}\right] \\ =&\frac{\left(\xi+4\eta+\zeta\right)\sigma}{6}. \end{aligned} \tag{20}$$

According to the equations (12), (15) and suitable nonnegative functions $\phi\left(\alpha\right)$, $\psi\left(\beta\right)$ and $\chi\left(\gamma\right)$ as $\phi\left(\alpha\right) = \frac{2\alpha}{\sigma}$, $\psi\left(\beta\right) = \frac{2(1-\beta)}{(1-\rho)}$ and $\chi\left(\gamma\right) = \frac{2(1-\gamma)}{(1-\tau)}$. Then the value of indeterminacy membership function of $\tilde{A}^{TNN}$ is

$$\begin{aligned}
V_{\pi}(\tilde{A}^{TNN}) =& \int_{\rho}^{1} \frac{(1-\beta)\left(\frac{(1-\beta)\eta+(\beta-\rho)\xi}{(1-\rho)} + \frac{(1-\beta)\eta+(\beta-\rho)\zeta}{(1-\rho)}\right)}{(1-\rho)} d\beta \\
=& \frac{1}{(1-\rho)^2} \int_{\rho}^{1} \left[2\eta(1-\beta) + (\beta-\rho)(\xi+\zeta)\right](1-\beta)\, d\beta \\
=& \frac{1}{(1-\rho)^2} \int_{\rho}^{1} \left[(1-\beta)^2(2\eta-\xi-\zeta) + (1-\rho)(\xi+\zeta)(1-\beta)\right] d\beta \\
=& \frac{1}{(1-\rho)^2} \left[\frac{(1-\rho)^3(2\eta-\xi-\zeta)}{3} + \frac{(1-\rho)^3(\xi+\zeta)}{2}\right] \\
=& \frac{(\xi+4\eta+\zeta)(1-\rho)}{6}.
\end{aligned} \tag{21}$$

According to the equations (13), (16) and suitable nonnegative functions $\phi(\alpha)$, $\psi(\beta)$ and $\chi(\gamma)$ as $\phi(\alpha) = \frac{2\alpha}{\sigma}$, $\psi(\beta) = \frac{2(1-\beta)}{(1-\rho)}$ and $\chi(\gamma) = \frac{2(1-\gamma)}{(1-\tau)}$. Then the value of falsity membership function of $\tilde{A}^{TNN}$ is

$$\begin{aligned}
V_{\nu}(\tilde{A}^{TNN}) =& \int_{\tau}^{1} \frac{(1-\gamma)\left(\frac{(1-\gamma)\eta+(\gamma-\tau)\xi}{(1-\tau)} + \frac{(1-\gamma)\eta+(\gamma-\tau)\zeta}{(1-\tau)}\right)}{(1-\tau)} d\gamma \\
=& \frac{1}{(1-\tau)^2} \int_{\tau}^{1} \left[2\eta(1-\gamma) + (\gamma-\tau)(\xi+\zeta)\right](1-\gamma)\, d\gamma \\
=& \frac{1}{(1-\tau)^2} \int_{\tau}^{1} \left[(1-\gamma)^2(2\eta-\xi-\zeta) + (1-\tau)(\xi+\zeta)(1-\gamma)\right] d\gamma \\
=& \frac{1}{(1-\tau)^2} \left[\frac{(1-\tau)^3(2\eta-\xi-\zeta)}{3} + \frac{(1-\tau)^3(\xi+\zeta)}{2}\right] \\
=& \frac{(\xi+4\eta+\zeta)(1-\tau)}{6}.
\end{aligned} \tag{22}$$

According to the equations (11), (17) and suitable nonnegative functions $\phi(\alpha)$, $\psi(\beta)$ and $\chi(\gamma)$ as $\phi(\alpha) = \frac{2\alpha}{\sigma}$, $\psi(\beta) = \frac{2(1-\beta)}{(1-\rho)}$ and $\chi(\gamma) = \frac{2(1-\gamma)}{(1-\tau)}$. Then the ambiguity of true membership function of $\tilde{A}^{TNN}$ is

$$\begin{aligned}
Amb_{\mu}(\tilde{A}^{TNN}) =& \int_{0}^{\sigma} \frac{2\alpha\left(\zeta - \frac{\alpha(\zeta-\eta)}{\sigma} - \xi - \frac{\alpha(\eta-\xi)}{\sigma} +\right)}{\sigma} d\alpha \\
=& \frac{1}{\sigma^2} \int_{0}^{\sigma} 2\alpha(\zeta-\xi)(\sigma-\alpha)\, d\alpha \\
=& \frac{2}{\sigma^2}(\zeta-\xi)\sigma^3\left(\frac{1}{2} - \frac{1}{3}\right) \\
=& \frac{(\zeta-\xi)\sigma}{3}.
\end{aligned} \tag{23}$$

According to the equations (12), (18) and suitable nonnegative functions $\phi(\alpha)$, $\psi(\beta)$ and $\chi(\gamma)$ as $\phi(\alpha) = \frac{2\alpha}{\sigma}$, $\psi(\beta) = \frac{2(1-\beta)}{(1-\rho)}$ and $\chi(\gamma) = \frac{2(1-\gamma)}{(1-\tau)}$. Then the ambiguity of indeterminacy membership function of $\tilde{A}^{TNN}$ is

$$\begin{aligned} Amb_{\pi}(\tilde{A}^{TNN}) &= \int_{\rho}^{1} \frac{2\left(1-\beta\right)\left[\frac{(1-\beta)\eta+(\beta-\rho)\zeta}{(1-\rho)} - \frac{(1-\beta)\eta+(\beta-\rho)\xi}{(1-\rho)}\right]}{(1-\rho)} d\beta \\ &= \frac{2}{\left(1-\rho\right)^{2}} \int_{\rho}^{1} \left(\beta-\rho\right)\left(\zeta-\xi\right)\left(1-\beta\right) d\beta \\ &= \frac{2\left(\zeta-\xi\right)}{\left(1-\rho\right)^{2}} \int_{\rho}^{1} \left[\left(1-\rho\right)\left(1-\beta\right) - \left(1-\beta\right)^{2}\right] d\beta \\ &= \frac{2\left(\zeta-\xi\right)\left(1-\rho\right)^{3}}{\left(1-\rho\right)^{2}} \left(\frac{1}{2}-\frac{1}{3}\right) \\ &= \frac{\left(\zeta-\xi\right)\left(1-\rho\right)}{3}. \end{aligned} \tag{24}$$

According to the equations (13), (19) and suitable nonnegative functions $\phi\left(\alpha\right)$, $\psi\left(\beta\right)$ and $\chi\left(\gamma\right)$ as $\phi\left(\alpha\right) = \frac{2\alpha}{\sigma}$, $\psi\left(\beta\right) = \frac{2(1-\beta)}{(1-\rho)}$ and $\chi\left(\gamma\right) = \frac{2(1-\gamma)}{(1-\tau)}$. Then the ambiguity of falsity membership function of $\tilde{A}^{TNN}$ is

$$\begin{aligned} Amb_{\nu}(\tilde{A}^{TNN}) &= \int_{\tau}^{1} \frac{2\left(1-\gamma\right)\left[\frac{(1-\gamma)\eta+(\gamma-\tau)\zeta}{(1-\tau)} - \frac{(1-\gamma)\eta+(\gamma-\tau)\xi}{(1-\tau)}\right]}{(1-\tau)} d\gamma \\ &= \frac{2}{\left(1-\tau\right)^{2}} \int_{\tau}^{1} \left(\gamma-\tau\right)\left(\zeta-\xi\right)\left(1-\gamma\right) d\gamma \\ &= \frac{2\left(\zeta-\xi\right)}{\left(1-\tau\right)^{2}} \int_{\tau}^{1} \left[\left(1-\tau\right)\left(1-\gamma\right) - \left(1-\gamma\right)^{2}\right] d\gamma \\ &= \frac{2\left(\zeta-\xi\right)\left(1-\tau\right)^{3}}{\left(1-\tau\right)^{2}} \left(\frac{1}{2}-\frac{1}{3}\right) \\ &= \frac{\left(\zeta-\xi\right)\left(1-\tau\right)}{3}. \end{aligned} \tag{25}$$

Theorem 3.1. *Let* $\tilde{A}^{TNN} = \langle\left(\xi_1, \eta_1, \zeta_1\right); \sigma_1, \rho_1, \tau_1\rangle$ *and* $\tilde{B}^{TNN} = \langle\left(\xi_2, \eta_2, \zeta_2\right); \sigma_2, \rho_2, \tau_2\rangle$ *be two single valued triangular neutrosophic numbers with* $\sigma_1 = \sigma_2$*,* $\rho_1 = \rho_2$ *and* $\tau_1 = \tau_2$ *then the following equalities hold*

(1) $V_{\mu}\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}\right) = V_{\mu}\left(\tilde{A}^{TNN}\right) + V_{\mu}\left(\tilde{B}^{TNN}\right)$.
(2) $V_{\pi}\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}\right) = V_{\pi}\left(\tilde{A}^{TNN}\right) + V_{\pi}\left(\tilde{B}^{TNN}\right)$.
(3) $V_{\nu}\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}\right) = V_{\nu}\left(\tilde{A}^{TNN}\right) + V_{\nu}\left(\tilde{B}^{TNN}\right)$.

Proof. Using Definition 2.3 and according to the given statement, we have

$$\tilde{A}^{TNN} + \tilde{B}^{TNN} = \langle\left(\xi_1+\xi_2, \eta_1+\eta_2, \zeta_1+\zeta_2\right); \sigma_1, \rho_1, \tau_1\rangle.$$

Thus by the definition of value of true membership function, we obtain

$$\begin{aligned} V_{\mu}\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}\right) &= \frac{\left[\left(\xi_1+\xi_2\right) + 4\left(\eta_1+\eta_2\right) + \left(\zeta_1+\zeta_2\right)\right]\sigma_1}{6} \\ &= \frac{\left(\xi_1+4\eta_1+\zeta_1\right)\sigma_1}{6} + \frac{\left(\xi_2+4\eta_2+\zeta_2\right)\sigma_2}{6} \end{aligned}$$

Therefore,

$$V_\mu\left(\tilde{A}^{TNN}+\tilde{B}^{TNN}\right)=V_\mu\left(\tilde{A}^{TNN}\right)+V_\mu\left(\tilde{B}^{TNN}\right).$$

In a similar manner the remaining results of the theorem can also be proved. □

Theorem 3.2. *Let* $\tilde{A}^{TNN}=\langle(\xi_1,\eta_1,\zeta_1);\sigma_1,\rho_1,\tau_1\rangle$ *and* $\tilde{B}^{TNN}=\langle(\xi_2,\eta_2,\zeta_2);\sigma_2,\rho_2,\tau_2\rangle$ *be two single valued triangular neutrosophic numbers with* $\sigma_1=\sigma_2$, $\rho_1=\rho_2$ *and* $\tau_1=\tau_2$ *then the following equalities hold*

(1) $Amb_\mu\left(\tilde{A}^{TNN}+\tilde{B}^{TNN}\right)=Amb_\mu\left(\tilde{A}^{TNN}\right)+Amb_\mu\left(\tilde{B}^{TNN}\right)$.
(2) $Amb_\pi\left(\tilde{A}^{TNN}+\tilde{B}^{TNN}\right)=Amb_\pi\left(\tilde{A}^{TNN}\right)+Amb_\pi\left(\tilde{B}^{TNN}\right)$.
(3) $Amb_\nu\left(\tilde{A}^{TNN}+\tilde{B}^{TNN}\right)=Amb_\nu\left(\tilde{A}^{TNN}\right)+Amb_\nu\left(\tilde{B}^{TNN}\right)$.

Proof. Using Definition 2.3 and according to the given statement, we have

$$\tilde{A}^{TNN}+\tilde{B}^{TNN}=\langle(\xi_1+\xi_2,\eta_1+\eta_2,\zeta_1+\zeta_2);\sigma_1,\rho_1,\tau_1\rangle.$$

Thus by the definition of ambiguity of true membership function, we obtain

$$\begin{aligned}Amb_\mu\left(\tilde{A}^{TNN}+\tilde{B}^{TNN}\right)&=\frac{[(\zeta_1+\zeta_2)-(\xi_1+\xi_2)]\,\sigma_1}{3}\\&=\frac{(\zeta_1-\xi_1)\,\sigma_1}{3}+\frac{(\zeta_2-\xi_2)\,\sigma_2}{3}\end{aligned}$$

Therefore,

$$Amb_\mu\left(\tilde{A}^{TNN}+\tilde{B}^{TNN}\right)=Amb_\mu\left(\tilde{A}^{TNN}\right)+Amb_\mu\left(\tilde{B}^{TNN}\right).$$

In a similar manner the remaining results of the theorem can also be proved. □

4. Value index and ambiguity index of $\tilde{A}^{TNN}$

Let $\tilde{A}^{TNN}=\langle(\xi,\eta,\zeta);\sigma,\rho,\tau\rangle$ be a single valued triangular neutrosophic number then the value index and the ambiguity index for $\tilde{A}^{TNN}$ are defined as follows:

(1) Value Index:

$$V\left(\tilde{A}^{TNN},\lambda\right)=V_\pi(\tilde{A}^{TNN})+V_\mu(\tilde{A}^{TNN})+\lambda\left[V_\nu(\tilde{A}^{TNN})-V_\mu(\tilde{A}^{TNN})\right]. \tag{26}$$

(2) Ambiguity Index:

$$A\left(\tilde{A}^{TNN},\lambda\right)=Amb_\pi(\tilde{A}^{TNN})+Amb_\nu(\tilde{A}^{TNN})-\lambda\left[Amb_\nu(\tilde{A}^{TNN})-Amb_\mu(\tilde{A}^{TNN})\right]. \tag{27}$$

which are continuous non decreasing and non increasing functions of the parameter λ respectively. Here $\lambda\in[0,1]$ represents the decision maker's preference informations. $\lambda\in[0,\frac{1}{2})$ represents that the decision maker prefer uncertainty or negative feeling. $\lambda\in(\frac{1}{2},1]$ represents that the decision maker prefer certainty or positive feeling. $\lambda=\frac{1}{2}$ represents that the decision maker is indifferent between positive and negative feeling.

Theorem 4.1. *Let* $\tilde{A}^{TNN} = \langle(\xi_1, \eta_1, \zeta_1); \sigma_1, \rho_1, \tau_1\rangle$ *and* $\tilde{B}^{TNN} = \langle(\xi_2, \eta_2, \zeta_2); \sigma_2, \rho_2, \tau_2\rangle$ *be two single valued triangular neutrosophic numbers with* $\sigma_1 = \sigma_2$*,* $\rho_1 = \rho_2$ *and* $\tau_1 = \tau_2$ *then the following equalities hold*

(1) $V\left(\tilde{A}^{TNFN} + \tilde{B}^{TNN}, \lambda\right) = V\left(\tilde{A}^{TNN}, \lambda\right) + V\left(\tilde{B}^{TNN}, \lambda\right)$.

(2) $A\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}, \lambda\right) = A\left(\tilde{A}^{TNN}, \lambda\right) + A\left(\tilde{B}^{TNN}, \lambda\right)$.

Proof. According to equation (26), we can write

$$\begin{aligned} V\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}, \lambda\right) =& V_\pi\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}\right) + V_\mu\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}\right) \\ &+ \lambda\left[V_\nu\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}\right) - V_\mu\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}\right)\right]. \end{aligned}$$

Using Theorem 3.2, we obtain

$$\begin{aligned} V\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}, \lambda\right) =& V_\pi\left(\tilde{A}^{TNN}\right) + V_\pi\left(\tilde{B}^{TNN}\right) + V_\mu\left(\tilde{A}^{TNN}\right) + V_\mu\left(\tilde{B}^{TNN}\right) \\ &+ \lambda\left[V_\nu\left(\tilde{A}^{TNN}\right) + V_\nu\left(\tilde{B}^{TNN}\right) - V_\mu\left(\tilde{A}^{TNN}\right) - V_\mu\left(\tilde{B}^{TNN}\right)\right] \\ =& \left[V_\pi\left(\tilde{A}^{TNN}\right) + V_\mu\left(\tilde{A}^{TNN}\right) + \lambda\left(V_\nu\left(\tilde{A}^{TNN}\right) - V_\mu\left(\tilde{A}^{TNN}\right)\right)\right] \\ &+ \left[V_\pi\left(\tilde{B}^{TNN}\right) + V_\mu\left(\tilde{B}^{TNN}\right) + \lambda\left(V_\nu\left(\tilde{B}^{TNN}\right) - V_\mu\left(\tilde{B}^{TNN}\right)\right)\right] \\ =& V\left(\tilde{A}^{TNN}, \lambda\right) + V\left(\tilde{B}^{TNN}, \lambda\right). \end{aligned}$$

This completes the first part of the theorem.

Now, according to equation (27), we can write

$$\begin{aligned} A\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}, \lambda\right) =& Amb_\pi\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}\right) + Amb_\nu\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}\right) \\ &- \lambda\left[Amb_\nu\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}\right) - Amb_\mu\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}\right)\right]. \end{aligned}$$

Using Theorem 3.3, we have

$$\begin{aligned} A\left(\tilde{A}^{TNN} + \tilde{B}^{TNN}, \lambda\right) =& Amb_\pi\left(\tilde{A}^{TNN}\right) + Amb_\pi\left(\tilde{B}^{TNN}\right) + Amb_\nu\left(\tilde{A}^{TNN}\right) + Amb_\nu\left(\tilde{B}^{TNN}\right) \\ &- \lambda\left[Amb_\nu\left(\tilde{A}^{TNN}\right) + Amb_\nu\left(\tilde{B}^{TNN}\right) - Amb_\mu\left(\tilde{A}^{TNN}\right) - Amb_\mu\left(\tilde{B}^{TNN}\right)\right] \\ =& \left[Amb_\pi\left(\tilde{A}^{TNN}\right) + Amb_\nu\left(\tilde{A}^{TNN}\right) - \lambda\left(Amb_\nu\left(\tilde{A}^{TNN}\right) - Amb_\mu\left(\tilde{A}^{TNN}\right)\right)\right] \\ &+ \left[Amb_\pi\left(\tilde{B}^{TNN}\right) + Amb_\nu\left(\tilde{B}^{TNN}\right) - \lambda\left(Amb_\nu\left(\tilde{B}^{TNN}\right) - Amb_\mu\left(\tilde{B}^{TNN}\right)\right)\right] \\ =& A\left(\tilde{A}^{TNN}, \lambda\right) + A\left(\tilde{B}^{TNN}, \lambda\right). \end{aligned}$$

This completes the second part of the theorem. □

Remark 4.2. It is easily seen that the value index and the ambiguity index are nonnegative for a nonnegative single valued triangular neutrosophic number, i.e., $V\left(\tilde{A}^{TNN}, \lambda\right) \geq 0$

and $A\left(\tilde{A}^{TNN},\lambda\right)\geq 0$. Also the value index should be maximized and the ambiguity index should be minimized, furthermore as a summarized result we can easily seen that the relations $\max V\left(\tilde{A}^{TNN},\lambda\right)=V_\pi(\tilde{A}^{TNN})+V_\nu(\tilde{A}^{TNN})$ and $\min A\left(\tilde{A}^{TNN},\lambda\right)=Amb_\pi(\tilde{A}^{TNN})+Amb_\mu(\tilde{A}^{TNN})$ holds.

Remark 4.3. If we assume that the decision maker is indifferent between the certainty and uncertainty, i.e., $\lambda=\frac{1}{2}$, then the value index and ambiguity index are given by

$$V\left(\tilde{A}^{TNN},\frac{1}{2}\right)\equiv V\left(\tilde{A}^{TNN}\right)=V_\pi(\tilde{A}^{TNN})+\frac{1}{2}\left[V_\mu(\tilde{A}^{TNN})+V_\nu(\tilde{A}^{TNN})\right]. \tag{28}$$

$$A\left(\tilde{A}^{TNN},\frac{1}{2}\right)\equiv A\left(\tilde{A}^{TNN}\right)=Amb_\pi(\tilde{A}^{TNN})+\frac{1}{2}\left[Amb_\mu(\tilde{A}^{TNN})+Amb_\nu(\tilde{A}^{TNN})\right]. \tag{29}$$

Theorem 4.4. *Let $\tilde{A}^{TNN}=\langle(\xi_1,\eta_1,\zeta_1);\sigma_1,\rho_1,\tau_1\rangle$ and $\tilde{B}^{TNN}=\langle(\xi_2,\eta_2,\zeta_2);\sigma_2,\rho_2,\tau_2\rangle$ be two single valued triangular neutrosophic numbers and λ_1, λ_2 be any two nonnegative real numbers then the following equalities hold*

(1) $V_\mu\left(\lambda_1\tilde{A}^{TNN}+\lambda_2\tilde{B}^{TNN}\right)=\min(\sigma_1,\sigma_2)\left[\lambda_1\frac{V_\mu(\tilde{A}^{TNN})}{\sigma_1}+\lambda_2\frac{V_\mu(\tilde{B}^{TNN})}{\sigma_2}\right]$.

(2) $V_\pi\left(\lambda_1\tilde{A}^{TNN}+\lambda_2\tilde{B}^{TNN}\right)=\min(1-\rho_1,1-\rho_2)\left[\lambda_1\frac{V_\pi(\tilde{A}^{TNN})}{(1-\rho_1)}+\lambda_2\frac{V_\pi(\tilde{B}^{TNN})}{(1-\rho_2)}\right]$.

(3) $V_\nu\left(\lambda_1\tilde{A}^{TNN}+\lambda_2\tilde{B}^{TNN}\right)=\min(1-\tau_1,1-\tau_2)\left[\lambda_1\frac{V_\nu(\tilde{A}^{TNN})}{(1-\tau_1)}+\lambda_2\frac{V_\nu(\tilde{B}^{TNN})}{(1-\tau_2)}\right]$.

(4) $Amb_\mu\left(\lambda_1\tilde{A}^{TNN}+\lambda_2\tilde{B}^{TNN}\right)=\min(\sigma_1,\sigma_2)\left[\lambda_1\frac{Amb_\mu(\tilde{A}^{TNN})}{\sigma_1}+\lambda_2\frac{Amb_\mu(\tilde{B}^{TNN})}{\sigma_2}\right]$.

(5) $Amb_\pi\left(\lambda_1\tilde{A}^{TNN}+\lambda_2\tilde{B}^{TNN}\right)$
$=\min(1-\rho_1,1-\rho_2)\left[\lambda_1\frac{Amb_\pi(\tilde{A}^{TNN})}{(1-\rho_1)}+\lambda_2\frac{Amb_\pi(\tilde{B}^{TNN})}{(1-\rho_2)}\right]$.

(6) $Amb_\nu\left(\lambda_1\tilde{A}^{TNN}+\lambda_2\tilde{B}^{TNN}\right)$
$=\min(1-\tau_1,1-\tau_2)\left[\lambda_1\frac{Amb_\nu(\tilde{A}^{TNN})}{(1-\tau_1)}+\lambda_2\frac{Amb_\nu(\tilde{B}^{TNN})}{(1-\tau_2)}\right]$.

Proof. The above results can be easily proven by using Definition 2.3 and the equations (20) to (25). □

4.1. *De-neutrosophication*

Let $N(R)$ be the set of all single valued triangular neutrosophic numbers defined on the set of real numbers, then a linear de-neutrosophication function $F:N(R)\to R$ for single valued triangular neutrosophic numbers in terms of value index and ambiguity index can be defined as follows

$$F\left(\tilde{A}^{TNN}\right)=V\left(\tilde{A}^{TNN}\right)-A\left(\tilde{A}^{TNN}\right). \tag{30}$$

5. Mathematical model of a matrix game

A two person zero sum matrix game played by a maximizing player (as player I) and a minimizing player (as player II) having the pure strategy $i = \{1, 2, \ldots, m\}$ and $j = \{1, 2, \ldots, n\}$ respectively is denoted by $[a_{ij}]_{m\times n}$. Here a_{ij} is the pay-off value for the player I and its opposite is the pay-off value for player II, when they choose the strategies i and j respectively such that there exists the saddle point of the game. If the matrix game $[a_{ij}]_{m\times n}$ has no saddle point, i.e., $\max_i\{\min_j\{a_{ij}\}\} \neq \min_j\{\max_i\{a_{ij}\}\}$, then to solve such matrix games we adopt the mixed strategy sets S_1 and S_2 for the player I and II respectively, as $S_1 = \{X = (x_1, x_2, \ldots, x_m) \in R^m : x_i \geq 0, \forall i = 1, 2, \ldots m, and \sum_{i=1}^{m} x_i = 1\}$ and $S_2 = \{Y = (y_1, y_2, \ldots, y_n) \in R^n : y_j \geq 0, \forall j = 1, 2, \ldots n, and \sum_{j=1}^{n} y_j = 1\}$.

5.1. *Mathematical model of a neutrosophic matrix game*

The maximin and minimax principal for matrix games states that the player I choose such a strategy which maximize his minimum expected gain and the player II choose such a strategy which minimizes his maximum expected loss, thus for the neutrosophic matrix game, we have as

For player I

$$\begin{cases} \max\limits_{x_i}\{\min\{\sum_{i=1}^{m} \tilde{a}_{i1}^{TNN} x_i, \sum_{i=1}^{m} \tilde{a}_{i2}^{TNN} x_i, \ldots, \sum_{i=1}^{m} \tilde{a}_{in}^{TNN} x_i\}\} \\ \text{s.t.}, \sum_{i=1}^{m} x_i = 1 \\ \text{and}; x_i \geq 0, \forall i = 1, 2, \ldots, m \end{cases} \tag{31}$$

For player II

$$\begin{cases} \min\limits_{y_j}\{\max\{\sum_{j=1}^{n} \tilde{a}_{1j}^{TNN} y_j, \sum_{j=1}^{n} \tilde{a}_{2j}^{TNN} y_j, \ldots, \sum_{j=1}^{n} \tilde{a}_{mj}^{TNN} y_j\}\} \\ \text{s.t.}, \sum_{j=1}^{n} y_j = 1 \\ \text{and}; y_j \geq 0, \forall j = 1, 2, \ldots, n \end{cases} \tag{32}$$

Now, let $\min\{\sum_{i=1}^{m} \tilde{a}_{i1}^{TNN} x_i, \sum_{i=1}^{m} \tilde{a}_{i2}^{TNN} x_i, \ldots, \sum_{i=1}^{m} \tilde{a}_{in}^{TNN} x_i\} = \tilde{u}^{TNN}$ be the expected minimum gain for player I and $\max\{\sum_{j=1}^{n} \tilde{a}_{1j}^{TNN} y_j, \sum_{j=1}^{n} \tilde{a}_{2j}^{TNN} y_j, \ldots, \sum_{j=1}^{n} \tilde{a}_{mj}^{TNN} y_j\} = \tilde{v}^{TNN}$ be the expected maximum loss for player II. Then the problems (31) and (32) can be written as

For player I

$$\begin{cases} \max \tilde{u}^{TNN} \\ \text{s.t.}, \sum_{i=1}^{m} \tilde{a}_{i1}^{TNN} x_i \succeq \tilde{u}^{TNN} \\ \sum_{i=1}^{m} \tilde{a}_{i2}^{TNN} x_i \succeq \tilde{u}^{TNN} \\ \ldots \\ \sum_{i=1}^{m} \tilde{a}_{in}^{TNN} x_i \succeq \tilde{u}^{TNN} \\ \sum_{i=1}^{m} x_i = 1 \\ \text{and}; x_i \geq 0, \forall i = 1, 2, \ldots, m \end{cases} \tag{33}$$

For player II

$$\begin{cases} \min \tilde{v}^{TNN} \\ \text{s.t.}, \sum_{j=1}^{n} \tilde{a}_{1j}^{TNN} y_j \preceq \tilde{v}^{TNN} \\ \sum_{j=1}^{n} \tilde{a}_{2j}^{TNN} y_j \preceq \tilde{v}^{TNN} \\ \dots \\ \sum_{j=1}^{n} \tilde{a}_{mj}^{TNN} y_j \preceq \tilde{v}^{TNN} \\ \sum_{j=1}^{n} y_j = 1 \\ \text{and}; y_j \geq 0, \forall j = 1, 2, \dots, n \end{cases} \tag{34}$$

Here $\tilde{u}^{TNN} = \langle (u_1, u_2, u_3) ; \sigma, \rho, \tau \rangle$ and $\tilde{v}^{TNN} = \langle (v_1, v_2, v_3) ; \sigma', \rho', \tau' \rangle$ are the single valued triangular neutrosophic numbers as expected minimum gain and expected maximum loss respectively. And $\succeq$ and $\preceq$ denotes the neutrosophic versions of the order relation $\geq$ and $\leq$ on the set of real numbers and has linguistic interpretation as 'essentially greater than or equal' and 'essentially less than or equal' respectively. The problems (33) and (34) are known as the neutrosophic linear programming problems for the player I and II respectively and can be written in the standard form as

For player I

$$\begin{cases} \max \tilde{u}^{TNN} \\ \text{s.t.}, \sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i \succeq \tilde{u}^{TNN}, \forall j = 1, 2, \dots, n \\ \sum_{i=1}^{m} x_i = 1 \\ \text{and}; x_i \geq 0, \forall i = 1, 2, \dots, m \end{cases} \tag{35}$$

For player II

$$\begin{cases} \min \tilde{v}^{TNN} \\ \text{s.t.}, \sum_{j=1}^{n} \tilde{a}_{ij}^{TNN} y_j \preceq \tilde{v}^{TNN}, \forall i = 1, 2, \dots, m \\ \sum_{j=1}^{n} y_j = 1 \\ \text{and}; y_j \geq 0, \forall j = 1, 2, \dots, n \end{cases} \tag{36}$$

Now, utilizing the de-neutrosophication function $F : N(R) \rightarrow R$ defined by the equation (30), the above neutrosophic linear programming problems (35) and (36) can be transformed into the crisp linear programming problems for the player I and II respectively as follows

For player I

$$\begin{cases} \max F\left(\tilde{u}^{TNN}\right) \\ \text{s.t.}, F\left(\sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i\right) \geq F\left(\tilde{u}^{TNN}\right), \forall j = 1, 2, \dots, n \\ \sum_{i=1}^{m} x_i = 1 \\ \text{and}; x_i \geq 0, \forall i = 1, 2, \dots, m \end{cases} \tag{37}$$

For Player II

$$\begin{cases} \min F\left(\tilde{v}^{TNN}\right) \\ \text{s.t.}, F\left(\sum_{j=1}^{n} \tilde{a}_{ij}^{TNN} y_j\right) \le F\left(\tilde{v}^{TNN}\right), \forall i=1,2,\ldots,m \\ \sum_{j=1}^{n} y_j = 1 \\ \text{and}; y_j \ge 0, \forall j=1,2,\ldots,n \end{cases} \tag{38}$$

Using equations (28) to (30) the above crisp linear programming problems (37) and (38) for player I and II respectively, can be written as

For player I

$$\begin{cases} \max V_\pi(\tilde{u}^{TNN}) + \frac{1}{2}\left[V_\mu(\tilde{u}^{TNN}) + V_\nu(\tilde{u}^{TNN})\right] - Amb_\pi(\tilde{u}^{TNN}) \\ -\frac{1}{2}\left[Amb_\mu(\tilde{u}^{TNN}) + Amb_\nu(\tilde{u}^{TNN})\right] \\ \text{s.t.}, V_\pi(\sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i) + \frac{1}{2}\left[V_\mu(\sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i) + V_\nu(\sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i)\right] \\ -Amb_\pi(\sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i) - \frac{1}{2}\left[Amb_\mu(\sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i) + Amb_\nu(\sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i)\right] \\ \ge V_\pi(\tilde{u}^{TNN}) + \frac{1}{2}\left[V_\mu(\tilde{u}^{TNN}) + V_\nu(\tilde{u}^{TNN})\right] - Amb_\pi(\tilde{u}^{TNN}) \\ -\frac{1}{2}\left[Amb_\mu(\tilde{u}^{TNN}) + Amb_\nu(\tilde{u}^{TNN})\right], \forall j=1,2,\ldots,n \\ \sum_{i=1}^{m} x_i = 1 \\ \text{and}; x_i \ge 0, \forall i=1,2,\ldots,m \end{cases} \tag{39}$$

For player II

$$\begin{cases} \min V_\pi(\tilde{v}^{TNN}) + \frac{1}{2}\left[V_\mu(\tilde{v}^{TNN}) + V_\nu(\tilde{v}^{TNN})\right] - Amb_\pi(\tilde{v}^{TNN}) \\ -\frac{1}{2}\left[Amb_\mu(\tilde{v}^{TNN}) + Amb_\nu(\tilde{v}^{TNN})\right] \\ \text{s.t.}, V_\pi(\sum_{j=1}^{n} \tilde{a}_{ij}^{TNN} y_j) + \frac{1}{2}\left[V_\mu(\sum_{j=1}^{n} \tilde{a}_{ij}^{TNN} y_j) + V_\nu(\sum_{j=1}^{n} \tilde{a}_{ij}^{TNN} y_j)\right] \\ -Amb_\pi(\sum_{j=1}^{n} \tilde{a}_{ij}^{TNN} y_j) - \frac{1}{2}\left[Amb_\mu(\sum_{j=1}^{n} \tilde{a}_{ij}^{TNN} y_j) + Amb_\nu(\sum_{j=1}^{n} \tilde{a}_{ij}^{TNN} y_j)\right] \\ \le V_\pi(\tilde{v}^{TNN}) + \frac{1}{2}\left[V_\mu(\tilde{v}^{TNN}) + V_\nu(\tilde{v}^{TNN})\right] - Amb_\pi(\tilde{v}^{TNN}) \\ -\frac{1}{2}\left[Amb_\mu(\tilde{v}^{TNN}) + Amb_\nu(\tilde{v}^{TNN})\right], \forall i=1,2,\ldots,m \\ \sum_{j=1}^{n} y_j = 1 \\ \text{and}; y_j \ge 0, \forall j=1,2,\ldots,n \end{cases} \tag{40}$$

The problems (39) and (40) can also be reformed as

For player I

$$\begin{cases} \max V_\pi(\tilde{u}^{TNN}) - Amb_\pi(\tilde{u}^{TNN}) \\ +\frac{1}{2}\left[V_\mu(\tilde{u}^{TNN}) - Amb_\mu(\tilde{u}^{TNN}) + V_\nu(\tilde{u}^{TNN}) - Amb_\nu(\tilde{u}^{TNN})\right] \\ \text{s.t.}, V_\pi(\sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i) - Amb_\pi(\sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i) \\ +\frac{1}{2}\left[V_\mu(\sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i) - Amb_\mu(\sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i)\right] \\ +\frac{1}{2}\left[V_\nu(\sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i) - Amb_\nu(\sum_{i=1}^{m} \tilde{a}_{ij}^{TNN} x_i)\right] \ge V_\pi(\tilde{u}^{TNN}) - Amb_\pi(\tilde{u}^{TNN}) \\ +\frac{1}{2}\left[V_\mu(\tilde{u}^{TNN}) - Amb_\mu(\tilde{u}^{TNN}) + V_\nu(\tilde{u}^{TNN}) - Amb_\nu(\tilde{u}^{TNN})\right], \forall j=1,2,\ldots,n \\ \sum_{i=1}^{m} x_i = 1 \\ \text{and}; x_i \ge 0, \forall i=1,2,\ldots,m \end{cases} \tag{41}$$

For player II

$$\begin{cases} \min V_\pi(\tilde{v}^{TNN}) - Amb_\pi(\tilde{v}^{TNN}) \\ +\frac{1}{2}\left[V_\mu(\tilde{v}^{TNN}) - Amb_\mu(\tilde{v}^{TNN}) + V_\nu(\tilde{v}^{TNN}) - Amb_\nu(\tilde{v}^{TNN})\right] \\ \text{s.t.}, V_\pi(\sum_{j=1}^{n}\tilde{a}_{ij}^{TNN}y_j) - Amb_\pi(\sum_{j=1}^{n}\tilde{a}_{ij}^{TNN}y_j) \\ +\frac{1}{2}\left[V_\mu(\sum_{j=1}^{n}\tilde{a}_{ij}^{TNN}y_j) - Amb_\mu(\sum_{j=1}^{n}\tilde{a}_{ij}^{TNN}y_j)\right] \\ +\frac{1}{2}\left[V_\nu(\sum_{j=1}^{n}\tilde{a}_{ij}^{TNN}y_j) - Amb_\nu(\sum_{j=1}^{n}\tilde{a}_{ij}^{TNN}y_j)\right] \le V_\pi(\tilde{v}^{TNN}) - Amb_\pi(\tilde{v}^{TNN}) \\ +\frac{1}{2}\left[V_\mu(\tilde{v}^{TNN}) - Amb_\mu(\tilde{v}^{TNN}) + V_\nu(\tilde{v}^{TNN}) - Amb_\nu(\tilde{v}^{TNN})\right], \forall i = 1,2,\ldots,m \\ \sum_{j=1}^{n} y_j = 1 \\ \text{and}; y_j \ge 0, \forall j = 1,2,\ldots,n \end{cases} \tag{42}$$

The problems (41) and (42) further can be written in the following manner by using the expected minimum gain and expected maximum loss $\tilde{u}^{TNN} = \langle (u_1, u_2, u_3); \sigma, \rho, \tau\rangle$ and $\tilde{v}^{TNN} = \langle (v_1, v_2, v_3); \sigma', \rho', \tau'\rangle$ as

For player I

$$\begin{cases} \max \frac{(u_1+4u_2+u_3)(1-\rho)}{6} - \frac{(u_3-u_1)(1-\rho)}{3} \\ +\frac{1}{2}\left[\frac{(u_1+4u_2+u_3)\sigma}{6} - \frac{(u_3-u_1)\sigma}{3} + \frac{(u_1+4u_2+u_3)(1-\tau)}{6} - \frac{(u_3-u_1)(1-\tau)}{3}\right] \\ \text{s.t.}, \min\limits_{i}(1-\rho_{ij})\left(\sum_{i=1}^{m}\frac{V_\pi(\tilde{a}_{ij}^{TNN})x_i}{(1-\rho_{ij})} - \sum_{i=1}^{m}\frac{Amb_\pi(\tilde{a}_{ij}^{TNN})x_i}{(1-\rho_{ij})}\right) \\ +\frac{\min\limits_{i}(\sigma_{ij})}{2}\left(\sum_{i=1}^{m}\frac{V_\mu(\tilde{a}_{ij}^{TNN})x_i}{(\sigma_{ij})} - \sum_{i=1}^{m}\frac{Amb_\mu(\tilde{a}_{ij}^{TNN})x_i}{(\sigma_{ij})}\right) \\ +\frac{\min\limits_{i}(1-\tau_{ij})}{2}\left(\sum_{i=1}^{m}\frac{V_\nu(\tilde{a}_{ij}^{TNN})x_i}{(1-\tau_{ij})} - \sum_{i=1}^{m}\frac{Amb_\nu(\tilde{a}_{ij}^{TNN})x_i}{(1-\tau_{ij})}\right) \ge \frac{(u_1+4u_2+u_3)(1-\rho)}{6} - \frac{(u_3-u_1)(1-\rho)}{3} \\ +\frac{1}{2}\left[\frac{(u_1+4u_2+u_3)\sigma}{6} - \frac{(u_3-u_1)\sigma}{3} + \frac{(u_1+4u_2+u_3)(1-\tau)}{6} - \frac{(u_3-u_1)(1-\tau)}{3}\right], \forall j = 1,2,\ldots,n \\ \sum_{i=1}^{m} x_i = 1 \\ \text{and}; x_i \ge 0, \forall i = 1,2,\ldots,m \end{cases} \tag{43}$$

For player II

$$\begin{cases} \min \frac{(v_1+4v_2+v_3)(1-\rho')}{6} - \frac{(v_3-v_1)(1-\rho')}{3} \\ +\frac{1}{2}\left[\frac{(v_1+4v_2+v_3)\sigma'}{6} - \frac{(v_3-v_1)\sigma'}{3} + \frac{(v_1+4v_2+v_3)(1-\tau')}{6} - \frac{(v_3-v_1)(1-\tau')}{3}\right] \\ \text{s.t.}, \min\limits_{j}(1-\rho_{ij}')\left(\sum_{j=1}^{n}\frac{V_\pi(\tilde{a}_{ij}^{TNN})y_j}{(1-\rho_{ij}')} - \sum_{j=1}^{n}\frac{Amb_\pi(\tilde{a}_{ij}^{TNN})y_j}{(1-\rho_{ij}')}\right) \\ +\frac{\min\limits_{j}(\sigma_{ij}')}{2}\left(\sum_{j=1}^{n}\frac{V_\mu(\tilde{a}_{ij}^{TNN})y_j}{(\sigma_{ij}')} - \sum_{j=1}^{n}\frac{Amb_\mu(\tilde{a}_{ij}^{TNN})y_j}{(\sigma_{ij}')}\right) \\ +\frac{\min\limits_{j}(1-\tau_{ij}')}{2}\left(\sum_{j=1}^{n}\frac{V_\nu(\tilde{a}_{ij}^{TNN})y_j}{(1-\tau_{ij}')} - \sum_{j=1}^{n}\frac{Amb_\nu(\tilde{a}_{ij}^{TNN})y_j}{(1-\tau_{ij}')}\right) \le \frac{(v_1+4v_2+v_3)(1-\rho')}{6} - \frac{(v_3-v_1)(1-\rho')}{3} \\ +\frac{1}{2}\left[\frac{(v_1+4v_2+v_3)\sigma'}{6} - \frac{(v_3-v_1)\sigma'}{3} + \frac{(v_1+4v_2+v_3)(1-\tau')}{6} - \frac{(v_3-v_1)(1-\tau')}{3}\right], \forall i = 1,2,\ldots,m \\ \sum_{j=1}^{n} y_j = 1 \\ \text{and}; y_j \ge 0, \forall j = 1,2,\ldots,n \end{cases} \tag{44}$$

For convenience, let

$$\frac{(u_1+4u_2+u_3)\,\sigma}{6}-\frac{(u_3-u_1)\,\sigma}{3}=L_1 \tag{45}$$

$$\frac{(u_1+4u_2+u_3)\,(1-\rho)}{6}-\frac{(u_3-u_1)\,(1-\rho)}{3}=M_1 \tag{46}$$

$$\frac{(u_1+4u_2+u_3)\,(1-\tau)}{6}-\frac{(u_3-u_1)\,(1-\tau)}{3}=N_1 \tag{47}$$

$$\frac{(v_1+4v_2+v_3)\,\sigma'}{6}-\frac{(v_3-v_1)\,\sigma'}{3}=L_2 \tag{48}$$

$$\frac{(v_1+4v_2+v_3)\,(1-\rho_{ij}{'})}{6}-\frac{(v_3-v_1)\,(1-\rho_{ij}{'})}{3}=M_2 \tag{49}$$

$$\frac{(v_1+4v_2+v_3)\,(1-\tau_{ij}{'})}{6}-\frac{(v_3-v_1)\,(1-\tau_{ij}{'})}{3}=N_2 \tag{50}$$

Then the problems (43) and (44) reduces as

For player I

$$\begin{cases} \max \frac{1}{2}L_1+M_1+\frac{1}{2}N_1 \\ \text{s.t.}, \min\limits_i (1-\rho_{ij})\left(\sum_{i=1}^m \frac{V_\pi(\tilde{a}_{ij}^{TNN})x_i}{(1-\rho_{ij})}-\sum_{i=1}^m \frac{Amb_\pi(\tilde{a}_{ij}^{TNN})x_i}{(1-\rho_{ij})}\right) \\ +\frac{\min\limits_i(\sigma_{ij})}{2}\left(\sum_{i=1}^m \frac{V_\mu(\tilde{a}_{ij}^{TNN})x_i}{(\sigma_{ij})}-\sum_{i=1}^m \frac{Amb_\mu(\tilde{a}_{ij}^{TNN})x_i}{(\sigma_{ij})}\right) \\ +\frac{\min\limits_i(1-\tau_{ij})}{2}\left(\sum_{i=1}^m \frac{V_\nu(\tilde{a}_{ij}^{TNN})x_i}{(1-\tau_{ij})}-\sum_{i=1}^m \frac{Amb_\nu(\tilde{a}_{ij}^{TNN})x_i}{(1-\tau_{ij})}\right) \geq \frac{1}{2}L_1+M_1+\frac{1}{2}N_1, \forall j=1,2,\ldots,n \\ \sum_{i=1}^m x_i=1 \\ \text{and}; x_i\geq 0, \forall i=1,2,\ldots,m \end{cases} \tag{51}$$

For player II

$$\begin{cases} \min \frac{1}{2}L_2+M_2+\frac{1}{2}N_2 \\ \text{s.t.}, \min\limits_j (1-\rho_{ij}{'})\left(\sum_{j=1}^n \frac{V_\pi(\tilde{a}_{ij}^{TNN})y_j}{(1-\rho_{ij}{'})}-\sum_{j=1}^n \frac{Amb_\pi(\tilde{a}_{ij}^{TNN})y_j}{(1-\rho_{ij}{'})}\right) \\ +\frac{\min\limits_j(\sigma_{ij}{'})}{2}\left(\sum_{j=1}^n \frac{V_\mu(\tilde{a}_{ij}^{TNN})y_j}{(\sigma_{ij}{'})}-\sum_{j=1}^n \frac{Amb_\mu(\tilde{a}_{ij}^{TNN})y_j}{(\sigma_{ij}{'})}\right) \\ +\frac{\min\limits_j(1-\tau_{ij}{'})}{2}\left(\sum_{j=1}^n \frac{V_\nu(\tilde{a}_{ij}^{TNN})y_j}{(1-\tau_{ij}{'})}-\sum_{j=1}^n \frac{Amb_\nu(\tilde{a}_{ij}^{TNN})y_j}{(1-\tau_{ij}{'})}\right) \leq \frac{1}{2}L_2+M_2+\frac{1}{2}N_2, \forall i=1,2,\ldots,m \\ \sum_{j=1}^n y_j=1 \\ \text{and}; y_j\geq 0, \forall j=1,2,\ldots,n \end{cases} \tag{52}$$

6. **Numerical example**

Consider a two person zero sum matrix game whose pay-offS are single valued triangular neutrosophic numbers as follows

$$\tilde{A}^{TNN}=\begin{bmatrix} \tilde{a}_{11}^{TNN} & \tilde{a}_{12}^{TNN} \\ \tilde{a}_{21}^{TNN} & \tilde{a}_{22}^{TNN} \end{bmatrix}$$

Here $\tilde{a}_{11}^{TNN}=\langle(175,180,190);0.6,0.4,0.2\rangle$, $\tilde{a}_{12}^{TNN}=\langle(150,156,158);0.6,0.35,0.1\rangle$, $\tilde{a}_{21}^{TNN}=\langle(80,90,100);0.9,0.5,0.1\rangle$, $\tilde{a}_{22}^{TNN}=\langle(175,180,190);0.6,0.4,0.2\rangle$. According to the problems (51) and (52) as explained in the mathematical procedure for a two person zero sum neutrosophic matrix game, we have

For player I

$$\begin{cases}
\max \frac{1}{2}L_1+M_1+\frac{1}{2}N_1\\
\text{s.t.}, \min\left(1-\rho_{11},1-\rho_{21}\right)\left(\frac{V_\pi(\tilde{a}_{11}^{TNN})x_1}{(1-\rho_{11})}+\frac{V_\pi(\tilde{a}_{21}^{TNN})x_2}{(1-\rho_{21})}-\frac{Amb_\pi(\tilde{a}_{11}^{TNN})x_1}{(1-\rho_{11})}-\frac{Amb_\pi(\tilde{a}_{21}^{TNN})x_2}{(1-\rho_{21})}\right)\\
+\frac{\min(\sigma_{11},\sigma_{21})}{2}\left(\frac{V_\mu(\tilde{a}_{11}^{TNN})x_1}{(\sigma_{11})}+\frac{V_\mu(\tilde{a}_{21}^{TNN})x_2}{(\sigma_{21})}-\frac{Amb_\mu(\tilde{a}_{11}^{TNN})x_1}{(\sigma_{11})}-\frac{Amb_\mu(\tilde{a}_{21}^{TNN})x_2}{(\sigma_{21})}\right)\\
+\frac{\min(1-\tau_{11},1-\tau_{21})}{2}\left(\frac{V_\nu(\tilde{a}_{11}^{TNN})x_1}{(1-\tau_{11})}+\frac{V_\nu(\tilde{a}_{21}^{TNN})x_2}{(1-\tau_{21})}-\frac{Amb_\nu(\tilde{a}_{11}^{TNN})x_1}{(1-\tau_{11})}-\frac{Amb_\nu(\tilde{a}_{21}^{TNN})x_2}{(1-\tau_{21})}\right)\\
\geq \frac{1}{2}L_1+M_1+\frac{1}{2}N_1\\
\min\left(1-\rho_{12},1-\rho_{22}\right)\left(\frac{V_\pi(\tilde{a}_{12}^{TNN})x_1}{(1-\rho_{12})}+\frac{V_\pi(\tilde{a}_{22}^{TNN})x_2}{(1-\rho_{22})}-\frac{Amb_\pi(\tilde{a}_{12}^{TNN})x_1}{(1-\rho_{12})}-\frac{Amb_\pi(\tilde{a}_{22}^{TNN})x_2}{(1-\rho_{22})}\right)\\
+\frac{\min(\sigma_{12},\sigma_{22})}{2}\left(\frac{V_\mu(\tilde{a}_{12}^{TNN})x_1}{(\sigma_{12})}+\frac{V_\mu(\tilde{a}_{22}^{TNN})x_2}{(\sigma_{22})}-\frac{Amb_\mu(\tilde{a}_{12}^{TNN})x_1}{(\sigma_{12})}-\frac{Amb_\mu(\tilde{a}_{22}^{TNN})x_2}{(\sigma_{22})}\right)\\
+\frac{\min(1-\tau_{12},1-\tau_{22})}{2}\left(\frac{V_\nu(\tilde{a}_{12}^{TNN})x_1}{(1-\tau_{12})}+\frac{V_\nu(\tilde{a}_{22}^{TNN})x_2}{(1-\tau_{22})}-\frac{Amb_\nu(\tilde{a}_{12}^{TNN})x_1}{(1-\tau_{12})}-\frac{Amb_\nu(\tilde{a}_{22}^{TNN})x_2}{(1-\tau_{22})}\right)\\
\geq \frac{1}{2}L_1+M_1+\frac{1}{2}N_1\\
x_1+x_2=1\\
\text{and}; x_1,x_2,L_1,M_1,N_1\geq 0
\end{cases}$$

For player II

$$\begin{cases}
\min \frac{1}{2}L_2+M_2+\frac{1}{2}N_2\\
\text{s.t.}, \min\left(1-\rho_{11}',1-\rho_{12}'\right)\left(\frac{V_\pi(\tilde{a}_{11}^{TNN})y_1}{(1-\rho_{11}')}+\frac{V_\pi(\tilde{a}_{12}^{TNN})y_2}{(1-\rho_{12}')}-\frac{Amb_\pi(\tilde{a}_{11}^{TNN})y_1}{(1-\rho_{11}')}-\frac{Amb_\pi(\tilde{a}_{12}^{TNN})y_2}{(1-\rho_{12}')}\right)\\
+\frac{\min(\sigma_{11}',\sigma_{12}')}{2}\left(\frac{V_\mu(\tilde{a}_{11}^{TNN})y_1}{(\sigma_{11}')}+\frac{V_\mu(\tilde{a}_{12}^{TNN})y_2}{(\sigma_{12}')}-\frac{Amb_\mu(\tilde{a}_{11}^{TNN})y_1}{(\sigma_{11}')}-\frac{Amb_\mu(\tilde{a}_{12}^{TNN})y_2}{(\sigma_{12}')}\right)\\
+\frac{\min(1-\tau_{11}',1-\tau_{12}')}{2}\left(\frac{V_\nu(\tilde{a}_{11}^{TNN})y_1}{(1-\tau_{11}')}+\frac{V_\nu(\tilde{a}_{12}^{TNN})y_2}{(1-\tau_{12}')}-\frac{Amb_\nu(\tilde{a}_{11}^{TNN})y_1}{(1-\tau_{11}')}-\frac{Amb_\nu(\tilde{a}_{12}^{TNN})y_2}{(1-\tau_{12}')}\right)\\
\leq \frac{1}{2}L_2+M_2+\frac{1}{2}N_2\\
\min\left(1-\rho_{21}',1-\rho_{22}'\right)\left(\frac{V_\pi(\tilde{a}_{21}^{TNN})y_1}{(1-\rho_{21}')}+\frac{V_\pi(\tilde{a}_{22}^{TNN})y_2}{(1-\rho_{22}')}-\frac{Amb_\pi(\tilde{a}_{21}^{TNN})y_1}{(1-\rho_{21}')}-\frac{Amb_\pi(\tilde{a}_{22}^{TNN})y_2}{(1-\rho_{22}')}\right)\\
+\frac{\min(\sigma_{21}',\sigma_{22}')}{2}\left(\frac{V_\mu(\tilde{a}_{21}^{TNN})y_1}{(\sigma_{21}')}+\frac{V_\mu(\tilde{a}_{22}^{TNN})y_2}{(\sigma_{22}')}-\frac{Amb_\mu(\tilde{a}_{21}^{TNN})y_1}{(\sigma_{21}')}-\frac{Amb_\mu(\tilde{a}_{22}^{TNN})y_2}{(\sigma_{22}')}\right)\\
+\frac{\min(1-\tau_{21}',1-\tau_{22}')}{2}\left(\frac{V_\nu(\tilde{a}_{21}^{TNN})y_1}{(1-\tau_{21}')}+\frac{V_\nu(\tilde{a}_{22}^{TNN})y_2}{(1-\tau_{22}')}-\frac{Amb_\nu(\tilde{a}_{21}^{TNN})y_1}{(1-\tau_{21}')}-\frac{Amb_\nu(\tilde{a}_{22}^{TNN})y_2}{(1-\tau_{22}')}\right)\\
\leq \frac{1}{2}L_2+M_2+\frac{1}{2}N_2\\
y_1+y_2=1\\
\text{and}; y_1,y_2,L_2,M_2,N_2\geq 0
\end{cases}$$

Hence, we obtain

For player I

$$\begin{cases}
\max \frac{1}{2}L_1+M_1+\frac{1}{2}N_1\\
\text{s.t.}, 211x_1+100x_2\geq 0.5L_1+M_1+0.5N_1\\
198.4667x_1+228.5833x_2\geq 0.5L_1+M_1+0.5N_1\\
x_1+x_2=1\\
\text{and}; x_1,x_2,L_1,M_1,N_1\geq 0
\end{cases}$$

For player II

$$\begin{cases} \min \frac{1}{2}L_2 + M_2 + \frac{1}{2}N_2 \\ \text{s.t.}, 228.5833y_1 + 198.4667y_2 \leq 0.5L_2 + M_2 + 0.5N_2 \\ 100y_1 + 211y_2 \leq 0.5L_2 + M_2 + 0.5N_2 \\ y_1 + y_2 = 1 \\ \text{and}; y_1, y_2, L_2, M_2, N_2 \geq 0 \end{cases}$$

Using standard simplex method we obtain that the optimal strategies for the player I and II are $X = (0.9112, 0.0888)^T$ and $Y = (0.0888, 0.9112)^T$ respectively. The minimum expected gain as single valued triangular neutrosophic number for player I is $\langle (152.2200, 158.1312, 160.8416)\,; 0.6, 0.4, 0.2\rangle$, while the maximum expected loss as single valued triangular neutrosophic number for player II is $\langle (166.5640, 172.0080, 180.0080)\,; 0.6, 0.5, 0.2\rangle$, when they choose the optimal strategies as $X = (0.9112, 0.0888)^T$ and $Y = (0.0888, 0.9112)^T$ respectively.

7. Conclusion

We have investigated a two-person zero-sum matrix game in a neutrosophic environment with single-valued triangular neutrosophic numbers as pay-offs. A ranking or de-neutrosophication, based on value and ambiguity index using α- cut, β- cut, and γ- cut is developed. A pair of neutrosophic linear programming problems estimated by the max-min approach of optimality of the two-person zero-sum matrix game is converted into another pair of crisp linear programming problems. Strategies and values of the matrix game are obtained by providing a numerical example.

The primary results of this study are pointed as:

- The relative properties and cut sets are developed for single-valued triangular neutrosophic numbers.
- Expressions for values and ambiguities are derived for single-valued triangular neutrosophic numbers.
- Related theorems for value and ambiguity indices are stated and proved.
- De-neutrosophication concept based on value and ambiguity index is derived.
- Established a mathematical model corresponding to neutrosophic matrix game.
- A numerical example is provided and verified to illustrate the theoretical establishments.

In the future, we can extend the recommended method for different types of neutrosophic numbers as an interval-valued neutrosophic number, bipolar neutrosophic number, and single-valued trapezoidal neutrosophic numbers.

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