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n-Refined Neutrosophic Modules

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Abstract: This paper introduces the concept of n-refined neutrosophic module as a new generalization of neutrosophic modules and refined neutrosophic modules respectively and as a new algebraic application of n-refined neutrosophic set. It studies elementary properties of these modules. Also, This work discusses some corresponding concepts such as weak/strong n-refined neutrosophic modules, n-refined neutrosophic homomorphisms, and kernels.

Keywords: n-Refined weak neutrosophic module, n-Refined strong neutrosophic module, n-Refined neutrosophic homomorphism.

1. Introduction

In 1980s the international movement called paradoxism, based on contradictions in science and literature, was founded by Smarandache, who then extended it to neutrosophy, based on contradictions and their neutrals. [30]

Neutrosophy as a new branch of philosophy studies origin, nature, and indeterminacies, it was founded by F. Smarandache and became a useful tool in algebraic structures. Many neutrosophic algebraic structures were defined and studied such as neutrosophic groups, neutrosophic rings, neutrosophic vector spaces, and neutrosophic modules [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15]. In 2013 Smarandache proposed a new idea, when he extended the neutrosophic set to refined [n-valued] neutrosophic set, i.e. the truth value T is refined/split into types of sub-truths such as (T_1, T_2, …,) similarly indeterminacy I is refined/split into types of sub-indeterminacies (I_1, I_2, …,) and the falsehood F is refined/split into sub-falsehood (F_1, F_2,..,) [17,18].
Recently, there are increasing efforts to study the neutrosophic generalized structures and spaces such as refined neutrosophic modules, spaces, equations, and rings [5,14,21,22,23,24]. Smarandache et.al introduced the concept of n-refined neutrosophic ring [20], and n-refined neutrosophic vector space [19] by using n-refined neutrosophic set concept. Also, neutrosophic sets played an important role in applied science such as health care, industry, and optimization [25,26,27,28].
In this paper we give a new concept based on n-refined neutrosophic set, where we define and study the concept of n-refined neutrosophic modules, submodules, and homomorphisms as a generalization of similar concepts in the case of neutrosophic and refined neutrosophic modules [13,14]. Also, we discuss some elementary properties.

For our purpose we use multiplication operation (defined in [20]) between indeterminacies

$I_1, I_2, \dots, I_n$ as follows:

$I_m I_s = I_{\min(m,s)}$.

All rings considered through this paper are commutative.

2. Preliminaries

Definition 2.1: [20]

Let (R,+,×) be a ring and $I_k; 1 \le k \le n$ be n indeterminacies. We define

R_n(I)={$a_0 + a_1 I + \dots + a_n I_n \,;\, a_i \in R$} to be n-refined neutrosophic ring.

Definition 2.2: [20]

(a) Let R_n(I) be an n-refined neutrosophic ring and P = $\sum_{i=0}^{n} P_i I_i$ = { $a_0 + a_1 I + \dots + a_n I_n : a_i \in P_i$},

where P_i is a subset of R, we define P to be an AH-subring if P_i is a subring of R for all .

AHS-subring is defined by the condition $P_i = P_j$ for all i, j.

(b) P is an AH-ideal if P_i is an two sides ideal of R for all i, the AHS-ideal is defined by the condition

$P_i = P_j$ for all i, j.

(c) The AH-ideal P is said to be null if $P_i = R \text{ or } P_i = \{0\}$ for all i.

Definition 2.3 :[10]

Let (V , + , .) be a vector space over the field K then (V(I) , + ,.) is called a weak neutrosophic vector space over the field K , and it is called a strong neutrosophic vector space if it is a vector space over the neutrosophic field K(I).

Definition 2.4: [13]

Let (M,+,.) be a module over the ring R then (M(I),+,.) is called a weak neutrosophic module over the ring R, and it is called a strong neutrosophic module if it is a module over the neutrosophic ring R(I).

Elements of M(I) have the form $x + yI; x, y \in M$, i.e M(I) can be written as $M(I) = M + MI$.

Definition 2.5: [13]

Let M(I) be a strong neutrosophic module over the neutrosophic ring R(I) and W(I) be a non empty set of M(I), then W(I) is called a strong neutrosophic submodule if W(I) itself is a strong neutrosophic module.

Definition 2.6: [13]

Let U(I) and W(I) be two strong neutrosophic submodules of M(I) and let $f: U(I) \to W(I)$, we say that f is a neutrosophic vector space homomorphism if

(a) $f(I) = I$.

(b) f is a module homomorphism.

3. Main concepts and results

Definition 3.1 :

Let (M,+,.) be a module over the ring R, we say that

$M_n(I) = M + MI_1 + \cdots + MI_n = \{x_0 + x_1 I_1 + \cdots + x_n I_n; x_i \in M\}$ is a weak n-refined neutrosophic module over the ring R. Elements of $M_n(I)$ are called n-refined neutrosophic vectors, elements of R are called scalars.

If we take scalars from the n-refined neutrosophic ring $R_n(I)$, we say that $M_n(I)$ is a strong n-refined neutrosophic module over the n-refined neutrosophic ring $M_n(I)$. Elements of $M_n(I)$ are called n-refined neutrosophic scalars.

Remark 3.2:

If we take n=1 we get the classical neutrosophic module.

Addition on $M_n(I)$ is defined as:

$\sum_{i=0}^{n} a_i I_i + \sum_{i=0}^{n} b_i I_i = \sum_{i=0}^{n} (a_i + b_i) I_i$.

Multiplication by a scalar $m \in R$ is defined as:

$m.\sum_{i=0}^{n} a_i I_i = \sum_{i=0}^{n} (m.a_i) I_i$.

Multiplication by an n-refined neutrosophic scalar $m = \sum_{i=0}^{n} m_i I_i \in R_n(I)$ is defined as:

$\sum_{i=0}^{n} m_i I_i \,.\sum_{i=0}^{n} a_i I_i = \sum_{i=0}^{n} (m_i.a_i) I_i I_j$.

Where $a_i \in M, m_i \in R, I_i I_j = I_{\min(i,j)}$.

Theorem 3.3 :

Let (M,+,.) be a module over the ring R. Then a weak n-refined neutrosophic module $M_n(I)$ is a module over the ring R. A strong n-refined neutrosophic module is a module over the n-refined neutrosophic ring R_n(I).

Proof:

It is similar to that of Theorem 5 in [9].

Example 3.4:

Let $M = Z_2$ be the finite module of integers modulo 2 over itself, we have:

(a) The corresponding weak 2-refined neutrosophic module over the ring Z_2 is

$M_n(I) = \{0,1, I_1, I_2, I_1 + I_2, 1 + I_1 + I_2, 1 + I_1, 1 + I_2\}$.

Definition 3.5:

Let $M_n(I)$ be a weak n-refined neutrosophic module over the ring R, a nonempty subset $W_n(I)$ is called a weak n-refined neutrosophic module of $M_n(I)$ if $W_n(I)$ is a submodule of $M_n(I)$ itself.

Definition 3.6:

Let $M_n(I)$ be a strong n-refined neutrosophic module over the n-refined neutrosophic ring $R_n(I)$, a nonempty subset $W_n(I)$ is called a strong n-refined neutrosophic submodule of $M_n(I)$ if $W_n(I)$ is a submodule of $M_n(I)$ itself.

Theorem 3.7:

Let $M_n(I)$ be a weak n-refined neutrosophic module over the ring R, $W_n(I)$ be a nonempty subset of $M_n(I)$. Then $W_n(I)$ is a weak n-refined neutrosophic submodule if and only if:

$x + y \in W_n(I), m.x \in W_n(I)$ for all $x, y \in W_n(I), m \in R$.

Proof:

It holds directly from the fact that is $W_n(I)$ is a submodule of $M_n(I)$.

Theorem 3.8:

Let $M_n(I)$ be a strong n-refined neutrosophic module over the n-refined neutrosophic ring $R_n(I)$, $W_n(I)$ be a nonempty subset of $M_n(I)$. Then $W_n(I)$ is a strong n-refined neutrosophic submodule if and only if:

$x + y \in W_n(I), m.x \in W_n(I)$ for all $x, y \in W_n(I), m \in R_n(I)$.

Proof:

It holds directly from the fact that is $W_n(I)$ is a submodule of $M_n(I)$ over the n-refined neutrosophic ring $R_n(I)$.

Example 3.9:

$M = R^2$ is a module over the ring R, $W =< (0,1) >$ is a submodule of M,

$R_2^2(I) = \{(a,b) + (m,s)I_1 + (k,t)I_2; a,b,m,s,k,t \in R\}$ is the corresponding weak/strong 2-refined neutrosophic module.

$W_2(I) = \{a_0 + a_1I_1 + a_2I_2\} = \{(0,x) + (0,y)I_1 + (0,z)I_2; x,y,z \in R\}$ is a weak 2-refined neutrosophic submodule of the weak 2-refined neutrosophic module $R_2^2(I)$ over the ring R.

$W_2(I) = \{a_0 + a_1I_1 + a_2I_2\} = \{(0,x) + (0,y)I_1 + (0,z)I_2; x,y,z \in R\}$ is a strong 2-refined neutrosophic submodule of the strong 2-refined neutrosophic module $R_2^2(I)$ over the n-refined neutrosophic ring $R_2(I)$.

Definition 3.10:

Let $M_n(I)$ be a weak n-refined neutrosophic module over the ring R, x be an arbitrary element of $M_n(I)$, we say that x is a linear combination of $\{x_1, x_2, \dots, x_m\} \leq M_n(I)$ is

$x = a_1x_1 + a_2x_2 + \cdots + a_mx_m;\ a_i \in R, x_i \in M_n(I)$.

Example 3.11:

Consider the weak 2-refined neutrosophic module in Example 3.11,

$x = (0,2) + (1,3)I_1 \in \mathrm{R}_2^2(\mathrm{I})$, we have

$x = 2(0,1) + 1.(1,0)I_1 + 3(0,1)I_1$

i.e x is a linear combination of the set $\{(0,1), (1,0)I_1, (0,1)I_1\}$ over the ring R.

Definition 3.12:

Let $M_n(I)$ be a strong n-refined neutrosophic module over the n-refined neutrosophic ring $R_n(I)$, x be an arbitrary element of $M_n(I)$, we say that x is a linear combination of $\{x_1, x_2, \dots, x_m\} \le M_n(I)$ is

$x = a_1x_1 + a_2x_2 + \dots + a_mx_m$: $a_i \in R_n(I), x_i \in M_n(I)$.

Example 3.12:

Consider the strong 2-refined neutrosophic module

$R_2^2(I) = \{(a,b) + (m,s)I_1 + (k,t)I_2; a,b,m,s,k,t \in R\}$ over the 2-refined neutrosophic ring $R_2(I)$,

=$(1 + I_1)(1,1)I_2 + I_2(2,1)I_1 = (1,1)I_1 + (1,1)I_2 + (2,1)I_1 = x$, hence x is a linear $x = (3,2)I_1 + (1,1)I_2$

combination of the set

$\{(2,1)I_1, (1,1)I_2\}$

over the 2-refined neutrosophic ring $R_2(I)$.

Definition 3.15:

Let $X = \{x_1, \dots, x_m\}$ be a subset of a weak n-refined neutrosophic module $M_n(I)$ over the ring R, X is a weak linearly independent set if $\sum_{i=0}^{m} a_i x_i = 0 \; implies \; a_i = 0; \; a_i \in R$.

Definition 3.16:

Let $X = \{x_1, \dots, x_m\}$ be a subset of a strong n-refined neutrosophic module $M_n(I)$ over the n-refined neutrosophic ring $R_n(I)$, X is a weak linearly independent set if

$\sum_{i=0}^{m} a_i x_i = 0 \; implies \; a_i = 0; \; a_i \in R_n(I)$.

Definition 3.17:

Let $M_n(I), W_n(I)$ be two strong n-refined neutrosophic modules over the n-refined neutrosophic ring $R_n(I)$, let $f: M_n(I) \to U_n(I)$ be a well defined map. It is called a strong n-refined neutrosophic homomorphism if:

$f(a.x + b.y) = a.f(x) + b.f(y)$ for all $x, y \in M_n(I), a, b \in R_n(I)$.

A weak n-refined neutrosophic homomorphism can be defined as the same.

Definition 3.18:

Let $f: M_n(I) \to U_n(I)$ be a weak/strong n-refined neutrosophic homomorphism, we define:

(a) $Ker(f) = \{x \in M_n(I); f(x) = 0\}$.

(b) $Im(f) = \{y \in U_n(I);\ \exists x \in M_n(I) and\ y = f(x)\}$.

Theorem 3.19:

Let $f: M_n(I) \to U_n(I)$ be a weak n-refined neutrosophic homomorphism. Then

(a) $Ker(f)$ is a weak n-refined neutrosophic submodule of M_n(I).

(b) $Im(f)$ is a weak n-refined neutrosophic submodule of $U_n(I)$.

Proof:

(a) f is a module homomorphism since $M_n(I), U_n(I)$ are modules, hence $Ker(f)$ is a submodule of the module $M_n(I)$, thus $Ker(f)$ is a weak n-refined neutrosophic submodule of $M_n(I)$.

(b) Holds by similar argument.

Theorem 3.20:

Let $f: M_n(I) \to U_n(I)$ be a strong n-refined neutrosophic homomorphism. Then

(a) $Ker(f)$ is a strong n-refined neutrosophic submodule of M_n(I).

(b) $Im(f)$ is a strong n-refined neutrosophic submodule of $U_n(I)$.

Proof:

(a) f is a module homomorphism since $M_n(I), U_n(I)$ are modules over the n-refined neutrosophic ring $R_n(I)$, hence $Ker(f)$ is a submodule of the module $M_n(I)$, thus $Ker(f)$ is a strong n-refined neutrosophic submodule of $M_n(I)$.

(b) Holds by similar argument.

Theorem 3.21:

Let $f: M_n(I) \to U_n(I)$ be a strong n-refined neutrosophic homomorphism. Then

(a) $Ker(f)$ is a strong n-refined neutrosophic submodule of M_n(I).

(b) $Im(f)$ is a strong n-refined neutrosophic submodule of $U_n(I)$.

Proof:

(a) f is a module homomorphism since $M_n(I), U_n(I)$ are modules over the n-refined neutrosophic ring $R_n(I)$, hence $Ker(f)$ is a submodule of the module $M_n(I)$, thus $Ker(f)$ is a strong n-refined neutrosophic submodule of $M_n(I)$.

(b) Holds by similar argument.

Example 3.22:

Let $R_2^2(I) = \{x_0 + x_1I_1 + x_2I_2;\ x_0, x_1, x_2 \in R^2\}$, $R_2^3(I) = \{y_0 + y_1I_1 + y_2I_2;\ y_0, y_1, y_2 \in R^3\}$ be two weak 2-refined neutrosophic modules over the ring of real numbers R. Consider $f: R_2^2(I) \to R_2^3(I)$, where $f[(a,b) + (m,n)I_1 + (k,s)I_2] = (a,0,0) + (m.0.0)I_1 + (k,0,0)I_2$, f is a weak 2-refined neutrosophic homomorphism over the ring R.

$Ker(f) = \{(0,b) + (0,n)I_1 + (0,s)I_2; b,n,s \in R\}$.

$Im(f) = \{(a,0,0) + (m,0,0)I_1 + (k,0,0)I_2; a,m,k \in R\}$.

Example 3.23:

Let $W_2(I) =< (0,0,1)I_1 >= \{q.(0,0,a)I_1; a \in R, q \in R_2(I)\}$,

$U_2(I) =< (0,1,0)I_1 >= \{q.(0,a,0)I_1; a \in R; q \in R_2(I)\}$ be two strong 2-refined neutrosophic modules of the strong 2-refined neutrosophic module $R_2^3(I)$ over 2-refined neutrosophic ring R_2(I). Define

$f: W_2(I) \to U_2(I); f[q(0,0,a)I_1] = q(0,a,0)I_1;\ q \in R_2(I)$.

f is a strong 2-refined neutrosophic homomorphism:

Let $A = q_1(0,0,a)I_1, B = q_2(0,0,b)I_1 \in W_2(I);\ q_1, q_2 \in R_2(I)$, we have

$A + B = (q_1 + q_2)(0,0,a+b)I_1, f(A+B) = (q_1 + q_2).(0,a+b,0)I_1 = f(A) + f(B)$.

Let $m = c + dI_1 + eI_2 \in R_2(I)$ be a 2-refined neutrosophic scalar, we have

$m.A = c.q_1(0,0,a)I_1 + d.q_1(0,0,a)I_1I_1 + e.q_1(0,0,a)I_2I_1 = q_1(0,0,c.a + d.a + e.a)I_1$,

$f(m.A) = q_1(0,c.a + d.a + e.a,0)I_1 = m.f(A)$, hence f is a strong 2-refined neutrosophic homomorphism.

$Ker(f) = (0,0,0) + (0,0,0)I_1 + (0,0,0)I_2$.

$.Im(f) = U_2(I)$

5. Conclusion

In this paper, we continuo the efforts about defining and studying n-refined neutrosophic algebraic structures, where we have introduced the concept of weak/strong n-refined neutrosophic module. Also, some related concepts such as weak/strong n-refined neutrosophic submodule, n-refined neutrosophic homomorphism have been presented and studied.

Future research

Authors hope that some corresponding notions will be studied in future such as weak/strong n-refined neutrosophic basis of n-refined neutrosophic modules, and AH-submodules.

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