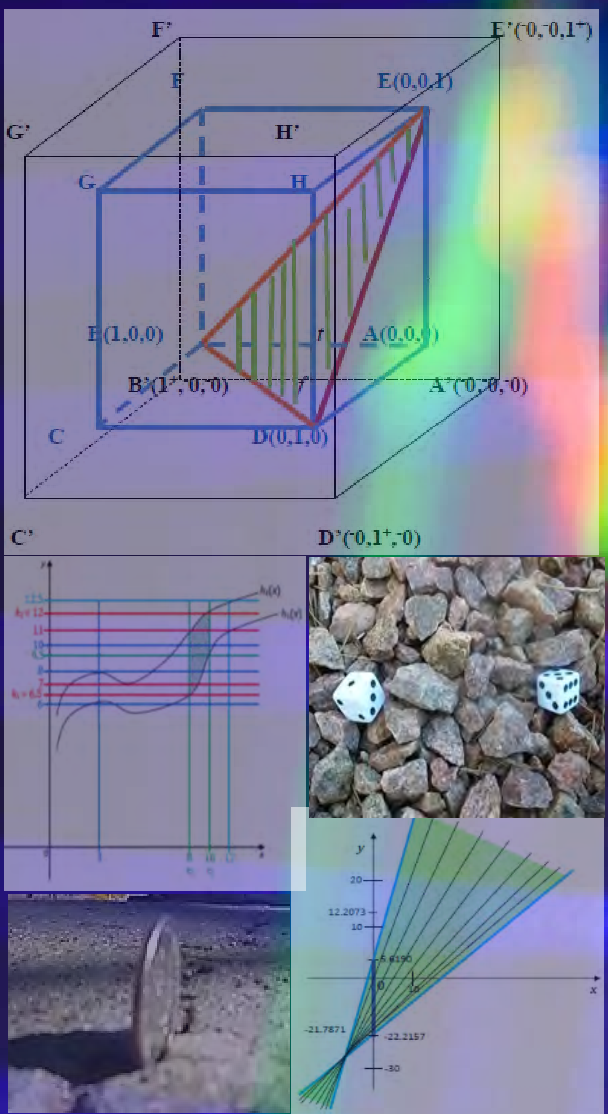
Volume 35, 2020

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering



<A> <neutA> <antiA>

Florentin Smarandache . Mohamed Abdel-Basset
Editors-in-Chief

ISSN 2331-6055 (Print)
ISSN 2331-608X (Online)

Neutrosophic Science
International Association (NSIA)

ISSN 2331-6055 (print) ISSN 2331-608X (online)

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering

University of New Mexico

ISSN 2331-6055 (print) ISSN 2331-608X (online)

University of New Mexico

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The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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ISSN 2331-6055 (print) ISSN 2331-608X (online)

University of New Mexico

NSS

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering

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March 20, 2019

Prof. Florentin Smarandache
Univ New Mexico, Gallup Campus

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ISSN 2331-6055 (print) NSS ISSN 2331-608X (online)
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ISSN 2331-6055 (print) NSS ISSN 2331-608X (online)
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COVID-X: Novel Health-Fog Framework Based on Neutrosophic Classifier for Confrontation Covid-19

Ibrahim Yasser [1,*], Abeer Twakol [2], A. A. Abd El-Khalek [3], Ahmed Samrah [4] and A. A. Salama [5]

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Abstract: The newly identified Coronavirus pneumonia, subsequently termed COVID-19, is highly transmittable and pathogenic with no clinically approved antiviral drug or vaccine available for treatment. Technological developments like edge computing, fog computing, Internet of Things (IoT), and Big Data have gained importance due to their robustness and ability to provide diverse response characteristics based on target application. In this paper, we present a novel Health-Fog framework universal system to automatically assist the early diagnosis, treatment, and preventive of people with COVID-19 in an efficient manner. Achieving an empirical of the proposed framework which mix between deep learning and Neutrosophic classifiers in the task of classifying COVID-19. There are some proposed applications based on the proposed COVID-X framework such as smart mask, smart medical suit, safe spacer, and Medical Mobile Learning (MML) will be presented. Computer-aided diagnosis systems could assist in the early detection of COVID-19 abnormalities and help to monitor the progression of the disease, potentially reduce mortality rates.

Keywords: Coronavirus Pneumonia; COVID-19; Intelligent Medical System; Fog Computing; Health-Fog; Neutrosophic; Deep Learning; Computer-Aided Diagnosis.

1. Introduction

The Coronavirus disease 2019-2020 pandemic (COVID-19) poses unprecedented challenges for governments and societies around the world. In addition to medical measures, non-pharmaceutical measures have proven to be critical for delaying and containing the spread of the virus. This includes (aggressive) testing and tracing, bans on large gatherings, school and university closures, international and domestic mobility restrictions and physical isolation, up to total lockdowns of regions and countries. However, effective and rapid decision-making during all stages of the pandemic requires reliable and timely data not only about infections, but also about human behavior, especially on mobility and physical co-presence of people [1]. There are growing privacy concerns about the ways governments use data to respond to the COVID-19 crisis. As new technologies emerge that aim to collect, disseminate and use data in order to support the fight against COVID-19, we need to ensure they respect ethical best practices. Even in times of crisis, we need to comply with data privacy regulations and ensure that the data is used ethically. One way to do that is to establish independent ethical committees or data trusts. Their role will be to create data governance mechanisms to find the balance between competing public interests, while protecting individual privacy. Examples of such rules include setting up clear guidelines on the purpose and timeline for the use of the data, defining clear processes for the access, processing and termination of use of personal data at the end of the crisis. Tracking a patient from symptoms, lab results and treatments can help a hospital understand how a disease is progressing through a community, how effective treatments are and what isn't working [2].

Technological developments like edge computing, fog computing, Internet of Things (IoT), and Big Data have gained importance due to their robustness and ability to provide diverse response characteristics based on target application. These emerging technologies provide storage, computation, and communication to edge devices, which facilitate and enhance mobility, privacy, security, low latency, and network bandwidth so that fog computing can perfectly match latency-sensitive or real-time applications [3]. Healthcare is one of the prominent application areas that requires accurate and real-time results, and people have introduced Fog Computing in this field which leads to a positive progress. With Fog computing, we bring the resources closer to the users thus decreasing the latency and thereby increasing the safety measure. Getting quicker results implies fast actions for critical COVID- 19 patients. But faster delivery of results is not enough as with such delicate data we cannot compromise with the accuracy of the result [4]. One way to obtain high accuracies is by using state-of-the-art analysis software typically those that employ deep learning and their variants trained on a large dataset. Deep learning techniques showed in the last years promising results to accomplish radiological tasks by automatic analyzing multimodal medical images [5]. Deep convolutional neural networks (DCNNs) are one of the powerful deep learning architectures and have been widely applied in many practical applications such as pattern recognition and image classification in an intuitive way [6]. DCNNs are able to handle four manners as follow [7]: 1) training the neural network weights on very large available datasets; 2) fine-tuning the network weights of a pre-trained DCNN based on small datasets; 3) Applying unsupervised pre-training to initialize the network weights before putting DCNN models in an application; and 4) using pre-trained DCNN is also called an off-the-shelf CNN being used as a feature extractor. Convolutional neural networks are sensitive to unknown noisy condition in the test phase and so their performance degrades for the noisy data classification task including noisy recognition. In this research, a convolutional neural network (CNN) model with data uncertainty handling; referred as NCNN (Neutrosophic Convolutional Neural Network); is proposed for classification task. The Neutrosophic is a new view of Modeling , designed to effectively deal underlying doubts in the real world, as it came to replace binary logic that recognized right and wrong by introducing a third neutral case which could be interpreted as non-specific or uncertain. Founded by Florentin Smarandache [8], he presented it in 1999 as a generalization of fuzzy logic. As an extension of this, A. A. Salama introduced the Neutrosophic crisp sets Theory as a generalization of crisp sets theory [9] and developed, inserted and formulated new concepts in the fields of mathematics, statistics, and computer science and information systems through Neutrosophic [10-12]. Neutrosophic has grown significantly in recent years through its application in measurement, sets and graphs and in many scientific and practical fields [13- 17].

In this work, a proposed novel COVID- X framework was developed as universal Health-Fog system for automatic diagnosis, treatment, and preventive of people with COVID-19 in an efficient manner using deep learning, Neutrosophic and IoT. Health-Fog provides healthcare as a fog service and efficiently manages the data of COVID-19 patients which is coming from different IoT devices. Health-Fog provides this service by using the proposed framework and demonstrates application enablement and engineering simplicity for leveraging fog resources to achieve the same.

In the following, the contributions of this paper are summarized:

- Building altogether a novel framework universal system to automatically assist the early diagnosis, treatment, and preventive of people with COVID-19 in an efficient manner.
- Proposed a generic system architecture for development of ensemble NCNN on fog computing
- Achieving an empirical of the proposed framework which mix between deep learning and Neutrosophic classifiers in the task of classifying COVID-19.
- The proposed COVID-X framework supports interdisciplinary researchers to continue developing advanced artificial intelligence techniques to fight the COVID-19 outbreak.
- This study demonstrated the useful applications of deep learning models to classify COVID-19 based on the proposed COVID-X framework such as smart mask, smart medical suit, safe spacer, and Medical Mobile Learning (MML). These applications are the next milestone of this research work.

The rest of this paper is structured as follows. Section 2 presents the related works. Section 3 gives a review on the state-of-the-art deep convolutional neural network models as image classifiers. Also, a detailed description of the COVIDX-Net framework is presented. Experimental results and comparative performance of the proposed deep learning classifiers are investigated and discussed in section 4. Finally, limitations and this study is concluded with the main prospects in sections 4, 5.

2. Related Work

Some studies have shown the use of imaging techniques such as X-rays or Computed Tomography (CT-scans) for finding characteristic symptoms of the novel corona virus in these imaging techniques. Hemdan et al. [18] developed a deep learning framework, COVIDX-Net, to diagnose COVID- 19 in X-Ray Images. A comparative study of different deep learning architectures including VGG19, DenseNet201, ResNetV2, InceptionV3, InceptionResNetV2, Xception and MobileNetV2 is provided by authors. Barstugan et al. [19] proposed a machine learning approach for COVID-19 classification from CT images. Kassani et al. [20] presented a feature extractor-based deep learning and machine learning classifier approach for computer-aided diagnosis (CAD) of COVID-19 pneumonia. Loey et al. [21] presented a detection model based on GAN network with deep transfer learning for COVID-19 detection in limited chest X-ray images. Table 1 compares the proposed model (HealthFog) with existing models. Recent studies suggest the use of chest radiography in the epidemic areas for the initial screening of COVID-19 [22]. Therefore, the screening of radiography images can be used as an alternate to the PCR method, which exhibit higher sensitivity in some cases [23]. Nevertheless, the main bottleneck that the radiologists experience in analyzing radiography images is the visual scanning of the subtle insights. This entails the use of intelligent approaches that can automatically extract useful insights from the chest X-rays those are characteristics of COVID-19.

Table 1: Comparison of existing models

Work	Fog Computing	IoT	Neutrosophic	Deep Learning	Dataset	Diagnosis	Healthcare applications
Hemdan et al. [18]				✓	✓	✓	
Barstugan et al. [19]				✓	✓	✓	
Kassani et al. [20]				✓	✓	✓	
Loey et al. [21]			✓	✓	✓	✓	
Proposed work	✓	✓	✓	✓	✓	✓	✓

3. Proposed COVID_X Description framework

Fog and Cloud computing paradigms have emerged as a backbone of modern economy and utilize Internet to provide on-demand services to users [24]. Both of these domains have captured significant attention of industries and academia. In this section will proposed a new deep learning framework for automatically identifying the status of COVID-19 extend support to emerging application paradigms such as IoT, Fog computing, Edge, and Big Data through service and infrastructure. The data generated from Things layer can vary in size, for instance, the data sent from sensors. The diversity in data packages size influence the behavior of Fog node during the processing event, thus, data packages will require more time to process than light data packages. Therefore, in the proposed model, there is a distinction processing tasks. In addition, the fog nodes were adopted collaboration framework to achieve the minimal request processing time for heavy data packages. In Figure 1 the collaboration concept was elaborated and the distinction different processing tasks received from Things layer. In addition, in this framework the advance approach was adopted to identify the suitable treatment process, such as, Fog reputation to process specific type of data (e.g., health data).

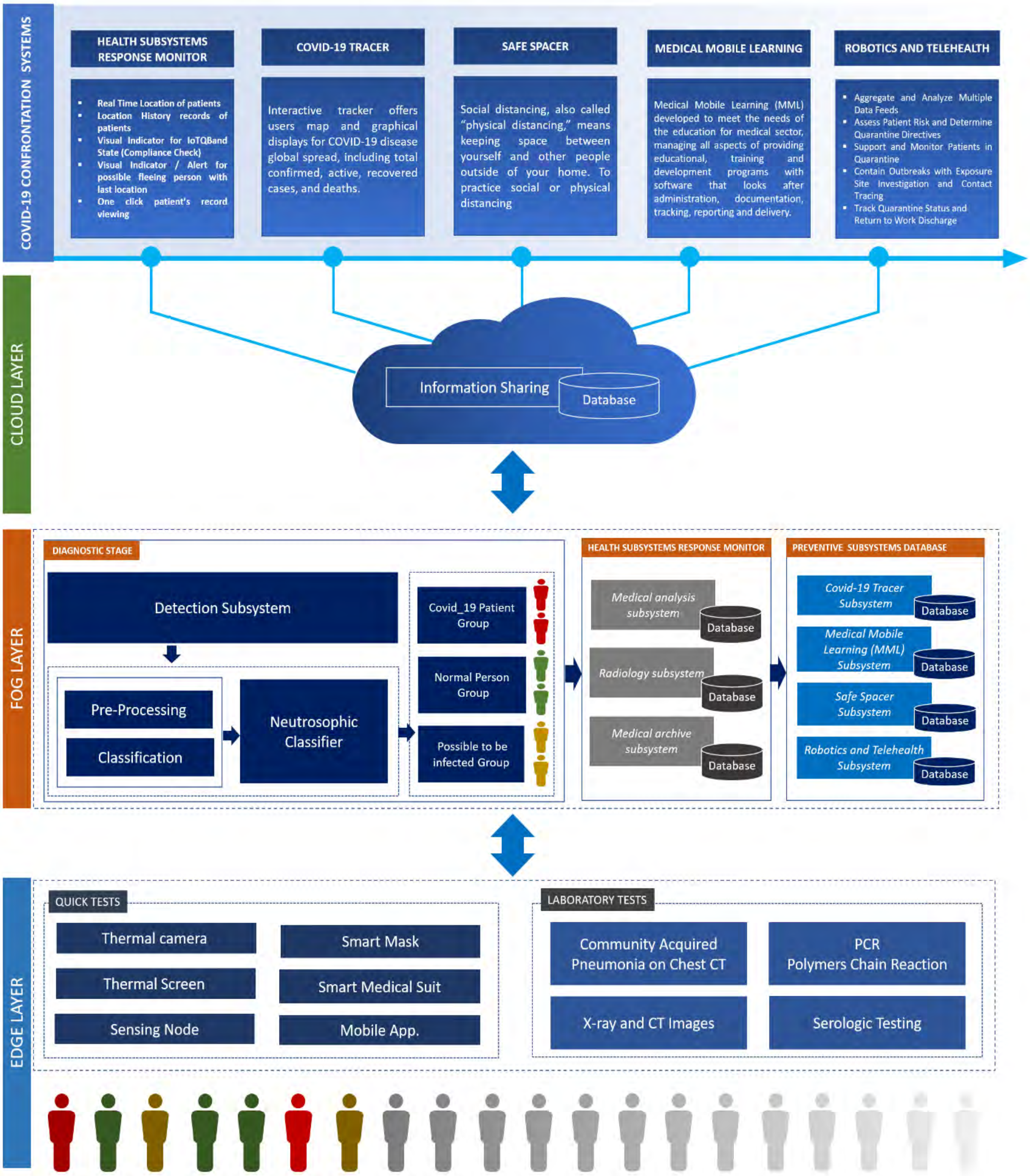

Figure 1. Proposed framework universal system for confrontation covid-19

3.1 Edge Layer:

The edge layer (perception layer), is the starting point of the IoT structure where data is been generated. This layer contains the networked Things (i.e., wireless sensors) such as heart-rate, blood-oxygen and etc., which operate to feed the system with patient symptoms data. Each Thing device/object in this layer is facilitated with communication protocol (such as IEEE 802.15.4, WiFi, Bluetooth, MQTT, and etc.) in which permit the Thing node to transmit the generated data to Fog nodes over the IoT network. In our proposed architecture, A TN denoted by T, is defined as a six-tuple: $T = \langle T_{id}, T_{st}, \tau_i, \mathcal{L}, \mathcal{H}, \mathcal{I}[q] \rangle$ where, T_{id} is an integer representing the unique ID of the TN, $T_{st} = \{0,1\}$, defines whether the node is in active state or not, (τ_i) indicates the type of event that a node senses. $(\mathcal{L})$ is refer to the geo-spatial location of a TN. $(\mathcal{H})$ is represented the specifications of an edge device. $\mathcal{I}[q]$ is a linear data structure, such as a 1-D array (with q elements) that stores the instance IDs of the application instances running on the device. These tuples are essential to represent the Thing node over the IoT network.

3.1.1 Thermal Screen

The smart helmet can also detect high body's temperature in the crowds and send the measured data to be displayed on a phone application. Smart Helmet system work is presented in Figure 2. As the high body temperature of people is one of the very common symptoms, a real time monitoring system of the screening process that automatically appearing the thermal image of temperature of people is needed. So, the diagnosis of the screening process will be less time consuming and less human interactions that might cause the spreading of the coronavirus faster. It can be concluded that the remote sensing procedures, which provide an assortment of ways to identify, sense, and monitoring of coronavirus, give an awesome promise and potential in order to fulfil the demands from the healthcare system [25].

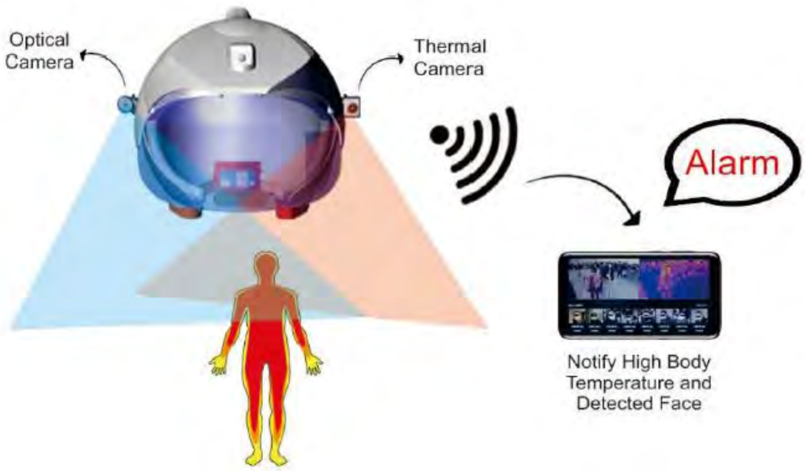

Figure 2. Smart Helmet system work

3.1.2 Sensing Node

Smart City and Intelligent Transportation System (ITS) as shown in Figure 3 offer a futuristic vision of smart, secure and safe experience to the end user, and at the same time efficiently manage the sparse resources and optimize the efficiency of city operations. However, outbreaks and pandemics like COVID-19 have revealed limitations of the existing deployments, therefore, architecture, applications and technology systems need to be developed for swift and timely enforcement of guidelines, rules and government orders to contain such future outbreaks. The proposed architecture and AI assisted applications discussed in the article can be used to effectively and timely enforce social distancing community measures, and optimize the use of resources in critical situations. It offers a conceptual overview and serves as a steppingstone to extensive research and deployment of automated data driven technologies in smart city and intelligent transportation systems [26].

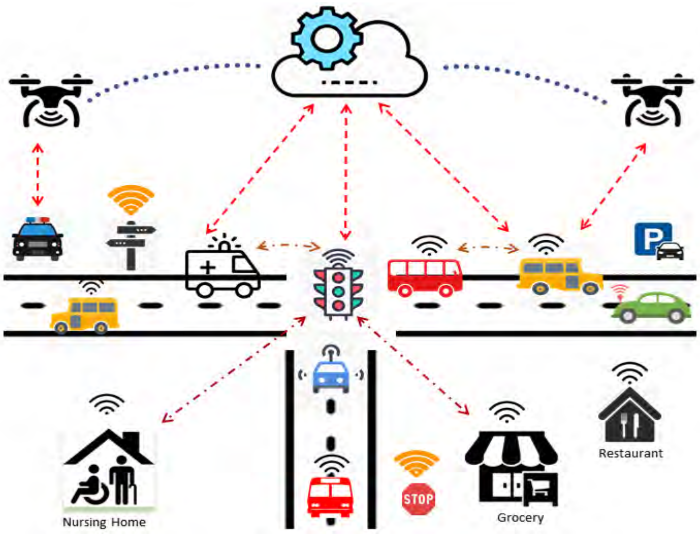

Figure 3. Smart City and ITS Architecture.

3.1.3 Smart Mask

Smart mask can be developed that can record air quality among other features. The Smart Mask is more than your average face mask, as its name suggests. Figure 4 shows the proposed Smart Mask, can record air quality information thanks to various sensors and electronics. Additionally, it can inform wearers of possible changes in lung capacity. While this may prove useful in areas of poor air quality,

Figure 4. The proposed Smart Mask

Specifications; Type: Head-mounted, rated voltage: DC 5V, rated power: 0.4W, Charging time: 2 hours Standby time: 5~8 hours, Filtering effect: 95%, Protection level: KN95, Function: Dustproof, anti-haze, anti-pollen, anti-tail gas, etc. Feature; Unique ventilation design, a plurality of holes, excellent permeability, Exhale, the valve is opened without resistance, air breathing valve, air resistance is smaller, smooth breathing, uses efficient and low-resistance filter material, combined with the smart electric air supply module to provide fresh air into the mask. The edge is protected by 3D sponge for effective sealing. best protection: The allergy mask separation of 98% of the dust, chemicals, smoke and particles, it can be used for dust, anti-vehicle exhaust, anti-pollen allergy, PM2.5, for cycling, hiking, skiing and other outdoor activities. High-performance breathing valve that reduces heat and moisture build-up for smoother breathing. Built-in adjustable nose clip for a good fit and comfort with the face. Charge once for 5~8-hour endurance to ensure commuting. KN95 industrial safety protection level. Low noise.one mask can be used for 5-8 days. Can be reused and Comfortable ear band made of soft cotton, easy to wear and remove ear loop design.

3.1.4 Smart Medical Suit

The nature of Health care workers job puts them health care at an increased risk of catching any communicable disease, including COVID-19. Where they spend a lot of time up close with the patient doing high risk activities, those high-risk activities include things like placing patients on ventilators or collecting samples of sputum from their lungs. That's why it's so important that they achieve the highest level of protective equipment. The proposed smart medical suits is showed in Figure 5.

Figure 5. The proposed Smart Medical Suit.

3.1.5 Mobile App.

The new MobileDetect COVID-19 test kit in Figure 6 was planned to launch in April 2020. The currently available free MobileDetect App for Apple and Android smartphone and tablet platforms will be updated with the additional COVID-19 testing capability upon launch. Due to the novel design incorporating simplistic operation along with credible field-testing capability, the COVID-19 test kits can be used by federal, state, local response, medical agencies and are also planned to be available to the general public [27].

Figure 6. MobileDetect Application.

3.1.6 X-ray and CT Images

Medical imaging is also playing a critical in monitoring the progression of the disease and patient care. Extracting features from radiology modalities is an essential step in training machine learning models since the model performance directly depends on the quality of extracted features. Figure 7. Illustrates the visual features extracted by VGGNet architecture from an X-ray image of a COVID-19 positive patient. Motivated by the success of deep learning models in computer vision, the focus of this research is to provide an extensive comprehensive study on the classification of COVID-19 pneumonia in chest X-ray and CT imaging using features extracted by the state-of-the-art deep CNN architectures and trained on machine learning algorithms [20].

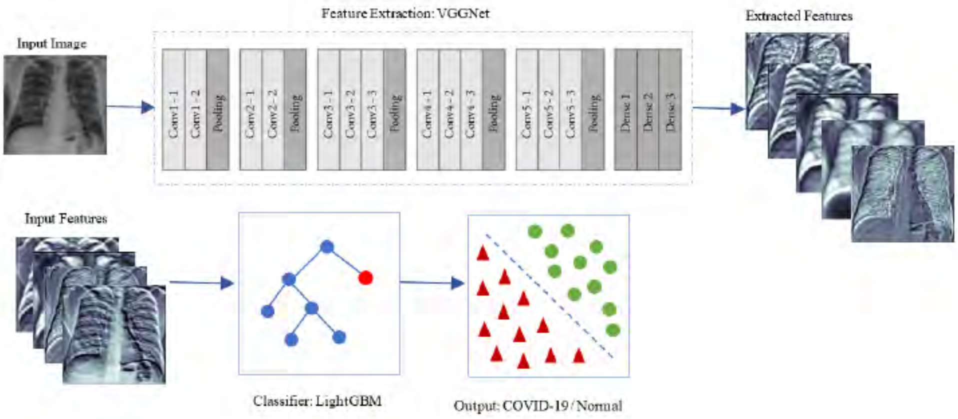

Figure 7. Framework of the method with VGGNet as feature extractor.

3.1.7 Community Acquired Pneumonia on Chest CT

In this study, a 3D deep learning framework was proposed for the detection of COVID-19 as shown in Figure 8. This framework is able to extract both 2D local and 3D global representative features. Deep learning has achieved superior performance in the field of radiology. RT-PCR is considered as the reference standard; however, it has been reported that chest CT could be used as a reliable and rapid approach for screening of COVID-19 [28]

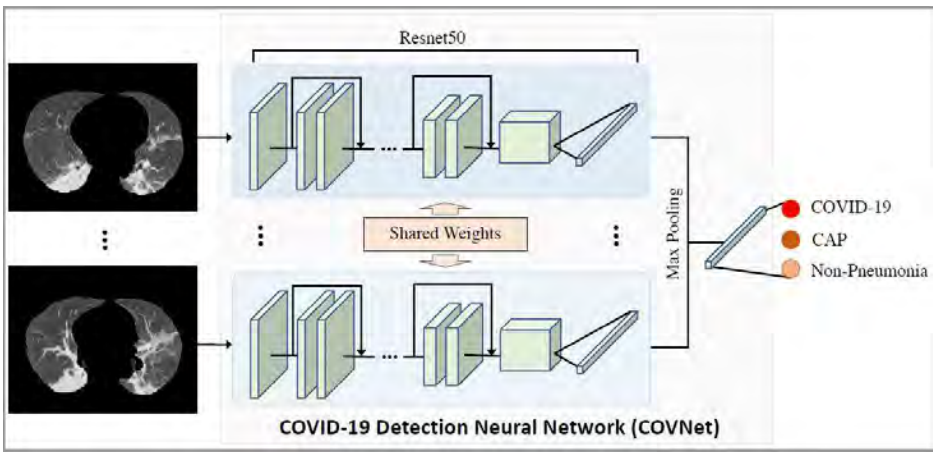

Figure 8. COVID-19 detection neural network (COVNet) architecture.

3.2 Fog Layer:

The Fog layer contains number of decentralized nodes in each given location. This layer handles the primary refining, compute, and processing of data generated in the Things layer. Fog nodes aim to improve efficiency of IoT applications, thus, Fog has the potential to reduce the amount of data transmitted to the Cloud layer and minimizing the requests-response time for IoT applications. This is often required to enhance the Quality of Service (QoS), such as reducing latency and improve network bandwidth. For example, in reference to our scenario the Fog will receive patient's data from their wearable, analyze the data according to predetermined artificial intelligent training, and make outcome available to caregiver over the dashboard and notify cloud with outcome for complex analysis.

3.2.1 Data pre-processing

Covid-19 tested data e.g. the images within the dataset were collected from multiple imaging clinics with different equipment and image acquisition parameters; therefore, considerable variations exist in images' intensity. The proposed method in this study avoids extensive pre-processing steps to improve the generalization ability of the convolution neural network (CNN) architecture. This helps to make the model more robust to noise, artifacts and variations in input images during feature extraction phase. Hence, we only employed two standard pre-processing steps in training deep learning models to optimize the training process [29].

3.2.2 Neutrosophic Classifier

Neutrosophic classifier: a classifier that would use Neutrosophic logic principles and Neutrosophic sets for the classification. Neutrosophic classifier incorporates a simple, Neutrosophic rule based approach like: IF X and Y THEN Z, for solving problem rather than attempting to model a system mathematically similar to fuzzy classifier [30]. Designing of Neutrosophic classification inference system using fuzzy methodology is based on the principles of Mamdani fuzzy inference method [25]. Figure 9 gives the block diagram representation of a Neutrosophic classification system using fuzzy logic toolbox of Matlab. Values of T, I and F Neutrosophic components are independent of each other. So using fuzzy logic toolbox of Matlab, three FIS have been designed: one for Neutrosophic truth component, second for Neutrosophic indeterminacy component and third for Neutrosophic falsity component. Though the working of these components are independent of each other but a correlation is drawn between membership functions of Neutrosophic T, I and F components so as to capture the truthness, indeterminacy and falsity of the input as well as the output.

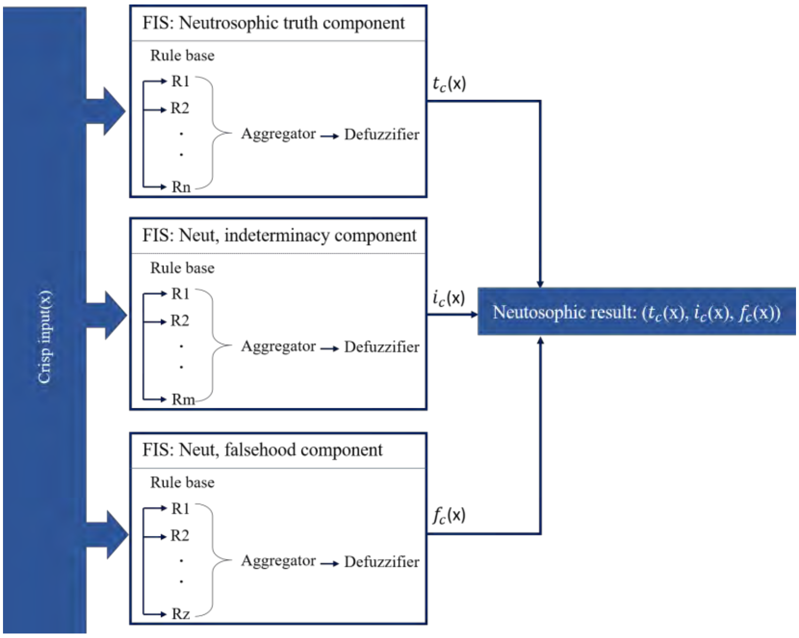

Figure 9. Block diagram for a Neutrosophic components

Neutrosophic Rule-based Classification System (NRCS) which is a rule based system where Neutrosophic logic is used as a tool for representing different forms of knowledge about the problem at hand, as well as for modeling the interactions and relationships that exist between its variables [23]. The generic structure of a NRCSs shown in Figure 10.

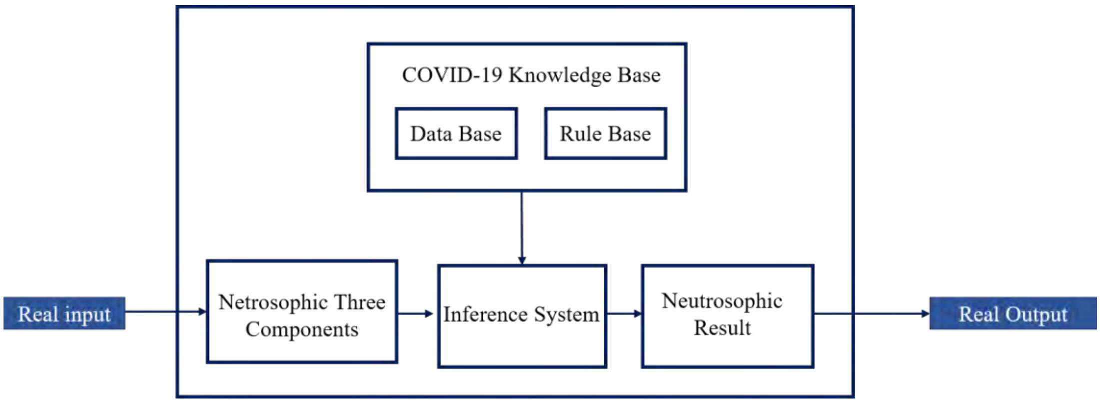

Figure 10. Basic structure of a Neutrosophic Rule-Based Classification System

Let U be a universe of discourse and W is a set in U which composed of bright pixels. A Neutrosophic images P_{NS} is characterized by three sub sets T, I, and F. that can be defined as T is the degree of membership, I is the degree of indeterminacy, and F is the degree of non-membership. In the image, a pixel P in the image is described as P(T,I,F) that belongs to W by its t% is true in the bright pixel, i% is the indeterminate and f% is false where t varies in T, i varies in I, and f varies in F.

The pixelp(i,j)in the image domain, is transformed to

$$NDP_{NS}(i,j) = \{T(i,j), I(i,j), F(i,j)\} \tag{1}$$

Where belongs to white set, belongs to indeterminate set and belongs to non-white set. Which can be defined as [31]:

$$P_{NS}(i,j) = \{T(i,j), I(i,j), F(i,j)\} \tag{2}$$

$$T(i,j) = \frac{\overline{g(\iota,\jmath)} - \bar{g}_{min}}{\bar{g}_{max} - \bar{g}_{min}} \tag{3}$$

$$I(i,j) = 1 - \frac{H_o(i,j) - H_o}{H_{o_{max}} - H_{o_{min}}} \tag{4}$$

$$F(i,j) = 1 - T(i,j) \tag{5}$$

$$H_o(i,j) = abs(g(i,j) - \overline{g(\iota,\jmath)} \tag{6}$$

Where $\overline{g(\iota,\jmath)}$ represents the local mean value of the pixels of window size, and $H_o(i,j)$ which can be defined as the homogeneity value of T at (i,j), that described by the absolute value of difference between intensity $g(i,j)$ and its local mean value $\overline{g(\iota,\jmath)}$.

The Content Based Image Retrieval (CBIR) goal is to retrieve images relevant to a query images which selected by a user. The image in CBIR is described by extracted low-level visual features, such as color, texture and shape. Retrieval System for images embedded in the Neutrosophic domain. In this first phase, extract a set of features to represent the content of each image in the training database. In the second phase, a similarity measurement is used to determine the distance between the image under consideration (query image), and each image in the training database, using their feature vectors constructed in the first phase. Hence, the N most similar images are retrieved. Several distance metrics were suggested for both content and texture image retrieval, respectively. In this paper, we are using a Neutrosophic version of the Euclidean distance, which was presented in [31]. For any two Neutrosophic Sets, the Content Based Image Retrieval (CBIR) goal is to retrieve images relevant to a query images which selected by a user. The image in CBIR is described by extracted low-level visual features, such as color, texture and shape. Retrieval System for images embedded in the Neutrosophic domain. In this first phase, extract a set of features to represent the content of each image in the training database. In the second phase, a similarity measurement is used to determine the distance between the image under consideration (query image), and each image in the training database, using their feature vectors constructed in the first phase. Hence, the N most similar images are retrieved. Several distance metrics were suggested for both content and texture image retrieval, respectively. In this paper, we are using a Neutrosophic version of the Euclidean distance, which was presented in [31]. For any two Neutrosophic Sets,

$$A = \{T_A(x), I_A(x), F_A(x)), x \in U\} \; and \tag{7}$$

$$B = \{T_B(x), I_B(x), F_B(x)), x \in U\} \; in \tag{8}$$

$$U = \{u_1, u_2, u_3, \dots, u_n\} \; then \tag{9}$$

The Neutrosophic Euclidean distance is equal to

$$d(A,B) = \sqrt{\sum_{i=1}^{n} ((T_A(x_i) - T_B(x_i))^2 + ((I_A(x_i) - I_B(x_i))^2 + ((F_A(x_i) - F_B(x_i))^2} \tag{10}$$

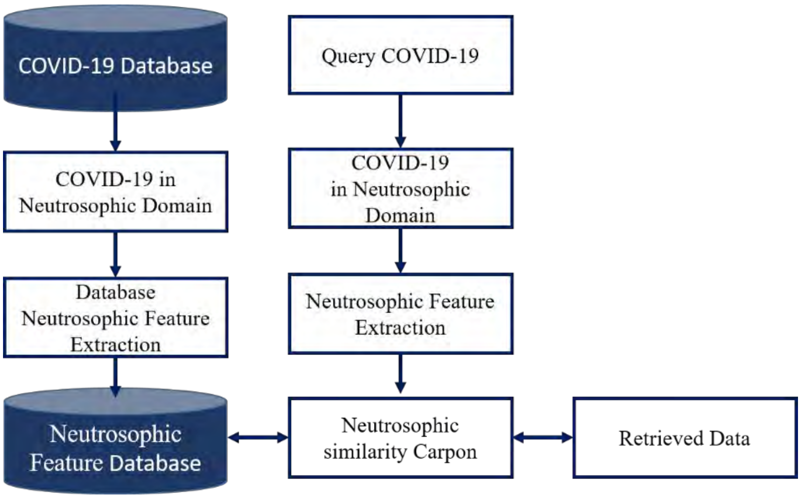

Figure 11. Neutrosophic COVID-19 image classifier Architecture

The algorithm for the proposed system is given below which presented in Figure 11:

1. Convert each image in the database from spatial domain to Neutrosophic domain.
2. Create a database containing various COVID-19.
3. Extract Texture feature of COVID-19 in the database.
4. Construct a combined feature vector for T, I, F and Stored in another database called Featured Database.
5. Find the distance between feature vectors of query COVID-19 and that of featured databases.
6. Sort the distance and Retrieve the N-top most similar.

The RNN structure replaces the traditional neuron by two neurons (lower neuron, upper neuron) to represent lower and upper approximations of each attribute in the CTG data set, its structure formed from 4 layers input, 2 hidden and output layers. The hidden layers have rough neurons, which overlap and exchange information between each other, While the input and output layers consists of traditional neurons as in Figure 12 [32].

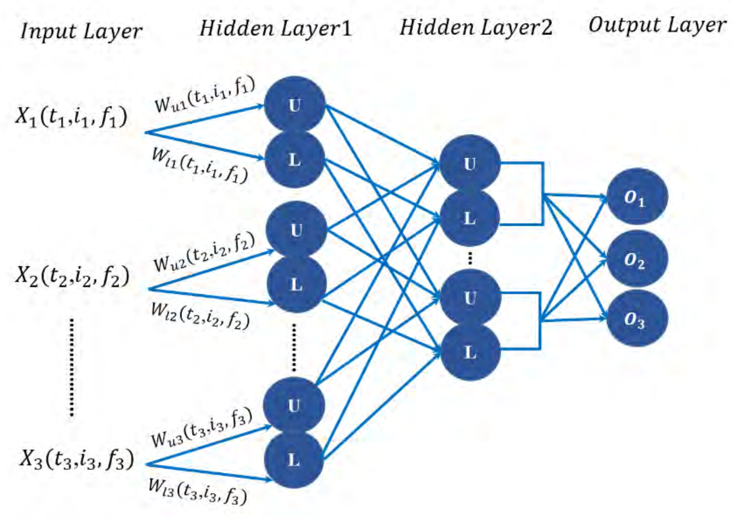

Figure 12. Rough Neural Network (RNN) structure.

Input layer is composed of neuron for each data attribute. The output layer represents the three FHR classes, the hidden layers rough neurons are determined by the Baum-Haussler rule [33].

$$N_{hn} = \frac{N_{ts} \times Te}{N_i + N_o} \tag{11}$$

Where N_{hn} is the number of hidden neurons, N_{ts} is the number of training samples, Te is the tolerance error, N_i is the number of inputs (attributes or features), and N_o is the number of the output.During training process, the normalized input data is multiplied by its weight and computed in sigmoid activation function.

$$f(x) = \frac{1}{1 + e^{-\lambda x}} \tag{12}$$

Step II: Training phase

1. Initialize random (upper, lower) weights of network
2. Feed forward of attribute values and multiply in both direction (Uw, Lw)
3. Compute (IU, IL) of hidden layers by relations:

$$I_{Ln} = \sum_{J=1}^{n} W_{Lnj} O_{nj} \tag{13}$$

$$I_{Un} = \sum_{J=1}^{n} W_{Unj} O_{nj} \tag{14}$$

4. Compute (OU, OL) of hidden layers by relations:

$$O_{Ln} = Min(f(I_{Ln}), f(I_{Un})) \tag{15}$$

5. Check fetus according to comparing between actual output (T) and output value (O), where output represent by

$$O = O_{Ln} + O_{Un} \tag{16}$$

6. If output is error, then use back propagation algorithm, and compute error.

$$\Delta = T - O \tag{17}$$

7. Update (upper, lower) weights of network by derivation of activation function:

New weight = old weight + (Δ * η *derivative* activation of (input)) (18)

where η is learning rate of model

8. Repeat 5, 6, 7, 8 and 8.1 until reduction error as possible as.

Step III: Testing phase Classify new sample of objects and determine the accuracy rate of the model by using relation Accuracy = 1–absolute error, also calculate time consumption in model processing. The proposed model for neutrosophic algorithms and source codes based on the works presented in [34-37] and others.

3.2.3 Classification Performance Analysis

In order to evaluate the performance for each deep learning model in the COVID-X, different metrics have been applied in this study to measure the true and/or misclassification of diagnosed COVID-19 in the tested X-ray images as follow. First, the cross validation estimator was used and resulted in a confusion matrix as illustrated in Table 2. The confusion matrix has four expected outcomes as follows. True Positive (TP) is a number of anomalies and has been identified with the right diagnosis. True Negative (TN) is an incorrectly measured number of regular instances. False Positive (FP) is a collection of regular instances that are classified as an anomaly diagnosis FP. False Negative (FN) is a list of anomalies observed as an ordinary diagnosis [18].

Table 2. Confusion Matrix.

	Predicted Positive	**Predicted Negative**
Actual Positive	True Positive (TP)	False Negative (FN)
Actual Negative	False Positive (FP)	True Negative (TN)

After calculating the values of possible outcomes in the confusion matrix, the following performance metrics can be calculated.

A) Accuracy: Accuracy is the most important metric for the results of our deep learning classifiers, as given in (1). It is simply the summation of true positives and true negatives divided by the total values of confusion matrix components. The most reliable model is the best but it is important to ensure that there are symmetrical datasets with almost equal false positive values and false adverse values. Therefore, the above components of the confusion matrix must be calculated to assess the classification quality of our proposed COVIDX-Net framework.

$$\text{Accuracy}(\%) = \frac{TP + TN}{TP + FP + TN + FN}\, 100\% \tag{19}$$

B) Precision: Precision is represented in (2) to give relationship between the true positive predicted values and full positive predicted values.

$$\text{Precision} = \frac{TP}{TP + FP} \tag{20}$$

C) ***Recall:*** In (3), recall or sensitivity is the ratio between the true positive values of prediction and the summation of predicted true positive values and predicted false negative values.

$$\text{Recall} = \frac{TP}{TP + FN} \tag{21}$$

D) ***F1-score:*** F1-score is an overall measure of the model's accuracy that combines precision and recall, as represented in (4). F1-score is the twice of the ratio between the multiplication to the summation of precision and recall metrics.

$$\text{F1} - \text{score} = 2\left(\frac{Precision \times Recall}{Precision + Recall}\right) \tag{22}$$

3.3 Cloud Layer:

Cloud or data-centres layer is the top layer of the IoT architecture in which enabling omnipresent, convenient, and proper network access to shared resources (e.g., storage, and services) over the IoT network. Thus, Cloud perform the heavy services of healthcare data analysis and processing that Fog cannot perform.

3.3.1 Covid-19 Tracer

Interactive tracker offers users map and graphical displays for COVID-19 disease global spread, including total confirmed, active, recovered cases, and deaths. The live dashboard pulls data from the proposed framework as well as the centers for disease control to show all confirmed and suspected cases of COVID-19, along with recovered patients and deaths. The data is visualized through a real-time graphic information system (GIS) as shows in Figure 13 [38].

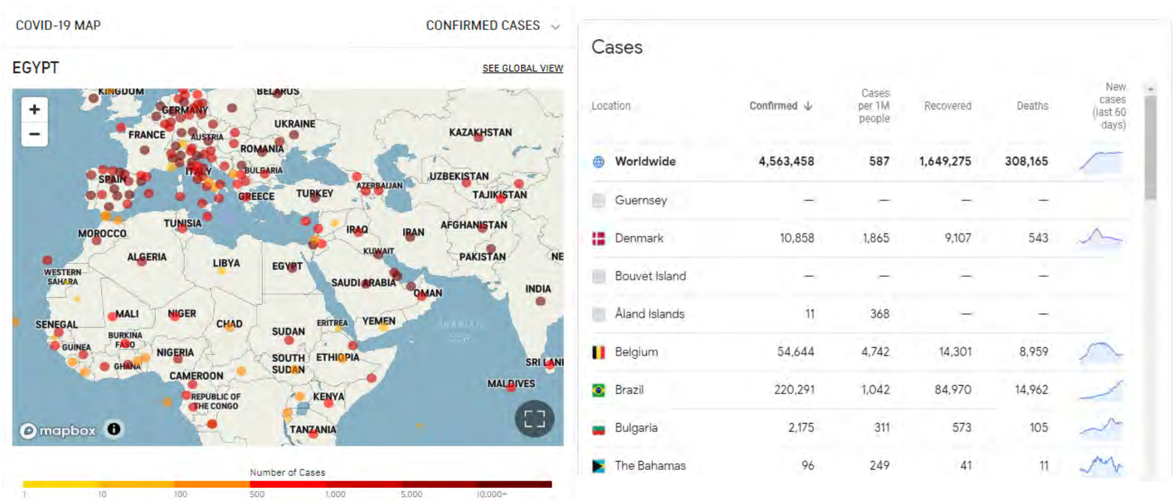

Figure 13. COVID-19 Tracer

3.3.2 Safe Spacer

Limiting face-to-face contact with others is the best way to reduce the spread of coronavirus disease 2020 (COVID-19). Safe spacer, also called "social distancing," means keeping space between yourself and other people outside of your home. The proposed safe spacer was showed in Figure 14. To practice social or physical distancing using Ultra-wideband technology, Safe Spacer runs wirelessly on a rechargeable battery and precisely senses when other devices come within 2m/6ft, alerting wearers with a choice of visual, vibrating or audio alarm. Each device features a unique ID tag and built-in memory to optionally associate with workers' names for tracing any unintentional contact. To maintain high privacy standards, no data except the device's ID and proximity is stored. For advanced workplace use, an optional iOS/Android app allows human resources or safety departments to associate IDs to specific workers, log and export daily tracing without collecting sensitive data, configure the alarms, set custom distance/alert thresholds and more.

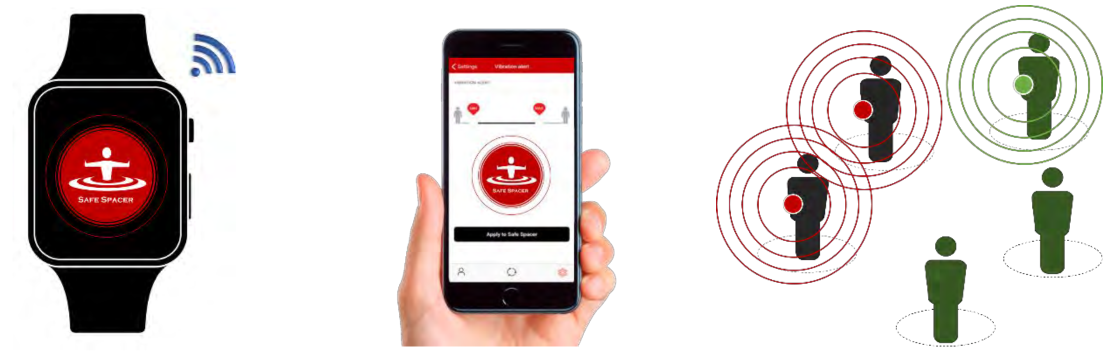

Figure 14. The proposed safe spacer

3.3.3 Health System Response Monitor

The COVID-19 Health System Response Monitor (HSRM) assists healthcare organizations and governments assess patient risk profiles and connects them with virtual care capabilities. It has been designed in response to the COVID-19 outbreak to collect and organize up-to-date information responding to the crisis. It focuses primarily on the responses of health systems but also captures wider public health initiatives. It can be presented the main subsystem in medical system as following:

- ***Medical analysis subsystem*** It records the results of the tests for the patients either manually or automatically by connecting the analytical devices to the system It provides a set of statistics such as: the number of analyzes required by a particular laboratory in a specific period and the number of analyzes that have already been done - analyzes of a particular patient divided according to his medical visits This system is linked to a database that includes all medical analyzes divided by type (chemistry - hematology - microbiology - immunology - pathology) and it is related to a set of applications that record the analyzes of each laboratory and the standard data for these analyzes (Normal Value) according to the kit used in the lab.
- ***Radiology subsystem.*** It records the data of the examination staff, showing the type of radiation required for each of them, along with some clinical data about some of the rays, such as CT-rays and records the radiology report. It contains a system Picture Archiving and Communication System (PACS) that links the radiology devices to the system so that the x-rays are sent to the x-ray. It provides a set of statistics, such as: the number of radiation transferred to a particular x-ray department in a specific period, the number of radiation already done, and the number of x-rays sent. This system is linked to a database that includes all the rays divided by type (therapeutic - diagnostic) or (ultrasound - CT scan - resonance) and it is linked to a set of applications that record the radiation of each section and the standard report for each radiator, as well as determining the work schedule for each section rays.
- ***Medical archive subsystem.*** It provides a set of statistics, such as: the numbers of patients attending a specific clinic in a specific period classified by type or age group or geographically distributed in the governorate, center or city. The system scans patient documents, whether paper documents or x-ray films, with scanners with special specifications. These documents

are stored as part of patient data on dedicated servers. The system contains the ability to record the type of document (x-rays-tests-good checks-surgeries -...) and the document history and some other data that can be used to create statistics for these documents can be added. The system contains a special viewer to display these documents with special capabilities for dealing with these images such as enlarging, reducing or rotating the images. The Digitizer can be used so that x-ray films are stored in the form of dicom files which is the same format that x-ray devices output so that they can be viewed through the PACS Viewer.

3.3.4 Medical Mobile Learning subsystem

Medical Mobile Learning (MML) is an unavoidable alternative during COVID-19. It developed to meet the needs of the education for medical sector, managing all aspects of providing educational, training and development programs with software that looks after administration, documentation, tracking, reporting and delivery. MML denote learning involving the use of a mobile device. It has several advantages and benefits. First, this teaching method can occur at anyplace, anytime, and anywhere and the learning process is not limited to one particular place. Besides, it allows doctors to personalize instruction and allow to self-regulate learning. Generally, mobile learning can helps doctors to develop technological skills, conversational skills, find answers to their questions for any update for COVID-19, develop a sense of collaboration, allow knowledge sharing, and hence leverage their learning.

3.3.5 Robotics and Telehealth system

Health systems broadly, to encompass the full continuum between public health (population-based services) and medical care (delivered to individual patients). When we think about digital transformation in healthcare, we usually think about some new software doctors are using or a new medical imaging machine. However, since doctors are now scrambling to contain the COVID-19 pandemic, they have to do so without endangering themselves as well. The proposed robotics and telehealth system shown in Figure 15. This is where robotics comes in instead of going into the room to see the patient, a robot goes in, and the doctors operate it via an iPad from the other side of the door—this digital innovation in healthcare currently being used in hospitals in Washington and other states. In fact, the robot even has a stethoscope to take the patients' vitals [39].

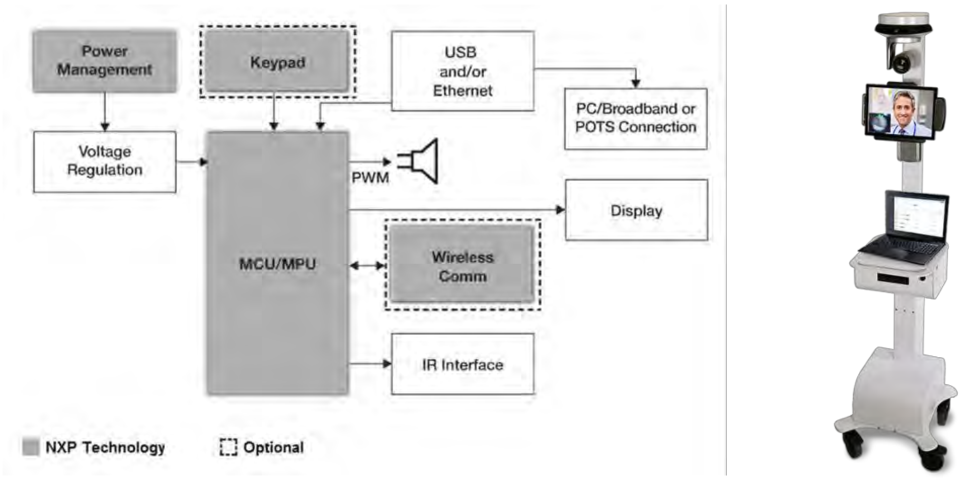

Figure 15. The proposed robotics and telehealth system

4. Limitations

This research is interested in aspects related to Fog computing applied to the healthcare area. In this sense, this paper focuses on the characteristics of fog computing architectures directly related to healthcare, disregarding models. This research is limited in availability of data makes it difficult to process due to the limited hardware availability. Interoperability, data processing, CPU management, memory and disk resources, and big data issues are still weaknesses in architectures that require a large number of heterogeneous devices such as healthcare applications.

5. Conclusion and Future Works

Infectious COVID-19 disease shocked the world and is still threating the lives of billions of people. In this study, a new CVOID-X framework has been proposed to automatically identify or COVID-19 based on deep learning classifiers. Technological developments like edge computing, fog computing, IoT, and Big Data have gained importance due to their robustness. In this retrospective and multi-center study, a deep learning model, COVID-19 detection neural network using Neautrosophic classifier, was developed to extract visual features from volumetric exams for the detection of COVID-19. The proposed system facilitates communication between people and medical centers so that the appropriate COVID-19 patient can be reached just on time. It also integrates the information scattered among different medical centers and health organizations across the country to confrontation COVID-19 Stakeholders are able to use the confrontation as an applications installed on their smartphones or as wearable devices. So the diagnosis of the screening process will be less time consuming and less human interactions that might cause the spreading of the coronavirus faster. It can be concluded that the remote sensing procedures, which provide an assortment of ways to identify, sense, and monitoring of COVID-19, give an awesome promise and potential in order to fulfil the demands from the healthcare system. As part of the future work, the proposed framework can be stimulated and analysis the results for every Thing device/object in Edge layer presented in this work. Moreover, to obtain the most accurate feature which is an essential component of learning, MobileNet, DenseNet, Xception, ResNet, InceptionV3, InceptionRes- NetV2, VGGNet, NASNet will be applied amongst a pool of deep convolutional neural networks. Furthermore, the proposed framework can also be extended towards other important domains of healthcare such as diabetes, cancer and hepatitis, which can provide efficient services to corresponding patients.

Acknowledgments: We would like to thank Prof. Florentin Smarandache [Department of Mathematics, University of New Mexico, USA] for helping us to understand Neutrosophic approach. In addition, we like to show gratitude to Prof. Mohamed A. Mohamed [Dean of the Faculty of Engineering, Mansoura University, Egypt] for his helping and advising during the research.

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Received: Apr 10, 2020. Accepted: July 1, 2020

Introduction to Decision Making for Neutrosophic Environment "Study on the Suez Canal Port, Egypt"

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Abstract: Paper aims to use the programming codes in calculating the values of neutrosophic grades and their representation in proving the certainty and uncertainty associated with the data of navigational projects development in the Suez Canal, Egypt. Added to, we reach a more descriptive of the data in terms of certainty and uncertainty, and that is through the neutrosophic representation of both the total revenue and the revenues of the Suez Canal from the transit carriers and ships. Finally, we will present a study of the decision-making process regarding the better investment in the Suez Canal. Is it investing in the oil tankers or investing in cargo ships, as this is done based on neutrosophic data. This will be done by studying optimistic, pessimistic, and remorse entrances to the neutrosophic data, to see which oil tankers or cargo ships offer better returns to the Canal.

Keywords: Neutrosophic categories; neutrosophic analysis; Neutrosophic data; Suez Canal; Neutrosophic information models; Decision Making.

1. Introduction

In real-life problems, the data associated are often imprecise, or non-deterministic. Not all real data can be precise because of their fuzzy nature. Imprecision can be of many types: non-matching data values, imprecise queries, inconsistent data misaligned schemas, etc. The fundamental concepts of neutrosophic set, introduced by Smarandache in [2, 3] and Salama et al. in [2-19]. Decision-making method developed on the accuracy of the information resulting from the neutrosophic data processing. The data has converted from the classic situation using the neutrosophic technique, which helps in the process of decision-making. Thus, we can rank all alternatives and make a better choice according to the degrees of certainty, uncertainty, and impartiality. Paper is limited to the data for the ships crossing the Suez Canal Port, Egypt, such as the oil tankers, cargo ships, passenger ships and rescue ships from 1976 to 2019, because they are considered the most important main types that cross the Suez Canal, due to the nature and characteristics of each of them, and this requires special attention to that types of ships

1.1 Preliminaries & Related Works

We recollect some relevant basic preliminaries, and in particular, the work of Smarandache in [2, 3] and Salama et al. [16]. The data was relied on the bulletins of the Suez Canal Authority Egypt, in [1].

2- Proposed frameworks

In 2014, Salama et al. [16] designed and implemented an object oriented programming [OOP] to deal with neutrosophic data operations.

The following are neutrosophic package class, some software algorithms and codes designed to generate neutrosophic data related to projects for the development of the navigation of the Suez Canal, Egypt:

1) The following diagram represent the neutrosophic structure.

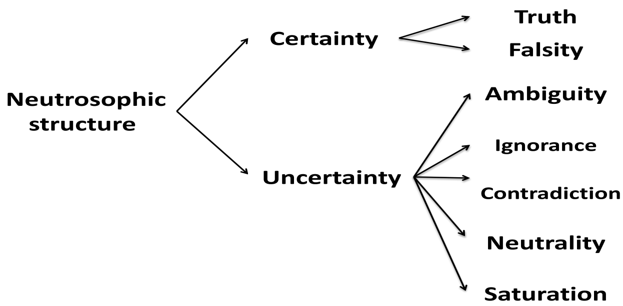

Figure 1. Neutrosophic Data Structure

2) The following diagram represent the neutrosophic Package

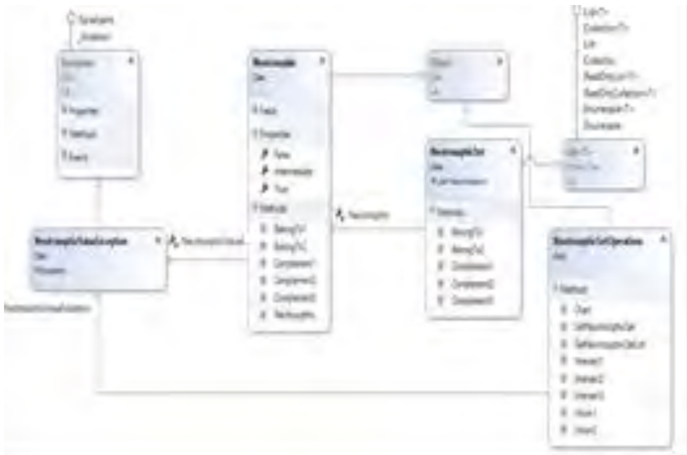

Figure 2: Neutrosophic Package Class Diagram.

3) The first input parameter to the neutrosophic variable has three-neutrosophic components membership function, indeterminacy and non-membership of data is illustrated in Figure 3.

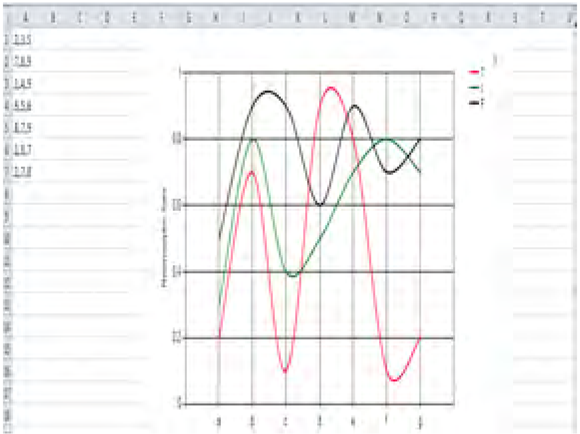

Figure 3: Neutrosophic Chart .

4) Some Neutrosophic codes

```
    using System;
using System.Collections.Generic;
using System.Linq;
using System.Text;
namespace RibbonCustomize
{
    class NeutrosophicValueException:Exception
    {
        public NeutrosophicValueException()
            : base("Neutrosophic value must be between 0 and 1")
        {
        }
    }
    class NeutrosophicSet:List<Neutrosophic>
    {
        public NeutrosophicSet Complement1()
        {
            NeutrosophicSet complementSet = new NeutrosophicSet();
            foreach (Neutrosophic n in this)
            {
                complementSet.Add(n.Complement1());
            }
            return complementSet;
        }
        public NeutrosophicSet Complement2()
        {
            NeutrosophicSet complementSet = new NeutrosophicSet();
            foreach (Neutrosophic n in this)
            {
                complementSet.Add(n.Complement2());
            }
            return complementSet;
        }
        public NeutrosophicSet Complement3()
        {
```

```
            NeutrosophicSet complementSet = new NeutrosophicSet();
            foreach (Neutrosophic n in this)
            {
                complementSet.Add(n.Complement3());
            }
            return complementSet;
        }
        public Boolean

BelongTo1(NeutrosophicSet nSet)
        {
            for (int i = 0; i < this.Count; i++)
            {
                if (!this[i].BelongTo1(nSet[i]))
                    return false;
            }
            return true;
        }
        public Boolean BelongTo2(NeutrosophicSet nSet)
        {
            for (int i = 0; i < this.Count; i++)
            {
                if (!this[i].BelongTo2(nSet[i]))
                    return false;
            }
            return true;
        }
      }

    class Neutrosophic
    {
        double t, i, f;

        public Neutrosophic(double t,double i,double f)
        {
            T = t;
            I = i;
            F = f;
        }
        public double T
        {
            get
            {
                return   Convert.ToDouble( Math.Round( t,4));
            }
            set
            {
                if (t < 0 || t > 1)
                    throw new NeutrosophicValueException();
                t = value;
            }
```

3- Neutrosophic Data Related to Projects for the Development of the Navigation Channel of the Suez Canal

In this section, software algorithms present the values of the neutrosophic grades (Membership, Indeterminacy, Non-membership) associated with the most important variables for the waterway development projects are introduced. Which in the future helps in the process of support and decision-making through the neutrosophic environment. The following tables represent for neutrosophic fuzzy data related to the development of Suez Canal projects.

3-1 Neutrosophic construction for the revenue of the oil tankers

The following table shows the neutrosophic functions membership, indeterminacy and non-membership for the revenue from oil tankers

membership	indeterminacy	non-membership
0.988740191	0.00461327	0.011259809
0.997541134	0.000366208	0.002458866
0.99372791	0.003386564	0.00627209
0.994747051	0.00352199	0.005252949
0.996362317	0.002416395	0.003637683
0.9992945	0.000439248	0.0007055
0.996371092	0.001113029	0.003628908
0.998656708	9.414E-05	0.001343292
0.996205932	0.002065225	0.003794068
0.996018585	0.003331682	0.003981415
0.998642595	0.001125642	0.001357405
0.997179889	0.001817804	0.002820111
0.997190918	0.00099931	0.002809082
0.99900649	0.000160861	0.00099351
0.998159997	0.001783886	0.001840003

0.998825512	0.000389082	0.001174488
0.999178991	9.72807E-05	0.000821009
0.997687655	0.000459245	0.002312345
0.997405507	0.001358576	0.002594493
0.996925695	0.000896872	0.003074305
0.998220312	4.8633E-05	0.001779688
0.99850752	0.00109835	0.00149248
0.999384376	0.000163505	0.000615624
0.999094749	0.000322558	0.000905251
0.999216422	0.00043813	0.000783578
0.999683732	0.00026184	0.000316268
0.999261561	0.000391302	0.000738439
0.998155283	0.000963051	0.001844717
0.998749522	0.000918046	0.001250478
0.997990694	0.000537374	0.002009306
0.997900945	0.000606896	0.002099055
0.99841285	0.000707733	0.00158715
0.998114783	0.001177655	0.001885217
0.998274559	0.001384855	0.001725441
0.998516734	0.000662619	0.001483266
0.998157819	0.001816153	0.001842181
0.998596055	0.000746927	0.001403945

0.999123659	3.86185E-05	0.000876341
0.999701164	0.000102857	0.000298836

3-2 Neutrosophic construction for the revenue of the casting cargo ships

The following table shows the neutrosophic functions membership, indeterminacy and non-membership for the revenue for casting cargo ships.

membership	indeterminacy	non-membership
0.979597358	0.011222455	0.020402642
0.99977598	0.000190899	0.00022402
0.987841221	0.011233986	0.012158779
0.988722391	0.004594681	0.011277609
0.999042398	2.94809E-05	0.000957602
0.996013645	0.003510839	0.003986355
0.993315242	0.001470175	0.006684758
0.999288912	0.000617127	0.000711088
0.997280058	0.001194023	0.002719942
0.997839878	0.001434591	0.002160122
0.999313108	0.000285615	0.000686892
0.998390123	0.000424146	0.001609877
0.998744516	1.66601E-05	0.001255484
0.996685415	0.002629738	0.003314585
0.999159935	0.000624388	0.000840065
0.999362719	0.000625203	0.000637281
0.998118538	0.00074875	0.001881462

0.999081238	0.000843198	0.000918762
0.998767326	0.000237711	0.001232674
0.996110421	0.00035633	0.003889579
0.996214035	0.001895643	0.003785965
0.997622584	0.001778035	0.002377416
0.997971546	0.001753866	0.002028454
0.996205565	0.001369003	0.003794435
0.99738413	0.002252678	0.00261587
0.99737923	0.00066253	0.00262077
0.99901595	0.000512284	0.00098405
0.998766167	0.001011506	0.001233833
0.999572232	0.000179547	0.000427768
0.99961459	0.000176656	0.00038541
0.9986817	0.000122479	0.0013183
0.998326458	0.000859449	0.001673542
0.998183893	0.000195493	0.001816107
0.99938128	0.000356578	0.00061872
0.999648021	0.000346284	0.000351979
0.998516736	0.000578491	0.001483264
0.99840945	0.000915206	0.00159055
0.998917136	6.82709E-05	0.001082864

0.999421069	2.40589E-05	0.000578931

3-3 Neutrosophic construction for the total revenue

The following table shows the neutrosophic functions membership, indeterminacy and non-membership for the total Revenue.

membership	**indeterminacy**	**non-membership**
0.99453	0.68563	0.00547
0.994298	0.258039	0.005702
0.999356	0.694458	0.000644
0.998015	0.759271	0.001985
0.998491	0.182653	0.001509
0.99755	0.842113	0.00245
0.999763	0.4457	0.000237
0.99826	0.747169	0.00174
0.999419	0.657663	0.000581
0.997668	0.176678	0.002332
0.999044	0.469494	0.000956
0.998904	0.408643	0.001096
0.999146	0.936579	0.000854
0.999439	0.72134	0.000561
0.998743	0.156889	0.001257
0.999089	0.752516	0.000911
0.999257	0.014837	0.000743

0.999994	0.553359	6.25E-06
0.998467	0.60711	0.001533
0.998461	0.670432	0.001539
0.998631	0.026247	0.001369
0.999688	0.585905	0.000312
0.99864	0.069339	0.00136
0.999698	0.843087	0.000302
0.99874	0.135408	0.00126
0.999383	0.682067	0.000617
0.999497	0.336895	0.000503
0.998713	0.076038	0.001287
0.99899	0.042053	0.00101
0.9991	0.299292	0.0009
0.9996	0.725316	0.0004
0.9998	0.08398	0.0002
0.9999	0.040729	1E-04
0.99994	0.497875	6E-05
0.99997	0.528715	3E-05
0.97811	0.4454	0.02189
0.9888	0.390296	0.0112
0.9999	0.715649	1E-04

0.99999	0.947433	1E-05
0.99999	0.243405	1E-05
0.999996	0.724688	4E-06
0.999998	0.512305	2E-06
0.999999	0.092309	1.5E-06

3-4 Neutrosophic construction for the oil tankers load sizes

The following table shows the neutrosophic functions membership, indeterminacy and non-membership for the oil tanker load sizes.

membership	indeterminacy	non-membership
0.999999683	1.67646E-07	3.1713E-07
0.999988666	4.86643E-06	1.13338E-05
0.999986858	1.139E-05	1.31419E-05
0.99999763	1.71931E-06	2.37026E-06
0.999998331	7.01436E-08	1.66936E-06
0.999994502	9.67939E-07	5.49785E-06
0.999999965	8.26009E-09	3.45204E-08
0.999995597	3.43245E-06	4.40344E-06
0.999992593	2.66158E-06	7.40721E-06
0.99999205	3.97483E-06	7.95034E-06
0.999995382	2.24479E-06	4.61806E-06
0.999996817	2.23691E-06	3.18252E-06
0.999996942	1.23941E-06	3.05795E-06
0.999994138	5.19437E-06	5.86174E-06

0.999994783	3.67107E-06	5.21695E-06
0.999998407	1.42519E-06	1.59336E-06
0.999999916	1.99228E-08	8.41154E-08
0.999998438	1.13182E-06	1.5621E-06
0.999994327	4.0787E-07	5.67272E-06
0.999994318	6.75615E-07	5.68204E-06
0.999988657	1.12072E-05	1.13432E-05
0.999999955	3.06172E-08	4.47353E-08
0.999997963	3.01945E-07	2.03699E-06
0.999987344	1.14491E-05	1.26559E-05
0.999995549	2.2139E-06	4.45134E-06
0.999992198	5.62777E-08	7.80226E-06
0.999994384	4.76185E-07	5.61551E-06
0.999994926	4.56611E-06	5.07413E-06
0.999994185	2.79521E-06	5.81518E-06
0.999993616	7.78117E-07	6.3837E-06
0.999995882	1.66929E-06	4.11835E-06
0.999996945	2.47541E-06	3.05471E-06
0.999993363	2.91819E-06	6.63674E-06
0.999997621	1.61063E-06	2.3791E-06
0.999994426	5.07718E-06	5.57402E-06

0.999994968	3.88646E-06	5.03216E-06
0.99999355	3.73223E-06	6.44966E-06
0.999998549	6.62626E-07	1.45134E-06
0.999998329	3.01809E-07	1.67063E-06
0.999996023	2.93175E-06	3.97664E-06
0.999995523	2.20011E-07	4.47681E-06
0.999995884	3.7598E-07	4.11594E-06
0.999996923	2.20412E-06	3.07715E-06

3-5 Neutrosophic construction for the sizes of tonnage of the cargo ships casting

The following table shows the neutrosophic functions membership, indeterminacy and non-membership of the sizes of tonnage for cargo ships casting.

membership	**indeterminacy**	**non-membership**
0.999980808	2.68364E-06	1.91925E-05
0.999975151	8.10953E-06	2.48487E-05
0.999998995	4.16959E-07	1.00491E-06
0.999995727	2.63953E-06	4.27268E-06
0.999979618	1.46253E-05	2.03821E-05
0.999988567	7.58596E-07	1.14333E-05
0.999984127	1.55296E-05	1.58727E-05
0.999988002	3.75304E-06	1.19978E-05
0.99999423	2.93423E-07	5.76964E-06
0.999982442	1.56162E-05	1.75578E-05

0.999999894	7.69112E-08	1.05778E-07
0.99999368	4.93127E-06	6.31956E-06
0.999989252	7.55723E-06	1.07479E-05
0.999993027	1.09647E-06	6.97259E-06
0.999996135	2.52924E-06	3.86463E-06
0.999994335	2.3808E-06	5.66547E-06
0.99999625	2.07892E-06	3.74972E-06
0.999996134	3.71619E-06	3.86632E-06
0.999997895	6.74216E-07	2.10478E-06
0.999988647	2.45364E-06	1.13533E-05
0.999993827	2.64785E-06	6.17332E-06
0.999993905	2.80495E-06	6.09518E-06
0.999997398	2.14679E-06	2.6023E-06
0.999991676	2.11799E-06	8.32415E-06
0.999996363	1.73944E-06	3.63679E-06
0.99999949	2.87001E-07	5.10254E-07
0.999999624	1.45097E-07	3.76357E-07
0.999997374	3.98441E-07	2.62619E-06
0.999992259	6.33327E-07	7.74082E-06
0.99999455	5.82952E-07	5.44981E-06
0.999999272	5.9419E-07	7.28466E-07

0.999992703	5.71205E-06	7.29738E-06
0.999999875	9.66849E-08	1.24589E-07
0.999990182	4.31384E-06	9.81787E-06
0.999999648	8.96058E-08	3.5191E-07
0.999993379	2.80686E-06	6.6212E-06
0.999993584	5.30479E-06	6.4156E-06
0.999998612	1.25861E-06	1.38826E-06

3-6 Neutrosophic construction for the total transit ship sizes

The following table shows the neutrosophic functions membership, indeterminacy and non-membership for the total transit ship sizes

membership	indeterminacy	non-membership
0.999990592	0.081595643	9.40827E-06
0.999993233	0.872542965	6.76657E-06
0.999997969	0.957965187	2.03095E-06
0.999996298	0.240681564	3.70197E-06
0.999996123	0.58812965	3.87712E-06
0.999994838	0.603066597	5.16243E-06
0.999995702	0.817902648	4.29796E-06
0.99999889	0.907832847	1.10985E-06
0.999996164	0.459277585	3.83603E-06
0.999995355	0.206840716	4.6451E-06
0.999995881	0.606598419	4.11885E-06

0.999997577	0.566837025	2.42347E-06
0.999999767	0.458398662	2.33325E-07
0.999998119	0.737134899	1.88071E-06
0.999996708	0.021642474	3.29234E-06
0.999996875	0.364123252	3.12526E-06
0.999999958	0.845535234	4.21479E-08
0.999995951	0.109050085	4.04896E-06
0.999994263	0.09590703	5.73726E-06
0.999998665	0.146926882	1.33461E-06
0.999996595	0.174293442	3.40508E-06
0.999998719	0.991957998	1.28056E-06
0.999995953	0.261600592	4.04712E-06
0.999998779	0.883490079	1.2211E-06
0.999999576	0.259355637	4.23945E-07
0.999997016	0.104312996	2.9843E-06
0.999995259	0.226220789	4.74055E-06
0.999999689	0.297662813	3.10824E-07
0.999999362	0.133707397	6.38072E-07
0.999999712	0.070726333	2.88346E-07
0.999998716	0.681388333	1.28386E-06
0.999999381	0.635096147	6.19429E-07

0.99999932	0.476381568	6.80231E-07
0.999996216	0.270553455	3.78429E-06
0.999997277	0.230908485	2.72295E-06
0.999998198	0.837658751	1.80196E-06
0.999996651	0.734244669	3.34864E-06
0.999999684	0.674576687	3.15575E-07

4 - Graphic Representation for Data in the Neutrosophic Environment

4-1 Neutrosophic functions of the Suez Canal revenues

The following graph shows the neutrosophic functions membership, indeterminacy and non-membership of the Suez Canal revenues from oil tankers

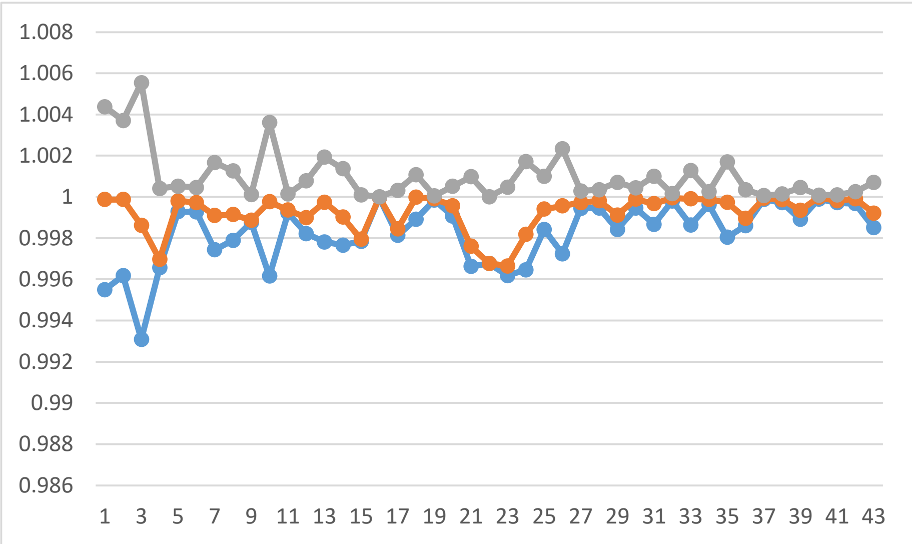

Fig.1, The neutrosophic functions of the Suez Canal revenue collected from oil tankers. (1976: 2019)

4 -2 Neutrosophic functions of the Suez Canal revenue collected from bulk cargo ships

The following graph shows the neutrosophic functions membership, indeterminacy and non-membership of the Suez Canal revenues received from bulk cargo ships.

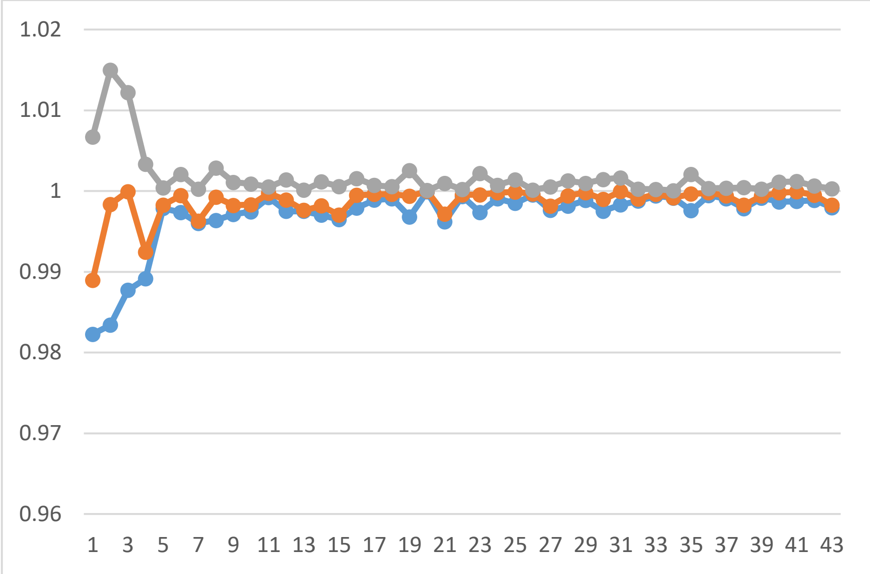

Fig.2. Neutrosophic functions of the Suez Canal revenue collected from bulk cargo ships. (1976: 2019)

4 -3 Neutrosophic functions of the Suez Canal revenue (total revenue)

The following graph shows the neutrosophic functions membership, indeterminacy and non-membership of the Suez Canal total revenue.

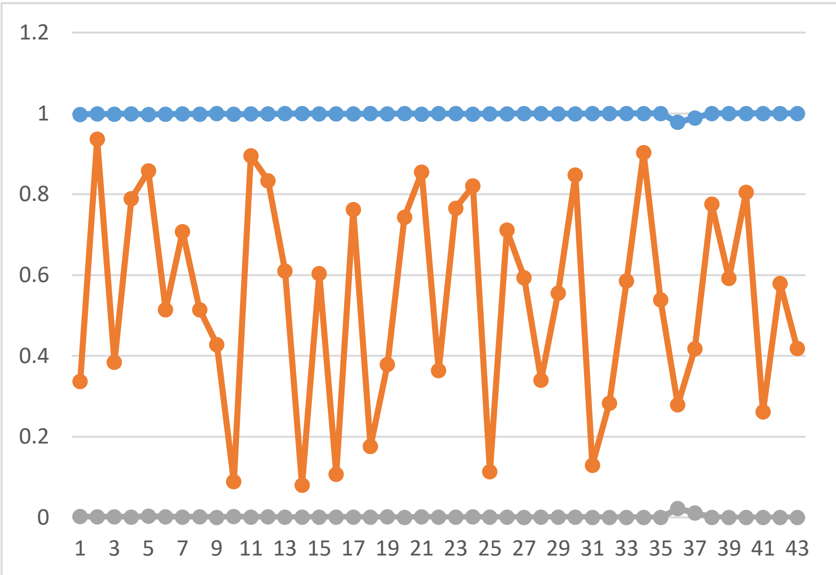

Fig.4. Neutrosophic functions of the Suez Canal revenues in million dollars (total revenue).

4 -4 Neutrosophic functions for volumes of shiploads

The following figure shows the neutrosophic functions membership, indeterminacy and non-membership of tonnage of tankers crossing the channel

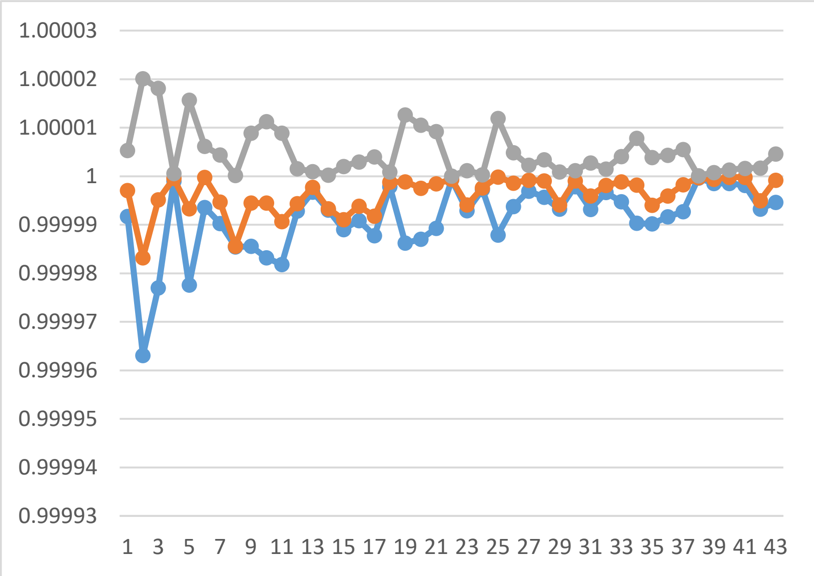

Fig.5. Neutrosophic functions for the volumes of tonnage of tankers crossing the channel.

4 - 5 Neutrosophic functions of tonnage of cargo vessels casting trans-channel

The following figure shows the neutrosophic functions membership, indeterminacy and non-membership of the tonnage of cargo ships for casting trans-shipment vessels.

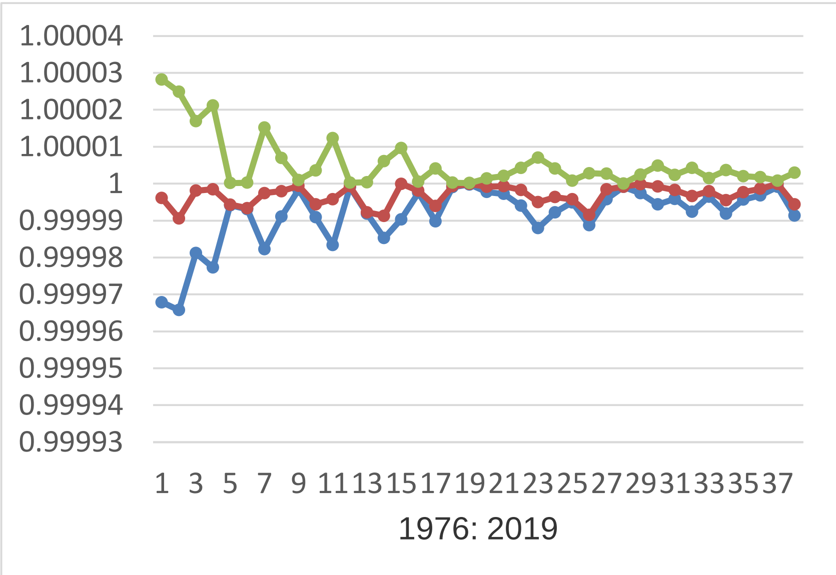

Fig.6. Neutrosophic functions of tonnage of cargo vessels casting trans-channel.

4 - 6 Neutrosophic functions of the tonnage of vessels transiting the channel

The following figure shows the neutrosophic functions membership, indeterminacy and non-membership of the sizes of tonnage of ships crossing the channel.

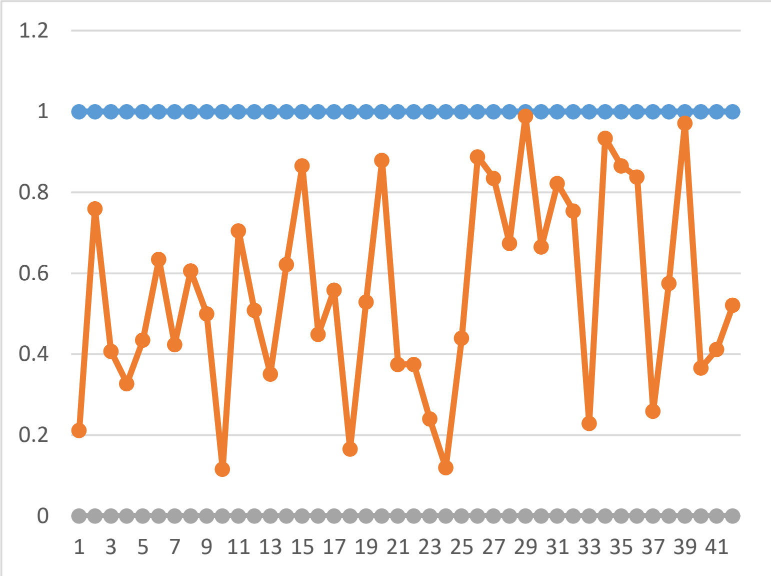

Fig.7. Neutrosophic functions of the sizes of tonnage of transit ships.

5 Decision Making for Neutrosophic Environment

Here we will present a study of the decision-making process regarding the better investment in the Suez Canal. Is it investing in oil tankers or investing in cargo ships, as this is done based on the previous neutrosophic data. This will be done by studying optimistic, pessimistic, and remorse entrances to the neutrosophic data, to see which oil tankers or cargo ships offer better returns to the Canal.

Study of entrances:

i. **The Optimistic entrance:**

We know that this entrance depends on evaluating the alternatives, in preparation for choosing the alternative that guarantees the best possible returns under optimistic natural states. Without any consideration for the pessimistic cases of this alternative. Which we express by the term (Max, Max). So that the first "Max" indicates the highest value, and the second "Max" denotes the optimistic natural state:

	Max Max
oil tankers	Max (0.999701164,0.000102857, 0.000298836) = 0.999701164
cargo ships	Max(0.99977598, 0.000190899, 0.00022402) = 0.99977598

Thus, according to the optimistic entrance, investing in the **cargo ships** is the best alternative considering that it includes the highest possible return, which is (0.99977598).

ii. **The conservative (pessimistic) entrance:**

We know that this entrance depends on evaluating alternatives. As a prelude to choosing the alternative, that guarantees the best possible returns in the light of pessimistic natural states.

Without regard for optimistic cases of that alternative. It is called the term (Max, Min), where "Max" means the highest value here, but it is related to the second part of the term "Min", which means the pessimistic natural state:

	Max Min
oil tankers	Max (0.988740191 ,0.00461327, 0.011259809) = 0.988740191
cargo ships	Max(0.979597358, 0.011222455, 0.020402642) =0.979597358

According to this entrance, investing in oil tankers is the best alternative, as it guarantees the highest possible return is (0.988740191).

iii. The entrance to remorse:

This entrance is not optimistic or pessimistic, but rather an intermediate entrance. It depends on the evaluation of the alternatives as a prelude to choosing the alternative that contains the least missed opportunities.

Choosing the most appropriate alternative in the light of this entrance requires creating a new matrix, as follows, we replace the alternative that achieves the highest value with a value of zero, given that there are no missed opportunities for this alternative.

	Highest neutrosophic return	**Lowest neutrosophic return**
oil tankers	(0.999701164,0.000102857, 0.000298836)	(0.988740191 ,0.00461327, 0.011259809)
cargo ships	(0.99977598, 0.000190899, 0.00022402)	(0.979597358, 0.011222455, 0.020402642)

	Highest neutrosophic return	**Lowest neutrosophic return**
oil tankers	(0.000074816,0.000088042, -0.000074816)	(0,0,0)
cargo ships	(0,0,0)	(0.009142833, -0.006609185, -0.009142833)

We subtract the highest value in the event of high return from the rest of the values present in this normal state. The same applies to the case of low return, and we subtract the highest value in the case of low return from the rest of the values found in this case.

Then now we create a short matrix that includes the highest missed opportunity values for each alternative, as follows:

	Missed opportunities
oil tankers	(0.000074816, 0.000088042, -0.000074816)
cargo ships	(0.009142833, -0.006609185, -0.009142833)

Consequently, according to this entrance, the appropriate alternative is oil tankers as it contains the least missed opportunities.

From the study of the previous three entrances in the light of the neutrosophic logic, we have different options for decision according to the entrances. This matter we can view positively as it enriches the decision-making process and is only a reflection of the circumstances of the decision-maker and the views that affect him.

6. Conclusion and Future Work:

Neutrosophic techniques as a generalization of crisp and fuzzy techniques that may better model imperfect information, which is omnipresent in any conscious decision making. In neutrosophic system, each attack is determined by membership, indeterminacy and non-membership degrees. In this paper, we have designed a program to generate neutrosophic grades for the most important variables of the waterway of the Suez Canal. In future studies we will design a statistical model to support and make decisions using the neutrosophic statistics. The future importance of the research paper is the use of neutrosophic in proposing a model for optimal decision-making in the neutrosophic environment.
The study aims at the possibility of proposing a general framework to support decision-making to maximize the profitability of the Suez Canal Authority by crossing ships using the neutrosophic analysis of navigation traffic data.
This is achieved through a set of objectives, as follows:

1. Neutrosophic analysis through the generation of organic functions with three degrees, for the navigation traffic in the Suez Canal.
2. Neutrosophic analysis of the numbers and volumes of tonnage of oil tankers transiting the Suez Canal through neutrosophic data.
3. Studying neutrosophic triple vehicles to predict future tanker and ship volumes.
4. Using the neutrosophic method to predict the value of revenues.

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Received: Apr 11, 2020. Accepted: July 2, 2020

Neutrosophic Vague Binary BCK/BCI-algebra

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Abstract: Ineradicable hindrances of the existing mathematical models widespread from probabilities to soft sets. These difficulties made up way for the opening of "neutrosophic set model'. Set theory of 'vague' values is an already established branch of mathematics. Complex situations which arose in problem solving, demanded more accurate models. As a result, 'neutrosophic vague' came into screen. At present, research works in this area are very few. But it is on the way of its moves. Algebra and topology are well connected, as algebra and geometry. So, anything related to geometric topology is equally important in algebraic topology too. Separate growth of algebra and topology will slow down the development of each branch. And in one sense it is imperfect! In this paper a new algebraic structure, BCK/BCI is developed for 'neutrosophic' and to 'neutrosophic vague' concept with 'single' and 'double' universe. It's sub-algebra, different kinds of ideals and cuts are developed in this paper with suitable examples where necessary. Several theorems connected to this are also got verified.

Keywords: Vague H - ideal, neutrosophic vague binary BCK/BCI - algebra, neutrosophic vague binary BCK/BCI – subalgebra, neutrosophic vague binary BCK/BCI - ideal, neutrosophic vague binary BCK/BCI p- ideal, neutrosophic vague binary BCK/BCI q - ideal, neutrosophic vague binary BCK/BCI a-ideal, neutrosophic vague binary BCK/BCI H - ideal, neutrosophic vague binary BCK/BCI - cut

Notations: NVBS : neutrosophic vague binary set, NVBSS : neutrosophic vague binary subset, NVBI : neutrosophic vague binary ideal, N BCK/BCI - algebra : neutrosophic BCK/BCI-algebra, NV BCK/BCI - algebra : neutrosophic vague BCK/BCI-algebra, NVB BCK/BCI - algebra : neutrosophic vague binary BCK/BCI - algebra, N BCK/BCI - subalgebra : neutrosophic BCK/BCI - subalgebra, NV BCK/BCI - subalgebra : neutrosophic vague BCK/BCI - subalgebra, NVB BCK/BCI – subalgebra : neutrosophic vague binary BCK/BCI - subalgebra, N BCK/BCI - ideal : neutrosophic BCK/BCI –ideal, NV BCK/BCI - ideal : neutrosophic vague BCK/BCI - ideal , NVB BCK/BCI- ideal : neutrosophic vague binary BCK/BCI - ideal, NVB BCK/BCI p-ideal : neutrosophic vague binary BCK/BCI p-ideal, NVB BCK/BCI q - ideal : neutrosophic vague binary BCK/BCI q - ideal, NVB BCK/BCI a - ideal : neutrosophic vague binary BCK/BCI a - ideal, NVB BCK/BCI H - ideal : neutrosophic vague binary BCK/BCI H - ideal

1. Introduction

Before 1990's, mathematicians and researchers made use of different mathematical models for problem solving viz. , Probability theory, Hard set theory, Fuzzy set theory, Rough set theory, Intuitionistic Fuzzy set theory etc., for problem solving. In 1993, W. L. Gau and D. J. Buehrer [16] introduced vague sets, with "truth and false" membership values as measurement tools. In 1995, Florentin Smarandache [13] introduced, "Neutrosophic set theory", in which an additional data 'uncertainty', is also got added besides 'truth and false'. In 2015, Shawkat Alkhazaleh [45] introduced 'Neutrosophic Vague' set theory, by inserting vague values, to each neutrosophic value –'truth, uncertainty & false'. With its several operations, he gave a rich explanation about the concept, in his pioneer paper itself. Neutrosophic set's main difference with Neutrosophic Vague is, with its outlook as an "interval" (imposed with certain conditions). An algebraic structure is a universal set with a set of operations applicable to that set, together with a set of axioms to be satisfied. BCK/BCI-logical algebra- is a new type of algebraic structure developed in 1966, by Yasuyuki Imai and Kiyoshi Iséki [48]. It is now found to be an active research area. MV-algebras, Boolean algebras etc. are some t-related logical algebras which extend to BCK-algebra. BCK-algebra further extends to BCI-algebra. In 2015, Samy M. Mostafa and Reham Ghanem [42] gave cubic structures of medial ideal on BCI- algebras. Paper introduced cubic medial – ideal, and it illustrates a relation between cubic medial – ideal and cubic medial BCI – ideal. Homomorphism and Cartesian product of this concept have been duly verified. In 2017, M. Kaviyarasu and K. Indira [22] gave a review on BCI/BCK-algebras and its developmental scenario. In 1999, Khalid and Ahmad [25] introduced fuzzy H- ideals in BCI-algebras. In 2007, Kordi and Moussavi [26] gave a detailed study on fuzzy ideals of BCI-algebras. In 2012, Borumand Saeid. A, Prince Williams. D. R and Kuchaki Rafsanjani [10] gave a preliminary note on anti-fuzzy BCK/BCI-subalgebra. Paper mainly contributed on generalized notion of fuzzy BCK/BCI-algebra. In 2018, based on hyper fuzzy structure, Young Bae Jun, Seok-Zun Song and Seon Jeong Kim [50] introduced length - fuzzy subalgebras (length -k-fuzzy; k = $1\leq k\leq 4$) in BCK/BCI- algebras. In 2018, Anas Al-Masarwah and Abd Ghafur Ahmad [4] discussed some properties of bipolar fuzzy H-ideals in BCK/BCI--algebra. In 2019, Anas Al-Masarwah, Abd Ghafur Ahmad [5] introduced m-Polar fuzzy subalgebras, m-polar fuzzy closed ideal and m-polar fuzzy commutative ideal of m-polar fuzzy sets. They also investigated their several characterizations and theorems. In 2019, Alcheikh. M & Anas Sabouh [3] proved several theorems connected to the already existing notions of fuzzy ideal, anti-fuzzy ideal and anti-fuzzy p-ideal of BCK-algebra. In 1983, Hu. Q. P and Li. X [19] defined BCH-algebra as a generalization of BCK/BCI-algebra. In 2001, Muhammad Anwar Chaudhary and Hafiz Fakhar-ud-din [34] studied some classes of BCH- algebras. In 1998, Jun. Y. B, Roh. E. H and Kim. H. S [21] introduced BH -algebra as a generalization of BCH/BCI/BCK-algebra and they discussed it's ideals and homomorphisms. In 2001, Qun Zhang, Young Bae Jun and Eun Hwan Roh [41] studied the connection of BH- algebras with 'BCH' and 'BCK/BCI'-algebras. They defined BH_1-algebra and normal BH - algebra. In 1996, Neggers. J and Kim. H. S [38] introduce d-algebras as a generalization of BCK-algebras and proved that oriented digraphs correspond to class of edge d-algebras. They also gave several notions of d-algebra with examples and also defined direct product and direct sum of d-algebras. In 1996, Stanley Gudder [46] introduced D-algebras as a generalization of D-poset (without assuming a partial order in D poset). He explained (interval) effect algebras, based on group structure and proved several lemmas and theorems regarding to this in a deep manner. In 2012, Muhammad Anwar Chaudhry and Faisal Ali [32] introduced multipliers in d-algebras. He remarked with example that every BCK-algebra is a d-algebra but the converse does not hold, in general. He defined positive implicative d - algebra and proved related theorems. In 2005, Akram. M and Dar. K. H [2] defined Fuzzy d-algebras, Fuzzy d-ideals, Fuzzy d-subalgebras, Fuzzy d-homomorphisms. In 2014, S. R. Barbhulya. K. Dutta. Choudhury [43] defined $(\varepsilon, \varepsilon\vee q)$- fuzzy ideals of d - algebra, it's cartesian product, homomorphism and also investigated a few theorems. In 2002, Neggers. J and Kim. H. S [39] introduced B – algebra which is closely related to BCH/BCI/BCK - algebras. Using a digraph on algebras, they gave a connection between B - algebras and groups. They also defined commutative B - algebras. In 2006, BM - algebras are introduced by Kim. C. B and Kim. H. S [24] as a specialization of B - algebras. They proved BM - algebra as a proper subclass of B - algebras. They showed that BM - algebra is equivalent to a 0 - commutative B - algebra. In 2011, A. Borumand Saeid and A. Zarandi [11] applied Vague Set theory to BM - algebras and discussed its cuts, Artinian and Noetherian concepts. In 2017, Arsham Borumand Saeid, Hee Sik Kim and Akbar Rezaei [7] introduced BI - algebras as a generalization of (dual) implication algebra. They defined ideals and congruence relations in BI - algebras. In 2006, Hee Sik Kim and Young Hee Kim [18] studied a generalization of BCK – algebras known as BE - algebras. They discussed its filter and self - distributive property along with some theorems. Various structures formed within short periods of time, along with smaller or bigger changes are B, BE, BF, $\mathrm{BF_1}$, $\mathrm{BF_2}$, BG, BI , BL, BM, BN, BO, BP, BQ, BZ, CI, Coxeter - algebra, FL, $\mathrm{FL_{ew}}$ (bounded integral commutative residuated lattice), GK, HW, KU, PS, Q, QS, QP, RG, TM, TP, TU, BCC (or BIK), BCL, BBG, $\mathrm{SB}L_{\neg}$, Smarandache BCH - algebra , SU, UP, Z etc. In 2006, group theory of vague sets is introduced by Hakimuddin Khan, Musheer Ahmad and Ranjit Biswas [17]. In 2008, Lee. K. J, So. K. S and Bang. K. S [27] introduced vague BCK/BCI- algebras with several theorems and propositions. It was one of the pioneer work in the area of BCK/BCI - algebraic structure with vague sets. Notion of vague ideals are introduced with properties. A condition for a vague set to become a vague ideal is also provided. Several characteristics for vague ideal are investigated and established. Arsham Borumand Saeid [6] also introduced vague BCK/BCI- algebras in 2008, but his work has been published in 2009. He discussed on cuts, subalgebras and their related theorems of vague BCK/BCI – algebra. In 2017, Jafari. A, Mariapresenti. L and Arockiarani. I [20] discussed on vague direct product in BCK- algebra. In 2002, Neggers. J and Kim Hee Sik [40] introduced, $\beta-$algebra as generalization of BCK-algebras. In 2016, B. Nageswararao, N. Ramakrishna, T. Eswarlal [37] introduced vague $\beta-$algebras, vague $\beta-$ideals, translation operators on vague $\beta-$algebras, translation operators on vague $\beta-$ideals, vague $\beta-$ideal extension of vague $\beta-$algebra etc. In 2013, Yun Sun Hwang and Sun Shin Ahn [52] developed vague p-ideals and vague a-ideals in BCI-algebras. In 2006, neutrosophic algebraic structures are introduced by Vasantha Kandasamy. W. B and Florentin Smarandache [47]. Neutrosophic group structure, neutrosophic ring structure etc., with lots of theorems and propositions are investigated. Based on this, in 2015, A. A. A. Agboola and B. Davvaz [1] introduced neutrosophic BCI/BCK- algebras and their elementary properties. In 2018, Young Bae Jun, Seok - Zun Song, Florentin Smarandache and Hashem Bordbar [51] discussed neutrosophic quadruple BCK/BCI-algebras. Paper consists of the newly defined definition of neutrosophic quadruple BCK/BCI-number, neutrosophic quadruple BCK/BCI-ideals etc., with proper verification of inter-connected notions. In 2019, Muhiuddin. G, Smarandache. F, Young Bae Jun, [35] gave a new idea - neutrosophic Quadruple ideals in neutrosophic Quadruple BCI- algebras. In 2018, Seon Jeong Kim, Seok-Zun Song and Young Bae Jun [44] discussed generalizations of neutrosophic subalgebras in BCK/BCI--algebras based on neutrosophic points. In 2018, Muhammad Akram, Hina Gulzar, Florentin Smarandache and Saeid Broumi [34] defined single-valued neutrosophic topological K-algebras and investigated some of the properties like C_5-connected, super-connected, compact and hausdorff. They also investigated the image and pre-image of these algebras under homomorphism. In 2018, Young Bae Jun, Florentin Smarandache, Mehmat AliÖztürk [49] introduced commutative falling neutrosophic ideals in BCK-algebras. In 2017, Bijan Daavavaz, Samy M. Mostafa and Fatema F. Kareem [8] developed Neutrosophic ideals of neutrosophic KU - algebras. In 2018, Muhiuddin. G, Bordbar. H, Smarandache. F, Jun. Y. B, [36] gave certain results on $(\varepsilon, \varepsilon)$ – neutrosophic subalgebras and ideals in BCK/BCI- algebras. They defined commutative $(\varepsilon, \varepsilon)$ neutrosophic ideal and commutative falling neutrosophic ideal for a BCK-algebra. In 2019, Chul Hwan Park [12] developed neutrosophic ideals of subtraction algebras. Khadaeman. S, Zahedi. M. M, Borzooei. R. A, Jun. Y. B [23] developed neutrosophic hyper BCK-ideals in 2019. Neutrosophic sets handle uncertainty in a remarkable way. But its generalization named as plithogenic set handles uncertainty in a more powerful level than neutrosophic! Application works using plithogenic are very few as it is a recently introduced work in set theory. But, in 2019, Mohamed Abdel-Basset and Rehab Mohamed [31], used a plithogenic TOPSIS-CRISIS method to sustainability supply chain risk management in telecommunication industry. Problem is well and systematically explained with adequate assistance of diagrams like bar diagram, pie diagram etc. In 2019, Mohamed Abdel-Basset, Mai Mohamed, Mohamed Elhoseny, Le Hoang Son, Francisco Chiclana, Abd El-Nasser H Zaied [28] pointed out some draw backs of

Dice and Jaccard similarity measures in bipolar neutrosophic set with examples. They provided a cosine similarity measure and weighted cosine similarity measure methods for 'bipolar and interval-valued bipolar'- neutrosophic set. They used the above method for diagnosing bipolar disorder diseases. A computational algorithm for MADM (Multi Attribute Decision Making) has also given in the paper. In 2019, using 'neutrosophic sets', Mohamed Abdel-Basset, Mumtaz Ali, Asma Atef [30] framed a resource levelling problem to construction projects. To improve work efficiency and to minimize cost were underlying principle. For calculating activity durations, trapezoidal neutrosophic numbers were used in this model. In 2019, Mohamed Abdel-Basset, Mumtaz Ali, Asma Atef [29] designed uncertainty assessments of linear Time-Cost Tradeoffs using neutrosophic sets. In 2020, Florentin Smarandache [14] introduced neutro - algebra as a generalization of partial algebra with examples and showed their differences. Points of odds between universal algebra, neutro - algebra and anti-algebra are well explained in the paper. Neutro - functions are more useful when range or domain is not clear. Several applications to neutro functions are given with a well explanation. In 2020, Bordbar. H, Mohseni Takallo. M, Borzooei. R. A, Young Bae Jun [9], defined BMBJ - neutrosophic subalgebra in BCI/BCK – algebras. Authors introduced BMBJ neutrosophic set as a generalization of neutrosophic set. Its subalgebra, images, translations, S - extension and its application to BCI/BCK – algebra are defined and explained. Neutrosophic vague binary sets are developed by Francina Shalini. A and Remya. P. B [15] in 2019. Authors developed a neutrosophic vague set with 2 universes and discussed its properties.

In this paper, BCK/BCI-algebraic structure is introduced to neutrosophic vague binary sets and it is simply called as neutrosophic vague binary BCK/BCI - algebra. It's ideal, neutrosophic vague binary BCK/BCI – ideal is also developed. Moreover, different neutrosophic vague binary BCK/BCI-ideals like neutrosophic vague binary BCK/BCI p-ideal, neutrosophic vague binary BCK/BCI q-ideal, neutrosophic vague binary BCK/BCI a-ideal and neutrosophic vague binary BCK/BCI H-ideal are also developed and compared. Neutrosophic vague binary BCK/BCI - subalgebra, neutrosophic vague binary BCK/BCI-cut and their relationships, properties and several theorems are also investigated and illustrated with examples.

Without algebra we can't even imagine mathematics. In one sense, geometry and algebra are equally important in mathematics. Even a layman can understand geometry because it deals with lines and shapes. It's applicational use in day to day life can't neglect. But algebra is like a silent player. In geometry, for finding out the solutions to lot of situations like, to get co-ordinates of centroid or to find out solution space to equations which represents lines, ellipse, hyperbolas, etc. - common way is to adopt the method of algebra. Study of surfaces is the main concept behind topology. Topological objects can bend, twist or stretch but are not allowed to tear, since there it loses its continuity. As a result, topological objects will become non – topological! Automatically they admit lack of homeomorphism in these situations. Geometrical nature of topology needs the assistance of algebra in several circumstances. This inevitable need of a mixed strategy, produced a new branch of mathematics called 'algebraic topology'. So developmental moments in any branch connected to topology from basic sets to neutrosophic sets via "fuzzy, rough, intuitionistic fuzzy, vague, interval mathematics, soft"- will equally demand the developments of it's counterpart-algebra. Thus both of them developed equally and produced vivid outputs like fuzzy BCK/BCI algebra, intuitionistic fuzzy BCK/BCI-algebra, rough BCK/BCI-algebra, vague BCK/BCI- algebra, soft BCK/BCI algebra and so on. So to stabilize neutrosophic branch, developments in various algebraic structures like BCK/BCI, BCH, BH etc are very critical and essential. This work will be important to neutrosophic due to its 'easy way approach' than [1] to reach to the same destination.

Method given in [1] is equally good but the concept of generating element is a little bit perplexing. Since [1] is closely connected to [47], it will be helpful, to verify lot of deep ideas given in [47]. Neutrosophic 'group and loop' concepts are well defined with examples and explanations in

[47]. It is to be noted that, as per [47], neutrosophic group does not possess a direct group structure, but it always contains one! Neutrosophic vague is a mixed form of neutrosophic and vague. It draws every positives and negatives of both the aforementioned sets. Numerical calculations for 'neutrosophic' are more than 'vague' due to its additional component - uncertainty. In real life problems, complex situations demand a more clear and easily accessible method to use with - 'neutrosophic, neutrosophic vague or neutrosophic vague binary'- set values. In group theory or ring theory algebraic structure is formed in such a manner that it includes set itself as a first member of the structure, then provide various algebraic operations as example shows : $(Z, +_4)$, (mZ, +), $(R, +, \ .)$ etc. Vague BCK/BCI algebraic-structure is defined as $(\mathrm{U}, *, 0)$ by enclosing only universal set and by omitting the corresponding vague set A. But in this context, universal set and vague set are simultaneously essential and available: since problem is being to be checked for a 'vague BCK/BCI-algebra and not for mere BCK/BCI-algebra' ! Our conclusion is that, being a core object in taking a decision to the question 'vague BCK/BCI-algebra or not? ' : inclusion of vague set, 'inside the structure' is important. It will avoid more confusions while doing theoretical work! Same thing is referable to fuzzy BCK/BCI-algebra, intuitionistic BCK/BCI-algebra, neutrosophic BCK/BCI-algebra and so on. This will be useful and applicable to all other existing structures like BCH, BH, B etc., with uncertain sets.

It is hoped that, when comparing to [1], concept developed in this paper, will be more useful to common people, since it uses values directly and hence easily accessible. This method depends on vague BCK/BCI paper [6]. In this paper our primary interest is to develop BCK/BCI-algebraic concept to neutrosophic vague binary sets. For this neutrosophic BCK/BCI algebraic concept and neutrosophic vague BCK/BCI-algebraic concept are needed as a base. Since it is not developed yet, in this paper, those are also developed with neutrosophic vague binary! An alternative structure approach, to vague BCK/BCI-algebra mentioned in [6] can be given as follows: A vague BCK/BCI-algebra is a structure $\mathfrak{B}_{\mathrm{A}} = \left(\mathrm{A}, \mathrm{U}^{\mathfrak{B}_{\mathrm{A}}} = (\mathrm{U}, *, 0), *, 0\right)$=$\left(\mathrm{A}, \mathrm{U}^{\mathfrak{B}_{\mathrm{A}}}, *, 0\right)$, where A is the vague set under consideration and $\mathrm{U}^{\mathfrak{B}_{\mathrm{A}}} = (\mathrm{U}, *, 0)$ is the underlying BCK/BCI-algebraic structure for A with universal set U, binary operation "$*$" and with constant "0". Similarly, when A becomes fuzzy set, the structure got is fuzzy BCK/BCI-algebraic. For theoretical applications, new approach is found to be more helpful and clear. Throughout this paper, new structure is used for neutrosophic/neutrosophic vague/ neutrosophic vague binary BCK/BCI-algebra.

Primary objective of this work is to develop a BCK/BCI-algebraic structure to neutrosophic vague binary set. Along with, care is taken, to use this novel concept, in 'theoretical applications'. Secondary objective is kept as the formation of various ideals to this new concept and their verification in theory part.

Paper consists of 8 sections. 1st section, provides an introduction, in which literature review has given. 2nd paragraph gives a general format of the work. 3rd paragraph explains why this work is essential to neutrosophic branch. 4th paragraph, points out 2 limitations of existing approaches. 5th paragraph mentions the alternative approaches to the limitations. 6th paragraph gives 2 objectives for the work. 7th paragraph, clearly explains how the paper is organized. 8th paragraph, summarizes all contributions of this paper in bullets. 2nd section of the paper describes materials for the work. In 3rd, neutrosophic vague binary/ neutrosophic vague/ neutrosophic BCK/BCI -algebras are developed. In 4th, neutrosophic vague binary BCK/BCI-subalgebra and neutrosophic vague binary BCK/BCI-ideal are developed. In 5th section, various neutrosophic vague binary BCK/BCI-ideals are formed and compared using a table. In 6[th] section, neutrosophic vague binary/ neutrosophic vague/ neutrosophic BCK/BCI - cuts are defined. In 7[th] section, propositions and lemmas related to this novel concept are discussed as a theoretical application. In 8th section, a conclusion to the paper is given.

Contributions in this paper are given in bullets below:

- Vague H-ideal
- Neutrosophic Vague Binary BCK/BCI-algebra
- Neutrosophic Vague Binary BCK/BCI-subalgebra
- Neutrosophic Vague Binary BCK/BCI-ideal
- Neutrosophic Vague Binary BCK/BCI- p ideal
  Neutrosophic Vague Binary BCK/BCI- q ideal
  Neutrosophic Vague Binary BCK/BCI- a ideal
  Neutrosophic Vague Binary BCK/BCI- H ideal
- Neutrosophic vague binary BCK/BCI- cut

2. Preliminaries

Some preliminaries are given in this section

Definition 2.1 [45] (Neutrosophic Vague Set)

A neutrosophic vague set A_{NV} (NVS in short) on the universe of discourse X can be written as $A_{NV} = \{\langle x ; \hat{T}_{A_{NV}}(X) ; \hat{I}_{A_{NV}}(X), \hat{F}_{A_{NV}}(X)\rangle ; x \in X\}$ whose truth-membership, indeterminacy-membership and falsity-membership functions are defined as

$\hat{T}_{A_{NV}}(x) = [T^-, T^+]$, $\hat{I}_{A_{NV}}(x) = [I^-, I^+]$ and $\hat{F}_{A_{NV}}(x) = [F^-, F^+]$

where (1) $T^+ = (1- F^-)$; $F^+ = (1- T^-)$ and

(2) $-0 \leq T^- + I^- + F^- \leq 2^+$

$-0 \leq T^+ + I^+ + F^+ \leq 2^+$

Definition 2.2 [15] (Neutrosophic Vague Binary Set)

A neutrosophic vague binary set (NVBS in short) M_{NVB} over a common universe $\{U_1 = \{x_j / 1 \leq j \leq n\}; U_2 = \{y_k/1 \leq k \leq p\}\}$ is an object of the form

$$M_{NVB} = \left\{\langle \frac{\hat{T}_{M_{NVB}}(x_j), \ \hat{I}_{M_{NVB}}(x_j), \ \hat{F}_{M_{NVB}}(x_j)}{x_j} ; \ \forall\, x_j \in U_1\rangle \langle \frac{\hat{T}_{M_{NVB}}(y_k), \ \hat{I}_{M_{NVB}}(y_k), \ \hat{F}_{M_{NVB}}(y_k)}{y_k} ; \ \forall\, y_k \in U_2\rangle\right\}$$

is defined as, ($\forall\, x_j \in U_1$ & $\forall\, y_k \in U_2$)

$\hat{T}_{M_{NVB}}(x_j) = [T^-(x_j), T^+(x_j)]$, $\hat{I}_{M_{NVB}}(x_j) = [I^-(x_j), I^+(x_j)]$ and $\hat{F}_{M_{NVB}}(x_j) = [F^-(x_j), F^+(x_j)]$ and

$\hat{T}_{M_{NVB}}(y_k) = [T^-(y_k), T^+(y_k)]$, $\hat{I}_{M_{NVB}}(y_k) = [I^-(y_k), I^+(y_k)]$ and $\hat{F}_{M_{NVB}}(y_k) = [F^-(y_k), F^+(y_k)]$

where (1) $T^+(x_j) = 1- F^-(x_j)$; $F^+(x_j) = 1- T^-(x_j)$ & $T^+(y_k) = 1- F^-(y_k)$; $F^+(y_k) = 1- T^-(y_k)$

(2) $^-0 \leq T^-(x_j)+I^-(x_j)+F^-(x_j) \leq 2^+$; $^-0 \leq T^-(y_k)+I^-(y_k)+F^-(y_k) \leq 2^+$ **or** $^-0 \leq T^-(x_j)+I^-(x_j)+F^-(x_j)+T^-(y_k)+I^-(y_k)+F^-(y_k) \leq 4^+$

$^-0 \leq T^+(x_j)+I^+(x_j)+F^+(x_j) \leq 2^+$; $^-0 \leq T^+(y_k)+I^+(y_k)+F^+(y_k) \leq 2^+$ **or** $^-0 \leq T^+(x_j)+I^+(x_j)+F^+(x_j)+T^+(y_k)+I^+(y_k)+F^+(y_k) \leq 4^+$

(3) $T^-(x_j), I^-(x_j), F^-(x_j) : V(U_1) \longrightarrow [0,1]$ and $T^-(y_k), I^-(y_k), F^-(y_k) : V(U_2) \longrightarrow [0,1]$

$T^+(x_j), I^+(x_j), F^+(x_j) : V(U_1) \longrightarrow [0,1]$ and $T^+(y_k), I^+(y_k), F^+(y_k) : V(U_2) \longrightarrow [0,1]$

Here $V(U_1), V(U_2)$ denotes power set of vague sets on U_1, U_2 respectively

Definition 2.3 [48] (BCI-algebra)

Let X be a non-empty set with a binary operation $*$ and a constant 0. Then (X, $*$, 0) is called a BCI-algebra if it satisfies the following conditions:

(i) $((x*y) *(x*z)) *(z*y) = 0$

(ii) $((x * (x * y)) * y = 0$
(iii) $(x * x) = 0$
(iv) $(x * y) = 0$ and $(y * x) = 0$ imply $x = y$, for all $x, y, z \in X$

Remark 2.4 [48]

We can define a partial ordering $\leq$ by $x \leq y$ if and only if $(x * y) = 0$

Remark 2.5 [48] (BCK – algebra)

If a BCI-algebra X satisfies $(0 * x) = 0$ for all $x \in X$, then we say that X is a BCK- algebra

Remark 2.6 [48]

A BCI-algebra X has the following properties:
(i) $(x * 0) = x$; $\forall x \in X$
(ii) $(x * y) * z = (x * z) * y$; $\forall x, y, z \in X$
(iii) $0 * (x * y) = (0 * x) * (0 * y)$; $\forall x, y \in X$
(iv) $x * (x * (x * y)) = (x * y)$; $\forall x, y \in X$
(v) $x \leq y \Rightarrow (x * z) \leq (y * z)$; $(z * y) \leq (z * x)$; $\forall x, y, z \in X$
(vi) $(x * z) * (y * z) \leq (x * y)$; $\forall x, y, z \in X$
(vii) $0 * (0 * ((x * z) * (y * z))) = ((0 * y) * (0 * x))$; $\forall x, y, z \in X$
(viii) $0 * (0 * (x * y)) = ((0 * y) * (0 * x))$; $\forall x, y \in X$

Definition 2.7 [2, 4, 6, 25, 52] (Sub-algebra, Ideal, p-ideal, q-ideal, a-ideal, H-ideal)

Any non-empty subset I of a BCK/BCI- algebra X is called,
- subalgebra /ideal /p-ideal /q-ideal /a-ideal /H-ideal - of X, if it satisfies the axioms given table:

	Condition 1	Condition 2
Subalgebra of X	Nil	$(x * \mathbf{y}) \in I$; $\forall x, y \in I$
BCK/BCI-subalgebra of X	Nil	$(x * \mathbf{y}) \in I$; $\forall x, y \in I$
μ be a fuzzy BCK/BCI-algebra of X	Nil	$\mu(\mathbf{x} * \mathbf{y}) \geq \mathbf{min}\{\mu(\mathbf{x}) * \mu(\mathbf{y})\}$; $\forall \mathbf{x}, \mathbf{y} \in \mathbf{X}$; μ **be a fuzzy set in a BCK/BCI – algebra X**
A be a vague BCK/BCI-algebra of X	Nil	$\mathbf{V_A}(\mathbf{x} * \mathbf{y}) \succcurlyeq \mathbf{r\,min}\{\mathbf{V_A}(\mathbf{x}), \mathbf{V_A}(\mathbf{y})\}$; $\forall \mathbf{x}, \mathbf{y} \in \mathbf{X}$; **i. e.,** $\mathbf{t_A}(\mathbf{x} * \mathbf{y}) \geq \mathbf{min}\{\mathbf{t_A}(\mathbf{x}), \mathbf{t_A}(\mathbf{y})\}$; $\mathbf{1 - f_A}(\mathbf{x} * \mathbf{y}) \geq \mathbf{min}\{\mathbf{1 - f_A}(\mathbf{x}), \mathbf{1 - f_A}(\mathbf{y})\}$ **A be a vague set in a BCK/BCI – algebra X**
Ideal of X	$0 \in I$	$(x * y) \in I$ & $y \in I \Rightarrow x \in \mathbf{I}$; $\forall \mathbf{x} \in \mathbf{I}$, $\forall \mathbf{y} \in X$
μ be a fuzzy BCI- ideal of X	$\mu(0) \geq \mu(x)$	$\mu(\mathbf{x}) \geq \mathbf{min}\{\mu(\mathbf{x} * \mathbf{y}), \mu(\mathbf{y})\}$; $\forall \mathbf{x}, \mathbf{y} \in \mathbf{X}$; μ **be a fuzzy set in a BCI – algebra X**
BCK/BCI - Ideal of X	$0 \in I$	$(x * y) \in I$ & $y \in I \Rightarrow x \in \mathbf{I}$; $\forall \mathbf{x} \in \mathbf{I}$, $\forall \mathbf{y} \in X$
p - ideal of X	$0 \in I$	$[(x * z) * (y * z)] \in \mathbf{I}$ & $y \in \mathbf{I} \Rightarrow x \in \mathbf{I}$; $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in X$
q - ideal of X	$0 \in I$	$[x * (y * z)] \in \mathbf{I}$ & $y \in \mathbf{I} \Rightarrow (x * z) \in \mathbf{I}$; $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in X$
a - ideal of X	$0 \in I$	$[(x * z) * (0 * y)] \in \mathbf{I}$ & $z \in \mathbf{I} \Rightarrow (y * x) \in \mathbf{I}$; $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in X$
H - ideal of X	$0 \in I$	$[x * (y * z)] \in \mathbf{I}$ & $y \in \mathbf{I} \Rightarrow (x * z) \in \mathbf{I}$; $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in X$
μ be a fuzzy BCK/BCI-ideal of X	$\mu(0) \geq \mu(x)$	$\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$; $\forall x, y \in X$; μ be a fuzzy set in a BCK/BCI – algebra X
vague BCI-ideal of X	$V_A(0) \geq V_A(x)$	$V_A(x) \geq r\min\{V_A(x * y), V_A(y)\}$; $\forall x, y \in X$; A- vague set in X

Remark 2.8 [6] (r min & r max)

Let D[0, 1] denote the family of all closed sub-intervals of [0, 1]. Now we define the refined minimum (briefly, *r min*) and an order "≤" on elements $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2]$ of D[0, 1] as : *r min* $(D_1, D_2) = [min\{a_1, a_2\}, min\{b_1, b_2\}]$. Similarly, we can define $\geq$, = and *r max*. Then the concept of r min and r max could be extended to define r inf and r sup of infinite number of elements of D [0, 1]. It is a known fact that L = {D [0, 1], *r inf*, *r sup*, ≤} is a lattice with universal bounds [0, 0] and [1, 1].

Definition 2.9 [6] (Vague–cuts)

Let A be a vague set of a universe X with the true-membership function t_A and false-membership function f_A. The (α, β)-cut of the vague set A is a crisp subset $A_{(\alpha, \beta)}$ of the set X given by

$A_{(\alpha,\beta)} = \{x \in X / V_A(x) \geq [\alpha, \beta]\}$ where $\alpha \leq \beta$. Clearly, $A_{(0,0)} = X$. The (α, β)-cuts are also called vague-cuts of the vague set A.

The α-cut of the vague set A is a crisp subset A_α of the set X given by $A_\alpha = A_{(\alpha,\alpha)}$.
Note that $\mathbf{A_0} = X$ and if $\alpha \geq \beta$ then $A_\beta \subseteq A_\alpha$ and $A_{(\beta,\alpha)} = A_\alpha$.
Equivalently, we can define the α-cut as $A_\alpha = \{x \in X / \mathbf{t_A(x)} \geq \alpha\}$

Definition 2.10 [50]
Given a non-empty set X, let $B_K(X)$ and $B_I(X)$ denote the collection of all BCK-algebras and all BCI-algebras, respectively. Also, $B(X) := B_K(X) \cup B_I(X)$.

For any $(X, *, 0) \in B(X)$, a fuzzy structure (X, μ) over $(X, *, 0)$ is called a

- Fuzzy subalgebra of $(X, *, 0)$ with type 1 (briefly, 1-fuzzy subalgebra of $(X, *, 0)$ if
  $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$; $\forall\ \mathbf{x}, \mathbf{y} \in X$

- Fuzzy subalgebra of $(X, *, 0)$ with type 2 (briefly, 2 - fuzzy subalgebra of $(X, *, 0)$ if
  $\mu(x * y) \leq \min\{\mu(x), \mu(y)\}$; $\forall\ \mathbf{x}, \mathbf{y} \in X$

- Fuzzy subalgebra of $(X, *, 0)$ with type 3 (briefly, 3-fuzzy subalgebra of $(X, *, 0)$ if
  $\mu(x * y) \geq \max\{\mu(x), \mu(y)\}$; $\forall\ \mathbf{x}, \mathbf{y} \in X$

- Fuzzy subalgebra of $(X, *, 0)$ with type 4 (briefly, 4-fuzzy subalgebra of $(X, *, 0)$
  $\mu(x * y) \leq \max\{\mu(x), \mu(y)\}$; $\forall\ \mathbf{x}, \mathbf{y} \in X$

3. Neutrosophic vague binary BCK/BCI-algebra

In this section neutrosophic BCK/BCI-algebra is developed first, based on paper [6]. Neutrosophic BCK/BCI-algebraic structure developed in this paper is a little bit different from the definition given in paper [1]. Concept is extended to neutrosophic vague sets and to neutrosophic vague binary sets.

Definition 3.1 (Neutrosophic BCK/BCI-algebra)
A neutrosophic BCK/BCI-algebra is a structure $\mathfrak{B}_{M_N} = \left(M_N,\ U^{\mathfrak{B}_{M_N}} = (U, *, 0),\ *, 0\right) = \left(M_N, U^{\mathfrak{B}_{M_N}},\ *, 0\right)$ where,

(1) M_N is a non-empty neutrosophic set
(2) $U^{\mathfrak{B}_{M_N}} = (U,\ *,\ 0)$ is the underlying BCK/BCI- algebraic structure, to the neutrosophic set M_N with a universal set U, a binary operation "$*$" & a constant "0" . It satisfies the following axioms :

(i) $((u_x * u_y) * (u_x * u_z)) * (u_z * u_y) = 0$ (ii) $((u_x * (u_x * u_y)) * u_y = 0$ (iii) $(u_x * u_x) = 0$

(iv) $(u_x * u_y) = 0$ and $(u_y * u_x) = 0$ imply $u_x = u_y$, $\forall\ u_x, u_y, u_z \in U$ (v) $(0 * u_x) = 0\ \forall\ u_x \in U$

(3) "$*$" and "0" are taken as defined in (2)

which satisfies the following condition,
$N_{M_N}(u_x * u_y) \succcurlyeq r\min\{N_{M_N}(u_x), N_{M_N}(u_y)\}$; $\forall\ u_x, u_y \in U$. That is,
$T_{M_N}(u_x * u_y) \geq \min\{T_{M_N}(u_x), T_{M_N}(u_y)\}$; $I_{M_N}(u_x * u_y) \leq \max\{I_{M_N}(u_x), I_{M_N}(u_y)\}$; $F_{M_N}(u_x * u_y) \leq \max\{F_{M_N}(u_x), F_{M_N}(u_y)\}$

Definition 3.2. (Neutrosophic vague BCK/BCI-algebra)

A neutrosophic vague BCK/BCI - algebra is a structure,

$\mathfrak{B}_{M_{NV}} = \left(M_{NV}, U^{\mathfrak{B}_{M_{NV}}} = (U, *, 0), *, 0\right) = \left(M_{NV}, U^{\mathfrak{B}_{M_{NV}}}, *, 0\right)$, where

(1) M_{NV} is a non-empty neutrosophic vague set

(2) $U^{\mathfrak{B}_{M_{NV}}} = (U, *, 0)$ is the underlying BCK/BCI- algebraic structure to the neutrosophic vague set M_{NV} with a universal set U, a binary operation "$*$" & a constant "0" satisfies the following axioms:

(i) $((u_x * u_y) * (u_x * u_z)) * (u_z * u_y) = 0$ (ii) $((u_x * (u_x * u_y)) * u_y = 0$ (iii) $(u_x * u_x) = 0$

(iv) $(u_x * u_y) = 0$ and $(u_y * u_x) = 0$ imply $u_x = u_y$, $\forall\ u_x, u_y, u_z \in U$

(v) $(0 * u_x) = 0\ \forall\ u_x \in U$

(3) "$*$" and "0" are taken as defined in $U^{\mathfrak{B}_{M_{NV}}}$

which satisfies the following condition,

$$NV_{M_{NV}}(u_x * u_y) \succcurlyeq r\min\{NV_{M_{NV}}(u_x), NV_{M_{NV}}(u_y)\} \quad ; \quad \forall\ u_x, u_y \in U\ . \quad \text{That is,}$$

$\hat{T}_{M_{NV}}(u_x * u_y) \geq \min\{\hat{T}_{M_{NV}}(u_x), \hat{T}_{M_{NV}}(u_y)\}$; $\hat{I}_{M_{NV}}(u_x * u_y) \leq \max\{\hat{I}_{M_{NV}}(u_x), \hat{I}_{M_{NV}}(u_y)\}$; $\hat{F}_{M_{NV}}(u_x * u_y) \leq \max\{\hat{F}_{M_{NV}}(u_x), \hat{F}_{M_{NV}}(u_y)\}$

General Outline

Let $U = \{0, u_p^1, u_p^2, u_p^3, ----, ----, u_p^k, ----, ----\ u_p^i\}$ be a universal set with algebraic structure $U^{\mathfrak{B}_{M_{NV}}} = (U, *, 0)$ where $*$ is the given binary operation and 0 is the constant . Let $U^{\mathfrak{B}_{M_{NV}}}$ forms a BCK/BCI-algebra. Corresponding Cayley table is given below:

$*$	**0**	u_p^1	u_p^2	----	----	u_p^k	----	----	u_p^i
0	0	0	0	0	0	0	----	0	0
u_p^1	u_p^1	0	----	----	----	----	----	----	----
u_p^2	u_p^2	----	0	----	----	----	----	----	----
----	----	----	----	0	----	----	----	----	----
----	----	----	----	----	0	----	----	----	----
u_p^k	u_p^k	----	----	----	----	0	----	----	----
----	----	----	----	----	----	----	0	----	----
----	----	----	----	----	----	----	----	----	----
u_p^i	u_p^i	----	----	----	----	----	----	----	0

By taking U as underlying set, form a neutrosophic vague set $\mathbf{M_{NV}}$ with neutrosophic vague membership grades, for any $\mathbf{u_p^k} \in U$,

$$\hat{\mathbf{T}}_{\mathbf{M_{NV}}}(\mathbf{u_p^k}) = \begin{cases} [\alpha_1, \alpha_2]\ ;\ \mathbf{u_p^k} = \mathbf{0} \\ [\alpha_3, \alpha_4]\ ;\ \mathbf{u_p^k} \neq \mathbf{0} \end{cases} \quad ; \quad \hat{\mathbf{I}}_{\mathbf{M_{NV}}}(\mathbf{u_p^k}) = \begin{cases} [\beta_1, \beta_2]\ ;\ \mathbf{u_p^k} = \mathbf{0} \\ [\beta_3, \beta_4]\ ;\ \mathbf{u_p^k} \neq \mathbf{0} \end{cases} ; \quad \hat{\mathbf{F}}_{\mathbf{M_{NV}}}(\mathbf{u_p^k}) = \begin{cases} [\gamma_1, \gamma_2]\ ;\ \mathbf{u_p^k} = \mathbf{0} \\ [\gamma_3, \gamma_4]\ ;\ \mathbf{u_p^k} \neq \mathbf{0} \end{cases}$$

$\therefore$ Corresponding neutrosophic vague set is,

$$\mathbf{M_{NV}} = \left\{\left\langle \frac{[\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]}{\mathbf{0}}, \frac{[\alpha_3,\alpha_4],[\beta_3,\beta_4],[\gamma_3,\gamma_4]}{u_p^1}, \frac{[\alpha_3,\alpha_4],[\beta_3,\beta_4],[\gamma_3,\gamma_4]}{u_p^2}, --, \frac{[\alpha_3,\alpha_4],[\beta_3,\beta_4],[\gamma_3,\gamma_4]}{u_p^k}, --, \frac{[\alpha_3,\alpha_4],[\beta_3,\beta_4],[\gamma_3,\gamma_4]}{u_p^i} \right\rangle\right\}$$

Algebraic structure $\mathfrak{B}_{\mathbf{M_{NV}}} = (\mathbf{M_{NV}}, \mathbf{U}^{\mathfrak{B}_{M_{NV}}}, *, 0)$ is called a neutrosophic vague BCK/BCI-algebra if it satisfies, $\mathbf{NV_{M_{NV}}(u_p^k * u_q^k) \succcurlyeq r\min\{NV_{M_{NV}}(u_p^k), NV_{M_{NV}}(u_q^k)\}}$; $\forall\ \mathbf{u_p^k, u_q^k \in U}$

Remark 3.3

Different neutrosophic vague membership grades are also applicable. It is explained in the general outline of definition 3.4

Definition 3.4 (Neutrosophic vague binary BCK/BCI- algebra)

A neutrosophic vague binary BCK/BCI- algebra is a structure,
$\mathfrak{B}_{M_{NVB}} = \left(M_{NVB},\ U^{\mathfrak{B}_{M_{NVB}}} = (U,\ *,\ 0),\ *,\ 0\right) = \left(M_{NVB},\ U^{\mathfrak{B}_{M_{NVB}}},\ *,\ 0\right)$, where

(1) M_{NVB} is a non-empty neutrosophic vague binary set

(2) $U^{\mathfrak{B}_{M_{NVB}}} = \left(U = \{U_1 \cup U_2\},\ *,\ 0\right)$ is the underlying BCK/BCI - algebraic structure to the neutrosophic vague binary set M_{NVB} with a universal set $U = \{U_1 \cup U_2\}$ [where U_1 and U_2 are universes of M_{NVB} & "$\cup$" is the usual set-theoretic union], a binary operation "$*$" & a constant "0" satisfies the following axioms:

(i) $((u_x * u_y) * (u_x * u_z)) * (u_z * u_y) = 0$ (ii) $((u_x * (u_x * u_y)) * u_y = 0$ (iii) $(u_x * u_x) = 0$

(iv) $(u_x * u_y) = 0$ and $(u_y * u_x) = 0$ imply $u_x = u_y\ \forall\ u_x, u_y, u_z \in U$ (v) $(0 * u_x) = 0\ \forall\ u_x \in U$

(3) "$*$" and "0" are same as defined in $U^{\mathfrak{B}_{M_{NVB}}}$

which satisfies the following condition,

$NVB_{M_{NVB}}(u_x * u_y) \geqslant r\min\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\},\ \forall\ u_x, u_y \in U = \{U_1 \cup U_2\}$. That is, $\hat{T}_{M_{NVB}}(u_x * u_y) \geq \min\{\hat{T}_{M_{NVB}}(u_x), \hat{T}_{M_{NVB}}(u_y)\}$; $\hat{I}_{M_{NVB}}(u_x * u_y) \leq \max\{\hat{I}_{M_{NVB}}(u_x), \hat{I}_{M_{NVB}}(u_y)\}$; $\hat{F}_{M_{NVB}}(u_x * u_y) \leq \max\{\hat{F}_{M_{NVB}}(u_x), \hat{F}_{M_{NVB}}(u_y)\}$

Remark 3.5

(i) Every NVB BCK-algebra is NVB BCI–algebra too. Generally, converse not true! (proved: Theorem 7.3). So distinguishing between structures of these two are important! To denote NVB BCK-algebra, following structures can be used: $\mathfrak{B}^{BCK}_{M_{NVB}} = \left(M_{NVB}, U^{\mathfrak{B}^{BCK}_{M_{NVB}}},\ *,\ 0\right)$ or simply as $\mathfrak{B}^{K}_{M_{NVB}} = \left(M_{NVB}, U^{\mathfrak{B}^{K}_{M_{NVB}}},\ *,\ 0\right)$. Similarly, to denote NVB BCI – algebra, following structures can be used : $\mathfrak{B}^{BCI}_{M_{NVB}} = \left(M_{NVB}, U^{\mathfrak{B}^{BCI}_{M_{NVB}}},\ *,\ 0\right)$ or simply as $\mathfrak{B}^{I}_{M_{NVB}} = \left(M_{NVB}, U^{\mathfrak{B}^{I}_{M_{NVB}}},\ *,\ 0\right)$.

(ii) For NVB BCK algebra, notation for NVB BCK/BCI − algebra, i.e., $\mathfrak{B}_{M_{NVB}} = (M_{NVB}, U^{\mathfrak{B}_{M_{NVB}}},\ *,\ 0)$ is used in this paper instead of using, those given in remark 3.5 (i)

(iii) Similarly structures for:

Neutrosophic :

N BCK – algebra : $\mathfrak{B}^{BCK}_{M_N} = \left(M_N, U^{\mathfrak{B}^{BCK}_{M_N}},\ *,\ 0\right)$ or $\mathfrak{B}^{K}_{M_N} = \left(M_N, U^{\mathfrak{B}^{K}_{M_N}},\ *,\ 0\right)$ or $\mathfrak{B}_{M_N} = (M_N, U^{\mathfrak{B}_{M_N}},\ *,\ 0)$

N BCI – algebra : $\mathfrak{B}^{BCI}_{M_N} = \left(M_N, U^{\mathfrak{B}^{BCI}_{M_N}},\ *,\ 0\right)$ or simply as $\mathfrak{B}^{I}_{M_N} = \left(M_N, U^{\mathfrak{B}^{I}_{M_N}},\ *,\ 0\right)$

Neutrosophic vague:

NV BCK – algebra : $\mathfrak{B}^{BCK}_{M_{NV}} = \left(M_{NV}, U^{\mathfrak{B}^{BCK}_{M_{NV}}},\ *,\ 0\right)$ or $\mathfrak{B}^{K}_{M_{NV}} = \left(M_{NV}, U^{\mathfrak{B}^{K}_{M_{NV}}},\ *,\ 0\right)$ or $\mathfrak{B}_{M_{NV}} = (M_{NV}, U^{\mathfrak{B}_{M_{NV}}},\ *,\ 0)$

NV BCI – algebra : $\mathfrak{B}^{BCI}_{M_{NV}} = \left(M_{NV}, U^{\mathfrak{B}^{BCI}_{M_{NV}}},\ *,\ 0\right)$ or simply as $\mathfrak{B}^{I}_{M_{NV}} = \left(M_{NV}, U^{\mathfrak{B}^{I}_{M_{NV}}},\ *,\ 0\right)$

General Outline

Let $\mathbf{U_1} = \{0,\ \mathbf{u^1_p},\ \mathbf{u^2_p},\ \mathbf{u^3_p},$ ---------, $\mathbf{u^i_p}\}$ and $\mathbf{U_2} = \{0,\ \mathbf{u^1_q},\ \mathbf{u^2_q},\ \mathbf{u^3_q},$ ---------, $\mathbf{u^j_q}\}$ be two universes under consideration. Let the combined universe $U = \{\mathbf{U_1} \cup \mathbf{U_2}\} = \{0,\ \mathbf{u^1_p},\ \mathbf{u^2_p},\ \mathbf{u^3_p},$ ----, $\mathbf{u^i_p},\ \mathbf{u^1_q},\ \mathbf{u^2_q},\ \mathbf{u^3_q},$ ----, $\mathbf{u^j_q}\} = \{\mathbf{0},\ \mathbf{u^1_r},\ \mathbf{u^2_r},\ \mathbf{u^3_r},$ ---------, $\mathbf{u^k_r}\}$ (obtained by recording once, the common elements) be a set with a binary operation $*$ and constant 0 . Let $\mathbf{U^{\mathfrak{B}_{M_{NVB}}}} = (U = \{\mathbf{U_1} \cup \mathbf{U_2}\},\ *,\ 0)$ forms a BCK/BCI-algebra. By taking U = $\{\mathbf{U_1} \cup \mathbf{U_2}\}$ as underlying set, form a neutrosophic vague binary set $\mathbf{M_{NVB}}$. Let neutrosophic vague binary membership grades are as follows:

for any $\mathbf{u_p^k} \in \mathbf{U_2}$: $\begin{cases} \hat{\mathbf{T}}_{\mathbf{M_{NVB}}}(\mathbf{u_p^k}) = \begin{cases} [\alpha_1^0, \alpha_2^0] \ ; \ \mathbf{u_p^k} = \mathbf{0} \\ [\alpha_1^1, \alpha_2^1] \ ; \ \mathbf{u_p^k} = \mathbf{u_p^1} \\ [\alpha_1^2, \alpha_2^2] \ ; \ \mathbf{u_p^k} = \mathbf{u_p^2} \\ ----- \\ ----- \\ [\alpha_1^i, \alpha_2^i] \ ; \ \mathbf{u_p^k} = \mathbf{u_p^i} \end{cases} \\ \hat{\mathbf{I}}_{\mathbf{M_{NVB}}}(\mathbf{u_p^k}) = \begin{cases} [\beta_1^0, \beta_2^0] \ ; \ \mathbf{u_p^k} = \mathbf{0} \\ [\beta_1^1, \beta_2^1] \ ; \ \mathbf{u_p^k} = \mathbf{u_p^1} \\ [\beta_1^2, \beta_2^2] \ ; \ \mathbf{u_p^k} = \mathbf{u_p^2} \\ ----- \\ ----- \\ [\beta_1^i, \beta_2^i] \ ; \ \mathbf{u_p^k} = \mathbf{u_p^i} \end{cases} \\ \hat{\mathbf{F}}_{\mathbf{M_{NVB}}}(\mathbf{u_p^k}) = \begin{cases} [\gamma_1^0, \gamma_2^0] \ ; \ \mathbf{u_p^k} = \mathbf{0} \\ [\gamma_1^1, \gamma_2^1] \ ; \ \mathbf{u_p^k} = \mathbf{u_p^1} \\ [\gamma_1^2, \gamma_2^2] \ ; \ \mathbf{u_p^k} = \mathbf{u_p^2} \\ ----- \\ ----- \\ [\gamma_1^i, \gamma_2^i] \ ; \ \mathbf{u_p^k} = \mathbf{u_p^i} \end{cases} \end{cases}$ & for any $\mathbf{u_q^k} \in \mathbf{U_2}$: $\begin{cases} \hat{\mathbf{T}}_{\mathbf{M_{NVB}}}(\mathbf{u_q^k}) = \begin{cases} [\delta_1^0, \delta_2^0] \ ; \ \mathbf{u_q^k} = \mathbf{0} \\ [\delta_1^1, \delta_2^1] \ ; \ \mathbf{u_q^k} = \mathbf{u_q^1} \\ [\delta_1^2, \delta_2^2] \ ; \ \mathbf{u_q^k} = \mathbf{u_q^2} \\ ----- \\ ----- \\ [\delta_1^j, \delta_2^j] \ ; \ \mathbf{u_q^k} = \mathbf{u_q^j} \end{cases} \\ \hat{\mathbf{I}}_{\mathbf{M_{NVB}}}(\mathbf{u_q^k}) = \begin{cases} [\rho_1^0, \rho_2^0] \ ; \ \mathbf{u_q^k} = \mathbf{0} \\ [\rho_1^1, \rho_2^1] \ ; \ \mathbf{u_q^k} = \mathbf{u_q^1} \\ [\rho_1^2, \rho_2^2] \ ; \ \mathbf{u_q^k} = \mathbf{u_q^2} \\ ----- \\ ----- \\ [\rho_1^j, \rho_2^j] \ ; \ \mathbf{u_q^k} = \mathbf{u_q^j} \end{cases} \\ \hat{\mathbf{F}}_{\mathbf{M_{NVB}}}(\mathbf{u_q^k}) = \begin{cases} [\vartheta_1^0, \vartheta_2^0] \ ; \ \mathbf{u_q^k} = \mathbf{0} \\ [\vartheta_1^1, \vartheta_2^1] \ ; \ \mathbf{u_q^k} = \mathbf{u_q^1} \\ [\vartheta_1^2, \vartheta_2^2] \ ; \ \mathbf{u_q^k} = \mathbf{u_q^2} \\ ----- \\ ----- \\ [\vartheta_1^j, \vartheta_2^j] \ ; \ \mathbf{u_q^k} = \mathbf{u_q^j} \end{cases} \end{cases}$

From this neutrosophic vague binary set $\mathbf{M_{NVB}}$, form neutrosophic vague binary membership grade for U = $\{\mathbf{U_1} \cup \mathbf{U_2}\}$ = $\{\mathbf{0},\ \mathbf{u_r^1},\ \mathbf{u_r^2}, ---, \mathbf{u_r^k}\}$ as :

$$\text{for any } \mathbf{u_r^k} \in \mathbf{U} : \mathbf{NVB_{M_{NVB}}}(\mathbf{u_r^k}) = \begin{cases} \mathbf{NVB_{M_{NVB}}}(\mathbf{u_p^k} = \mathbf{0}) \cup \mathbf{NVB_{M_{NVB}}}(\mathbf{u_q^k} = \mathbf{0})\ ; \mathbf{u_r^k} \in \mathbf{U_1};\ \mathbf{u_r^k} \in \mathbf{U_2}\ \mathbf{u_r^k} = \mathbf{0} \\ \mathbf{NVB_{M_{NVB}}}(\mathbf{u_p^k})\ ;\ \mathbf{u_r^k} \in \mathbf{U_1}; \mathbf{u_r^k} \notin \mathbf{U_2};\ \mathbf{u_r^k} \neq \mathbf{0} \\ \mathbf{NVB_{M_{NVB}}}(\mathbf{u_q^k})\ ;\ \mathbf{u_r^k} \notin \mathbf{U_1},\ \mathbf{u_r^k} \in \mathbf{U_2};\ \mathbf{u_r^k} \neq \mathbf{0} \\ \mathbf{NVB_{M_{NVB}}}(\mathbf{u_p^k}) \cup \mathbf{NVB_{M_{NVB}}}(\mathbf{u_q^k})\ ;\ \mathbf{u_r^k} \in \mathbf{U_1};\ \mathbf{u_r^k} \in \mathbf{U_2}\ ;\ \mathbf{u_r^k} \neq \mathbf{0} \end{cases}$$

i.e., for any $\mathbf{u_r^k} \in \mathbf{U}$:

$$\begin{cases} \hat{\mathbf{T}}_{\mathbf{M_{NVB}}}(\mathbf{u_r^k}) = \begin{cases} \mathbf{max}\{\hat{\mathbf{T}}_{\mathbf{M_{NVB}}}(\mathbf{u_p^k} = \mathbf{0}), \hat{\mathbf{T}}_{\mathbf{M_{NVB}}}(\mathbf{u_q^k} = \mathbf{0})\} = \mathbf{max}\{[\alpha_1^0, \alpha_2^0], [\delta_1^0, \delta_2^0]\};\ \mathbf{u_r^k} \in \mathbf{U_1};\ \mathbf{u_r^k} \in \mathbf{U_2}\ \mathbf{u_r^k} = \mathbf{0} \\ \hat{\mathbf{T}}_{\mathbf{M_{NVB}}}(\mathbf{u_p^k})\ ;\ \mathbf{u_r^k} \in \mathbf{U_1}; \mathbf{u_r^k} \notin \mathbf{U_2};\ \mathbf{u_r^k} \neq \mathbf{0} \\ \hat{\mathbf{T}}_{\mathbf{M_{NVB}}}(\mathbf{u_q^k})\ ;\ \mathbf{u_r^k} \notin \mathbf{U_1},\ \mathbf{u_r^k} \in \mathbf{U_2};\ \mathbf{u_r^k} \neq \mathbf{0} \\ \mathbf{max}\{\hat{\mathbf{T}}_{\mathbf{M_{NVB}}}(\mathbf{u_p^k}), \hat{\mathbf{T}}_{\mathbf{M_{NVB}}}(\mathbf{u_q^k})\};\ \mathbf{u_r^k} \in \mathbf{U_1};\ \mathbf{u_r^k} \in \mathbf{U_2}\ ;\ \mathbf{u_r^k} \neq \mathbf{0} \end{cases} \\ \hat{\mathbf{I}}_{\mathbf{M_{NVB}}}(\mathbf{u_r^k}) = \begin{cases} \mathbf{min}\{\hat{\mathbf{I}}_{\mathbf{M_{NVB}}}(\mathbf{u_p^k} = \mathbf{0}), \hat{\mathbf{I}}_{\mathbf{M_{NVB}}}(\mathbf{u_q^k} = \mathbf{0})\} = \mathbf{max}\{[\beta_1^0, \beta_2^0],\ [\rho_1^0, \rho_2^0]\}; \mathbf{u_r^k} \in \mathbf{U_1};\ \mathbf{u_r^k} \in \mathbf{U_2}\ \mathbf{u_r^k} = \mathbf{0} \\ \hat{\mathbf{I}}_{\mathbf{M_{NVB}}}(\mathbf{u_p^k})\ ;\ \mathbf{u_r^k} \in \mathbf{U_1}; \mathbf{u_r^k} \notin \mathbf{U_2};\ \mathbf{u_r^k} \neq \mathbf{0} \\ \hat{\mathbf{I}}_{\mathbf{M_{NVB}}}(\mathbf{u_q^k})\ ;\ \mathbf{u_r^k} \notin \mathbf{U_1},\ \mathbf{u_r^k} \in \mathbf{U_2};\ \mathbf{u_r^k} \neq \mathbf{0} \\ \mathbf{min}\{\hat{\mathbf{I}}_{\mathbf{M_{NVB}}}(\mathbf{u_p^k}), \hat{\mathbf{I}}_{\mathbf{M_{NVB}}}(\mathbf{u_q^k})\};\ \mathbf{u_r^k} \in \mathbf{U_1};\ \mathbf{u_r^k} \in \mathbf{U_2}\ ;\ \mathbf{u_r^k} \neq \mathbf{0} \end{cases} \\ \hat{\mathbf{F}}_{\mathbf{M_{NVB}}}(\mathbf{u_r^k}) = \begin{cases} \mathbf{min}\{\hat{\mathbf{F}}_{\mathbf{M_{NVB}}}(\mathbf{u_p^k} = \mathbf{0}), \hat{\mathbf{F}}_{\mathbf{M_{NVB}}}(\mathbf{u_q^k} = \mathbf{0})\} = \mathbf{min}\{[\beta_1^0, \beta_2^0],\ [\rho_1^0, \rho_2^0]\}; \mathbf{u_r^k} \in \mathbf{U_1}; \mathbf{u_r^k} \in \mathbf{U_2};\ \mathbf{u_r^k} = \mathbf{0} \\ \hat{\mathbf{F}}_{\mathbf{M_{NVB}}}(\mathbf{u_p^k})\ ;\ \mathbf{u_r^k} \in \mathbf{U_1}; \mathbf{u_r^k} \notin \mathbf{U_2};\ \mathbf{u_r^k} \neq \mathbf{0} \\ \hat{\mathbf{F}}_{\mathbf{M_{NVB}}}(\mathbf{u_q^k})\ ;\ \mathbf{u_r^k} \notin \mathbf{U_1},\ \mathbf{u_r^k} \in \mathbf{U_2};\ \mathbf{u_r^k} \neq \mathbf{0} \\ \mathbf{min}\{\hat{\mathbf{F}}_{\mathbf{M_{NVB}}}(\mathbf{u_p^k}), \hat{\mathbf{F}}_{\mathbf{M_{NVB}}}(\mathbf{u_q^k})\};\ \mathbf{u_r^k} \in \mathbf{U_1};\ \mathbf{u_r^k} \in \mathbf{U_2}\ ;\ \mathbf{u_r^k} \neq \mathbf{0} \end{cases} \end{cases}$$

Corresponding Cayley table is given by:

$*$	0	$\mathbf{u_r^1}$	$\mathbf{u_r^2}$	---	$\mathbf{u_r^k}$
0	0	0	0	0	0
$\mathbf{u_r^1}$	$\mathbf{u_r^1}$	0	---	---	---
$\mathbf{u_r^2}$	$\mathbf{u_r^2}$	---	0	---	---
---	---	---	---	0	---
$\mathbf{u_r^k}$	$\mathbf{u_r^k}$	---	---	---	0

Algebraic structure $\mathfrak{B}_{\mathbf{M_{NVB}}}$ = $(\mathbf{M_{NVB}},\ \mathbf{U}^{\mathfrak{B}_{\mathbf{M_{NVB}}}},\ *,\ \mathbf{0})$ is called a NVB BCK/BCI-algebra, if it satisfies:

$$\mathbf{NVB_{M_{NVB}}}(\mathbf{u_r^k} * \mathbf{u_s^k}) \succcurlyeq \mathbf{r\,min}\{\mathbf{NVB_{M_{NVB}}}(\mathbf{u_r^k}), \mathbf{NVB_{M_{NVB}}}(\mathbf{u_s^k})\}\ ;\quad \forall\quad \mathbf{u_r^k},\ \mathbf{u_s^k} \in \mathbf{U}$$

$$\hat{\mathbf{T}}_{\mathbf{M_{NVB}}}(\mathbf{u_r^k} * \mathbf{u_s^k}) \geq\ \mathbf{min}\,\{\hat{\mathbf{T}}_{\mathbf{M_{NVB}}}(\mathbf{u_r^k}), \hat{\mathbf{T}}_{\mathbf{M_{NVB}}}(\mathbf{u_s^k})\}\ ;\ \hat{\mathbf{I}}_{\mathbf{M_{NVB}}}(\mathbf{u_r^k} * \mathbf{u_s^k}) \leq\ \mathbf{max}\,\{\hat{\mathbf{I}}_{\mathbf{M_{NVB}}}(\mathbf{u_r^k}), \hat{\mathbf{I}}_{\mathbf{M_{NVB}}}(\mathbf{u_s^k})\}\ ;\ \hat{\mathbf{F}}_{\mathbf{M_{NVB}}}(\mathbf{u_r^k} * \mathbf{u_s^k}) \leq\ \mathbf{max}\,\{\hat{\mathbf{F}}_{\mathbf{M_{NVB}}}(\mathbf{u_r^k}), \hat{\mathbf{F}}_{\mathbf{M_{NVB}}}(\mathbf{u_s^k})\}$$

Remark 3.6

(i) Neutrosophic vague binary membership grade of common elements of $\mathbf{U_1}$ and $\mathbf{U_2}$ is got by taking their neutrosophic vague binary union.

For eg., let $\mathbf{U_1} = \{\mathbf{0}, \mathbf{1}\}$ and $\mathbf{U_2} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$ be two universes; $\therefore \{\mathbf{U_1} \cup \mathbf{U_2}\} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$; $\mathbf{U_1} \cap \mathbf{U_2} = \{\mathbf{0}, \mathbf{1}\}$

$\therefore\ \mathbf{NVB_{M_{NVB}}(0)} = \mathbf{NVB^{U_1}_{M_{NVB}}(0)} \cup \mathbf{NVB^{U_2}_{M_{NVB}}(0)}$; $\mathbf{NVB^{U_1}_{M_{NVB}}(0)}$ is the neutrosophic vague binary membership grade of 0 in universe 1. Similarly, to other common elements.

(ii) It is to be noted that, neutrosophic vague binary membership grade of 0 is not same in $\mathbf{U_1}$, $\mathbf{U_2}$ generally. Similarly, to other common elements!

Example 3.7

Let $\mathbf{U_1} = \{\mathbf{0}, \mathbf{a}\}$ and let $\mathbf{U_2} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$ be the universes under consideration. Combined universe $\mathbf{U} = \{\mathbf{U_1} \cup \mathbf{U_2}\} = \{\mathbf{0}, \mathbf{a}, \mathbf{1}, \mathbf{2}\}$ with $(\mathbf{U_1} \cap \mathbf{U_2}) = \{0\}$. Cayley table to the binary operation $*$ for U is given as:

*	0	a	1	2
0	0	0	0	0
a	a	0	0	1
1	1	a	0	1
2	2	2	2	0

Clearly, $\mathbf{U}^{\mathfrak{B}_{\mathbf{M_{NVB}}}} = (\mathbf{U} = \{\mathbf{U_1} \cup \mathbf{U_2}\},\ *,\ \mathbf{0})$ is a BCK/BCI-algebra. Let a non-empty neutrosophic vague binary set $\mathbf{M_{NVB}}$ with underlying set $\mathbf{U}$, is given as:

$$\mathbf{M_{NVB}} = \left\{\left\langle \frac{[0.3,0.8],[0.1,0.3],[0.2,0.7]}{0}, \frac{[0.2,0.3],[0.2,0.5],[0.7,0.8]}{a} \right\rangle, \left\langle \frac{[0.1,0.7],[0.7,0.8],[0.3,0.9]}{0}, \frac{[0.2,0.6],[0.5,0.7],[0.4,0.8]}{1}, \frac{[0.2,0.6],[0.5,0.7],[0.4,0.8]}{2} \right\rangle\right\}$$

$\Rightarrow\quad \forall\ \ \mathbf{u^k_p} \in \mathbf{U_1}$ and $\forall\ \ \mathbf{u^k_q} \in \mathbf{U_2}$,

$$\hat{\mathbf{T}}_{\mathbf{M_{NVB}}}(\mathbf{u^k_p}) = \begin{cases} [0.3, 0.8] & \text{if } \mathbf{u^k_p} = 0 \\ [0.2, 0.3] & \text{if } \mathbf{u^k_p} = a \text{ or } \mathbf{u^k_p} \neq 0 \end{cases} ; \hat{\mathbf{I}}_{\mathbf{M_{NVB}}}(\mathbf{u^k_p}) = \begin{cases} [0.1, 0.3] & \text{if } \mathbf{u^k_p} = 0 \\ [0.2, 0.5] & \text{if } \mathbf{u^k_p} = a \text{ or } \mathbf{u^k_p} \neq 0 \end{cases} ; \hat{\mathbf{F}}_{\mathbf{M_{NVB}}}(\mathbf{u^k_p}) = \begin{cases} [0.2, 0.7] & \text{if } \mathbf{u^k_p} = 0 \\ [0.7, 0.8] & \text{if } \mathbf{u^k_p} = a \text{ or } \mathbf{u^k_p} \neq 0 \end{cases}$$

$$\hat{\mathbf{T}}_{\mathbf{M_{NVB}}}(\mathbf{u^k_q}) = \begin{cases} [0.1, 0.7]; \text{ if } \mathbf{u^k_q} = 0 \\ [0.2, 0.6]; \text{ if } \mathbf{u^k_q} = \{1,2\} \text{ or } \mathbf{u^k_q} \neq 0 \end{cases} ; \hat{\mathbf{I}}_{\mathbf{M_{NVB}}}(\mathbf{u^k_q}) = \begin{cases} [0.7, 0.8]; \text{ if } \mathbf{u^k_q} = 0 \\ [0.5, 0.7]; \text{ if } \mathbf{u^k_q} = \{1,2\} \text{ or } \mathbf{u^k_q} \neq 0 \end{cases} ; \hat{\mathbf{F}}_{\mathbf{M_{NVB}}}(\mathbf{u^k_q}) = \begin{cases} [0.3, 0.9]; \text{ if } \mathbf{u^k_q} = 0 \\ [0.4, 0.8]; \text{ if } \mathbf{u^k_q} = \{1,2\} \text{ or } \mathbf{u^k_q} \neq 0 \end{cases}$$

$\therefore \mathbf{NVB_{M_{NVB}}(0)} = ([0.3, 0.8], [0.1, 0.3], [0.2, 0.7]) \cup ([0.1, 0.7], [0.7, 0.8], [0.3, 0.9]) = ([0.3, 0.8], [0.1, 0.3], [0.2, 0.7])$

$\mathbf{NVB_{M_{NVB}}(a)} = ([0.2, 0.3], [0.2, 0.5], [0.7, 0.8])$ [since a is not a common element]

$\mathbf{NVB_{M_{NVB}}(1)} = \mathbf{NVB_{M_{NVB}}(2)} = ([0.2, 0.6], [0.5, 0.7], [0.4, 0.8])$; [since 1 and 2 are not a common element]

$$\Rightarrow \mathbf{NVB_{M_{NVB}}(u^k_r)} = \begin{cases} [0.3, 0.8], [0.1, 0.3], [0.2, 0.7]; & \mathbf{u^k_r} = \mathbf{0} \\ [0.2, 0.3], [0.2, 0.5], [0.7, 0.8]; & \mathbf{u^k_r} = \{\mathbf{a}\} \text{ and } \mathbf{u^k_r} \neq \mathbf{0} \\ [0.2, 0.6], [0.5, 0.7], [0.4, 0.8]; & \mathbf{u^k_r} = \{\mathbf{1}, \mathbf{2}\} \text{ and } \mathbf{u^k_r} \neq \mathbf{0} \end{cases} \quad ; (\text{for any } \mathbf{u^k_r} \in \mathrm{U})$$

It is clear after verification that, $\mathfrak{B}_{\mathbf{M_{NVB}}} = \left(\mathbf{M_{NVB}}, \mathbf{U}^{\mathfrak{B}_{\mathbf{M_{NVB}}}}, *, \mathbf{0}\right)$ is a NVB BCK/BCI- algebra.

Remark. 3.8

(1) If $\mathbf{U_1} \subseteq \mathbf{U_2}$ then $\mathbf{U} = \mathbf{U_2}$ (2) If $\mathbf{U_2} \subseteq \mathbf{U_1}$ then $\mathbf{U} = \mathbf{U_1}$

(2) The symbols $\succcurlyeq$ and $\not\succcurlyeq$ does not imply our usual $\geq$ or $\ngeq$

(3) In a Cayley table,

(i) principal diagonal elements of a BCK/BCI-algebra U is always zero, since $(\mathbf{x} * \mathbf{x}) = \mathbf{0}, \forall\, \mathbf{x} \in \mathbf{U}$

(ii) Using the property $(\mathbf{x} * \mathbf{0}) = \mathbf{x}$; $\forall\ \mathbf{x} \in \mathbf{U}$ of BCI- algebra, it is clear that $(\mathbf{0} * \mathbf{0}) = \mathbf{0}$

Every BCK-algebra is a BCI-algebra. Hence the above is true for BCK-algebra also

(iii) Body of first column of Cayley table for a BCI- algebra will be an exact copy of column of operands, by using the property $(\mathbf{x} * \mathbf{0}) = \mathbf{x}\ \forall\ \mathbf{x} \in \mathbf{U}$. But $\mathbf{1^{st}}$ row need not be!

(iv) Above is true for a BCK- algebra also, since every BCK- algebra is a BCI-algebra. In addition, for a BCK- algebra, body of first row takes only 0, using the property $(\mathbf{0} * \mathbf{x}) = \mathbf{0}$; $\forall\ \mathbf{x} \in \mathbf{U}$

Binary Operation $*$	**Row of operands (**Elements of $\mathbf{U}$)
Column of operands (Elements of $\mathbf{U}$**)**	Body of Cayley table (occupy with elements got after binary operation taken via column vise row operations)

4. Neutrosophic vague binary BCK/BCI-subalgebra & Neutrosophic vague binary BCK/BCI-ideal

In this section N/NV/NVB BCK/BCI- subalgebra (neutrosophic/neutrosophic vague/neutrosophic vague binary BCK/BCI – subalgebra) & N/NV/NVB BCK/BCI– ideal (neutrosophic/neutrosophic vague/neutrosophic vague binary BCK/BCI – ideal) are developed. Priority is given for developing sub-algebraic and ideal concepts to neutrosophic vague binary BCK/BCI- algebra [NVB BCK/BCI-algebra]. For neutrosophic and neutrosophic vague, things are similar.

Definition 4.1 (Neutrosophic vague binary BCK/BCI-subalgebra)

A NVBSS $\mathbf{P_{NVB}}$ of a NVB BCK/BCI-algebra $\mathfrak{B}_{\mathbf{M_{NVB}}} = \left(\mathbf{M_{NVB}},\ \mathbf{U}^{\mathfrak{B}_{\mathbf{M_{NVB}}}} = (\mathbf{U},\ *,\ \mathbf{0}),\ *,\ \mathbf{0}\right)$ is called NVB- BCK/BCI - subalgebra of $\mathfrak{B}_{\mathbf{M_{NVB}}}$ if,

$$\mathbf{NVB_{P_{NVB}}}(\mathbf{u_x} * \mathbf{u_y}) \succcurlyeq \mathbf{r\ min}\left\{\mathbf{NVB_{P_{NVB}}}(\mathbf{u_x}), \mathbf{NVB_{P_{NVB}}}(\mathbf{u_y})\right\} \quad ; \quad \forall \quad \mathbf{u_x}, \mathbf{u_y} \in \mathbf{U}$$

$\hat{\mathbf{T}}_{\mathbf{P_{NVB}}}(\mathbf{u_x} * \mathbf{u_y}) \geq \mathbf{min}\{\hat{\mathbf{T}}_{\mathbf{P_{NVB}}}(\mathbf{u_x}), \hat{\mathbf{T}}_{\mathbf{P_{NVB}}}(\mathbf{u_y})\}$; $\hat{\mathbf{I}}_{\mathbf{P_{NVB}}}(\mathbf{u_x} * \mathbf{u_y}) \leq \mathbf{max}\{\hat{\mathbf{I}}_{\mathbf{P_{NVB}}}(\mathbf{u_x}), \hat{\mathbf{I}}_{\mathbf{P_{NVB}}}(\mathbf{u_y})\}$; $\hat{\mathbf{F}}_{\mathbf{P_{NVB}}}(\mathbf{u_x} * \mathbf{u_y}) \leq \mathbf{max}\{\hat{\mathbf{F}}_{\mathbf{P_{NVB}}}(\mathbf{u_x}), \hat{\mathbf{F}}_{\mathbf{P_{NVB}}}(\mathbf{u_y})\}$

Definition 4.2 (Neutrosophic vague binary BCK/BCI- Ideal)

A non-empty NVBSS $\mathbf{P_{NVB}}$ of a NVB BCK/BCI-algebra, $\mathfrak{B}_{\mathbf{M_{NVB}}} = \left(\mathbf{M_{NVB}},\ \mathbf{U}^{\mathfrak{B}_{\mathbf{M_{NVB}}}},\ *,\ \mathbf{0}\right)$ is called a NVB BCK/BCI- ideal of $\mathfrak{B}_{\mathbf{M_{NVB}}}$ if

(i) $\mathbf{NVB_{P_{NVB}}}(\mathbf{0}) \succcurlyeq \mathbf{NVB_{P_{NVB}}}(\mathbf{u_k})$; for any $\mathbf{u_k} \in \mathbf{U}$
i.e., $\hat{\mathbf{T}}_{\mathbf{P_{NVB}}}(\mathbf{0}) \geq \hat{\mathbf{T}}_{\mathbf{P_{NVB}}}(\mathbf{u_k})$; $\hat{\mathbf{I}}_{\mathbf{P_{NVB}}}(\mathbf{0}) \leq \hat{\mathbf{I}}_{\mathbf{P_{NVB}}}(\mathbf{u_k})$; $\hat{\mathbf{F}}_{\mathbf{P_{NVB}}}(\mathbf{0}) \leq \hat{\mathbf{F}}_{\mathbf{P_{NVB}}}(\mathbf{u_k})$

(ii) $\mathbf{NVB_{P_{NVB}}}(\mathbf{u_a}) \succcurlyeq \mathbf{r\ min}\left\{\mathbf{NVB_{P_{NVB}}}(\mathbf{u_a} * \mathbf{u_b}), \mathbf{NVB_{P_{NVB}}}(\mathbf{u_b})\right\}$; for any $\mathbf{u_a}, \mathbf{u_b} \in \mathbf{U}$
$\hat{\mathbf{T}}_{\mathbf{P_{NVB}}}(\mathbf{u_a}) \geq \mathbf{min}\{\hat{\mathbf{T}}_{\mathbf{P_{NVB}}}(\mathbf{u_a} * \mathbf{u_b}), \hat{\mathbf{T}}_{\mathbf{P_{NVB}}}(\mathbf{u_b})\}$; $\hat{\mathbf{I}}_{\mathbf{P_{NVB}}}(\mathbf{u_a}) \leq \mathbf{max}\{\hat{\mathbf{I}}_{\mathbf{P_{NVB}}}(\mathbf{u_a} * \mathbf{u_b}), \hat{\mathbf{I}}_{\mathbf{P_{NVB}}}(\mathbf{u_b})\}$; $\hat{\mathbf{F}}_{\mathbf{P_{NVB}}}(\mathbf{u_a}) \leq \mathbf{max}\{\hat{\mathbf{F}}_{\mathbf{P_{NVB}}}(\mathbf{u_a} * \mathbf{u_b}), \hat{\mathbf{F}}_{\mathbf{P_{NVB}}}(\mathbf{u_b})\}$

Remark 4.3
For NVB BCK – ideal underlying structure will confine to BCK -algebra and for NVB BCI – ideal it will confine to BCK -algebra. For different ideals mentioned in definition 5.2, the same principle follows.
Remark 4.4
Similarly, for neutrosophic and neutrosophic vague. Only difference is with sets $\mathbf{P_N}$, $\mathbf{P_{NV}}$ instead of $\mathbf{P_{NVB}}$ in above definitions taken in order. It is trivial. Moreover, instead of $\mathbf{U} = \{\mathbf{U_1} \cup \mathbf{U_2}\}$, for both of them U is applied.

5. Various neutrosophic vague binary BCK/BCI-ideals

In this section vague H-ideal is developed first. Then p-ideal, q-ideal, a-ideal and H-ideal are developed for NVB BCK/BCI-algebra $\mathfrak{B}_{\mathbf{M_{NVB}}} = \left(\mathbf{M_{NVB}}, \mathbf{U}^{\mathfrak{B}_{\mathbf{M_{NVB}}}} = (\mathbf{U} = \{\mathbf{U_1} \cup \mathbf{U_2}\},\ *,\ \mathbf{0}),\ *,\ \mathbf{0}\right)$
Definition 5.1 (Vague H-ideal)

A vague set A of X is called a vague H - ideal of a BCI – algebra X if it satisfies

(i) $\mathbf{V_A}(\mathbf{0}) \succcurlyeq \mathbf{V_A}(\mathbf{x})$ $(\forall\ \mathbf{x} \in \mathbf{X})$;

i.e., $\begin{cases} \mathbf{t_A(0)} \geq \mathbf{t_A(x)} \\ \mathbf{1 - f_A(0)} \geq \mathbf{1 - f_A(x)} \end{cases}$ i.e., $\begin{cases} \mathbf{t_A(0)} \geq \mathbf{t_A(x)} \\ \mathbf{f_A(0)} \leq \mathbf{f_A(x)} \end{cases}$ and

(ii) $\mathbf{V_A(x*z)} \succcurlyeq \mathbf{r\,min}\{\mathbf{V_A(x*(y*z)), V_A(y)}\}$; ($\forall$ $\mathbf{x, y, z} \in \mathbf{X}$); i.e., $\begin{cases} \mathbf{t_A(x*z)} \geq \mathbf{min}\{\mathbf{t_A(x*(y*z)), t_A(y)}\} \\ \mathbf{1 - f_A(x*z)} \geq \mathbf{min}\{\mathbf{1 - f_A(x*(y*z)), 1 - f_A(y)}\} \end{cases}$

Definition 5.2 (Comparison of different NVB BCK/BCI- ideals)

Let $\mathfrak{B}_{\mathbf{M_{NVB}}} = \left(\mathbf{M_{NVB}}, \mathbf{U}^{\mathfrak{B}_{\mathbf{M_{NVB}}}} = \left(\mathbf{U} = \{\mathbf{U_1} \cup \mathbf{U_2}\}, *, \mathbf{0}\right), *, \mathbf{0}\right)$ be a NVB BCK/BCI-algebra. Conditions for a non-empty NVBSS $\mathbf{P_{NVB}}$ of $\mathfrak{B}_{\mathbf{M_{NVB}}}$ to become a neutrosophic vague binary BCK/BCI - p ideal, neutrosophic vague binary BCK/BCI - q ideal, neutrosophic vague binary BCK/BCI - a ideal and neutrosophic vague binary BCK/BCI - H ideal are given in the table below:

	Condition (1) ; ($\forall$ $\mathbf{u_k} \in \mathbf{U}$)	Condition (2) ; (for any $\mathbf{u_a}, \mathbf{u_b}, \mathbf{u_c} \in \mathbf{U}$)
NVB BCK/BCI **p**-ideal	$\mathbf{NVB_{P_{NVB}}(0)} \succcurlyeq \mathbf{NVB_{P_{NVB}}(u_k)}$	$\mathbf{NVB_{P_{NVB}}(u_a)} \succcurlyeq \mathbf{r\,min}\{\mathbf{NVB_{P_{NVB}}((u_a*u_c)*(u_b*u_c)), NVB_{P_{NVB}}(u_b)}\}$
NVB BCK/BCI **q**-ideal	$\mathbf{NVB_{P_{NVB}}(0)} \succcurlyeq \mathbf{NVB_{P_{NVB}}(u_k)}$	$\mathbf{NVB_{P_{NVB}}(u_a*u_c)} \succcurlyeq \mathbf{r\,min}\left\{\mathbf{NVB_{P_{NVB}}}\left(\left(\mathbf{u_a}*(\mathbf{u_b}*\mathbf{u_c})\right)\right), \mathbf{NVB_{P_{NVB}}(u_b)}\right\}$
NVB BCK/BCI **a**-ideal	$\mathbf{NVB_{P_{NVB}}(0)} \succcurlyeq \mathbf{NVB_{P_{NVB}}(u_k)}$	$\mathbf{NVB_{P_{NVB}}(u_b*u_a)} \succcurlyeq \mathbf{r\,min}\left\{\mathbf{NVB_{P_{NVB}}}\left(\left((\mathbf{u_a}*\mathbf{u_c})*(\mathbf{0}*\mathbf{u_b})\right)\right), \mathbf{NVB_{P_{NVB}}(u_c)}\right\}$
NVB BCK/BCI **H**-ideal	$\mathbf{NVB_{P_{NVB}}(0)} \succcurlyeq \mathbf{NVB_{P_{NVB}}(u_k)}$	$\mathbf{NVB_{P_{NVB}}(u_a*u_c)} \succcurlyeq \mathbf{r\,min}\left\{\mathbf{NVB_{P_{NVB}}}\left(\left(\mathbf{u_a}*(\mathbf{u_b}*\mathbf{u_c})\right)\right), \mathbf{NVB_{P_{NVB}}(u_b)}\right\}$

6. Neutrosophic vague binary BCK/BCI - cuts

In this section N BCK/BCI-cut, NV BCK/BCI-cut and NVB BCK/BCI-cut are developed

Definition 6.1 (Neutrosophic BCK/BCI-$(\alpha, \beta, \gamma)-$ cut or Neutrosophic BCK/BCI-cut)
Let the neutrosophic set M_N is a N BCK/BCI-algebra with algebraic structure $\mathfrak{B}_{M_N} = \left(M_N, U^{\mathfrak{B}_{M_N}}, *, 0\right)$. Truth membership function, indeterminacy membership function and false membership function of M_N are T_{M_N}, I_{M_N}, F_{M_N} respectively. A neutrosophic BCK/BCI (α, β, γ) - cut of $\mathfrak{B}_{M_N}$ is a crisp subset $M_{N(\alpha,\beta,\gamma)}$ of the neutrosophic set M_N given by:

$M_{N(\alpha,\beta,\gamma)} = \{u_k \in U \,/\, N_{M_N}(u_k) \succcurlyeq (\alpha, \beta, \gamma)\,;\ \text{with } \alpha, \beta, \gamma \in [0,1]\}$
$= \{u_k \in U / T_{M_N}(u_k) \geq \alpha\,;\ I_{M_N}(u_k) \leq \beta\,;\ F_{M_N}(u_k) \leq \gamma\,;\ \text{with } \alpha, \beta, \gamma \in [0,1]\}$

Definition 6.2 (**Neutrosophic Vague BCK/BCI $([\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2])$ - cut or Neutrosophic Vague BCK/BCI- cut)**
Let the neutrosophic vague set M_{NV} is a NV BCK/BCI-algebra with algebraic structure $\mathfrak{B}_{M_{NV}} = \left(M_{NV}, U^{\mathfrak{B}_{M_{NV}}} = (U, *, 0), *, 0\right)$. Truth membership function, indeterminacy membership function and false membership function of M_{NV} are $\hat{T}_{M_{NV}}$, $\hat{I}_{M_{NV}}$, $\hat{F}_{M_{NV}}$ respectively. A neutrosophic vague BCK/BCI $([\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2])$ - cut of $\mathfrak{B}_{M_{NV}}$ is a crisp subset $M_{NV([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2])}$ of the neutrosophic vague set M_{NV} given by :

$M_{NV([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2])}$
$= \{u_k \in U \,/\, NV_{M_{NV}}(u_k) \succcurlyeq ([\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2])\,;$ where $\alpha_1 \leq \alpha_2$, $\beta_1 \leq \beta_2$, $\gamma_1 \leq \gamma_2$; with $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in [0,1]\}$
$= \{u_k \in U /\ \hat{T}_{M_{NV}}(u_k) \geq [\alpha_1, \alpha_2]\,; \hat{I}_{M_{NV}}(u_k) \leq [\beta_1, \beta_2];\ \hat{F}_{M_{NV}}(u_k) \leq [\gamma_1, \gamma_2]\}$;

i.e., $T^-(u_k) \geq \alpha_1$ and $T^+(u_k) \geq \alpha_2$; $I^-(u_k) \leq \beta_1$ and $I^+(u_k) \leq \beta_2$; $F^-(u_k) \leq \gamma_1$ and $F^+(u_k) \leq \gamma_2$

Definition 6.3 (Neutrosophic Vague Binary BCK/BCI $\langle([\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2]), ([\delta_1, \delta_2], [\rho_1, \rho_2], [\vartheta_1, \vartheta_2])\rangle$ - cut or Neutrosophic Vague Binary BCK/BCI- cut)

Let the NVBS M_{NVB} is a NVB BCK/BCI-algebra with algebraic structure,
$\mathfrak{B}_{M_{NVB}} = \left(M_{NVB}, U^{\mathfrak{B}_{M_{NVB}}} = (U = \{U_1 \cup U_2\}, *, 0), *, 0\right)$. Truth membership function, indeterminacy membership function and false membership function of M_{NVB} are $\hat{T}_{M_{NVB}}$, $\hat{I}_{M_{NVB}}$, $\hat{F}_{M_{NVB}}$ respectively. A neutrosophic vague binary BCK/BCI $\langle([\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2]), ([\delta_1, \delta_2], [\rho_1, \rho_2], [\vartheta_1, \vartheta_2])\rangle$ - cut of $\mathfrak{B}_{M_{NVB}}$ is a crisp subset $M_{NVB\langle([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]),([\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2])\rangle}$ of the NVBS M_{NVB} given by :

$$M_{NVB\langle([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]),([\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2])\rangle} = \{u_k \in U / NVB_{M_{NVB}}(u_k) \succcurlyeq \begin{cases} [\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2]\ ;\ \text{if } u_k \in U_1 \\ [\delta_1, \delta_2], [\rho_1, \rho_2], [\vartheta_1, \vartheta_2]\ ;\ \text{if } u_k \in U_2 \\ [\chi_1, \chi_2], [\phi_1, \phi_2], [\pi_1, \pi_2]\ ;\ \text{if } u_k \in U_1 \cap U_2 \end{cases}$$

$\max\{[\alpha_1, \alpha_2], [\delta_1, \delta_2]\} = [\chi_1, \chi_2]$ (say); $\min[\beta_1, \beta_2], [\rho_1, \rho_2]\} = [\phi_1, \phi_2]$ (say); $\min\{[\gamma_1, \gamma_2], [\vartheta_1, \vartheta_2] = [\pi_1, \pi_2]$ (say) ;

with $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \rho_1, \rho_2, \vartheta_1, \vartheta_2, \chi_1, \chi_2, \phi_1, \phi_2, \pi_1, \pi_2 \in [0, 1]$

and $\alpha_1 \leq \alpha_2$, $\beta_1 \leq \beta_2$, $\gamma_1 \leq \gamma_2$; $\delta_1 \leq \delta_2$, $\rho_1 \leq \rho_2$, $\vartheta_1 \leq \vartheta_2$, $\chi_1 \leq \chi_2$, $\phi_1 \leq \phi_2$; $\pi_1 \leq \pi_2$

i.e., $\hat{T}_{M_{NVB}}(u_k) \geq [\alpha_1, \alpha_2]$; $\hat{I}_{M_{NVB}}(u_k) \leq [\beta_1, \beta_2]$; $\hat{F}_{M_{NVB}}(u_k) \leq [\gamma_1, \gamma_2]$
$\hat{T}_{M_{NVB}}(u_k) \geq [\delta_1, \delta_2]$; $\hat{I}_{M_{NVB}}(u_k) \leq [\rho_1, \rho_2]$; $\hat{F}_{M_{NVB}}(u_k) \leq [\vartheta_1, \vartheta_2]$
$\hat{T}_{M_{NVB}}(u_k) \geq [\chi_1, \chi_2]$; $\hat{I}_{M_{NVB}}(u_k) \leq [\phi_1, \phi_2]$; $\hat{F}_{M_{NVB}}(u_k) \leq [\pi_1, \pi_2]$

i.e., $T^-(u_k) \geq \alpha_1$ and $T^+(u_k) \geq \alpha_2$; $I^-(u_k) \leq \beta_1$ and $I^+(u_k) \leq \beta_2$; $F^-(u_k) \leq \gamma_1$ and $F^+(u_k) \leq \gamma_2$
$T^-(u_k) \geq \delta_1$ and $T^+(u_k) \geq \delta_2$; $I^-(u_k) \leq \rho_1$ and $I^+(u_k) \leq \rho_2$; $F^-(u_k) \leq \vartheta_1$ and $F^+(u_k) \leq \vartheta_2$
$T^-(u_k) \geq \chi_1$ and $T^+(u_k) \geq \chi_2$; $I^-(u_k) \leq \phi_1$ and $I^+(u_k) \leq \phi_2$; $F^-(u_k) \leq \pi_1$ and $F^+(u_k) \leq \pi_2$

Remark 6.4

(i) (a) $M_{NVB}([0,0],[1,1],[1,1]) = U$
(b) $M_{NVB}\langle([0,0],[1,1],[1,1]),([0,0],[1,1],[1,1])\rangle = U = \{U_1 \cup U_2\}$

(ii) If $[\alpha_1, \alpha_2]$ and $[\delta_1, \delta_2]$ coincides; $[\beta_1, \beta_2]$ and $[\rho_1, \rho_2]$coincides; $[\gamma_1, \gamma_2]$ and $[\vartheta_1, \vartheta_2]$ coincides, then $(\langle[\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2]\rangle, \langle[\delta_1, \delta_2], [\rho_1, \rho_2], [\vartheta_1, \vartheta_2]\rangle) -$ cuts are called $(\langle[\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2]\rangle, \langle[\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2]\rangle)$ - cuts and is denoted by $M_{NVB([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2],[\delta_1,\delta_2])}$ instead of $M_{NVB(\langle[\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]\rangle,\langle[\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]\rangle)}$

(iii) If $(\langle[\alpha_1^*, \alpha_2^*], [\beta_1^*, \beta_2^*], [\gamma_1^*, \gamma_2^*]\rangle, \langle[\delta_1^*, \delta_2^*], [\rho_1^*, \rho_2^*], [\vartheta_1^*, \vartheta_2^*]\rangle) \geq (\langle[\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2]\rangle, \langle[\delta_1, \delta_2], [\rho_1, \rho_2], [\vartheta_1, \vartheta_2]\rangle)$ then $M_{NVB(\langle[\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]\rangle,\langle[\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2]\rangle)} \subseteq M_{NVB(\langle[\alpha_1^*,\alpha_2^*],[\beta_1^*,\beta_2^*],[\gamma_1^*,\gamma_2^*]\rangle,\langle[\delta_1^*,\delta_2^*],[\rho_1^*,\rho_2^*],[\vartheta_1^*,\vartheta_2^*]\rangle)}$

7. Application

In this section theoretical application of NVB BCK/BCI algebra is developed. Various theorems and propositions are found good to this concept.

Lemma 7.1

Every NVB BCI – algebra $\mathfrak{B}^{BCI}_{M_{NVB}}$ of a BCI -algebra $U^{\mathfrak{B}^{BCI}_{M_{NVB}}}$ satisfies:
$NVB_{M_{NVB}}(0) \succcurlyeq NVB_{M_{NVB}}(u_k)$; $\forall\ u_k \in U = \{U_1 \cup U_2\}$

Proof

For a $\mathfrak{B}^{BCI}_{M_{NVB}}$, underlying BCI - algebraic structure satisfies, $(u_k * u_k) = 0,\ \forall\ u_k \in U$

[By property (iii)of definition 2.3]

$\Rightarrow \forall\ u_k \in U,\ NVB_{M_{NVB}}(0) = NVB_{M_{NVB}}(u_k * u_k) \succcurlyeq r\min\{NVB_{M_{NVB}}(u_k), NVB_{M_{NVB}}(u_k)\} = NVB_{M_{NVB}}(u_k)$

[By definition 3.4]

Lemma 7.2

Every $\mathfrak{B}^{BCK}_{M_{NVB}}$ satisfies $NVB_{M_{NVB}}(0) \succcurlyeq NVB_{M_{NVB}}(u_k)$; $\forall\ u_k \in U$

Proof

For a $\mathfrak{B}^{BCK}_{M_{NVB}}$, underlying BCK- algebraic structure satisfies, an additional condition, $(0 * u_k) = 0, \forall u_k \in U$ besides $(u_k * u_k) = 0,\ \forall\ u_k \in U$; [By remark 2.5]
$\Rightarrow$ Additional to, $NVB_{M_{NVB}}(0) \succcurlyeq NVB_{M_{NVB}}(u_k)$; $\forall\ u_k \in U$ [by lemma 7.1], we get,
$NVB_{M_{NVB}}(0) = NVB_{M_{NVB}}(0 * u_k) \succcurlyeq r\min\{NVB_{M_{NVB}}(0), NVB_{M_{NVB}}(u_k)\}$; $\forall\ u_k \in U$
$\Rightarrow$ $NVB_{M_{NVB}}(0) \succcurlyeq r\min\{NVB_{M_{NVB}}(0), NVB_{M_{NVB}}(u_k)\}$; $\forall\ u_k \in U$, for $\mathfrak{B}^{BCK}_{M_{NVB}}$
& $r\min\{NVB_{M_{NVB}}(0), NVB_{M_{NVB}}(u_k)\}$ will depend upon the given NVBS M_{NVB}
$\Rightarrow$ $NVB_{M_{NVB}}(0) \succcurlyeq NVB_{M_{NVB}}(u_k)$ and $NVB_{M_{NVB}}(0) \succcurlyeq r\min\{NVB_{M_{NVB}}(0), NVB_{M_{NVB}}(u_k)\}$
Even if $r\min\{NVB_{M_{NVB}}(0), NVB_{M_{NVB}}(u_k)\}$ will depend upon the given NVBS, using lemma 7.1, $NVB_{M_{NVB}}(0) \succcurlyeq r\min\{NVB_{M_{NVB}}(0), NVB_{M_{NVB}}(u_k)\}$ will become $NVB_{M_{NVB}}(0) \succcurlyeq NVB_{M_{NVB}}(u_k)$
$\Rightarrow$ $NVB_{M_{NVB}}(0) \succcurlyeq NVB_{M_{NVB}}(u_k)$ and $NVB_{M_{NVB}}(0) \succcurlyeq NVB_{M_{NVB}}(u_k)$ $\forall\ u_k \in U$
So, combining both, for a $\mathfrak{B}^{BCK}_{M_{NVB}}$ too, $NVB_{M_{NVB}}(0) \succcurlyeq NVB_{M_{NVB}}(u_k)$; $\forall\ u_k \in U$

Remark 7.3

Every $\mathfrak{B}^{BCK}_{M_{NVB}}$/ $\mathfrak{B}^{BCI}_{M_{NVB}}$ satisfies: $NVB_{M_{NVB}}(0) \succcurlyeq NVB_{M_{NVB}}(u_k)$; $\forall\ u_k \in U$
i.e., Every NVB BCK/BCI – algebra satisfies: $NVB_{M_{NVB}}(0) \succcurlyeq NVB_{M_{NVB}}(u_k)$; $\forall\ u_k \in U$

Theorem 7.4

Every $\mathfrak{B}^{BCK}_{M_{NVB}}$ is a $\mathfrak{B}^{BCI}_{M_{NVB}}$. But converse not true, generally. i.e., every $\mathfrak{B}^{BCI}_{M_{NVB}}$ is not a $\mathfrak{B}^{BCK}_{M_{NVB}}$ generally.

Proof

For a fixed universal set U, underlying BCK – algebraic structure of $\mathfrak{B}^{BCK}_{M_{NVB}}$ consists the underlying BCI – structure of $\mathfrak{B}^{BCI}_{M_{NVB}}$ $\Rightarrow$ Every $\mathfrak{B}^{BCK}_{M_{NVB}}$ is $\mathfrak{B}^{BCI}_{M_{NVB}}$. But converse does not hold. It is illustrated with the case (i) of remark 7.5.

Remark 7.5

Following example illustrates both the cases:

Let $U_1 = \{0\}$ and let $U_2 = \{0, 1\}$ be the universes under consideration. Combined universe $U = \{U_1 \cup U_2\} = \{0, 1\}$ with $(U_1 \cap U_2) = \{0\}$.

$\therefore$ Cayley table to the binary operation $*$ for U is given as:

$*$	0	1
0	0	1
1	1	0

BCI-algebra [fig (i)]

$*$	0	1
0	0	0
1	1	0

BCK/BCI-algebra[fig(ii)]

Clearly, $\mathbf{U}^{\mathfrak{B}^{BCI}_{M_{NVB}}} = (\mathbf{U} = \{\mathbf{U_1} \cup \mathbf{U_2}\}, *, \mathbf{0})$ is a BCI-algebra [fig (i)].

$\mathbf{U}^{\mathfrak{B}^{BCK}_{M_{NVB}}} = (\mathbf{U} = \{\mathbf{U_1} \cup \mathbf{U_2}\}, *, \mathbf{0})$ is a BCI-algebra [fig (ii)].

Case (i) : Example for a $\mathfrak{B}^{BCI}_{M_{NVB}}$ which is a $\mathfrak{B}^{BCK}_{M_{NVB}}$

Let M_{NVB} be a non-empty NVBS with $U^{\mathfrak{B}^{BCI}_{M_{NVB}}}$ as underlying algebraic structure:

$M_{NVB} = \left\{\langle\frac{[0.1,0.8],[0.1,0.5],[0.2,0.9]}{0}\rangle, \langle\frac{[0.3,0.7],[0.2,0.4],[0.3,0.7]}{0}, \frac{[0.1,0.4],[0.3,0.5],[0.6,0.9]}{1}\rangle\right\}$; Here, $(U_1 \cap U_2) = \{0\}$

$NVB_{M_{NVB}}(0) = ([0.1, 0.8], [0.1, 0.5], [0.2, 0.9]) \cup ([0.3, 0.7], [0.2, 0.4], [0.3, 0.7]) = [0.3, 0.8], [0.1, 0.4], [0.2, 0.7]$

After verification, clearly M_{NVB} is a $\mathfrak{B}^{BCI}_{M_{NVB}}$. Next question is that, - "whether $\mathfrak{B}^{BCI}_{M_{NVB}}$ is a $\mathfrak{B}^{BCK}_{M_{NVB}}$ or not " ? $\therefore$ Additional condition to be satisfied is that, for a BCK-algebra is, $(0 * 1) = 0$ from Cayley table fig (ii). Correspondingly,

$NVB_{M_{NVB}}(0-1) \succcurlyeq r\min\{NVB_{M_{NVB}}(0), NVB_{M_{NVB}}(1)\} \Rightarrow NVB_{M_{NVB}}(0) \succcurlyeq r\min\{NVB_{M_{NVB}}(0), NVB_{M_{NVB}}(1)\}$

$\Rightarrow [0.3, 0.8], [0.1, 0.4], [0.2, 0.7] \succcurlyeq r\min\{[0.3, 0.8], [0.1, 0.4], [0.2, 0.7], [0.1, 0.4], [0.3, 0.5], [0.6, 0.9]\}$

$\Rightarrow [0.3, 0.8], [0.1, 0.4], [0.2, 0.7] \succcurlyeq [0.1, 0.4], [0.3, 0.5], [0.6, 0.9]$

Since additional condition got satisfied, $\mathfrak{B}^{BCI}_{M_{NVB}}$ is clearly a $\mathfrak{B}^{BCK}_{M_{NVB}}$.

Case (ii) : Example for a $\mathfrak{B}^{BCI}_{P_{NVB}}$ which is not a $\mathfrak{B}^{BCK}_{P_{NVB}}$

Take binary operation and Cayley table as taken in Case (i).

Consider another NVBS $\mathbf{P}_{NVB}$ with same conditions as in case (i)

$\mathbf{P}_{NVB} = \left\{\langle\frac{[\mathbf{0.1,0.5}],[\mathbf{0.2,0.5}],[\mathbf{0.5,0.9}]}{\mathbf{0}}\rangle, \langle\frac{[\mathbf{0.1,0.6}],[\mathbf{0.3,0.3}],[\mathbf{0.4,0.9}]}{\mathbf{0}}, \frac{[\mathbf{0.1,0.7}],[\mathbf{0.3,0.4}],[\mathbf{0.3,0.9}]}{\mathbf{1}}\rangle\right\}$

$NVB_{P_{NVB}}(0) = ([0.1, 0.5], [0.2, 0.5], [0.5, 0.9]) \cup ([0.1, 0.6], [0.3, 0.3], [0.4, 0.9]) = [0.1, 0.6], [0.2, 0.3], [0.4, 0.9]$

By verification P_{NVB} is a $\mathfrak{B}^{BCI}_{P_{NVB}}$. But in this case, additional condition not got satisfied:

$NVB_{P_{NVB}}(0 * 1) \not\succcurlyeq r\min\{NVB_{P_{NVB}}(0), NVB_{P_{NVB}}(1)\}$ [Since, $NVB_{P_{NVB}}(0) \not\succcurlyeq r\min\{NVB_{P_{NVB}}(0), NVB_{P_{NVB}}(1)\}$.

Since, $[0.1, 0.6], [0.2, 0.3], [0.4, 0.9] \not\succcurlyeq r\min\{[0.1, 0.8], [0.2, 0.3], [0.2, 0.9], [0.1, 0.7], [0.3, 0.4], [0.3, 0.9]\}$

Since, $[0.1, 0.6], [0.2, 0.3], [0.4, 0.9] \not\succcurlyeq [0.1, 0.7], [0.3, 0.4], [0.3, 0.9]]$

In this case, clearly, $\mathfrak{B}^{BCI}_{P_{NVB}}$ is not a $\mathfrak{B}^{BCK}_{P_{NVB}}$

Theorem 7.6

Intersection of two NVB BCK/BCI -algebra remains as a NVB BCK/BCI-algebra itself.

Proof

Let M_{NVB} and P_{NVB} be two NVB BCK/BCI -algebras with structures $\mathfrak{B}_{M_{NVB}} = (M_{NVB}, U^{\mathfrak{B}_{M_{NVB}}}, *, 0)$

and $\mathfrak{B}_{P_{NVB}} = (P_{NVB}, U^{\mathfrak{B}_{P_{NVB}}}, *, 0)$ respectively, with same universal sets U_1 and U_2. So, $\forall\ u_1, u_2 \in U$, $NVB_{(M_{NVB} \cap P_{NVB})}(u_1 * u_2) = r\min\{NVB_{M_{NVB}}(u_1 * u_2), NVB_{P_{NVB}}(u_1 * u_2)\}$
$\succcurlyeq r\min\{r\min\{NVB_{M_{NVB}}(u_1), NVB_{M_{NVB}}(u_2)\}, r\min\{NVB_{P_{NVB}}(u_1), NVB_{P_{NVB}}(u_2)\}\}$
$= r\min\{NVB_{(M_{NVB} \cap P_{NVB})}(u_1), NVB_{(M_{NVB} \cap P_{NVB})}(u_2)\}$
Therefore, $NVB_{(M_{NVB} \cap P_{NVB})}(u_1 * u_2) \succcurlyeq r\min\{NVB_{(M_{NVB} \cap P_{NVB})}(u_1), NVB_{(M_{NVB} \cap P_{NVB})}(u_2)\}$
$\Rightarrow (M_{NVB} \cap P_{NVB})$ is also a NVB BCK/BCI - algebra

Proposition 7.7

Every NVB BCI - ideal P_{NVB} of a $\mathfrak{B}^{BCI}_{M_{NVB}}$ satisfies:
(i) $u_a \leq u_b \Rightarrow NVB_{P_{NVB}}(u_a) \succcurlyeq NVB_{P_{NVB}}(u_b)$; $(\forall\ u_a, u_b \in U)$
(ii) $NVB_{P_{NVB}}(u_a * u_c) \succcurlyeq r\min\{NVB_{P_{NVB}}((u_a * u_b) * u_c), NVB_{P_{NVB}}(u_b)\}; \forall\ u_a, u_b, u_c \in U$

Proof

(i) Let $u_a, u_b \in U$ be such that $u_a \leq u_b$.
Since P_{NVB} is a NVB BCI - ideal of $\mathfrak{B}^{BCI}_{M_{NVB}}$
$\Rightarrow NVB_{P_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{P_{NVB}}(u_a * u_b), NVB_{P_{NVB}}(u_b)\}$, [By condition (2) of definition 4.2]
$= r\min\{NVB_{P_{NVB}}(0), NVB_{P_{NVB}}(u_b)\}$, take $(u_a * u_b) = 0$
$= NVB_{P_{NVB}}((u_b))$ [By lemma 7.1]
$\Rightarrow NVB_{P_{NVB}}(u_a) \succcurlyeq NVB_{P_{NVB}}(u_b)$
(ii) Let P_{NVB} be a NVB BCI - ideal of $\mathfrak{B}^{BCI}_{M_{NVB}}$
$\Rightarrow NVB_{P_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{P_{NVB}}(u_a * u_b), NVB_{P_{NVB}}(u_b)\}$; $\forall\ u_a, u_b \in U$
$\Rightarrow NVB_{P_{NVB}}(u_a * u_c) \succcurlyeq r\min\{NVB_{P_{NVB}}((u_a * u_c) * u_b), NVB_{P_{NVB}}(u_b)\}$;
[by putting $u_a = (u_a * u_c)$; $\forall\ u_a, u_b, u_c \in U$]
$\Rightarrow NVB_{P_{NVB}}(u_a * u_c) \succcurlyeq r\min\{NVB_{P_{NVB}}((u_a * u_b) * u_c), NVB_{P_{NVB}}(u_b)\}$;[By property (ii) of remark 2.6]

Lemma 7.8

Let P_{NVB} be a NVB BCI-ideal of $\mathfrak{B}^{BCI}_{M_{NVB}}$. Then, $NVB_{P_{NVB}}(0 * (0 * u_k)) \succcurlyeq NVB_{P_{NVB}}(u_k)$; $\forall\ u_k \in U$

Proof

$NVB_{P_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{P_{NVB}}(u_a * u_b), NVB_{P_{NVB}}(u_b)\}$; for any $u_a, u_b \in U$ [By definition 4.2]
Let $u_a = (0 * (0 * u_k))$ and $u_b = u_k$.
$\therefore$ For any $u_k \in U$, $NVB_{P_{NVB}}(0 * (0 * u_k)) \succcurlyeq r\min\left\{NVB_{P_{NVB}}\left((0 * (0 * u_k)) * u_k\right), NVB_{P_{NVB}}(u_k)\right\}$
$= r\min\{NVB_{P_{NVB}}((0 * u_k) * (0 * u_k)), NVB_{P_{NVB}}(u_k)\}$; [By property (ii) of remark 2.6]
$= r\min\{NVB_{P_{NVB}}(0 * (u_k * u_k)), NVB_{P_{NVB}}(u_k)\}$; [By property (iii) of remark 2.6]
$= r\min\{NVB_{P_{NVB}}(0 * 0), NVB_{P_{NVB}}(u_k)\}$; [By condition (iii) of definition 2.3]
$= r\min\{NVB_{P_{NVB}}(0), NVB_{P_{NVB}}(u_k)\}$; [By property (i) of remark 2.6]
$= NVB_{P_{NVB}}(u_k)$ [By lemma 7.1]
$\therefore$ It is concluded that, $NVB_{M_{NVB}}(0 * (0 * u_k)) \succcurlyeq NVB_{M_{NVB}}(u_k)$; $\forall\ u_k \in U$

Proposition 7.9

If the NVBS R_{NVB} of $\mathfrak{B}^{BCI}_{M_{NVB}}$ is a NVB BCI - ideal of $\mathfrak{B}^{BCI}_{M_{NVB}}$, then it satisfies: for any $u_a, u_b, u_c \in U$;
$(u_a * u_b) \leq u_c \Rightarrow NVB_{R_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{R_{NVB}}(u_b), NVB_{R_{NVB}}(u_c)\}$

Proof

Let R_{NVB} be a NVB BCI - ideal of $\mathfrak{B}^{BCI}_{M_{NVB}}$ with $(u_a * u_b) \leq u_c$ where $u_a, u_b, u_c \in U$
$\Rightarrow NVB_{R_{NVB}}(u_a * u_b) \succcurlyeq NVB_{R_{NVB}}(u_c)$ [By proposition 7.7]
Since R_{NVB} be a NVB BCI - ideal of $\mathfrak{B}^{BCI}_{M_{NVB}}$
$\Rightarrow NVB_{R_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{R_{NVB}}(u_a * u_b), NVB_{R_{NVB}}(u_b)\}$ for any $u_a, u_b \in U$
$\succcurlyeq r\min\{NVB_{R_{NVB}}(u_c), NVB_{R_{NVB}}(u_b)\} = r\min\{NVB_{R_{NVB}}(u_b), NVB_{R_{NVB}}(u_c)\}$
$\Rightarrow NVB_{R_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{R_{NVB}}(u_b), NVB_{R_{NVB}}(u_c)\}$ for any $u_a, u_b, u_c \in U$

Proposition 7.10

If the NVBS R_{NVB} of $\mathfrak{B}^{BCI}_{M_{NVB}}$ is a NVB BCI - algebra $\mathfrak{B}^{BCI}_{R_{NVB}}$ of then it satisfies for any $u_x, u_y, u_z \in U$
$(u_a * u_b) \leq u_c \Rightarrow NVB_{R_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{R_{NVB}}(u_b), NVB_{R_{NVB}}(u_c)\}$

Proof

Let R_{NVB} be a NVBS of $\mathfrak{B}^{BCI}_{M_{NVB}}$ with $(u_a * u_b) \leq u_c \Rightarrow NVB_{R_{NVB}}(u_c) \succcurlyeq NVB_{R_{NVB}}(u_a * u_b)$

R_{NVB} is a $\mathfrak{B}^{BCI}_{R_{NVB}}$ $\Rightarrow NVB_{R_{NVB}}(u_a * u_b) \succcurlyeq r\min\{NVB_{R_{NVB}}(u_a), NVB_{R_{NVB}}(u_b)\}$
$\Rightarrow NVB_{R_{NVB}}(u_c) \succcurlyeq NVB_{R_{NVB}}(u_a * u_b) \succcurlyeq r\min\{NVB_{R_{NVB}}(u_a), NVB_{R_{NVB}}(u_b)\}$
$\Rightarrow NVB_{R_{NVB}}(u_c) \succcurlyeq r\min\{NVB_{R_{NVB}}(u_a), NVB_{M_{NVB}}(u_b)\}$
$\Rightarrow NVB_{R_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{R_{NVB}}(u_c), NVB_{R_{NVB}}(u_b)\}$; [By putting $u_c = u_a$ &$u_a = u_c$]
$\Rightarrow NVB_{R_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{R_{NVB}}(u_b), NVB_{R_{NVB}}(u_c)\}$;

Theorem 7.11

Let S_{NVB} be both a NVB BCI–algebra $\mathfrak{B}^{BCI}_{S_{NVB}}$ and a NVB BCI-ideal of a NVB BCI - algebra $\mathfrak{B}^{BCI}_{S_{NVB}}$. Then $NVB_{S_{NVB}}(0 * u_k) \succcurlyeq NVB_{S_{NVB}}(u_k)$ for all $u_k \in U$

Proof

Let S_{NVB} be a NVB BCI- algebra $\mathfrak{B}^{BCI}_{S_{NVB}}$
$\Rightarrow NVB_{S_{NVB}}(u_a * u_b) \succcurlyeq r\min\{NVB_{S_{NVB}}(u_a), NVB_{S_{NVB}}(u_b)\}$; for all $u_a, u_b \in U$
$\Rightarrow NVB_{S_{NVB}}(0 * u_b) \succcurlyeq r\min\{NVB_{S_{NVB}}(0), NVB_{S_{NVB}}(u_b)\}$; [By putting $u_a = 0$]
$\Rightarrow NVB_{S_{NVB}}(0 * u_b) \succcurlyeq NVB_{S_{NVB}}(u_b)$ [By definition 4.2 (i)]
$\Rightarrow NVB_{S_{NVB}}(0 * u_k) \succcurlyeq NVB_{S_{NVB}}(u_k)$ [By putting $u_b= u_k$]
$\therefore$ For any $u_k \in U$, $NVB_{S_{NVB}}(0 * u_k) \succcurlyeq NVB_{S_{NVB}}(u_k)$

Proposition 7.12

Let T_{NVB} be a NVB BCI - ideal of a NVB BCI -algebra $\mathfrak{B}^{BCI}_{M_{NVB}}$.
If T_{NVB} satisfies $NVB_{T_{NVB}}(u_a * u_b) \succcurlyeq NVB_{T_{NVB}}((u_a * u_c) * (u_b * u_c))$ for all $u_a, u_b, u_c \in U$,
then T_{NVB} is a NVB BCI p - ideal of $\mathfrak{B}^{BCI}_{M_{NVB}}$

Proof

T_{NVB} be a NVB BCI - ideal of a NVB BCI - algebra $\mathfrak{B}^{BCI}_{M_{NVB}}$.
$\Rightarrow NVB_{T_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{T_{NVB}}(u_a * u_b), NVB_{T_{NVB}}(u_b)\}$ for all $u_a, u_b, u_c \in U$
$\Rightarrow NVB_{T_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{T_{NVB}}((u_a * u_c) * (u_b * u_c)), NVB_{M_{NVB}}(u_b)\}$ for all $u_a, u_b, u_c \in U$
[From given condition]
$\Rightarrow T_{NVB}$ is a NVB BCI – p ideal of $\mathfrak{B}^{BCI}_{M_{NVB}}$ [By definition 5.2]

Proposition 7.13

Any NVB BCI - ideal D_{NVB} of a NVB BCI -algebra $\mathfrak{B}^{BCI}_{M_{NVB}}$ is a NVB BCI -p ideal
$\Leftrightarrow NVB_{D_{NVB}}(u_a) \succcurlyeq NVB_{D_{NVB}}(0 * (0 * u_a))$; for all $u_a \in U$

Proof

Let D_{NVB} be a NVB BCI - ideal of a NVB BCI -algebra $\mathfrak{B}^{BCI}_{M_{NVB}}$. Also let D_{NVB} is a NVB BCI -p ideal.
$\therefore NVB_{D_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{D_{NVB}}((u_a * u_c) * (u_b * u_c)), NVB_{D_{NVB}}(u_b)\}$ for all $u_a, u_b, u_c \in U$
[By definition 5.2 of NVB BCI − p ideal]
Put $u_c = u_a$ and $u_b = 0$ in the above,
$\therefore NVB_{D_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{D_{NVB}}((u_a * u_a) * (0 * u_a)), NVB_{D_{NVB}}(0)\}$ for all $u_a, u_b, u_c \in U$
$\Rightarrow NVB_{D_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{D_{NVB}}(0 * (0 * u_a)), NVB_{D_{NVB}}(0)\}$ for all $u_a, u_b \in U$
[By condition (iii)of definition 2.3]
$= NVB_{D_{NVB}}(0 * (0 * u_a))$ for all $u_a \in U$ [By lemma 7.1]
$\Rightarrow NVB_{D_{NVB}}(u_a) \succcurlyeq NVB_{D_{NVB}}(0 * (0 * u_a))$; for all $u_a \in U$
Conversely, let a NVB BCI - ideal D_{NVB} of a NVB BCI - algebra $\mathfrak{B}^{BCI}_{M_{NVB}}$ satisfies the given condition, $NVB_{D_{NVB}}(u_a) \succcurlyeq NVB_{M_{NVB}}(0 * (0 * u_a))$; for all $u_a \in U$. By lemma 7.8,
" Let P_{NVB} be a NVB BCI-ideal of $\mathfrak{B}^{BCI}_{M_{NVB}}$. Then, $NVB_{P_{NVB}}(0 * (0 * u_k)) \succcurlyeq NVB_{P_{NVB}}(u_k)$; $\forall$ $u_k \in U$"
$\Rightarrow NVB_{D_{NVB}}((u_a * u_c) * (u_b * u_c)) \preccurlyeq NVB_{M_{NVB}}\left(0 * \left(0 * \left((u_a * u_c) * (u_b * u_c)\right)\right)\right)$
[By putting $u_k = (u_a * u_c) * (u_b * u_c)$ in lemma 7.8]
$= NVB_{D_{NVB}}((0 * u_b) * (0 * u_a))$ [By property (vii)of remark 2.6]
$= NVB_{D_{NVB}}\left(0 * \left(0 * (u_a * u_b)\right)\right)$; [By property (viii) of remark 2.6]
$= NVB_{D_{NVB}}(0 * (u_a * u_b))$; [By property (i) of remark 2.6]
$= NVB_{D_{NVB}}(u_a * u_b)$; [By property (i) of remark 2.6]
$\Rightarrow NVB_{D_{NVB}}(u_a * u_b) \succcurlyeq NVB_{D_{NVB}}((u_a * u_c) * (u_b * u_c))$

$\Rightarrow$ $NVB_{D_{NVB}}$ is a NVB BCI - p ideal [By proposition 7.12]

Theorem 7.14

Every NVB BCI - p ideal of a NVB BCI - algebra $\mathfrak{B}^{BCI}_{M_{NVB}}$ is a NVB BCI - ideal of $\mathfrak{B}^{BCI}_{M_{NVB}}$.

Proof

Let M_{NVB} be a NVB BCI - p ideal of a NVB BCI - algebra $\mathfrak{B}^{BCI}_{M_{NVB}}$. By definition,

$NVB_{M_{NVB}}(u_x) \succcurlyeq r\min\left\{NVB_{M_{NVB}}\left((u_x * u_z) * (u_y * u_z)\right), NVB_{M_{NVB}}(u_y)\right\}$ for all $u_x, u_y, u_z \in U$

Put $u_z = 0$ then the above becomes,

$NVB_{M_{NVB}}(u_x) \succcurlyeq r\min\left\{NVB_{M_{NVB}}\left((u_x * 0) * (u_y * 0)\right), NVB_{M_{NVB}}(u_y)\right\}$ for all $u_x, u_y \in U$

$= r\min\{NVB_{M_{NVB}}(u_x * u_y), NVB_{M_{NVB}}(u_y)\}$ for all $u_x, u_y \in U$

[By property (iii)of 2.3 &By property (i)of remark 2.6]

$\Rightarrow NVB_{M_{NVB}}(u_x) \succcurlyeq r\min\{NVB_{M_{NVB}}(u_x * u_y), NVB_{M_{NVB}}(u_y)\}$ for all $u_x, u_y \in U$

Obviously, M_{NVB} is a NVB BCI - ideal

Converse of this statement need not be true and it can be verified with an example and is trivial.

Theorem 7.15

Every NVB BCK/BCI H - ideal of a NVB BCK/BCI - algebra $\mathfrak{B}_{M_{NVB}}$ acts both as

(i) NVB BCK/BCI - ideal of $\mathfrak{B}_{M_{NVB}}$ (ii) NVB BCK/BCI - subalgebra $\mathfrak{B}_{M_{NVB}}$

Proof

Let I_{NVB} be a NVB BCK/BCI- H ideal of a NVB BCK/BCI – algebra $\mathfrak{B}_{M_{NVB}}$

(i) From definition of NVB BCK/BCI- H ideal,

$NVB_{I_{NVB}}(u_a * u_c) \succcurlyeq \min\{NVB_{I_{NVB}}(u_a * (u_b * u_c)), NVB_{I_{NVB}}(u_b)\}$ for all $u_a, u_b, u_c \in U$

Put $u_c = 0$

$NVB_{I_{NVB}}(u_a * 0) \succcurlyeq \min\{NVB_{I_{NVB}}(u_a * (u_b * 0)), NVB_{M_{NVB}}(u_b)\}$ for all $u_a, u_b, u_c \in U$

$\Rightarrow NVB_{I_{NVB}}(u_a) \succcurlyeq \min\{NVB_{I_{NVB}}(u_a * u_b), NVB_{M_{NVB}}(u_b)\}$ for all $u_a, u_b, u_c \in U$

[Using property (i)of remark 2.6]

Since I_{NVB} is a NVB BCK/BCI- H ideal $\Rightarrow NVB_{I_{NVB}}(0) \succcurlyeq NVB_{I_{NVB}}(u_k)$; for any $u_k \in U$

$\therefore$ I_{NVB} is a NVB BCK/BCI - ideal of $\mathfrak{B}_{M_{NVB}}$ [By definition 4.2]

(ii) Let I_{NVB} be a NVB BCK/BCI- H ideal of $\mathfrak{B}_{M_{NVB}}$

$\therefore NVB_{I_{NVB}}(u_a * u_c) \succcurlyeq r\min\{NVB_{I_{NVB}}(u_a * (u_b * u_c)), NVB_{I_{NVB}}(u_b)\}$; for all $u_a, u_b, u_c \in U$

$\Rightarrow$ $NVB_{I_{NVB}}(u_a * u_b) \succcurlyeq r\min\{NVB_{I_{NVB}}(u_a * (u_b * u_b)), NVB_{I_{NVB}}(u_b)\}$; [By putting $u_c = u_b$]

$\Rightarrow$ $NVB_{I_{NVB}}(u_a * u_b) \succcurlyeq r\min\{NVB_{I_{NVB}}(u_a * 0), NVB_{I_{NVB}}(u_b)\}$; [By condition (iii)of definition 2.3]

$\Rightarrow$ $NVB_{I_{NVB}}(u_a * u_b) \succcurlyeq r\min\{NVB_{I_{NVB}}(u_a), NVB_{I_{NVB}}(u_b)\}$; [By condition (i)of remark 2.6]

$\Rightarrow$ I_{NVB} be a NVB BCK/BCI - subalgebra of $\mathfrak{B}_{M_{NVB}}$

Theorem 7.16

P_{NVB} be a NVBS of a NVB BCK/BCI - algebra $\mathfrak{B}_{M_{NVB}}$. Then P_{NVB} is a NVB BCK/BCI -ideal of $\mathfrak{B}_{M_{NVB}}$ $\Leftrightarrow$ it satisfies the following conditions:

(i) $NVB_{P_{NVB}}(u_a * u_b) \succcurlyeq NVB_{P_{NVB}}(u_b)$; $(\forall\, u_a, u_b \in U)$

(ii) $NVB_{P_{NVB}}\left(u_a * \left((u_a * u_m) * u_n\right)\right) \succcurlyeq r\min\ \{NVB_{P_{NVB}}(u_m), NVB_{P_{NVB}}(u_n)\}$; $(\forall\, u_a, u_m, u_n \in U)$

Proof

Let P_{NVB} be NVB BCK/BCI - ideal of $\mathfrak{B}_{M_{NVB}}$. By definition,

$NVB_{P_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{P_{NVB}}(u_a * u_b), NVB_{P_{NVB}}(u_b)\}$; $\forall\ u_a, u_b \in U$

(i) Put $u_a = (u_a * u_b)$ and $u_b = u_a$ in the above,

$NVB_{P_{NVB}}(u_a * u_b) \succcurlyeq r\min\{NVB_{P_{NVB}}((u_a * u_b) * u_a), NVB_{P_{NVB}}(u_a)\}$

$\Rightarrow NVB_{P_{NVB}}(u_a * u_b) \succcurlyeq r\min\{NVB_{P_{NVB}}((u_a * u_a) * u_b), NVB_{P_{NVB}}(u_a)\}$; [By property (ii)of remark 2.6]

$\Rightarrow NVB_{P_{NVB}}(u_a * u_b) \succcurlyeq r\min\{NVB_{P_{NVB}}(0 * u_b), NVB_{P_{NVB}}(u_a)\}$; [By condition (iii)of definition 2.3]

$\Rightarrow NVB_{P_{NVB}}(u_a * u_b) \succcurlyeq r\min\{NVB_{P_{NVB}}(0 * u_b), NVB_{P_{NVB}}(u_b)\}$; [By assumption $u_a = u_b$]

$\Rightarrow$ $NVB_{P_{NVB}}(u_a * u_b) \succcurlyeq r\min\{NVB_{P_{NVB}}(0), NVB_{P_{NVB}}(u_b)\}$; [By remark 2.5]

$\Rightarrow$ $NVB_{P_{NVB}}(u_a * u_b) \succcurlyeq NVB_{P_{NVB}}(u_b)$ [Using lemma 7.1]

(ii) Consider, $\left(u_a * \left((u_a * u_m) * u_n\right)\right) * u_m$

$= (u_a * u_m) * \left((u_a * u_m) * u_n\right) \leq u_n$

$$\begin{bmatrix} \text{By condition (ii)of remark 2.3, } (x*(x*y))*y=0 \Rightarrow x*(x*y)\le y, \text{by remark 2.4} \\ Here, \left(u_a*((u_a*u_m)*u_n)\right)*u_m = (u_a*u_m)*((u_a*u_m)*u_n) = ((u_a*u_m)*(u_a*u_m))*u_n = 0*u_n = 0. \text{ So remark 2.4 is applicable in this case} \\ \text{Since } (u_a*u_m)*((u_a*u_m)*u_n)=0 \text{ we have } (u_a*u_m)*((u_a*u_m)*u_n)\le u_n \end{bmatrix}$$

Above can be written as, $\left(u_a*((u_a*u_m)*u_n)\right)*u_m \le u_n$

$\Rightarrow NVB_{P_{NVB}}\left(\left(u_a*((u_a*u_m)*u_n)\right)*u_m\right) \succcurlyeq NVB_{P_{NVB}}(u_n)$ [By proposition 7.7]

P_{NVB} is a NVB BCK/BCI -ideal of $\mathfrak{B}_{M_{NVB}} \Rightarrow NVB_{P_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{P_{NVB}}(u_a*u_b), NVB_{P_{NVB}}(u_b)\}$

Put $u_a = \left(u_a*((u_a*u_m)*u_n)\right)$ & $u_b = u_m$ in above,

$NVB_{P_{NVB}}\left(u_a*((u_a*u_m)*u_n)\right) \succcurlyeq r\min\left\{NVB_{P_{NVB}}\left(\left(u_a*((u_a*u_m)*u_n)\right)*u_m\right), NVB_{P_{NVB}}(u_m)\right\}$

$= r\min\{NVB_{P_{NVB}}(u_n), NVB_{P_{NVB}}(u_m)\}$ [proved above]

$\succcurlyeq r\min\{NVB_{P_{NVB}}(u_m), NVB_{P_{NVB}}(u_n)\}$

$\Rightarrow NVB_{M_{NVB}}\left(u_a*((u_a*u_m)*u_n)\right) \succcurlyeq r\min\{NVB_{P_{NVB}}(u_m), NVB_{P_{NVB}}(u_n)\}$; $[\forall u_a, u_m, u_n \in U]$

Conversely,

Let P_{NVB} be a NVBS of a NVB BCK/BCI – algebra $\mathfrak{B}_{M_{NVB}}$ satisfying, the given conditions,

(i) $NVB_{P_{NVB}}(u_a*u_b) \succcurlyeq NVB_{P_{NVB}}(u_a)$; $[\forall u_a, u_b \in U]$

(ii) $NVB_{P_{NVB}}\left(u_a*((u_a*u_m)*u_n)\right) \succcurlyeq r\min\{NVB_{P_{NVB}}(u_m), NVB_{P_{NVB}}(u_n)\}$; $[\forall u_a, u_m, u_n \in U]$

To prove condition (1) of a NVB BCK/BCI - ideal, take $u_b = u_a$ in (i) and (ii) respectively,

(i) $\Rightarrow NVB_{P_{NVB}}(u_a*u_a) \succcurlyeq NVB_{P_{NVB}}(u_a) \Rightarrow NVB_{P_{NVB}}(0) \succcurlyeq NVB_{P_{NVB}}(u_a)$; [By property (iii)of definition 2.3]

To prove condition (2) of a NVB BCK/BCI - ideal,

take, $NVB_{P_{NVB}}(u_a) = NVB_{P_{NVB}}(u_a*0)$ [By property(i)of remark 2.6]

$= NVB_{P_{NVB}}\left(u_a*((u_a*u_b)*(u_a*u_b))\right)$; [By property (iii)of definition 2.3]

$= NVB_{P_{NVB}}\left(u_a*((u_a*(u_a*u_b))*u_b)\right)$; [By property (ii) of remark 2.6]

$= NVB_{P_{NVB}}\left(u_a*((u_a*u_m)*u_n)\right)$; [By putting $(u_a*u_b) = u_m$ and $u_b = u_n$]

$\succcurlyeq r\min\{NVB_{P_{NVB}}(u_m), NVB_{P_{NVB}}(u_n)\}$; [By condition (ii) in the assumption]

$= r\min\{NVB_{P_{NVB}}(u_a*u_b), NVB_{P_{NVB}}(u_b)\}$; [By putting $(u_a*u_b) = u_m$ and $u_b = u_n$]

$\Rightarrow NVB_{P_{NVB}}(u_a) \succcurlyeq r\min\{NVB_{P_{NVB}}(u_a*u_b), NVB_{P_{NVB}}(u_b)\}$

$\therefore P_{NVB}$ is a NVB BCK/BCI - ideal of $\mathfrak{B}_{M_{NVB}}$

Theorem 7.17

Let M_{NVB} be a NVB BCK/BCI - algebra $\mathfrak{B}_{M_{NVB}}$. Then any NVB BCK/BCI - cut of M_{NVB} is a crisp NVB BCK/BCI - subalgebra of $\mathfrak{B}_{M_{NVB}}$

Proof

Let for any $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \rho_1, \rho_2, \vartheta_1, \vartheta_2 \in [0, 1]$,

$M_{NVB([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]),([\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2])}$ be a NVB BCK/BCI -cut of M_{NVB}.

Assume $u_x, u_y \in M_{NVB([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]),([\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2])}$

$\Rightarrow NVB_{M_{NVB}}(u_x) \ge [\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]$ & $NVB_{M_{NVB}}(u_x) \ge ([\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2])$

$NVB_{M_{NVB}}(u_y) \ge [\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]$ & $NVB_{M_{NVB}}(u_y) \ge ([\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2])$

$\Rightarrow \hat{T}_{M_{NVB}}(u_x) \ge [\alpha_1,\alpha_2]$; $\hat{I}_{M_{NVB}}(u_x) \le [\beta_1,\beta_2]$; $\hat{F}_{M_{NVB}}(u_x) \le [\gamma_1,\gamma_2]$ &

$\hat{T}_{M_{NVB}}(u_x) \ge [\delta_1,\delta_2]$; $\hat{I}_{M_{NVB}}(u_x) \le [\rho_1,\rho_2]$; $\hat{F}_{M_{NVB}}(u_x) \le [\vartheta_1,\vartheta_2]$

$\hat{T}_{M_{NVB}}(u_y) \ge [\alpha_1,\alpha_2]$; $\hat{I}_{M_{NVB}}(u_y) \le [\beta_1,\beta_2]$; $\hat{F}_{M_{NVB}}(u_y) \le [\gamma_1,\gamma_2]$

$\hat{T}_{M_{NVB}}(u_y) \ge [\delta_1,\delta_2]$; $\hat{I}_{M_{NVB}}(u_y) \le [\rho_1,\rho_2]$; $\hat{F}_{M_{NVB}}(u_y) \le [\vartheta_1,\vartheta_2]$

M_{NVB} is a NVB BCK/BCI -algebra $\mathfrak{B}_{M_{NVB}} \Rightarrow NVB_{M_{NVB}}(u_x*u_y) \succcurlyeq r\min\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\}$

$\Rightarrow \hat{T}_{M_{NVB}}(u_x*u_y) \ge \min\{\hat{T}_{M_{NVB}}(u_x), \hat{T}_{M_{NVB}}(u_y)\}$; $\hat{I}_{M_{NVB}}(u_x*u_y) \le \max\{\hat{I}_{M_{NVB}}(u_x), \hat{I}_{M_{NVB}}(u_y)\}$; $\hat{F}_{M_{NVB}}(u_x*u_y) \le \max\{\hat{F}_{M_{NVB}}(u_x), \hat{F}_{M_{NVB}}(u_y)\}$

$\Rightarrow (u_x*u_y) \in M_{NVB([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]),([\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2])}$

$\Rightarrow$ NVB BCK/BCI - cut $M_{NVB([\alpha_1,\alpha_2],[\beta_1,\beta_2],[\gamma_1,\gamma_2]),([\delta_1,\delta_2],[\rho_1,\rho_2],[\vartheta_1,\vartheta_2])}$ of M_{NVB} is a crisp NVB BCK/BCI - subalgebra of $\mathfrak{B}_{M_{NVB}}$

8. Conclusions

In this paper, two logical algebras viz., BCK and BCI are developed for neutrosophic vague binary sets. It's subalgebra, ideal and cuts are also got discussed. Different kinds of ideals like p ideal, q ideal, a ideal, H ideal for neutrosophic vague binary BCK/BCI -algebra have been investigated. Theorems and propositions related to this concept are verified. In this paper BCK/BCI-algebra for neutrosophic sets are firstly developed. Then it is extended to neutrosophic vague and to neutrosophic vague binary. Work can be further extended to higher concepts like its group, rings, filter, near-rings etc. Behavior differences of these two algebras in different algebraic notions have to be addressed more deeply and properly to get a correct vision. This area demands some more notice and filtering to find out its correct drawbacks. Further investigations will make it, to balance its moves to the correct direction. Medial BCI -algebra, commutative BCK-algebra, Associative BCI – algebra, BCK/BCI- homomorphisms, bounded commutative BCI-algebra are a few points have to be addressed and have to be analyzed more.

Funding: "This research received no external funding"

Acknowledgments: Authors are very grateful to the anonymous reviewers and would like to thank for their valuable and critical suggestions to structure the paper systematically and to remove the errors.

Conflicts of Interest: "The authors declare that they have no conflict of interest."

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Received: Apr 13, 2020. Accepted: July 3, 2020

Neutrosophic Sets and Systems, Vol. 35, 2020
University of New Mexico

Introduction to Topological Indices in Neutrosophic Graphs

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Abstract: Neutrosophic Graphs are graphs that follow three-valued logic. They may be considered a fuzzy graph, although in some cases, it is difficult to optimize and model them using fuzzy graphs. In this paper, the first and second Zagreb indices, the Harmonic index, the Randic′ index and the Connectivity index for these graphs are investigated and some of the theorems related to these indices are discussed and proven. These indices are also calculated for some specific types of Neutrosophic Graphs, such as regular Neutrosophic Graphs and regular complete Neutrosophic Graphs.

Keywords: Neutrosophic Graphs; Zagreb indices; Harmonic index; Randic′ index; Connectivity index

1. Introduction

Graph theory has many applications for modeling problems in various fields of computer science such as systems analysis, computer networks, transportation, operations research and economics. The vertices and edges of the graphs are used to represent objects and the relationships between them, respectively. Many of the optimization issues are caused by inaccurate information due to factors like lack of evidence, incomplete statistical data, and lack of sufficient information; this creates uncertainty in various issues. Classical Graphic Theory uses the basic concept of classical set theory, as proposed by Contour. In a classic graph, for each vertex or edge, there are two possibilities: either in the graph or not in the graph. Therefore, classical graphs cannot model uncertain optimization problems. Real-life issues are often unclear, making modeling by classical graphs difficult. Zadeh introduced the degree of membership/truth (T) in 1965 and defined the fuzzy set. Atanassov [14] introduced the degree of nonmembership/falsehood (F) in 1983 and defined the intuitionistic fuzzy set. Smarandache [15] introduced the degree of indeterminacy/neutrality (I) as an independent component in 1995 and defined the neutrosophic set on three components (T, I, F) [4].

Fuzzy set [1] is a generalized version of the classical set in which objects have different membership degrees. A fuzzy set gives the degree of different members between zero and one. Much work has already been done on fuzzy graphs, including the calculation of various topology indices, indicators such as Zagreb index, Randic′, harmonic, and so on. However, there is another class of graphs that is a broad case of fuzzy graphs. In this type of graph, known as neutrosophic graphs, in addition to the degree of accuracy of each membership function, the degree of its membership is uncertain, as well as its inaccuracy. So in many cases, it may be more logical to use this model than graphs in real-world problems.

Since that neutrosophic graphs are more efficient than fuzzy graphs for modeling real problems. Therefore, in this paper, we try for the first time to calculate some topological indices for this type of graph.

2. Preliminaries

This section, provides some definitions and theorems needed.

Definition 1. [13] Let $G=(N,M)$ be a single-valued Neutrosophic graph, where N is a Neutrosophic set on V and, M is a Neutrosophic set on E, which satisfy the following

$$T_M(u,v) \le min\big(T_N(u),T_N(v)\big),$$
$$I_M(u,v) \ge max\big(I_N(u),I_N(v)\big),$$
$$F_M(u,v) \ge max\big(F_N(u),F_N(v)\big),$$

Where u and v are two vertices of G, and $(u,v)\in E$ is an edge of G.

Definition 2. [2] Let $G=(N,M)$ be a Single-Valued Neutrosophic Graph and P is a path in G. P is a collection of different vertices, $v_0,v_1,v_2,\dots,v_n$ such that $\big(T_M(v_{i-1},v_i),I_M(v_{i-1},v_i),F_M(v_{i-1},v_i)\big)>0$ for $0\le i\le n$. P is a Neutrosophic cycle if $v_0=v_n$ and $n\ge 3$.

Definition 3. [2] Suppose $G=(N,M)$ a single-valued Neutrosophic graph. G is a connected Single-Valued Neutrosophic Graph if there exists no isolated vertex in G. ($v\in V_G$ is isolated vertex, if there exists no incident edge to the vertex v.)

Definition 4. [2] Let $G=(N,M)$ be a Single-Valued Neutrosophic Graph, and $v\in V$ is vertex of G. The degree of vertex v is the sum of the truth membership values, the sum of the indeterminacy membership values, and the sum of the falsity membership values of all the edges that are adjacent to vertex v. And is denoted by $d(v)$, that

$$d(v)=\big(d_T(v),d_I(v),d_F(v)\big)=\left(\sum_{\substack{v\in V\\ v\neq u}} T_M(v,u),\sum_{\substack{v\in V\\ v\neq u}} I_M(v,u),\sum_{\substack{v\in V\\ v\neq u}} F_M(v,u)\right).$$

Definition 5. [2] Let $G=(N,M)$ be a Single-Valued Neutrosophic Graph, and the d_m–degree of any vertex v in G is denoted as $d_m(v)$ where

$$d_m(v)=\left(\sum_{u\neq v\in V} T_M^m(u,v),\sum_{u\neq v\in V} I_M^m(u,v),\sum_{u\neq v\in V} F_M^m(u,v)\right)$$

Here, the path $v=v_0v_1\,v_2\,\dots\,v_n=u$ is the shortest path between the vertices v and u, when the length of this path is m.

Definition 6. [2] Let $G=(N,M)$ be a Single-Valued Neutrosophic Graph, G is a regular neutrosophic graph if it satisfies the following,

$$\sum_{v\neq u} T_M(v,u)=c,\qquad \sum_{v\neq u} I_M(v,u)=c,\qquad \sum_{v\neq u} F_M(v,u)=c,$$

Where c is a constant value.

3. Topological Indices in Neutrosophic Graphs

In the section, we introduce Topological Indices in Neutrosophic Graphs and provide a number of examples. We define Zagreb indices, Harmonic index, and Randic′ index, and in finally Connectivity index on the neutrosophic graphs.

3.1. Zagreb index of First and Second Kind in Neutrosophic Graphs

Definition 8. Let $G = (N, M)$ be the Neutrosophic Graph whit non-empty vertex set. The first Zagreb index is denoted by $M(G)$ and defined as

$$M(G) = \sum_{i=1}^{n} (T_N(u_i), I_N(u_i), F_N(u_i)) d_2(u_i), \qquad \forall\ u_i \in V.$$

Example 1. Consider the Neutrosophic Graph $G = (N, M)$ as shown in figure 1, with the vertex set $V = \{a, b, c\}$ such that $(T_N, I_N, F_N)(a) = (0.3, 0.6, 0.7)$, $(T_N, I_N, F_N)(b) = (0.3, 0.5, 0.6)$, and $(T_N, I_N, F_N)(c) = (0.4, 0.5, 0.6)$, The edge set contains $(T_M, I_M, F_M)(a, b) = (0.2, 0.6, 0.8)$, $(T_M, I_M, F_M)(b, c) = (0.2, 0.6, 0.7)$, and $(T_M, I_M, F_M)(a, c) = (02, 0.8, 0.9)$. We have,

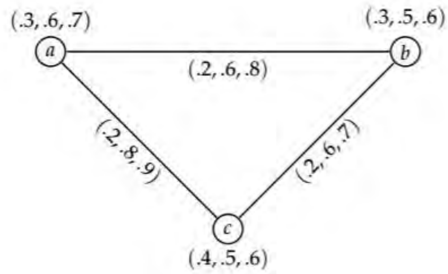

Figure 1. A neutrosophic graph with $V = \{a, b, c\}$

The first Zagreb index is

$$d(a) = (0.2 + 0.2, 0.6 + 0.8, 0.8 + 0.9) = (0.4, 1.4, 1.7),$$
$$d(b) = (0.2 + 0.2, 0.6 + 0.6, 0.8 + 0.7) = (0.4, 1.2, 1.5),$$
$$d(c) = (0.2 + 0.2, 0.8 + 0.6, 0.9 + 0.7) = (0.4, 1.4, 1.6).$$

Now, we have

$$d_2(a) = (0.04 + 0.04, 0.36 + 0.64, 0.64 + 0.81) = (0.08, 1, 1.45),$$
$$d_2(b) = (0.04 + 0.04, 0.36 + 0.36, 0.64 + 0.49) = (0.08, 0.72, 1.13),$$
$$d_2(c) = (0.04 + 0.04, 0.64 + 0.36, 0.81 + 0.49) = (0.08, 1, 1.3).$$

$$M(G) = \sum_{i=1}^{4} (T_N(u_i), I_N(u_i), F_N(u_i)) d_2(u_i)$$

$$= (0.3, 0.6, 0.7)(0.08, 1, 1.45) + (0.3, 0.5, 0.6)(0.08, 0.72, 1.13)$$
$$+ (0.4, 0.5, 0.6)(0.08, 1. 1.3)$$
$$= (0.024 + 0.6 + 1.015) + (0.024 + 0.36 + 0.678) + (0.032 + 0.5 + 0.78) = 4.013.$$

Definition 9. The second Zagreb index is denoted by $M^*(G)$ and defined as

$$M^*(G) = \frac{1}{2} \sum [(T_N(u_i), I_N(u_i), F_N(u_i)) d(u_i)][(T_N(v_j), I_N(v_j), F_N(v_j)) d(v_j)], \quad \forall i \neq j \ and\ (u_i, v_j) \in E.$$

Example 2. If G is the same Neutrosophic Graph as example 1, we have

$$M^*(G) = \frac{1}{2}[(0.3, 0.6, 0.7).(0.4, 1.4, 1.7) \times (0.3, 0.5, 0.6).(0.4, 1.2, 1.5) + (0.3, 0.6, 0.7).(0.4, 1.4, 1.7)$$
$$\times (0.4, 0.5, 0.6).(0.4, 1.4, 1.6) + (0.3, 0.5, 0.6).(0.4, 1.2, 1.5)$$
$$\times (0.4, 0.5, 0.6).(0.4, 1.4, 1.6)]$$
$$= \frac{1}{2}[(0.12 + 0.84 + 1.19) \times (0.12 + 0.6 + 0.9) + (0.12 + 0.84 + 1.19)$$
$$\times (0.16 + 0.7 + 0.96) + (0.12 + 0.6 + 0.9) \times (0.16 + 0.7 + 0.96)$$
$$= \frac{1}{2}[(2.15)(1.62) + (2.15)(1.82) + (1.62)(1.82)] = \frac{1}{2}(10.3444) = 5.1722.$$

Note 1. As we have seen, the value of $M^*(G)$ is less than the value of $M(G)$, and this is always the case.

Theorem 1. Let G is the Neutrosophic Graph and H is the Neutrosophic sub graph of G such that $H = G - u$ then $M(H) < M(G)$ and $M^*(H) < M^*(G)$.

Proof. Given that by omitting a vertex of G, a positive value, the sum is lost, so the proof is obvious.

□

Theorem 2. Let G be the regular neutrosophic graph. Then, we have

$$M(G) = c^2 \times \sum_{i=1}^{n} (T_N(u_i) + I_N(u_i) + F_N(u_i)), \qquad \forall\ u_i \in V.$$

Where $\sum_{v\neq u} T_M(v,u) = c$, $\sum_{v\neq u} I_M(v,u) = c$, $\sum_{v\neq u} F_M(v,u) = c$.

Proof. Given the degree of definition of each vertex,

$$d(v) = (d_T(v), d_I(v), d_F(v)) = \left(\sum_{\substack{v\in V\\ v\neq u}} T_M(v,u), \sum_{\substack{v\in V\\ v\neq u}} I_M(v,u), \sum_{\substack{v\in V\\ v\neq u}} F_M(v,u)\right).$$

On the other hand, for regular neutrosophic graphs, we know that

$$\sum_{v\neq u} T_M(v,u) = c, \qquad \sum_{v\neq u} I_M(v,u) = c, \qquad \sum_{v\neq u} F_M(v,u) = c,$$

Therefore

$$d(v) = (d_T(v), d_I(v), d_F(v)) = (c, c, c).$$

Now, by embedding the formula in the first Zagreb index, we will get the desired result. The proof is complete.

□

Theorem 3. Let G be the regular neutrosophic graph. Then, we have

$$M^*(G) = \frac{1}{2}(c^2) \sum [T_N(u_i) + I_N(u_i) + F_N(u_i)][T_N(v_j) + I_N(v_j) + F_N(v_j)],$$
$$\forall i \neq j \text{ and } (u_i, v_j) \in E,$$

Where $\sum_{v\neq u} T_M(v,u) = c$, $\sum_{v\neq u} I_M(v,u) = c$, $\sum_{v\neq u} F_M(v,u) = c$.

Proof. Assume G is a regular neutrosophic graph, using the second Zagreb index formula for G, we have $\forall\ i \neq j \text{ and } (u_i, v_j) \in E$,

$$M^*(G) = \frac{1}{2}\sum[(T_N(u_i), I_N(u_i), F_N(u_i))d(u_i)][(T_N(v_j), I_N(v_j), F_N(v_j))d(v_j)]$$
$$= \frac{1}{2}\sum[(T_N(u_i), I_N(u_i), F_N(u_i))d\big(d_T(u_i), d_I(u_i), d_F(u_i)\big)]$$
$$\times [(T_N(v_j), I_N(v_j), F_N(v_j))d\left(d_T(v_j), d_I(v_j), d_F(v_j)\right)]$$
$$= \frac{1}{2}\sum[(T_N(u_i), I_N(u_i), F_N(u_i)).(c,c,c)][(T_N(v_j), I_N(v_j), F_N(v_j)).(c,c,c)]$$
$$= \frac{1}{2}\sum[c.T_N(u_i) + c.I_N(u_i) + c.F_N(u_i)][c.T_N(v_j) + c.I_N(v_j) + c.F_N(v_j)]$$
$$= \frac{1}{2}\sum c[T_N(u_i) + I_N(u_i) + F_N(u_i)].c[T_N(v_j) + I_N(v_j) + F_N(v_j)]$$
$$= \frac{1}{2}c^2\sum[T_N(u_i) + I_N(u_i) + F_N(u_i)][T_N(v_j) + I_N(v_j) + F_N(v_j)].$$

The desired result was obtained.

□

3.2. Harmonic index in Neutrosophic Graphs

Definition 10. The Harmonic index of Neutrosophic Graph G is defined as

$$H(G) = \sum \frac{1}{(T_N(u_i), I_N(u_i), F_N(u_i))d(u_i) + (T_N(v_j), I_N(v_j), F_N(v_j))d(v_j)}, \quad \forall i \neq j \; and \; (u_i, v_j) \in E.$$

Example 3. We have the previous example,

$$H(G) = \frac{1}{(0.3, 0.6, 0.7)(0.4, 1.4, 1.7) + (0.3, 0.5, 0.6)(0.4, 1.2, 1.5)}$$
$$+ \frac{1}{(0.3, 0.6, 0.7)(0.4, 1.4, 1.7) + (0.4, 0.5, 0.6)(0.4, 1.4, 1.6)}$$
$$+ \frac{1}{(0.3, 0.5, 0.6)(0.4, 1.2, 1.5) + (0.4, 0.5, 0.6)(0.4, 1.4, 1.6)}$$
$$= \frac{1}{2.15 + 1.62} + \frac{1}{2.15 + 1.82} + \frac{1}{1.62 + 1.82} = \frac{1}{3.77} + \frac{1}{3.97} + \frac{1}{3.44} = 0.8078.$$

3.3. Randic′ index in Neutrosophic Graphs

Definition 11. The Randic′ index of Neutrosophic Graph G is defined as

$$R(G) = \sum((T_N(u_i), I_N(u_i), F_N(u_i))d(u_i)(T_N(v_j), I_N(v_j), F_N(v_j))d(v_j))^{\frac{-1}{2}}, \quad \forall i \neq j \; and \; (u_i, v_j) \in E.$$

Example 3. For above example, by simple calculations, it is easy to see that

$$R(G) = \frac{1}{\sqrt{(0.3, 0.6, 0.7).(0.4, 1.4, 1.7) \times (0.3, 0.5, 0.6).(0.4, 1.2, 1.5)}}$$
$$+ \frac{1}{\sqrt{(0.3, 0.6, 0.7).(0.4, 1.4, 1.7) \times (0.4, 0.5, 0.6).(0.4, 1.4, 1.6)}}$$
$$+ \frac{1}{\sqrt{(0.3, 0.5, 0.6).(0.4, 1.2, 1.5) \times (0.4, 0.5, 0.6).(0.4, 1.4, 1.6)}}$$
$$= \frac{1}{\sqrt{2.15 \times 1.62}} + \frac{1}{\sqrt{2.15 \times 1.82}} + \frac{1}{\sqrt{1.62 \times 1.82}} = 1.6237.$$

3.4. Connectivity index in Neutrosophic Graphs

Connectivity index is an important parameter in the graph. Using it, we can study and study some of the features of graph models.

Definition 12. Let $G = (N, M)$ be the Neutrosophic Graph. The connectivity index of G is defined by

$$CI(G) = \sum_{u_i,v_j\in V} (T_N(u_i), I_N(u_i), F_N(u_i))(T_N(v_j), I_N(v_j), F_N(v_j)) \times CONN_G(u_i, v_j).$$

Where $CONN_G(u_i, v_j)$ is the strength of connectedness between u_i and v_j.

Definition 13. The strength of connectedness between u_i and v_j is defined as

$$CONN_P(u_i, v_j) = \left(\min_{e\in P_{u_iv_j}} T_M(e), \max_{e\in P_{u_iv_j}} I_M(e), \max_{e\in P_{u_iv_j}} F_M(e)\right),$$

Where $P_{u_iv_j}$ is the path between u_i and v_j.

$$|CONN_P(u_i, v_j)| = 2\left(\min_{e\in P_{u_iv_j}} T_M(e)\right) - \left(\max_{e\in P_{u_iv_j}} I_M(e)\right) - \left(\max_{e\in P_{u_iv_j}} F_M(e)\right),$$

Then

$$CONN_G(u_i, v_j) = \max_P\{|CONN_P(u_i, v_j)|\}.$$

Example 4. For example, in the above figure, the strength of connectedness between:
a and b from the direct path $P_1 = ab$ is

$$CONN_{P_1}(a, b) = M_{ab} = (0.2, 0.6, 0.8),$$

From path $P_2 = acb$ is

$$CONN_{P_2}(a, b) = (min\{0.2, 0.2\}, \quad max\{0.8, 0.6\}, \quad max\{0.9, 0.7\}) = (0.2, 0.8, 0.9);$$

a and c from the direct path $P_1 = ac$ is

$$CONN_{P_1}(a, c) = M_{ac} = (0.2, 0.8, 0.9),$$

From path $P_2 = abc$ is

$$CONN_{P_2}(a, c) = (min\{0.2, 0.2\}, \quad max\{0.6, 0.6\}, \quad max\{0.8, 0.7\}) = (0.2, 0.6, 0.8);$$

b and c from the direct path $P_1 = bc$ is

$$CONN_{P_1}(b, c) = M_{bc} = (0.2, 0.6, 0.7),$$

From path $P_2 = bac$ is

$$CONN_{P_2}(b, c) = (min\{0.2, 0.2\}, \quad max\{0.6, 0.8\}, \quad max\{0.8, 0.9\}) = (0.2, 0.8, 0.9).$$

Then, we have for a and b

$$|CONN_{P_1}(a, b)| = 2 \times (0.2) - 0.6 - 0.8 = -1,$$
$$|CONN_{P_2}(a, b)| = 2 \times (0.2) - 0.8 - 0.9 = -1.3.$$

For a and c,

$$|CONN_{P_1}(a, c)| = 2 \times (0.2) - 0.8 - 0.9 = -1.3,$$
$$|CONN_{P_2}(a, c)| = 2 \times (0.2) - 0.6 - 0.8 = -1.$$

For b and c,

$$|CONN_{P_1}(b, c)| = 2 \times (0.2) - 0.6 - 0.7 = -0.9,$$
$$|CONN_{P_2}(b, c)| = 2 \times (0.2) - 0.8 - 0.9 = -1.3.$$

Since we have

$$CONN_G(a,b) = -1; \quad CONN_G(a,c) = -1; \quad CONN_G(b,c) = -0.9.$$

Then $CI(G)$ is calculated as follows

$$\begin{aligned}CI(G) = \sum_{u_i,v_j \in V} &(T_N(u_i), I_N(u_i), F_N(u_i))(T_N(v_j), I_N(v_j), F_N(v_j)) \times CONN_G(u_i,v_j)\\
&= (0.3, 0.6, 0.7).(0.3, 0.5, 0.6) \times (-1) + (0.3, 0.6, 0.7).(0.4, 0.5, 0.6) \times (-1)\\
&+ (0.3, 0.5, 0.6).(0.4, 0.5, 0.6) \times (-0.9)\\
&= (0.09 + 0.3 + 0.42)(-1) + (0.12 + 0.3 + 0.42)(-1) + (0.12 + 0.25 + 0.36)(-0.9)\\
&= (0.81)(-1) + (0.84)(-1) + (0.73)(-0.9) = -2.307.\end{aligned}$$

The connectivity index of G is equal -2.307, which the negative sing indicates the high level of false and indeterminacy information in the problem.

Theorem 4. Let G and H be the two Neutrosophic Graphs are isomorphic, then the topological indices values of two Neutrosophic Graphs are equal.

Proof. To prove, let $G = (V_G, N_G, M_G)$ and $H = (V_H, N_H, M_H)$ be isomorphic Neutrosophic Graphs. Hence there is an identity function $\mu_N: N_G(u) \to N_H(u^*)$, for all $u \in V_G$ there exist $u^* \in V_H$ as well as $\mu_M: M_G(u,v) \to M_H(u^*, v^*)$, then each vertex of G corresponds to an vertex in H, with the same membership value and the same edges. Hence, the Neutrosophic graph structure may differ but collection of vertices and edges are same gives the equal topological indices value.
□

Theorem 5. Let $G = (V_G, N_G, M_G)$, is a neutrosophic Graph and H is the neutrosophic sub graph of G, Such that H is made by removing edge $uv \in M_G$ from G. Then, we have, $CI(H) < CI(G)$ iff uv is a bridge.

Proof. To prove the first side of the theorem we consider two cases:

Case 1. Let uv be an edge with all three components having the least value, Therefore the edge uv will have no effect on the result. Then we have $CI(H) = CI(G)$.

Case 2. Now suppose that uv is an edge that has maximum components, so they will have an effect on $CONN_G(u,v)$. Therefore, by removing edge uv, the value of $CONN_G(u,v)$ will decrease, then we have $CI(H) < CI(G)$. Since the bridge is called the edge that has its deletion reducing the $CONN_G(u,v)$, However, uv is a bridge.

Conversely, given that uv is a bridge. According to the definition of bridge we have, for the edge uv, $CONN_G(u,v) > CONN_{G-uv}(u,v)$, So we conclude that, $CI(H) < CI(G)$.
□

4. Applications

Fuzzy set theory and intuitionistic fuzzy set theory are useful models for modelling problems in real life. But they may not be sufficient in modelling of indeterminate and inconsistent information encountered in real word. In cases where our information is incomplete or part of our information is incompatible with each other, depending on the features of the neutrosophic graphs, we can use them for modeling. However, neutrosophic graphs have many application in real life. For example, social network model, detection of a safe root for an Airline journey and military problems are application neutrosophic graph theory [4]. Note that to many applications that neutrosophic graphs have, obtaining topological indices can be a way to compare the different problems that are modeled by neutrosophic graphs. For example, by obtaining different indicators for the two social networks Telegram and Whatsapp, we can analyze some of the features of the network and their impact.

To see more applications of the neutrosophic graphs, you can refer to [5-12].

Here we refer to one of the applications of the connectivity index for an example of [4].

4.1. Optimal flight path for weather emergency landing

In this application, we use the concept of rough neutrosophic digraph for decision-making in real-life problems [4]. There, provided a formula for obtaining the desired result, and after performing the calculations, reached the desired result.

Now, using the connectivity index for different paths, it is possible to predict the optimal path for flying in weather emergency landing.

Suppose $V = \{Chicago(CH), Beijing(BJ), Lahore(LH), Paris(PA), Istanbul(IS)\}$, be the set of cities under consideration and R an equivalence relation on V, where equivalence classes represent cities having same characteristics.

Assume that a flight Boeing 747 of Pakistan International Airways (PIA) travels to these cities. In case of bad weather, the flight will be directed to the city with good weather condition among the cities under consideration.

Let

$$N = \{CH, 0.1, 0.2, 0.8), (BJ, 0.9, 0.7, 0.5), (LH, 0.8, 0.4, 0.3), (PA, 0.6, 0.5, 0.4), (IS, 0.2, 0.4, 0.6)\},$$

And

$$M = \{\big((BJ, CH), 0.1, 0.1, 0.3\big), \big((LH, CH), 0.1, 0.2, 0.3\big), \big((BJ, LH), 0.1, 0.3, 0.2\big),$$
$$\big((IS, BJ), 0.2, 0.1, 0.1\big), \big((PA, BJ), 0.1, 0.1, 0.4\big), \big((PA, LH), 0.2, 0.2, 0.3\big)\}.$$

Now, we obtain the connectivity index for all paths.

The direct path $BJ\ \_\ CH$

$$CONN_P(BJ, CH) = 2(0.1) - 0.1 - 0.3 = -0.2\,,$$

The direct path $BJ\ \_\ LH$

$$CONN_P(BJ, LH) = 2(0.1) - 0.3 - 0.2 = -0.3,$$

The direct path $LH\ \_\ CH$

$$CONN_P(LH, CH) = 2(0.1) - 0.2 - 0.3 = -0.3,$$

The direct path $IS\ \_\ BJ$

$$CONN_P(IS, BJ) = 2(0.2) - 0.1 - 0.1 = 0.2,$$

The direct path $PA\ \_\ BJ$

$$CONN_P(BJ, CH) = 2(0.1) - 0.1 - 0.4 = -0.3,$$

The direct path $PA\ \_\ LH$

$$CONN_P(PA, LH) = 2(0.2) - 0.2 - 0.3 = -0.1.$$

Hence, as expected from [4], the weather condition between Beijing and Istanbul is good, and Boeing 747 can use this path in case of weather emergency. We were able to achieve the desired result with much shorter calculations. Also, if needed, we can calculate the connectivity index for indirect paths and finally for neutrosophic graph.

For connectivity index of G we have,

$$CI(G) = \sum_{u_i, v_j \in V} (T_N(u_i), I_N(u_i), F_N(u_i))(T_N(v_j), I_N(v_j), F_N(v_j)) \times CONN_G(u_i, v_j)$$
$$= (0.63)(-0.2) + (0.76)(-0.3) + (0.4)(-0.3) + (1.09)(-0.3) + (0.8)(-0.1) + (0.76)(0.2) + (0.5)(-0.3) + (0.58)(-0.2) + (0.8)(-0.5) + (0.48)(-0.3) + (0.58)(-0.4) + (0.48)(-0.5)$$
$$= -0.126 - 0.228 - 0.12 - 0.327 - 0.08 + 0.152 - 0.15 - 0.116 - 0.4 - 0.144 - 0.232 - 0.24 = -1.783.$$

As you can see, the negative numerical connectivity index was obtained, which means that our incorrect information was less than our correct information.

Conclusion

In this paper, for the first time, some topological indices for neutrosophic graphs are defined. This topic has a lot of work to do, and it can also be used for its results on various issues related to this category of graphs. In the rest of our research and in future articles, we will address more of these theorems and their applications.

Funding: "This research received no external funding"

Acknowledgments:

The authors thank the reviewers for their constructive feedback.

Conflicts of Interest: "The authors declare no conflict of interest."

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Received: Apr 14, 2020. Accepted: July 4, 2020

Neutrosophic Sets and Systems, Vol. 35, 2020
University of New Mexico

Assessment of MCDM problems by TODIM using aggregated weights

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Abstract: Sustainability of sheep and goat production systems is a significant task for any organization that aims for long term goals. Housing and feeding selection for goat farming is the most important factor that should be considered before setting out the goat farm. The decision framework of housing selection should include environmental, social and human impact for the long term, rather than on short-term gains. In the selection process, various parameters are involved such as housing materials, area to prevent water stagnation, ventilation, enough space for the pen and run system, space for feeders and water troughs. Those parameters highlight the quality of housing in relation to aspects of traditional breeding provided by the organizations. However, the process of housing selection is often led by hands on experience which contains vague, ambiguous and uncertain decisions. To overcome this issue it is necessary to frame an efficient algorithm which could remove the entire barrier in the decision making process. In this paper we propose a neutrosophic multi-criteria decision making framework that combines the TODIM method with the SD- HNWA operator. The resulting multi-criteria decision analytical MCDM framework is then applied in selecting the best system in housing and feeding of goats at a mixed farming agrofarm in India. The proposed approach allows us to establish the neutrosophic based value function that measures the degree to which one alternative is superior to others by calculating accurate number of information in pair wise comparison in terms of gain and loss. The outcomes of the proposed method are compared with the use of the TOPSIS method to prove its efficiency and validate the results.

Keywords: MCDM, Hexagonal Neutrosophic numbers, Similarity Degree, Aggregated Weights, TODIM, TOPSIS.

1. Introduction

Live stock management is considered as one of the most important study topic as it plays a vital role in self employment for the younger generation with higher level of educational qualifications in a country like India, with a traditionally high rate of population growth. It is also considered as an employment intervention strategy for the younger generation for the self employment of the youth. Goats are among the main meat producing animal in India where it has huge domestic demand. As a result, goat production system in India is shifting to intensive system of management.The goat rearing using improved management practices concentrates on maximization of the returns from the view of the entrepreneur.

However, without any systematic study it is difficult to assess the economic viability of the goat farming, as the whole system is built upon nature. The good management practice in livestock management is the key for the resilience, social, economical and ecological sustainability and preservation of bio-diversity in pastoral eco-systems, especially in the rural areas where goat production plays a relevant role in the livehood for farmers. For example, Shalander [25] has proposed a multi-disciplinary project on transfer of technology for sustainable goat production in which he indicates that lack of technical knowledge in housing and feeding management system per capita income in goat rearing is not being up to the expected margin of the goat farmers. Biswas et al. [9] shows that the growth rate of goat feeder with supplements by additional concentrate with grazing was more when compared with the normal grazing goats.

In the real world, just like other decision making problem such as supplier selection or candidate selection, the challenge of uncertainty in the process of housing and feeding selection in live-stock management is inevitable owing to the fact that the consequences of events are not precisely known. In addition human judgmental analysis also contributes to its intricacy in the decision making analysis. To overcome this vagueness and intricacy in decision making this study aims to propose an integrated framework under neutrosophic environment to evaluate alternative choices in terms of management system of housing and feeding.

In this research the TODIM and TOPSIS methods will be applied in the processing of selecting such alternatives. The TODIM method (an acronym for Interactive Multi-Criteria Decision Making in Portuguese) is a discrete multi-criteria method founded on prospect theory which underlies a psychological theory in it, while in practice all other discrete multi-criteria methods assume that the decision maker always looks for the solution corresponding to the maximum of some global measure. In this way, the method is based on a descriptive theory, proved by empirical evidence, of how people effectively make decisions when they are under risk. The mathematical structure of TODIM allows measuring the degree to which one alternative is superior to others and then ranking the alternatives by computing the global value of each alternative. That structure is embedded in the paradigm of prospect theory. Gomes and Lima [18] first applied TODIM in its classical formulation as a tool for ranking projects based on the environmental impacts of alternative road standards in Brazil. A number of other applications of TODIM has appeared in the literature since then as it is commented in the section 2.2. Similarly, the TOPSIS method [23] is used to weight and compare alternatives against a set of criteria and then select the best one. The application of both TODIM and TOPSIS are then compared one against the other. The novelty of this framework lies in studying the behavioral risk analysis under neutrosophic environment as pointed out in the above paragraph.

The main contribution of this article is as follows

- A framework is designed that emphasizes the importance of shelter and feeding system for sustainable and productive goat farming.
- Two well established Multi-Criteria Decision Making (MCDM) methods dealing with imprecise information are applied to a quite important problem in India and compared.
- Relevant criteria and sub-criteria are defined for the alternatives to maintain accuracy and consistency in selecting the alternatives.

2. Literature review

2.1. Commercial goat farming

Raising animals lie upon a set of activities that are dependent upon biotic and socio-economic factors. Choudhary et al. [35] highlights that India is the rich in its repository of goat genetic resource with 28 recognized breeds with higher proportion of non-descriptive or mixed breeds. A study was undertaken by Patil et al. [28] to compare the grazing system and stall feeding system in goats in Gulbarga District in Karnataka which highlighted that in stall feeding system of goat rearing, goats are found healthier and weight gain was much faster than grazing system. Kumar [26] investigated on commercial goat farming in India and presented that planned management and technology based system would help in increasing the goat productivity in goat farming and bridge the demand-supply gap. Argüello [8] has presented a review on trends in goat research which talks about the pathology, reproduction, milk and cheese production and quality, production systems, nutrition, hair production, drugs knowledge and meat production.

2.2. Multi Criteria Decision Making

Zadeh [42] put forward the concept of fuzzy sets in 1965. Later the theory of fuzzy sets gradually developed in the further years. The theory of 'intuitionistic fuzzy set' [IFS] was proposed by Atanassov [10] in 1986. Intuitionistic fuzzy set [IFS] was extended to 'Interval intuitionistic fuzzy sets' [IIFS] by Atanassov and Gargov [11]. A number of researchers have contributed their research to the study of MCDM and a commendable accomplishment has been obtained in fuzzy sets. Smarandache [36] proposed neutrosophic set based on Neutrosophy in 1998. The neutrosophic theory takes into account the dynamic features of all limitations to handle uncertain, indeterminate situations. Abdel-Basset et al. [2] proposed uncertainty assessments of linear time-cost tradeoffs using neutrosophic set considering the neutrosophic activity duration of time-cost tradeoffs in project management such as the tradeoffs between the project completion time and the cost and the uncertain conditions of environment of projects. Abdel-Basset [6] developed and applied a novel decision making model for sustainable supply chain under uncertainty environment.

Wang et al. [38] developed 'Single Valued Neutrosophic Set' (SVNS) and proposed various properties of set-theoretic operators to deal with uncertain, indeterminate and inconsistent data. Ye [40] proposed trapezoidal neutrosophic number an extension from SVNS and trapezoidal fuzzy number and defined its score and accuracy function with aggregating operators in [41]. Smarandache [37] introduced the plithogenic set as generalization of crisp, fuzzy, intuitionistic fuzzy and neutrosophic sets whose elements are characterized by many attribute values which have corresponding contradiction degree values between each attribute value and the dominant attribute value. Abdel-Basset [1] developed an evaluation framework based on plithogenic set theory for smart disaster response systems in uncertainty environment that deals more effectively with disaster by the effective communication of the information provided by the sensors with the response teams.

Decision making situations in real life are much complicated when the decision makers (DMs) have to fit in the best alternatives with respect to the given multiple criteria. Biswas et al. [14] established TOPSIS strategy for (MCDM) in trapezoidal neutrosophic environment using the maximum deviation strategy and also developed an optimization model to obtain the weight of the attributes which are incompletely known or completely unknown. Abdel-Basset [5] proposed a decision making problem to solve a supply chain problem of inventory location using the best-worst method based on a novel plithogenic model.

Pramanik and Mallick [30] proposed a VIKOR method for group Decision Making Problem involving trapezoidal neutrosophic number and they adapted a problem of Investment Company from [16] and provided a comparative analysis. Mondal and Pramanik [29] proposed MCDM approach for teacher recruitment in higher education with unknown weights based on score and accuracy function, hybrid score and accuracy functions under simplified neutrosophic environment. Biswas et al. [12,13] developed a new methodology for neutrosophic MCDM with unknown weight information and a Cosine similarity measure based MCDM with trapezoidal fuzzy neutrosophic numbers. Abdel-Basset [3] designed resource levelling problem to minimize the cost of daily resource fluctuation in construction projects under neutrosophic environment to overcome the ambiguity caused by real world problems.

Based on observations of human behaviour, studies have found that human decision making is not completely rational under practical decision situations. After undertaking a number of surveys and experiments, Kahneman and Tversky [24] proposed Prospect theory partially the subject of the Nobel Prize for Economics awarded in 2002, which belongs to the field of cognitive psychology and describes how people make decision under conditions of risk.

Gomes and Lima [20] used the TODIM method in order to show how human judgements in practical multi-criteria analysis fit in to the framework of Prospect Theory and additive difference model. Gomes et al. [19] used the classical TODIM formulation to recommend alternatives for destination of natural gas reserves recently discovered in Santos Basin in Brazil. Gomes et al. [22] proposed a behavioural multi-criteria decision analysis by using the TODIM method with criteria interactions. Gomes and Rangel [21] developed a novel approach using TODIM method on rental evaluation of residential properties carried out together with real estate agents in the city of Volta Redonda, Brazil which has made many successful applications in selection problems. Zindani et al. [44] proposed a material selection approach using the TODIM method and applied it to find the best suited materials for two products, engine flywheel and metallic gear.Duarte [7] proposed the use of multi criteria decision analysis to valuation of six Brazilian banks by applying the fuzzy TODIM method.

Sang and Liu [34] developed the IT2 FSs-based TODIM method to green supplier selection for automobile manufacturers by introducing a new distance computing method. Wang et al. [39] proposed a likehood-based TODIM approach on multi-hesitant fuzzy linguistic information (MHFLSs) which is an extension of (HFLSs) for selection and evaluation of contractors in logistics outsourcing. Chakraborty and Chakraborty [15] used TODIM in identifying the most attractive and affordable under-construction housing project in the city of Kolkata in India. Rangel et al. [32] used TODIM a multi-criteria decision aiding method in the evaluation of the various types of access to the broadband internet available in Volta Redonda, Brazil. Candidate selection is a significant task for any organization that aims to select the most appropriate candidates who lead the firm forward through his strong organizational skill. To overcome this tough task Abdel-Basset [4] proposed a bipolar neutrosophic multi criteria decision making framework for professional selection that employs a collection of neutrosophic analytical network process and TOPSIS under bipolar neutrosophic numbers.

Lourenzutti and Krohling [27] combined TOPSIS and TODIM methods to propose the Hellinger distance in MCDM which serves as an illustration to both methods. Fan et al. [17] proposed an extension of TODIM (H-TODIM) to solve the hybrid MCDM problem in which attribute values have three forms crisp number, interval number and fuzzy number. Ren et al. [33] proposed a Pythagorean fuzzy TODIM approach to analyse MCDM problem. Qin et al. [31] proposed generalizing of the TODIM method under triangular intuitionistic fuzzy environment. Zhang et al. [43] proposed an extended multiple attribute group decision making based on the TODIM method to solve the MCDM problem in which the attribute values are expressed with neutrosophic number.

3. Preliminaries

3.1. Hexagonal Neutrosophic Weighted Aggregated Operator (HNWA)

Let $\widetilde{A}_1 = (a_1, b_1, c_1, d_1, e_1, f_1), (l_1, m_1, n_1, p_1, q_1, r_1), (u_1, v_1, w_1, x_1, y_1, z_1)$ be a collection of hexagonal neutrosophic numbers, then the HNWA: $\Omega^n \to \Omega$ is defined as follows

$$HNWA(\widetilde{A}_1, \widetilde{A}_2, ...\widetilde{A}_n) = \sum_{j=1}^{n} \omega_j \widetilde{A}_j$$

$$= \left\langle \begin{array}{c} \begin{pmatrix} 1-\prod_{j=1}^{n}(1-a_j)^{\omega_j}, 1-\prod_{j=1}^{n}(1-b_j)^{\omega_j}, 1-\prod_{j=1}^{n}(1-c_j)^{\omega_j}, \\ 1-\prod_{j=1}^{n}(1-d_j)^{\omega_j}, 1-\prod_{j=1}^{n}(1-e_j)^{\omega_j}, 1-\prod_{j=1}^{n}(1-f_j)^{\omega_j} \end{pmatrix} \\ \left(\prod_{j=1}^{n}(l_j)^{\omega_j}, \prod_{j=1}^{n}(m_j)^{\omega_j}, \prod_{j=1}^{n}(n_j)^{\omega_j}, \prod_{j=1}^{n}(p_j)^{\omega_j} \prod_{j=1}^{n}(q_j)^{\omega_j}, {}_{j=1}\prod_{j=1}^{n}(r_j)^{\omega_j} \right) \\ \left(\prod_{j=1}^{n}(u_j)^{\omega_j}, \prod_{j=1}^{n}(v_j)^{\omega_j}, \prod_{j=1}^{n}(w_j)^{\omega_j}, \prod_{j=1}^{n}(x_j)^{\omega_j}, \prod_{j=1}^{n}(y_j)^{\omega_j}, \prod_{j=1}^{n}(z_j)^{\omega_j} \right) \\ \text{where } w=(w_1,w_2,w_3,w_4......w_n)^T \text{ is a weight vector of } \widetilde{A}_j \text{ and } \sum_{j=1}^{n} w_j=1, w_j>0 \end{array} \right\rangle --(1)$$

3.2. Distance between two Hexagonal Neutrosophic numbers

Let $\widetilde{A}_1 = (a_1, b_1, c_1, d_1, e_1, f_1), (l_1, m_1, n_1, p_1, q_1, r_1), (u_1, v_1, w_1, x_1, y_1, z_1)$

$\widetilde{A}_2 = (a_2, b_2, c_2, d_2, e_2, f_2), (l_2, m_2, n_2, p_2, q_2, r_2), (u_2, v_2, w_2, x_2, y_2, z_2)$ be two hexagonal neutrosophic numbers then the weighted distance between $\widetilde{A}_1$ *and* $\widetilde{A}_2$ is defined as follows.

$$d(\widetilde{A}_1, \widetilde{A}_2) = \frac{1}{18}\begin{pmatrix} |a_1-a_2|+|b_1-b_2|+|c_1-c_2|+|d_1-d_2|+|e_1-e_2|+|f_1-f_2|+ \\ |l_1-l_2|+|m_1-m_2|+|n_1-n_2|+|p_1-p_2|+|q_1-q_2|+|r_1-r_2|+ \\ |u_1-u_2|+|v_1-v_2|+|w_1-w_2|+|x_1-x_2|+|y_1-y_2|+|z_1-z_2| \end{pmatrix} --(2)$$

3.3. Similarity Degree between two Hexagonal Neutrosophic numbers

Let $\widetilde{A}_1 = [(a_1, b_1, c_1, d_1, e_1, f_1), \ (l_1, m_1, n_1, p_1, q_1, r_1), (u_1, v_1, w_1, x_1, y_1, z_1)]$ *and*

$\widetilde{A}_2 = [(a_2, b_2, c_2, d_2, e_2, f_2), \ (l_2, m_2, n_2, p_2, q_2, r_2), (u_2, v_2, w_2, x_2, y_2, z_2)]$

be two hexagonal neutrosophic numbers and let

$\widetilde{A}_2^C = [(u_2, v_2, w_2, x_2, y_2, z_2), (1-l_2, 1-m_2, 1-n_2, 1-p_2, 1-q_2, 1-r_2), (a_2, b_2, c_2, d_2, e_2, f_2)]$ be the complement of $\widetilde{A}_2$ then the Degree of Similarity between $\widetilde{A}_1$ *and* $\widetilde{A}_2$ is defined as follows.

$$\theta(\widetilde{A}_1, \widetilde{A}_2) = \left\{ \frac{d(\widetilde{A}_1, \widetilde{A}_2^C)}{d(\widetilde{A}_1, \widetilde{A}_2) + d(\widetilde{A}_1, \widetilde{A}_2^C)} \right\} --------------- (3)$$

3.4. Hexagonal Neutrosophic Decision Matrix

Let $\widetilde{R} = (\widetilde{r}_{ij})_{m \times n}$ .If all $\widetilde{r}_{ij}$ are hexagonal neutrosophic number then

$$\widetilde{R} = \widetilde{r}_{ij} = \left[\left(\widetilde{a}_{ij}, \widetilde{b}_{ij}, \widetilde{c}_{ij}, \widetilde{d}_{ij}, \widetilde{e}_{ij}, \widetilde{f}_{ij}\right), \left(\widetilde{l}_{ij}, \widetilde{m}_{ij}, \widetilde{n}_{ij}, \widetilde{p}_{ij}, \widetilde{q}_{ij}, \widetilde{r}_{ij}\right), \left(\widetilde{u}_{ij}, \widetilde{v}_{ij}, \widetilde{w}_{ij}, \widetilde{x}_{ij}, \widetilde{y}_{ij}, \widetilde{z}_{ij}\right)\right]$$

is a hexagonal neutrosophic decision matrix.

3.5. Aggregated Hexagonal Neutrosophic Decision Matrix

Let $\widetilde{R}^{(k)} = (\widetilde{r}_{ij}^{(k)})_{m\times n}$ $(k = 1,2,3,.....t)$ be a't' neutrosophic decision matrix evaluated by the decision makers $DM_d\,(d = 1,2,3....m)$ respectively, then the aggregated hexagonal neutrosophic decision matrix $\widetilde{R} = (\widetilde{r}_{ij})_{m\times n}$ is defined as

$$\widetilde{r}_{ij} = \left[\left(\widetilde{a}_{ij}, \widetilde{b}_{ij}, \widetilde{c}_{ij}, \widetilde{d}_{ij}, \widetilde{e}_{ij}, \widetilde{f}_{ij}\right), \left(\widetilde{l}_{ij}, \widetilde{m}_{ij}, \widetilde{n}_{ij}, \widetilde{p}_{ij}, \widetilde{q}_{ij}, \widetilde{r}_{ij}\right), \left(\widetilde{u}_{ij}, \widetilde{v}_{ij}, \widetilde{w}_{ij}, \widetilde{x}_{ij}, \widetilde{y}_{ij}, \widetilde{z}_{ij}\right)\right]$$

$$(i = 1,2,3....m), (j = 1,2...n) \quad where \quad \widetilde{r} = \frac{1}{t}\sum_{k=1}^{t}\widetilde{r}_{ij}^{(k)}$$

3.6. Degree of Similarity

Let $\widetilde{R}^{(k)} = (\widetilde{r}_{ij}^{(k)})_{m\times n}$ $(k = 1,2,3,......t)$ be a't' neutrosophic decision matrix and $\widetilde{R}' = (\widetilde{r}'_{ij})_{m\times n}$ be their aggregated hexagonal neutrosophic decision matrix then

$$\theta(\widetilde{R}^{(k)}, \widetilde{R}') = \frac{1}{m\times n}\sum_{i=1}^{m}\sum_{j=1}^{n}\theta(\widetilde{r}_{ij}^{(k)}, \widetilde{r}'_{ij}) ----------------(4)$$

is called the degree of similarity between $\widetilde{R}^{(k)}$ and $\widetilde{R}'$

3.7. Determine the weight of experts using Degree of Similarity:

If the hexagonal neutrosophic decision matrix $\widetilde{R}^{(k)} = (\widetilde{r}_{ij}^{(k)})_{m\times n}$ $(k =,1,2,......t)$ are non-identical,then the weight vectors of the experts are expressed as follows.

$$w^{(k)} = \frac{(\theta(\widetilde{R}^{(k)}, \widetilde{R}))^{\alpha}}{\sum_{k=1}^{t}(\theta(\widetilde{R}^{(k)}, \widetilde{R}))^{\alpha}} ---------------------(5)$$

4. A Comparative Analysis of TODIM and TOPSIS Methods.

4.1. TODIM

To solve the MCDM problem with hexagonal neutrosophic information's we propose a hexagonal neutrosophic aggregation TODIM method based on prospect theory under the decision maker's behavioral risk and arithmetic mean operator.

Let $A_i = (A_1, A_2, ...A_m)$ be the alternatives, and $C_j = \{C_1, C_2, ..., C_n\}$ be the criteria.

Let $w=(w_1,w_2,....w_n)$ be the weights of $C_j, 0\le w_j\le 1,$ and $\sum_{j=1}^{n} w_j = 1.$ Let,

$$\widetilde{R}_k=\left(\widetilde{r}_{ij}{}^{(k)}\right)_{m\times n}=(T_{ij},I_{ij},F_{ij})_{m\times n}=(\widetilde{a}_{ij},\widetilde{b}_{ij},\widetilde{c}_{ij},\widetilde{d}_{ij},\widetilde{e}_{ij},\widetilde{f}_{ij}),(\widetilde{l}_{ij},\widetilde{m}_{ij},\widetilde{n}_{ij},\widetilde{p}_{ij},\widetilde{q}_{ij},\widetilde{r}_{ij}),(\widetilde{u}_{ij},\widetilde{v}_{ij},\widetilde{w}_{ij},\widetilde{x}_{ij},\widetilde{y}_{ij},\widetilde{z}_{ij})$$

be a hexagonal neutrosophic decision matrix, where $\widetilde{r}_{ij}=(T_{ij},I_{ij},F_{ij})$ is an attribute value given by the experts for the alternatives A_i with the criteria C_j, $T_{ij}\in[0,1], I_{ij}\in[0,1], F_{ij}\in[0,1]$,

$0\le T_{ij}+I_{ij}+F_{ij}\le 3\quad (i=1,2,......,m),(j=1,2,....,n)$

The proposed method is presented as follows.

Stage 1.

Step 1. Construct a decision matrix of dimension $m\times n$ by using the information provided by the decision maker for the alternatives A_i under the criteria C_j. The m^{th} hexagonal neutrosophic decision matrix denoted by the decision maker is defined as follows.

$$\widetilde{R}_k=\begin{bmatrix} \begin{matrix}(a_{11},b_{11},c_{11},d_{11},e_{11},f_{11})\\(l_{11},m_{11},n_{11},p_{11},q_{11},r_{11})\\(u_{11},v_{11},w_{11},x_{11},y_{11},z_{11})\end{matrix} & \cdots\ \cdots\ \cdots & \begin{matrix}(a_{1n},b_{1n},c_{1n},d_{1n},e_{1n},f_{1n})\\(l_{1n},m_{1n},n_{1n},p_{1n},q_{1n},r_{1n})\\(u_{1n},v_{1n},w_{1n},x_{1n},y_{1n},z_{1n})\end{matrix}\\ \vdots & \vdots\ \vdots\ \vdots & \vdots\\ \vdots & \vdots\ \vdots\ \vdots & \vdots\\ \vdots & \vdots\ \vdots\ \vdots & \vdots\\ \begin{matrix}(a_{m1},b_{m1},c_{m1},d_{m1},e_{m1},f_{m1})\\(l_{m1},m_{m1},n_{m1},p_{m1},q_{m1},r_{m1})\\(u_{m1},v_{m1},w_{m1},x_{m1},y_{m1},z_{m1})\end{matrix} & \cdots\ \cdots\ \cdots & \begin{matrix}(a_{mn},b_{mn},c_{mn},d_{m},e_{mn},f_{mn})\\(l_{mn},m_{mn},n_{mn},p_{mn},q_{mn},r_{mn})\\(u_{mn},v_{mn},w_{mn},x_{mn},y_{mn},z_{mn})\end{matrix}\end{bmatrix}$$

Step 2: Find the aggregated hexagonal neutrosophic decision matrix of all the three decision makers..The aggregated hexagonal neutrosophic decision matrix $\widetilde{R}'=(\widetilde{r}'_{ij})_{m\times n}$ is defined as given below.

$$\widetilde{r}'_{ij}=\left[\left(\bar{a}'_{ij},\bar{b}'_{ij},\bar{c}'_{ij},\bar{d}'_{ij},\bar{e}'_{ij},\bar{f}'_{ij}\right),\left(\bar{l}'_{ij},\bar{m}'_{ij},\bar{n}'_{ij},\bar{p}'_{ij},\bar{q}'_{ij},\bar{r}'_{ij}\right),\left(\bar{u}'_{ij},\bar{v}'_{ij},\bar{w}'_{ij},\bar{x}'_{ij},\bar{y}'_{ij},\bar{z}'_{ij}\right)\right]$$
$$(i=1,2,3.....,m),(j=1,2....,n)$$
$$where\ \bar{r}'=\frac{1}{t}\sum_{k=1}^{t}\widetilde{r}_{ij}^{(k)}$$

Step 3. Calculate the normalized hamming distance for each $(\widetilde{R},\widetilde{R}')$ using the equation (2)

Step 4. Calculate the Degree of Similarity between $\overline{A}_1$ *and* $\overline{A}'_2$ using equation (3) and (4)

Step 5. Calculate the weight vector $w^{(k)}$ using equation (5)

Step 6. Using equation (1) calculate HNWA operator

Step 7. Calculate the score value using the equation

$$S(A)=\frac{1}{3}\left[2+\frac{a+b+c+d+e+f}{6}-\frac{l+m+n+p+q+r}{6}-\frac{u+v+w+x+y+z}{6}\right]-----(6)$$

Step 8. Calculate the normalized hamming distance for the aggregated decision matrix using (2)

Step 9. When the aggregated matrix is brought into expression (7), matrix $\rho(A_i, A_p)$ will be derived .The function $\rho(A_i, A_p)$ is used to represent the degree to which alternative i is better than j. $\varepsilon_j(A_i, A_p)$ is the sum of the sub-function where $j=1....,n$. Sub-function $\varepsilon_j(A_i, A_p)$ indicates the degree to which i is better than j when a particular criteria c is given

$$\varepsilon_j(A_i,A_p)=\begin{cases}\sqrt{\dfrac{w_{jr}d(\tilde{r}_{ij}-\tilde{r}_{pj})}{\sum\limits_{j=1}^{n} w_{jr}}} & if \quad \tilde{r}_{ij}-\tilde{r}_{pj}\succ 0\\ 0 & if \quad \tilde{r}_{ij}-\tilde{r}_{pj}=0\\ \dfrac{-1}{\upsilon}\sqrt{\dfrac{\sum\limits_{j=1}^{n} w_{jr}d(\tilde{r}_{ij}-\tilde{r}_{pj})}{w_{jr}}} & if \quad \tilde{r}_{ij}-\tilde{r}_{pj}\prec 0\end{cases}---------------(7)$$

The parameter υ shows the dilution factor of the loss. If $\tilde{r}_{ij}-\tilde{r}_{pj}\succ 0$ then $\varepsilon_j(A_i, A_p)$ represents the gain and if $\tilde{r}_{ij}-\tilde{r}_{pj}\prec 0$ then $\varepsilon_j(A_i, A_p)$ represents the loss.

Step 10. On the basics of the above equation the overall dominance degree is obtained as

$$\rho_x=\sum_{j=1}^{n}\varepsilon_j(A_i,A_p),(i,p=1,2,....m)$$

Step 11. Calculate the aggregated dominance matrix

$$\rho(A_i,A_p)=\sum_{x=1}^{n}\lambda_x\rho_x(A_i,A_p),(i,p=1,2,....m)---------(8)$$

Step 12. Calculate the overall dominance degree matrix $\rho=[\rho(A_i, A_p)]_{m\times n}$

Step 13. Then the overall value of each A_i can be calculated using the equation

$$\rho(A_i)=\frac{\sum_{p=1}^{m}\rho(A_i,A_p)-\min_i\left\{\sum_{p=1}^{m}\rho(A_i,A_p)\right\}}{\max_i\left\{\sum_{p=1}^{m}\rho(A_i,A_p)\right\}-\min_i\left\{\sum_{p=1}^{m}\rho(A_i,A_p)\right\}}--------(9)$$

Step 14. Rank all alternatives and select the most desirable one in accordance with $\rho(A_i)$. The alternative with maximum value is the best one.

4.2. TOPSIS

Stage 2: Applying the information's derived from step 1 to 6 in stage 1, move on to step 7 of stage 2

Step 7: Let B_1 be the set of benefit attributes and B_2 be the set of cost attributes, of the alternatives respectively. Let B^+ be the hexagonal neutrosophic positive ideal solution and B^- be the hexagonal neutrosophic negative ideal solution. Then B^+ and B^- are defined as follows.

$$B^+=\left[\left\langle r_j^+=(1,1,1,1,1,1),(0,0,0,0,0,0),(0,0,0,0,0,0)\right\rangle j\in B_1\right],\left[\left\langle r_j^+=(0,0,0,0,0,0),(1,1,1,1,1,1),(1,1,1,1,1,1)\right\rangle j\in B_2\right]$$

$$B^-=\left[\left\langle r_j^-=(0,0,0,0,0,0),(1,1,1,1,1,1),(1,1,1,1,1,1)\right\rangle j\in B_1\right],\left[\left\langle r_j^-=,(1,1,1,1,1,1),(0,0,0,0,0,0),(0,0,0,0,0,0)\right\rangle j\in B_2\right]$$

Step 8: Calculate the separation measures, S_i^+ *and* S_i^- of each alternative from the hexagonal neutrosophic positive ideal solution and the hexagonal neutrosophic negative ideal solution as follows.

$$S_i^+=\frac{1}{n}\sum_{j=1}^{n}w_j d(r_{ij},r_j^+)-----------------------(10)$$

$$S_i^-=\frac{1}{n}\sum_{j=1}^{n}w_j d(r_{ij},r_j^-)-----------------------(11)$$

Step 9: Calculate the relative closeness coefficient of the hexagonal neutrosophic ideal solution. The relative closeness coefficient of the alternative A_i is given as follows.

$$C_i=\frac{S_i^-}{S_i^+ + S_i^-},0\le C_i\le 1----------------(12)$$

Step 10: Make a decision for selecting the preference alternative by ranking the closeness coefficient in the descending order of C_i to select the best choice.

5. Case Analysis:

In this section, a case study is represented for the proposed multi-criteria group decision-making method. This is related to assessing the best system of housing and feeding of goats in the existing

Goat farm rearing in which goats grow healthier, gain better body weight, and are safer on health grounds. A group of three decision-makers (D_1, D_2 and D_3) are requested to assess the four alternatives ($A_1$to A_4) with respect to the four criteria's, (C_1 to C_4) defined by this group of decision-makers to appraise the alternatives. These criteria and their definitions are represented as follows:

Alternatives:

A_1 - Stall feeding system with normal flooring (intensive system)

A_2 - Grazing system (extensive system)

A_3 - Elevated floor shed with rotational grazing system

A_4 - A part of both extensive and intensive grazing system

The consideration of the criteria and sub criteria's after a brief study on the previous literature review and discussion with the experts are stated below.

Criteria:

C_1 - Floor space requirements

(Covered area, Open area, Ventilation, Bedding, Confinement, Site location)

C_2 - Feeding (Feeder) and watering space requirement

(Feeder size, Fodder type, Quantity, Food Schedule, immunization feeder, feed storage room)

C_3 - Maintenance of health and sanitization

(Nutritional ratio, Vaccination, Climate pattern, Temperature, Supplementary feeding, Cleanliness)

C_4 - Productivity

(Capital, Typologies of farms, Technology integration, Agro climatic characteristics, Market value, Place of selling)

A questionnaire is prepared and handed over to the domain experts. These experts further graded the degree of the statement as given below.

Statement	Very high	High	Fair	Average	Medium	Satisfactory	Low	Very low	Not sure
Score	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1

Table 5.1. Rating scale used by experts

Solution.

Step 1. The judgment of the three decision makers for the alternatives A_i under the four criteria were presented using hexagonal neutrosophic number as shown in Table 5.2 .

	Criteria				
DMs	Alternatives	C_1	C_2	C_3	C_4
	A_1	[(.1,.2,.3,.4,.5,.6), (.1,.1,.1,.1,.1,.1), (.5,.6,.7,.8,.9,.9)]	[(.1,.2,.3,.4,.5,.6), (.1,.1,.1,.1,.1,.1), (.5,.6,.7,.8,.9,.9)]	[(.8,.8,.8,.8,.9,.9), (.3,.4,.5,.6,.7,.8), (.1,.1,.1,.1,.1,.1)]	[(.2,.3,.4,.5,.6,.7), (.1,.2,.2,.3,.3,.3), (.5,.5,.6,.6,.7,.7)]

D_1	A_2	[(.3,.4,.5,.6,.7,.8), (.2,.3,.4,.5,.5,.6), (.6,.6,.7,.8,.8,.8)]	[(.2,.2,.2,.2,.2,.2), (.5,.6,.7,.8,.8,.8), (.6,.6,.7,.7,.8,.8)]	[(.8,.8,.8,.9,.9,.9), (.2,.3,.3,.3,.4,.5), (.1,.1,.2,.2,.2,.2)]	[(.2,.3,.4,.5,.6,.7), (.1,.2,.2,.3,.3,.3), (.5,.5,.6,.6,.7,.7)]
	A_3	[(.4,.5,.6,.7,.8,.9), (.2,.3,.4,.6,.7,.8), (.2,.2,.2,.2,.2,.2)]	[(.1,.1.1,.1,.1,.1) ,(.3,.4,.5,.6,.7,.8) (.4,.4,.5,.5,.6,.6)]	[(.1..2,.3,.5,.6,.7), (.3,.3,.4,.5,.8,.9.) (.5,.5,.5,.6,.6,.6)]	[(.9,.9,.9,.9,.9,.9) (.1,.2,.3,.4,.5,.6) (.3,.4,.4,.5,.6,.7)]
	A_4	[(.1,.2,.2,.3,.3,.3) ,(.2,.2,.3,.3,.5,.5) (.4,.5,.6,.7,.8,.9,)]	[(.3,.3,.4,.4,.4,.5), (.1,.2,.2,.3,.4,.4) (.5,.5,.5,.6,.6,.6)]	[(.1,.2,.3,.4,.5,.6) ,(.3,.3,.4,.6,.6,.6) (.3,.4,.5,.5,.6,.6)]	[(.6,.6,.6,.7,.7,.8), (.5,.5,.5,.6,.6,.6) (.2,.2,.3,.3,.3,.3)]
D_2	A_1	[(.6,.7,.8,.9,.9,.9), (.1,.2,.3,.3,.4,.5) (.3,.4,.4,.4,.5,.5)]	[(.3,.4,.5,.6,.7,.8) ,(.2,.3,.4,.5,.6,.7) (.4,.5,.6,.7,.8,.9)]	[(.3,.4,.4,.4,.4,.4) ,(.2,.2,.2,.2,.2,.2) (.4,.4,.5,.5,.5,.5)]	[(.8,.8,.8,.8,.8,.8) ,(.2,.2,.3,.3,.4,.4) (.2,.3,.3,.4,.4,.5)]
	A_2	[(.5,.5,.5,.5,.5,.5), (.2,.3,.4,.5,.6,.7), (.5,.5,.5,.5,.5,.5)]	[(.4,.5,.6,.7,.8,.8), (.3,.3,.3,.4,.4,.4), (.2,.3,.4,.5,.6,.7)]	[(.5,.6,.6,.7,.7,.8), (.3,.4,.5,.6,.6,.7), (.5,.5,.6,.7,.8,.8)]	[(.8,.8,.9,.9,.9,.9), (.2,.2,.3,.3,.4,.4), (.2,.3,.3,.4,.4,.5)]
	A_3	[(.2,.3,.4,.5,.6,.7), (.6,.6,.6,.6,.6,.6), (.2,.2,.2,.3,.4,.5)]	[(.5,.6,.7,.8,.9,.9), (.3,.4,.4,.4,.4,.4), (.4,.4,.4,.5,.5,.5)]	[(.1,.3,.3,.3,.4,.4), (.3,.3,.3,.3,.3,.3), (.4,.5,.5,.6,.6,.6)]	[(.6,.7,.7,.8,.8,.8), (.1,.2,.3,.4,.5,.6), (.4,.4,.5,.5,.6,.6)]
	A_4	[(.3,.4,.4,.5,.5,.6), (.1,.1,.1,.2,.2,.2), (.1,.2,.3,.4,.5,.6)]	[(.4,.4,.5,.6,.7,.7), (.2,.2,.3,.3,.4,.4), (.1,.2,.3,.4,.4,.5)]	[(.8,.9,.9,.9,.9,.9), (.1,.1,.2,.2,.3,.3), (.1,.1,.1,.1,.1,.1)]	[(.5,.5,.5,.6,.6,.7), (.2,.2,.2,.3,.3,.3), (.3,.3,.4,.5,.6,.7)]
D_3	A_1	[(.1,.2,.2,.2,.3,.3), (.1,.1,.1,.2,.3,.4), (.5,.6,.7,.7,.7,.7)]	[(.4,.4,.5,.6,.7,.8), (.4,.5,.6,.6,.7,.7), (.3,.4,.4,.5,.5,.5)]	[(.6,.7,.8,.8,.9,.9), (.3,.3,.3,.4,.4,.5), (.4,.5,.6,.6,.6,.6)]	[(.3,.3,.4,.4,.4,.5), (.4,.5,.6,.7,.8,.9), (.3,.4,.4,.5,.5,.6)]
	A_2	[(.4,.5,.5,.6,.6,.7) ,(.4,.4,.5,.6,.6,.7) (.4,.5,.6,.7,.8,.9)]	[(.7,.7,.7,.8,.8,.8), (.5,.6,.6,.6,.6,.6), (.4,.4,.5,.5,.6,.6)]	[(.6,.7,.7,.8,.8,.8), (.4,.5,.5,.6,.6,.7), (.2,.3,.4,.5,.6,.7)]	[(.2,.2,.2,.3,.3,.3), (.2,.2,.2,.3,.4,.5), (.4,.5,.5,.6,.7,.8)]
	A_3	[(.6,.7,.7,.8,.9,.9), (.5,.6,.7,.7,.8,.8), (.3,.4,.5,.5,.6,.6)]	[(.4,.5,.6,.7,.8,.9), (.7,.7,.7,.8,.8,.8), (.1,.2,.3,.4,.5,.6)]	[(.8,.8,.8,.9,.9,.9), (.5,.5,.6,.6,.7,.7), (.2,.3,.3,.4,.4,.5)]	[(.6,.7,.7,.8,.8,.9), (.1,.1,.1,.1,.1,.1), (.2,.2,.2,.3,.3,.3)]
	A_4	[(.6,.6,.6,.7,.7,.7), (.4,.5,.6,.6,.7,.7), (.4,.5,.5,.5,.5,.6)]	[(.8,.8,.8,.9,.9,.9), (.4,.4,.4,.4,.4,.4), (.3,.4,.4,.5,.5,.5)]	[(.3,.4,.5,.6,.7,.8), (.5,.5,.5,.5,.5,.5), (.4,.5,.5,.6,.7,.8)]	[(.2,.3,.3,.3,.4,.4), (.6,.6,.6,.7,.7,.7), (.8,.8,.8,.9,.9,.9)]

Table 5.2 Opinion of decision makers on performance values

Step 2. Normalize the hexagonal neutrosophic decision matrix $\widetilde{R}^k = (\widetilde{r}_{ij}^k)_{m\times n}$ given by the experts $D_k\ (k=1,2,3)$ to get the matrix $\widetilde{R}' = (\widetilde{r}'_{ij})_{m\times n}$

Criteria				
Alternative	C_1	C_2	C_3	C_4

$\overline{A}'_1$	[(.3,.4,.5,.6,.7,.7), (.1,.1,.2,.2,.3,.3), (.4,.5,.6,.6,.7,.7)]	[(.3,.3,.4,.5,.6,.7), (.3,.4,.4,.5,.6,.6), (.4,.5,.6,.7,.7,.7)]	[(.6,.6,.7,.7,.7,.7), (.3,.3,.3,.4,.4,.5), (.3,.3,.4,.4,.4,.4)]	[(.4,.5,.5,.6,.6,.7), (.2,.3,.3,.4,.4,.4), (.4,.4,.4,.5,.5,.6)]
$\overline{A}'_2$	(.4,.5,.5,.6,.6,.7), (.3,.3,.4,.5,.6,.7), (.5,.5,.6,.7,.7,.7)]	[(.4,.5,.5,.6,.6,.6), (.4,.5,.5,.6,.6,.6), (.4,.4,.5,.6,.6,.7)]	[(.6,.7,.7,.8,.8,.8), (.3,.4,.4,.5,.6,.7), (.3,.3,.4,.5,.5,.6)]	[(.5,.5,.6,.6,.6,.7). (.2,.2,.3,.3,.4,.4), (.2,.3,.3,.4,.5,.6)]
$\overline{A}'_3$	[(.4,.5,.6,.7,.8,.8), (.4,.5,.5,.6,.6,.7), (.2,.3,.3,.3,.4,.4)]	[(.3,.4,.5,.5,.6,.6), (.4,.5,.5,.6,.6,.7), (.3,.3,.4,.5,.5,.6)]	[(.3,.4,.5,.6,.6,.7), (.4,.4,.4,.5,.6,.6), (.3,.4,.4,.5,.5,.6)]	[(.7,.8,.8,.8,.8,.9), (.1,.2,.2,.3,.4,.4), (.3,.3,.4,.4,.5,.5)]
$\overline{A}'_4$	[(.3,.4,.4,.5,.5,.5), (.2,.3,.3,.4,.5,.5), (.3,.4,.5,.6,.6,.7)]	[(.5,.5,.6,.6,.7,.7), (.2,.3,.3,.3,.4,.4), (.3,.4,.4,.5,.5,.5)]	[(.4,.5,.6,.6,.7,.8), (.3,.3,.4,.4,.5,.5), (.3,.3,.3,.4,.4,.5)]	[(.4,.5,.5,.5,.6,.6), (.4,.4,.4,.5,.5,.5), (.4,.4,.5,.6,.6,.6)]

Table 5.3 Normalized hexagonal neutrosophic decision matrix

Step 3.

Once the decision makers provide the decision matrix we calculate the relative weight of each criterion C_j Consider the weight of each criterion as $w = (0.15, 0.15, 0.20, 0.50)$

$$w_r = \max\{w_j / j = 1,2,....n\}$$
$$w_r = \max\{0.15, 0.15, 0.20, 0.50\}$$
$$w_r = 0.50$$

Since $w_r = 0.50$ then C_4 is the reference criterion and the reference criterion weight is 0.50. Then calculate the relative weights of the criterion $C_j (j = 1,2,3,4)$ as

$$w_{1r} = \frac{w_1}{w_r} = \frac{0.15}{0.50} = 0.3, w_{2r} = 0.3, w_{3r} = 0.4, w_{4r} = 1$$

The parameter υ the dilution factor of the loss is

$$\upsilon = \sum_{j=1}^{4} w_{jr} = 0.3 + 0.3 + 0.4 + 1 = 2$$

Step 4. Consider the alternative A_1 of DM_1 and the criteria C_1

Calculate the distance between $\overline{A}_1$ *and* $\overline{A}'_1$, $\overline{A}_1$ *and* $\overline{A}'^{C}_1$ *of* DM_1

$\overline{C}_1 = [(.1,.2,.3,.4,.5,.6),(.1,.1,.1,.1,.1,.1)\ (.5,.6,.7,.8,.9,.9)]$, $\overline{C}'_1 = [(.3,.4,.4,.5,.6,.6),(.1,.1,.2,.2,.3,.3)\ (.4,.5,.6,.6,.7,.7)]$

$\overline{C}'^{C}_1 = [(.4,.5,.6,.6,.7,.7)\ (.9,.9,.8,.8,.7,.7)\ (.3,.4,.4,.5,.6,.6)]$ $d(\overline{C}_1, \overline{C}'_1) = \frac{1}{18}(2.2)$, $d(\overline{C}_1, \overline{C}'^{C}_1) = \frac{1}{18}(6.9)$

Step 5. The Degree of Similarity between $\overline{A}_1$ *and* $\overline{A}'$ is defined as follows.

$$\theta(\overline{C}_1, \overline{C}_1') = \left\{ \frac{d(\overline{C}_1, \overline{C}_1'^{C})}{d(\overline{C}_1, \overline{C}_1') + d\left(\overline{C}_1, \overline{C}_1'^{C}\right)} \right\} = \frac{\frac{1}{18}(6.9)}{\frac{1}{18}(2.2 + 6.9)} = 0.76$$

Continuing the above process for all decision makers the consolidated Degree of Similarity is tabulated below.

Degree of Similarity	A_1 of DM_1	A_2 of DM_1	A_3 of DM_1	A_4 of DM_1	A_1 of DM_2	A_2 of DM_2	A_3 of DM_2	A_4 of DM_2	A_1 of DM_3	A_2 of DM_3	A_3 of DM_3	A_4 of DM_3
$\theta(\overline{C}_1, \overline{C}_1')$	**0.76**	**0.70**	**0.76**	**0.68**	**0.56**	**0.69**	**0.66**	**0.51**	**0.77**	**0.77**	**0.66**	**0.42**
$\theta(\overline{C}_1, \overline{C}_2')$	**0.68**	**0.53**	**0.53**	**0.57**	**0.74**	**0.56**	**0.50**	**0.81**	**0.45**	**0.57**	**0.59**	**0.72**
$\theta(\overline{C}_3, \overline{C}_3')$	**0.66**	**0.61**	**0.60**	**0.48**	**0.54**	**0.61**	**0.38**	**0.65**	**0.72**	**0.83**	**0.56**	**0.45**
$\theta(\overline{C}_4, \overline{C}_4')$	**0.62**	**0.57**	**0.81**	**0.50**	**0.68**	**0.78**	**0.80**	**0.51**	**0.44**	**0.54**	**0.80**	**0.45**

Table 5.4 Degree of Similarity between the alternatives compared with the criteria

Step 6. Calculate the weight vectors of the decision makers using degree of similarity

$$\theta(\widetilde{R}^{(k)}, \widetilde{R}') = \frac{1}{m \times n} \sum_{i=1}^{m} \sum_{j=1}^{n} \theta(\widetilde{r}_{ij}^{(k)}, \widetilde{r}_{ij}')$$

$$\theta(\widetilde{R}^{(1)}, \widetilde{R}') = \frac{10.524}{12} = .877, \theta(\widetilde{R}^{(2)}, \widetilde{R}') = \frac{10..097}{12} = 0.8814, \theta(\widetilde{R}^{(3)}, \widetilde{R}') = \frac{9.9000}{12} = 0.825$$

$$w^{(1)} = \frac{(\theta(\widetilde{R}^{(1)}, \widetilde{R}')}{\sum_{k=1}^{t}(\theta(\widetilde{R}^{(k)}, \widetilde{R}')} = \frac{.877}{2.59} = .34 \quad w^{(2)} = \frac{(\theta(\widetilde{R}^{(2)}, \widetilde{R}')}{\sum_{k=1}^{t}(\theta(\widetilde{R}^{(k)}, \widetilde{R}'))} = \frac{.8814}{2.59} = .33$$

$$w^{(3)} = \frac{(\theta(\widetilde{R}^{(2)}, \widetilde{R}')}{\sum_{k=1}^{t}(\theta(\widetilde{R}^{(k)}, \widetilde{R}'))} = \frac{.825}{2.59} = .33$$

Step 7. Using equation (1) HNWA the aggregated decision matrix is as follows.

	Criteria			
Alternative	C_1	C_2	C_3	C_4
$\overline{A}_1$	[(.4,.5,.6,.7,.8,.9), (0,.1,.2,.3,.4,.5), (.1,.1,.1,.1,.1,.1)]	[(0,.1,.2,.3,.4,.5) (0,.1,.2,.3,.4,.5) (.2,.3,.4,.5,.6,.7)]	[(.3,.4,.5,.6,.7,.8), (0,.1,.2,.3,.4,.5), (.1,.1,.1,.1,.1,.1)]	[(.3,.4,.5,.6,.7,.8), (.1,.1,.1,.1,.1,.1), (.1,.2,.3,.4,.5,.6)]
$\overline{A}_2$	[(.3,.4,.5,.6,.7,.8), (.1,.1,.2,.2,.3,.3), (0,.1,.1,.1,.2,.2)]	[(.1,.2,.3,.4,.5,.6), (0,.1,.1,.2,.3,.3), (0,.1,.1,.1,.2,.2)]	[(0,.1,.1,.2,.2,.2), (.1,.1,.1,.1,.1,.1), (.5,.6,.6,.7,.7,.8)]	[(.3,.4,.5,.5,.6,.6), (0,0,.1,.1,.1,.1), (.1,.1,.1,.1,.1,.1)]

$\overline{A}_3$	[(.1,.1,.1,.1,.2,.2), (.2,.2,.2,.2,.3,.3), (.4,.4,.5,.5,.6,.6)]	[(.2,.2,.2,.3,.3,.3), (0,.1,.2,.2,.3,.3), (.4,.5,.6,.7,.8,.9)]	[(.3,.4,.5,.6,.7,.8), (0,.1,.2,.3,.4,.5), (.1,.2,.2,.2,.3,.3)]	[(.1,.2,.3,.4,.5,.6), (.2,.2,.2,.2,.2,.2), (.4,.5,.6,.7,.8,.9)]
$\overline{A}_4$	[(.1,.2,.3,.4,.5,.6), (0,.1,.1,.1,.2,.4), (.1,.1,.2,.2,.4,.4)]	[(.5,.5,.5,.5,.5,.5), (.2,.3,.3,.4,.4,.5), (.1,.1,.1,.1,.2,.2)]	[(.4,.5,.5,.6,.7,.8), (.1,.1,.1,.1,.1,.1), (.1,.2,.3,.4,.4,.4)]	[(.1,.1,.1,.1,.1,.1), (0,.1,.1,.2,.3,.3), (.5,.5,.6,.6,.7,.8)]

Table. 5.5 Aggregated decision matrix

Step 8 . Calculate the score value using the equation (6)

$$S(A)=\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array}\begin{pmatrix} C_1 & C_2 & C_3 & C_4 \\ 0.59 & 0.48 & 0.48 & 0.62 \\ 0.49 & 0.47 & 0.65 & 0.66 \\ 0.58 & 0.54 & 0.56 & 0.73 \\ 0.55 & 0.64 & 0.67 & 0.54 \end{pmatrix}$$

Step 9. Using the score function we check for the conditions and find $\left[d\left(r_{ij},r_{pj}\right)\right]$

Here we consider $j=1, i=1,2,3,4$ *and* $p=1,2,3,4$ and check for the conditions in (7)

1) $\tilde{r}_{ij}>\tilde{r}_{pj}$ *or* 2) $\tilde{r}_{ij}=\tilde{r}_{pj}$ *or* 3) $\tilde{r}_{ij}<\tilde{r}_{pj}$ *for* $i=1,2,3,4$ $j=1$ *and* $p=1,2,3,4$

$$\left[d\left(r_{ij},r_{pj}\right)\right]_{m\times n}=\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array}\begin{bmatrix} C_1 & C_2 & C_3 & C_4 \\ 0 & 0.1222 & 0.2266 & 0.1055 \\ 0.1222 & 0 & 0.1666 & 0.111 \\ 0.2266 & 0.1666 & 0 & 0.2111 \\ 0.1055 & 0.111 & 0.2111 & 0 \end{bmatrix}$$

To construct the dominance matrix we check for $(\tilde{r}_{ij}\ is >,< or = to\ \tilde{r}_{pj})$

Since we have

$(r_{11}=r_{11})$, $\varepsilon_1(A_1,A_1)=0$ and as $(r_{11}>r_{21})$ $\varepsilon_1(A_1,A_2)=\sqrt{\dfrac{w_{1r}d(r_{11},r_{21})}{\sum_{j=1}^{4} w_{jr}}}=0.11055$

and $(r_{21}<r_{11})$, $\varepsilon_1(A_2,A_1)=\sqrt{\dfrac{\sum_{j=1}^{4} w_{jr}d(r_{21},r_{11})}{w_{1r}}}=-0.4401$

Using equation (7) calculate the dominance matrix $\varepsilon_1\left(A_i,A_p\right)$ as follows.

$$\left[\varepsilon_1\left(A_i,A_p\right)\right]_{m\times n}=\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array}\begin{bmatrix} C_1 & C_2 & C_3 & C_4 \\ 0 & 0.1105 & 0.1632 & 0.1027 \\ -0.4401 & 0 & -0.5155 & -0.4214 \\ -0.6523 & 0.1290 & 0 & 0.1452 \\ -0.4108 & 0.1053 & -0.5810 & 0 \end{bmatrix}$$

Similarly for the values $j = 2, i = 1,2,3,4 \ \ and \ \ p = 1,2,3,4$,

$j = 3, i = 1,2,3,4 \ and \ p = 1,2,3,4$ and $j = 4, i = 1,2,3,4 \ and \ p = 1,2,3,4$ the dominance matrix are calculated.

$$\left[\varepsilon_2\left(A_i, A_p\right)\right]_{m\times n} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array}\begin{bmatrix} C_1 & C_2 & C_3 & C_4 \\ 0 & 0.1393 & -0.2508 & -0.3154 \\ -0.2229 & 0 & -0.1977 & -0.3040 \\ 0.1624 & 0.1235 & 0 & -0.2598 \\ 0.1971 & 0.1900 & 0.1624 & 0 \end{bmatrix}$$

$$\left[\varepsilon_3\left(A_i, A_p\right)\right]_{m\times n} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array}\begin{bmatrix} C_1 & C_2 & C_3 & C_4 \\ 0 & 0.1581 & 0.1786 & 0.1624 \\ -0.2529 & 0 & 0.1624 & -0.2304 \\ -0.2858 & -0.2598 & 0 & -0.2665 \\ -0.2598 & 0.1440 & 0.1440 & 0 \end{bmatrix}$$

Step 10. On the basics of the above equation the overall dominance degree is obtained as

$$\rho(A_i, A_p) = \sum_{j=1}^{n} \varepsilon_j (A_i, A_p), (i, p = 1,2,....m)$$

$$\delta = \begin{bmatrix} 0 & 0.238 & -0.1331 & -0.1656 \\ -0.7789 & 0 & -0.7508 & -0.7298 \\ -0.5598 & 0.1927 & 0 & -0.0963 \\ -0.656 & 0.2133 & -0.8454 & 0 \\ & & & \end{bmatrix}$$

Now $\sum_{j=1}^{4} \varepsilon_j (A_i, A_p), (i, p = 1,2,....m$ are (0.2363,-2.2266,-0.4654,-1.000)

Step 12. Then the overall value of each A_i can be calculated using the equation (9)

$\rho(A_1) = 1.000, \quad \rho(A_2) = 0, \quad \rho(A_3) = 0.7150, \quad \rho(A_4) = 0.4980$

Step 13. Ranking the values of all alternatives $\rho(A_i)$ and selecting the most desirable alternatives in accordance with $\rho\left(A_i\right)$, among the four alternatives A_1 is the best choice and the ranking order is

$A_1 > A_3 > A_4 > A_2$

Stage 2.

Step 7. Floor space requirement C_1, Feeding (Feeder) and watering space requirement C_2 are benefiting type criteria $B_1 = \langle C_1, C_2 \rangle$. Maintenance of health and sanitization C_3 and Productivity C_4 are cost type $B_2 = \langle C_3, C_4 \rangle$ .The hexagonal neutrosophic positive-ideal solution B^+ and hexagonal neutrosophic negative-ideal solution B^- are obtained as follows

$$B^+ = \left[\left\langle \begin{matrix} (1,1,1,1,1,1)(0,0,0,0,0,0)(0,0,0,0,0,0),(1,1,1,1,1,1)(0,0,0,0,0,0)(0,0,0,0,0,0), \\ ((0,0,0,0,0,0)(1,1,1,1,1,1)(1,1,1,1,1,1)),((0,0,0,0,0,0)(1,1,1,1,1,1)(1,1,1,1,1,1)) \end{matrix} \right\rangle \right]$$

$$B^- = \left[\left\langle \begin{matrix} (0,0,0,0,0,0)(1,1,1,1,1,1)(1,1,1,1,1,1),\ (0,0,0,0,0,0)(1,1,1,1,1,1)(1,1,1,1,1,1) \\ (,(1,1,1,1,1,1)(0,0,0,0,0,0)(0,0,0,0,0,0)),((1,1,1,1,1,1)(0,0,0,0,0,0)(0,0,0,0,0,0)) \end{matrix} \right\rangle \right]$$

Step 8. The vector of the attribute weight is $w = (0.15, 0.15, 0.20, 0.50)$. By using equation (10) calculate the separation measure S_i^+ of the each alternative from the hexagonal neutrosophic positive ideal solution where $d(r_{ij}, r_j^+)$ is calculated using equation (2).

The calculated values are as follows

$S_1^+ = 0.1482$ $S_2^+ = 0.1408$ $S_3^+ = 0.1370$ $S_4^+ = 0.1164$

By using equation (11) calculate the separation measure S_i^- of the each alternative from the hexagonal neutrosophic negative ideal solution. The calculated values are as follows

$S_1^- = 0.0989$ $S_2^- = 0.1094$ $S_3^- = 0.1129$ $S_4^- = 0.1335$

Step 9. Using equation (12) calculate the relative closeness coefficient of the hexagonal neutrosophic ideal solution. The relative closeness coefficient values are as follows
$C_1 = 0.4002$ $C_2 = 0.4372$ $C_3 = 0.4517$ $C_4 = 0.5342$

Step 10. Rank the alternatives in the decreasing order of closeness coefficient values.

$$A_4 > A_3 > A_2 > A_1$$

6. Graphical Representation of the Comparative study

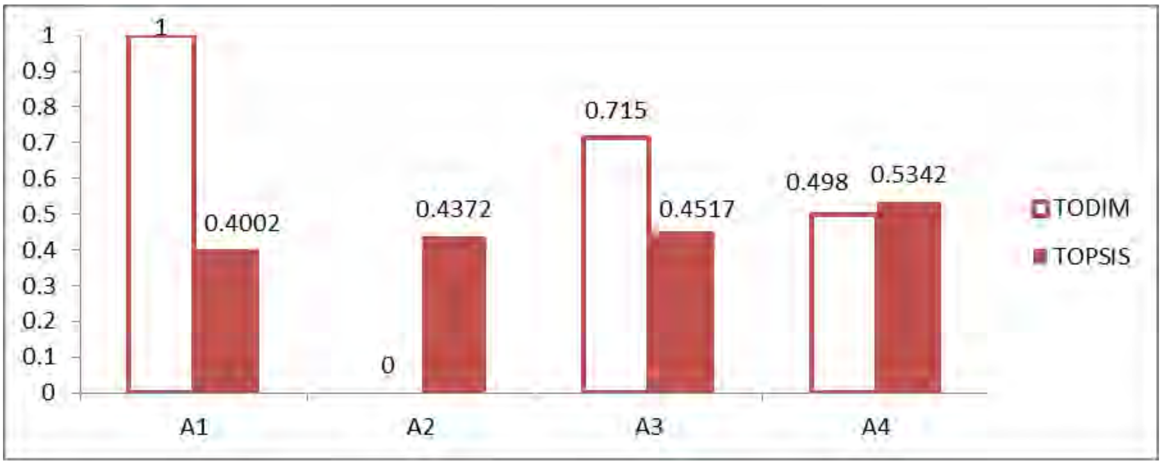

Figure 6.1 Ranking of the four alternatives using TODIM and TOPSIS

- The ranking results of TODIM show that A_1 is the best alternative with maximum global value $\rho(A_1)=1$ and the least global value is $\rho(A_2)=0$ The ranking of the four alternatives using TODIM is $A_1 > A_3 > A_4 > A_2$
- The ranking result using TOPSIS shows that A_4 is the best suited alternative as it ranking is in first position and A_1 is considered to be last as it takes fourth position in ranking .
- The ranking of the four alternatives using TOPSIS is $A_4 > A_3 > A_2 > A_1$.
- In both the methods A_3 take the same position and A_4 is in the third level in TODIM which is nearest to the ranking of TOPSIS. Similarly, A_2 is in the fourth level in TODIM which is very close to the ranking of TOPSIS.
- Both the MCDM ranking results shows that they are similar by large percentage which provides decision maker to increase the flexibility in choosing the optimal alternative.

Conclusion

The research presented in this article is an assessment study of the sustainability of commercial goat farming and its recent impact on self-employment for youth has been carried out in a context characterized by two MCDM methods, TODIM and TOPSIS. Using those methods the social, economic and ecological sustainability in housing and feeding systems of goat farming are evaluated by three experts and the evaluation was considered as hexagonal neutrosophic numbers in order to remove the ambiguity and increase the accuracy in the decision making process. Using the TODIM approach which is able to distinguish between risks based alternative and definite alternative in uncertain circumstances is analyzed .At the same time, by using the TOPSIS method the ranking is performed based on distance of each alternatives to its positive and negative ideal solutions. The ranking results of TODIM show a large percentage of similarity with ranking resulting from TOPSIS.The result shows that stall feeding system with normal flooring and a part with both intensive and extensive grazing system are best suited for sustainable commercial goat farming. This study may be applied in several other fields like livestock management systems with technology adaptation as well as in the economics of goat farming and other livestock sectors..

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Received: Apr 15, 2020. Accepted: July 5, 2020

Neutrosophic Sets and Systems, Vol. 35, 2020
University of New Mexico

MBJ – Neutrosophic β – Ideal of β – Algebra

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Abstract: This paper extends the concept of ideal of a β – algebra to MBJ – Neutrosophic β – Ideal of a β – algebra. Further discusses about the homomorphic image, inverse image, cartesian product and related results.

Keywords: Neutrosophic Set, MBJ – Neutrosophic Set, MBJ – Neutrosophic β – Ideals, Cartesian Product of MBJ – Neutrosophic β – algebra.

1. Introduction

Zadeh [21, 22] first presented the idea of Fuzzy Set by which shown a meaningful application in many fields and this theory is welcomed to handle the uncertainty. As a generalization of fuzzy set Atanassov [7, 11] introduced Intuitionistic Fuzzy Set which assigns a pair with membership degree and non – membership degree. The Interval Valued Fuzzy Set [6, 10, 12] represents the membership degree with interval values to reflect the uncertainty in assigning membership degree. As an extension for all elements in any set, Neutrosophic Fuzzy Set provides with truth, intermediate and false membership function by Smarandache, F [16, 17, 18] and is further developed to MBJ – Neutrosophic fuzzy set [19, 20] with truth membership function, intermediate interval valued membership function and false membership function.

Neggers and Kim [18] brought a new structure of algebra called β – algebra and Jun [17] dealt some related topics on β – algebra. The fusion of fuzzy with algebra and the notion was initiated by Rosenfeld [15]. Further many researchers correlated some algebras with fuzzy sets. Ansari [5, 8] initialized the fuzzy β – subalgebra of β – algebra and also introduced fuzzy β – ideal of β – algebra. With these inspirations, this paper extends to MBJ – Neutrosophic β – ideal of β – algebra and analyzed some result.

2. Preliminaries

In this section, some definitions and examples of β – algebra and fuzzy set are discussed.

2.1 Definition: [5, 8, 14] A non-empty set $(X, +, -, 0)$ is called a β – algebra if

i. $x - 0 = x$

ii. $(0 - x) + x = 0$

iii. $(x-y)-z = x-(z+y) \ \ \forall\, x,y,z \in X.$

2.2 Example: [9] The following Cayley's table is formed using a set $X = \{0,1,2,3,4,5\}$ with a constant 0 and two binary operations $+$ and $-$

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	4	5	2	3
2	2	5	0	4	3	1
3	3	4	5	0	1	2
4	4	3	1	2	5	0
5	5	2	3	1	0	4

-	0	1	2	3	4	5
0	0	1	2	3	5	4
1	1	0	4	5	3	2
2	2	5	0	4	1	3
3	3	4	5	0	2	1
4	4	3	1	2	0	5
5	5	2	3	1	4	0

$\therefore$ The set X is a β – algebra.

2.3 Definition: [5] A non – empty subset S of a β – algebra $(X,+,-,0)$ is known as β – subalgebra if

i. $x-y \in S$

ii. $x+y \in S \ \ \forall\, x,y \in S$

2.4 Example: Let $U_1 = \{0,2\}$ and $U_2 = \{0,1\}$ be any two subset of a β – algebra $X = \{(0,1,2,3,4,5),\ +,-,0\}$. Here U_1 is a β – subalgebra of X where as U_2 is not a β – subalgebra of X.

2.5 Definition: [8] A non – empty subset I of a β – algebra is said to be β – ideal of $(X,+,-,0)$ if it has the following conditions

i. $0 \in I$

ii. $x+y \in I$

iii. $x-y$ and $y \in I$ then $x \in I \ \forall\, x,y \in X$

2.6 Exercise: [12] Consider a β – algebra$(X,+,-,0)$ in the Cayley's table

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

-	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

The subset $I_1 = \{0,3\}$ of X is a β – ideal of X.

2.7 Definition: [5] A mapping $f: X \to Y$ is said to be a β – homomorphism where X and Y are two β – algebras with constant 0 and two binary operations $+$ and $-$ if

i. $f(x+y) = f(x) + f(y)$

ii. $f(x-y) = f(x) - f(y) \; \forall \, x, y \in X$.

2.8 Definition: [22] A Fuzzy Set in X is a mapping, $\rho : X \to [0,1]$ for each x in X, $\rho(x)$ is called the membership value of $x \in X$.

2.9 Definition: [7] A non – empty set X is said to be Intuitionistic Fuzzy Set and is defined by $A = \{ < x, \rho(x), \eta(x) > / x \in X\}$ where $\rho_A : X \to [0,1]$ is a membership function of A and $\eta_A : X \to [0,1]$ is a non – membership function of A with $0 \leq \rho_A(x) + \eta_A \leq 1$.

2.10 Definition: [6] An Interval Valued Fuzzy Set on X is represented as $A = \{(x, \bar{\rho}_A(x))\} \; x \in X$ where $\bar{\rho}_A : X \to D[0,1]$ where $D[0,1]$ is the family of all closed subintervals of $[0,1]$. Also $\bar{\rho}_A(x) = [\, \rho_A^L(x) \, , \, \rho_A^U(x)]$ where ρ_A^L and ρ_A^U are two fuzzy sets in X such that $\rho_A^L(x) \leq \rho_A^U(x) \; \forall \, x \in X$.

Remark: Now let us illustrate refined minimum ($rmin$) and refined maximum ($rmax$) of two elements in $D[0,1]$. Also characterized the symbols $\leq, \geq, =$ in case of two elements in $D[0,1]$.
Let $D_1 = [a_1, b_1]$ & $D_2 = [a_2, b_2] \in D[0,1]$ then
$rmin(D_1 \, , \, D_2) = [\min(a_1, a_2) \, , \min(b_1, b_2)]$
$rmax(D_1 \, , D_2) = [\max(a_1, a_2) \, , \max(b_1, b_2)]$.
For $D_i = [a_i, b_i] \in D[0,1]$ for $i = 1, 2, 3, ...$
$rsup_i(D_i) = [sup_i(b_i), sup_i(b_i)]$ & $rinf_i(D_i) = [inf_i(b_i), inf_i(b_i)]$
Now $D_1 \geq D_2$ if and only if $a_1 \geq a_2$, $b_1 \geq b_2$. Likewise, for $D_1 \leq D_2$ and $D_1 = D_2$ are defined.

2.11 Definition: [6] The representation of an Interval Valued Intuitionistic Fuzzy Set A on X is $A = \{ < x, \bar{\rho}_A(x), \bar{\eta}_A(x) > / x \in X\}$ where $\bar{\rho}_A : X \to D[0,1]$ and $\bar{\eta}_A : X \to D[0,1]$ where $\bar{\rho}_A(x) = [\, \rho_A^L(x) \, , \, \rho_A^U(x)]$ and $\bar{\eta}_A(x) = [\, \eta_A^L(x) \, , \eta_A^U(x)]$ with the condition that $0 \leq \rho_A^L(x) + \eta_A^L \leq 1$ and $0 \leq \rho_A^U(x) + \eta_A^U \leq 1$.

2.12 Definition: [16, 17] The definition of an Neutrosophic Fuzzy Set A on X is characterized by a Truth – membership function ρ_T , an indeterminacy membership function ξ_I , and a falsity – membership function η_F where ρ_T, ξ_I, η_F are subsets of $[0,1]$ that is ρ_T, $\xi_I, \eta_F : X \to [0,1]$. Thus, the Neutrosophic Set is defined as $A = \{ < x, \; \rho_T(x), \; \xi_I(x), \; \eta_F(x) > / x \in X\}$.

2.13 Definition: [19,20] The structure $A = \{ < x, \; \rho_T(x), \bar{\xi}_I(x), \; \eta_F(x) > / x \in X\}$ is called MBJ – Neutrosophic Set in X where ρ_T , $\eta_F : X \to [0,1]$ and $\bar{\xi}_I : X \to D[0,1]$ with $\rho_T(x)$ denotes the truth membership function , $\bar{\xi}_I(x)$ denotes an intermediate interval valued membership function and $\eta_F(x)$ denotes an false membership function.

2.14 Definition: An Fuzzy set is said to have a supremum property for any subset W of X there exists $x_0 \in W$ such that $\rho_A(x_0) = sup_{x \in W} \rho_A(x)$.

2.15 Definition: An Intuitionistic Fuzzy Set A is said to have a $sup - inf$ property for any subset W of X, there exists $x_0 \in W$ such that $\rho_A(x_0) = sup_{x \in W}\rho_A(x)$ and $\eta_A(x_0) = inf_{x \in W}\eta_A(x)$.

2.16 Definition: An Interval Valued Intuitionistic Fuzzy Set A in any set X is said to have $rsup - rinf$ property if for subset W of X there exists $x_0 \in W$ such that $\bar{\rho}_A(x_0) = rsup_{x \in W}\bar{\rho}_A(x)$ and $\bar{\eta}_A(x_0) = rinf_{x \in W}\bar{\eta}_A(x)$.

2.17 Definition: [19] An MBJ – Neutrosophic Fuzzy Set A in X has $sup - rsup - inf$ property if for subset W of X there exists $x_0 \in W$ such that $\rho_A(x_0) = sup_{x \in W}\rho_A(x)$; $\bar{\xi}_A(x_0) = rsup_{x \in W}\, \bar{\xi}_A(x)$; $\eta_A(x_0) = inf_{x \in W}\, \eta_A(x)$ respectively.

2.18 Definition: [12] An Interval Valued Fuzzy Set $A = \{< x, \bar{\rho}_A(x) >/x \in X\}$ in X is said to be Interval Valued Fuzzy β – ideal of X if

i. $\bar{\rho}_A(0) \geq \bar{\rho}_A(x)$
ii. $\bar{\rho}_A(x + y) \geq r\min\{\bar{\rho}_A(x), \bar{\rho}_A(y)\}$
iii. $\bar{\rho}_A(x) \geq \mathrm{rmin}\{\bar{\rho}_A(x - y), \bar{\rho}_A(y)\}\ \forall\, x, y \in X$.

2.19 Definition: An Intuitionistic Fuzzy Set $A = \{< x, \rho(x), \eta(x) >/x \in X\}$ in X is known as Intuitionistic Fuzzy β - ideal of X if

i. $\rho_A(0) \geq \rho_A(x)$; $\eta_A(0) \leq \eta_A(x)$
ii. $\rho_A(x + y) \geq \min\{\rho_A(x), \rho_A(y)\}$; $\eta_A(x + y) \leq \max\{\eta_A(x), \eta_A(y)\}$
iii. $\rho_A(x) \geq \min\{\rho_A(x - y), \rho_A(y)\}$; $\eta_A(x) \leq \max\{\eta_A(x - y), \eta_A(y)\}$

2.20 Definition: [19] Let X be a β – algebra and an MBJ Neutrosophic Set $A = \{\rho_A, \bar{\xi}_A, \eta_A\}$ is called an MBJ – Neutrosophic β – subalgebra of X if it satisfies

i. $\rho_A(x + y) \geq \min\{\rho_A(x), \rho_A(y)\}$; $\rho_A(x - y) \geq \min\{\rho_A(x), \rho_A(y)\}$
ii. $\bar{\xi}_A(x + y) \geq r\min\{\bar{\xi}_A(x), \bar{\xi}_A(y)\}$; $\bar{\xi}_A(x - y) \geq r\min\{\bar{\xi}_A(x), \bar{\xi}_A(y)\}$
iii. $\eta_A(x + y) \leq \max\{\eta_A(x), \eta_A(y)\}$; $\eta_A(x - y) \leq \max\{\eta_A(x), \eta_A(y)\}$

3 MBJ – Neutrosophic β – Ideal of β – Algebra

This part frames the structure of MBJ – Neutrosophic β – Ideal of β – Algebra and studied the related results.

3.1 Definition: Let $(X, +, -, 0)$ be a β – algebra. An MBJ – Neutrosophic Set $K = \{\rho_K, \bar{\xi}_K, \eta_K\}$ in X is called an MBJ – Neutrosophic β – Ideal of X if it satisfies the following conditions:

i. $\rho_K(0) \geq \rho_K(x)$
$\rho_K(x + y) \geq \min\{\rho_K(x), \rho_K(y)\}$
$\rho_K(x) \geq \min\{\rho_K(x - y), \rho_K(y)\}$

ii. $\bar{\xi}_K(0) \geq \bar{\xi}_K(x)$

$\bar{\xi}_K(x+y) \geq r\min\{\bar{\xi}_K(x), \bar{\xi}_K(y)\}$

$\bar{\xi}_K(x) \geq r\min\{\bar{\xi}_K(x-y), \bar{\xi}_K(y)\}$

iii. $\eta_K(0) \leq \eta_K(x)$

$\eta_K(x+y) \leq \max\{\eta_K(x), \eta_K(y)\}$

$\eta_K(x) \leq \max\{\eta_K(x-y), \eta_K(y)\} \quad \forall x, y \in X$

3.2 Example : A β – algebra X in example 2.6 defines a MBJ – Neutrosophic set as $\rho_A : X \to [0,1]$; $\bar{\xi}_A : X \to D[0,1]$ and $\eta_A : X \to [0,1]$ such that

$$\rho_A(x) = \begin{cases} 0.4, & x = 0 \\ 0.2, & x = 1,3 \\ 0.3, & x = 2 \end{cases}$$

$$\bar{\xi}_{K_A}(x) = \begin{cases} [0.3, 0.7] & x = 0 \\ [0.1, 0.5] & x = 1,3 \\ [0.2, 0.6] & x = 2 \end{cases}$$

$$\eta_A(x) = \begin{cases} 0.1, & x = 0 \\ 0.4, & x = 1,3 \\ 0.5, & x = 2 \end{cases}$$ is an MBJ – Neutrosophic β – Ideal of X.

3.3 Theorem: The intersection of any two MBJ – Neutrosophic β – Ideal of a β – algebra is also an MBJ – Neutrosophic β – Ideal.

Proof: Let $K_1 \& K_2$ be two MBJ – Neutrosophic β – Ideal of X.

Now, $(\rho_{K_1 \cap K_2})(0) \geq \min\{\rho_{K_1}(0), \rho_{K_2}(0)\}$

$= \min\{\rho_{K_1}(x), \rho_{K_2}(x)\}$

$= (\rho_{K_1 \cap K_2})(x)$

$(\rho_{K_1 \cap K_2})(x+y) \geq \min\{\rho_{K_1}(x+y), \rho_{K_2}(x+y)\}$

$= \min\{\min\{\rho_{K_1}(x), \rho_{K_1}(y)\}, \min\{\rho_{K_2}(x), \rho_{K_2}(y)\}\}$

$= \min\{\min\{\rho_{K_1}(x), \rho_{K_2}(x)\}, \min\{\rho_{K_1}(y), \rho_{K_2}(y)\}\}$

$= \min\{\rho_{K_1 \cap K_2}(x), \rho_{K_1 \cap K_2}(y)\}$

$\rho_{K_1 \cap K_2}(x) \geq \min\{\rho_{K_1}(x), \rho_{K_2}(x)\}$

$= \min\{\min\{\rho_{K_1}(x-y), \rho_{K_1}(y)\}, \min\{\rho_{K_2}(x-y), \rho_{K_2}(y)\}\}$

$= \min\{\min\{\rho_{K_1}(x-y), \rho_{K_2}(x-y)\}, \min\{\rho_{K_1}(y), \rho_{K_2}(y)\}\}$

$= \min\{\min\{\rho_{K_1 \cap K_2}(x-y), \rho_{K_1 \cap K_2}(y)\}\}$

$(\bar{\xi}_{K_1 \cap K_2})(0) \geq r\min\{\bar{\xi}_{K_1}(0), \bar{\xi}_{K_2}(0)\}$

$= r\min\{\bar{\xi}_{K_1}(x), \bar{\xi}_{K_2}(x)\}$

$= (\bar{\xi}_{K_1 \cap K_2})(x)$

$(\bar{\xi}_{K_1 \cap K_2})(x+y) \geq r\min\{\bar{\xi}_{K_1}(x+y), \bar{\xi}_{K_2}(x+y)\}$

$= \text{rmin}\{\text{rmin}\{\bar{\xi}_{K_1}(x), \bar{\xi}_{K_1}(y)\}, \text{rmin}\{\bar{\xi}_{K_2}(x), \bar{\xi}_{K_2}(y)\}\}$

$= \text{rmin}\{\text{rmin}\{\bar{\xi}_{K_1}(x), \bar{\xi}_{K_2}(x)\}, \text{rmin}\{\bar{\xi}_{K_1}(y), \bar{\xi}_{K_2}(y)\}\}$

$= \text{rmin}\{\bar{\xi}_{K_1 \cap K_2}(x), \bar{\xi}_{K_1 \cap K_2}(y)\}$

$\bar{\xi}_{K_1 \cap K_2}(x) \geq r\min\{\bar{\xi}_{K_1}(x), \bar{\xi}_{K_2}(x)\}$

$= \text{rmin } \{ \text{ rmin} \{ \bar{\xi}_{K_1}(x-y), \bar{\xi}_{K_1}(y) \}, \text{rmin} \{ \bar{\xi}_{K_2}(x-y), \bar{\xi}_{K_2}(y) \} \}$

$= \text{rmin } \{ \text{ rmin} \{ \bar{\xi}_{K_1}(x-y), \bar{\xi}_{K_2}(x-y) \}, \text{rmin} \{ \bar{\xi}_{K_1}(y), \bar{\xi}_{K_2}(y) \} \}$

$= \text{rmin } \{ \text{ rmin } \{ \bar{\xi}_{K_1 \cap K_2}(x-y), \bar{\xi}_{K_1 \cap K_2}(y) \} \}$

$(\eta_{K_1 \cap K_2})(0) \le \max\{ \eta_{K_1}(0), \eta_{K_2}(0) \}$

$= \max\{ \eta_{K_1}(x), \eta_{K_2}(x) \}$

$= (\eta_{K_1 \cap K_2})(x)$

$(\eta_{K_1 \cap K_2})(x+y) \le \max\{ \eta_{K_1}(x+y), \eta_{K_2}(x+y) \}$

$= \max \{ \max \{ \eta_{K_1}(x), \eta_{K_1}(y) \}, \max \{ \eta_{K_2}(x), \eta_{K_2}(y) \} \}$

$= \max \{ \max \{ \eta_{K_1}(x), \eta_{K_2}(x) \}, \max \{ \eta_{K_1}(y), \eta_{K_2}(y) \} \}$

$= \max \{ \eta_{K_1 \cap K_2}(x), \eta_{K_1 \cap K_2}(y) \}$

$\eta_{K_1 \cap K_2}(x) \le \max\{ \eta_{K_1}(x), \eta_{K_2}(x) \}$

$= \max \{ \max \{ \eta_{K_1}(x-y), \eta_{K_1}(y) \}, \max \{ \eta_{K_2}(x-y), \eta_{K_2}(y) \} \}$

$= \max \{ \max \{ \eta_{K_1}(x-y), \eta_{K_2}(x-y) \}, \max \{ \eta_{K_1}(y), \eta_{K_2}(y) \} \}$

$= \max \{ \max \{ \eta_{K_1 \cap K_2}(x-y), \eta_{K_1 \cap K_2}(y) \} \}$

Hence $K_1 \cap K_2$ is an MBJ – Neutrosophic β – Ideal of X.

3.4 Theorem: The intersection of any set of MBJ – Neutrosophic β – Ideal of a β – Algebra X is also an MBJ – Neutrosophic β – Ideal.

3.5 Theorem: Let $K = \{ \rho_K, \bar{\xi}_K, \eta_K \}$ be an MBJ – Neutrosophic β – Ideal. If $x \le y$ then $\rho_K(x) \ge \rho_K(y)$; $\bar{\xi}_K(x) \ge \bar{\xi}_K(y)$ and $\eta_K(x) \le \eta_K(y)$.

Proof: For any $x, y \in X$, $x \le y \Rightarrow x - y = 0$.

$\rho_K(x) \ge \min \{ \rho_K(x-y), \rho_K(y) \}$

$= \min \{ \rho_K(0), \rho_K(y) \}$

$= \rho_K(y)$

$\rho_K(x) \ge \rho_K(y)$

$\bar{\xi}_K(x) \ge \text{rmin } \{ \bar{\xi}_K(x-y), \bar{\xi}_K(y) \}$

$= \text{rmin } \{ \bar{\xi}_K(0), \bar{\xi}_K(y) \}$

$= \bar{\xi}_K(y)$

$\bar{\xi}_K(x) \ge \bar{\xi}_K(y)$

$\eta_K(x) \le \max \{ \eta_K(x-y), \eta_K(y) \}$

$= \max \{ \eta_K(0), \eta_K(y) \}$

$= \eta_K(y)$

$\eta_K(x) \le \eta_K(y)$.

3.6 Theorem: Let K be an MBJ – Neutrosophic β – Ideal of X whenever $x \le z + y$ then $\rho_K(x) \ge \min\{ \rho_K(z), \rho_K(y) \}$; $\bar{\xi}_K(x) \ge \text{rmin}\{ \bar{\xi}_K(z), \bar{\xi}_K(y) \}$ and $\eta_K(x) \le \max\{ \eta_K(z), \eta_K(y) \}$

Proof: For $x, y, z \in X$

$\rho_K(x) \ge \min \{ \rho_K(x-y), \rho_K(y) \}$

$= \min \{ \min \{ \rho_K((x-y)-z), \rho_K(z) \}, \rho_K(y) \}$

$= \min\{\min\{\rho_K(x-(z+y)), \rho_K(z)\}, \rho_K(y)\}$
$= \min\{\min\{\rho_K(0), \rho_K(z)\}, \rho_K(y)\}$
$\geq \min\{\rho_K(z), \rho_K(y)\}$

$\bar{\xi}_K(x) \geq \text{rmin}\{\bar{\xi}_K(x-y), \bar{\xi}_K(y)\}$
$= \text{rmin}\{\text{rmin}\{\bar{\xi}_K((x-y)-z), \bar{\xi}_K(z)\}, \bar{\xi}_K(y)\}$
$= \text{rmin}\{\text{rmin}\{\bar{\xi}_K(x-(z+y)), \bar{\xi}_K(z)\}, \bar{\xi}_K(y)\}$
$= \text{rmin}\{\text{rmin}\{\bar{\xi}_K(0), \bar{\xi}_K(z)\}, \bar{\xi}_K(y)\}$
$\geq r\min\{\bar{\xi}_K(z), \bar{\xi}_K(y)\}$

$\eta_K(x) \leq \max\{\eta_K(x-y), \eta_K(y)\}$
$= \max\{\max\{\eta_K((x-y)-z), \eta_K(z)\}, \eta_K(y)\}$
$= \max\{\max\{\eta_K(x-(z+y)), \eta_K(z)\}, \eta_K(y)\}$
$= \max\{\max\{\eta_K(0), \eta_K(z)\}, \eta_K(y)\}$
$\leq \max\{\eta_K(z), \eta_K(y)\}$

3.7 Theorem: Let $K = \{\rho_K, \bar{\xi}_K, \eta_K\}$ be an MBJ – Neutrosophic β – Ideal of X, then sets $X_{\rho_K} = \{x \in X : \rho_K(x) = \rho_K(0)\}$; $X_{\bar{\xi}_K} = \{x \in X : \bar{\xi}_K(x) = \bar{\xi}_K(0)\}$ and $X_{\eta_K} = \{x \in X : \eta_K(x) = \eta_K(0)\}$ are β – ideals of X.

Proof: Since $\rho_K(x) = \rho_K(0) \Rightarrow 0 \in X_{\rho_K}$

If $x - y, y \in X_{\rho_K}$

$\Rightarrow \rho_K(x-y) = \rho_K(0)$; $\rho_K(y) = \rho_K(0)$

Now, $\rho_K(x) \geq \min\{\rho_K(x-y), \rho_K(y)\}$
$= \min\{\rho_K(0), \rho_K(0)\}$
$= \rho_K(0)$

$\rho_K(x) \geq \rho_K(0)$

But $\rho_K(x) \leq \rho_K(0)$ implies $\rho_K(x) = \rho_K(0)$

$\Rightarrow x \in X_{\rho_K}$

$x - y, y \in X_{\rho_K} \Rightarrow x \in X_{\rho_K}$

$\therefore X_{\rho_K}$ is an β – Ideal of X

$\bar{\xi}_K(x) = \bar{\xi}_K(0) \Rightarrow 0 \in X_{\bar{\xi}_K}$

If $x - y, y \in X_{\bar{\xi}_K}$

$\Rightarrow \bar{\xi}_K(x-y) = \bar{\xi}_K(0)$; $\bar{\xi}_K(y) = \bar{\xi}_K(0)$

Now, $\bar{\xi}_K(x) \geq \text{rmin}\{\bar{\xi}_K(x-y), \bar{\xi}_K(y)\}$
$= \text{rmin}\{\bar{\xi}_K(0), \bar{\xi}_K(0)\}$
$= \bar{\xi}_K(0)$

$\bar{\xi}_K(x) \geq \bar{\xi}_K(0)$

But $\bar{\xi}_K(x) \leq \bar{\xi}_K(0)$ implies $\bar{\xi}_K(x) = \bar{\xi}_K(0)$

$\Rightarrow x \in X_{\bar{\xi}_K}$

$x - y, y \in X_{\bar{\xi}_K} \Rightarrow x \in X_{\bar{\xi}_K}$

$\therefore X_{\bar{\xi}_K}$ is an β – Ideal of X.

Similarly, X_{η_K} is also an β – Ideal of X.

3.8 Theorem: Suppose J is subset of X. An MBJ – Neutrosophic set $K = \{\rho_K, \bar{\xi}_K, \eta_K\}$ such that $\rho_K = \begin{cases} t, & x \in J \\ s, & x \notin J \end{cases}$; $\bar{\xi}_K = \begin{cases} \bar{t}, & x \in J \\ \bar{s} & x \notin J \end{cases}$ and $\eta_K = \begin{cases} \alpha, & x \in J \\ \beta, & x \notin J \end{cases}$ where $t, s, \alpha, \beta \in [0,1]$ and $\bar{t}, \bar{s} \in D[0,1]$ with $[t_0, t_1] \geq [s_0, s_1]$. Then the MBJ – Neutrosophic set $K = \{\rho_K, \bar{\xi}_K, \eta_K\}$ is an MBJ – Neutrosophic β – ideal of X if and only if J *is an* β – ideal of X.

Proof: Consider an MBJ – Neutrosophic set $K = \{\rho_K, \bar{\xi}_K, \eta_K\}$ is an MBJ - Neutrosophic β – ideal of X

i) $a)$ $\rho_K(0) \geq \rho_K(x) \quad \forall x \in X$

$\rho_K(0) = t \implies 0 \in J$

$b)$ For $x, y \in J$

$\implies \rho_K(x) = t = \rho_K(y)$

$\therefore \rho_K(x+y) \geq \min\{\rho_K(x), \rho_K(y)\}$

$= \min\{t, t\}$

$\rho_K(x+y) = t$

$\implies x + y \in J$

$c)$ For $x, y \in J$ if $x - y$ *and* $y \in J$

$\implies \rho_K(x-y) = t = \rho_K(y)$

$\therefore \rho_K(x) \geq \min\{\rho_K(x-y), \rho_K(y)\}$

$= \min\{t, t\} = t$

$\rho_K(x) = t$

$\implies x \in J$

ii) $a)$ $\bar{\xi}_K(0) \geq \bar{\xi}_K(x) \quad \forall x \in X$

$\bar{\xi}_K(0) = [t_0, t_1] \implies 0 \in J$

$b)$ For $x, y \in J$

$\implies \bar{\xi}_K(x) = [t_0, t_1] = \bar{\xi}_K(y)$

$\therefore \bar{\xi}_K(x+y) \geq \text{rmin}\{\bar{\xi}_K(x), \bar{\xi}_K(y)\}$

$= \text{rmin}\{[t_0, t_1], [t_0, t_1]\}$

$\bar{\xi}_K(x+y) = [t_0, t_1]$

$\implies x + y \in J$

$c)$ For $x, y \in J$ if $x - y$ *and* $y \in J$

$\implies \bar{\xi}_K(x-y) = [t_0, t_1] = \bar{\xi}_K(y)$

$\therefore \bar{\xi}_K(x) \geq \text{rmin}\{\bar{\xi}_K(x-y), \bar{\xi}_K(y)\}$

$= \text{rmin}\{[t_0, t_1], [t_0, t_1]\} = [t_0, t_1]$

$\bar{\xi}_K(x) = [t_0, t_1]$

$\implies x \in J$

iii) $a)$ $\eta_K(0) \leq \eta_K(x) \quad \forall x \in X$

$\eta_K(0) = \alpha \implies 0 \in J$

$b)$ For $x, y \in J$

$\implies \eta_K(x) = \alpha = \eta_K(y)$

$\therefore \eta_K(x+y) \leq \max\{\eta_K(x), \eta_K(y)\}$

$= \max\{\alpha, \alpha\}$

$\eta_K(x+y) = \alpha$

$\Rightarrow x + y \in J$

$c)$ For $x , y \in J$ if $x - y$ and $y \in J$

$\Rightarrow \eta_K(x - y) = \alpha = \eta_K(y)$

$\therefore \eta_K(x) \leq \max\{\eta_K(x - y) , \eta_K(y)\}$

$= \max\{ \alpha , \alpha \} = \alpha$

$\eta_K(x) = \alpha$

$\Rightarrow x \in J$

$\therefore$ J is an β – ideal of X

Conversely, assuming J is an β – ideal of X. Then

i) $a)$ If $0 \in J$

Implies $\rho_K(0) = t$

Also $\forall x \in X$, $Im(\rho_K) = [t , s]$ & $t > s$

$\Rightarrow \rho_K(0) \geq \rho_K(x)$ $\forall x \in X$

$b)$ For any $x , y \in J$

$\Rightarrow x + y \in J$

$\Rightarrow \rho_K(x) = \rho_K(x + y) = t = \rho_K(y)$

$= \min\{ \rho_K(x) , \rho_K(y)\}$

$\therefore \rho_K(x + y) \geq \min\{\rho_K(x) , \rho_K(y)\}$

c) For any $x , y \in J$

If $x - y$ and $y \in J \Rightarrow x \in J$

$\rho_K(x) = t = \min[t, t] = \min\{ \rho_K(x - y) , \rho_K(y) \}$

ii) $a)$ If $0 \in J$

Implies $\bar{\xi}_K(0) = \bar{t}$

Also $\forall x \in X$, $Im\left(\bar{\xi}_K\right) = [\bar{t} , \bar{s}]$ & $\bar{t} > \bar{s}$

$\Rightarrow \bar{\xi}_K(0) \geq \bar{\xi}_K(x)$ $\forall x \in X$

$b)$ For any $x , y \in J$

$\Rightarrow x + y \in J$

$\Rightarrow \bar{\xi}_K(x) = \bar{\xi}_K(x + y) = \bar{t} = \bar{\xi}_K(y)$

$= \text{rmin}\{ \bar{\xi}_K(x) , \bar{\xi}_K(y)\}$

$\therefore \bar{\xi}_K(x + y) \geq \text{rmin}\left\{\bar{\xi}_K(x) , \bar{\xi}_K(y)\right\}$

c) For any $x , y \in J$

If $x - y$ and $y \in J \Rightarrow x \in J$

$\bar{\xi}_K(x) = \bar{t} = \text{rmin}[\bar{t}, \bar{t}] = \text{rmin}\{ \bar{\xi}_K(x - y) , \bar{\xi}_K(y) \}$

iii) $a)$ If $0 \in J$

Implies $\eta_K(0) = \alpha$

Also $\forall x \in X$, $Im(\eta_K) = [\alpha , \beta]$& $\alpha < \beta$

$\Rightarrow \eta_K(0) \leq \eta_K(x)$ $\forall x \in X$

$b)$ For any $x , y \in J$

$\Rightarrow x + y \in J$

$\Rightarrow \eta_K(x) = \eta_K(x + y) = \alpha = \eta_K(y)$

$= \max\{ \eta_K(x) , \eta_K(y)\}$

$\therefore \eta_K(x + y) \leq \max\{\eta_K(x) , \eta_K(y)\}$

c) For any $x, y \in J$

If $x - y$ and $y \in J \Rightarrow x \in J$

$\eta_K(x) = \alpha = \max[\alpha, \alpha] = \max\{\eta_K(x-y), \eta_K(y)\}$

$\therefore K$ is an MBJ – β – Ideal of X.

3.9 Definition: Let $K = \{< x,\ \rho_K(x),\ \bar{\xi}_K(x),\ \eta_K(x) > / x \in X\}$ be an MBJ- Neutrosophic Set in X and $f : X \to Y$ be a mapping then the image of K under f, $f(K)$ is defined as $f(K) = \{< x,\ f_{sup}(\rho_K),\ f_{rsup}(\bar{\xi}_K),\ f_{inf}\eta_K(x) > / x \in Y\}$ where $f_{sup}(\rho_K)(y) = sup_{x \in f^{-1}(y)}\rho_K(x)$; $f_{rsup}(\bar{\xi}_K)(y) = rsup_{x \in f^{-1}(y)}\bar{\xi}_K(x)$ and $f_{inf}(\eta_K)(y) = inf_{x \in f^{-1}(y)}\eta_K(x)$

3.10 Definition: Let $f : X \to Y$ be a function and let K and L be two MBJ – Neutrosophic β – Ideal in $X \& Y$ respectively then the preimage of L under f is defined by $f^{-1}(L) = \{x,\ f^{-1}(\rho_K(x)),\ f^{-1}(\bar{\xi}_K(x)), f^{-1}(\eta_K(x)) > / x \in X\}$ such that

$f^{-1}(\rho_K(x)) = \rho_K(f(x))$; $f^{-1}\left(\bar{\xi}_K(x)\right) = \bar{\xi}_K(f(x))$ and $f^{-1}(\eta_K(x)) = \eta_K(f(x))$.

3.11 Theorem: Let $f : X \to Y$ be an onto homomorphism of β - algebra. Suppose K is an MBJ – Neutrosophic β – Ideal of Y, then the preimage of $f^{-1}(K)$ is an MBJ – Neutrosophic β – Ideal of X.

Proof: Suppose K be an MBJ - Neutrosophic β - ideal of Y

i) For $x \in X$

$$f^{-1}(\rho_K(0)) = \rho_K(f(0))$$
$$= \rho_K(0)$$
$$\geq \rho_K(x)$$

For some $x, y \in X$

$$f^{-1}(\rho_K)(x+y) = \rho_K(f(x+y))$$
$$= \rho_K(f(x)+f(y))$$
$$\geq min\{\rho_K(f(x)), \rho_K(f(y))\}$$
$$= \min\{f^{-1}(\rho_K(x)), f^{-1}(\rho_K(y))\}$$

$$f^{-1}(\rho_K)(x) = \rho_K(f(x))$$
$$\geq min\{\rho_K(f(x)-f(y)), \rho_K(f(y))\}$$
$$= min\{\rho_K(f(x-y)), \rho_K(f(y))\}$$
$$= min\{f^{-1}(\rho_K(x-y)), f^{-1}(\rho_K(y))\}$$

ii) $$f^{-1}\left(\bar{\xi}_K(0)\right) = \bar{\xi}_K(f(0))$$
$$= \bar{\xi}_K(0)$$
$$\geq \bar{\xi}_K(x)$$

For some $x, y \in X$

$$f^{-1}(\bar{\xi}_K)(x+y) = \bar{\xi}_K(f(x+y))$$
$$= \bar{\xi}_K(f(x)+f(y))$$
$$\geq rmin\{\bar{\xi}_K(f(x)), \bar{\xi}_K(f(y))\}$$
$$= \mathrm{rmin}\{f^{-1}\left(\bar{\xi}_K(x)\right), f^{-1}\left(\bar{\xi}_K(y)\right)\}$$

$f^{-1}(\bar{\xi}_K)(x) = \bar{\xi}_K\,(f(x))$

$\geq rmin\,\{\,\bar{\xi}_K\,(f(x) - f(y))\,, \bar{\xi}_K\,(f(y))\,\}$

$= rmin\,\{\,\bar{\xi}_K\,(f(x-y))\,, \bar{\xi}_K\,(f(y))\,\}$

$= rmin\,\{\,f^{-1}(\bar{\xi}_K(x-y))\,,\ f^{-1}(\bar{\xi}_K(y))\,\}$

iii) $f^{-1}(\eta_K(0)) = \eta_K\,(f(0))$

$= \eta_K\,(0)$

$\leq \eta_K\,(x)$

For some $x\,,y \in X$

$f^{-1}(\eta_K)(x+y) = \eta_K\,(f(x+y))$

$= \eta_K\,(f(x) + f(y))$

$\leq max\,\{\,\eta_K\,(f(x))\,, \eta_K\,(f(y))\,\}$

$= \max\{\,f^{-1}(\eta_K(x)), f^{-1}(\eta_K(y))\}$

$f^{-1}(\eta_K)(x) = \eta_K\,(f(x))$

$\leq max\,\{\,\eta_K\,(f(x) - f(y))\,,\ \eta_K\,(f(y))\,\}$

$= max\,\{\,\eta_K\,(f(x-y))\,, \eta_K\,(f(y))\,\}$

$= max\,\{\,f^{-1}(\eta_K(x-y))\,,\ f^{-1}(\eta_K(y))\,\}$

Hence $f^{-1}(K)$ is an MBJ – β – Ideal of X.

3.12 Theorem: Let $f : X \rightarrow X$ be an endomorphism on X. If K is an MBJ – Neutrosophic β – Ideal of X then $f(K) = \{< x,\ \rho_f(x) = \rho(f(x))\,,\ \bar{\xi}_f(x) = \bar{\xi}(f(x)),\ \eta_f(x) = \eta(f(x)) >/x \in X\,\}$ is an MBJ – Neutrosophic β – Ideal of X.

Proof: Suppose K be an MBJ – Neutrosophic β - ideal of X. Then,

i) $\rho_f(0) = \rho(f(0))$

$= \rho(0) \geq \rho(x)\ \forall\, x \in X$

$\rho_f(x+y) = \rho(f(x+y))$

$= \rho(f(x) + f(y))$

$= \min\,\{\ \rho(f(x)) + \rho(f(y))\,\}$

$= \min\,\{\,\rho_f(x)\,,\ \rho_f(y)\,\}\ \forall\, x\,,y \in X$

Also, $\rho_f(x) = \rho(f(x))$

$\geq \min\,\{\ \rho(f(x) - f(y))\,,\ \rho(f(y)))\,\}$

$= \min\,\{\ \rho(f(x-y))\,,\ \rho(f(y))\,\}$

$= \min\,\{\,\rho_f(x-y)\,,\ \rho_f(y)\,\}$

ii) $\bar{\xi}_f(0) = \bar{\xi}(f(0))$

$= \bar{\xi}(0) \geq \bar{\xi}(x)\ \forall\, x \in X$

$\bar{\xi}_f(x+y) = \bar{\xi}(f(x+y))$

$= \bar{\xi}(f(x) + f(y))$

$= \mathrm{rmin}\,\{\ \bar{\xi}\left(f(x)) + \bar{\xi}(f(y)\right)\,\}$

$= \mathrm{rmin}\,\{\,\bar{\xi}_f(x)\,,\ \bar{\xi}_f(y)\,\}\ \forall\, x\,,y \in X$

Also, $\bar{\xi}_f(x) = \bar{\xi}(f(x))$

$\geq \text{rmin} \{ \bar{\xi}\left(f(x) - f(y)\right), \bar{\xi}\left(f(y)\right) \}$

$= \text{rmin} \{ \bar{\xi}\left(f(x - y)\right), \bar{\xi}\left(f(y)\right) \}$

$= \text{rmin} \{ \bar{\xi}_f(x - y), \bar{\xi}_f(y) \}$

iii) $\eta_f(0) = \eta(f(0))$

$= \eta(0) \leq \eta(x) \ \forall x \in X$

$\eta_f(x + y) = \eta(f(x + y))$

$= \eta(f(x) + f(y))$

$= \max \{ \eta(f(x)) + \eta(f(y)) \}$

$= \max \{ \eta_f(x), \eta_f(y) \} \ \forall x, y \in X$

Also, $\eta_f(x) = \eta(f(x))$

$\leq \max \{ \eta(f(x) - f(y)), \eta(f(y))) \}$

$= \max \{ \eta(f(x - y)), \eta(f(y)) \}$

$= \max \{ \eta_f(x - y), \eta_f(y) \}$

$\therefore f(K)$ is an MBJ – β – Ideal of X.

3.13 Theorem: Let $f : X \to Y$ be a homomorphism of β – algebra. If K is an MBJ – Neutrosophic β – Ideal of X, with sup – rsup – inf property and $\ker(f) \subseteq X_K$ then the image of the set K, $f(K)$ is an MBJ – Neutrosophic β – ideal of Y.

Proof: Suppose K is an MBJ – Neutrosophic β – Ideal of X, with sup – rsup – inf property and $\ker(f) \subseteq X_K$ then

i) $f(\rho_K)(0) = sup_{x \in f^{-1}(0)}\{ \rho_K(x) \}$

$= \rho_K(0)$

$\geq \rho_K(x) \ \forall x \in X$

Hence, $f(\rho_K)(0) = sup_{x \in f^{-1}(0)}\{ \rho_K(x) \}$

$= f(\rho_K)(y) \ \forall y \in Y$

Let $y_1, y_2 \in Y$

Then there exists $x_1, x_2 \in X$ such that $f(x_1) = y_1, f(x_2) = y_2$.

$f(\rho_K)(y_1 + y_2) = \sup \{ \rho_K(x_1 + x_2) : x \in f^{-1}(y_1 + y_2) \}$

$\geq \sup \{ \rho_K(x_1 + x_2) : x_1 \in f^{-1}(y_1) \ \& \ x_2 \in f^{-1}(y_2) \}$

$\geq \sup\{\min\{ \rho_K(x_1), \rho_K(x_2)\}, x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}$

$\geq \min\{\sup\{ \rho_K(x_1) : x_1 \in f^{-1}(y_1)\}, \sup\{ \rho_K(x_2) : x_2 \in f^{-1}(y_2)\}\}$

$= \min\{ sup_{x_1 \in f^{-1}(y_1)}\{ \rho_K(x_1) \}, sup_{x_2 \in f^{-1}(y_2)}\{ \rho_K(x_2) \}$

$= \min\{ f(\rho_K)(y_1), f(\rho_K)(y_2)\}$

Suppose that for some $y_1, y_2 \in Y$

Then $f(\rho_K)(y_1) \leq \min\{ f(\rho_K)(y_1 - y_2), f(\rho_K)(y_2)\}$

Since f is onto $\exists \ x_1, x_2 \in X$ such that $f(x_1) = y_1 \ \& \ f(x_2) = y_2$

$f(\rho_K)\left(f(x_1)\right) < \min\{ f(\rho_K)\left(f(x_1) - f(x_2)\right), f(\rho_K)\left(f(x_2)\right) \}$

$= \min\{ f(\rho_K)\left(f(x_1 - x_2)\right), f(\rho_K)\left(f(x_2)\right) \}$

$< \min\{ f^{-1}(f(\rho_K)) (x_1 - x_2), f^{-1}(f(\rho_K))(x_2)\}$

$\rho_K(x_1) < \min\{ \rho_K(x_1 - x_2), \ \rho_K(x_2)\}$

ii) $f(\bar{\xi}_K)(0) = rsup_{x \in f^{-1}(0)}\{ \bar{\xi}_K(x) \}$

$= \bar{\xi}_K(0)$

$\geq \bar{\xi}_K(x) \ \forall \ x \in X$

Hence, $f(\bar{\xi}_K)(0) = rsup_{x \in f^{-1}(0)}\{ \bar{\xi}_K(x) \}$

$= f(\bar{\xi}_K)(y) \ \forall \ y \in Y$

Let $f(x_1) = y_1, f(x_2) = y_2$.

$f(\bar{\xi}_K)(y_1 + y_2) = r\sup \{ \bar{\xi}_K(x_1 + x_2) : x \in f^{-1}(y_1 + y_2) \}$

$\geq \text{rsup} \{ \bar{\xi}_K(x_1 + x_2) : x_1 \in f^{-1}(y_1) \ \& \ x_2 \in f^{-1}(y_2) \}$

$\geq \text{rsup}\{\text{rmin}\{ \bar{\xi}_K(x_1), \bar{\xi}_K(x_2)\}, x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}$

$\geq \text{rmin}\{ \text{rsup}\{ \bar{\xi}_K(x_1) : x_1 \in f^{-1}(y_1)\}, \text{rsup}\{ \bar{\xi}_K(x_2) : x_2 \in f^{-1}(y_2)\}\}$

$= \text{rmin}\{ rsup_{x_1 \in f^{-1}(y_1)}\{ \bar{\xi}_K(x_1) \}, rsup_{x_2 \in f^{-1}(y_2)}\{ \bar{\xi}_K(x_2) \} \}$

$= \text{rmin}\{ f(\bar{\xi}_K)(y_1), \ f(\bar{\xi}_K)(y_2)\}$

For $y_1, y_2 \in Y$

$f(\bar{\xi}_K)(y_1) \leq \text{rmin}\{ f(\bar{\xi}_K)(y_1 - y_2), \ f(\bar{\xi}_K)(y_2)\}$

$f(\bar{\xi}_K)(f(x_1)) < \text{rmin}\{ f(\bar{\xi}_K)(f(x_1) - f(x_2)), \ f(\bar{\xi}_K)(f(x_2)) \}$

$= \text{rmin}\{ f(\bar{\xi}_K)(f(x_1 - x_2)), \ f(\bar{\xi}_K)(f(x_2)) \}$

$< \text{rmin}\{ f^{-1}(f(\bar{\xi}_K)) (x_1 - x_2), f^{-1}(f(\bar{\xi}_K))(x_2)\}$

$\bar{\xi}_K(x_1) < \text{rmin}\{ \bar{\xi}_K(x_1 - x_2), \ \bar{\xi}_K(x_2) \}$

iii) $f(\eta_K)(0) = inf_{x \in f^{-1}(0)}\{ \eta_K(x) \}$

$= \eta_K(0)$

$\leq \eta(x) \ \forall \ x \in X$

Hence, $f(\eta_K)(0) = inf_{x \in f^{-1}(0)}\{ \eta_K(x) \}$

$= f(\eta_K)(y) \ \forall \ y \in Y$

Let $f(x_1) = y_1, f(x_2) = y_2$.

$f(\eta_K)(y_1 + y_2) = \inf \{ \eta_K(x_1 + x_2) : x \in f^{-1}(y_1 + y_2) \}$

$\leq \inf \{ \eta_K(x_1 + x_2) : x_1 \in f^{-1}(y_1) \ \& \ x_2 \in f^{-1}(y_2) \}$

$\leq \inf \{\max\{ \eta_K(x_1), \eta_K(x_2)\}, x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}$

$\leq \max\{ \inf \{ \eta_K(x_1) : x_1 \in f^{-1}(y_1)\}, \inf \{ \eta_K(x_2) : x_2 \in f^{-1}(y_2)\} \}$

$= \max\{ inf_{x_1 \in f^{-1}(y_1)}\{ \eta_K(x_1) \}, inf_{x_2 \in f^{-1}(y_2)}\{ \eta_K(x_2) \}$

$= \max\{ f(\eta_K)(y_1), \ f(\eta_K)(y_2)\}$

For $y_1, y_2 \in Y$

$f(\eta_K)(y_1) \leq \max\{ f(\eta_K)(y_1 - y_2), \ f(\eta_K)(y_2)\}$

$f(\eta_K)(f(x_1)) < \max\{ f(\eta_K)(f(x_1) - f(x_2)), \ f(\eta_K)(f(x_2)) \}$

$= \max\{ f(\eta_K)(f(x_1 - x_2)), \ f(\eta_K)(f(x_2)) \}$

$< \max\{ f^{-1}(f(\eta)) (x_1 - x_2), f^{-1}(f(\eta_K))(x_2)\}$

$\eta_K(x_1) < \max\{ \eta_K(x_1 - x_2), \ \eta_K(x_2)\}$

Thus, $f(K)$ is an MBJ – β – ideal of Y.

3.14 Theorem: Let $f : X \to Y$ be an onto homomorphism of β – algebra. If K is an MBJ – Neutrosophic β – ideal of X, with $\ker(f) \subseteq X_K$ then $f^{-1}\big(f(K)\big) = K$.

Proof: To prove $f^{-1}\big(f(K)\big) = K$.

It's necessary to prove

$$f^{-1}\big(f(\rho_K)\big)(x) = \rho_K(x)\ ;\ f^{-1}\left(f\big(\bar{\xi}_K\big)\right)(x) = \bar{\xi}_K(x) \text{ and } f^{-1}\big(f(\eta_K)\big)(x) = \eta_K(x).$$

For $x \in X ; f(x) = y$

i) Now, $f^{-1}\big(f(\rho_K)\big)(x) = f(\rho_K)(f(x))$

$= f(\rho_K)(y)$

$= sup_{x \in f^{-1}(y)}\{\ \rho_K(x)\ \}$

For $x' \in X, x' \in f^{-1}(y) \Rightarrow f(x') = y$

$f(x') = f(x)$

$\Rightarrow f(x') - f(x) = 0$

$f(x' - x) = 0$

This implies $x' - x \in Ker\ f$

$x' - x \in X_{\rho_K}$

$\rho_K(x' - x) = \rho_K(0)$

$\rho_K(x') \geq \min\{\rho_K(x' - x), \rho_K(x)\}$

$= \min\{\rho_K(0), \rho_K(x)\}$

$= \rho_K(x)$

$\rho_K(x') \geq \rho_K(x)$ and similarly, $\rho_K(x) \geq \rho_K(x')$

Therefore, $\rho_K(x') = \rho_K(x)$

$f^{-1}\big(f(\rho_K)\big)(x) = f(\rho_K)(f(x))$

$= f(\rho_K)(f(x'))$

$= sup_{x \in f^{-1}(y)}\{\ \rho_K(x')\ \}$

$= \rho_K(x)$

$f^{-1}\big(f(\rho_K)\big)(x) = \rho_K(x)$

ii) $f^{-1}\left(f\big(\bar{\xi}_K\big)\right)(x) = f\big(\bar{\xi}_K\big)(f(x))$

$= f\big(\bar{\xi}_K\big)(y)$

$= rsup_{x \in f^{-1}(y)}\{\ \bar{\xi}_K(x)\ \}$

For $x' \in X, x' \in f^{-1}(y) \Rightarrow f(x') = y$

$f(x') = f(x)$

$\Rightarrow f(x') - f(x) = 0$

$f(x' - x) = 0$

This implies $x' - x \in Ker\ f$

$x' - x \in X_{\bar{\xi}_K}$

$\bar{\xi}_K(x' - x) = \bar{\xi}_K(0)$

$\bar{\xi}_K(x') \geq \text{rmin}\{\bar{\xi}_K(x' - x), \bar{\xi}_K(x)\}$

$= \text{rmin}\{\bar{\xi}_K(0), \bar{\xi}_K(x)\}$

$= \bar{\xi}_K(x)$

$\bar{\xi}_K(x') \geq \bar{\xi}_K(x)$ and similarly, $\bar{\xi}_K(x) \geq \bar{\xi}_K(x')$

Therefore, $\bar{\xi}_K(x') = \bar{\xi}_K(x)$

$f^{-1}\left(f(\bar{\xi}_K)\right)(x) = f(\bar{\xi}_K)(f(x))$

$= f(\bar{\xi}_K)(f(x'))$

$= rsup_{x \in f^{-1}(y)}\{ \bar{\xi}_K(x') \}$

$= \bar{\xi}_K(x)$

$f^{-1}\left(f(\bar{\xi}_K)\right)(x) = \bar{\xi}_K(x)$

iii) Proceeding in the same way,

$f^{-1}(f(\eta_K))(x) = f(\eta_K)(f(x))$

$= f(\eta_K)(y)$

$= inf_{x \in f^{-1}(y)}\{ \eta_K(x) \}$

For $x' \in X, x' \in f^{-1}(y) \Rightarrow f(x') = y$

$f(x') = f(x)$

$\Rightarrow f(x') - f(x) = 0$

$f(x' - x) = 0$

This implies $x' - x \in Ker\, f$

$x' - x \in X_{\eta_K}$

$\eta_K(x' - x) = \eta_K(0)$

$\eta_K(x') \leq \max\{\eta_K(x' - x), \eta_K(x)\}$

$= \max\{\eta_K(0), \eta_K(x)\}$

$= \eta_K(x)$

$\eta_K(x') \geq \eta_K(x)$ and similarly, $\eta_K(x) \geq \eta_K(x')$

Therefore, $\eta_K(x') = \eta_K(x)$

$f^{-1}(f(\eta_K))(x) = f(\eta_K)(f(x))$

$= f(\eta_K)(f(x'))$

$= inf_{x \in f^{-1}(y)}\{ \eta_K(x') \}$

$= \eta_K(x)$

$f^{-1}(f(\eta_K))(x) = \eta_K(x)$

Therefore, all these conditions are proved and hence $f^{-1}(f(K)) = K$.

4 Cartesian Product of MBJ – Neutrosophic β – Ideal

This section introduces the cartesian product of MBJ – Neutrosophic β – ideal and discusses few associated results.

4.1.Definition: Let $K = \{ < x, \rho_K(x), \bar{\xi}_K(x), \eta_K(x) > / x \in X \}$ and $L = \{ < y, \rho_K(y), \bar{\xi}_K(y), \eta_K(y) > / y \in Y \}$ be two MBJ – Neutrosophic sets $X \,\&\, Y$ respectively. The Cartesian product of K and L is denoted by $K \times L$ is defined as $K \times L = \{ < (x, y), \rho_{K\times L}(x,y), \bar{\xi}_{K\times L}(x,y), \eta_{K\times L}(x,y) > / (x,y) \in X \times Y\}$ where $\rho_{K\times L} : X \times Y \to [0,1]$; $\bar{\xi}_{K\times L} : X \times Y \to D[0,1]$ and $\eta_{K\times L} : X \times Y \to [0,1]$. $\rho_{K\times L}(x,y) =$

$\min\{\rho_K(x), \rho_L(y)\}$; $\bar{\xi}_{K\times L}(x,y) = \text{rmin}\{\bar{\xi}_K(x), \bar{\xi}_L(y)\}$ and $\eta_{K\times L}(x,y) = \max\{\eta_K(x), \eta_L(y)\}$

4.2 Theorem: If K and L be two MBJ – Neutrosophic β – Ideal of X & Y respectively then $K \times L$ is an MBJ – Neutrosophic β – Ideal of $X \times Y$.

Proof: Let $K = \{< x, \rho_K(x), \bar{\xi}_K(x), \eta_K(x) >/x \in X\}$ and $L = \{< y, \rho_K(y), \bar{\xi}_K(y), \eta_K(y) >/y \in Y\}$ be two MBJ – Neutrosophic sets X & Y.

Take $(x,y) \in X \times Y$

i) $\rho_{K\times L}(0,0) = \min\{\rho_K(0,0), \rho_L(0,0)\}$

$$\geq \min\{\min\{\rho_K(0), \rho_K(0)\}, \min\{\rho_L(0), \rho_L(0)\}\}$$
$$= \min\{\min\{\rho_K(x), \rho_K(y)\}, \min\{\rho_L(x), \rho_L(y)\}\}$$
$$= \min\{\min\{\rho_K(x), \rho_L(x)\}, \min\{\rho_K(y), \rho_L(y)\}\}$$
$$= \min\{\rho_{K\times L}(x), \rho_{K\times L}(y)\}$$
$$\geq \rho_{K\times L}(x,y)$$

Take $(u,v)) \in X \times Y$ where $u = (x_1, y_1), v = (x_2, y_2)$

$$\rho_{K\times L}(u+v) = \rho_{K\times L}((x_1, y_1) + (x_2, y_2))$$
$$= \rho_{K\times L}((x_1 + x_2), (y_1 + y_2))$$
$$= \min\{\rho_K(x_1 + x_2), \rho_L(y_1 + y_2)\}$$
$$\geq \min\{\min\{\rho_K(x_1), \rho_K(x_2)\}, min\{\rho_L(y_1), \rho_L(y_2)\}\}$$
$$= \min\{\min\{\rho_K(x_1), \rho_L(y_1)\}, min\{(\rho_K(x_2), \rho_L(y_2)\}\}$$
$$= \min\{\rho_{K\times L}(x_1, y_1), \rho_{K\times L}((x_2, y_2))\}$$
$$\geq \min\{\rho_{K\times L}(u), \rho_{K\times L}(v)\}$$

$$\rho_{K\times L}(u) = \rho_{K\times L}(x_1, y_1)$$
$$= \min\{\rho_{K\times L}(x_1), \rho_{K\times L}(y_1)\}$$
$$\geq \min\{\min\{\rho_K(x_1 - x_2), \rho_K(x_2)\}, min\{\rho_L(y_1 - y_2), \rho_L(y_2)\}\}$$
$$= \min\{\min\{\rho_K(x_1 - x_2), \rho_L(y_1 - y_2))\}, min\{\rho_L(x_2), \rho_L(y_2)\}\}$$
$$= \min\{\rho_{K\times L}((x_1, y_1) - (x_2, y_2)), \rho_{K\times L}(x_2, y_2)\}$$
$$\geq \min\{\rho_{K\times L}(u-v), \rho_{K\times L}(v)\}$$

ii) $\bar{\xi}_{K\times L}(0,0) = \text{rmin}\{\bar{\xi}_K(0,0), \bar{\xi}_L(0,0)\}$

$$\geq r\text{min}\{\text{rmin}\{\bar{\xi}_K(0), \bar{\xi}_K(0)\}, \text{rmin}\{\bar{\xi}_L(0), \bar{\xi}_L(0)\}\}$$
$$= r\text{min}\{\text{rmin}\{\bar{\xi}_K(x), \bar{\xi}_K(y)\}, \text{rmin}\{\bar{\xi}_L(x), \bar{\xi}_L(y)\}\}$$
$$= r\text{min}\{\text{rmin}\{\bar{\xi}_K(x), \bar{\xi}_L(x)\}, \text{rmin}\{\bar{\xi}_K(y), \bar{\xi}_L(y)\}\}$$
$$= \text{rmin}\{\bar{\xi}_{K\times L}(x), \bar{\xi}_{K\times L}(y)\}$$
$$\geq \bar{\xi}_{K\times L}(x,y)$$

$$\bar{\xi}_{K\times L}(u+v) = \bar{\xi}_{K\times L}((x_1, y_1) + (x_2, y_2))$$
$$= \bar{\xi}_{K\times L}((x_1 + x_2), (y_1 + y_2))$$
$$= r\text{min}\{\bar{\xi}_K(x_1 + x_2), \bar{\xi}_L(y_1 + y_2)\}$$
$$\geq r\text{min}\{\text{rmin}\{\bar{\xi}_K(x_1), \bar{\xi}_K(x_2)\}, rmin\{\bar{\xi}_L(y_1), \bar{\xi}_L(y_2)\}\}$$
$$= r\text{min}\{r\min\{\bar{\xi}_K(x_1), \bar{\xi}_L(y_1)\}, rmin\{(\bar{\xi}_K(x_2), \bar{\xi}_L(y_2)\}\}$$
$$= \text{rmin}\{\bar{\xi}_{K\times L}(x_1, y_1), \bar{\xi}_{K\times L}((x_2, y_2))\}$$
$$\geq \text{rmin}\{\bar{\xi}_{K\times L}(u), \bar{\xi}_{K\times L}(v)\}$$

$$\bar{\xi}_{K\times L}(u) = \bar{\xi}_{K\times L}(x_1, y_1)$$

$= \text{rmin}\{ \xi_{K\times L}(x_1\ ), \xi_{K\times L}(y_1)\}$
$\geq r\text{min}\{\text{rmin}\{ \xi_K(x_1 - x_2), \xi_K(x_2)\}, rmin\{\xi_L(y_1 - y_2), \xi_L(y_2)\}\}$
$= r\text{min}\{\text{rmin}\{ \xi_K(x_1 - x_2), \xi_L(y_1 - y_2)\}\}, rmin\{\xi_L(x_2), \xi_L(y_2)\}\}$
$= \text{rmin}\{ \xi_{K\times L}((x_1, y_1) - (x_2, y_2)), \xi_{K\times L}(x_2, y_2)\}$
$\geq \text{rmin}\{ \xi_{K\times L}(u - v), \xi_{K\times L}(v)\}$

i) $\eta_{K\times L}(0,0) = \max\{ \eta_K(0,0), \eta_L(0,0)\}$
$\leq \max\{\max\{\eta_K(0), \eta_K(0)\}, \max\{\eta_L(0), \eta_L(0)\}\}$
$= \max\{\max\{\eta_K(x), \eta_K(y)\}, \max\{\eta_L(x), \eta_L(y)\}\}$
$= \max\{\max\{\eta_K(x), \eta_L(x)\}, \max\{\eta_K(y), \eta_L(y)\}\}$
$= \max\{ \eta_{K\times L}(x), \eta_{K\times L}(y)\}$
$\leq \eta_{K\times L}(x, y)$

$\eta_{K\times L}(u + v) = \eta_{K\times L}((x_1, y_1) + (x_2, y_2))$
$= \eta_{K\times L}((x_1 + x_2), (y_1 + y_2))$
$= \max\{ \eta_K(x_1 + x_2), \eta_L(y_1 + y_2)\}$
$\leq \max\{\max\{ \eta_K(x_1), \eta_K(x_2)\}, max\{\eta_L(y_1), \eta_L(y_2)\}\}$
$= \max\{\max\{ \eta_K(x_1), \eta_L(y_1)\}, max\{(\eta_K(x_2), \eta_L(y_2)\}\}$
$= \max\{ \eta_{K\times L}(x_1, y_1), \eta_{K\times L}((x_2, y_2))\}$
$\leq \max\{ \eta_{K\times L}(u), \eta_{K\times L}(v)\}$

$\eta_{K\times L}(u) = \eta_{K\times L}(x_1, y_1)$
$= \max\{ \eta_{K\times L}(x_1\ ), \eta_{K\times L}(y_1)\}$
$\leq \max\{\max\{ \eta_K(x_1 - x_2), \eta_K(x_2)\}, max\{\eta_L(y_1 - y_2), \eta_L(y_2)\}\}$
$= \max\{\max\{ \eta_K(x_1 - x_2), \eta_L(y_1 - y_2)\}\}, max\{\eta_L(x_2), \eta_L(y_2)\}\}$
$= \max\{ \eta_{K\times L}((x_1, y_1) - (x_2, y_2)), \eta_{K\times L}(x_2, y_2)\}$
$\leq \max\{ \eta_{K\times L}(u - v), \eta_{K\times L}(v)\}$

Hence $K \times L$ is an MBJ – Neutrosophic β – Ideal of $X \times Y$.

4.3 Theorem: If $K_1, K_2, \ldots\ldots K_n$ be an MBJ – Neutrosophic β – Ideals of $X_1, X_2, \ldots X_n$ respectively, then $\prod_{i=1}^{n} K_i$ is also a MBJ – Neutrosophic β – Ideal of $\prod_{i=1}^{n} X_i$.

Proof: By induction on Theorem 4.2,

i) $\prod_{i=1}^{n} \rho_{K_i}(0) \geq \prod_{i=1}^{n} \rho_{K_i}(x_i)$
$\prod_{i=1}^{n} \rho_{K_i}(x_i + y_i) \geq \min\{ \prod_{i=1}^{n} \rho_{K_i}(x_i), \prod_{i=1}^{n} \rho_{K_i}(y_i) \}$
$\prod_{i=1}^{n} \rho_{K_i}(x_i) \geq \min\{ \prod_{i=1}^{n} \rho_{K_i}(x_i - y_i), \prod_{i=1}^{n} \rho_{K_i}(y_i) \}$

ii) $\prod_{i=1}^{n} \bar{\xi}_{K_i}(0) \geq \prod_{i=1}^{n} \bar{\xi}_{K_i}(x_i)$
$\prod_{i=1}^{n} \bar{\xi}_{K_i}(x_i + y_i) \geq r\text{min}\{ \prod_{i=1}^{n} \bar{\xi}_{K_i}(x_i), \prod_{i=1}^{n} \bar{\xi}_{K_i}(y_i) \}$
$\prod_{i=1}^{n} \bar{\xi}_{K_i}(x_i) \geq r\text{min}\{ \prod_{i=1}^{n} \bar{\xi}_{K_i}(x_i - y_i), \prod_{i=1}^{n} \bar{\xi}_{K_i}(y_i) \}$

iii) $\prod_{i=1}^{n} \eta_{K_i}(0) \leq \prod_{i=1}^{n} \eta_{K_i}(x_i)$
$\prod_{i=1}^{n} \eta_{K_i}(x_i + y_i) \leq \max\{ \prod_{i=1}^{n} \eta_{K_i}(x_i), \prod_{i=1}^{n} \eta_{K_i}(y_i) \}$
$\prod_{i=1}^{n} \eta_{K_i}(x_i) \leq \max\{ \prod_{i=1}^{n} \eta_{K_i}(x_i - y_i), \prod_{i=1}^{n} \eta_{K_i}(y_i) \}$

Hence the proof is clear.

4.4 Theorem: For the MBJ – Neutrosophic subsets $K \; and \; L$ of $X \,\&\, Y$, if $K \times L$ is an MBJ – Neutrosophic β – ideal of $X \times Y$ then

i) $\rho_K(0) \geq \rho_L(y) \,\&\, \rho_L(0) \geq \rho_K(x)$

ii) $\bar{\xi}_K(0) \geq \bar{\xi}_L(y) \,\&\, \bar{\xi}_L(0) \geq \bar{\xi}_K(x)$

iii) $\eta_K(0) \leq \eta_L(y) \,\&\, \eta_L(0) \leq \eta_K(x)$

Proof: Let $K \,\&\, L$ be MBJ – Neutrosophic subsets of $X \,\&\, Y$ with $K \times L$ is an MBJ – Neutrosophic β – ideal of $X \times Y$.

Suppose $\rho_L(y) \geq \rho_K(0)$ and $\rho_K(x) \geq \rho_L(0)$ for some $x \in X, y \in Y$.

$$\begin{aligned}\rho_{K\times L}(x,y) &= \min\{\rho_K(x), \rho_L(y)\}\\ &\geq \min\{\rho_L(0), \rho_K(0)\}\\ &= \rho_{K\times L}(0,0)\end{aligned}$$

which is a contradiction.

Thus, $\rho_K(0) \geq \rho_L(y) \,\&\, \rho_L(0) \geq \rho_K(x)$

Similarly, $\bar{\xi}_L(y) \geq \bar{\xi}_K(0)$ and $\bar{\xi}_K(x) \geq \bar{\xi}_L(0)$ for some $x \in X, y \in Y$.

$$\begin{aligned}\bar{\xi}_{K\times L}(x,y) &= r\min\{\bar{\xi}_K(x), \bar{\xi}_L(y)\}\\ &\geq \mathrm{rmin}\{\bar{\xi}_L(0), \bar{\xi}_K(0)\}\\ &= \bar{\xi}_{K\times L}(0,0)\end{aligned}$$

Now, $\eta_L(y) \leq \eta_K(0)$ and $\eta_K(x) \leq \eta_L(0)$

$$\begin{aligned}\eta_{K\times L}(x,y) &= \max\{\eta_K(x), \eta_L(y)\}\\ &\leq \max\{\eta_L(0), \eta_K(0)\}\\ &= \eta_{K\times L}(0,0)\end{aligned}$$

Hence the condition is satisfied.

4.5 Theorem: Let $K \,\&\, L$ be two MBJ – Neutrosophic β – ideals of $X \,\&\, Y$ such that $K \times L$ is an MBJ – Neutrosophic β – ideals of $X \times Y$. Then, either K is an MBJ - β – ideals of X or L is an MBJ – Neutrosophic β – ideals of Y.

Proof: By using the above theorem

i) We consider $\rho_K(0) \geq \rho_L(y)$ then

$\rho_{K\times L}(0,y) \geq \min\{\rho_K(0), \rho_L(y)\}$ …… (1)

Given $K \times L$ is an MBJ – Neutrosophic β – ideals of $X \times Y$

$\rho_{K\times L}\big((x_1,y_1),(x_2,y_2)\big) \geq \min\{\rho_{K\times L}((x_1,y_1)-(x_2,y_2)), \rho_{K\times L}(x_2,y_2)\}$

$\because \; \rho_{K\times L}((x_1,y_1)-(x_2,y_2)) \geq \min\{\rho_{K\times L}(x_1,y_1), \rho_{K\times L}(x_2,y_2)\}$

$\rho_{K\times L}(x_1,y_1) \geq \min\{\rho_{K\times L}((x_1-x_2),(y_1-y_2)), \rho_{K\times L}(x_2,y_2)\}$ ……. (2)

Now,

$\rho_{K\times L}((x_1-x_2),(y_1-y_2)) \geq \min\{\rho_{K\times L}(x_1,y_1), \rho_{K\times L}(x_2,y_2)\}$ ……(3)

Put $x_1 = x_2 = 0$ in Equation (2 & 3)

$\rho_{K\times L}(0,y_1) \geq \min\{\rho_{K\times L}(0,(y_1-y_2)), \rho_{K\times L}(0,y_2)\}$ and

$\rho_{K\times L}\big(0,(y_1-y_2)\big) \geq \min\{\rho_{K\times L}(0,y_1), \rho_{K\times L}(0,y_2)\}$ ……..(4)

From (1) & (4)

$\rho_L(y_1) \geq \min\{\rho_L(y_1-y_2), \rho_L(y_2)\}$ and

$\rho_L(y_1-y_2) \geq \min\{\rho_L(y_1), \rho_L(y_2)\}$

ii) Consider $\bar{\xi}_K(0) \geq \bar{\xi}_L(y)$. Then

$\bar{\xi}_{K\times L}(0, y) \geq r\min\{\bar{\xi}_K(0), \bar{\xi}_L(y)\}$ …… (5)

$\bar{\xi}_{K\times L}\big((x_1, y_1), (x_2, y_2)\big) \geq r\min\{\bar{\xi}_{K\times L}((x_1, y_1) - (x_2, y_2)), \bar{\xi}_{K\times L}(x_2, y_2)\}$

$\because \bar{\xi}_{K\times L}((x_1, y_1) - (x_2, y_2)) \geq \text{rmin}\ \{ \bar{\xi}_{K\times L}(x_1, y_1), \bar{\xi}_{K\times L}(x_2, y_2) \}$

$\bar{\xi}_{K\times L}(x_1, y_1) \geq \text{rmin}\{ \bar{\xi}_{K\times L}((x_1 - x_2), (y_1 - y_2)), \bar{\xi}_{K\times L}(x_2, y_2) \}$ …….(6)

Now,

$\bar{\xi}_{K\times L}((x_1 - x_2), (y_1 - y_2)) \geq \text{rmin}\{ \bar{\xi}_{K\times L}(x_1, y_1), \bar{\xi}_{K\times L}(x_2, y_2) \}$ …(7)

Put $x_1 = x_2 = 0$ in Equation (6 & 7)

$\bar{\xi}_{K\times L}(0, y_1) \geq \text{rmin}\{ \bar{\xi}_{K\times L}(0, (y_1 - y_2)), \bar{\xi}_{K\times L}(0, y_2) \}$ and

$\bar{\xi}_{K\times L}\big(0, (y_1 - y_2)\big) \geq \text{rmin}\{ \bar{\xi}_{K\times L}(0, y_1), \bar{\xi}_{K\times L}(0, y_2)\}$ ……..(8)

From (5) & (7)

$\bar{\xi}_L(y_1) \geq \text{rmin}\{ \bar{\xi}_L(y_1 - y_2), \bar{\xi}_L(y_2) \}$ and

$\bar{\xi}_L(y_1 - y_2) \geq \text{rmin}\{ \bar{\xi}_L(y_1), \bar{\xi}_L(y_2)\}$

iii) As in the same way if we proceed, we get

$\eta_L(y_1) \leq \max\{ \eta_L(y_1 - y_2), \eta_L(y_2) \}$ and

$\eta_L(y_1 - y_2) \leq \max\{ \eta_L(y_1), \eta_L(y_2)\}$

$\therefore$ B is an MBJ - β – ideals of Y.

5. Conclusion

This paper presents the characterization of MBJ – Neutrosophic β – Ideal of β – algebra. In depth, the study analysed the homomorphic image, pre – image, cartesian product and related results. The concept can be explored to other substructures of a β – algebra.

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Received: Apr 16, 2020. Accepted: July 6, 2020

A Study of Social Media linked MCGDM Skill under Pentagonal Neutrosophic Environment in the Banking Industry

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ABSTRACT:

Social media is a new observable fact in computer-based technology and neutrosophic theory. Researchers are now thinking of the power of social media in banks as it is the fastest expanding online noticeable fact and banks with poor presence in social media are facing identity crisis under uncertainty fields. Through social media we can share ideas and information through establishing virtual networks. Initially it was evident that people used it for personal interaction with friends and relatives but with changing time it is established that business houses and financial institutions including Banks are using this popular technology to reach out to the prospective customers. Especially in the banking industry digital communication is becoming most popular and powerful as here consumers' interface is obligatory. Online communication has become a powerful medium between banks and consumers. The power of social media is to connect and share information with people across globe. Social Media in Indian Bank is not only a medium of advertising but it also helps the Banks to be a part of their customers' life as this relation involves conversation beyond business under neutrosophic environment. The aim of this study is to find out the best social media as per users' preference and explore its impact on Banks' business in pentagonal neutrosophic (PNN) arena by increasing customer satisfaction and augment customer relationship management in banking industry.

Keywords: Social Media, Customer Relationship Management, Customer satisfaction, Banking Industry, PNN.

1. INTRODUCTION

1.1 Social Media:

The traditional marketing media consisting of radio, print, television etc offered a shotgun approach as they represent communication in One to Many modes which we may call as Passive Approach. However Social media marketing is following Many to Many mode, may be called Active Approach with the power of implementation of Word Of Mouth. They are interactive in nature and believe in peer to peer relationship [1] (Githa Heggde, 2018)

The substantial and considerable use of social media for last few years has elucidated that it is amongst a few powerful weapons that has shown tremendous impacts on social life of human beings and has hastened the mingling of people with each other. Previously, it was an encumbrance for us to keep ourselves in touch with all those who were a little distant from us. Things have apparently changed and social networking sites can take every credit for this prodigious platform which enabled people to create their own identity. Whether it is about uploading personal posts, surfing across the globe, getting all the indispensable information or even if one wants to express their cavernous feelings then social media can act as a gullible platform for everyone. At times a few of our problematic situations, disturbing sentiments need to get some succor and support by our loved ones. At times only a single post of ours explains everything about what we are actually feeling. Social media and its comprehensive enhancement is undeniable reality in this modern era. Verily speaking social networking sites has made our socialization a bit easier with the rest of the world. Data and statistics distinctly show the massive use of social media. Social Media has grown tremendously due to increase in penetration of Internet Connectivity and easy availability of smart phones and mobile gadgets. The conventional use of social media has changed from mere entertainment to opportunities for trade and commerce. An estimate confirms that nearly two third of Internet users are active on social media as well and this number is expected to cross approx three billion by the end of 2020.

1.2 Social Media Platforms:

Social Media is a blur of likes, tweets, shares, posts and contents. It has spread its wings in every corner of the world. The numbers are staggering. 70% of the total internet users are now using social media as per the research [2] (Bullas, 2014). In a research by Pew Research, 2014 [3] (DUGGAN, 2014) , it is established that globally people are getting addicted to social media regardless of age, gender and profession.

There are a variety of technological driven services in social media like sharing of pictures, videos and audios, blogging, social games, social networking, business networks, reviews and much more. Social media consists of a variety of internet-based mediums that enable users to network, share content, interact with each other, and create communities around common interests. Social media is therefore the media that we use to be sociable online and it can be divided into three main categories:

- Messaging and communication, e,g. blogging and micro-blogging such as Twitter.
- Communities and social groups, e.g. Facebook
- Photo and video sharing, e.g. YouTube

Statistically speaking, number of people using social media has considerably increased. The number of people across globe who uses social media has extended 3 billion. As per a report Face book reported 1.871, Whatsapp a billion and Instagram 600 million active users in January 2017 due to the intensified use of social media.

1.3 Social Media in Bank:

The bank with no social media marketing strategy is at a risk of being left behind its competitors as social media is playing a big role in marketing field. The tremendous growth and popularity of this medium is forcing banks to learn different social media platforms available to them and their customers, different strategies to be adapted for proper selection of right social media channel so as to reach out to maximum customers and improve their business. [4] (L, 2010) Banks are opting social media channels due to the following main points:

- To increase engagement with customers
- To enhance their brand image by connecting with customers
- To find out ways to distinguish themselves with competitors
- To reduce cost as implementation of social media channels are less expensive in comparison to the traditional marketing methods and with higher results
- To boost innovations as through proper market research through social media more customized products/services can be incorporated
- To increase revenue as satisfies customers result in more business which in turn brings revenue

The advancement faced by the banking sector today in the field of digitalization is an amalgamation of social media and the wise users of this powerful tool which helps innumerable people in their everyday work. With the help of digital feed people can access different social media sites like Face book, Twitter, YouTube, Instagram etc to expand required knowledge about different products and services offered by banks.

Many people opined that the new generations with proper knowledge of digital technologies are more prone to use of social media but our response rate of seniors above 50 years was good. It was observed that this number is gradually on the rise. Customers are an integral part of Banking Industry and social media is an easiest and fastest way to reach to existing and prospective customers. All the leading banks worldwide are trying to create business opportunities through enhancing their creativity and innovative capacities. Through social media Banks can inform their customers about their product & services offered in a most unique, attractive and innovative way. It also helps the customers to consider sensibly about their investments and eradicate all the complexity involved with the traditional banking processes. Traditional banks focused on providing services to customers through different strategies such as advertising, direct mail or face to face whereas banks and other financial institutions' is focusing on establishing relations with customers through continue digital interaction vide different social media channels. By this continuous interaction through social media Banks can discover customers' interests, feelings and behavior. Customers of today look ahead to personalized services and they need to be heard and answered promptly. Banks may fulfill their expectations through different digital media platforms like face book, twitter and you tube instead of face to face interactions between customers and managers. Bank's Monitoring centers may follow comments, posts and tweets on social Medias which can give a broad standpoint of customer insight about products and services banks can achieve an accurate perceptive of customers. Today Banks need to have an effective presence in social media due to the customers' anticipation and their obsession for the same.

Now a day social media has become crucial tool for banks. Banks are using the platform to discover and keep up the relation with customers, motivating sales through advertisement and sales endorsement, guess change in consumer behavior and follow their trends and finally providing customized services and support. Social media also helps in building customer relationships through its reliability programs. It is an emerging concept in marketing especially in relation to Banking Industry. Banks have now realized the influence of this medium over traditional form of marketing strategy as it is the fastest growing online trend. Its influence has increased to the power that the Banks with no social media connections are at a risk of being left out from competitors.

1.4 Survey of Uncertainty & Neutrosophic Theory:

In this current epoch, vagueness theory plays a vital position in social sciences and management fields. Initially, it was discovered by prof. L.A Zadeh [5] & further, advancement of triangular [6], trapezoidal [7], pentagonal [8], hexagonal [9], heptagonal [10] fuzzy number are established by distinct researchers. It was extended by Prof. Attasonov [11] incorporating the idea of intuitionistic fuzzy & further by Prof. Smarandache [12] discovering the concept of neutrosophic set. Nowadays, researchers from distinct area are specializing in neutrosophic idea and advanced lots of exciting articles in this domain. Recently, categorization of triangular [13], trapezoidal [14], pentagonal [15-18] neutrosophic numbers has been developed by Chakraborty et.al. Recently, some MCGDM based articles [19-23] are established in this neutrosophic arena which plays an essential impact in this research domain.

2. Literature Review and Preliminaries:

This study focuses on identifying the services provided by banks through social media and measuring its effect on customer satisfaction. The study also tries to find out the ways through which Bank support the customers with the help of social media and the problems faced by the customers to approach banks through social media.

Different customer services that can be provided by banks are;

- Sharing of financial offers and upcoming promotions
- Posting of education information and financial guidance
- Allow clients to post reviews, complaints and suggestions
- Reward them for recommending them

These virtual services are giving same level of personal interaction which was normally found in physical banking as well but the advantage is that clients need not physically visit the banks. The bank provides different services like Corporate banking, Investment banking, Asset Management, Treasury services, Retail Banking etc. With the growth of information technology and advent of Internet now banks are also using online banking. Internet banking is a convenient virtual banking activity that is available for all the customers of the banks with easy and secured access to their accounts. [2] Justified that now a day's social media is being regularly accessed by almost 72% of the internet users. Social Media helps the customers in providing utmost customer satisfaction through obtaining real time comments, suggestions, complaints and addressing them instantly.

2.1 Safety & Reliability as Social Media Attribute:

Users' Safety & Reliability is an important tool in consideration of social media channels. Data should be handled without breaching the users' privacy and data protection should be enormously scrutinized. The most grounded measure that needs to be taken is to make undaunted quality of one's privacy whoever has affiliated with the social media channel [24] (Senthil Kumar N*, 2016). Many a time's users' share their personal data intentionally and sometimes unknowingly. Often data are extracted from them extrinsically by offering them some payback , for e.g, Location-Based Social Network Services (LBSNS) like Google Latitude can trace the location of a person and his/her friends [25] (Paul Lowry, 2011).

According to the safety analytics viewpoint, many people supervise the benefits and threats associated while unveiling their credentials. It is often observed that customers are ready to forego some privacy for a satisfactory range of danger. But reliability may be attacked significantly if personal information is not utilized rationally and unvaryingly [26] (Patrick Van Eecke, 2010). Proper implementation of security settings may improvise the Safety & Reliability of users' data as per their will [27] (Gail-Joon Ahn, 2011). Hence the quality of services provided by the social media platforms, in terms of Safety & Reliability becomes an important criterion for its selection.

2.2 Responsiveness & Effectiveness as Social Media Attribute:

Responsiveness and Effectiveness of a social media site is measured by the internet speed, expediency, response time etc with which customers access and use bank's social media sites. [28]

(Frederic Marimon, 2012). Efficiency of a bank's social media is observed by timely and convenient completion of all required interaction [29] (Chung Tin Fah, 2012). Social media can enhance the conventional personnel–client bonding with an effective technological knowledge-based relationship [30] (Rahimi & Me, 2016) .

Prompt responses can effectively be done in social media by providing customers relevant and quick information as & when required. It is surely required for enhancement of quick responses to customers' queries for the improvement of e-services and improved customer satisfaction [31] (Chinedu-Okeke & Obi, 2016). Banks can provide unique banking experience to their clients by giving them services combined with technology [32] (Kalia, 2013). Banks may respond to its customers' query effectively through its social media sites but it needs to carefully monitor its personnel's' response on social media sites to assess effectiveness of its response.

2.3 Ease of Use & Customer's Satisfaction as Social Media Attribute:

Social media platform should fulfill the customers' requirement and should be easy to be used with minimum response time. Customers generally choose the media which is easy to operate. By ease of use it means the service reliability and methods to use relevant information provided on a bank's social media websites [33] (Emel Kursunluoglu, 2015). Customers need punctual response for acknowledgement of their complaints. The satisfaction dimension concentrates on evaluating the banks promptness in responding to customers' requirements [34] (Ajimon George, 2013). For getting customer loyalty the banks create user generated customized content for getting the customers' satisfaction dimension [35] (Norman Gwangwava, 2014). Customer's confidence on Bank's social media platform to the extent their requirements are satisfied is termed as fulfillment or satisfaction. Recently, several articles are established [36-40] in this research arena which plays an essential role in research domain.

Definition 2.4: Neutrosophic Set: A set $\widetilde{neuS}$ in the universal discourse X, symbolically denoted by x, it is called a neutrosophic set if $\widetilde{neuS} = \{\langle x; [T_{\widetilde{neuS}}(x), I_{\widetilde{neuS}}(x), F_{\widetilde{neuS}}(x)] \rangle : x \in X\}$, where $T_{\widetilde{neuS}}(x): X \to]-0,1+[$ is said to be the true membership function, which has the degree of belongingness, $I_{\widetilde{neuS}}(x): X \to]-0,1+[$ is said to be the indeterminacy membership, having degree of uncertainty, and $F_{\widetilde{neuS}}(x): X \to]-0,1+[$ is said to be the incorrect membership, which has the degree of non-belongingness of the decision maker. $T_{\widetilde{neuS}}(x), I_{\widetilde{neuS}}(x) \& F_{\widetilde{neuS}}(x)$ exhibits the following relation:

$$-0 \le Sup\{T_{\widetilde{neuS}}(x)\} + Sup\{I_{\widetilde{neuS}}(x)\} + Sup\{F_{\widetilde{neuS}}(x)\} \le 3+.$$

Definition 2.5: Single Typed Neutrosophic Number: Single Typed Neutrosophic Number$(\tilde{n})$is denoted as $\tilde{n} = \langle[(u^1, v^1, w^1, x^1); \alpha], [(u^2, v^2, w^2, x^2); \beta], [(u^3, v^3, w^3, x^3); \gamma]\rangle$

where $\alpha, \beta, \gamma \in [0,1]$, where $(\varphi_{\tilde{n}}): \mathbb{R} \to [0, \alpha]$, $(\gamma_{\tilde{n}}): \mathbb{R} \to [\beta, 1]$ and $(\delta_{\tilde{n}}): \mathbb{R} \to [\gamma, 1]$ is given as:

$$\varphi_{\tilde{n}}(\in) = \begin{cases} \epsilon_{\widetilde{nl}}(\in) & u^1 \leq \in \leq v^1 \\ \alpha & v^1 \leq \in \leq w^1 \\ \epsilon_{\widetilde{nu}}(\in) & w^1 \leq \in \leq x^1 \\ 0 & otherwise \end{cases}, \pounds_{\tilde{n}}(\in) = \begin{cases} \gamma_{\widetilde{nl}}(\in) & u^2 \leq \in \leq v^2 \\ \beta & v^2 \leq \in \leq w^2 \\ \gamma_{\widetilde{nu}}(\in) & w^2 \leq \in \leq x^2 \\ 1 & otherwise \end{cases},$$

$$\delta_{\tilde{n}}(\in) = \begin{cases} \mu_{\widetilde{nl}}(\in) & u^3 \leq \in \leq v^3 \\ \gamma & v^3 \leq \in \leq w^3 \\ \mu_{\widetilde{nu}}(\in) & w^3 \leq \in \leq x^3 \\ 1 & otherwise \end{cases}$$

2.6 Definition: Single-Valued Pentagonal Neutrosophic Number: A Single-Valued Pentagonal Neutrosophic Number $(\tilde{S})$ is defined

as$\tilde{s} = \langle[(m^1, n^1, o^1, p^1, q^1); \pi], [(m^2, n^2, o^2, p^2, q^2); \rho], [(m^3, n^3, o^3, p^3, q^3); \sigma]\rangle$,

where $\pi, \rho, \sigma \in [0,1]$. The accuracy membership function$(\tau_{\tilde{s}}): R \to [0, \pi]$, the indeterminacy membership function $(\iota_{\tilde{s}}): R \to [\rho, 1]$ and the falsity membership function $(\varepsilon_{\tilde{s}}): R \to [\sigma, 1]$ are given as:

$$\tau_{\tilde{s}}(x) = \begin{cases} \tau_{\widetilde{sl1}}(x) & m^1 \leq x < n^1 \\ \tau_{\widetilde{sl2}}(x) & n^1 \leq x < o^1 \\ \mu & x = o^1 \\ \tau_{\widetilde{sr2}}(x) & o^1 \leq x < p^1 \\ \tau_{\widetilde{sr1}}(x) & p^1 \leq x < q^1 \\ 0 & O.W \end{cases}, \iota_{\tilde{s}}(x) = \begin{cases} \iota_{\widetilde{sl1}}(x) & m^2 \leq x < n^2 \\ \iota_{\widetilde{sl2}}(x) & n^2 \leq x < o^2 \\ \vartheta & x = o^2 \\ \iota_{\widetilde{sr2}}(x) & o^2 \leq x < p^2 \\ \iota_{\widetilde{sr1}}(x) & p^2 \leq x < q^2 \\ 1 & O.W \end{cases}$$

$$\varepsilon_{\tilde{s}}(x) = \begin{cases} \varepsilon_{\widetilde{sl1}}(x) & m^3 \leq x < n^3 \\ \varepsilon_{\widetilde{sl2}}(x) & n^3 \leq x < o^3 \\ \vartheta & x = o^3 \\ \varepsilon_{\widetilde{sr2}}(x) & o^3 \leq x < p^3 \\ \varepsilon_{\widetilde{sr1}}(x) & p^3 \leq x < q^3 \\ 1 & O.W \end{cases}$$

2.7 Proposed Score Function:

Score function of a PNN completely depends on the value of truth, falsity and hesitation membership indicator degree. The necessity of score function is to draw comparison or transfer a PNN into a crisp number. In this section we will generate a score function as follows. For any Pentagonal Single typed Neutrosophic Number (PSNN)

$\tilde{A}_{Pt} = (s_1, s_2, s_3, s_4, s_5; \pi, \mu, \sigma)$

We define the score function as $S_{Pt} = \frac{1}{15}(s_1 + s_2 + s_3 + s_4 + s_5) \times (2 + \pi - \sigma - \mu)$

2.7.1 Relationship between any two pentagonal neutrosophic fuzzy numbers:

Let us consider any two pentagonal neutrosophic fuzzy number defined as follows

$A_{Pt1} = (\pi_{Pt1}, \mu_{Pt1}, \sigma_{Pt1})$,$A_{Pt2} = (\pi_{Pt2}, \mu_{Pt2}, \sigma_{Pt2})$

1) $S_{Pt1} > S_{Pt2}$,$A_{Pt1} > A_{Pt2}$
2) $S_{Pt1} < S_{Pt2}$,$A_{Pt1} < A_{Pt2}$
3) $S_{Pt1} = S_{Pt2}$,$A_{Pt1} = A_{Pt2}$

2.8 Basic Operations:

Let $\widetilde{m}_1 = < (m_1, m_2, m_3, m_4, m_5); \pi_{\widetilde{m}_1}, \mu_{\widetilde{m}_1}, \sigma_{\widetilde{m}_1} >$ and $\widetilde{m}_2 = < (n_1, n_2, n_3, n_4, n_5); \pi_{\widetilde{n}_2}, \mu_{\widetilde{n}_2}, \sigma_{\widetilde{n}_2} >$

be two IPFNs and $\alpha \geq 0$. Then the following operational relations hold:

2.8.1 $\widetilde{m}_1 + \widetilde{m}_2 = < (m_1+n_1, m_2+n_2, m_3+n_3, m_4+n_4, m_5+n_5); \pi_{\widetilde{m}_1} + \pi_{\widetilde{n}_2} - \pi_{\widetilde{m}_1}\ \pi_{\widetilde{n}_2}, \mu_{\widetilde{m}_1} \cdot \mu_{\widetilde{n}_2}, \sigma_{\widetilde{m}_1} \cdot \sigma_{\widetilde{n}_2} >$

2.8.2 $\widetilde{m}_1\ \widetilde{m}_2 = < (m_1 n_1, m_2 n_2, m_3 n_3, m_4 n_4, m_5 n_5); \pi_{\widetilde{m}_1}\ \pi_{\widetilde{n}_2}, \mu_{\widetilde{m}_1} +, \mu_{\widetilde{n}_2} - \mu_{\widetilde{m}_1} \mu_{\widetilde{n}_2}, \sigma_{\widetilde{m}_1} + \sigma_{\widetilde{n}_2} - \sigma_{\widetilde{m}_1} \sigma_{\widetilde{n}_2} >$

2.8.3 $\alpha\widetilde{m}_1 = < (\alpha m_1, \alpha m_2, \alpha m_3, \alpha m_4, \alpha m_5); 1 - (1 - \pi_{\widetilde{m}_1})^\alpha, \mu_{\widetilde{m}_1}{}^\alpha, \sigma_{\widetilde{m}_1}{}^\alpha) >$

2.8.4 $\widetilde{m}_1{}^\alpha = < (m_1{}^\alpha, m_2{}^\alpha, m_3{}^\alpha, m_4{}^\alpha, m_5{}^\alpha); \pi_{\widetilde{m}_1}{}^\alpha, (1 - \mu_{\widetilde{m}_1})^\alpha, (1 - \sigma_{\widetilde{m}_1})^\alpha >$

3. OBJECTIVE OF THE STUDY:

- To understand the factors affecting acceptance of Social Media Banking Technology across Gender.
- To understand the best suitable social media channel for Banking Industry as per customers' preference.

4. RESEARCH METHODOLOGY

The data have been collected from various respondents working in different organizations categorized mainly as education sector, service sectors as banks, hospitals, etc. engineering works and Government and Public sector companies in the Kolkata metro area. The study consisted of 94 respondents. A five point Likert scale is used where 5 indicates strongly agree, and 1 indicates strongly disagree. 40.43% respondents are female and 59.57% are male. Age wise respondents below the age <25 was 29.79 %, between 25 – 45 yrs was 52.13%, and >45 yrs was 18.08%

Research Instrument: The questinnaire is mainly focussed on: Social Media platforms used by the banks and users adaptability of the same.

TABLE 4.1 DEMOGRAPPHIC DETAILS OF RESPONDENTS

CHARACTERISTICS		FREQUENCY	%
GENDER	MALE	56	59.57
	FEMALE	38	40.43
AGE	<25	28	29.79
	25-45	49	52.13
	>45	17	18.08

SOURCE: QUESTIONNAIRE

Table 4.2 Indicate acceptance of Online Banking Technology across Gender

GENDER	ATTRIBUTES	Safety & Reliability					Responsiveness & Effectiveness					Ease of Use & Customer's Satisfaction				
		SD	D	N	A	SA	SD	D	N	A	SA	SD	D	N	A	SA
M	TWITTER	5	20	28	34	7	3	18	39	28	6	6	24	29	29	6
		5.32	21.28	29.79	36.17	7.45	3.19	19.15	41.49	29.79	6.38	6.38	25.53	30.85	30.85	6.38
	FACEBOOK	3	24	26	30	11	5	19	45	20	5	4	22	39	20	9
		3.19	25.53	27.66	31.91	11.70	5.32	20.21	47.87	21.28	5.32	4.26	23.40	41.49	21.28	9.57
	YOU TUBE	5	18	23	39	9	7	15	42	26	4	5	22	29	31	7
		5.32	19.15	24.47	41.49	9.57	7.45	15.96	44.68	27.66	4.26	5.32	23.40	30.85	32.98	7.45
F	TWITTER	9	31	26	24	4	9	21	44	16	4	3	26	38	21	6
		9.57	32.98	27.66	25.53	4.26	9.57	22.34	46.81	17.02	4.26	3.19	27.66	40.43	22.34	6.38
	FACEBOOK	8	23	22	34	7	5	19	34	30	6	6	21	34	25	8
		8.51	24.47	23.40	36.17	7.45	5.32	20.21	36.17	31.91	6.38	6.38	22.34	36.17	26.60	8.51
	YOU TUBE	7	32	20	29	6	4	30	31	24	5	7	27	28	25	7
		7.45	34.04	21.28	30.85	6.38	4.26	31.91	32.98	25.53	5.32	7.45	28.72	29.79	26.60	7.45

Table 4.3 Indicate acceptance of Social Media in Online Banking Technology across Age Gap

GENDER	ATTRIBUTES	Safety & Reliability	Responsiveness & Effectiveness	Ease of Use & Customer's Satisfaction
<25	TWITTER	16	6	6
		57.14	21.43	21.43
	FACEBOOK	16	7	5
		57.14	25.00	17.86
	YOU TUBE	17	5	6
		60.71	17.86	21.43
25-45	TWITTER	29	11	8
		59.18	22.45	16.33
	FACEBOOK	29	14	6

		59.18	28.57	12.24
	YOU TUBE	31	12	8
		63.27	24.49	16.33
>45	TWITTER	9	5	3
		52.94	29.41	17.65
	FACEBOOK	8	4	5
		47.06	23.53	29.41
	YOU TUBE	10	4	3
		58.82	23.53	17.65

5. Multi-Criteria Group Decision Making Problem in Pentagonal Neutrosophic Environment

In this current decade, researchers are very much interested in doing MCGDM problem in different fields. Its main goal of this problem is to find out the best option among finite number of different options in presence of distinct attributes, different decision maker's choice and hesitation in human thinking.

5.1 Illustration of the MCGDM problem

Let $G = \{ G_1, G_2, G_3 \dots \dots \dots . . G_m\}$ is the distinct alternative set and $H = \{ H_1, H_2, H_3 \dots \dots \dots . . H_n\}$ is the distinct attribute set respectively. Let $\omega = \{ \omega_1, \omega_2, \omega_3 \dots \dots \dots . . \omega_n\}$ be the weight set associated with the decision maker $D = \{ D_1, D_2, D_3 \dots \dots \dots . . D_K\}$ and each $\omega \geq 0$ and also satisfies the relation $\sum_{i=1}^{m} \omega_i = 1$. Also, weight vector of the attribute function is defined as $\delta = \{\delta_1, \delta_2, \delta_3 \dots \dots \dots . . \delta_k\}$ where each $\delta_i \geq 0$ and also satisfies the relation $\sum_{i=1}^{k} \delta_i = 1$.

5.2 Normalisation Algorithm of MCGDM Problem:

Step 1: Framework of Decision Matrices

Here, we considered all decision matrices according to the decision maker's choice related with finite alternatives and finite attribute functions. It is noted that the member's y_{ij} for each matrices are of triangular fuzzy numbers. Thus, the final matrix is defined as follows:

$$X^K = \begin{pmatrix} . & R_1 & R_2 & R_3 & . & . & . & R_n \\ P_1 & y_{11}^k & y_{12}^k & y_{13}^k & . & .. & . & y_{1n}^k \\ P_2 & y_{21}^k & y_{22}^k & y_{23}^k & . & . & . & y_{2n}^k \\ P_3 & . & . & . & . & . & . & . \\ . & \vdots & . & . & . & . & . & . \\ P_m & y_{m1}^k & y_{m2}^k & y_{m3}^k & . & . & . & y_{mn}^k \end{pmatrix} \ldots\ldots\ldots\ldots\ldots(4.1)$$

Step 2: Framework of Standardized Decision matrix

We consider the following skill of normalization to obtain the standardized decision matrix where

$V^*=(\bar{y}_{ij})_{mn}$ in which the entity $\bar{y}_{ij}$ = ([$\bar{y}_{ij}^1, \bar{y}_{ij}^2, \bar{y}_{ij}^3, \bar{y}_{ij}^4, \bar{y}_{ij}^5$];$\bar{T}_{pt}, \bar{I}_{pt}, \bar{F}_{pt}$) is formulated as

$\bar{Y}_{ij}$= ([$\frac{y_{ij}^1}{S}, \frac{y_{ij}^2}{S}, \frac{y_{ij}^3}{S}, \frac{y_{ij}^4}{S}, \frac{y_{ij}^5}{S}$];$T_{Pt}, I_{Pt}, F_{Pt}$) where S= $\sqrt{{y_{ij}^1}^2 + {y_{ij}^2}^2 + {y_{ij}^3}^2 + {y_{ij}^4}^2 + {y_{ij}^5}^2}$.

Hence the new matrix becomes,

$$M^K = \begin{pmatrix} . & R_1 & R_2 & R_3 & . & . & . & R_n \\ P_1 & Y_{11}^k & Y_{12}^k & Y_{13}^k & . & .. & . & Y_{1n}^k \\ P_2 & Y_{21}^k & Y_{22}^k & Y_{23}^k & . & . & . & Y_{2n}^k \\ P_3 & . & . & . & . & . & . & . \\ . & \vdots & . & . & . & . & . & . \\ P_m & Y_{m1}^k & Y_{m2}^k & Y_{m3}^k & . & . & . & Y_{mn}^k \end{pmatrix} \ldots\ldots\ldots\ldots\ldots(5.2)$$

Step 3: Framework of Single Decision matrix

To formulate a single group decision matrix M we utilized these logical operations of PNN [2.8] $S_{ij}^{'} = \{\sum_{i=1}^{k} \omega_i M^i\}$ $where\ \omega_i$ are the weights of the decision makers for individual decision matrix M^i. So, the matrix becomes as follows:

$$M = \begin{pmatrix} . & R_1 & R_2 & R_3 & . & . & . & R_n \\ P_1 & S_{11}^{'} & S_{12}^{'} & S_{13}^{'} & . & .. & . & S_{1n}^{'} \\ P_2 & S_{21}^{'} & S_{22}^{'} & S_{23}^{'} & . & . & . & S_{2n}^{'} \\ P_3 & . & . & . & . & . & . & . \\ . & \vdots & \vdots & \vdots & . & . & . & . \\ P_m & S_{m1}^{'} & S_{m2}^{'} & S_{m3}^{'} & . & . & . & S_{mn}^{'} \end{pmatrix} \ldots\ldots\ldots\ldots\ldots (5.3)$$

Step 4: Framework of Final matrix

To make the final decision matrix we used the logical operation [2.8] for different weights of the attribute values and also finally operated $R_1^{'} = R_1 + R_2 + \cdots + R_n$ and converted the total matrix into a Colum matrix form, finally we get the decision matrix as,

$$M = \begin{pmatrix} . & R_1 \\ P_1 & S''_{11} \\ P_2 & S''_{21} \\ \vdots & \vdots \\ P_m & S''_{m1} \end{pmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.4)$$

Step 5: Ranking

Now, by considering the Score value (2.7) and converting the matrix (5.3) into crisp form, so that we could evaluate the best alternative corresponding to the best attributes.

5.3 Flowchart:

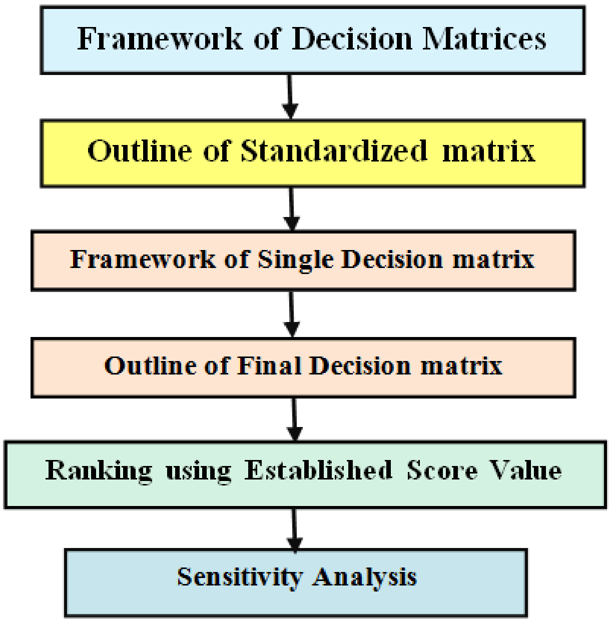

Figure 5.3.1: Flowchart for the problem

5.4 Illustrative Example: Here, we constructed a social media selection problem based on the questionnaire table from which we have three different social medias are available. The problem is to find out the best social media platform among these after computing the decision maker's opinion and maintain the attribute weights properly for this problem. Generally, social media platforms are related with the attributes like safety & reliability, Responsiveness & Effectiveness, Ease of Use & Customer's Satisfaction of the system. Keeping these points in mind decision maker's (Male/Female) gives their opinion in hesitation arena and using verbal phrase we set the problem in pentagonal neutrosophic domain. According to their suggestions we constructed the distinct decision matrices in PNN environment as shows below:

$G_1 = Twitter, G_2 = Facebook, G_3 = Youtube$ are the alternatives.

$$H_1 = \ Safety \ \& \ Reliability,$$

$$H_2 = Responsiveness \; \& \; Effectivness \, ,$$

$H_3 = Ease \; of \; Use \; \& \; Customer's \; Satisfaction$ are the attributes.

According to our problem there are two distinct decision makers are available in our environment, $D_1 = Male's Opinion, \; D_2 = Female's \; Opinion$ having weight distribution $D = \{\, 0.55, 0.45\}$ and the weight vector related with the attribute function $\delta = \{0.32, 0.30, 0.38\}$.

5.5 List of Verbal Phrase

No.	Quantitative Attributes	Verbal phrase
1	$Safety \; \& \; Reliability$	Strongly Agree (SA), Agree(A), Neutral(N), Disagree(D), Strongly Disagree (SD)
2	$Responsiveness \; \& \; Effectivness$	Strongly Agree (SA), Agree(A), Neutral(N), Disagree(D), Strongly Disagree (SD)
3	$Ease \; of \; Use \; \& \; Customer's \; Satisfaction$	Strongly Agree (SA), Agree(A), Neutral(N), Disagree(D), Strongly Disagree (SD)

Step 1

According to the decision maker's opinion from the questionnaire table we constructed the decision matrices are as follows:

$$D^1 = \begin{pmatrix} . & H_1 & H_2 & H_3 \\ G_1 & < 5.3, 7.4, 21.3, 29.8, 36.2; 0.6, 0.7, 0.6 > & < 3.2, 6.38, 19.1, 29.8, 41.5; 0.7, 0.5, 0.7 > & < 6.3, 6.4, 25.5, 30.8, 31.0; 0.7, 0.4, 0.5 > \\ G_2 & < 3.2, 11.7, 25.5, 27.7, 31.9; 0.7, 0.6, 0.7 > & < 5.3, 5.35, 20.2, 21.3; 0.8, 0.6, 0.7 > & < 4.3, 9.8, 21.9, 23.4, 41.5; 0.6, 0.7, 0.4 > \\ G_3 & < 5.3, 9.6, 19.2, 24.5, 41.8; 0.8, 0.3, 0.6 > & < 4.3, 7.5, 15.9, 27.7, 44.7; 0.7, 0.4, 0.6 > & < 5.3, 7.5, 23.4, 30.8, 32.9; 0.7, 0.3, 0.5 > \end{pmatrix}$$

$$Male's \; opinion$$

$$D^2 = \begin{pmatrix} . & H_1 & H_2 & H_3 \\ G_1 & < 4.26, 9.58, 25.5, 27.7, 32.9; 0.6, 0.7, 0.5 > & < 4.3, 9.6, 17.1, 22.3, 46.8; 0.7, 0.8, 0.3 > & < 3.2, 6.4, 22.3, 27.7, 40.4; 0.3, 0.5, 0.6 > \\ G_2 & < 7.45, 8.5, 23.4, 24.5, 36.2; 0.6, 0.5, 0.4 > & < 5.3, 6.4, 20.2, 31.9, 36.2; 0.6, 0.5, 0.3 > & < 6.4, 8.5, 22.4, 26.6, 36.2; 0.8, 0.8, 0.4 > \\ G_3 & < 6.38, 7.45, 21.28, 30.8, 34.1; 0.7, 0.5, 0.7 > & < 4.3, 5.3, 25.5, 31.9, 33.0; 0.8, 0.6, 0.7 > & < 7.4, 7.6, 26.6, 28.8, 29.8; 0.5, 0.7, 0.6 > \end{pmatrix}$$

$$Female's \; Opinion$$

Step 2: Framework of Standardized decision matrix

$$D^1 = \begin{array}{c} \\ G_1 \\ G_2 \\ G_3 \end{array} \begin{array}{ccc} H_1 & H_2 & H_3 \end{array}$$

$$= \begin{pmatrix} . & H_1 & H_2 & H_3 \\ G_1 & <0.10,0.141,0.407,0.57,0.692;0.6,0.7,0.6,> & <0.058,0.116,0.347,0.542,0.754;0.7,0.5,0.7> & <0.123,0.124,0.496,0.599,0.608;0.7,0.4,0.5> \\ G_2 & <0.063,0.023,0.502,0.545,0.628;0.7,0.6,0.7> & <0.096,0.098,0.368,0.459,0.798;0.8,0.6,0.7> & <0.080,0.183,0.409,0.437,0.776;0.6,0.7,0.4> \\ G_3 & <0.099,0.18,0.36,0.46,0.785;0.8,0.3,0.6> & <0.077,0.135,0.286,0.498,0.804;0.7,0.4,0.6> & <0.102,0.146,0.454,0.598,0.639;0.7,0.3,0.5> \end{pmatrix}$$

$$Male's\ opinion$$

$$D^2$$

$$= \begin{pmatrix} . & H_1 & H_2 & H_3 \\ G_1 & <0.083,0.188,0.499,0.542,0.644;0.6,0.7,0.5> & <0.077,0.173,0.308,0.401,0.842;0.7,0.8,0.3> & <0.058,0.118,0.411,0.51,0.744;0.3,0.5,0.6> \\ G_2 & <0.146,0.167,0.459,0.48,0.709;0.6,0.5,0.4> & <0.1,0.121,0.381,0.602,0.683;0.6,0.5,0.3> & <0.125,0.166,0.437,0.518,0.705;0.8,0.8,0.4> \\ G_3 & <0.124,0.144,0.412,0.597,0.661;0.7,0.5,0.7> & <0.081,0.1,0.481,0.602,0.623;0.8,0.6,0.7> & <0.147,0.151,0.528,0.571,0.591;0.5,0.7,0.6> \end{pmatrix}$$

$$Female's\ Opinion$$

Step 3: Framework of weighted Single Decision matrix

$$D^1$$

$$= \begin{pmatrix} . & H_1 & H_2 & H_3 \\ G_1 & <0.0923,0.1621,0.448,0.557,0.670;0.6,0.7,0.5> & <0.0665,0.1416,0.3294,0.4785,0.794;0.7,0.5,0.3> & <0.0937,0.1213,0.4577,0.559,0.669;0.7,0.4,0.5> \\ G_2 & <0.101,0.0878,0.483,0.5157,0.664;0.7,0.5,0.4> & <0.0978,0.1083,0.3738,0.523,0.746;0.8,0.6,0.7> & <0.10025,0.1753,0.4216,0.4734,0.744;0.6,0.7,0.4> \\ G_3 & <0.1103,0.1638,0.3834,0.52165,0.7292;0.8,0.3,0.6> & <0.0788,0.11923,0.3737,0.5448,0.722;0.7,0.4,0.6> & <0.122,0.148,0.487,0.586,0.617;0.7,0.3,0.5> \end{pmatrix}$$

Step 4: Framework of Final Single Decision matrix

$$M = \begin{pmatrix} <1.2974,1.559,2.184,2.376,2.597;0.65,0.52,0.43> \\ <1.392,1.476,2.212,2.336,2.609;0.69,0.39,0.43> \\ <1.406,1.564,2.188,2.386,2.574;0.62,0.54,0.46> \end{pmatrix}$$

Step 4: Ranking

Now, we consider the established Score function (2.7), to convert the pentagonal neutrosophic numbers into crisp one, thus we get the final ideal decision matrix as

$$M = \begin{pmatrix} <1.135> \\ <1.249> \\ <1.093> \end{pmatrix}$$

Thus, ranking of the social media service is as $G_2 > G_1 > G_3$.

5.6 Results and Sensitivity Analysis

To understand how the attribute weights of each criterion affecting the relative matrix and their ranking a sensitivity analysis is done. The below table is the evaluation table which shows the sensitivity results.

Decision Maker's Weight	Final Decision Matrix	Ordering
<(0.55, 0.45)>	$\begin{pmatrix} <1.135> \\ <1.249> \\ <1.093> \end{pmatrix}$	$G_2 > G_1 > G_3$
<(0.45, 0.55)>	$\begin{pmatrix} <1.024> \\ <1.132> \\ <1.048> \end{pmatrix}$	$G_2 > G_3 > G_1$
<(0.48, 0.52)>	$\begin{pmatrix} <1.035> \\ <1.185> \\ <1.042> \end{pmatrix}$	$G_2 > G_3 > G_1$
<(0.52, 0.48)>	$\begin{pmatrix} <1.078> \\ <1.202> \\ <1.044> \end{pmatrix}$	$G_2 > G_1 > G_3$
<(0.5, 0.5)>	$\begin{pmatrix} <1.062> \\ <1.195> \\ <1.046> \end{pmatrix}$	$G_2 > G_1 > G_3$

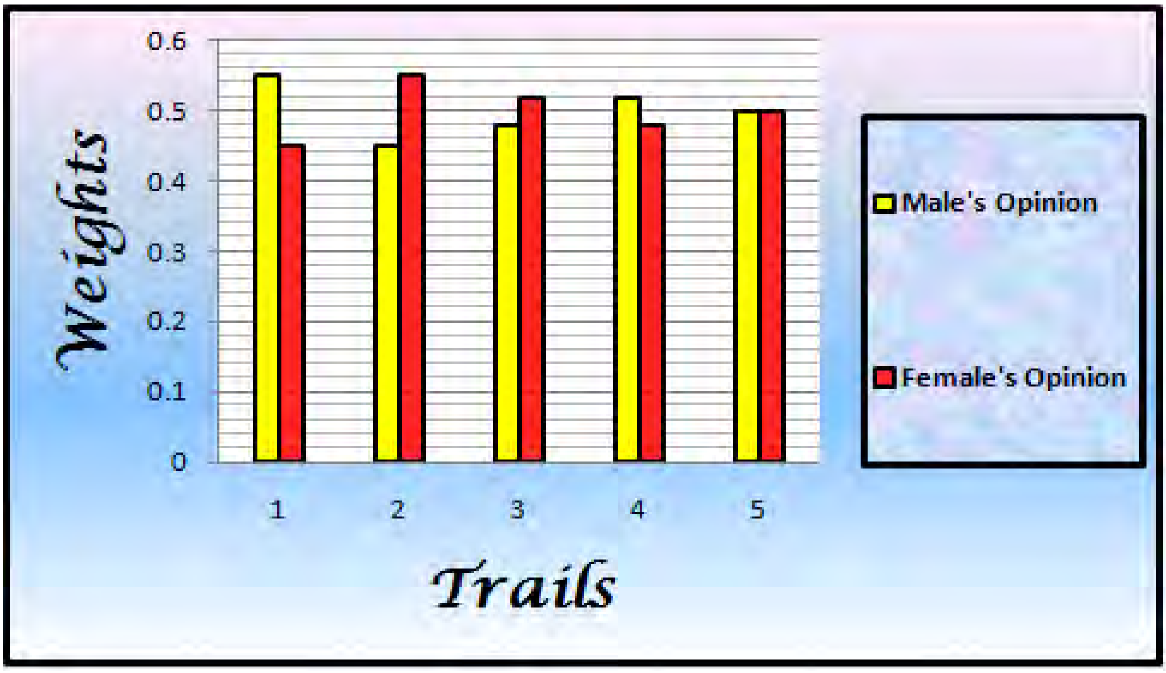

Figure 5.6.1: Sensitivity analysis table on Decision Maker's Weight.

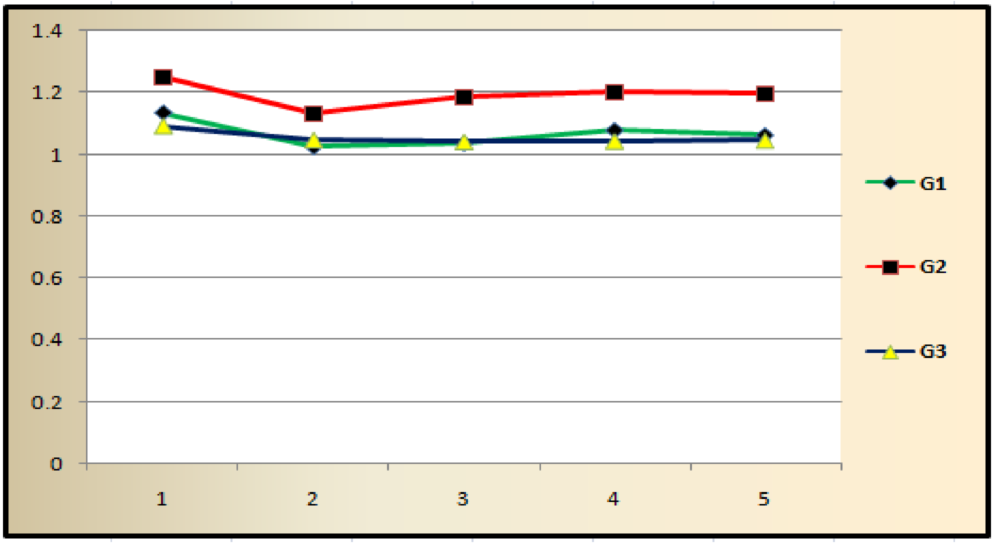

Figure 5.6.2: Best Alternative Social Media Service Table

5.7 Comparison Table

We compared this proposed work with the established works proposed by the researchers to find the best social media and it is noticed that in each cases the alternative G_2 becomes the best social media service. The comparative table given as follows:

Approach	*Ranking*
(Chakraborty et al.) [18]	$G_2 > G_1 > G_3$
(Biswas et al.) [41]	$G_2 > G_3 > G_1$
Our Proposed	$G_2 > G_1 > G_3$

6. Implication:

Different social media platforms are available for communication with customers and digital marketing like face book, twitter, Google plus, linked in, you tube etc. This study was primarily done to identify the best suited social media platform for Banking Industry especially for customers of West Bengal. We wanted to discover the right social media platform based on different attributes as desired by customers. The perception of neutrosophy plays a critical role in designing mathematical calculations. In this research work, we set the MCGDM problem in PNN environment using the realistic data set. Applying the verbal phrases we formulate the MCGDM problem and hence applied our logical operations of PNN on it to get the best alternatives. Finally, sensitivity analysis is also performed here to which has a crucial impact in the ranking results. This novel thought will help the other researchers in doing MCGDM problem from realistic data in social media platform.

There are a lot of researches already done in social media implementation in Banking Industry. However many results are still unknown. Our work is to explore the idea in the following points:

- Defining the attributes necessary for social media platform for Banking Industry in West Bengal.
- Discovering the best suitable social media site for Banking Industry in West Bengal as per customers' preference.
- Finding the best social media site which satisfies customers and generate revenue by increasing business.
- The graphical representation of adaptation of social media platform based on its attributes.
- Covert the problem into PNN environment using verbal phrases.
- Apply proposed MCGDM method in PNN arena.
- Sensitivity analysis for Ranking in different cases.

7. Discussion

The main focus of this study was to find out the best social media platform for Banks. In total 94 respondents were asked varied questions and their choices and preferences about use of social media in banks. Three parameters focusing their requirement were fixed as Safety, Efficiency and Ease of use. The study examined different social media platform like Messaging and communication, e,g. Twitter, Communities and social groups, e.g. Face book and Photo and video sharing, e.g. YouTube. Face book was found to be most preferred channels both by the male and female considering all the three factors. However other two channels have different opinion based on different factors. In the sample considered here men respondents are more than women; most of the respondents are under 45 years of age and they frequently uses social media. Both men and women are equally boasting the use of social media however the worldwide trend also applied here as it was observed that youngsters are dominating the social media sites. Social media mainly has not only impacted the life of youngsters but it has also become drastically momentous since last ten years across all age groups. It was also observed that awareness about the use of social media for banking transactions is comparatively low in this region. It is agreed that Banks must publicize the use of social media as an important tool for banking transactions. Social media has proven to be the fastest communication mode and banks may use it for satisfying the ever increasing customized needs of its customers. The more satisfied customers would result in more improved business for banks. Moreover in the long run these satisfied customers would foster the brand loyalty and customer loyalty would further result in improved customer relationship management.

Quantification of social media quality and its effects has got very less attention in the state. It is accepted that the overall social media quality should be measured by banks to satisfy customers.

Long tern connectivity with banks will improve if the services experienced by customers are satisfactory. Better customer satisfaction in turn will bring customer loyalty. The objective of adaption of social media for banking sector is not merely for likes and shares but it goes beyond that. It is more of creating brand awareness and brand advocacy. Hence Banks should design their social media strategy focusing realistic goals.

8. Findings:

Customers basically want three things from Banks like, better and responsive services, easier way to bank and most importantly they want to be understood. Customers do not want generic ads and offers, they want products and services tailored to them and will exchange data in order to receive this. All the above is possible if the banks implement social media methodically and keep a proper follow up for the same. As of now it is the best, easier and fastest responsive way to communicate with customers. The following findings were done:

- Face book is most preferred social media medium in comparison to other options like you tube and twitter etc. considering all the three attributes
- After applying pentagonal neutrosophic numbers into crisp one, we get the final ideal decision matrix which gives the ranking of the social media as follows, Facebook>Twitter>YouTube.
- In spite of changing the weight age of attributes Face book remains the most preferred choice across gender.
- The three different attributes like Security, Efficiency and Ease of use have a strong impact on overall customers' satisfaction which resulted in selection of Bank's social media platform
- Banks profit margin would be boosted with the help of proper implementation of social media strategies. This will increase customers' base without expansion of physical branches which will result in reduction in cost..

9. Conclusions:

It may be concluded that Social Medias can greatly influence and enhance the function which is being carried out in banks. This research found out that almost big banks in the state are using social media for banking operations. On the questionnaire received from respondents the main concern or obstacle for using social media was Security and privacy issues. Almost majority preferred social media in terms of its efficiency and ease of use. Face book was found to be most acceptable mode compare to any other media across gender and age. Majority of the respondents showed positive indications for use of social media for banking operations in case of higher security. Hence we can conclude that customers are willing to accept the social media for banking operations if Banks take complete care of their security and privacy of data. Therefore for banks in West Bengal all conditions are met and it is up to the Banks' policy of achieving the highest security in order to help the customers to adapt the transactional social media. Our forecast is that transactional social media will become more acceptable and popular in banking industry in coming years.

Our future study includes more questionnaire collection and feedback received from customers and banks to analysis the functionality of transactional social media and to suggest the ways to improve the same.

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Received: Apr 17, 2020 Accepted: July 7, 2020.

Neutrosophic Sets and Systems, Vol. 35, 2020
University of New Mexico

Neutrosophic Multiset Topological Space

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Abstract: In this article we have investigated some properties of netrosophic multiset topology. The behavior of compactness and connectedness in netrosophic multiset topology, continuous function on netrosophic multiset topology etc have been examined. Neutrosophic multiset is a generalization of multisets and neutrosophic sets. Several properties of neutrosophic topological space in view of neutrosophic multiset topological space have been studied.

Keywords: Neutrosophic Multiset; Neutrosophic Minimal set; Neutrosophic Maximal set; Neutrosophic Multiset topology; Compactness, Connectedness; Continuous Neutrosophic Multiset; Separation axioms; Distance function.

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1. Introduction

In recent years, multisets and neutrosophic sets have become a subject of great interest for researchers. Mathematicians always like to solve a complicated problem in a simple way and to find out the most feasible solution. Neutrosophy has been introduced and studied by Smarandache [13, 15] as a new branch of philosophy. Recently various papers published on neutrosophic topology and many researchers doing very well, neutrosophic decision making had been studied in [15, 17]. Algebraic properties of neutrosophic set studied in [9, 13], Neutrosophic Bipolar Vague Soft Set, and its property studied in [9]. Smarandache generalizes intuitionistic fuzzy sets (IFSs) and other kinds of sets to neutrosophic sets (NSs). In Smarandache [12, 13], some distinctions between NSs and IFSs are underlined. decision-making problem, algebraic property one can analysis by topological property connectedness and compactness property that property can help to take the decision into a more reliable way. Smarandache [13, 14, 15] also defined various notions of neutrosophic topologies on the non-standard interval. The logic of the neutrosophic set is very clear and its utilization on topology is very beneficial for many standard problems like diagnosis of bipolar disorder diseases group decision making and analytical property and evaluation Hospital medical care systems etc. [1, 9, 13]. The relation between the intuitionistic fuzzy topology (IFT) on an IFS and the neutrosophic topology are also analyzed by Smarandache.

Multiset theory was introduced by Bilzard [3]. Later on multiset topological space was studied by many researcher Shravan and Tripathy [17, 18, 19]. The purpose of this paper is to construct a new generalization of topological space called the neutrosophic multiset topological space. The possible application of neutrosophic multiset topological space has been studied. For the different types of behavior of objects in nature sometimes set theory and multiset theory fails to describe some particular situation. Sometimes it is observed that Neutrosophic Multiset can be described in an easier way to handle such cases. Neutrosophic set and topological space have been studied by Salama and Alblowi [10, 11]. The concept of multiset topological space has been applied for studying different properties of spatial objects. In this article we have used multiset neutrosophic topological space for studying various spatial topological properties, like closeness connectedness and the completeness property and its application further in various fields.

2. Materials and Methods

We procure some existing definitions in this paper, one may refer to Smarandache ([13], [15]) and S. Alias, et.al [2].

We define functions T, F and I from X to [0, 1]. Where T is membership value, F fails membership value and I is the indeterminacy value.

The definition of neutrosophic multiset was first define by Smarandache [12] as follows.

Definition 2.1. [12] A Neutrosophic Multiset is a neutrosophic set where one or more elements are repeated with the same neutrosophic components, or with different neutrosophic components.

Definition 2.2. The Empty neutrosophic multiset is denoted by N_Θ and define by

$N_\Theta = \{<x_{(0,1,1)}> : \forall x \in X\}$ where x can be repeated.

Definition 2.3. The Whole neutrosophic multiset is denoted by W_X and define by

$W_X = \{<x_{(1,0,0)}> : \forall x \in X\}$ where x can be repeated.

The power set of neutrosophic multiset is denoted by $P(X)$.

The collection of all possible subsets of X is called the power set of the netrosophic multiset.

Definition 2.4. Let $A = \{(x_{<T_{A(x)}, I_{A(x)}, F_{A(x)}>}): x \in X\}$ be a neutrosophic multiset on X then the compliment of A is denoted by A^c and define by

$A^c = \{(x_{<F_{A(x)}, 1-I_{A(x)}, T_{A(x)}>}): x \in X\}$.

Where x can be repeated based on its multiplicity and the corresponding T, F, I values may or may not be equal.

Definition 2.5. The intersection of *NM* sets are defined by $A \cap B = \{x: x \in A \text{ and } x \in B\}$.

Definition 2.6. The union of *NM* sets are defined by $A \cup B = \{ x: x \in A \text{ or } x \in B\}$.

Definition 2.7. In the *NM* sets $A \subset B$ if $x \in A$ implies that $x \in B$.

Definition 2.8. Cardinality of a *NM* set A is denote the number of elements in a set A which is define by card(A).

Definition 2.9. The Cartesian product of two neutrosophic multiset is defined by $A \times B = \{(x, y) : x \in A$ and $y \in B\}$.

Definition 2.10 The difference of two *NM* sets A and B is the collection of members such that all members belong to A but not in B.

Now we introduce two new types of operation maximal union *NM* set and minimal intersection *NM* set.

Definition 2.11. Let X be a non-empty set, and neutrosophic multiset A and B in the form $A = \{(x_{<T_{A(x)}, I_{A(x)}, F_{A(x)}>}): x \in X\}$ and $B = \{(x_{<T_{B(x)}, I_{B(x)}, F_{B(x)}>}): x \in X\}$, then the operations of maximal union and minimal intersection *NM* set relation are defined as follows:

1. $(A \cup B)_{\max} = \{(x_{< T_{(A \cup B)_{max}}(x), I_{(A \cup B)_{max}}(x)} F_{(A \cup B)_{max}}(x)): x \in X\}$, where $T_{(A \cup B)_{max}}(x) = \max\{T_{A(x)}, T_{B(x)}\}$, $F_{(A \cup B)_{max}}(x) = \min\{T_{A(x)}, T_{B(x)}\}$ and $I_{(A \cup B)_{max}}(x) = \min\{I_{A(x)}, I_{B(x)}\}$.

2. $(A \cap B)_{\min} = \{(x_{<T_{(A \cap B)_{min}}(x), I_{(A \cap B)_{min}}(x)} F_{(A \cap B)_{min}}(x)>) : x \in X\}$, where $T_{(A \cap B)_{min}}(x) = \min\{T_{A(x)}, T_{B(x)}\}$ and $F_{(A \cap B)_{min}}(x) = \max\{F_{A(x)}, F_{B(x)}\}$ and $I_{(A \cap B)_{min}}(x) = \max\{I_{A(x)}, I_{B(x)}\}$.

Example 2.1. Let $X = \{x, y, z, t\}$ and $A = \{ x_{<0.7, 0.2, 0.3>}, x_{<0.7, 0.2, 0.3>}, y_{<0.3, 0.2, 0.7>}, y_{<0.9, 0.3, 0.1>}, z_{<0.0, 1, 1>}, t_{<0.5, 0.7, 0.5>}\}$, $B = \{ x_{<0.7, 0.2, 0.3>}, x_{<0.8, 0.5, 0.2>}, y_{<0.3, 0.2, 0.7>}, y_{<0.3, 0.2, 0.7>}, z_{<0.7, 0.8, 0.3>}, t_{<0.0, 1, 1>}\}$ be neutrosophic multisets, then the maximal union and minimal intersection are

$(A \cup B)_{\text{Max}} = \{x_{<0.8, 0.2, 0.2>}, y_{<0.9, 0.2, 0.1>}, z_{<0.7, 0.7, 0.3>}, t_{<0.5, 0.7, 0.5>}\}$ and

$(A \cap B)_{Min} = \{x_{<0.7, 0.5, 0.3>}, y_{<0.3, 0.3, 0.7>}, z_{<0.0, 1, 1>}, t_{<0.0, 1, 1>}\}$

We formulate the following results without proof.

Result 2.1. Union of any family of neutrosophic multisets is always a neutrosophic multiset.

Result 2.2. Intersection of any family of neutrosophic multisets is always a neutrosophic multiset.

Result 2.3. The compliment of a neutrosophic multiset is always a neutrosophic multiset.

Result 2.4. Every neutrosophic set is a neutrosophic multiset but not necessarily conversely.

Example 2.2. Let $A = \{8_{\langle 0.6, 0.3, 0.2\rangle}, 8_{\langle 0.6, 0.3, 0.2\rangle}, 8_{\langle 0.4, 0.1, 0.3\rangle}, 7_{\langle 0.2, 0.7, 0.0\rangle}\}$.
Here A is a neutrosophic multiset but not a neutrosophic set.

Result 2.5. Let $\{A_j : j \in \Lambda\}$ be an arbitrary family of NM set in X, then the arbitrary maximal union and arbitrary minimal intersection is also a NM set.

Remark 2.1. A neutrosophic multiset is a natural generalization of multiset as well as Cantor set.

We introduced neutrosophic multiset topological space and study some of its properties.

Definition 2.12. Let X be neutrosophic multiset and a non-empty family τ subsets of W_X is said to be neutrosophic multiset topological space if the following axioms hold:

1. $N_\Theta, W_X \in \tau$.
2. $A \cap B \in \tau$, for $A, B \in \tau$.
3. $\bigcup_{i \in \Lambda} A_i \in \tau$, for $\forall \{A_i : i \in \Lambda\} \in \tau$

In this case the pair (W_X, τ) is called a neutrosophic multiset topological space (*NMTS* in short) and any neutrosophic multiset in τ is known as open neuterosophic multiset (*ONMS* in short) in W_X. The elements of τ^c are called closed neutrosophic multisets, otherwise a neutrosophic set F is closed if and only if its complement F^c is an open neutrosophic multiset.

Definition 2.13. Let (W_X, τ_1) and (W_X, τ_2) be two neutrosophic multiset topological spaces on W_X. Then τ_1 is said be contained in τ_2 that is if $\tau_1 \subseteq \tau_2$ i.e, $A\in \tau_2$ for each $A\in \tau_1$. In this case, we also say that τ_1 is coarser than τ_2.

Definition 2.14. Let (W_X, τ) be a neutrosophic multiset topological space on W_X. A non-empty family of subsets β of X is called neutrosophic multiset basis of the neutrosophic multiset topological space W_X if any element of τ can be express as the union of the element of β.

Remark 2.2. As usual, basis of a neutrosophic multiset topological space is not unique.

Definition 2.15. Let (W_X, τ) be a neutrosophic multiset topological space with base β. The interior of the neutrosophic multiset A is the union of basis element of τ which is contained in A and it is denoted by $NM_{int}A$, i.e, $NM_{int}(A) = \{ \cup\beta_i : \beta_i \subseteq A$ and $\beta_i \in \beta\}$.

Definition 2.16. Let (W_X, τ) be a neutrosophic multiset topological space. The closure of the neutrosophic multiset A is the intersection of all closed neutrosophic multiset containing the set A it is denoted by $NM_{Cl}(A)$, i.e, $NM_{Cl}(A) = \{\cap F_i : A \subseteq F_i$ and $F_i^c \in \tau \}$.

In view of the definitions, we formulate the following result.

Proposition 2.1. Let (W_X, τ) be a neutrosophic multiset topological space and A, B be two neutrosophic multiset on W_X, then the following property hold:

1. $NM_{int}A \subseteq A$.
2. $A \subseteq B \Rightarrow NM_{int}(A) \subseteq NM_{int}(B)$.
3. $A \subseteq NM_{Cl}(A)$.
4. $A \subseteq B \Rightarrow NM_{Cl}(A) \subseteq NM_{Cl}(B)$
5. $NM_{int}(NM_{int}(A)) = NM_{int}(A)$.
6. $NM_{Cl}(NM_{Cl}(A)) = NM_{Cl}(A)$.
7. $NM_{Cl}(A\cup B) = NM_{Cl}(A) \cup NM_{Cl}(B)$.
8. $NM_{int}(W_X) = W_X$.
9. $NM_{Cl}(N_\Theta) = N_\Theta$.

Definition 2.17. Let (W_X, τ) be a neutrosophic multiset topological space a non-empty set S is called a subbasis if the finite intersection of the elements of S can form a basis for τ.

Definition 2.18. Let (W_X, τ) be a neutrosophic multiset topological space a point $P \in A \subseteq W_X$ is said to be a limit point of A if for every basis element β containing p contains one element of A other than p, i.e, $\beta \cap A \neq N_\Theta$.

3. Results

3.1. Compactness, Connectedness and Continuous map.

Definition 3.1.1. Let (W_X, τ) be a neutrosophic multiset topological space. A neutrosophic multiset A is said to be disjoint if $\exists$ two neutrosophic multisubsets B, C such that $B \cap C = N_\Theta$ and A = $B \cup C$.

Definition 3.1.2. Let (W_X, τ) be a neutrosophic multiset topological space. The space W_X is said to be connected if W_X cannot be express as the union of two disjoint neutrosophic multisets.

Definition 3.1.3. Let (W_X, τ) be a neutrosophic multiset topological space. The space W_X is said to be compact if every open cover of W_X has a finite subcover.

Proposition 3.1.1. Every finite neutrosophic multiset topological space is compact.

Definition 3.1.4. Let (W_X, τ_1) and (W_X, τ_2) be two neutrosophic multiset topological space. The *NMS* function $f : (W_X, \tau_1) \rightarrow (W_X, \tau_2)$ is said to be continuous if for each open neutrosophic multiset V of τ_2 the neutrosophic multiset $f^{-1}(V)$ is an open submset of τ_1.

Proposition 3.1.2. Let f be a continuous function from a *NMS* topological space (W_X, τ_1) to another *NMS* topological space (W_X, τ_2), the function f is said to be a homomorphism if $f(A \cup B) = f(A) \cup f(B)$ where $A, B \in \tau_1$ and $f(A)$, $f(B) \in \tau_2$.

3.2. Separation axioms on neutrosophic multiset.

We have defined disjoint neutrosophic multiset, connectedness, compactness and the continuous image of neutrosophic multiset topological space. In this section we define separation axioms on *NMS* topological space.

In the *NMS* a singleton set $\{p\}$ is define by $\{p\} = \{x_{<T_{(x)}, I_{(x)}, F_{(x)}>} : T_{(x)} \neq 0$, when $x = p$ otherwise $T_{(x)} = 0$, $I_{(x)} = 1 - T_{(x)}$ and $F_{(x)} = 1 - T_{(x)}$ for all $x \in W_X\}$.

Where x can be occurs more than one times it's depends on its multiplicity and then T, F, I value may or may not be equal.

Definition 3.2.1. Let (W_X, τ) be a neutrosophic multiset topological space. If there exist only two open neutrosophic multiset in (W_X, τ) is called indiscrete *NMS* topological space.

Definition 3.2.2. Let (W_X, τ) be a neutrosophic multiset topological space. If every singleton neutrosophic multiset is an open *NMS* set then (W_X, τ) is called discrete *NMS* topological space.

Definition 3.2.3. Let (W_X, τ) be a neutrosophic multiset topological space. If for every two distinct *NMS* singleton sets, $\{x_1\}$; $\{x_2\}$ then there exist $V, U \in \tau$ such that$\{x_1\} \subseteq V$ and $\{x_2\} \not\subset V$ or $\{x_2\} \subseteq U$ and $\{x_1\} \not\subset U$. Hence, (W_X, τ) is $NMST_o$-space. i.e., there exists τ-open *NMS* which contains one of them but not the other.

Definition 3.2.4. Let (W_X, τ) be a neutrosophic multiset topological space. If for every two distinct *NMS* singleton sets, $\{x_1\}$; $\{x_2\}$ then there exist $V, U \in \tau$ such that$\{x_1\} \subseteq V$ and $\{x_2\} \not\subset V$ and $\{x_2\} \subseteq U$ and $\{x_1\} \not\subset U$. Hence, (W_X, τ) is $NMST_1$-space.

Definition 3.2.5. Let (W_X, τ) be a neutrosophic multiset topological space. If for every two distinct *NMS* singleton sets, $\{x_1\}$; $\{x_2\}$ then there exist $V, U \in \tau$ such that$\{x_1\} \subseteq V$ and $\{x_2\} \not\subset V$ and $\{x_2\} \subseteq U$ and $\{x_1\} \not\subset U$ and $U \cap V = N_\Theta$. Hence, (W_X, τ) is $NMST_2$-space.

Proposition 3.2.1. Every $NMST_2$-space is $NMST_1$-space but it is not necessarily conversely.

Example 3.2.1. In co-finite neutrosophic multiset topological space is not a $NMST_2$-space but when the space has the finite neutrosophic multiset topology then it is $NMST_1$-space.

So when we consider the infinite neutrosophic multiset topology we can get our desire result.

Proposition 3.2.2. Every $NMST_1$-space is $NMST_0$-space but it is not necessarily conversely.

Example 3.2.2. Let $W_X = \{x_{<0.5, 0.7, 0.5>}, x_{<0.5, 0.7, 0.5>}, y_{<0.3, 0.4, 0.7>}\}$ and $\tau = \{ W_X, N_\Theta, \{y\}\}$.

Here (W_X, τ) is a $NMST_0$-space but it is not a $NMST_1$.

Proposition 3.2.3. Every $NMST_2$-space is $NMST_0$-space but it is not necessarily conversely.

Example 3.2.3. Since every $NMST_0$-space is not a $NMST_1$-space and every $NMST_1$-space is not a $NMST_2$-space so every $NMST_0$-space is not a $NMST_2$-space.

Proposition 3.2.4. Every discrete NMS topological space is $NMST_2$-space.

3.3. Distance function on NMS.

In this section we are going to define a distance function on Neutrosophic set. Since in Neutrosophic set we have defined Neutrosophic elements, Neutrosophic subset so it is natural to ask, can we measure the distance between two Neutrosophic points or two Neutrosophic sets or is there any distance between a Neutrosophic point to a Neutrosophic set?

The distance function between multiset points is defined by Shravan and Tripathy [12], based on the multiplicity and the elements.

The Neutrosophic point p of a Neutrosophic multiset W_X is define by $p = \{(T_{(x)}, \mathrm{I}_{(x)}, \mathrm{F}_{(x)}): T_{(x)} \neq 0$, when $x=p$, otherwise $T_{(x)}= 0$, $\mathrm{I}_{(x)} = 1\text{-}T_{(x)}$ and $\mathrm{F}_{(x)} = 1\text{-}T_{(x)}$ for all $x \in W_X\}$.

Note: The Neutrosophic point p can have multiple time it's depends on its multiplicity.

Definition 3.3.1. Let x, y be two Neutrosophic points on a Neutrosophic set W_X. The distance between the points is denoted by $\bar{d}(x, y)$ and is define by $\bar{d}(x, y) = \sup\{|x\text{-}y|, |T_{(x)} - T_{(y)}|, |\mathrm{I}_{(x)} - \mathrm{I}_{(y)}|, |\mathrm{F}_{(x)} - \mathrm{F}_{(y)}|\}$, where the distance function $\bar{d}$ is define by, $\bar{d}: W_X \rightarrow R^+ \cup \{0\}$.

Definition 3.3.2. Let x be a Neutrosophic point and A be a subset on a Neutrosophic set W_X. The distance between the point x and set the A is denoted by $\bar{d}(x, A)$ and is define by $\bar{d}(x, B)$ =inf-sup{$|x-y_i|$, $|T_{(x)} - T_{(y)_i}|$, $|I_{(x)} - I_{(y)_i}|$, $|F_{(x)} - F_{(y)_i}|$: for all $y_i \in A$}.

Definition 3.3.3. Let A, B be two Neutrosophic subset of a Neutrosophic set W_X the distance between the sets A and set B is denoted by $\bar{d}(A, B)$ and is define by $\bar{d}(A, B)$ =inf sup{$|x_i - y_i|$, $|T_{(x)_i} - T_{(y)_i}|$, $|I_{(x)_i} - I_{(y)_i}|$, $|F_{(x)_i} - F_{(y)_i}|$: $\forall x_i \in A$, and $y_i \in B$}.

From the definition 5.1, 5.2 and 5.3 we can define another definition of matric space on a Neutrosophic multiset.

Definition 3.3.4. A non-empty Neutrosophic set W_X is said to be a Neutrosophic metric space with the distance function $\bar{d}$: $W_X X W_X \rightarrow R_+ \cup \{0\}$, if W_X satisfy following:

1. $\bar{d}(x,y) \geq 0$, $\forall x, y \in W_X$.
2. $\bar{d}(x,y)=0$, *iff* $x=y$ and $T_{(x)_i} = T_{(y)_i}$, $F_{(x)_i} = F_{(y)_i}$, $I_{(x)_i} = I_{(y)_i}$.
3. $\bar{d}(x,y) = \bar{d}(y,x)$, $\forall x,y \in W_X$
4. $\bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z)$, $\forall x,y,z \in W_X$

Theorem 3.3.1. If $\bar{d}^1$ and $\bar{d}^2$ be two Neutrosophic metric spaces then $\bar{d}$ = max{$\bar{d}^1$, $\bar{d}^2$} is also a Neutrosophic metric space.

Theorem 3.3.2. If $\bar{d}^1$ and $\bar{d}^2$ be two Neutrosophic metric space then $\bar{d}$ = min{$\bar{d}^1$, $\bar{d}^2$} is not a Neutrosophic metric space.

The proof of the above two theorem is obvious using the concept of general matric space.

4. Applications

The work done in this paper is based on the application of neutrosophic sets in multiset topological space. These can be further applicable for the development of neutrosophic topology separation axioms on neutrosophic multiset topology and neutrosophic multisets.

5. Conclusions

In this paper we have established some properties of the neutrosophic multiset topological space such as compactness and connectedness, continuous function on netrosophic multiset topology, separation axioms on neutrosophic multiset topology. Also we have introduced the notion of the distance function in neutrosophic multiset and examined some properties. This paper can be useful for further development of neutrosophic multiset theory and neutrosophic topology.

Acknowledgments: The authors express their sincere thanks to Prof. F. Smarandache, Department of Mathematics, University of New Mexico, USA for his comments, those improved the presentation of the article.

Conflicts of Interest: The authors declare that no conflicting interest is involved in this article.

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Received: Apr 18, 2020. Accepted: July 8, 2020

Impact of Social Media in Banking Sector under Triangular Neutrosophic Arena Using MCGDM Technique

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Abstract: This paper aims to uncover the position of social media in customer relationship management (CRM) in banking industry in West Bengal (W.B) under neutrosophic environment. It also tries to identify the attributes that influence the adaptation of different social media platforms for marketing by Banks and finally its use in CRM approaches. The scope of this research is, however, limited to the West Bengal (India) state. In this study a qualitative in-depth questionnaire has been used in presence of impreciseness. Three case studies were developed, which explained the adaptation and implementation of social media in retail banks in W.B. The responses, gathered through in-depth interviews with top bank officials and estimated data from official web sites of the banks have been used for MCGDM and sensitivity analysis. Different attributes like Safety & Privacy, Effectiveness & Efficiency and Fulfillment & Responsiveness have a significant impact on the overall service quality perception for Banks using social media and its platforms. We have performed comparative analysis with the established method to find out the best social media platform under neutrosophic environment in WB's banking Industry. Successful implementation of these platforms would then ensure Customer Loyalty and effective CRM. It was also noted that customers mainly refrain from Banking through social media due to safety and privacy concerns. The study was done to suggest betterment of social media marketing performance for banks in WB in presence of uncertainty. It recommended managers to continuously monitor the overall service quality of social media platforms as they lead to customer loyalty and CRM.

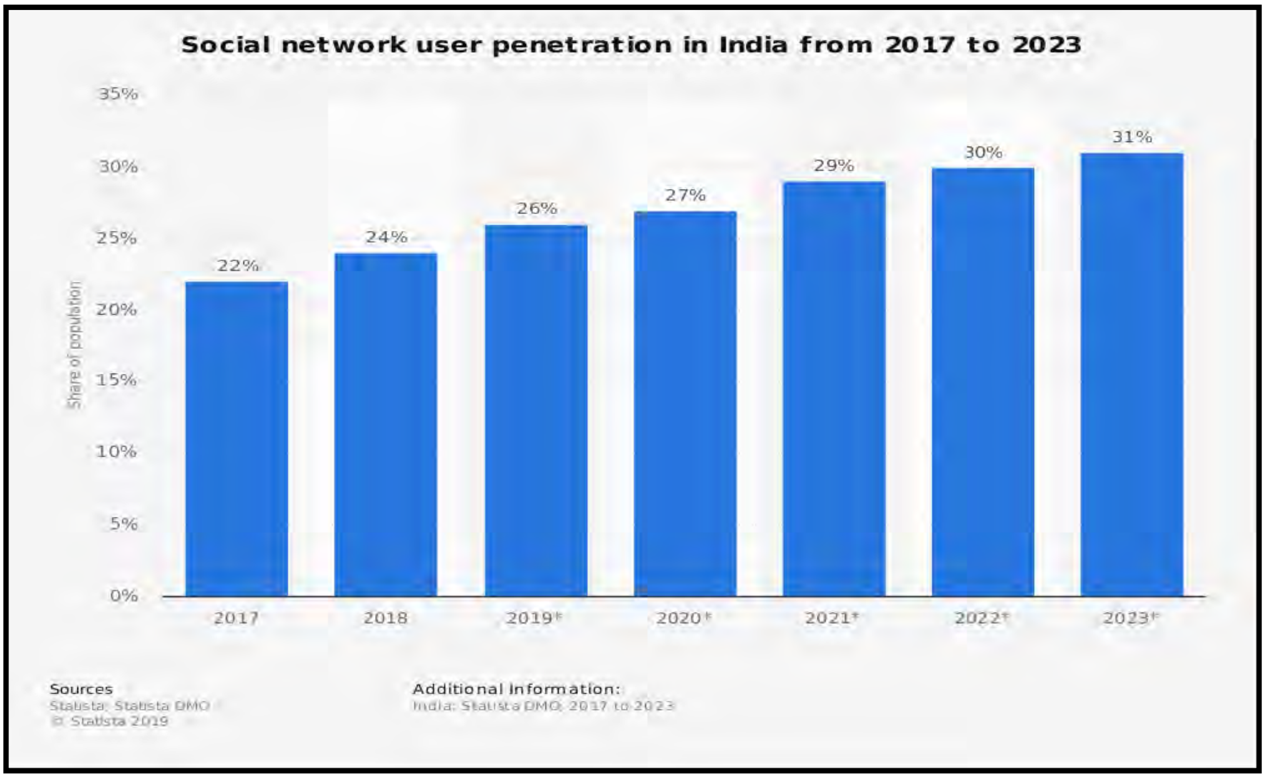
Keywords: West Bengal, Social media, Customer loyalty, Service quality, Customer Relationship Management, Neutrosophic, CRM, Retail banking.

1. INTRODUCTION:

1.1. SOCIAL MEDIA: Social Media is a communication platform that facilitates communication via virtual networks. It is a virtual medium which is designed to aid people to share contents, pictures, videos, and views swiftly and in real-time through websites and applications. The ability to share photos, opinions, events, etc instantaneously has transformed the way we communicate and, also, the way we do business. It provides the facility of continuously communicating with a large number of people at a time. The revolution of Social media and its increasing impact has transformed its old conventional image of amusement to an opportunity to work and trade. This vibrant use of social media has affected almost every business sectors either positively or negatively. It has changed the way business was done and Marketing has taken a new shift after this. Social media offers different ways to promote business either through organic marketing (free) or by paid marketing. Web 2.0 technologies are the stage of Internet expansion where static web pages were converted to user generated content [1]. The business communication is enhanced to a new height via online mode through Social media [2]. According to [3] People share a lot of information about their personal lives, their needs and preferences on social media and it may assist the institutions to design their marketing policies. Based on the above data it can be said that the social media set-up facilitate in building virtual group for individuals with similar mind-set, hobbies, work culture etc [4]. Therefore, use of social networking could assist Banks build up their brand awareness and brand loyalty which ultimately help in customer acquirement and retention [5]. Communication between clients and Banks has improved a lot after successful implementation of Internet mainly because it has eliminated geographical hindrances [6]. Now it has almost become mandatory for all the banks to adapt social media for getting customer loyalty and effective CRM.

1.2. **Social media statistics in India:** India is the 2nd largest country in the world in terms of Population with over 1.36 billion people.

- India currently has a population of 1,369,566,180 - this is 17.1% of the world's total population
- Median age is 27.1 years - it's a young country
- Life expectancy is 69 years
- Internet penetration is low in India - yet, in December 2018, 566 million users were online in India. Out of this - 493 million are regular users of the internet. (source: livemint).
- At the end of 2018, the number of social media users in India stood at 326.1 million. (statista)
- At the end of 2019, this number has been estimated to grow to 351.4 million.
- On average, Indian users spend 2.4 hours on social media a day (slightly below the global average of 2.5 hours a day). (Source: The Hindu)
- 290 million active social media users in India access social networks through their mobile devices. (Source: Hootsuite)



India: social network penetration 2017-2023

Based on customer's requirement and rapid market the number of social media sites is increasing day by day to cater to the needs of different audience groups. Before choosing social media platform, it is essential for banks to realize the available social media platforms and location of their customer base in these Medias. Some of the social media categories are as follows:

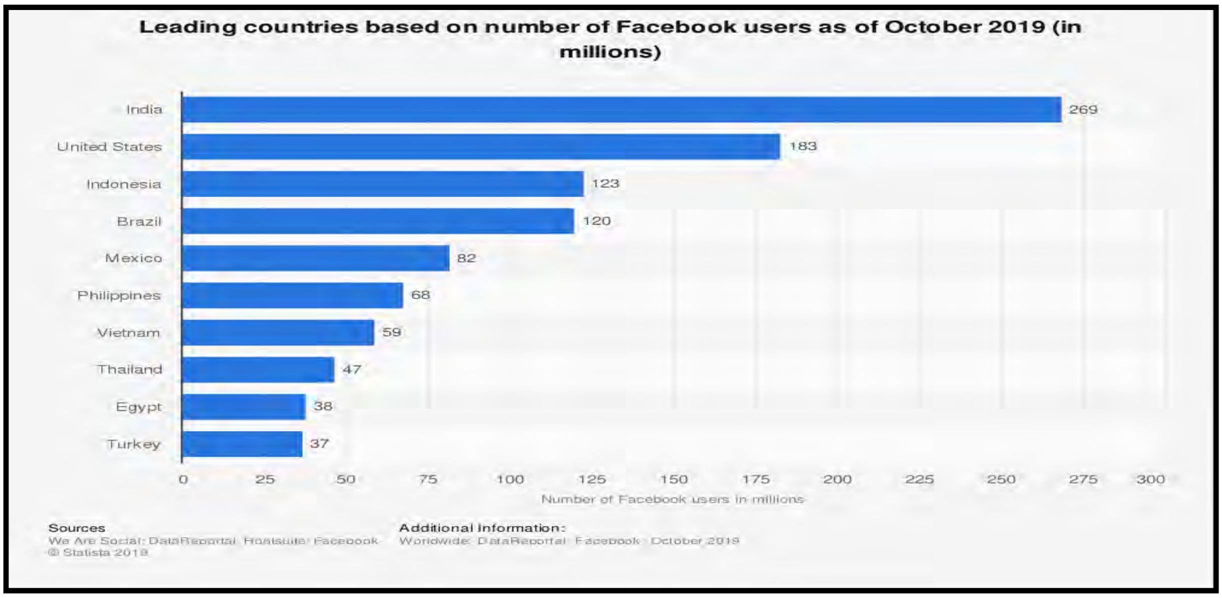
1.2.1 Communities and social groups:

"We build technologies to give people the power to connect with friends and family, find communities and grow businesses"- face book

These sites allow connecting people of similar interests and background. This is used to share information and events to large number of customers and building relationship by regular interaction. Banks may also pose their brand on social network as an expert information source. This may also be used for educating and training customers regarding different products and services provided by banks.

Face book Statistics in India:

- India ranks first in terms of face book users. Currently is has 269 million active users in India (Source: Investopedia)
- The largest user group by age on Face book is 18-24 years, with a massive 97.2 million users.



Face book usage penetration in India from 2015 to 2023

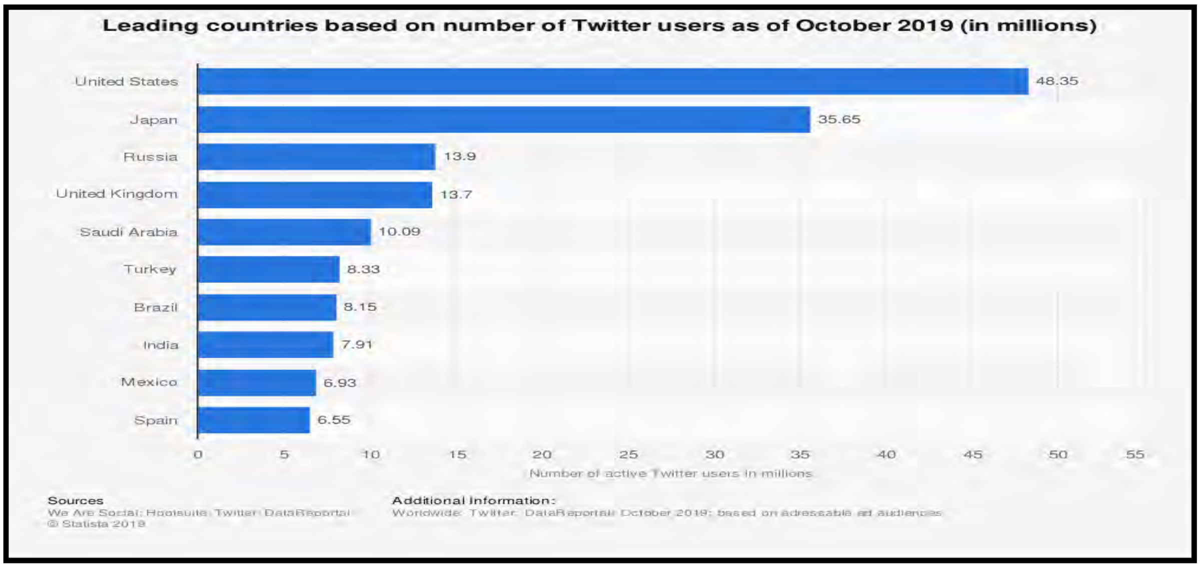
Messaging and communication: (e.g. blogging and micro-blogging such as Twitter):

"Follow everything from breaking news and entertainment, to sports, politics, and everyday interests. Then, join the conversation"- Twitter

Blogging and Micro Blogging are used for creating online communities where customers can seek out information and answers to their questions. It is used to listen and resolve customer queries/issues in banking world. It creates a vast online, viral, and word of mouth, which is optimal for establishing brand loyalty and monitoring reputation.

Twitter Statistics in India:

- India has 7.75 million users on Twitter. (Source: statista)
- 18% of social media users in India look at Twitter as a source of news. (Source: Reuters)
- Twitter usage unlike other platforms is actually decreasing = 2.2% per quarter (Source: Digital 2019 report from Hootsuite)

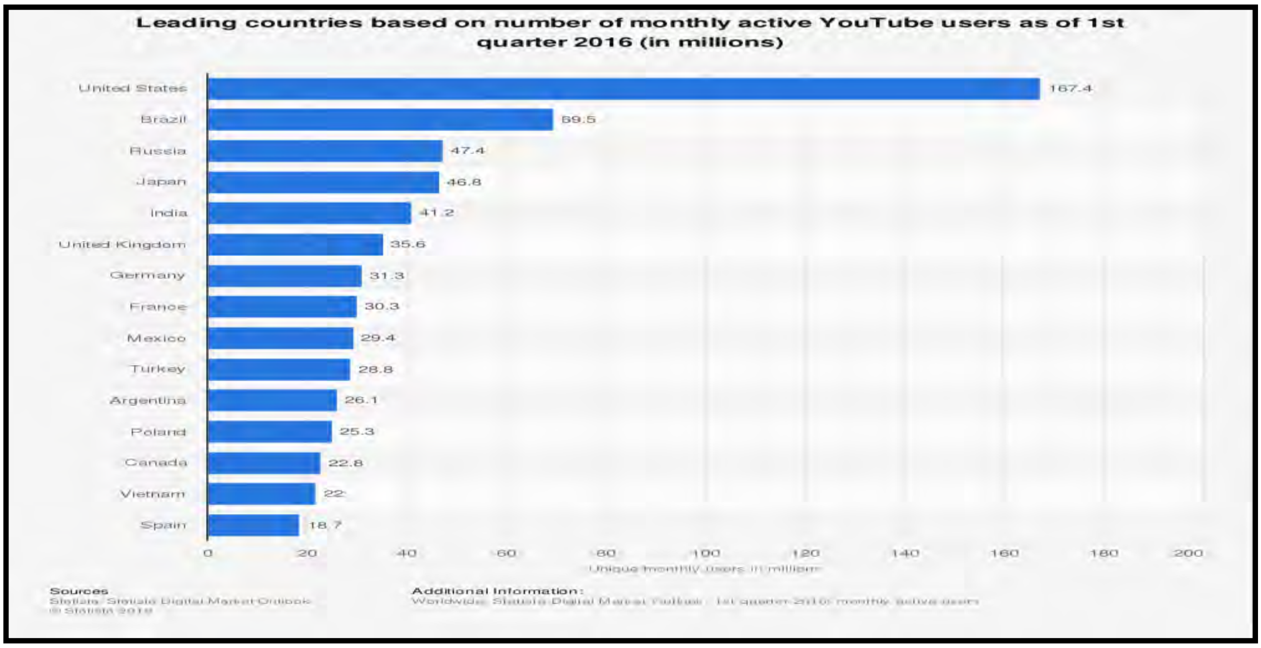


Content Communities: (Photo and video sharing, e.g. YouTube):

"Enjoy the videos and music you love, upload original content, and share it all with friends, family, and the world" – YouTube. They are content specific. These could be used for brand promotion, engaging customer through sharing pictures, videos etc.

You Tube statistics in India

- As per Google announcement, as of August 2018, there were 245 million active You Tube users in India.
- This figure is predicted to double over the next two years.
- Online video accounts for 75% of data traffic in the country – and with 4G networks improving; this is likely to further increase.



The literature on the banking sector has abundant references to online and electronic services (e.g. e-banking), but has paid relatively little attention to the adoption and use of social media [7-9].

1.3. BANKING AND SOCIAL MEDIA

Banking sector is the backbone of any emerging economy. Banks are instrumental in implementing the economic reforms. Any revolution in the banking sector because of the acceptance of technology is bound to have a broad impact on an economy's growth. These days, banks are seeking unconventional ways to provide and differentiate amongst their various services. Customers now demand a facility to conduct their banking activities at any time and place according to their convenience [10].

Banking sector is the backbone of any emerging economy. Banks are instrumental in implementing the economic reforms. Any revolution in the banking sector because of the acceptance of technology is bound to have a broad impact on an economy's growth. These days, banks are seeking unconventional ways to provide and differentiate amongst their various services. Customers now demand a facility to conduct their banking activities at any time and place according to their convenience [11].

Social media has changed the entire gamut of business and marketing and Banking Industry is no exception to this because here the Customer Interaction is a must. Today Social media is universal and pervasive, so banks can rely on it. Digital communication is becoming a strong communication medium between Banks and customers. This media is proving itself indispensable in connecting to the potential clients. By allowing transfer of money, getting credit and even simply opening a bank account, it has improved customer services which in turn are improving the customer relationship. Assessing people's sentiments is a very significant and staggering job, particularly in case of service industry. Social media has a unique ability to create and sustain associations with customers, creating better Customer relations. Hence banks need to consider social media as an integral part of their overall marketing strategy [12].

People use Face book, Twitter, YouTube, Instagram, LinkedIn etc to understand different information regarding the different products and services provided by banks only after understanding the facilities and prospects of various social media platforms. Banks are using this network to inform their customers about their products and upgrade them according to customers' feedback. On the other hand, there is the talk of turnover in social networks. Also, purchases can be made through social networks.

Physical Banking opted tactics like advertising, direct mail or face to face communication for customer interaction so far but now the approaches have changed from providing customer service to affiliation and long term relationship with customers. For doing it, banks need to diagnose customers' interests, emotions and behavior and with help of social media this analysis are being done easily. Today, customers expect that they should be heard and answered and receive the services they need through social media.

Social Medias can greatly affect the reputation and the brand image of the banks. Banks need a transparent understanding of the key elements in the development of social media and adopt a road map and a strategy. The banks may use the following pathway in social media to listen to the customers.

- **As Is:** Banks need to understand the customers' requirements initially by analyzing their data in social networks.
- **Listen:** The next step would be to analyze the data carefully. Then the bank should design and provide support as per their expectation,
- **Engage:** Information can be collected through customers and through feedback taken Bank's can fulfill the customers' needs.
- **Optimize:** In the last step bank should attract fans and increase the loyalty of existing customers by using customers' feedback and analyzing their interactions with each other.

In a media landscape increasingly dominated by social media, Bank's marketing strategy for these platforms can make or break its success as a brand. Banks need to hold their social media efforts to high standard, creating custom made strategies that build their brand, win customers, and yield high ROI. Therefore social media techniques have become essential communication tools for banks to communicate with people across globe. Banks are adapting social media because they are finding it difficult to fight with traditional banking methods such as interest rates and product differentiation to attract new clients and sustain the existing ones. In today's aggressive atmosphere customer loyalty can be gained through allocation of finer service quality to ensure maximum customer satisfaction [13].The purpose of this study is thus, to explore the implication of social media on service quality perception and client loyalty in the banking industry of West Bengal. Social media service quality can be used to boost customers' loyalty by Banks in the India banking industry [14]. There are limited studies on social media service quality and client loyalty for Indian Banking industry. This study will contribute towards reducing the knowledge gap between impact of social media on service quality and customers' loyalty. These attributes so discussed would be able to improve the quality of social media performance.

The article is structured as follows: The next section will provide a discussion on the use of social media in the Indian banking industry, followed by a discussion on the methodology that was used for data collection, and a presentation of the results. The last section provides the study's findings and conclusion.

2. Literature Review: Indian Banks have started using social media in their regular operations in various capacities a little lately and are at different stages of maturity. As of April 2013, some private banks provide regular updates on the latest offers and allow basic customer operations through popular social media sites. A large private bank in India hosted Face book application on its secure servers allowing balance amount check, cheque book request, stop payment, etc. Some of the private banks are using their social media websites to provide their customers, distinct offers, detailed product information and consumer care services. With some banks taking the lead by setting example, the others also have started following their footsteps. In a survey by the Financial Brand newsletter in July 2013, it was established that ICICI, Axis and HDFC Banks are among the top 10 Banks with Social Media presence. Of late public sector banks have also started using this media in a grand way. As per present scenario, Indian banks can no longer live in denial by avoiding and not using Social Media if they do not want threatening their own business. The Indian banking industry has envisaged some social media channels to attract tech-savvy clients and improve customer services to bring customer loyalty [15]. The use of social media in India has gained its importance.

2.1 Social Media Safety & Privacy: Privacy refers to the extent by which the customers' details are protected by bank's social media platform [16].Banks need to give their customers enough confidence to use their social media accounts so that they may perceive that their personal information will be secured and not to be misused by banks [17]. Banks can build new healthy relationship with customers if the privacy is perceived positively by customers [18]. The information get disclosed and shared through social media so easily, that it has raised doubts about its privacy among the users [19]. Maintenance of privacy in bank's social media channel has been a big challenge for the banking industry. The main challenge is to monitor and control the posts in these sites [20]. A proper privacy setting of social media site is very essential in banks because privacy invasion may lead to theft of personal identification and may lead to criminal proceedings. In case of low security features hackers may hack the social media sites and/or may clone the original, befooling customers and duping them [21].

2.2 Social Media Efficiency & Effectiveness: Effectiveness refers to the ease of use, internet speed, expediency etc with which customers may access and use bank's social media sites [22]. Effectiveness measures the efficiency of bank's social media and it estimates the speed of accessing and working on the bank's social media sites to ensure timely and convenient completion of all required interaction [23]. Social media can augment the conventional personnel–client bonding with an effective technological knowledge-based relationship [24].

Today's customers need prompt responses and it can effectively be done in social media by providing them relevant and quick information as & when required. It is surely required for enhancement of quick responses to customers' queries for the improvement of e-services and clients' improved customer satisfaction [25]. Banks can provide unique banking experience to their clients by giving them services combined with technology. Hence the primary task of the bank is to find out and respond to customers' queries effectively on Bank's social media sites. By monitoring the response of bank personnel on social media sites, Banks need to assess the service quality. As per the above discussion we can make the following hypothesis:

2.3 Social Media Fulfillment & Responsiveness: Fulfillment concentrate on the service truthfulness and ease of use of relevant information provided on a bank's social media websites [26]. Customers need prompt response and acknowledgement of their complaints or suggestions. The fulfillment dimension concentrates on evaluating the banks promptness in responding to customers' requirements [27]. For getting customer loyalty the banks create user generated customized content for getting the Fulfillment dimension [28]. Hence Fulfillment refers to the customer's confidence on Bank's social media platform to the extent their requirements are fulfilled.

2.4 Theory of Vagueness and Multi-Criteria Decision-Making Problem (MCDM): Due to the complication of detached things and hesitation in human thinking, [29] manifested a remarkable perception of neutrosophic set theory, which has been widely applied on disjunctive arenas of science and engineering. Recently, researchers developed pentagonal [30], Hexagonal [31], Heptagonal [32] fuzzy numbers in research domain. Researchers also established some useful techniques [33-35] which linked the hesitant number and the crisp number in real life scenario. In this era, MCDM is the paramount topic in decision scientific research. Recently, it is more essential in such problems where a group of criteria is apprised. For such problems involving multi-criteria group, decision-making problems (MCGDM) have come into existence. In this current epoch, several works has been already published in this arena. [36] Introduced MCDM skill in Pythagorean fuzzy set field, [37] focused on linguistic aggregation operators based on MCGDM problem, [38] surveyed intuitionistic interval fuzzy information and applied it in MCGDM problem, [39] derived MCGDM methodology using type-2 neutrosophic linguistic judgments, [40] manifested the idea of MCGDM in human resource development arena, [41] developed MCGDM skill in thermal enovation of masonry buildings field,[42] introduced best-Worst-Method and ELECTRE Method using MCGDM, [43] applied MCGDM in garage location selection based civil engineering problems, [44] derived decision making method in intuitionistic neutrosophic environment, [45] utilized MCDM in bipolar neutrosophic set arena, [46] wielded MCGDM in entropy based problem, [47] used MCGDM in smart phone selection problem, [48] developed MCGDM in selection of advanced manufacturing technology in neutrosophic set, [49] derived attribute based MCDM in linguistic variable in intuitionistic fuzzy set.

Motivated by Smarandache's neutrosophic theory [52], researchers established several articles [53-62] in this domain and it is fruitfully applied in various field of mathematics. Also, a few new techniques are manifested in neutrosophic theory which can grab and solve MCDM, MCGDM problems in disjunctive domain. In this phenomenon, Vikor [63], TOPSIS [64], MOORA [65], GRA [66] skills are developed to solve decision making problems using some suitable and logical operators in neutrosophic theory. So, in case of social science related hesitant data, decision making problem becomes one of the key topics in neutrosophic ambient.

In this research article, we consider a triangular neutrosophic based MCGDM technique to select the best social media for online marketing in banking sector. Here, we collect all the information's from different banks based on their online marketing report. But, we observed that these data's are fluctuating and filled with lots of hesitations. Now, due to the presence of impreciseness we need to improve our general established method. Thus, we have introduced triangular neutrosophic number to tackle this system for better results. Additionally, we also incorporate different weights in distinct attribute functions as well as decision maker's choice. Finally, we performed a sensitivity analysis and comparative study which reflects different case studies in disjunctive scenario.

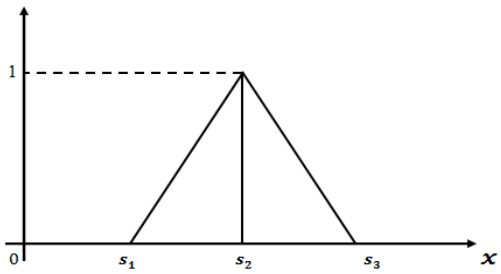
2.5 Preliminaries:

Definition 2.5.1: Fuzzy Set: A set $\tilde{\mathrm{F}}$, generally defined as $\tilde{\mathrm{F}} = \left\{\left(\alpha, \mu_{\tilde{\mathrm{F}}}(\alpha)\right) : \alpha \in \mathrm{S}, \mu_{\tilde{\mathrm{F}}}(\alpha) \in [0,1]\right\}$, denoted by the pair$\left(\alpha, \mu_{\tilde{\mathrm{F}}}(\alpha)\right)$, where α belongs to the crisp set F and $\mu_{\tilde{\mathrm{S}}}(\alpha)$ belongs to the interval$[0, 1]$, then set $\tilde{\mathrm{S}}$ is called a fuzzy set.

Definition 2.5.2: Triangular Fuzzy Number: A triangular fuzzy number $\tilde{\mathrm{A}} = (\mathrm{s}_1, \mathrm{s}_2, \mathrm{s}_3)$ should satisfy the following condition

(1) $\mu_{\tilde{\mathrm{A}}}(\mathrm{x})$ is a continuous function which is in the interval [0,1]
(2) $\mu_{\tilde{\mathrm{A}}}(\mathrm{x})$ is strictly increasing and continuous function on the intervals $[\mathrm{s}_1, \mathrm{s}_2]$.
(3) $\mu_{\tilde{\mathrm{A}}}(\mathrm{x})$ is strictly decreasing and continuous function on the intervals$[\mathrm{s}_2, \mathrm{s}_3]$.

Definition 2.5.3: Linear Triangular Fuzzy Number (TFN): A linear triangular fuzzy number can be written as $\tilde{\mathrm{A}}_{\mathrm{TFN}} = (\mathrm{s}_1, \mathrm{s}_2, \mathrm{s}_3)$ whose membership function is defined as follows:



$$\mu_{\tilde{A}_{TFN}}(x) = \begin{cases} \frac{x-s_1}{s_2-s_1} & if\ s_1 \le x \le s_2 \\ 1 & if\ \ x = s_2 \\ \frac{s_3-x}{s_3-s_2} & if s_2 \le x \le s_3 \\ 0 & Elsewhere \end{cases}$$

Figure 2.5.3.1: Graphical Representation of Linear Triangular Fuzzy Number

Definition 2.5.4: Neutrosophic Set: [52] A set $\widetilde{neuS}$ in the universal discourse X, symbolically denoted by x, it is called a neutrosophic set if $\widetilde{neuS} = \{\langle x; [T_{\widetilde{neuS}}(x), I_{\widetilde{neuS}}(x), \mathrm{F}_{\widetilde{neuS}}(x)]\rangle : x \in X\}$, where $T_{\widetilde{neuS}}(x): X \to]-0,1+[$ is said to be the true membership function, which has the degree of belongingness, $I_{\widetilde{neuS}}(x): X \to]-0,1+[$ is said to be the indeterminacy membership, having degree of uncertainty, and $F_{\widetilde{neuS}}(x): X \to]-0,1+[$ is said to be the incorrect membership, which has the degree of non-belongingness of the decision maker. $T_{\widetilde{neuS}}(x), I_{\widetilde{neuS}}(x) \& F_{\widetilde{neuS}}(x)$ exhibits the following relation:

$$-0 \le Sup\{T_{\widetilde{neuS}}(x)\} + Sup\{I_{\widetilde{neuS}}(x)\} + Sup\{F_{\widetilde{neuS}}(x)\} \le 3+.$$

2.5.5: Triangular Single Valued Neutrosophic number: [33] A Triangular Single Valued Neutrosophic number is defined as $\tilde{A}_{Neu} = (p_1, p_2, p_3; q_1, q_2, q_3; r_1, r_2, r_3)$ whose truth membership, indeterminacy and falsity membership is defined as follows:

$$T_{\tilde{A}_{Neu}}(x) = \begin{cases} \frac{x-p_1}{p_2-p_1} & when\ p_1 \le x < p_2 \\ 1 & when\ x = p_2 \\ \frac{p_3-x}{p_3-p_2} & when\ p_2 < x \le p_3 \\ 0 & otherwise \end{cases}, \quad I_{\tilde{A}_{Neu}}(x) = \begin{cases} \frac{q_2-x}{q_2-q_1} & when\ q_1 \le x < q_2 \\ 0 & when\ x = q_2 \\ \frac{x-q_2}{q_3-q_2} & when\ q_2 < x \le q_3 \\ 1 & otherwise \end{cases}$$

$$F_{\tilde{A}_{Neu}}(x) = \begin{cases} \frac{r_2-x}{r_2-r_1} & when\ r_1 \le x < r_2 \\ 0 & when\ x = r_2 \\ \frac{x-r_2}{r_3-r_2} & when\ r_2 < x \le r_3 \\ 1 & otherwise \end{cases}$$

Where, $0 \le T_{\tilde{A}_{Neu}}(x) + I_{\tilde{A}_{Neu}}(x) + F_{\tilde{A}_{Neu}}(x) \le 3,\ x \in \tilde{A}_{Neu}$

2.5.6: Score Function: [50] If $\tilde{A}_{Neu} = (p_1, p_2, p_3; \pi, \rho, \sigma)$ be a triangular neutrosophic number then its score function is defined as $S_C = \frac{1}{8}(p_1 + p_2 + p_3) \times (2 + \pi - \rho - \sigma)$ and accuracy value is defined as,

$$A_C = \frac{1}{8}(p_1 + p_2 + p_3) \times (2 + \pi - \rho + \sigma)$$

3. Purpose/ Objectives of the Study:

1. To understand the factors affecting the customers' attitude towards acceptance of Social Media Channels,
2. To help Banks understand the impact of Social Media Channels on customer satisfaction and customer loyalty.

4. Research Methodology:

The data have been collected from various respondents working in different organizations categorized mainly as education sector, service sectors as banks, hospitals, etc. engineering works and Government and Public sector companies in the Kolkata metro area. The study consisted of 234 respondents whose income is above 15,000 per month as it is assumed that those people at least above Rs. 15000 earning/ month will be transacting more through online mode and can afford a smart phone. We have used a five point Likert scale where 5 indicates strongly agree, and 1 indicates strongly disagree. 64.9% respondents are male and 35.1% are female.

Research Instrument*:* Demographic Profile is the independent variable in this paper. Technology acceptance model by Ajzen & Fishbein, 1980, Davis, 1989 and Ajzen, 1991 are used for validating questionnaire. The questionnaire is mainly focused on: Social Media platforms used by the banks and attributes affecting the users' adaptability of the same.

TABLE 4.1.1 DEMOGRAPPHIC DETAILS OF RESPONDENTS

CHARACTERISTICS	TYPES	FREQUENCY	%
GENDER	MALE	135	57.69
	FEMALE	99	42.31

AGE	<25	75	32.05
	25-40	154	65.81
	>40	5	02.14
OCCUPATION	EMPLOYED	92	39.32
	UNEMPLOYED	22	9.40
	PROFESSIONAL	14	5.98
	STUDENT	95	40.60
	BUSINESS	10	4.27
	OTHERS	1	0.43
SOCIAL MEDIA PLATFORM	FACEBOOK	132	56.41
	TWITTER	47	20.09
	YOUTUBE	55	23.50
HOURS OF SURFING THROUGH SOCIAL MEDIA	DAILY	149	63.68
	WEEKLY	13	5.55
	MONTHLY	6	2.56
	VERY RARE	66	28.21

Table 4.1.2 Indicate acceptance of Social Media based on various attributes

BANK	PLATFORM	SAFETY & PRIVACY (%)	EFFICIENCY & EFFECTIVENESS (%)	FULFILLMENT & RESPONSIVENESS (%)
1	FACEBOOK	10	65	54
	TWITTER	6	16	50
	YOUTUBE	5	26	28
2	FACEBOOK	15	76	56
	TWITTER	12	37	26
	YOUTUBE	21	24	15
3	FACEBOOK	23	29	45
	TWITTER	13	15	16
	YOUTUBE	45	9	7

4.1 Multi-Criteria Group Decision Making Problem in Triangular Neutrosophic Environment

One of the most dependable, logistical and widely used topic in this recent era is Multi criteria decision making problem. Its main objective is to find out the finest option among finite number of different alternatives based on finite unlike attribute values. Its execution process was quiet tough to estimate in triangular neutrosophic environment. To handle this MCGDM problem an algorithm was developed using some mathematical operator and de-fuzzification technique.

4.1.1 Illustration of the MCGDM problem

We consider the problem as follows:

Let $P = \{ P_1, P_2, P_3 \dots\dots\dots .. P_m\}$ is the distinct alternative set and $R = \{ R_1, R_2, R_3 \dots\dots\dots .. R_n\}$ is the distinct attribute set respectively. Let $\omega = \{ \omega_1, \omega_2, \omega_3 \dots\dots\dots .. \omega_n\}$ be the weight set associated with the attributes R where each $\omega \geq 0$ and also satisfies the relation $\sum_{i=1}^{n} \omega_i = 1$. We also consider the set of decision maker $D = \{ D_1, D_2, D_3 \dots\dots\dots .. D_K\}$ associated with alternatives whose weight vector is defined as $\Delta = \{\Delta_1, \Delta_2, \Delta_3 \dots\dots\dots .. \Delta_k\}$ where each $\Delta_i \geq 0$ and also satisfies the relation $\sum_{i=1}^{k} \Delta_i = 1$.

4.1.2 Normalisation Algorithm of MCGDM Problem:

Step 1: Framework of Decision Matrices

Here, we considered all decision matrices according to the decision maker's choice related with finite alternatives and finite attribute functions. It is noted that the member's y_{ij} for each matrices are of triangular neutrosophic numbers. Thus, the final matrix is defined as follows:

$$X^K = \begin{pmatrix} . & R_1 & R_2 & R_3 & . & . & . & R_n \\ P_1 & y_{11}^k & y_{12}^k & y_{13}^k & . & .. & . & y_{1n}^k \\ P_2 & y_{21}^k & y_{22}^k & y_{23}^k & . & . & . & y_{2n}^k \\ P_3 & \cdot & \cdot & \cdot & . & . & . & \cdot \\ . & \vdots & . & . & . & . & . & . \\ P_m & y_{m1}^k & y_{m2}^k & y_{m3}^k & . & . & . & y_{mn}^k \end{pmatrix} \dots\dots\dots\dots\dots\dots...(4.1)$$

Step 2: Framework of normalised matrix

To formulate a single group decision matrix X we utilized this logical operation $y_{ij}^{'} = \{\sum_{i=1}^{k} \omega_i X^i\}$ for individual decision matrix X^i. hence, the final matrix becomes as follows:

$$X = \begin{pmatrix} . & R_1 & R_2 & R_3 & . & . & . & R_n \\ P_1 & y_{11}^{'} & y_{12}^{'} & y_{13}^{'} & . & .. & . & y_{1n}^{'} \\ P_2 & y_{21}^{'} & y_{22}^{'} & y_{23}^{'} & . & . & . & y_{2n}^{'} \\ P_3 & \cdot & \cdot & \cdot & . & . & . & \cdot \\ . & \vdots & \vdots & \vdots & . & . & . & . \\ P_m & y_{m1}^{'} & y_{m2}^{'} & y_{m3}^{'} & . & . & . & y_{mn}^{'} \end{pmatrix} \dots\dots\dots\dots\dots\dots\dots(4.2)$$

Step 3: Framework of Final matrix

To formulate the final decision matrix we utilized the logical operation $y_{ij}^{''} = \{ \sum_{i=1}^{n} \Delta_i y_{ci}^{'} , c = 1,2 \dots . m\}$ for each individual Colum and finally, we get the decision matrix as,

$$X = \begin{pmatrix} . & R_1 \\ P_1 & y''_{11} \\ P_2 & y''_{21} \\ \vdots & \vdots \\ P_m & y''_{m1} \end{pmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.3)$$

Step 4: Ranking

Now, by considering the score and accuracy value (2.5.6) and converting the matrix (4.3) into crisp form, so that we could evaluate the best alternative corresponding to the best attributes.

4.1.3 Flowchart:

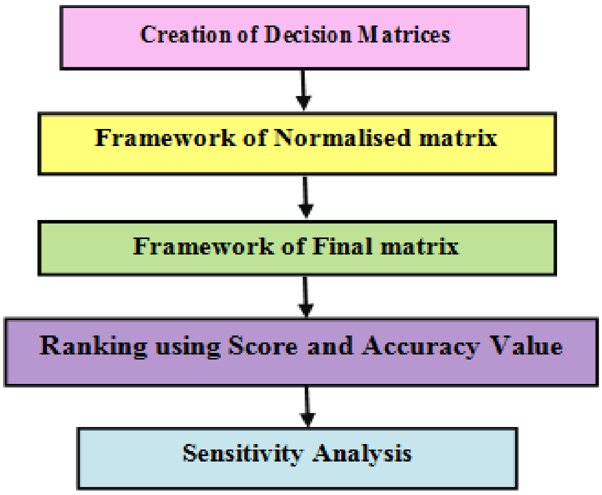


Figure 4.1.3.1: Flowchart for the problem

4.1.4 Illustrative Example:

Here, we constructed a social media selection problem in which we have considered three different social media services. Among these different social media platforms we want to select the best social media service in a logical way. Normally, social media services are fully dependent on the attributes like Safety & Privacy, efficiency & effectiveness and fulfilment & responsiveness of the system. Keeping these points in mind different banks provided some realistic information in which vagueness was present. Thus, we considered the data in the form of triangular neutrosophic number and according to their suggestions we constructed the distinct decision matrices in triangular neutrosophic environment as shows below: $P_1 = Facebook, P_2 = Twitter, P_3 = Youtube$ are the alternatives.$R_1 = Safety \,\&\, Privacy, R_2 = Efficiency \,\&\, Effectivness\ , R_3 = Fulfillment \,\&\, Reponsiveness$ are the attributes.

Let us select four distinct decision makers from our environment, $D_1 = Bank\ 1, D_2 = Bank\ 2, D_3 = Bank\ 3$ having weight distribution $D = \{\, 0.35, 0.33, 0.32 \,\}$ and the weight vector related with the attribute function $\Delta = \{0.32, 0.35, 0.33\}$.

Step 1

According to the decision maker's opinion the decision matrices are shown as follows:

$$D^1 = \begin{pmatrix} . & R_1 & R_2 & R_3 \\ P_1 & <8.5,10,11;0.8,0.5,0.4> & <62,65,67;0.7,0.4,0.5> & <51,54,57;0.6,0.5,0.5> \\ P_2 & <3,6,8;0.6,0.4,0.5> & <13,16,18;0.7,0.3,0.4> & <47,50,54;0.5,0.2,0.3> \\ P_3 & <3,5,7;0.5,0.3,0.2> & <23,26,30;0.6,0.3,0.4> & <24,28,30;0.4,0.6,0.7> \end{pmatrix}$$

$$Bank\ 1\ opinion$$

$$D^2 = \begin{pmatrix} . & R_1 & R_2 & R_3 \\ P_1 & <12,15,17;0.6,0.4,0.3> & <72,76,79;0.5,0.6,0.4> & <53,56,60;0.6,0.4,0.5> \\ P_2 & <10,12,15;0.5,0.4,0.3> & <35,37,39;0.5,0.2,0.3> & <24,26,29;0.5,0.4,0.5> \\ P_3 & <18,21,25;0.5,0.6,0.4> & <21,24,27;0.5,0.3,0.4> & <11,15,18;0.8,0.5,0.4> \end{pmatrix}$$

$$Bank\ 2\ opinion$$

$$D^3 = \begin{pmatrix} . & R_1 & R_2 & R_3 \\ P_1 & <21,23,25;0.6,0.4,0.5> & <26,29,31;0.6,0.4,0.5> & <41,45,47;0.7,0.3,0.2> \\ P_2 & <10,13,17;0.5,0.2,0.3> & <12,15,19;0.7,0.5,0.5> & <14,16,18;0.8,0.5,0.4> \\ P_3 & <42,45,49;0.6,0.4,0.5> & <6,9,13;0.6,0.4,0.5> & <5,7,10;0.4,0.2,0.3> \end{pmatrix}$$

$$Bank\ 3\ opinion$$

Step 2: Framework of Normalised decision matrix

$$M = \begin{pmatrix} . & R_1 & R_2 & R_3 \\ P_1 & <13.65,15.81,17.46;0.8,0.4,0.3> & <53.78,57.11,59.44;0.7,0.4,0.4> & <48.46,51.78,54.79;0.7,0.3,0.2> \\ P_2 & <7.55,10.22,13.19;0.6,0.2,0.3> & <19.94,22.61,25.25;0.7,0.2,0.3> & <28.85,31.2,34.23;0.8,0.2,0.3> \\ P_3 & <20.43,23.08,26.38;0.6,0.3,0.2> & <16.9,19.9,23.57;0.6,0.3,0.4> & <13.63,16.99,19.64;0.8,0.2,0.3> \end{pmatrix}$$

Step 3: Framework of Final matrix

$$M = \begin{pmatrix} <39.18,42.13,44.47;0.74,0.36,0.26> \\ <18.92,21.48,24.35;0.68,0.2,0.3> \\ <16.95,19.96,23.17;0.7,0.25,0.32> \end{pmatrix}$$

Step 4: Ranking

Now, we consider the score and Accuracy function technique (2.5.6), to convert the triangular neutrosophic numbers into crisp one, thus we get the final ideal decision matrix as

$$M = \begin{pmatrix} <33.34> \\ <17.65> \\ <16.01> \end{pmatrix}$$

Thus, ranking of the social media service is as $P_1 > P_2 > P_3$.

4.1.5 Results and Sensitivity Analysis

To understand how the attribute weights of each criterion affecting the relative matrix and their ranking a sensitivity analysis is done. The basic idea of sensitivity analysis is to exchange weights of the attribute values keeping the rest of the terms are fixed. The below table is the evaluation table which shows the sensitivity results.

Attribute Weight	Final Decision Matrix	Ordering
<($\mathbf{0.4}, \mathbf{0.3}, \mathbf{0.3}$)>	$\begin{pmatrix} < 28.26 > \\ < 15.56 > \\ < 14.42 > \end{pmatrix}$	$P_1 > P_2 > P_3$
<($\mathbf{0.3}, \mathbf{0.4}, \mathbf{0.3}$)>	$\begin{pmatrix} < 31.45 > \\ < 16.42 > \\ < 16.20 > \end{pmatrix}$	$P_1 > P_2 > P_3$
<($\mathbf{0.3}, \mathbf{0.3}, \mathbf{0.4}$)>	$\begin{pmatrix} < 30.54 > \\ < 16.44 > \\ < 17.30 > \end{pmatrix}$	$P_1 > P_3 > P_2$
<($\mathbf{0.32}, \mathbf{0.35}, \mathbf{0.33}$)>	$\begin{pmatrix} < 33.34 > \\ < 17.65 > \\ < 16.01 > \end{pmatrix}$	$P_1 > P_2 > P_3$
<($\mathbf{0.37}, \mathbf{0.32}, \mathbf{0.31}$)>	$\begin{pmatrix} < 35.62 > \\ < 16.23 > \\ < 15.45 > \end{pmatrix}$	$P_1 > P_2 > P_3$

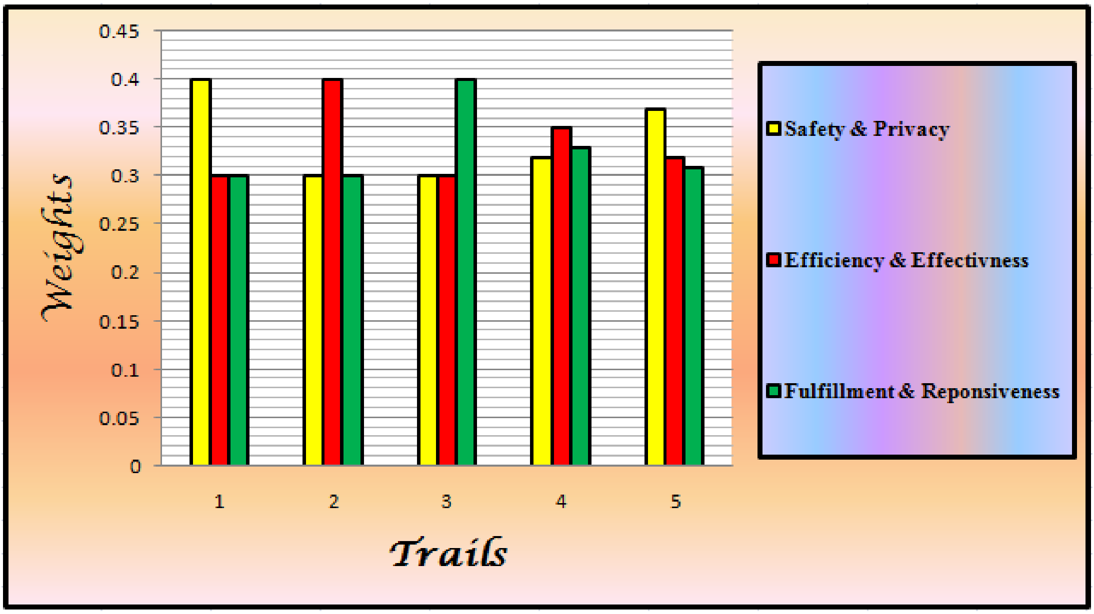


Figure 4.1.5.1: Sensitivity analysis table on attribute function.

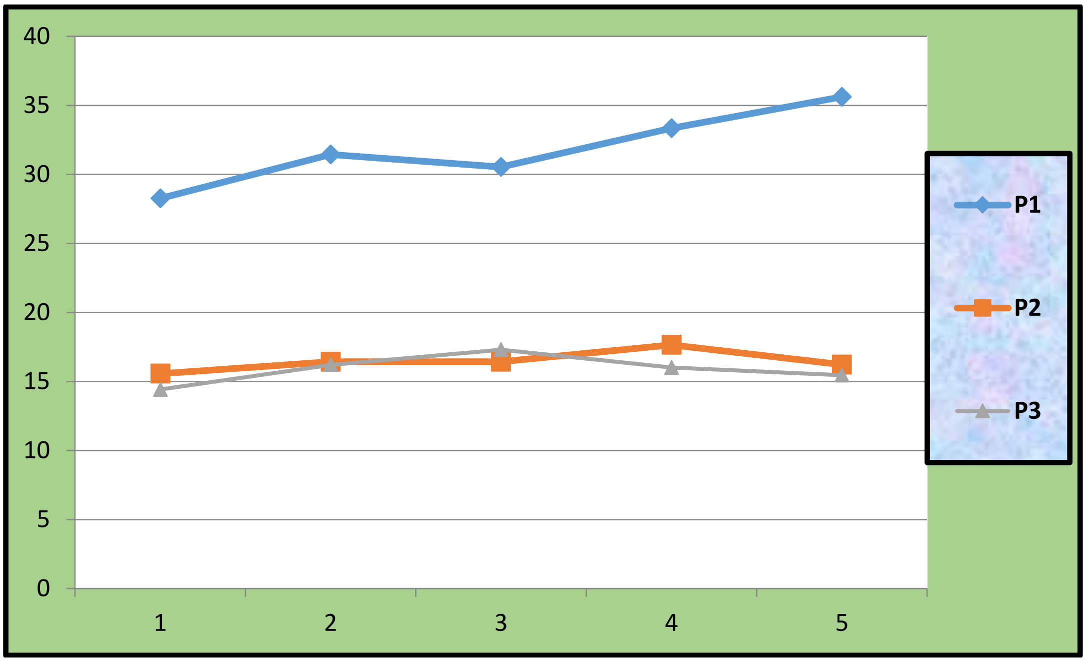


Figure 4.1.5.2: Best Alternative Social Media Service Table

4.1.6 Comparison Table

We compared this proposed work with the established works proposed by the researchers to find the best social media and it is noticed that in each cases P_1 (facebook) becomes the best social media service. The comparison table given as follows:

Approach	Ranking
(Deli, Ali, & Smarandache, 2015) [51]	$P_1 > P_2 > P_3$
(H.Garg, 2016) [36]	$P_1 > P_3 > P_2$
Our Proposed	$P_1 > P_2 > P_3$

5. Implication:

There are a lot of social media sites like face book, twitter, Google plus, linked in, you tube etc. available for online marketing. This study was primarily done to identify the impact of social media marketing especially in Banking Industry based on different social media attributes. We wanted to discover the right social media platform best suited for Banking Industry in West Bengal. The perception of vagueness plays a vital role in designing mathematical calculations. In this study we wanted to check the functionality of this system to find out the impact of different social media attributes on its acceptance in Online banking system in WB. Later we pioneered some more fascinating outcome on score and exactness function.

There are a lot of researches already done in social media implementation in Banking Industry. However many results are still unknown. Our work is to explore the idea in the following points:

- Defining the attributes necessary for social media platform for Banking Industry in West Bengal.
- Discovering the best suitable social media site for Banking Industry in West Bengal.
- The graphical representation of adaptation of social media platform based on its attributes.
- Application of Triangular neutrosophic number based MCGDM problem for selection of social media platforms.

Discussion

This study was done primarily to understand the perceptions of the people of West Bengal to use social media for their banking transactions. The study examined the three different types of websites i.e. Face book, Twitter and You Tube individually using three different attributes: Safety & Privacy, Efficiency & Effectiveness and Fulfillment & Responsiveness.

The study yielded new viewpoints that are useful to both academicians and Banks. This study showed that the selection of social media for Banking depends on various attributes which differs based on customers' perception.

All the three social media considered in this paper is different in nature. Communications & Social groups like Face book, Messaging & Communication like Twitter, and Content & Communication like You tube. Publicity in these three different social media sites differ both in content and context.

In the sample considered here men respondents are more than women; most of the respondents are under 40 years of age and they frequently uses social media. Like the worldwide trend here also it was observed that youngsters are dominating the social media sites. Social media mainly has impacted the life of youngsters. It has become radically significant since last ten years and it has attracted all age groups.

In West Bengal banking industry very less attention has been given to the measurement of social media quality and its effects. It is agreed that Banks must consider the overall social media quality measurement to satisfy customers. If the services experienced by customers are satisfactory, then it will induce them for long tern connectivity with banks. Long term connectivity with improved customer satisfaction in turn will bring customer loyalty.

Adaption of social media for banking industry is something beyond likes, comments and shares. The main aim of adaption of social media is brand awareness, creation of leads and ultimately conversions and finally brand advocacy. Banks should design their social media strategy considering their pragmatic goals. Once the goals are set it is important to find their KPIs (Key Performance Indicator) before implementing social media campaigns. A KPI is a quantifiable measurement to evaluate their campaign in relation to their defined goals. The common social media KPIs for banks can include Leads generation (through email signups or fulfilling some contact forms), Conversions (account sign ups, deposits), Referral traffic (from social media to website), Brand Advocacy (Like, comment and share)

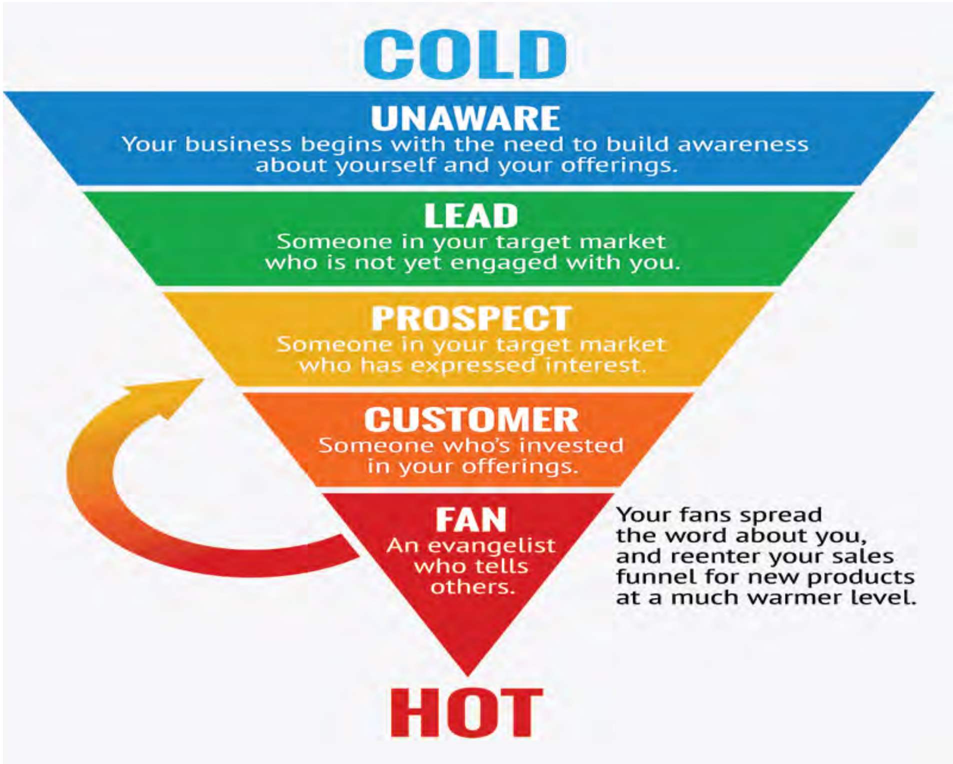


Figure 5.1: Example of Social Media KPI

6. Findings:

- All the three websites; Face book, Twitter and YouTube have gained attention among the social media users in India, but Face book is the widely used social media website.
- Banks are mostly using all international brands of social media channels for their operations due to lack of availability of good national social media networks. There is a great chance of development of some social media channels locally by the Govt.
- Bank's Social media Privacy drastically influences the endorsement of social media platform in the banking industry of West Bengal.
- Social media Efficiency appreciably control the acceptance of bank's social media platform in the West Bengal Banking Industry.
- Social media Fulfillment extensively influences the acceptance of social media platform in the West Bengal banking industry.

- Customers' prefer a bank that proposes them an experience that comprises all their service needs.
- All the three mentioned attributes have significant impact on overall customers' satisfaction which resulted in selection of Bank's social media platform
- Social media privacy appreciably persuades overall customer decision in selecting Banks social media sites in West Bengal banking industry. The study findings discovered that customers worth the social media privacy highly in banking operations.
- Face book is most preferred platform for all demography regardless of age, gender and occupation for all the Banks services.
- For You Tube and Twitter websites, people have different perceptions and choices depending on different Banks.
- Banks may augment their profit margin by increased customers' base through implementing proper social media strategies and reduction in cost due to lesser no of physical branches.

7. Conclusions:

In this current era, the West Bengal Banking Industry has conventionally been a high contact service submission. As implementation of social media reduces direct human interaction, hence there arises the need of continuous evaluation of service quality offered by Banks' social media sites and monitoring client's perception on it. It was observed that clients were satisfied with the traditional banking; still their expectations have grown bigger after introduction of e-services including social media.

This study concluded that the following attributes of social media like Safety & Privacy, efficiency & effectiveness and fulfillment & responsiveness have a significant influence on the service quality of social media in the West Bengal Banking Industry under neutrosophic environment. It was observed that customers mainly focuses on the attributes and service quality of Bank's social media, hence it is suggested that West Bengal Banking sector may priorities social media factors in their marketing mixes. Additionally, comparison analysis is done with the established methods and sensitivity analysis is performed in MCGDM technique under triangular neutrosophic arena. Finally it was concluded that successful implementation of social media in banking industry generates customer satisfaction and long term association which in turn converts to customer loyalty.

Further, researchers can apply this conception of triangular neutrosophic number in various fields like social business problem, diagnoses problem, mathematical modeling, pattern recognition problem, industrial problem, banking problem, marketing policy problem etc.

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Received: Apr 19, 2020 Accepted: July 9, 2020.

Neutrosophic Sets and Systems, Vol.35, 2020

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A Generalized Neutrosophic Solid Transportation Model with Insufficient Supply

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Abstract. The classical transportation problem and the solid transportation problem are special types of linear programming problems which are very important in Operations Research. In this paper, a solid transportation model is described, where the total supply of goods is insufficient to fulfil the total demand of goods, due to which the supplier company tries to obtain the required remaining goods from another source. An expression is derived to determine the import plan. The parameters of the model are considered to be uncertain and imprecise and are taken as trapezoidal neutrosophic numbers. The paper gives a general formulation of such a model and an algorithm is proposed to solve the model. The main objective function of the model present in the manuscript is to minimize the total cost. A formula is provided to check the degree of sufficiency of such a solution. The model is elucidated with a numerical example and its solution shows its efficiency and optimality in practical aspect. Finally, the paper provides a brief discussion about the computational time and some relative points of research.

Keywords: Solid Transportation Model, Insufficient supply, Trapezoidal Neutrosophic Number, Ranking function.

1 Introduction

Transportation is the movement of humans, animals, commodities, etc. from one location to another. Modes of transport include air, land (rail and road), water cable, pipeline and space. The field can be divided into infrastructure, vehicles and operations. Transportation is important because it enables trade between people, which is essential for the development of civilizations. It is a key component of growth and globalization.

The transportation problem (TP) was first forwarded by Hitchcock [1] in 1941. It is a popular type of problem in Operations Research where the decision maker wants to find the optimal way to transport goods from source warehouses to destination warehouses. So, there are two types of constraints, namely source constraints and demand constraints. But, real systems may contain other type of constraints too such as product type constraints or transportation mode constraints. This gives a third dimension to the transportation problem and converts the classical transportation problem into the solid transportation problem (STP).

The STP was first stated by Schell [2] in 1955 and later, in 1962, it was formally introduced by Haley [3]. In this paper, we consider that different types of conveyances are required for shipping goods and so, the third type of constraints here are the conveyance constraints.

The classical theories of Mathematics cannot solve problems which simulate real life situations. The information is imprecise and uncertain in nature. To deal with vague information, the fuzzy set theory was introduced by Zadeh [4] in 1965. But, fuzzy sets cannot represent imprecise information efficiently as they only consider the truth membership values of the data. Then, Atanassov [5, 6] introduced the concept of intuitionistic fuzzy sets, where the data are represented by their membership and non-membership values. But, they can only handle incomplete information, not indeterminate or inconsistent information.

Smarandache [8] proposed the concept of neutrosophic set theory by adding an independent indeterminacy membership. The neutrosophic set theory generalizes the concepts of classical set theory, fuzzy set theory, intuitionistic fuzzy set theory, and so on, since it considers all three aspects of decision-making, viz. “agree”, “disagree” and “not sure”.Basset et al. [24] used Neutrosophic theory to solve transition difficulties of Internet of Things identifying some challenge affectingthe process by non-traditional methods. In the article [26], an advance type of Neutrosophic set called type-2 Neutrosophic number are defined with TOPSIS method. A green supply chain model is developed incorporated with neutrosophic set and robust ranking technique and its performance is shown in decision making process [25].

Various researchers like Jiménez and Verdegay [7], Yang and Liu [9], Hussain and Kumar [10], Kundu et al. [11], Singh and Yadav [13], Das et al. [14], Giri et al. [16], Das et al. [18], Aggarwal and Gupta [19], etc. have studied the classical and solid transportation models in different fuzzy and intuitionistic fuzzy environments. A supply chain model is formulated based on some importance matrices based on economic, environment, social aspect as well as information gathering [23]. A hybrid pliogenic decision making approach is developed in this regard. Basset et al. [22] developed an evaluation model to show the performance and efficiency of medical care system with pliogenic set.

In this paper, a mathematical model is developed for the solid transportation model. The model is considered in neutrosophic environment so that we can address the fact of truth, indeterminacy and falsity arises in the data due to factors like unawareness of the scale of the problem, imperfection in data, poor forecasting, etc. As the concept of neutrosophic set theory is relatively new, a few of article is available dealing the transportation or solid transportation models with neutrosophic parameters in literature. A few of them in this context are by Thamaraiselvi and Santhi [15], and Rizk-Allah et al. [21].

The mathematical model present in this paperdescribes a transportation model shipping a homogeneous product from some source warehouses to some destination warehouses by means of heterogeneous conveyances. It is assumed that the conveyances have the necessary overall capacity to transport the whole demanded quantity of the commodity. In this research work, it is considered that the source warehouses do not have the sufficient quantity of goods to supply at a time and they fall short of some amount. At that time, the supplier decides to import the goods from another source. Again, if this new source does not have the requisite amount of goods, it imports the remaining amount from another source, and so on. This process is continued until the fulfilment of the total demand. It terminates after a certain number of sources, since the total original demand of goods is a fixed quantity. The paper addresses the general notion of the situation and also the presence of uncertainties in the data.

The main contribution of the paper is to develop the mathematical model for solid transportation plan to satisfy the demand of customer with insufficient supply of source point.The main objective function of the model is to minimize the total cost. In this research work, parameters of the model are considered in neutrosophic environment. Consideration of neutrosophic number gives an ideal approach of a decision making process dealing the uncertainty with truth, false and in determinant state of information. In this regard, trapezoidal neutrosophic number is used in this STP model. A proposition is provided to establish the relation between the import goods and the cost which define a degree of insufficiency. Hereby, a solution algorithm is given in his manuscript. A numerical example is also shown to discuss the performance of the model.

In this paper, Section 2 contains some preliminary definitions and concepts regarding the model. Section 3 describes the model and gives a general formulation of the model. Section 4 is all about the solution approach to the problem, concerned with the model and Section 5 helps in understanding the model with the help of a numerical example and its solution by the given procedure. Finally, Section 6 briefly discusses the model along with the computational time of the solution process, exemplified by the numerical example. It also suggests some relative points of research and is followed by the conclusion.

2 Preliminaries

In this section, we recall some important definitions and concepts.

2.1 Single-valued neutrosophic set [20]

Let X be a non-empty set. Then a single-valued neutrosophic (SVN) set $\tilde{A}$ of X is defined as

$$\tilde{A}=\{\langle x, T_{\tilde{A}}(x), I_{\tilde{A}}(x), F_{\tilde{A}}(x)\rangle \mid x \in X\},$$

where $T_{\tilde{A}}(x)$, $I_{\tilde{A}}(x)$, $F_{\tilde{A}}(x) \in [0, 1]$ and $0 \le T_{\tilde{A}}(x) + I_{\tilde{A}}(x) + F_{\tilde{A}}(x) \le 3$, $\forall x \in X$. $T_{\tilde{A}}(x)$, $I_{\tilde{A}}(x)$ and $F_{\tilde{A}}(x)$ respectively represent truth membership, indeterminacy membership and falsity membership degrees of x in $\tilde{A}$.

2.2 Trapezoidal neutrosophic number [20]

A trapezoidal neutrosophic number (TNN) $\tilde{A}$ is a neutrosophic set in $\boldsymbol{R}$ with the following truth, indeterminacy and falsity membership functions:

$$T_{\tilde{A}}(x) = \begin{cases} \alpha_{\tilde{A}}\left(\dfrac{x-a_1}{a_2-a_1}\right) & for\ a_1 \le x \le a_2, \\ \alpha_{\tilde{A}} & for\ a_2 \le x \le a_3, \\ \alpha_{\tilde{A}}\left(\dfrac{a_4-x}{a_4-a_3}\right) & for\ a_3 \le x \le a_4, \\ 0 & otherwise, \end{cases}$$

$$I_{\tilde{A}}(x) = \begin{cases} \dfrac{a_2 - x + \theta_{\tilde{A}}(x-a_1')}{a_2-a_1'} & for\ a_1' \le x \le a_2, \\ \theta_{\tilde{A}} & for\ a_2 \le x \le a_3, \\ \dfrac{x - a_3 + \theta_{\tilde{A}}(a_4'-x)}{a_4'-a_3} & for\ a_3 \le x \le a_4', \\ 1 & otherwise, \end{cases}$$

$$F_{\tilde{A}}(x) = \begin{cases} \dfrac{a_2 - x + \beta_{\tilde{A}}(x-a_1'')}{a_2-a_1''} & for\ a_1'' \le x \le a_2, \\ \beta_{\tilde{A}} & for\ a_2 \le x \le a_3, \\ \dfrac{x - a_3 + \beta_{\tilde{A}}(a_4''-x)}{a_4''-a_3} & for\ a_3 \le x \le a_4'', \\ 1 & otherwise, \end{cases}$$

where $\alpha_{\tilde{A}}$, $\theta_{\tilde{A}}$ and $\beta_{\tilde{A}}$ represent the maximum degree of truthiness, minimum degree of indeterminacy and minimum degree of falsity respectively, $\alpha_{\tilde{A}}, \theta_{\tilde{A}}, \beta_{\tilde{A}} \in [0, 1]$. Also, $a_1'' \le a_1 \le a_1' \le a_2 \le a_3 \le a_4' \le a_4 \le a_4''$.

The membership functions of trapezoidal neutrosophic number are shown in Fig. 1.

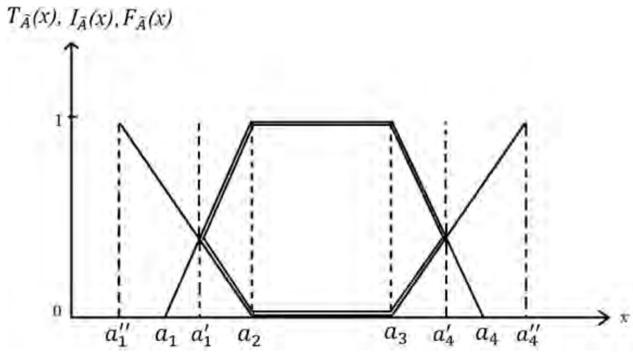

Figure 1: Truth, indeterminacy and falsity membership functions of trapezoidal neutrosophic number.

2.3 Ranking function [20]

A ranking function of neutrosophic numbers is a function $\Re : N(R) \rightarrow \boldsymbol{R}$, where $N(R)$ is a set of neutrosophic numbers defined on the set of real numbers, which convert each neutrosophic number into the real line.

Let $\tilde{A} = \langle\langle a_1, a_2, a_3, a_4\rangle; \alpha_{\tilde{A}}, \theta_{\tilde{A}}, \beta_{\tilde{A}}\rangle$ and $\tilde{B} = \langle\langle b_1, b_2, b_3, b_4\rangle; \alpha_{\tilde{B}}, \theta_{\tilde{B}}, \beta_{\tilde{B}}\rangle$ betwo trapezoidal neutrosophic numbers.

- If $\Re(\tilde{A}) > \Re(\tilde{B})$, then $\tilde{A} \gtrsim \tilde{B}$,
- If $\Re(\tilde{A}) < \Re(\tilde{B})$, then $\tilde{A} \lesssim \tilde{B}$,
- If $\Re(\tilde{A}) = \Re(\tilde{B})$, then $\tilde{A} \approx \tilde{B}$.

3 Description and formulation of model

Real life situations regarding transportation of commodities are complex which give rise to various transportation models. This paper discusses one such situation where the primary "supplier" company (say, Y_1) has shortage of goods to meet the adequate demand of the primary "purchaser" company (say, Y_0).

It may happen that the total required amount of goods cannot be produced due to shortage of time or lack of raw materials or some other factors to fulfill the total demand. So, Company Y_1 decides to import the remaining amount of goods from another company (say, Y_2) and then transport the aggregate amount to Company Y_0. Again, it may happen that Company Y_2 faces the same problem, where it is unable to fulfill the total demand of Company Y_1. So, Company Y_2 imports the remaining amount from another company (say, Y_3). The chain continues until Company Y_N (say) fulfills the total demand of Company Y_{N-1} (say). The process surely terminates, since the total original demand of Company Y_0 is a finite quantity. Here, N is at least 2.

While stating its demand, Company Y_0 may not be sure about the exact quantity of goods it needs. This may be due to the nature of the commodities, uncertain market trend and business scope, etc. Similarly, due to possible production and technical issues, the supply quantity of goods may be uncertain. Also, uncertainty may arise in determining the costs of transportation and the exact capacities of the conveyances due to road issues, weather issues, etc. So, here, all of these parameters in all the *N* steps are considered as trapezoidal neutrosophic numbers.

3.1 Assumptions

- The total supply (in stock) of Company Y_P from its origin warehouses is insufficient to fulfill the total demand of the destination warehouses of Company Y_{P-1} (p = 1, 2, …, N – 1).
- Company Y_{P-1} is indifferent to the arrangement of goods by Company Y_P and Company Y_{P+1} is indifferent to the use of the goods imported by Company Y_P (p = 1, 2, …, N – 1).
- Company Y_P does not have any extra warehouse to import goods. It imports the remaining amount of goods to its existing warehouses (p = 1, 2, …, N – 1).
- The warehouses of company Y_P have the capacity to hold the remaining amount, but the whole amount cannot be stored in a single warehouse and is transported to each of the warehouses in parts (p = 1, 2, …, N – 1).
- The total conveyance capacity of Company Y_P is greater than or equal to the total demand of Company Y_{P-1} (p = 1, 2, …, N).
- Company Y_N can supply the remaining quantity of goods, demanded (required) by Company Y_{N-1}, from its warehouses sufficiently. So, the model saturates in the N^{th} step and thus it is an N-step model.

3.2 Notations

- $c_{ijk}^{(p)}$: Per unit cost of transportation from the i^{th} origin warehouse to the j^{th} destination warehouse by the k^{th} conveyance in the p^{th} step.
- $x_{ijk}^{(p)}$: Amount of goods to be transported from the i^{th} origin warehouse to the j^{th} destination warehouse by the k^{th} conveyance in the p^{th} step.

- : Amount by which the supply falls short in the p^{th} step.
- $A_i^{(p)}$: Original amount of supply of the i^{th} origin in the p^{th} step.
- $a_i^{(p)}$: Total amount of supply of the i^{th} origin in the p^{th} step.
- $b_j^{(p)}$: Amount of demand of the j^{th} destination in the p^{th} step.
- $e_k^{(p)}$: Capacity of the k^{th} conveyance in the p^{th} step.
- m_p : Number of origin warehouses in the p^{th} step.
- m_{p-1} : Number of destination warehouses in the p^{th} step.
- K_p : Number of conveyances in the p^{th} step.
- $AC_i^{(p)}$: Average per unit cost of transportation from the i^{th} origin in the p^{th} step.
- $H^{(p)}$: Harmonic mean of $AC_i^{(p)}$'s ($i = 1, 2, \ldots, m_p$) in the p^{th} step.

3.3 Formulation

The model is formulated mathematically as follows:

$$Min\ z^{(p)} = \sum_{i=1}^{m_p}\sum_{j=1}^{m_{p-1}}\sum_{k=1}^{k_p} c_{ijk}^{(p)} x_{ijk}^{(p)} \quad ; \; p = 1,2,\ldots\ldots,N \tag{1}$$

Subject to

$$\sum_{j=1}^{m_{p-1}}\sum_{k=1}^{k_p} x_{ijk}^{(p)} \le a_i^{(p)}; \quad p = 1,2,\ldots\ldots,N; \qquad i = 1,2,\ldots\ldots\ldots,m_p \tag{2}$$

$$\sum_{i=1}^{m_p}\sum_{k=1}^{k_p} x_{ijk}^{(p)} \ge b_j^{(p)}; \quad p = 1,2,\ldots\ldots,N; \qquad j = 1,2,\ldots\ldots\ldots,m_{p-1} \tag{3}$$

$$\sum_{i=1}^{m_p}\sum_{j=1}^{m_{p-1}} x_{ijk}^{(p)} \le e_k^{(p)}; \quad p = 1,2,\ldots\ldots,N; \qquad k = 1,2,\ldots\ldots\ldots,k_p \tag{4}$$

$$\text{and} \quad x_{ijk}^{(p)} \ge 0 \quad \forall\ \text{p, i, j, k} \tag{5}$$

where

$$a_i^{(p)} = A_i^{(p)} + b_i^{(p+1)} \quad p = 1,2,\ldots\ldots\ldots,n-1; \quad i = 1,2,\ldots\ldots\ldots\ldots,m_p \tag{6}$$

$$a_i^{(N)} = A_i^{(N)} i = 1,2,\ldots\ldots\ldots\ldots,m_N \tag{7}$$

$$b_j^{(p)} = \left\lfloor \frac{x_s^{(p-1)} H^{(p-1)}}{AC_j^{(p-1)} m_{p-1}} \right\rfloor; \quad p = 2,3,\ldots\ldots\ldots\ldots,N; \quad j = 1,2,\ldots\ldots m_{p-1} \tag{8}$$

$$x_s^{(p)} = \sum_{j=1}^{m_{p-1}} b_j^{(p)} - \sum_{i=1}^{m_p} A_i^{(p)} > 0; \quad p = 1,2,\ldots\ldots.N-1 \tag{9}$$

$$AC_i^{(p)} = \sum_{j=1}^{m_{p-1}}\sum_{k=1}^{k_p} c_{ijk}^{(p)}; \; p = 1,2,\ldots\ldots,N-1; \quad i = 1,2,\ldots\ldots\ldots,m_p \tag{10}$$

$$H^{(p)} = \frac{m_p}{\sum_{i=1}^{m_p}\frac{1}{AC_i^{(p)}}}; \quad p = 1,2,\ldots\ldots.N-1 \tag{11}$$

As it can be seen, there are N objective functions in (1) for N steps (p = 1, 2, …, N) of the model. Here, the value of N is always a finite natural number greater than or equal to 2. (2), (3) and (4) are the supply, demand and conveyance constraints respectively. The non-negativity constraints (5) are must, since the quantity of goods is always non-negative.

Here, all the parameters and the decision variables$x_{ijk}^{(p)}$ are taken as trapezoidal neutrosophic numbers. But, $x_{ijk}^{(p)}$denote quantities of goods to be transported and in reality, any manager or decision maker would want to obtain the crisp optimal solution of the problem through considering vague, imprecise and inconsistent information while defining the problem.

Equation (8) is used to calculate $b_j^{(p)}$'s (crisp values) after all the given parameters are converted into their corresponding crisp values by a suitable ranking function. So, (8) becomes

$$b_j^{(p)} = \left\lceil \frac{x_s^{(p-1)} H^{(p-1)}}{AC_j^{(p-1)} m_{p-1}} \right\rceil; \qquad p = 2, 3, \dots, N, \qquad j = 1, 2, \dots, m_{p-1}. \tag{12}$$

Proposition 3.3.1

If the import plan due to insufficient supply for each supplier Company Y_p (p = 1, 2, …, N – 1) is – "import the highest quantity of goods from Y_{p+1} to that warehouse *j*from which the average per unit cost of transportation of goods to Y_{p-1} is minimum", then the import plan (quantity of goods to be imported to each warehouse *j*) is mathematically given by:

$$b_j^{(p)} = \left\lceil \frac{x_s^{(p-1)} H^{(p-1)}}{AC_j^{(p-1)} m_{p-1}} \right\rceil; \qquad p = 2, 3, \dots, N, \qquad j = 1, 2, \dots, m_{p-1}.$$

Proof:

Here, $b_j^{(p)}$'s denote the demands of the destination warehouses in the p^{th} step, which are also the origin warehouses in the $(p-1)^{th}$ step. In the p^{th} step, we want to import the highest quantity of goods to that warehouse *j*from which the average per unit cost of transportation of goods A $AC_j^{(p-1)}$minimum. So, $b_j^{(p)}$in-versely proportional to $AC_j^{(p-1)}$i.e.,

$$b_j^{(p)} \propto \frac{1}{AC_j^{(p-1)}}$$

i.e.,
$$b_j^{(p)} = \kappa \frac{1}{AC_j^{(p-1)}},$$

where κ is the proportionality constant.

Now, $\text{total demand } = x_s^{(p-1)}$

i.e.,$\sum_{j=1}^{m_{p-1}} b_j^{(p)} = x_s^{(p-1)}$

i.e.,$\sum_{j=1}^{m_{p-1}} K \frac{1}{AC_j^{(p-1)}} = x_s^{(p-1)}$

i.e., $K = \frac{x_s^{(p-1)}}{\sum_{j=1}^{m_{p-1}} \frac{1}{AC_j^{(p-1)}}}$

But,$H^{(p-1)} = \frac{m_{p-1}}{\sum_{i=1}^{m_{p-1}} \frac{1}{AC_i^{(p-1)}}} = \frac{m_{p-1}}{\sum_{j=1}^{m_{p-1}} \frac{1}{AC_j^{(p-1)}}}$

Therefore, $$K = \frac{x_s^{(p-1)} H^{(p-1)}}{m_{p-1}}$$

And so, $$b_j^{(p)} = \frac{x_s^{(p-1)} H^{(p-1)}}{AC_j^{(p-1)} m_{p-1}} \tag{13}$$

Now, while calculating $b_j^{(p)}$'s using (13), rounding off their decimals may result in loss of significant quantity of $x_s^{(p-1)}$'s. So, we use the ceiling function and hence, obtain

$$b_j^{(p)} = \left\lceil \frac{x_s^{(p-1)} H^{(p-1)}}{AC_j^{(p-1)} m_{p-1}} \right\rceil; \qquad p = 2, 3, \dots, N, \qquad j = 1, 2, \dots, m_{p-1}.$$

4 Solution approach

An approach is suggested to find the optimal solution of such problems. The step by step procedure is as follows:

Step I. Collect the information for a given problem as trapezoidal neutrosophic numbers from the decision makers with the information that we always want to maximize the truth degree and minimize the indeterminacy and falsity degrees of the data.

Step II. Construct the neutrosophic solid transportation table of the given problem for $p = 1$.

Step III. Convert all the trapezoidal neutrosophic numbers into their equivalent crisp values by the use of the ranking function, proposed by M. Abdel-Basset et al. [20], which is given by:

$$\Re(\tilde{a}) = \frac{a^l + a^u - 3(a^{m1} + a^{m2})}{2} + \textit{confirmation degree},$$

or mathematically,

$$\Re(\tilde{a}) = \frac{a^l + a^u - 3(a^{m1} + a^{m2})}{2} + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}}), \tag{14}$$

where $T_{\tilde{a}}$, $I_{\tilde{a}}$ and $F_{\tilde{a}}$ are respectively the truth, indeterminacy and falsity degrees of the trapezoidal neutrosophic number $\tilde{a} = (a^l, a^u, a^{m1}, a^{m2})$. Here, a^l, a^u, a^{m1} and a^{m2} are the lower bound, upper bound, first median and second median values of $\tilde{a}$ respectively, which can be obtained from the form $\tilde{a} = (a_1, a_2, a_3, a_4)$ by the transformations: $a^l = a_2$, $a^u = a_3$, $a^{m1} = a_2 - a_1$ and $a^{m2} = a_4 - a_3$.

Step IV. Compute $x_s^{(p)}$ for $p = 1$ using the crisp form of (9).

Step V. Calculate $b_j^{(p)}$'s for $p = 2$ using (12).

Step VI. Construct the crisp solid transportation table for $p = 2$ using the ranking function (14) for the values of supply and conveyance capacity.

Step VII. Repeat Steps (IV – VI) until some $x_s^{(p-1)}$ (say, for $p = N$) is found, which can be totally satisfied by the next supplier company (Y_N).

Step VIII. Calculate $b_j^{(p)}$'s for $p = N$ using (12).

Step IX. Construct the crisp solid transportation table for $p = N$ using the ranking function (14) for the values of supply and conveyance capacity.

Step X. Solve the crisp solid transportation table for $p = N$ using a standard method as used for solving a general crisp STP and obtain the optimal solution for this step.

Step XI. Compute the new crisp values of supply for $p = N - 1$ using the crisp form of (6) and similarly, solve the table for $p = N - 1$ as solved for $p = N$.

Step XII. Repeat Step XI and similarly, solve the tables for $p = N - 2, N - 3, \dots, 1$.

Step XIII. Conclude the solution with the degree of sufficiency η, which is defined as:

$$\eta = \frac{1}{N-1} \sum_{p=1}^{N-1} \eta^{(p)}, \tag{15}$$

where

$$\eta^{(p)} = \frac{\sum_{i=1}^{m_p} a_i^{(p)} - \sum_{i=1}^{m_p} A_i^{(p)}}{x_s^{(p)}} - 1 \tag{16}$$

The proposed solution approach in the paper is a first of its kind. The solution approach is depicted as follows:

Transportation Possible if $X \geq Y$

Transportation Not Possible if $X < Y$

The model proposed in the paper makes the transportation possible in the second case (shown above).

5Numerical example

Suppose, Company Y_1 has to transport a commodity (e.g., wheat) to Company Y_0. But, it falls short of some amount and wants to import the required amount from Company Y_2. Similarly, Company Y_2 does not have the sufficient amount of wheat to fulfill the total demand of Company Y_1, so Company Y_2 imports the required amount from Company Y_3. It is assumed that Company Y_3 has the right amount of wheat to fulfill the total demand of Company Y_2.

The neutrosophic data for the transportations $Y_1 \to Y_0$, $Y_2 \to Y_1$ and $Y_3 \to Y_2$ are given in Table 1, Table 2 and Table 3 respectively. For the sake of simplicity, $(T_{\tilde{a}}, I_{\tilde{a}}, F_{\tilde{a}})$ is taken as (0.9, 0.1, 0.1) for all the trapezoidal neutrosophic numbers. The costs are considered in INR and the commodity is measured in kilograms.

$Y_1 \to Y_0$							Conveyance Capacity
	$E_1^{(1)}$			$E_1^{(1)}$			(3150,4500,170,185)
		$E_2^{(1)}$			$E_2^{(1)}$		(4100,5200,200,180)
			$E_3^{(1)}$			$E_3^{(1)}$	(2900,3850,175,190)
	$D_1^{(1)}$			$D_2^{(1)}$			Supply
$O_1^{(1)}$	(50,65,7,6)	(40,60,7,7)	(90,110,8,9)	(80,100,6,8)	(40,55,5,7)	(55,70,4,3)	(1300,1600,140,170)
$O_2^{(1)}$	(70,80,6,9)	(80,95,6,4)	(65,75,4,6)	(30,45,7,5)	(50,70,3,5)	(65,85,6,7)	(1650,2000,165,150)
$O_3^{(1)}$	(60,80,6,5)	(45,60,4,5)	(70,85,9,6)	(60,85,6,7)	(75,95,8,5)	(95,115,7,4)	(1050,1400,120,135)
Demand	(4500, 5700,250,230)			(5000,6500,245,260)			

Table 1: Neutrosophic data table for $Y_1 \to Y_0$.

$Y_2 \to Y_1$							Conveyance Capacity
	$E_1^{(2)}$		$E_1^{(2)}$		$E_1^{(2)}$		(3400,4150,150,130)
		$E_2^{(2)}$		$E_2^{(2)}$		$E_2^{(2)}$	(3250,3900,155,140)
	$D_1^{(2)}$		$D_2^{(2)}$		$D_3^{(2)}$		Supply
$O_1^{(2)}$	(90,110,10,8)	(50,70,4,6)	(40,55,5,7)	(70,85,9,6)	(35,55,4,5)	(75,85,8,6)	(1000,1300,120,135)

$\boldsymbol{O_2^{(2)}}$	(75,90,7,8)	(65,80,6,5)	(85,95,6,6)	(50,60,4,3)	(60,80,4,6)	(60,85,9,7)	(1250,1500,145,160)
$\boldsymbol{O_3^{(2)}}$	(80,90,6,7)	(90,105,9,8)	(55,75,8,7)	(75,90,5,5)	(100,120,5,7)	(40,55,6,5)	(1000,1350,105,125)
$\boldsymbol{O_4^{(2)}}$	(30,50,4,4)	(70,85,5,6)	(80,95,5,7)	(60,80,4,6)	(65,75,8,7)	(50,65,4,3)	(1200,1450,110,100)
Demand	( )		( )		( )		

Table 2: Neutrosophic data table for $Y_2 \rightarrow Y_1$.

$Y_3 \rightarrow Y_2$									**Conveyance Capacity**
	$\boldsymbol{E_1^{(3)}}$		$\boldsymbol{E_1^{(3)}}$		$\boldsymbol{E_1^{(3)}}$		$\boldsymbol{E_1^{(3)}}$		(1650,1950,135,140)
		$\boldsymbol{E_2^{(3)}}$		$\boldsymbol{E_2^{(3)}}$		$\boldsymbol{E_2^{(3)}}$		$\boldsymbol{E_2^{(3)}}$	(1800,2200,140,150)
	$\boldsymbol{D_1^{(3)}}$		$\boldsymbol{D_2^{(3)}}$		$\boldsymbol{D_3^{(3)}}$		$\boldsymbol{D_4^{(3)}}$		**Supply**
$\boldsymbol{O_1^{(3)}}$	(85,95,7,8)	(35,50,4,5)	(60,75,8,6)	(70,80,9,7)	(50,60,3,6)	(50,70,4,6)	(80,95,7,5)	(30,50,4,6)	(1150,1350,105,100)
$\boldsymbol{O_2^{(3)}}$	(70,85,6,7)	(45,60,6,5)	(50,70,4,6)	(40,60,4,5)	(80,95,8,6)	(60,70,4,6)	(50,65,4,3)	(85,95,8,7)	(800,1100,90,95)
$\boldsymbol{O_3^{(3)}}$	(50,70,5,4)	(65,75,7,4)	(85,95,7,7)	(75,90,9,6)	(45,60,5,4)	(55,70,5,5)	(35,55,4,5)	(50,65,5,6)	(1400,1700,115,135)
Demand	( )		( )		( )		( )		

Table 3: Neutrosophic data table for $Y_3 \rightarrow Y_2$.

The crisp tables are solved with LINGO 17.0 software. Table 4, Table 5 and Table 6 show the optimal crisp solutions for $Y_3 \rightarrow Y_2$, $Y_2 \rightarrow Y_1$ and $Y_1 \rightarrow Y_0$ respectively.

The minimum values of the objective functions are given below:

$z^{(1)}$: ₹308719.43
$z^{(2)}$: ₹216642.65
$z^{(3)}$: ₹90474.05

The degree of sufficiency η is found out to be 0.00062.

$Y_3 \rightarrow Y_2$									**Conveyance Capacity**
	$\boldsymbol{E_1^{(3)}}$		$\boldsymbol{E_1^{(3)}}$		$\boldsymbol{E_1^{(3)}}$		$\boldsymbol{E_1^{(3)}}$		1388.2
		$\boldsymbol{E_2^{(3)}}$		$\boldsymbol{E_2^{(3)}}$		$\boldsymbol{E_2^{(3)}}$		$\boldsymbol{E_2^{(3)}}$	1565.7
	$\boldsymbol{D_1^{(3)}}$		$\boldsymbol{D_2^{(3)}}$		$\boldsymbol{D_3^{(3)}}$		$\boldsymbol{D_4^{(3)}}$		**Supply**
$\boldsymbol{O_1^{(3)}}$	0 (68.20)	737 (29.70)	0 (47.20)	0 (51.70)	0 (42.20)	0 (45.70)	0 (70.20)	184.7 (25.70)	943.2
$\boldsymbol{O_2^{(3)}}$	0 (58.70)	0 (36.70)	0 (45.70)	644 (37.20)	0 (67.20)	0 (50.70)	0 (47.70)	0 (68.20)	673.2
$\boldsymbol{O_3^{(3)}}$	0 (47.20)	0 (54.20)	0 (69.70)	0 (60.70)	585 (39.70)	0 (48.20)	517.3 (32.20)	0 (41.70)	1175.7
Demand	737		644		585		702		

Table 4: Optimal solution table for $Y_3 \rightarrow Y_2$.

$Y_2 \rightarrow Y_1$							**Conveyance Capacity**
	$\boldsymbol{E_1^{(2)}}$		$\boldsymbol{E_1^{(2)}}$		$\boldsymbol{E_1^{(2)}}$		3355.7
		$\boldsymbol{E_2^{(2)}}$		$\boldsymbol{E_2^{(2)}}$		$\boldsymbol{E_2^{(2)}}$	3133.2
	$\boldsymbol{D_1^{(2)}}$		$\boldsymbol{D_2^{(2)}}$		$\boldsymbol{D_3^{(2)}}$		**Supply (new)**
$\boldsymbol{O_1^{(2)}}$	0 (73.70)	0 (45.70)	1087.9 (30.20)	0 (55.70)	417.3 (32.20)	0 (59.70)	1505.2
$\boldsymbol{O_2^{(2)}}$	0	507.3	0	1052.1	0	0	1562.2

	(60.70)	(56.70)	(72.70)	(45.20)	(55.70)	(49.20)	
$O_3^{(2)}$	0 (66.20)	0 (72.70)	0 (43.20)	0 (68.20)	0 (92.70)	1415.7 (31.70)	1415.7
$O_4^{(2)}$	1712.7 (28.70)	0 (61.70)	0 (70.20)	0 (55.70)	0 (48.20)	0 (47.70)	1712.7
Demand	2220		2140		1833		

Table 5: Optimal solution table for $Y_2 \rightarrow Y_1$.

$Y_1 \rightarrow Y_0$							**Conveyance Capacity**
	$E_1^{(1)}$			$E_1^{(1)}$			3293.2
		$E_2^{(1)}$			$E_2^{(1)}$		4080.7
			$E_3^{(1)}$			$E_3^{(1)}$	2828.2
	$D_1^{(1)}$			$D_2^{(1)}$			**Supply (new)**
$O_1^{(1)}$	0 (38.70)	1705.7 (29.70)	0 (75.20)	0 (69.70)	1500 (30.20)	0 (52.70)	3205.7
$O_2^{(1)}$	0 (53.20)	0 (73.20)	0 (55.70)	3293.2 (20.20)	0 (48.70)	200 (56.20)	3493.2
$O_3^{(1)}$	0 (54.20)	875 (39.70)	1800 (55.70)	0 (53.70)	0 (66.20)	0 (89.20)	2676.2
Demand	4380.7			4993.2			

Table 6: Optimal solution table for $Y_1 \rightarrow Y_0$.

6Discussion

The model, discussed in this paper, is a very interesting solid transportation model, which can be useful in the business sector. It can be safely concluded that the problem of insufficient supply that arises in the model, is dealt with effectively, as the degree of sufficiency η is positive for the given example and is very close to 0 (zero). η should always be non-negative and the closer η is to zero, the more sufficient the solution is for the model.

The numerical example given above is a 3-step model with fewer amounts of data. The computational time for this problem is not too high, but in real systems, the data is greater in amount and so, higher will be the computational time. The computational time T for an N-step model may roughly be given by the expression:

$$T = Ca(n) + Co(n) + Sol(n),$$

where

Ca(n) is the total time component for calculatio $b_j^{(p)}$ s ’s,

Co(n) is the total time component for conversion of the trapezoidal neutrosophic numbers into crisp values,

Sol(n) is the total time component for solving the crisp data,

and all these components depend on the amount of data n, each of them varying directly with n.

Clearly, the amount of data n consists of N heterogeneous components. So, *Ca(n)*has N – 1 sub-components, *Co(n)* has N sub-components and *Sol(n)* also has N sub-components. For the given numerical example, *Ca(n)* has 2 components, while *Co(n)* and *Sol(n)* have 3 components each.

Equation (12) is a key part of the solution method and a point of research for constructing a more efficient model. The degree of sufficiency η is evidently dependent on (12). The ranking function (14) converts the trapezoidal neutrosophic numbers into their equivalent crisp values effectively by considering the degrees of all three aspects of decision, but efforts can be made to construct a better ranking function to get more accurate crisp models and better results. The model may also be considered with a time constraint (along with some time penalty) for each supplier. Also, as we know that the notion of neutrosophic set theory is relatively new and it broadly covers all the aspects of decision making, so there is a good potential for its extensive research and applications in complex logistic systems.

7 Conclusion and future scope

The solid transportation problem is a significant problem in Operations Research, where the primary goal is to transport commodities from some source warehouses to some destination warehouses via different modes of conveyance. This paper formulates a model, where the source cannot fulfill the total demand and brings in the required amount from another source, which in turn, if unable to supply the necessary amount, brings in the remaining amount from another source, and so on. An expression is derived to provide the distribution of demand of the deficient quantity of goods among the importing warehouses. The paper also considers the impreciseness and uncertainty that may exist in the data and takes the input as trapezoidal neutrosophic numbers. An approach is presented to solve the model and the quality of the solution is checked with the degree of sufficiency. Also, the computational time is shown for the model and it is believed that the model is useful and has an interesting scope.

In this manuscript, the mathematical model has considered the minimization of cost as an objective function. But it is very important to complete the fulfilment of demand of customers as early as possible. Therefore, for future research, one can consider the minimization of time as an objective function. In the business purpose, profit is essential to grow. In this regard, maximization of profit can be treated as an objective function for further study. In this manuscript, uncertainty is used in terms of trapezoidal neutrosophic number. But in the direction of future research, one can use different parameters e.g. uncertain number, fuzzy number, type-2 fuzzy number etc. To solve the problem, genetic algorithm can be developed in future research.

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Received: Apr 20, 2020. Accepted: July 10, 2020

Neutrosophic Sets and Systems, Vol. 35, 2020
University of New Mexico

Generalized b Closed Sets and Generalized b Open Sets in Fuzzy Neutrosophic bi-Topological Spaces

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Abstract: In this paper, the authors study and introduce a new class of sets called generalized b-closed sets and its complement generalized b-open sets via fuzzy neutrosophic bi-topological spaces. Also, we prove some theorem related to this definitions. Then, we investigate the relations between the new defined sets by hand and some other fuzzy neutrosophic sets on the other hand. Some applications and many examples are presented and discussed.in fuzzy neutrosophic bi-topological spaces.

Keywords: fuzzy neutrosophic set; fuzzy neutrosophic bi-topology; fuzzy neutrosophic b-open set; fuzzy neutrosophic b-closed set; fuzzy neutrosophic generalized b-closed sets.

1. Introduction

At the beginning use of the concept of fuzzy sets "FS" was submitted by L. Zadeh's conference paper in 1965 [1] where each element had a degree of membership. Then many extension done by several studies. Intuitionistic fuzzy set "IFS" was one of the extension proved and known by K .Atanassov in 1983 [2- 4], when he has proved the degree of membership of an item of any set in"FS" and added a degree of non-membership in "IFS". Then many studies are being on the generalizations of the notion of "IFS", one of them proved was by F. Smarandache in 2005 [5,6], when he developed something else in membership and added indeterminacy membership between the last two membership and non-membership which were known in "IFS" and called it neutrosophic sets "NSs". After that, A Salama et.al. in 2014 [7,8] introduced neutrosophic topological spaces "NTSs".

The term of neutrosophic sets "NSs" was defined with membership, non-membership not specified degree. In the last three year ago, Veereswari [9] submitted his paper in fuzzy neutrosophic topological spaces "FNTSs" to be the solution and representation of the problems different fields where he takes all values between the closed interval 0 and 1 instead of the unitary non-standard interval]-0,1+[in NSs.

In this work, as generalized of the work of R.K. Al-Hamido [10] and the last papers which studied by F. Mohammed [11-13], we have identified a new category of fuzzy neutrosophic sets

"FNSs" called fuzzy neutrosophic generalized b-closed sets in fuzzy neutrosophic bi-topological spaces. Finally, on the basis of our manster's we will discuss some new characteristics and apply it. Finally, there are many application of NSs in many fields see [14-19], so before we ended our work we added some applications based in our new sets via fuzzy neutrosophic bi-topological spaces.

2. Preliminaries:

In this part of our study, we will refer to some basic definitions and operations which are useful in our work.

Definition 2.1 [9]: Let U be a non-empty fixed set. The fuzzy neutrosophic set "FNS" μ_N is an object having the form $\mu_N=\{< u, \lambda\mu_N(u), \gamma\mu_N(u), V\mu_N(u) > : u\epsilon U\}$ where the functions $\lambda\mu_N(u), \gamma\mu_N(u), V\mu_N(u)$: $U\rightarrow[0,1]$ denote the degree of membership function (namely $\lambda\mu_N(u)$), the degree of indeterminacy function (namely $\gamma\mu_N(u)$) and the degree of non-membership function (namely $V\mu_N(u)$) respectively of each element uϵ U to the set μ_N and $0 \leq \lambda_{\mu N}(u)+ \gamma_{\mu N}(u) +V_{\mu N}(u) \leq 3$, for each u$\in$ U.

Remark 2.2: FNS $\mu_N = \{< u, \lambda_{\mu N}(u), \sigma_{\mu N}(u), V_{\mu N}(u) >: u \in U\}$ can be identified to an ordered triple $< u, \lambda_{\mu N}, \sigma_{\mu N}, V_{\mu N} >$ *in* $[0,1]$ *on* U.

Lemma 2.3 [9]: Let U be a non-empty set and the "FNS" μ_N and γ_N be in the form $\mu_N = \{< u, \lambda_{\mu N}, \sigma_{\mu N}, V_{\mu N} > \}$ and $\gamma_N=\{< u, \lambda_{\gamma N}, \sigma_{\gamma N}, V_{\gamma N} >\}$ on U. Then,

i. $\mu_N \subseteq \gamma_N$ *iff* $\lambda_{\mu N} \leq \lambda_{\gamma N}, \sigma_{\mu N} \leq \sigma_{\gamma N}$ *and* $V_{\mu N} \geq V_{\gamma N}$,

ii. $\mu_N = \gamma_N$ *iff* $\mu_N \subseteq \gamma_N$ *and* $\gamma_N \subseteq \mu_N$,

iii. $(\mu_N)^c =\{< u, V_{\mu N}, 1-\sigma_{\mu N}, \lambda_{\mu N} >\}$,

iv. $\mu_N \cup \gamma_N =\{< u, Mx(\lambda_{\mu N}, \lambda_{\gamma N}), Mx(\sigma_{\mu N}, \sigma_{\gamma N}), Mn(V_{\mu N}, V_{\gamma N}) >\}$,

v. $\mu_N \cap \gamma_N =\{< u, Mn(\lambda_{\mu N}, \lambda_{\gamma N}), Mn(\sigma_{\mu N}, \sigma_{\gamma N}), Mx(V_{\mu N}, V_{\gamma N}) >\}$,

vi. $0_N = \{< u, 0, 0, 1>\}$ *and* $1_N = \{ < u, 1, 1, 0 >\}$.

Definition 2.4 [9]: Fuzzy neutrosophic topology (for short, FNT) on a non-empty set U is a family T_N of fuzzy neutrosophic subset in U satisfying the following axioms:

i. $0_N, 1_N \in T_N$,

ii. $\mu_{N1} \cap \mu_{N2} \in T_N \ \forall \mu_{N1}, \mu_{N2} \in T_N$,

iii. $\cup \mu_{Nj} \in T_N, \ \forall \{ \mu_{Nj} : j \in J\} \subseteq T_N$.

In this case the pair (U, T_N) is called fuzzy neutrosophic topological space (for short, FNTS). The elements of T_N are called fuzzy neutrosophic-open sets (for short, FN-OS). The complement of FN-OS in the FNTS (U, T_N) is called fuzzy neutrosophic- closed set (for short, FN-CS).

Definition 2.5 [9]: Let (U, T_N) is FNTS and $\mu_N = < u, \lambda_{\mu N}, \sigma_{\mu N}, V_{\mu N} >$ is FNS in U. Then the fuzzy neutrosophic-closure (for short, FN-Cl) and the fuzzy neutrosophic-interior (for short, FN-In) of μ_N are defined by:

$$FN\text{-}Cl\ (\mu_N) = \cap \{ \gamma_N : \gamma_N \text{ is FN-CS in } U \text{ and } \mu_N \subseteq \gamma_N \},$$

$$FN\text{-}In\,(\mu_N) = \cup\{\gamma_N : \gamma_N \text{ is FN-OS in } U \text{ and } \gamma_N \subseteq \mu_N\}.$$

Now, the FN-Cl (μ_N) is FN-CS and FN-In(μ_N) is FN-OS in U.

Further,

i. μ_N is FN-CS in U iff $FN\text{-}Cl(\mu_N) = \mu_N$,

ii. μ_N is FN-OS in U iff $FN\text{-}In(\mu_N) = \mu_N$.

Definition 2.6: Let (U_N, T_{N1}, T_{N2}) is FNTS and $\mu_N =< u, \lambda_{\mu N}, \sigma_{\mu N}, V_{\mu N} >$ is FNS in U_N. Then the fuzzy neutrosophic semi-closure (resp. fuzzy neutrosophic Pre-closure and fuzzy neutrosophic α-closure) of μ_N and denoted by FN-SCl (μ_N) (resp. $FN\text{-}PCl(\mu_N)$ and $FN\text{-}\alpha Cl(\mu_N)$ are defined by:

$$FN\text{-}SCl(\mu_N) = \cap\{\gamma_N : \gamma_N \text{ is FN-SCS set in } U \text{ and } \mu_N \subseteq \gamma_N\} = \mu_N \cup FN\text{-}In(FN\text{-}Cl(\mu_N)),$$

$$FN\text{-}PCl(\mu_N) = \cap\{\gamma_N : \gamma_N \text{ is FN-PCS set in } U \text{ and } \mu_N \subseteq \gamma_N\} = \mu_N \cup FN\text{-}Cl(FN\text{-}In(\mu_N)),$$

$$FN\text{-}\alpha\,Cl(\mu_N) = \cap\{\gamma_N : \gamma_N \text{ is FN-}\alpha\text{CS set in } U \text{ and } \mu_N \subseteq \gamma_N\} = \mu_N \cup FN\text{-}Cl\,(FN\text{-}In(FN\text{-}Cl(\mu_N))),$$

Definition 2.7 [11, 12]: FNS λ_N in FNTS (U, T_N) is called**:**

i. Fuzzy neutrosophic-regular open set (FN-ROS) if $\mu_N = FN\text{-}In(FN\text{-}Cl(\mu_N))$,

ii. Fuzzy neutrosophic-regular closed set (FN-RCS) if $\mu_N = FN\text{-}Cl(FN\text{-}In(\mu_N)$,

iii. Fuzzy neutrosophic-semi open set (FN-SOS) if $\mu_N \subseteq FN\text{-}Cl(FN\text{-}In(\mu_N))$,

iv. Fuzzy neutrosophic-semi closed set(FN-SCS) if $FN\text{-}In(FN\text{-}Cl(\mu_N)) \subseteq \mu_N$,

v. Fuzzy neutrosophic pre-open set(FN-POS) if $\mu_N \subseteq FN\text{-}In(FN\text{-}Cl(\mu_N))$,

vi. Fuzzy neutrosophic pre-closed set(FN-PCS) if $FN\text{-}Cl(FN\text{-}In(\mu_N)) \subseteq \mu_N$,

vii. Fuzzy neutrosophic-α-open set(FN-αOS) if $\mu_N \subseteq FN\text{-}In(FN\text{-}Cl(FN\text{-}In(\mu_N)))$,

viii. Fuzzy neutrosophic-α-closed set(FN-αCS) if FN-Cl(FN-In(FN-Cl(μ_N))) $\subseteq \mu_N$,

ix. Fuzzy neutrosophic generalized closed set (FN-GCS) if $FN\text{-}Cl(K \subseteq N)$ whenever K $\subseteq$ N and N is a FN-OS,

x. Fuzzy neutrosophic generalized pre closed set (FN-GPCS) if $FN\text{-}PCl(K) \subseteq N$, whenever K$\subseteq$ N and N is a FN-OS,

xi. Fuzzy neutrosophic α generalized closed set (FN-αGCS) *if* $FN\alpha\text{-}Cl(K) \subseteq N$ whenever K$\subseteq$ N and N is a FN-OS,

xii. Fuzzy neutrosophic generalized semi closed set (FN-GSCS) if $FN\text{-}SCl(K) \subseteq N$, whenever K$\subseteq$ N and N is a FN-OS.

Definition 2.8 [13]: A fuzzy neutrosophic set K in FNTs (U, T_N) is called fuzzy neutrosophic b-closed set (FN-b-CS) set if and only if $FN\text{-}In(FN\text{-}Cl\,(K)) \vee FN\text{-}Cl(FN\text{-}In\,(K)) \leq K$.

Definition 2.9 [13]: Let U_N be a non-empty set and (U, T_{N1}), (U, T_{N2}) be two topological spaces then, the triple (U_N, T_{N1}, T_{N2}) is a fuzzy neutrosophic bi-topological space (for short, FN-bi-TS).

3. Generalized b-Open Sets and Generalized b-Closed Sets in Fuzzy Neutrosophic bi-Topological Spaces

In this section, we generalized our work [13] and study the concept of generalized b-closed sets and generalized b-open sets based of fuzzy neutrosophic bi- topological spaces and introduced it after giving the definition of fuzzy neutrosophic bi- topological spaces as follows:

Definition 3. 1: Let U be a non-empty set and T_{N1}, T_{N2} be two topologies on FNTS (U, T_N), then the triple (U, T_{N1}, T_{N2}) is a fuzzy neutrosophic bi- topological space (for short, FN-bi-TS).

Definition 3.2: Let U be a non-empty set and T_{N1}, T_{N2} be two topologies on FNTS (U, T_N). A subset A of U is called fuzzy neutrosophic open set (for short, FN-OS) set if $A \in T_{N1} \cup T_{N2}$. A is called fuzzy neutrosophic closed set (for short, FN-CS) if 1_N-A is FN-OS.
Note: In this work we refer to $T_{N1} \cup T_{N2}$ by T_N.

Example 3.3: Let $U = \{ k_1, k_2\}$, $T_{N1} =\{0_N, 1_N\}$, $T_{N2} = \{0_N, 1_N, E_1\}$ and, $T_N =\{ 0_N, E_1, 1_N\}$ be a FN-bi-TS on U,
Where, $E_1 = < u, (k_{1(0.2)}, k_{1(0.5)}, k_{1(0.8)}), (k_{2(0.3)}, k_{2(0.5)}, k_{2(0.7)}) >$.
Then the neutrosophic set $Z = < u, (k_{1(0.7)}, k_{1(0.5)}, k_{1(0.3)}), (k_{2(0.6)}, k_{2(0.5)}, k_{2(0.4)}) >$ is a FN-b-CS in U.

Definition 3.4: Let (U, T_N) be any FN-bi-TS and $\mu_N = < u, \lambda_{\mu N}, \sigma_{\mu N}, V_{\mu N} >$ be FNS in U. Then the fuzzy neutrosophic-b-closure (for short, FN-bCl) and the fuzzy neutrosophic-b-interior (for short, FN-bIn) of μ_N are defined by:

$$FN\text{-}bCl\,(\mu_N) = \cap \{ \gamma_N : \gamma_N \text{ is FN-bCS in U and } \mu_N \subseteq \gamma_N \},$$
$$FN\text{-}bIn\,(\mu_N) = \cup \{ \gamma_N : \gamma_N \text{ is FN-bOS in U and } \gamma_N \subseteq \mu_N \}.$$

Definition 3.5: Let (U,T_N) be a FN-bi-TS, then, for each $\mu_1, \lambda_1 \in I^U$ the fuzzy set μ_1 is called fuzzy neutrosophic- generalized b-open set (for short, FN-gb-OS) set *if* $\mu_1 \leq FN\text{-}bIn\,(\lambda_1)$ such that $\mu_1 \leq \lambda_1$ and μ_1 is FN-CS.

Theorem 3.6: A fuzzy neutrosophic set Z of FN-bi-TS (U, T_N) is a FN-gb-OS *iff* $N \subseteq FN\text{-}bIn(Z)$ whenever N is a FN-CS and $N \subseteq Z$.
Proof: Necessity : Suppose Z is a FN-gb-OS in FN-bi-TS (U, T_N) and let E be a FN-CS and $N \subseteq Z$.
Then $H^c = 1_N$-H is a FN-OS in U such that $Z^c = 1_N\text{-}Z \subseteq N^c = 1_N\text{-}N$
$\Rightarrow 1_N$-Z is a FN-gb-CS and FN-bCl(1_N-Z) $\subseteq 1_N$-N ,
Hence, (1_N-FN-bIn(Z)) $\subseteq 1_N$-N $\Rightarrow N \subseteq$ FN-bIn(Z).
Sufficiency: Let Z be any FNS of U and let $N \subseteq$ FN-bIn(Z) whenever, N is a FN-CS and $N \subseteq Z$.

Theorem 3.7: Let (U, T_N) be FN-bi-TS, then:

(1) Every FN-CS is a FN-gb-CS,

(2) Every FN-αCS is a FN-gb-CS,

(3) Every FN-PCS is a FN-gb-CS,

(4) Every FN-b-CS is a FN-gb-CS,

(5) Every FN-RCS is a FN-gb-CS,

(6) Every FN-GCS is a FN-gb-CS,

(7) Every FN-αGCS is a FN-gb-CS,

(8) Every FN-GPCS is a FN-gb-CS

(9) Every FN-SCS is a FN-gb-CS.

(10) Every FN-GSCS is FN-gb-CS.

Proof : (1): Let Z $\subseteq$ N and N be a FN-CS in FN-bi-TS (U, T_N) with FN-bCl(Z) $\subseteq$ FN-Cl(Z).
But, FN-bCl(Z) = Z $\subseteq$ N. Therefore, Z is a FN-gb-CS in FN-bi-TS (U, T_N).

(2): Let Z $\subseteq$ N and N $\in$ T_N,$\Longrightarrow$ Z is a FN-αCl(Z) = Z. Therefore, FN-bCl(Z) $\subseteq$ FN-αCl(Z) = Z $\subseteq$ N.
Hence, Z is a FN-gb-CS in FN-bi-TS (U, T_N).

(3): Let Z $\subseteq$ N and N $\in$ T_N .
Since Z is a FN-PCS, and FN-Cl(FN-In(Z)) $\subseteq$ Z.
Therefore, FNCl(FN-In(Z)) $\cap$ FN- In(FN-Cl(Z)) $\subseteq$ FN-Cl(Z) $\cap$ FN-Cl(FN-In(Z)) $\subseteq$ Z.
$\Longrightarrow$ FN-bCl(Z) $\subseteq$ N. Hence, Z is a FN-gb-CS in U.

(4): Let Z $\subseteq$ N and N be a FN-OS in FN-bi-TS (U, T_N)
$\Longrightarrow$ Z is a FN-b-CS and FN-bCl(Z) = Z.
Therefore, FN-bCl(Z) = Z $\subseteq$ N. Hence, Z is a FN-gb-CS in FN-bi-TS (U, T_N).

(5): Let Z $\subseteq$ N and N $\in$ T_N and let Z be a FN-RCS.
But, FN-Cl(FN-In(Z)) = Z $\Longrightarrow$ FN-Cl(Z) = FN-Cl(FN-In(Z)). Therefore, FN-Cl(Z) = Z.
Hence, Z is a FN-CS in U. By (1), we get Z is a FN- gb-CS in FN-bi-TS (U, T_N).

(6): Let Z $\subseteq$ N and N $\in$ T_N $\Longrightarrow$ Z is a FN-GCS, FN-Cl(Z) $\subseteq$ N.
Therefore, FN-bCl(Z) $\subseteq$ FN-Cl(Z).
But, FN-bCl(Z) $\subseteq$ N. Hence, Z is a FN-gb-CS in FN-bi-TS (U, T_N).

(7): Let Z $\subseteq$ N and N $\in$ T_N $\Longrightarrow$ Z is a FN-αGCS.
But, FN-αCl(Z) $\subseteq$ N. Therefore, FNbCl(Z) $\subseteq$ FN-αCl(Z),
So, FN-bCl(Z) $\subseteq$ N. Hence, Z is a FN-gb-CS in FN-bi-TS (U, T_N).

(8): Let Z $\subseteq$ N and N $\in$ T_N $\Longrightarrow$ Z is a FN-gp-CS and FN-PCl(Z) $\subseteq$ N.
Therefore, FNbCl(Z) $\subseteq$ FN-pCl(Z), so FN-bCl(Z) $\subseteq$ N.

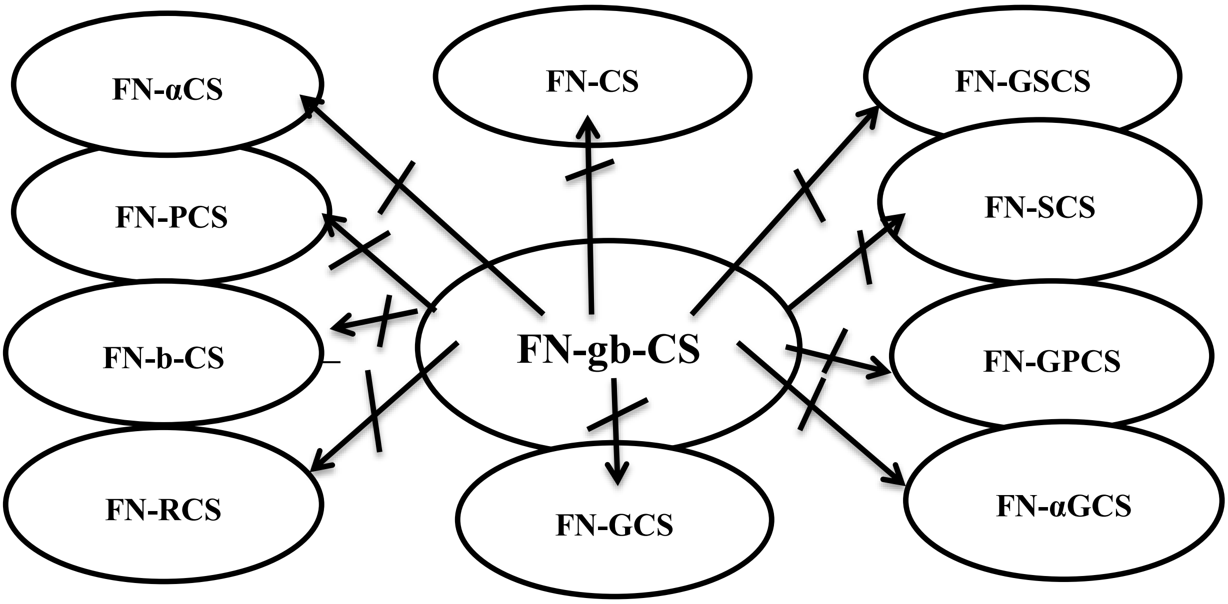
Hence, Z is a FN-gb-clos. set in FN-bi-TS (U, T_N).

(9): Let Z $\subseteq$ N and N $\in$ T_N $\Rightarrow$ Z is a FN-SCS.

But, FN-bCl(Z) $\subseteq$ FN-SCl(Z) $\subseteq$ N. Therefore, Z is a FN-gb-CS in FN-bi-TS (U, T_N).

(10): Obivious

Proposition 3.8: The converse of theorem 3.7 is not true in general for all cases and we can see it in (Diagram 1.)



(Diagram 1)

Example 3.9: (i): Let $U = \{ k_1, k_2 \}$, $T_{N1} = \{ 0_N, 1_N \}$, $T_{N2} = \{0_N, 1_N, E_1\}$.

Then, $T_N = \{ 0_N, E_1, 1_N \}$ is a FN-bi-TS on U ,

1- Take $E_1 = < u, (k_{1(0.3)}, k_{1(0.5)}, k_{1(0.6)}), (k_{2(0.2)}, k_{2(0.5)}, k_{2(0.7)}) >$.

Then, the FNS $"Z" = < u, (k_{1(0.5)}, k_{1(0.5)}, k_{1(0.4)}), (k_{2(0.6)}, k_{2(0.5)}, k_{2(0.3)})$ is a FN-gb-CS but, not a FN-CS in U $\Rightarrow$ FN-Cl("Z") = E_1 $\neq$ "Z".

2- Let $E_1 = < u, (k_{1(0.3)}, k_{1(0.5)}, k_{1(0.6)}), (k_{2(0.2)}, k_{2(0.5)}, k_{2(0.8)}) >$ **.** Then, the FNS "Z" = $< u, (k_{1(0.5)}, k_{1(0.5)}, k_{1(0.3)}), (k_{2(0.6)}, k_{2(0.5)}, k_{2(0.3)}) >$ is a FN-αCS in U$\Longrightarrow$ $FN\text{-}Cl(FN\text{-}Cl(Z)) = 1_N\text{-} E_1 \nsubseteq$ "Z".

3- Let $E_1 = < u, (k_{1(0.9)}, k_{1(0.5)}, k_{1(0.8)}), (k_{2(0.3)}, k_{2(0.5)}, k_{2(0.7)}) >$.

Then, the FNS $"Z" = < u, (k_{1(0.4)}, k_{1(0.5)}, k_{1(0.6)}), (k_{2(0.5)}, k_{2(0.5)}, k_{2(0.5)}) >$ is a FN-gb-CS but, not a FN-PCS in U $\Rightarrow$ FN-Cl(FN-In("Z")) = E_1 $\nsubseteq$ "Z".

4- Let $E_1 = < u, (k_{1(0.6)}, k_{1(0.5)}, k_{1(0.4)}), (k_{2(0.8)},k_{2(0.5)},k_{2(0.2)}) >$.

Then, the FNS $"Z" = < u, (k_{1(0.8)}, k_{1(0.5)}, k_{1(0.2)}), (k_{2(0.9)}, k_{2(0.5)}, k_{2(0.1)}) >$ is a FN-gb-CS but, not a FN-b-CS in FN-bi-TS (U, T_N). $\Longrightarrow$ FN-RCS is a FN-gb-CS but, not a FN-b-CS in FN-bi-TS (U, T_N),

$\Longrightarrow$ FN-Cl(FN-In("Z")) $\cap$ FN-In(FN-Cl("Z")) $= 1_N \not\subseteq$ "Z".

5- Let $E_1 = < u, (k_{1(0.2)}, k_{1(0.5)}, k_{1(0.8)}), (k_{2(0.4)}, k_{2(0.5)}, k_{2(0.6)}) >$.

Then, the *FNS* $"Z" = < u, (k_{1(0.7)}, k_{1(0.5)}, k_{1(0.3)}), (k_{2(0.5)}, k_{2(0.5)}, k_{2(0.5)}) >$ is a FN-gb-CS but, not a FN-RCS in *FN-bi-TS* $(U, T_N) \Longrightarrow$ *FN-Cl(FN-In("Z"))* $= 1_N - E_1 \neq$ "Z".

6- Let $E_1 = < u, (k_{1(0.2)}, k_{1(0.5)}, k_{1(0.8)}), (k_{2(0.4)}, k_{2(0.5)}, k_{2(0.6)}) >$.

Then, the FNS $"Z" = < u, (k_{1(0.1)}, k_{1(0.5)}, k_{1(0.8)}), (k_{2(0.3)}, k_{2(0.5)}, k_{2(0.7)}) >$ is a FN-gb-CS but, not a FN-GCS in *FN-bi-TS* (U, T_N). $\Longrightarrow$ *FN-Cl("Z")* $= E_1^c \not\subseteq E_1$.

7- Let $E_1 = < u, (k_{1(0.5)}, k_{1(0.5)}, k_{1(0.4)}), (k_{2(0.5)}, k_{2(0.5)}, k_{2(0.5)}) >$.

Then, the FNS $"Z" = < u, (k_{1(0.5)}, k_{1(0.5)}, k_{1(0.5)}), (k_{2(0.3)}, k_{2(0.3)}, k_{2(0.7)}) >$ is a FN-gb-CS but, not a FN-αGCS in FN-bi-TS (U, T_N). $\Longrightarrow$ FN-Cl(FN-In(FN-Cl("Z"))) $= 1_N \not\subseteq E_1$.

8- Let $E_1 = < u, (k_{1(0.9)}, k_{1(0.5)}, k_{1(0.1)}), (k_{2(0.7)}, k_{2(0.5)}, k_{2(0.2)}) >$.

Then, the FNS $"Z" = < u, (k_{1(0.7)}, k_{1(0.5)}, k_{1(0.3)}), (k_{2(0.6)}, k_{2(0.5)}, k_{2(0.4)}) >$ is a FN-gb-CS but, not a FN-SCS in FN-bi-TS (U, T_N), $\Longrightarrow$ *FN-In(FN-Cl("Z"))* $= 1_N \not\subseteq$ "Z".

9- Let $E_1 = < u, (k_{1(0.8)}, k_{1(0.5)}, k_{1(0.6)}), (k_{2(0.0)}, k_{2(0.5)}, k_{2(0.1)}) >$.

Then, the *FNS* $"Z" = < u, (k_{1(0.6)}, k_{1(0.5)}, k_{1(0.5)}), (k_{2(0.2)}, k_{2(0.5)}, k_{2(0.3)}) >$ is a FN-gb-CS but, not a FN-GSCS in FN-bi-TS (U, T_N), $\Longrightarrow$ *FN-In(FN-Cl("Z"))* $= 1_N \not\subseteq$ "Z".

10- Let $U = \{ k1, k2 \}, T_{N1} = \{0_N, E_1\}, T_{N2} = \{0_N, 1_N, E_1, E_2\} = T_N$ be a FN-bi-TS on U.

Where, $E_1 = < u, (k_{1(0.2)}, k_{1(0.5)}, k_{1(0.8)}), (k_{2(0.3)}, k_{2(0.5)}, k_{2(0.7)}) >$,

$E_2 = < u, (k_{1(0.4)}, k_{1(0.5)}, k_{1(0.6)}), (k_{2(0.5)}, k_{2(0.5)}, k_{2(0.5)}) >$.

Then, the *FNS* $"Z" = < u, (k_{1(0.4)}, k_{1(0.5)}, k_{1(0.6)}), (k_{2(0.5)}, k_{2(0.5)}, k_{2(0.5)}) >$ *is a FN-gb-CS but, not a FN-GPCS in U* $\Longrightarrow$ *FN-PCl("Z)* $= 1_N - E_2 \not\subseteq E_2$.

Theorem 3.10: The union of any two FN-gb-CS need not be a FN-gb-CS in general as seen from the following example:

Example 3.11: Let $U = \{ k_1, k_2 \}, T_{N1} = \{ 0_N, E_1\}$ and $T_{N2} = \{ 0_N, 1_N, E_1\} = T_N$ be a FNT on U, where $E_1 = < u, (k_{1(0.6)}, k_{1(0.5)}, k_{1(0.4)}), (k_{2(0.8)}, k_{2(0.5)}, k_{2(0.2)}) >$.

Then, the FNS $"Z" = < u, (k_{1(0.1)}, k_{1(0.5)}, k_{1(0.9)}), (k_{2(0.8)}, k_{2(0.5)}, k_{2(0.2)}) >$,

$M = < u, (k_{1(0.6)}, k_{1(0.5)}, k_{1(0.4)}), (k_{2(0.7)}, k_{2(0.5)}, k_{2(0.3)}) >$ is a FN-gb-CS but, $Z \cap M$ is not a FN-gb-CS in U

$\Longrightarrow$ *FN-bCl("Z"∩M)* $= 1_N \not\subseteq E_1$.

Theorem 3.12: If Z is a FN-gb-CS in FN-bi-TS (U, T_N) , such that Z ⊆ M ⊆ FN-bCl(Z) then, M is a FN-gb-CS in (U, T_N)

Proof : Let M be any FNS in a FN-bi-TS (U, T_N), such that M ⊆ N and N ∈ T_N ⟹ Z ⊆ N , since Z is a FN-gb-CS and FN-bCl(Z) ⊆ N.

By hypothesis, we have FN-bCl(M) ⊆ FNbCl(FN-bCl(Z)) = FN-bCl(Z) ⊆ N.

Hence, M is FN-gb-CS in U.

Theorem 3.13: If Z is a FN-b-OS and FN-gb-CS in FN-bi-TS (U, T_N), then Z is a FN-b-CS.

Proof : Since Z is a FN-b-OS and FN-gb-CS in FN-bi-TS (U, T_N) such that FN-bCl(Z) ⊆ Z.

But, Z ⊆ FN-bCl(Z) .

Thus, FN-bCl(Z) = Z and hence, Z is FN-b-CS in FN-bi-TS (U, T_N).

Definition 3.14: A fuzzy neutrosophic set Z is said to be a fuzzy neutrosophic generalized b open set (FN-gb-OS) in FN-bi-TS (U, T_N). If the complement 1_N-Z is a FN-gb-CS in U. The family of all FN-gb-OS of FN-bi-TS (U, T_N) is denoted by FN-gb-O (U).

Example 3.15: Let $U = \{ k_1, k_2 \}, T_{N1} = \{0_N, E_1\}, T_{N2} = \{0_N, 1_N, E_1\} = T_N$ be FN-bi-TS on U, where

$E_1 = < u, (k_{1(0.3)}, k_{1(0.5)}, k_{1(0.7)}), (k_{2(0.4)}, k_{2(0.5)}, k_{2(0.6)}) >$.

Then, the FNS $Z = < u, (k_{1(0.4)}, k_{1(0.5)}, k_{1(0.6)}), (k_{2(0.5)}, k_{2(0.5)}, k_{2(0.5)}) >$ is a FN-gb-OS in U.

4. Some Applications of Generalized b-Closed Sets in Fuzzy Neutrosophic bi-Topological Spaces

In [14] they propose two models for solving Neutrosophic Goal Programming Problem (NGPP), and in [15-19], we can see many applications of neutrosophic so, we will try in our study to give some application of our new studies concepts.

Definition 4.1: A FN-bi-TS (U, T_N) is called:

i. a fuzzy neutrosophic b$\frac{1}{2}$ space (for short, FN-b$\frac{1}{2}$S) if every FN-bCS is a FN-CS.

ii. a fuzzy neutrosophic gb$\frac{1}{2}$ space (for short, FN-gb$\frac{1}{2}$S) if every FN-gb-CS is a FN-CS.

iii. a fuzzy neutrosophic gbU$_b$ space (FN-gb$_b$S) if every FN-gb-CS is a FN-b-CS.

Theorem 4.2: Every FN-gb$\frac{1}{2}$S is a FN-gbU$_b$ S in any FN-bi-TS (U, T_N),.

Proof : Let (U, T_N) be a FN-gb$\frac{1}{2}$S and let Z be any FN-gb-CS in FN-bi-TS (U, T_N), By hypothesis, Z is a FN-CS in U.

Since every FN-CS is a FN-b-CS in U. Hence, (U, T_N) , is a FN-gbU$_b$ S.

The converse of above theorem need not be true in general as seen from the following example:

Example 4.3: Let $U = \{ k_1, k_2 \}$, $T_{N1} = T_N = \{0_N, 1_N, E_1\}$ and $T_{N2} = \{ 0_N, 1_N\}$ be a FNT on U, where,
$E1 = < u, (k_{1(0.9)}, k_{1(0.5)}, k_{1(0.9)}), (k_{2(0.1)}, k_{2(0.5)}, k_{2(0.1)}) >$.

Then, the FNS "Z" $= < u, (k_{1(0.2)}, k_{1(0.5)}, k_{1(0.3)}), (k_{2(0.8)}, k_{2(0.5)}, k_{2(0.7)}) >$ is a FN-gbU$_b$ S but, not a FN-gb$\frac{1}{2}$ S.

Theorem 4.4: Let (U, T_N) be a FN-bi-TS and (U, T_N). A FN-gb$\frac{1}{2}$S. Then we have the following statement:

i- Any union of FN-gb-CS is a FN-gb-CS.
ii- Any intersection of any FN-gb-OS is a FN-gb-OS.

Proof : (i) Let $\{N_i\}_{i \in J}$ be a collection of FN-gb-CS in a FN-gb$\frac{1}{2}$S, (U, T_N).

Therefore, every FN-gb-CS is a FN-CS.
But, the union of FN-CS is a FN-CS. Hence, the union of FN-gb-CS is a FN-gb-CS in U.

(ii) It can be proved by taking complement in **(i)**.

Theorem 4.5: A FN-bi-TS (U, T_N) is a FN-gbU$_b$ S if and only if FN-gb(U) = FNb-O (U)
Proof : **Necessity** : Let "Z" be a FN-gb-OS in a FN-bi-TS (U, T_N). Then, 1_N-Z is a FN-gb-CS.
By hypothesis , 1_N-Z is a FN-b-CS in U. Therefore, Z is a FN-b-OS
Hence, FN-gb-O(U) = FNb-O (U).
Sufficiency : Let Z be a FN-gb-CS in any FN-bi-TS (U, T_N). Then, 1_N-Z is a FN-gb-OS in U.
By hypothesis , 1_N-Z is a FN-b-OS in U.
Therefore, Z is a FN-b-CS in U. Hence, (U, T_N) is a FN-gbU$_b$S.

Theorem 4.8: A FN-bi-TS (U, T_N) is a FN-gb$\frac{1}{2}$ if and only if FN-gb-O(U) = FN-O(U).

Proof : **Necessity** : Let Z be a FN-gb-OS in a FN-bi-TS (U, T_N). Then 1_N-Z is a FN-gb-CS in U.
By hypothesis, 1_N-Z is a FN-CS in U. Therefore, Z is a FN-OS in U.
Hence, FN-gb-O(U) = FN-O(U)
Sufficiency : Let Z be a FN-gb-CS. Then, 1_N-Z is a FN-gb-OS in U. By hypothesis, 1_N-Z is a FN-OS in U. Therefore, Z is a FN-CS in U. Hence, (U, T_N) is a FN-gb$\frac{1}{2}$.

5. Conclusions

In this paper, the new concept of a new class of sets was studied and called fuzzy neutrosophic generalized b-closed sets and its complement fuzzy neutrosophic generalized b-open sets. We investigated the relations between fuzzy neutrosophic generalized b closed sets and other fuzzy neutrosophic sets such as α closed sets, regular closed sets, semi closed sets pre closed sets, generalized closed sets, b closed sets, α generalized closed sets and semi generalized closed sets based of fuzzy neutrosophic bi-topological spaces and applied some new spaces to be applications of the new defined sets.

6. Acknowledgements

The authors would like to thanks the reviewers for their valuable suggestions to improve the paper and get it as in this design.

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Received: Apr 21, 2020. Accepted: July 11, 2020

Neutrosophic Sets and Systems, Vol. 35, 2020
University of New Mexico

Neutrosophic Soft Rough Topology and its Applications to Multi-Criteria Decision-Making

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Abstract: In this manuscript, we introduce the notion of neutrosophic soft rough topology (NSR-topology) defined on neutrosophic soft rough set (NSR-set). We define certain properties of NSR-topology including NSR-interior, NSR-closure, NSR-exterior, NSR-neighborhood, NSR-limit point, and NSR-bases. Furthermore, we aim to develop some multi-criteria decision-making (MCDM) methods based on NSR-set and NSR-topology to deal with ambiguities in the real-world problems. For this purpose, we establish algorithm 1 for suitable brand selection and algorithm 2 to determine core issues to control crime rate based on NSR-lower approximations, NSR-upper approximations, matrices, core, and NSR-topology.

Keywords: Neutrosophic soft rough (NSR) set, NSR-topology, NSR-interior, NSR-closure, NSR-exterior, NSR-neighborhood, NSR-limit point, NSR-bases, Multi-criteria group decision making.

1. Introduction

The limitations of existing research are recognized in the field of management, social sciences, operational research, medical, economics, artificial intelligence, and decision-making problems. These limitations can be dealt with the Fuzzy set [1], rough set [2, 3], neutrosophic set [4, 5], soft set [6], and different hybrid structures of these sets. Rough set theory was initiated by Pawlak [2], which is an effective mathematical model to deal with vagueness and imprecise knowledge. Its boundary region gives the concept of vagueness, which can be interpreted by using the vagueness of Frege's idea. He invented that vagueness can be dealt with the upper and lower approximations of precise set using any equivalence relation. In the real life, rough set theory has many applications in different fields such as social sciences, operational research, medical, economics, and artificial intelligence, etc. Many real-world problems have neutrosophy in their nature and cannot handle by using fuzzy or intuitionistic fuzzy set theory. For example, when we are dealing with conductors and non-conductors there must be a possibility having insulators. For this purpose, Smarandache [4, 5] inaugurated the neutrosophic set theory as a generalization of fuzzy and intuitionistic fuzzy set theory. The neutrosophic set yields the value from real standard or non-standard subsets of $]^-0, 1^+[$. It is difficult to utilize these values in daily life science and technology problems. Therefore, the concept of a single-valued neutrosophic set, which takes value from the subset of [0,1], as defined by Wang et al. [7]. The beauty of this set is that it gives the membership grades for truth, indeterminacy and falsity for the corresponding attribute. All the grades are independent of each other and provide information about the three shades of an arbitrary attribute. Smarandache [8] extended the neutrosophic set respectively to neutrosophic Overset (when some neutrosophic components are >1), Neutrosophic Underset (when some neutrosophic components are <0), and to Neutrosophic Offset (when some neutrosophic components are off the interval $[0,1]$, i.e. some neutrosophic components are >1 and other neutrosophic components <0). In 2016, Smarandache introduced the Neutrosophic Tripolar Set and Neutrosophic Multipolar Set, also the Neutrosophic Tripolar Graph and Neutrosophic Multipolar Graph [8].

The soft set is a mathematical model to deal ambiguities and imprecisions in parametric manners. This is another abstraction of the crisp set theory. In 1999, Molodtsov [9] worked on parametrizations of the universal set and invented a parameterized family of subsets of the universal set called soft set. In recent years, many mathematicians worked on different hybrid structures of the fuzzy and rough sets. Ali et al. [10, 11] established some novel operations in the soft sets, rough soft sets and, fuzzy soft set theory. Aktas and Çağman [12] introduced various results on soft sets and soft groups. Bakier et al. [13] introduced the idea of soft rough topology. Çağman et al. [14] introduced various results on soft topology. Chen [15] worked on parametrizations reduction of soft sets and gave its applications in decision-making. Feng et al. [16, 17] established various results on soft set, fuzzy set, rough set and soft rough sets with the help of illustrations. Hashmi et al. [18] introduced the notion of m-polar neutrosophic set and m-polar neutrosophic topology and their applications to multi-criteria decision-making (MCDM) in medical diagnosis and clustering analysis. Hashmi and Riaz [19] introduced a novel approach to the census process by using Pythagorean m-polar fuzzy Dombi's aggregation operators. Kryskiewicz [20] introduced the rough set approach to incomplete information systems. Karaaslan and Çağman [21] introduced bipolar soft rough sets and presented their applications in decision-making. Kumar and Garg [22] introduced the TOPSIS method based on the connection number of set pair analyses under an interval-valued intuitionistic fuzzy set environment. Maji et al. [23, 24, 25] worked on some results of a soft set and gave its applications in decision-making problems. He also invented the idea of a neutrosophic soft set and gave various results to intricate the concept with numerous applications. Naeem et al. [26] introduced the novel concept of Pythagorean m-polar fuzzy sets and the TOPSIS method for the selection of advertisement mode. Peng and Garg [27] introduced algorithms for interval-valued fuzzy soft sets in emergency decision making based on WDBA and CODAS with new information measures. Peng and Yang [28] presented some results for Pythagorean fuzzy sets. Peng et al. [29] introduced Pythagorean fuzzy information measures and their applications. Peng et al. [30] introduced a Pythagorean fuzzy soft set and its application. Peng and Dai [31] introduced certain approaches to single-valued neutrosophic MADM based on MABAC, TOPSIS and, new similarity measure with score function. Marei [32] invented some more results on neutrosophic soft rough sets and worked on its modifications. Pei and Miao [33] worked on the information system using the idea of a soft set. Quran et al. [34] introduced a novel approach to neutrosophic soft rough set under uncertainty. Riaz et al. [35] introduced soft rough topology with its applications to group decision making.

Riaz and Hashmi [36] introduced the notion of linear Diophantine fuzzy Set (LDFS) and its applications towards the MCDM problem. Linear Diophantine fuzzy Set (LDFS) is superior to IFS, PFS and, q-ROFS. Riaz and Hashmi [37] introduced novel concepts of soft rough Pythagorean m-Polar fuzzy sets and Pythagorean m-polar fuzzy soft rough sets with application to decision-making. Riaz and Tehrim [38] established the idea of cubic bipolar fuzzy ordered weighted geometric aggregation operators and, their application using internal and external cubic bipolar fuzzy data. They presented various illustrations and decision-making applications of these concepts by using different algorithms. Roy and Maji [39] introduced a fuzzy soft set-theoretic approach to decision-making problems. Salama [40] investigated some topological properties of rough sets with tools for data mining. Shabir and Naz [41] worked on soft topological spaces and presented their applications. Thivagar et al. [42] presented some mathematical innovations of a modern topology in medical events. Xueling et al. [43] presented some decision-making methods based on certain hybrid soft set models. Zhang et al. [44, 45, 46] established fuzzy soft β-covering based fuzzy rough sets, fuzzy soft coverings based fuzzy rough sets and, covering on generalized intuitionistic fuzzy rough sets with their applications to multi-attribute decision-making (MADM) problems. Broumi et al. [47] established the concept of rough neutrosophic sets. Christianto et al. [48] introduced the idea about the extension of standard deviation notion with neutrosophic interval and quadruple neutrosophic numbers. Adeleke et al. [49, 50] invented the concepts of refined eutrosophic rings I and refined neutrosophic rings II. Parimala et al. [51] worked on $\alpha\omega$-closed sets and its connectedness in terms of neutrosophic topological spaces. Ibrahim et al. [52] introduced the neutrosophic subtraction algebra and neutrosophic subraction semigroup.

The neutrosophic soft rough set and neutrosophic soft rough topology have many applications in MCDM problems. This hybrid erection is the most efficient and flexible rather than other constructions. It is constructed with a combination of neutrosophic, soft and, rough set theory. The interesting point in this structure is that by using this idea, we can deal with those type of models which have roughness, neutrosophy and, parameterizations in their nature.

The motivation of this extended and hybrid work is presented step by step in the whole manuscript. This model is generalized form and use to collect data at a large scale and applicable in medical, engineering, artificial intelligence, agriculture and, other daily life problems. In the future, this work can be gone easily for other approaches and different types of hybrid structures.

The layout of this paper is systematized as follows. Section 2, implies some basic ideas including soft set, rough set, neutrosophic set, neutrosophic soft set and, neutrosophic soft rough set. We elaborate on these ideas with the help of illustrations. In Section 3, we establish neutrosophic soft rough topology (NSR-topology) with some examples. We introduce some topological structures on NSR-topology named NSR-interior, NSR-closure, NSR-exterior, NSR-neighborhood, NSR-limit point and, NSR-bases. In Section 4 and 5, we present multi-criteria decision-making problems by using two different algorithms on NSR-set and NSR-topology. We use the idea of upper and lower approximations for NSR-set and construct algorithms using NSR-sets and NSR-bases We discuss the optimal results obtained from both algorithms and present a comparitive analysis of proposed approach with some existing approaches. Finally, the conclusion of this research is summarized in section 6.

2. Preliminaries

This section presents some basic definitions including soft set, rough set, neutrosophic soft set, and neutrosophic soft rough set .

Definition 2.1 [18]

Let U be the universal set. Let $\mathrm{I(U)}$ is collection of subsets of U. A pair $(\Theta, \mathfrak{A})$ is said to be a soft set over the universe U, where $\mathfrak{A} \subseteq \mathrm{E}$ and $\Theta: \mathfrak{A} \rightarrow \mathrm{I(U)}$ is a set-valued function. We denote soft set as $(\Theta, \mathfrak{A})$ or $\Theta_{\mathfrak{A}}$ and mathematically write it as

$$\Theta_{\mathfrak{A}} = \{(\xi, \Theta(\xi)): \xi \in \mathfrak{A}, \Theta(\xi) \in \mathrm{I(U)}\}.$$

For any $\xi \in \mathfrak{A}$, $\Theta(\xi)$ is ξ-approximate elements of soft set $\Theta_{\mathfrak{A}}$.

Definition 2.2 [21]

Let U be the initial universe and $\mathrm{Y} \subseteq \mathrm{U}$. Then, lower, upper, and boundary approximations of Y are defined as

$$\mathfrak{R}_{\mathring{a}}(\mathrm{Y}) = \cup_{g \in \mathrm{U}} \{\mathfrak{R}(g): \mathfrak{R}(g) \subseteq \mathrm{Y}\},$$

$$\mathfrak{R}^{\mathring{a}}(\mathrm{Y}) = \cup_{g \in \mathrm{U}} \{\mathfrak{R}(g): \mathfrak{R}(g) \cap \mathrm{Y} \neq \emptyset\},$$

and

$$\mathrm{B}_{\mathfrak{R}}(\mathrm{Y}) = \mathfrak{R}^{\mathring{a}}(\mathrm{Y}) - \mathfrak{R}_{\mathring{a}}(\mathrm{Y}),$$

respectively. Where $\mathfrak{R}$ is an indiscernibility relation $\mathfrak{R} \subseteq \mathrm{U} \times \mathrm{U}$ which indicates our information about elements of U. The set Y is said to be defined if $\mathfrak{R}^{\mathring{a}}(\mathrm{Y}) = \mathfrak{R}_{\mathring{a}}(\mathrm{Y})$. If $\mathfrak{R}^{\mathring{a}}(\mathrm{Y}) \neq \mathfrak{R}_{\mathring{a}}(\mathrm{Y})$ i.e $\mathrm{B_R(Y)} \neq \emptyset$, the set Y is rough set w.r.t $\mathfrak{R}$.

Definition 2.3 [41] Let U be the initial universe. Then, a neutrosophic set N on the universe U is defined as

$$\mathrm{N} = \{< g, \mathfrak{T}_{\mathrm{N}}(g), \mathfrak{I}_{\mathrm{N}}(g), \mathfrak{F}_{\mathrm{N}}(g) >: g \in \mathrm{U}\}, \text{where}$$

$$^{-}0 \leq \mathfrak{T}_{\mathrm{N}}(g) + \mathfrak{I}_{\mathrm{N}}(g) + \mathfrak{F}_{\mathrm{N}}(g) \leq 3^{+}, \text{where}$$

$$\mathfrak{T}, \mathfrak{I}, \mathfrak{F}: \mathrm{U} \rightarrow]^{-}0, 1^{+}[.$$

Where $\mathfrak{T}, \mathfrak{I}$ and $\mathfrak{F}$ represent the degree of membership, degree of indeterminacy and degree of non-membership for some $g \in \mathrm{U}$, respectively.

Definition 2.4 [16] Let U be an initial universe and E be a set of parameters. Suppose $\mathfrak{A} \subset \mathrm{E}$, and let $\mathcal{J}(\mathrm{U})$ represents the set of all neutrosophic sets of U. The collection $(\Phi, \mathfrak{A})$ is said to be the neutrosophic soft set over U, where Φ is a mapping given by

$$\Phi: \mathfrak{A} \rightarrow \mathcal{J}(\mathrm{U}).$$

The set containing all neutrosophic soft sets over U is denoted by $\mathrm{NS_U}$.

Example 2.5 Consider $\mathrm{U} = \{g_1, g_2, g_3, g_4, g_5\}$ be set of objects and attribute set is given by $\mathfrak{A} = \{\xi_1, \xi_2, \xi_3, \xi_4\} = \mathrm{E} = \mathfrak{A}$, where

The neutrosophic soft set represented as $\Phi_{\mathfrak{A}}$. Consider a mapping $\Phi: \mathfrak{A} \rightarrow \mathrm{I(U)}$ such that

$$\Phi(\xi_1) = \{< g_1, 0.7,0.7,0.3 >, < g_2, 0.5,0.7,0.7 >, < g_3, 0.7,0.5,0.2 >, < g_4, 0.7,0.4,0.4 >, < g_5, 0.9,0.3,0.4 >\},$$
$$\Phi(\xi_2) = \{< g_1, 0.9,0.5,0.4 >, < g_2, 0.7,0.3,0.5 >, < g_3, 0.9,0.2,0.4 >, < g_4, 0.9,0.3,0.3 >, < g_5, 0.9,0.4,0.3 >\},$$
$$\Phi(\xi_3) = \{< g_1, 0.8,0.5,0.4 >, < g_2, 0.7,0.5,0.4 >, < g_3, 0.8,0.3,0.6 >, < g_4, 0.6,0.3,0.7 >, < g_5, 0.8,0.4,0.5 >\},$$
$$\Phi(\xi_4) = \{< g_1, 0.9,0.7,0.5 >, < g_2, 0.8,0.7,0.7 >, < g_3, 0.8,0.7,0.5 >, < g_4, 0.8,0.6,0.7 >, < g_5, 1.0,0.6,0.7 >\}.$$

The tabular representation of neutrosophic soft set $\mathrm{K} = (\Phi, \mathfrak{A})$ is given in Table 1.

$(\Phi,\mathfrak{A})$	g_1	g_2	g_3	g_4	g_5
ξ_1	(0.7,0.7,0.3)	(0.5,0.7,0.7)	(0.7,0.5,0.2)	(0.7,0.4,0.4)	(0.9,0.3,0.4)
ξ_2	(0.9,0.5,0.4)	(0.7,0.3,0.5)	(0.9,0.2,0.4)	(0.9,0.3,0.3)	(0.9,0.4,0.3)
ξ_3	(0.8,0.5,0.4)	(0.7,0.5,0.4)	(0.8,0.3,0.6)	(0.6,0.3,0.7)	(0.8,0.4,0.5)
ξ_4	(0.9,0.7,0.5)	(0.8,0.7,0.7)	(0.8,0.7,0.5)	(0.8,0.6,0.7)	(1.0,0.6,0.7)

Table 1: Neutrosophic soft set $(\Phi,\mathfrak{A})$

Definition 2.6 Let $(\Phi,\mathfrak{A})$ be a neutrosophic soft set on a universe U. For some elements $g \in U$, a neutrosophic right neighborhood, regarding $\xi \in \mathfrak{A}$ is interpreted as follows;

$$g_\xi = \{g_i \in U: \mathfrak{T}_\xi(g_i) \geq \mathfrak{T}_\xi(g), \mathfrak{I}_\xi(g_i) \geq \mathfrak{I}_\xi(g), \mathfrak{F}_\xi(g_i) \leq \mathfrak{F}_\xi(g)\}.$$

Definition 2.7 Let $(\Phi,\mathfrak{A})$ be a neutrosophic soft set over a universe U. For some elements $g \in U$, a neutrosophic right neighborhood regarding all parameters $\mathfrak{A}$ is interpreted as follows;

$$g]_{\mathfrak{A}} = \cap \{g_{\xi_i}: \xi_i \in \mathfrak{A}\}.$$

Example 2.8 Consider Example 2.5 then we find the following neutosophic right neighborhood regarding all parameters $\mathfrak{A}$ as

$$g_{1\xi_1} = g_{1\xi_2} = g_{1\xi_3} = g_{1\xi_4} = \{g_1\}, g_{2\xi_1} = g_{2\xi_3} = \{g_1, g_2\}, g_{2\xi_2} = \{g_1, g_2, g_4, g_5\}, g_{2\xi_4} = \{g_1, g_2, g_3\}, g_{3\xi_1} = g_{3\xi_4} = \{g_1, g_3\}, g_{3\xi_2} = \{g_1, g_3, g_4, g_5\}, g_{3\xi_3} = \{g_1, g_3, g_5\}, g_{4\xi_1} = \{g_1, g_3, g_4\}, g_{4\xi_2} = \{g_4, g_5\}, g_{4\xi_3} = U, g_{4\xi_4} = U, g_{5\xi_1} = g_{5\xi_2} = g_{5\xi_4} = \{g_5\}, g_{5\xi_3} = \{g_1, g_5\}.$$

It follows that,

$g_1]_{\mathfrak{A}} = \{g_1\}$,
$g_2]_{\mathfrak{A}} = \{g_1, g_2\}$,
$g_3]_{\mathfrak{A}} = \{g_1, g_3\}$,
$g_4]_{\mathfrak{A}} = \{g_4\}$,
$g_5]_{\mathfrak{A}} = \{g_5\}$.

Definition 2.9 Let $(\Phi,\mathfrak{A})$ be a neutrosophic soft set over U. For any $X \subseteq U$, neutrosophic soft lower ($\underline{apr}_{NSR}$) approximation, neutrosophic soft upper ($\overline{apr}_{NSR}$) approximation, and neutrosophic soft boundary (B_{NSR}) approximation of X are defined as

$$\underline{apr}_{NSR}(X) = \cup \{g]_{\mathfrak{A}}: g \in U, g]_{\mathfrak{A}} \subseteq X\}$$

$$\overline{apr}_{NSR}(X) = \cup \{g]_{\mathfrak{A}}: g \in U, g]_{\mathfrak{A}} \cap X \neq \emptyset\}$$

$$B_{NSR}(X) = \overline{apr}_{NSR}(X) - \underline{apr}_{NSR}(X),$$

respectively. If $\underline{apr}_{NSR}(X) = \overline{apr}_{NSR}(X)$ then X is neutrosophic soft definable set.

Example 2.10 Consider Example 2.5, If $X = \{g_1\} \subseteq U$, then $\underline{apr}_{NSR}(X) = \{g_1\}$ and $\overline{apr}_{NSR}(X) = \{g_1, g_2, g_3\}$. Since its clear $\underline{apr}_{NSR}(X) \neq \overline{apr}_{NSR}(X)$, so X is neutrosophic soft rough set on U.

3 Neutrosophic Soft Rough Topology

In this section, we introduce and study the idea of neutrosophic soft rough topology and its related properties. Concepts of (NSR)-open set, (NSR)-closed set, (NSR)-closure, (NSR)-interior, (NSR)-exterior, (NSR)-neighborhood, (NSR)-limit point, and (NSR)-bases are defined.

Definition 3.1 Let U be the initial space, $\mathfrak{Y} \subseteq \mathrm{U}$ and $\mathrm{G} = (\mathrm{U}, \mathrm{K})$ be a neutrosophic soft approximation space, where $\mathrm{K} = (\Phi, \mathfrak{A})$ is a neutrosophic soft set. The upper and lower approximations are calculated on the basis of neutrosophic soft approximation space and neighborhoods. Then, the collection

$$\tau_{\mathrm{NSR}}(\mathfrak{Y}) = \{\mathrm{U}, \emptyset, \underline{\mathrm{apr}}_{\mathrm{NSR}}(\mathfrak{Y}), \overline{\mathrm{apr}}_{\mathrm{NSR}}(\mathfrak{Y}), \mathrm{B}_{\mathrm{NSR}}(\mathfrak{Y})\}$$

is called neutrosophic soft rough topology (NSR-topology) which guarantee the following postulates:

- U and $\emptyset$ belongs to $\tau_{\mathrm{NSR}}(\mathfrak{Y})$.
- Union of members of $\tau_{\mathrm{NSR}}(\mathfrak{Y})$ belongs to $\tau_{\mathrm{NSR}}(\mathfrak{Y})$.
- Finite Intersection of members of $\tau_{\mathrm{NSR}}(\mathfrak{Y})$ belongs to $\tau_{\mathrm{NSR}}(\mathfrak{Y})$.

Then $(\mathrm{U}, \tau_{\mathrm{NSR}}(\mathfrak{Y}), \mathrm{E})$ is said to be NSR-topological space, if $\tau_{\mathrm{NSR}}(\mathfrak{Y})$ is Neutrosophic soft rough topology.

Note that Neutrosophic soft rough topology is based on lower and upper approximations of neutrosophic soft rough set.

Example 3.2 From Example 2.5, if $\mathfrak{Y} = \{g_2, g_4\} \subseteq \mathrm{U}$, we obtain $\underline{\mathrm{apr}}_{\mathrm{NSR}}(\mathfrak{Y}) = \{g_4\}$, $\overline{\mathrm{apr}}_{\mathrm{NSR}}(\mathfrak{Y}) = \{g_1, g_2, g_4\}$ and $\mathrm{B}_{\mathrm{NSR}}(\mathfrak{Y}) = \{g_1, g_2\}$. Then,

$$\tau_{\mathrm{NSR}}(\mathfrak{Y}) = \{\mathrm{U}, \emptyset, \{g_4\}, \{g_1, g_2, g_4\}, \{g_1, g_2\}\}$$

is a NSR-topology.

Definition 3.3 Let $(\mathrm{U}, \tau_{\mathrm{NSR}}(\mathfrak{Y}), \mathrm{E})$ be an NSR-topological space. Then, the members of $\tau_{\mathrm{NSR}}(\mathfrak{Y})$ are called NSR-open sets. An NSR-set is said to be an NSR-closed set if its complement belongs to $\tau_{\mathrm{NSR}}(\mathfrak{Y})$.

Proposition 3.4 Consider $(\mathrm{U}, \tau_{\mathrm{NSR}}(\mathfrak{Y}), \mathrm{E})$ as NSR-space over U. Then,

- U and $\emptyset$ are NSR-closed sets.
- The intersection of any number of NSR-closed sets is an NSR-closed set over U.
- The finite union of NSR-closed sets is an NSR-closed set over U.

Proof. The proof is straightforward.

Definition 3.5 Let $(\mathrm{U}, \tau_{\mathrm{NSR}}(\mathfrak{Y}), \mathrm{E})$ be an NSR-space over U and $\tau_{\mathrm{NSR}}(\mathfrak{Y}) = \{\mathrm{U}, \emptyset\}$. Then, τ_{NSR} is called NSR-indiscrete topology on U w.r.t $\mathfrak{Y}$ and corresponding space is said to be an NSR-indiscrete space over U.

Definition 3.6 Let $(\mathrm{U}, \tau_{\mathrm{NSR}}(\mathfrak{Y}), \mathrm{E})$ is an NSR-topological space and $\mathrm{A} \subseteq \mathrm{B} \subseteq \mathrm{U}$. Then, the collection $\tau_{\mathrm{NSR}_\mathrm{A}} = \{\mathrm{B}_\mathrm{i} \cap \mathrm{A} : \mathrm{B}_\mathrm{i} \in \tau_{\mathrm{NSR}}, \mathrm{i} \in \mathrm{L} \subseteq \mathrm{N}\}$ is called NSR-subspace topology on A. Then, $(\mathrm{A}, \tau_{\mathrm{NSR}_\mathrm{A}})$ is called an NSR-topological subspace of $(\mathrm{B}, \tau_{\mathrm{NSR}})$.

Definition 3.7 Let $(\mathrm{U}, \tau_{\mathrm{NSR}'}(\mathfrak{Y}), \mathrm{E})$ and $(\mathrm{U}, \tau_{\mathrm{NSR}}(\mathfrak{X}), \mathrm{E})$ be two NSR-topological spaces. $\tau_{\mathrm{NSR}'}(\mathfrak{Y})$ is finer than $\tau_{\mathrm{NSR}}(\mathfrak{X})$, if $\tau_{\mathrm{NSR}'}(\mathfrak{Y}) \supseteq \tau_{\mathrm{NSR}}(\mathfrak{X})$.

Definition 3.8 Let $(\mathrm{U}, \tau_{\mathrm{NSR}}(\mathfrak{Y}), \mathrm{E})$ be a NSR-topological space and $\beta_{\mathrm{NSR}} \subseteq \tau_{\mathrm{NSR}}$. If we can write members of τ_{NSR} as the union of members of β_{NSR}, then β_{NSR} is called NSR-basis for the NSR-topology τ_{NSR}.

Proposition 3.9 If $\tau_{\mathrm{NSR}}(\mathfrak{Y})$ is an NSR-topology on U w.r.t $\mathfrak{Y}$ the the collection

$$\beta_{\mathrm{NSR}} = \{\mathrm{U}, \underline{\mathrm{apr}}_{\mathrm{NSR}}(\mathfrak{Y}), \mathrm{B}_{\mathrm{NSR}}(\mathfrak{Y})\}$$

is a base for $\tau_{\mathrm{NSR}}(\mathfrak{Y})$

Theorem 3.10 Let $(\mathrm{U}, \tau_{\mathrm{NSR}}(\mathfrak{Y}), \mathrm{E})$ and $(\mathrm{U}, \tau_{\mathrm{NSR}'}(\mathfrak{Y}'), \mathrm{E})$ be two NSR-topological spaces w.r.t $\mathfrak{Y}$ and $\mathfrak{Y}'$ respectively. Let β_{NSR} and $\beta_{\mathrm{NSR}'}$ be NSR-bases for τ_{NSR} and $\tau_{\mathrm{NSR}'}$, respectively. If $\beta_{\mathrm{NSR}'} \subseteq \beta_{\mathrm{NSR}}$ then τ_{NSR} is finer than $\tau_{\mathrm{NSR}'}$ and $\tau_{\mathrm{NSR}'}$ is weaker than τ_{NSR}.

Theorem 3.11 Let $(U, \tau_{NSR}(\mathfrak{Y}), E)$ be an NSR-topological space. If β_{NSR} is an NSR-basis for τ_{NSR}. Then, the collection $\beta_{NSR_B} = \{A_i \cap B : A_i \in \beta_{NSR}, i \in I \subseteq \mathbb{N}\}$ is an NSR-basis for the NSR-subspace topology on B.

Proof. Consider $A_i \in \tau_{NSR_B}$. By definition of NSR-subspace topology, $C = D \cap B$,where $D \in \tau_{NSR}$. Since $D \in \tau_{NSR}$, it follows that $D = \cup_{A_i \in \beta_{NSR}} A_i$. Therefore,

$$C = (\cup_{A_i \in \beta_{NSR}} A_i) \cap B = \cup_{A_i \in \beta_{NSR}} (A_i \cap B).$$

3.1 Main Results

We present some results of neutrosophic soft rough topology including NSR-interior, NSR-exterior, NSR-closure, NSR-frontier, NSR-neighbourhood and NSR-limit point. These are some topological properties of NSR-topology and can be used to prove various results related to NSR-topological spaces.

Definition 3.12 Let $(U, \tau_{NSR}(\mathfrak{Y}), E)$ be an NSR-topological space w.r.t $\mathfrak{Y}$, where $T \subseteq U$ be an arbitrary subset. The NSR-interior of T is union of all NSR-open subsets of T and we denote it as $Int_{NSR}(T)$.

We verify that $Int_{NSR}(T)$ is the largest NSR-open set contained by T.

Theorem 3.13 Let $(U, \tau_{NSR}(\mathfrak{Y}), E)$ be a NSR-topological space over U w.r.t $\mathfrak{Y}$, S and T are NSR-sets over U. Then

- $Int_{NSR}(\emptyset) = \emptyset$ and $Int_{NSR}(U) = U$,
- $Int_{NSR}(S) \subseteq S$,
- S is NSR-open set $\Leftrightarrow Int_{NSR}(S) = S$,
- $Int_{NSR}(Int_{NSR}(S)) = Int_{NSR}(S)$,
- $S \subseteq T$ implies $Int_{NSR}(S) \subseteq Int_{NSR}(T)$,
- $Int_{NSR}(S) \cup Int_{NSR}(T) \subseteq Int_{NSR}(S \cup T)$,
- $Int_{NSR}(S) \cap Int_{NSR}(T) = Int_{NSR}(S \cap T)$.

Proof. (i) and (ii) are obvious.

(iii) First, suppose that $Int_{NSR}(S) = S$. Since $Int_{NSR}(S)$ is an NSR-open set, it follows that S is NSR-open set. For the converse, if S is a NSR-open set, then the largest NSR-open set that is contained in S is S itself. Thus, $Int_{NSR}(S) = S$.

(iv) Since $Int_{NSR}(S)$ is an NSR-open set, by part (iii) we get $Int_{NSR}(Int_{NSR}(S)) = Int_{NSR}(S)$.

(v) Suppose that $S \subseteq T$. By (ii) $Int_{NSR}(S) \subseteq S$. Then $Int_{NSR}(S) \subseteq T$. Since $Int_{NSR}(S)$ is NSR-open set contained by T. So by definition of NSR-interior $Int_{NSR}(S) \subseteq Int_{NSR}(T)$.

(vi) By using (ii) $Int_{NSR}(S) \subseteq S$ and $Int_{NSR}(T) \subseteq T$. Then, $Int_{NSR}(S) \cup Int_{NSR}(T) \subseteq S \cup T$. Since $Int_{NSR}(S) \cup Int_{NSR}(T)$ is an NSR-open, it follows that $Int_{NSR}(S) \cup Int_{NSR}(T) \subseteq Int_{NSR}(S \cup T)$.

(vii) By using (ii) $Int_{NSR}(S) \subseteq S$ and $Int_{NSR}(T) \subseteq T$. Then, $Int_{NSR}(S) \cap Int_{NSR}(T) \subseteq S \cap T$. Since $Int_{NSR}(S) \cap Int_{NSR}(T)$ is NSR-open, it follows that $Int_{NSR}(S) \cap Int_{NSR}(T) \subseteq Int_{NSR}(S \cap T)$. For the converse, $S \cap T \subseteq S$ also $S \cap T \subseteq T$. Then, $Int_{NSR}(S \cap T) \subseteq Int_{NSR}(S)$ and $Int_{NSR}(S \cap T) \subseteq Int_{NSR}(T)$. Hence $Int_{NSR}(S \cap T) \subseteq Int_{NSR}(S) \cap Int_{NSR}(T)$.

Definition 3.14 Let $(U, \tau_{NSR}(\mathfrak{Y}), E)$ be an NSR-topological space w.r.t $\mathfrak{Y}$, where $\mathfrak{Y} \subseteq U$. Let $T \subseteq U$. Then, NSR-exterior of T is defined as $Int_{NSR}(T^c)$, where T^c is complement of T. NSR-exterior of T is denoted by $Ext_{NSR}(T)$.

Definition 3.15 Let $(U, \tau_{NSR}(\mathfrak{Y}), E)$ be an NSR-topological space w.r.t $\mathfrak{Y}$, where $\mathfrak{Y} \subseteq U$. Let $T \subseteq U$. Then, NSR-closure of T is defined to be intersection of all NSR-closed supersets of T and is denoted by $Cl_{NSR}(T)$.

Example 3.16 Consider the NSR-topology given in Example 3.2, taking $T = \{g_1, g_2, g_3\}$, so $T^c = \{g_4, g_5\}$. Then $Int_{NSR}(T) = \{g_1, g_2\}$, $Ext_{NSR(T)} = Int_{NSR}(T^c) = \{g_4\}$ and $Cl_{NSR}(\mathrm{T}) = \{g_1, g_2, g_3, g_5\}$.

Theorem 3.17 Let $(U, \tau_{NSR}(\mathfrak{Y}), E)$ be a NSR-topological space over U w.r.t $\mathfrak{Y}$, S and T are NSR-sets over U. Then

- $Cl_{NSR}(\emptyset) = \emptyset$ and $Cl_{NSR}(U) = U$,
- $S \subseteq Cl_{NSR}(S)$,
- S is NSR-closed set $\Leftrightarrow S = Cl_{NSR}(S)$,
- $Cl_{NSR}(Cl_{NSR}(S)) = Cl_{NSR}(S)$,
- $S \subseteq T$ implies $Cl_{NSR}(S) \subseteq Cl_{NSR}(T)$,
- $Cl_{NSR}(S \cup T) = Cl_{NSR}(S) \cup Cl_{NSR}(T)$,
- $Cl_{NSR}(S \cap \mathrm{T}) \subseteq Cl_{NSR}(S) \cap Cl_{NSR}(T)$.

Proof. (i) and (ii) are straightforward.

(iii) First, consider $S = Cl_{NSR}(S)$. Since $Cl_{NSR}(S)$ is an NSR-closed set, so S is an NSR-closed set over U. For the converse, suppose that S be an NSR-closed set over U. Then, S is NSR-closed superset of S. So that $S = Cl_{NSR}(S)$.

(iv) By definition $Cl_{NSR}(S)$ is always NSR-closed set. Therefore, by part (iii) we have $Cl_{NSR}(Cl_{NSR}(S)) = Cl_{NSR}(S)$.

(v) Let $S \subseteq T$. By (ii) $T \subseteq Cl_{NSR}(T)$. Then, $S \subseteq Cl_{NSR}(T)$. Since $Cl_{NSR}(T)$ is a NSR-closed superset of S, it follows that $Cl_{NSR}(S) \subseteq Cl_{NSR}(T)$.

(vi) Since $S \subseteq S \cup T$ and $T \subseteq S \cup T$, by part (v), $Cl_{NSR}(S) \subseteq Cl_{NSR}(S \cup T)$ and $Cl_{NSR}(T) \subseteq Cl_{NSR}(S \cup T)$. Hence $Cl_{NSR}(S) \cup Cl_{NSR}(S) \subseteq Cl_{NSR}(S \cup T)$. For the converse, let $S \subseteq Cl_{NSR}(S)$ and $T \subseteq Cl_{NSR}(T)$. Then, $S \cup T \subseteq Cl_{NSR}(S) \cup CL_{NSR}(T)$. Since $Cl_{NSR}(S) \cup Cl_{NSR}(T)$ is a NSR-closed superset of $S \cup T$. Thus, $Cl_{NSR}(S \cup T) = Cl_{NSR}(S) \cup Cl_{NSR}(T)$.

(vii) Since $S \cap T \subseteq S$ and $S \cap T \subseteq T$, by part(5) $Cl_{NSR}(S \cap T) \subseteq Cl_{NSR}(S)$ and $Cl_{NSR}(S \cap T) \subseteq Cl_{NSR}(T)$. Thus, we obtain $Cl_{NSR}(S \cap T) \subseteq Cl_{NSR}(S) \cap Cl_{NSR}(T)$.

Definition 3.18 Let $(U, \tau_{NR}(\mathfrak{Y}), E)$ be a NSR-topological space w.r.t $\mathfrak{Y}$, where $\mathfrak{Y} \subseteq U$. Let $T \subseteq U$. Then, NSR-frontier or NSR-boundary of T is denoted by $F_{r_{NSR}}(T)$ or $b_{NSR}(T)$ and mathematically defined as

$$F_{r_{NSR}}(T) = Cl_{NSR}(T) \cap Cl_{NSR}(T^c).$$

Clearly NSR-frontier $F_{r_{NSR}}(T)$ is an NSR-closed set.

Example 3.19 Consider the NSR-topology given in Example 3.2, taking $T = \{g_1, g_2, g_3\}$, so $T^c = \{g_4, g_5\}$. Then, $Cl_{NSR}(T) = \{g_1, g_2, g_3, g_5\}$ and $Cl_{NSR}(T^c) = \{g_3, g_4, g_5\}$.

$$F_{r_{NSR}}(T) = Cl_{NSR}(T) \cap Cl_{NSR}(T^c) = \{g_3, g_5\}$$

Definition 3.20 Let $(U, \tau_{NR}(\mathfrak{Y}), E)$ be an NSR-topological space. A subset X of U is said to be NSR-neighborhood of $g \in U$ if there exist an NSR-open set W_g containing g so that

$$g \in W_g \subseteq X.$$

Definition 3.21 The set of all the NSR-limit points of S is known as NSR-derived set of S and is denoted by S^d_{NSR}.

4 NSR-set in multi-criteria decision-making

In this section, we present an idea for multi-criteria decision-making method based on the neutrosophic soft rough sets $NSR-set$.

Let $U = \{g_1, g_2, g_3, \dots, g_m\}$ is the set of objects under observation, E be the set of criteria to analyze the objects in U. Let $\mathfrak{A} = \{\xi_1, \xi_2, \xi_3, \dots, \xi_n\} \subseteq E$ and $(\Phi, \mathfrak{A})$ be a neutrosophic soft set over U. Suppose that $H = \{P_1, P_2, \dots, P_k\}$ be a set of experts, $\mathfrak{Y}_1, \mathfrak{Y}_2, \dots, \mathfrak{Y}_k$ are subsets of U which indicate results of initial evaluations of experts $P_1, P_2, \dots, P_k$, respectively and $\mathfrak{T}_1, \mathfrak{T}_2, \dots \mathfrak{T}_r \in NS_U$ are real results that previously obtained for same or similar problems in different times or different places.

Definition 4.1 Let $\underline{apr}_{NSR_{\mathfrak{T}_q}}(\mathfrak{Y}_j), \overline{apr}_{NSR_{\mathfrak{T}_q}}(\mathfrak{Y}_j)$ be neutrosophic soft lower and upper approximations of $\mathfrak{Y}_j (j = 1,2,\dots,k)$ related to $\mathfrak{T}_q (q = 1,2,\dots,r)$. Then,

$$\underline{a} = \begin{pmatrix} \underline{n}_1^1 & \underline{n}_2^1 & \cdots & \underline{n}_k^1 \\ \underline{n}_1^2 & \underline{n}_2^2 & \cdots & \underline{n}_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ \underline{n}_1^r & \underline{n}_2^r & \cdots & \underline{n}_k^r \end{pmatrix} \tag{1}$$

$$\overline{a} = \begin{pmatrix} \overline{n}_1^1 & \overline{n}_2^1 & \cdots & \overline{n}_k^1 \\ \overline{n}_1^2 & \overline{n}_2^2 & \cdots & \overline{n}_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ \overline{n}_1^r & \overline{n}_2^r & \cdots & \overline{n}_k^r \end{pmatrix} \tag{2}$$

are called neutrosophic soft lower and neutrosophic upper approximations matrices, respectively, and represented by $\underline{a}$ and $\overline{a}$. Here

$$\underline{n}_j^q = (\underline{g}_{1j}^q, \underline{g}_{2j}^q, \dots, \underline{g}_{nj}^q) \tag{3}$$

$$\overline{n}_j^q = (\overline{g}_{1j}^q, \overline{g}_{2j}^q, \dots, \overline{g}_{nj}^q) \tag{4}$$

Where

$$\underline{g}_{ij}^q = \begin{pmatrix} 1, g_i \in \underline{apr}_{NSR_{\mathfrak{T}_q}}(\mathfrak{Y}_j) \\ 0, g_i \not\in \underline{apr}_{NSR_{\mathfrak{T}_q}}(\mathfrak{Y}_j) \end{pmatrix}$$

and

$$\overline{g}_{ij}^q = \begin{pmatrix} 1, g_i \in \overline{apr}_{NSR_{\mathfrak{T}_q}}(\mathfrak{Y}_j) \\ 0, g_i \not\in \overline{apr}_{NSR_{\mathfrak{T}_q}}(\mathfrak{Y}_j) \end{pmatrix}$$

Definition 4.2 Let $\underline{n}$ and $\overline{n}$ be neutrosophic soft lower and neutrosophic upper approximations matrices based on $\underline{apr}_{NSR_{\mathfrak{T}_q}}(\mathfrak{Y})_j$, $\overline{apr}_{NSR_{\mathfrak{T}_q}}(\mathfrak{Y})_j$ for $q = 1,2,\ldots r$ and $j = 1,2,\ldots,k$. Neutrosophic soft lower approximation vector represented by $(\underline{n})$ and neutrosophic soft upper approximation vector represented by $(\overline{n})$ are defined by, respectively,

$$\underline{n} = \bigoplus_{j=1}^{k} \bigoplus_{q=1}^{r} \underline{n}_j^q \tag{5}$$

$$\overline{n} = \bigoplus_{j=1}^{k} \bigoplus_{q=1}^{r} \overline{n}_j^q \tag{6}$$

Here the operation $\oplus$ represents the vector summation.

Definition 4.3 Let $\underline{n}$ and $\overline{n}$ be neutrosophic soft $\mathfrak{T}_q -$ lower approximation vector and neutrosophic soft $\mathfrak{T}_q -$ upper approximation vector, respectively. Then, vector summation $\underline{n} \oplus \overline{n} = (w_1, w_2, \ldots, w_n)$ is called decision vector.

Definition 4.4 Let $\underline{n} \oplus \overline{n} = (w_1, w_2, \ldots, w_n)$ be the decision vector. Then, each w_i is called a weighted number of $g_i \in U$ and g_i is called an optimum element of U if it weighted number is maximum of $w_i \forall i \in I_n$. In this case, if there are more then one optimum elements of U, select one of them.

Algorithm 1 for neutrosophic soft rough set:

Input

Step-1: Take initial evaluations $\mathfrak{Y}_1, \mathfrak{Y}_2, \ldots, \mathfrak{Y}_k$ of experts $P_1, P_2, \ldots, P_k$.

Step-2: Construct $\mathfrak{T}_1, \mathfrak{T}_2, \ldots \mathfrak{T}_r$ neutrosophic soft sets using real results.

Step-3: Compute $\underline{apr}_{NSR_{\mathfrak{T}_q}}(\mathfrak{Y}_j)$ and $\overline{apr}_{NSR_{\mathfrak{T}_q}}(\mathfrak{Y}_j)$ for each $q = 1,2,\ldots,r$ and $j = 1,2,\ldots,k$.

Step-4: Construct neutrosophic soft lower and neutrosophic soft upper approximations matrices $\underline{a}$ and $\overline{a}$.

Step-5: Compute $\underline{n}$ and $\overline{n}$,

Step-6: Compute $\underline{n} \oplus \overline{n}$,

Output

Step-7: Select $max_{i \in I_n} w_i$.

The flow chart of proposed algorithm 1 is represented in Figure.1

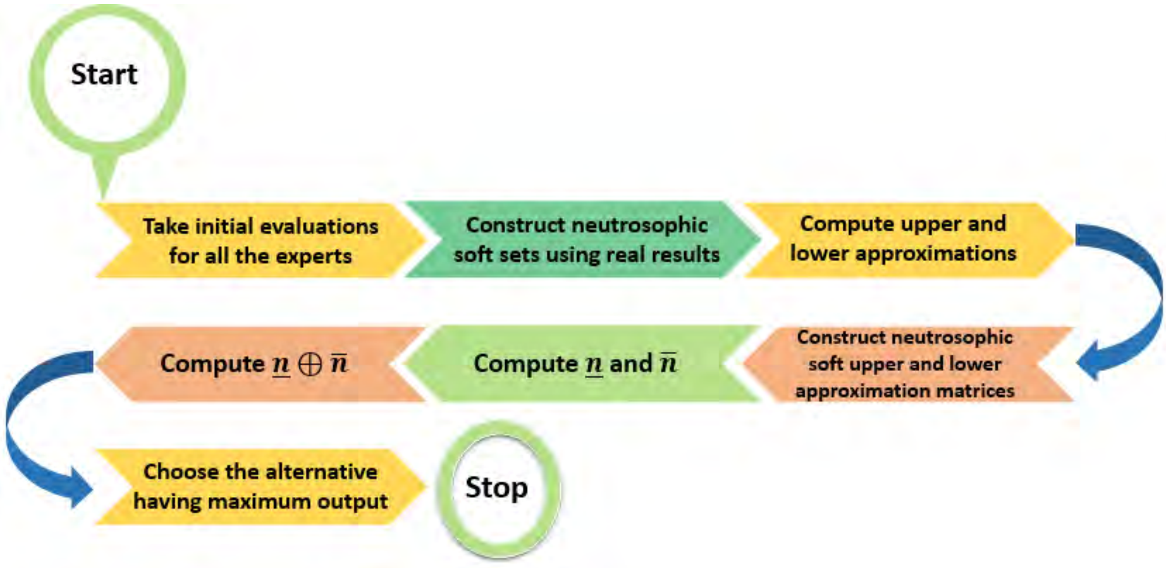

Fig 1: Flow chart diagram of proposed algorithm 1 for NSR-set.

Example 4.5 In finance company three finance experts P_1, P_2, P_3 want to make investment one of the clothing brand

$$\{g_1 = Jor, g_2 = Aero, g_3 = Chan, g_4 = Li, g_5 = Srk\}.$$

The set of parameters include the following parameters

$$\mathfrak{A} = \{\xi_1 = Market\ Share, \xi_2 = Acknowledgement, \xi_3 = Uniqueness, \xi_4 = Economical\ Magnification\}$$

*Step*1: $\mathfrak{Y}_1 = \{g_1, g_2, g_4\}, \mathfrak{Y}_2 = \{g_1, g_3, g_5\}, \mathfrak{Y}_3 = \{g_2, g_4, g_5\}$ are primary evaluations of experts P_1, P_2, P_3, respectively.

*Step*2: Neutrosophic soft sets $\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3$ are the actual results in individual three periods and tabular representations of these neutrosophic soft sets are given in Table 2, Table 3 and Table 4, respectively.

$\mathfrak{T}_1$	ξ_1	ξ_2	ξ_3	ξ_4
g_1	(0.6,0.6,0.2)	(0.8,0.4.0.3)	(0.7,0.4,0.3)	(0.8,0.6,0.4)
g_2	(0.4,0.6,0.6)	(0.6,0.2,0.4)	(0.6,0.4,0.3)	(0.7,0.6,0.6)
g_3	(0.6,0.4,0.2)	(0.8,0.1,0.3)	(0.7,0.2,0.5)	(0.7,0.6,0.4)
g_4	(0.6,0.3,0.3)	(0.8,0.2,0.2)	(0.5,0.2,0.6)	(0.7,0.5,0.6)
g_5	(0.8,0.2,0.3)	(0.8,0.3,0.2)	(0.7,0.3,0.4)	(0.9,0.5,0.7)

Table 2: Neutrosophic soft set $\mathfrak{T}_1$

$\mathfrak{T}_2$	ξ_1	ξ_2	ξ_3	ξ_4
g_1	(0.6,0.4,0.2)	(0.8,0.1,0.3)	(0.7,0.2,0.5)	(0.7,0.6,0.4)
g_2	(0.4,0.6,0.6)	(0.6,0.2,0.4)	(0.6,0.4,0.3)	(0.7,0.6,0.6)
g_3	(0.8,0.2,0.3)	(0.8,0.3,0.2)	(0.7,0.3,0.4)	(0.9,0.5,0.7)
g_4	(0.6,0.3,0.3)	(0.8,0.2,0.2)	(0.5,0.2,0.6)	(0.7,0.5,0.6)
g_5	(0.6,0.6,0.2)	(0.8,0.4.0.3)	(0.7,0.4,0.3)	(0.8,0.6,0.4)

Table 3: Neutrosophic soft set $\mathfrak{T}_2$

$\mathfrak{T}_3$	ξ_1	ξ_2	ξ_3	ξ_4
g_1	(0.6,0.6,0.2)	(0.8,0.4.0.3)	(0.7,0.4,0.3)	(0.8,0.6,0.4)
g_2	(0.6,0.3,0.3)	(0.8,0.2,0.2)	(0.5,0.2,0.6)	(0.7,0.5,0.6)
g_3	(0.6,0.4,0.2)	(0.8,0.1,0.3)	(0.7,0.2,0.5)	(0.7,0.6,0.4)
g_4	(0.4,0.6,0.6)	(0.6,0.2,0.4)	(0.6,0.4,0.3)	(0.7,0.6,0.6)
g_5	(0.8,0.2,0.3)	(0.8,0.3,0.2)	(0.7,0.3,0.4)	(0.9,0.5,0.7)

Table 4: Neutrosophic soft set $\mathfrak{T}_3$

The tabular representation of the neutrosophic right neighborhoods of $\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3$ are given in Table5, Table 6 and Table 7 respectively.

Neighborhoods of $\mathfrak{T}_1$	
$g_1]_{\mathfrak{A}}$	$\{g_1\}$
$g_2]_{\mathfrak{A}}$	$\{g_1, g_2\}$
$g_3]_{\mathfrak{A}}$	$\{g_1, g_3\}$
$g_4]_{\mathfrak{A}}$	$\{g_4\}$
$g_5]_{\mathfrak{A}}$	$\{g_5\}$

Table 5: Neutrosophic right neighborhoods of $\mathfrak{T}_1$ w.r.t set $\mathfrak{A}$

Neighborhoods of $\mathfrak{T}_2$	
$g_1]_{\mathfrak{A}}$	$\{g_1, g_5\}$
$g_2]_{\mathfrak{A}}$	$\{g_2, g_5\}$
$g_3]_{\mathfrak{A}}$	$\{g_3\}$
$g_4]_{\mathfrak{A}}$	$\{g_4\}$
$g_5]_{\mathfrak{A}}$	$\{g_5\}$

Table 6: Neutrosophic right neighborhoods of $\mathfrak{T}_2$ w.r.t set $\mathfrak{A}$

Neighborhoods of $\mathfrak{T}_3$	
$g_1]_{\mathfrak{A}}$	$\{g_1\}$
$g_2]_{\mathfrak{A}}$	$\{g_2\}$
$g_3]_{\mathfrak{A}}$	$\{g_1, g_3\}$
$g_4]_{\mathfrak{A}}$	$\{g_1, g_4\}$
$g_5]_{\mathfrak{A}}$	$\{g_5\}$

Table 7: Neutrosophic right neighborhoods of $\mathfrak{T}_3$ w.r.t set $\mathfrak{A}$

$Step3$: Next we find $\underline{apr}_{NSR_{\mathfrak{T}_1}}$ and $\overline{apr}_{NSR_{\mathfrak{T}_1}}$ for each $\mathfrak{Y}_j$, where $j = 1,2,3$.

$$\underline{apr}_{NSR_{\mathfrak{T}_1}}(\mathfrak{Y}_1) = \{g_1, g_2, g_4\},$$
$$\overline{apr}_{NSR_{\mathfrak{T}_1}}(\mathfrak{Y}_1) = \{g_1, g_2, g_3, g_4\},$$
$$\underline{apr}_{NSR_{\mathfrak{T}_1}}(\mathfrak{Y}_2) = \{g_1, g_3, g_5\},$$
$$\overline{apr}_{NSR_{\mathfrak{T}_1}}(\mathfrak{Y}_2) = \{g_1, g_2, g_3, g_5\},$$
$$\underline{apr}_{NSR_{\mathfrak{T}_1}}(\mathfrak{Y}_3) = \{g_4, g_5\},$$
$$\overline{apr}_{NSR_{\mathfrak{T}_1}}(\mathfrak{Y}_3) = \{g_1, g_2, g_3, g_4, g_5\}$$

Similarly we find $\underline{apr}_{NSR_{\mathfrak{T}_2}}, \overline{apr}_{NSR_{\mathfrak{T}_2}}$ and $\underline{apr}_{NSR_{\mathfrak{T}_3}}$, $\overline{apr}_{NSR_{\mathfrak{T}_3}}$ corresponding to each $\mathfrak{Y}_j$, where $j =$

1,2,3.

$$\underline{apr}_{NSR_{\mathfrak{T}_2}}(\mathfrak{Y}_1) = \{g_4\},$$
$$\overline{apr}_{NSR_{\mathfrak{T}_2}}(\mathfrak{Y}_1) = \{g_1, g_2, g_4, g_5\},$$
$$\underline{apr}_{NSR_{\mathfrak{T}_2}}(\mathfrak{Y}_2) = \{g_1, g_3, g_5\},$$
$$\overline{apr}_{NSR_{\mathfrak{T}_2}}(\mathfrak{Y}_2) = \{g_1, g_2, g_3, g_5\},$$
$$\underline{apr}_{NSR_{\mathfrak{T}_2}}(\mathfrak{Y}_3) = \{g_4, g_5\},$$
$$\overline{apr}_{NSR_{\mathfrak{T}_2}}(\mathfrak{Y}_3) = \{g_1, g_2, g_4, g_5\}$$

and

$$\underline{apr}_{NSR_{\mathfrak{T}_3}}(\mathfrak{Y}_1) = \{g_1, g_2, g_4\},$$
$$\overline{apr}_{NSR_{\mathfrak{T}_3}}(\mathfrak{Y}_1) = \{g_1, g_2, g_3, g_4\},$$
$$\underline{apr}_{NSR_{\mathfrak{T}_3}}(\mathfrak{Y}_2) = \{g_1, g_3, g_5\},$$
$$\overline{apr}_{NSR_{\mathfrak{T}_3}}(\mathfrak{Y}_2) = \{g_1, g_3, g_4, g_5\},$$
$$\underline{apr}_{NSR_{\mathfrak{T}_3}}(\mathfrak{Y}_3) = \{g_2, g_5\},$$
$$\overline{apr}_{NSR_{\mathfrak{T}_3}}(\mathfrak{Y}_3) = \{g_1, g_2, g_4, g_5\}$$

$Step4$: Neutrosophic soft lower approximation matrix and neutrosophic soft upper approximation matrix are obtained as follows:

$$\underline{a} = \begin{pmatrix} (1,1,0,1,0) & (1,0,1,0,1) & (0,0,0,1,1) \\ (0,0,0,1,0) & (1,0,1,0,1) & (0,0,0,1,1) \\ (1,1,0,1,0) & (1,0,1,0,1) & (0,1,0,0,0) \end{pmatrix} \tag{7}$$

$$\overline{a} = \begin{pmatrix} (1,1,1,1,0) & (1,1,1,0,1) & (1,1,1,1,1) \\ (1,1,0,1,1) & (1,1,1,0,1) & (1,1,0,1,1) \\ (1,1,1,1,0) & (1,0,1,1,1) & (1,1,0,1,1) \end{pmatrix} \tag{8}$$

$Step5$: Using Eqs. 7 and 8, neutrosophic soft lower approximation vector and neutrosophic soft upper approximation vector are obtained as follows:

$$\underline{n} = (5,3,3,5,5)$$

$$\overline{n} = (9,8,6,7,7)$$

$Step6$: Decision vector is obtained as $\underline{n} \oplus \overline{n} = (14,11,9,12,12)$.

$Step7$: Since $max_{i \in I_n} w_i = w_1 = 14$, optimal clothing brand is $g_1 = Jor$.

5 NSR-topology in multi-criteria decision-making

In this section, we use the concept of NSR-topology in multi-criteria decision-making. The idea of core in the picking of attributes to the rough set was introduced by Thivagar in [45]. In the following definition, we develop this idea of core to the NSR-set.

Definition 5.1 Let U be the set of objects, $K = (\Phi, \mathfrak{A})$ is the neutrosophic soft set and $G = (U, K)$ is the the corresponding neutrosophic soft approximation space. Let $\mathfrak{R}$ be an indiscernibility relation. Let τ_{NSR} be an NSR-topology on U and β_{NSR} be the basis defined for τ_{NSR}. Let $\mathfrak{N}$ be the subset of $\mathfrak{A}$, is said to be core of $\mathfrak{R}$ if $\beta_{\mathfrak{N}} \neq \beta_{NSR-(s)}$ for each 's' in $\mathfrak{N}$. i.e. a core of $\mathfrak{R}$ is the subset of attributes with the condition that if we remove any element from $\mathfrak{N}$ it will affect the classification power of the attributes.

Algorithm 2 for neutrosophic soft rough topology:

Input

Step-1: Consider initial universe U, set of attributes $\mathfrak{A}$ which can be classified into division $\mathbb{D}$ of decision attributes, $\mathbb{C}$ of condition attributes and an indiscernibility relation $\mathfrak{R}$ on U. Construct the neutrosophic soft set in tabular form corresponding to $\mathbb{C}$ condition attributes and a subset $\mathfrak{Y}$ of U. The columns indicate the elements of universe, rows represent the attributes and entries of table give attribute values.

Output

Step-2: Classify set $\mathfrak{Y}$ and find the NSR-approximation subsets $\left(\underline{\mathfrak{R}}_G(\mathfrak{Y}), \overline{\mathfrak{R}}_G(\mathfrak{Y})\right) and\ B_G(\mathfrak{Y})$ w.r.t $\mathfrak{R}$.

Step-3: Define Neutrosophic Soft Rough Topology $\tau_{\mathfrak{R}}$ on U and find basis β_{NSR}.

Step-4: By removing an attribute ξ from $\mathbb{C}$, find again the NSR-approximation subsets $(\underline{\mathfrak{R}}_G(\mathfrak{Y}), \overline{\mathfrak{R}}_G(\mathfrak{Y})), B_G(\mathfrak{Y}))$ w.r.t $\mathfrak{R} on \mathbb{C} - (\xi)$.

Step-5: Generate $NSR - topology\ \tau_{NSR-(\xi)}$ on U,define its basis $\beta_{NSR-(\xi)}$.

Step-6: Repeat step 4 and step 5 for each attribute in $\mathbb{C}$.

Step-7: The attributes for which $\beta_{NSR-(\xi)} \neq \beta_{NSR}$ gives the $core(\mathfrak{R})$.

The flow chart diagram of proposed algorithm 2 is represented as Figure 2.

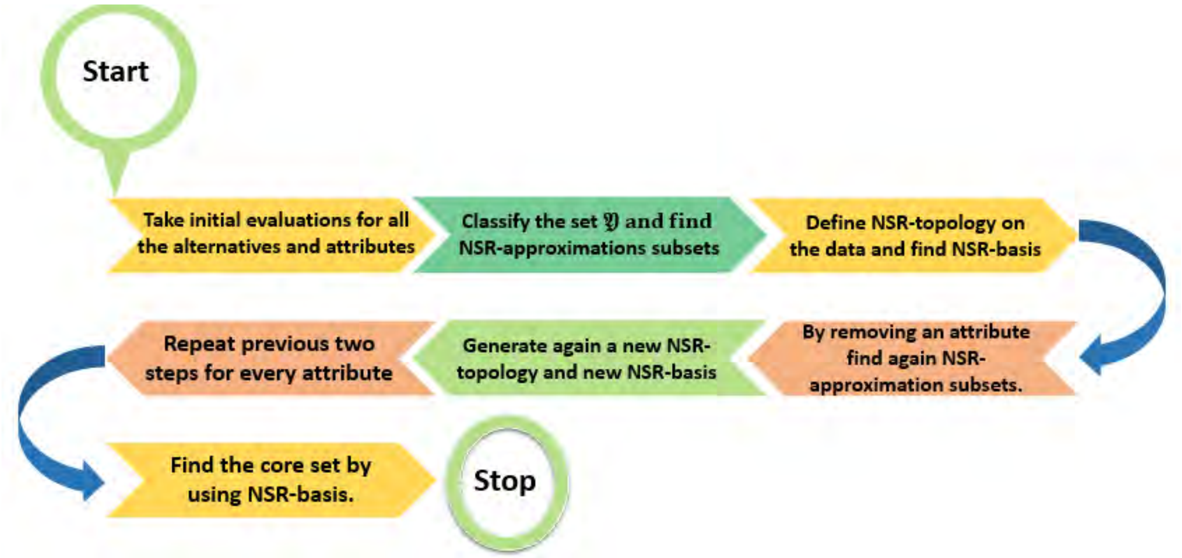

Fig 2: The flow chart diagram of algorithm 2 for NSR-topology.

Example 5.2 Here we consider the problem of Crime rate in developing countries of Asia, Crime is an unlawful act punishable by a state or other authority. In other words, we can say that a crime is an act harmful not only to some individual but also to a community, society or the state. A developing country is a country with a less developed industrial base and a low Human Development Index (HDI) relative to other countries. Developing countries are facing so many issues including high crime rate. This is the fundamental reason of emerging questions in our mind, that why the crime rate is higher in developing countries?

We apply the concept of NSR-topology in Crime rate of developing countries of Asia. Consider the following information table which shows data about 5 developing countries. The rows of the table represent the objects(countries). Let $U = \{g_1 = Bangladesh, g_2 = Afghanistan, g_3 = SriLanka, g_4 = Nepal, g_5 = Pakistan\}$ be the set of developing countries and $\mathfrak{A} = \{\xi_1, \xi_2, \xi_3, \xi_4\}$, where ξ_1 stands for `corruption', $\xi_2$ stands for `poverty ', ξ_3 stands for `self actualization' and $\xi_4$ stands for `lack of education'. Let $K = (\Phi, \mathfrak{A})$ is the neutrosophic soft set over U shown by Table 8,corresponding soft approximation space $G = (U, K)$.

K	ξ_1	ξ_2	ξ_3	ξ_4	Crime Rate
g_1	(0.6,0.6,0.2)	(0.8,0.4.0.3)	(0.7,0.4,0.3)	(0.8,0.6,0.4)	High
g_2	(0.4,0.6,0.6)	(0.6,0.2,0.4)	(0.6,0.4,0.3)	(0.7,0.6,0.6)	Medium
g_3	(0.6,0.4,0.2)	(0.8,0.1,0.3)	(0.7,0.2,0.5)	(0.7,0.6,0.4)	Medium
g_4	(0.6,0.3,0.3)	(0.8,0.2,0.2)	(0.5,0.2,0.6)	(0.7,0.5,0.6)	High
g_5	(0.8,0.2,0.3)	(0.8,0.3,0.2)	(0.7,0.3,0.4)	(0.9,0.5,0.7)	High

Table 8: Neutrosophic soft set $K = (\Phi, \mathfrak{A})$

The tabular representation of neutrosophic right neighborhoods of K w.r.t set $\mathfrak{A}$ is given Table 9.

Neighborhoods of K	
$g_1]_{\mathfrak{A}}$	$\{g_1\}$
$g_2]_{\mathfrak{A}}$	$\{g_1, g_2\}$
$g_3]_{\mathfrak{A}}$	$\{g_1, g_3\}$

$g_4]_{\mathfrak{A}}$	$\{g_4\}$
$g_5]_{\mathfrak{A}}$	$\{g_5\}$

Table 9: Neutrosophic right neighborhoods of K w.r.t set $\mathfrak{A}$

For $\mathfrak{Y} = \{g_1, g_3, g_5\}$ and indiscernibility relation 'Crime rate' we have $\underline{\mathfrak{R}}_G(\mathfrak{Y}) = \{g_1, g_3, g_5\}$, $\overline{\mathfrak{R}}_G(\mathfrak{Y}) = \{g_1, g_2, g_3, g_5\}$ and $B_G(\mathfrak{Y}) = \{g_2\}$.

So we define NSR-topology as $\tau_{NSR}(\mathfrak{Y}) = \{U, \emptyset, \{g_1, g_3, g_5\}, \{g_1, g_2, g_3, g_5\}, \{\mathrm{g}_2\}\}$ and its basis $\beta_{NSR} = \{U, \{g_1, g_3, g_5\}, \{g_2\}\}$.

If we remove the attribute `Corruption', then the tabular representation of neutrosophic right neighborhoods of K w.r.t set $\mathfrak{A} - \xi_1$ is given Table 10.

Neighborhoods of K	
$g_1]_{\mathfrak{A}-\xi_1}$	$\{g_1\}$
$g_2]_{\mathfrak{A}-\xi_1}$	$\{g_1, g_2\}$
$g_3]_{\mathfrak{A}-\xi_1}$	$\{g_1, g_3\}$
$g_4]_{\mathfrak{A}-\xi_1}$	$\{g_4\}$
$g_5]_{\mathfrak{A}-\xi_1}$	$\{g_5\}$

Table 10: Neutrosophic right neighborhoods of K w.r.t set $\mathfrak{A} - \xi_1$

we have

$$\tau_{NSR-\xi_1}(\mathfrak{Y}) = \{U, \emptyset, \{g_1, g_3, g_5\}, \{g_1, g_2, g_3, g_5\}, \{g_2\}\}$$

is a NSR-topology and its basis is

$$\beta_{NSR} - \xi_1 = \{U, \{g_1, g_3, g_5\}, \{g_2\}\} = \beta_{NSR}.$$

If we remove the attribute `poverty', then the tabular representation of neutrosophic right neighborhoods of K w.r.t set $\mathfrak{A} - \xi_2$ is given Table 11.

Neighborhoods of K	
$g_1]_{\mathfrak{A}-\xi_2}$	$\{g_1\}$
$g_2]_{\mathfrak{A}-\xi_2}$	$\{g_1, g_2\}$
$g_3]_{\mathfrak{A}-\xi_2}$	$\{g_1, g_3\}$
$g_4]_{\mathfrak{A}-\xi_2}$	$\{g_1, g_3, g_4\}$
$g_5]_{\mathfrak{A}-\xi_2}$	$\{g_5\}$

Table 11: Neutrosophic right neighborhoods of K w.r.t set $\mathfrak{A} - \xi_2$

We have an NSR-topology and its base as follows:

$$\tau_{NSR-\xi_2}(Y) = \{U, \emptyset, \{g_1, g_3, g_5\}, \{g_2, g_4\}\}$$

and $$\beta_{NSR} - \xi_2 = \{U, \{g_1, g_3, g_5\}, \{g_2, g_4\}\} \neq \beta_{NSR},$$

respectively. If we remove the attribute 'self actualization', then the tabular representation of neutrosophic right neighborhoods of K w.r.t set $\mathfrak{A} - \xi_3$ is given Table 12.

Neighborhoods of K	
$g_1]_{\mathfrak{A}-\xi_3}$	$\{g_1\}$
$g_2]_{\mathfrak{A}-\xi_3}$	$\{g_1, g_2\}$

$g_3]_{\mathfrak{A}-\xi_3}$	$\{g_1, g_3\}$
$g_4]_{\mathfrak{A}-\xi_3}$	$\{g_4\}$
$g_5]_{\mathfrak{A}-\xi_3}$	$\{g_5\}$

Table 12: Neutrosophic right neighborhoods of K w.r.t set $\mathfrak{A} - \xi_3$

We have an NSR-topology and its base as follows:

$$\tau_{NSR-\xi_3}(Y) = \{U, \emptyset, \{g_1, g_3, g_5\}, \{g_1, g_2, g_3, g_5\}, \{g_2\}\}$$

and

$$\beta_{NSR} - \xi_3 = \{U, \emptyset, \{g_1, g_3, g_5\}, \{g_2\} = \beta_{NSR}\},$$

respectively. If we remove the attribute `lack of education', then the tabular representation of neutrosophic right neighborhoods of K w.r.t set $\mathfrak{A} - \xi_4$ is given Table 13.

Neighborhoods of K	
$g_1]_{\mathfrak{A}-\xi_4}$	$\{g_1\}$
$g_2]_{\mathfrak{A}-\xi_4}$	$\{g_1, g_2\}$
$g_3]_{\mathfrak{A}-\xi_4}$	$\{g_1, g_3\}$
$g_4]_{\mathfrak{A}-\xi_4}$	$\{g_4\}$
$g_5]_{\mathfrak{A}-\xi_4}$	$\{g_5\}$

Table 13: Neutrosophic right neighborhoods of K w.r.t set $\mathfrak{A} - \xi_4$

We have an NSR-topology and its base as follows:

$$\tau_{NSR-\xi_4}(Y) = \{U, \emptyset, \{g_1, g_3, g_5\}, \{g_1, g_2, g_3, g_5\}, \{g_2\}\}$$

and

$$\beta_{NSR} - \xi_4 = \{U, \emptyset, \{g_1, g_3, g_5\}, \{g_2\} = \beta_{NSR}\},$$

respectively. Thus, $CORE(NSR) = \{\xi_2\}$, i.e., `poverty' is the deciding attributes of the Crime Rate in developing countries of Asia.

Discussion and comparitive analysis 5.3 In this section, we discuss our results obtained from both numerical examples and present a comparative analysis of proposed topological space to some existing topological spaces. Table 14 describes the comparison of both proposed algorithms based on NSR-sets and NSR-topology. The algorithm 1 is used to find the optimal decision about the set of alternatives and establish the ranking order between them. We can choose the best and worst alternative from the given input information. While algorithm 2 is used to choose the most relevant and significant attribute to which one can observe the specific characteristic of the alternatives. This is called the CORE of the problem, which is an essential part of the decision-making difficulty. Both algorithms have their own merits and can be used to solve decision-making problems in medical, artificial intelligence, business, agriculture, engineering, etc.

Proposed Algorithms	Choice values	Final Decision	Selection criteria
Algorithm 1 (NSR-sets)	$g_1 \succ g_4 \succ g_5 \succ g_2 \succ g_3$	g_1	Based on alternatives
Algorithm 2 (NSR-topology)	$CORE(NSR) = \{\xi_2\}$	$\xi_2 =$ poverty	Based on attributes

Table 14: Comparison of prooposed algorithms

Now we present a soft comparative analysis of proposed approach with some existing approaches. In Table 15, we describe the comparison and discuss about their advantages and limitations.

Set theories	Information about Indeterminacy part	Upper and lower approximations with boundary region	Parameterizations	Advantages	Limitations
Fuzzy sets [1]	No	No	No	Deal with the hesitations.	Do not collect any information about the indeterminacy of input data.
Neutrosophic sets [4, 5]	Yes	No	No	Deal with the data having indeterminacy information.	Do not deal with the roughness and parameterizations.
Rough sets [2, 3]	No	Yes	No	Deal with the roughness of input information and create upper, lower and boundary regions.	Do not give any information about the parameterizations.
Soft sets [6]	No	No	Yes	Deal with the uncertainity with parameterizations.	Do not provide information about the roughness of data.
Soft rough sets [17]	No	Yes	Yes	Deal with uncertainities and roughness of data.	Do not give information about the indeterminacy part of problem.
Rough neutrosophic sets [47]	Yes	Yes	No	Deal with the roughness having indeterminacy information.	Do not deal with the parameterizations.
Neutrosophic soft rough sets and topology (proposed)	Yes	Yes	Yes	Provide the data of indeterminacy part and remove roughness under parameterizations without any loss of information.	Effective but heavy calculations as compared to above existing theories.

Table 15: Comparitive analysis of proposed approach with some exsting theories.

6. Conclusion

Most of the issues in decision-making problems are associated with uncertain, imprecise and, multipolar information, which cannot be tackled properly through the fuzzy set. So to overcome this particular deficiency rough set was introduced by Pawlak, which deals with the vagueness of input data. This research implies the novel approach of neutrosophic soft rough set (NSR-set) with neutrosophic soft rough topology (NSR-topology). We presented various topological structures of NSR-topology named as NSR-interior, NSR-closure, NSR-exterior, NSR-neighborhood, NSR-limit point and, NSR-bases with numerous examples. We established two novel algorithms to deal with multi-criteria decision-making (MCDM) problems under NSR-data. One is based on NSR-sets and the other is based on NSR-topology with NSR-bases. This research is more efficient and flexible than the other approaches. The proposed algorithms are simple and easy to understand which can be applied easily on whatever type of alternatives and measures. Both algorithms are flexible and easily altered according to the different situations, inputs and, outputs. In the future, we will extend our work to solve the MCDM problems by using TOPSIS, AHP, VIKOR, ELECTRE family and, PROMETHEE family using different hybrid structures of fuzzy and rough sets.

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Received: Apr 22, 2020. Accepted: July 12, 2020

However, crisp graphs do not represent any system because the world is now full of imprecise data. The idea of fuzziness was used first to define the fuzzy graph [2] by Kaufmann (1973).

Fuzzy graph [3] theory was developed by Rosenfeld (1975). In the same time, Yeh and Bang (1975) introduced various connectedness concepts in fuzzy graphs [4]. Also, $\mu - length$ of a path and $\mu - distance$ in a fuzzy graph [3] was introduced by Rosenfeld (1975). Hence Bhattacharya (1987) introduced the idea of eccentricity and centre in the fuzzy graph [5] using $\mu - distance$. Also, the properties of $\mu - distance$ [6] were developed by Sunitha and Vijayakumar (1998). Bhutani and Rosenfeld (2003) introduced the concepts of $g - distance$ in fuzzy graphs [7, 8] and eccentricity, centre etc. [9] were also developed. There were further studies on $g - distance$ [10] by Linda and Sunitha (2012).

Day to day, there were developments on fuzzy graphs. Akram (2011) introduced bipolar fuzzy graphs [11] and the interval-valued fuzzy graph [12] were introduced by Akram and Dudek (2011). Samanta and Pal (2013, 2015) introduced fuzzy k-competition graphs, p-competition graphs [13] and also introduced fuzzy planar graph [14]. Tom and Sunitha (2015) introduced a new definition of the length of a path and strong sum distance in fuzzy graphs [15]. There are many research works on fuzzy graphs. But in all these fuzzy graphs, edge membership value is less than its vertex membership values. To remove this limitation, Samanta and Sarkar (2016) introduced a generalized fuzzy graph [16].

As a generalization of fuzzy set and intuitionistic set theory, Smarandache (1998) introduced the concepts of neutrosophic set [17] that consist of a degree of truth membership, falsity membership and indeterminacy membership. In reality, every uncertainty has some possibility, some risk and some neutral factors. Neutrosophic graphs include all three notions properly. Thus any uncertainty/ ambiguity of networks can be represented by neutrosophic graphs. Broumi et al. (2016) introduced the notion of a single-valued neutrosophic graph [18] as a generalization of fuzzy graphs. After that, there are several research works on neutrosophic graphs [19,20]. Akram and Siddique (2017) introduced the neutrosophic competition graphs [21]. Hence Das et al. (2020) proposed generalized neutrosophic competition graphs [22] with applications to economic competitions among some countries.

Abdel-Basset (2019) utilized the neutrosophic theory to solve the transition difficulties of IoT-based enterprises [23]. Also, there are many real-life applications including evaluation of the green supply chain management practices [24], evaluation Hospital medical care systems based on pathogenic sets [25], decision-making approach with quality function deployment for selecting supply chain sustainability metrics [26], intelligent medical decision support model based on soft computing and IoT [27]. Chakraborty (2020) introduced pentagonal neutrosophic number in shortest path problem [28] and a new score function of the pentagonal neutrosophic number and its application in networking problem [29]. Das and Edalatpanah (2020) proposed a new ranking function of the triangular neutrosophic number and its use in integer programming [30]. The remaining study can be found in [31-40].

The rest of the paper is organized as follows. In Section 2, we discuss the contribution of the study. In section 3, we study some preliminaries related to graph theory. In Section 4, we introduce the sum distance in a neutrosophic graph with some properties. In Section 5, we introduce eccentricity, radius and diameter in a neutrosophic graph with properties. In Section 6, we discuss an application to a travelling salesman problem. In section 7, we conclude the study with future directions.

The gist of contributions of authors (Table 1) are arranged below.

Authors	Year	Contributions
Rosenfeld	1975	Introduce $\mu-length$ of a path and $\mu-distance$in a fuzzy graph.
Bhattacharya	1987	Introduce eccentricity and centre in the fuzzy graph.
Bhutani and Rosenfeld	2003	$g-distance$ in fuzzy graphs and developed eccentricity, centre etc.
Linda and Sunitha	2011	Studied on $g-distance$ in fuzzy graphs.
Tom and Sunitha	2015	Introduce length of a path and strong sum distance in fuzzy graphs.
Das et al.	This paper	Introduce sum distance in neutrosophic graph and eccentricity, radius etc. are studied. An application is illustrated.

Table 1. Contributions of authors

2. Major contributions of the study

The neutrosophic graph is a generalization of the fuzzy graph. The contributions of the study are below.

- This study introduces the concepts of the weight of edges of a neutrosophic graph and weighted sum distance in neutrosophic graph.
- Also the eccentricity, diameter and radius are defined with some properties.
- At last, an application of sum distance in the neutrosophic graph to a travelling salesman problem is illustrated.

3. Preliminaries

Definition 1. A *graph* is an ordered pair (V, E) such that V is the set of vertices and $E \subseteq V \times V$ is the set of edges between vertices. A *path of length* n is a sequence $v_0e_1v_1e_2 \dots \dots . e_nv_n$ where $v_0, v_1, \dots . . , v_n$ are distinct vertices and $e_1, e_2, \dots . , e_n$ are distinct edges. The *distance* between the vertices u and v is the minimum length of the path between u and v. The *eccentricity* of a vertex is the maximum distance to any vertex in the graph. The *radius* of a graph is the minimum eccentricities of all vertices, and the *diameter* of a graph is the maximum eccentricities of vertices.

Definition 2.[3] A *fuzzy graph* G is a triplet (V, σ, μ) in which V is the set of vertices, $\sigma: V \to [0,1]$ and $\mu: V \times V \to [0,1]$ such that $\mu(x,y) \leq \sigma(x) \wedge \sigma(y)$ where $\sigma(x)$ represents the membership value of x and $\mu(x,y)$ represents the membership value of edge (x,y).

Definition 3.[15] *Length* $L(P)$ *of a path* $P: v_0 e_1 v_1 e_2 \dots \dots . e_n v_n$ in a connected fuzzy graph $G: (V, \sigma, \mu)$ is given by $L(P) = \sum_{i=1}^{n} \mu(e_i)$ where $\mu(e_i)$ represents membership values of edges e_i.

Definition 4.[15] The strong sum distance between vertices u *and* v is the minimum length of all paths between vertices u *and* v.

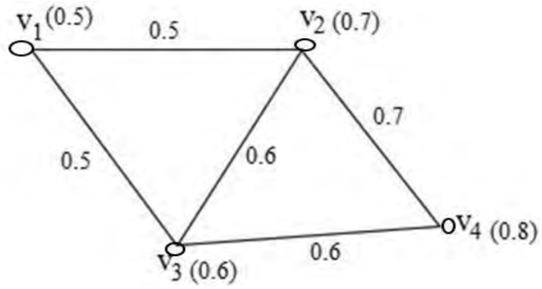

Figure 1. Example of a fuzzy graph

Example 1. The fuzzy graph (Fig.1) has four vertices with five edges. There are three paths from vertex v_1 to vertex v_4. The paths are $P_1: v_1 - v_2 - v_3 - v_4$, $P_2: v_1 - v_2 - v_4$, $P_3: v_1 - v_3 - v_4$. Then $L(P_1) = 1.7, L(P_2) = 1.2, L(P_3) = 1.1$ *and*the strong sum distance between vertices v_1 and v_4 is 1.1.

Definition 5.[18] A graph $G = (\mathrm{V}, E)$ where $E \subseteq V \times V$ is said to be neutrosophic graph if

i) there exist functions $\rho_T: V \to [0,1], \rho_F: V \to [0,1]$ *and* $\rho_I: V \to [0,1]$ such that

$$0 \leq \rho_T(v_i) + \rho_F(v_i) + \rho_I(v_i) \leq 3 \text{ for all } v_i \in V \ (i = 1,2,3, \dots, n)$$

where $\rho_T(v_i), \rho_F(v_i), \ \rho_I(v_i)$ denote the degree of true membership, degree of falsity membership and degree of indeterminacy membership of the vertex $v_i \in V$ respectively.

ii) there exist functions $\mu_T: E \to [0,1], \mu_F: E \to [0,1]$ *and* $\mu_I: E \to [0,1]$ such that

$$\mu_T(v_i, v_j) \leq \min \ [\rho_T(v_i), \rho_T(v_j)]$$

$$\mu_F(v_i, v_j) \geq max[\rho_F(v_i), \rho_F(v_j)]$$

$$\mu_I(v_i, v_j) \geq max[\rho_I(v_i), \rho_I(v_j)]$$

$$\text{and } 0 \leq \mu_T(v_i, v_j) + \mu_F(v_i, v_j) + \ \mu_I(v_i, v_j) \leq 3 \text{ for all } (v_i, v_j) \in E$$

where $\mu_T(v_i, v_j)$, $\mu_F(v_i, v_j)$, $\mu_I(v_i, v_j)$ denote the degree of true membership, degree of falsity membership and degree of indeterminacy membership of the edge $(v_i, v_j) \in E$ respectively.

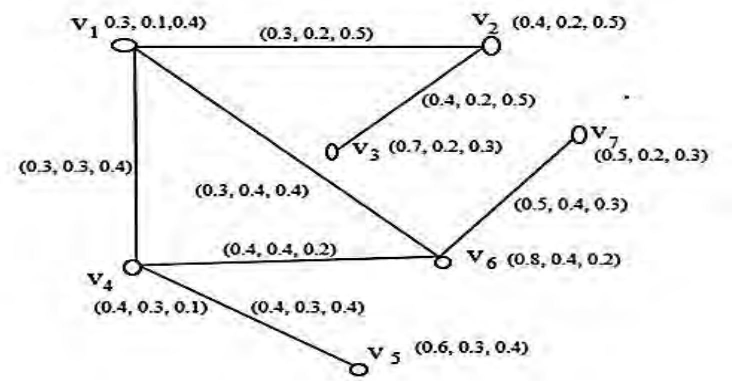

Figure 2. A neutrosophic graph

4. Weighted sum distance in the neutrosophic graph

In the neutrosophic graph, membership values of edges are in neutrosophic nature. So we cannot compare among edges in a neutrosophic graph. To overcome it, we define weight function that maps from the membership value of edges to a crisp value lies between 0 and 1.

Definition 6. Consider a function ω: $[0,1] \times [0,1] \times [0,1] \to [0,1]$ defined by

$$\omega_{ij}(t, i, f) = w_1 t(1 - f) + w_2 i \text{ where } t, i, f, w_1, w_2 \text{ are the numbers} \in [0,1].$$

The weight of an edge (v_i, v_j) in a neutrosophic graph is a number between 0 and 1 which is obtained from the image of the function ω for corresponding membership value $\left(T_E(v_i, v_j), F_E(v_i, v_j), I_E(v_i, v_j)\right)$ of the edge and it is denoted by ω_{ij}.

Note: This function indicates the overall impression of true , falsity and indeterminacy values. Suppose, in one network, generally predictions are always true of some facts. Then w_1 must be higher value and close to 1. Similarly for the other cases.

Example 2. Weight ω_{23} of edge (v_2, v_3) in the neutrosophic graph (Fig.2) 0.38 where $w_1 = \frac{2}{3}$ and $w_2 = \frac{1}{3}$.

Definition 7. Let $P: u_0 - u_1 - u_2 - \cdots \ldots \ldots - u_n$ be any path in a neutrosophic graph $G = (\mathrm{V}, E)$. Then the length of the path p is the sum of the weights of the edges of the path P in $G = (\mathrm{V}, E)$.

$$L_N(p) = \sum_{\substack{0 \leq i < n \\ i<j}} \omega_{ij} \,,$$

where ω_{ij} is the weight of edge between vertices u_i and u_j.

Example 3. Length of the path $v_1 - v_4 - v_6 - v_7$ in the neutrosophic graph (Fig.2) is 0.8 where $w_1 = \frac{2}{3}$ and $w_2 = \frac{1}{3}$.

Definition 8. Let $G = (\mathrm{V}, E)$ be a neutrosophic graph and P be the collection of all paths between two nodes u *and* v *i.e.* P = $\{p_i\,, i = 1, 2, 3, \ldots \ldots, n\}$. Then the weighted distance between the nodes u *and* v is denoted by $d_N(u, v)$ and is defined by

$$d_N(u,v) = min\{L_N(p_i): p_i \in P,\ i = 1,2,3, \dots \dots, n\},$$
where $L_N(p_i)$ is the length of the path p_i.

Example 4. Sum distance between the nodes v_1 and v_7 in the neutrosophic graph (Fig.2) is 0.55 where $w_1 = \frac{2}{3}$ and $w_2 = \frac{1}{3}$. .

Theorem 1. Let $G = (V, E)$ be a neutrosophic graph and $d_N(u,v)$ be weighted sum distance between any two nodes $u\ and\ v$. Then $\forall u, v, w \in V$

i) $d_N(u,v) \geq 0$
ii) $d_N(u,v) = 0\ if\ and\ only\ if\ u = v$
iii) $d_N(u,v) = d_N(v,u)$
iv) $d_N(u,v) \leq d_N(u,w) + d_N(w,v)$.

Proof. (i) It clears from the definition that $d_N(u,v) \geq 0$.

(ii) It clears from the definition that $d_N(u,v) = 0\ if\ and\ only\ if\ u = v$.

(iii) $d_N(u,v)$denotes the strong sum distance from u to v. Then there exists a path whose length is minimum among all the path between u to v. Hence the length should be the same from v to u. So $d_N(u,v) = d_N(v,u)$.

(iv) Let p be a path $u - w$ such that $L_N(p) = d_N(u,w)$ and q be a path $w - v$ such that $L_N(q) = d_N(w,v)$. Then $u - v$ is a walk and it is a strong path whose length is at most $d_N(u,w) + d_N(w,v)$. Thus $d_N(u,v) \leq d_N(u,w) + d_N(w,v)$.

5. Eccentricity, Radius and Diameter

The parameters eccentricity, radius and diameter are crucial in graph theory. We studied these important parameters in neutrosophic graph considering the concepts of sum distance. The relations among radius, diameters, eccentricity and distance are studied as follows.

Definition 9. The eccentricity $e_N(u)$ of a node, u is the distance from u to the furthest node in the neutrosophic graph G. Thus

$$e_N(u) = max\{d_N(u,v): \forall v \in V\}.$$

Example 5. Consider a neutrosophic graph (Fig.3). The eccentricity $e_N(v_1)$ of the vertex v_1 is calculated by the following:

$$e_N(v_1) = \max\{d_N(v_1,v_2), d_N(v_1,v_3), d_N(v_1,v_4)\}$$
$$= \max\{0.33, 0.41, 0.87\} = 0.87$$

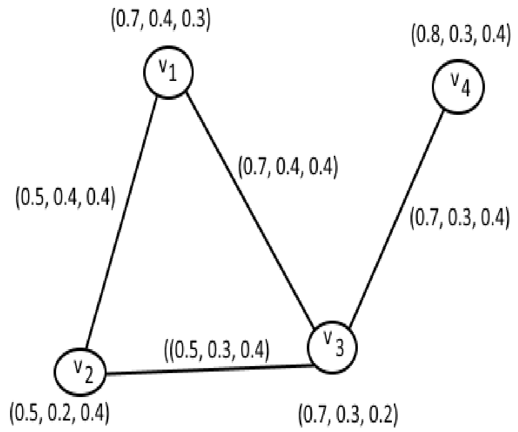

Fig. 3. An example of a neutrosophic graph

Theorem 2. Let $G = (V,E)$ be a connected neutrosophic graph and u, v be any two nodes of G. Then $|e_N(u) - e_N(v)| \leq d_N(u,v)$.

Proof. Let $u, v \in G$ be two nodes such that $e_N(u) \geq e_N(v)$ and $x \in G$ be anode such that $e_N(u) = d_N(u,x)$. Then $d_N(u,x) \leq d_N(u,v) + d_N(v,x)$, by theorem 3.7 (iv). Also $d_N(v,x) \leq e_N(v)$. Thus $e_N(u) = d_N(u,x) \leq d_N(u,v) + e_N(v)$, this implies $0 \leq e_N(u) - e_N(v) \leq d_N(u,v)$. Similarly, if we take $e_N(u) \leq e_N(v)$, we will get $-d_N(u,v) \leq e_N(u) - e_N(v)$. Thus $|e_N(u) - e_N(v)| \leq d_N(u,v)$.

Definition 10. The radius $r_N(G)$ of a neutrosophic graph, G is the minimum among all eccentricity of nodes. Thus

$$r_N(G) = min\{e_N(u): \forall u \in G\}.$$

Example 6. Consider the neutrosophic graph (Fig. 3). The radius $r_N(G)$ of the graph G is calculated by the following:

$$r_N(G) = \min\{e_N(v_1), e_N(v_2), e_N(v_3), e_N(v_4)\}$$
$$= \min\{0.87, 0.83, 0.46, 0.87\} = 0.46$$

Definition 11. The diameter $d_N(G)$ of a neutrosophic graph, G is the maximum among all eccentricity of nodes. Thus

$$d_N(G) = max\{e_N(u): \forall u \in G\}.$$

Example 7. Consider the neutrosophic graph (Fig.3). The diameter $d_N(G)$ of graph G is calculated by the following:

$$d_N(G) = \max\{e_N(v_1), e_N(v_2), e_N(v_3), e_N(v_4)\}$$
$$= \max\{0.87, 0.83, 0.46, 0.87\} = 0.87$$

Definition 12. A node in a neutrosophic graph is called a central node if its eccentricity is equal to the radius of the graph. Thus for a central node u,

$$e_N(u) = r_N(G).$$

Example 8. Consider the neutrosophic graph (Fig. 3). The node $v_3$is a central node, since

the eccentricity $e_N(v_3) =$ the radius $r_N(G)$

Definition 13. A node in a neutrosophic graph is called a peripheral node if its eccentricity is equal to the diameter of the graph. Thus for a peripheral node u,

$$e_N(u) = d_N(G).$$

Example 9. Consider the neutrosophic graph (Fig. 3). The nodes v_1 and v_4 are peripheral nodes, since the eccentricity $e_N(v_1) =$ the diameter $d_N(G) =$ the eccentricity $e_N(v_4)$.

Theorem 3. Let $G = (V, E)$ be a connected neutrosophic graph with $r_N(G)$ and $d_N(G)$ be the radius and diameter respectively, then $r_N(G) \leq d_N(G) \leq 2r_N(G)$.

Proof. From the definition, it follows that $r_N(G) \leq d_N(G)$. Let $u, v, w \in V$ such that u be central node i.e. $e_N(u) = r_N(G)$ and v, w be peripheral node i.e. $e_N(v) = e_N(w) = d_N(G)$. Now $d_N(v, w) \leq d_N(v, u) + d_N(u, w)$, by theorem (iv). This implies $d_N(G) \leq r_N(G) + r_N(G) = 2r_N(G)$. Thus $d_N(G) \leq 2r_N(G)$. Therefore, $r_N(G) \leq d_N(G) \leq 2r_N(G)$.

6. Application to travelling salesman problem

Suppose there are few places in a city and roads connect the places. Hence the places and roads together form a network. But the problem is to find a way that a salesman can visit all the planes once with the lowest travelling cost. Now the travelling cost is directly proportional to the road distance travel by salesman. But all the roads are not in the same smooth conditions to measure road distance in practical. So the real travelling distance with cost may be effected the bad road, non-pucca roads, water path etc. Thus to calculate the path distance, it is generally ignored the current condition of the paths. The true value indicates the expected distance on good road. The falcity indicates the current false parameter like general traffic on the routes, muddied on road. Indeterminacy includes delay due to road construction, political movement and any other factors like water path. Therefore Travelling salesman problem should be presented by neutrosophic environment. Hence the travelling distance between the places should be taken as neutrosophic value.

6.1. Steps to find the sum distance of the travelling salesman problem.

To find the minimum travelling cost in travelling salesman problem in the neutrosophic environment, all the necessary steps are given below as an algorithm.

Step-1: Input all edge membership values between the places.

Step-2: Evaluate the weight of edges.

Step-3: Find all the Hamiltonian cycles between the requird places.

Step-4: Evaluate length of the said cycles.

Step-5: Find the minimum length among the cycles.

6.2. Numerical example

Suppose there are four places and six roads are connecting the places in a city. A salesman wants to visit all the places once and returning back to the starting place. The problem is to find a cycle with minimum cost of travelling.

The edge membership values (Table 2) between the places are given in the figure 4 where the membership value $(0.5, 0.3, 0.2)$ between two places P_1 and P_2 represent that distance of good road between P_1 and P_2 is 0.5, distance of bad road between P_1 and P_2 is 0.3 and distance of non-constructed road between P_1 and P_2 is 0.2 and similar for the other values.

Places	Distance between places
$P_1 - P_2$	$(0.5, 0.3, 0.2)$
$P_1 - P_3$	$(0.4, 0.6, 0.4)$
$P_1 - P_4$	$(0.8, 0.3, 0.2)$
$P_4 - P_2$	$(0.5, 0.3, 0.6)$
$P_3 - P_2$	$(0.6, 0.2, 0.3)$
$P_4 - P_3$	$(0.7, 0.2, 0.4)$

Table 2. Distace between two places

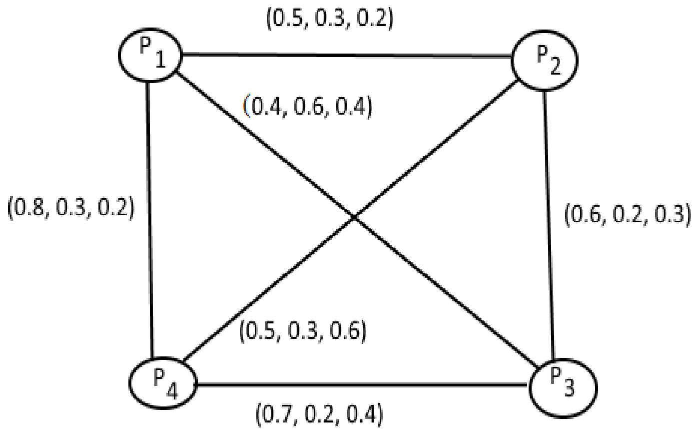

Figure 4: A graph among four cities.

The weight ω_{ij} between the places P_i and P_j are given below:

$$\omega_{12}(P_1, P_2) = 0.3, \qquad \omega_{13}(P_1, P_3) = 0.24,$$
$$\omega_{14}(P_1, P_4) = 0.44, \qquad \omega_{23}(P_2, P_3) = 0.42,$$
$$\omega_{24}(P_2, P_4) = 0.43, \qquad \omega_{34}(P_3, P_4) = 0.51.$$

There are four possible cycles to visit all the places once from starting point to that point. These cycles are:

$$C_1: v_1 - v_2 - v_3 - v_4 - v_1$$

$$C_2: v_1 - v_2 - v_4 - v_3 - v_1$$

$$C_3: v_1 - v_3 - v_2 - v_4 - v_1$$

$$C_4: v_1 - v_3 - v_4 - v_2 - v_1$$

The length $L_N(C_i)$ travelled by the salesman for the above cycles C_i are:

$$L_N(C_1) = 1.67,\ L_N(C_2) = 1.48\ ,\ L_N(C_3) = 1.53,\ L_N(C_4) = 1.48$$

Since the value 1.48 is minimum length for the cycles C_2 and C_4, hence these cycles give the minimum travelling cost to the salesman.

7. Conclusions

In this article, sum distance, eccentricity, radius etc. in a neutrosophic graph has been developed. Some definitions, examples and theorems give a clear idea about the proposed study. A neutrosophic graph is recently a very important topic. There are many scopes to research on that topic. One can develop this study to the generalized neutrosophic graph. The real application inthe travelling salesman problem has been illustrated with a numerical example. This idea also gives us to develop future research in neutrosophic graphs.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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Received: Apr 23, 2020. Accepted: July 13, 2020

Neutrosophic Sets and Systems, Vol. 35, 2020
University of New Mexico

Exploration of the Factors Causing Autoimmune Diseases using Fuzzy Cognitive Maps with Concentric Neutrosophic Hypergraphic Approach

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Abstract: Neutrosophic sets are comprehensively used in decision making environment. The manifestation of neutrosophic sets in concentric hypergraphs is proposed in this research work. The intention of developing a decision making model using the combination of Fuzzy Cognitive Maps and concentric neutrosophic hypergraph is to rank the core factors of decision making problem and find the inter relational impacts. This proposed model is validated with the exploration of the causative factors of autoimmune diseases. The proposed model is highly compatible as it assists in determining the core factors and their inter association. This model will certainly benefit the decision maker at all managerial levels to design optimal decisions.

Keywords: Autoimmune disease, fuzzy cognitive maps, neutrosophic hypergraphs, optimal decision making

1. Introduction

Westernization the cause of modernization has unlocked the portals of cultural, behavioural and environmental changes of the people which greatly influence the biological system of human and this also lays the core reason for the outbreak of novel diseases. Presently the people of the world are characterized by multicultural and multi technological adoption. The integration and the association between people of varied culture have brought diverse implications on the external and internal environment of the human. Not just the social interactions contribute to such modifications; also the technological advancement and the work space of an individual cause a varied range of changes in the mankind. The tendency of manhood repelling from indigenous practices is the gateway for several health woes. The health system of the human is getting affected by several factors and especially the vulnerable target group is the women. In recent days, the people are chained by diseases of various kinds, even the economy of the nation face huge falls due to the effect of epidemic diseases, amidst such miserable situations, the immunity of the human is the only armed force against these viruses, but if the immune system fails to be defensive in nature and if it joins hand with the external invaders the entire human health system collapses and it ends in fatality. This is the characterization of auto immune diseases and the women are greatly affected by these diseases. It is highly a dreadful circumstance to tackle the consequences of these self-destructing diseases. The autoimmune diseases predominantly affecting the women are Rheumatoid Arthritis, Multiple Sclerosis, Systemic lupus Erythematosus, Grave's disease, Hashimoto's thyroiditis and Myasthenia gravis. Presently the rate of occurrences of such diseases is at its pinnacle and the medical experts are investigating the ways and means of its mitigation. [1]

Generally the women are highly susceptible to these autoimmune diseases as the immune system gets weakened during pre and post pregnancy stages. This scenario has gained the medical concerns and medical researchers are on their study, to render support to it, this paper aims to underlie the core factors contributing to autoimmune diseases in women and to find the inter association between the core factors. Optimal decisions can be made by applying scientific methods in the process of decision making process. The entire scenario of decision making must be modeled based on decisive factors of the study. One of the realistic tools of decision making is fuzzy cognitive maps (FCM), introduced by Kosko [2], later several academicians extended this FCM tool based on the requirements. FCM is a directed graph representing the casual relationship between factors considered for study. The nodes and the edges of the graph represent the study factors and their association. The weights [-1,1] represent the nature of the association. The integration of FCM with other graphic structures was initiated by Nivetha and Pradeepa [3]. The hypergraphic and fuzzy hypergraphic approaches with FCM unlocked the construction of concentric fuzzy hypergraphs and its integration with FCM [4,5]. This field of integrated FCM with fuzzy hypergraphs has made the researchers explore by introducing various types of concentric fuzzy hypergraphs.

In this research work, a fuzzy cognitive map with concentric neutrosophic hypergraphic approach is introduced. The notion of neutrosophic fuzzy sets and neutrosophic logic was first coined by Smarandache [6] and presently many researchers are highly interested to carry out their research in this field, the concepts of neutrosophic is applied in almost all types of decision making tools. Neutrosophic sets, play significant role in making decisions in uncertain environment as it provides space for the pragmatic representation of the expert's opinion. Abdel Basset et al [7]developed a decision making model for evaluating the framework for smart disaster response system in an uncertain environment, neutrosophic sets are used for uncertainty assessments of linear time-cost tradeoffs [8]; resource levelling problem[9] in construction project was modeled under neutrosophic environment. The concept of neutrosophic sets was extended to bipolar neutrosophic representation [10] and it is used in multi criteria decision making framework for professional selection. Das et al [11] developed neutrosophic fuzzy matrices and algebraic operation that had some utility in decision making. Plithogenic sets, the extension of neutrosophic sets are used in solving supply chain problem with the development of a novel plithogenic model [12]. Such massive applications of neutrosophic sets in decision making and its robust nature triggered the idea of integrating neutrosophic sets to concentric hypergraphs. To the best of our knowledge, the integration of neutrosophic concentric fuzzy hypergraphs with FCM has not been instituted and so this is a new arena of research towards optimal decision making.

Fuzzy Cognitive Maps are more useful in determining the association between study factors, if the number of study factors is less, FCM's are highly compatible, but if the number of factors is more, then comparative analysis between the factors is difficult and tedious, to resolve such crisis, the core factors of the problem are to be decided and then the inter association between the core factors can be determined easily. To find the core factors, the intervention of various experts is mandatory, based on which the factors can be ranked and the core factors are decided based on the rank positions of the factors. This eases the process of making decisions as it helps in filtering the non- core factors. Generally in medicinal environment, the medical experts analyze the factors contributing to diseases, initially the causative factors taken for study will be more in number, but the factors have to drop at each stage of their research to find the prime causative factors. In the process of factor filtration, the expert's opinions play a vital role. The role of each causative factor of a disease cannot be certainly express but representation using neutrosophic sets makes it possible and more meaningful. Thus the integration of FCM with concentric neutrosophic hypergraph will help to tackle the difficulties in handling large number of study factors.

The paper is structured as follows: section 2consists of the methodology in which the algorithm of finding optimal decision is presented; section 3 comprises of the adaptation of the proposed model to the decision making problem; section 4 discusses the results and the last section summarizes the research work.

2. Methodology and its application

The steps in making optimal decisions is presented as an algorithm as follows,

Step 1: The expert's opinion of the study factors are represented by concentric fuzzy hypergraphs with neutrosophic fuzzy sets representations of the envelope.

Step 2: The score values of the neutrosophic fuzzy sets are determined.

Step 3: The factors are ranked based on the score values.

Step 4: The core factors are determined based on the ranking positions.

Step5: The inter association between the core factors is obtained based on the conventional FCM procedure.

The case histories of patients belonging to women gender suffering from autoimmune diseases are taken as the source of data collection and the factors contributing to the occurrence of auto immune disease in women [13] are presented below based on the medical expert's opinion and data obtained from questionnaire.

F1. Excess presence of VGLL3 (Vestigial Like Family Member 3) in skin cells
F2. Changes in the gene system
F3. Exposure to ultraviolet radiation from sunlight
F4. Acquaintance with organic mercury
F5. Alteration in food habits
F6. Gene-Environment interface
F7. Fluctuations in sex hormones
F8. Modifications in Nutritional diet
F9. Post pregnancy impacts
F10. Genetic vulnerability
F11. Genetic differences in immunity

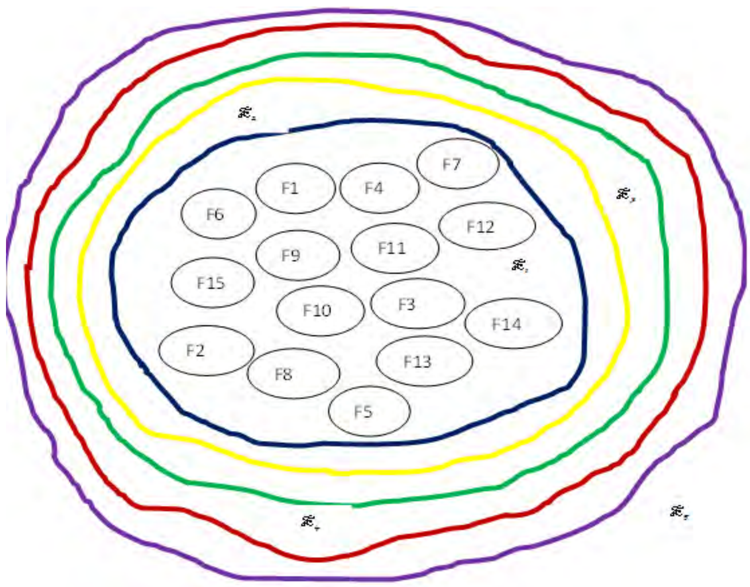

Fig.3.1.Concentric Neutrosophic Fuzzy Hypergraphic representation

The concentric neutrosophic fuzzy hyper envelopes with neutrosophic representations of the expert's opinion are presented below in Table 3.1.

Table 3.1 Representations of Expert's opinion

Experts	F1	F2	F3	F4	F5	F6	F7	F8	F9	F10	F11
E1	(0.3,0.2, 0.8)	(0.5,0.2, 0.3)	(0.4,0.1, 0.5)	(0.3,0.4, 0.6)	(0.8,0.1, 0.2)	(0.7,0.2, 0.3)	(0.7,0.3, 0.4)	(0.7,0.2, 0.3)	(0.3,0.2, 0.8)	(0.5,0.2, 0.3)	(0.5,0.2, 0.3)
E2	(0.2,0.2, 0.9)	(0.4,0.3, 0.5)	(0.5,0.2, 0.3)	(0.2,0.2, 0.9)	(0.7,0.2, 0.3)	(0.6,0.2, 0.3)	(0.7,0.5, 0.4)	(0.6,0.2, 0.3)	(0.4,0.3, 0.5)	(0.6,0.2, 0.3)	(0.8,0.3, 0.2)
E3	(0.3,0.4, 0.6)	(0.3,0.5, 0.6)	(0.4,0.3, 0.5)	(0.3,0.2, 0.8)	(0.8,0.3, 0.2)	(0.9,0.2, 0.3)	(0.9,0.1, 0.3)	(0.6,0.2, 0.3)	(0.3,0.5, 0.6)	(0.7,0.3, 0.4)	(0.6,0.2, 0.3)
E4	(0.5,0.2, 0.3)	(0.2,0.2, 0.9)	(0.5,0.2, 0.3)	(0.4,0.4, 0.6)	(0.7,0.1, 0.2)	(0.7,0.3, 0.4)	(0.6,0.2, 0.3)	(0.7,0.1, 0.2)	(0.2,0.2, 0.9)	(0.6,0.2, 0.3)	(0.4,0.3, 0.5)
E5	(0.2,0.5 ,0.6)	(0.3,0.2, 0.8)	(0.6,0.2, 0.3)	(0.5,0.2, 0.3)	(0.6,0.2, 0.3)	(0.8,0.1, 0.2)	(0.6,0.2, 0.3)	(0.9,0.2, 0.3)	(0.4,0.4, 0.6)	(0.5,0.2, 0.3)	(0.7,0.3, 0.4)

The score values of the factors are presented in Table 3.2 and it is represented graphically in Fig.3.2

Table 3.2 Score values of the Factors

F1	F2	F3	F4	F5	F6	F7	F8	F9	F10	F11
0.571	0.571	0.546	0.538	0.667	0.783	0.756	0.573	0.445	0.636	0.667
7	7	8	9	5	1	2	6	10	3	4

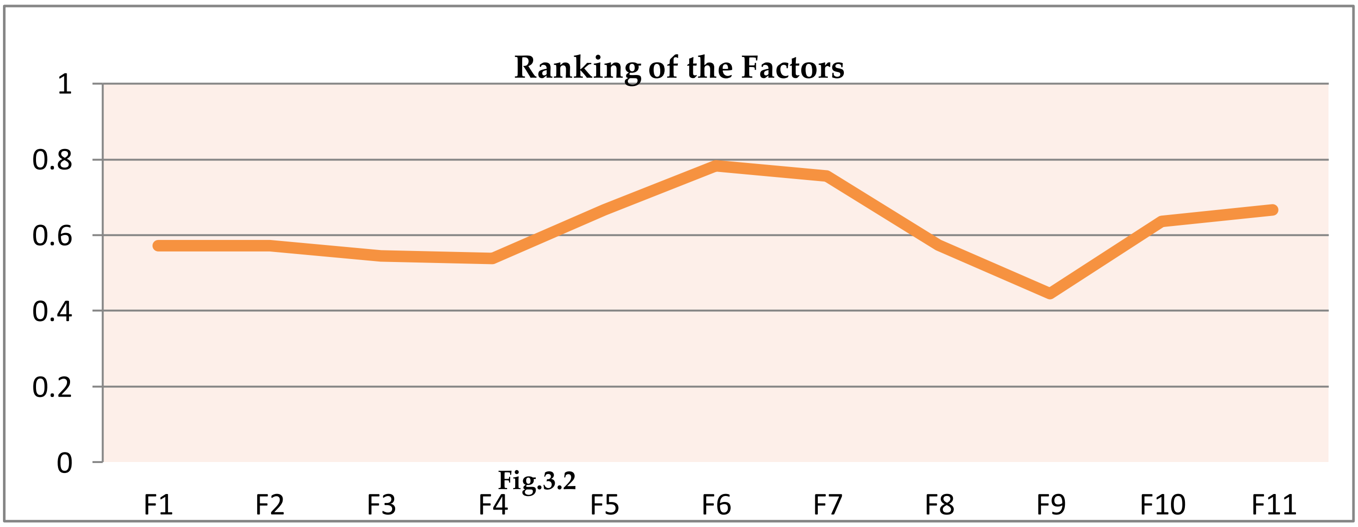

Fig.3.2

Based on the scores, the following factors are considered as the core factors and their inter association is expressed as linguistic variables, which then later quantified by heptagonal fuzzy numbers.

HP1. Alteration in food habits

HP2. Gene-Environment interface

HP3. Fluctuations in sex hormones
HP4. Genetic vulnerability
HP5. Genetic differences in immunity
The connection matric between the factors, based on the expert's opinion

$$
\begin{array}{c} \\ HP1 \\ HP2 \\ HP3 \\ HP4 \\ HP5 \end{array}
\begin{pmatrix}
HP1 & HP2 & HP3 & HP4 & HP5 \\
0 & M & H & L & L \\
L & 0 & M & H & H \\
L & M & 0 & M & L \\
L & M & H & 0 & M \\
L & M & M & H & 0
\end{pmatrix}
$$

The modified matrix based on the values of quantification in Table 3.3

Linguistic Variable	**Heptagonal Weight**	**Membership value**
Low	(0,0.1,0.2,0.3,0.35,0.4,0.45)	**0.26**
Medium	(0.4,0.45,0.5,0.55,0.6,0.65,0.7)	**0.55**
High	(0.65,0.7,0.8,0.9,1,1,1)	**0.86**

$$
\begin{array}{c} \\ HP1 \\ HP2 \\ HP3 \\ HP4 \\ HP5 \end{array}
\begin{pmatrix}
HP1 & HP2 & HP3 & HP4 & HP5 \\
0 & 0.55 & 0.86 & 0.26 & 0.26 \\
0.26 & 0 & 0.55 & 0.86 & 0.86 \\
0.26 & 0.55 & 0 & 0.55 & 0.26 \\
0.26 & 0.55 & 0.86 & 0 & 0.55 \\
0.26 & 0.55 & 0.55 & 0.86 & 0
\end{pmatrix}
$$

The interrelationship between the factors is determined by the similar application of FCM methodology [9-10] and it is presented graphically in Fig 3.2

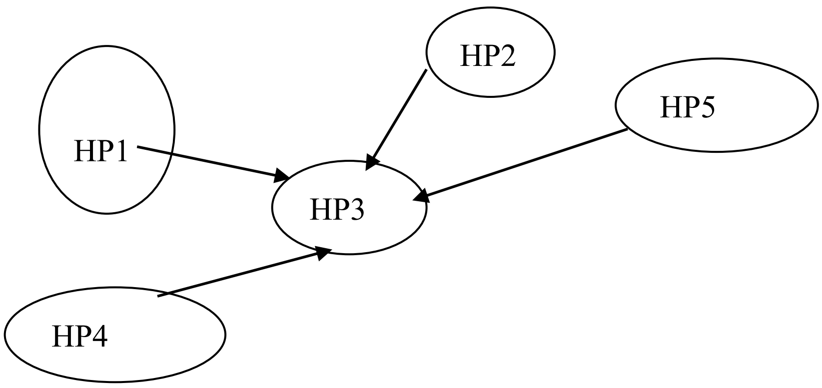

Fig.3.2 FCM representation of the inter association of the core factors

4. Results and Discussion

Fig. 3.2 clearly states the factor, fluctuations in sex hormone is the core causative factor of auto immune diseases. The findings of this research will certainly assist the medical experts to ascertain the causes of the auto immune disease in women and give treatment in accordance to it. Hormonal imbalance is quite common in the life of the women as they undergo various stages of puberty, maternity, menopause, but still proper medications has to be given to avoid the risks of such fatal diseases. The representation of the imprecise data in the form neutrosophic sets is the pragmatic reflection of the expert's opinion, as the factors contributing to the diseases are quite uncertain. The degree of truth values, indeterminacy and false values are indeed very essential in making optimal decisions.

5. Conclusion

The proposed decision making tool with the integration of FCM and concentric neutrosophic fuzzy hypergraphs is a highly feasible tool to obtain optimal decisions. The difficulty in handling several factors in FCM is reduced and this integrated approach facilitate the determination of inter association between the factors. This method of decision making can be extended to other kinds of concentric fuzzy hypergraphs with various representations. Plithogenic sets representation is the future extension of this proposed research work.

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Received: Apr 24, 2020. Accepted: July 14, 2020

Neutrosophic Sets and Systems, Vol. 35, 2020
University of New Mexico

Neutrosophy Logic and its Classification: An Overview

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Abstract: Over the past few years, neutrosophy has gained an exponential growth and has attracted a good number of researchers especially those who focus on soft computing based uncertainty computation. This paper presents the various techniques in neutrosophy. The various techniques are discussed lucidly which help a naïve researcher in this field to understand the on-going researches and establish a strong base. We have summarized the previous work carried out in the field of neutrosophic logic, set, measure, and also classification techniques in neutrosophy and the relevant research work has been discussed. Further, various applications in the field of neutrosophy are elaborated. The major contributions of the existing research in neutrosophy is reviewed and presented from different perspectives. The development of newer algorithms for solving the problems of neutrosophy will provide impetus to the existing research in this field.

Keywords: Neutrosophy, indeterminacy, neutrosophic logic

1. Introduction

Neutrosophy, having emerged as a generalization to fuzzy logic is being used in the research area in a number of fields like logics, set theory and others. Florentin Smarandache, in 1980, introduced this new field of philosophy which deals with the uncertainties and indeterminacy in the data. He defines neutrosophy as the science which deals with neutralities. This field takes into consideration the dawn, kind and scope of such neutralities and how they interact with various ideational spectra. The fundamentals of the study of the logic of neutrosophy, probability in neutrosophy, sets in neutrosophy and the statistics is given by neutrosophy. Various researchers have incorporated the idea of Neutrosophic Logic (NL), Neutrosophic Cognitive Maps (NCM) and other technologies in areas such as Information system application, IT, Decision Support System Application, Physics, Healthcare, Social Sciences etc. In 2019, F. Smarandache, introduced the concept of Neutrosociology[1], which is the amalgamation of sociology and neutrosophic methods. In [3], an improved method using clustering using k-means was incorporated for performing image segmentation using neutrosophic logic. In [4], the authors presented a way of correcting the uncertainties that arise in discursive analysis by applying Neutrosophy Theory in relation with sentiment analysis. In [5], the authors gave a framework to see how mental models could be analyzed using neutrosophic logic. In [6], [9], [10], [11], [15], [16] and [17], Neutrosophy was used to deal with the uncertainties and indeterminacy in situation analysis. In [25], the evaluation of the smart disaster response systems in times of ambiguity has been done using a framework. The degrees of contradiction in the evaluation criteria have been addressed with the help of plithogenic set theory which checks the uncertainty environment. In [26], to tackle time scheduling in projects, a framework has been given to minimize the cost of projects in environments which are ambiguous. Neutrosophic theory has been used to consider the dynamic features of all parameters. In [29], a resource levelling model to minimize the costs of daily resource fluctuations is given, using neutrosophic set, with the aim of tackling the issues of uncertainty in the problems of the real world. In [30], the authors have given a framework for professional selection by making use of neutrosophic multi-criteria decision making, in an attempt to check the vagueness and ambiguity in the selection process. In [31], a case study of Thailand's sugar industry has been done to validate the model proposed, using the plithogenic decision making perspective for evaluating supply chain sustainability. In this paper, we have reviewed the neutrosophic technologies that have been incorporated in various researches all over the world. The figure 1 depicts the workflow.

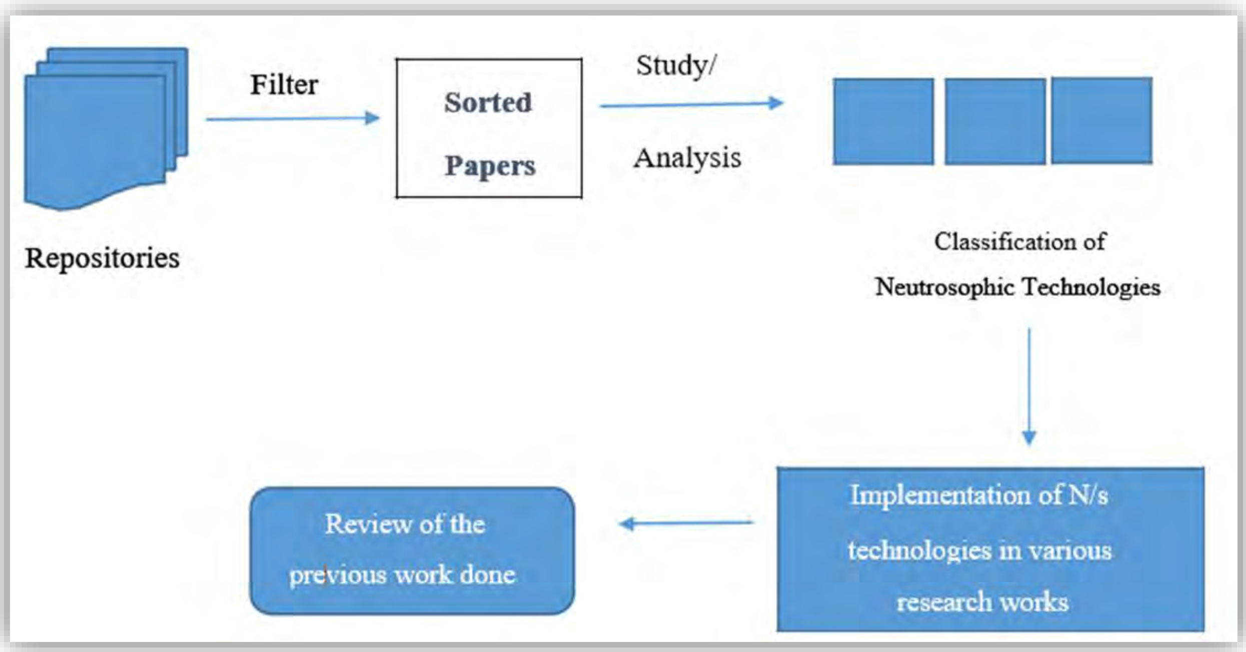

Figure 1. Block diagram for the process of the research carried out in the manuscript

2. Background Study

Florentin Smarandache [2019] in his book, Introduction to Neutrosophic Sociology (Neutrosociology) discussed Sociological Forecasting, Neutrosophic Social norms and situations which cannot be solved in the classical way. He discussed neutrosophic Grand Theories to find abstract ideas about concrete facts in large social groups. He has also discussed Neutrosophic Big Data, IoT and Neutrosophic Microsociology in this book. [1]

Aasim Zafar, Mohd Anas Wajid [2019] used the concept of neutrosophy to study the reasons of criminal behavior in Nigeria. They found that out of various factor taken by the researchers, some were excluded because they were found to be indeterminate. To show how such factors did actually contribute to the criminal behavior, they modelled the situation mathematically using FCM's and NCM's, where the former stands for Fuzzy cognitive Maps and the latter stands for Neutrosophic

Cognitive Maps. They further conclude how NCM is more effective than FCM in dealing with uncertainties and indeterminacy in situation analysis. They further concluded that if the indeterminate factors were taken, it could improve the efficiency and accuracy of the mathematical models using the concept of Neutrosophic Cognitive Maps. [6]

V Christiano, F Smarandache [2019] reviewed the seven applications of Neutrosophic Logic. They have used logical analysis based on Neutrosophic Logic. They further suggest that NL theory could be applied in Psychology pertaining to different cultures, forming theories in the field of economics, resolution of conflicts, philosophy of science and other fields like applied mathematics, economics and physics. [7]

Nancy El-Hefenawy, et al. [2016] reviewed the application of Neutrosophic Sets. They suggest that there exist a number of application in fields such as in decision making systems, IT, various information systems. This paper presented some important areas of neutrosophic sets, logic in neutrosophy, neutrosophy related measures and a neutrosophic set of a single value (SVNS). They further suggest that these could produce a new algorithm for tackling any neutrosophic problem. These can help also to solve any fuzzy problem using neutrosophic algorithm. [8]

S Pramanik, S Chackrabarti [2013], studied the issues which were faced by the construction workers in West Bengal and used the technique- neutrosophic cognitive maps in order to find the solutions for it. They discussed the major problems faced by the workers and based on the opinions of the experts and after considering the indeterminacy factor, they formulated the NCM. [9]

Anne-Laure Jousselme, et al. [2003], proposed a discussion on how uncertainty plays a role in situation analysis. They gave an overall understanding of the principal typologies of uncertainty which were found in the literature of the recent times. They discuss that besides richness and ambiguity of the language which is natural is the reason for varied uncertainty conceptions, it is also a result of the not-so-simple physical nature of the information. They further define some concepts to better understand uncertainty and the benefits that are sought. [10]

Vasantha K, W. B.; Smarandache, Florentin [2004], used NCM to study and analyze the social aspects of laborers who had migrated from different place and were suffering from HIV/AIDS in the rural areas of Tamil Nadu. They made use of the Relational Maps in neutrosophy (NRM) and defined some new neutrosophic tools which they adopted in the study and analysis of this issue. They further gave a sketch of some sixty laborers who were infected with HIV/AIDS. [11]

K Pérez-Teruel, M Leyva-Vázquez [2014], gave a structure with the help of which they analyzed the mental models and did their elicitation using neutrosophic logic. To show the applicability of the project, they showed an illustrative example. They discuss a framework for the processing of indeterminacy and uncertainty in mental models. [12]

Mustafa Mamat et al. [2012], used an approach based on fuzzy linear programming for the planning of a balanced diet. They discussed the causes of disease-related lifestyle and eating disorders which are critical issues in the world. They calculated the nutrient amount in food to be taken by the Fuzzy Linear Programming Approach and considered it to estimate the nutritional requirements for an individual on a daily basis. They further suggest that this planning could help in preventing the eating disorders and certain disease-related lifestyle. [13].

Igor Bagány and Márta Takács [2017] discussed the correlations in a number of factors involved in education system in a way that the functionality could be modelled. They do so to examine the education system in an effective manner. They further employed the fuzzy cognitive map (FCM) technology, because it helps in determination of qualitative description of the given parameters and relationships [14].

S No.	Author and Year	Primary Contribution	References
1.	Florentin Smarandache (2019)	◎ Sociological Forecasting, Neutrosophic Social norms Neutrosophic Grand Theories ◎ Neutrosophic Big Data, IoT and Neutrosophic Microsociology.	[1]
2.	Victor Christiano and F Smarandache (2019)	Applications of neutrosophy in : ◎ Psychology with respect to cultures ◎ Forming theories in economics Resolving conflicts.	[7]
3.	Nancy El-Hefenawy et al. (2016)	◎ Decision support system, IT, information system ◎ Some important notions pertaining to Neutrosophy.	[8]
4.	Aasim Zafar and Mohd Anas Wajid (2019)	◎ NCM to model the criminal behavior in Nigeria. ◎ Indeterminate factors, if considered improve the accuracy and efficiency of the model.	[6]
5.	Surapati Pramanik and Sourendranath Chackrabarti (2013)	◎ NCM for the issues related to laborers in West Bengal.	[9]
6.	Anne-Laure Jousselme (2003)	◎ Role of uncertainty in situation analysis	[10]
7.	Vasantha K, W. B. and Smarandache, F (2004)	◎ Analyzing the social aspects, using NCM, of those laborers who had migrated and suffer from HIV-AIDS.	[11]
8.	KPTeruel and ML Vázquez (2014)	◎ A framework for the analysis of mental models based on NL (neutrosophic logic).	[12]
9.	M Mamat et al. (2012)	◎ An approach incorporating FLP for a balanced diet planning.	[13]
10.	Igor Bagány and Márta Takács (2017)	◎ FCM for finding correlations in a number of factors involved in education system in a way that the functionality could be modelled.	[14]
11.	Shuqi Xue et al. (2014)	◎ The information processing model which focuses on the behavior of the human brain, with respect to cognition.	[15]
12.	Dr.M.Albert William et al. (2013)	◎ NCM for analyzing the risk factors for Breast Cancer	[16]
13.	K Mondal and S Pramanik	◎ NCM for analyzing the issues faced by Hijra community in West Bengal.	[17]

14.	Abdel-Basset et al. (2020)	• Smart disaster response systems in uncertainty environments • Plithogenic Decision Making approach (Supply Chain Sustainability) • Bipolar Neutrosophic Multi-Criteria Decision Making Framework (Professional Selection) • Neutrosophic Set for assessing uncertainty of linear time-cost tradeoffs • Resource levelling model based on neutrosophic set	[25], [26], [29], [30], [31].

Shuqi Xue et al. [2014], described the information processing model which is based on the behavior of the human brain, with respect to cognition. They proposed that the two methods of modelling a situation cognitively are representing and reasoning about situation analysis with the help of Ontology and the use of FCM, in order to formulate a Situation analysis framework. The presented approach of FCM is for a systematic analysis of the Situation Analysis theory; it provides an understanding of how the working of its elements. [15]

Dr.M.Albert William et al. [2013] analyzed the risk factors for breast cancer using NCMs. Based on the expert's opinion, they had chosen certain factors as the main nodes for obtaining a neutrosophic directed graph. They had analyzed the risk factors and their solutions and discussed how certain factors are crucial for the development of the disease [16]. However few softcomputing approaches have been used in [27, 28] K Mondal and S Pramanik [2014] have studied the situation of the hijra community in West Bengal and addressed their issues using NCMs. On the basis of the experts' opinion as well as the idea of indeterminacy, they have formulated the NCM [17].

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3. Classification of Neutrosophic Techniques:

Various researchers have studied the concept of neutrosophy and applied various techniques to address different problems of indeterminacy. Some techniques are given below:

a) Neutrosophic Cognitive Maps

b) Neutrosophic Logic

c) Neutrosophic Set

d) Neutrosophic Measure

e) Single Valued Neutrosophic Set

3.1. Neutrosophic Cognitive Maps (NCM):

Florentin Smarandache introduced the idea of NCM. They are considered to be an addendum of the Fuzzy Cognitive Maps with the difference being in the fact that, the values of indeterminacy are included. Various real life situations contain the factor of indeterminacy which cannot be modeled using existing methods. To show how indeterminacy affects the situation under consideration, NCMs have proven to be an important tool.

Definition:

It is a directed graph which has concepts (as in, any policy/event) and causalities where the former is for nodes and the latter is for the edges. It is a representation of a relationship between concepts. A simple NCM can be defined as those which have edge weights or causalities from the set {-1, 0, 1, I}.

Let the two nodes of the NCM be denoted by Ai and Aj. The effect of one node on the other is represented with the help of a directed edge from Ai to Aj, which is called connections. The weightage is assigned to each edge with a number in the set {-1, 1, 0, I}. We assume that eij is the weight assigned to the directed edge Ai Aj, eij belongs to {-1, 0, 1, I}. The following table II shows the value of eij and the effect it has on the corresponding edges:

Table II: Value of eij and its effect on corresponding edges

Value of e_{ij}	**Effect of A_i on A_j**
$e_{ij}=0$	No effect.
$e_{ij}=1$	Increase (or decrease) in A_i causes increase (or decrease) in A_j.
$e_{ij}=-1$	Increase (or decrease) in A_i causes decrease (or increase) in A_j.
$e_{ij}=I$	The effect of A_i on A_j is indeterminate.

Many researchers have incorporated the concept of NCMs in their work. NCMs are an effective way to deal with uncertainties and indeterminacy in Situation Analysis. They have shown how indeterminate factors if taken into consideration could enhance the efficiency and accuracy of the mathematical models using the concept of Neutrosophic Cognitive Maps.

Dr. M. Albert William et al. (2013) have analyzed the risk factors of Breast Cancer and their solutions with the help of Neutrosophic cognitive maps (NCMs). They have taken some twelve factors as the main nodes for their study. With the help of corresponding adjacency matrix related to the neutrosophic directed graph, they model the situation with the help of certain mathematical calculations.

Dr A. Kalaichelvi and L. Gomathy (2011) have analyzed the issues that the girl students had to face who got married while studying, with the help of Neutrosophic Cognitive Maps (NCM's). they collected the data from some hundred students in different courses in various colleges in Tamil Nadu, India. They identified certain factors on the basis of the generated opinions by those who were considered. In this way, they assessed what the effect of one factor would be on the other.

Surapati Pramanik et al. studied the issues faced by the laborers in the construction industry in West Bengal on the basis of NCM's. They identified some major problems and on the basis of the opinion of the expert and the factor of indeterminacy, they formulated the NCM. Then, they studied how the state vectors would affect the two matrices i.e; the connection matrix and neutrosophic adjacency matrix.

Aasim Zafar and M Anas Wajid studied the various factors which led to criminal behavior in Nigeria. They analyzed the situation of crime there and found out that the prominent researchers who had been monitoring the situation there cited certain causes like family breakdown, corruption, poverty etc as the reasons for criminal behavior. However, they do not take into account factors like inadequate equipment, NGOs, underemployment because these are considered to be indeterminate factors. They used NCMs to shows that these indeterminate factors were actually related to the crime in Nigeria. They further conclude that the accuracy and efficiency of mathematical models can be enhanced using NCMs if indeterminate factors are taken into consideration.

3.2. Neutrosophic Logic (NL):

It is also called Smarandache logic. The fuzzy logic is generalized on the basis of Neutrosophy and it gives rise to something called Neutrosophic logic. It says that a proposition could be take three values: true (t), false (f) and indeterminate (I) and each of these are the values from the range of [T, I, F]. There is an introduction of a certain idea of 'indeterminacy' because of the parameters which are not expected and therefore, concealed in some statements. NL is the analysis of the partition in a triad. It includes the membership degrees of truthfulness T, falsity F and indeterminacy I. Figure 2 illustrates the following.

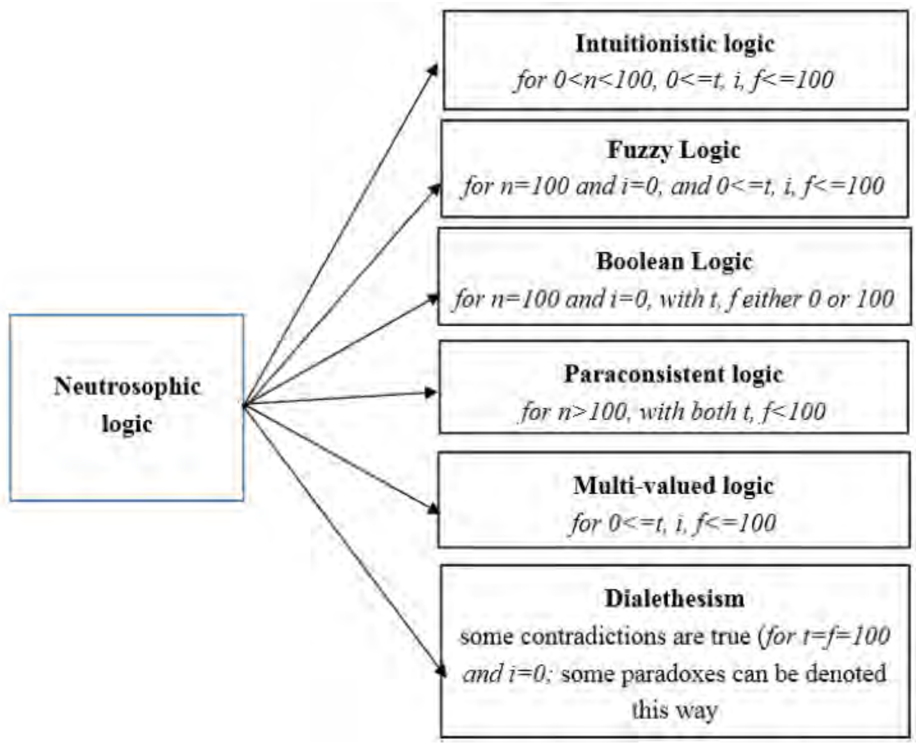

Figure 2. Neutrosophic logic and its relationship with intuitionistic logic

Florentin Smarandache in 2003 has written a paper to give an understanding of the Neutrosophic Logic (NL). He has also pointed out the differences between the Intuitionistic Fuzzy set and the neutrosophic set. [20]

Karina Pérez-Teruel and Maikel Leyva-Vázquez have analyzed the mental models and did their elicitation using NL. To show the applicability of the project, they showed an illustrative example. They discuss a model for the understanding the effect of indeterminacy and uncertainty in such models. [5]

Florentin Smarandache and Luige Vlădâreanu in 2011, have introduced the concept of NL and set operators. They have described the dynamics of a robot mathematically and how neutrosophic science is applicable to robotics [8].

3.3. Neutrosophic Set (NS):

Neutrosophic set is defined as the area of neutrosophy that is associated with the study of the dawn, scope and type of neutralities, and how they interact with various analytical spectra. [8]

Smarandache defined neutrosophic set as: Let the space of points be denoted by (Y). Let the general element in (Y) be denoted by (y). A NS (B) in (Y) has three membership functions (MF): truth MF -T B(y), an indeterminacy MF- I B(y) and a falsity MF- F B(y). The functions TB(y), I B(y), and F B(y) are real subsets of [0−, 1+] (they could be real standard or nonstandard).

That is:

$$T_B(y): Y \to [0-, 1+],\ I_B(y): \quad Y \to [0-, 1+] \text{ and } F_B(y): \quad Y \to [0-, 1+]$$

There is no limiting condition on the sum of T B(y), I B(y) and F B(y), so 0− ≤ sup T B(y) +sup I B(y) +sup F B(y) ≤ 3+.

Neutrosophic Sets have been used in various research works. Some examples are:
F. Smarandache, in [7] wrote about the Schrödinger's Cat Theory. He said that at one moment, the photon's quantum state could be in more than one place. It meant that one particular element might or might not belong to a set or a place at one time. It also refers to the fact that an element (a quantum state) has a possibility of belonging to two contrasting sets (or places) at one time.

In [26], to tackle time scheduling in projects, a framework has been given to minimize the cost of projects in environments which are ambiguous. Neutrosophic set theory has been used to consider the dynamic features of all parameters.

In K. Atanassov, Fuzzy Sets and Systems (2005), neutrosophic sets could also be used to relate an image with information that is not certain, using a new tool; the information could have been applied to some technique wherein the processing of images takes place. The examples are in the field of image segmentation, thresholding and removing the noise. Neutrosophic sets find their real life example in terms of philosophical application. They could also be used to calculate the truth-value in some theories of philosophy of Zen doctrine.

3.4. Neutrosophic Measure (NM):

The classical measure is generalized for such a case where the space has some factor of uncertainty or indeterminacy. The imprecise probabilities and the classical ones are generalized with the help of neutrosophic probability. There are a number of rules of the classical probability that are defined in the way that they are in unison with those of neutrosophy [8].

Let an item be defined as <B>. <B> could be any thought, feature, hypothesis, concept etc. Let <anti B> be the inverse of <B>; while <neut B> be none of the two: <B> and <anti B>, having some sense of neutrality (or indeterminacy) in relation to <B>. For example, if <B> = rain, then <anti B>= no rain, while <neut B> = no idea. Let <B> represent the truth value of a notion, then <anti B> represents its falsehood, while <neut B> represents its degree of indeterminacy.

If <B> = it will rain tomorrow, <anti B> = it will not rain tomorrow, while <neut B> = not knowing if it will rain or not/cloudy/humid day. We think of the measure to be null {m (anti B) =0} when the case does not prevail. When <neut B> does not prevail, the measure is written as null {m (neutB) = 0} [8].

3.5. Single Valued Neutrosophic Sets (SVNS) :

It is the instance of a NS which gives an additional possibility for the representation of uncertainty or indeterminacy, imprecision, incompleteness or inconsistency in some details which is present in the real world. The use of information that is not determinate and consistent could be suitably used in applications which include scientific and engineering domains. [9][10]

Let X define the space of points (objects). Let the collective elements in X be denoted by x (Wang et al., 2010). A Single Valued Neutrosophic Set, A in X is described by three membership functions (MF): truth MF TA(x), falsity MF FA(x) and an indeterminacy MF IA(x).

For every point x in X, the three MF's: TA(x), IA(x), FA(x) belong to the interval [0, 1].

SVNS, when continuous is written mathematically as [9,10]:

$$A = \int_X \frac{\langle T_A(x), I_A(x), F_A(x) \rangle}{x}, \quad x \in X.$$

SVNS, when discrete is written mathematically as:

$$A = \sum_{i=1}^{n} \frac{\langle T_A(x_i), I_A(x_i), F_A(x_i) \rangle}{x_i}, \quad x_i \in X.$$

Jun Ye, in [25], has presented the correlation and correlation coefficient of SVNSs, based on the extension of the connection of intuitionistic fuzzy sets (IFS's). Further, the use of correlation coefficient or similarity measure in cosine (both weighted) is suggested for the decision-making method. The options are evaluated on the basis of the criteria with the help of the membership degrees of truth, falsehood and indeterminacy under the SVNS environment.

M Abdel-Basset et al. in their paper, have analyzed the role of SVNS's and rough sets in smart city. They have proposed a framework for dealing with information that is incomplete and imperfect with the help of theories of SVNS and rough set. This combination of these two sets will take into account all aspects of uncertainty, imprecision of data and information and make lives of the citizens of the smart cities better with the introduction of services and decisions. They have focused mainly on making a framework of all kinds of imperfection that could possibly happen in smart cities [24].

4. Application Areas of Neutrosophy:

1. Cultural Psychology
2. Socio-economic theorizing
3. Information System Application
4. Decision Support Systems
5. IT Application
6. Healthcare and related areas
7. Situation Analysis
8. Sociological Forecasting
9. Supply chain Sustainability
10. Project Management

- In cultural psychology, NL theory can be used to reconcile the issues in socio-economic theorizing (collectivism vs individualism).
- In socio-economic theorizing, the conflicts arising out of human tensions could be reconciled, as in the conflicts between the two different perspectives i.e.; fermions and bosons, capitalism and socialism.
- In the deep problem of philosophy of science, NL theory can be implemented wherein it suggests that whenever there are two sides which oppose each other, a choice is always there to find the part that is neutral, so that the two opposite sides could be reconciled.
- In the field of cosmology, the NL analyses the underlying cause of changes of neutralities and opposites. It concludes that there is a possibility that there had been some start, in addition to some lasting background also, which they could be the 'primordial fluid'.
- In American football game, an attempt to score a goal involves an infinite sort of events that could happen. So, there is a possibility that NL could be expanded some states which could be more than three.
- In gravitation, this perspective could help find a middle-course between the two kinds of forces (pull and push), by keeping in view the fact that both the forces are in action. [11]

So, many fields of science are being improved with the help of the theory of neutrosophic logic. This theory is applicable in different research areas as well- in applied mathematics, social sciences, economics, and physics.

<u>More Applications:</u>

- In Information System Application (Neutrosophic Database, Analysis of the social networks, systems which deal with e-learning, in finding the middle course in the information of financial markets).

- In Information Technology Application (Neutrosophic Security, NCM's for Situation

Analysis, In Robotics).

- Decision Support System (in markets related to finances, management of risks, expert systems related to neutrosophy, linguistic variables in neutrosophy).

In short, it has applications in any field related to science or even human-centered, where inconsistency, incompleteness, indeterminacy is present. In general terms, where <neut A> (i.e; sense of neutrality in relation to item <A>) occurs [11].

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5. Conclusion and Future work:

Neutrosophy is an important field of research nowadays as it deals with uncertainties which cannot be taken into consideration using conventional modeling methods. There is indeterminacy in almost all aspects of this world; neutrosophy is doing its bit to make sense of the unknown. This paper presents a review of the technologies used in neutrosophy and the researches which have incorporated these concepts as well. Various applications of neutrosophy in many fields such as information system, information technology, decision support system and others are given. The future work holds the potential to develop newer algorithms for solving any problem of neutrosophy, which can also help in solving any fuzzy problems. The algorithms in the multi-criteria decision making problems which are based on neutrosophic theory are being used to solve practical applications in other areas such as medical diagnosis, financial market information, robotics, security, information fusion system, expert system and bioinformatics.

Conflicts of Interest: "The authors declare no conflict of interest."

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Received: Apr 25, 2020. Accepted: July 16, 2020

Neutrosophic Sets and Systems, Vol.35, 2020

University of New Mexico

MADM Using m-Generalized q-Neutrosophic Sets

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Abstract: Although the single valued neutrosophic sets (SVNSs) are effective tool to express uncertain information and are superior to the fuzzy sets, intuitionistic fuzzy sets, Pythagorean fuzzy sets and q-rung orthopair fuzzy sets, there is not yet reported an operation which can provide desirable generality and flexibility under single valued neutrosophic environment, although many operations have been developed earlier to meet above such eventualities. So, the primary aim of this paper is to propose the concept of m-generalized q-neutrosophic sets (*mGqNSs*) as a further generalization of fuzzy sets, intuitionistic fuzzy sets, Pythagorean fuzzy sets and q-rung orthopair fuzzy sets, single valued neutrosophic sets, n-hyperspherical neutrosophic sets and single valued spherical neutrosophic sets. Under the m-generalized q-neutrosophic environment, we develop some new operational laws and study their properties. Using these operations, we define m-generalized q-neutrosophic weighted aggregation operators. The distinguished features of these proposed weighted aggregation operators are studied in detail. Furthermore, based on these proposed operators, a MADM (multi-attribute decision making) approach is developed. Finally, an illustrative example is provided to show the feasibility and effectiveness of the proposed approach.

Keywords: Single valued neutrosophic set, m-generalized q-neutrosophic set, m-generalized q-neutrosophic weighted averaging aggregation operator (*mGqNWAA*), m-generalized q-neutrosophic weighted geometric aggregation operator (*mGqNWGA*), score value, decision making.

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1. Introduction

Multi-attribute decision making (MADM) is basically a process of selecting an optimal alternative from a set of chosen ones. In our daily life, we come across various types of multi-attribute decision making problems. Therefore, all of us need to learn the techniques to make decisions. The area of decision making problems has attracted the interest of many researchers. Many authors have worked in this field by utilizing various approaches. All the traditional decision making processes involve crisp data set but in many real life problems, data may not be in crisp form always. Fuzzy set theory is one such extremely useful tool that helps us to deal with non-crisp data. In 1965, Lotfi A. Zadeh **[1]** first published the famous research paper on fuzzy sets that originated due to mainly the inclusion of vague human assessments in computing problems and it can deal with uncertainty, vagueness, partially trueness, impreciseness, Sharpless boundaries etc. Basically, the theory of fuzzy set is founded on the concept of relative graded membership which deals with the partial belongings of an element in a set in order to process inexact information. Later on, fuzzy sets have been generalized to intuitionistic fuzzy sets **[2]** by adding a non-membership function by Atanassov in 1986 in order to deal with problems that possess incomplete information. In the context of fuzzy sets or intuitionistic fuzzy sets, it is known that the membership (or non-membership) value of an element in a set admits a unique value in the closed interval [0,1]. However, the application range of intuitionistic fuzzy set is narrow because it has the constraint that sum of membership degree and non-membership degree of an element is not greater than one. But, in complex decision-making problems, decision makers/experts may choose the preferences in such a way that the above condition gets violated. For instance, if an expert gives his preference with membership degree 0.8 and non-membership degree 0.7, then clearly their sum is 1.5, which is greater than 1. Therefore, this situation can't be not properly handled by the intuitionistic fuzzy sets. To solve this problem, Yager **[3, 4]** introduced the nonstandard fuzzy set named as Pythagorean fuzzy sets with membership degree ζ and non-membership degree ϑ with the condition $\zeta^2 + \vartheta^2 \leq 1$. Obviously, the Pythagorean fuzzy sets accommodate more uncertainties than the intuitionistic fuzzy sets. Yager **[5]** defined q-rung orthopair fuzzy sets (*q-ROFSs*) by enlarging the scope of Pythagorean fuzzy sets. The q-rung orthopair fuzzy sets allows the result of the qth power of the membership grade plus the qth power of the non-membership grade to be limited in interval [0,1]. If q=1, the q-rung orthopair fuzzy set transforms into the intuitionistic fuzzy set; if q=2, the q-rung orthopair fuzzy set transforms into the Pythagorean fuzzy set, which means that the q-rung orthopair fuzzy sets are extensions of intuitionistic fuzzy sets and Pythagorean fuzzy sets.

In 1999, Smarandache **[6]** introduced the notion neutrsophic set as a generalization of the classical set, fuzzy set, intuitionistic fuzzy set, Pythagorean fuzzy set and q-rung orthopair fuzzy set. The characterization of this neutrosophic set is explicitly done by truth-membership function, indeterminacy membership function and falsity membership function. The concept of single valued neutrosophic set was developed by Wang et al. **[7]** as an extension of fuzzy sets, Pythagorean fuzzy sets, q-rung orthopair fuzzy sets, intuitionistic fuzzy sets, single valued spherical neutrosophic sets **[8]**, n-hyperspherical neutrosophic sets **[8]**. The possible applications of neutrosophic sets and single valued neutrosophic sets on image segmentation have been studied in Gou and Cheng **[9]**, Gou and Sensur **[10]**. Also, we find their probable infliction on clustering analysis in Karaaslan **[11]** and on medical diagnosis problems in Ansari et al. **[12]** respectively. Furthermore, the subject of the neutrosophic set theory has been practiced in Wang et al. **[13]**, Gou et al. **[14]**, Ye **[15]**, Sun et al. **[16]**, Ye **[17-19]** and Abdel Basset et al. **[20, 21]**. Some recent studies on this area can be found in **[22-37]**.

The growing capacity of decision complexity induces the real-life decision-making problems that indulge both generality and flexibility of the operations used. Some of the basic operations of single valued spherical neutrosophic sets fail to generalize the basic operations of fuzzy sets, intuitionistic fuzzy sets, Pythagorean fuzzy sets and q-rung orthopair fuzzy sets. Getting inspired and provoked with this fact, in this paper, we have tried to propose a new concept called "*m*-generalized *q*-neutrosophic sets (*mGqNSs*)" and develop some aggregation operators in *m*-generalized *q*-neutrosophic environment to deal with *MADM* problems. The aims in this article are pursued below:

(1) To propose the concept of *m*-generalized *q*-neutrosophic sets (*mGqNSs*) as a further generalization of fuzzy sets, Pythagorean fuzzy sets, q-rung orthopair fuzzy sets, intuitionistic fuzzy sets, single valued neutrosophic sets, n-hyperspherical neutrosophic sets and single valued spherical neutrosophic sets.

(2) To define few operations between the *m*-generalized *q*-neutrosophic numbers.

(3) To develop the weighted aggregation operators such as *m*-generalized *q*-neutrosophic weighted averaging aggregation operator (*mGqNWAA*) and *m*-generalized *q*-neutrosophic weighted geometric aggregation operator (*mGqNWGA*) and study their properties.

(4) To propose a multi-attribute decision making method based on the m-generalized q-neutrosophic weighted aggregation operators.

To do so, the rest of the article is arranged as follows:

In section 2, we review some basic concepts. In Section 3, we first define m-generalized q-neutrosophic sets (*mGqNSs*) and *m*-generalized *q*-neutrosophic numbers (*mGqNNs*) and then propose few operations between the mGqNNs. Furthermore, we introduce the score of a *mGqNN* to ranking the *mGqNNs*. In section 4, we propose two types of m-generalized q-neutrosophic weighted aggregation operators to aggregate the m-generalized q-neutrosophic information. In section 5, based on the *m*-generalized *q*-neutrosophic weighted aggregation operators and score of *mGqNNs*, we develop a multi attribute decision making approach, in which the evaluation values of alternatives on the attribute are represented in terms of *mGqNNs* and the alternatives are ranked according to the values of the score of *mGqNNs* to select the best (most desirable) one. Also, we present a practical example to demonstrate the application and effectiveness of the proposed method. In final section, we present the conclusion of the study.

2. Preliminaries:

In this section, first we recall some basic notions that are relevant to our study.

2.1 Definition: **[7]** A single-valued neutrosophic set ς on the universe set U is given by

$$\varsigma = \{< x, \xi(x), \vartheta(x), \eta(x) >: x \in U\}$$

where the functions $\xi,\vartheta,\eta:U\to[0,1]$ satisfy the condition $0\le\xi(x)+\vartheta(x)+\eta(x)\le 3$ for every $x\in U$. The functions $\xi(x),\vartheta(x),\eta(x)$ define the degree of truth-membership, indeterminacy-membership and falsity-membership, respectively of $x\in U$.

2.2 Definition: **[7]** Suppose ς and ς' be two single-valued neutrosophic sets on U and are given by

$\varsigma=\{<x,\xi(x),\vartheta(x),\eta(x)>:x\in U\}$ and $\varsigma'=\{<x,\xi'(x),\vartheta'(x),\eta'(x)>:x\in U\}$. Then

(i) $\varsigma\subseteq\varsigma'$ if and only if $\xi(x)\le\xi'(x),\vartheta(x)\ge\vartheta'(x),\eta(x)\ge\eta'(x)\ \forall x\in U.$

(ii) $\varsigma^c=\{<x,\eta(x),1-\vartheta(x),\xi(x)>:x\in U\}$

(iii) $\varsigma\cup\varsigma'=\{<x,\max(\xi(x),\xi'(x)),\min(\vartheta(x),\vartheta'(x)),\ \min(\eta(x),\eta'(x))>:\ x\in U\}$.

(iv) $\varsigma\cap\varsigma'=\{<x,\min(\xi(x),\xi'(x)),\max(\vartheta(x),\vartheta'(x)),\ \max(\eta(x),\eta'(x))>:\ x\in U\}$.

3. *m*-GENERALIZED *q*-NEUTROSOPHIC SETS:

In this section first we define a *m*-generalized *q*-neutrosophic set as a further generalization of fuzzy set, Pythagorean fuzzy set, *q*-rung orthopair fuzzy set, intuitionistic fuzzy set, single valued neutrosophic set, single valued *n*-hyperspherical neutrosophic set and single valued spherical neutrosophic set. Then we present few operations between the *m*-generalized *q*-neutrosophic numbers.

3.1 Definition: Suppose *U* is a universe set and $x\in U$. A *m*-generalized *q*-neutrosophic set (*mGqNs*) in *U* is described as:

$$\psi=\{<x,\xi(x),\vartheta(x),\eta(x)>:x\in U\}$$

where $\xi,\vartheta,\eta:U\to[0,r]\ (o<r\le 1)$ are functions such that $0\le\xi(x),\vartheta(x),\eta(x)\le 1$ and

$$0\le(\xi(x))^{\frac{qm}{3}}+(\vartheta(x))^{\frac{qm}{3}}+(\eta(x))^{\frac{qm}{3}}\le\frac{3}{m}\quad(m,q\ge 1)\ .$$

Here $\xi(x),\vartheta(x),\eta(x)$ represent *m*-generalized truth membership, *m*-generalized indeterminacy membership and *m*-generalized falsity membership respectively of $x\in U$. The triplet $\psi=<\xi,\vartheta,\eta>$ is termed as *m*-generalized *q*-neutrosophic number (*mGqNN* for short).

In particular,

(i) when *m*=*r*=1 and *q*=3, ψ reduces to a single valued neutrosophic set **[7]**.

(ii) when *m*=3, *r*=*q*=1 and $\eta(x)=0\ \forall x\in U$, ψ reduces to an intuitionistic fuzzy set **[2]**.

(iii) when *m*=3, *r*=*q*=1 and $\eta(x)=\vartheta(x)=0\ \forall x\in U$, ψ reduces to a fuzzy set **[1]**.

(iv) when *m*=3, *r*= 1 and $\eta(x)=0\ \forall x\in U$, ψ reduces to a q-Rung orthopair fuzzy set **[5]**.

(v) when *m*=3, *r*= 1, *q*=2 and $\eta(x)=0\ \forall x\in U$, ψ reduces to a Pythagorean fuzzy set **[3, 4]**.

(vi) For $r=\sqrt[n]{3}$, *m*=1 and *q*=3*n* $(n\ge 1)$, ψ reduces to a single valued n-hyperspherical neutrosophic set **[8]**.

(vii) For $r=\sqrt{3}$, *m*=1 and *q*=6, ψ reduces to a single valued spherical neutrosophic set **[8]**.

Next we define few operations between *m*-generalized *q*-neutrosophic numbers.

3.2 Definition: Suppose $\psi_1=<\xi_1,\vartheta_1,\eta_1>$ and $\psi_2=<\xi_2,\vartheta_2,\eta_2>$ be two *m*-generalized *q*-neutrosophic numbers defined on *U* and λ be any real number >0. We define

(i) $\psi_1 \oplus \psi_2 = \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right) \left(\frac{3}{m} - \xi_2^{\frac{qm}{3}} \right) \right)^{\frac{3}{qm}}, \vartheta_1 \vartheta_2, \eta_1 \eta_2 \right\rangle$

(ii) $\psi_1 \otimes \psi_2 = \left\langle \xi_1 \xi_2, \left(\frac{3}{m} - \left(\frac{3}{m} - \vartheta_1^{\frac{qm}{3}} \right) \left(\frac{3}{m} - \vartheta_2^{\frac{qm}{3}} \right) \right)^{\frac{3}{qm}}, \left(\frac{3}{m} - \left(\frac{3}{m} - \eta_1^{\frac{qm}{3}} \right) \left(\frac{3}{m} - \eta_2^{\frac{qm}{3}} \right) \right)^{\frac{3}{qm}} \right\rangle$

(iii) $\lambda * \psi_1 = \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{\lambda} \right)^{\frac{3}{qm}}, \vartheta_1^{\lambda}, \eta_1^{\lambda} \right\rangle$

(iv) $\lambda \circ \psi_1 = \left\langle \xi_1^{\lambda}, \left(\frac{3}{m} - \left(\frac{3}{m} - \vartheta_1^{\frac{qm}{3}} \right)^{\lambda} \right)^{\frac{3}{qm}}, \left(\frac{3}{m} - \left(\frac{3}{m} - \eta_1^{\frac{qm}{3}} \right)^{\lambda} \right)^{\frac{3}{qm}} \right\rangle$

3.3 Theorem: Suppose $\psi_1 = <\xi_1, \vartheta_1, \eta_1>$ and $\psi_2 = <\xi_2, \vartheta_2, \eta_2>$ be two *m*-generalized *q*-neutrosophic numbers defined on *U* and $\lambda, \lambda_1, \lambda_2$ be any three real numbers >0. Then

(i) $\psi_1 \oplus \psi_2 = \psi_2 \oplus \psi_1$

(ii) $\psi_1 \otimes \psi_2 = \psi_2 \otimes \psi_1$

(iii) $\lambda * (\psi_1 \oplus \psi_2) = (\lambda * \psi_1) \oplus (\lambda * \psi_2)$

(iv) $\lambda \circ (\psi_1 \otimes \psi_2) = (\lambda \circ \psi_1) \otimes (\lambda \circ \psi_2)$

(v) $(\lambda_1 + \lambda_2) * \psi_1 = (\lambda_1 * \psi_1) \oplus (\lambda_2 * \psi_1)$

(vi) $(\lambda_1 + \lambda_2) \circ \psi_1 = (\lambda_1 \circ \psi_1) \otimes (\lambda_2 \circ \psi_1)$

Proof: (i), (ii) are straight forward.

(iii) We have, $\psi_1 \oplus \psi_2 = \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right) \left(\frac{3}{m} - \xi_2^{\frac{qm}{3}} \right) \right)^{\frac{3}{qm}}, \vartheta_1 \vartheta_2, \eta_1 \eta_2 \right\rangle$.

$\therefore \lambda * (\psi_1 \oplus \psi_2)$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right) \left(\frac{3}{m} - \xi_2^{\frac{qm}{3}} \right) \right) \right)^{\lambda} \right)^{\frac{3}{qm}}, (\vartheta_1 \vartheta_2)^{\lambda}, (\eta_1 \eta_2)^{\lambda} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{\lambda} \left(\frac{3}{m} - \xi_2^{\frac{qm}{3}} \right)^{\lambda} \right)^{\frac{3}{qm}}, \vartheta_1^{\lambda} \vartheta_2^{\lambda}, \eta_1^{\lambda} \eta_2^{\lambda} \right\rangle$$

On the other hand, we have,

$(\lambda * \psi_1) \oplus (\lambda * \psi_2)$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{\lambda} \right)^{\frac{3}{qm}}, \vartheta_1^{\lambda}, \eta_1^{\lambda} \right\rangle \oplus \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_2^{\frac{qm}{3}} \right)^{\lambda} \right)^{\frac{3}{qm}}, \vartheta_2^{\lambda}, \eta_2^{\lambda} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{\lambda} \right) \right) \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_2^{\frac{qm}{3}} \right)^{\lambda} \right) \right) \right)^{\frac{3}{qm}}, \vartheta_1^{\lambda} \vartheta_2^{\lambda}, \eta_1^{\lambda} \eta_2^{\lambda} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{\lambda} \left(\frac{3}{m} - \xi_2^{\frac{qm}{3}} \right)^{\lambda} \right)^{\frac{3}{qm}}, \vartheta_1^{\lambda} \vartheta_2^{\lambda}, \eta_1^{\lambda} \eta_2^{\lambda} \right\rangle$$

Thus, we get, $\lambda * (\psi_1 \oplus \psi_2) = (\lambda * \psi_1) \oplus (\lambda * \psi_2)$.

(iv) Similar to (iii)

(v) We have,

$$(\lambda_1 + \lambda_2) * \psi_1 = \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{\lambda_1 + \lambda_2} \right)^{\frac{3}{qm}}, \vartheta_1^{\lambda_1 + \lambda_2}, \eta_1^{\lambda_1 + \lambda_2} \right\rangle$$

On the other hand,

$(\lambda_1 * \psi_1) \oplus (\lambda_2 * \psi_1)$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{\lambda_1} \right)^{\frac{3}{qm}}, \vartheta_1^{\lambda_1}, \eta_1^{\lambda_1} \right\rangle \oplus \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{\lambda_2} \right)^{\frac{3}{qm}}, \vartheta_1^{\lambda_2}, \eta_1^{\lambda_2} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{\lambda_1} \right) \right) \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{\lambda_2} \right) \right) \right)^{\frac{3}{qm}}, \vartheta_1^{\lambda_1} \vartheta_1^{\lambda_2}, \eta_1^{\lambda_1} \eta_1^{\lambda_2} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{\lambda_1+\lambda_2} \right)^{\frac{3}{qm}}, \vartheta_1^{\lambda_1+\lambda_2}, \eta_1^{\lambda_1+\lambda_2} \right\rangle$$

Thus we get, $(\lambda_1 + \lambda_2) * \psi_1 = (\lambda_1 * \psi_1) \oplus (\lambda_2 * \psi_1)$.

(vi) Similar to (v).

3.4 Definition: The score of the *mGqNN* $\psi = <\xi, \vartheta, \eta>$ is defined as: $S(\psi) = \dfrac{2 + \xi - \vartheta - \eta}{3}$.

The ranking method for ranking the *mGqNNs* is given below:

If $\psi = <\xi, \vartheta, \eta>$ and $\psi' = <\xi', \vartheta', \eta'>$ be two *mGqNNs*, then

(I) if $S(\psi) > S(\psi')$, then $\psi > \psi'$

(II) if $S(\psi) = S(\psi')$, then $\psi = \psi'$

4. *m*-GENERALIZED *q*-NEUTROSOPHIC WEIGHTED AGGREGATION OPERATORS:

In this section first we define *m*-generalized *q*-neutrosophic weighted averaging aggregation operator (*mGqNWAA*) and *m*-generalized *q*-neutrosophic weighted geometric aggregation operator (*mGqNWGA*) and study their basic properties.

4.1 Definition: Suppose $\psi_k = <\xi_k, \vartheta_k, \eta_k> (k = 1,2,3,.......,n)$ be a collection of *mGqNNs* defined on the universe set *U*. Then a *m*-generalized *q*-neutrosophic weighted averaging aggregation operator (*mGqNWAA* for short) is given as $mGqNWAA : \Theta^n \to \Theta$ and is defined as:

$$mGqNWAA(\psi_1, \psi_2, \psi_3, ..., \psi_n) = (w_1 * \psi_1) \oplus (w_2 * \psi_2) \oplus (w_3 * \psi_3) \oplus .... \oplus (w_n * \psi_n)$$

where Θ is the collection of all *mGqNNs* defined on the universe set *U*, $w = (w_1, w_2, w_3, ..., w_n)^T$ is the weight vector of $(\psi_1, \psi_2, \psi_3, ..., \psi_n)$ such that $w_k \geq 0 (k = 1,2,3,....,n)$ and $\sum_{k=1}^{n} w_k = 1$.

On the basis of the operational rules of the *mGqNNs*, we can get the aggregation result as described as Theorem 4.2.

4.2 Theorem: Suppose $\psi_k = <\xi_k, \vartheta_k, \eta_k> (k = 1,2,3,.......,n)$ be a collection of *mGqNNs* defined on the universe set *U* and $w = (w_1, w_2, w_3, ..., w_n)^T$ is the weight vector of $(\psi_1, \psi_2, \psi_3, ..., \psi_n)$ such that

$w_k \geq 0 (k = 1,2,3,....,n)$ and $\sum_{k=1}^{n} w_k = 1$. Then $mGqNWAA(\psi_1, \psi_2, \psi_3, ..., \psi_n)$ is also a *mGqNN*.

Moreover, we have,

$$mGqNWAA(\psi_1, \psi_2, \psi_3, ..., \psi_n) = \left\langle \left(\frac{3}{m} - \prod_{k=1}^{n} \left(\frac{3}{m} - \xi_k^{\frac{qm}{3}} \right)^{w_k} \right)^{\frac{3}{qm}}, \prod_{k=1}^{n} \vartheta_k^{w_k}, \prod_{k=1}^{n} \eta_k^{w_k} \right\rangle.$$

Proof:

The first part of the theorem can be proved easily. To show the rest part, let us use the method of mathematical induction on *n*.

Step-1: For *n*=1, the proof is straight forward. So first take *n*=2.

Then,

$$mGqNWAA(\psi_1, \psi_2)$$

$$= (w_1 * \psi_1) \oplus (w_2 * \psi_2)$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{w_1} \right)^{\frac{3}{qm}}, \vartheta_1^{w_1}, \eta_1^{w_1} \right\rangle \oplus \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_2^{\frac{qm}{3}} \right)^{\lambda_2} \right)^{\frac{3}{qm}}, \vartheta_1^{w_2}, \eta_1^{w_2} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{w_1} \right) \right) \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_2^{\frac{qm}{3}} \right)^{w_2} \right) \right) \right)^{\frac{3}{qm}}, \vartheta_1^{w_1} \vartheta_1^{w_2}, \eta_1^{w_1} \eta_1^{w_2} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{w_1} \left(\frac{3}{m} - \xi_2^{\frac{qm}{3}} \right)^{w_2} \right)^{\frac{3}{qm}}, \vartheta_1^{w_1} \vartheta_1^{w_2}, \eta_1^{w_1} \eta_1^{w_2} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \prod_{k=1}^{2} \left(\frac{3}{m} - \xi_k^{\frac{qm}{3}} \right)^{w_k} \right)^{\frac{3}{qm}}, \prod_{k=1}^{2} \vartheta_k^{w_k}, \prod_{k=1}^{2} \eta_k^{w_k} \right\rangle.$$

Thus the result is true for *n*=2.

Step-2: Suppose that the result is true for *n*=*p* i.e;

$$mGqNWAA(\psi_1, \psi_2, \psi_3, ..., \psi_p) = \left\langle \left(\frac{3}{m} - \prod_{k=1}^{p} \left(\frac{3}{m} - \xi_k^{\frac{qm}{3}} \right)^{w_k} \right)^{\frac{3}{qm}}, \prod_{k=1}^{p} \vartheta_k^{w_k}, \prod_{k=1}^{p} \eta_k^{w_k} \right\rangle$$

Step-3: Take *n*=*p*+1.Then we have,

$$mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_{p+1})$$
$$=((w_1*\psi_1)\oplus(w_2*\psi_2)\oplus(w_3*\psi_3)\oplus.........\oplus(w_p*\psi_p))\oplus(w_{p+1}*\psi_{p+1})$$
$$=\left\langle\left(\frac{3}{m}-\prod_{k=1}^{p}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}},\prod_{k=1}^{p}\vartheta_k^{w_k},\prod_{k=1}^{p}\eta_k^{w_k}\right\rangle$$
$$\oplus\left\langle\left(\frac{3}{m}-\left(\frac{3}{m}-\xi_{p+1}^{\frac{qm}{3}}\right)^{w_{p+1}}\right)^{\frac{3}{qm}},\vartheta_{p+1}^{w_{p+1}},\eta_{p+1}^{w_{p+1}}\right\rangle$$
$$=\left\langle\left(\frac{3}{m}-\left(\frac{3}{m}-\left(\frac{3}{m}-\prod_{k=1}^{p}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\right)\right)\left(\frac{3}{m}-\left(\frac{3}{m}-\left(\frac{3}{m}-\xi_{p+1}^{\frac{qm}{3}}\right)^{w_{p+1}}\right)\right)\right)^{\frac{3}{qm}},\right.$$
$$\left.\vartheta_{p+1}^{w_{p+1}}\prod_{k=1}^{p}\vartheta_k^{w_k},\eta_{p+1}^{w_{p+1}}\prod_{k=1}^{p}\eta_k^{w_k}\right\rangle$$
$$=\left\langle\left(\frac{3}{m}-\prod_{k=1}^{p}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\times\left(\frac{3}{m}-\xi_{p+1}^{\frac{qm}{3}}\right)^{w_{p+1}}\right)^{\frac{3}{qm}},\vartheta_{p+1}^{w_{p+1}}\prod_{k=1}^{p}\vartheta_k^{w_k},\eta_{p+1}^{w_{p+1}}\prod_{k=1}^{p}\eta_k^{w_k}\right\rangle$$
$$=\left\langle\left(\frac{3}{m}-\prod_{k=1}^{p+1}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}},\prod_{k=1}^{p+1}\vartheta_k^{w_k},\prod_{k=1}^{p+1}\eta_k^{w_k}\right\rangle$$

Thus the result is true for n=p+1 also. Hence, by the method of induction, the result is true for all n.

Let us explore some more results related to $mGqNWAA$ operator in the form of theorems 4.3-4.6.

4.3 Theorem: Suppose $\psi_k=<\xi_k,\vartheta_k,\eta_k>(k=1,2,3,.......,n)$ be a collection of m-Gq-NNs defined on the universe set U and $w=(w_1,w_2,w_3,.......,w_n)^T$ is the weight vector of $(\psi_1,\psi_2,\psi_3,........,\psi_n)$ such that $w_k\geq 0(k=1,2,3,....,n)$ and $\sum_{k=1}^{n}w_k=1$. Then for $\psi_0=<\xi_0,\vartheta_0,\eta_0>\in\Theta$ (where Θ is the collection of all *mGqNNs* defined on the universe set *U*), we have $mGqNWAA(\psi_0\oplus\psi_1,\psi_0\oplus\psi_2,\psi_0\oplus\psi_3,........,\psi_0\oplus\psi_n)=\psi_0\oplus mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n)$.

Proof:

$$\psi_0 \oplus \psi_k = \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_0^{\frac{qm}{3}} \right) \left(\frac{3}{m} - \xi_k^{\frac{qm}{3}} \right) \right)^{\frac{3}{qm}}, \vartheta_0 \vartheta_k, \eta_0 \eta_k \right\rangle \ (k = 1, 2, 3, \ldots\ldots, n)$$

$$\therefore mGqNWAA(\psi_0 \oplus \psi_1, \psi_0 \oplus \psi_2, \psi_0 \oplus \psi_3, \ldots\ldots, \psi_0 \oplus \psi_n)$$

$$= \left\langle \left(\frac{3}{m} - \prod_{k=1}^{n} \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_0^{\frac{qm}{3}} \right) \left(\frac{3}{m} - \xi_k^{\frac{qm}{3}} \right) \right) \right)^{w_k} \right)^{\frac{3}{qm}}, \prod_{k=1}^{n} (\vartheta_0 \vartheta_k)^{w_k}, \prod_{k=1}^{n} (\eta_0 \eta_k)^{w_k} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_0^{\frac{qm}{3}} \right)^{\sum_{k=1}^{n} w_k} \prod_{k=1}^{n} \left(\frac{3}{m} - \xi_k^{\frac{qm}{3}} \right)^{w_k} \right)^{\frac{3}{qm}}, \vartheta_0^{\sum_{k=1}^{n} w_k} \prod_{k=1}^{n} \vartheta_k^{w_k}, \eta_0^{\sum_{k=1}^{n} w_k} \prod_{k=1}^{n} \eta_k^{w_k} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_0^{\frac{qm}{3}} \right) \prod_{k=1}^{n} \left(\frac{3}{m} - \xi_k^{\frac{qm}{3}} \right)^{w_k} \right)^{\frac{3}{qm}}, \vartheta_0 \prod_{k=1}^{n} \vartheta_k^{w_k}, \eta_0 \prod_{k=1}^{n} \eta_k^{w_k} \right\rangle$$

On the other hand, $\psi_0 \oplus mGqNWAA(\psi_1, \psi_2, \psi_3, \ldots\ldots, \psi_n)$

$$= < \xi_0, \vartheta_0, \eta_0 > \oplus \left\langle \left(\frac{3}{m} - \prod_{k=1}^{n} \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{w_k} \right)^{\frac{3}{qm}}, \prod_{k=1}^{n} \vartheta_k^{w_k}, \prod_{k=1}^{n} \eta_k^{w_k} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_0^{\frac{qm}{3}} \right) \prod_{k=1}^{n} \left(\frac{3}{m} - \left(\frac{3}{m} - \prod_{k=1}^{n} \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{w_k} \right) \right) \right)^{\frac{3}{qm}}, \vartheta_0 \prod_{k=1}^{n} \vartheta_k^{w_k}, \eta_0 \prod_{k=1}^{n} \eta_k^{w_k} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_0^{\frac{qm}{3}} \right) \prod_{k=1}^{n} \left(\frac{3}{m} - \xi_k^{\frac{qm}{3}} \right)^{w_k} \right)^{\frac{3}{qm}}, \vartheta_0 \prod_{k=1}^{n} \vartheta_k^{w_k}, \eta_0 \prod_{k=1}^{n} \eta_k^{w_k} \right\rangle$$

Hence,

$$mGqNWAA(\psi_0 \oplus \psi_1, \psi_0 \oplus \psi_2, \psi_0 \oplus \psi_3, \ldots\ldots, \psi_0 \oplus \psi_n) = \psi_0 \oplus mGqNWAA(\psi_1, \psi_2, \psi_3, \ldots\ldots, \psi_n)$$

.

4.4 Theorem: *(Idempotency)* Suppose $\psi_k = < \xi_k, \vartheta_k, \eta_k > (k = 1, 2, 3, \ldots\ldots, n)$ be a collection of m-Gq-NNs defined on the universe set U and $w = (w_1, w_2, w_3, \ldots\ldots, w_n)^T$ is the weight vector of

$(\psi_1,\psi_2,\psi_3,........,\psi_n)$ such that $w_k \geq 0 (k=1,2,3,....,n)$ and $\sum_{k=1}^{n} w_k = 1$. If $\psi_0 =< \xi_0,\vartheta_0,\eta_0 >\in \Theta$ (where Θ is the collection of all *mGqNNs* defined on the universe set *U*) such that $\psi_k = \psi_0 \, \forall\, k=1,2,3,.......,n$, then we have $mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n)=\psi_0$.

Proof: We have, $mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n)$

$$=\left\langle \left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}, \prod_{k=1}^{n}\vartheta_k^{\,w_k}, \prod_{k=1}^{n}\eta_k^{\,w_k} \right\rangle$$

$$=\left\langle \left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_0^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}, \prod_{k=1}^{n}\vartheta_0^{\,w_k}, \prod_{k=1}^{n}\eta_0^{\,w_k} \right\rangle$$

$$=\left\langle \left(\frac{3}{m}-\left(\frac{3}{m}-\xi_0^{\frac{qm}{3}}\right)^{\sum_{k=1}^{n} w_k}\right)^{\frac{3}{qm}}, \vartheta_0^{\sum_{k=1}^{n} w_k}, \prod_{k=1}^{n}\eta_0^{\sum_{k=1}^{n} w_k} \right\rangle$$

$$=\left\langle \left(\frac{3}{m}-\left(\frac{3}{m}-\xi_0^{\frac{qm}{3}}\right)\right)^{\frac{3}{qm}}, \vartheta_0, \eta_0 \right\rangle =< \xi_0,\vartheta_0,\eta_0 >= \psi_0$$

4.5 Theorem: *(Monotonocity)* Suppose $\psi_k =< \xi_k,\vartheta_k,\eta_k >$ and $\psi_k' =< \xi_k',\vartheta_k',\eta_k' >$ $(k=1,2,3,.......,n)$ be two collections of *mGqNNs* defined on the universe set *U* and $w=(w_1,w_2,w_3,.......,w_n)^T$ is the weight vector of $(\psi_1,\psi_2,\psi_3,........,\psi_n)$ as well as $(\psi_1',\psi_2',\psi_3',........,\psi_n')$ such that $w_k \geq 0 (k=1,2,3,....,n)$ and $\sum_{k=1}^{n} w_k = 1$. If $\xi_k \geq \xi_k', \vartheta_k \leq \vartheta_k', \eta_k \leq \eta_k' >$ $(k=1,2,3,.......,n)$, then $mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n) \geq mGqNWAA(\psi_1',\psi_2',\psi_3',........,\psi_n')$.

Proof: Since $\xi_k \geq \xi_k', \vartheta_k \leq \vartheta_k', \eta_k \leq \eta_k' >$ for all k,

so $\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k} \leq \prod_{k=1}^{n}\left(\frac{3}{m}-\xi_k'^{\frac{qm}{3}}\right)^{w_k}, \prod_{k=1}^{n}\vartheta_k^{\,w_k} \leq \prod_{k=1}^{n}\vartheta_k'^{\,w_k}, \prod_{k=1}^{n}\eta_k^{\,w_k} \leq \prod_{k=1}^{n}\eta_k'^{\,w_k}$

$$\Rightarrow\left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}\geq\left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_k'^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}},\prod_{k=1}^{n}\vartheta_k^{w_k}\leq\prod_{k=1}^{n}\vartheta_k'^{w_k},$$

$$\prod_{k=1}^{n}\eta_k^{w_k}\leq\prod_{k=1}^{n}\eta_k'^{w_k}$$

$$\Rightarrow\left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}-\left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_k'^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}-\left(\prod_{k=1}^{n}\vartheta_k^{w_k}-\prod_{k=1}^{n}\vartheta_k'^{w_k}\right)$$

$$-\left(\prod_{k=1}^{n}\eta_k^{w_k}-\prod_{k=1}^{n}\eta_k'^{w_k}\right)\geq 0$$

$$\Rightarrow\frac{2+\left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}-\prod_{k=1}^{n}\vartheta_k^{w_k}-\prod_{k=1}^{n}\eta_k^{w_k}}{3}$$

$$-\frac{2+\left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_k'^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}-\prod_{k=1}^{n}\vartheta_k'^{w_k}-\prod_{k=1}^{n}\eta_k'^{w_k}}{3}\geq 0$$

$$\Rightarrow S(mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n))\geq S(mGqNWAA(\psi_1',\psi_2',\psi_3',........,\psi_n'))$$

Hence by definition of score value and ranking method, we have, $mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n)\geq mGqNWAA(\psi_1',\psi_2',\psi_3',........,\psi_n')$.

4.6 Theorem: *(Boundedness)* Suppose $\psi_k=<\xi_k,\vartheta_k,\eta_k>(k=1,2,3,.......,n)$ be a collection of *mGqNNs* defined on the universe set *U* and $w=(w_1,w_2,w_3,.......,w_n)^T$ is the weight vector of $(\psi_1,\psi_2,\psi_3,........,\psi_n)$ such that $w_k\geq 0(k=1,2,3,....,n)$ and $\sum_{k=1}^{n}w_k=1$. Let us define two *mGqNNs* by: $\psi^-=<\min_{1\leq k\leq n}\xi_k,\max_{1\leq k\leq n}\vartheta_k,\max_{1\leq k\leq n}\eta_k>,\psi^+=<\max_{1\leq k\leq n}\xi_k,\min_{1\leq k\leq n}\vartheta_k,\min_{1\leq k\leq n}\eta_k>$

Then, $\psi^-\leq mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n)\leq\psi^+$.

Proof: Suppose $mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n)=<\xi,\vartheta,\eta>$.

Then we have, $\frac{3}{m}-\max_{1\leq k\leq n}\xi_k^{\frac{qm}{3}}\leq\frac{3}{m}-\xi_k^{\frac{qm}{3}}\leq\frac{3}{m}-\min_{1\leq k\leq n}\xi_k^{\frac{qm}{3}}$

$$\Rightarrow\left(\frac{3}{m}-\max_{1\leq k\leq n}\xi_k^{\frac{qm}{3}}\right)^{w_k}\leq\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\leq\left(\frac{3}{m}-\min_{1\leq k\leq n}\xi_k^{\frac{qm}{3}}\right)^{w_k}$$

$$\Rightarrow \left(\frac{3}{m} - \prod_{k=1}^{n} \left(\frac{3}{m} - \min_{1 \le k \le n} \xi_k^{\frac{qm}{3}} \right)^{w_k} \right)^{\frac{3}{qm}} \le \left(\frac{3}{m} - \prod_{k=1}^{n} \left(\frac{3}{m} - \xi_k^{\frac{qm}{3}} \right)^{w_k} \right)^{\frac{3}{qm}} \le \left(\frac{3}{m} - \prod_{k=1}^{n} \left(\frac{3}{m} - \max_{1 \le k \le n} \xi_k^{\frac{qm}{3}} \right)^{w_k} \right)^{\frac{3}{qm}}$$

$$\Rightarrow \left(\frac{3}{m} - \left(\frac{3}{m} - \min_{1 \le k \le n} \xi_k^{\frac{qm}{3}} \right)^{\sum_{k=1}^{n} w_k} \right)^{\frac{3}{qm}} \le \xi \le \left(\frac{3}{m} - \left(\frac{3}{m} - \max_{1 \le k \le n} \xi_k^{\frac{qm}{3}} \right)^{\sum_{k=1}^{n} w_k} \right)^{\frac{3}{qm}}$$

$$\Rightarrow \min_{1 \le k \le n} \xi_k \le \xi \le \max_{1 \le k \le n} \xi_k$$

Again, $\prod_{k=1}^{n} (\min_{1 \le k \le n} \vartheta_k)^{w_k} \le \prod_{k=1}^{n} \vartheta_k^{w_k} \le \prod_{k=1}^{n} (\max_{1 \le k \le n} \vartheta_k)^{w_k}$

$$\Rightarrow (\min_{1 \le k \le n} \vartheta_k)^{\sum_{k=1}^{n} w_k} \le \prod_{k=1}^{n} \vartheta_k^{w_k} \le (\max_{1 \le k \le n} \vartheta_k)^{\sum_{k=1}^{n} w_k} \Rightarrow \min_{1 \le k \le n} \vartheta_k \le \vartheta \le \max_{1 \le k \le n} \vartheta_k$$

Similarly, we can get, $\min_{1 \le k \le n} \eta_k \le \eta \le \max_{1 \le k \le n} \eta_k$.

Hence $\min_{1 \le k \le n} \xi_k - \max_{1 \le k \le n} \vartheta_k - \max_{1 \le k \le n} \eta_k \le \xi - \vartheta - \eta \le \max_{1 \le k \le n} \xi_k - \min_{1 \le k \le n} \vartheta_k - \min_{1 \le k \le n} \eta_k$

$$\Rightarrow \frac{2 + \min_{1 \le k \le n} \xi_k - \max_{1 \le k \le n} \vartheta_k - \max_{1 \le k \le n} \eta_k}{3} \le \frac{2 + \xi - \vartheta - \eta}{3} \le \frac{2 + \max_{1 \le k \le n} \xi_k - \min_{1 \le k \le n} \vartheta_k - \min_{1 \le k \le n} \eta_k}{3}$$

$$\Rightarrow S(\psi^-) \le S(mGqNWAA(\psi_1, \psi_2, \psi_3, ..., \psi_n)) \le S(\psi^+)$$

Therefore by definition of score value and ranking method, we have, $\psi^- \le mGqNWAA(\psi_1, \psi_2, \psi_3, ..., \psi_n) \le \psi^+$.

4.7 Definition: Suppose $\psi_k = < \xi_k, \vartheta_k, \eta_k > (k = 1, 2, 3, ..., n)$ be a collection of *mGqNNs* defined on the universe set *U*. Then a *m*-generalized *q*-neutrosophic weighted geometric aggregation operator (*mGqNWGA* for short) is given as $mGqNWGA : \Theta^n \to \Theta$ and is defined as:

$$mGqNWGA(\psi_1, \psi_2, \psi_3, ..., \psi_n) = (w_1 \circ \psi_1) \otimes (w_2 \circ \psi_2) \otimes (w_3 \circ \psi_3) \otimes .... \otimes (w_n \circ \psi_n)$$

where Θ is the collection of all *mGqNNs* defined on the universe set *U*, $w = (w_1, w_2, w_3, ..., w_n)^T$ is the weight vector of $(\psi_1, \psi_2, \psi_3, ..., \psi_n)$ such that $w_k \ge 0 (k = 1, 2, 3, ..., n)$ and $\sum_{k=1}^{n} w_k = 1$.

On the basis of the operational rules of the *mGqNNs*, we can get the aggregation result as described as Theorem 4.2.

4.8 Theorem: Suppose $\psi_k = < \xi_k, \vartheta_k, \eta_k > (k = 1, 2, 3, ..., n)$ be a collection of *mGqNNs* defined on the universe set *U* and $w = (w_1, w_2, w_3, ..., w_n)^T$ is the weight vector of $(\psi_1, \psi_2, \psi_3, ..., \psi_n)$ such that $w_k \ge 0 (k = 1, 2, 3, ..., n)$ and $\sum_{k=1}^{n} w_k = 1$. Then $mGqNWGA(\psi_1, \psi_2, \psi_3, ..., \psi_n)$ is also a *mGqNN*. Moreover, we have, $mGqNWGA(\psi_1, \psi_2, \psi_3, ..., \psi_n)$

$$=\left\langle \prod_{k=1}^{n}\xi_k^{w_k}, \left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\vartheta_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}, \left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\eta_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}} \right\rangle.$$

Proof: Similar to theorem 4.2.

4.9 Theorem: Suppose $\psi_k =< \xi_k, \vartheta_k, \eta_k > (k=1,2,3,.......,n)$ be a collection of *mGqNNs* defined on the universe set *U* and $w=(w_1,w_2,w_3,.......,w_n)^T$ is the weight vector of $(\psi_1,\psi_2,\psi_3,........,\psi_n)$ such that $w_k \geq 0 (k=1,2,3,....,n)$ and $\sum_{k=1}^{n} w_k = 1$. Then for $\psi_0 =< \xi_0, \vartheta_0, \eta_0 > \in \Theta$ (where Θ is the collection of all *mGqNNs* defined on the universe set *U*), we have
$mGqNWGA(\psi_0 \otimes \psi_1, \psi_0 \otimes \psi_2, \psi_0 \otimes \psi_3, ..., \psi_0 \otimes \psi_n) = \psi_0 \otimes mGqNWGA(\psi_1,\psi_2,\psi_3,........,\psi_n)$.

Proof: Similar to theorem 4.3.

4.10 Theorem: *(Idempotency)* Suppose $\psi_k =< \xi_k, \vartheta_k, \eta_k > (k=1,2,3,.......,n)$ be a collection of *mGqNNs* defined on the universe set *U* and $w=(w_1,w_2,w_3,.......,w_n)^T$ is the weight vector of $(\psi_1,\psi_2,\psi_3,........,\psi_n)$ such that $w_k \geq 0 (k=1,2,3,....,n)$ and $\sum_{k=1}^{n} w_k = 1$. If $\psi_0 =< \xi_0, \vartheta_0, \eta_0 > \in \Theta$ (where Θ is the collection of all *mGqNNs* defined on the universe set *U*) such that $\psi_k = \psi_0 \,\forall\, k=1,2,3,.......,n$, then we have $mGqNWGA(\psi_1,\psi_2,\psi_3,........,\psi_n) = \psi_0$.
Proof: Similar to theorem 4.4.

4.11 Theorem: *(Monotonocity)* Suppose $\psi_k =< \xi_k, \vartheta_k, \eta_k >$ and $\psi_k' =< \xi_k', \vartheta_k', \eta_k' >$ $(k=1,2,3,.......,n)$ be two collections of *mGqNNs* defined on the universe set *U* and $w=(w_1,w_2,w_3,.......,w_n)^T$ is the weight vector of $(\psi_1,\psi_2,\psi_3,........,\psi_n)$ as well as $(\psi_1',\psi_2',\psi_3',........,\psi_n')$ such that $w_k \geq 0 (k=1,2,3,....,n)$ and $\sum_{k=1}^{n} w_k = 1$. If $\xi_k \geq \xi_k', \vartheta_k \leq \vartheta_k', \eta_k \leq \eta_k' >$ $(k=1,2,3,.......,n)$, then $mGqNWGA(\psi_1,\psi_2,\psi_3,........,\psi_n) \geq$ $mGqNWGA(\psi_1',\psi_2',\psi_3',........,\psi_n')$.
Proof: Similar to theorem 4.5.

4.12 Theorem: *(Boundedness)* Suppose $\psi_k =< \xi_k, \vartheta_k, \eta_k > (k=1,2,3,.......,n)$ be a collection of *mGqNNs* defined on the universe set *U* and $w=(w_1,w_2,w_3,.......,w_n)^T$ is the weight vector of $(\psi_1,\psi_2,\psi_3,........,\psi_n)$ such that $w_k \geq 0 (k=1,2,3,....,n)$ and $\sum_{k=1}^{n} w_k = 1$. Let us define two *mGqNNs* by: $\psi^- =< \min_{1\leq k\leq n} \xi_k, \max_{1\leq k\leq n} \vartheta_k, \max_{1\leq k\leq n} \eta_k >, \psi^+ =< \max_{1\leq k\leq n} \xi_k, \min_{1\leq k\leq n} \vartheta_k, \min_{1\leq k\leq n} \eta_k >$

Then $\psi^- \leq mGqNWGA(\psi_1,\psi_2,\psi_3,........,\psi_n) \leq \psi^+$.

Proof: Similar to theorem 4.6.

5. MULTI ATTRIBUTE DECISION MAKING:

Consider a multi-attribute decision making problem which consists of *m* different alternatives A_1, A_2, ..., A_l which are evaluated under the set of *n* different attributes C_1, C_2, ..., C_n. Assume that an expert evaluates the given alternatives $A_i\,(i=1,2,...,l)$ under the attribute $C_j\,(j=1,2,...,n)$ and the evaluation result is presented by the form of *m*-generalized *q*-neutrosophic numbers $\zeta_{ij}=\left\langle \xi_{ij},\vartheta_{ij},\delta_{ij}\right\rangle$ such that $0\le \xi_{ij},\vartheta_{ij},\delta_{ij}\le 1$ and $0\le\left(\xi_{ij}\right)^{\frac{qm}{3}}+\left(\vartheta_{ij}\right)^{\frac{qm}{3}}+\left(\delta_{ij}\right)^{\frac{qm}{3}}\le\frac{3}{m}$ where $i=1,2,...,l;\,j=1,2,...,n$. Further assume that $w_j\,(j=1,2,...,n)$ is the weight of the attribute such C_j such that $w_j>0(j=1,2,...,n)$ and $\sum_{j=1}^{n}w_j=1$.

Then to determine the most desirable alternative (s), the proposed operators are utilized to develop a multi-attribute decision making with *m*-generalized *q*-neutrosophic information, which involves the following steps:

Step-1 Arrange the rating values of the expert in the form of decision matrix $\tilde{D}=\left(\zeta_{ij}\right)_{l\times n}=\left(\left\langle \xi_{ij},\vartheta_{ij},\delta_{ij}\right\rangle\right)_{l\times n}$.

Step-2: Construct aggregated *m*-generalized *q*-neutrosophic decision matrix. In order to do that, the proposed operators can be utilized as follows:

Let $\tilde{R}=\left(\tilde{R}_i\right)_{l\times 1}$ be the aggregated *m*-generalized *q*-neutrosophic decision matrix, where

$$\tilde{R}_i=mGqNWAA\left(\zeta_{i1},\zeta_{i2},.....,\zeta_{in}\right)=\left(w_1*\zeta_{i1}\right)\oplus\left(w_2*\zeta_{i2}\right)\oplus..........\oplus\left(w_n*\zeta_{in}\right)$$

OR

$$\tilde{R}_i=mGqNWGA\left(\zeta_{i1},\zeta_{i2},.....,\zeta_{in}\right)=\left(w_1\circ\zeta_{i1}\right)\otimes\left(w_2\circ\zeta_{i2}\right)\otimes..........\otimes\left(w_n\circ\zeta_{in}\right)$$

Step-3: Calculate the score values $S\left(\tilde{R}_i\right)\,(i=1,2,...,l)$ of *m*-generalized *q*-neutrosophic numbers $\tilde{R}_i(i=1,2,...,m)$.

Step-4: Rank all the alternatives $A_i\,(i=1,2,...,l)$ and hence select the most desirable alternative(s).

- **CASE STUDY**:

We consider a multi attribute decision making problem adapted from **[15, 17, 18, 19]** to demonstrate the application of the proposed decision making method.

"Suppose there is an investment company that wants to invest a sum of money in the best option available. There is a panel with four possible alternatives in which to invest the money: (i) A_1 is a car company, (ii) A_2 is a food company, (iii) A_3 is a computer company and (iv) A_4 is an arms company. The investment company must take a decision according to the following attributes:

(1) C_1 is the risk,

(2) C_2 is the growth and

(3) C_3 is the environmental impact.

The attribute weight vector is given as: $w=(0.35, 0.25, 0.40)^T$. The four alternatives $A_i\,(i=1,2,3,4)$ are to be evaluated using the *m*-generalized *q*-neutrosophic information by some decision makers or experts under the attributes $C_j\,(j=1,2,3)$**"**.

Step-1: The rating values of the expert(s) are given in the form of the following decision matrix $\tilde{D}$:

	c_1	c_2	c_3
A_1	<0.3, 0.1, 0.4>	<0.5, 0.3, 0.4>	<0.3, 0.2, 0.6>
A_2	<0.8, 0.2, 0.3>	<0.7, 0.1, 0.3>	<0.7, 0.2 0.2>
A_3	<0.5, 0.4, 0.3>	<0.6, 0.3, 0.4>	<0.5, 0.1, 0.3>
A_4	<0.6, 0.1, 0.2>	<0.7, 0.1, 0.2>	<0.3, 0.2, 0.3>

Step-2: Using the operator $mGqNWAA$, we construct the aggregated *m*-generalized *q*-neutrosophic decision matrix $\tilde{R}$ given below (taking *m*=3 and *q*=3):

A_1	<0.374405104, 0.173657007, 0.470431609>
A_2	<0.741650663, 0.168179283, 0.2550849>
A_3	<0.529784239, 0.213796854, 0.322237098>
A_4	<0.56691263, 0.131950791, 0.235215805>

Step-3: The score values of the alternatives are calculated as:

S(A_1)=0.5767, S(A_2)=0.7727, S(A_3)=0.6645, S(A_4)=0.7332

Step-4: The ranking order of the alternatives are: $A_2 > A_4 > A_3 > A_1$ which coincides with the ranking order determined by Jun Ye **[15, 17, 18, 19]** and hence the most desirable alternative is A_2.

Now if we want to utilize the $mGqNWGA$ operator instead of $mGqNWAA$ operator, then the steps for solving the multi attribute decision making problem are as follows:

Step-1: The rating values of the expert(s) are given in the form of the following decision matrix $\tilde{D}$:

	c_1	c_2	c_3
A_1	<0.3, 0.1, 0.4>	<0.5, 0.3, 0.4>	<0.3, 0.2, 0.6>
A_2	<0.8, 0.2, 0.3>	<0.7, 0.1, 0.3>	<0.7, 0.2 0.2>
A_3	<0.5, 0.4, 0.3>	<0.6, 0.3, 0.4>	<0.5, 0.1, 0.3>
A_4	<0.6, 0.1, 0.2>	<0.7, 0.1, 0.2>	<0.3, 0.2, 0.3>

Step-2: Using the operator $mGqNWGA$, we construct the aggregated *m*-generalized *q*-neutrosophic decision matrix $\tilde{R}$ given below (taking *m*=3 and *q*=3):

A_1	<0.34086581, 0.2179417, 0.504033104>
A_2	<0.73349173, 0.184246871, 0.268903022>
A_3	<0.52331757, 0.310497774, 0.331365356>
A_4	<0.472580665, 0.156129905, 0.250101937>

Step-3: The score values of the alternatives are calculated as:

$S(A_1)=0.5396$, $S(A_2)=0.7601$, $S(A_3)=0.6271$, $S(A_4)=0.6887$

Step-4: The ranking order of the alternatives are: $A_2 > A_4 > A_3 > A_1$ which also coincides with the ranking order determined by Jun Ye **[15, 17, 18, 19]** and hence the most desirable alternative is still A_2.

6. CONCLUSIONS:

In this paper, the notion of *m*-generalized *q*- neutrosophic sets is proposed and the basic properties of *m*-generalized *q*- neutrosophic numbers (*mGqNNs* for short) are presented. Also, various types of operations between the *mGqNNs* are discussed. Then, two types of *m*-generalized *q*- neutrosophic weighted aggregation operators are proposed to aggregate the *m*-generalized *q*- neutrosophic information. Furthermore, score of *a mGqNN* is proposed to ranking the *mGqNNs*. Utilizing the *m*-generalized *q*- neutrosophic weighted aggregation operators and score of *a mGqNN*, a multi attribute decision making method is developed, in which the evaluation values of alternatives on the attribute are represented in terms of *mGqNNs* and the alternatives are ranked according to the values of the score of *mGqNNs* to select the most desirable one. Finally, a practical example for investment decision making is presented to demonstrate the application and effectiveness of the proposed method. The advantage of the proposed method is that it is more suitable for solving multi attribute decision making problems because *m*-generalized *q*-neutrosophic sets (*mGqNS*s) are extensions of fuzzy sets, Pythagorean fuzzy sets, *q*-rung orthopair fuzzy sets, intuitionistic fuzzy sets, single valued neutrosophic sets, *n*-hyperspherical neutrosophic sets and single valued spherical neutrosophic sets.

FUNDING: This research received no external funding.

ACKNOWLEDGEMENTS: Nil.

CONFLICTS OF INTEREST: The authors declare no conflict of interest.

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Received: April 10, 2020. Accepted: July 1, 2020

Neutrosophic Sets and Systems, Vol. 35, 2020

University of New Mexico

HESITANT Triangular Neutrosophic Numbers and Their Applications to MADM

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ABSTRACT: Hesitant neutrosophic sets can accomodate more uncertainty compare to hesitant fuzzy sets and hesitant intuitionistic sets. On the other hand, triangular neutrosophic numbers are often used by the decision makers to evaluate their opinion in multi-attribute group decision making problems. Based on the combination of triangular neutrosophic numbers and hesitant neutrosophic sets, in this paper, we propose hesitant triangular neutrosophic numbers. Also, we discuss various types of operations between them including some properties. Then, we propose various types of hesitant triangular neutrosophic weighted aggregation operators to aggregate the hesitant triangular neutrosophic information. Furthermore, we introduce score of hesitant triangular neutrosophic numbers to ranking the hesitant triangular neutrosophic numbers. Based on the hesitant triangular neutrosophic weighted aggregation operators and score of hesitant triangular neutrosophic numbers, we develop a multi attribute decision making (MADM) approach, in which the evaluation values of alternatives on the attribute are represented in terms of hesitant triangular neutrosophic numbers and the alternatives are ranked according to the values of the score of hesitant triangular neutrosophic numbers to select the most desirable one. Finally, we give a practical example, including a comparision study with the other existing method, for enterprise resource planning system selection to verify the application and effectiveness of the proposed method.

Keywords: Neutrosophic sets**,** hesitant triangular neutrosophic numbers, aggregation operators, score value, decision making.

1. INTRODUCTION

In our real life, most of the mathematical problems do not contain exact or complete information about the given mathematicalmodeling. Therefore, fuzzy set theory by introduced Zadeh **[01]** is a proper tool to process inexact information because it allows the partial belongings of an element in a set with a membership function. Atanassov **[02]** generalized fuzzy sets to intuitionistic fuzzy sets by adding a non-membership function to overcome problems that contain incomplete information. In case of fuzzy sets and intuitionistic fuzzy sets, the membership (or non-membership) value of an element in a set is a unique value in the closed interval [0, 1]. But since 2009, researchers begin to investigate, what if the membership (non-membership) value of an element in a set is a discrete finite subset of [0, 1]. In order to tackle this situation, Torra **[03]** proposed the concept of a hesitant fuzzy set, which as an extension of a fuzzy set arises from our hesitation among a few different values lying between the number 0 and 1. Thus the hesitant fuzzy set can more accurately reflect the people's hesitancy in stating their preferences over objectives compared to the fuzzy set and its classical extensions. Beg and Rashid **[04]** introduced the concept of intuitionistic hesitant fuzzy sets by merging the concept of intuitionistic fuzzy sets and hesitant fuzzy sets.Various researchers have analyzed the decision making problems under fuzzy, hesitant fuzzy, intuitionistic fuzzy and intuitionistic hesitant fuzzy environment in Li **[05]**, Ye **[06]**, Xia and Xu **[07]**, Xu and Xia **[08]**, Wei et al. **[09]**, Xu and Xia **[10]**, Xu and Xia **[11]**, Xu and Zhang **[12]**, Chen et al. **[13]**, Qian et al. **[14]**, Yu **[15]**, Yu **[16]**, Ye **[17]**, Shi et al. **[18]**, Pathinathan and Johnson **[19]**, Joshi and Kumar **[20]**, Liu **[21]**, Nehi **[22]**, Zhang **[23]**, Chen and Huang **[24]**, Yang et al. **[25]**, Lan et al. **[26]** and Zhang et al. **[27]**.

Although intuitionistic fuzzy sets naturally include hesitancy degree to handle uncertain information, it cannot manage indeterminate information properly because it is dependent on memebership and non-membership degrees. To handle this situation, Smarandache **[28]** introduced the neutrosophic set which is basically a powerful general formal framework that generalizes the concept of the classical set, fuzzy set, intuitionistic fuzzy set. A neutrosophic set is characterized explicitly by truth-membership function, indeterminacy-membership function and falsity membership function and it has applications on image segmentation in Gou and Cheng **[29],** Gou and Sensur **[30]**, on clustering analysis in Karaaslan **[31]**, on medical diagnosis problem in Ansari et al. **[32]** etc.The neutrosophic set theory have also studied in Wang et al. **[33]**, Wang et al. **[34]**, Gou et al. **[35]**, Ye **[36]**, Sun et al. **[37]**, Ye **[38]** and Abdel Basset et al. **[39]**. The neutrosophic set cannot represent uncertain, imprecise, incomplete and inconsistent information with a few different values assigned by truth-membership degree, indeterminacy-membership degree and falsity-membership degree due to doubts of decision maker. In such a situation, all the decision making algorithms based on neutrosophic sets are difficult to use for such a decision making problem with three kinds of hesitancy information that exists in the real world. To overcome this situation, Ye **[40]** introduced the concept of hesitant neutrosophic sets which is characterized by three membership degrees, namely-truth membership degrees, indeterminacy membership degrees and falsity membership degrees which is a few different values lying between the number 0 and 1.

Aggregation operators play a vital role in many fields such as decision making, supply chain, personnel evaluation and financial investment to solve multi-criteria group decision making problems. A series of aggregation operatorsin Xia et al. **[41]**, Wang et al. **[42]**, Zhao et al. **[43]**, and Peng **[44]** were developed based on fuzzy and hesitant fuzzy information and those were applied in solving decision-making problems. Xu **[45]**, Wan and Dong **[46]**, Wan et al. **[47]** and Xu and Yager **[48]** presented an averaging and geometric aggregation operators for aggregating the different intuitionistic fuzzy sets based information. Wang and Liu **[49]** proposed some Einstein weighted geometric operators for intuitionistic fuzzy sets. Liu et al. **[50]** proposed some generalized neutrosophic number Hamacher aggregation operators. Liu and Wang **[51]** defined few neutrosophic normalized, weighted Bonferroni mean operators.Chen and Ye **[52]** used single-valued neutrosophic dombi weighted aggregation operators for solving a multiple attribute decision-making problem. Some more aggregation operators on neutrosophic environment can be found in Zhao et al. **[53]**, Liu and Shi **[54]** and Liu and Tang **[55]**.

Since Smarandache put forward the concept of neutrosophic sets, the neutrosophic number is given by Şubaş **[56]** subsequently, and it has been made much deeper by many authors in Abdel-Basset **[57]**. As a special neutrosophic number,Şubaş gave two special forms of single valued neutrosophic numbers such as single valued trapezoidal neutrosophic numbers and single valued triangular neutrosophic numbers on the real number set R. Now the theory of neutrsphic number has become the fundamental of neutrosophic decision making. For example; Deli and Şubaş **[58]** introduced the concepts of cut sets of neutrosophic numbers and also they applied to single valued trapezoidal neutrosophic numbers and triangular neutrosophic numbers. Finally they presented a ranking method by defining the values and ambiguities ofneutrosophic numbers. Also, by using the value and ambiguity index, Biswas et al. **[59]** presented a multi-attribute decision making method. Broumi et al. **[60]** gave an application shortest path problem under triangular fuzzy neutrosophic numbers. Deli and Şubaş **[61]** developed an approach to handle multicriteriadecision making problems under the single valued triangular neutrosophic numbers. Also, they presented some new geometric operators including weighted geometric operator, ordered weighted geometric operator and ordered hybrid weighted geometric operator. Ye **[62]**, Biswas et al. **[63]** and Deli **[64]** proposed some weighted arithmetic operators and weighted geometric operators to present some multi attribute decision making methods. Karaaslan **[65]** introduced Gaussiansingle valued neutrosophic numbers and applied to a multi attribute decision making. Öztürk **[66]** and Deli and Öztürk **[67, 68]** initiated concept of distance measure based on cut sets, magnitude function, 1. and 2. centroid point and 1. and 2. score function. Deli **[69]** defined concept of centroid point based on single valued trapezoidal neutrosophic numbers and examine several useful properties. Also, he developed hamming ranking value and Euclidean ranking value of single valued trapezoidal neutrosophic numbers. Chakraborty et al. **[70]** presented a decision making method by introducing different forms of triangular neutrosophic numbers including de-neutrosophication techniques. Fan et al. **[71]** defined linguistic neutrosophic number Einstein sum, linguistic neutrosophic number Einstein product, and linguistic neutrosophic number Einstein exponentiation operations based on the Einstein operation and used them to develop some MADM problems. Garg and Nancy **[72]** introduced some linguistic single valued neutrosophic power aggregation operators and presented their applications to group decision making process. Zhao et al. **[73]** developed induced choquet integral aggregation operators with single valued neutrosophic uncertain linguistic numbers. Recently, Deli and Karaaslan **[74]** defined generalized trapezoidal hesitant fuzzy numbers and Deli **[75]** presented a TOPSIS method formulti-criteria decision making problems by using the numbers. Some more trapezoidal/triangular hesitant fuzzy numbers can be found in Zhang et al. **[76]** and Ye **[77]**.

Motivated by the idea of triangular neutrosophic number, hesitant neutrosophic set and aggregation operators, the aim of this present article is:

(1) To present the idea of hesitant triangular neutrosophic numbers.

(2) To define few operations between hesitant triangular neutrosophic numbers and study their basic properties.

(3) To develop a few weighted aggregation operators such as hesitant triangular neutrosophic weighted arithmetic aggregation operator of type-1, hesitant triangular neutrosophic weighted arithmetic aggregation operator of type-2, hesitant triangular neutrosophic weighted geometric aggregation operator of type-1 and hesitant triangular neutrosophic weighted geometric aggregation operator of type-2.

(4) To propose a decision making method based on the hesitant triangular neutrosophic weighted aggregation operators to handle multicriteria decision making problems with hesitant triangular neutrosophic information.

To do so, the rest of the article is arranged as follows:

In section 2, we review some basic concepts. In Section 3, we propose hesitant triangular neutrosophic number and illustrate it with an example. Also, we discuss various types of operations between them including some properties. In section 4, we propose various types of hesitant triangular neutrosophic weighted aggregation operators to aggregate the hesitant triangular neutrosophic information. Furthermore, we introduce the score of a hesitant triangular neutrosophic number to ranking the hesitant triangular neutrosophic numbers. In section 5, based on the hesitant triangular neutrosophic weighted aggregation operators and score of hesitant triangular neutrosophic numbers, we develop a multi attribute decision making approach, in which the evaluation values of alternatives on the attribute are represented in terms of hesitant triangular neutrosophic numbers and the alternatives are ranked according to the values of the score of hesitant triangular neutrosophic numbers to select the best (most desirable) one. Also, we present a practical example for enterprise resource planning system selection to demonstrate the application and effectiveness of the proposed method. Section 6 is devoted for comparative study. In final section, we present the conclusion of the study.

2. PRELIMINARIES:

A neutrosophic set is a part of neutrosophy which studies the origin, nature and scope of neutralities as well as their interactions with different ideational spectra and is a powerful general formal framework that generalizes the traditional mathematical tools such as fuzzy sets and intuitionistic fuzzy sets.

Definition 1: [34] A single-valued neutrosophic set A on universe set E is given by

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in E\}$$

where $T_A: E \to [0,1]$, $I_A: E \to [0,1]$, and $F_A: E \to [0,1]$ satisfy the condition $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$, for every $x \in E$. The functions T_A, I_A, and F_A define the degree of truth-membership function, indeterminacy-membership function and falsity-membership function, respectively.

Definition 2: [52] $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in E\}$ and $B = \{\langle x, T_B(x), I_B(x), F_B(x)\rangle : x \in E\}$ be two single-valued neutrosophic sets and $\lambda \neq 0$. Then,

1. $$A+B=\{< x, 1-\frac{1}{1+\left\{\left(\frac{T_A(x)}{1-T_A(x)}\right)^p+\left(\frac{T_B(x)}{1-T_B(x)}\right)^p\right\}^{\frac{1}{p}}}, \frac{1}{1+\left\{\left(\frac{1-I_A(x)}{I_A(x)}\right)^p+\left(\frac{1-I_B(x)}{I_B(x)}\right)^p\right\}^{\frac{1}{p}}}, \frac{1}{1+\left\{\left(\frac{1-F_A(x)}{F_A(x)}\right)^p+\left(\frac{1-F_B(x)}{F_B(x)}\right)^p\right\}^{\frac{1}{p}}} >: x \in E\}$$

2. $$A\times B=\{< x, \frac{1}{1+\left\{\left(\frac{1-T_A(x)}{T_A(x)}\right)^p+\left(\frac{1-T_B(x)}{T_B(x)}\right)^p\right\}^{\frac{1}{p}}}, 1-\frac{1}{1+\left\{\left(\frac{I_A(x)}{1-I_A(x)}\right)^p+\left(\frac{I_B(x)}{1-I_B(x)}\right)^p\right\}^{\frac{1}{p}}}, 1-\frac{1}{1+\left\{\left(\frac{F_A(x)}{1-F_A(x)}\right)^p+\left(\frac{F_B(x)}{1-F_B(x)}\right)^p\right\}^{\frac{1}{p}}} >: x \in E\}$$

3. $$\lambda.A=\{< x, 1-\frac{1}{1+\left\{\lambda\left(\frac{T_A(x)}{1-T_A(x)}\right)^p\right\}^{\frac{1}{p}}}, \frac{1}{1+\left\{\lambda\left(\frac{1-I_A(x)}{I_A(x)}\right)^p\right\}^{\frac{1}{p}}}, \frac{1}{1+\left\{\lambda\left(\frac{1-F_A(x)}{F_A(x)}\right)^p\right\}^{\frac{1}{p}}} >: x \in E\}$$

4. $A^{\lambda}=\{<x,\dfrac{1}{1+\left\{\lambda\left(\dfrac{1-T_A(x)}{T_A(x)}\right)^p\right\}^{\frac{1}{p}}},\ 1-\dfrac{1}{1+\left\{\lambda\left(\dfrac{I_A(x)}{1-I_A(x)}\right)^p\right\}^{\frac{1}{p}}},\ 1-\dfrac{1}{1+\left\{\lambda\left(\dfrac{F_A(x)}{1-F_A(x)}\right)^p\right\}^{\frac{1}{p}}}>: x\in E\}$

By combining single-valued neutrosophic sets and hesitant fuzzy sets, Ye (2015a) introduced the single-valued neutrosophic hesitant fuzzy set as a further generalization of the concepts of fuzzy set, intuitionistic fuzzy set, single-valued neutrosophic set. He also developed single-valued neutrosophic hesitant fuzzy weighted averaging operator and single-valued neutrosophic hesitant fuzzy weighted geometric operator and applied them to solve a multiple-attribute decision- making problem.

Definition 3: **[40]** A hesitant neutrosophicset on universe set E is given by

$$N=\{\langle x,\tilde{T}_N(x),\tilde{I}_N(x),\tilde{F}_N(x)\rangle : x\in E\}$$

in which $\tilde{T}_N(x),\tilde{I}_N(x)$ and $\tilde{F}_N(x)$ are three sets of some values in [0,1], denoting the possible truth-membership hesitant degrees, indeterminacy-membership hesitant degrees, and falsity-membership hesitant degrees of the element $x\in E$ to the set N, respectively, with the conditions $0\le \delta\ ,\gamma\ ,\eta\ \le 1$ and $0\le \delta^{+}+\gamma^{+}+\eta^{+}\le 3$, where
$\delta\in\tilde{T}_N(x)$, $\gamma\in\tilde{I}_N(x)$, $\eta\ \in\tilde{F}_N(x)$, $\delta^{+}\in\tilde{T}_N^{+}(x)=\cup_{\ \delta\in\tilde{T}_N(x)}\max\{\delta\}$, $\gamma^{+}\in$
$\tilde{I}_N^{+}(x)=\cup_{\gamma\in\tilde{I}_N(x)}\max\{\gamma\}$, and $\eta^{+}\ \in\tilde{F}_N^{+}(x)=\cup_{\eta\ \in\tilde{F}_N(x)}\max\{\eta\ \}$, for $x\in E$.

For $N_1=\{\langle x,\tilde{T}_{N_1}(x),\tilde{I}_{N_1}(x),\tilde{F}_{N_1}(x)\rangle : x\in E\}$ and $N_2=\{\langle x,\tilde{T}_{N_2}(x),\tilde{I}_{N_2}(x),\tilde{F}_{N_2}(x)\rangle : x\in E\}$ be two hesitant neutrosophicsets and $\lambda\neq 0$. Then,

1. $N_1\oplus N_1=\{<x,\tilde{T}_{N_1}(x)\oplus\tilde{T}_{N_2}(x),\tilde{I}_{N_1}(x)\oplus\tilde{I}_{N_2}(x),\tilde{F}_{N_1}(x)\oplus\tilde{F}_{N_2}(x)>: x\in X\}$
$=\bigcup_{\delta_1\in\tilde{T}_{N_1}(x),\delta_2\in\tilde{T}_{N_2}(x),\gamma_1\in\tilde{I}_{N_1}(x),\gamma_2\in\tilde{I}_{N_2}(x),\eta_1\in\tilde{F}_{N_1}(x),\eta_2\in\tilde{F}_{N_2}(x)}\{<x,\{\delta_1+\delta_2-\delta_1.\delta_2\},\{\gamma_1.\gamma_2\},\{\eta_1\eta_2\}>: x\in X\}$

2. $N_1\otimes N_1=\{<x,\tilde{T}_{N_1}(x)\otimes\tilde{T}_{N_2}(x),\tilde{I}_{N_1}(x)\otimes\tilde{I}_{N_2}(x),\tilde{F}_{N_1}(x)\otimes\tilde{F}_{N_2}(x)>: x\in X\}$
$=\bigcup_{\delta_1\in\tilde{T}_{N_1}(x),\delta_2\in\tilde{T}_{N_2}(x),\gamma_1\in\tilde{I}_{N_1}(x),\gamma_2\in\tilde{I}_{N_2}(x),\eta_1\in\tilde{F}_{N_1}(x),\eta_2\in\tilde{F}_{N_2}(x)}\{<x,\{\delta_1.\delta_2\},\{\gamma_1+\gamma_2-\gamma_1.\gamma_2\},\{\eta_1+\eta_2-\eta_1\eta_2\}>: x\in X\}$

3. $\lambda.N_1=\bigcup_{\delta_1\in\tilde{T}_{N_1}(x),\gamma_1\in\tilde{I}_{N_1}(x),\eta_1\in\tilde{F}_{N_1}(x)}\{<x,\{1-(1-\delta_1)^{\lambda}\},\{\gamma_1^{\ \lambda}\},\{\eta_1^{\ \lambda}\}>: x\in X\}(\lambda>0)$

4. $N_1^{\ \lambda}=\bigcup_{\delta_1\in\tilde{T}_{N_1}(x),\gamma_1\in\tilde{I}_{N_1}(x),\eta_1\in\tilde{F}_{N_1}(x)}\{<x,\{\delta_1^{\ \lambda}\},\{1-(1-\gamma_1)^{\lambda}\},\{1-(1-\eta_1)^{\lambda}\}>: x\in X\}(\lambda>0)$

Definition 4: [56] Let $a_1\le b_1\le c_1$ such that $a_1,b_1,c_1\in R$. A triangular neutrosophic number $\tilde{A}=\langle(a_1,b_1,c_1);w_{\tilde{A}},u_{\tilde{A}},y_{\tilde{A}}\rangle$ is a special neutrosophic set on the real number set R, whose truth-membership function $\mu_{\tilde{A}}:R\to[0,w_{\tilde{A}}]$, indeterminacy-membership function $\nu_{\tilde{A}}:R\to[u_{\tilde{A}},1]$ and falsity-membership function $\lambda_{\tilde{A}}:R\to[y_{\tilde{A}},1]$ are given as follows;

$$\mu_{\tilde{A}}(x)=\begin{cases}\dfrac{(x-a_1)w_{\tilde{A}}}{b_1-a_1}, & a_1\le x\le b_1\\ \dfrac{(c_1-x)w_{\tilde{A}}}{c_1-b_1}, & b_1\le x\le c_1\\ 0, & otherwise\end{cases}\quad,\quad \nu_{\tilde{A}}(x)=\begin{cases}\dfrac{b_1-x+u_{\tilde{A}}(x-a_1)}{b_1-a_1}, & a_1\le x\le b_1\\ \dfrac{x-b_1+u_{\tilde{A}}(c_1-x)}{c_1-b_1}, & b_1\le x\le c_1\\ 1, & otherwise\end{cases}$$

$$\lambda_{\tilde{A}}(x)=\begin{cases}\dfrac{b_1-x+y_{\tilde{A}}(x-a_1)}{b_1-a_1}, & a_1\le x\le b_1\\ \dfrac{x-b_1+y_{\tilde{A}}(c_1-x)}{c_1-b_1}, & b_1\le x\le c_1\\ 1, & otherwise\end{cases}$$

Since triangular neutrosophic numbers (**[56]**, **[58]**) is a special case of trapozidial neutrosophic numbers (Ye 2017), operations of trapozidial neutrosophic numbers (Ye 2015b, 2017) based on algebraic sum and algebraic product for triangular neutrosophic numbers can be given as;

If $\tilde{A}=\langle(a_1,b_1,c_1);w_{\tilde{A}},u_{\tilde{A}},y_{\tilde{A}}\rangle$ and $\tilde{B}=\langle(a_2,b_2,c_2);w_{\tilde{B}},u_{\tilde{B}},y_{\tilde{B}}\rangle$ be two triangular neutrosophic numbers and $\gamma\neq 0$, then we have

1. $\tilde{A}+\tilde{B}=\langle(a_1+a_1,b_1+b_2,c_1+c_2);w_{\tilde{A}}+w_{\tilde{B}}-w_{\tilde{A}}.w_{\tilde{B}},u_{\tilde{A}}.u_{\tilde{B}},y_{\tilde{A}}.y_{\tilde{B}}\rangle$
2. $\tilde{A}.\tilde{B}=\langle(a_1a_2,b_1b_2,c_1c_2);w_{\tilde{A}}.w_{\tilde{B}},u_{\tilde{A}}+u_{\tilde{B}}-u_{\tilde{A}}.u_{\tilde{B}},y_{\tilde{A}}+y_{\tilde{B}}-y_{\tilde{A}}.y_{\tilde{B}}\rangle$
3. $\lambda\tilde{A}=\langle\ (\gamma a_1,\gamma b_1,\gamma c_1);1-(1-w_{\tilde{A}})^{\lambda},u_{\tilde{A}}{}^{\lambda},y_{\tilde{A}}{}^{\lambda}\rangle$
4. $\tilde{A}^{\lambda}=\langle\ (a_1{}^{\lambda},b_1{}^{\lambda},c_1{}^{\lambda});w_{\tilde{A}}{}^{\lambda},1-(1-u_{\tilde{A}})^{\lambda},1-(1-y_{\tilde{A}})^{\lambda}\rangle$

Definition 5: [56] Let $\tilde{A}=\langle(a,b,c);w_{\tilde{A}},u_{\tilde{A}},y_{\tilde{A}}\rangle$ be atriangular neutrosophic number. Then,score function of $\tilde{A}$, is denoted by $S_Y(\tilde{A})$, is defined as:

$$S_Y(\tilde{A})=\frac{1}{8}[a+b+c]\times(2+\mu_{\tilde{A}}-\nu_{\tilde{A}}-\gamma_{\tilde{A}})$$

Definition 6: [61] Let $\tilde{A}_j=\langle(a_j,b_j,c_j,);w_{\tilde{A}_j},u_{\tilde{A}_j},y_{\tilde{A}_j}\rangle$ $(j=1,2,\dots,n)$ be a collection of triangular neutrosophic numbers. Then,

1. Triangular neutrosophic weighted arithmetic operatoris defined as;

$$N_{ao}(\tilde{A}_1,\tilde{A}_2,\dots,\tilde{A}_n)=\sum_{j=1}^{n}w_j\tilde{A}_j$$

2. Triangular neutrosophic weighted geometric operatoris defined as;

$$N_{go}(\tilde{A}_1,\tilde{A}_2,\dots,\tilde{A}_n)=\prod_{j=1}^{n}\tilde{A}_j{}^{w_j}$$

where, $w=(w_1,w_2,\dots,w_n)^T$ is a weight vector associated with the N_{ao} or N_{go} operator, for every j $(j=1,2,\dots,n)$ and $w_j\in[0,1]$with$\sum_{j=1}^{n}w_j=1$.

3. HESITANT TRIANGULAR NEUTROSOPHIC NUMBERS:

In this section, the concept of a hesitant triangular neutrosophic number is presented on the basis of the combination of triangular neutrosophic numbers and hesitant fuzzy sets as a further generalization of the concep ttriangular neutrosophic numbers. A hesitant triangular neutrosophic number is a special hesitant neutrosophic set on the real number R, whose truth-membership function, indeterminacy-membership function and falsity-membership function are expressed by several possible functions.

Definition 7. Let $a_1\le b_1\le c_1$ such that $a_1,b_1,c_1\in R$, $w^i_{\tilde{a}}\in[0,1](i\in I_m=\{1,2,...,m\})$, $u^i_{\tilde{a}}\in[0,1](i\in I_n=\{1,2,...,n\})$ and $y^i_{\tilde{a}}\in[0,1](i\in I_k=\{1,2,...,k\})$. A hesitant triangular neutrosophic number $\tilde{a}=<(a_1,b_1,c_1);\{w^i_{\tilde{a}}:i\in I_m\},\{u^j_{\tilde{a}}:j\in I_n\},\{y^l_{\tilde{a}}:l\in I_k\}>$ is a special hesitant neutrosophic set on the real number R, whose truth-membership functions $\mu^{HTRI}_{\tilde{a}}:R\to[0,w^i_{\tilde{a}}](i\in I_m)$, indeterminacy-membership function $\gamma^{HTRI}_{\tilde{a}}:R\to[0,u^j_{\tilde{a}}](j\in I_n)$ and falsity-membership function $\delta^{HTRI}_{\tilde{a}}:R\to[0,y^l_{\tilde{a}}](l\in I_k)$ are given as follows;

$$\mu_{\tilde{a}}^{HTRI}=\begin{cases}\bigcup_{\{\alpha:\alpha\in\{w_{\tilde{a}}^{i}:i\in I_m\}\}}\{\frac{(x-a_1)\alpha}{b_1-a_1}\}, & a_1\le x<b_1\\ \{\alpha:\alpha\in\{w_{\tilde{a}}^{i}:i\in I_m\}\}, & x=b_1\\ \bigcup_{\{\alpha:\alpha\in\{w_{\tilde{a}}^{i}:i\in I_m\}\}}\{\frac{(c_1-x)\alpha}{c_1-b_1}\}, & b_1<x\le c_1\\ \{0\}, & \text{otherwise}\end{cases},$$

$$\gamma_{\tilde{a}}^{HTRI}=\begin{cases}\bigcup_{\{\beta:\beta\in\{u_{\tilde{a}}^{j}:j\in I_n\}\}}\{\frac{b_1-x+\beta(x-a_1)}{b_1-a_1}\}, & a_1\le x<b_1\\ \{\beta:\beta\in\{u_{\tilde{a}}^{j}:j\in I_n\}\}, & x=b_1\\ \bigcup_{\{\beta:\beta\in\{u_{\tilde{a}}^{j}:j\in I_n\}\}}\{\frac{x-b_1+\beta(c_1-x)}{c_1-b_1}\}, & b_1<x\le c_1\\ \{1\}, & \text{otherwise}\end{cases}$$

$$\delta_{\tilde{a}}^{HTRI}=\begin{cases}\bigcup_{\{\lambda:\lambda\in\{y_{\tilde{a}}^{l}:l\in I_k\}\}}\{\frac{b_1-x+\lambda(x-a_1)}{b_1-a_1}\}, & a_1\le x<b_1\\ \{\lambda:\lambda\in\{y_{\tilde{a}}^{l}:l\in I_k\}\}, & x=b_1\\ \bigcup_{\{\lambda:\lambda\in\{y_{\tilde{a}}^{l}:l\in I_k\}\}}\{\frac{x-b_1+\lambda(c_1-x)}{c_1-b_1}\}, & b_1<x\le c_1\\ \{1\}, & \text{otherwise}\end{cases}$$

Example 8. $\tilde{a}=<(1,2,5);\{0.8,0.9\},\{0.4,0.5,0.6\},\{0.4\}>$ is a hesitant triangular neutrosophic number whose truth membership function, indeterminacy membership function and falsity membership functionare given respectively by:

$$\mu_{\tilde{a}}^{HTRI}(x)=\begin{cases}\{0.8(x-1),0.9(x-1)\}, & 1\le x<2\\ \{0.8,0.9\}, & x=2\\ \{0.8\frac{(5-x)}{3},0.9\frac{(5-x)}{3}\}, & 2<x\le 5\\ \{0\}, & \text{otherwise}\end{cases},\qquad \delta_{\tilde{a}}^{HTRI}(x)=\begin{cases}\{1.6-0.6x\}, & 1\le x<2\\ \{0.4\}, & x=2\\ \{0.2x\}, & 2<x\le 3\\ \{1\}, & \text{otherwise}\end{cases}$$

$$\gamma_{\tilde{a}}^{HTRI}(x)=\begin{cases}\{1.6-0.6x,1.5-0.5x,1.4-0.4x\}, & 1\le x<2\\ \{0.4,0.5,0.6\}, & x=2\\ \{\frac{0.6x}{3},0.\frac{5x+0.5}{3},\frac{0.4x+1}{3}\}, & 2<x\le 5\\ \{1\}, & \text{otherwise}\end{cases}$$

4. OPERATIONS ON HESITANT TRIANGULAR NEUTROSOPHIC NUMBERS:

In this section, we introduce various operations between hesitant triangular neutrosophic numbers and demonstrate their basic properties.

Definition 9. Let $\tilde{a}=<(a_1,b_1,c_1);\{w_{\tilde{a}}^{i}:i\in I_{m_1}\},\{u_{\tilde{a}}^{j}:j\in I_{n_1}\},\{y_{\tilde{a}}^{l}:l\in I_{k_1}\}>$ and $\tilde{b}=<(a_2,b_2,c_2);\{w_{\tilde{b}}^{i}:i\in I_{m_2}\},\{u_{\tilde{b}}^{j}:j\in I_{n_2}\},\{y_{\tilde{b}}^{l}:l\in I_{k_2}\}>$ be two hesitant triangular neutrosophic numbers and $\sigma>0$, then

1. $\tilde{a}\oplus'\tilde{b}=<(a_1+a_2,b_1+b_2,c_1+c_2);\{\alpha_1+\alpha_2-\alpha_1\alpha_2:\alpha_1\in\{w_{\tilde{a}}^{i}:i\in I_{m_1}\},\alpha_2\in\{w_{\tilde{b}}^{i}:i\in I_{m_2}\}\},$

$$\{\beta_1\beta_2:\beta_1\in\{u_{\tilde{a}}^{j}:j\in I_{n_1}\},\beta_2\in\{u_{\tilde{b}}^{j}:j\in I_{n_2}\}\},\ \{\lambda_1\lambda_2:\lambda_1\in\{y_{\tilde{a}}^{l}:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde{b}}^{l}:l\in I_{k_2}\}\}>$$

2. $a \otimes' \tilde{b} =< (a_1a_2, b_1b_2, c_1c_2); \{\alpha_1\alpha_2 : \alpha_1 \in \{w_{\tilde{a}}^i : i \in I_{m_1}\}, \alpha_2 \in \{w_{\tilde{b}}^i : i \in I_{m_2}\}\}, \{\beta_1 + \beta_2 - \beta_1\beta_2 :$

$\beta_1 \in \{u_{\tilde{a}}^j : j \in I_{n_1}\}, \beta_2 \in \{u_{\tilde{b}}^j : j \in I_{n_2}\}\}, \{\lambda_1 + \lambda_2 - \lambda_1\lambda_2 : \lambda_1 \in \{y_{\tilde{a}}^l : l \in I_{k_1}\}, \lambda_2 \in \{y_{\tilde{b}}^l : l \in I_{k_2}\}\} >$

3. $\sigma \odot' \tilde{a} =< (\sigma a, \sigma b, \sigma c); \{1-(1-\alpha)^\sigma : \alpha \in \{w_{\tilde{a}}^i : i \in I_{m_1}\}\}, \{\beta^\sigma : \beta \in \{u_{\tilde{a}}^j : j \in I_{n_1}\}\}, \{\lambda^\sigma : \lambda \in \{y_{\tilde{a}}^l : l \in I_{k_1}\}\} >$

4. $\sigma *' \tilde{a} =< (a^\sigma, b^\sigma, c^\sigma); \{\alpha^\sigma : \alpha \in \{w_{\tilde{a}}^i : i \in I_{m_1}\}\}, \{1-(1-\beta)^\sigma : \beta \in \{u_{\tilde{a}}^j : j \in I_{n_1}\}\},$

$\{1-(1-\lambda)^\sigma : \lambda \in \{y_{\tilde{a}}^l : l \in I_{k_1}\}\} >$

Theorem 10. Let $\tilde{a} =< (a_1, b_1, c_1); \{w_{\tilde{a}}^i : i \in I_{m_1}\}, \{u_{\tilde{a}}^j : j \in I_{n_1}\}, \{y_{\tilde{a}}^l : l \in I_{k_1}\} >$, $\tilde{b} =< (a_2, b_2, c_2); \{w_{\tilde{b}}^i : i \in I_{m_2}\}, \{u_{\tilde{b}}^j : j \in I_{n_2}\}, \{y_{\tilde{b}}^l : l \in I_{k_2}\} >$ and $\tilde{c} =< (a_3, b_3, c_3); \{w_{\tilde{c}}^i : i \in I_{m_3}\}, \{u_{\tilde{c}}^j : j \in I_{n_3}\}, \{y_{\tilde{c}}^l : l \in I_{k_3}\} >$ be three hesitant triangular neutrosophic numbers and $\sigma, \sigma_1, \sigma_2 > 0$, then

1. $\tilde{a} \oplus' \tilde{b} = \tilde{b} \oplus' \tilde{a}$
2. $\tilde{a} \otimes' \tilde{b} = \tilde{b} \otimes' \tilde{a}$
3. $.\tilde{a} \oplus' (\tilde{b} \oplus' \tilde{c}) = (\tilde{a} \oplus' \tilde{b}) \oplus' \tilde{c}$
4. $\tilde{a} \otimes' (\tilde{b} \otimes' \tilde{c}) = (\tilde{a} \otimes' \tilde{b}) \otimes' \tilde{c}$
5. $\sigma \odot' (\tilde{a} \oplus' \tilde{b}) = (\sigma \odot' \tilde{a}) \oplus' (\sigma \odot' \tilde{b})$
6. $\sigma *' (\tilde{a} \otimes' \tilde{b}) = (\sigma *' \tilde{a}) \otimes' (\sigma *' \tilde{b})$
7. $\sigma *' (\tilde{a} \otimes' \tilde{b}) = (\sigma *' \tilde{a}) \otimes' (\sigma *' \tilde{b})$
8. $(\sigma_1 + \sigma_2) *' \tilde{a} = (\sigma_1 *' \tilde{a}) \otimes' (\sigma_2 *' \tilde{a})$

Proof: 1-2 straight forward.

3. $\tilde{a} \oplus' (\tilde{b} \oplus' \tilde{c}) =< (a_1, b_1, c_1); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} > \oplus' < a_2 + a_3, b_2 + b_3, c_2 + c_3 \;; \{\alpha_2 + \alpha_3 - \alpha_2\alpha_3 : \alpha_2 \in \{w_{\tilde{b}}^i : i \in I_{m_2}\},$

$$\begin{aligned}
&\alpha_3 \in \{w_{\tilde{c}}^i : i \in I_{m_3}\}\}, \{\beta_2\beta_3 : \beta_2 \in \{u_{\tilde{b}}^j : j \in I_{n_2}\}, \beta_3 \in \{u_{\tilde{c}}^j : j \in I_{n_3}\}\}, \{\lambda_2\lambda_3 : \lambda_2 \in \{y_{\tilde{b}}^l : l \in I_{k_2}\},\\
&\lambda_3 \in \{y_{\tilde{c}}^l : l \in I_{k_3}\}\} >\\
&=< (a_1 + (a_2 + a_3), b_1 + (b_2 + b_3), c_1 + (c_2 + c_3)); \{\alpha_1 + (\alpha_2 + \alpha_3 - \alpha_2\alpha_3) - \alpha_1(\alpha_2 + \alpha_3 - \alpha_2\alpha_{31}) :\\
&\alpha_1 \in \{w_{\tilde{a}}^i : i \in I_{m_1}\}, \alpha_2 \in \{w_{\tilde{b}}^i : i \in I_{m_2}\}, \alpha_3 \in \{w_{\tilde{c}}^i : i \in I_{m_3}\}\}, \{\beta_1(\beta_2\beta_3) : \beta_1 \in \{u_{\tilde{a}}^j : j \in I_{n_1}\}, \beta_2 \in \{u_{\tilde{b}}^j :\\
&j \in I_{n_2}\}, \beta_3 \in \{u_{\tilde{c}}^j : j \in I_{n_3}\}\}, \{\lambda_1(\lambda_2\lambda_3) : \lambda_1 \in \{y_{\tilde{a}}^l : l \in I_{k_1}\}, \lambda_2 \in \{y_{\tilde{b}}^l : l \in I_{k_2}\}, \lambda_3 \in \{y_{\tilde{c}}^l : l \in I_{k_3}\}\} >\\
&=< (a_1 + a_2 + a_3, b_1 + b_2 + b_3, c_1 + c_2 + c_3); \{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_2\alpha_3 - \alpha_1\alpha_2 + \alpha_1\alpha_3 - \alpha_1\alpha_2\alpha_3) :\\
&\alpha_1 \in \{w_{\tilde{a}}^i : i \in I_{m_1}\}, \alpha_2 \in \{w_{\tilde{b}}^i : i \in I_{m_2}\}, \alpha_3 \in \{w_{\tilde{c}}^i : i \in I_{m_3}\}\}, \{\beta_1\beta_2\beta_3 : \beta_1 \in \{u_{\tilde{a}}^j : j \in I_{n_1}\},\\
&\beta_2 \in \{u_{\tilde{b}}^j : j \in I_{n_2}\}, \beta_3 \in \{u_{\tilde{c}}^j : j \in I_{n_3}\}\}, \{\lambda_1\lambda_2\lambda_3 : \lambda_1 \in \{y_{\tilde{a}}^l : l \in I_{k_1}\}, \lambda_2 \in \{y_{\tilde{b}}^l : l \in I_{k_2}\},\\
&\lambda_3 \in \{y_{\tilde{c}}^l : l \in I_{k_3}\}\} > \qquad (1)
\end{aligned}$$

and

$$\begin{aligned}
(\tilde{a} \oplus' \tilde{b}) \oplus' \tilde{c} &=< (a_1 + a_2, b_1 + b_2, c_1 + c_2); \{\alpha_1 + \alpha_2 - \alpha_1\alpha_2 : \alpha_1 \in \{w_{\tilde{a}}^i : i \in I_{m_1}\}, \alpha_2 \in \{w_{\tilde{b}}^i : i \in I_{m_2}\}\},\\
&\{\beta_1\beta_2 : \beta_1 \in \{u_{\tilde{a}}^j : j \in I_{n_1}\}, \beta_2 \in \{u_{\tilde{b}}^j : j \in I_{n_2}\}\}, \{\lambda_1\lambda_2 : \lambda_1 \in \{y_{\tilde{a}}^l : l \in I_{k_1}\}, \lambda_2 \in \{y_{\tilde{b}}^l : l \in I_{k_2}\}\} >\\
&\oplus' < (a_3, b_3, c_3); \{w_{\tilde{c}}^i : i \in I_{m_3}\}, \{u_{\tilde{c}}^j : j \in I_{n_3}\}, \{y_{\tilde{c}}^l : l \in I_{k_3}\} >\\
&=< ((a_1 + a_2) + a_3, (b_1 + b_2) + b_3, (c_1 + c_2) + c_3); \{(\alpha_1 + \alpha_2 - \alpha_1\alpha_2) + \alpha_3 - (\alpha_1 + \beta_1 - \alpha_1\beta_1)\alpha_3 :\\
&\alpha_1 \in \{w_{\tilde{a}}^i : i \in I_{m_1}\}, \alpha_2 \in \{w_{\tilde{b}}^i : i \in I_{m_2}\}, \alpha_3 \in \{w_{\tilde{c}}^i : i \in I_{m_3}\}\}, \{(\beta_1\beta_2)\beta_3 : \beta_1 \in \{u_{\tilde{a}}^j : j \in I_{n_1}\}, \beta_2 \in \{u_{\tilde{b}}^j :\\
&j \in I_{n_2}\} \beta_3 \in \{u_{\tilde{c}}^j : j \in I_{n_3}\}\}, \{(\lambda_1\lambda_2)\lambda_3 : \lambda_1 \in \{y_{\tilde{a}}^l : l \in I_{k_1}\}, \lambda_2 \in \{y_{\tilde{b}}^l : l \in I_{k_2}\}, \lambda_3 \in \{y_{\tilde{c}}^l : l \in I_{k_3}\}\} >\\
&=< (a_1 + a_2 + a_3, b_1 + b_2 + b_3, c_1 + c_2 + c_3); \{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_2\alpha_3 - \alpha_1\alpha_2 + \alpha_1\alpha_3 - \alpha_1\alpha_2\alpha_3) :\\
&\alpha_1 \in \{w_{\tilde{a}}^i : i \in I_{m_1}\}, \alpha_2 \in \{w_{\tilde{b}}^i : i \in I_{m_2}\}, \alpha_3 \in \{w_{\tilde{c}}^i : i \in I_{m_3}\}\}, \{\beta_1\beta_2\beta_3 : \beta_1 \in \{u_{\tilde{a}}^j : j \in I_{n_1}\},\\
&\beta_2 \in \{u_{\tilde{b}}^j : j \in I_{n_2}\}, \beta_3 \in \{u_{\tilde{c}}^j : j \in I_{n_3}\}\}, \{\lambda_1\lambda_2\lambda_3 : \lambda_1 \in \{y_{\tilde{a}}^l : l \in I_{k_1}\}, \lambda_2 \in \{y_{\tilde{b}}^l : l \in I_{k_2}\},\\
&\lambda_3 \in \{y_{\tilde{c}}^l : l \in I_{k_3}\}\} > \qquad (2)
\end{aligned}$$

Hence from eq. 1-2, we have, $\tilde{a} \oplus' (\tilde{b} \oplus' \tilde{c}) = (\tilde{a} \oplus' \tilde{b}) \oplus' \tilde{c}$.

4. Proof is similar to 3

5. $\sigma \odot' (\tilde{a} \oplus' \tilde{b}) = \sigma \odot' < (a_1+a_2, b_1+b_2, c_1+c_2); \{\alpha_1+\alpha_2-\alpha_1\alpha_2 : \alpha_1 \in \{w_{\tilde{a}}^i : i \in I_{m_1}\}, \alpha_2 \in \{w_{\tilde{b}}^i : i \in I_{m_2}\}\},$
$\{\beta_1\beta_2 : \beta_1 \in \{u_{\tilde{a}}^j : j \in I_{n_1}\}, \beta_2 \in \{u_{\tilde{b}}^j : j \in I_{n_2}\}\}, \{\lambda_1\lambda_2 : \lambda_1 \in \{y_{\tilde{a}}^l : l \in I_{k_1}\}, \lambda_2 \in \{y_{\tilde{b}}^l : l \in I_{k_2}\}\} >$
$= < (\sigma(a_1+a_2), \sigma(b_1+b_2), \sigma(c_1+c_2)); \{1-(1-(\alpha_1+\alpha_2-\alpha_1\alpha_2))^\sigma : \alpha_1 \in \{w_{\tilde{a}}^i : i \in I_{m_1}\},$
$\alpha_2 \in \{w_{\tilde{b}}^i : i \in I_{m_2}\}\}, \{(\beta_1\beta_2)^\sigma : \beta_1 \in \{u_{\tilde{a}}^j : j \in I_{n_1}\}, \beta_2 \in \{u_{\tilde{b}}^j : j \in I_{n_2}\}\}, \{(\lambda_1\lambda_2)^\sigma :$
$\lambda_1 \in \{y_{\tilde{a}}^l : l \in I_{k_1}\}, \lambda_2 \in \{y_{\tilde{b}}^l : l \in I_{k_2}\}\} >$
$= < (\sigma a_1+\sigma a_2, \sigma b_1+\sigma b_2, \sigma c_1+\sigma c_2); \{1-(1-\alpha_1)^\sigma(1-\alpha_2)^\sigma : \alpha_1 \in \{w_{\tilde{a}}^i : i \in I_{m_1}\}, \alpha_2 \in \{w_{\tilde{b}}^i : i \in I_{m_2}\}\},$
$\{\beta_1^\sigma\beta_2^\sigma : \beta_1 \in \{u_{\tilde{a}}^j : j \in I_{n_1}\}, \beta_2 \in \{u_{\tilde{b}}^j : j \in I_{n_2}\}\}, \{\lambda_1^\sigma\lambda_2^\sigma : \lambda_1 \in \{y_{\tilde{a}}^l : l \in I_{k_1}\}, \lambda_2 \in \{y_{\tilde{b}}^l : l \in I_{k_2}\}\} > \quad (3)$

and

$(\sigma \odot' \tilde{a}) \oplus' (\sigma \odot' \tilde{b}) = < (\sigma a_1, \sigma b_1, \sigma c_1); \{1-(1-\alpha_1)^\sigma : \alpha_1 \in \{w_{\tilde{a}}^i : i \in I_{m_1}\}\}, \{\beta_1^\sigma : \beta_1 \in \{u_{\tilde{a}}^j : j \in I_{n_1}\}\},$
$\{\lambda_1^\sigma : \lambda_1 \in \{y_{\tilde{a}}^l : l \in I_{k_1}\}\} > \oplus' < (\sigma a_2, \sigma b_2, \sigma c_2); \{1-(1-\alpha_2)^\sigma : \alpha_2 \in \{w_{\tilde{b}}^i : i \in I_{m_2}\}\},$
$\{\beta_2^\sigma : \beta_2 \in \{u_{\tilde{b}}^j : j \in I_{n_2}\}\}, \{\lambda_2^\sigma : \lambda_2 \in \{y_{\tilde{b}}^l : l \in I_{k_2}\}\} >$
$= < (\sigma a_1+\sigma a_2, \sigma b_1+\sigma b_2, \sigma c_1+\sigma c_2); \{(1-(1-\alpha_1)^\sigma)+(1-(1-\alpha_2)^\sigma)-$
$(1-(1-\alpha_1)^\sigma)(1-(1-\alpha_2)^\sigma) : \alpha_1 \in \{w_{\tilde{a}}^i : i \in I_{m_1}\}, \alpha_2 \in \{w_{\tilde{b}}^i : i \in I_{m_2}\}\}, \{\beta_1^\sigma\beta_2^\sigma : \beta_1 \in$
$\{u_{\tilde{a}}^j : j \in I_{n_1}\}, \beta_2 \in \{u_{\tilde{b}}^j : j \in I_{n_2}\}\}, \{\lambda_1^\sigma\lambda_2^\sigma : \lambda_1 \in \{y_{\tilde{a}}^l : l \in I_{k_1}\}, \lambda_2 \in \{y_{\tilde{b}}^l : l \in I_{k_2}\}\} >$
$= < (\sigma a_1+\sigma a_2, \sigma b_1+\sigma b_2, \sigma c_1+\sigma c_2); \{1-(1-\alpha_1)^\sigma(1-\alpha_2)^\sigma : \alpha_1 \in \{w_{\tilde{a}}^i : i \in I_{m_1}\},$
$\alpha_2 \in \{w_{\tilde{b}}^i : i \in I_{m_2}\}\}, \{\beta_1^\sigma\beta_2^\sigma : \beta_1 \in \{u_{\tilde{a}}^j : j \in I_{n_1}\}, \beta_2 \in \{u_{\tilde{b}}^j : j \in I_{n_2}\}\}, \{\lambda_1^\sigma\lambda_2^\sigma :$
$\lambda_1 \in \{y_{\tilde{a}}^l : l \in I_{k_1}\}, \lambda_2 \in \{y_{\tilde{b}}^l : l \in I_{k_2}\}\} > \quad (4)$

Hence from eq. 3-4, we have $\sigma \odot' (\tilde{a} \oplus' \tilde{b}) = (\sigma \odot' \tilde{a}) \oplus' (\sigma \odot' \tilde{b})$.

6. Proof is similar to 5

7. $(\sigma_1+\sigma_2) \odot' \tilde{a} = < ((\sigma_1+\sigma_2)a_1, (\sigma_1+\sigma_2)b_1, (\sigma_1+\sigma_2)c_1); \{1-(1-\alpha_1)^{\sigma_1+\sigma_2} : \alpha_1 \in \{w_{\tilde{a}}^i : i \in I_{m_1}\}\}, \{\beta_1^{\sigma_1+\sigma_2} :$
$\beta_1 \in \{u_{\tilde{a}}^j : j \in I_{n_1}\}\}, \{\lambda_1^{\sigma_1+\sigma_2} : \lambda_1 \in \{y_{\tilde{a}}^l : l \in I_{k_1}\}\} >$
$= < (\sigma_1 a_1+\sigma_2 a_1, \sigma_1 b_1+\sigma_2 b_1, \sigma_1 c_1+\sigma_2 c_1); \{1-(1-\alpha_1)^{\sigma_1}(1-\alpha_1)^{\sigma_2} : \alpha_1 \in w_{\tilde{a}}\},$
$\{\beta_1^{\sigma_1}\beta_1^{\sigma_2} : \beta_1 \in \{u_{\tilde{a}}^j : j \in I_{n_1}\}\}, \{\lambda_1^{\sigma_1}\lambda_1^{\sigma_2} : \lambda_1 \in \{y_{\tilde{a}}^l : l \in I_{k_1}\}\} > \quad (5)$

And

$(\sigma_1 \odot' \tilde{a}) \oplus' (\sigma_2 \odot' \tilde{a}) = < (\sigma_1 a_1, \sigma_1 b_1, \sigma_1 c_1); \{1-(1-\alpha_1)^{\sigma_1} : \alpha_1 \in \{w_{\tilde{a}}^i : i \in I_{m_1}\}\}, \{\beta_1^{\sigma_1} : \beta_1 \in \{u_{\tilde{a}}^j : j \in I_{n_1}\}\},$
$\{\lambda_1^{\sigma_1} : \lambda_1 \in \{y_{\tilde{a}}^l : l \in I_{k_1}\}\} > \oplus' < (\sigma_2 a_1, \sigma_2 b_1, \sigma_2 c_1); \{1-(1-\alpha_1)^{\sigma_2} : \alpha_1 \in \{w_{\tilde{a}}^i :$
$i \in I_{m_1}\}\}, \{\beta_1^{\sigma_2} : \beta_1 \in \{u_{\tilde{a}}^j : j \in I_{n_1}\}\}, \{\lambda_1^{\sigma_2} : \lambda_1 \in \{y_{\tilde{a}}^l : l \in I_{k_1}\}\} >$
$= < (\sigma_1 a_1+\sigma_2 a_1, \sigma_1 b_1+\sigma_2 b_1, \sigma_1 c_1+\sigma_2 c_1); \{(1-(1-\alpha_1)^{\sigma_1})+(1-(1-\alpha_1)^{\sigma_2})-$
$(1-(1-\alpha_1)^{\sigma_1})(1-(1-\alpha_1)^{\sigma_2}) : \alpha_1 \in \{w_{\tilde{a}}^i : i \in I_{m_1}\}\}, \{\beta_1^{\sigma_1}\beta_1^{\sigma_2} : \beta_1 \in \{u_{\tilde{a}}^j : j \in I_{n_1}\}\},$
$\{\lambda_1^{\sigma_1}\lambda_1^{\sigma_2} : \lambda_1 \in \{y_{\tilde{a}}^l : l \in I_{k_1}\}\} >$
$= < (\sigma_1 a_1+\sigma_2 a_1, \sigma_1 b_1+\sigma_2 b_1, \sigma_1 c_1+\sigma_2 c_1); \{1-(1-\alpha_1)^{\sigma_1}(1-\alpha_1)^{\sigma_2} : \alpha_1 \in w_{\tilde{a}}\},$
$\{\beta_1^{\sigma_1}\beta_1^{\sigma_2} : \beta_1 \in \{u_{\tilde{a}}^j : j \in I_{n_1}\}\}, \{\lambda_1^{\sigma_1}\lambda_1^{\sigma_2} : \lambda_1 \in \{y_{\tilde{a}}^l : l \in I_{k_1}\}\}\} \quad (6)$

Hencefrom eq. 5-6, we have $(\sigma_1+\sigma_2) \odot' \tilde{a} = (\sigma_1 \odot' \tilde{a}) \oplus' (\sigma_2 \odot' \tilde{a})$.

8. Proof is similar to 7

Definition 11. Let $\tilde{a} = < (a_1, b_1, c_1); \{w_{\tilde{a}}^i : i \in I_{m_1}\}, \{u_{\tilde{a}}^j : j \in I_{n_1}\}, \{y_{\tilde{a}}^l : l \in I_{k_1}\} >$ and $\tilde{b} = < (a_2, b_2, c_2); \{w_{\tilde{b}}^i : i \in I_{m_2}\}, \{u_{\tilde{b}}^j : j \in I_{n_2}\}, \{y_{\tilde{b}}^l : l \in I_{k_2}\} >$ be two hesitant triangular neutrosophic numbers and $\sigma > 0$, then

1. $\tilde{a} \oplus'' \tilde{b} = <(a_1+a_2, b_1+b_2, c_1+c_2); \{1-\frac{1}{1+\left\{\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2+\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2\right\}^{\frac{1}{2}}} : \alpha_1 \in \{w_{\tilde{a}}^i : i \in I_{m_1}\}, \alpha_2 \in \{w_{\tilde{b}}^i : i \in I_{m_2}\}\},$

$\{\frac{1}{1+\left\{\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2+\left(\frac{1-\beta_2^2}{\beta_2^2}\right)^2\right\}^{\frac{1}{2}}} : \beta_1 \in \{u_{\tilde{a}}^j : j \in I_{n_1}\}, \beta_2 \in \{u_{\tilde{b}}^j : j \in I_{n_2}\}\},$

$\{\frac{1}{1+\left\{\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2+\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2\right\}^{\frac{1}{2}}} : \lambda_1 \in \{y_{\tilde{a}}^l : l \in I_{k_1}\}, \lambda_2 \in \{y_{\tilde{b}}^l : l \in I_{k_2}\}\} >$

2. $\tilde{a} \otimes'' \tilde{b} = <(a_1a_2, b_1b_2, c_1c_2); \{\frac{1}{1+\left\{\left(\frac{1-\alpha_1^2}{\alpha_1^2}\right)^2+\left(\frac{1-\alpha_2^2}{\alpha_2^2}\right)^2\right\}^{\frac{1}{2}}} : \alpha_1 \in \{w_{\tilde{a}}^i : i \in I_{m_1}\}, \alpha_2 \in \{w_{\tilde{b}}^i : i \in I_{m_2}\}\},$

$\{1-\frac{1}{1+\left\{\left(\frac{\beta_1^2}{1-\beta_1^2}\right)^2+\left(\frac{\beta_2^2}{1-\beta_2^2}\right)^2\right\}^{\frac{1}{2}}} : \beta_1 \in \{u_{\tilde{a}}^j : j \in I_{n_1}\}, \beta_2 \in \{u_{\tilde{b}}^j : j \in I_{n_2}\}\},$

$\{1-\frac{1}{1+\left\{\left(\frac{\lambda_1^2}{1-\lambda_1^2}\right)^2+\left(\frac{\lambda_2^2}{1-\lambda_2^2}\right)^2\right\}^{\frac{1}{2}}} : \lambda_1 \in \{y_{\tilde{a}}^l : l \in I_{k_1}\}, \lambda_2 \in \{y_{\tilde{b}}^l : l \in I_{k_2}\}\}\} >$

3. $\sigma \odot'' \tilde{a} = <(\sigma a_1, \sigma b_1, \sigma c_1); \{1-\frac{1}{1+\left\{\sigma\left(\frac{\alpha^2}{1-\alpha^2}\right)^2\right\}^{\frac{1}{2}}} : \alpha_1 \in \{w_{\tilde{a}}^i : i \in I_{m_1}\}\}, \{\frac{1}{1+\left\{\sigma\left(\frac{1-\beta^2}{\beta^2}\right)^2\right\}^{\frac{1}{2}}} : \beta_1 \in \{u_{\tilde{a}}^j : j \in I_{n_1}\}\},$

$\{\frac{1}{1+\left\{\sigma\left(\frac{1-\lambda^2}{\lambda^2}\right)^2\right\}^{\frac{1}{2}}} : \lambda_1 \in \{y_{\tilde{a}}^l : l \in I_{k_1}\}\} >$

4. $\sigma *'' \tilde{a} = <(a_1^{\sigma}, b_1^{\sigma}, c_1^{\sigma}); \{\frac{1}{1+\left\{\sigma\left(\frac{1-\alpha^2}{\alpha^2}\right)^2\right\}^{\frac{1}{2}}} : \alpha_1 \in \{w_{\tilde{a}}^i : i \in I_{m_1}\}\}, \{1-\frac{1}{1+\left\{\sigma\left(\frac{\beta^2}{1-\beta^2}\right)^2\right\}^{\frac{1}{2}}} : \beta_1 \in \{u_{\tilde{a}}^j : j \in I_{n_1}\}\},$

$\{1-\frac{1}{1+\left\{\sigma\left(\frac{\lambda^2}{1-\lambda^2}\right)^2\right\}^{\frac{1}{2}}} : \lambda_1 \in \{y_{\tilde{a}}^l : l \in I_{k_1}\}\} >$

Theorem 12. Let $\tilde{a} = <(a_1, b_1, c_1); \{w_{\tilde{a}}^i : i \in I_{m_1}\}, \{u_{\tilde{a}}^j : j \in I_{n_1}\}, \{y_{\tilde{a}}^l : l \in I_{k_1}\} >$,
$\tilde{b} = <(a_2, b_2, c_2); \{w_{\tilde{b}}^i : i \in I_{m_2}\}, \{u_{\tilde{b}}^j : j \in I_{n_2}\}, \{y_{\tilde{b}}^l : l \in I_{k_2}\} >$ and
$\tilde{c} = <(a_3, b_3, c_3); \{w_{\tilde{c}}^i : i \in I_{m_3}\}, \{u_{\tilde{c}}^j : j \in I_{n_3}\}, \{y_{\tilde{c}}^l : l \in I_{k_3}\} >$ be three hesitant triangular neutrosophic numbers and $\sigma, \sigma_1, \sigma_2 > 0$, then

1. $\tilde{a} \oplus'' \tilde{b} = \tilde{b} \oplus'' \tilde{a}$
2. $\tilde{a} \otimes'' \tilde{b} = \tilde{b} \otimes'' \tilde{a}$
3. $\tilde{a} \oplus'' (\tilde{b} \oplus'' \tilde{c}) = (\tilde{a} \oplus'' \tilde{b}) \oplus'' \tilde{c}$
4. $\tilde{a} \otimes'' (\tilde{b} \otimes'' \tilde{c}) = (\tilde{a} \otimes'' \tilde{b}) \otimes'' \tilde{c}$
5. $\sigma \odot'' (\tilde{a} \oplus'' \tilde{b}) = (\sigma \odot'' \tilde{a}) \oplus'' (\sigma \odot'' \tilde{b})$
6. $\sigma *'' (\tilde{a} \otimes'' \tilde{b}) = (\sigma *'' \tilde{a}) \otimes'' (\sigma *'' \tilde{b})$
7. $(\sigma_1 + \sigma_2) \odot'' \tilde{a} = (\sigma_1 \odot'' \tilde{a}) \oplus'' (\sigma_2 \odot'' \tilde{a})$
8. $(\sigma_1 + \sigma_2) *'' \tilde{a} = (\sigma_1 *'' \tilde{a}) \otimes'' (\sigma_2 *'' \tilde{a})$

Proof:1.-2. Straight forward.

3. $\tilde{a} \oplus'' (\tilde{b} \oplus'' \tilde{c})$

$$= < (a_1, b_1, c_1); \{w_{\tilde{a}}^i : i \in I_{m_1}\}, \{u_{\tilde{a}}^j : j \in I_{n_1}\}, \{y_{\tilde{a}}^l : l \in I_{k_1}\} > \oplus'' < (a_2 + a_3, b_2 + b_3, c_2 + c_3);$$

$$\{1 - \frac{1}{1 + \left\{\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2 + \left(\frac{\alpha_3^2}{1-\alpha_3^2}\right)^2\right\}^{\frac{1}{2}}} : \alpha_2 \in \{w_{\tilde{b}}^i : i \in I_{m_2}\}, \alpha_3 \in \{w_{\tilde{c}}^i : i \in I_{m_3}\}\},$$

$$\{\frac{1}{1 + \left\{\left(\frac{1-\beta_2^2}{\beta_2^2}\right)^2 + \left(\frac{1-\beta_3^2}{\beta_3^2}\right)^2\right\}^{\frac{1}{2}}} : \beta_2 \in \{u_{\tilde{b}}^j : j \in I_{n_2}\}, \beta_3 \in \{u_{\tilde{c}}^j : j \in I_{n_3}\}\},$$

$$\{\frac{1}{1 + \left\{\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2 + \left(\frac{1-\lambda_3^2}{\lambda_3^2}\right)^2\right\}^{\frac{1}{2}}} : \lambda_2 \in \{y_{\tilde{b}}^l : l \in I_{k_2}\}, \lambda_3 \in \{y_{\tilde{c}}^l : l \in I_{k_3}\}\} >$$

$$= < (a_1 + (a_2 + a_3), b_1 + (b_2 + b_3), c_1 + (c_2 + c_3));$$

$$\{1 - \frac{1}{1 + \left\{\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2 + \left(\frac{1 - \frac{1}{1 + \left\{\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2 + \left(\frac{\alpha_3^2}{1-\alpha_3^2}\right)^2\right\}^{\frac{1}{2}}}}{1 - \left(1 - \frac{1}{1 + \left\{\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2 + \left(\frac{\alpha_3^2}{1-\alpha_3^2}\right)^2\right\}^{\frac{1}{2}}}\right)}\right)^2\right\}^{\frac{1}{2}}} : \alpha_1 \in \{w_{\tilde{a}}^i : i \in I_{m_1}\}, \alpha_2 \in \{w_{\tilde{b}}^i : i \in I_{m_2}\}, \alpha_3 \in \{w_{\tilde{c}}^i : i \in I_{m_3}\}\},$$

$$\{\frac{1}{1+\left\{\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2+\left(\frac{1-\frac{1}{1+\left\{\left(\frac{1-\beta_2^2}{\beta_2^2}\right)^2+\left(\frac{1-\beta_3^2}{\beta_3^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\left(\frac{1-\beta_2^2}{\beta_2^2}\right)^2+\left(\frac{1-\beta_3^2}{\beta_3^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}}:\beta_1\in\{u_{\tilde a}^j:j\in I_{n_1}\},\beta_2\in\{u_{\tilde b}^j:j\in I_{n_2}\},\beta_3\in\{u_{\tilde c}^j:j\in I_{n_3}\}\},$$

$$\{\frac{1}{1+\left\{\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2+\left(\frac{1-\frac{1}{1+\left\{\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2+\left(\frac{1-\lambda_3^2}{\lambda_3^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2+\left(\frac{1-\lambda_3^2}{\lambda_3^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde a}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde b}^l:l\in I_{k_2}\},\lambda_3\in\{y_{\tilde c}^l:l\in I_{k_3}\}\}>$$

$$=<(a_1+a_2+a_3,b_1+b_2+b_3,c_1+c_2+c_3);\{1-\frac{1}{1+\left\{\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2+\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2+\left(\frac{\alpha_3^2}{1-\alpha_3^2}\right)^2\right\}^{\frac{1}{2}}}:$$

$$\alpha_1\in\{w_{\tilde a}^i:i\in I_{m_1}\},\alpha_2\in\{w_{\tilde b}^i:i\in I_{m_2}\},\alpha_3\in\{w_{\tilde c}^i:i\in I_{m_3}\}\},$$

$$\{\frac{1}{1+\left\{\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2+\left(\frac{1-\beta_2^2}{\beta_2^2}\right)^2+\left(\frac{1-\beta_3^2}{\beta_3^2}\right)^2\right\}^{\frac{1}{2}}}:\beta_1\in\{u_{\tilde a}^j:j\in I_{n_1}\},\beta_2\in\{u_{\tilde b}^j:j\in I_{n_2}\},\ \beta_3\in\{u_{\tilde c}^j:j\in I_{n_3}\}\},$$

$$\{\frac{1}{1+\left\{\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2+\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2+\left(\frac{1-\lambda_3^2}{\lambda_3^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde a}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde b}^l:l\in I_{k_2}\},\lambda_3\in\{y_{\tilde c}^l:l\in I_{k_3}\}\}> \quad (7)$$

$$(\tilde a\oplus''\tilde b)\oplus''\tilde c$$

$$=<(a_1+a_2,b_1+b_2,c_1+c_2);\{1-\frac{1}{1+\left\{\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2+\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2\right\}^{\frac{1}{2}}}:\alpha_1\in\{w_{\tilde a}^i:i\in I_{m_1}\},\alpha_2\in\{w_{\tilde b}^i:i\in I_{m_2}\}\},$$

$$\{\frac{1}{1+\left\{\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2+\left(\frac{1-\beta_2^2}{\beta_2^2}\right)^2\right\}^{\frac{1}{2}}}:\beta_1\in\{u_{\tilde a}^j:j\in I_{n_1}\},\beta_2\in\{u_{\tilde b}^j:j\in I_{n_2}\}\},$$

$$\{\frac{1}{1+\left\{\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2+\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde a}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde b}^l:l\in I_{k_2}\}\}>\oplus''<(a_3,b_3,c_3);w_{\tilde c},u_{\tilde c},y_{\tilde c}>$$

$$=<((a_1+a_2)+a_3,(b_1+b_2)+b_3,(c_1+c_2)+c_3);$$

$$\{1-\frac{1}{\left[1+\left\{\left(\frac{\alpha_3^2}{1-\alpha_3^2}\right)^2+\left(\frac{1-\frac{1}{1+\left\{\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2+\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2\right\}^{\frac{1}{2}}}}{1-\left(1-\frac{1}{1+\left\{\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2+\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2\right\}^{\frac{1}{2}}}\right)}\right)^2\right\}\right]^{\frac{1}{2}}}:\alpha_1\in\{w_{\tilde a}^i:i\in I_{m_1}\},\alpha_2\in\{w_{\tilde b}^i:i\in I_{m_2}\},\alpha_3\in\{w_{\tilde c}^i:i\in I_{m_3}\}\},$$

$$\{\frac{1}{\left[1+\left\{\left(\frac{1-\beta_3^2}{\beta_3^2}\right)^2+\left(\frac{1-\frac{1}{1+\left\{\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2+\left(\frac{1-\beta_2^2}{\beta_2^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2+\left(\frac{1-\beta_2^2}{\beta_2^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}\right]^{\frac{1}{2}}}:\beta_1\in\{u_{\tilde a}^j:j\in I_{n_1}\},\beta_2\in\{u_{\tilde b}^j:j\in I_{n_2}\},\beta_3\in\{u_{\tilde c}^j:j\in I_{n_3}\}\},$$

$$\{\frac{1}{\left[1+\left\{\left(\frac{1-\lambda_3^2}{\lambda_3^2}\right)^2+\left(\frac{1-\frac{1}{1+\left\{\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2+\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2+\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}\right]^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde a}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde b}^l:l\in I_{k_2}\},\lambda_3\in\{y_{\tilde c}^l:l\in I_{k_3}\}\}>$$

$$=<(a_1+a_2+a_3,b_1+b_2+b_3,c_1+c_2+c_3);\{1-\frac{1}{1+\left\{\left(\frac{\alpha_3^2}{1-\alpha_3^2}\right)^2+\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2+\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2\right\}^{\frac{1}{2}}}:\alpha_1\in\{w_{\tilde a}^i:i\in I_{m_1}\},$$

$$\alpha_2\in\{w_{\tilde b}^i:i\in I_{m_2}\},\alpha_3\in\{w_{\tilde c}^i:i\in I_{m_3}\}\},\{\frac{1}{1+\left\{\left(\frac{1-\beta_3^2}{\beta_3^2}\right)^2+\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2+\left(\frac{1-\beta_2^2}{\beta_2^2}\right)^2\right\}^{\frac{1}{2}}}:$$

$$\beta_1\in\{u_{\tilde a}^j:j\in I_{n_1}\},\beta_2\in\{u_{\tilde b}^j:j\in I_{n_2}\},\beta_3\in\{u_{\tilde c}^j:j\in I_{n_3}\}\},\{\frac{1}{1+\left\{\left(\frac{1-\lambda_3^2}{\lambda_3^2}\right)^2+\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2+\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2\right\}^{\frac{1}{2}}}:$$

$$\lambda_1\in\{y_{\tilde a}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde b}^l:l\in I_{k_2}\},\lambda_3\in\{y_{\tilde c}^l:l\in I_{k_3}\}\}> \tag{8}$$

Hence from eq. 7-8, we have, $\tilde a\oplus''(\tilde b\oplus''\tilde c)=(\tilde a\oplus''\tilde b)\oplus''\tilde c$.

4. Proof is similar to 3.

5. $\sigma \odot'' (\tilde{a} \oplus'' \tilde{b})$

$$= \sigma \odot'' < (a_1 + a_2, b_1 + b_2, c_1 + c_2); \{1 - \frac{1}{1 + \left\{ \left(\frac{\alpha_1^2}{1-\alpha_1^2} \right)^2 + \left(\frac{\alpha_2^2}{1-\alpha_2^2} \right)^2 \right\}^{\frac{1}{2}}} : \alpha_1 \in \{w_{\tilde{a}}^i : i \in I_{m_1}\}, \alpha_2 \in \{w_{\tilde{b}}^i : i \in I_{m_2}\}\},$$

$$\{ \frac{1}{1 + \left\{ \left(\frac{1-\beta_1^2}{\beta_1^2} \right)^2 + \left(\frac{1-\beta_2^2}{\beta_2^2} \right)^2 \right\}^{\frac{1}{2}}} : \beta_1 \in \{u_{\tilde{a}}^j : j \in I_{n_1}\}, \beta_2 \in \{u_{\tilde{b}}^j : j \in I_{n_2}\}\},$$

$$\{ \frac{1}{1 + \left\{ \left(\frac{1-\lambda_1^2}{\lambda_1^2} \right)^2 + \left(\frac{1-\lambda_2^2}{\lambda_2^2} \right)^2 \right\}^{\frac{1}{2}}} : \lambda_1 \in \{y_{\tilde{a}}^l : l \in I_{k_1}\}, \lambda_2 \in \{y_{\tilde{b}}^l : l \in I_{k_2}\}\} >$$

$$= < (\sigma a_1 + \sigma a_2, \sigma b_1 + \sigma b_2, \sigma c_1 + \sigma c_2); \{1 - \frac{1}{1 + \left\{ \sigma \left(\frac{1 - \frac{1}{1 + \left\{ \left(\frac{\alpha_1^2}{1-\alpha_1^2} \right)^2 + \left(\frac{\alpha_2^2}{1-\alpha_2^2} \right)^2 \right\}^{\frac{1}{2}}}}{\frac{1}{1 + \left\{ \left(\frac{\alpha_1^2}{1-\alpha_1^2} \right)^2 + \left(\frac{\alpha_2^2}{1-\alpha_2^2} \right)^2 \right\}^{\frac{1}{2}}}} \right)^2 \right\}^{\frac{1}{2}}} : \alpha_1 \in \{w_{\tilde{a}}^i : i \in I_{m_1}\},$$

$\alpha_2 \in \{w_{\tilde{b}}^i : i \in I_{m_2}\}\},$

$$\{ \frac{1}{1 + \left\{ \sigma \left(\frac{1 - \frac{1}{1 + \left\{ \left(\frac{1-\beta_1^2}{\beta_1^2} \right)^2 + \left(\frac{1-\beta_2^2}{\beta_2^2} \right)^2 \right\}^{\frac{1}{2}}}}{\frac{1}{1 + \left\{ \left(\frac{1-\beta_1^2}{\beta_1^2} \right)^2 + \left(\frac{1-\beta_2^2}{\beta_2^2} \right)^2 \right\}^{\frac{1}{2}}}} \right)^2 \right\}^{\frac{1}{2}}} : \beta_1 \in \{u_{\tilde{a}}^j : j \in I_{n_1}\}, \beta_2 \in \{u_{\tilde{b}}^j : j \in I_{n_2}\}\},$$

$$\{ \frac{1}{1 + \left\{ \sigma \left(\frac{1 - \frac{1}{1 + \left\{ \left(\frac{1-\lambda_1^2}{\lambda_1^2} \right)^2 + \left(\frac{1-\lambda_2^2}{\lambda_2^2} \right)^2 \right\}^{\frac{1}{2}}}}{\frac{1}{1 + \left\{ \left(\frac{1-\lambda_1^2}{\lambda_1^2} \right)^2 + \left(\frac{1-\lambda_2^2}{\lambda_2^2} \right)^2 \right\}^{\frac{1}{2}}}} \right)^2 \right\}^{\frac{1}{2}}} : \lambda_1 \in \{y_{\tilde{a}}^l : l \in I_{k_1}\}, \lambda_2 \in \{y_{\tilde{b}}^l : l \in I_{k_2}\}\} >$$

$$=<(\sigma a_1+\sigma a_2,\sigma b_1+\sigma b_2,\sigma c_1+\sigma c_2);\{1-\frac{1}{1+\left\{\sigma\left(\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2+\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2\right)\right\}^{\frac{1}{2}}}:\alpha_1\in\{w_{\tilde a}^i:i\in I_{m_1}\},$$

$$\alpha_2\in\{w_{\tilde b}^i:i\in I_{m_2}\}\},\{\frac{1}{1+\left\{\sigma\left(\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2+\left(\frac{1-\beta_2^2}{\beta_2^2}\right)^2\right)\right\}^{\frac{1}{2}}}:\beta_1\in\{u_{\tilde a}^j:j\in I_{n_1}\},\beta_2\in\{u_{\tilde b}^j:j\in I_{n_2}\}\},$$

$$\{\frac{1}{1+\left\{\sigma\left(\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2+\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2\right)\right\}^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde a}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde b}^l:l\in I_{k_2}\}\}>$$

$$=<(\sigma a_1+\sigma a_2,\sigma b_1+\sigma b_2,\sigma c_1+\sigma c_2);\{1-\frac{1}{1+\left\{\left(\frac{1-\frac{1}{1+\left\{\sigma\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sigma\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2+\left(\frac{1-\frac{1}{1+\left\{\sigma\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sigma\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}}:$$

$$\alpha_1\in\{w_{\tilde a}^i:i\in I_{m_1}\},\alpha_2\in\{w_{\tilde b}^i:i\in I_{m_2}\}\},\{\frac{1}{1+\left\{\left(\frac{1-\frac{1}{1+\left\{\sigma\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sigma\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2+\left(\frac{1-\frac{1}{1+\left\{\sigma\left(\frac{1-\beta_2^2}{\beta_2^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sigma\left(\frac{1-\beta_2^2}{\beta_2^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}}:$$

$$\beta_1\in\{u_{\tilde a}^j:j\in I_{n_1}\},\beta_2\in\{u_{\tilde b}^j:j\in I_{n_2}\}\},\{\frac{1}{1+\left\{\left(\frac{1-\frac{1}{1+\left\{\sigma\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sigma\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2+\left(\frac{1-\frac{1}{1+\left\{\sigma\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sigma\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}}:$$

$$\lambda_1\in\{y_{\tilde a}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde b}^l:l\in I_{k_2}\}\}>$$

$$=<(\sigma a_1,\sigma b_1,\sigma c_1);\{1-\frac{1}{1+\left\{\sigma\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2\right\}^{\frac{1}{2}}}:\alpha_1\in\{w_{\tilde a}^i:i\in I_{m_1}\}\},\{\frac{1}{1+\left\{\sigma\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2\right\}^{\frac{1}{2}}}:\beta_1\in\{u_{\tilde a}^j:j\in I_{n_1}\}\},$$

$$\{\frac{1}{1+\left\{\sigma\left(\frac{1-\lambda_1{}^2}{\lambda_1{}^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\}\}>\oplus''<\sigma a_2,\sigma b_2,\sigma c_2\ ;\{1-\frac{1}{1+\left\{\sigma\left(\frac{\alpha_2{}^2}{1-\alpha_2{}^2}\right)^2\right\}^{\frac{1}{2}}}:\alpha_2\in\{w_{\tilde{b}}^i:i\in I_{m_2}\}\},$$

$$\{\frac{1}{1+\left\{\sigma\left(\frac{1-\beta_2{}^2}{\beta_2{}^2}\right)^2\right\}^{\frac{1}{2}}}:\beta_2\in\{u_{\tilde{b}}^j:j\in I_{n_2}\}\},\{\frac{1}{1+\left\{\sigma\left(\frac{1-\lambda_2{}^2}{\lambda_2{}^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_2\in\{y_{\tilde{b}}^l:l\in I_{k_2}\}\}>$$

$$=(\sigma\odot''\tilde{a})\oplus''(\sigma\odot''\tilde{b})$$

6. Proof is similar to 5

7. $(\sigma_1+\sigma_2)\odot''\tilde{a}$

$$=<((\sigma_1+\sigma_2)a_1,(\sigma_1+\sigma_2)b_1,(\sigma_1+\sigma_2)c_1);\{1-\frac{1}{1+\left\{(\sigma_1+\sigma_2)\left(\frac{\alpha_1{}^2}{1-\alpha_1{}^2}\right)^2\right\}^{\frac{1}{2}}}:\alpha_1\in\{w_{\tilde{a}}^i:i\in I_{m_1}\}\},$$

$$\{\frac{1}{1+\left\{(\sigma_1+\sigma_2)\left(\frac{1-\beta_1{}^2}{\beta_1{}^2}\right)^2\right\}^{\frac{1}{2}}}\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\}\},\{\frac{1}{1+\left\{(\sigma_1+\sigma_2)\left(\frac{1-\lambda_1{}^2}{\lambda_1{}^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde{a}}^l:l\in I_{k_1}\}$$

$$=<((\sigma_1+\sigma_2)a_1,(\sigma_1+\sigma_2)b_1,(\sigma_1+\sigma_2)c_1);$$

$$\{1-\frac{1}{1+\left\{\left(\frac{1-\frac{1}{1+\left\{\sigma_1\left(\frac{\alpha_1{}^2}{1-\alpha_1{}^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sigma_1\left(\frac{\alpha_1{}^2}{1-\alpha_1{}^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2+\left(\frac{1-\frac{1}{1+\left\{\sigma_2\left(\frac{\alpha_1{}^2}{1-\alpha_1{}^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sigma_2\left(\frac{\alpha_1{}^2}{1-\alpha_1{}^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}}:\alpha_1\in\{w_{\tilde{a}}^i:i\in I_{m_1}\}\},$$

$$\{\frac{1}{1+\left\{\left(\frac{1-\frac{1}{1+\left\{\sigma_1\left(\frac{1-\beta_1{}^2}{\beta_1{}^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sigma_1\left(\frac{1-\beta_1{}^2}{\beta_1{}^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2+\left(\frac{1-\frac{1}{1+\left\{\sigma_2\left(\frac{1-\beta_1{}^2}{\beta_1{}^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sigma_2\left(\frac{1-\beta_1{}^2}{\beta_1{}^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}}:\beta_1\in\{u_{\tilde{a}}^j:j\in I_{n_1}\}\},$$

$$\{\frac{1}{1+\left\{\left(\frac{1-\frac{1}{1+\left\{\sigma_1\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sigma_1\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2+\left(\frac{1-\frac{1}{1+\left\{\sigma_2\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sigma_2\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde a}^l:l\in I_{k_1}\}\}>$$

$$=<(\sigma_1 a_1,\sigma_1 b_1,\sigma_1 c_1);\{1-\frac{1}{1+\left\{\sigma_1\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2\right\}^{\frac{1}{2}}}:\alpha_1\in\{w_{\tilde a}^i:i\in I_{m_1}\}\},\{\frac{1}{1+\left\{\sigma_1\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2\right\}^{\frac{1}{2}}}:\beta_1\in\{u_{\tilde a}^j:j\in I_{n_1}\}\},$$

$$\{\frac{1}{1+\left\{\sigma_1\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde a}^l:l\in I_{k_1}\}\}>\oplus''<(\sigma_2 a_1,\sigma_2 b_1,\sigma_2 c_1);\{1-\frac{1}{1+\left\{\sigma_2\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2\right\}^{\frac{1}{2}}}:$$

$$\alpha_1\in\{w_{\tilde a}^i:i\in I_{m_1}\}\},\{\frac{1}{1+\left\{\sigma_2\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2\right\}^{\frac{1}{2}}}:\beta_1\in\{u_{\tilde a}^j:j\in I_{n_1}\}\},\{\frac{1}{1+\left\{\sigma_2\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde a}^l:l\in I_{k_1}\}\}>$$

$$=(\sigma_1\odot''\tilde a)\oplus''(\sigma_2\odot''\tilde a)$$

8. Proof is similar to 7.

5. HESITANT TRIANGULAR NEUTROSOPHIC WEIGHTED AGGREGATION OPERATORS:

This section deals with various types of hesitant triangular neutrosophic weighted aggregation operators along with their basic properties.

Definition 13: Let $\tilde a_j=<(a_j,b_j,c_j);\{w_{\tilde a_j}^i:i\in I_{m_j}\},\{u_{\tilde a_j}^r:r\in I_{n_j}\},\{y_{\tilde a_j}^l:l\in I_{k_j}\}>$ (j=1, 2, 3,….., n) be a collection of hesitant triangular neutrosophic numbers. Then the hesitant triangular neutrosophic weighted arithmetic aggregation operator of type-1 ($HTNWAAO_{T_1}$ for short) is defined as:

$$HTNWAAO_{T_1}\ \tilde a_1,\tilde a_2,\tilde a_3,......,\tilde a_n\ =(w_1\odot'\tilde a_1)\oplus'(w_2\odot'\tilde a_2)\oplus'(w_3\odot'\tilde a_3)\oplus'........\oplus'(w_n\odot'\tilde a_n)$$

where w_j is the weight of $\tilde a_j$ (j=1,2,3,……, n) such that $w_j\ge 0$ and $\sum_{j=1}^{n}w_j=1$.

Theorem 14: Let $\tilde a_j=<(a_j,b_j,c_j);\{w_{\tilde a_j}^i:i\in I_{m_j}\},\{u_{\tilde a_j}^r:r\in I_{n_j}\},\{y_{\tilde a_j}^l:l\in I_{k_j}\}>$ (j=1, 2, 3,….., n) be a collection of hesitant triangularneutrosophic numbers. Then $HTNWAAO_{T_1}\ \tilde a_1,\tilde a_2,\tilde a_3,......,\tilde a_n$ is a hesitant triangular neutrosophic number and

$$HTNWAAO_{T_1}(\tilde a_1,\tilde a_2,\tilde a_3,......,\tilde a_n)=<(\sum_{j=1}^{n}w_ja_j,\sum_{j=1}^{n}w_jb_j,\sum_{j=1}^{n}w_jc_j),\{1-\prod_{j=1}^{n}(1-\alpha_{jk})^{w_j}:$$

$$\alpha_j\in\{w_{\tilde a_j}^i:i\in I_{m_j}\}\},\{\prod_{j=1}^{n}\beta_{jm}{}^{w_j}:\beta_j\in\{u_{\tilde a_j}^r:r\in I_{n_j}\}\},\{\prod_{j=1}^{n}\lambda_{jr}{}^{w_j}:\lambda_j\in\{y_{\tilde a_j}^l:l\in I_{k_j}\}\}>$$

where w_j is the weight of $\tilde{a}_j$ (j=1,2,3,……,n) such that $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$.

Proof: Let us prove the result using the method of mathematical induction. For n=2, $HTNWAO_{T_1}(\tilde{a}_1, \tilde{a}_2)$

$$= (w_1 \odot' \tilde{a}_1) \oplus' (w_2 \odot' \tilde{a}_2)$$

$$=< (w_1 a_1, w_1 b_1, w_1 c_1); \{1-(1-\alpha_1)^{w_1} : \alpha_1 \in \{w^i_{\tilde{a}_1} : i \in I_{m_1}\}\}, \{\beta_1^{w_1} : \beta_1 \in \{u^r_{\tilde{a}_1} : r \in I_{n_1}\}\},$$
$$\{\lambda_1^{w_1} : \lambda_1 \in \{y^l_{\tilde{a}_1} : l \in I_{k_1}\}\} > \oplus' < (w_2 a_2, w_2 b_2, w_2 c_2); \{1-(1-\alpha_2)^{w_2} : \alpha_2 \in \{w^i_{\tilde{a}_2} : i \in I_{m_2}\}\},$$
$$\{\beta_2^{w_2} : \beta_2 \in \{u^r_{\tilde{a}_2} : j \in I_{n_2}\}\}, \{\lambda_2^{w_2} : \lambda_2 \in \{y^l_{\tilde{a}_2} : l \in I_{k_2}\}\} >$$
$$=< (w_1 a_1 + w_2 a_2, w_1 b_1 + w_2 b_2, w_1 c_1 + w_2 c_2); (1-(1-\alpha_1)^{w_1}) + (1-(1-\alpha_2)^{w_2}) -$$
$$\{(1-(1-\alpha_1)^{w_1}) \times (1-(1-\alpha_2)^{w_2}) : \alpha_1 \in \{w^i_{\tilde{a}_1} : i \in I_{m_1}\}, \alpha_2 \in \{w^i_{\tilde{a}_2} : i \in I_{m_2}\}\}, \{\beta_1^{w_1} \beta_2^{w_2} :$$
$$\beta_1 \in \{u^r_{\tilde{a}_1} : r \in I_{n_1}\}, \beta_2 \in \{u^r_{\tilde{a}_2} : j \in I_{n_2}\}\}, \{\lambda_1^{w_1} \lambda_2^{w_2} : \lambda_1 \in \{y^l_{\tilde{a}_1} : l \in I_{k_1}\}, \lambda_2 \in \{y^l_{\tilde{a}_2} : l \in I_{k_2}\}\} >$$
$$=< (w_1 a_1 + w_2 a_2, w_1 b_1 + w_2 b_2, w_1 c_1 + w_2 c_2); \{1-(1-\alpha_1)^{w_1}(1-\alpha_2)^{w_2} : \alpha_1 \in \{w^i_{\tilde{a}_1} : i \in I_{m_1}\},$$
$$\alpha_2 \in \{w^i_{\tilde{a}_2} : i \in I_{m_2}\}\}, \{\beta_1^{w_1} \beta_2^{w_2} : \beta_1 \in \{u^r_{\tilde{a}_1} : r \in I_{n_1}\}, \beta_2 \in \{u^r_{\tilde{a}_2} : j \in I_{n_2}\}\}, \{\lambda_1^{w_1} \lambda_2^{w_2} :$$
$$\lambda_1 \in \{y^l_{\tilde{a}_1} : l \in I_{k_1}\}, \lambda_2 \in \{y^l_{\tilde{a}_2} : l \in I_{k_2}\}\} >$$
$$=< (\sum_{j=1}^{2} w_j a_j, \sum_{j=1}^{2} w_j b_j, \sum_{j=1}^{2} w_j c_j); \{1-\prod_{j=1}^{2}(1-\alpha_j)^{w_j} : \alpha_j \in \{w^i_{\tilde{a}_j} : i \in I_{m_j}\}\}, \{\prod_{j=1}^{2} \beta_j^{w_j} : \beta_j \in \{u^r_{\tilde{a}_j} : r \in I_{n_j}\}\},$$
$$\{\prod_{j=1}^{2} \lambda_j^{w_j} : \lambda_j \in \{y^l_{\tilde{a}_j} : l \in I_{k_j}\}\} >$$

Thus the result is true for n=2. Let us assume that the result is true for n=s. Then $HTNWAO_{T_1}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, ..., \tilde{a}_s)$

$$=< (\sum_{j=1}^{s} w_j a_j, \sum_{j=1}^{s} w_j b_j, \sum_{j=1}^{s} w_j c_j); \{1-\prod_{j=1}^{s}(1-\alpha_j)^{w_j} : \alpha_j \in \{w^i_{\tilde{a}_j} : i \in I_{m_j}\}\}, \{\prod_{j=1}^{s} \beta_j^{w_j} : \beta_j \in \{u^r_{\tilde{a}_j} : r \in I_{n_j}\}\},$$
$$\{\prod_{j=1}^{s} \lambda_j^{w_j} : \lambda_j \in \{y^l_{\tilde{a}_j} : l \in I_{k_j}\}\} >$$

Now for n=s+1, we have, $HTNWAO_{T_1}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, ..., \tilde{a}_{s+1})$

$$=< (\sum_{j=1}^{s} w_j a_j, \sum_{j=1}^{s} w_j b_j, \sum_{j=1}^{s} w_j c_j); \{1-\prod_{j=1}^{s}(1-\alpha_j)^{w_j} : \alpha_j \in \{w^i_{\tilde{a}_j} : i \in I_{m_j}\}\}, \prod_{j=1}^{s} \beta_j^{w_j} : \{\beta_j \in \{u^r_{\tilde{a}_j} : r \in I_{n_j}\}\},$$
$$\{\prod_{j=1}^{s} \lambda_j^{w_j} : \lambda_j \in \{y^l_{\tilde{a}_j} : l \in I_{k_j}\}\} > \oplus' (w_{s+1} \odot \tilde{a}_{s+1})$$
$$=< (\sum_{j=1}^{s} w_j a_j, \sum_{j=1}^{s} w_j b_j, \sum_{j=1}^{s} w_j c_j), \{1-\prod_{j=1}^{s}(1-\alpha_j)^{w_j} : \alpha_j \in \{w^i_{\tilde{a}_j} : i \in I_{m_j}\}\}, \prod_{j=1}^{s} \beta_j^{w_j} : \beta_j \in \{u^r_{\tilde{a}_j} : r \in I_{n_j}\}\},$$
$$\{\{\prod_{j=1}^{s} \lambda_j^{w_j} : \lambda_j \in \{y^l_{\tilde{a}_j} : l \in I_{k_j}\}\} > \oplus' < (w_{s+1} a_{s+1}, w_{s+1} b_{s+1}, w_{s+1} c_{s+1}); \{1-(1-\alpha_{(s+1)})^{w_{s+1}} :$$
$$\alpha_{s+1} \in \{w^i_{\tilde{a}_{s+1}} : i \in I_{m_{s+1}}\}\} \{\beta_{s+1}^{w_{s+1}} : \beta_{s+1} \in \{u^r_{\tilde{a}_{s+1}} : r \in I_{n_{s+1}}\}\}, \{\lambda_{s+1}^{w_{s+1}} : \lambda_{s+1} \in \{y^l_{\tilde{a}_{s+1}} : l \in I_{k_{s+1}}\}\} >$$
$$=< (\sum_{j=1}^{s} w_j a_j + w_{s+1} a_{s+1}, \sum_{j=1}^{s} w_j b_j + w_{s+1} b_{s+1}, \sum_{j=1}^{s} w_j c_j + w_{s+1} c_{s+1});$$
$$\{(1-\prod_{j=1}^{s}(1-\alpha_j)^{w_j}) + (1-(1-\alpha_{(s+1)})^{w_{s+1}}) - (1-\prod_{j=1}^{s}(1-\alpha_j)^{w_j}) \times (1-(1-\alpha_{(s+1)})^{w_{s+1}}) : \alpha_j \in \{w^i_{\tilde{a}_j} : i \in I_{m_j}\},$$

$$\alpha_{s+1} \in \{w^i_{\tilde{a}_{s+1}} : i \in I_{m_{s+1}}\}\}, \{(\prod_{j=1}^{s} \beta_j^{w_j})\beta_{s+1}^{w_{s+1}} : \beta_j \in \{u^r_{\tilde{a}_j} : r \in I_{n_j}\}, \beta_{s+1} \in \{u^r_{\tilde{a}_{s+1}} : r \in I_{n_{s+1}}\}\},$$

$$\{(\prod_{j=1}^{s} \lambda_j^{w_j})\lambda_{s+1}^{w_{s+1}} : \lambda_j \in \{y^l_{\tilde{a}_j} : l \in I_{k_j}\}, \lambda_{s+1} \in \{y^l_{\tilde{a}_{s+1}} : l \in I_{k_{s+1}}\}\} >$$

$$= < (\sum_{j=1}^{s} w_j a_j + w_{s+1}a_{s+1}, \sum_{j=1}^{s} w_j b_j + w_{s+1}b_{s+1}, \sum_{j=1}^{s} w_j c_j + w_{s+1}c_{s+1}); \{1 - ((\prod_{j=1}^{s}(1-\alpha_j)^{w_j})\alpha_{s+1}^{w_{s+1}}) : \alpha_j \in \{w^i_{\tilde{a}_j} : i \in I_{m_j}\},$$

$$\alpha_{s+1} \in \{w^i_{\tilde{a}_{s+1}} : i \in I_{m_{s+1}}\}\}, \{(\prod_{j=1}^{s} \beta_j^{w_j})\beta_{s+1}^{w_{s+1}} : \beta_j \in \{u^r_{\tilde{a}_j} : r \in I_{n_j}\}, \beta_{s+1} \in \{u^r_{\tilde{a}_{s+1}} : r \in I_{n_{s+1}}\}\},$$

$$\{(\prod_{j=1}^{s} \lambda_j^{w_j})\lambda_{s+1}^{w_{s+1}} : \lambda_j \in \{y^l_{\tilde{a}_j} : l \in I_{k_j}\}, \lambda_{s+1} \in \{y^l_{\tilde{a}_{s+1}} : l \in I_{k_{s+1}}\}\} >$$

$$= < (\sum_{j=1}^{s+1} w_j a_j, \sum_{j=1}^{s+1} w_j b_j, \sum_{j=1}^{s+1} w_j c_j), \{1 - \prod_{j=1}^{s+1}(1-\alpha_j)^{w_j} : \alpha_j \in \{w^i_{\tilde{a}_j} : i \in I_{m_j}\}\}, \{\prod_{j=1}^{s+1} \beta_j^{w_j} : \beta_j \in \{u^r_{\tilde{a}_j} : r \in I_{n_j}\}\},$$

$$\{\prod_{j=1}^{s+1} \lambda_j^{w_j} : \lambda_j \in \{y^l_{\tilde{a}_j} : l \in I_{k_j}\}\} >$$

Thus the result is true for *n*=*s*+1 also. Hence by the principle of mathematical induction, the result is true for any natural number *n*.

Theorem 15: Let $\tilde{a}_j = < (a_j, b_j, c_j); \{w^i_{\tilde{a}_j} : i \in I_{m_j}\}, \{u^r_{\tilde{a}_j} : r \in I_{n_j}\}, \{y^l_{\tilde{a}_j} : l \in I_{k_j}\} >$ (j=1, 2, 3,….., *n*) be a collection of hesitant triangular neutrosophic numbers. Then for any hesitant triangular neutrosophic number $\tilde{\theta}$, we have,

(i) $HTNWAAO_{T_1}\ \tilde{\theta} \oplus' \tilde{a}_1, \tilde{\theta} \oplus' \tilde{a}_2, \tilde{\theta} \oplus' \tilde{a}_3, ..., \tilde{\theta} \oplus' \tilde{a}_n\ = \tilde{\theta} \oplus' HTNWAAO_{T_1}\ \tilde{a}_1, \tilde{a}_2, \tilde{a}_3, ..., \tilde{a}_n$

(ii) $HTNWAAO_{T_1}\ \tilde{a}_1, \tilde{a}_2, \tilde{a}_3, ..., \tilde{a}_n\ = \tilde{\theta}$ if $\tilde{a}_j = \tilde{\theta}$ for each j

Proof: Straight Forward.

Definition 16: Let $\tilde{a}_j = < (a_j, b_j, c_j); \{w^i_{\tilde{a}_j} : i \in I_{m_j}\}, \{u^r_{\tilde{a}_j} : r \in I_{n_j}\}, \{y^l_{\tilde{a}_j} : l \in I_{k_j}\} >$ (j=1, 2, 3,….., n) be a collection of hesitant triangular neutrosophic numbers. Then the hesitant triangular neutrosophic weighted geometric aggregation operator of type-1 ($HTNWGAO_{T_1}$ for short) is defined as:

$$HTNWGAO_{T_1}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, ..., \tilde{a}_n) = (w_1 *' \tilde{a}_1) \otimes' (w_2 *' \tilde{a}_2) \otimes' (w_3 *' \tilde{a}_3) \otimes' .... \otimes' (w_n *' \tilde{a}_n)$$

where w_j is the weight of $\tilde{a}_j$ (j=1,2,3,……,n) such that $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$.

Theorem 17: Let $\tilde{a}_j = < (a_j, b_j, c_j); \{w^i_{\tilde{a}_j} : i \in I_{m_j}\}, \{u^r_{\tilde{a}_j} : r \in I_{n_j}\}, \{y^l_{\tilde{a}_j} : l \in I_{k_j}\} >$ (j=1,2,3,…..,n) be a collection of hesitant triangularneutrosophic numbers. Then $HTNWGAO_{T_1}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, ..., \tilde{a}_n)$ is a hesitant triangular neutrosophic number and $HTNWGAO_{T_1}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, ..., \tilde{a}_n)$

$$= < (\prod_{j=1}^{n} a_j^{w_j}, \prod_{j=1}^{n} b_j^{w_j}, \prod_{j=1}^{n} c_j^{w_j}); \prod_{j=1}^{n} \alpha_j^{w_j} : \alpha_j \in \{w^i_{\tilde{a}_j} : i \in I_{m_j}\}\}, \{1 - \prod_{j=1}^{n}(1-\beta_j)^{w_j} : \beta_j \in \{u^r_{\tilde{a}_j} : r \in I_{n_j}\}\},$$

$$\{1 - \prod_{j=1}^{n}(1-\lambda_j)^{w_j} : \lambda_j \in \{y^l_{\tilde{a}_j} : l \in I_{k_j}\}\} >$$

where w_j is the weight of $\tilde{a}_j$ (j=1,2,3,……,n) such that $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$.

Proof: Similar to the proof of Theorem **14.**

Theorem 18: Let $\tilde{a}_j = < (a_j, b_j, c_j); \{w^i_{\tilde{a}_j} : i \in I_{m_j}\}, \{u^r_{\tilde{a}_j} : r \in I_{n_j}\}, \{y^l_{\tilde{a}_j} : l \in I_{k_j}\} >$ (j=1, 2, 3,….., *n*) be a collection of hesitant triangular neutrosophic numbers. Then for any hesitant triangular neutrosophic number $\tilde{\theta}$, we have,

(i) $HTNWGAO_{T_1}\left(\tilde{\theta}\oplus'\tilde{a}_1,\tilde{\theta}\oplus'\tilde{a}_2,\tilde{\theta}\oplus'\tilde{a}_3,......,\tilde{\theta}\oplus'\tilde{a}_n\right)=\tilde{\theta}\oplus' HTNWGAO_{T_1}\left(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3,......,\tilde{a}_n\right)$

(ii) $HTNWGAO_{T_1}\left(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3,......,\tilde{a}_n\right)=\tilde{\theta}$ if $\tilde{a}_j=\tilde{\theta}$ for each j

Proof: Straight forward .

Definition 19: Let $\tilde{a}_j=<(a_j,b_j,c_j);\{w^i_{\tilde{a}_j}:i\in I_{m_j}\},\{u^r_{\tilde{a}_j}:r\in I_{n_j}\},\{y^l_{\tilde{a}_j}:l\in I_{k_j}\}>$ (j=1, 2, 3,….., n) be a collection of hesitant triangular neutrosophic numbers. Then the hesitant triangular neutrosophic weighted arithmetic aggregation operator of type-2 is denoted by $HTNWAAO_{T_2}(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3,......,\tilde{a}_n)$ and is defined by:

$$HTNWAAO_{T_2}(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3,......,\tilde{a}_n)=(w_1\odot''\tilde{a}_1)\oplus''(w_2\odot''\tilde{a}_2)\oplus''(w_3\odot''\tilde{a}_3)\oplus''........\oplus''(w_n\odot''\tilde{a}_n)$$

where w_j is the weight of $\tilde{a}_j$ (j=1,2,3,……,n) such that $w_j\geq 0$ and $\sum_{j=1}^{n}w_j=1$.

Theorem 20: Let $\tilde{a}_j=<(a_j,b_j,c_j);\{w^i_{\tilde{a}_j}:i\in I_{m_j}\},\{u^r_{\tilde{a}_j}:r\in I_{n_j}\},\{y^l_{\tilde{a}_j}:l\in I_{k_j}\}>$ (j=1, 2, 3,….., n) be a collection of hesitant triangular neutrosophic numbers. Then $HTNWAAO_{T_2}(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3,......,\tilde{a}_n)$ is a hesitant triangular neutrosophic number and $HTNWAAO_{T_2}(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3,......,\tilde{a}_n)$

$$=<(\sum_{j=1}^{n}w_ja_j,\sum_{j=1}^{n}w_jb_j,\sum_{j=1}^{n}w_jc_j),\{1-\frac{1}{1+\left\{\sum_{j=1}^{n}w_j\left(\frac{\alpha_j^2}{1-\alpha_j^2}\right)^2\right\}^{\frac{1}{2}}}:\alpha_j\in\{w^i_{\tilde{a}_j}:i\in I_{m_j}\}\},\{\frac{1}{1+\left\{\sum_{j=1}^{n}w_j\left(\frac{1-\beta_j^2}{\beta_j^2}\right)^2\right\}^{\frac{1}{2}}}:$$

$$\beta_j\in\{u^r_{\tilde{a}_j}:r\in I_{n_j}\}\},\{\frac{1}{1+\left\{\sum_{j=1}^{n}w_j\left(\frac{1-\lambda_j^2}{\lambda_j^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_j\in\{y^l_{\tilde{a}_j}:l\in I_{k_j}\}\}>$$

where w_j is the weight of $\tilde{a}_j$ (j=1,2,3,……,n) such that $w_j\geq 0$ and $\sum_{j=1}^{n}w_j=1$.

<u>**Proof:**</u>For n=2, we have,

$HTNWAAO_{T_2}(\tilde{a}_1,\tilde{a}_2)$

$=(w_1\odot''\tilde{a}_1)\oplus''(w_2\odot''\tilde{a}_2)$

$$=<(w_1a_1,w_1b_1,w_1c_1);\{1-\frac{1}{1+\left\{w_1\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2\right\}^{\frac{1}{2}}}:\alpha_1\in\{w^i_{\tilde{a}_1}:i\in I_{m_1}\}\},\frac{1}{1+\left\{w_1\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2\right\}^{\frac{1}{2}}}:\{\beta_1\in\{u^r_{\tilde{a}_1}:r\in I_{n_1}\}\},$$

$$\{\frac{1}{1+\left\{w_1\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_1\in\{y^l_{\tilde{a}_1}:l\in I_{k_1}\}\}>\oplus''<(w_2a_2,w_2b_2,w_2c_2);\{1-\frac{1}{1+\left\{w_2\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2\right\}^{\frac{1}{2}}}:$$

$$\{\alpha_2\in\{w^i_{\tilde{a}_2}:i\in I_{m_2}\}\},\{\frac{1}{1+\left\{w_2\left(\frac{1-\beta_2^2}{\beta_2^2}\right)^2\right\}^{\frac{1}{2}}}:\beta_2\in u_{\tilde{a}_2}\},\{\frac{1}{1+\left\{w_2\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_2\in\{y^l_{\tilde{a}_2}:l\in I_{k_2}\}\}>$$

$$=<(w_1a_1+w_2a_2,w_1b_1+w_2b_2,w_1c_1+w_2c_2);$$

$$\{1-\frac{1}{1+\left\{\left(\frac{1-\frac{1}{1+\left\{w_1\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2\right\}^{\frac{1}{2}}}}{1-1+\frac{1}{1+\left\{w_1\left(\frac{\alpha_{1k}^2}{1-\alpha_1^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2+\left(\frac{1-\frac{1}{1+\left\{w_2\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2\right\}^{\frac{1}{2}}}}{1-1+\frac{1}{1+\left\{w_2\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}}:\alpha_1\in\{w_{\tilde{a}_1}^i:i\in I_{m_1}\},\alpha_2\in\{w_{\tilde{a}_2}^i:i\in I_{m_2}\}\},$$

$$\{\frac{1}{1+\left\{\left(\frac{1-\frac{1}{1+\left\{w_1\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{w_1\left(\frac{1-\beta_{1m}^2}{\beta_{1m}^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2+\left(\frac{1-\frac{1}{1+\left\{w_2\left(\frac{1-\beta_2^2}{\beta_2^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{w_2\left(\frac{1-\beta_{2m}^2}{\beta_{2m}^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}}:\beta_1\in\{u_{\tilde{a}_1}^r:r\in I_{n_1}\},\beta_2\in\{u_{\tilde{a}_2}^r:j\in I_{n_2}\}\},$$

$$\{\frac{1}{1+\left\{\left(\frac{1-\frac{1}{1+\left\{w_1\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{w_1\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2+\left(\frac{1-\frac{1}{1+\left\{w_2\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{w_2\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde{a}_1}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde{a}_2}^l:l\in I_{k_2}\}\}>$$

$$=<(w_1a_1+w_2a_2,w_1b_1+w_2b_2,w_1c_1+w_2c_2);\{1-\frac{1}{1+\left\{w_1\left(\frac{\alpha_1^2}{1-\alpha_1^2}\right)^2+w_2\left(\frac{\alpha_2^2}{1-\alpha_2^2}\right)^2\right\}^{\frac{1}{2}}}:\alpha_1\in\{w_{\tilde{a}_1}^i:i\in I_{m_1}\},$$

$$\alpha_2\in\{w_{\tilde{a}_2}^i:i\in I_{m_2}\}\},\{\frac{1}{1+\left\{w_1\left(\frac{1-\beta_1^2}{\beta_1^2}\right)^2+w_2\left(\frac{1-\beta_2^2}{\beta_2^2}\right)^2\right\}^{\frac{1}{2}}}:\beta_1\in\{u_{\tilde{a}_1}^r:r\in I_{n_1}\},\beta_2\in\{u_{\tilde{a}_2}^r:j\in I_{n_2}\}\},$$

$$\{\frac{1}{1+\left\{w_1\left(\frac{1-\lambda_1^2}{\lambda_1^2}\right)^2+w_2\left(\frac{1-\lambda_2^2}{\lambda_2^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_1\in\{y_{\tilde{a}_1}^l:l\in I_{k_1}\},\lambda_2\in\{y_{\tilde{a}_2}^l:l\in I_{k_2}\}\}>$$

$$=<(\sum_{j=1}^{2}w_ja_j,\sum_{j=1}^{2}w_jb_j,\sum_{j=1}^{2}w_jc_j);\{1-\frac{1}{1+\left\{\sum_{j=1}^{2}w_j\left(\frac{\alpha_j^2}{1-\alpha_j^2}\right)^2\right\}^{\frac{1}{2}}}:\alpha_j\in\{w_{\tilde{a}_j}^i:i\in I_{m_j}\}\},\{\frac{1}{1+\left\{\sum_{j=1}^{2}w_j\left(\frac{1-\beta_j^2}{\beta_j^2}\right)^2\right\}^{\frac{1}{2}}}:$$

$$\beta_j \in \{u^r_{\tilde{a}_j} : r \in I_{n_j}\}\}, \{\frac{1}{1+\left\{\sum_{j=1}^{2} w_j \left(\frac{1-\lambda_j^2}{\lambda_j^2}\right)^2\right\}^{\frac{1}{2}}} : \lambda_j \in \{y^l_{\tilde{a}_j} : l \in I_{k_j}\}\} >$$

Thus the result is true for n=2. Let us assume that the result is true for n=s.

Then we have, $HTNWAAO_{T_2}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, ..., \tilde{a}_s)$

$$= < (\sum_{j=1}^{s} w_j a_j, \sum_{j=1}^{s} w_j b_j, \sum_{j=1}^{s} w_j c_j), \{1 - \frac{1}{1+\left\{\sum_{j=1}^{s} w_j \left(\frac{\alpha_j^2}{1-\alpha_j^2}\right)^2\right\}^{\frac{1}{2}}} : \alpha_j \in \{w^i_{\tilde{a}_j} : i \in I_{m_j}\}\}, \{\frac{1}{1+\left\{\sum_{j=1}^{s} w_j \left(\frac{1-\beta_j^2}{\beta_j^2}\right)^2\right\}^{\frac{1}{2}}} :$$

$$\beta_j \in \{u^r_{\tilde{a}_j} : r \in I_{n_j}\}\}, \{\frac{1}{1+\left\{\sum_{j=1}^{s} w_j \left(\frac{1-\lambda_j^2}{\lambda_j^2}\right)^2\right\}^{\frac{1}{2}}} : \lambda_j \in \{y^l_{\tilde{a}_j} : l \in I_{k_j}\}\} >$$

Now for n=s+1, we have, $HTNWAO_{T_2}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, ..., \tilde{a}_{s+1})$

$$= < (\sum_{j=1}^{s} w_j a_j, \sum_{j=1}^{s} w_j b_j, \sum_{j=1}^{s} w_j c_j); \{1 - \frac{1}{1+\left\{\sum_{j=1}^{s} w_j \left(\frac{\alpha_j^2}{1-\alpha_j^2}\right)^2\right\}^{\frac{1}{2}}} : \alpha_j \in \{w^i_{\tilde{a}_j} : i \in I_{m_j}\}\},$$

$$\{\frac{1}{1+\left\{\sum_{j=1}^{s} w_j \left(\frac{1-\beta_j^2}{\beta_j^2}\right)^2\right\}^{\frac{1}{2}}} : \beta_j \in \{u^r_{\tilde{a}_j} : r \in I_{n_j}\}\}, \{\frac{1}{1+\left\{\sum_{j=1}^{s} w_j \left(\frac{1-\lambda_j^2}{\lambda_j^2}\right)^2\right\}^{\frac{1}{2}}} : \lambda_j \in \{y^l_{\tilde{a}_j} : l \in I_{k_j}\}\} >$$

$$\oplus'' < (w_{s+1} a_{s+1}, w_{s+1} b_{s+1}, w_{s+1} c_{s+1}); 1 - \frac{1}{1+\left\{w_{s+1} \left(\frac{\alpha_{(s+1)}^2}{1-\alpha_{(s+1)}^2}\right)^2\right\}^{\frac{1}{2}}} : \alpha_{s+1} \in \{w^i_{\tilde{a}_{s+1}} : i \in I_{m_{s+1}}\}\},$$

$$\{\frac{1}{1+\left\{w_{s+1} \left(\frac{1-\beta_{(s+1)}^2}{\beta_{(s+1)}^2}\right)^2\right\}^{\frac{1}{2}}} : \beta_{s+1} \in \{u^r_{\tilde{a}_{s+1}} : r \in I_{n_{s+1}}\}\}, \{\frac{1}{1+\left\{w_{s+1} \left(\frac{1-\lambda_{(s+1)}^2}{\lambda_{(s+1)}^2}\right)^2\right\}^{\frac{1}{2}}} : \lambda_{s+1} \in \{y^l_{\tilde{a}_{s+1}} : l \in I_{k_{s+1}}\}\} >$$

$$= < (w_1 a_1 + w_2 a_2, w_1 b_1 + w_2 b_2, w_1 c_1 + w_2 c_2),$$

$$\{1 - \frac{1}{1+\left\{\left(\frac{1-\frac{1}{1+\left\{\sum_{j=1}^{s} w_j \left(\frac{\alpha_{jk}^2}{1-\alpha_j^2}\right)^2\right\}^{\frac{1}{2}}}}{1-1+\frac{1}{1+\left\{\sum_{j=1}^{s} w_j \left(\frac{\alpha_j^2}{1-\alpha_j^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2 + \left(\frac{1-\frac{1}{1+\left\{w_{s+1} \left(\frac{\alpha_{(s+1)}^2}{1-\alpha_{(s+1)}^2}\right)^2\right\}^{\frac{1}{2}}}}{1-1+\frac{1}{1+\left\{w_{s+1} \left(\frac{\alpha_{(s+1)}^2}{1-\alpha_{(s+1)}^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}} : \alpha_j \in \{w^i_{\tilde{a}_j} : i \in I_{m_j}\}, \alpha_{s+1} \in \{w^i_{\tilde{a}_{s+1}} : i \in I_{m_{s+1}}\}\},$$

$$\{\frac{1}{1+\left\{\left(\frac{1-\frac{1}{1+\left\{\sum_{j=1}^{s}w_j\left(\frac{1-\beta_j^2}{\beta_j^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sum_{j=1}^{s}w_j\left(\frac{1-\beta_j^2}{\beta_j^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2+\left(\frac{1-\frac{1}{1+\left\{w_{s+1}\left(\frac{1-\beta_{(s+1)}^2}{\beta_{(s+1)}^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{w_{s+1}\left(\frac{1-\beta_{(s+1)}^2}{\beta_{(s+1)}^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}}:\beta_j\in\{u_{\tilde{a}_j}^r:r\in I_{n_j}\},\beta_{s+1}\in\{u_{\tilde{a}_{s+1}}^r:r\in I_{n_{s+1}}\}\},$$

$$\{\frac{1}{1+\left\{\left(\frac{1-\frac{1}{1+\left\{\sum_{j=1}^{s}w_j\left(\frac{1-\lambda_j^2}{\lambda_j^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{\sum_{j=1}^{s}w_j\left(\frac{1-\lambda_j^2}{\lambda_j^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2+\left(\frac{1-\frac{1}{1+\left\{w_{s+1}\left(\frac{1-\lambda_{(s+1)}^2}{\lambda_{(s+1)}^2}\right)^2\right\}^{\frac{1}{2}}}}{\frac{1}{1+\left\{w_{s+1}\left(\frac{1-\lambda_{(s+1)}^2}{\lambda_{(s+1)}^2}\right)^2\right\}^{\frac{1}{2}}}}\right)^2\right\}^{\frac{1}{2}}}:\lambda_j\in\{y_{\tilde{a}_j}^l:l\in I_{k_j}\},\lambda_{s+1}\in\{y_{\tilde{a}_{s+1}}^l:$$

$$l\in I_{k_{s+1}}\}\}>$$

$$=<(\sum_{j=1}^{s+1}w_ja_j,\sum_{j=1}^{s+1}w_jb_j,\sum_{j=1}^{s+1}w_jc_j),\{1-\frac{1}{1+\left\{\sum_{j=1}^{s}w_j\left(\frac{\alpha_j^2}{1-\alpha_j^2}\right)^2+w_{s+1}\left(\frac{\alpha_{(s+1)}^2}{1-\alpha_{(s+1)}^2}\right)^2\right\}^{\frac{1}{2}}}:$$

$$\alpha_j\in\{w_{\tilde{a}_j}^i:i\in I_{m_j}\},\alpha_{s+1}\in\{w_{\tilde{a}_{s+1}}^i:i\in I_{m_{s+1}}\}\},$$

$$\{\frac{1}{1+\left\{\sum_{j=1}^{s}w_j\left(\frac{1-\beta_j^2}{\beta_j^2}\right)^2+w_{s+1}\left(\frac{1-\beta_{(s+1)}^2}{\beta_{(s+1)}^2}\right)^2\right\}^{\frac{1}{2}}}:\beta_j\in\{u_{\tilde{a}_j}^r:r\in I_{n_j}\},\beta_{s+1}\in\{u_{\tilde{a}_{s+1}}^r:r\in I_{n_{s+1}}\}\},$$

$$\{\frac{1}{1+\left\{\sum_{j=1}^{s}w_j\left(\frac{1-\lambda_j^2}{\lambda_j^2}\right)^2+w_{s+1}\left(\frac{1-\lambda_{(s+1)}^2}{\lambda_{(s+1)}^2}\right)^2\right\}^{\frac{1}{2}}}:\lambda_j\in\{y_{\tilde{a}_j}^l:l\in I_{k_j}\},\lambda_{s+1}\in\{y_{\tilde{a}_{s+1}}^l:l\in I_{k_{s+1}}\}\}>$$

$$=<(\sum_{j=1}^{s+1}w_ja_j,\sum_{j=1}^{s+1}w_jb_j,\sum_{j=1}^{s+1}w_jc_j);\{1-\frac{1}{1+\left\{\sum_{j=1}^{s+1}w_j\left(\frac{\alpha_j^2}{1-\alpha_j^2}\right)^2\right\}^{\frac{1}{2}}}:\alpha_j\in\{w_{\tilde{a}_j}^i:i\in I_{m_j}\},\alpha_{s+1}\in\{w_{\tilde{a}_{s+1}}^i:i\in I_{m_{s+1}}\}\},$$

$$\{\frac{1}{1+\left\{\sum_{j=1}^{s+1}w_j\left(\frac{1-\beta_j^2}{\beta_j^2}\right)^2\right\}^{\frac{1}{2}}}:\beta_j\in\{u_{\tilde{a}_j}^r:r\in I_{n_j}\},\beta_{s+1}\in\{u_{\tilde{a}_{s+1}}^r:r\in I_{n_{s+1}}\}\},\{\frac{1}{1+\left\{\sum_{j=1}^{s+1}w_j\left(\frac{1-\lambda_j^2}{\lambda_j^2}\right)^2\right\}^{\frac{1}{2}}}:$$

$$\lambda_j\in\{y_{\tilde{a}_j}^l:l\in I_{k_j}\},\lambda_{s+1}\in\{y_{\tilde{a}_{s+1}}^l:l\in I_{k_{s+1}}\}\}>$$

Thus the result is true for $n=s+1$ also. Hence by the principle of mathematical induction, the result is true for any natural number n.

Theorem 21: Let $\tilde{a}_j = <(a_j, b_j, c_j); \{w^i_{\tilde{a}_j} : i \in I_{m_j}\}, \{u^r_{\tilde{a}_j} : r \in I_{n_j}\}, \{y^l_{\tilde{a}_j} : l \in I_{k_j}\}>$ (j=1, 2, 3,….., n) be a collection of hesitant triangular neutrosophic numbers. Then for any hesitant triangular neutrosophic number $\tilde{\theta}$, we have,

(i) $HTNWAAO_{T_2}\ \tilde{\theta} \oplus'' \tilde{a}_1, \tilde{\theta} \oplus'' \tilde{a}_2, \tilde{\theta} \oplus'' \tilde{a}_3, ..., \tilde{\theta} \oplus'' \tilde{a}_n\ = \tilde{\theta} \oplus'' HTNWAAO_{T_2}\ \tilde{a}_1, \tilde{a}_2, \tilde{a}_3, ..., \tilde{a}_n$

(ii) $HTNWAAO_{T_2}\ \tilde{a}_1, \tilde{a}_2, \tilde{a}_3, ..., \tilde{a}_n\ = \tilde{\theta}$ if $\tilde{a}_j = \tilde{\theta}$ for each j

Proof: Straight forward .

Definition 22: Let $\tilde{a}_j = <(a_j, b_j, c_j); \{w^i_{\tilde{a}_j} : i \in I_{m_j}\}, \{u^r_{\tilde{a}_j} : r \in I_{n_j}\}, \{y^l_{\tilde{a}_j} : l \in I_{k_j}\}>$ (j=1, 2, 3,….., n) be a collection of hesitant triangular neutrosophic numbers. Then the hesitant triangular neutrosophicweighted geometric aggregation operator of type-2 is denoted by $HTNWGAO_{T_2}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, ..., \tilde{a}_n)$ and is defined by:

$$HTNWGAO_{T_2}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, ..., \tilde{a}_n) = (w_1 *'' \tilde{a}_1) \otimes'' (w_2 *'' \tilde{a}_2) \otimes'' (w_3 *'' \tilde{a}_3) \otimes'' .... \otimes'' (w_n *'' \tilde{a}_n)$$

where w_j is the weight of $\tilde{a}_j$ (j=1,2,3,……, n) such that $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$.

Theorem 23: Let $\tilde{a}_j = <(a_j, b_j, c_j); \{w^i_{\tilde{a}_j} : i \in I_{m_j}\}, \{u^r_{\tilde{a}_j} : r \in I_{n_j}\}, \{y^l_{\tilde{a}_j} : l \in I_{k_j}\}>$ (j=1, 2, 3,….., n) be a collection of hesitant triangular neutrosophic numbers. Then $HTNWGAO_{T_2}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, ..., \tilde{a}_n)$ is a hesitant triangular neutrosophic number and

$$HTNWGAO_{T_2}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, ..., \tilde{a}_n)$$
$$= <(\prod_{j=1}^{n} a_j^{w_j}, \prod_{j=1}^{n} b_j^{w_j}, \prod_{j=1}^{n} c_j^{w_j}); \{\frac{1}{1+\left\{\sum_{j=1}^{n} w_j \left(\frac{1-\alpha_j^2}{\alpha_j^2}\right)^2\right\}^{\frac{1}{2}}} : \alpha_j \in \{w^i_{\tilde{a}_j} : i \in I_{m_j}\}\}, \{1 - \frac{1}{1+\left\{\sum_{j=1}^{n} w_j \left(\frac{\beta_j^2}{1-\beta_j^2}\right)^2\right\}^{\frac{1}{2}}} :$$
$$\beta_j \in \{u^r_{\tilde{a}_j} : r \in I_{n_j}\}\},\ \{1 - \frac{1}{1+\left\{\sum_{j=1}^{n} w_j \left(\frac{\lambda_j^2}{1-\lambda_j^2}\right)^2\right\}^{\frac{1}{2}}} : \lambda_j \in \{y^l_{\tilde{a}_j} : l \in I_{k_j}\}>$$

where w_j is the weight of $\tilde{a}_j$ (j=1,2,3,……,n) such that $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$.

Proof: Similar to the proof of Theorem 20.

Theorem 24: Let $\tilde{a}_j = <(a_j, b_j, c_j); \{w^i_{\tilde{a}_j} : i \in I_{m_j}\}, \{u^r_{\tilde{a}_j} : r \in I_{n_j}\}, \{y^l_{\tilde{a}_j} : l \in I_{k_j}\}>$ (j=1, 2, 3,….., n) be a collection of hesitant triangular neutrosophic numbers. Then for any hesitant triangular neutrosophic number $\tilde{\theta}$, we have,

(i) $HTNWGAO_{T_2}\ \tilde{\theta} \otimes'' \tilde{a}_1, \tilde{\theta} \otimes'' \tilde{a}_2, \tilde{\theta} \otimes'' \tilde{a}_3, ..., \tilde{\theta} \otimes'' \tilde{a}_n\ = \tilde{\theta} \otimes'' HTNWGAO_{T_2}\ \tilde{a}_1, \tilde{a}_2, \tilde{a}_3, ..., \tilde{a}_n$

(ii) $HTNWGAO_{T_2}\ \tilde{a}_1, \tilde{a}_2, \tilde{a}_3, ..., \tilde{a}_n\ = \tilde{\theta}$ if $\tilde{a}_j = \tilde{\theta}$ for each j

Proof: Straight forward .

Definition 25: Let $\tilde{a} = <(a_1, b_1, c_1); \{w^i_{\tilde{a}} : i \in I_m\}, \{u^j_{\tilde{a}} : j \in I_n\}, \{y^l_{\tilde{a}} : l \in I_k\}>$ be a hesitant triangular neutrosophic number. Then the score of $\tilde{a}$ is defined by:

$$S(\tilde{a})=\frac{1}{3\times\max\{a_1,b_1,c_1\}}(a_1+b_1+c_1)\times\left[2+\frac{1}{m}\sum_{j=1}^{m}\alpha_j-\frac{1}{n}\sum_{j=1}^{n}\beta_j-\frac{1}{k}\sum_{j=1}^{k}\lambda_j\right]$$

Where $\alpha_j\in\{w^i_{\tilde{a}_j}:i\in I_{m_j}\},\beta_j\in\{u^r_{\tilde{a}_j}:r\in I_{n_j}\},\lambda_j\in\{y^l_{\tilde{a}_j}:l\in I_{k_j}\}$

If $\tilde{a}_j=<(a_j,b_j,c_j);\{w^i_{\tilde{a}_j}:i\in I_{m_j}\},\{u^r_{\tilde{a}_j}:r\in I_{n_j}\},\{y^l_{\tilde{a}_j}:l\in I_{k_j}\}>(j=1,2)$ be two hesitant triangular neutrosophic numbers, then, the comparison method is given as;

I. If $S(\tilde{a}_1)>S(\tilde{a}_2)$ then $\tilde{a}_1\succ\tilde{a}_2$

II. If $S(\tilde{a}_1)=S(\tilde{a}_2)$ then $\tilde{a}_1=\tilde{a}_2$

5. APPLICATION OF HESITANT TRAPEZOIDAL NEUTROSOPHIC NUMBERS:

In this section, we apply the weighted aggregation operators and the score function of hesitant triangular neutrosophic numbers to the multi-attribute decision-making problem with hesitant triangular neutrosophic information.

Let $X=\{A_1,A_2,A_3,......,A_m\}$ be a set of alternatives, $A=\{c_1,c_2,c_3,......,c_n\}$ be a set of attributes and $w=\{w_1,w_2,w_3,......,w_n\}$ be a set of weights (w_j is the weight of attribute c_j (j=1,2,3,……,n) such that $w_j\geq 0$ and $\sum_{j=1}^{n}w_j=1$.) In this case, the characteristic of the alternative $A_i(i=1,2,...,m)$ on attribute c_j $(j=1,2,...,n)$ is represented by the following form of a hesitant triangular neutrosophic number:

$A_{ij}=<(a_{ij},b_{ij},c_{ij});\{w^p_{\tilde{a}_{ij}}:p\in I_{m_{ij}}\},\{u^r_{\tilde{a}_{ij}}:r\in I_{n_{ij}}\},\{y^l_{\tilde{a}_{ij}}:l\in I_{k_{ij}}\}>$.

Now, we construct a multi-attribute decision making method by the following algorithm:

- **ALGORITHM:**

Step-1: Express the evaluation results of the expert based on the alternative $A_i(i=1,2,...,m)$ on attribute c_j $(1,2,\dots,n)$ in terms of hesitant triangular neutrosophic numbers x_{ij} as a $m\times n$ Table.

Step-2: Compute the aggregation values $g_i^{T_k^A}$ $(i=1,2,...,m)$ $(k=1,2)$ *or* $g_i^{T_k^G}$ $(i=1,2,...,m)$ $(k=1,2)$ of $A_i(i=1,2,...,m)$ as;

$$g_i^{T_k^A}=HTNWAAO_{T_k}(A_{i1},A_{i2},...,A_{in})\ (i=1,2,...,m)\ (k=1,2)$$

or

$$g_i^{T_k^G}=HTNWGAO_{T_k}(A_{i1},A_{i2},...,A_{in})\ (i=1,2,...,m)\ (k=1,2)$$

Step-3: Calculate the score values of $g_i^{T_k^A}$ $(i=1,2,...,m)$ $(k=1,2)$ *or* $g_i^{T_k^G}$ $(i=1,2,...,m)$ $(k=1,2)$ of $A_i(i=1,2,...,m)$ based on Definition 25.

Step-4: Rank the alternatives by using definition 25.

Example 22:

Let us consider a decision making problem adapted from Wei et al. (2017). Suppose an organisation plans to implement enterprise resource planning (ERP) system. The first step is to form a project team that consists of CIO and two senior representatives from user departments. By collecting all possible information about ERP vendors and systems, project team chooses five potential ERP systems A_i (i=1, 2, 3, 4, 5) as candidates. The company employs some external professional organizations (or experts) to aid this decision making. The project team selects four attributes to evaluate the alternatives: function and technology c_1, strategic fitness c_2, vendor's ability c_3 and vendor's reputation c_4. The five possible ERP systems A_i (i=1, 2, 3, 4, 5) are to be evaluate during the hesitant triangular neutrosophic numbers by the decision makers under the above four attributes whose weighting vector is $w=(0.3,0.3,0.2,0.2)$.

Step-1: We express the initial evaluation results of the expertforfivepossiblealternativesbased on fourattributesbythe form of hesitant triangular neutrosophic numbers, as shownin Table 1.

Table 1: The evaluation result by the expert is shown in the below table

	c_1	c_2	c_3	c_4
A_1	<(0.3,0.4,0.2); {0.5,0.6},{0.1,0.3},{0.4, 0.7}>	<(0.1,0.5,0.6);{0.3},{0.4, 0.5},{0.3,0.6}>	<(0.5,0.6,0.7);{0.2,0.4} ,{0.5},{0.3,0.4}>	<(0.4,0.6,0.7);{0.8},{0.6, 0.7},{0.1,0.2}>
A_2	<(0.2,0.3,0.4); {0.2}, {0.5,0.7},{0.4}>	<(0.4,0.6,0.6);{0.7,0.9},{ 0.2},{0.5,0.7}>	<(0.5,0.7,0.9);{0.1,0.2} , {0.6,0.8},{0.5,0.6}>	<(0.2,0.3,0.5);{0.6},{0.4, 0.5},{0.2,0.4}>
A_3	<(0.6,0.6,0.7);0.1,0.4},{0 .6,0.8},{0.4,0.5,0.7}>	<(0.3,0.4,0.4);{0.6, 0.7},{0.5},{0.5,0.8}>	<(0.1,0.2,0.3);{0.2,0.3} ,{0.4,0.6},{0.5,0.6}>	<(0.5,0.5,0.6);{0.3,0.5}, {0.3,0.5},{0.2}>
A_4	<(0.1,0.3,0.3); {0.7}, {0.6,0.9},{0.4}	<(0.5,0.5,0.5);{0.1,0.2},{ 0.4,0.7},{0.2,0.5}>	<(0.4,0.5,0.6);{0.2,0.6} ,{0.2,0.4},{0.3,0.4}>	<(0.1,0.1,0.2);{0.3,0.5}, {0.7,0.8},{0.1,0.2}>
A_5	<(0.5,0.6,0.6); {0.3,0.6}, {0.4,0.5},{0.2,0.4}>	<(0.2,0.3,0.3); {0.3,0.5},{0.4,0.5},{0.4, 0.6}>	<(0.7,0.8,0.8);{0.5,0.7} ,{0.3,0.6},{0.1,0.3}>	<(0.2,0.3,0.5); {0.4,0.5},{0.2,0.3},{0.6, 0.7,0.8}>

Step-2: We compute the aggregationvalues $g_i^{T_1^A}\ (i=1,2,...,5)$ of $A_i (i=1,2,...,5)$ as;

$g_1^{T_1^A} = HTNWAAO_{T_1}(A_{11}, A_{12}, A_{13}, A_{14})$

= <(0.30,0.51,0.52); {0.494124,0.522409, 0.526880, 0.553333}, {0.299254,0.308624,0.319973,0.329992,0.416080, 0.429108, 0.444888,0.458818},{0.262529, 0.301566,0.278077,0.319426, 0.323211, 0.371272, 0.342353, 0.393260,0.310519,0.356693, 0.328909,0.377818,0.382294, 0.439141,0.412567,0.465148}>

$g_2^{T_1^A} = HTNWAAO_{T_1}(A_{21}, A_{22}, A_{23}, A_{24})$

= <(0.32,0.47,0.58); {0.468719,0.481089,0.617891,0.626787},{0.376740,0.393934, 0.399052,0.417264,0.416754, {0.389321,0.447212,0.403779, 0.435774,0.441436,0.461583},0.463821,0.430672,0.494712, 0.446666,0.513085}>

$g_3^{T_1^A} = HTNWAAO_{T_1}(A_{31}, A_{32}, A_{33}, A_{34})$

= <(0.39,0.44,0.51);{0.419636, 0.457406,0.434930,0.471704,0.467623,0.502270,0.481653, 0.515387,0.344568, 0.387223, 0.361840,0.403371,0.398762, 0.437891, 0.414606,0.452704},{0.456007,0.505058,0.494527, 0.547722,0.497111, 0.550583, 0.539102,0.597092},{0.389321, 0.448274,0.479310,0.460489, 0.416275,0.530219,0.403779, 0.464922,0.497111,0.477590, 0.431735,0.549910}>

$g_4^{T_1^A} = HTNWAAO_{T_1}(A_{41}, A_{42}, A_{43}, A_{44})$

= <(0.39,0.44,0.51);{0.398762,0.437891, 0.476592,0.510656,0.419636,0.457406, 0.494764,0.527644},{0.439833, 0.451737,0.505235,0.518910,0.520235,0.534315, 0.597593,0.613767,0.496724,0.510168, 0.570586,0.586030,0.587525, 0.603427, 0.674889,0.693156},{0.232461,0.267027,0.282842,0.246228,0.306007,0.351510, 0.372328,0.324130}>

$g_5^{T_1^A} = HTNWAAO_{T_1}(A_{51}, A_{52}, A_{53}, A_{54})$

= <(0.39,0.49,0.53);{0.365425,0.388147,0.427055, 0.447570,0.426353,0.446894, 0.482066,0.500612,0.463497,0.482708,0.515603,0.532948,0.515010, 0.532376,0.562112,0.577792},{0.328749, 0.356519,0.377634,0.409533,0.351510,0.381202,0.403779,0.437887,0.375847, 0.407595,0.431735,0.468204}, {0.267027,0.275388,0.282842,0.332644,0.343059,0.352345, 0.301566,0.311008,0.319426,0.375670, 0.387433, 0.397920,0.328749,0.339042,0.348219, 0.409533,0.422356,0.433787,0.371272,0.382897, 0.393260,0.462505, 0.476986,0.489897}>

Step-3: Wecalculate the score values of $g_i^{T_k^A}\,(i=1,2,...,5)$ of $A_i\,(i=1,2,...,5)$ as

$$S(A_1)=\frac{(0.30+0.51+0.52)}{3\times0.52}\times[2+\frac{1}{4}(0.494124+0.522409+\ 0.526880+\ 0.553333)$$
$$-\frac{1}{8}(0.299254+0.308624+0.319973+0.329992+0.416080+0.429108+0.444888+0.458818)$$
$$-\frac{1}{16}(0.262529+0.301566+0.278077+0.319426+0.323211+0.371272+0.342353+0.393260$$
$$+0.310519+0.356693+0.328909+0.377818+0.382294+0.439141+0.412567+0.465148)$$
$$=\frac{1.33}{1.56}\times[2+0.524186-0.375842-0.354048]$$
$$=1.5297,$$

Similarly, we have; $S(A_2)=1.3244$, $S(A_3)=1.2687$, $S(A_4)=1.4110$, $S(A_5)=1.5235$.

Step-4: Since $S(A_1)>S(A_5)>S(A_4)>S(A_2)>S(A_3)$, So $A_1\succ A_5\succ A_4\succ A_2\succ A_3$.

Thus we conclude that A_1 is the best (most desirable) ERP system. On the other hand, if we apply the other proposed weighted aggregation operators such as $HTNWGAO_{T_1}$, $HTNWAAO_{T_2}$, $HTNWGAO_{T_2}$ for computing the best alternative(s), then step 2 of the proposed approach has been executed for each weighted aggregation operators and hence their corresponding hesitant triangular neutrosophic number has been constructed. Finally, based on these, the score values of the aggregated hesitant triangular neutrosophic numbers are computed and ranking has been done which are summarized in table-2. We can conclude from table-2 that although the ranking orders of the alternatives are slightly different; the best (most desirable) alternative is still A_1 in all cases.

Table-2: Ranking order of alternatives

Weighted aggregation operators	Ranking	Best alternative
$HTNWAAO_{T_1}$	$A_1\succ A_5\succ A_4\succ A_2\succ A_3$	A_1
$HTNWGAO_{T_1}$	$A_1\succ A_5\succ A_2\succ A_4\succ A_3$	A_1
$HTNWAAO_{T_2}$	$A_1\succ A_2\succ A_5\succ A_3\succ A_4$	A_1
$HTNWGAO_{T_2}$	$A_1\succ A_2\succ A_3\succ A_5\succ A_4$	A_1

6. COMPARATIVE STUDY:

In order to compare the performance of the proposed method with some existing methods (Ye 2013a, Ye 2014, Ye 2015a, Ye 2015b, Liu 2016, Abdel-Basset et al. 2017, Wei et al. 2017), a comparative study is presented and their corresponding final ranking are summarized in table 3. From table-3, it is clear that although the ranking order of the alternatives are slightly different, but the best (most desirable) alternative is the same as found in the existing approaches (Ye 2013a, Ye 2014, Ye 2015a, Ye 2015b, Liu 2016, Abdel-Basset et al. 2017). Thus, our proposed method can be suitably utilized to solve the multi attribute decision making problems than the other existing methods due to the fact that more fuzziness and uncertainties are involved in our proposed approach.

Table 3: Comparative study

Existing approach	Ranking	Our proposed method	Ranking	Best alternative
Ye **[36]**	$A_2 \succ A_4 \succ A_3 \succ A_1$	$HTNWAAO_{T_1}$	$A_2 \succ A_4 \succ A_3 \succ A_1$	A_2
		$HTNWGAO_{T_1}$	$A_2 \succ A_3 \succ A_4 \succ A_1$	A_2
		$HTNWAAO_{T_2}$	$A_2 \succ A_4 \succ A_1 \succ A_3$	A_2
		$HTNWGAO_{T_2}$	$A_2 \succ A_1 \succ A_4 \succ A_3$	A_2
Ye **[17]**	$A_2 \succ A_4 \succ A_3 \succ A_1$	$HTNWAAO_{T_1}$	$A_2 \succ A_4 \succ A_3 \succ A_1$	A_2
		$HTNWGAO_{T_1}$	$A_2 \succ A_3 \succ A_4 \succ A_1$	A_2
		$HTNWAAO_{T_2}$	$A_2 \succ A_4 \succ A_1 \succ A_3$	A_2
		$HTNWGAO_{T_2}$	$A_2 \succ A_1 \succ A_4 \succ A_3$	A_2
Ye **[40]**	$A_2 \succ A_4 \succ A_3 \succ A_1$	$HTNWAAO_{T_1}$	$A_2 \succ A_4 \succ A_3 \succ A_1$	A_2
		$HTNWGAO_{T_1}$	$A_2 \succ A_3 \succ A_4 \succ A_1$	A_2
		$HTNWAAO_{T_2}$	$A_2 \succ A_4 \succ A_1 \succ A_3$	A_2
		$HTNWGAO_{T_2}$	$A_2 \succ A_1 \succ A_4 \succ A_3$	A_2
Ye **[38]**	$A_4 \succ A_2 \succ A_3 \succ A_1$	$HTNWAAO_{T_1}$	$A_4 \succ A_2 \succ A_3 \succ A_1$	A_4
		$HTNWGAO_{T_1}$	$A_4 \succ A_3 \succ A_2 \succ A_1$	A_4
		$HTNWAAO_{T_2}$	$A_4 \succ A_2 \succ A_1 \succ A_3$	A_4
		$HTNWGAO_{T_2}$	$A_4 \succ A_1 \succ A_2 \succ A_3$	A_4
Liu **[21]**	$A_4 \succ A_2 \succ A_3 \succ A_1$	$HTNWAAO_{T_1}$	$A_4 \succ A_2 \succ A_3 \succ A_1$	A_4
		$HTNWGAO_{T_1}$	$A_4 \succ A_3 \succ A_2 \succ A_1$	A_4
		$HTNWAAO_{T_2}$	$A_4 \succ A_2 \succ A_1 \succ A_3$	A_4
		$HTNWGAO_{T_2}$	$A_4 \succ A_1 \succ A_2 \succ A_3$	A_4
Abdel-Basset et al. **[57]**	$A_4 \succ A_2 \succ A_3 \succ A_1$	$HTNWAAO_{T_1}$	$A_4 \succ A_2 \succ A_3 \succ A_1$	A_4
		$HTNWGAO_{T_1}$	$A_4 \succ A_3 \succ A_2 \succ A_1$	A_4
		$HTNWAAO_{T_2}$	$A_4 \succ A_2 \succ A_1 \succ A_3$	A_4
		$HTNWGAO_{T_2}$	$A_4 \succ A_1 \succ A_2 \succ A_3$	A_4

7. CONCLUSION

In this paper, hesitant triangular neutrosophic numbers and their basic properties are presented. Also, various types of operations between the hesitant triangular neutrosophic numbers are discussed. Then, various types of hesitant triangular neutrosophic weighted aggregation operators are proposed to aggregate the hesitant triangular neutrosophic information. Furthermore, score of hesitant triangular neutrosophic numbers is proposed to ranking the hesitant triangular neutrosophic numbers.Based on the hesitant triangular neutrosophic weighted aggregation operators and score of hesitant triangular neutrosophic numbers, a multi attribute decision making method is developed, in which the evaluation values of alternatives on the attribute are represented in terms of hesitant triangular neutrosophic numbers and the alternatives are ranked according to the values of the score of hesitant triangular neutrosophic numbers to select the most desirable one. Finally, a practical example for enterprise resource planning (ERP) system selection is presented to demonstrate the application and effectiveness of the proposed method. The advantage of the proposed method is that it is more suitable for solving multi attribute decision making problems with hesitant triangular neutrosophic information because hesitant triangular neutrosophic numbers can handle indeterminate and inconsistent information and are the extensions of hesitant triangular fuzzy numbers, hesitant triangular intuitionistic fuzzy numbersas well as triangular neutrosophic numbers.

In the future, we will develop another approach called linguistic hesitant triangular neutrosophic number as a further generalization of it and this will be applied in different practical problems.

FUNDING: This research received no external funding.

ACKNOWLEDGEMENTS: Nil.

CONFLICTS OF INTEREST: The authors declare no conflict of interest.

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Received: April 11, 2020. Accepted: July 2, 2020

Neutrosophic Sets and Systems, Vol. 35, 2020

University of New Mexico

Translative and Multiplicative Interpretation of Neutrosophic Cubic Set

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Abstract: In this paper, we introduce the idea of neutrosophic cubic translation (NCT) and neutrosophic cubic multiplication (NCM) and provide entirely new type of conditions for neutrosophic cubic translation and neutrosophic cubic multiplication on BF-algebra. This is the new kind of approach towards translation and multiplication which involves the indeterminacy membership function. We also define neutrosophic cubic magnified translation (NCMT) on BF-algebra which handles the neutrosophic cubic translation and neutrosophic cubic multiplication at the same time on membership function, indeterminacy membership function and non-membership function. We present the examples for better understanding of neutrosophic cubic translation, neutrosophic cubic multiplication, and neutrosophic cubic magnified translation, and investigate significant results of BF-ideal and BF-subalgebra by applying the ideas of NCT, NCM and NCMT. Intersection and union of neutrosophic cubic BF-ideals are also explained through this new type of translation and multiplication.

Keywords: BF-algebra, neutrosophic cubic translation, neutrosophic cubic multiplication, neutrosophic cubic BF ideal, neutrosophic cubic BF subalgebra, neutrosophic cubic magnified translation.

1. Introduction

Zadeh [1] presented the theory of fuzzy set in 1965. Fuzzy idea has been applied to different algebraic structures like groups, rings, modules, vector spaces and topologies. In this way, Iseki and Tanaka [2] introduced the idea of BCK-algebra in 1978. Iseki [3] introduced the idea of BCI-algebra in 1980 and it is obvious that the class of BCK-algebra is a proper sub class of the class of BCI-algebra. Lee et al. [4] studied the fuzzy translation, (normalized, maximal) fuzzy extension and fuzzy multiplication of fuzzy subalgebra in BCK/BCI-algebra. Link among fuzzy translation, (normalized, maximal) fuzzy extension and fuzzy multiplication are also discussed. Ansari and Chandramouleeswaran [5] introduced the idea of fuzzy translation, fuzzy extension and fuzzy multiplication of fuzzy β ideal of β-algebra and investigated some of their properties. Satyanarayana [6] introduced the concepts of fuzzy ideals, fuzzy implicative ideals and fuzzy p-ideals in BF-algebras and investigate some of its properties. Andrzej [7] defined the BF-algebra. Lekkoksung [8] focused on fuzzy magnified translation in ternary hemirings, which is a extension of BCI / BCK/Q / KU / d-algebra. Senapati et al. [9] have done much work on intuitionistic fuzzy H-ideal in BCK/BCI-algebra. Jana et al. [10] wrote on intuitionistic fuzzy G-algebra. Senapati et al. [11] studied fuzzy translations of fuzzy H-ideals in BCK/BCI-algebra. Atanassov [12] introduced the intuitionistic fuzzy sets. Senapati [13] investigated the relationship among intuitionistic fuzzy translation, intuitionistic fuzzy extension and intuitionistic fuzzy multiplication in B-algebra. Kim and Jeong [14] defined the intuitionistic fuzzy structure of B-algebra. Senapati et al. [15] introduced the cubic subalgebras and cubic closed ideals of B-algebras. Senapati et al. [16] discussed the fuzzy dot subalgebra and fuzzy dot ideal of B-algebras. Priya and Ramachandran [17] worked on fuzzy translation and fuzzy multiplication in PS-algebra. Chandramouleeswaran et al. [18] worked on fuzzy translation and fuzzy multiplication in BF/BG-algebra. Jana et al. [19] studied the cubic G-subalgebra of G-algebra. Smarandache [20,21] extended the intuitionistic fuzzy set, paraconsistent set, and intuitionistic set to the neutrosophic set through Several examples. Jun et al. [22] studied the Cubic set and apply the idea of cubic sets in group and gave the definition of cubic subgroups. Saeid and Rezvani [23] introduced and studied fuzzy BF-algebra, fuzzy BF-subalgebras, level subalgebras,fuzzy topological BF-algebra. Jun et al. [24] defined the neutrosophic cubic set introduced truth-internal and truth-external and discuss the many properties. Jun et al. [25] investigated the commutative falling neutrosophic ideals in BCK-algebra. C. H. Park [26] defined the neutrosophic ideal in subtraction algebra and studied it through several properties, he also discussed conditions for a neutrosophic set to be a neutrosophic ideal along with properties of neutrosophic ideal. Khalid et al. [27] investigated the neutrosophic soft cubic subalgebra through significant characteristic like P-union, R-intersection etc. Khalid et al. [28] interestinly investigated the intuitionistic fuzzy translation and multiplication through subalgebra and ideals. Khalid et al. [29] defined the T-neutrosophic cubic set and studied this set through ideals and subalgebras and investigated many results.

The purpose of this paper is to introduce the idea of neutrosophic cubic translation (**NCT**), neutrosophic cubic multiplication **(NCM)** and neutrosophic cubic magnified translation **(NCMT)** on BF-algebra. In second section we discuss some fundamental definitions which are used to develop the paper. In third's first subsection we discuss the neutrosophic cubic translation (**NCT)** and neutrosophic cubic multiplication **(NCM)** of BF subalgebra. In second subsection we discuss the neutrosophic cubic translation (**NCT)** and neutrosophic cubic multiplication **(NCM)** of BF ideal. In third subsection we discuss the neutrosophic cubic magnified translation **(NCMT)** of BF ideal and BF subalgebra.

2 Preliminaries

First we discuss some definitions which are used to present this paper.

Definition 2.1 [3] An algebra $(Y,*,0)$ of type (2,0) is called a BCI-algebra if it satisfies the following conditions:

i) $(t_1 * t_2) * (t_1 * t_3) \leq (t_3 * t_2)$,

ii) $t_1 * (t_1 * t_2) \leq t_2$,

iii) $t_1 \leq t_1$,

iv) $t_1 \leq t_2$ and $t_2 \leq t_1 \Rightarrow t_1 = t_2$,

v) $t_1 \leq 0 \Rightarrow t_1 = 0$, where $t_1 \leq t_2$ is defined by $t_1 * t_2 = 0$, for all $t_1, t_2, t_3 \in Y$.

Definition 2.2 [1] An algebra $(Y,*,0)$ of type (2,0) is called a BCK-algebra if it satisfies the following conditions:

i) $(t_1 * t_2) * (t_1 * t_3) \leq (t_3 * t_2)$,

ii) $t_1 * (t_1 * t_2) \leq t_2$,

iii) $t_1 \leq t_1$,

iv) $t_1 \leq t_2$ and $t_2 \leq t_1 \Rightarrow t_1 = t_2$,

v) $0 \leq t_1 \Rightarrow t_1 = 0$, where $t_1 \leq t_2$ is defined by $t_1 * t_2 = 0$, for all $t_1, t_2, t_3 \in Y$.

Definition 2.3 [7] A nonempty set Y with a constant 0 and a binary operation $*$ is called BF–algebra when it fulfills these axioms.

i) $t_1 * t_1 = 0$

ii) $t_1 * 0 = 0$

iii) $0 * (t_1 * t_2) = t_2 * t_1$ for all $t_1, t_2 \in Y$.

A BF-algebra is denoted by $(Y,*,0)$.

Definition 2.4 [7] Let S be a nonempty subset of BF-algebra Y, then S is called a BF-subalgebra of Y if $t_1 * t_2 \in S$, for all $t_1, t_2 \in S$.

Definition 2.5 [6] Let Y ba a BF-algebra and I is a subset of Y, then I is called a BF ideal of Y if it satisfies the following conditions:

i) $0 \in I$,

ii) $t_2 * t_1 \in I$ and $t_2 \in I \rightarrow t_1 \in I$.

Definition 2.6 [6] Let Y be a BF-algebra. A fuzzy set B of Y is called a fuzzy BF ideal of Y if it satisfies the following conditions:

i) $\kappa(0) \geq \kappa(t_1)$,

ii) $\kappa(t_1) \geq \min\{\kappa(t_2 * t_1), \kappa(t_2)\}$, for all $t_1, t_2 \in Y$.

Definition 2.7 [1] Let Y be a group of objects denoted generally by t_1. Then a fuzzy set B of Y is defined as $B = \{< t_1, \kappa_B(t_1) > | t_1 \in Y\}$, where $\kappa_B(t_1)$ is called the membership value of t_1 in B and $\kappa_B(t_1) \in [0,1]$.

Definition 2.8 [23] A fuzzy set B of BF-algebra Y is called a fuzzy PS subalgebra of Y if $\kappa(t_1 * t_2) \geq \min\{\kappa(t_1), \kappa(t_2)\}$, for all $t_1, t_2 \in Y$.

Definition 2.9 [4,5] Let a fuzzy subset B of Y and $\alpha \in [0,1 - \sup\{\kappa_B(t_1)|t_1 \in Y\}]$. A mapping $(\kappa_B)^T_\alpha|Y \in [0,1]$ is said to be a fuzzy α translation of κ_B if it satisfies $(\kappa_B)^T_\alpha(t_1) = \kappa_B(t_1) + \alpha$, for all $t_1 \in Y$.

Definition 2.9 [4,5] Let a fuzzy subset B of Y and $\alpha \in [0,1]$. A mapping $(\kappa_B)^M_\alpha: Y \to [0,1]$ is said to be a fuzzy α multiplication of B if it satisfies $(\kappa_B)^M_\alpha(t_1) = \alpha.(\kappa_B)(t_1)$, for all $t_1 \in Y$.

Definition 2.10 [12] An intuitionistic fuzzy set **(IFS)** B over Y is an object having the form $B = \{\langle t_1, \kappa_B(t_1), \upsilon_B(t_1)\rangle | t_1 \in Y\}$, where $\kappa_B(t_1): Y \to [0,1]$ and $\upsilon_B(t_1)|Y \to [0,1]$, with the condition $0 \le \kappa_B(t_1) + \upsilon_B(t_1) \le 1$, for all $t_1 \in Y$. $\kappa_B(t_1)$ and $\upsilon_B(t_1)$ represent the degree of membership and the degree of non-membership of the element t_1 in the set B respectively.

Definition 2.11 [12] Let $B = \{\langle t_1, \kappa_B(t_1), \upsilon_B(t_1)\rangle | t_1 \in Y\}$ and $B = \{\langle t_1, \kappa_B(t_1), \upsilon_B(t_1)\rangle | t_1$

$\in Y\}$ be two **IFSs** on Y. Then intersection and union of A and B are indicated by $A \cap B$ and $A \cup B$ respectively and are given by

$$A \cap B = \{\langle t_1, \min(\kappa_A(t_1), \kappa_B(t_1)), \max(\upsilon_A(t_1), \upsilon_B(t_1))\rangle | t_1 \in Y\},$$

$$A \cup B = \{\langle t_1, \max(\kappa_A(t_1), \kappa_B(t_1)), \min(\upsilon_A(t_1), \upsilon_B(t_1))\rangle | t_1 \in Y\}.$$

Definition 2.12 [14] An **IFS** $B = \{\langle t_1, \kappa_B(t_1), \upsilon_B(t_1)\rangle | t_1 \in Y\}$ of Y is called an **IFSU** of Y if it satisfies these two conditions:

(i) $\kappa_B(t_1 * t_2) \ge \min\{\kappa_B(t_1), \kappa_B(t_2)\}$,

(ii) $\upsilon_B(t_1 * t_2) \le \max\{\upsilon_B(t_1), \upsilon_B(t_2)\}$, for all $t_1, t_2 \in Y$.

Definition 2.13 An **IFS** $B = \{\langle t_1, \kappa_B(t_1), \upsilon_B(t_1)\rangle | t_1 \in Y\}$ of Y is said to be an **IFID** of Y if it satisfies these three conditions:

(i) $\kappa_B(0) \ge \kappa_B(t_1), \upsilon_B(0) \le \upsilon_B(t_1)$,

(ii) $\kappa_B(t_1) \ge \min\{\kappa_B(t_1 * t_2), \kappa_B(t_2)\}$,

(iii) $\upsilon_B(t_1) \le \max\{\upsilon_B(t_1 * t_2), \upsilon_B(t_2)\}$, for all $t_1, t_2 \in Y$.

Definition 2.14 [8] Let κ be a fuzzy subset of Y, $\alpha \in [0,T]$ and $\beta \in [0,1]$. A mapping $\kappa^{M\,T}_{\beta\,\alpha}|Y \to [0,1]$ is said to be a fuzzy magnified $\beta\alpha$ translation of κ if it satisfies: $\kappa^{M\,T}_{\beta\,\alpha}(t_1) = \beta.\kappa(t_1) + \alpha$ for all $t_1 \in Y$.

Jun et al. [22,24]introduced neutrosophic cubic set and investigated several properties.

Definition 2.15 [24] Suppose X be a nonempty set. A neutrosophic cubic set in X is pair $\mathcal{C} = (\kappa, \sigma)$ where $\kappa = \{\langle t_1; \kappa_E(t_1), \kappa_I(t_1), \kappa_N(t_1)\rangle \,| t_1 \in X\}$ is an interval neutrosophic set in X and $\sigma = \{\langle t_1; \sigma_E(t_1), \sigma_I(t_1), \sigma_N(t_1)\rangle \,| t_1 \in X\}$ is a neutrosophic set in X.

Definition 2.16 [15] Let $C = \{\langle t_1, \kappa(t_1), \sigma(t_1)\rangle\}$ be a cubic set, where $\kappa(t_1)$ is an interval-valued fuzzy set in X, $\sigma(t_1)$ is a fuzzy set in X. Then C is cubic subalgebra under binary operation "$*$", if following axioms are satisfied:

i) $\kappa(t_1 * t_2) \ge \mathrm{rmin}\{\kappa(t_1), \kappa(t_2)\}$,

ii) $\sigma(t_1 * t_2) \le \max\{\sigma(t_1), \sigma(t_2)\}\ \forall\ t_1, t_2 \in X$.

Definition 2.17 [28] Let $A=(\kappa_A,\upsilon_A)$ be an **IFS** of G-algebra and let $\alpha\in[0,¥]$. An object of the form $A^T_\alpha=((\kappa_A)^T_\alpha,(\upsilon_A)^T_\alpha)$ is called an intuitionistic fuzzy α-translation (**IFAT**) of A when $(\kappa_A)^T_\alpha(t_1)=\kappa_A(t_1)+\alpha$ and $(\upsilon_A)^T_\alpha(t_1)=\upsilon_A(t_1)-\alpha$ for all $t_1\in Y$.

3 Translative and Multiplicative Interpretation of Neutrosophic Cubic Set

For our simplicity, we use the notation $B=(\kappa_{T,I,F},\upsilon_{T,I,F})$ for the NCS $B=\{\langle t_1,\kappa_{T,I,F}(t_1),\upsilon_{T,I,F}(t_1)\rangle|t_1\in Y\}$. In this paper, we used $٦=[1,1]-\text{rsup}\{\kappa_{\{T,I\}}(t_1)|t_1\in Y\}$, $¥=\text{rinf}\{\kappa_F(t_1)|t_1\in Y\}$, $\Gamma=1-\sup\{\upsilon_{\{T,I\}}(t_1)|t_1\in Y\}$, $£=\inf\{\upsilon_F(t_1)|t_1\in Y$ for any NCS $B=(\kappa_{T,I,F},\upsilon_{T,I,F})$ of Y.

3.1 Translative and Multiplicative Interpretation of Neutrosophic Cubic Subalgebra

Definition 3.1.1 Let $B=(\kappa_{T,I,F},\upsilon_{T,I,F})$ be a **NCS** of Y and for $\kappa_{T,I,F}$, $\alpha,\beta\in[[0,0],٦]$ and $\gamma\in[[0,0],¥]$, where for $\upsilon_{T,I,F}$, $\alpha,\beta\in[0,\Gamma]$ and $\gamma\in[0,£]$. An object of the form $B^T_{\alpha,\beta,\gamma}=((\kappa_{T,I,F})^T_{\alpha,\beta,\gamma},(\upsilon_{T,I,F})^T_{\alpha,\beta,\gamma})$ is called a **NCT** of B, when $(\kappa_T)^T_\alpha(t_1)=\kappa_B(t_1)+\alpha$, $(\kappa_I)^T_\beta(t_1)=\kappa_B(t_1)+\beta$, $(\kappa_F)^T_\gamma(t_1)=\kappa_F(t_1)-\gamma$ and $(\upsilon_T)^T_\alpha(t_1)=\upsilon_T(t_1)+\alpha$, $(\upsilon_I)^T_\beta(t_1)=\upsilon_B(t_1)+\beta$, $(\upsilon_F)^T_\gamma(t_1)=\upsilon_B(t_1)-\gamma$ for all $t_1\in Y$.

Example 3.1.1 Let $Y=\{0,1,2\}$ be a BF-algebra with the following Cayley table:

*	0	1	2
0	0	1	2
1	0	0	1
2	0	2	0

Let $B=(\kappa_{T,I,F},\upsilon_{T,I,F})$ be a **NCS** of Y is defined as

$$\kappa_T(t_1)=\begin{cases}[0.1,0.3] & \text{if } t_1=0\\ [0.4,0.7] & \text{if otherwise}\end{cases}$$

$$\kappa_I(t_1)=\begin{cases}[0.2,0.4] & \text{if } t_1=0\\ [0.5,0.7] & \text{if otherwise}\end{cases}$$

$$\kappa_F(t_1)=\begin{cases}[0.4,0.6] & \text{if } t_1=0\\ [0.3,0.8] & \text{if otherwise}\end{cases}$$

and

$$\upsilon_T(t_1)=\begin{cases}0.1 & \text{if } t_1=0\\ 0.4 & \text{if otherwise}\end{cases}$$

$$\upsilon_I(t_1)=\begin{cases}0.2 & \text{if } t_1=0\\ 0.3 & \text{if otherwise}\end{cases}$$

$$\upsilon_F(t_1) = \begin{cases} 0.5 & \text{if } t_1 = 0 \\ 0.7 & \text{if otherwise.} \end{cases}$$

Then B is a neutrosophic cubic subalgebra. Here we choose for $\upsilon_{T,I,F}$, $\alpha = 0.01, \beta = 0.02,\ \gamma = 0.03$, and for $\kappa_{T,I,F}$, $\alpha = [0.1,0.2],\ \beta = [0.2,0.25],\ \gamma = [0.2,0.3]$ then the mapping $B^T\!: Y \to [0,1]$ is given by

$$(\kappa_T)^T_{[0.1,0.2]}(t_1) = \begin{cases} [0.2, 0.5] & \text{if } t_1 = 0 \\ [0.5, 0.9] & \text{if otherwise} \end{cases}$$

$$(\kappa_I)^T_{[0.2,0.25]}(t_1) = \begin{cases} [0.4, 0.7] & \text{if } t_1 = 0 \\ [0.7, 0.95] & \text{if otherwise} \end{cases}$$

$$(\kappa_F)^T_{[0.2,0.3]}(t_1) = \begin{cases} [0.2, 0.3] & \text{if } t_1 = 0 \\ [0.1, 0.5] & \text{if otherwise} \end{cases}$$

and

$$(\upsilon_T)^T_{0.01}(t_1) = \begin{cases} 0.11 & \text{if } t_1 = 0 \\ 0.41 & \text{if otherwise.} \end{cases}$$

$$(\upsilon_I)^T_{0.02}(t_1) = \begin{cases} 0.22 & \text{if } t_1 = 0 \\ 0.32 & \text{if otherwise.} \end{cases}$$

$$(\upsilon_F)^T_{0.03}(t_1) = \begin{cases} 0.47 & \text{if } t_1 = 0 \\ 0.67 & \text{if otherwise,} \end{cases}$$

which imply $(\kappa_T)^T_{[0.1,0.2]}(t_1) = \kappa_T(t_1) + [0.1,0.2]$, $(\kappa_I)^T_{[0.2,0.25]}(t_1) = \kappa_I(t_1) + [0.2,0.25\ \ ]$, $(\kappa_F)^T_{[0.2,0.3]}(t_1) = \kappa_F(t_1) - [0.2,0.3]$ and $(\upsilon_T)^T_{0.01}(t_1) = \upsilon_T(t_1) + 0.01$, $(\upsilon_I)^T_{0.02}(t_1) = \upsilon_I(t_1) + 0.02$, $(\upsilon_F)^T_{0.03}(t_1) = \upsilon_F(t_1) - 0.03$ for all $t_1 \in Y$. Hence B^T is a neutrosophic cubic translation.

Theorem 3.1.1 Let B be a **NCSU** of Y and for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ٦]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$. Then **NCT** $B^T_{\alpha,\beta,\gamma}$ of B is a **NCSU** of Y.

Proof. Assume $t_1, t_2 \in Y$. Then

$$(\kappa_T)^T_\alpha(t_1 * t_2) = \kappa_T(t_1 * t_2) + \alpha$$

$$\geq \mathrm{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\} + \alpha$$

$$= \mathrm{rmin}\{\kappa_T(t_1) + \alpha, \kappa_T(t_2) + \alpha\}$$

$$(\kappa_T)^T_\alpha(t_1 * t_2) = \mathrm{rmin}\{(\kappa_T)^T_\alpha(t_1), (\kappa_T)^T_\alpha(t_2)\},$$

$$(\kappa_I)^T_\beta(t_1 * t_2) = \kappa_I(t_1 * t_2) + \beta$$

$$\geq \mathrm{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\} + \beta$$

$$= \mathrm{rmin}\{\kappa_I(t_1) + \beta, \kappa_I(t_2) + \beta\}$$

$$(\kappa_I)^T_\beta(t_1 * t_2) = \text{rmin}\{(\kappa_I)^T_\beta(t_1), (\kappa_I)^T_\beta(t_2)\},$$

$$(\kappa_F)^T_\gamma(t_1 * t_2) = \kappa_F(t_1 * t_2) - \gamma$$

$$\geq \text{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\} - \gamma$$

$$= \text{rmin}\{\kappa_F(t_1) - \gamma, \kappa_F(t_2) - \gamma\}$$

$$(\kappa_F)^T_\gamma(t_1 * t_2) = \text{rmin}\{(\kappa_F)^T_\gamma(t_1), (\kappa_F)^T_\gamma(t_2)\}$$

and

$$(\upsilon_T)^T_\alpha(t_1 * t_2) = \upsilon_T(t_1 * t_2) + \alpha$$

$$\leq \max\{\upsilon_T(t_1), \upsilon_T(t_2)\} + \alpha$$

$$= \max\{\upsilon_T(t_1) + \alpha, \upsilon_T(t_2) + \alpha\}$$

$$(\upsilon_T)^T_\alpha(t_1 * t_2) = \max\{(\upsilon_T)^T_\alpha(t_1), (\upsilon_T)^T_\alpha(t_2)\},$$

$$(\upsilon_I)^T_\beta(t_1 * t_2) = \upsilon_I(t_1 * t_2) + \beta$$

$$\leq \max\{\upsilon_I(t_1), \upsilon_I(t_2)\} + \beta$$

$$= \max\{\upsilon_I(t_1) + \beta, \upsilon_I(t_2) + \beta\}$$

$$(\upsilon_I)^T_\beta(t_1 * t_2) = \max\{(\upsilon_I)^T_\beta(t_1), (\upsilon_I)^T_\beta(t_2)\},$$

$$(\upsilon_F)^T_\gamma(t_1 * t_2) = \upsilon_F(t_1 * t_2) - \gamma$$

$$\leq \max\{\upsilon_F(t_1), \upsilon_F(t_2)\} - \gamma$$

$$= \max\{\upsilon_F(t_1) - \gamma, \upsilon_F(t_2) - \gamma\}$$

$$(\upsilon_F)^T_\gamma(t_1 * t_2) = \max\{(\upsilon_F)^T_\gamma(t_1), (\upsilon_F)^T_\gamma(t_2)\}.$$

Hence **NCT** $B^T_{\alpha,\beta,\gamma}$ of B is a **NCSU** of Y.

Theorem 3.1.2 Let B be a **NCS** of Y such that **NCT** $B^T_{\alpha,\beta,\gamma}$ of B is a **NCSU** of Y for some $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ℸ]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$. Then B is a **NCSU** of Y.

Proof. Let $B^T_{\alpha,\beta,\gamma} = ((\kappa_{T,I,F})^T_{\alpha,\beta,\gamma}, (\upsilon_{T,I,F})^T_{\alpha,\beta,\gamma})$ be a **NCSU** of Y for some $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ℸ]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$ and $t_1, t_2 \in Y$. Then

$$\kappa_T(t_1 * t_2) + \alpha = (\kappa_T)^T_\alpha(t_1 * t_2)$$

$$\geq \text{rmin}\{(\kappa_T)^T_\alpha(t_1), (\kappa_T)^T_\alpha(t_2)\}$$

$$= \text{rmin}\{\kappa_T(t_1) + \alpha, \kappa_T(t_2) + \alpha\}$$

$$\kappa_T(t_1 * t_2) + \alpha = \text{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\} + \alpha,$$

$$\kappa_I(t_1 * t_2) + \beta = (\kappa_I)^T_\beta(t_1 * t_2)$$

$$\geq \text{rmin}\{(\kappa_I)^T_\beta(t_1), (\kappa_I)^T_\beta(t_2)\}$$

$= \mathrm{rmin}\{\kappa_I(t_1) + \beta, \kappa_I(t_2) + \beta\}$

$\kappa_I(t_1 * t_2) + \beta = \mathrm{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\} + \beta,$

$\kappa_F(t_1 * t_2) - \gamma = (\kappa_F)^T_\gamma(t_1 * t_2)$

$\geq \mathrm{rmin}\{(\kappa_F)^T_\gamma(t_1), (\kappa_F)^T_\gamma(t_2)\}$

$= \mathrm{rmin}\{\kappa_F(t_1) - \gamma, \kappa_F(t_2) - \gamma\}$

$\kappa_F(t_1 * t_2) - \gamma = \mathrm{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\} - \gamma$

and

$\upsilon_T(t_1 * t_2) + \alpha = (\upsilon_T)^T_\alpha(t_1 * t_2)$

$\leq \max\{(\upsilon_T)^T_\alpha(t_1), (\upsilon_T)^T_\alpha(t_2)\}$

$= \max\{\upsilon_T(t_1) + \alpha, \upsilon_B(t_2) + \alpha\}$

$\upsilon_T(t_1 * t_2) + \alpha = \max\{\upsilon_T(t_1), \upsilon_T(t_2)\} + \alpha,$

$\upsilon_I(t_1 * t_2) + \beta = (\upsilon_I)^T_\beta(t_1 * t_2)$

$\leq \max\{(\upsilon_I)^T_\beta(t_1), (\upsilon_I)^T_\beta(t_2)\}$

$= \max\{\upsilon_I(t_1) + \beta, \upsilon_B(t_2) + \beta\}$

$\upsilon_I(t_1 * t_2) + \beta = \max\{\upsilon_I(t_1), \upsilon_I(t_2)\} + \beta,$

$\upsilon_F(t_1 * t_2) - \gamma = (\upsilon_F)^T_\gamma(t_1 * t_2)$

$\leq \max\{(\upsilon_F)^T_\gamma(t_1), (\upsilon_F)^T_\gamma(t_2)\}$

$= \max\{\upsilon_F(t_1) - \gamma, \upsilon_B(t_2) - \gamma\}$

$\upsilon_F(t_1 * t_2) - \gamma = \max\{\upsilon_F(t_1), \upsilon_F(t_2)\} - \gamma,$

which imply $\kappa_T(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\}$, $\kappa_I(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\}$, $\kappa_F(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\}$, and $\upsilon_T(t_1 * t_2) \leq \max\{\upsilon_T(t_1), \upsilon_T(t_2)\}$, $\upsilon_I(t_1 * t_2) \leq \max\{\upsilon_I(t_1), \upsilon_I(t_2)\}$, $\upsilon_F(t_1 * t_2) \leq \max\{\upsilon_F(t_1), \upsilon_F(t_2)\}$, for all $t_1, t_2 \in Y$. Hence B is a **NCSU** of Y.

Definition 3.1.2 Let B be a **NCS** of Y and $\delta \in [0,1]$. An object having the form $B^M_\delta = (((\kappa_T)^M_\delta, (\kappa_I)^M_\delta, (\kappa_F)^M_\delta), ((\upsilon_T)^M_\delta, (\upsilon_I)^M_\delta, (\upsilon_F)^M_\delta))$ is called a **NCM** of B, when $(\kappa_T)^M_\delta(t_1) = \delta.\kappa_T(t_1)$, $(\kappa_I)^M_\delta(t_1) = \delta.\kappa_I(t_1)$, $(\kappa_F)^M_\delta(t_1) = \delta.\kappa_F(t_1)$ and $(\upsilon_T)^M_\delta(t_1) = \delta.\upsilon_T(t_1)$, $(\upsilon_I)^M_\delta(t_1) = \delta.\upsilon_I(t_1)$,$(\upsilon_F)^M_\delta(t_1) = \delta.\upsilon_F(t_1)$ for all $t_1 \in Y$.

Example 3.1.2 Let $Y = \{0,1,2\}$ be a BF-algebra with the following Cayley table:

*	0	1	2
0	0	1	2
1	0	0	1
2	0	2	0

Let $B = (\kappa_{T,I,F}, \upsilon_{T,I,F})$ be a **NCS** of Y is defined as

$$\kappa_T(t_1) = \begin{cases} [0.1, 0.3] & \text{if } t_1 = 0 \\ [0.4, 0.7] & \text{if otherwise} \end{cases}$$

$$\kappa_I(t_1) = \begin{cases} [0.2, 0.4] & \text{if } t_1 = 0 \\ [0.5, 0.7] & \text{if otherwise} \end{cases}$$

$$\kappa_F(t_1) = \begin{cases} [0.4, 0.6] & \text{if } t_1 = 0 \\ [0.3, 0.8] & \text{if otherwise} \end{cases}$$

and

$$\upsilon_T(t_1) = \begin{cases} 0.1 & \text{if } t_1 = 0 \\ 0.4 & \text{if otherwise} \end{cases}$$

$$\upsilon_I(t_1) = \begin{cases} 0.2 & \text{if } t_1 = 0 \\ 0.3 & \text{if otherwise} \end{cases}$$

$$\upsilon_F(t_1) = \begin{cases} 0.5 & \text{if } t_1 = 0 \\ 0.7 & \text{if otherwise.} \end{cases}$$

Then B is a neutrosophic cubic subalgebra, choose $\delta = 0.01$ for υ and $\delta = [0.1,0.2]$ for κ then the mapping $B_\delta^M|Y \rightarrow [0,1]$ is given by

$$(\kappa_T)_{[0.1,0.2]}^M(t_1) = \begin{cases} [0.01, 0.06] & \text{if } t_1 = 1 \\ [0.04, 0.14] & \text{if otherwise,} \end{cases}$$

$$(\kappa_I)_{[0.1,0.2]}^M(t_1) = \begin{cases} [0.02, 0.08] & \text{if } t_1 = 1 \\ [0.05, 0.14] & \text{if otherwise,} \end{cases}$$

$$(\kappa_F)_{[0.1,0.2]}^M(t_1) = \begin{cases} [0.04, 0.12] & \text{if } t_1 = 1 \\ [0.03, 0.16] & \text{if otherwise} \end{cases}$$

and

$$(\upsilon_T)_{0.01}^M(t_1) = \begin{cases} 0.001 & \text{if } t_1 = 0 \\ 0.004 & \text{if otherwise,} \end{cases}$$

$$(\upsilon_I)_{0.01}^M(t_1) = \begin{cases} 0.002 & \text{if } t_1 = 0 \\ 0.003 & \text{if otherwise,} \end{cases}$$

$$(\upsilon_F)_{0.01}^M(t_1) = \begin{cases} 0.005 & \text{if } t_1 = 0 \\ 0.007 & \text{if otherwise,} \end{cases}$$

which imply $(\kappa_T)^M_{[0.1,0.2]}(t_1) = \kappa_T(t_1).[0.1,0.2]$, $(\kappa_I)^M_{[0.1,0.2]}(t_1) = \kappa_I(t_1).[0.1,0.2]$, $(\kappa_F)^M_{[0.1,0.2]}(t_1) = \kappa_F(t_1).[0.1,0.2]$ and $(\upsilon_T)^M_{0.01}(t_1) = \upsilon_T(t_1).(0.01)$, $(\upsilon_I)^M_{0.01}(t_1) = \upsilon_I(t_1).(0.01)$, $(\upsilon_F)^M_{0.01}(t_1) = \upsilon_F(t_1).(0.01)$ for all $t_1 \in Y$. Hence B^M_δ is a neutrosophic cubic multiplication.

Theorem 3.1.3 Let B be a **NCS** of Y such that **NCM** B^M_δ of B is a **NCSU** of Y for some $\delta \in [0,1]$. Then B is a **NCSU** of Y.

Proof. Assume B^M_δ of B is a **NCSU** of Y for some $\delta \in [0,1]$. Now for all $t_1, t_2 \in Y$, we have

$$\kappa_T(t_1 * t_2).\delta = (\kappa_T)^M_\delta(t_1 * t_2)$$
$$\geq \mathrm{rmin}\{(\kappa_T)^M_\delta(t_1), (\kappa_T)^M_\delta(t_2)\}$$
$$= \mathrm{rmin}\{\kappa_T(t_1).\delta, \kappa_T(t_2).\delta\}$$
$$\kappa_T(t_1 * t_2).\delta = \mathrm{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\}.\delta,$$

$$\kappa_I(t_1 * t_2).\delta = (\kappa_I)^M_\delta(t_1 * t_2)$$
$$\geq \mathrm{rmin}\{(\kappa_I)^M_\delta(t_1), (\kappa_I)^M_\delta(t_2)\}$$
$$= \mathrm{rmin}\{\kappa_I(t_1).\delta, \kappa_I(t_2).\delta\}$$
$$\kappa_I(t_1 * t_2).\delta = \mathrm{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\}.\delta,$$

$$\kappa_F(t_1 * t_2).\delta = (\kappa_F)^M_\delta(t_1 * t_2)$$
$$\geq \mathrm{rmin}\{(\kappa_F)^M_\delta(t_1), (\kappa_F)^M_\delta(t_2)\}$$
$$= \mathrm{rmin}\{\kappa_F(t_1).\delta, \kappa_F(t_2).\delta\}$$
$$\kappa_F(t_1 * t_2).\delta = \mathrm{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\}.\delta$$

and

$$\upsilon_T(t_1 * t_2).\delta = (\upsilon_T)^M_\delta(t_1 * t_2)$$
$$\leq \max\{(\upsilon_T)^M_\delta(t_1), (\upsilon_T)^M_\delta(t_2)\}$$
$$= \max\{\upsilon_T(t_1).\delta, \upsilon_T(t_2).\delta\}$$
$$\upsilon_T(t_1 * t_2).\delta = \max\{\upsilon_T(t_1), \upsilon_T(t_2)\}.\delta,$$

$$\upsilon_I(t_1 * t_2).\delta = (\upsilon_I)^M_\delta(t_1 * t_2)$$
$$\leq \max\{(\upsilon_I)^M_\delta(t_1), (\upsilon_I)^M_\delta(t_2)\}$$
$$= \max\{\upsilon_I(t_1).\delta, \upsilon_I(t_2).\delta\}$$
$$\upsilon_I(t_1 * t_2).\delta = \max\{\upsilon_I(t_1), \upsilon_I(t_2)\}.\delta,$$

$$\upsilon_F(t_1 * t_2).\delta = (\upsilon_F)^M_\delta(t_1 * t_2)$$

$\leq \max\{(\upsilon_F)^M_\delta(t_1), (\upsilon_F)^M_\delta(t_2)\}$

$= \max\{\upsilon_F(t_1).\delta, \upsilon_F(t_2).\delta\}$

$\upsilon_F(t_1 * t_2).\delta = \max\{\upsilon_F(t_1), \upsilon_F(t_2)\}.\delta,$

which imply $\kappa_T(t_1 * t_2) \geq \text{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\}$, $\kappa_I(t_1 * t_2) \geq \text{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\}$, $\kappa_F(t_1 * t_2) \geq \text{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\}$ and $\upsilon_T(t_1 * t_2) \leq \max\{\upsilon_T(t_1), \upsilon_T(t_2)\}$, $\upsilon_I(t_1 * t_2) \leq \max\{\upsilon_I(t_1), \upsilon_I(t_2)\}$, $\upsilon_F(t_1 * t_2) \leq \max\{\upsilon_F(t_1), \upsilon_F(t_2)\}$ for all $t_1, t_2 \in Y$. Hence B is a **NCSU** of Y.

Theorem 3.1.4 Let B be a **NCSU** of Y for $\delta \in [0,1]$. Then **NCM** B^M_δ of B is a **NCSU** of Y.

Proof. Assume $t_1, t_2 \in Y$. Then

$(\kappa_T)^M_\delta(t_1 * t_2) = \delta.\kappa_T(t_1 * t_2)$

$\geq \delta.\text{rmin}\{(\kappa_T)(t_1), (\kappa_T)(t_2)\}$

$= \text{rmin}\{\delta.\kappa_T(t_1), \delta.\kappa_T(t_2)\}$

$= \text{rmin}\{(\kappa_T)^M_\delta(t_1), (\kappa_T)^M_\delta(t_2)\}$

$(\kappa_T)^M_\delta(t_1 * t_2) \geq \text{rmin}\{(\kappa_T)^M_\delta(t_1), (\kappa_T)^M_\delta(t_2)\},$

$(\kappa_I)^M_\delta(t_1 * t_2) = \delta.\kappa_I(t_1 * t_2)$

$\geq \delta.\text{rmin}\{(\kappa_I)(t_1), (\kappa_I)(t_2)\}$

$= \text{rmin}\{\delta.\kappa_I(t_1), \delta.\kappa_I(t_2)\}$

$= \text{rmin}\{(\kappa_I)^M_\delta(t_1), (\kappa_I)^M_\delta(t_2)\}$

$(\kappa_I)^M_\delta(t_1 * t_2) \geq \text{rmin}\{(\kappa_I)^M_\delta(t_1), (\kappa_I)^M_\delta(t_2)\},$

$(\kappa_F)^M_\delta(t_1 * t_2) = \delta.\kappa_F(t_1 * t_2)$

$\geq \delta.\text{rmin}\{(\kappa_F)(t_1), (\kappa_F)(t_2)\}$

$= \text{rmin}\{\delta.\kappa_F(t_1), \delta.\kappa_F(t_2)\}$

$= \text{rmin}\{(\kappa_F)^M_\delta(t_1), (\kappa_F)^M_\delta(t_2)\}$

$(\kappa_F)^M_\delta(t_1 * t_2) \geq \text{rmin}\{(\kappa_F)^M_\delta(t_1), (\kappa_F)^M_\delta(t_2)\}$

and

$(\upsilon_T)^M_\delta(t_1 * t_2) = \delta.\upsilon_T(t_1 * t_2)$

$\leq \delta.\max\{(\upsilon_T)(t_1), (\upsilon_T)(t_2)\}$

$= \max\{\delta.\upsilon_T(t_1), \delta.\upsilon_T(t_2)\}$

$= \max\{(\kappa_B)^M_\delta(t_1), (\kappa_B)^M_\delta(t_2)\}$

$(\upsilon_T)^M_\delta(t_1 * t_2) \leq \max\{(\upsilon_T)^M_\delta(t_1), (\upsilon_T)^M_\delta(t_2)\},$

$$(\upsilon_I)^M_\delta(t_1 * t_2) = \delta.\upsilon_I(t_1 * t_2)$$
$$\leq \delta.\max\{(\upsilon_I)(t_1), (\upsilon_I)(t_2)\}$$
$$= \max\{\delta.\upsilon_I(t_1), \delta.\upsilon_I(t_2)\}$$
$$= \max\{(\kappa_B)^M_\delta(t_1), (\kappa_B)^M_\delta(t_2)\}$$
$$(\upsilon_I)^M_\delta(t_1 * t_2) \leq \max\{(\upsilon_I)^M_\delta(t_1), (\upsilon_I)^M_\delta(t_2)\},$$

$$(\upsilon_F)^M_\delta(t_1 * t_2) = \delta.\upsilon_F(t_1 * t_2)$$
$$\leq \delta.\max\{(\upsilon_F)(t_1), (\upsilon_F)(t_2)\}$$
$$= \max\{\delta.\upsilon_F(t_1), \delta.\upsilon_F(t_2)\}$$
$$= \max\{(\kappa_B)^M_\delta(t_1), (\kappa_B)^M_\delta(t_2)\}$$
$$(\upsilon_F)^M_\delta(t_1 * t_2) \leq \max\{(\upsilon_F)^M_\delta(t_1), (\upsilon_F)^M_\delta(t_2)\},$$

which imply $\kappa_T(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\}$, $\kappa_I(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\}$, $\kappa_F(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\}$ and $\upsilon_T(t_1 * t_2) \leq \max\{\upsilon_T(t_1), \upsilon_T(t_2)\}$, $\upsilon_I(t_1 * t_2) \leq \max\{\upsilon_I(t_1), \upsilon_I(t_2)\}$, $\upsilon_F(t_1 * t_2) \leq \max\{\upsilon_F(t_1), \upsilon_F(t_2)\}$ for all $t_1, t_2 \in Y$. Hence B^M_δ is a **NCSU** of Y.

3.2 Translative and Multiplicative Interpretation of Neutrosophic Cubic Ideal

In this section, neutrosophic cubic translation of **NCID**, neutrosophic cubic multiplication of **NCID**, union and intersection of neutrosophic cubic translation of **NCID** are investigated through some results.

Theorem 3.2.1 If **NCT** $B^T_{\alpha,\beta,\gamma}$ of B is a neutrosophic cubic BF ideal, then it fulfills the conditions $(\kappa_T)^T_\alpha(t_1 * (t_2 * t_1)) \geq (\kappa_T)^T_\alpha(t_2)$, $(\kappa_I)^T_\beta(t_1 * (t_2 * t_1)) \geq (\kappa_I)^T_\beta(t_2)$, $(\kappa_F)^T_\gamma(t_1 * (t_2 * t_1)) \geq (\kappa_F)^T_\gamma(t_2)$ and $(\upsilon_T)^T_\alpha(t_1 * (t_2 * t_1)) \leq (\upsilon_T)^T_\alpha(t_2)$, $(\upsilon_I)^T_\beta(t_1 * (t_2 * t_1)) \leq (\upsilon_I)^T_\beta(t_2)$, $(\upsilon_F)^T_\gamma(t_1 * (t_2 * t_1)) \leq (\upsilon_F)^T_\gamma(t_2)$.

Proof. Let **NCT** $B^T_{\alpha,\beta,\gamma}$ of B be a neutrosophic cubic BF ideal. Then

$$(\kappa_T)^T_\alpha(t_1 * (t_2 * t_1)) = \kappa_T(t_1 * (t_2 * t_1)) + \alpha$$
$$\geq \mathrm{rmin}\{\kappa_T(t_2 * (t_1 * (t_2 * t_1))) + \alpha, \kappa_T(t_2) + \alpha\}$$
$$= \mathrm{rmin}\{\kappa_T(0) + \alpha, \kappa_T(t_2) + \alpha\}$$
$$= \mathrm{rmin}\{(\kappa_T)^T_\alpha(0), (\kappa_T)^T_\alpha(t_2)\}$$
$$(\kappa_T)^T_\alpha(t_1 * (t_2 * t_1)) = (\kappa_T)^T_\alpha(t_2),$$

$$(\kappa_I)^T_\alpha(t_1 * (t_2 * t_1)) = \kappa_I(t_1 * (t_2 * t_1)) + \beta$$
$$\geq \mathrm{rmin}\{\kappa_I(t_2 * (t_1 * (t_2 * t_1))) + \beta, \kappa_I(t_2) + \beta\}$$
$$= \mathrm{rmin}\{\kappa_I(0) + \beta, \kappa_I(t_2) + \beta\}$$

$$= \mathrm{rmin}\{(\kappa_I)^T_\beta(0), (\kappa_I)^T_\beta(t_2)\}$$

$$(\kappa_I)^T_\beta(t_1 * (t_2 * t_1)) = (\kappa_I)^T_\beta(t_2),$$

$$(\kappa_F)^T_\gamma(t_1 * (t_2 * t_1)) = \kappa_F(t_1 * (t_2 * t_1)) - \gamma$$

$$\geq \mathrm{rmin}\{\kappa_F(t_2 * (t_1 * (t_2 * t_1))) - \gamma, \kappa_F(t_2) - \gamma\}$$

$$= \mathrm{rmin}\{\kappa_F(0) - \gamma, \kappa_F(t_2) - \gamma\}$$

$$= \mathrm{rmin}\{(\kappa_F)^T_\gamma(0), (\kappa_F)^T_\gamma(t_2)\}$$

$$(\kappa_F)^T_\gamma(t_1 * (t_2 * t_1)) = (\kappa_F)^T_\gamma(t_2)$$

and

$$(\upsilon_T)^T_\alpha(t_1 * (t_2 * t_1)) = \upsilon_T(t_1 * (t_2 * t_1)) + \alpha$$

$$\leq \max\{\upsilon_T(t_2 * (t_1 * (t_2 * t_1))) + \alpha, \upsilon_T(t_2) + \alpha\}$$

$$= \max\{\upsilon_T(0) + \alpha, \upsilon_T(t_2) + \alpha\}$$

$$= \max\{(\upsilon_T)^T_\alpha(0), (\upsilon_T)^T_\alpha(t_2)\}$$

$$(\upsilon_T)^T_\alpha(t_1 * (t_2 * t_1)) = (\upsilon_T)^T_\alpha(t_2),$$

$$(\upsilon_I)^T_\alpha(t_1 * (t_2 * t_1)) = \upsilon_I(t_1 * (t_2 * t_1)) + \beta$$

$$\leq \max\{\upsilon_I(t_2 * (t_1 * (t_2 * t_1))) + \beta, \upsilon_I(t_2) + \beta\}$$

$$= \max\{\upsilon_I(0) + \beta, \upsilon_I(t_2) + \beta\}$$

$$= \max\{(\upsilon_I)^T_\beta(0), (\upsilon_I)^T_\beta(t_2)\}$$

$$(\upsilon_I)^T_\beta(t_1 * (t_2 * t_1)) = (\upsilon_I)^T_\beta(t_2),$$

$$(\upsilon_F)^T_\gamma(t_1 * (t_2 * t_1)) = \upsilon_F(t_1 * (t_2 * t_1)) - \gamma$$

$$\leq \max\{\upsilon_F(t_2 * (t_1 * (t_2 * t_1))) - \gamma, \upsilon_F(t_2) - \gamma\}$$

$$= \max\{\upsilon_F(0) - \gamma, \upsilon_F(t_2) - \gamma\}$$

$$= \max\{(\upsilon_F)^T_\gamma(0), (\upsilon_F)^T_\gamma(t_2)\}$$

$$(\upsilon_F)^T_\gamma(t_1 * (t_2 * t_1)) = (\upsilon_F)^T_\gamma(t_2).$$

Hence $(\kappa_T)^T_\alpha(t_1 * (t_2 * t_1)) \geq (\kappa_T)^T_\alpha(t_2)$, $(\kappa_I)^T_\beta(t_1 * (t_2 * t_1)) \geq (\kappa_I)^T_\beta(t_2)$, $(\kappa_F)^T_\gamma(t_1 * (t_2 * t_1)) \geq (\kappa_F)^T_\gamma(t_2)$ and $(\upsilon_T)^T_\alpha(t_1 * (t_2 * t_1)) \leq (\upsilon_T)^T_\alpha(t_2)$, $(\upsilon_I)^T_\beta(t_1 * (t_2 * t_1)) \leq (\upsilon_I)^T_\beta(t_2)$, $(\upsilon_F)^T_\gamma(t_1 * (t_2 * t_1)) \leq (\upsilon_F)^T_\gamma(t_2)$.

Theorem 3.2.2 Let B be a **NCID** of Y and for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ℸ]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$. Then **NCT** $B^T_{\alpha,\beta,\gamma}$ of B is a **NCID** of Y.

Proof. Let B be a **NCID** of Y and for $\kappa_{T,I,F}$, $\alpha, \beta \in \left[[0,0], ℸ\right]$ and $\gamma \in \left[[0,0], ¥\right]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$. Then $(\kappa_T)^T_\alpha(0) = \kappa_T(0) + \alpha \geq \kappa_T(t_1) + \alpha = (\kappa_T)^T_\alpha(t_1)$, $(\kappa_I)^T_\beta(0) = \kappa_I(0) + \beta \geq$

$\kappa_I(t_1) + \beta = (\kappa_I)^T_\beta(t_1)$, $(\kappa_F)^T_\gamma(0) = \kappa_F(0) - \gamma \geq \kappa_F(t_1) - \gamma = (\kappa_F)^T_\gamma(t_1)$ and $(\upsilon_T)^T_\alpha(0) = \upsilon_T(0) + \alpha \leq \upsilon_T(t_1) + \alpha = (\upsilon_T)^T_\alpha(t_1)$, $(\upsilon_I)^T_\beta(0) = \upsilon_I(0) + \beta \leq \upsilon_I(t_1) + \beta = (\upsilon_I)^T_\beta(t_1)$, $(\upsilon_F)^T_\gamma(0) = \upsilon_F(0) - \gamma \leq \upsilon_F(t_1) - \gamma = (\upsilon_F)^T_\gamma(t_1\ )$. So

$$(\kappa_T)^T_\alpha(t_1) = \kappa_T(t_1) + \alpha$$

$$\geq \mathrm{rmin}\{\kappa_T(t_1 * t_2), \kappa_T(t_2)\} + \alpha$$

$$= \mathrm{rmin}\{\kappa_T(t_1 * t_2) + \alpha, \kappa_T(t_2) + \alpha\}$$

$$(\kappa_T)^T_\alpha(t_1) = \mathrm{rmin}\{(\kappa_T)^T_\alpha(t_1 * t_2), (\kappa_T)^T_\alpha(t_2)\},$$

$$(\kappa_I)^T_\beta(t_1) = \kappa_I(t_1) + \beta$$

$$\geq \mathrm{rmin}\{\kappa_I(t_1 * t_2), \kappa_I(t_2)\} + \beta$$

$$= \mathrm{rmin}\{\kappa_I(t_1 * t_2) + \beta, \kappa_I(t_2) + \beta\}$$

$$(\kappa_I)^T_\beta(t_1) = \mathrm{rmin}\{(\kappa_I)^T_\beta(t_1 * t_2), (\kappa_I)^T_\beta(t_2)\},$$

$$(\kappa_F)^T_\alpha(t_1) = \kappa_F(t_1) - \gamma$$

$$\geq \mathrm{rmin}\{\kappa_F(t_1 * t_2), \kappa_F(t_2)\} - \gamma$$

$$= \mathrm{rmin}\{\kappa_F(t_1 * t_2) - \gamma, \kappa_F(t_2) - \gamma\}$$

$$(\kappa_F)^T_\gamma(t_1) = \mathrm{rmin}\{(\kappa_F)^T_\gamma(t_1 * t_2), (\kappa_F)^T_\gamma(t_2)\}$$

and

$$(\upsilon_T)^T_\alpha(t_1) = \upsilon_T(t_1) + \alpha$$

$$\leq \max\{\upsilon_T(t_1 * t_2), \upsilon_T(t_2)\} + \alpha$$

$$= \max\{\upsilon_T(t_1 * t_2) + \alpha, \upsilon_T(t_2) + \alpha\}$$

$$(\upsilon_T)^T_\alpha(t_1) = \max\{(\upsilon_T)^T_\alpha(t_1 * t_2), (\upsilon_T)^T_\alpha(t_2)\},$$

$$(\upsilon_I)^T_\beta(t_1) = \upsilon_I(t_1) + \beta$$

$$\leq \max\{\upsilon_I(t_1 * t_2), \upsilon_I(t_2)\} + \beta$$

$$= \max\{\upsilon_I(t_1 * t_2) + \beta, \upsilon_I(t_2) + \beta\}$$

$$(\upsilon_I)^T_\beta(t_1) = \max\{(\upsilon_I)^T_\beta(t_1 * t_2), (\upsilon_I)^T_\beta(t_2)\},$$

$$(\upsilon_F)^T_\gamma(t_1) = \upsilon_F(t_1) - \gamma$$

$$\leq \max\{\upsilon_F(t_1 * t_2), \upsilon_F(t_2)\} - \gamma$$

$$= \max\{\upsilon_F(t_1 * t_2) - \gamma, \upsilon_F(t_2) - \gamma\}$$

$$(\upsilon_F)^T_\gamma(t_1) = \max\{(\upsilon_F)^T_\gamma(t_1 * t_2), (\upsilon_F)^T_\gamma(t_2)\},$$

for all $t_1, t_2 \in Y$ and for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ٦]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$. Hence $B^T_{\alpha,\beta,\gamma}$ of B is a **NCID** of Y.

Theorem 3.2.3 Let B be a neutrosophic cubic set of Y such that **NCT** $B^T_{\alpha,\beta,\gamma}$ of B is a **NCID** of Y for all $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ٦]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$. Then B is a **NCID** of Y.

Proof. Suppose $B^T_{\alpha,\beta,\gamma}$ is a **NCID** of Y, where for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ٦]$ and $\gamma \in [[0,0], ¥]$, and for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$ and $t_1, t_2 \in Y$. Then

$$\kappa_T(0) + \alpha = (\kappa_T)^T_\alpha(0) \geq (\kappa_T)^T_\alpha(t_1) = \kappa_T(t_1) + \alpha,$$
$$\kappa_I(0) + \beta = (\kappa_I)^T_\beta(0) \geq (\kappa_I)^T_\beta(t_1) = \kappa_I(t_1) + \beta,$$
$$\kappa_F(0) - \gamma = (\kappa_F)^T_\gamma(0) \geq (\kappa_F)^T_\gamma(t_1) = \kappa_F(t_1) - \gamma,$$

and

$$\upsilon_T(0) + \alpha = (\upsilon_T)^T_\alpha(0) \leq (\upsilon_T)^T_\alpha(t_1) = \upsilon_T(t_1) + \alpha,$$
$$\upsilon_I(0) + \beta = (\upsilon_I)^T_\beta(0) \leq (\upsilon_I)^T_\beta(t_1) = \upsilon_I(t_1) + \beta$$
$$\upsilon_F(0) - \gamma = (\upsilon_F)^T_\gamma(0) \leq (\upsilon_F)^T_\gamma(t_1) = \upsilon_F(t_1) - \gamma,$$

which imply $\kappa_T(0) \geq \kappa_T(t_1)$, $\kappa_I(0) \geq \kappa_I(t_1)$, $\kappa_F(0) \geq \kappa_F(t_1)$ and $\upsilon_T(0) \leq \upsilon_T(t_1)$, $\upsilon_I(0) \leq \upsilon_I(t_1)$, $\upsilon_F(0) \leq \upsilon_F(t_1)$, now

$$\kappa_T(t_1) + \alpha = (\kappa_T)^T_\alpha(t_1) \geq \mathrm{rmin}\{(\kappa_T)^T_\alpha(t_1 * t_2), (\kappa_T)^T_\alpha(t_2)\}$$
$$= \mathrm{rmin}\{\kappa_T(t_1 * t_2) + \alpha, \kappa_T(t_2) + \alpha\}$$
$$\kappa_T(t_1) + \alpha = \mathrm{rmin}\{\kappa_T(t_1 * t_2), \kappa_T(t_2)\} + \alpha,$$

$$\kappa_I(t_1) + \beta = (\kappa_I)^T_\beta(t_1) \geq \mathrm{rmin}\{(\kappa_I)^T_\beta(t_1 * t_2), (\kappa_I)^T_\beta(t_2)\}$$
$$= \mathrm{rmin}\{\kappa_I(t_1 * t_2) + \beta, \kappa_I(t_2) + \beta\}$$
$$\kappa_I(t_1) + \beta = \mathrm{rmin}\{\kappa_I(t_1 * t_2), \kappa_I(t_2)\} + \beta,$$

$$\kappa_F(t_1) - \gamma = (\kappa_F)^T_\gamma(t_1) \geq \mathrm{rmin}\{(\kappa_F)^T_\gamma(t_1 * t_2), (\kappa_F)^T_\gamma(t_2)\}$$
$$= \mathrm{rmin}\{\kappa_F(t_1 * t_2) - \gamma, \kappa_F(t_2) - \gamma\}$$
$$\kappa_F(t_1) - \gamma = \mathrm{rmin}\{\kappa_F(t_1 * t_2), \kappa_F(t_2)\} - \gamma,$$

and

$$\upsilon_T(t_1) + \alpha = (\upsilon_T)^T_\alpha(t_1) \leq \max\{(\upsilon_T)^T_\alpha(t_1 * t_2), (\upsilon_T)^T_\alpha(t_2)\}$$
$$= \max\{\upsilon_T(t_1 * t_2) + \alpha, \upsilon_T(t_2) + \alpha\}$$

$\upsilon_T(t_1) + \alpha = \max\{\upsilon_T(t_1 * t_2), \upsilon_T(t_2)\} + \alpha,$

$\upsilon_I(t_1) + \beta = (\upsilon_I)^T_\beta(t_1) \leq \max\{(\upsilon_I)^T_\beta(t_1 * t_2), (\upsilon_I)^T_\beta(t_2)\}$

$= \max\{\upsilon_I(t_1 * t_2) + \beta, \upsilon_I(t_2) + \beta\}$

$\upsilon_I(t_1) + \beta = \max\{\upsilon_I(t_1 * t_2), \upsilon_I(t_2)\} + \beta,$

$\upsilon_F(t_1) - \gamma = (\upsilon_F)^T_\gamma(t_1) \leq \max\{(\upsilon_F)^T_\gamma(t_1 * t_2), (\upsilon_F)^T_\gamma(t_2)\}$

$= \max\{\upsilon_F(t_1 * t_2) - \gamma, \upsilon_F(t_2) - \gamma\}$

$\upsilon_F(t_1) - \gamma = \max\{\upsilon_F(t_1 * t_2), \upsilon_F(t_2)\} - \gamma,$

which imply $\kappa_T(t_1) \geq \mathrm{rmin}\{\kappa_T(t_1 * t_2), \kappa_T(t_2)\}$, $\kappa_I(t_1) \geq \mathrm{rmin}\{\kappa_I(t_1 * t_2), \kappa_I(t_2)\}$, $\kappa_F(t_1) \geq \mathrm{rmin}\{\kappa_F(t_1 * t_2), \kappa_F(t_2)\}$ and $\upsilon_T(t_1) \leq \max\{\upsilon_T(t_1 * t_2), \upsilon_T(t_2)\}$, $\upsilon_I(t_1) \leq \max\{\upsilon_I(t_1 * t_2), \upsilon_I(t_2)\}$, $\upsilon_F(t_1) \leq \max\{\upsilon_F(t_1 * t_2), \upsilon_F(t_2)\}$ for all $t_1, t_2 \in Y$. Hence B is a **NCID** of Y.

Theorem 3.2.4 Let B be a **NCID** of Y for some $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ℸ]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$. Then **NCT** $B^T_{\alpha,\beta,\gamma}$ of B is a **NCSU** of Y.

Proof. Assume $t_1, t_2 \in Y$. Then

$(\kappa_T)^T_\alpha(t_1 * t_2) = \kappa_T(t_1 * t_2) + \alpha$

$\geq \mathrm{rmin}\{\kappa_T(t_2 * (t_1 * t_2)), \kappa_T(t_2)\} + \alpha$

$= \mathrm{rmin}\{\kappa_T(0), \kappa_T(t_2)\} + \alpha$

$\geq \mathrm{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\} + \alpha$

$= \mathrm{rmin}\{\kappa_T(t_1) + \alpha, \kappa_T(t_2) + \alpha\}$

$(\kappa_T)^T_\alpha(t_1 * t_2) = \mathrm{rmin}\{(\kappa_T)^T_\alpha(t_1), (\kappa_T)^T_\alpha(t_2)\}$

$(\kappa_T)^T_\alpha(t_1 * t_2) \geq \mathrm{rmin}\{(\kappa_T)^T_\alpha(t_1), (\kappa_T)^T_\alpha(t_2)\},$

$(\kappa_I)^T_\beta(t_1 * t_2) = \kappa_I(t_1 * t_2) + \beta$

$\geq \mathrm{rmin}\{\kappa_I(t_2 * (t_1 * t_2)), \kappa_I(t_2)\} + \beta$

$= \mathrm{rmin}\{\kappa_I(0), \kappa_I(t_2)\} + \beta$

$\geq \mathrm{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\} + \beta$

$= \mathrm{rmin}\{\kappa_I(t_1) + \beta, \kappa_I(t_2) + \beta\}$

$(\kappa_I)^T_\beta(t_1 * t_2) = \mathrm{rmin}\{(\kappa_I)^T_\beta(t_1), (\kappa_I)^T_\beta(t_2)\}$

$(\kappa_I)^T_\beta(t_1 * t_2) \geq \mathrm{rmin}\{(\kappa_I)^T_\beta(t_1), (\kappa_I)^T_\beta(t_2)\},$

$(\kappa_F)^T_\gamma(t_1 * t_2) = \kappa_F(t_1 * t_2) - \gamma$

$$\geq \mathrm{rmin}\{\kappa_F(t_2 * (t_1 * t_2)), \kappa_F(t_2)\} - \gamma$$

$$= \mathrm{rmin}\{\kappa_F(0), \kappa_F(t_2)\} - \gamma$$

$$\geq \mathrm{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\} - \gamma$$

$$= \mathrm{rmin}\{\kappa_F(t_1) - \gamma, \kappa_F(t_2) - \gamma\}$$

$$(\kappa_F)_\gamma^T(t_1 * t_2) = \mathrm{rmin}\{(\kappa_F)_\gamma^T(t_1), (\kappa_F)_\gamma^T(t_2)\}$$

$$(\kappa_F)_\gamma^T(t_1 * t_2) \geq \mathrm{rmin}\{(\kappa_F)_\gamma^T(t_1), (\kappa_F)_\gamma^T(t_2)\}$$

and

$$(\upsilon_T)_\alpha^T(t_1 * t_2) = \upsilon_T(t_1 * t_2) + \alpha$$

$$\leq \max\{\upsilon_T(t_2 * (t_1 * t_2)), \upsilon_T(t_2)\} + \alpha$$

$$= \max\{\upsilon_T(0), \upsilon_T(t_2)\} + \alpha$$

$$\leq \max\{\upsilon_T(t_1), \upsilon_T(t_2)\} + \alpha$$

$$= \max\{\upsilon_T(t_1) + \alpha, \upsilon_T(t_2) + \alpha\}$$

$$(\upsilon_T)_\alpha^T(t_1 * t_2) = \max\{(\upsilon_T)_\alpha^T(t_1), (\upsilon_T)_\alpha^T(t_2)\}$$

$$(\upsilon_T)_\alpha^T(t_1 * t_2) \leq \max\{(\upsilon_T)_\alpha^T(t_1), (\upsilon_T)_\alpha^T(t_2)\},$$

$$(\upsilon_I)_\beta^T(t_1 * t_2) = \upsilon_I(t_1 * t_2) + \beta$$

$$\leq \max\{\upsilon_I(t_2 * (t_1 * t_2)), \upsilon_I(t_2)\} + \beta$$

$$= \max\{\upsilon_I(0), \upsilon_I(t_2)\} + \beta$$

$$\leq \max\{\upsilon_I(t_1), \upsilon_I(t_2)\} + \beta$$

$$= \max\{\upsilon_I(t_1) + \beta, \upsilon_I(t_2) + \beta\}$$

$$(\upsilon_I)_\beta^T(t_1 * t_2) = \max\{(\upsilon_I)_\beta^T(t_1), (\upsilon_I)_\beta^T(t_2)\}$$

$$(\upsilon_I)_\beta^T(t_1 * t_2) \leq \max\{(\upsilon_I)_\beta^T(t_1), (\upsilon_I)_\beta^T(t_2)\},$$

$$(\upsilon_F)_\gamma^T(t_1 * t_2) = \upsilon_F(t_1 * t_2) - \gamma$$

$$\leq \max\{\upsilon_F(t_2 * (t_1 * t_2)), \upsilon_F(t_2)\} - \gamma$$

$$= \max\{\upsilon_F(0), \upsilon_F(t_2)\} - \gamma$$

$$\leq \max\{\upsilon_F(t_1), \upsilon_F(t_2)\} - \gamma$$

$$= \max\{\upsilon_F(t_1) - \gamma, \upsilon_F(t_2) - \gamma\}$$

$$(\upsilon_F)_\gamma^T(t_1 * t_2) = \max\{(\upsilon_F)_\gamma^T(t_1), (\upsilon_F)_\gamma^T(t_2)\}$$

$$(\upsilon_F)_\gamma^T(t_1 * t_2) \leq \max\{(\upsilon_F)_\gamma^T(t_1), (\upsilon_F)_\gamma^T(t_2)\}.$$

Hence $B_{\alpha,\beta,\gamma}^T$ is a **NCSU** of Y.

Theorem 3.2.5 If **NCT** $B^{T}_{\alpha,\beta,\gamma}$ of B is a **NCID** of Y for some $\kappa_{T,I,F}$, $\alpha,\beta \in [[0,0],ך]$ and $\gamma \in [[0,0],¥]$, and for $\upsilon_{T,I,F}$, $\alpha,\beta \in [0,\Gamma]$ and $\gamma \in [0,£]$. Then B is a **NCSU** of Y.

Proof. Suppose $B^{T}_{\alpha,\beta,\gamma}$ of B is a **NCID** of Y. Then

$$(\kappa_T)(t_1 * t_2) + \alpha = (\kappa_T)^T_\alpha(t_1 * t_2)$$

$$\geq \mathrm{rmin}\{(\kappa_T)^T_\alpha(t_2 * (t_1 * t_2)), (\kappa_T)^T_\alpha(t_2)\}$$

$$= \mathrm{rmin}\{(\kappa_T)^T_\alpha(0), (\kappa_T)^T_\alpha(t_2)\}$$

$$\geq \mathrm{rmin}\{(\kappa_T)^T_\alpha(t_1), (\kappa_T)^T_\alpha(t_2)\}$$

$$= \mathrm{rmin}\{\kappa_T(t_1) + \alpha, \kappa_T(t_2) + \alpha\}$$

$$(\kappa_T)(t_1 * t_2) + \alpha = \mathrm{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\} + \alpha,$$

$$(\kappa_I)(t_1 * t_2) + \beta = (\kappa_I)^T_\beta(t_1 * t_2)$$

$$\geq \mathrm{rmin}\{(\kappa_I)^T_\beta(t_2 * (t_1 * t_2)), (\kappa_I)^T_\beta(t_2)\}$$

$$= \mathrm{rmin}\{(\kappa_I)^T_\beta(0), (\kappa_I)^T_\beta(t_2)\}$$

$$\geq \mathrm{rmin}\{(\kappa_I)^T_\beta(t_1), (\kappa_I)^T_\beta(t_2)\}$$

$$= \mathrm{rmin}\{\kappa_I(t_1) + \beta, \kappa_I(t_2) + \beta\}$$

$$(\kappa_I)(t_1 * t_2) + \beta = \mathrm{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\} + \beta,$$

$$(\kappa_F)(t_1 * t_2) - \gamma = (\kappa_F)^T_\gamma(t_1 * t_2)$$

$$\geq \mathrm{rmin}\{(\kappa_F)^T_\gamma(t_2 * (t_1 * t_2)), (\kappa_F)^T_\gamma(t_2)\}$$

$$= \mathrm{rmin}\{(\kappa_F)^T_\gamma(0), (\kappa_F)^T_\gamma(t_2)\}$$

$$\geq \mathrm{rmin}\{(\kappa_F)^T_\gamma(t_1), (\kappa_F)^T_\gamma(t_2)\}$$

$$= \mathrm{rmin}\{\kappa_F(t_1) - \gamma, \kappa_F(t_2) - \gamma\}$$

$$(\kappa_F)(t_1 * t_2) - \gamma = \mathrm{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\} - \gamma$$

$\Rightarrow \kappa_T(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\}, \kappa_I(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\}$ and $\kappa_F(t_1 * t_2)$
$\geq \mathrm{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\}$ and now

$$(\upsilon_T)(t_1 * t_2) + \alpha = (\upsilon_T)^T_\alpha(t_1 * t_2)$$

$$\leq \max\{(\upsilon_T)^T_\alpha(t_2 * (t_1 * t_2)), (\upsilon_T)^T_\alpha(t_2)\}$$

$$= \max\{(\upsilon_T)^T_\alpha(0), (\upsilon_T)^T_\alpha(t_2)\}$$

$$\leq \max\{(\upsilon_T)^T_\alpha(t_1), (\upsilon_T)^T_\alpha(t_2)\}$$

$$= \max\{\upsilon_T(t_1) + \alpha, \upsilon_T(t_2) + \alpha\}$$

$$(\upsilon_T)(t_1 * t_2) + \alpha = \max\{\upsilon_T(t_1), \upsilon_T(t_2)\} + \alpha,$$

$(\upsilon_I)(t_1 * t_2) + \beta = (\upsilon_I)^T_\beta(t_1 * t_2)$

$\leq \max\{(\upsilon_I)^T_\beta(t_2 * (t_1 * t_2)), (\upsilon_I)^T_\beta(t_2)\}$

$= \max\{(\upsilon_I)^T_\beta(0), (\upsilon_I)^T_\beta(t_2)\}$

$\leq \max\{(\upsilon_I)^T_\beta(t_1), (\upsilon_I)^T_\beta(t_2)\}$

$= \max\{\upsilon_I(t_1) + \beta, \upsilon_I(t_2) + \beta\}$

$(\upsilon_I)(t_1 * t_2) + \beta = \max\{\upsilon_I(t_1), \upsilon_I(t_2)\} + \beta,$

$(\upsilon_F)(t_1 * t_2) - \gamma = (\upsilon_F)^T_\gamma(t_1 * t_2)$

$\leq \max\{(\upsilon_F)^T_\gamma(t_2 * (t_1 * t_2)), (\upsilon_F)^T_\gamma(t_2)\}$

$= \max\{(\upsilon_F)^T_\gamma(0), (\upsilon_F)^T_\gamma(t_2)\}$

$\leq \max\{(\upsilon_F)^T_\gamma(t_1), (\upsilon_F)^T_\gamma(t_2)\}$

$= \max\{\upsilon_F(t_1) - \gamma, \upsilon_F(t_2) - \gamma\}$

$(\upsilon_F)(t_1 * t_2) - \gamma = \max\{\upsilon_F(t_1), \upsilon_F(t_2)\} - \gamma$

$\Rightarrow \upsilon_T(t_1 * t_2) \leq \max\{\upsilon_T(t_1), \upsilon_T(t_2)\}, \upsilon_I(t_1 * t_2) \leq \max\{\upsilon_I(t_1), \upsilon_I(t_2)\}$ and $\upsilon_F(t_1 * t_2) \leq \max\{\upsilon_F(t_1), \upsilon_F(t_2)\}$. Hence B is a **NCSU** of Y.

Theorem 3.2.6 Intersection of any two neutrosophic cubic translations of a neutrosophic cubic BF ideals B of Y is a neutrosophic cubic BF ideal of Y.

Proof. Suppose $B^T_{\alpha,\beta,\gamma}$ and $B^T_{\alpha',\beta',\gamma'}$ are two neutrosophic cubic translations of neutrosophic cubic BF ideal B and C of Y respectively, where for $B^T_{\alpha,\beta,\gamma}$, for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ℸ]$, $\gamma \in [[0,0], ¥]$, for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$, $\gamma \in [0, £]$ and for $B^T_{\alpha',\beta',\gamma'}$, for $\kappa_{T,I,F}$ $\alpha', \beta' \in [[0,0], ℸ]$, $\gamma' \in [[0,0], ¥]$, for $\upsilon_{T,I,F}$, $\alpha', \beta' \in [0, \Gamma]$, $\gamma' \in [0, £]$ and $\alpha \leq \alpha'$, $\beta \leq \beta'$, $\gamma \leq \gamma'$ as we know that, $B^T_{\alpha,\beta,\gamma}$ and $B^T_{\alpha',\beta',\gamma'}$ are neutrosophic cubic BF ideals of Y. So

$((\kappa_T)^T_\alpha \cap (\kappa_T)^T_{\alpha'})(t_1) = \mathrm{rmin}\{(\kappa_T)^T_\alpha(t_1), (\kappa_T)^T_{\alpha'}(t_1)\}$
$= \mathrm{rmin}\{\kappa_T(t_1) + \alpha, \kappa_T(t_1) + \alpha'\}$
$= \kappa_T(t_1) + \alpha$
$((\kappa_T)^T_\alpha \cap (\kappa_T)^T_{\alpha'})(t_1) = (\kappa_T)^T_\alpha(t_1),$

$((\kappa_I)^T_\beta \cap (\kappa_I)^T_{\beta'})(t_1) = \mathrm{rmin}\{(\kappa_I)^T_\beta(t_1), (\kappa_I)^T_{\beta'}(t_1)\}$
$= \mathrm{rmin}\{\kappa_I(t_1) + \beta, \kappa_I(t_1) + \beta'\}$
$= \kappa_I(t_1) + \beta$
$((\kappa_I)^T_\beta \cap (\kappa_I)^T_{\beta'})(t_1) = (\kappa_I)^T_\beta(t_1),$

$$((\kappa_F)^T_\gamma \cap (\kappa_F)^T_{\gamma'})(t_1) = \mathrm{rmin}\{(\kappa_F)^T_\gamma(t_1), (\kappa_F)^T_{\gamma'}(t_1)\}$$
$$= \mathrm{rmin}\{\kappa_F(t_1) - \gamma, \kappa_F(t_1) - \gamma'\}$$
$$= \kappa_F(t_1) - \gamma'$$
$$((\kappa_F)^T_\gamma \cap (\kappa_F)^T_{\gamma'})(t_1) = (\kappa_F)^T_{\gamma'}(t_1)$$

and

$$((\upsilon_T)^T_\alpha \cap (\upsilon_T)^T_{\alpha'})(t_1) = \max\{(\upsilon_T)^T_\alpha(t_1), (\upsilon_T)^T_{\alpha'}(t_1)\}$$
$$= \max\{\upsilon_T(t_1) + \alpha, \upsilon_T(t_1) + \alpha'\}$$
$$= \upsilon_T(t_1) + \alpha'$$
$$((\upsilon_T)^T_\alpha \cap (\upsilon_T)^T_{\alpha'})(t_1) = (\upsilon_T)^T_{\alpha'}(t_1),$$

$$((\upsilon_I)^T_\beta \cap (\upsilon_I)^T_{\beta'})(t_1) = \max\{(\upsilon_I)^T_\beta(t_1), (\upsilon_I)^T_{\beta'}(t_1)\}$$
$$= \max\{\upsilon_I(t_1) + \beta, \upsilon_I(t_1) + \beta'\}$$
$$= \upsilon_I(t_1) + \beta'$$
$$((\upsilon_I)^T_\beta \cap (\upsilon_I)^T_{\beta'})(t_1) = (\upsilon_I)^T_{\beta'}(t_1),$$

$$((\upsilon_F)^T_\gamma \cap (\upsilon_F)^T_{\gamma'})(t_1) = \max\{(\upsilon_F)^T_\gamma(t_1), (\upsilon_F)^T_{\gamma'}(t_1)\}$$
$$= \max\{\upsilon_F(t_1) - \gamma, \upsilon_F(t_1) - \gamma'\}$$
$$= \upsilon_F(t_1) - \gamma$$
$$((\upsilon_F)^T_\gamma \cap (\upsilon_F)^T_{\gamma'})(t_1) = (\upsilon_F)^T_\gamma(t_1).$$

Hence $B^T_{\alpha,\beta,\gamma} \cap B^T_{\alpha',\beta',\gamma'}$ is a neutrosophic cubic BF ideal of Y.

Theorem 3.2.7 Union of any two neutrosophic cubic translations of a neutrosophic cubic BF ideals B of Y is a neutrosophic cubic BF ideal of Y.

Proof. Suppose $B^T_{\alpha,\beta,\gamma}$ and $B^T_{\alpha',\beta',\gamma'}$ are two neutrosophic cubic translations of neutrosophic cubic BF ideal B of Y respectively, where for $B^T_{\alpha,\beta,\gamma}$, for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ℸ]$, $\gamma \in [[0,0], ¥]$, for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$, $\gamma \in [0, £]$ and for $B^T_{\alpha\prime,\beta\prime,\gamma\prime}$, for $\kappa_{T,I,F}$ $\alpha', \beta' \in [[0,0], ℸ]$, $\gamma' \in [[0,0], ¥]$, for $\upsilon_{T,I,F}$, $\alpha', \beta' \in [0, \Gamma]$, $\gamma' \in [0, £]$ and $\alpha \geq \alpha'$, $\beta \geq \beta'$, $\gamma \geq \gamma'$ as we know that, $B^T_{\alpha,\beta,\gamma}$ and $B^T_{\alpha',\beta',\gamma'}$ are neutrosophic cubic BF ideals of Y. Then

$$((\kappa_T)^T_\alpha \cup (\kappa_T)^T_{\alpha'})(t_1) = \mathrm{rmax}\{(\kappa_T)^T_\alpha(t_1), (\kappa_T)^T_{\alpha'}(t_1)\}$$
$$= \mathrm{rmax}\{\kappa_T(t_1) + \alpha, \kappa_T(t_1) + \alpha'\}$$
$$= \kappa_T(t_1) + \alpha$$
$$((\kappa_T)^T_\alpha \cup (\kappa_T)^T_{\alpha'})(t_1) = (\kappa_T)^T_\alpha(t_1),$$

$$((\kappa_I)^T_\beta \cup (\kappa_I)^T_{\beta'})(t_1) = \mathrm{rmax}\{(\kappa_I)^T_\beta(t_1), (\kappa_I)^T_{\beta'}(t_1)\}$$
$$= \mathrm{rmax}\{\kappa_I(t_1) + \beta, \kappa_I(t_1) + \beta'\}$$
$$= \kappa_I(t_1) + \beta$$
$$((\kappa_I)^T_\beta \cup (\kappa_I)^T_{\beta'})(t_1) = (\kappa_I)^T_\beta(t_1),$$

$$((\kappa_F)^T_\gamma \cup (\kappa_F)^T_{\gamma'})(t_1) = \mathrm{rmax}\{(\kappa_F)^T_\gamma(t_1), (\kappa_F)^T_{\gamma'}(t_1)\}$$
$$= \mathrm{rmax}\{\kappa_F(t_1) - \gamma, \kappa_F(t_1) - \gamma'\}$$
$$= \kappa_F(t_1) - \gamma'$$
$$((\kappa_F)^T_\gamma \cup (\kappa_F)^T_{\gamma'})(t_1) = (\kappa_F)^T_{\gamma'}(t_1)$$

and

$$((\upsilon_T)^T_\alpha \cup (\upsilon_T)^T_{\alpha'})(t_1) = \min\{(\upsilon_T)^T_\alpha(t_1), (\upsilon_T)^T_{\alpha'}(t_1)\}$$
$$= \min\{\upsilon_T(t_1) + \alpha, \upsilon_T(t_1) + \alpha'\}$$
$$= \upsilon_T(t_1) + \alpha'$$
$$((\upsilon_T)^T_\alpha \cup (\upsilon_T)^T_{\alpha'})(t_1) = (\upsilon_T)^T_{\alpha'}(t_1),$$

$$((\upsilon_I)^T_\beta \cup (\upsilon_I)^T_{\beta'})(t_1) = \min\{(\upsilon_I)^T_\beta(t_1), (\upsilon_I)^T_{\beta'}(t_1)\}$$
$$= \min\{\upsilon_I(t_1) + \beta, \upsilon_I(t_1) + \beta'\}$$
$$= \upsilon_I(t_1) + \beta'$$
$$((\upsilon_I)^T_\beta \cup (\upsilon_I)^T_{\beta'})(t_1) = (\upsilon_I)^T_\beta(t_1),$$

$$((\upsilon_F)^T_\gamma \cup (\upsilon_F)^T_{\gamma'})(t_1) = \min\{(\upsilon_F)^T_\gamma(t_1), (\upsilon_F)^T_{\gamma'}(t_1)\}$$
$$= \min\{\upsilon_F(t_1) - \gamma, \upsilon_F(t_1) - \gamma'\}$$
$$= \upsilon_F(t_1) - \gamma$$
$$((\upsilon_F)^T_\gamma \cup (\upsilon_F)^T_{\gamma'})(t_1) = (\upsilon_F)^T_\gamma(t_1)$$

Hence $B^T_{\alpha,\beta,\gamma} \cup B^T_{\alpha',\beta',\gamma'}$ is a neutrosophic cubic BF ideal of Y.

Theorem 3.2.8 Let B be a **NCS** of Y such that **NCM** B^M_δ of B is a **NCID** of Y for $\delta \in (0,1]$ then B is a **NCID** of Y.

Proof. Suppose that B^M_δ is a NCID of Y for $\delta \in (0,1]$ and $t_1, t_2 \in Y$. Then $\delta.\kappa_T(0) = (\kappa_T)^M_\delta(0) \geq (\kappa_T)^M_\delta(t_1) = \delta.\kappa_T(t_1)$, so $\kappa_T(0) \geq \kappa_T(t_1)$,$\delta.\kappa_I(0) = (\kappa_I)^M_\delta(0) \geq (\kappa_I)^M_\delta(t_1) = \delta.\kappa_I(t_1)$, so $\kappa_I(0) \geq \kappa_I(t_1)$, $\delta.\kappa_F(0) = (\kappa_F)^M_\delta(0) \geq (\kappa_F)^M_\delta(t_1) = \delta.\kappa_F(t_1)$, so $\kappa_F(0) \geq \kappa_F(t_1)$ and $\delta.\upsilon_T(0) = (\upsilon_T)^M_\delta(0) \leq (\upsilon_T)^M_\delta(t_1) = \delta.\upsilon_T(t_1)$, so $\upsilon_T(0) \leq \upsilon_T(t_1)$, $\delta.\upsilon_I(0) = (\upsilon_I)^M_\delta(0) \leq (\upsilon_I)^M_\delta(t_1) = \delta.\upsilon_I(t_1)$, so $\upsilon_I(0) \leq \upsilon_I(t_1)$, $\delta.\upsilon_F(0) = (\upsilon_F)^M_\delta(0) \leq (\upsilon_F)^M_\delta(t_1) = \delta.\upsilon_F(t_1)$, so $\upsilon_F(0) \leq \upsilon_F(t_1)$. Now

$$\delta.\kappa_T(t_1) = (\kappa_T)^M_\delta(t_1)$$
$$\geq \mathrm{rmin}\{(\kappa_T)^M_\delta(t_1 * t_2), (\kappa_T)^M_\delta(t_2)\}$$
$$= \mathrm{rmin}\{\delta.\kappa_T(t_1 * t_2), \delta.\kappa_T(t_2)\}$$
$$\delta.\kappa_T(t_1) = \delta.\mathrm{rmin}\{\kappa_T(t_1 * t_2), \kappa_T(t_2)\},$$

$$\delta.\kappa_I(t_1) = (\kappa_I)^M_\delta(t_1)$$

$\geq \text{rmin}\{(\kappa_I)^M_\delta(t_1 * t_2), (\kappa_I)^M_\delta(t_2)\}$

$= \text{rmin}\{\delta.\kappa_I(t_1 * t_2), \delta.\kappa_I(t_2)\}$

$\delta.\kappa_I(t_1) = \delta.\text{rmin}\{\kappa_I(t_1 * t_2), \kappa_I(t_2)\},$

$\delta.\kappa_F(t_1) = (\kappa_F)^M_\delta(t_1)$

$\geq \text{rmin}\{(\kappa_F)^M_\delta(t_1 * t_2), (\kappa_F)^M_\delta(t_2)\}$

$= \text{rmin}\{\delta.\kappa_F(t_1 * t_2), \delta.\kappa_F(t_2)\}$

$\delta.\kappa_F(t_1) = \delta.\text{rmin}\{\kappa_F(t_1 * t_2), \kappa_F(t_2)\},$

so $\kappa_T(t_1) \geq \text{rmin}\{\kappa_T(t_1 * t_2), \kappa_T(t_2)\}, \kappa_I(t_1) \geq \text{rmin}\{\kappa_I(t_1 * t_2), \kappa_I(t_2)\}$ and $\kappa_F(t_1) \geq \text{rmin}\{\kappa_F(t_1 * t_2), \kappa_F(t_2)\}$ and also

$\delta.\upsilon_T(t_1) = (\upsilon_T)^M_\delta(t_1)$

$\leq \max\{(\upsilon_T)^M_\delta(t_1 * t_2), (\upsilon_T)^M_\delta(t_2)\}$

$= \max\{\delta.\upsilon_T(t_1 * t_2), \delta.\upsilon_T(t_2)\}$

$\delta.\upsilon_T(t_1) = \delta.\max\{\upsilon_T(t_1 * t_2), \upsilon_T(t_2)\},$

$\delta.\upsilon_I(t_1) = (\upsilon_I)^M_\delta(t_1)$

$\leq \max\{(\upsilon_I)^M_\delta(t_1 * t_2), (\upsilon_I)^M_\delta(t_2)\}$

$= \max\{\delta.\upsilon_I(t_1 * t_2), \delta.\upsilon_I(t_2)\}$

$\delta.\upsilon_I(t_1) = \delta.\max\{\upsilon_I(t_1 * t_2), \upsilon_I(t_2)\},$

$\delta.\upsilon_F(t_1) = (\upsilon_F)^M_\delta(t_1)$

$\leq \max\{(\upsilon_F)^M_\delta(t_1 * t_2), (\upsilon_F)^M_\delta(t_2)\}$

$= \max\{\delta.\upsilon_F(t_1 * t_2), \delta.\upsilon_F(t_2)\}$

$\delta.\upsilon_F(t_1) = \delta.\max\{\upsilon_F(t_1 * t_2), \upsilon_F(t_2)\},$

so $\upsilon_T(t_1) \leq \max\{\upsilon_T(t_1 * t_2), \upsilon_T(t_2)\}, \upsilon_I(t_1) \leq \max\{\upsilon_I(t_1 * t_2), \upsilon_I(t_2)\}$ and $\upsilon_F(t_1) \leq \max\{\upsilon_F(t_1 * t_2), \upsilon_F(t_2)\}$. Hence B is a **NCID** of Y.

Theorem 3.2.9 If B is a **NCID** of Y, then **NCM** B^M_δ of B is a **NCID** of Y, for all $\delta \in (0,1]$.

Proof. Let B be a **NCID** of Y and $\delta \in (0,1]$. Then we have $(\kappa_T)^M_\delta(0) = \delta.\kappa_T(0) \geq \delta.\kappa_T(t_1) \rightarrow (\kappa_T)^M_\delta(0) = (\kappa_T)^M_\delta(t_1)$, $(\kappa_I)^M_\delta(0) = \delta.\kappa_I(0) \geq \delta.\kappa_I(t_1) \rightarrow (\kappa_I)^M_\delta(0) = (\kappa_I)^M_\delta\ (\ t_1)$, $(\kappa_F)^M_\delta(0) = \delta.\kappa_F(0) \geq \delta.\kappa_F(t_1) \rightarrow (\kappa_F)^M_\delta(0) = (\kappa_F)^M_\delta(t_1)$ and $(\upsilon_T)^M_\delta(0) = \delta.\upsilon_T(0) \leq \delta.\upsilon_T(t_1) \rightarrow (\upsilon_T)^M_\delta(0) = (\upsilon_T)^M_\delta(t_1)$,

$(\upsilon_I)^M_\delta(0) = \delta.\upsilon_I(0) \le \delta.\upsilon_I(t_1) \to (\upsilon_I)^M_\delta(0) = (\upsilon_I)^M_\delta(t_1)$, $(\upsilon_F)^M_\delta(0) = \delta.\upsilon_F(0) \le \delta.\upsilon_F(t_1) \to (\upsilon_F)^M_\delta(0) = (\upsilon_F)^M_\delta(t_1)$.

Now

$$(\kappa_T)^M_\delta(t_1) = \delta.\kappa_T(t_1)$$
$$\ge \delta.\mathrm{rmin}\{\kappa_T(t_1 * t_2), \kappa_T(t_2)\}$$
$$= \mathrm{rmin}\{\delta.\kappa_T(t_1 * t_2), \delta.\kappa_T(t_2)\}$$
$$(\kappa_T)^M_\delta(t_1) = \mathrm{rmin}\{(\kappa_T)^M_\delta(t_1 * t_2), (\kappa_T)^M_\delta(t_2)\}$$
$$(\kappa_T)^M_\delta(t_1) \ge \mathrm{rmin}\{(\kappa_T)^M_\delta(t_1 * t_2), (\kappa_T)^M_\delta(t_2)\},$$

$$(\kappa_I)^M_\delta(t_1) = \delta.\kappa_I(t_1)$$
$$\ge \delta.\mathrm{rmin}\{\kappa_I(t_1 * t_2), \kappa_I(t_2)\}$$
$$= \mathrm{rmin}\{\delta.\kappa_I(t_1 * t_2), \delta.\kappa_I(t_2)\}$$
$$(\kappa_I)^M_\delta(t_1) = \mathrm{rmin}\{(\kappa_I)^M_\delta(t_1 * t_2), (\kappa_I)^M_\delta(t_2)\}$$
$$(\kappa_I)^M_\delta(t_1) \ge \mathrm{rmin}\{(\kappa_I)^M_\delta(t_1 * t_2), (\kappa_I)^M_\delta(t_2)\},$$

$$(\kappa_F)^M_\delta(t_1) = \delta.\kappa_F(t_1)$$
$$\ge \delta.\mathrm{rmin}\{\kappa_F(t_1 * t_2), \kappa_F(t_2)\}$$
$$= \mathrm{rmin}\{\delta.\kappa_F(t_1 * t_2), \delta.\kappa_F(t_2)\}$$
$$(\kappa_F)^M_\delta(t_1) = \mathrm{rmin}\{(\kappa_F)^M_\delta(t_1 * t_2), (\kappa_F)^M_\delta(t_2)\}$$
$$(\kappa_F)^M_\delta(t_1) \ge \mathrm{rmin}\{(\kappa_F)^M_\delta(t_1 * t_2), (\kappa_F)^M_\delta(t_2)\}$$

and

$$(\upsilon_T)^M_\delta(t_1) = \delta.\upsilon_T(t_1)$$
$$\le \delta.\max\{\upsilon_T(t_1 * t_2), \upsilon_T(t_2)\}$$
$$= \max\{\delta.\upsilon_T(t_1 * t_2), \delta.\upsilon_T(t_2)\}$$
$$(\upsilon_T)^M_\delta(t_1) = \max\{(\upsilon_T)^M_\delta(t_1 * t_2), (\upsilon_T)^M_\delta(t_2)\}$$
$$(\upsilon_T)^M_\delta(t_1) \le \max\{(\upsilon_T)^M_\delta(t_1 * t_2), (\upsilon_T)^M_\delta(t_2)\},$$

$$(\upsilon_I)^M_\delta(t_1) = \delta.\upsilon_I(t_1)$$
$$\le \delta.\max\{\upsilon_I(t_1 * t_2), \upsilon_I(t_2)\}$$
$$= \max\{\delta.\upsilon_I(t_1 * t_2), \delta.\upsilon_I(t_2)\}$$
$$(\upsilon_I)^M_\delta(t_1) = \max\{(\upsilon_I)^M_\delta(t_1 * t_2), (\upsilon_I)^M_\delta(t_2)\}$$
$$(\upsilon_I)^M_\delta(t_1) \le \max\{(\upsilon_I)^M_\delta(t_1 * t_2), (\upsilon_I)^M_\delta(t_2)\},$$

$(\upsilon_F)^M_\delta(t_1) = \delta.\upsilon_F(t_1)$

$\leq \delta.\max\{\upsilon_F(t_1 * t_2), \upsilon_F(t_2)\}$

$= \max\{\delta.\upsilon_F(t_1 * t_2), \delta.\upsilon_F(t_2)\}$

$(\upsilon_F)^M_\delta(t_1) = \max\{(\upsilon_F)^M_\delta(t_1 * t_2), (\upsilon_F)^M_\delta(t_2)\}$

$(\upsilon_F)^M_\delta(t_1) \leq \max\{(\upsilon_F)^M_\delta(t_1 * t_2), (\upsilon_F)^M_\delta(t_2)\}$.

Hence B^M_δ of B is a **NCID** of Y, for all $\delta \in (0,1]$.

Theorem 3.2.10 Let B be a **NCID** of Y and $\delta \in [0,1]$ then **NCM** B^M_δ of B is a **NCSU** of Y.

Proof. Suppose $t_1, t_2 \in Y$. Then

$(\kappa_T)^M_\delta(t_1 * t_2) = \delta.\kappa_T(t_1 * t_2)$

$\geq \delta.\mathrm{rmin}\{\kappa_T(t_2 * (t_1 * t_2)), \kappa_T(t_2)\}$

$= \delta.\mathrm{rmin}\{\kappa_T(0), \kappa_T(t_2)\}$

$\geq \delta.\mathrm{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\}$

$= \mathrm{rmin}\{\delta.\kappa_T(t_1), \delta.\kappa_T(t_2)\}$

$(\kappa_T)^M_\delta(t_1 * t_2) = \mathrm{rmin}\{(\kappa_T)^M_\delta(t_1), (\kappa_T)^M_\delta(t_2)\}$

$(\kappa_T)^M_\delta(t_1 * t_2) \geq \mathrm{rmin}\{(\kappa_T)^M_\delta(t_1), (\kappa_T)^M_\delta(t_2)\}$,

$(\kappa_I)^M_\delta(t_1 * t_2) = \delta.\kappa_I(t_1 * t_2)$

$\geq \delta.\mathrm{rmin}\{\kappa_I(t_2 * (t_1 * t_2)), \kappa_I(t_2)\}$

$= \delta.\mathrm{rmin}\{\kappa_I(0), \kappa_I(t_2)\}$

$\geq \delta.\mathrm{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\}$

$= \mathrm{rmin}\{\delta.\kappa_I(t_1), \delta.\kappa_I(t_2)\}$

$(\kappa_I)^M_\delta(t_1 * t_2) = \mathrm{rmin}\{(\kappa_I)^M_\delta(t_1), (\kappa_I)^M_\delta(t_2)\}$

$(\kappa_I)^M_\delta(t_1 * t_2) \geq \mathrm{rmin}\{(\kappa_I)^M_\delta(t_1), (\kappa_I)^M_\delta(t_2)\}$,

$(\kappa_F)^M_\delta(t_1 * t_2) = \delta.\kappa_F(t_1 * t_2)$

$\geq \delta.\mathrm{rmin}\{\kappa_F(t_2 * (t_1 * t_2)), \kappa_F(t_2)\}$

$= \delta.\mathrm{rmin}\{\kappa_F(0), \kappa_F(t_2)\}$

$\geq \delta.\mathrm{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\}$

$= \mathrm{rmin}\{\delta.\kappa_F(t_1), \delta.\kappa_F(t_2)\}$

$(\kappa_F)^M_\delta(t_1 * t_2) = \mathrm{rmin}\{(\kappa_F)^M_\delta(t_1), (\kappa_F)^M_\delta(t_2)\}$

$$(\kappa_F)^M_\delta(t_1 * t_2) \geq \text{rmin}\{(\kappa_F)^M_\delta(t_1), (\kappa_F)^M_\delta(t_2)\}$$

and

$$(\upsilon_T)^M_\delta(t_1 * t_2) = \delta.\upsilon_T(t_1 * t_2)$$
$$\leq \delta.\max\{\upsilon_T(t_2 * (t_1 * t_2)), \upsilon_T(t_2)\}$$
$$= \delta.\max\{\upsilon_T(0), \upsilon_T(t_2)\}$$
$$\leq \delta.\max\{\upsilon_T(t_1), \upsilon_T(t_2)\}$$
$$= \max\{\delta.\upsilon_T(t_1), \delta.\upsilon_T(t_2)\}$$
$$(\upsilon_T)^M_\delta(t_1 * t_2) = \max\{(\upsilon_T)^M_\delta(t_1), (\upsilon_T)^M_\delta(t_2)\}$$
$$(\upsilon_T)^M_\delta(t_1 * t_2) \leq \max\{(\upsilon_T)^M_\delta(t_1), (\upsilon_T)^M_\delta(t_2)\},$$

$$(\upsilon_I)^M_\delta(t_1 * t_2) = \delta.\upsilon_I(t_1 * t_2)$$
$$\leq \delta.\max\{\upsilon_I(t_2 * (t_1 * t_2)), \upsilon_I(t_2)\}$$
$$= \delta.\max\{\upsilon_I(0), \upsilon_I(t_2)\}$$
$$\leq \delta.\max\{\upsilon_I(t_1), \upsilon_I(t_2)\}$$
$$= \max\{\delta.\upsilon_I(t_1), \delta.\upsilon_I(t_2)\}$$
$$(\upsilon_I)^M_\delta(t_1 * t_2) = \max\{(\upsilon_I)^M_\delta(t_1), (\upsilon_I)^M_\delta(t_2)\}$$
$$(\upsilon_I)^M_\delta(t_1 * t_2) \leq \max\{(\upsilon_I)^M_\delta(t_1), (\upsilon_I)^M_\delta(t_2)\},$$

$$(\upsilon_F)^M_\delta(t_1 * t_2) = \delta.\upsilon_F(t_1 * t_2)$$
$$\leq \delta.\max\{\upsilon_F(t_2 * (t_1 * t_2)), \upsilon_F(t_2)\}$$
$$= \delta.\max\{\upsilon_F(0), \upsilon_F(t_2)\}$$
$$\leq \delta.\max\{\upsilon_F(t_1), \upsilon_F(t_2)\}$$
$$= \max\{\delta.\upsilon_F(t_1), \delta.\upsilon_F(t_2)\}$$
$$(\upsilon_F)^M_\delta(t_1 * t_2) = \max\{(\upsilon_F)^M_\delta(t_1), (\upsilon_F)^M_\delta(t_2)\}$$
$$(\upsilon_F)^M_\delta(t_1 * t_2) \leq \max\{(\upsilon_F)^M_\delta(t_1), (\upsilon_F)^M_\delta(t_2)\}.$$

Hence B^M_δ is a **NCSU** of Y.

Theorem 3.2.11 If the **NCM** B^M_δ of B is a **NCID** of Y, for $\delta \in (0,1]$. Then B is a neutrosophic cubic BF-subalgebra of Y.

Proof. Assume B^M_δ of B is a **NCID** of Y. Then

$$\delta.(\kappa_T)(t_1 * t_2) = (\kappa_T)^M_\delta(t_1 * t_2)$$
$$\geq \text{rmin}\{(\kappa_T)^M_\delta(t_2 * (t_1 * t_2)), (\kappa_T)^M_\delta(t_2)\}$$
$$= \text{rmin}\{(\kappa_T)^M_\delta(0), (\kappa_T)^M_\delta(t_2)\}$$

$$\geq \mathrm{rmin}\{(\kappa_T)_\delta^M(t_1), (\kappa_T)_\delta^M(t_2)\}$$

$$= \mathrm{rmin}\{\delta.\kappa_T(t_1), \delta.\kappa_T(t_2)\}$$

$$\delta.(\kappa_T)(t_1 * t_2) = \delta.\mathrm{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\}$$

$$\Rightarrow \kappa_T(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\},$$

$$\delta.(\kappa_I)(t_1 * t_2) = (\kappa_I)_\delta^M(t_1 * t_2)$$

$$\geq \mathrm{rmin}\{(\kappa_I)_\delta^M(t_2 * (t_1 * t_2)), (\kappa_I)_\delta^M(t_2)\}$$

$$= \mathrm{rmin}\{(\kappa_I)_\delta^M(0), (\kappa_I)_\delta^M(t_2)\}$$

$$\geq \mathrm{rmin}\{(\kappa_I)_\delta^M(t_1), (\kappa_I)_\delta^M(t_2)\}$$

$$= \mathrm{rmin}\{\delta.\kappa_I(t_1), \delta.\kappa_I(t_2)\}$$

$$\delta.(\kappa_I)(t_1 * t_2) = \delta.\mathrm{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\}$$

$$\Rightarrow \kappa_I(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\},$$

$$\delta.(\kappa_F)(t_1 * t_2) = (\kappa_F)_\delta^M(t_1 * t_2)$$

$$\geq \mathrm{rmin}\{(\kappa_F)_\delta^M(t_2 * (t_1 * t_2)), (\kappa_F)_\delta^M(t_2)\}$$

$$= \mathrm{rmin}\{(\kappa_F)_\delta^M(0), (\kappa_F)_\delta^M(t_2)\}$$

$$\geq \mathrm{rmin}\{(\kappa_F)_\delta^M(t_1), (\kappa_F)_\delta^M(t_2)\}$$

$$= \mathrm{rmin}\{\delta.\kappa_F(t_1), \delta.\kappa_F(t_2)\}$$

$$\delta.(\kappa_F)(t_1 * t_2) = \delta.\mathrm{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\}$$

$$\Rightarrow \kappa_F(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\}$$

and

$$\delta.(\upsilon_T)(t_1 * t_2) = (\upsilon_T)_\delta^M(t_1 * t_2)$$

$$\leq \max\{(\upsilon_T)_\delta^M(t_2 * (t_1 * t_2)), (\upsilon_T)_\delta^M(t_2)\}$$

$$= \max\{(\upsilon_T)_\delta^M(0), (\upsilon_T)_\delta^M(t_2)\}$$

$$\leq \max\{(\upsilon_T)_\delta^M(t_1), (\upsilon_T)_\delta^M(t_2)\}$$

$$= \max\{\delta.\upsilon_T(t_1), \delta.\upsilon_T(t_2)\}$$

$$\delta.(\upsilon_T)(t_1 * t_2) = \delta.\max\{\upsilon_T(t_1), \upsilon_T(t_2)\}$$

$$\Rightarrow \upsilon_T(t_1 * t_2) \leq \max\{\upsilon_T(t_1), \upsilon_T(t_2)\},$$

$$\delta.(\upsilon_I)(t_1 * t_2) = (\upsilon_I)_\delta^M(t_1 * t_2)$$

$$\leq \max\{(\upsilon_I)_\delta^M(t_2 * (t_1 * t_2)), (\upsilon_I)_\delta^M(t_2)\}$$

$$= \max\{(\upsilon_I)_\delta^M(0), (\upsilon_I)_\delta^M(t_2)\}$$

$\leq \max\{(\upsilon_I)^M_\delta(t_1), (\upsilon_I)^M_\delta(t_2)\}$

$= \max\{\delta.\upsilon_I(t_1), \delta.\upsilon_I(t_2)\}$

$\delta.(\upsilon_I)(t_1 * t_2) = \delta.\max\{\upsilon_I(t_1), \upsilon_I(t_2)\}$

$\Rightarrow \upsilon_I(t_1 * t_2) \leq \max\{\upsilon_I(t_1), \upsilon_I(t_2)\}$,

$\delta.(\upsilon_F)(t_1 * t_2) = (\upsilon_F)^M_\delta(t_1 * t_2)$

$\leq \max\{(\upsilon_F)^M_\delta(t_2 * (t_1 * t_2)), (\upsilon_F)^M_\delta(t_2)\}$

$= \max\{(\upsilon_F)^M_\delta(0), (\upsilon_F)^M_\delta(t_2)\}$

$\leq \max\{(\upsilon_F)^M_\delta(t_1), (\upsilon_F)^M_\delta(t_2)\}$

$= \max\{\delta.\upsilon_F(t_1), \delta.\upsilon_F(t_2)\}$

$\delta.(\upsilon_F)(t_1 * t_2) = \delta.\max\{\upsilon_F(t_1), \upsilon_F(t_2)\}$

$\Rightarrow \upsilon_F(t_1 * t_2) \leq \max\{\upsilon_F(t_1), \upsilon_F(t_2)\}$.

Hence B is a **NCSU** of Y.

Theorem 3.2.12 Intersection of any two neutrosophic cubic multiplications of a **NCID** B of Y is a **NCID** of Y.

Proof. Suppose B^M_δ and $B^M_{\delta'}$ are neutrosophic cubic multiplications of **NCID** B of Y, where $\delta, \delta' \in (0,1]$ and $\delta \leq \delta'$, as we know that B^M_δ and $B^M_{\delta'}$ are **NCIDs** of Y. Then

$((\kappa_T)^M_\delta \cap (\kappa_T)^M_{\delta'})(t_1) = \mathrm{rmin}\{(\kappa_T)^M_\delta(t_1), (\kappa_T)^M_{\delta'}(t_1)\}$

$= \mathrm{rmin}\{\kappa_T(t_1).\delta, \kappa_T(t_1).\delta'\}$

$= \kappa_T(t_1).\delta$

$((\kappa_T)^M_\delta \cap (\kappa_T)^M_{\delta'})(t_1) = (\kappa_T)^M_\delta(t_1)$,

$((\kappa_I)^M_\delta \cap (\kappa_I)^M_{\delta'})(t_1) = \mathrm{rmin}\{(\kappa_I)^M_\delta(t_1), (\kappa_I)^M_{\delta'}(t_1)\}$

$= \mathrm{rmin}\{\kappa_I(t_1).\delta, \kappa_I(t_1).\delta'\}$

$= \kappa_I(t_1).\delta$

$((\kappa_I)^M_\delta \cap (\kappa_I)^M_{\delta'})(t_1) = (\kappa_I)^M_\delta(t_1)$,

$((\kappa_F)^M_\delta \cap (\kappa_F)^M_{\delta'})(t_1) = \mathrm{rmin}\{(\kappa_F)^M_\delta(t_1), (\kappa_F)^M_{\delta'}(t_1)\}$

$= \mathrm{rmin}\{\kappa_F(t_1).\delta, \kappa_F(t_1).\delta'\}$

$= \kappa_F(t_1).\delta$

$((\kappa_F)^M_\delta \cap (\kappa_F)^M_{\delta'})(t_1) = (\kappa_F)^M_\delta(t_1)$

and

$((\upsilon_T)_\delta^M \cap (\upsilon_T)_{\delta'}^M)(t_1) = \max\{(\upsilon_T)_\delta^M(t_1), (\upsilon_T)_{\delta'}^M(t_1)\}$

$= \max\{\upsilon_T(t_1).\delta, \upsilon_T(t_1).\delta'\}$

$= \upsilon_T(t_1).\delta'$

$((\upsilon_T)_\delta^M \cap (\upsilon_T)_{\delta'}^M)(t_1) = (\upsilon_T)_{\delta'}^M(t_1),$

$((\upsilon_I)_\delta^M \cap (\upsilon_I)_{\delta'}^M)(t_1) = \max\{(\upsilon_I)_\delta^M(t_1), (\upsilon_I)_{\delta'}^M(t_1)\}$

$= \max\{\upsilon_I(t_1).\delta, \upsilon_I(t_1).\delta'\}$

$= \upsilon_I(t_1).\delta'$

$((\upsilon_I)_\delta^M \cap (\upsilon_I)_{\delta'}^M)(t_1) = (\upsilon_I)_{\delta'}^M(t_1),$

$((\upsilon_F)_\delta^M \cap (\upsilon_F)_{\delta'}^M)(t_1) = \max\{(\upsilon_F)_\delta^M(t_1), (\upsilon_F)_{\delta'}^M(t_1)\}$

$= \max\{\upsilon_F(t_1).\delta, \upsilon_F(t_1).\delta'\}$

$= \upsilon_F(t_1).\delta'$

$((\upsilon_F)_\delta^M \cap (\upsilon_F)_{\delta'}^M)(t_1) = (\upsilon_F)_{\delta'}^M(t_1).$

Hence $B_\delta^M \cap B_{\delta'}^M$ is **NCID** of Y.

Theorem 3.2.13 Union of any two neutrosophic cubic multiplications of a **NCID** B of Y is a **NCID** of Y.

Proof. Suppose B_δ^M and $B_{\delta'}^M$ are neutrosophic cubic multiplications of **NCID** B of Y, where $\delta, \delta' \in (0,1]$ and $\delta \leq \delta'$, as we know that B_δ^M and $B_{\delta'}^M$ are **NCIDs** of Y. Then

$((\kappa_T)_\delta^M \cup (\kappa_T)_{\delta'}^M)(t_1) = \mathrm{rmax}\{(\kappa_T)_\delta^M(t_1), (\kappa_T)_{\delta'}^M(t_1)\}$

$= \mathrm{rmax}\{\kappa_T(t_1).\delta, \kappa_T(t_1).\delta'\}$

$= \kappa_T(t_1).\delta'$

$((\kappa_T)_\delta^M \cup (\kappa_T)_{\delta'}^M)(t_1) = (\kappa_T)_{\delta'}^M(t_1),$

$((\kappa_I)_\delta^M \cup (\kappa_I)_{\delta'}^M)(t_1) = \mathrm{rmax}\{(\kappa_I)_\delta^M(t_1), (\kappa_I)_{\delta'}^M(t_1)\}$

$= \mathrm{rmax}\{\kappa_I(t_1).\delta, \kappa_I(t_1).\delta'\}$

$= \kappa_I(t_1).\delta'$

$((\kappa_I)_\delta^M \cup (\kappa_I)_{\delta'}^M)(t_1) = (\kappa_I)_{\delta'}^M(t_1),$

$((\kappa_F)_\delta^M \cup (\kappa_F)_{\delta'}^M)(t_1) = \mathrm{rmax}\{(\kappa_F)_\delta^M(t_1), (\kappa_F)_{\delta'}^M(t_1)\}$

$= \mathrm{rmax}\{\kappa_F(t_1).\delta, \kappa_F(t_1).\delta'\}$

$= \kappa_F(t_1).\delta'$

$((\kappa_F)_\delta^M \cup (\kappa_F)_{\delta'}^M)(t_1) = (\kappa_F)_{\delta'}^M(t_1)$

and

$$((\upsilon_T)^M_\delta \cup (\upsilon_T)^M_{\delta'})(t_1) = \min\{(\upsilon_T)^M_\delta(t_1), (\upsilon_T)^M_{\delta'}(t_1)\}$$
$$= \min\{\upsilon_T(t_1).\delta, \upsilon_T(t_1).\delta'\}$$
$$= \upsilon_T(t_1).\delta$$
$$((\upsilon_T)^M_\delta \cup (\upsilon_T)^M_{\delta'})(t_1) = (\upsilon_T)^M_\delta(t_1),$$

$$((\upsilon_I)^M_\delta \cup (\upsilon_I)^M_{\delta'})(t_1) = \min\{(\upsilon_I)^M_\delta(t_1), (\upsilon_I)^M_{\delta'}(t_1)\}$$
$$= \min\{\upsilon_I(t_1).\delta, \upsilon_I(t_1).\delta'\}$$
$$= \upsilon_I(t_1).\delta$$
$$((\upsilon_I)^M_\delta \cup (\upsilon_I)^M_{\delta'})(t_1) = (\upsilon_I)^M_\delta(t_1),$$

$$((\upsilon_F)^M_\delta \cup (\upsilon_F)^M_{\delta'})(t_1) = \min\{(\upsilon_F)^M_\delta(t_1), (\upsilon_F)^M_{\delta'}(t_1)\}$$
$$= \min\{\upsilon_F(t_1).\delta, \upsilon_F(t_1).\delta'\}$$
$$= \upsilon_F(t_1).\delta$$
$$((\upsilon_F)^M_\delta \cup (\upsilon_F)^M_{\delta'})(t_1) = (\upsilon_F)^M_\delta(t_1).$$

Hence $B^M_\delta \cup B^M_{\delta'}$ is **NCID** of Y.

3.3 Magnified Translative Interpretation of Neutrosophic Cubic Subalgebra and Neutrosophic Cubic Ideal

In this section, we define the notion of neutrosophic cubic magnified translation **NCMT** and investigate some results.

Definition 3.3.1 Let $B = (\kappa_{T,I,F}, \upsilon_{T,I,F})$ be a **NCS** of Y and for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ךּ]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$ and for all $\delta \in [0,1]$. An object having the form $B^{M\,T}_{\delta\,\alpha,\beta,\gamma} = \{(\kappa_{T,I,F})^{M\,T}_{\delta\,\alpha,\beta,\gamma}, (\upsilon_{T,I,F})^{M\,T}_{\delta\,\alpha,\beta,\gamma}\}$ is said to be a **NCMT** of B, when $(\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1) = \delta.\kappa_T(t_1) + \alpha$, $(\kappa_I)^{M\,T}_{\delta\,\beta}(t_1) = \delta.\kappa_I(t_1) + \beta$, $(\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1) = \delta.\kappa_F(t_1)$ - γ and $(\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1) = \delta.\upsilon_T(t_1) + \alpha$, $(\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1) = \delta.\upsilon_I(t_1) + \beta$, $(\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1) = \delta.\upsilon_F(t_1)$-$\gamma$ for all $t_1 \in Y$.

Example 3.3.1 Let $Y = \{0,1,2\}$ be a BF-algebra as defined in Example 3.2.1. A **NCS** $B = (\kappa_{T,I,F}, \upsilon_{T,I,F})$ of Y is defined as

$$\kappa_T(t_1) = \begin{cases} [0.1, 0.3] & \text{if } t_1 = 0 \\ [0.4, 0.7] & \text{if otherwise} \end{cases}$$

$$\kappa_I(t_1) = \begin{cases} [0.2, 0.4] & \text{if } t_1 = 0 \\ [0.5, 0.7] & \text{if otherwise} \end{cases}$$

$$\kappa_F(t_1) = \begin{cases} [0.4, 0.6] & \text{if } t_1 = 0 \\ [0.5, 0.8] & \text{if otherwise} \end{cases}$$

and

$$\upsilon_T(t_1) = \begin{cases} 0.1 & \text{if } t_1 = 0 \\ 0.4 & \text{if otherwise} \end{cases}$$

$$\upsilon_I(t_1) = \begin{cases} 0.2 & \text{if } t_1 = 0 \\ 0.3 & \text{if otherwise} \end{cases}$$

$$\upsilon_F(t_1) = \begin{cases} 0.5 & \text{if } t_1 = 0 \\ 0.7 & \text{if otherwise.} \end{cases}$$

Then B is a neutrosophic cubic subalgebra, for $\upsilon_{T,I,F}$ choose $\delta = 0.1, \alpha = 0.02, \beta = 0.03, \gamma = 0.04$ and for $\kappa_{T,I,F}$ choose $\delta = [0.1,0.4], \alpha = [0.03,0.07], \beta = [0.04,0.08], \gamma = [0.02,0.06]$ then the mapping $B^{M\,T}_{(0.1)\,(\alpha,\beta,\gamma)}|Y \to [0,1]$ is given by

$$(\kappa_T)^{M}_{[0.1,0.4]\,[0.03,0.07]}(t_1) = \begin{cases} [0.04, 0.19] & \text{if } t_1 = 1 \\ [0.07, 0.35] & \text{if otherwise} \end{cases}$$

$$(\kappa_I)^{M\,T}_{[0.1,0.4]\,[0.04,0.08]}(t_1) = \begin{cases} [0.06, 0.24] & \text{if } t_1 = 1 \\ [0.09, 0.36] & \text{if otherwise} \end{cases}$$

$$(\kappa_F)^{M\,T}_{[0.1,0.4]\,[0.02,0.06]}(t_1) = \begin{cases} [0.02, 0.18] & \text{if } t_1 = 1 \\ [0.03, 0.26] & \text{if otherwise} \end{cases}$$

and

$$(\upsilon_T)^{M}_{0.1,0.02}(t_1) = \begin{cases} 0.03 & \text{if } t_1 = 1 \\ 0.06 & \text{if otherwise} \end{cases}$$

$$(\upsilon_I)^{M\,T}_{0.1,0.03}(t_1) = \begin{cases} 0.05 & \text{if } t_1 = 1 \\ 0.06 & \text{if otherwise} \end{cases}$$

$$(\upsilon_F)^{M\,T}_{0.1.0.04}(t_1) = \begin{cases} 0.01 & \text{if } t_1 = 1 \\ 0.03 & \text{if otherwise,} \end{cases}$$

which imply $(\kappa_T)^{M\,T}_{[0.1,0.4][0.03,0.07]}(t_1) = [0.1,0.4].\kappa_T(t_1) + [0.03,0.07]$, $(\kappa_I)^{M\,T}_{[0.1,0.4][0.04,0.08]}(t_1) = [0.1,0.4].\kappa_T(t_1) + [0.04,0.08]$, $(\kappa_F)^{M\,T}_{[0.1,0.4][0.02,0.06]}(t_1) = [0.1,0.4].\kappa_F(t_1) - [0.02,0.06]$ and $(\upsilon_T)^{M\,T}_{(0.1)(0.02)}(t_1) = (0.1).\upsilon_T(t_1) + 0.02$, $(\upsilon_I)^{M\,T}_{(0.1)(0.03)}(t_1) = (0.1).\upsilon_T(t_1) + 0.03$, $(\nu_F)^{M\,T}_{(0.1)\,(0.04)}(t_1) = (0.1).\nu_F(t_1) - 0.04$ for all $t_1 \in Y$. Hence $B^{M\,T}$ is a neutrosophic cubic magnified translation.

Theorem 3.3.1 Let B be a neutrosophic cubic subset of Y such that for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ℸ]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$ and $\delta \in [0,1]$ and a mapping $B^{M\,T,I,F}_{\delta\,\alpha,\beta,\gamma}|Y \to [0,1]$ be a **NCMT** of B. If B is **NCSU** of Y, then $B^{M\,T,I,F}_{\delta\,\alpha,\beta,\gamma}$ is a **NCSU** of Y.

Proof. Let B be a neutrosophic cubic subset of Y such that for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ℸ]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$ and $\delta \in [0,1]$ and a mapping $B^{M\,T,I,F}_{\delta\,\alpha,\beta,\gamma}|Y \to [0,1]$ be a **NCMT** of B. Suppose B is a **NCSU** of Y. Then $\kappa_T(t_1 * t_2) \geq \text{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\}, \kappa_I(t_1 * t_2) \geq \text{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\}, \kappa_F(t_1 * t_2) \geq \text{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\}$ and $\upsilon_T(t_1 * t_2) \leq \max\{\upsilon_T(t_1), \upsilon_T(t_2)\}, \upsilon_I(t_1 * t_2) \leq \max\{\upsilon_I(t_1), \upsilon_I(t_2)\}, \upsilon_F(t_1 * t_2) \leq \max\{\upsilon_F(t_1), \upsilon_F(t_2)\}$. Now

$$(\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1 * t_2) = \delta.\kappa_T(t_1 * t_2) + \alpha$$

$$\geq \delta.\text{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\} + \alpha$$

$= \mathrm{rmin}\{\delta.\kappa_T(t_1) + \alpha, \delta.\kappa_T(t_2) + \alpha\}$

$(\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1 * t_2) = \mathrm{rmin}\{(\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1), (\kappa_T)^{M\,T}_{\delta\,\alpha}(t_2)\}$

$(\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1 * t_2) \geq \mathrm{rmin}\{(\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1), (\kappa_T)^{M\,T}_{\delta\,\alpha}(t_2)\},$

$(\kappa_I)^{M\,T}_{\delta\,\beta}(t_1 * t_2) = \delta.\kappa_I(t_1 * t_2) + \beta$

$\geq \delta.\mathrm{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\} + \beta$

$= \mathrm{rmin}\{\delta.\kappa_I(t_1) + \beta, \delta.\kappa_I(t_2) + \beta\}$

$(\kappa_I)^{M\,T}_{\delta\,\beta}(t_1 * t_2) = \mathrm{rmin}\{(\kappa_I)^{M\,T}_{\delta\,\beta}(t_1), (\kappa_I)^{M\,T}_{\delta\,\beta}(t_2)\}$

$(\kappa_I)^{M\,T}_{\delta\,\beta}(t_1 * t_2) \geq \mathrm{rmin}\{(\kappa_I)^{M\,T}_{\delta\,\beta}(t_1), (\kappa_I)^{M\,T}_{\delta\,\beta}(t_2)\},$

$(\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2) = \delta.\kappa_F(t_1 * t_2) - \gamma$

$\geq \delta.\mathrm{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\} - \gamma$

$= \mathrm{rmin}\{\delta.\kappa_F(t_1) - \gamma, \delta.\kappa_F(t_2) - \gamma\}$

$(\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2) = \mathrm{rmin}\{(\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1), (\kappa_F)^{M\,T}_{\delta\,\gamma}(t_2)\}$

$(\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2) \geq \mathrm{rmin}\{(\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1), (\kappa_F)^{M\,T}_{\delta\,\gamma}(t_2)\}$

and

$(\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1 * t_2) = \delta.\upsilon_T(t_1 * t_2) + \alpha$

$\leq \delta.\max\{\upsilon_T(t_1), \upsilon_T(t_2)\} + \alpha$

$= \max\{\delta.\upsilon_T(t_1) + \alpha, \delta.\upsilon_T(t_2) + \alpha\}$

$(\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1 * t_2) = \max\{(\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1), (\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_2)\}$

$(\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1 * t_2) \leq \max\{(\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1), (\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_2)\},$

$(\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1 * t_2) = \delta.\upsilon_I(t_1 * t_2) + \beta$

$\leq \delta.\max\{\upsilon_I(t_1), \upsilon_I(t_2)\} + \beta$

$= \max\{\delta.\upsilon_I(t_1) + \beta, \delta.\upsilon_I(t_2) + \beta\}$

$(\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1 * t_2) = \max\{(\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1), (\upsilon_I)^{M\,T}_{\delta\,\beta}(t_2)\}$

$(\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1 * t_2) \leq \max\{(\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1), (\upsilon_I)^{M\,T}_{\delta\,\beta}(t_2)\},$

$(\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2) = \delta.\upsilon_F(t_1 * t_2) - \gamma$

$\leq \delta.\max\{\upsilon_F(t_1), \upsilon_F(t_2)\} - \gamma$

$= \max\{\delta.\upsilon_F(t_1) - \gamma, \delta.\upsilon_F(t_2) - \gamma\}$

$(\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2) = \max\{(\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1), (\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_2)\}$

$(\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2) \leq \max\{(\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1), (\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_2)\}.$

Hence **NCMT** $B^{M\,T}_{\delta\,\alpha,\beta,\gamma}$ is a **NCSU** of Y.

Theorem 3.3.2 Let B be a **NCS** of Y such that and for $\kappa_{T,I,F}$, $\alpha,\beta \in [[0,0],ר]$ and $\gamma \in [[0,0],¥]$, where for $\upsilon_{T,I,F}$, $\alpha,\beta \in [0,\Gamma]$ and $\gamma \in [0,£]$ and $\delta \in [0,1]$ and a mapping $B^{M\,T}_{\delta\,\alpha,\beta,\gamma}: Y \to [0,1]$ be a **NCMT** of B. If $B^{M\,T}_{\delta\,\alpha,\beta,\gamma}$ is **NCSU** of Y. Then B is a **NCSU** of Y.

Proof. Let B be a neutrosophic cubic subset of Y, where $\alpha,\beta,\gamma \in [0,¥]$, $\delta \in [0,1]$ and a mapping $B^{M\,T}_{\delta\,\alpha,\beta,\gamma}: Y \to [0,1]$ be a **NCMT** of B. Suppose $B^{M\,T}_{\delta\,\alpha,\beta,\gamma} = \{(\kappa_B)^{M\,T,I,F}_{\delta\,\alpha,\beta,\gamma}, (\upsilon_B)^{M\,T,I,F}_{\delta\,\alpha,\beta,\gamma}\}$ is a **NCSU** of Y, then

$$\delta.\kappa_T(t_1 * t_2) + \alpha = (\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1 * t_2)$$
$$\geq \text{rmin}\{(\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1), (\kappa_T)^{M\,T}_{\delta\,\alpha}(t_2)\}$$
$$= \text{rmin}\{\delta.\kappa_T(t_1) + \alpha, \delta.\kappa_T(t_2) + \alpha\}$$
$$\delta.\kappa_T(t_1 * t_2) + \alpha = \delta.\text{rmin}\{\kappa_T(t_2), \kappa_T(t_1)\} + \alpha,$$

$$\delta.\kappa_I(t_1 * t_2) + \beta = (\kappa_I)^{M\,T}_{\delta\,\beta}(t_1 * t_2)$$
$$\geq \text{rmin}\{(\kappa_I)^{M\,T}_{\delta\,\beta}(t_1), (\kappa_I)^{M\,T}_{\delta\,\beta}(t_2)\}$$
$$= \text{rmin}\{\delta.\kappa_I(t_1) + \beta, \delta.\kappa_I(t_2) + \beta\}$$
$$\delta.\kappa_I(t_1 * t_2) + \beta = \delta.\text{rmin}\{\kappa_I(t_2), \kappa_I(t_1)\} + \beta,$$

$$\delta.\kappa_F(t_1 * t_2) - \gamma = (\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2)$$
$$\geq \text{rmin}\{(\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1), (\kappa_F)^{M\,T}_{\delta\,\gamma}(t_2)\}$$
$$= \text{rmin}\{\delta.\kappa_F(t_1) - \gamma, \delta.\kappa_F(t_2) - \gamma\}$$
$$\delta.\kappa_F(t_1 * t_2) - \gamma = \delta.\text{rmin}\{\kappa_F(t_2), \kappa_F(t_1)\} - \gamma,$$

and

$$\delta.\upsilon_T(t_1 * t_2) + \alpha = (\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1 * t_2)$$
$$\leq \max\{(\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1), (\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_2)\}$$
$$= \max\{\delta.\upsilon_T(t_1) + \alpha, \delta.\upsilon_T(t_2) + \alpha\}$$
$$\delta.\upsilon_T(t_1 * t_2) + \alpha = \delta.\max\{\upsilon_T(t_2), \upsilon_T(t_1)\} + \alpha,$$

$$\delta.\upsilon_I(t_1 * t_2) + \beta = (\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1 * t_2)$$
$$\leq \max\{(\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1), (\upsilon_I)^{M\,T}_{\delta\,\beta}(t_2)\}$$
$$= \max\{\delta.\upsilon_I(t_1) + \beta, \delta.\upsilon_I(t_2) + \beta\}$$
$$\delta.\upsilon_I(t_1 * t_2) + \beta = \delta.\max\{\upsilon_I(t_2), \upsilon_I(t_1)\} + \beta,$$

$$\delta.\upsilon_F(t_1 * t_2) - \gamma = (\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2)$$

$$\leq \max\{(\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1), (\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_2)\}$$

$$= \max\{\delta.\upsilon_F(t_1) - \gamma, \delta.\upsilon_F(t_2) - \gamma\}$$

$$\delta.\upsilon_F(t_1 * t_2) - \gamma = \delta.\max\{\upsilon_F(t_2), \upsilon_F(t_1)\} - \gamma,$$

which imply $\kappa_T(t_1 * t_2) \geq r\min\{\kappa_T(t_1), \kappa_T(t_2)\}$, $\kappa_I(t_1 * t_2) \geq r\min\{\kappa_I(t_1), \kappa_I(t_2)\}$, $\kappa_F(t_1 * t_2) \geq r\min\{\kappa_F(t_1), \kappa_F(t_2)\}$ and $\upsilon_T(t_1 * t_2) \leq \max\{\upsilon_T(t_1), \upsilon_T(t_2)\}$,$\upsilon_I(t_1 * t_2) \leq \max\{\upsilon_I(t_1), \upsilon_I(t_2)\}$,$\upsilon_F(t_1 * t_2) \leq \max\{\upsilon_F(t_1), \upsilon_F(t_2)\}$ for all $t_1, t_2 \in Y$. Hence B is a **NCSU** of Y.

Theorem 3.3.3 If B is a **NCID** of Y. Then **NCMT** $B^{M\,T}_{\delta\,\alpha,\beta,\gamma}$ of B is a **NCID** of Y for all $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ל]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$ and $\delta \in (0,1]$.

Proof. Suppose $B = (\kappa_{T,I,F}, \upsilon_{T,I,F})$ is a **NCID** of Y. Then

$$(\kappa_T)^{M\,T}_{\delta\,\alpha}(0) = \delta.\kappa_T(0) + \alpha$$

$$\geq \delta.\kappa_T(t_1) + \alpha$$

$$(\kappa_T)^{M\,T}_{\delta\,\alpha}(0) = (\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1),$$

$$(\kappa_I)^{M\,T}_{\delta\,\beta}(0) = \delta.\kappa_I(0) + \beta$$
$$\geq \delta.\kappa_I(t_1) + \beta$$
$$(\kappa_I)^{M\,T}_{\delta\,\beta}(0) = (\kappa_I)^{M\,T}_{\delta\,\beta}(t_1),$$

$$(\kappa_F)^{M\,T}_{\delta\,\gamma}(0) = \delta.\kappa_F(0) - \gamma$$
$$\geq \delta.\kappa_F(t_1) - \gamma$$
$$(\kappa_F)^{M\,T}_{\delta\,\gamma}(0) = (\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1)$$

and

$$(\upsilon_T)^{M\,T}_{\delta\,\alpha}(0) = \delta.\upsilon_T(0) + \alpha$$

$$\leq \delta.\upsilon_T(t_1) + \alpha$$

$$(\upsilon_T)^{M\,T}_{\delta\,\alpha}(0) = (\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1),$$

$$(\upsilon_I)^{M\,T}_{\delta\,\beta}(0) = \delta.\upsilon_I(0) + \beta$$
$$\leq \delta.\upsilon_I(t_1) + \beta$$
$$(\upsilon_I)^{M\,T}_{\delta\,\beta}(0) = (\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1),$$

$$(\upsilon_F)^{M\,T}_{\delta\,\gamma}(0) = \delta.\upsilon_F(0) - \gamma$$
$$\leq \delta.\upsilon_F(t_1) - \gamma$$
$$(\upsilon_F)^{M\,T}_{\delta\,\gamma}(0) = (\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1)$$

Now

$$(\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1) = \delta.\kappa_T(t_1) + \alpha$$

$$\geq \delta.\mathrm{rmin}\{\kappa_T(t_1 * t_2), \kappa_T(t_2)\} + \alpha$$

$$= \mathrm{rmin}\{\delta.\kappa_T(t_1 * t_2) + \alpha, \delta.\kappa_T(t_2) + \alpha\}$$

$$(\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1) = \mathrm{rmin}\{(\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1 * t_2), (\kappa_T)^{M\,T}_{\delta\,\alpha}(t_2)\}$$

$$\Rightarrow (\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1) \geq \mathrm{rmin}\{(\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1 * t_2), (\kappa_T)^{M\,T}_{\delta\,\alpha}(t_2)\},$$

$$(\kappa_I)^{M\,T}_{\delta\,\beta}(t_1) = \delta.\kappa_I(t_1) + \beta$$

$$\geq \delta.\mathrm{rmin}\{\kappa_I(t_1 * t_2), \kappa_I(t_2)\} + \beta$$

$$= \mathrm{rmin}\{\delta.\kappa_I(t_1 * t_2) + \beta, \delta.\kappa_I(t_2) + \beta\}$$

$$(\kappa_I)^{M\,T}_{\delta\,\beta}(t_1) = \mathrm{rmin}\{(\kappa_I)^{M\,T}_{\delta\,\beta}(t_1 * t_2), (\kappa_I)^{M\,T}_{\delta\,\beta}(t_2)\}$$

$$\Rightarrow (\kappa_I)^{M\,T}_{\delta\,\beta}(t_1) \geq \mathrm{rmin}\{(\kappa_I)^{M\,T}_{\delta\,\beta}(t_1 * t_2), (\kappa_I)^{M\,T}_{\delta\,\beta}(t_2)\},$$

$$(\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1) = \delta.\kappa_F(t_1) - \gamma$$

$$\geq \delta.\mathrm{rmin}\{\kappa_F(t_1 * t_2), \kappa_F(t_2)\} - \gamma$$

$$= \mathrm{rmin}\{\delta.\kappa_F(t_1 * t_2) - \gamma, \delta.\kappa_F(t_2) - \gamma\}$$

$$(\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1) = \mathrm{rmin}\{(\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2), (\kappa_F)^{M\,T}_{\delta\,\gamma}(t_2)\}$$

$$\Rightarrow (\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1) \geq \mathrm{rmin}\{(\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2), (\kappa_F)^{M\,T}_{\delta\,\gamma}(t_2)\}$$

and

$$(\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1) = \delta.\upsilon_T(t_1) + \alpha$$

$$\leq \delta.\max\{\upsilon_T(t_1 * t_2), \upsilon_T(t_2)\} + \alpha$$

$$= \max\{\delta.\upsilon_T(t_1 * t_2) + \alpha, \delta.\upsilon_T(t_2) + \alpha\}$$

$$(\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1) = \max\{(\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1 * t_2), (\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_2)\}$$

$$\Rightarrow (\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1) \leq \max\{(\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1 * t_2), (\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_2)\},$$

$$(\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1) = \delta.\upsilon_I(t_1) + \beta$$

$$\leq \delta.\max\{\upsilon_I(t_1 * t_2), \upsilon_I(t_2)\} + \beta$$

$$= \max\{\delta.\upsilon_I(t_1 * t_2) + \beta, \delta.\upsilon_I(t_2) + \beta\}$$

$$(\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1) = \max\{(\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1 * t_2), (\upsilon_I)^{M\,T}_{\delta\,\beta}(t_2)\}$$

$$\Rightarrow (\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1) \le \max\{(\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1 * t_2), (\upsilon_I)^{M\,T}_{\delta\,\beta}(t_2)\},$$

$$(\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1) = \delta.\upsilon_F(t_1) - \gamma$$

$$\le \delta.\max\{\upsilon_F(t_1 * t_2), \upsilon_F(t_2)\} - \gamma$$

$$= \max\{\delta.\upsilon_F(t_1 * t_2) - \gamma, \delta.\upsilon_F(t_2) - \gamma\}$$

$$(\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1) = \max\{(\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2), (\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_2)\}$$

$$\Rightarrow (\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1) \le \max\{(\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2), (\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_2)\},$$

for all $t_1, t_2 \in Y$ and all for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ך]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$ and $\delta \in (0,1]$. Hence $B^{M\,T}_{\delta\,\alpha,\beta,\gamma}$ of B is a **NCID** of Y.

Theorem 3.3.3 If B is a neutrosophic cubic set of Y such that **NCMT** $B^{M\,T}_{\delta\,\alpha,\beta,\gamma}$ of B is a **NCID** of Y for all for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ך]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$ and $\delta \in (0,1]$, then B is a **NCID** of Y.

Proof. Suppose **NCMT** $B^{M\,T}_{\delta\,\alpha,\beta,\gamma}$ is a **NCID** of Y for some $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ך]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$ and $\delta \in (0,1]$ and $t_1, t_2 \in Y$. Then

$$\delta.\kappa_T(0) + \alpha = (\kappa_T)^{M\,T}_{\delta\,\alpha}(0)$$

$$\ge (\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1)$$

$$\delta.\kappa_T(0) + \alpha = \delta.\kappa_T(t_1) + \alpha,$$

$$\delta.\kappa_I(0) + \beta = (\kappa_I)^{M\,T}_{\delta\,\beta}(0)$$

$$\ge (\kappa_I)^{M\,T}_{\delta\,\beta}(t_1)$$

$$\delta.\kappa_I(0) + \beta = \delta.\kappa_I(t_1) + \beta,$$

$$\delta.\kappa_F(0) - \gamma = (\kappa_F)^{M\,T}_{\delta\,\gamma}(0)$$

$$\ge (\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1)$$

$$\delta.\kappa_F(0) - \gamma = \delta.\kappa_F(t_1) - \gamma,$$

and

$$\delta.\upsilon_T(0) + \alpha = (\upsilon_T)^{M\,T}_{\delta\,\alpha}(0)$$

$$\le (\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1)$$

$$\delta.\upsilon_T(0) + \alpha = \delta.\upsilon_T(t_1) + \alpha,$$

$$\delta.\upsilon_I(0) + \beta = (\upsilon_I)^{M\,T}_{\delta\,\beta}(0)$$

$$\le (\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1)$$

$$\delta.\upsilon_I(0) + \beta = \delta.\upsilon_I(t_1) + \beta,$$

$$\delta.\upsilon_F(0) - \gamma = (\upsilon_F)^{M\,T}_{\delta\,\gamma}(0)$$
$$\leq (\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1)$$
$$\delta.\upsilon_F(0) - \gamma = \delta.\upsilon_F(t_1) - \gamma,$$

which imply $\kappa_T(0) \geq \kappa_T(t_1)$,$\kappa_I(0) \geq \kappa_I(t_1)$,$\kappa_F(0) \geq \kappa_F(t_1)$ and $\upsilon_T(0) \leq \upsilon_T(t_1)$, $\upsilon_I(0) \leq \upsilon_I(t_1)$, $\upsilon_F(0) \leq \upsilon_F(t_1)$. Now, we have

$$\delta.\kappa_T(t_1) + \alpha = (\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1)$$
$$\geq \mathrm{rmin}\{(\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1 * t_2), (\kappa_T)^{M\,T}_{\delta\,\alpha}(t_2)\}$$
$$= \mathrm{rmin}\{\delta.\kappa_T(t_1 * t_2) + \alpha, \delta.\kappa_T(t_2) + \alpha\}$$
$$\delta.\kappa_T(t_1) + \alpha = \delta.\mathrm{rmin}\{\kappa_T(t_1 * t_2), \kappa_T(t_2)\} + \alpha,$$

$$\delta.\kappa_I(t_1) + \beta = (\kappa_I)^{M\,T}_{\delta\,\beta}(t_1)$$
$$\geq \mathrm{rmin}\{(\kappa_I)^{M\,T}_{\delta\,\beta}(t_1 * t_2), (\kappa_I)^{M\,T}_{\delta\,\beta}(t_2)\}$$
$$= \mathrm{rmin}\{\delta.\kappa_I(t_1 * t_2) + \beta, \delta.\kappa_I(t_2) + \beta\}$$
$$\delta.\kappa_I(t_1) + \beta = \delta.\mathrm{rmin}\{\kappa_I(t_1 * t_2), \kappa_I(t_2)\} + \beta,$$

$$\delta.\kappa_F(t_1) - \gamma = (\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1)$$
$$\geq \mathrm{rmin}\{(\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2), (\kappa_F)^{M\,T}_{\delta\,\gamma}(t_2)\}$$
$$= \mathrm{rmin}\{\delta.\kappa_F(t_1 * t_2) - \gamma, \delta.\kappa_F(t_2) - \gamma\}$$
$$\delta.\kappa_F(t_1) - \gamma = \delta.\mathrm{rmin}\{\kappa_F(t_1 * t_2), \kappa_F(t_2)\} - \gamma$$

and

$$\delta.\upsilon_T(t_1) + \alpha = (\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1)$$
$$\leq \max\{(\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1 * t_2), (\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_2)\}$$
$$= \max\{\delta.\upsilon_T(t_1 * t_2) + \alpha, \delta.\upsilon_T(t_2) + \alpha\}$$
$$\delta.\upsilon_T(t_1) + \alpha = \delta.\max\{\upsilon_T(t_1 * t_2), \upsilon_T(t_2)\} + \alpha,$$

$$\delta.\upsilon_I(t_1) + \beta = (\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1)$$
$$\leq \max\{(\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1 * t_2), (\upsilon_I)^{M\,T}_{\delta\,\beta}(t_2)\}$$
$$= \max\{\delta.\upsilon_I(t_1 * t_2) + \beta, \delta.\upsilon_I(t_2) + \beta\}$$
$$\delta.\upsilon_I(t_1) + \beta = \delta.\max\{\upsilon_I(t_1 * t_2), \upsilon_I(t_2)\} + \beta,$$

$$\delta.\upsilon_F(t_1) - \gamma = (\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1)$$
$$\leq \max\{(\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2), (\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_2)\}$$
$$= \max\{\delta.\upsilon_F(t_1 * t_2) - \gamma, \delta.\upsilon_F(t_2) - \gamma\}$$

$$\delta.\upsilon_F(t_1) - \gamma = \delta.\max\{\upsilon_F(t_1 * t_2), \upsilon_F(t_2)\} - \gamma$$

which imply $\kappa_T(t_1) \geq \text{rmin}\{\kappa_T(t_1 * t_2), \kappa_T(t_2)\}$, $\kappa_I(t_1) \geq \text{rmin}\{\kappa_I(t_1 * t_2), \kappa_I(t_2)\}$, $\kappa_F(t_1) \geq \text{rmin}\{\kappa_F(t_1 * t_2), \kappa_F(t_2)\}$ and $\upsilon_T(t_1) \leq \max\{\upsilon_T(t_1 * t_2), \upsilon_T(t_2)\}$,$\upsilon_I(t_1) \leq \max\{\upsilon_I(t_1 * t_2), \upsilon_I(t_2)\}$,$\upsilon_F(t_1) \leq \max\{\upsilon_F(t_1 * t_2), \upsilon_F(t_2)\}$ for all $t_1, t_2 \in Y$. Hence B is a **NCID** of Y.

Theorem 3.3.4 Intersection of any two **NCMT** of a **NCID** B of Y is a **NCID** of Y.

Proof. Suppose $B^{M\,T}_{\delta\,\alpha,\beta,\gamma}$ and $B^{M\,T}_{\delta'\,\alpha',\beta',\gamma'}$ are two **NCMTs** of **NCID** B ofY, where for $B^{M\,T}_{\alpha,\beta,\gamma}$, for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ٦]$, $\gamma \in [[0,0], ¥]$, for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$, $\gamma \in [0, £]$ and for $B^{T}_{\alpha',\beta',\gamma'}$, for $\kappa_{T,I,F}$ $\alpha', \beta' \in [[0,0], ٦]$, $\gamma' \in [[0,0], ¥]$, for $\upsilon_{T,I,F}$, $\alpha', \beta' \in [0, \Gamma]$, $\gamma' \in [0, £]$. Assume $\alpha \leq \alpha', \beta \leq \beta', \gamma \leq \gamma'$ and $\delta = \delta'$. Then by Theorem 3.3.3, $B^{M\,T}_{\delta\,\alpha,\beta,\gamma}$ and $B^{M\,T}_{\delta'\,\alpha',\beta',\gamma'}$ are **NCIDs** of Y. So

$$((\kappa_T)^{M\,T}_{\delta\,\alpha} \cap (\kappa_T)^{M\,T}_{\delta'\,\alpha'})(t_1) = \text{rmin}\{(\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1), (\kappa_T)^{M\,T}_{\delta'\,\alpha'}(t_1)\}$$
$$= \text{rmin}\{\delta.\kappa_T(t_1) + \alpha, \delta'.\kappa_T(t_1) + \alpha'\}$$
$$= \delta.\kappa_T(t_1) + \alpha$$
$$((\kappa_T)^{M\,T}_{\delta\,\alpha} \cap (\kappa_T)^{M\,T}_{\delta'\,\alpha'})(t_1) = (\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1),$$

$$((\kappa_I)^{M\,T}_{\delta\,\beta} \cap (\kappa_I)^{M\,T}_{\delta'\,\beta'})(t_1) = \text{rmin}\{(\kappa_I)^{M\,T}_{\delta\,\beta}(t_1), (\kappa_I)^{M\,T}_{\delta'\,\beta'}(t_1)\}$$
$$= \text{rmin}\{\delta.\kappa_I(t_1) + \beta, \delta'.\kappa_I(t_1) + \beta'\}$$
$$= \delta.\kappa_I(t_1) + \beta$$
$$((\kappa_I)^{M\,T}_{\delta\,\beta} \cap (\kappa_I)^{M\,T}_{\delta'\,\beta'})(t_1) = (\kappa_I)^{M\,T}_{\delta\,\beta}(t_1),$$

$$((\kappa_F)^{M\,T}_{\delta\,\gamma} \cap (\kappa_F)^{M\,T}_{\delta'\,\gamma'})(t_1) = \text{rmin}\{(\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1), (\kappa_F)^{M\,T}_{\delta'\,\gamma'}(t_1)\}$$
$$= \text{rmin}\{\delta.\kappa_F(t_1) - \gamma, \delta'.\kappa_F(t_1) - \gamma'\}$$
$$= \delta'.\kappa_F(t_1) - \gamma'$$
$$((\kappa_F)^{M\,T}_{\delta\,\gamma} \cap (\kappa_F)^{M\,T}_{\delta'\,\gamma'})(t_1) = (\kappa_F)^{M\,T}_{,\delta'\,\gamma'}(t_1)$$

and

$$((\upsilon_T)^{M\,T}_{\delta\,\alpha} \cap (\upsilon_T)^{M\,T}_{\delta'\,\alpha'})(t_1) = \max\{(\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1), (\upsilon_T)^{M\,T}_{\delta'\,\alpha'}(t_1)\}$$
$$= \max\{\delta.\upsilon_T(t_1) + \alpha, \delta'.\upsilon_T(t_1) + \alpha'\}$$
$$= \delta'.\upsilon_T(t_1) + \alpha'$$
$$((\upsilon_T)^{M\,T}_{\delta\,\alpha} \cap (\upsilon_T)^{M\,T}_{\delta'\,\alpha'})(t_1) = (\upsilon_T)^{M\,T}_{\delta'\alpha'}(t_1),$$

$$((\upsilon_I)^{M\,T}_{\delta\,\beta} \cap (\upsilon_I)^{M\,T}_{\delta'\,\beta'})(t_1) = \max\{(\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1), (\upsilon_I)^{M\,T}_{\delta'\,\beta'}(t_1)\}$$
$$= \max\{\delta.\upsilon_I(t_1) + \beta, \delta'.\upsilon_I(t_1) + \beta'\}$$
$$= \delta'.\upsilon_I(t_1) + \beta'$$
$$((\upsilon_I)^{M\,T}_{\delta\,\beta} \cap (\upsilon_I)^{M\,T}_{\delta'\beta'})(t_1) = (\upsilon_I)^{M\,T}_{\delta'\,\beta'}(t_1),$$

$$((\upsilon_F)^{M\,T}_{\delta\,\gamma} \cap (\upsilon_F)^{M\,T}_{\delta'\,\gamma'})(t_1) = \max\{(\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1), (\upsilon_F)^{M\,T}_{\delta'\,\gamma'}(t_1)\}$$
$$= \max\{\delta.\upsilon_F(t_1) - \gamma, \delta'.\upsilon_F(t_1) - \gamma'\}$$
$$= \delta.\upsilon_F(t_1) - \gamma$$
$$((\upsilon_F)^{M\,T}_{\delta\,\gamma} \cap (\upsilon_F)^{M\,T}_{\delta'\,\gamma'})(t_1) = (\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1).$$

Hence $B^{M\,T}_{\delta\,\alpha,\beta,\gamma} \cap B^{M\,T}_{\delta'\alpha',\beta',\gamma'}$ is **NCID** of Y.

Theorem 3.3.5 Union of any two **NCMT** $B^{M\,T}_{\delta\,\alpha,\beta,\gamma}$ of a **NCID** B of Y is a **NCID** of Y.

Proof. Suppose $B^{M\,T}_{\delta\,\alpha,\beta,\gamma}$ and $B^{M\,T}_{\delta'\alpha',\beta',\gamma'}$ are two NCMTs of NCID B of Y, where for $B^{M\,T}_{\alpha,\beta,\gamma}$, for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ℸ], \gamma \in [[0,0], ¥]$, for $\upsilon_{T,I,F}, \alpha, \beta \in [0, \Gamma], \gamma \in [0, £]$ and for $B^{T}_{\alpha',\beta',\gamma'}$, for $\kappa_{T,I,F}$ $\alpha', \beta' \in [[0,0], ℸ], \gamma' \in [[0,0], ¥]$, for $\upsilon_{T,I,F}, \alpha', \beta' \in [0, \Gamma]$, $\gamma' \in [0, £]$. Assume $\alpha \geq \alpha', \beta \geq \beta', \gamma \geq \gamma'$ and $\delta = \delta'$. Then by Theorem 3.3.3, $B^{M\,T}_{\delta\,\alpha,\beta,\gamma}$ and $B^{M\,T}_{\delta'\,\alpha',\beta',\gamma'}$ are NCIDs of Y. So

$$((\kappa_T)^{M\,T}_{\delta\,\alpha} \cup (\kappa_T)^{M\,T}_{\delta'\,\alpha'})(t_1) = \mathrm{rmax}\{(\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1), (\kappa_T)^{M\,T}_{\delta'\alpha'}(t_1)\}$$
$$= \mathrm{rmax}\{\delta.\kappa_T(t_1) + \alpha, \delta'.\kappa_T(t_1) + \alpha'\}$$
$$= \delta.\kappa_T(t_1) + \alpha$$
$$((\kappa_T)^{M\,T}_{\delta\,\alpha} \cup (\kappa_T)^{M\,T}_{\delta'\alpha'})(t_1) = (\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1),$$

$$((\kappa_I)^{M\,T}_{\delta\,\beta} \cup (\kappa_I)^{M\,T}_{\delta'\beta'})(t_1) = \mathrm{rmax}\{(\kappa_I)^{M\,T}_{\delta\,\beta}(t_1), (\kappa_I)^{M\,T}_{\delta'\beta'}(t_1)\}$$
$$= \mathrm{rmax}\{\delta.\kappa_I(t_1) + \beta, \delta'.\kappa_I(t_1) + \beta'\}$$
$$= \delta.\kappa_I(t_1) + \beta$$
$$((\kappa_I)^{M\,T}_{\delta\,\beta} \cup (\kappa_I)^{M\,T}_{\delta'\,\beta'})(t_1) = (\kappa_I)^{M\,T}_{\delta\,\beta}(t_1),$$

$$((\kappa_F)^{M\,T}_{\delta\,\gamma} \cup (\kappa_F)^{M\,T}_{\delta'\,\gamma'})(t_1) = \mathrm{rmax}\{(\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1), (\kappa_F)^{M\,T}_{\delta'\,\gamma'}(t_1)\}$$
$$= \mathrm{rmax}\{\delta.\kappa_F(t_1) - \gamma, \delta'.\kappa_F(t_1) - \gamma'\}$$
$$= \delta'.\kappa_F(t_1) - \gamma'\}$$
$$((\kappa_F)^{M\,T}_{\delta\,\gamma} \cup (\kappa_F)^{M\,T}_{\delta'\,\gamma'})(t_1) = (\kappa_F)^{M\,T}_{\delta'\,\gamma'}(t_1)$$

and

$$((\upsilon_T)^{M\,T}_{\delta\,\alpha} \cup (\upsilon_T)^{M\,T}_{\delta'\,\alpha'})(t_1) = \min\{(\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1), (\upsilon_T)^{M\,T}_{\delta'\,\alpha'}(t_1)\}$$
$$= \min\{\delta.\upsilon_T(t_1) + \alpha, \delta'.\upsilon_T(t_1) + \alpha'\}$$
$$= \delta'.\upsilon_T(t_1) + \alpha'$$
$$((\upsilon_T)^{M\,T}_{\delta\,\alpha} \cup (\upsilon_T)^{M\,T}_{\delta'\,\alpha'})(t_1) = (\upsilon_T)^{M\,T}_{\delta'\alpha'}(t_1),$$

$$((\upsilon_I)^{M\,T}_{\delta\,\beta} \cup (\upsilon_I)^{M\,T}_{\delta'\,\beta'})(t_1) = \min\{(\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1), (\upsilon_I)^{M\,T}_{\delta'\,\beta'}(t_1)\}$$
$$= \min\{\delta.\upsilon_I(t_1) + \beta, \delta'.\upsilon_I(t_1) + \beta'\}$$
$$= \delta'.\upsilon_I(t_1) + \beta'$$

$$((\upsilon_I)^{M\,T}_{\delta\,\beta} \cup (\upsilon_I)^{M\,T}_{\delta'\,\beta'})(t_1) = (\upsilon_I)^{M\,T}_{\delta'\,\beta'}(t_1),$$

$$((\upsilon_F)^{M\,T}_{\delta\,\gamma} \cup (\upsilon_F)^{M\,T}_{\delta'\,\gamma'})(t_1) = \min\{(\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1), (\upsilon_F)^{M\,T}_{\delta'\,\gamma'}(t_1)\}$$
$$= \min\{\delta.\upsilon_F(t_1) - \gamma, \delta'.\upsilon_F(t_1) - \gamma'\}$$
$$= \delta.\upsilon_F(t_1) - \gamma$$
$$((\upsilon_F)^{M\,T}_{\delta\,\gamma} \cup (\upsilon_F)^{M\,T}_{\delta'\,\gamma'})(t_1) = (\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1).$$

Hence $B^{M\,T}_{\delta\,\alpha,\beta,\gamma} \cup B^{M\,T}_{\delta'\,\alpha',\beta',\gamma'}$ is **NCID** of Y.

4. Conclusion

In this paper, we defined neutrosophic cubic translation,, neutrosophic cubic multiplication and neutrosophic cubic magnified translation for neutrosophic cubic set on BF-algebra. We provided the new sort of different conditions for neutrosophic cubic translation, neutrosophic cubic multiplication and neutrosophic cubic magnified translation and proved with examples. Moreover, for better understanding we investigated many results for NCT, NCM and NCMT using the subalgebra and ideals. For future work, translation and multiplication can be applied on neutrosophic cubic soft set and T-neutrosophic cubic set.

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Received: Apr 13, 2020. Accepted: July 4 2020

Neutrosophic Sets and Systems, Vol. 35, 2020
University of New Mexico

Neutrosophic Generalized Homeomorphism

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Abstract: The idea of neutrosophic generalized homeomorphism is presented in neutrosophic topological spaces. In addition to this, neutrosophic generalized closed and open mappings are also presented. At the same time, their characterizations are discussed by establishing their related attributes.
Keywords: GN-closed set**,** GN-closed map, GN-open map, Neutrosophic g-homeomorphism, Neutrosophic g*-homeomorphism.

1. Introduction

Neutrosophic sets were initially established as a generality of intuitionistic fuzzy sets [10] by Smarandache [18] such that the membership, the non-membership, and the indeterminacy degrees are considered. In analogy with more unsure philosophy, the neutrosophic set discharge agreement with an indeterminacy condition. The neutrosophic conception has a broad scope of real-time requests in the fields of [1-9] Artificial Intelligence, Computer Science, Information Systems, Decision Making, Uncertainty assessments of linear time-cost tradeoffs, Applied Mathematics, and solving the supply chain problem. Salama et al. [15, 16] adapted the notion of the neutrosophic set to operate in neutrosophic topological spaces (NTSs in short) and pioneered generalized neutrosophic set and topological spaces. In [11], generalized neutrosophic closed set (in short, GNCS) is defined and using this generalized neutrosophic continuous (GN-continuous), and generalized neutrosophic irresolute (in short, GN-irresolute) functions are defined. Recently in [12, 13], the perception of generalized α-contra continuous and neutrosophic almost α-contra-continuous functions are introduced. Parimala M et al. [14] introduced and studied the thought of Neutrosophic homeomorphism and Neutrosophic $\alpha\psi$ homeomorphism in Neutrosophic topological spaces. This paper aspires to overly enunciate the thought of neutrosophic generalized homeomorphism (in short, neutrosophic g-homeomorphism) in NTSs by utilizing GN-continuous function and study some of their properties. We have also provided the idea of generalized neutrosophic closed and open mappings by establishing some of their characterizations. Besides, neutrosophic g*-homeomorphism is also presented and establish its relation with the neutrosophic g-homeomorphism.

2. Preliminaries

Definition 2.1 [15]: A neutrosophic topology (in short,N-topology) on $X \neq \emptyset$ is a family ξ of N-sets in X satisfying the laws given below:
(i) $0_N, 1_N \in \xi$,

(ii) $W_1 \cap W_2 \in \xi$ being $W_1, W_2 \in \xi$,
(iii) $\cup W_i \in \xi$ for arbitrary family $\{W_i | i \in \Lambda\} \subseteq \xi$.

In this situation the ordered pair (X, ξ) or simply X is termed as NTS and each NS in ξ is named as neutrosophic open set (in short, NOS). The complement $\overline{\Lambda}$ of an N-open set Λ in X is known as neutrosophic closed set (briefly, NCS) in X.

Definition 2.2 [15]: Let Λ be an NS in an NTS (X, ξ). Thereupon

(i) $Nint(\Lambda) = \cup\{G | G$ is a NOS in X and $G \subseteq \Lambda\}$ is termed as neutrosophic interior (in brief $Nint$) of Λ;
(ii) $Ncl(\Lambda) = \cap\{G | G$ is an NCS in X and $G \supseteq \Lambda\}$ is termed as neutrosophic closure (shortly Ncl) of Λ.

Definition 2.3 [11]: Allow (X, ξ) be a NTS. A NS Λ in (X, ξ) is termed as generalized neutrosophic closed set (in short GNCS) if $Ncl(\Lambda) \subseteq \Gamma$ whenever $\Lambda \subseteq \Gamma$ and Γ is a NOS. The complement of a GNCS is generalized neutrosophic open set (in short GNOS).

Definition 2.4 [11]: Let (X, ξ) be NTS and B be a NS in X. Then neutrosophic generalized closure is defined as, $GNcl(B) = \cap\{G: G$ is a GNCS in X and $B \subseteq G\}$.

Definition 2.5 [11, 17]: A map $\eta: X \to Y$ is said to be

(i) neutrosophic closed (in short, NC-map) if the image of every NCS in X is a NCS in Y.
(ii) neutrosophic continuous (in short, N-continuous) if inverse image of every NCS in Y is a NCS in X.
(iii) generalized neutrosophic continuous (in short, GN-continuous) if inverse image of every NCS in Y is a GNCS in X.
(iv) generalized neutrosophic irresolute (in short, GN-irresolute) if inverse image of every GNCS in Y is a GNCS in X.

Definition 2.6 [14]: A bijection $g: X \to Y$ is called a neutrosophic homeomorphism if g and g^{-1} are neutrosophic continuous.

3. Neutrosophic Generalized Homeomorphism

Definition 3.1: A bijection $\eta: X \to Y$ is named as neutrosophic generalized homeomorphism (in short neutrosophic g-homeomorphism) if η and η^{-1} are GN-continuous.

Proposition 3.2: Every neutrosophic homeomorphism is a neutrosophic g-homeomorphism.

Proof: Consider a bijection mapping $\eta: X \to Y$ be a neutrosophic homeomorphism, in which η as well as η^{-1} are N-continuous. We have each N-continuous mapping is GN-continuous, so η and η^{-1} are GN-continuous. Hence, η is neutrosophic g-homeomorphism.

Remark 3.3: The next illustration makes clear that the opposite of the above proposition is not valid.

Example 3.4: Let $X = \{p, q, r\}$, $\xi = \{0_N, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, 1_N\}$ be a N-topology on X.
$\mathcal{A}_1 = \langle x, (0.2,0.1,0.1), (0.2,0.1,0.1), (0.3,0.5,0.5) \rangle$, $\mathcal{A}_2 = \langle x, (0.1,0.2,0.2), (0.4,0.3,0.3), (0.3,0.3,0.3) \rangle$,
$\mathcal{A}_3 = \langle x, (0.2,0.2,0.2), (0.2,0.1,0.1), (0.3,0.3,0.3) \rangle$, $\mathcal{A}_4 = \langle x, (0.1,0.1,0.1), (0.4,0.3,0.3), (0.3,0.5,0.5) \rangle$,
and let $Y = \{p, q, r\}$, $\sigma = \{0_N, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, 1_N\}$ be a neutrosophic topology on Y.
$\mathcal{B}_1 = \langle y, (0.3,0.3,0.3), (0.2,0.1,0.1), (0.2,0.2,0.2) \rangle$, $\mathcal{B}_2 = \langle y, (0.2,0.2,0.2), (0.1,0.1,0.1), (0.3,0.3,0.3) \rangle$,
$\mathcal{B}_3 = \langle y, (0.3,0.3,0.3), (0.1,0.1,0.1), (0.2,0.1,0.1) \rangle$, $\mathcal{B}_4 = \langle y, (0.2,0.2,0.2), (0.2,0.1,0.1), (0.3,0.3,0.3) \rangle$.
Define $\eta: (X, \xi) \to (Y, \sigma)$ by $\eta(p) = p, \eta(q) = q$ and $\eta(r) = r$. Then η is neutrosophic g-homeomorphism but not neutrosophic homeomorphism.

Definition 3.5: A mapping $\eta: X \to Y$ is generalized neutrosophic closed (in short, GNC-map) if the image $\eta(Q)$ is GNCS in Y for every NCS Q in X.

Definition 3.6: A mapping $\eta: X \to Y$ is generalized neutrosophic open (in short, GNO-map) if the image $\eta(R)$ is GNOS in Y for every NOS R in X.

Proposition 3.7: Every NC-mapping is a GNC-mapping.
Proof: Consider $\eta: X \to Y$ is a NC-mapping, so as Q is an NCS in X. As η is NC- mapping, $\eta(Q)$ is NCS in Y. Since each NCS is GNCS. Therefore, $\eta(Q)$ is a GNCS in Y. Hence, η is GNC-mapping.

Remark 3.8: The opposite of the above proposition is not valid as indicated.

Example 3.9: Let $X = \{p, q, r\}$, $\xi = \{0_N, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, 1_N\}$ be a N-topology on X.
$\mathcal{A}_1 = \langle x, (0.2,0.1,0.1), (0.2,0.1,0.1), (0.3,0.5,0.5) \rangle$, $\mathcal{A}_2 = \langle x, (0.1,0.2,0.2), (0.4,0.3,0.3), (0.3,0.3,0.3) \rangle$,
$\mathcal{A}_3 = \langle x, (0.2,0.2,0.2), (0.2,0.1,0.1), (0.3,0.3,0.3) \rangle$, $\mathcal{A}_4 = \langle x, (0.1,0.1,0.1), (0.4,0.3,0.3), (0.3,0.5,0.5) \rangle$,
and let $Y = \{p, q, r\}$, $\sigma = \{0_N, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, 1_N\}$ be a neutrosophic topology on Y.
$\mathcal{B}_1 = \langle y, (0.3,0.3,0.3), (0.2,0.1,0.1), (0.2,0.2,0.2) \rangle$, $\mathcal{B}_2 = \langle y, (0.2,0.2,0.2), (0.1,0.1,0.1), (0.3,0.3,0.3) \rangle$,
$\mathcal{B}_3 = \langle y, (0.3,0.3,0.3), (0.1,0.1,0.1), (0.2,0.1,0.1) \rangle$, $\mathcal{B}_4 = \langle y, (0.2,0.2,0.2), (0.2,0.1,0.1), (0.3,0.3,0.3) \rangle$.
Define $\eta: (X, \xi) \to (Y, \sigma)$ by $\eta(p) = p, \eta(q) = q$ and $\eta(r) = r$. Then η is GNC-mapping but not NC-mapping.

Proposition 3.10: A map $\eta: X \to Y$ is a GNC-mapping if the image of each NOS in X is GNOS in Y.
Proof: Let R be a NOS in X. Hence $\overline{R}$ is a NCS in X. As η is GNC-mapping, $\eta(\overline{R})$ is a GNCS in Y. Since $\eta(\overline{R}) = \overline{(\eta(R))}$, $\eta(R)$ is a GNOS in Y.

Proposition 3.11: Let $\eta: X \to Y$ be a bijective mapping, then the next assertions are same:
(i) η is GNO-mapping.
(ii) η is GNC-mapping.
(iii) η^{-1} is GN-continuous.

Proof: $(i) \to (ii)$. Suppose that η is GNO-mapping. Then, P is a NOS in X, then image $\eta(P)$ is GNOS in Y. Here, P is NCS in X, then $X - P$ is a NOS in X. By prediction, $\eta(X - P)$ is a GNOS in Y. Hence, $Y - \eta(X - P)$ is a GNCS in Y. Hence, η is a GNC-mapping.
$(ii) \to (iii)$. Let R be an NCS in X. By (ii), $\eta(R)$ is GNCS in Y. Therefore, $\eta(R) = (\eta^{-1})^{-1}(R)$, so η^{-1} is a GNCS in Y. Hence, η^{-1} is a GN-continuous.
$(iii) \to (i)$. Let Q be a NOS in X. By (iii), $(\eta^{-1})^{-1}(Q) = \eta(Q)$ is GNO-mapping.

Proposition 3.12: Let $\eta: (X, \xi) \to (Y, \sigma)$ be a bijective mapping. If η is GN-continuous, thereupon the declarations are identical:
(i) η is GNC-mapping.
(ii) η is GNO-mapping.
(iii) η^{-1} is neutrosophic g-homeomorphism.
Proof: $(i) \to (ii)$. Presume that η is bijective as well as a GNC-mapping. So, η^{-1} is a GN-continuous mapping. As we have every NOS is GNOS in Y. Hence, η is GNO-mapping.
$(ii) \to (iii)$. Consider a bijective NO-mapping η. Furthermore, η^{-1} is a GN-continuous mapping. Accordingly, η and η^{-1} are GN-continuous. Hence, η is neutrosophic g-homeomorphism.
$(iii) \to (i)$. Let η be neutrosophic g-homeomorphism, then η and η^{-1} are GN-continuous. As each NCS in X is a GNCS in Y, therefore η is a GNC-mapping.

Definition 3.13 [19]: Let (X, ξ) be an NTS said to be a as neutrosophic-$T_{1/2}$ (in short N-$T_{1/2}$) space if every GNCS is NCS in X.

Proposition 3.14: Let $\eta: (X, \xi) \to (Y, \sigma)$ be neutrosophic g-homeomorphism, then η is neutrosophic homoemorphism if X and Y are N-$T_{1/2}$ space.
Proof: Consider that D is an NCS in Y, then $\eta^{-1}(D)$ is a GNCS in X due to the assumption. Since X is N-$T_{1/2}$ space, $\eta^{-1}(D)$ is NCS in X. Then, η is GN-continuous. By hypothesis η^{-1} is GN-continuous. Let H be a NCS in X. $(\eta^{-1})^{-1}(H) = \eta(H)$ is a NCS in Y, by preassumption. As Y is N-$T_{1/2}$ space, $\eta(H)$ is a NCS in Y. Hence, η^{-1} is N-continuous. Therefore, η is a neutrosophic homeomorphism.

Proposition 3.15: Let $\eta: X \to Y$ and $\mu: Y \to Z$ be GNC-mappings where X and Z are NTSs and Y is N-$T_{1/2}$ space, then $(\mu \, o \, \eta)$ is GNC-mapping.
Proof: Let R be a NCS in X. As η is GNC-map and $\eta(R)$ is a GNCS in Y, by assumption, $\eta(R)$ is a NCS in Y. Since μ is GNC-map, then $\mu(\eta(R))$ is a GNCS in X and Z and $\mu(\eta(R)) = (\mu \, o \, \eta)(R)$. Therefore, $(\mu \, o \, \eta)$ is GNC-map.

Proposition 3.16: Let $\mu: X \to Y$ and $\lambda: Y \to Z$ be NTSs, then the following hold:
(i) If $(\lambda \, o \, \mu)$ is GNO-map and μ is N-continuous, then λ is GNO-map.
(ii) If $(\lambda \, o \, \mu)$ is GNO-map and μ is GN-continuous, then λ is GNO-map.

Proof: (i) Let K be NOS in Y. Then, $\mu^{-1}(K)$ is a NOS in X. Since $(\lambda\, o\, \mu)$ GNO-map and $(\lambda\, o\, \mu)\mu^{-1}(K) = \lambda\left(\mu\left(\mu^{-1}(K)\right)\right) = \lambda(K)$ is GN-open in Z, hence λ is GN-open map.

(ii) Let K be NOS in X. Then, $\lambda\,(\,\mu(K)\,)$ is a NOS in Z. Hence, $\lambda^{-1}(\lambda\,(\mu(K)) = \mu(K)$ is GNOS in Y. Therefore μ is GNO- map.

4. Neutrosophic g*-Homeomorphism

Definition 4.1: A bijection $\mu: X \to Y$ is called neutrosophic g*-homeomorphism if μ and μ^{-1} are GN-irresolute mappings.

Proposition 4.2: Every neutrosophic g*-homeomorphism is a neutrosophic g-homeomorphism.

Proof: A map μ is a neutrosophic g*-homeomorphism. Predict that K is a NCS in Y. So it is a GNCS in Y. By pressumption, $\mu^{-1}(K)$ is a GNCS in X. Accordingly, μ is GN-continuous mapping. Therefore, μ and μ^{-1} are GN-continuous mappings. Henec, μ is a neutrosophic g-homeomorphism.

Remark 4.3: The example is given to show that the reverese of the above proposition is not possible.

Example 4.4: Let $X = \{p, q, r\}$, $\xi = \{0_N, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, 1_N\}$ be a N-topology on X.
$\mathcal{A}_1 = \langle x, (0.2,0.1,0.1), (0.2,0.1,0.1), (0.3,0.5,0.5)\rangle$, $\mathcal{A}_2 = \langle x, (0.1,0.2,0.2), (0.4,0.3,0.3), (0.3,0.3,0.3)\rangle$,
$\mathcal{A}_3 = \langle x, (0.2,0.2,0.2), (0.2,0.1,0.1), (0.3,0.3,0.3)\rangle$, $\mathcal{A}_4 = \langle x, (0.1,0.1,0.1), (0.4,0.3,0.3), (0.3,0.5,0.5)\rangle$,
and let $Y = \{p, q, r\}$, $\sigma = \{0_N, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, 1_N\}$ be a neutrosophic topology on Y.
$\mathcal{B}_1 = \langle y, (0.3,0.3,0.3), (0.2,0.1,0.1), (0.2,0.2,0.2)\rangle$, $\mathcal{B}_2 = \langle y, (0.2,0.2,0.2), (0.1,0.1,0.1), (0.3,0.3,0.3)\rangle$,
$\mathcal{B}_3 = \langle y, (0.3,0.3,0.3), (0.1,0.1,0.1), (0.2,0.1,0.1)\rangle$, $\mathcal{B}_4 = \langle y, (0.2,0.2,0.2), (0.2,0.1,0.1), (0.3,0.3,0.3)\rangle$.
Define $\eta: (X, \xi) \to (Y, \sigma)$ by $\eta(p) = p, \eta(q) = q$ and $\eta(r) = r$. Then η is neutrosophic g-homeomorphism but not neutrosophic g*-homeomorphism.

Proposition 4.5: If $\mu: X \to Y$ and $\lambda: Y \to Z$ are neutrosophic g*-homeomorphisms, then $(\lambda\, o\, \mu)$ is a neutrosophic g*-homeomorphism.

Proof: Consider μ and λ as neutrosophic g*-homeomorphisms. Predict K is a GNCS in Z. Thereupon, by the presumption, $\lambda^{-1}(K)$ is a GNCS in Y. Hence, by hypothesis, $\mu^{-1}(\lambda^{-1}(K))$ is a GNCS in X. Hence, $(\lambda\, o\, \mu)$ is a GN-irresolute mapping. Now, consider H be a GNCS in X. Then, by the presumption, $\mu(H)$ is a GNCS in Y. So, by hypothesis, $\lambda(\,\mu(H))$ is a GNCS in Z. This implies that $(\lambda\, o\, \mu)$ is a GN-irresolute mapping. Therefore, $(\lambda\, o\, \mu)$ is neutrosophic g*-homeomorphism.

Proposition 4.6: If $\mu: X \to Y$ is a neutrosophic g*-homeomorophism, then $NGcl\left(\mu^{-1}(K)\right) = \mu^{-1}(NGcl(K))$ for each NS K in Y.

Proof: As μ is neutrosophic g*-homeomorphism, then μ is GN-irresolute mapping. Let K be a NS in Y. Clearly, $NGcl(K)$ is GNCS in X. This proves that $GNcl(K)$ is GNCS in X. Since $\mu^{-1}(K) \subseteq$

$\mu^{-1}(GNcl(K))$, then $GNcl(\mu^{-1}(K)) \subseteq GNcl\left(\mu^{-1}(GNcl(K))\right) = \mu^{-1}(GNcl(K))$. Therefore, $GNcl(\mu^{-1}(K)) \subseteq \mu^{-1}(GNcl(K))$.

Let μ be neutrosophic g*-homeomorphism. μ^{-1} is a GN-irresolute mapping. Consider NS $\mu^{-1}(K)$ in X, which implies that $GNcl(\mu^{-1}(K))$ is GNCS in X. Therefore, $GNcl(\mu^{-1}(K))$ is a GNCS in X. This implies that $(\mu^{-1})^{-1}(GNcl(\mu^{-1}(K)) = \mu(GNcl(\mu^{-1}(K)))$ is a GNCS in Y. This proves that $K = (\mu^{-1})^{-1}(\mu^{-1}(K)) \subseteq (\mu^{-1})^{-1}\left(GNcl(\mu^{-1}(K))\right) = \mu(GNcl(\mu^{-1}(K)))$, since μ^{-1} is GN-irresolute mapping. Hence, $\mu^{-1}(GNcl(K)) \subseteq \mu^{-1}\left(\mu\left(GNcl(\mu^{-1}(K))\right)\right) = GNcl(\mu^{-1}(K))$.

That is, $\mu^{-1}(GNcl(K)) \subseteq GNcl(\mu^{-1}(K))$. Hence, $GNcl(\mu^{-1}(K)) = \mu^{-1}(GNcl(K))$.

5. Conclusions

We have introduced neutrosophic generalized homeomorphism in neutrosophic topological space using GN-contiuous functions. Some characterizations have been provided to illustrate how far topological structures are conserved by the new neutrosophic notion defined. Furthermore, neutrosophic g*-homeomorphism, neutrosophic generalized open and closed mappings are also studied. The study demonstrated neutrosophic g*-homeomorphisms and also proved some of their related attributes. Also, the relation between generalized neutrosophic closed mappings and other existed Neutrosophic closed mappings in Neutrosophic topological spaces were established and derived some of their related attributes. Examples are given wherever necessary.

In future, we can carry out the further rsearch on neutrosophic g-compactness, neutrosophic g-connectedness and neutrosophic almost g-contra continuous functions.

Funding: This research received no external funding.

Acknowledgments: The authors are highly grateful to the Referees for their constructive suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

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Received: Apr 15, 2020. Accepted: July 3 2020

Neutrosophic Sets and Systems, Vol. 35, 2020
University of New Mexico

Triangular Neutrosophic Based Production Reliability Model of Deteriorating Item with Ramp Type Demand under Shortages and Time Discounting

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Abstract: An economic production quantity model with triangular neutrosophic environment has been developed for deteriorating items with ramp type demand rate and reliability dependent unit production. The main objective of this paper is to determine the most cost effective production to generate better quality items under time discounting. Additionally, it is considered that the deterioration function deals with three parameters Weibull's distribution under finite time horizon. Moreover, it also considered the effect of shortages which are partially backordered and partially lost in sale. Here the reliability of the production process along with the production period is considered as decision variables. A numerical example is studied in both crisp and neutrosophic environment and a comparative analysis is performed here. It is observed that the model performs better in triangular neutrosophic arena rather than crisp domain. Finally, a sensitivity analysis of optimal solution is observed for some parameters and some crucial decision is taken with managerial insight.

Keywords: Ramp-type demand, Finite time horizon, Time-value of money, Reliability, Triangular Neutrosophic number.

1. Introduction

In market economy system, for a single product, many items are produced by the different manufacturing companies. The manufacturers are trying to give wide variety of option to the customer to gain competitive advantages over their competitors. But customers choose those items which have high reliability i.e. better in quality, and lower in cost. The companies require advanced planning many years prior to the sale target date in order to minimize the total cost and maximize the profit. Thus the facts like variation in the reliability of the production process, demand rate of an item, deterioration and shortages are in growing interest. In case of classical EPQ model the basic assumptions are that the production set-up cost is fixed and the item produced are of perfect quality. All the manufacturing sectors want to produce perfect quality item, but in reality the product quality are not always perfect because there may be machine breakdown, labor problem, etc. The product quality is directly affected by the reliability of production process. In addition to that, the classical models also consider an ideal case that the demand and quality of the items remains unaffected by time and replenishment is done instantaneously. However in reality these assumptions do not hold. The inventories are often replenished periodically at certain production rate. Even if the items are purchased it takes days to sell the item so the items remained stored and hence the item deteriorates and their value reduces with time. Cheng [1] proposed a general equation for relationship between production set up cost and process reliability and flexibility. Later it was used by (Leung [2]; Bag et al. [3]) in their respective models studied on fuzzy random demand with flexibility and reliability on production process. Sarkar [4] analyzed an EMQ model with reliability in an imperfect production process. Many researchers (like Gomez et al. [5]; Cai et al. [6]) worked for production quality, tracking production control, etc. Pan and Li [7] worked with stochastic production system for deteriorating item with some environmental constrains. Rathore [8] explored a production reliability model with advertisement related demand. The paper considers reliability in unit production cost in order to identify the product quality with minimum total cost.

Traditionally in inventory models, the researchers have assumed constant demand pattern in their deterministic models, but in reality demand has specific patterns which depicts the real scenarios in market. There are various types of demand rates such as linear or quadratic function of time, exponentially increasing or decreasing, price and stock dependent, etc. If the demand is linearly dependent on time i.e., demand as well as the vending increases and decreases in growth and decline phase respectively. Researchers have manifested these demands in their respective papers (Hariga [9], Bose et al. [10], etc). Demand of the item depending on price and stocking amount of the items with optimal replenishment policy for non-instantaneous deteriorating items with partial backlogging was discussed by Wu et al. [11]. Alfares [12] worked on stock dependent demand. Chung and Wee [13] organized an inventory model for stock dependent selling rate with deterioration under replenishment plan. Pal et al. [14] has developed a inventory model with price and stock depended demand rate for deteriorating item under inflation and delay in payment. In this field, some remarkable researches were done by Yang et al. [15]. It was observed that for seasonal and fashionable products the nature of demand is increasing-steady-decreasing. But for newly launched fashion goods and cosmetics, garments, etc. the demand rate increases linearly with time and then it become constant. Thus to understand the concept of such a demand, the ramp type function of time was introduced. (Skouri et al. [16], Luo [17], Manna and Chaudhari [18]) worked with ramp type demand rate with time dependent deterioration. Pal et al. [19] considered the EOQ model with ramp type demand under finite time horizon.

As the effect of deterioration cannot be ignored so many researchers worked on it (Skouri et al. [20], Jaggi et al. [21], etc.). Generally, deterioration means spoilage or damage obsolescence, etc. which cannot be used further for its original purpose. Medicine, blood banks, etc. are difficult to preserve and they have some expiry date i.e., products maximum life time is time bounded. Electronic products become obsolete as technology changes; new fashion depreciates the clothing value over time; all these are also considered as deterioration. It has been observed that the delinquency in the life expectancy drugs, deterioration of roasted ground coffee, corn seeds, frozen food, pasteurized milk, refrigerated meat, ice creams, and leakage failure of the batteries can be expressed in terms of Weibull's distribution. Wu [22] presented an inventory model with ramp type demand and Weibull's distribution deterioration under partial backlogging. Many researcher such as Skouri et al. [23], Sharma and Chaudhury [24], etc. worked with this type of deterioration. Mandal [25] discussed an inventory model with Weibull's distributed deterioration with ramp type demand rate. A common characteristic in most of these models are that they does not allows shortages. Widyadana et al. [26] developed an EOQ model for deteriorating items with planned backorder level. Wee et al. al. [27] worked with shortages and finite time horizon for deteriorating items. Yang [28] developed an inventory model with deterioration as three parameter Weibull's distribution in two ware house system. Recently Pal and Chakraborty [29] have worked on non- instantaneous deteriorating items under shortage, Rahaman et al. [30] worked on arbitrary ordered generalized EPQ model with and without deterioration. In this paper shortages is also considered where the part of the unsatisfied demand are backordered and part of the sales are lost.

As the amount of the money available at the present time is worth more than that of the same amount in the future due to its potential earning capacity. So it is necessary to consider the effect of time value of money in today's inventory where forecasting is required. To consider the effect of time value of money, a finite time horizon for planning the replenishment cycle is considered. From the last few decades we have observed that the economic situation of most countries has changes so it would be unrealistic to ignore the effect of time value of money. Hariga [31] developed the effect of inflation and time value of money for time dependent demand. Hou [32] considered a model for deteriorating items and stock-dependent demand rate with shortages and time discounting. Dash et al. [33] worked on EPQ model for declined quadratic demand with time value of money and shortages. Thus the paper considers time value of money specially when investment and forecasting are considered.

In this current century, vagueness theory plays a crucial role in different Öeld of mathematical modeling and engineering problems. The theory of impreciseness was first invented by Zadeh [34]. Difference between crisp set and fuzzy set is shown briefly in this article by considering membership gradation and its formulation. Demonstration of triangular [35], trapezoidal [36], pentagonal [37] fuzzy number has already been developed by the researchers. In 1983 and later in 1986 Attasonov [38, 39] manifested a remarkable idea of intuitionistic fuzzy set where membership and non-membership functions are both considered together. Further, triangular intuitionistic [40, 41], trapezoidal intuitionistic [42] number has been introduced in this intuitionistic fuzzy research arena. After that, in 1998 Smarandache [43] established an amazing concept of neutrosophic fuzzy set where three disjunctive kinds of membership functions has been considered namely i) truthness ii) falseness iii) indeterminacy. Due to the presence of hesitation factor in fuzzy arena, neutrosophic number becomes more logical and scientific significance in research work. In this current era, researchers from different arena are focusing on neutrosophic concept and developed lots of interesting articles in this domain. Illustration of triangular, trapezoidal neutrosophic number has been introduced day by day and recently in 2018 Chakraborty et.al [44, 45] classifies different form of triangular and trapezoidal neutrosophic number and de-neutrosophication technique for crispification. Further, bipolarization of triangular bipolar number has been developed by

Chakraborty et.al [46] and also Maity et.al [47] manifested the concept of heptagonal dense fuzzy number related EOQ based model in 2018. Recently, Mullai [48] introduced EOQ model in neutrosophic domain and Mondal et.al [49] manifested optimization of EOQ Model with limited storage capacity by neutrosophic Geometric Programming application. Also, Majumdar et.al [50] focused on EPQ Model of deteriorating Items under partial trade credit financing and demand declining market in neutrosophic environment. Some useful articles [51-58] are also developed by the researchers in the neutrosophic arena recently. As developments goes on, some researchers [59-62] have extended the idea of neutrosophic set into plithogenic set and applied it in MCDM, MADM and optimization technique supply chain based model. Currently, several researchers from distinct fields focused on triangular neutrosophic number related to operation research models. As uncertainty prevails in various parameters such as inflation, holding cost, purchase cost so we have developed an EPQ under ramp type demand and considered the hesitation in those parameters by considering those parameter as neutrosophic number. Finally we compare the model in crisp and neutrosophic domain and observe that the model works better in neutrosophic arena.

Previously the researchers have worked on ramp type demand with two parameter Weibull's distribution as deterioration. But in this paper we have considered ramp type demand with three parameter Weibull's distribution. In addition the model assumes that the product qualities are never perfect and it is the function of reliability of the production process so the production of items depend on the reliability of the items i.e., if the items are highly reliable then there is more demand in the market and hence its production should be more in order to fulfill the demand. In this model we also have considered finite planning horizon to observe the effect of time value of money under shortage. The shortage items are partially backlogged or partially lost in sales, which cannot be ignored. Also under this complicated scenario no work has been done by considering holding cost, purchase cost and inflation as triangular neutrosophic number.

The rest of the paper is organized as follows: In section 2 we have presented some assumptions and notations and some definition of neutrosophic number that we have used in this paper. In this section we have defined few terminologies related to triangular neutrosophic number and also have formulated the model. In section 3 we have analyzed and optimized of the model. In Section 4 we have discussed the de-neutrosophication of the triangular neutrosophic number. In section 5 we present the numerical example and its mathematical analysis which is shown graphically. It is observed that the model works better in neutrosophic domain. In section 6 we present sensitivity analysis of some parameters. Finally in section 7 a concluding remark is stated along with its future extension.

2. Mathematical formulation of the inventory model

In this model we have considered ramp type demand with deterioration as three parameter Weibull distributions, shortages, lost in sales under the influence of time discounting in finite planning horizon. The finite time horizon has been considered to evaluate the effect of inflation on the total cost for a finite period. The paper also considered reliability in production of items. The proposed model is graphically shown in figure-1.

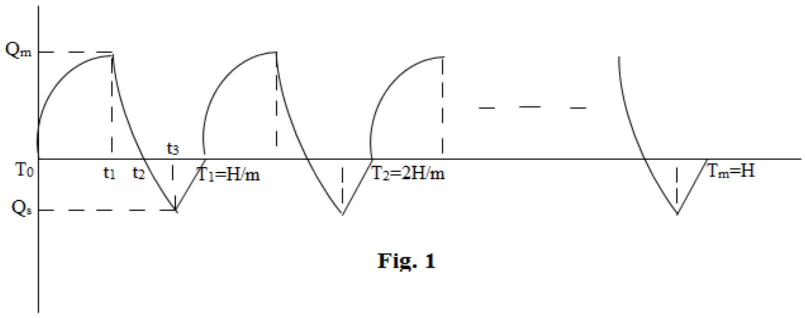

Fig. 1

The production process starts from t=0 and ends $t=t_1$. The production has occurred along with the demand in the market and at $t=t_1$ the inventory level is maximum, Q_m. From $t=t_1$ to $t=t_2$ the inventory level decreases and at time $t=t_2$, the inventory level reaches zero. Now during $[t_2,t_3]$ the model undergoes shortage with partial backlog and partial lost in sales. Only the backlogged items are replaced by the next replenishment. During $[t_3,T_1]$ production resumes to overcome the shortage (i.e., for backlogged items). Thus the total number of backlogged items is replaced in the next replenishment and the cycle repeats.

Notations

The notations used in this paper are as follows:

G Demand rate,

P Production rate,

p Unit production cost,

$\varrho(t)$ Time distribution for deterioration of the item,

k Discount rate,

h Inventory carrying cost per unit item per unit time,

d Deterioration cost per unit per unit time,

S Set-up cost for one replenishment cycle.

c_1 Purchase cost per unit item,

c_2 Shortage cost,

c_3 Penalty cost of a lost sale including loss of profit,

r Production process reliability (a decision variable)

B Fraction of backorder ($0<B\leq1$),

T Replenishment cycle,

H Finite Planning horizon,

m No. of replenishment during the planning horizon i.e., m=(H/T),

T_j Time between start and end of j^{th} replenishment cycle i.e., $T_0=0, T_1=T, T_2=2T, \ldots, T_m=mT=H$,

Q_m Maximum quantity of inventory,

Q_s Maximum quantity of inventory after shortage.

Assumptions

The assumptions which are considered in this model are as follows.

1. A ramp type demand rate G=f(t) is a function of time $f(t) = R[t - (t-\mu)H(t-\mu)], R > 0$ and $H(t)$ is a Heaviside function $H(t-\mu) = \begin{cases} 1 & if\ t \geq \mu \\ 0 & if\ t < \mu \end{cases}$

2. A function of three parameter Weibull's distribution of time is used to represent deterioration of the item is $\rho(t) = \alpha\beta(t-\gamma)^{\beta-1}, 0 < \alpha < 1, \beta \geq 1, -\infty < \gamma < \infty$ actually in this model $T_j < \gamma < T_{j+1}, i = 0,1,2,\ldots,m, where\ \alpha\ (0 < \alpha < 1)$ is a scaling parameter, β is the shape parameter and γ is the location parameter i.e., items shelf-time and t is the time of deterioration.

3. Deterioration begins as it reaches the inventory.

4. One item is considered in the prescribed time cycle.

5. Demand during shortage is partially lost and partially backordered.

6. Time discounting effect is considered under finite time horizon.

7 Production rate is greater than demand rate so P=σf(t) is the production rate where σ >1.

8. μ is less than production time.

9. The unit production cost is inversely proportional to the demand rate (G) and directly proportional to production reliability (r), so the unit production cost is $p = aG^{-b}r^c$, where b(>1) is called price elasticity and a,c (>0) are scaling parameters.

10. The reliability r means, r% of all the item produced are of acceptable quality that can fulfill the demand.

Few assumptions taken above are the basic assumption used in classical inventory model for deteriorating item with shortages. The first assumption states that the demand rate linearly increases with time when t<μ and then become steady i.e., constant at and after t≥μ. We can see this type of demand in newly launched items like fashionable products, electronic items, etc. The demand increases with time during the initial stage i.e., [0,μ]. After some time the demand become constant, this continues for some period i.e., in the time interval [μ,T_1]. Then the cycle ends. Again the next cycle starts with another new brand item and it will follow the same pattern of demand and production i.e., increasing and then steady and then stops. The finite time horizon has been considered to evaluate the effect of the time value of money on the total cost. Thus to understand the concept of value of future money in present date (which actually decreases due to time discounting rate) we need to consider a finite time horizon where its effect will be observed. The last assumption is mainly based on the unit variable production which is dependent on demand and process reliability. When the demand of an item increases then the production/purchase cost per unit item decreases and hence the unit production cost reduces which is inversely proportional to demand. Again the reliability of the produced items increases by using high quality raw material, technologically advanced machinery, quality control inspections, etc. Thus to produce high reliable product the production cost per unit item increases.

3. Neutrosophic number and its De-neutrosophication technique

Definition 3.1 (Neutrosophic Set [5]) A set $\tilde{S}$ in the universal discourse X, it is said to be a neutrosophic set if $\tilde{S} = \{\langle x; [\pi_{\tilde{S}}(x), \theta_{\tilde{S}}(x), \eta_{\tilde{S}}(x)]\rangle : x \in X\}, where\ \pi_{\tilde{S}}(x): X \rightarrow]-0,1+[$ is called the truth membership function, $\theta_{\tilde{S}}(x): X \rightarrow]-0,1+[$ is called the hesitation membership function, and $\eta_{\tilde{S}}(x): X \rightarrow]-0,1+[$ is called the false membership function of the decision maker, where $\pi_{\tilde{S}}(x), \theta_{\tilde{S}}(x), \eta_{\tilde{S}}(x)$ satisfies the following condition: $0 \leq Sup\{\pi_{\tilde{S}}(x)\} + Sup\{\theta_{\tilde{S}}(x)\} + Sup\{\eta_{\tilde{S}}(x)\} \leq 3$.

Definition 3.2 (Single-Valued Neutrosophic Set) A Neutrosophic set $\tilde{S}$ in the above definition 2.1 is also known as single-Valued Neutrosophic Set $sig(\tilde{S})$ if x is a single-valued independent variable.
$sig(\tilde{S}) = \{< x; [\pi_{sig(\tilde{S})}(x), \theta_{sig(\tilde{S})}(x), \eta_{sig(\tilde{S})}(x)]\rangle : x \in X\}$, where $\pi_{sig(\tilde{S})}(x), \theta_{sig(\tilde{S})}(x), \eta_{sig(\tilde{S})}(x)$ represent the concept of truth, hesitation and falsity memberships function respectively.

Definition 3.2.1: (Neutro-normal) Let us consider three points, for which p,q,r for which, $\pi_{sig(\tilde{S})}(p) = 1$, $\theta_{sig(\tilde{S})}(q) = 1, \eta_{sig(\tilde{S})}(r) = 1$ then the $sig(\tilde{S})$ is defined as neutro-normal.

Definition 3.2.2: (Neutro-convex) A $sig(\tilde{S})$ is called neutro-convex if the following condition holds:

$$(i)\pi_{sig(\tilde{S})}(\lambda\alpha + (1-\lambda)\beta) \geq min(\pi_{sig(\tilde{S})}(\alpha), \pi_{sig(\tilde{S})}(\beta))$$
$$(ii)\theta_{sig(\tilde{S})}(\lambda\alpha + (1-\lambda)\beta) \geq min(\theta_{sig(\tilde{S})}(\alpha), \theta_{sig(\tilde{S})}(\beta)),$$
$$(iii)\eta_{sig(\tilde{S})}(\lambda\alpha + (1-\lambda)\beta) \geq min(\eta_{sig(\tilde{S})}(\alpha), \eta_{sig(\tilde{S})}(\beta))$$

$where\ \alpha, \beta \in R, and\ \lambda \in [0,1]$

Definition 3.3 (Triangular Single Valued Neutrosophic Number) A triangular Single Valued Neutrosophic Number ($\tilde{S}$) is defined as $\tilde{S} =< (m_1, m_2, m_3: \mu), (n_1, n_2, n_3: \vartheta), (p_1, p_2, p_3: \zeta) >, where\ \mu, \vartheta, \zeta \in [0,1]$. Here the truth membership function $\pi_{\tilde{S}}: R \rightarrow [0, \mu]$, the hesitation membership function $\theta_{\tilde{S}}: R \rightarrow [\vartheta, 1]$ and the falsity membership function $\eta_{\tilde{S}}: R \rightarrow [\zeta, 1]$ are defined as follows:

$$\pi_{\tilde{S}}(x) = \begin{cases} \delta_{\tilde{S}l}(x), & m_1 \leq x < m_2 \\ \mu, & x = m_2 \\ \delta_{\tilde{S}r}(x), & m_2 < x \leq m_3 \\ 0, & otherwise \end{cases} \quad \theta_{\tilde{S}(x)} = \begin{cases} \varepsilon_{\tilde{S}l}(x), n_1 \leq x < n_2 \\ \vartheta, \quad x = n_2 \\ \varepsilon_{\tilde{S}r}(x), n_2 < x \leq n_3 \\ 1, \quad otherwise \end{cases} \quad \eta_{\tilde{S}}(x) = \begin{cases} l_{\tilde{S}l}(x), p_1 \leq x < p_2 \\ \vartheta, \quad x = p_2 \\ l_{\tilde{S}r}(x), p_2 < x \leq p_3 \\ 1, \quad otherwise \end{cases}$$

De-neutrosophication of triangular single valued neutrosophic number: In this model we have applied removal area technique to evaluate the de-neutrosophication value of triangular single valued neutrosophic number
$\tilde{S} =< (m_1, m_2, m_3: \mu), (n_1, n_2, n_3: \vartheta), (p_1, p_2, p_3: \zeta) >$ as done by (Chakraborty, et. al.). The de-neutrosophic form of $\tilde{S}$ is given as $neuD_{\tilde{S}} = \left(\frac{m_1+2m_2+m_3+n_1+2n_2+n_3+p_1+2p_2+p_3}{12}\right)$

4. Proposed model

Thus the inventory level for the proposed model at any time t over [0,T] is described mathematically by the following equations:

$\frac{dQ(t)}{dt} + \rho(t)Q(t) = rP - G = (r\sigma - 1)Rt, \quad 0 \le t \le \mu$ (1)

$\frac{dQ(t)}{dt} + \rho(t)Q(t) = (r\sigma - 1)R\mu, \quad \mu \le t \le t_1$ (2)

$\frac{dQ(t)}{dt} + \rho(t)Q(t) = -G = -R\mu, \quad t_1 \le t \le t_2$ (3)

$\frac{dQ(t)}{dt} = -BG = -BR\mu, \quad t_2 \le t \le t_3$ (4)

$\frac{dQ(t)}{dt} = rP - G = rK - R\mu = (r\sigma - 1)R\mu, t_3 \le t \le T_1$ (5)

with boundary conditions

$$Q(0) = 0, Q(\mu) = I, Q(t_1) = Q_m, Q(t_2) = 0, Q(t_3) = -Q_s \text{ and } Q(T_1) = 0,$$

$$where\ I = (r\sigma - 1)R[\left(\frac{\mu^2}{2}\right) + \left(\frac{\alpha\gamma}{\beta+1}\right)(\mu-\gamma)^{\beta+1} + \left(\frac{\alpha}{\beta+2}\right)(\mu-\gamma)^{\beta+2} + \frac{(-1)^{\beta}\gamma^{\beta+2}}{(\beta+1)(\beta+2)}]$$

4.1 Mathematical Analysis of the proposed model

From the above differential equations [1, 2, 3, 4, 5] and using the assumptions and the boundary conditions we obtain the inventory level of the proposed inventory model as follows:

$$Q(t) = (r\sigma - 1)R[\left(\frac{t^2}{2}\right) - \left(\frac{\alpha t^2}{2}\right)(t-\gamma)^{\beta} + \left(\frac{\alpha\gamma}{\beta+1}\right)(t-\gamma)^{\beta+1} + \left(\frac{\alpha}{\beta+2}\right)(t-\gamma)^{\beta+2} + \left(\frac{(-1)^{\{\beta\}}\gamma^{\beta+2}}{(\beta+1)(\beta+2)}\right)]$$

$Q(t) = (r\sigma - 1)R[t\mu - \left(\frac{\mu^2}{2}\right) + \mu\alpha(t-\gamma)^{\beta}\left(\left(\frac{t-\gamma}{\beta+1}\right) - t + \left(\frac{\mu}{2}\right)\right) - \left(\frac{\alpha}{(\beta+1)(\beta+2)}\right)\{(\mu-\gamma)^{\beta+2} - (-1)^{\beta}\gamma^{\beta+2}\}]$ (6)

$Q(t) = R\mu[t_1 - t + \left(\frac{\alpha}{\beta+1}\right)\{(t_1-\gamma)^{\beta+1} - (t-\gamma)^{\beta+1}\} + \alpha(t-t_1)(t-\gamma)^{\beta}] + Q_m(1 - \alpha(t-\gamma)^{\beta} + \alpha(t_1-\gamma)^{\beta}), t_1 \le t \le t_2$ (7)

$Q(t) = -BR\mu(t - t_2), \quad t_2 \le t \le t_3$ (8)

$Q(t) = (r\sigma - 1)R\mu(t - t_3) - Q_s, \quad t_3 \le t \le T_1$ (9)

Now using $Q(t_2)=0$ and eq.(6) we get the maximum amount inventory Q_m,

$Q_m = R\mu[t_2 - t_1 + \left(\frac{\alpha}{\beta+1}\right)(t_2-\gamma)^{\beta+1} - \alpha(t_1-\gamma)^{\beta}(\left(\frac{t^1-\gamma}{\beta+1}\right) + t_2 - t_1)]$ (10)

Now using eq.(8), eq.(9) and the relation $Q(t_3)=-Q_s$ we get the maximum shortages in the inventory level,

$Q_s = BR\mu(t_3 - t_2)$ (11)

Inventory carrying cost or holding cost:

$$HC = h\left[\int_0^{\mu} Q(t)dt + \int_{\mu}^{t_1} Q(t)dt + \int_{t_1}^{t_2} Q(t)dt\right]$$

$$= h[(r\sigma - 1)R\{\left(\frac{\mu^4}{6}\right) - \left(\frac{\alpha\beta\mu(\mu-\gamma)^{\beta+3}}{2(\beta+2)(\beta+3)}\right)(\left(\frac{\gamma(\beta+5)}{\beta+1}\right) + \mu + \left(\frac{((-1)^{\beta}\alpha\mu\gamma^{\beta+2})}{(\beta+1)(\beta+2)}\right)\left(\mu - \left(\frac{\gamma}{\beta+3}\right)\right)$$

$$+ \left(\frac{\alpha\mu\gamma^2(\mu-\gamma)^{\beta+1}}{2(\beta+1)}\right) + \left(\frac{\mu t_1}{2}\right)(t_1 - \mu) - \left(\frac{\alpha\beta\mu(t_1-\gamma)^{\beta+2}}{(\beta+1)(\beta+2)}\right) + R\mu\{-\left(\frac{(t_2-t_1)^2}{2}\right)$$

$$+\left(\frac{\alpha\mu\left(\left(\frac{\mu}{2}\right)-\gamma\right)}{\beta+1}\right)\left[(t_1-\gamma)^{\beta+1}-(\mu-\gamma)^{\beta+1}\right]+\left(\frac{(\alpha(\mu-\gamma)^{\beta+2})}{\beta+2}\right)\left(\mu-\left(\frac{t_1}{\beta+1}\right)\right)+\left(\frac{(-1)^{\beta}\alpha\gamma^{(\beta+2)}}{(\beta+1)(\beta+2)}\right)(t_1-\mu)\}+$$

$$\left(\frac{\alpha(t_1-\gamma)(t_2-\gamma)}{\beta+1}\right)\left[(t_1-\gamma)^{\beta}-(t_2-\gamma)^{\beta}\right]+Q_m\left(t_2-t_1-\left(\frac{\alpha(t_2-\gamma)^{\beta+1}}{\beta+1}\right)+\alpha(t_1-\gamma)^{\beta}\left(\left(\frac{t_1-\gamma}{\beta+1}\right)-t_2-t_1\right)\right)\}] \quad (12)$$

Production cost: The unit production cost depends on demand and process reliability. When the demand of an item increases then the production/purchase cost of the item decreases hence the unit production cost reduces i.e., production / purchase cost varies inversely with demand. The process reliability level r means only r% of the produced items is of acceptable quality which can be used to meet demand.
The unit production cost $p = aD^{-b}r^{c}$ $where\ a, b, c > 0\ and\ b \neq 2.$

The cost of production in $[t, t+dt]$ $is\ Kpdt = \sigma D.aD^{-b}r^{c}dt = \left(\frac{\sigma ar^{c}}{D^{b-1}}\right)dt.$

Since the production occurs $[0,t_1]$ and $[t_3,T_1]$ so the production cost (PDC) is given as follows.

Production cost (PDC)= $\int_0^{\mu}\left(\frac{\sigma ar^{c}}{D^{b-1}}\right)dt+\int_{\mu}^{t_1}\left(\frac{\sigma ar^{c}}{D^{b-1}}\right)dt+\int_{t_3}^{T_1}\left(\frac{\sigma ar^{c}}{D^{b-1}}\right)dt$

$$=\sigma ar^{c}\left[\int_0^{\mu}(Rt)^{1-b}dt+\int_{\mu}^{t_1}(R\mu)^{1-b}dt+\int_{t_3}^{T_1}(R\mu)^{1-b}dt\right]$$

$$=\left(\frac{\sigma ar^{c}R^{1-b}}{2-b}\right)\left[(b-1)\mu^{2-b}+(2-b)\mu^{1-b}(t_1+T_1-t_3)\right], b\neq 2 \quad (13)$$

Deterioration cost: The total no. of deteriorated items in $[0,T_1]$ is same as deterioration in $[0,t_2]$ as there is no deterioration of items during the period $[t_2,T_1]$.
D_1=Total no. of deteriorated items in $[0,t_2]$
=r×Production in $[0,\mu]$+r×Production in $[\mu,t_1]$-Demand in $[0,\mu]$-Demand in $[\mu,t_2]$

$$=r\sigma\int_0^{\mu}Rtdt+r\sigma\int_{\mu}^{t_1}R\mu dt-\int_0^{\mu}Rtdt-\int_{\mu}^{t_2}R\mu dt$$

$$=\left(\frac{1}{2}\right)Rr\mu\sigma(2t_1-\mu)-\left(\frac{1}{2}\right)R\mu(2t_2-\mu)$$

$$\therefore\ Deterioration\ cost\ (DC)=(dD_1)=\left(\frac{R\mu d}{2}\right)(r\sigma(2t_1-\mu)-(2t_2-\mu)) \quad (14)$$

Purchase cost: Since there is shortages in our model so the producer has to purchase raw material not only during $[0,t_1]$ but also in $[t_3, T_1]$. So we have to calculate purchase cost during the above two period.

$$PC=c_1\sigma\left(\int_0^{\mu}Rtdt+r\sigma\int_{\mu}^{t_1}R\mu dt+\int_{t_3}^{T_1}R\mu dt\right)=c_1\sigma R\mu\left(t_1+T_1-t_3-\left(\frac{\mu}{2}\right)\right) \quad (15)$$

Shortage cost: Since the model undergoes shortages so we observe shortages during $[t_2, T_1]$.

$$SC=c_2\int_{t_2}^{t_3}-Q(t)dt+c_2\int_{t_3}^{T_1}-Q(t)dt=\left(\frac{c^2R\mu}{2}\right)\left[B(t_3-t_2)^2+(r\sigma-1)(T_1-t_3)^2\right] \quad (16)$$

Lost cost: Due to urgency of demand the consumer opt to another shop so there is a chance for loss in sale during the shortages period $[t_2, t_3]$. Thus the lost cost for one replenishment interval is (LC).

$$LC=c_3(1-B)\int_{t_2}^{t_3}R\mu dt=c_3(1-B)R\mu(t_3-t_2) \quad (17)$$

The present value of ***total cost*** is (TC):

$$TC = (DC + PC + HC + LC + PDC + SC)\sum_{i=1}^{m} e^{-(i-1)kT} \approx (DC + PC + LC + SC + PDC + HC)\left(\frac{1-e^{-kmT}}{1-e^{-kT}}\right)$$

$$= R\mu\Bigg[\left(\tfrac{d}{2}\right)\left(r\sigma(2t_1-\mu)-(2t_2-\mu)\right)+c_1\sigma\left(t_1+T_1-t_3-\tfrac{\mu}{2}\right)+c_3(1-B)(t_3-t_2)+\left(\tfrac{c_2}{2}\right)[B(t_3-t_2)^2+$$

$$(r\sigma-1)(T_1-t_3)^2]+\left(\tfrac{\sigma a r^{c}{}_R{}^{-b}}{2-b}\right)[(b-1)\mu^{1-b}+(2-b)\mu^{-b}(t_1+T_1-t_3)]+h\Bigg[(r\sigma-1)\Bigg\{\xi+\left(\tfrac{t_1}{2}\right)(t_1-\mu)-$$

$$\left(\tfrac{\alpha\{\mu\beta(t_1-\gamma)^{\beta+2}-t_1(\mu-\gamma)^{\beta+2}+(-1)^{(\beta)}\alpha\gamma^{\beta+2}t_1\}}{\mu(\beta+1)(\beta+2)}\right)+\tfrac{\left(\alpha\left(\frac{\mu}{2}-\gamma\right)\right)(t_1-\gamma)^{\beta+1}}{\beta+1}\Bigg\}-\left(\tfrac{(t_2-t_1)^2}{2}\right)+\left(\tfrac{\alpha[(t_1-\gamma)^{\beta+1}(t_2-\gamma)-(t_1-\gamma)(t_2-\gamma)^{\beta+1}]}{\beta+1}\right)+$$

$$Q_m\left(t_2-t_1-\left(\tfrac{\alpha(t_2-\gamma)^{\beta+1}}{\beta+1}\right)+\alpha(t_1-\gamma)^{\beta}\left(\left(\tfrac{t_1-\gamma}{\beta+1}\right)-t_2-t_1\right)\right)\Bigg]\Bigg]\left(\tfrac{1-e^{-kmT}}{1-e^{-kT}}\right) \tag{18}$$

$Where$

$$b \neq 2, \qquad \xi = \frac{\mu^3}{6} - \frac{\alpha\beta(\mu-\gamma)^{\beta+3}}{2(\beta+2)(\beta+3)}\left(\frac{\gamma(\beta+5)}{\beta+1}+\mu\right) + \frac{(-1)^{\beta}\alpha\gamma^{\beta+2}}{(\beta+1)(\beta+2)}(\mu - \frac{\gamma}{\beta+3} - 1) + \frac{\alpha(\mu-\gamma)^{\beta+1}(\gamma^2+2\gamma-\mu)}{2(\beta+1)} + \frac{\alpha(\mu-\gamma)^{\beta+2}}{\beta+2}$$

We observe that TC is a function of t_1,t_2,t_3 and m. But for the sake of simplicity we simplified t_2 and t_3 in terms of t_1 and r.

Considering eq.(7), eq.(8) and the condition $Q(t_1)=Q_m$ we get t_2 in terms of t_1, and r. Expanding the exponential terms and neglecting the second and higher order terms of α and after simplifying the above two equations we get,

$$t_2 = (r\sigma-1)\left[\tfrac{\mu}{2}-\tfrac{\alpha}{\mu(\beta+1)(\beta+2)}\{(\mu-\gamma)^{\beta+2}-(-1)^{\beta}\gamma^{\beta+2}\}\right]+r\sigma\left[t_1+\tfrac{\alpha(t_1-\gamma)^{\beta+1}}{\beta+1}\right] \tag{19}$$

Also considering (11), and $Q(T_1)=0$, we get t_3 in terms of t_1,and r.

$$BP\mu(t_3-t_2) = (\gamma-1)R\mu(T_1-t_3)$$

$$t_3 = \tfrac{1}{B+r\sigma-1}((r\sigma-1)T_1+Bt_2) \tag{20}$$

Thus the total cost TC is function of t_1, r and m.

Optimization process

The following technique is derived to obtain the optimal value of t_1, r and m.

Step 1: Start by choosing a discrete value of m, a positive integer number.

Step 2: Take the partial derivative of total cost $TC(t_1, r, m)$ with respect to t_1 and r and equate it to zero, the necessary condition for optimality is $\frac{\partial TC(t_1,r,m)}{\partial t_1} = 0 \ and \ \frac{\partial TC(t_1,r,m)}{\partial r} = 0$.

Step 3: For different values of m, Obtain the optimum value of the time taken t_1^* and reliability r^* from the above two equation. Then substituting the value of t_1^*, r^* and m in equation [18] and obtain $TC(t_1^*,r^*,m)$

Step 4: Repeat step 2 and step 3 for different values of m and obtain the $TC(t_1^*,r^*,m)$. The minimum value of TC is obtained for optimum value of m^*. Thus (t_1^*,r^*,m^*) and $TC(t_1^*,r^*,m^*)$ are the optimal solution of our model. It satisfies the following condition:

$$\Delta TC(t_1^*, r^*, m^* - 1) < 0 < \Delta TC(t_1^*, r^*, m^* + 1)$$

Where $\Delta TC(t_1^*, r^*, m^*) = TC(t_1^*, r^*, m^* + 1) - TC(t_1^*, r^*, m^*)$

Step 5: To confirm that the objective function is convex, the derived value of $TC(t_1^*, r^*, m^*)$ must satisfy the sufficient condition:

$$\begin{pmatrix} \frac{\partial^2 TC(t_1,r)}{\partial t_1^2} & \frac{\partial^2 TC(t_1,r)}{\partial r \partial t_1} \\ \frac{\partial^2 TC(t_1,r)}{\partial t_1 \partial r} & \frac{\partial^2 TC(t_1,r)}{\partial r^2} \end{pmatrix} > 0 \; and \; \frac{\partial^2 TC(t_1,r)}{\partial t_1^2} > 0 \; or \; \frac{\partial^2 TC(t_1,r)}{\partial r^2} > 0 \qquad (21)$$

Since TC^* is very complicated with high powers so it is not possible to show the analytic validity of eq.(21). For this reason the above inequality is assessed by a numerical example.

4.2 Effect of Neutrosophication of parameter in proposed inventory model

Neutrosophic number actually deals with the conception of three different kinds of membership function related with real life scenario. It consists of truth, hesitation and falseness of an imprecise number. In this model we have considered purchase cost (c_1), holding cost (h) and inflation (k) as neutrosophic fuzzy number since in reality all the parameters are uncertain and contains a dilemma in decision maker's mind. So we try to manifest the model by introducing neutrosophication in the above cost and rates, and thus observe the effect of the above by comparing it with crisp model. The neutrosophic form of holding cost, purchase cost and inflation are represented by $\tilde{h}$, $\tilde{c_1}$ and $\tilde{k}$. Thus

$$\tilde{h} = < (h_1 - \varepsilon_1, h_1, h_1 + \varepsilon_2{:}\, \mu), (h_2 - \varepsilon_1, h_2, h_2 + \varepsilon_2{:}\, \vartheta), (h_3 - \varepsilon_1, h_3, h_3 + \varepsilon_2{:}\, \zeta) >,$$

$$\tilde{c_1} = < (c_{11} - \varepsilon_1, c_{11}, c_{11} + \varepsilon_2{:}\, \mu), (c_{12} - \varepsilon_1, c_{12}, c_{12} + \varepsilon_2{:}\, \vartheta), (c_{13} - \varepsilon_1, c_{13}, c_{13} + \varepsilon_2{:}\, \zeta) >,$$

$$\tilde{k} = < (k_1 - \varepsilon_1, k_1, k_1 + \varepsilon_2{:}\, \mu), (k_2 - \varepsilon_1, k_2, k_2 + \varepsilon_2{:}\, \vartheta), (k_3 - \varepsilon_1, k_3, k_3 + \varepsilon_2{:}\, \zeta) >$$

$where\ \mu, \vartheta, \zeta \in [0,1]\ and\ 0 < \varepsilon_1, \varepsilon_2 < 1.$

This neutrosophic fuzzy number is implemented in this model and thus the total cost obtain using this neutrosophic number is

$$TC_{neu}(\tilde{h}, \tilde{c_1}, \tilde{k}) = R\mu \Bigg[\left(\frac{d}{2}\right)\left(r\sigma(2t_1 - \mu) - (2t_2 - \mu)\right) + \tilde{c_1}\sigma\left(t_1 + T_1 - t_3 - \frac{\mu}{2}\right) + c_3(1 - B)(t_3 - t_2) +$$

$$\left(\frac{c_2}{2}\right)[B(t_3 - t_2)^2 + (r\sigma - 1)(T_1 - t_3)^2] + \left(\frac{\sigma a r^{c} R^{-b}}{2-b}\right)[(b-1)\mu^{1-b} + (2-b)\mu^{-b}(t_1 + T_1 - t_3)] + \tilde{h}\Bigg[(r\sigma - 1)\Bigg\{\xi +$$

$$\left(\frac{t_1}{2}\right)(t_1 - \mu) - \left(\frac{\alpha\{\mu\beta(t_1-\gamma)^{\beta+2} - t_1(\mu-\gamma)^{\beta+2} + (-1)^{(\beta)}\alpha\gamma^{\beta+2}t_1\}}{\mu(\beta+1)(\beta+2)}\right) + \frac{\left(\alpha\left(\frac{\mu}{2}-\gamma\right)\right)(t_1-\gamma)^{\beta+1}}{\beta+1}\Bigg\} - \left(\frac{(t_2-t_1)^2}{2}\right) +$$

$$\left(\frac{\alpha[(t_1-\gamma)^{\beta+1}(t_2-\gamma)-(t_1-\gamma)(t_2-\gamma)^{\beta+1}]}{\beta+1}\right)+Q_m\left(t_2-t_1-\left(\frac{\alpha(t_2-\gamma)^{\beta+1}}{\beta+1}\right)+\alpha(t_1-\gamma)^{\beta}\left(\left(\frac{t_1-\gamma}{\beta+1}\right)-t_2-t_1\right)\right)\Bigg]\Bigg]\left(\frac{1-e^{-\tilde{k}mT}}{1-e^{-\tilde{k}T}}\right) \quad (22)$$

Using removal area technique (Chakraborty et. al. [3]) the de- neutrosophic numbers are

$$\widetilde{h_{neuD}}=\frac{h_1+h_2+h_3}{3}-\frac{\varepsilon_1+\varepsilon_2}{4}, \widetilde{(c_1)_{neuD}}=\frac{c_{11}+c_{12}+c_{13}}{3}-\frac{\varepsilon_1+\varepsilon_2}{4}, and\ \widetilde{k_{neuD}}=\frac{k_1+k_2+k_3}{3}-\frac{\varepsilon_1+\varepsilon_2}{4}.$$

So we substitute the value of $h_{neuD}, (c_1)_{neuD}$ and k_{neuD} and obtain the total cost in neutrosophic domain.

Thus by de-neutrosophication we get

$$TC_{neu}(\tilde{h},\tilde{c_1},\tilde{k})=R\mu\Bigg[\left(\frac{d}{2}\right)\left(r\sigma(2t_1-\mu)-(2t_2-\mu)\right)+\widetilde{(c_1)_{neuD}}\sigma\left(t_1+T_1-t_3-\frac{\mu}{2}\right)+c_3(1-B)(t_3-t_2)+\left(\frac{c_2}{2}\right)[B(t_3-t_2)^2+(r\sigma-1)(T_1-t_3)^2]+\left(\frac{\sigma a r^{c}{}_{R}{}^{-b}}{2-b}\right)[(b-1)\mu^{1-b}+(2-b)\mu^{-b}(t_1+T_1-t_3)]+\widetilde{h_{neuD}}\Bigg[(r\sigma-1)\Bigg\{\xi+\left(\frac{t_1}{2}\right)(t_1-\mu)-\left(\frac{\alpha\{\mu\beta(t_1-\gamma)^{\beta+2}-t_1(\mu-\gamma)^{\beta+2}+(-1)^{(\beta)}\alpha\gamma^{\beta+2}t_1\}}{\mu(\beta+1)(\beta+2)}\right)+\frac{\left(\alpha\left(\frac{\mu}{2}-\gamma\right)\right)(t_1-\gamma)^{\beta+1}}{\beta+1}\Bigg\}-\left(\frac{(t_2-t_1)^2}{2}\right)+\left(\frac{\alpha[(t_1-\gamma)^{\beta+1}(t_2-\gamma)-(t_1-\gamma)(t_2-\gamma)^{\beta+1}]}{\beta+1}\right)+Q_m\left(t_2-t_1-\left(\frac{\alpha(t_2-\gamma)^{\beta+1}}{\beta+1}\right)+\alpha(t_1-\gamma)^{\beta}\left(\left(\frac{t_1-\gamma}{\beta+1}\right)-t_2-t_1\right)\right)\Bigg]\Bigg]\left(\frac{1-e^{-\widetilde{k_{neuD}}mT}}{1-e^{-\widetilde{k_{neuD}}T}}\right) \quad (23)$$

5. Numerical Example

The model is illustrated by an example. A new brand item follows the demand rate as ramp type function of time where the produced items are directly affected by reliability(r) of production process. The manufacturer maintains the production rate 1.3 times the demand rate where demand factor is considered as 12 unit per cycle. Also the items deteriorate with time is in the form of $\alpha\beta(t-\gamma)^{\beta-1}$, (where γ = 0.6 unit and α=0.001,β=1) which cost 1$ per unit time. The purchase cost of the raw material of the item is 3.5$ per unit item and 100$ is used for setting up for the production cycle. To hold the item in store the retailer has to pay 0.4$ per unit item. During shortages, which cost 3.2$, let 0.75 fraction of stock demand get backordered as the rest sales are lost. The cost for penalty (lost in sell) is 15$. The model is considered under 15 years of planning horizon with various replenishment cycle i.e., m=2,3,4,5 and discounting rate of inflation as 12%.

Therefore, the data considered to illustrate the models are as follows:

$$c_1=3.5, c_2=3.2, c_3=15, h=0.4, d=1, B=0.75, H=15, T=H/m, \mu=1.2, \sigma=1.3, \alpha=0.001, \beta=1, \gamma=0.6, a=3, b=0.8, c=2, k=0.12, R=12, S=100.$$

Table 1: Optimal solution of inventory model for different replenishment

m	T in year	t_1^* in year	t_2^* in year	t_3^* in year	reliability (r^*)	TC^*
2	7.5	7.175	7.452	7.454	0.799	806.54
3	5	4.571	4.883	4.894	0.828	738.13
4	3.75	3.249	3.579	3.603	0.864	717.58
5*	3*	2.451*	2.789*	2.83*	0.909*	715.26*

From the table 1 it is observed that the optimal solution is obtained (i.e., total cost is minimum) if we consider short replenishment cycles. This is realistic because if we decrease the time of the production then it produces less items and hence the total cost of the inventory decreases. It is also observed the better quality items are produced at shorter replenishment cycle i.e., the reliability (r) of the items increases in shorter production or replenishment cycle. This occurs because if we take small cycle then at the end of each cycle their is maintenance in production system happens regularly and thus the reliability of the items increases.

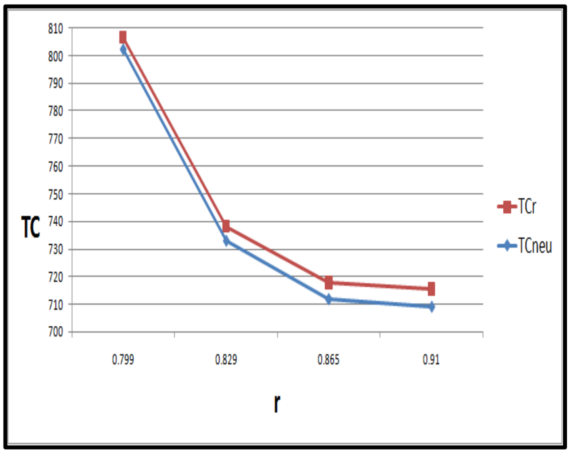

Figure 2: Graphical presentation of production cycle vs total cost

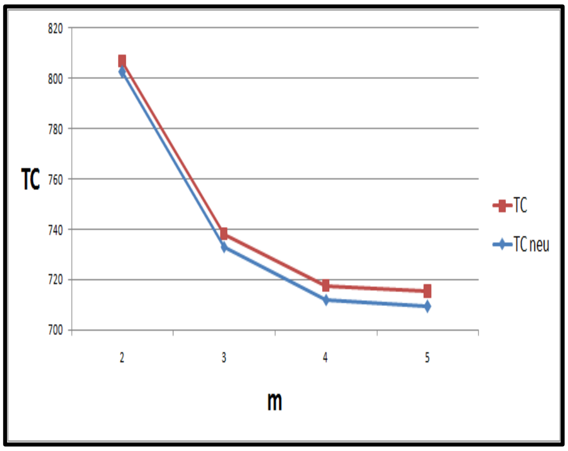

Figure 3: Graphical presentation of reliability vs total cost.

We observe from the figure 2 that for smaller production cycle (i.e., for large value of m), the optimal total cost (TC) decreases with optimal cost at m = 5.

In figure 3 we observe that the as reliability (r) increases then the total cost (TC) decreases. This holds because as reliability increases the demand of the item in the market increases as a result the cost per unit item decreases and hence the total cost decreases.

The above result is desirable because in the competitive market the business strategies of the manufacturer is to work in small cycle and producing highly reliable items at less cost.

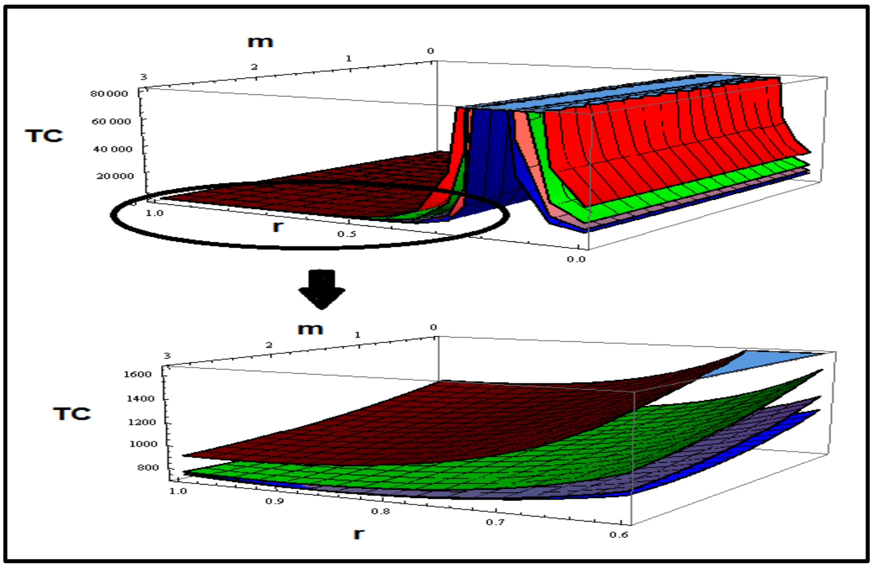

Figure 4: Graphical representation of total cost vs reliability and production time

Figure 4 gives the 3-dimensional plot of the total cost, reliability and no. of replenishment cycle in crisp model. In this figure we observe that reliability (r) increases for large value of m where the total cost (TC) decreases, i.e., highly reliable items are produced during small replenishment cycle at less cost, which is desirable in producer-oriented EPQ model. This is obvious as, in small cycle, the machinery gets upgraded and ameliorated eventually at the end of each cycle, and hence better quality of items are produced at much faster rate and thus cost per unit items decreases and hence the total costing of the inventory decreases.

In reality few parameters are uncertain and thus there is a dilemma in decision maker's mind. Thus instead of considering the model in crisp domain let us consider the model in neutrosophic domain and examine the same example as above. Here we have considered purchase cost (c_1), holding cost (h) and inflation (k) as triangular neutrosophic fuzzy number. Thus the neutrosophic numbers of the above parameters are $k_1 = 0.125, k_2 = 0.118, k_3 = 0.132, h_1 = 0.38, h_2 = 0.4, h_3 = 0.42, c_{11} = 2.5, c_{12} = 2.45. c_{13} = 2.55, \varepsilon_1 = 0.005, \varepsilon_2 = 0.007$.

$$Then, \widetilde{c_1} =< (2.495,2.5,2.507),(2.445,2.45,2.457),(2.545,2.55,2.557) >,$$

$$\widetilde{h} =< (0.375,0.38,0.387),(0.395,0.4,0.407),(0.415,0.42,0.427) > and$$

$$\widetilde{k} =< (0.12,0.125,0.132),(0.113,0.118,0.125),(0.127,0.132,0.139) >.$$

Thus we obtained table 2 under neutrosophic arena for the optimal solution of the model for different replenishment cycle.

Table 2: Optimal time and cost of inventory model under neutrosophic domain

m	T in year	t_1^* in year	t_2^* in year	t_3^* in year	reliability (r^*)	TC^*
2	7.5	7.172	7.448	7.451	0.799	802.45
3	5	4.569	4.886	4.897	0.829	733
4	3.75	3.247	3.58	3.605	0.865	711.87
5^*	3^*	2.449^*	2.789^*	2.83^*	0.91^*	709.11^*

Thus if we compare table 1 and table 2 it is observed that the total cost (TC) decreases if we consider the model in neutrosophic arena. This is desirable as few parameters has hesitation factor in decision maker's mind and thus this model under neutrosophic domain gives us better result.

6. Sensitivity Analysis

The retailer should be aware of the effect in the total cost for any changes in the parameter. In order to examine the implications of these changes, the sensitivity analysis will be helpful for decision-making. Using the numerical example as given in the preceding section, we perform the sensitivity analysis by changing few crisp parameters by -10%, -5%, 5% and 10% by taking one parameter at time and keeping the other parameter fixed. As per Table 1 we observe that optimal solution is obtained when we consider small replenishment cycle. So we perform the sensitivity analysis for m=5.

Table 3. Sensitivity analysis of some parameters

Parameters	**Change (%)**	**t_1^* in year**	**t_2^* in year**	**t_3^* in year**	**reliability (r^*)**	**TC^***	**% change of TC^***
	-10	2.642	2.933	2.44	0.878	677.95	-5.5
c_1	-5	2.557	2.872	2.894	0.892	696.78	-2.65
	5	2.313	2.677	2.748	0.932	733.29	2.46
	10	2.122	2.517	2.64	0.968	750.71	4.72
	-10	2.398	2.776	2.825	0.93	711.92	-0.47
h	-5	2.425	2.761	2.808	0.91	713.67	-0.22
	5	2.475	2.796	2.834	0.9	716.7	0.2
	10	2.498	2.803	2.838	0.892	718.01	0.38
	-10	2.451	2.789	2.83	0.909	687.65	-4.02
S	-5	2.451	2.789	2.83	0.909	701.45	-1.97
	5	2.451	2.789	2.83	0.909	729.06	1.89
	10	2.451	2.789	2.83	0.909	742.87	3.72
	-10	2.617	2.903	2.918	0.975	681.84	-4.9
σ	-5	2.533	2.844	2.871	0.939	698.61	-2.38
	5	2.364	2.728	2.786	0.883	731.72	2.25
	10	2.264	2.66	2.742	0.865	747.95	4.37
	-10	2.398	2.771	2.819	0.923	681.47	-4.96
μ	-5	2.425	2.781	2.825	0.916	698.63	-2.38
	5	2.475	2.798	2.836	0.903	731.35	2.2
	10	2.498	2.805	2.841	0.897	746.92	4.24
	-10	2.451	2.789	2.83	0.909	750.84	4.74
k	-5	2.451	2.789	2.83	0.909	732.63	2.37
	5	2.451	2.789	2.83	0.909	698.69	-2.37
	10	2.451	2.789	2.83	0.909	682.87	-4.74
	-10	2.41	2.781	2.828	0.926	671.72	-6.48
R	-5	2.431	2.785	2.829	0.917	693.48	-3.14
	5	2.471	2.794	2.832	0.901	737.05	2.96
	10	2.49	2.799	2.834	0.894	758.86	5.74

From the above table 3 it is observed that the model is highly sensitive to purchase cost, demand rate factor (R), moderately sensitive to setup cost, σ, μ, inflation (k) and less sensitive to holding cost. It is also noted that the model is insensitive to the shortage cost, lost in sale cost and deterioration cost. That means deterioration is not going to affect the model as much.

(i) The model is highly sensitive to purchase cost i.e., if we increase purchase cost (c_1), the total cost increases. It is also noted that as the purchase cost increases, the reliability increases and production time decreases which means if we buy good quality raw material then we have better quality of finished good at less manufacturing time. Again the total cost TC increases with increase in demand factor R. This is obvious because if demand increases means more items are produced and hence the production time and production cost also increases which leads to increase in total cost.

(ii) The model is moderately sensitive to set up cost (S), σ, μ, inflation (k). Investing more money for upgradation of machineries, i.e., by increasing in set up cost (S), the total cost increases. It is noted that in our model the set up cost does not depends on reliability and production time. Again with the increase in production rate (σ) and production time (μ), the total cost increases. This is true because, if production time increases then more items are produced also if we increase the production rate then we have more finished good at less manufacturing time and thus in both the case the total cost increases. Also the toal cost decreases with increase in inflation (k). This is obvious because with the increase in inflation the time value of money increases and thus the total cost decreases in present day.

(iii) It is noticed that as the holding cost (h) is a less sensitive parameter. With the increase in holding cost, the total cost increases. It is also observed that the production time also increases with increase in holding cost. It means that the items has to be held for longer time with high value of holding cost then obviously the total cost will increase.

It has been observed that there are various parameters which are very less sensitive hence it is not included in the table.

7. Concluding remarks

This paper developed an EPQ model for deteriorating item with reliability in production process and ramp type demand rate under crisp and neutrosophic domain. The model also considers shortages where part of the items gets backlogged and part of the sales are lost. The model coincides with practical situations since we have considered the effect of time value of money under finite time horizon. Also the model optimizes by considering the reliability of production process, as the reliability of production process increases, the total cost decreases. This model is cost effective because highly reliable items are obtained at less cost and which is desirable in managerial point of view. It is also observed that the highly reliable items are produced in small cycles. The paper also compares the model under two different environment, crisp and neutrosophic, and it is observed that the model works better in neutrosophic domain as compare to crisp environment. In this paper we have done sensitivity analysis in crisp environment to illustrate our example and we have noted that the minimum value of total cost is obtained for short replenishment cycle. This work could be extended by considering multi-layer supply chain lot sizing model with manufacturer end, retailer end under neutrosophic environment. Also we can extend this same model and can compare the model with neutrosophic number and hybrid plithogenic decision-making method.

Further, in the forthcoming research, people can fruitfully execute and apply the idea of triangular neutrosophic into distinct research arenas like structural modeling, diagnostic problems, realistic modeling, recruitment based problems, pattern recognition etc.

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Received: Apr 18, 2020 Accepted: July 12, 2020.

Operations on Neutrosophic Vague Graphs

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Abstract: Neutrosophic graph is a mathematical tool to hold with imprecise and unspecified data. In this manuscript, the operations on neutrosophic vague graphs are introduced. Moreover, Cartesian product, lexicographic product, cross product, strong product and composition of neutrosophic vague graphs are investigated. The proposed concepts are demonstrated with suitable examples.

Keywords: Neutrosophic vague graph, Operations of neutrosophic vague graph, Cartesian product, Cross product, Strong product

1. Introduction

In a classical graph, any vertex or edge have two situations, namely, it is either in the graph or it is not in the graph and it is not sufficient to model uncertain optimization problems. Therefore, real-life problems are not suitable to model using classical graphs. Hence the fuzzy set arises, which is an extension of classical set; here the objects have varying membership degrees. Vague sets are regarded as a special case of context-dependent fuzzy sets. At first, vague set theory was investigated by Gau and Buehrer [36] that is an extension of fuzzy set theory. The classical fuzzy set handles only the membership degree, but intuitionistic fuzzy handles independent membership degree and non-membership degree for any element with the only requirement is that the sum of non-membership and membership degree values is not greater than one [16].

On the other hand, to hold this indeterminate and inconsistent information, the neutrosophic set is introduced by F. Smarandache and has been studied extensively (see [31]-[35]). Neutrosophic set and related notions have weird applications in many different fields. In the definition of neutrosophic set, the indeterminacy value is quantified explicitly and truth-membership, false-membership and indeterminacy-membership are stated as exactly independent provided sum of these values belonging to 0 and 3. Neutrosophic soft rough graphs with applications are established in [10]. Neutrosophic soft relations and neutrosophic refined relations with their properties are studied in [15, 20]. Single valued neutrosophic graph are studied in [17, 18]. Some types of neutrosophic graphs and co-neutrosophic graphs are discussed in [23]. Neutrosophic vague set is first initiated in [11]. Al-Quran and Hassan in [7] introduced the notion of neutrosophic vague soft expert set as a generalization of neutrosophic vague set and soft expert set in order to revise the application in decision-making in real-life problems. Intuitionistic bipolar neutrosophic set and its application to graphs are established in [28]. Further, neutrosophic vague graphs are investigated in

[27]. Motivated by the articles [11, 27, 28, 29], we introduce the concept of operations on neutrosophic vague graphs. The main contributions in this manuscript are given below:

- Operations on neutrosophic vague graphs are established. In Section 2, basic definitions regarding to neutrosophic vague graphs are explained with an example.
- In Section 3, Cartesian product, lexicographic product, cross product, strong product and composition of neutrosophic vague graph are illustrated with examples. Finally, a conclusion is elaborated with future direction.

2. Preliminaries

In this section, basic definitions and example are given, which is used to prove the main results.

Definition 2.1 [36] A vague set $\mathbb{A}$ on a non empty set $\mathbb{X}$ is a pair $(\mathbb{T}_{\mathbb{A}}, \mathbb{F}_{\mathbb{A}})$, where $\mathbb{T}_{\mathbb{A}}: \mathbb{X} \to [0,1]$ and $\mathbb{F}_{\mathbb{A}}: \mathbb{X} \to [0,1]$ are true membership and false membership functions, respectively, such that

$$0 \leq \mathbb{T}_{\mathbb{A}}(x) + \mathbb{F}_{\mathbb{A}}(x) \leq 1 \text{ for every } x \in \mathbb{X}.$$

Let $\mathbb{X}$ and $\mathbb{Y}$ be two non-empty sets. A vague relation R of $\mathbb{X}$ to $\mathbb{Y}$ is a vague set R on $\mathbb{X} \times \mathbb{Y}$ that is $R = (\mathbb{T}_R, \mathbb{F}_R)$, where $\mathbb{T}_R: \mathbb{X} \times \mathbb{Y} \to [0,1]$, $\mathbb{F}_R: \mathbb{X} \times \mathbb{Y} \to [0,1]$ and satisfies the condition:

$$0 \leq \mathbb{T}_R(x,y) + \mathbb{F}_R(x,y) \leq 1 \text{ for any } x, y \in \mathbb{X}.$$

Definition 2.2 [12] Let $\mathbb{G}^* = (\mathbb{V}, \mathbb{E})$ be a graph. A pair $\mathbb{G} = (\mathbb{J}, \mathbb{K})$ is called a vague graph on $\mathbb{G}^*$, where $\mathbb{J} = (\mathbb{T}_{\mathbb{J}}, \mathbb{F}_{\mathbb{J}})$ is a vague set on $\mathbb{V}$ and $\mathbb{K} = (\mathbb{T}_{\mathbb{K}}, \mathbb{F}_{\mathbb{K}})$ is a vague set on $\mathbb{E} \subseteq \mathbb{V} \times \mathbb{V}$ such that for each $xy \in \mathbb{E}$,

$$\mathbb{T}_{\mathbb{K}}(xy) \leq \min\{\mathbb{T}_{\mathbb{J}}(x), \mathbb{T}_{\mathbb{J}}(y)\} \text{ and } \mathbb{F}_{\mathbb{K}}(xy) \geq \max\{\mathbb{F}_{\mathbb{J}}(x), \mathbb{F}_{\mathbb{J}}(y)\}.$$

Definition 2.3 [31] A Neutrosophic set $\mathbb{A}$ is contained in another neutrosophic set $\mathbb{B}$, (i.e) $\mathbb{A} \subseteq \mathbb{B}$ if $\forall x \in \mathbb{X}, \mathbb{T}_{\mathbb{A}}(x) \leq \mathbb{T}_{\mathbb{B}}(x), \mathbb{I}_{\mathbb{A}}(x) \geq \mathbb{I}_{\mathbb{B}}(x)$and $\mathbb{F}_{\mathbb{A}}(x) \geq \mathbb{F}_{\mathbb{B}}(x)$.

Definition 2.4 [20, 31] Let $\mathbb{X}$ be a space of points (objects), with generic elements in $\mathbb{X}$ denoted by x. A single valued neutrosophic set $\mathbb{A}$ in $\mathbb{X}$ is characterised by truth-membership function $\mathbb{T}_{\mathbb{A}}(x)$, indeterminacy-membership function $\mathbb{I}_{\mathbb{A}}(x)$ and falsity-membership-function $\mathbb{F}_{\mathbb{A}}(x)$,

For each point x in $\mathbb{X}$, $\mathbb{T}_{\mathbb{A}}(x), \mathbb{I}_{\mathbb{A}}(x), \mathbb{F}_{\mathbb{A}}(x) \in [0,1]$. Also

$$\mathbb{A} = \{\langle x, \mathbb{T}_{\mathbb{A}}(x), \mathbb{I}_{\mathbb{A}}(x), \mathbb{F}_{\mathbb{A}}(x)\rangle\} \text{ and } 0 \leq \mathbb{T}_{\mathbb{A}}(x), +\mathbb{I}_{\mathbb{A}}(x) + \mathbb{F}_{\mathbb{A}}(x) \leq 3.$$

Definition 2.5 [6, 18] A neutrosophic graph is defined as a pair $\mathbb{G}^* = (\mathbb{V}, \mathbb{E})$ where

(i) $\mathbb{V} = \{v_1, v_2, .., v_n\}$ such that $\mathbb{T}_1 : \mathbb{V} \to [0,1]$, $\mathbb{I}_1 : \mathbb{V} \to [0,1]$ and $\mathbb{F}_1 : \mathbb{V} \to [0,1]$ denote the degree of truth-membership function, indeterminacy function and falsity-membership function, respectively, and

$$0 \leq \mathbb{T}_1(v) + \mathbb{I}_1(v) + \mathbb{F}_1(v) \leq 3,$$

(ii) $\mathbb{E} \subseteq \mathbb{V} \times \mathbb{V}$ where $\mathbb{T}_2 : \mathbb{E} \to [0,1]$, $\mathbb{I}_2 : \mathbb{E} \to [0,1]$ and $\mathbb{F}_2 : \mathbb{E} \to [0,1]$ are such that

$$\mathbb{T}_2(uv) \leq \min\{\mathbb{T}_1(u), \mathbb{T}_1(v)\},$$

$$\mathbb{I}_2(uv) \leq \min\{\mathbb{I}_1(u), \mathbb{I}_1(v)\},$$

$$\mathbb{F}_2(uv) \leq \max\{\mathbb{F}_1(u), \mathbb{F}_1(v)\}$$

$$\text{and } 0 \leq \mathbb{T}_2(uv) + \mathbb{I}_2(uv) + \mathbb{F}_2(uv) \leq 3, \forall uv \in \mathbb{E}.$$

Definition 2.6 [11] A Neutrosophic Vague Set $\mathbb{A}_{NV}$ (NVS in short) on the universe of discourse $\mathbb{X}$ written as

$$\mathbb{A}_{NV} = \{\langle x, \widehat{\mathbb{T}}_{\mathbb{A}_{NV}}(x), \widehat{\mathbb{I}}_{\mathbb{A}_{NV}}(x), \widehat{\mathbb{F}}_{\mathbb{A}_{NV}}(x)\rangle, x \in \mathbb{X}\},$$

whose truth-membership, indeterminacy membership and falsity-membership function are defined as

$$\widehat{\mathbb{T}}_{\mathbb{A}_{NV}}(x) = [\mathbb{T}^-(x), \mathbb{T}^+(x)], \widehat{\mathbb{I}}_{\mathbb{A}_{NV}}(x) = [\mathbb{I}^-(x), \mathbb{I}^+(x)] \text{and } \widehat{\mathbb{F}}_{\mathbb{A}_{NV}}(x) = [\mathbb{F}^-(x), \mathbb{F}^+(x)],$$

where $\mathbb{T}^+(x) = 1 - \mathbb{F}^-(x), \mathbb{F}^+(x) = 1 - \mathbb{T}^-(x)$, and $0 \le \mathbb{T}^-(x) + \mathbb{I}^-(x) + \mathbb{F}^-(x) \le 2$.

Definition 2.7 [11] The complement of NVS $\mathbb{A}_{NV}$ is denoted by $\mathbb{A}^c_{NV}$ and it is defined by

$$\widehat{\mathbb{T}}^c_{\mathbb{A}_{NV}}(x) = [1 - \mathbb{T}^+(x), 1 - \mathbb{T}^-(x)],$$
$$\widehat{\mathbb{I}}^c_{\mathbb{A}_{NV}}(x) = [1 - \mathbb{I}^+(x), 1 - \mathbb{I}^-(x)],$$
$$\widehat{\mathbb{F}}^c_{\mathbb{A}_{NV}}(x) = [1 - \mathbb{F}^+(x), 1 - \mathbb{F}^-(x)].$$

Definition 2.8 [11] Let $\mathbb{A}_{NV}$ and $\mathbb{B}_{NV}$ be two NVSs of the universe $\mathbb{U}$. If for all $u_i \in \mathbb{U}$,

$$\widehat{\mathbb{T}}_{\mathbb{A}_{NV}}(u_i) \le \widehat{\mathbb{T}}_{\mathbb{B}_{NV}}(u_i), \widehat{\mathbb{I}}_{\mathbb{A}_{NV}}(u_i) \ge \widehat{\mathbb{I}}_{\mathbb{B}_{NV}}(u_i), \widehat{\mathbb{F}}_{\mathbb{A}_{NV}}(u_i) \ge \widehat{\mathbb{F}}_{\mathbb{B}_{NV}}(u_i),$$

then the NVS, $\mathbb{A}_{NV}$ are included in $\mathbb{B}_{NV}$, denoted by $\mathbb{A}_{NV} \subseteq \mathbb{B}_{NV}$ where $1 \le i \le n$.

Definition 2.9 [11] The union of two NVSs , $\mathbb{A}_{NV}$ and $\mathbb{B}_{NV}$, is a NVSs, $\mathbb{D}_{NV}$, written as $\mathbb{D}_{NV} = \mathbb{A}_{NV} \cup \mathbb{B}_{NV}$ whose truth-membership function, indeterminacy-membership function and false-membership function are related to those of $\mathbb{A}_{NV}$ and $\mathbb{B}_{NV}$ by

$$\widehat{\mathbb{T}}_{\mathbb{D}_{NV}}(x) = [\max(\mathbb{T}^-_{\mathbb{A}_{NV}}(x), \mathbb{T}^-_{\mathbb{B}_{NV}}(x)), \max(\mathbb{T}^+_{\mathbb{A}_{NV}}(x), \mathbb{T}^+_{\mathbb{B}_{NV}}(x))]$$
$$\widehat{\mathbb{I}}_{\mathbb{D}_{NV}}(x) = [\min(\mathbb{I}^-_{\mathbb{A}_{NV}}(x), \mathbb{I}^-_{\mathbb{B}_{NV}}(x)), \min(\mathbb{I}^+_{\mathbb{A}_{NV}}(x), \mathbb{I}^+_{\mathbb{B}_{NV}}(x))]$$
$$\widehat{\mathbb{F}}_{\mathbb{D}_{NV}}(x) = [\min(\mathbb{F}^-_{\mathbb{A}_{NV}}(x), \mathbb{F}^-_{\mathbb{B}_{NV}}(x)), \min(\mathbb{F}^+_{\mathbb{A}_{NV}}(x), \mathbb{F}^+_{\mathbb{B}_{NV}}(x))].$$

Definition 2.10 [11] The intersection of two NVSs, $\mathbb{A}_{NV}$ and $\mathbb{B}_{NV}$ is a NVSs, $\mathbb{D}_{NV}$, written as $\mathbb{D}_{NV} = \mathbb{A}_{NV} \cap \mathbb{B}_{NV}$, whose truth-membership function, indeterminacy-membership function and false-membership function are related to those of $\mathbb{A}_{NV}$ and $\mathbb{B}_{NV}$ by

$$\widehat{\mathbb{T}}_{\mathbb{D}_{NV}}(x) = [\min(\mathbb{T}^-_{\mathbb{A}_{NV}}(x), \mathbb{T}^-_{\mathbb{B}_{NV}}(x)), \min(\mathbb{T}^+_{\mathbb{A}_{NV}}(x), \mathbb{T}^+_{\mathbb{B}_{NV}}(x))]$$
$$\widehat{\mathbb{I}}_{\mathbb{D}_{NV}}(x) = [\max(\mathbb{I}^-_{\mathbb{A}_{NV}}(x), \mathbb{I}^-_{\mathbb{B}_{NV}}(x)), \max(\mathbb{I}^+_{\mathbb{A}_{NV}}(x), \mathbb{I}^+_{\mathbb{B}_{NV}}(x))]$$
$$\widehat{\mathbb{F}}_{\mathbb{D}_{NV}}(x) = [\max(\mathbb{F}^-_{\mathbb{A}_{NV}}(x), \mathbb{F}^-_{\mathbb{B}_{NV}}(x)), \max(\mathbb{F}^+_{\mathbb{A}_{NV}}(x), \mathbb{F}^+_{\mathbb{B}_{NV}}(x))].$$

Definition 2.11 [27] Let $G^* = (R, S)$ be a graph. A pair $\mathbb{G} = (\mathbb{A}, \mathbb{B})$ is called a neutrosophic vague graph (NVG) on G^* or a neutrosophic vague graph where $\mathbb{A} = (\widehat{\mathbb{T}}_{\mathbb{A}}, \widehat{\mathbb{I}}_{\mathbb{A}}, \widehat{\mathbb{F}}_{\mathbb{A}})$ is a neutrosophic vague set on R and $\mathbb{B} = (\widehat{\mathbb{T}}_{\mathbb{B}}, \widehat{\mathbb{I}}_{\mathbb{B}}, \widehat{\mathbb{F}}_{\mathbb{B}})$is a neutrosophic vague set $S \subseteq R \times R$ where

(1) $R = \{v_1, v_2, \ldots, v_n\}$ such that $\mathbb{T}^-_{\mathbb{A}}: R \to [0,1], \mathbb{I}^-_{\mathbb{A}}: R \to [0,1], \mathbb{F}^-_{\mathbb{A}}: R \to [0,1]$ which satisfies the condition $\mathbb{F}^-_{\mathbb{A}} = [1 - \mathbb{T}^+_{\mathbb{A}}]$,

$\mathbb{T}^+_{\mathbb{A}}: R \to [0,1], \mathbb{I}^+_{\mathbb{A}}: R \to [0,1], \mathbb{F}^+_{\mathbb{A}}: R \to [0,1]$ which satisfies the condition $\mathbb{F}^+_{\mathbb{A}} = [1 - \mathbb{T}^-_{\mathbb{A}}]$

denotes the degree of truth membership function, indeterminacy membership and falsity membership of the element $v_i \in R$, and

$$0 \le \mathbb{T}^-_{\mathbb{A}}(v_i) + \mathbb{I}^-_{\mathbb{A}}(v_i) + \mathbb{F}^-_{\mathbb{A}}(v_i) \le 2$$
$$0 \le \mathbb{T}^+_{\mathbb{A}}(v_i) + \mathbb{I}^+_{\mathbb{A}}(v_i) + \mathbb{F}^+_{\mathbb{A}}(v_i) \le 2.$$

(2) $S \subseteq R \times R$ where

$$\mathbb{T}^-_{\mathbb{B}}: R \times R \to [0,1],\ \mathbb{I}^-_{\mathbb{B}}: R \times R \to [0,1],\ \mathbb{F}^-_{\mathbb{B}}: R \times R \to [0,1]$$
$$\mathbb{T}^+_{\mathbb{B}}: R \times R \to [0,1],\ \mathbb{I}^+_{\mathbb{B}}: R \times R \to [0,1],\ \mathbb{F}^+_{\mathbb{B}}: R \times R \to [0,1]$$

denotes the degree of truth membership function, indeterminacy membership and falsity membership of the element $v_i v_j \in S$, respectively and such that,

$$0 \le \mathbb{T}^-_{\mathbb{B}}(v_i v_j) + \mathbb{I}^-_{\mathbb{B}}(v_i v_j) + \mathbb{F}^-_{\mathbb{B}}(v_i v_j) \le 2$$
$$0 \le \mathbb{T}^+_{\mathbb{B}}(v_i v_j) + \mathbb{I}^+_{\mathbb{B}}(v_i v_j) + \mathbb{F}^+_{\mathbb{B}}(v_i v_j) \le 2.$$

such that

$$\mathbb{T}^-_{\mathbb{B}}(v_iv_j) \le \min\{\mathbb{T}^-_{\mathbb{A}}(v_i), \mathbb{T}^-_{\mathbb{A}}(v_j)\}$$
$$\mathbb{I}^-_{\mathbb{B}}(v_iv_j) \le \min\{\mathbb{I}^-_{\mathbb{A}}(v_i), \mathbb{I}^-_{\mathbb{A}}(v_j)\}$$
$$\mathbb{F}^-_{\mathbb{B}}(v_iv_j) \le \max\{\mathbb{F}^-_{\mathbb{A}}(v_i), \mathbb{F}^-_{\mathbb{A}}(v_j)\}$$

and similarly

$$\mathbb{T}^+_{\mathbb{B}}(v_iv_j) \le \min\{\mathbb{T}^+_{\mathbb{A}}(v_i), \mathbb{T}^+_{\mathbb{A}}(v_j)\}$$
$$\mathbb{I}^+_{\mathbb{B}}(v_iv_j) \le \min\{\mathbb{I}^+_{\mathbb{A}}(v_i), \mathbb{I}^+_{\mathbb{A}}(v_j)\}$$
$$\mathbb{F}^+_{\mathbb{B}}(v_iv_j) \le \max\{\mathbb{F}^+_{\mathbb{A}}(v_i), \mathbb{F}^+_{\mathbb{A}}(v_j)\}.$$

Example 2.12 Consider a neutrosophic vague graph $G = (R, S)$ such that $\mathbb{A} = \{a, b, c\}$ and $\mathbb{B} = \{ab, bc, ca\}$ are defined by

$$\hat{a} = T[0.5,0.6], I[0.4,0.3], F[0.4,0.5], \qquad \hat{b} = T[0.4,0.6], I[0.7,0.3], F[0.4,0.6],$$
$$\hat{c} = T[0.4,0.4], I[0.5,0.3], F[0.6,0.6]$$
$$a^- = (0.5,0.4,0.4), b^- = (0.4,0.7,0.4), c^- = (0.4,0.5,0.6)$$
$$\mathbf{a^+ = (0.6, 0.3, 0.5)},\ \mathbf{b^+ = (0.6, 0.3, 0.6)},\ \mathbf{c^+ = (0.4, 0.3, 0.6)}.$$

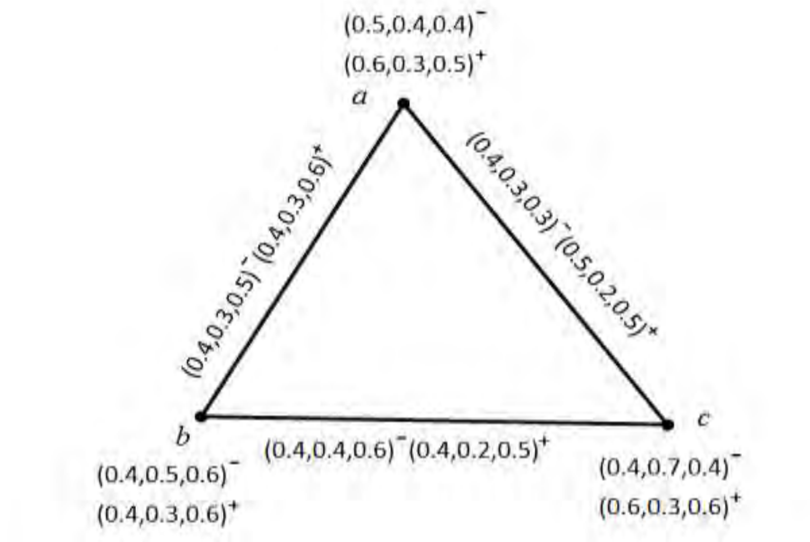


Figure 1: NEUTROSOPHIC VAGUE GRAPH

3. Operations on Neutrosophic Vague Graphs

In this section, the results on operations of neutrosophic vague graphs with example are established.

Definition 3.1 The Cartesian product of two NVGs G_1 and G_2 is denoted by the pair $G_1 \times G_2 = (R_1 \times R_2, S_1 \times S_2)$ and defined as

$$T^-_{A_1 \times A_2}(kl) = T^-_{A_1}(k) \wedge T^-_{A_2}(l)$$
$$I^-_{A_1 \times A_2}(kl) = I^-_{A_1}(k) \wedge I^-_{A_2}(l)$$
$$F^-_{A_1 \times A_2}(kl) = F^-_{A_1}(k) \vee F^-_{A_2}(l)$$
$$T^+_{A_1 \times A_2}(kl) = T^+_{A_1}(k) \wedge T^+_{A_2}(l)$$
$$I^+_{A_1 \times A_2}(kl) = I^+_{A_1}(k) \wedge I^+_{A_2}(l)$$
$$F^+_{A_1 \times A_2}(kl) = F^+_{A_1}(k) \vee F^+_{A_2}(l),$$

for all $(k,l) \in R_1 \times R_2$.

The membership value of the edges in $G_1 \times G_2$ can be calculated as,

$$(1)\ T^-_{B_1\times B_2}(kl_1)(kl_2) = T^-_{A_1}(k) \wedge T^-_{B_2}(l_1l_2)$$
$$T^+_{B_1\times B_2}(kl_1)(kl_2) = T^+_{A_1}(k) \wedge T^+_{B_2}(l_1l_2),$$

$$(2)\ I^-_{B_1\times B_2}(kl_1)(kl_2) = I^-_{A_1}(k) \wedge I^-_{B_2}(l_1l_2)$$
$$I^+_{B_1\times B_2}(kl_1)(kl_2) = I^+_{A_1}(k) \wedge I^+_{B_2}(l_1l_2),$$

$$(3)\ F^-_{B_1\times B_2}(kl_1)(kl_2) = F^-_{A_1}(k) \vee F^-_{B_2}(l_1l_2)$$
$$F^+_{B_1\times B_2}(kl_1)(kl_2) = F^+_{A_1}(k) \vee F^+_{B_2}(l_1l_2),$$

for all $k \in R_1, l_1l_2 \in S_2$.

$$(4)\ T^-_{B_1\times B_2}(k_1l)(k_2l) = T^-_{A_2}(l) \wedge T^-_{B_2}(k_1k_2)$$
$$T^+_{B_1\times B_2}(k_1l)(k_2l) = T^+_{A_2}(l) \wedge T^+_{B_2}(k_1k_2),$$

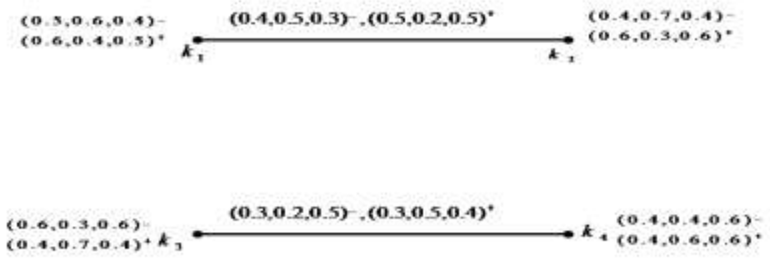
$$(5)\ I^-_{B_1\times B_2}(k_1l)(k_2l) = I^-_{A_2}(l) \wedge I^-_{B_2}(k_1k_2)$$
$$I^+_{B_1\times B_2}(k_1l)(k_2l) = I^+_{A_2}(l) \wedge I^+_{B_2}(k_1k_2),$$

$$(6)\ F^-_{B_1\times B_2}(k_1l)(k_2l) = F^-_{A_2}(l) \vee F^-_{B_2}(k_1k_2)$$
$$F^+_{B_1\times B_2}(k_1l)(k_2l) = F^+_{A_2}(l) \vee F^+_{B_2}(k_1k_2),$$

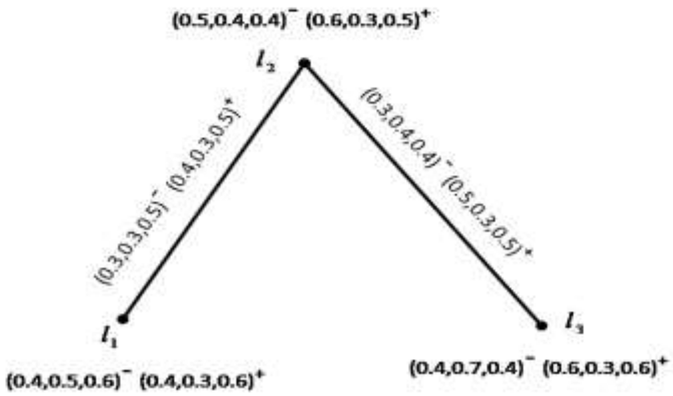
for all $k_1k_2 \in S_1, l \in R_2$.

Example 3.2 Consider $G_1 = (R_1, S_1)$ and $G_2 = (R_2, S_2)$ are two NVGs of $G = (R, S)$, as represented in Figure 2, now we get $G_1 \times G_2$ as follows see Figure 3.

$$\hat{k}_1 = T[0.5,0.6], I[0.6,0.4], F[0.4,0.5], \hat{k}_2 = T[0.4,0.6], I[0.7,0.3], F[0.4,0.6],$$
$$\hat{k}_3 = T[0.6,0.4], I[0.3,0.7], F[0.6,0.4], \hat{k}_4 = T[0.4,0.4], I[0.4,0.6], F[0.6,0.6]$$
$$\hat{l}_1 = T[0.4,0.4], I[0.5,0.3], F[0.6,0.6], \hat{l}_2 = T[0.5,0.6], I[0.4,0.3], F[0.4,0.5],$$
$$\hat{l}_3 = T[0.4,0.6], I[0.7,0.3], F[0.4,0.6]$$
$$k_1^- = (0.5,0.6,0.4), k_2^- = (0.4,0.7,0.4), k_3^- = (0.6,0.3,0.6), k_4^- = (0.4,0.4,0.6)$$
$$k_1^+ = (0.6,0.4,0.5), k_2^+ = (0.6,0.3,0.6), k_3^+ = (0.4,0.7,0.4), k_4^- = (0.4,0.6,0.6)$$
$$l_1^- = (0.4,0.5,0.6), l_2^- = (05,0.4,0.4), l_3^- = (0.4,0.7,0.4)$$
$$\mathbf{l_1^+ = (0.4, 0.3, 0.6), l_2^+ = (0.6, 0.3, 0.5), l_3^+ = (0.6, 0.3, 0.6)}.$$

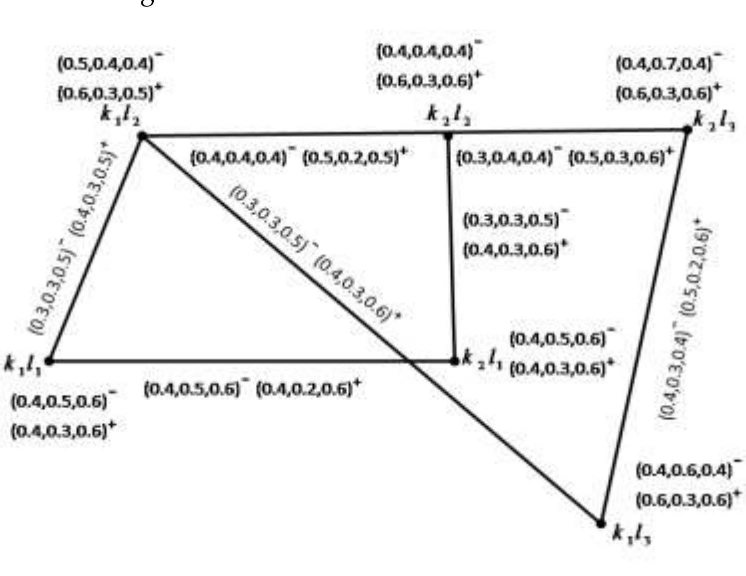


G_1



G_2

Figure 2: NEUTROSOPHIC VAGUE GRAPH



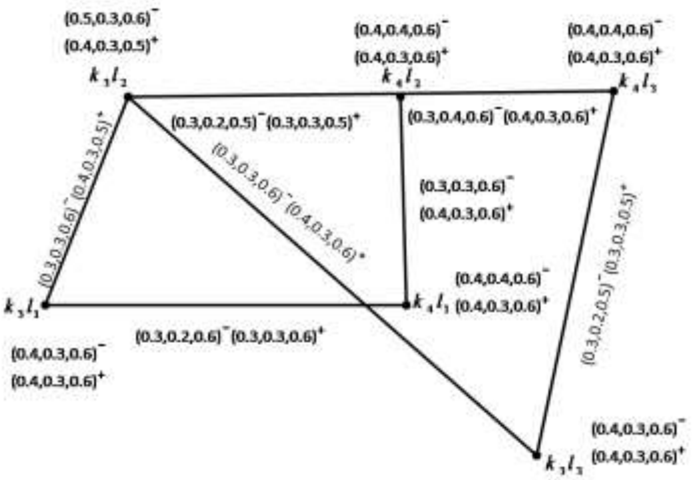


Figure 3: CARTESIAN PRODUCT OF NEUTROSOPHIC VAGUE GRAPH

Theorem 3.3 The Cartesian product $G_1 \times G_2 = (R_1 \times R_2, S_1 \times S_2)$ of two NVG G_1 and G_2 is also the NVG of $G_1 \times G_2$.

Proof. We consider two cases.

Case 1: for $k \in R_1, l_1l_2 \in S_2$,

$$\begin{aligned}
\hat{T}_{(B_1\times B_2)}((kl_1)(kl_2)) &= \hat{T}_{A_1}(k) \wedge \hat{T}_{B_2}(l_1l_2) \\
&\leq \hat{T}_{A_1}(k) \wedge [\hat{T}_{A_2}(l_1) \wedge \hat{T}_{A_2}(l_2)] \\
&= [\hat{T}_{A_1}(k) \wedge \hat{T}_{A_2}(l_1)] \wedge [\hat{T}_{A_1}(k) \wedge \hat{T}_{A_2}(l_2)] \\
&= \hat{T}_{(A_1\times A_2)}(k, l_1) \wedge \hat{T}_{(A_1\times A_2)}(k, l_2)
\end{aligned}$$

$$\begin{aligned}
\hat{I}_{(B_1\times B_2)}((kl_1)(kl_2)) &= \hat{I}_{A_1}(k) \wedge \hat{I}_{B_2}(l_1l_2) \\
&\leq \hat{I}_{A_1}(k) \wedge [\hat{I}_{A_2}(l_1) \wedge \hat{I}_{A_2}(l_2)] \\
&= [\hat{I}_{A_1}(k) \wedge \hat{I}_{A_2}(l_1)] \wedge [\hat{I}_{A_1}(k) \wedge \hat{I}_{A_2}(l_2)] \\
&= \hat{I}_{(A_1\times A_2)}(k, l_1) \wedge \hat{I}_{(A_1\times A_2)}(k, l_2)
\end{aligned}$$

$$\begin{aligned}
\hat{F}_{(B_1\times B_2)}((kl_1)(kl_2)) &= \hat{F}_{A_1}(k) \vee \hat{F}_{B_2}(l_1l_2) \\
&\leq \hat{F}_{A_1}(k) \vee [\hat{F}_{A_2}(l_1) \vee \hat{F}_{A_2}(l_2)] \\
&= [\hat{F}_{A_1}(k) \vee \hat{F}_{A_2}(l_1)] \vee [\hat{F}_{A_1}(k) \vee \hat{F}_{A_2}(l_2)] \\
&= \hat{F}_{(A_1\times A_2)}(k, l_1) \vee \hat{F}_{(A_1\times A_2)}(k, l_2)
\end{aligned}$$

for all $kl_1, kl_2 \in G_1 \times G_2$.

Case 2: for $k \in R_2, l_1l_2 \in S_1$.

$$\begin{aligned}
\hat{T}_{(B_1\times B_2)}((l_1k)(l_2k)) &= \hat{T}_{A_2}(k) \wedge \hat{T}_{B_1}(l_1l_2) \\
&\leq \hat{T}_{A_2}(k) \wedge [\hat{T}_{A_1}(l_1) \wedge \hat{T}_{A_1}(l_2)] \\
&= [\hat{T}_{A_2}(k) \wedge \hat{T}_{A_1}(l_1)] \wedge [\hat{T}_{A_2}(k) \wedge \hat{T}_{A_1}(l_2)] \\
&= \hat{T}_{(A_1\times A_2)}(l_1, k) \wedge \hat{T}_{(A_1\times A_2)}(l_2, k)
\end{aligned}$$

$$\hat{I}_{(B_1\times B_2)}((l_1k)(l_2k)) = \hat{I}_{A_2}(k) \wedge \hat{I}_{B_1}(l_1l_2)$$

$$\leq \hat{I}_{A_2}(k) \wedge [\hat{I}_{A_1}(l_1) \wedge \hat{I}_{A_1}(l_2)]$$
$$= [\hat{I}_{A_2}(k) \wedge \hat{I}_{A_1}(l_1)] \wedge [\hat{I}_{A_2}(k) \wedge \hat{I}_{A_1}(l_2)]$$
$$= \hat{I}_{(A_1\times A_2)}(l_1,k) \wedge \hat{I}_{(A_1\times A_2)}(l_2,k)$$

$$\hat{F}_{(B_1\times B_2)}((l_1k)(l_2k)) = \hat{F}_{A_2}(k) \vee \hat{F}_{B_1}(l_1l_2)$$
$$\leq \hat{F}_{A_2}(k) \vee [\hat{F}_{A_1}(l_1) \vee \hat{F}_{A_1}(l_2)]$$
$$= [\hat{F}_{A_2}(k) \vee \hat{F}_{A_1}(l_1)] \vee [\hat{F}_{A_2}(k) \vee \hat{F}_{A_1}(l_2)]$$
$$= \hat{F}_{(A_1\times A_2)}(l_1,k) \vee \hat{F}_{(A_1\times A_2)}(l_2,k)$$

for all $l_1k, l_2k \in G_1 \times G_2$.

Definition 3.4 The Cross product of two NVGs G_1 and G_2 is denoted by the pair $G_1 \star G_2 = (R_1 \star R_2, S_1 \star S_2)$ and is defined as

$$\text{(i)}T^-_{A_1\star A_2}(kl) = T^-_{A_1}(k) \wedge T^-_{A_2}(l)$$
$$I^-_{A_1\star A_2}(kl) = I^-_{A_1}(k) \wedge I^-_{A_2}(l)$$
$$F^-_{A_1\star A_2}(kl) = F^-_{A_1}(k) \vee F^-_{A_2}(l)$$
$$T^+_{A_1\star A_2}(kl) = T^+_{A_1}(k) \wedge T^+_{A_2}(l)$$
$$I^+_{A_1\star A_2}(kl) = I^+_{A_1}(k) \wedge I^+_{A_2}(l)$$
$$F^+_{A_1\star A_2}(kl) = F^+_{A_1}(k) \vee F^+_{A_2}(l),$$

for all $k, l \in R_1 \star R_2$.

$$\text{(ii)}T^-_{(B_1\star B_2)}(k_1l_1)(k_2l_2) = T^-_{B_1}(k_1k_2) \wedge T^-_{B_2}(l_1l_2)$$
$$I^-_{(B_1\star B_2)}(k_1l_1)(k_2l_2) = I^-_{B_1}(k_1k_2) \wedge I^-_{B_2}(l_1l_2)$$
$$F^-_{(B_1\star B_2)}(k_1l_1)(k_2l_2) = F^-_{B_1}(k_1k_2) \vee F^-_{B_2}(l_1l_2)$$
$$\text{(iii)}T^+_{(B_1\star B_2)}(k_1l_1)(k_2l_2) = T^+_{B_1}(k_1k_2) \wedge T^+_{B_2}(l_1l_2)$$
$$I^+_{(B_1\star B_2)}(k_1l_1)(k_2l_2) = I^+_{B_1}(k_1k_2) \wedge I^+_{B_2}(l_1l_2)$$
$$F^+_{(B_1\star B_2)}(k_1l_1)(k_2l_2) = F^+_{B_1}(k_1k_2) \vee F^+_{B_2}(l_1l_2),$$

for all $k_1k_2 \in S_1, l_1l_2 \in S_2$.

Example 3.5 Consider $\mathbf{G_1} = (\mathbf{R_1}, \mathbf{S_1})$ and $\mathbf{G_2} = (\mathbf{R_2}, \mathbf{S_2})$ as two NVG of $\mathbf{G} = (\mathbf{R}, \mathbf{S})$ respectively, (see Figure 2). We obtain the cross product of $\mathbf{G_1} \star \mathbf{G_2}$ as follows (see Figure 4).

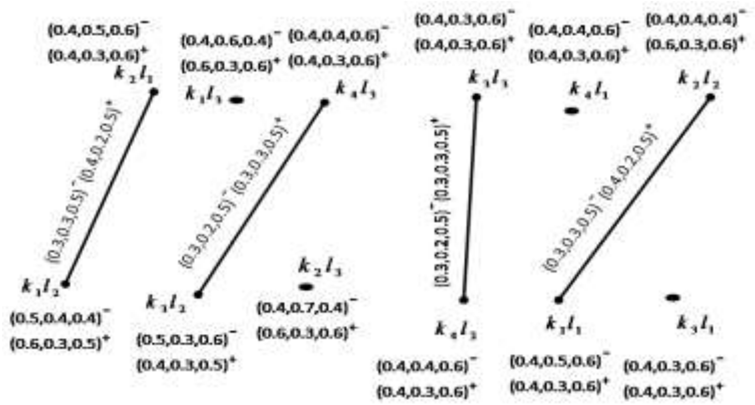


Figure 4: CROSS PRODUCT OF NEUTROSOPHIC VAGUE GRAPH

Theorem 3.6 The cross product $G_1 \star G_2 = (R_1 \star R_2, S_1 \star S_2)$ of two NVG G_1 and G_2 is an the NVG of $G_1 \star G_2$.

Proof. For all $k_1l_1, k_2l_2 \in G_1 \star G_2$

$$\begin{aligned}\widehat{T}_{(B_1\star B_2)}((k_1l_1)(k_2l_2)) &= \widehat{T}_{B_1}(k_1k_2) \wedge \widehat{T}_{B_2}(l_1l_2)\\ &\le [\widehat{T}_{A_1}(k_1) \wedge \widehat{T}_{A_1}(k_2)] \wedge [\widehat{T}_{A_2}(l_1) \wedge \widehat{T}_{A_2}(l_2)]\\ &= [\widehat{T}_{A_1}(k_1) \wedge \widehat{T}_{A_2}(l_1)] \wedge [\widehat{T}_{A_1}(k_2) \wedge \widehat{T}_{A_2}(l_2)]\\ &= \widehat{T}_{(A_1\star A_2)}(k_1l_1) \wedge \widehat{T}_{(A_1\star A_2)}(k_2, l_2)\end{aligned}$$

$$\begin{aligned}\hat{I}_{(B_1\star B_2)}((k_1l_1)(k_2l_2)) &= \hat{I}_{B_1}(k_1k_2) \wedge \hat{I}_{B_2}(l_1l_2)\\ &\le [\hat{I}_{A_1}(k_1) \wedge \hat{I}_{A_1}(k_2)] \wedge [\hat{I}_{A_2}(l_1) \wedge \hat{I}_{A_2}(l_2)]\\ &= [\hat{I}_{A_1}(k_1) \wedge \hat{I}_{A_2}(l_1)] \wedge [\hat{I}_{A_1}(k_2) \wedge \hat{I}_{A_2}(l_2)]\\ &= \hat{I}_{(A_1\star A_2)}(k_1l_1) \wedge \hat{I}_{(A_1\star A_2)}(k_2, l_2)\end{aligned}$$

$$\begin{aligned}\widehat{F}_{(B_1\star B_2)}((k_1l_1)(k_2l_2)) &= \widehat{F}_{B_1}(k_1k_2) \vee \widehat{F}_{B_2}(l_1l_2)\\ &\le [\widehat{F}_{A_1}(k_1) \vee \widehat{F}_{A_1}(k_2)] \vee [\widehat{F}_{A_2}(l_1) \vee \widehat{F}_{A_2}(l_2)]\\ &= [\widehat{F}_{A_1}(k_1) \vee \widehat{F}_{A_2}(l_1)] \vee [\widehat{F}_{A_1}(k_2) \vee \widehat{F}_{A_2}(l_2)]\\ &= \widehat{F}_{(A_1\star A_2)}(k_1l_1) \vee \widehat{F}_{(A_1\star A_2)}(k_2, l_2).\end{aligned}$$

This completes the proof.

Definition 3.7 The lexicographic product of two NVGs G_1 and G_2 is denoted by the pair $G_1 \bullet G_2 = (R_1 \bullet R_2, S_1 \bullet S_2)$ and defined as

$$\begin{aligned}&\text{(i)}T^-_{(A_1\bullet A_2)}(kl) = T^-_{A_1}(k) \wedge T^-_{A_2}(l)\\ &I^-_{(A_1\bullet A_2)}(kl) = I^-_{A_1}(k) \wedge I^-_{A_2}(l)\\ &F^-_{(A_1\bullet A_2)}(kl) = F^-_{A_1}(k) \vee F^-_{A_2}(l)\\ &T^+_{(A_1\bullet A_2)}(kl) = T^+_{A_1}(k) \wedge T^+_{A_2}(l)\\ &I^+_{(A_1\bullet A_2)}(kl) = I^+_{A_1}(k) \wedge I^+_{A_2}(l)\\ &F^+_{(A_1\bullet A_2)}(kl) = F^+_{A_1}(k) \vee F^+_{A_2}(l),\end{aligned}$$

for all $kl \in R_1 \times R_2$

$$\begin{aligned}&\text{(ii)}T^-_{(B_1\bullet B_2)}(kl_1)(kl_2) = T^-_{A_1}(k) \wedge T^-_{B_2}(l_1l_2)\\ &I^-_{(B_1\bullet B_2)}(kl_1)(kl_2) = I^-_{A_1}(k) \wedge I^-_{B_2}(l_1l_2)\\ &F^-_{(B_1\bullet B_2)}(kl_1)(kl_2) = F^-_{A_1}(k) \vee F^-_{B_2}(l_1l_2)\\ &T^+_{(B_1\bullet B_2)}(kl_1)(kl_2) = T^+_{A_1}(k) \wedge T^+_{B_2}(l_1l_2)\\ &I^+_{(B_1\bullet B_2)}(kl_1)(kl_2) = I^+_{A_1}(k) \wedge I^+_{B_2}(l_1l_2)\\ &F^+_{(B_1\bullet B_2)}(kl_1)(kl_2) = F^+_{A_1}(k) \vee F^+_{B_2}(l_1l_2),\end{aligned}$$

for all $k \in R_1, l_1l_2 \in S_2$.

$$\begin{aligned}&\text{(iii)}T^-_{(B_1\bullet B_2)}(k_1l_1)(k_2l_2) = T^-_{B_1}(k_1k_2) \wedge T^-_{B_2}(l_1l_2)\\ &I^-_{(B_1\bullet B_2)}(k_1l_1)(k_2l_2) = I^-_{B_1}(k_1k_2) \wedge I^-_{B_2}(l_1l_2)\\ &F^-_{(B_1\bullet B_2)}(k_1l_1)(k_2l_2) = F^-_{B_1}(k_1k_2) \vee F^-_{B_2}(l_1l_2)\\ &T^+_{(B_1\bullet B_2)}(k_1l_1)(k_2l_2) = T^+_{B_1}(k_1k_2) \wedge T^+_{B_2}(l_1l_2)\\ &I^+_{(B_1\bullet B_2)}(k_1l_1)(k_2l_2) = I^+_{B_1}(k_1k_2) \wedge I^+_{B_2}(l_1l_2)\\ &F^+_{(B_1\bullet B_2)}(k_1l_1)(k_2l_2) = F^+_{B_1}(k_1k_2) \vee F^+_{B_2}(l_1l_2), \quad \text{for all } k_1k_2 \in S_1, l_1l_2 \in S_2.\end{aligned}$$

Example 3.8 The lexicographic product of NVG $G_1 = (R_1, S_1)$ and $G_2 = (R_2, S_2)$ shown in Figure 2 is defined as $G_1 \bullet G_2 = (R_1 \bullet R_2, S_1 \bullet S_2)$ and is presented in Figure 5.

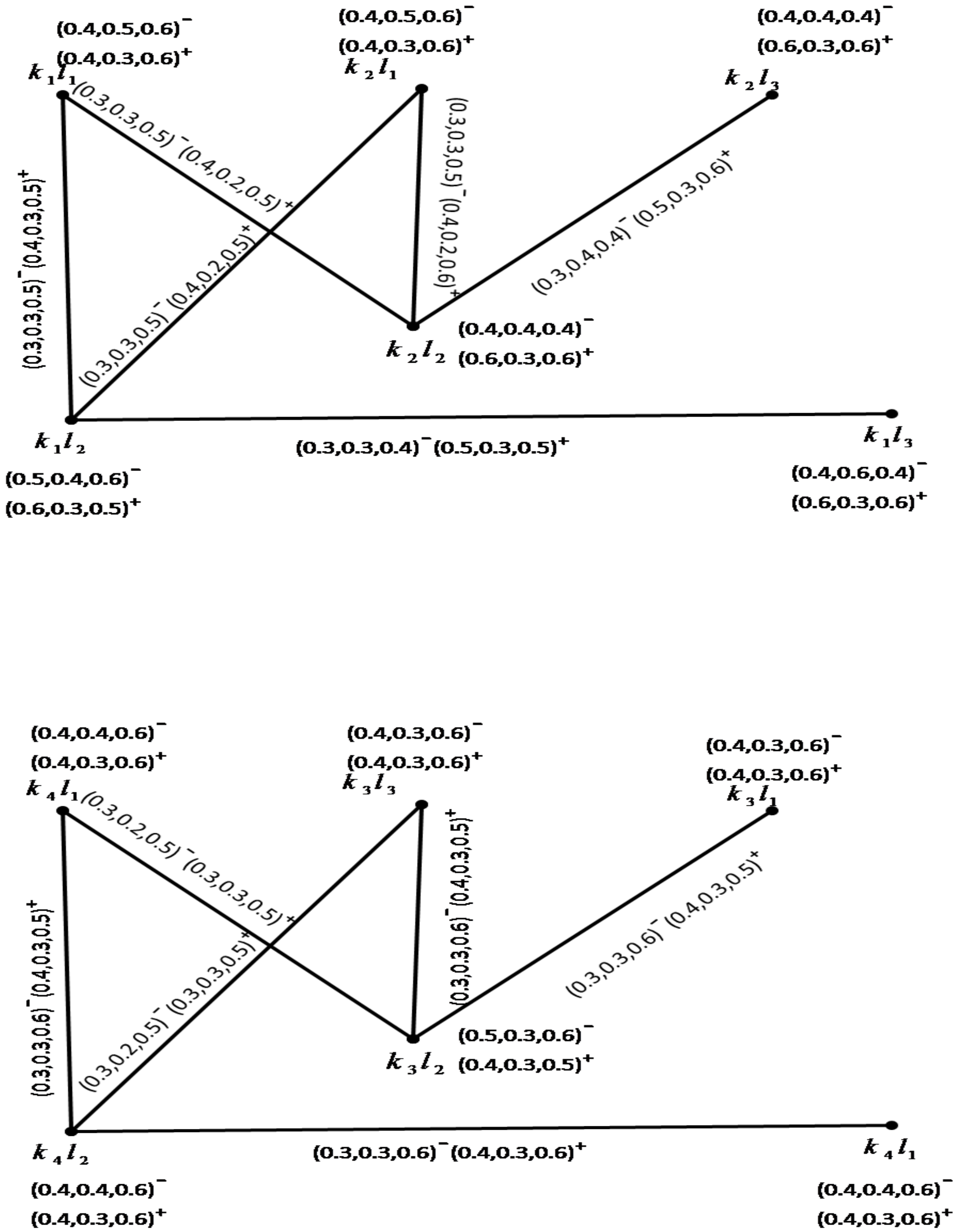


Figure 5: LEXICOGRAPHIC PRODUCT OF NEUTROSOPHIC VAGUE GRAPH

Theorem 3.9 The lexicographic product $G_1 \bullet G_2 = (R_1 \bullet R_2, S_1 \bullet S_2)$ of two NVG G_1 and G_2 is the NVG of $G_1 \bullet G_2$.

Proof. We have two cases.

Case 1: For $k \in R_1, l_1l_2 \in S_2$,

$$\begin{aligned}\hat{T}_{(B_1\bullet B_2)}((kl_1)(kl_2)) &= \hat{T}_{A_1}(k) \wedge \hat{T}_{B_2}(l_1l_2)\\ &\leq \hat{T}_{A_1}(k) \wedge [\hat{T}_{A_2}(l_1) \wedge \hat{T}_{A_2}(l_2)]\\ &= [\hat{T}_{A_1}(k) \wedge \hat{T}_{A_2}(l_1)] \wedge [\hat{T}_{A_1}(k) \wedge \hat{T}_{A_2}(l_2)]\\ &= \hat{T}_{(A_1\bullet A_2)}(k, l_1) \wedge \hat{T}_{(A_1\bullet A_2)}(k, l_2)\end{aligned}$$

$$\begin{aligned}\hat{I}_{(B_1\bullet B_2)}((kl_1)(kl_2)) &= \hat{I}_{A_1}(k) \wedge \hat{I}_{B_2}(l_1l_2)\\ &\leq \hat{I}_{A_1}(k) \wedge [\hat{I}_{A_2}(l_1) \wedge \hat{I}_{A_2}(l_2)]\\ &= [\hat{I}_{A_1}(k) \wedge \hat{I}_{A_2}(l_1)] \wedge [\hat{I}_{A_1}(k) \wedge \hat{I}_{A_2}(l_2)]\\ &= \hat{I}_{(A_1\bullet A_2)}(k, l_1) \wedge \hat{I}_{(A_1\bullet A_2)}(k, l_2)\end{aligned}$$

$$\begin{aligned}\hat{F}_{(B_1\bullet B_2)}((kl_1)(kl_2)) &= \hat{F}_{A_1}(k) \vee \hat{F}_{B_2}(l_1l_2)\\ &\leq \hat{F}_{A_1}(k) \vee [\hat{F}_{A_2}(l_1) \vee \hat{F}_{A_2}(l_2)]\\ &= [\hat{F}_{A_1}(k) \vee \hat{F}_{A_2}(l_1)] \vee [\hat{F}_{A_1}(k) \vee \hat{F}_{A_2}(l_2)]\\ &= \hat{F}_{(A_1\bullet A_2)}(k, l_1) \vee \hat{F}_{(A_1\bullet A_2)}(k, l_2)\end{aligned}$$

for all $kl_1, kl_2 \in S_1 \times S_2$.

Case 2: For all $k_1l_1 \in S_1, k_2l_2 \in S_2$,

$$\begin{aligned}\hat{T}_{(B_1\bullet B_2)}((k_1l_1)(k_2l_2)) &= \hat{T}_{B_1}(k_1k_2) \wedge \hat{T}_{B_2}(l_1l_2)\\ &\leq [\hat{T}_{A_1}(k_1) \wedge \hat{T}_{A_1}(k_2)] \wedge [\hat{T}_{A_2}(l_1) \wedge \hat{T}_{A_2}(l_2)]\\ &= [\hat{T}_{A_1}(k_1) \wedge \hat{T}_{A_2}(l_1)] \wedge [\hat{T}_{A_1}(k_2) \wedge \hat{T}_{A_2}(l_2)]\\ &= \hat{T}_{(A_1\bullet A_2)}(k_1l_1) \wedge \hat{T}_{(A_1\bullet A_2)}(k_2, l_2)\end{aligned}$$

$$\begin{aligned}\hat{I}_{(B_1\bullet B_2)}((k_1l_1)(k_2l_2)) &= \hat{I}_{B_1}(k_1k_2) \wedge \hat{I}_{B_2}(l_1l_2)\\ &\leq [\hat{I}_{A_1}(k_1) \wedge \hat{I}_{A_1}(k_2)] \wedge [\hat{I}_{A_2}(l_1) \wedge \hat{I}_{A_2}(l_2)]\\ &= [\hat{I}_{A_1}(k_1) \wedge \hat{I}_{A_2}(l_1)] \wedge [\hat{I}_{A_1}(k_2) \wedge \hat{I}_{A_2}(l_2)]\\ &= \hat{I}_{(A_1\bullet A_2)}(k_1l_1) \wedge \hat{I}_{(A_1\bullet A_2)}(k_2, l_2)\end{aligned}$$

$$\begin{aligned}\hat{F}_{(B_1\bullet B_2)}((k_1l_1)(k_2l_2)) &= \hat{F}_{B_1}(k_1k_2) \vee \hat{F}_{B_2}(l_1l_2)\\ &\leq [\hat{F}_{A_1}(k_1) \vee \hat{F}_{A_1}(k_2)] \vee [\hat{F}_{A_2}(l_1) \vee \hat{F}_{A_2}(l_2)]\\ &= [\hat{F}_{A_1}(k_1) \vee \hat{F}_{A_2}(l_1)] \vee [\hat{F}_{A_1}(k_2) \vee \hat{F}_{A_2}(l_2)]\\ &= \hat{F}_{(A_1\bullet A_2)}(k_1l_1) \vee \hat{F}_{(A_1\bullet A_2)}(k_2, l_2)\end{aligned}$$

for all $k_1, l_1 \in k_2, l_2 \in R_1 \bullet R_2$.

Definition 3.10 The strong product of two NVG G_1 and G_2 is denoted by the pair $G_1 \boxtimes G_2 = (R_1 \boxtimes R_2, S_1 \boxtimes S_2)$ and defined as

$$\begin{aligned}&(i) T^{-}_{(A_1\boxtimes A_2)}(kl) = T^{-}_{A_1}(k) \wedge T^{-}_{A_2}(l)\\ &\quad I^{-}_{(A_1\boxtimes A_2)}(kl) = I^{-}_{A_1}(k) \wedge I^{-}_{A_2}(l)\\ &\quad F^{-}_{(A_1\boxtimes A_2)}(kl) = F^{-}_{A_1}(k) \vee F^{-}_{A_2}(l)\end{aligned}$$

$$T^{+}_{(A_1 \boxtimes A_2)}(kl) = T^{+}_{A_1}(k) \wedge T^{+}_{A_2}(l)$$
$$I^{+}_{(A_1 \boxtimes A_2)}(kl) = I^{+}_{A_1}(k) \wedge I^{+}_{A_2}(l)$$
$$F^{+}_{(A_1 \boxtimes A_2)}(kl) = F^{+}_{A_1}(k) \vee F^{+}_{A_2}(l)$$

for all $kl \in R_1 \boxtimes R_2$

$$\text{(ii)}T^{-}_{(B_1 \boxtimes B_2)}(kl_1)(kl_2) = T^{-}_{A_1}(k) \wedge T^{-}_{B_2}(l_1l_2)$$
$$I^{-}_{(B_1 \boxtimes B_2)}(kl_1)(kl_2) = I^{-}_{A_1}(k) \wedge I^{-}_{B_2}(l_1l_2)$$
$$F^{-}_{(B_1 \boxtimes B_2)}(kl_1)(kl_2) = F^{-}_{A_1}(k) \vee F^{-}_{B_2}(l_1l_2)$$
$$T^{+}_{(B_1 \boxtimes B_2)}(kl_1)(kl_2) = T^{+}_{A_1}(k) \wedge T^{+}_{B_2}(l_1l_2)$$
$$I^{+}_{(B_1 \boxtimes B_2)}(kl_1)(kl_2) = I^{+}_{A_1}(k) \wedge I^{+}_{B_2}(l_1l_2)$$
$$F^{+}_{(B_1 \boxtimes B_2)}(kl_1)(kl_2) = F^{+}_{A_1}(k) \vee F^{+}_{B_2}(l_1l_2),$$

for all $k \in R_1, l_1l_2 \in S_2$.

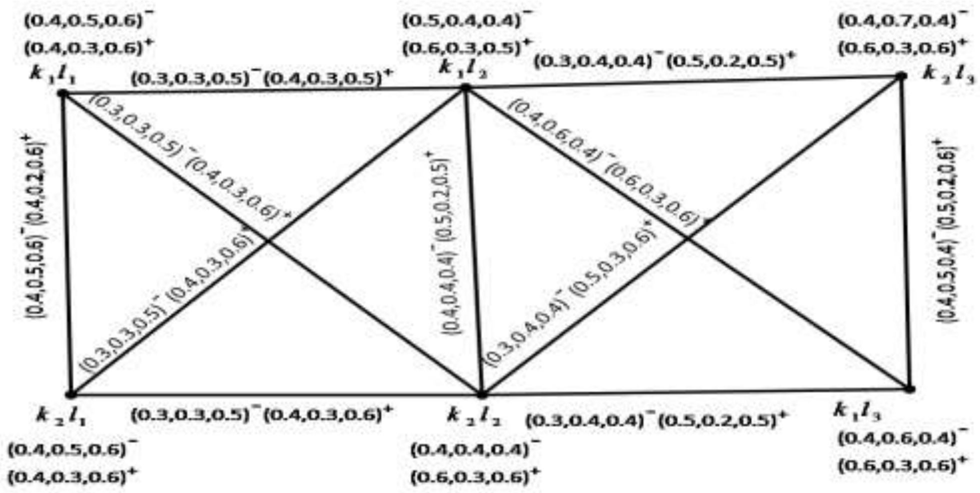
$$\text{(iii)}T^{-}_{B_1 \boxtimes B_2}(k_1l)(k_2l) = T^{-}_{A_2}(l) \wedge T^{-}_{B_2}(k_1k_2)$$
$$I^{-}_{B_1 \boxtimes B_2}(k_1l)(k_2l) = I^{-}_{A_2}(l) \wedge I^{-}_{B_2}(k_1k_2)$$
$$F^{-}_{B_1 \boxtimes B_2}(k_1l)(k_2l) = F^{-}_{A_2}(l) \vee F^{-}_{B_2}(k_1k_2)$$
$$T^{+}_{B_1 \boxtimes B_2}(k_1l)(k_2l) = T^{+}_{A_2}(l) \wedge T^{+}_{B_2}(k_1k_2)$$
$$I^{+}_{B_1 \boxtimes B_2}(k_1l)(k_2l) = I^{+}_{A_2}(l) \wedge I^{+}_{B_2}(k_1k_2)$$
$$F^{+}_{B_1 \boxtimes B_2}(k_1l)(k_2l) = F^{+}_{A_2}(l) \vee F^{+}_{B_2}(k_1k_2),$$

for all $k_1k_2 \in S_1, l \in R_2$.

$$\text{(iv)}T^{-}_{(B_1 \boxtimes B_2)}(k_1l_1)(k_2l_2) = T^{-}_{B_1}(k_1k_2) \wedge T^{-}_{B_2}(l_1l_2)$$
$$I^{-}_{(B_1 \boxtimes B_2)}(k_1l_1)(k_2l_2) = I^{-}_{B_1}(k_1k_2) \wedge I^{-}_{B_2}(l_1l_2)$$
$$F^{-}_{(B_1 \boxtimes B_2)}(k_1l_1)(k_2l_2) = F^{-}_{B_1}(k_1k_2) \vee F^{-}_{B_2}(l_1l_2)$$
$$T^{+}_{(B_1 \boxtimes B_2)}(k_1l_1)(k_2l_2) = T^{+}_{B_1}(k_1k_2) \wedge T^{+}_{B_2}(l_1l_2)$$
$$I^{+}_{(B_1 \boxtimes B_2)}(k_1l_1)(k_2l_2) = I^{+}_{B_1}(k_1k_2) \wedge I^{+}_{B_2}(l_1l_2)$$
$$F^{+}_{(B_1 \boxtimes B_2)}(k_1l_1)(k_2l_2) = F^{+}_{B_1}(k_1k_2) \vee F^{N}_{B_2}(l_1l_2),$$

for all $k_1k_2 \in S_1, l_1l_2 \in S_2$.

Example 3.11 The strong product of NVG $G_1 = (R_1, S_1)$ and $G_2 = (R_2, S_2)$ shown in Figure 2 is defined as $G_1 \boxtimes G_2 = (S_1 \boxtimes S_2, T_1 \boxtimes T_2)$ and is presented in Figure 6.



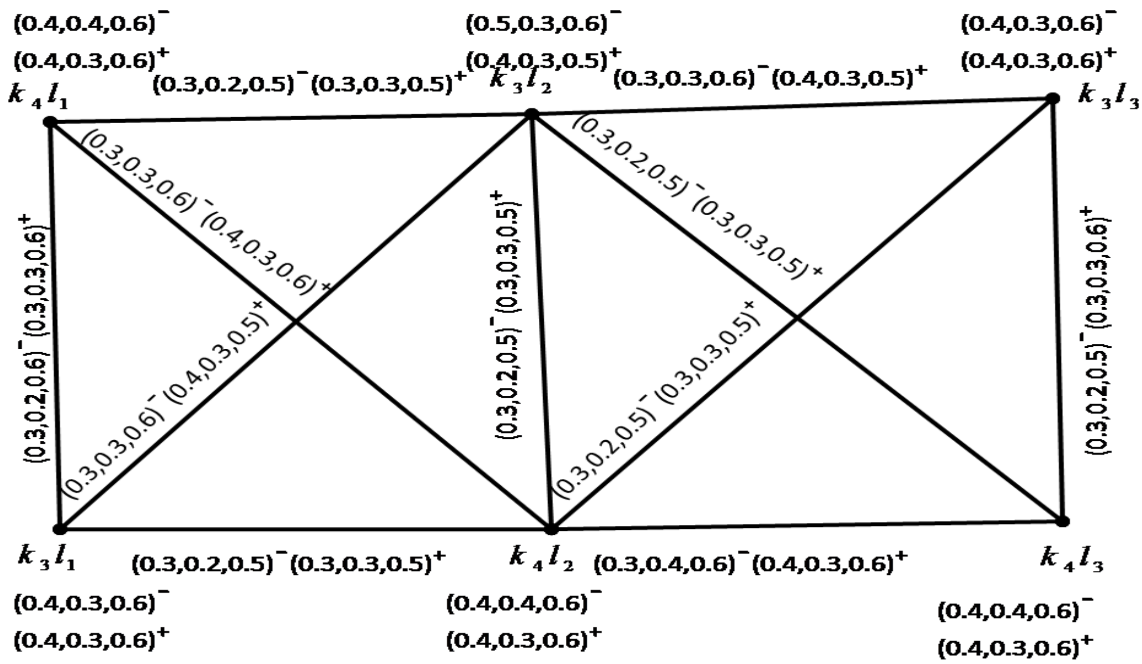


Figure 6: STRONG PRODUCT OF NEUTROSOPHIC VAGUE GRAPH

Theorem 3.12 The strong product $G_1 \boxtimes G_2 = (R_1 \boxtimes R_2, S_1 \boxtimes S_2)$ of two NVG G_1 and G_2 is a NVG of $G_1 \boxtimes G_2$.

Proof. There are three cases:

Case 1: for $k \in R_1, l_1l_2 \in S_2$,

$$\begin{aligned}\hat{T}_{(B_1\boxtimes B_2)}((kl_1)(kl_2)) &= \hat{T}_{A_1}(k) \wedge \hat{T}_{B_2}(l_1l_2)\\ &\leq \hat{T}_{A_1}(k) \wedge [\hat{T}_{A_2}(l_1) \wedge \hat{T}_{A_2}(l_2)]\\ &= [\hat{T}_{A_1}(k) \wedge \hat{T}_{A_2}(l_1)] \wedge [\hat{T}_{A_1}(k) \wedge \hat{T}_{A_2}(l_2)]\\ &= \hat{T}_{(A_1\boxtimes A_2)}(k, l_1) \wedge \hat{T}_{(A_1\boxtimes A_2)}(k, l_2)\end{aligned}$$

$$\begin{aligned}\hat{I}_{(B_1\boxtimes B_2)}((kl_1)(kl_2)) &= \hat{I}_{A_1}(k) \wedge \hat{I}_{B_2}(l_1l_2)\\ &\leq \hat{I}_{A_1}(k) \wedge [\hat{I}_{A_2}(l_1) \wedge \hat{I}_{A_2}(l_2)]\\ &= [\hat{I}_{A_1}(k) \wedge \hat{I}_{A_2}(l_1)] \wedge [\hat{I}_{A_1}(k) \wedge \hat{I}_{A_2}(l_2)]\\ &= \hat{I}_{(A_1\boxtimes A_2)}(k, l_1) \wedge \hat{I}_{(A_1\boxtimes A_2)}(k, l_2)\end{aligned}$$

$$\begin{aligned}\hat{F}_{(B_1\boxtimes B_2)}((kl_1)(kl_2)) &= \hat{F}_{A_1}(k) \vee \hat{F}_{B_2}(l_1l_2)\\ &\leq \hat{F}_{A_1}(k) \vee [\hat{F}_{A_2}(l_1) \vee \hat{F}_{A_2}(l_2)]\\ &= [\hat{F}_{A_1}(k) \vee \hat{F}_{A_2}(l_1)] \vee [\hat{F}_{A_1}(k) \vee \hat{F}_{A_2}(l_2)]\\ &= \hat{F}_{(A_1\boxtimes A_2)}(k, l_1) \vee \hat{F}_{(A_1\boxtimes A_2)}(k, l_2),\end{aligned}$$

for all $kl_1, kl_2 \in R_1 \boxtimes R_2$.

Case 2: for $k \in R_2, l_1l_2 \in S_1$,

$$\hat{T}_{(B_1\boxtimes B_2)}((l_1k)(l_2k)) = \hat{T}_{A_2}(k) \wedge \hat{T}_{B_1}(l_1l_2)$$

$$\leq \hat{T}_{A_2}(k) \wedge [\hat{T}_{A_1}(l_1) \wedge \hat{T}_{A_1}(l_2)]$$
$$= [\hat{T}_{A_2}(k) \wedge \hat{T}_{A_1}(l_1)] \wedge [\hat{T}_{A_2}(k) \wedge \hat{T}_{A_1}(l_2)]$$
$$= \hat{T}_{(A_1 \boxtimes A_2)}(l_1, k) \wedge \hat{T}_{(A_1 \boxtimes A_2)}(l_2, k)$$

$$\hat{I}_{(B_1 \boxtimes B_2)}((l_1k)(l_2k)) = \hat{I}_{A_2}(k) \wedge \hat{I}_{B_1}(l_1l_2)$$
$$\leq \hat{I}_{A_2}(k) \wedge [\hat{I}_{A_1}(l_1) \wedge \hat{I}_{A_1}(l_2)]$$
$$= [\hat{I}_{A_2}(k) \wedge \hat{I}_{A_1}(l_1)] \wedge [\hat{I}_{A_2}(k) \wedge \hat{I}_{A_1}(l_2)]$$
$$= \hat{I}_{(A_1 \boxtimes A_2)}(l_1, k) \wedge \hat{I}_{(A_1 \boxtimes A_2)}(l_2, k)$$

$$\hat{F}_{(B_1 \boxtimes B_2)}((l_1k)(l_2k)) = \hat{F}_{A_2}(k) \vee \hat{F}_{B_1}(l_1l_2)$$
$$\leq \hat{F}_{A_2}(k) \vee [\hat{F}_{A_1}(l_1) \vee \hat{F}_{A_1}(l_2)]$$
$$= [\hat{F}_{A_2}(k) \vee \hat{F}_{A_1}(l_1)] \vee [\hat{F}_{A_2}(k) \vee \hat{F}_{A_1}(l_2)]$$
$$= \hat{F}_{(A_1 \boxtimes A_2)}(l_1, k) \vee \hat{F}_{(A_1 \boxtimes A_2)}(l_2, k)$$

for all $l_1k, l_2k \in R_1 \boxtimes R_2$.

Case 3: for $k_1, k_2 \in S_1, l_1l_2 \in S_2$

$$\hat{T}_{(B_1 \boxtimes B_2)}((k_1l_1)(k_2l_2)) = \hat{T}_{B_1}(k_1k_2) \wedge \hat{T}_{B_2}(l_1l_2)$$
$$\leq [\hat{T}_{A_1}(k_1) \wedge \hat{T}_{A_1}(k_2)] \wedge [\hat{T}_{A_2}(l_1) \wedge \hat{T}_{A_2}(l_2)]$$
$$= [\hat{T}_{A_1}(k_1) \wedge \hat{T}_{A_2}(l_1)] \wedge [\hat{T}_{A_1}(k_2) \wedge \hat{T}_{A_2}(l_2)]$$
$$= \hat{T}_{(A_1 \boxtimes A_2)}(k_1l_1) \wedge \hat{T}_{(A_1 \boxtimes A_2)}(k_2, l_2)$$

$$\hat{I}_{(B_1 \boxtimes B_2)}((k_1l_1)(k_2l_2)) = \hat{I}_{B_1}(k_1k_2) \wedge \hat{I}_{B_2}(l_1l_2)$$
$$\leq [\hat{I}_{A_1}(k_1) \wedge \hat{I}_{A_1}(k_2)] \wedge [\hat{I}_{A_2}(l_1) \wedge \hat{I}_{A_2}(l_2)]$$
$$= [\hat{I}_{A_1}(k_1) \wedge \hat{I}_{A_2}(l_1)] \wedge [\hat{I}_{A_1}(k_2) \wedge \hat{I}_{A_2}(l_2)]$$
$$= \hat{I}_{(A_1 \boxtimes A_2)}(k_1l_1) \wedge \hat{I}_{(A_1 \boxtimes A_2)}(k_2, l_2)$$

$$\hat{F}_{(B_1 \boxtimes B_2)}((k_1l_1)(k_2l_2)) = \hat{F}_{B_1}(k_1k_2) \vee \hat{F}_{B_2}(l_1l_2)$$
$$\leq [\hat{F}_{A_1}(k_1) \vee \hat{F}_{A_1}(k_2)] \vee [\hat{F}_{A_2}(l_1) \vee \hat{F}_{A_2}(l_2)]$$
$$= [\hat{F}_{A_1}(k_1) \vee \hat{F}_{A_2}(l_1)] \vee [\hat{F}_{A_1}(k_2) \vee \hat{F}_{A_2}(l_2)]$$
$$= \hat{F}_{(A_1 \boxtimes A_2)}(k_1l_1) \vee \hat{F}_{(A_1 \boxtimes A_2)}(k_2, l_2),$$

for all $l_1k_1, l_2k_1 \in R_1 \boxtimes R_2$.

Definition 3.13 The composition of two NVG G_1 and G_2 is denoted by the pair $G_1 \circ G_2 = (R_1 \boxtimes R_2, S_1 \circ S_2)$ and defined as

$$\text{(i)}\, T^-_{(A_1 \circ A_2)}(kl) = T^-_{A_1}(k) \wedge T^-_{A_2}(l)$$
$$I^-_{(A_1 \circ A_2)}(kl) = I^-_{A_1}(k) \wedge I^-_{A_2}(l)$$
$$F^-_{(A_1 \circ A_2)}(kl) = F^-_{A_1}(k) \vee F^-_{A_2}(l)$$
$$T^+_{(A_1 \circ A_2)}(kl) = T^+_{A_1}(k) \wedge T^+_{A_2}(l)$$
$$I^+_{(A_1 \circ A_2)}(kl) = I^+_{A_1}(k) \wedge I^+_{A_2}(l)$$
$$F^+_{(A_1 \circ A_2)}(kl) = F^+_{A_1}(k) \vee F^+_{A_2}(l)$$

for all $kl \in R_1 \circ R_2$.

$$\text{(ii)}\, T^-_{(B_1 \circ B_2)}(kl_1)(kl_2) = T^-_{A_1}(k) \wedge T^-_{B_2}(l_1l_2)$$

$$I^-_{(B_1\circ B_2)}(kl_1)(kl_2) = I^-_{A_1}(k) \wedge I^-_{B_2}(l_1l_2)$$
$$F^-_{(B_1\circ B_2)}(kl_1)(kl_2) = F^-_{A_1}(k) \vee F^-_{B_2}(l_1l_2)$$
$$T^+_{(B_1\circ B_2)}(kl_1)(kl_2) = T^+_{A_1}(k) \wedge T^+_{B_2}(l_1l_2)$$
$$I^+_{(B_1\circ B_2)}(kl_1)(kl_2) = I^+_{A_1}(k) \wedge I^+_{B_2}(l_1l_2)$$
$$F^+_{(B_1\circ B_2)}(kl_1)(kl_2) = F^+_{A_1}(k) \vee F^+_{B_2}(l_1l_2),$$

for all $k \in R_1, l_1l_2 \in S_2$.

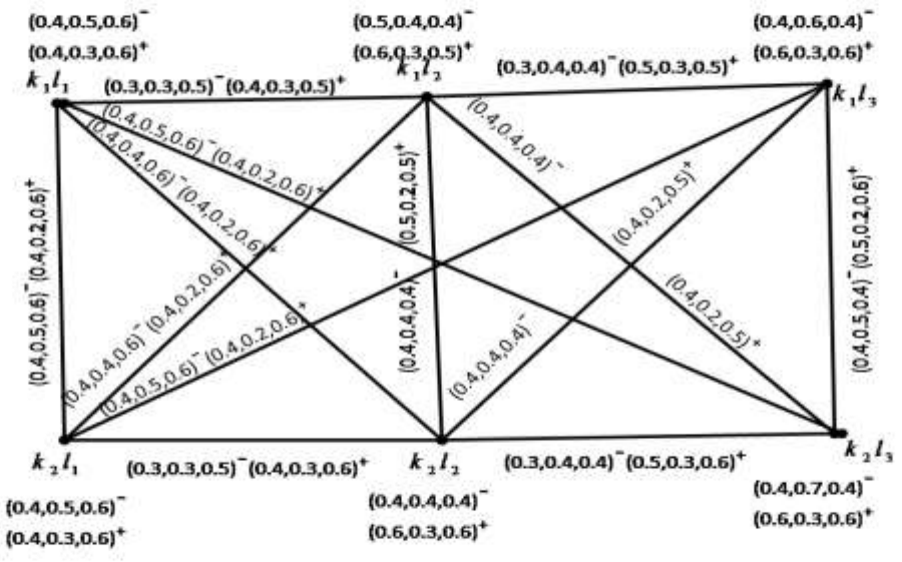
$$\text{(iii)}T^-_{B_1\circ B_2}(k_1l)(k_2l) = T^-_{A_2}(l) \wedge T^-_{B_2}(k_1k_2)$$
$$I^-_{B_1\circ B_2}(k_1,l)(k_2,l) = I^-_{A_2}(l) \wedge I^-_{B_2}(k_1k_2)$$
$$F^-_{B_1\circ B_2}(k_1,l)(k_2,l) = F^-_{A_2}(l) \vee F^-_{B_2}(k_1k_2)$$
$$T^+_{B_1\circ B_2}(k_1,l)(k_2,l) = T^+_{A_2}(l) \wedge T^+_{B_2}(k_1k_2)$$
$$I^+_{B_1\circ B_2}(k_1,l)(k_2,l) = I^+_{A_2}(l) \wedge I^+_{B_2}(k_1k_2)$$
$$F^+_{B_1\circ B_2}(k_1,l)(k_2,l) = F^+_{A_2}(l) \vee F^+_{B_2}(k_1k_2),$$

for all $k_1k_2 \in S_1, l \in R_2$.

$$\text{(iv)}T^-_{(B_1\circ B_2)}(k_1l_1)(k_2l_2) = T^-_{B_1}(k_1k_2) \wedge T^-_{A_2}(l_1) \wedge T^-_{A_2}(l_2)$$
$$I^-_{(B_1\circ B_2)}(k_1l_1)(k_2l_2) = I^-_{B_1}(k_1k_2) \wedge I^-_{A_2}(l_1) \wedge I^-_{A_2}(l_2)$$
$$F^-_{(B_1\circ B_2)}(k_1l_1)(k_2l_2) = F^-_{B_1}(k_1k_2) \vee F^-_{A_2}(l_1) \vee F^-_{A_2}(l_2)$$
$$T^+_{(B_1\circ B_2)}(k_1l_1)(k_2l_2) = T^-_{B_1}(k_1k_2) \wedge T^+_{A_2}(l_1) \wedge T^+_{A_2}(l_2)$$
$$I^+_{(B_1\circ B_2)}(k_1l_1)(k_2l_2) = I^+_{B_1}(k_1k_2) \wedge I^+_{A_2}(l_1) \wedge I^+_{A_2}(l_2)$$
$$F^+_{(B_1\circ B_2)}(k_1l_1)(k_2l_2) = F^+_{B_1}(k_1k_2) \vee F^+_{A_2}(l_1) \vee F^+_{A_2}(l_2),$$

for all $k_1k_2 \in S_1, l_1l_2 \in S_2$.

Example 3.14 The composition of NVG $G_1 = (R_1, S_1)$ and $G_2 = (R_2, S_2)$ shown in Figure 2 is defined as $G_1 \circ G_2 = (R_1 \circ R_2, S_1 \circ S_2)$ and is presented in Figure 7.



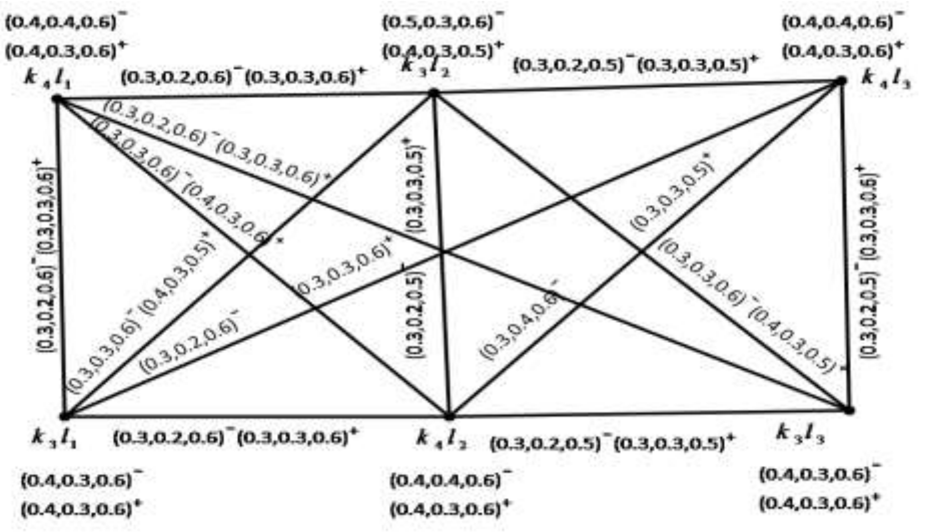


Figure 7: COMPOSITION OF NEUTROSOPHIC VAGUE GRAPH

Theorem 3.15 Composition $G_1 \circ G_2 = (R_1 \circ R_2, S_1 \circ S_2)$ of two NVG G_1 and G_2 is the NVG of $G_1 \circ G_2$.

Proof. We divide the proof into three cases:

Case:1 For $k \in R_1, l_1l_2 \in S_2$,

$$\begin{aligned}\widehat{T}_{(B_1\circ B_2)}\big((kl_1)(kl_2)\big) &= \widehat{T}_{A_1}(k) \wedge \widehat{T}_{B_2}(l_1l_2)\\ &\le \widehat{T}_{A_1}(k) \wedge [\widehat{T}_{A_2}(l_1) \wedge \widehat{T}_{A_2}(l_2)]\\ &= [\widehat{T}_{A_1}(k) \wedge \widehat{T}_{A_2}(l_1)] \wedge [\widehat{T}_{A_1}(k) \wedge \widehat{T}_{A_2}(l_2)]\\ &= \widehat{T}_{(A_1\circ A_2)}(k, l_1) \wedge \widehat{T}_{(A_1\circ A_2)}(k, l_2)\end{aligned}$$

$$\begin{aligned}\hat{I}_{(B_1\circ B_2)}((kl_1)(kl_2)) &= \hat{I}_{A_1}(k) \wedge \hat{I}_{B_2}(l_1l_2)\\ &\le \hat{I}_{A_1}(k) \wedge [\hat{I}_{A_2}(l_1) \wedge \hat{I}_{A_2}(l_2)]\\ &= [\hat{I}_{A_1}(k) \wedge \hat{I}_{A_2}(l_1)] \wedge [\hat{I}_{A_1}(k) \wedge \hat{I}_{A_2}(l_2)]\\ &= \hat{I}_{(A_1\circ A_2)}(k, l_1) \wedge \hat{I}_{(A_1\circ A_2)}(k, l_2)\end{aligned}$$

$$\begin{aligned}\widehat{F}_{(B_1\circ B_2)}((kl_1)(kl_2)) &= \widehat{F}_{A_1}(k) \vee \widehat{F}_{B_2}(l_1l_2)\\ &\le \widehat{F}_{A_1}(k) \vee [\widehat{F}_{A_2}(l_1) \vee \widehat{F}_{A_2}(l_2)]\\ &= [\widehat{F}_{A_1}(k) \vee \widehat{F}_{A_2}(l_1)] \vee [\widehat{F}_{A_1}(k) \vee \widehat{F}_{A_2}(l_2)]\\ &= \widehat{F}_{(A_1\circ A_2)}(k, l_1) \vee \widehat{F}_{(A_1\circ A_2)}(k, l_2)\end{aligned}$$

for all $kl_1, kl_2 \in R_1 \circ R_2$.

Case 2: for $k \in R_2, l_1l_2 \in S_1$,

$$\begin{aligned}\widehat{T}_{(B_1\circ B_2)}((l_1k)(l_2k)) &= \widehat{T}_{A_2}(k) \wedge \widehat{T}_{B_1}(l_1l_2)\\ &\le \widehat{T}_{A_2}(k) \wedge [\widehat{T}_{A_1}(l_1) \wedge \widehat{T}_{A_1}(l_2)]\\ &= [\widehat{T}_{A_2}(k) \wedge \widehat{T}_{A_1}(l_1)] \wedge [\widehat{T}_{A_2}(k) \wedge \widehat{T}_{A_1}(l_2)]\\ &= \widehat{T}_{(A_1\circ A_2)}(l_1, k) \wedge \widehat{T}_{(A_1\circ A_2)}(l_2, k)\end{aligned}$$

$$\begin{aligned}
\hat{I}_{(B_1\circ B_2)}((l_1k)(l_2k)) &= \hat{I}_{A_2}(k) \wedge \hat{I}_{B_1}(l_1l_2) \\
&\leq \hat{I}_{A_2}(k) \wedge [\hat{I}_{A_1}(l_1) \wedge \hat{I}_{A_1}(l_2)] \\
&= [\hat{I}_{A_2}(k) \wedge \hat{I}_{A_1}(l_1)] \wedge [\hat{I}_{A_2}(k) \wedge \hat{I}_{A_1}(l_2)] \\
&= \hat{I}_{(A_1\circ A_2)}(l_1, k) \wedge \hat{I}_{(A_1\circ A_2)}(l_2, k)
\end{aligned}$$

$$\begin{aligned}
\hat{F}_{(B_1\circ B_2)}((l_1k)(l_2k)) &= \hat{F}_{A_2}(k) \vee \hat{F}_{B_1}(l_1l_2) \\
&\leq \hat{F}_{A_2}(k) \vee [\hat{F}_{A_1}(l_1) \vee \hat{F}_{A_1}(l_2)] \\
&= [\hat{F}_{A_2}(k) \vee \hat{F}_{A_1}(l_1)] \vee [\hat{F}_{A_2}(k) \vee \hat{F}_{A_1}(l_2)] \\
&= \hat{F}_{(A_1\circ A_2)}(l_1, k) \vee \hat{F}_{(A_1\circ A_2)}(l_2, k), \quad \text{for all } l_1k, l_2k \in R_1 \circ R_2.
\end{aligned}$$

Case 3: For $k_1k_2 \in S_1, l_1, l_2 \in R_2$ such that $l_1 \neq l_2$,

$$\begin{aligned}
\hat{T}_{(B_1\circ B_2)}((k_1l_1)(k_2l_2)) &= \hat{T}_{B_1}(k_1, k_2) \wedge \hat{T}_{A_2}(l_1) \wedge \hat{T}_{A_2}(l_2) \\
&\leq [\hat{T}_{A_1}(k_1) \wedge \hat{T}_{A_1}(k_2)] \wedge [\hat{T}_{A_2}(l_1) \wedge \hat{T}_{A_2}(l_2)] \\
&= [\hat{T}_{A_1}(k_1) \wedge \hat{T}_{A_2}(l_1)] \wedge [\hat{T}_{A_1}(k_2) \wedge \hat{T}_{A_2}(l_2)] \\
&= \hat{T}_{(A_1\circ A_2)}(k_1l_1) \wedge \hat{T}_{(A_1\circ A_2)}(k_2l_2)
\end{aligned}$$

$$\begin{aligned}
\hat{I}_{(B_1\circ B_2)}((k_1l_1)(k_2l_2)) &= \hat{I}_{B_1}(k_1, k_2) \wedge \hat{I}_{A_2}(l_1) \wedge \hat{I}_{A_2}(l_2) \\
&\leq [\hat{I}_{A_1}(k_1) \wedge \hat{I}_{A_1}(k_2)] \wedge [\hat{I}_{A_2}(l_1) \wedge \hat{I}_{A_2}(l_2)] \\
&= [\hat{I}_{A_1}(k_1) \wedge \hat{I}_{A_2}(l_1)] \wedge [\hat{I}_{A_1}(k_2) \wedge \hat{I}_{A_2}(l_2)] \\
&= \hat{I}_{(A_1\circ A_2)}(k_1l_1) \wedge \hat{I}_{(A_1\circ A_2)}(k_2l_2)
\end{aligned}$$

$$\begin{aligned}
\hat{F}_{(B_1\circ B_2)}((k_1l_1)(k_2l_2)) &= \hat{F}_{B_1}(k_1, k_2) \vee \hat{F}_{A_2}(l_1) \vee \hat{F}_{A_2}(l_2) \\
&\leq [\hat{F}_{A_1}(k_1) \vee \hat{F}_{A_1}(k_2)] \vee [\hat{F}_{A_2}(l_1) \vee \hat{F}_{A_2}(l_2)] \\
&= [\hat{F}_{A_1}(k_1) \vee \hat{F}_{A_2}(l_1)] \vee [\hat{F}_{A_1}(k_2) \vee \hat{F}_{A_2}(l_2)] \\
&= \hat{F}_{(A_1\circ A_2)}(k_1l_1) \vee \hat{F}_{(A_1\circ A_2)}(k_2l_2), \quad \text{for all } k_1l_1, k_2l_2 \in R_1 \circ R_2.
\end{aligned}$$

Conclusion

Graph theory is an extremely useful tool in studying and modeling several applications in computer science, engineering, genetics, decision-making, economics, etc. An extension of intuitionistic fuzzy graph is regarded as a single-valued neutrosophic graph which is very useful to formulate the appropriate real life situation. In this research article, the operations on neutrosophic vague graphs have been established. Moreover, Cartesian product, lexicographic product, cross product, strong product and composition of neutrosophic vague graph have been investigated and the given concepts are demonstrated through examples. Furthermore, in future, we are able to investigate the domination number and isomorphic properties of the NVGs.

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Received: Apr 10, 2020. Accepted: July 2 2020

Neutrosophic Sets and Systems, Vol. 35, 2020
University of New Mexico

Partner selection in Virtual enterprises using the Interval Neutrosophic fuzzy approach

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Abstract: With the rapid development of the Internet, information technology, and globalization of the economy, Some small and medium-sized companies know that they cannot compete with their limited capacity alone. As a result, they are beginning to seek collaboration and a collective approach to meet the dynamic needs of customers and increase their power for competition in the market. Virtual enterprise is a temporary platform for working with different companies that share their core tasks to meet customer’s demand. Partner selection is a major issue in the formation of a virtual organization. This is especially difficult due to the uncertainties regarding information, market dynamics, customer expectations, and rapidly changing technology, with highly random decision making. As a generalization of fuzzy sets and intuitionistic fuzzy sets, Neutrosophic sets are created to show the uncertain, and inconsistent information available in the real world. The main purpose of this paper is to identify and select partners in the formation of Virtual Enterprises under uncertainty and contradictory factors using the extended VIKOR group decision making technique using the Interval Neutrosophic fuzzy approach. For this purpose, after identifying the factors affecting partner selection, the factors are weighted using the Maximizing deviation method and the partners are ranked using this method. Finally, a sensitivity analysis for assessing the validity of the method is also presented. The results show that the Willingness to share information criterion is the most important partner selection criterion in this enterprise.

Keywords: Virtual Enterprise, Partner Selection, Interval Neutrosophic Numbers, Group Decision Making, Uncertainty, VIKOR.

1. Introduction

With the globalization of the market and the economy, the rapid development of the use of the Internet and information technologies, faster product updates and market needs have become more uncertain and personalized [1]. Globally, companies are increasingly in need of the competence of other companies to meet growing customers’ demands [2]. Therefore, it is difficult to adapt the traditional business model to the new market environment. At the same time, companies need to maintain lower costs and shorter delivery cycles, that this challenges old organizational form [3]. In fact, a enterprise cannot meet the rapid market changes by integrating internal resources and competencies alone [1]. As such, many companies are attracting partners to absorb opportunities in emerging markets to share costs, reduce development time, and utilize the effectiveness of design, production, and marketing skills within and outside companies [4]. With the rapid growth of competition in the global industry, a dynamic virtual enterprise (VE) approach will be needed to meet market needs for quality, responsiveness and customer satisfaction [5]. VE is created to address a specific opportunity in a fast-paced and simultaneous market, creating a collaborative work environment for managing and using a set of resources provided by companies. Business partners are all connected to share their skills, and take advantage of the rapidly changing opportunities in a dynamic network [6]. In fact, through the VE framework, each VE partner brings its expertise for implementing the original project, [7] and each partner focuses on its own core competence. This increases the ability of the organization to meet the unpredictable demands of customers [8]. Therefore, by maintaining the agility of the entire structure, this collaboration will deliver high quality products based on customer's specific needs [7]. In this alliance, the links are made easier by computer technology, [4] and eventually when the market opportunity is over, the VE will be dissolved [5].

Compared to the traditional organizational form, VE is considered a low cost, high responsive and adaptive organization and members of this alliance can share cost, risk, technology, and key competition with each other, through which members can gain win-win policy. However, many issues arise throughout the life cycle of a VE, including how we can find the right partners, which is a key issue for the core enterprise in the VE development phase, and this issue has been considered by many researchers [3]. As the VE environment continues to grow in size and complexity, the importance of managing such complexities increases [5]. In a virtual enterprise (VE), choosing a partner is very important because of the short life of these organizations (temporary alliances) and the absence of formal mechanisms (contracts) to ensure participants' responsibility [9].

The complexity of the partner selection process is reinforced by the fact that there are several centralized internal and external organizational factors that have both tangible and intangible characteristics and should be incorporated into the decision analysis for this selection process [8]. Like all decision-making issues, partner selection involves tangible and intangible paradox specifications under conflicting or incomplete information [10]. Therefore, it is important to select the most appropriate companies while there may be dozens of volunteer companies involved in the project [7]. The multitude of factors that are considered when choosing partners for a business opportunity such as cost, quality, trust and delivery time cannot be expressed by the same size or scale [11]. In practice, partner selection should consider higher levels of uncertainty and risk as a way of addressing uncontrolled factors: such as price or demand fluctuations, lack of enough knowledge sharing among VE members, resource constraints, and incomplete information about candidates and their performance [12].

The multi-attribute group decision making (MAGDM) approach is to provide a comprehensive solution by evaluating and ranking alternatives based on contrasting features based on decision makers' (DM) preferences [13]. Decision-making is often about the optimal choice between a set of options, considering the impact of many criteria. In the past five decades, Multi criteria decision making method (MCDM) has become one of the most important and key ways of solving complex decision problems, despite of various criteria and options. In MCDM problems, the characteristics of dependence, opposition, and interaction are ambiguous between decision criteria, which obscures the degree of membership [14]. In fact, it is difficult for DMs due to the uncertainty of the information and the many constraints such as time pressure, lack of awareness, and problems of data extraction and so on to express their preferences numerically in many complex realities [15]. The fuzzy set theories or the intuitionistic fuzzy theories are used to overcome this obstacle. However, these sets are not always suitable [14]. The fuzzy set (FS) has only one member and cannot display complex information and the intuitionistic fuzzy set, which includes membership and non-membership degree, can only manage incomplete information, and cannot deal with inconsistent information, and degree of indeterminate membership at IFS has always been ignored [16]. Smarandache recommended Neutrosophic set (NS) by adding an indefinite membership function based on IFS. In NS, the degree of accuracy, lack of reliability, and the degree of inaccuracy are completely independent [17].

The Neutrosophic set is becoming a scientific tool and has attracted the attention of many scientists and academic researchers to develop and improve the Neutrosophic method [14]. Abdel-Basset et al. (2020) considered inventory location problem, They applied the best-worst method (BWM) to find the weight of these criteria and propose a combination of plithogenic aggregation operations, and the BWM to solve MCDM problems [18]. Veerappan et al (2020) considered Multi-Aspect Decision-Making Process in Equity Investment Using Neutrosophic Soft Matrices [19]. Abdel-Basset and Mohamed (2020) proposed a combination of plithogenic multi-criteria decision-making approach based on the TOPSIS and Criteria Importance Through Inter-criteria Correlation (CRITIC) methods for sustainable supply chain risk management [20]. Abdel-Basset et al. (2020) provided a new hybrid neutrosophic MCDM framework that employs a collection of neutrosophic ANP, and TOPSIS under bipolar neutrosophic numbers for professional selection [21]. Edalatpanah and Smarandache proposed an input-oriented DEA model with simplified neutrosophic numbers and present a new strategy to solve it [22]. Abdel-Basset et al. (2020) applied a combination of quality function deployment (QFD) with plithogenic aggregation operations for Selecting Supply Chain Sustainability Metrics [23].

In this paper, we combine the Interval Neutrosophic Numbers (NS) set and the VIKOR method to select a partner in a virtual enterprise. One of the best ways to solve decision problems with inconsistent and unbelievable criteria is the VIKOR approach. VIKOR can be an effective tool for decision making when the decision maker is unable to identify and express the superiority of a problem at the time it is started and designed [24]. For this purpose, the criteria for selecting the partner were first identified by the experts and then their opinions about each of the candidate partners were collected according to the effective factors. Finally, partner rating and selection are performed using the VIKOR method, which is based on the concurrent planning of multivariate decision problems and evaluates issues with inappropriate, and incompatible criteria, in the Interval Neutrosophic environment. The innovation of this paper is that the Interval Neutrosophic set is used to express the evaluation of information, and partner selection in virtual enterprise will be implemented under an Interval Neutrosophic environment. Since the weight of the criteria varies with the mental state, and no specific information is available, in this paper the weight of the criteria is determined using the Maximizing deviation method under the Interval Neutrosophic environment.

2. Research literature and Related studies

The widespread development of Internet technologies in the late twentieth century has led to the dramatic formation and enhancement of the virtual environment in the employment sector, and virtual enterprises, virtual sectors and a series of virtual businesses have expanded. Information on so-called virtual companies was first provided in the early 1990s by Steven L. Goldman, Rocer N. Nagel and David B. Greenberger, and William H. Davidow and Michael S. Malone. The innovative technology market enables companies to form temporary partnerships, and the creation of such links through the Internet leads to the formation of Virtual Enterprises [25]. Member companies in such a virtual enterprise, rather than being independent companies and focusing on their own business goals, work together to share their information about their capabilities, programs and cost structures, to improve their technical, logistical, financial and other activities in order to compete [4]. The short-term goal of a VE is primarily to increase productivity, reduce inventory and total cycle time. The long-term goal is to increase customer satisfaction, market share, and profit levels for all members. Failure to cooperate may result in a delay in delivery, poor customer service, and inventory creation, and so on [26]. The success of this mission depends on all the organizations that work together as a unit. Because everyone gives its own core strengths or competencies to the virtual enterprise. In other words, the competitive advantage gained by a virtual enterprise depends on each other and their ability to integrate with each other. The key factor in forming a virtual enterprise is choosing agile, competent and consistent partners [27]. The life cycle of a VE consists of four stages: creation, operation, evolution, and dissolution [28]. In the creation phase, when an organization wins a large contract project and is unable to complete it with its proper capacity, it seeks out potential partners and negotiates with them through its information infrastructures and VE will be created. At the operation stage, after signing contracts between the partners, VE manages the process of production or execution of the project. At the development stage, the VE is configured to meet the resource requirements when the project is changed, and at the dissolution stage, when the project is completed, the VE will be eventually dissolved [29]. Obviously, the first step, namely the selection of partners, is crucial to the success of the VE [30]. The main difference between a regular supplier selection issue and a partner selection issue in a VE is the expected duration of the relationship. In fact, companies in a VE rarely have the time to implement, and develop all the features needed for successful relationships. They therefore emphasize on the fact that partner selection is definitely an important step in VE development [12]. Determining the right criteria and evaluating all of the influencing factors in partner selection is difficult. There are many factors that must be considered during decision making. Some are qualitative, such as friendship, credibility, and reliability, and others are quantitative, such as cost, and delivery time. It is very costly and time-consuming to evaluate each partner and identify the most desirable ones [26].

There is an extensive literature on partner selection in VE, each offering a new approach for evaluating and selecting the most appropriate partners among the set of organizations. Sha and Che (2004) develop a partner selection and production distribution planning problem with a new partner selection Model based on Analytical Hierarchy Process (AHP), multi-attribute utility theory (MAUT), and integer programming (IP), for Virtual integration (VI) with multiple criteria. The AHP and MAUT methods are used to evaluate and weight each partner's candidate, and the IP model applies this weigh to find the best potential partners and provide the right distribution plan for the selected partners [31]. Sarkis et al. (2007) present a practical paradigm that can be used by organizations to help form agile virtual companies using ANP method [8]. Ye and Li (2009) proposed two group decision models for spatial decision making to solve the problem of partner selection under incomplete information. The first model is a technique for preferring the order with similarity with ideal Solution (TOPSIS) for group decision making based on degree of deviation. The second approach is TOPSIS group decision-making based on risk factor [28]. Crispim and Sousa (2009) propose an exploratory process to help the decision maker to acquire knowledge about the network in order to identify the criteria and companies that provide the needs of a project very well. This process involves a multi-objective meta-heuristic search algorithm designed to find a good approximation of the PARETO front and a fuzzy TOPSIS algorithm to rank the configuration of VE options. Preliminary computational results clearly showed the potential of this approach for practical applications [9]. Ye (2010) investigated the problem of partner selection in partial and uncertain information environments and used the extended TOPSIS technique for group decision making with intuitive fuzzy numbers with interval values for problem solving [32].

Liu et al (2016) proposed a partner selection method based on distance multipliers preferences with approximate compatibility. In this paper, using a (n - 1) pairwise comparison, a new partner selection method is proposed, which introduces a new concept of approximate compatibility for multidimensional preferential relationships [27]. Nikghadam et al. (2016) designed a customer-based algorithm to select a partner in a virtual enterprise. In this study, customers were classified into three categories: passive, standard and assertive. Three different approaches; fuzzy logic-FAHP TOPSIS and ideal programming were used for each type of customer, respectively. The results confirm that adopting this algorithm not only helps VE to select the most appropriate partners based on customer preferences, but also adapts its model to each customer's attitude. As a result, the overall flexibility of the system significantly improves [7]. Polyantchikov et al. (2017) performed virtual enterprise formation in the context of a sustainable partner network using methodologies such as Analytical Hierarchy Process (AHP), fuzzy AHP approach and TOPSIS method [33]. Huang et al. (2018) studied the problem of partner selection for virtual production companies facing an uncertain environment and using the gray system theory studied uncertainty at the start of a project, in the completion time, in shipping time, and also studied the cost. They used the chaotic particle swarm optimization (CPSO) algorithm to solve the problem [30].

Meng et al. (2019) in their paper presented Interval Neutrosophic Preferred Relations and examined its application with numerical examples in virtual partner selection. The algorithm presented in this paper is based on group decision-making based on INPRs which can be applied to address incomplete and inconsistent INPRs [3]. Chen and Goh (2019) sought a cooperative partner selection mechanism from the perspective of dual-factor theory. They proposed a new framework for problem solving and cooperative partner selection. This framework uses the degree of compatibility of the triangular fuzzy soft set (TFSS) to measure the level of participation, and a broad TODIM based on TFSS to measure the degree of influence on the individual level [34]. Ionescu (2020) reviews the most prominent approaches to solving partner selection problems and discuss some of the most documented methods and algorithms for VO creation and reconfiguration [35]. Zhao et al. (2020) studied a multi-objective virtual enterprise partner selection model with relative superiority parameter in fuzzy environment. In this paper, the completion time and delivery time were fuzzily processed [36] . Wan and Dong (2020) applied the group decision making (GDM) problems with interval-valued Atanassov intuitionistic fuzzy preference relations (IV-AIFPRs) and developed a novel method for solving a virtual enterprise partner selection problem [37].

These papers use different methods and techniques to select partners in virtual enterprises. Many of these studies make use of fixed weights of the criteria, and consider a limited set of uncertainties. They do not make sensitivity analysis to examine solutions, and are, in general, very time-consuming or too complex to be understood by the DM. However, in practice, there are multiple uncertainties in the VE partner selection problem and to assign precise weights to criteria becoming more critical when the number of criteria increases and when the VE life cycle is rather short. In this paper, the weight of the criteria is determined using the maximizing deviation method under the Interval Neutrosophic environment. and combine the Interval Neutrosophic Numbers (NS) set and the VIKOR method have considerable potential to this problem. Neutrosophic sets are very powerful and successful in overcoming situations and cases in uncertainty, vagueness, and imprecision. This model is easy to understand and use, and flexible, and tolerant with inconsistent and inaccurate information. Additionally, the procedure proposed in this work overcomes some of the shortcomings of decision-support tools and provides automatic sensitivity analysis on the results.

On the other hand, many factors should be taken into consideration when selecting partners of a VE By studying the research literature, the most important factors influencing partner selection in Virtual Enterprises can be classified according to Table 1. These factors are the most popular and most influential factors in choosing a partner in a virtual enterprise.

Table 1. Criteria for partner selection

Criteria	**Reference**
Cost	[28], [9], [32],[12], [4],[27], [30], [10], [38], [2], [39]
Time	[28], [32], [12], [10], [2]
Trust	[28], [32], [10], [34], [3]
Risk	[28], [32], [12], [9] ,[10], [40]
Quality	[28], [9], [33], [27], [10], [26], [38], [39], [6]
Productivity & Performance history	[9], [33], [7], [26], [2]
Market entrance capability	[9], [12]
Knowledge and managerial experience	[9], [33], [34]
Age of the organisation	[9], [12]
Competency & technical expertise	[9], [33], [3]
Information and communication technology resources	[9], [33]
Price	[12], [33], [7], [26], [6]
Delivery	[12], [33], [7], [30], [26], [39]
Customer service	[12], [7], [27], [26], [38], [2], [6]
Geographical location	[33], [26], [34]
The financial stability	[27], [34], [38], [6]

Willingness to share information	[12], [34]
Tardiness penalty	[4], [27]
Technology capability	[34], [26], [38], [34],
Reputation and position in industry	[3], [26], [38], [33]
IT infrastructure	[38], [26]

3. Methedology

This research is applied in terms of purpose and quantitative in terms of variables. In the partner selection process, decision makers are usually unsure of their preferences [41]. Because information about candidates and their performance is incomplete and unclear. In terms of data collection, selecting and evaluating partners is difficult due to the complex interactions between different entities, and because of their preferences they may be inaccessible based on incomplete or partial information. To address this issue under a multi-criteria perspective, several types of information (numerical, interval, qualitative and binary) are used to facilitate the expression of preferences or the evaluation of stakeholders in decision making [12]. In this paper, Interval Neutrosophic numbers are used to express the preferences of experts. In this regard, First, the effective criteria influencing the choice of partner are selected, and then experts express their opinion about candidates with the competence of linguistic terms according to the effectiveness criteria. After converting the experts' opinions to Interval Neutrosophic numbers, the weight of the criteria is calculated using the maximum deviation method. In the second step, expert opinions on each company integrate using the interval neutrosophic weighted average operator. Finally, rankings of companies perform by using the Vikor fuzzy interval neutrosophic method. The general framework of proposed method presented in Fig 1.

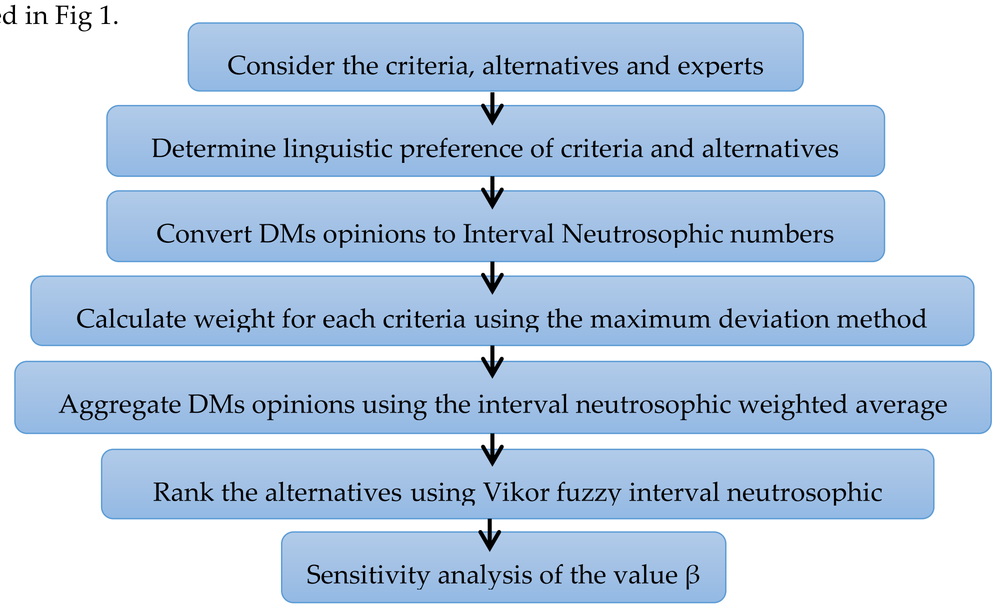

Fig 1. A general framework of proposed method

3.1. Interval Neutrosophic fuzzy set

In the real world, decision information is often incomplete, uncertain, and inconsistent. In order to process this type of information, Smarandache introduced Neutrosophic set (NS) from a philosophical perspective by adding independent indeterminacy-membership, which is an extension of the fuzzy set (FS), the fuzzy set with interval values, the intuitionistic fuzzy set, and so on [42]. Smarandache believed that these types of sets not only had the degree of membership and the degree of non-membership, but also consider the degree of non-determination and lack of compatibility [16]. The new theory of Neutrosophic sets allows to work with the "Knowledge of neural thought". In fact, Neutrosophic sets are generalizations of fuzzy logic and allow to deal with more complex uncertainty models. In "classical" fuzzy sets, each element is defined by a degree of membership, and the available methods are controlled by fuzzy sets [43]. The fuzzy set cannot express neutral state, meaning neither support nor opposition. To overcome this defect, Atanassov introduced the concept of the Intuitionistic Fuzzy Set (IFS). Compared to The fuzzy set, the intuitionistic fuzzy set can simultaneously express three modes of support, opposition, and neutrality. Although the FS and IFS have been developed and publicized, they cannot address the uncertain and inconsistent issues of real decision-making. To solve this problem, Neutrosophic (NS) sets have been suggested [44]. Unlike The intuitionistic fuzzy sets, which depend on the degree of uncertainty on membership and non-membership, by the Neutrosophic logic the value of the indeterminate membership is independent of the degree of truth and falsehood [43]. Neutrosophic logic is flexible and tolerant with inconsistent and inaccurate data. This logic is based on natural language and is made up of specialized knowledge. The concept of the Neutrosophic set provides an alternative approach in the case of inaccuracies in the decisions made by deterministic sets or traditional fuzzy sets, and where the information provided is inadequate for finding it inaccurate [45]. Neutrosophic sets are powerful and successful in overcoming situations and in an inadequate information environment, uncertainty, ambiguity and inaccuracy [14]. A Neutrosophic set A with an A value in X is expressed by 1.

$$\boldsymbol{A} = \{\boldsymbol{x}(\boldsymbol{T_A}(\boldsymbol{x}), \boldsymbol{I_A}(\boldsymbol{x}), \boldsymbol{F_A}(\boldsymbol{x})) | \boldsymbol{x} \in \boldsymbol{X}\} \tag{1}$$

With Neutrosophic set logic, every aspect of the problem is represented by the degree of the truth membership ($T_A(x)$), the degree of the indeterminate membership ($I_A(x)$) and the degree of the false membership ($F_A(x)$) according to 1.

For each x, $T_A(x) + I_A(x) + F_A(x) \in [0,1]$ and the sum of these memberships is less than or equal to three [46]. Thus, Neutrosophic sets provide a means of expressing DM preferences and priorities, and fully determine membership performance in situations where DM comments are subject to the indeterminate membership or lack of information [14].

$$\boldsymbol{0} \leq \boldsymbol{T_A}(\boldsymbol{x}) + \boldsymbol{I_A}(\boldsymbol{x}) + \boldsymbol{F_A}(\boldsymbol{x}) \leq \boldsymbol{3} \tag{2}$$

Sometimes the degree of truth, falsehood, and uncertainty of a particular sentence is not precisely defined in real terms, but is determined by several possible interval values [47]. Thus, the Interval Neutrosophic Set (INS) was introduced by Wang et al (2005). [48]. As a special case of Neutrosophic sets, the Interval Neutrosophic Set (INS) can be used to address uncertain and inconsistent information in decision making [3]. Wang et al showed Interval Neutrosophic (INS) assemblages with distance membership, the degree of non-membership, and degree of hesitant (The indeterminate membership) as follows.

$$\boldsymbol{x} = ([\boldsymbol{T^L}, \boldsymbol{T^u}], [\boldsymbol{I^L}, \boldsymbol{I^u}], [\boldsymbol{F^L}, \boldsymbol{F^u}]) \tag{3}$$

The Interval Neutrosophic set can be simpler to express incomplete, uncertain, and contradictory information [49], and is flexible and practical for dealing with decision problems. Compared to other fuzzy set expansions, INS has the following advantages. (A) Compared to The fuzzy set, INS can simultaneously express positive, negative, and hesitant judgments of DM using membership degree, non-membership degree, and degree of hesitation. (B) Compared with The Intuitionistic fuzzy sets, INS independently express the degree of positive, negative, and uncertain judgments. That DMs have more flexibility to express their uncertain and contradictory information [3].

3.2. Interval Neutrosophic Fuzzy VIKOR Method

VIKOR is an effective decision making method that selects the optimal option with group utility maximization and individual regret minimization. And it is used as one of the applied MCDM techniques to solve a discrete decision problem with disproportionate criteria with different and conflicting units of measurement [50]. This method was proposed by Opricovic (1998) to solve the problem of multi-criteria decision making in an incompatible and inconsistent criteria environment [43]. VIKOR is an efficient tool for finding the compromise solution from a set of conflicting criteria. Where compromise means an agreement made with mutual concessions [51]. That can help decision makers to make a final decision [52]. The VIKOR method is based on the specific property of being close to the ideal solution. One of the features of this method is that the options are evaluated according to all defined criteria (performance matrix) and the stability analysis of the intervals shows the stability of the weight [53]. The effectiveness of this approach becomes more apparent when the decision maker is not able to express his/her preferences and uses agreed solutions to solve the problems. An agreed solution is a justified solution that is close to the ideal solution and that decision makers accept because of the maximum utility of the group [50].

Suppose the rating of options $P_i = \{p_1, p_2, \dots, p_m\}$ is given as f_{ij} with respect to criteria of $C_j = \{c_1, c_2, \dots, c_n\}$. $w_j = \{w_1, w_2, \dots, w_n\}$ is the weight vector of the criteria. The formula for measuring distance on P_i options is based on equation (4).

$$\boldsymbol{L_{\alpha,i}} = \left(\sum_{j=1}^{n} \left(\boldsymbol{w_j} \frac{\boldsymbol{f_j^* - f_{ij}}}{\boldsymbol{f_j^* - f_j^-}} \right)^{\alpha} \right)^{\frac{1}{\alpha}}, \mathbf{1} \le \boldsymbol{\alpha} \le \infty, \qquad \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \dots, \boldsymbol{m} \tag{4}$$

where $f_j^* = max_i f_{ij}$ and $f_j^- = min_i f_{ij}$ are the ideal and anti-ideal points, respectively [53].

Let $W = \{w_1, w_2, \dots, w_n\}$ be the weight of the criteria, $0 \le W_j \le 1$ and $\sum_1^n W_j = 1$. If the set of decision makers be $E = \{E_1, E_2, \dots, E_t\}$ and the weight of decision makers be $\sigma = \{\sigma_1, \sigma_2, \dots, \sigma_t\}$, $0 \le \sigma_k \le 1$ and $\sum_{k=1}^{t} \sigma_k = 1$.

Suppose that $\widetilde{R}_k = (\tilde{r}_{ij}^k)_{m\times n} = \left(\left[T_{ij}^{L(k)}, T_{ij}^{R(k)}\right], \left[I_{ij}^{L(k)}, I_{ij}^{R(k)}\right], \left[F_{ij}^{L(k)}, F_{ij}^{R(k)}\right]\right)_{m\times n}$ is the Matrix of decision of Interval Neutrosophic Numbers, $e_k \in E$.

$$\left[T_{ij}^{L(k)}, T_{ij}^{R(k)}\right], \left[I_{ij}^{L(k)}, I_{ij}^{R(k)}\right], \left[F_{ij}^{L(k)}, F_{ij}^{R(k)}\right] \subseteq [0,1], \tag{5}$$

$$0 \le T_{ij}^{R(k)} + I_{ij}^{R(k)} + F_{ij}^{R(k)} \le 3\,, \tag{6}$$

$$i = 1,2, \dots, m, \qquad j = 1,2, \dots, n, \qquad k = 1,2, \dots, t \tag{7}$$

The steps of the VIKOR method for multi-criteria group decision-making problems of the Interval Neutrosophic set are as follows [13].

Step 1. Convert Evaluation Information to the Interval Neutrosophic Number Set

Step 2. Calculate the weight of the criteria

Since the weight of the benchmarks may be completely unknown, the benchmark weight is calculated using the Maximizing deviation method. According to this view, if the criterion values of all alternatives to a particular attribute are quantitative deviations, quantitative weight can be assigned to this criterion. Otherwise, the criterion that causes the deviation to be greater should be weightier. In particular, if the criterion values of all the different options are equal to a given property, the weight of such a criterion may be zero [49]. The weight of the criteria is thus calculated using the equation (8) [48].

$$w_j = \frac{\sum_{k=1}^{t} \sigma_t \sum_{i=1}^{m} \sum_{s=1}^{m} (\Delta_{upv})}{\sum_{j=1}^{n} \sum_{k=1}^{t} \sigma_t \sum_{i=1}^{m} \sum_{s=1}^{m} (\Delta_{upv})} \tag{8}$$

$$\Delta_{upv} = \left|T_{ij}^{L} - T_{sj}^{L}\right| + \left|I_{ij}^{L} - I_{sj}^{L}\right| + \left|F_{ij}^{L} - F_{sj}^{L}\right| + \left|T_{ij}^{R} - T_{sj}^{R}\right| + \left|I_{ij}^{R} - I_{sj}^{R}\right| + \left|F_{ij}^{R} - F_{sj}^{R}\right| \tag{9}$$

Step 3. Using $\widetilde{\mathrm{R}}_{\mathrm{k}}$ and calculating the interval neutrosophic number weighted averaging (INNWA) operator [47]

$$INNWA_{\sigma}(E_1, E_2, \dots, E_t) = \langle \left[1 - \prod_{i=1}^{t}\left(1 - T_{E_i}^{L}\right)^{\sigma_i}, 1 - \prod_{i=1}^{t}\left(1 - T_{E_i}^{R}\right)^{\sigma_i}\right], \left[\prod_{i=1}^{t}\left(I_{E_i}^{L}\right)^{\sigma_i}, \prod_{i=1}^{t}\left(I_{E_i}^{R}\right)^{\sigma_i}\right], \left[\prod_{i=1}^{t}\left(F_{E_i}^{L}\right)^{\sigma_i}, \prod_{i=1}^{t}\left(F_{E_i}^{R}\right)^{\sigma_i}\right]\rangle \tag{10}$$

Step 4. Define the solution of positive and negative ideals (R^+ and R^-)

$$\tilde{R}^{+} = \left(\left[T_j^{L+}, T_j^{R+}\right], \left[I_j^{L+}, I_j^{R+}\right], \left[F_j^{L+}, F_j^{R+}\right]\right) \tag{11}$$

$$\tilde{R}^{-} = \left(\left[T_j^{L-}, T_j^{R-}\right], \left[I_j^{L-}, I_j^{R-}\right], \left[F_j^{L-}, F_j^{R-}\right]\right) \tag{12}$$

For positive and incremental criteria

$$\left(\left[T_j^{L+}, T_j^{R+}\right], \left[I_j^{L+}, I_j^{R+}\right], \left[F_j^{L+}, F_j^{R+}\right]\right) = \left(\left[max_i T_{ij}^{L}, max_i T_{ij}^{R}\right], \left[min_i I_{ij}^{L}, min_i I_{ij}^{R}\right], \left[min_i F_{ij}^{L}, min_i F_{ij}^{R}\right]\right) \tag{13}$$

$$\left(\left[T_j^{L-}, T_j^{R-}\right], \left[I_j^{L-}, I_j^{R-}\right], \left[F_j^{L-}, F_j^{R-}\right]\right) = \left(\left[min_i T_{ij}^{L}, min_i T_{ij}^{R}\right], \left[max_i I_{ij}^{L}, max_i I_{ij}^{R}\right], \left[max_i F_{ij}^{L}, max_i F_{ij}^{R}\right]\right) \tag{14}$$

For negative and decreasing criteria

$$\left(\left[T_j^{L+}, T_j^{R+}\right], \left[I_j^{L+}, I_j^{R+}\right], \left[F_j^{L+}, F_j^{R+}\right]\right) = \left(\left[min_i T_{ij}^{L}, min_i T_{ij}^{R}\right], \left[max_i I_{ij}^{L}, max_i I_{ij}^{R}\right], \left[max_i F_{ij}^{L}, max_i F_{ij}^{R}\right]\right) \tag{15}$$

$$\left(\left[T_j^{L-}, T_j^{R-}\right], \left[I_j^{L-}, I_j^{R-}\right], \left[F_j^{L-}, F_j^{R-}\right]\right) = \left(\left[max_i T_{ij}^{L}, max_i T_{ij}^{R}\right], \left[min_i I_{ij}^{L}, min_i I_{ij}^{R}\right], \left[min_i F_{ij}^{L}, min_i F_{ij}^{R}\right]\right) \tag{16}$$

Step 5. Calculate the indicators of maximum group utility (Γ_i) and minimum individual regret (Z_i)

$$\Gamma_i = \sum_{j=1}^{n} \frac{w_j \times d\left(\left(\left[T_j^{L+}, T_j^{R+}\right], \left[I_j^{L+}, I_j^{R+}\right], \left[F_j^{L+}, F_j^{R+}\right]\right), \left(\left[T_j^{L}, T_j^{R}\right], \left[I_j^{L}, I_j^{R}\right], \left[F_j^{L}, F_j^{R}\right]\right)\right)}{d\left(\left(\left[T_j^{L+}, T_j^{R+}\right], \left[I_j^{L+}, I_j^{R+}\right], \left[F_j^{L+}, F_j^{R+}\right]\right), \left(\left[T_j^{L-}, T_j^{R-}\right], \left[I_j^{L-}, I_j^{R-}\right], \left[F_j^{L-}, F_j^{R-}\right]\right)\right)} \tag{17}$$

$$Z_i = max_j \left\{ \frac{w_j \times d\left(\left([T_j^{L+}, T_j^{R+}], [I_j^{L+}, I_j^{R+}], [F_j^{L+}, F_j^{R+}]\right), \left([T_j^{L}, T_j^{R}], [I_j^{L}, I_j^{R}], [F_j^{L}, F_j^{R}]\right)\right)}{d\left(\left([T_j^{L+}, T_j^{R+}], [I_j^{L+}, I_j^{R+}], [F_j^{L+}, F_j^{R+}]\right), \left([T_j^{L-}, T_j^{R-}], [I_j^{L-}, I_j^{R-}], [F_j^{L-}, F_j^{R-}]\right)\right)} \right\} \quad (18)$$

$$d(A, B) = \frac{1}{6}\left(|T_A^L - T_B^L| + |I_A^L - I_B^L| + |F_A^L - F_B^L| + |T_A^R - T_B^R| + |I_A^R - I_B^R| + |F_A^R - F_B^R|\right) \quad (19)$$

Step 6. Calculate VIKOR Index (Q_i)

$$Q_i = \beta \frac{(\Gamma_i - \Gamma_i^*)}{(\Gamma_i^- - \Gamma_i^*)} + (1 - \beta) \frac{(Z_i - Z_i^*)}{(Z_i^- - Z_i^*)} \quad (20)$$

$$\Gamma_i^* = min_i \Gamma_i, \qquad \Gamma_i^- = max_i \Gamma_i \quad (21)$$

$$Z_i^* = min_i Z_i, \qquad Z_i^- = max_i Z_i \quad (22)$$

Step 7. Rank the options based on Q_i, Γ_i and Z_i in accordance with the classic VIKOR ranking rule

Step 8. The compromise solution must meet one of the following conditions:

(A) Acceptable advantage in the sense that a compromise solution must be significantly different from its next solution: $Q(A^2) - Q(A^1) \geq DQ = \frac{1}{m-1}$ Where A^1 and A^2 are the first and second choices in the ordered list and m is the number of options.

(B) Acceptable consistency in the decision-making process means that the adaptive solution chosen must have Group utility maximization and at least individual impact: A^1 should be the best rank in Γ_i and Z_i. This is the compromise solution throughout the decision-making process.

If the above conditions for a compromise solution are not met, a set of adaptation strategies is provided instead of one.

Step 9. A set of compromise solutions is obtained if one of the conditions is not satisfied.

A^1 and A^2 are compromise solutions if only condition 2 is not met. Or A^1, A^2 and ... A^M are compromise solutions if condition 1 is not fulfilled, by the constraint $Q(A^M) - Q(A^1) < DQ$ decides for maximum M [54].

4. Case study

A company in the online sales of various products has been selected as the numerical example of this research. The company supplies products to various suppliers and sends them to its customers. Due to limited resources and limited resources, the company cannot independently complete the entire project. Therefore, the company intends to select an optimal partner from the project candidates for the transport sector of the company and create a dynamic virtual enterprise alliance to collectively complete the entire project. In the issue of partner selection, first by studying the research literature, the most important criteria affecting partner selection in different domains were identified in accordance with Table (1). Then, 8 experts from the company with expertise in virtual enterprise and partner selection and with over 5 years' experience were selected 13 criteria of the most important partner selection criteria in the transport sector of the company according to Table (2).

Table 2. Criteria linguistic assessments for partner selection by experts

Criteria		Partner 1				Partner 2			
		E_1	E_2	E_3	E_4	E_1	E_2	E_3	E_4
Cost	C_1	VH	H	VH	VH	H	VH	VH	VH
Delivery	C_2	H	L	H	M	L	M	L	M
Trust	C_3	VH	H	VH	M	H	H	VH	M
Risk	C_4	L	VL	M	VL	M	M	M	VL
Quality	C_5	H	H	H	VH	H	H	VH	H
Reputation and position in industry	C_6	H	VH	VH	H	VH	H	VH	VH
Customer service	C_7	H	M	M	VH	H	VH	M	VH
Knowledge and managerial experience	C_8	H	H	M	H	H	L	M	L
Technology capability	C_9	M	VL	L	VL	M	VL	L	L
Information and communication technology resources	C_{10}	VH	H	VH	H	H	VH	VH	H
Willingness to share information	C_{11}	M	M	L	VL	M	VL	H	L
Competency & technical expertise	C_{12}	M	H	M	M	VH	H	VH	H
IT infrastructure	C_{13}	VH	H	H	H	VH	M	H	M

After defining effective criteria in the Partner selection of the transport sector, 4 experts of the company expressed their opinion about the 4 candidates with the competence of linguistic terms according to the effective criteria. Table (2) gives some examples of expert opinions.

4.1. Findings

After gathering the experts' opinions in the form of linguistic terms, they first converted to Interval Neutrosophic numbers using Table 3.

Table 3. Transformations between numerical ratings and INSs

Linguistic terms	INSs
VH	{[0.9,1],[0,0.1],[0,0.1]
H	{[0.75,0.85],[0.05,0.15],[0.15,0.25]}
M	{[0.55,0.65],[0.15,0.25],[0.35,0.45]}
L	{[0.35,0.45],[0.25,0.35],[0.55,0.65]}
VL	{[0.15,0.25],[0.35,0.45],[0.75,0.85]

Next, using these observations, the weighting of the criteria was calculated using the maximum deviation and correlation technique (8) according to Table (4). The results show that the Willingness to share information criterion with a weight of 0.113 is the most important partner selection criterion in this company. This illustrates the importance of the quality of information shared. As such, it is important for Virtual Enterprises to collaborate effectively with the information sharing organization for optimal collaboration. And keeping in touch with other partners, such as finding out where and when to deliver the goods, and keeping the customer informed of the delivery and planning process of the company to ship other products will ultimately lead to better overall company performance. Competency & technical expertise is ranked second and reflects the importance of technical and practical expertise from the point of view of company experts in choosing a virtual partner. Reputation and position in the industry are of third importance for the company and the background, reputation and position of the company in the industry and among the competitors can be an effective choice. The notable point in this company is that the cost criterion (lowest-weighted) is the last priority. This indicates the importance of other criteria for cost, and the company tends to be more costly in choosing the optimal partner.

Table 4. Weight of criteria

Criteria	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}
Weight	0.048	0.095	0.051	0.086	0.049	0.096	0.084	0.085	0.056	0.067	0.113	0.098	0.071

Given the group decision-making of choosing a virtual partner, it is necessary to integrate expert opinions on each company. For this purpose, using the Interval Neutrosophic Weighted Average Operator for each candidate company, the relation of 10 decision matrices of consensus of expert opinions is calculated. The Consensus Decision Matrix of Business Partner 1 is in the form of Interval Neutrosophic Numbers as shown in Table 5. The same applies to other business partners.

Table 5. The decision matrix $\tilde{R}_1$

	T^+	T^-	I^+	I^-	F^+	F^-
C_1	0.874257	1	0	0.110668	0	0.125743
C_2	0.632293	0.743461	0.098399	0.210643	0.256539	0.367707
C_3	0.816858	1	0	0.139158	0	0.183142
C_4	0.321982	0.426361	0.260341	0.364845	0.573639	0.678018
C_5	0.801182	1	0	0.13554	0	0.198818
C_6	0.841886	1	0	0.122474	0	0.158114
C_7	0.733258	1	0	0.174982	0	0.266742
C_8	0.710427	0.81461	0.065804	0.170433	0.18539	0.289573
C_9	0.321982	0.426361	0.260341	0.364845	0.573639	0.678018
C_{10}	0.841886	1	0	0.122474	0	0.158114
C_{11}	0.421652	0.525878	0.210643	0.314985	0.474122	0.578348
C_{12}	0.611497	0.716813	0.113975	0.220028	0.283187	0.388503
C_{13}	0.801182	1	0	0.13554	0	0.198818

Finally, the VIKOR fuzzy Interval Neutrosophic method and the equations of 10 to 23 rankings of the four transport companies were performed. After calculating the performance and distance from the ideal level of options and obtaining the indicators of maximum group utility (G_i) and minimum individual regret (Z_i) and the value of VIKOR index (Q_i), the final ranking of options was done according to Table 5. Accordingly, the least Q value is chosen as the best option.

Table 6. Sorting results

Partner	Partner 1	Partner 2	Partner 3	Partner 4	The ranking order
Γ_i	0.383959	0.355709	0.471911	0.666381	$P_2>P_1>P_3> P_4$
Z_i	0.100536	0.085379	0.113432	0.098186	$P_2>P_4> P_1>P_3$
Q_i	0.31562	0	0.687017	0.728261	$P_2>P_1>P_3> P_4$

Thus Business Partner 2 with $Q_2 = 0$ is selected as the best virtual partner. This result is now examined by two conditions. $0.31562 - 0 < \mathrm{DQ} = \frac{1}{4-1} = 0.333$ Hence the first condition is not applicable. Since option A_2 has the best rank in G_i and Z_i ($\beta = 0.5$), so the second condition holds. Since only the second condition is in place, the options are rated $P_2 \sim P_1> P_3> P_4$, and both A_2 and A_1 are eventually selected and get top rankings.

In the relationships of the Neutrosophic fuzzy VIKOR method, β is defined as the weight of most criteria strategy, or most group utility, and is usually considered to be 0.5. However, the β value may affect the value of the VIKOR index. For this purpose, calculations for different values of β are performed according to Table 7, and the applicability and stability of the proposed method are investigated.

Table 6. Sensitivity analysis of the value β

β	Q_1	Q_2	Q_3	Q_4	**Rank order**
0	0.540309	0	1	0.456522	$P_2> P_4>P_1>P_3$
0.2	0.450433	0	0.874807	0.565217	$P_2>P_1 >P_4>P_3$
0.4	0.360558	0	0.749613	0.673913	$P_2>P_1> P_4>P_3$
0.5	0.31562	0	0.687017	0.728261	$P_2>P_1>P_3> P_4$
0.6	0.270682	0	0.62442	0.782609	$P_2>P_1>P_3> P_4$
0.8	0.180807	0	0.499227	0.891304	$P_2>P_1>P_3> P_4$
1	0.090931	0	0.374034	1	$P_2>P_1>P_3> P_4$

Weight sensitivity analysis of the majority (β) strategy indicates that the firm manager can select the appropriate group (β) value to reflect the decision maker priority. If the manager prefers to eliminate Group utility maximization, it supports β = 1 and uses the G marker. Conversely, if the decision maker pays more attention to regret thinking, then β = 0 and the value of Z is accepted. Figure (2) shows the effect of changing β on Q_i. In different values of β, trading partner 2 and 1 are ranked first and second, respectively, with values below 0.5 third partner and values above 0.5 partner 4 last.

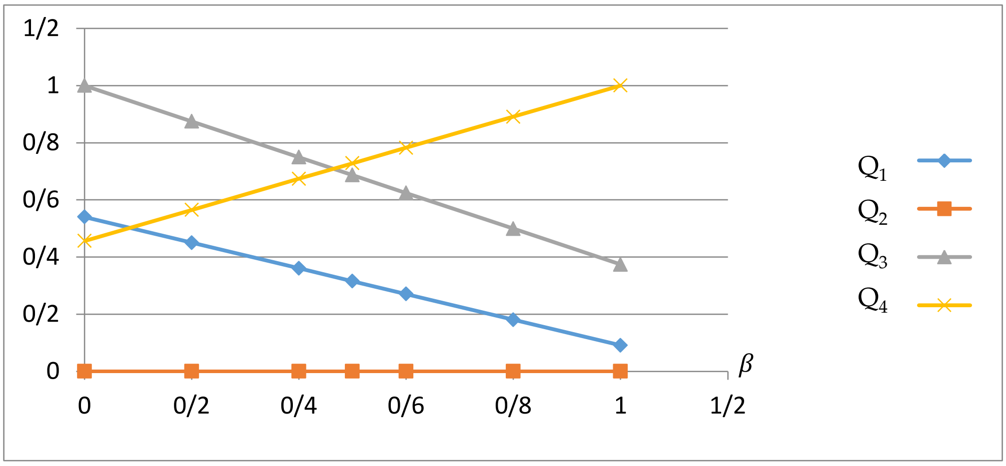

Fig 2. Sensitivity analysis of the value β for each alternative with INSs

Figure 3 shows the spider diagram of the sensitivity analysis and the effect of the β parameter change on the VIKOR index. Partner rankings in this chart are centered outward, and Partner 2 in the chart is ranked first in all β values, and Partner 2 is not second only to value β = 0. This chart shows the gap between the partners. At point β = 0, business partner 4 ranks second. While in other values of β the first partner is at this rank. The spider diagram shows that the distance between these two partners is very small at this point, and the Q value of Partner 1 is only slightly different from Partner 4, and the stability of this ratio can be confirmed. But for the third and fourth partner the subject is slightly different and when the β value is greater than 0.5 the rating changes and the distance between the two graphs is noticeable indicating the influence of individual views of the group. Accordingly, when the group views are more important, the third partner is ranked third and in the smaller values of β the individual opinions are more important. The fourth partner ranks third. The impact of the importance of group versus individual views on this ranking is clearly illustrated by the decrease and increase in the distance between the third and fourth partner graphs in Figure 3.

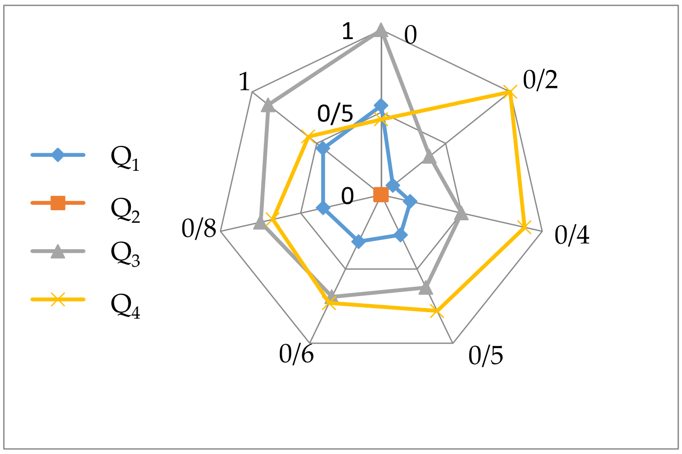

Fig 3. Spider chart of the value β for each alternative with INSs

Sensitivity analysis showed that the value of the parameter β did not significantly influence the results of the selection of the best partner. Therefore, the ranking results obtained using the proposed method for INS are reliable and effective.

5. Conclusions

In today's business environment, competition is focused on innovation, speed, and flexibility. A new business model is needed to help companies gain competitive benefit in the volatile market [55]. Increasing complexity has allowed any business to reconfigure itself to meet its needs, and opportunities and remain in a highly competitive environment, because they do not have all the skills and resources needed to meet new market demand. Virtual enterprise (VE) has been proposed as a new organizational approach to meet the requirements of low cost, high quality, fast responsiveness, and greater customer satisfaction to be adapted with this rapidly changing environment [56]. The criteria for choosing a partner in Virtual Enterprises vary depending on the type of activity. In this paper, firstly, by studying the research literature and using the experts' opinions, 13 criteria affecting the selection of a partner in the transport sector of virtual enterprise were identified. How to choose the right partners for success in Virtual Enterprises (VE) is very important and has received a great deal of attention from researchers and experts. Given the different types of uncertainty in the real environment, decision makers are usually not sure when choosing a partner because the information on the candidates is incomplete and unclear. In addition, some of the features of decision making are subjective and qualitative. In many cases decision makers are unable to express their decisions about candidates in precise quantities. For this purpose, in the second step, the partner selection problem with VIKOR method is used to form a VE under Interval Neutrosophic environment. The VIKOR method considers the boundary rationality of decision makers, and makes more rational decisions. Interval Neutrosophic Numbers are used to address problems with uncertain, incomplete and inconsistent information. This method helps to reduce the mentality of decision makers. In this paper, the method of weighting the maximum deviation in the Neutrosophic environment is used in the absence of benchmark information, which can be very useful in deciding issues with inconsistent and uncertain criteria. The Partner Selection Process In this paper, we have designed a new combination and comprehensive classification of partner evaluation criteria in the context of the virtual enterprise. The proposed approach can effectively reduce the subjectivity and uncertainty of the multi-criteria decision-making problem and rely on the underlying data to make the evaluation result more objective and reliable. Also, by improving the existing method of weight calculation, the Maximizing deviation method can effectively guarantee the consistency of the judgments and simplify the weighting function in cases where the information is incomplete or there is no metric weight information. Expanding the VIKOR method to Interval Neutrosophic numbers can effectively counteract uncertainty assessment information. Without increasing mental states, it retains more decision information and makes Partner selection in the virtual enterprise more scientific. The results of the weight sensitivity analysis of the group utility strategy (β) show that the business firm is selected as the best partner for all β values according to the identified effective factors. Ranked second in trading partner 1 for all values of β, with only zero for trading partner 4. The β parameter is determined by the degree of agreement of the decision maker, and the larger the β, the greater the group's views (too much agreement) and the smaller the β, the greater the individual's opinions (little agreement). In this paper, the rankings are slightly different for the smaller β values as illustrated in Fig. 2, with the trading partner 4 being ranked second and the trading partner 3 last. But one still remains the top partner.

In the actual decision making, there is much qualitative information that can be expressed by uncertain linguistic variables. Interval Neutrosophic numbers can easily express uncertain and contradictory information in the real world, and by combining multi-criteria decision-making techniques to make the paradoxical features more scientific and reasonable. In this paper, the VIKOR method is developed to deal with uncertain linguistic information in the Interval Neutrosophic environment. In this method, the criterion values are presented as Interval Neutrosophic numbers. Neutrosophic set with interval value is used to express incomplete knowledge of the expert group and to prevent loss of information. However, the approach proposed for selecting the best partner in Virtual Enterprises has advantages in terms of selection criteria. But the main limitation is the lack of quantitative data and the limited number of respondents in the study. With increasing awareness of Virtual Enterprises, effective benchmarks should be developed according to the field of business activity, and other weighting techniques such as AHP, ANP and artificial intelligence techniques can be used in combination with VIKOR. Other ranking methods such as AHP and TOPSIS can be used in combination with the Neutrosophic environment. Optimization techniques can also be applied to partner selection in Virtual Enterprises. The proposed model can be applied to other decision-making issues such as supplier selection, risk assessment. Also, comparison of model results with other uncertainty modeling techniques can be suggested. Finally, the robustness of the proposed model can be tested through scenario analysis and uncertainty analysis.

Funding: This research received no external funding

Conflicts of Interest: The authors declare no conflict of interest.

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Received: Apr 25, 2020. Accepted: July 15 2020

Basic operations on hypersoft sets and hypersoft point

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Abstract: The aim of this paper is to initiat formal study of hypersoft sets. We first, present basic operations like union, intersection and difference of hypersoft sets; basic ingrediants for topological structures on the collection of hypersoft sets. Moreover we introduce hypersoft points in different envorinments like fuzzy hypersoft set, intuitionistic fuzzy hypersoft set, neutrosophic hypersoft, plithogenic hypersoft set, and give some basic properties of hypersoft points in these envorinments. We expect that this will constitue an appropriate framework of hypersoft functions and the study of hypersoft function spaces. Examples are provided to explain the newly defined concepts.

Keywords: soft set; hypersoft set; set operations on hypersoft sets; hypersoft point; fuzzy hypersoft set; intuitionistic fuzzy hypersoft set; neutrosophic hypersoft; plithogenic hypersoft set.

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1. Introduction

Molodtsov [16] defined soft set as a mathematical tool to deal with uncertainties associated with real world problems. Soft set theory has application in decision making, demand analysis, forecasting, information sciences and other disciplines (see for example, [13, 14, 15, 17, 18, 19, 20, 21, 22, 23]). Plithogenic and neutrosophic hypersoft sets theory is being applied successfully in decision making problems (see, [2, 3, 4, 5, 6, 7, 8, 9,10,11,12]).

By definition, a soft set can be identified by a pair (F, A), where F stands for a multivalued function defined on the set of parameters A.

Smarandache [1] extended the notion of a soft set to the hypersoft set by replacing the function F with a multi-argument function defined on the Cartesian product of n different set of parameters. This concept is more flexible than soft set and more suitable in the context of decision making problems.

We expect the notion of hypersoft set will attract the attention of researchers working on soft set theory and its diverse applications. The purpose of this paper is to initiate a formal investigation in this new area of research.

As a first step, we present the basic operations like union, intersection and difference of hypersoft sets. Moreover we introduce hypersoft points and some basic properties of these points which may provide the foundation for the hypersoft functions and hence the hypersoft fixed point theory.

2. Operations on hypersoft sets

In this section, we define basic operations on hypersoft sets. Smarandache defined the hypersoft set in the following manner:

Definition 1 *[1] Let U be a universe of discourse, $P(U)$ the power set U and $E_1, E_2, \dots, E_n$ the pairwise disjoint sets of parameters. Let A_i be the nonempty subset of E_i for each $i = 1,2,\dots,n$. A hypersoft set can be identified by the pair $(F, A_1 \times A_2 \times \cdots \times A_n)$, where:*

$$F: A_1 \times A_2 \times \cdots \times A_n \to P(U).$$

For sake of simplicity, we write the symbols $\mathbf{E}$ for $E_1 \times E_2 \times \cdots \times E_n$, $\mathbf{A}$ for $A_1 \times A_2 \times \cdots \times A_n$ and $\boldsymbol{\alpha}$ for an element of the set $\mathbf{A}$. We also suppose that none of the set A_i is empty.

Definition 2 *[1] A hypersoft set;*
on a crisp universe of discourse U_C is called Crisp Hypersoft set (or simply "hypersoft set");
on a fuzzy universe of discourse U_F is called Fuzzy Hypersoft set.
on a Intuitionistic Fuzzy universe of discourse U_{IF} is called Intuitionistic Fuzzy Hypersoft set;
on a Neutrosophic universe of discourse U_N is called Neutrosophic Hypersoft Set;
on a Plithogenic universe of discourse U_P is called Plithogenic Hypersoft Set.

The nature of $F(\boldsymbol{\alpha})$ is determined by the nature of universe of discourse. Therefore $P(U)$ depends upon the nature of universe. We denote $\mathcal{H}(U_*, \mathbf{E})$ by the family of all *-hypersoft sets over $(U_*, \mathbf{E})$, where $*$ can take any value in the set $\{C, F, IF, N, P\}$, where symbols C, F, IF, N, P denote Crisp, Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Plithogenic sets, respectively.

The following are the basic operations on *-hypersoft set.

Definition 3 *Let U_* be a universe of discourse and $\boldsymbol{A}$ a subset of $\boldsymbol{E}$. Then $(F, \boldsymbol{A})$ is called*

1. *a null *-hypersoft set if for each parameter $\boldsymbol{\alpha} \in \boldsymbol{A}$, $F(\boldsymbol{\alpha})$ is an 0_*. We will denote it by $\Phi_{\boldsymbol{A}}$.*
2. *an absolute *-hypersoft set if for each parameter $\boldsymbol{\alpha} \in \boldsymbol{A}$, $F(\boldsymbol{\alpha}) = U_*$. We will denote it by $\widetilde{U}_{\boldsymbol{A}}$.*

Remark 1 *We consider $0_C = \varnothing$ for empty set, $0_F = \{\frac{x}{0}, x \in U_F\}$ for null fuzzy set, $0_{IF} = \{\frac{x}{<0,1>}, x \in U_{IF}\}$ for null intuitionistic fuzzy set, $0_N = \{\frac{x}{<0,1,1>}, x \in U_N\}$ for null neutrosophic set. However, in case of plithogenic set, we have the following notations:*

- *Null plithgenic crisp set*

$$0_{PC} = \{x(0,0,\dots,0), \text{forall} x \in U_P\}.$$

- *Universal plithgenic crisp set*

$$1_{PC} = \{x(1,1,\dots,1), \text{forall} x \in U_P\}.$$

Note that null plithgenic fuzzy set will be same as null plithgenic crisp set and universal plithgenic fuzzy set will be the same as universal plithgenic crisp set.

- *Null plithgenic intuitionistic fuzzy set*

$$0_{PIF} = \{x((0,1),(0,1),\ldots,(0,1)), \text{for all} x \in U_P\}.$$

- *Universal plithgenic intuitionistic fuzzy set*

$$1_{PIF} = \{x((1,0),(1,0),\ldots,(1,0)), \text{for all} x \in U_P\}.$$

- *Null plithgenic neutrosophic set*

$$0_{PN} = \{x((0,1,1),(0,1,1),\ldots,(0,1,1)), \text{for all} x \in U_P\}.$$

- *Universal plithgenic neutrosophic set*

$$1_{PN} = \{x((1,0,0),(1,0,0),\ldots,(1,0,0)), \text{for all} x \in U\}.$$

Definition 4 *Let $(F,\boldsymbol{A})$ and $(G,\boldsymbol{B})$ be two *-hypersoft sets over U_*. Then union of $(F,\boldsymbol{A})$ and $(G,\boldsymbol{B})$ is denoted by $(H,\boldsymbol{C}) = (F,\boldsymbol{A}) \tilde{\cup} (G,\boldsymbol{B})$ with $\boldsymbol{C} = C_1 \times C_2 \times \cdots \times C_n$, where $C_i = A_i \cup B_i$ for $i = 1,2,\ldots,n$, and H is defined by*

$$H(\boldsymbol{\alpha}) = \begin{cases} F(\boldsymbol{\alpha}), & \text{if} \boldsymbol{\alpha} \in \mathbf{A} - \mathbf{B} \\ G(\boldsymbol{\alpha}), & \text{if} \boldsymbol{\alpha} \in \mathbf{B} - \mathbf{A} \\ F(\boldsymbol{\alpha}) \cup_* G(\boldsymbol{\alpha}), & \text{if} \boldsymbol{\alpha} \in \mathbf{A} \cap \mathbf{B}, \\ 0_*, & \text{else,} \end{cases}$$

where $\boldsymbol{\alpha} = (c_1, c_2, \ldots, c_n) \in C$.

Remark 2 *Note that, in the case of union of two hypersoft sets the set of parameters is a Cartesian product of sets of parameters whereas in the case of union of two soft sets the set of parameter is just the union of sets of parameters.*

Definition 5 *Let $(F,\boldsymbol{A})$ and $(G,\boldsymbol{B})$ be two *-hypersoft sets over U_*. Then intersection of $(F,\boldsymbol{A})$ and $(G,\boldsymbol{B})$ is denoted by $(H,\boldsymbol{C}) = (F,\boldsymbol{A}) \tilde{\cap} (G,\boldsymbol{B})$, where $\boldsymbol{C} = C_1 \times C_2 \times \cdots \times C_n$ is such that $C_i = A_i \cap B_i$ for $i = 1,2,\ldots,n$ and H is defined as*

$$H(\boldsymbol{\alpha}) = F(\boldsymbol{\alpha}) \cap_* G(\boldsymbol{\alpha}),$$

*where $\boldsymbol{\alpha} = (c_1, c_2, \ldots, c_n) \in \boldsymbol{C}$. If C_i is an empty set for some i, then $(F,\boldsymbol{A}) \tilde{\cap} (G,\boldsymbol{B})$ is defined to be a null *-hypersoft set.*

Definition 6 *Let $(F,\boldsymbol{A})$ and $(G,\boldsymbol{B})$ be two *-hypersoft sets over U_*. Then $(F,\boldsymbol{A})$ is called a *-hypersoft subset of $(G,\boldsymbol{B})$ if $\boldsymbol{A} \subseteq \boldsymbol{B}$, and $F(\boldsymbol{\alpha}) \subseteq_* G(\boldsymbol{\alpha})$ for all $\boldsymbol{\alpha} \in \boldsymbol{A}$. We denote this by $(F,\boldsymbol{A}) \tilde{\subseteq} (G,\boldsymbol{B})$. Thus $(F,\boldsymbol{A})$ and $(G,\boldsymbol{B})$ are said to equal if $(F,\boldsymbol{A}) \tilde{\subseteq} (G,\boldsymbol{B})$ and $(F,\boldsymbol{A}) \tilde{\supseteq} (G,\boldsymbol{B})$.*

Definition 7 *Let $(F,\boldsymbol{A})$ and $(G,\boldsymbol{B})$ be two *-hypersoft sets over U_*. Then *-hypersoft difference of $(F,\boldsymbol{A})$ and $(G,\boldsymbol{B})$, denoted by $(H,\boldsymbol{C}) = (F,\boldsymbol{A}) \tilde{\setminus} (G,\boldsymbol{B})$, where $\boldsymbol{C} = C_1 \times C_2 \times \cdots \times C_n$ is such that $C_i = A_i \cap B_i$ for $i = 1,2,\ldots,n$, and H is defined by*

$$H(\boldsymbol{\alpha}) = F(\boldsymbol{\alpha}) \setminus_* G(\boldsymbol{\alpha}),$$

where $\boldsymbol{\alpha} = (c_1, c_2, \ldots, c_n) \in \mathbf{C}$. If C_i is an empty set for some i then $(F,\mathbf{A}) \tilde{\setminus} (G,\mathbf{B})$ is defined to be $(F,\mathbf{A})$.

Definition 8 *The complement of a *-hypersoft set $(F,\boldsymbol{A})$ is denoted as $(F,\boldsymbol{A})^c$ and is defined by $(F,\boldsymbol{A})^c = (F^c,\boldsymbol{A})$ where $F^c(\boldsymbol{\alpha})$ is the *-complemet of $F(\boldsymbol{\alpha})$ for each $\boldsymbol{\alpha} \in \boldsymbol{A}$.*

Example 1 Let $\mathrm{U} = \{\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4\}$. Define the attributes sets by:

$$E_1 = \{a_{11}, a_{12}\}, E_2 = \{a_{21}, a_{22}\}, E_3 = \{a_{31}, a_{32}\}.$$

Suppose that

$$A_1 = \{a_{11}, a_{12}\}, A_2 = \{a_{21}, a_{22}\}, A_3 = \{a_{31}\}, \text{and}$$
$$B_1 = \{a_{11}\}, B_2 = \{a_{21}, a_{22}\}, B_3 = \{a_{31}, a_{32}\}$$

that is,. $A_i, B_i \subseteq E_i$ for each $i = 1,2,3$.

Let the crisp hypersoft sets $(F, \mathbf{A})$ and $(G, \mathbf{B})$ be defined by

$$(F, \mathbf{A}) = \{((a_{11}, a_{21}, a_{31}), \{x_1, x_2\}), ((a_{11}, a_{22}, a_{31}), \{x_2\}),$$
$$((a_{12}, a_{21}, a_{31}), \{x_3, x_4\}), ((a_{12}, a_{22}, a_{31}), \{x_1, x_4\})\}.$$

and

$$(G, \mathbf{B}) = \{((a_{11}, a_{21}, a_{31}), \{x_2, x_3\}), ((a_{11}, a_{22}, a_{31}), \{x_2\}),$$
$$((a_{11}, a_{21}, a_{32}), \{x_1, x_4\}), ((a_{11}, a_{22}, a_{32}), \{x_3, x_4\})\}.$$

We have excluded those $\boldsymbol{\alpha} \in \mathbf{A}$ for which $F(\boldsymbol{\alpha})$ is an empty set (similarly for those $\boldsymbol{\beta} \in \mathbf{B}$ for which $G(\boldsymbol{\beta})$ is an empty set).

Then the union and intersections of $(F, \mathbf{A})$ and $(G, \mathbf{B})$ are given by:

$$(F, \mathbf{A}) \,\tilde{\cup}\, (G, \mathbf{B}) = \{((a_{11}, a_{21}, a_{31}), \{x_1, x_2, x_3\}), ((a_{11}, a_{22}, a_{31}), \{x_2\}),$$
$$((a_{12}, a_{21}, a_{31}), \{x_3, x_4\}), ((a_{12}, a_{22}, a_{31}), \{x_1, x_4\}),$$
$$((a_{11}, a_{21}, a_{32}), \{x_1, x_4\}), ((a_{11}, a_{22}, a_{32}), \{x_3, x_4\}),$$
$$((a_{12}, a_{21}, a_{32}), 0_C), ((a_{12}, a_{22}, a_{32}), 0_C)\};$$

and

$$(F, \mathbf{A}) \,\tilde{\cap}\, (G, \mathbf{B}) = \{((a_{11}, a_{21}, a_{31}), \{x_2\}), ((a_{11}, a_{22}, a_{31}), \{x_2\})\}.$$

The differences $(F, \mathbf{A}) \tilde{\setminus} (G, \mathbf{B})$ and $(G, \mathbf{B}) \tilde{\setminus} (F, \mathbf{A})$ are the following

$$(F, \mathbf{A}) \tilde{\setminus} (G, \mathbf{B}) = \{((a_{11}, a_{21}, a_{31}), \{x_1\}), ((a_{11}, a_{22}, a_{31}), 0_C)\};$$
$$(G, \mathbf{B}) \tilde{\setminus} (F, \mathbf{A}) = \{((a_{11}, a_{21}, a_{31}), \{x_3\}), ((a_{11}, a_{22}, a_{31}), 0_C)\}.$$

Example 2 Let $\mathrm{U} = \{\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4\}$. Define the attributes sets by:

$$E_1 = \{a_{11}, a_{12}\}, E_2 = \{a_{21}, a_{22}\}, E_3 = \{a_{31}, a_{32}\}.$$

Suppose that

$$A_1 = \{a_{11}, a_{12}\}, A_2 = \{a_{21}, a_{22}\}, A_3 = \{a_{31}\}, \text{and}$$
$$B_1 = \{a_{11}\}, B_2 = \{a_{21}, a_{22}\}, B_3 = \{a_{31}, a_{32}\}$$

are subsets of E_i for each $i = 1,2,3$, that is,. $A_i, B_i \subseteq E_i$ for each i.

Let the fuzzy hypersoft sets $(F, \mathbf{A})$ and $(G, \mathbf{B})$ be defined by

$$(F, \mathbf{A}) = \{((a_{11}, a_{21}, a_{31}), \{\tfrac{x_1}{0.5}, \tfrac{x_2}{0.7}\}), ((a_{11}, a_{22}, a_{31}), \{\tfrac{x_2}{0.3}\}),$$
$$((a_{12}, a_{21}, a_{31}), \{\tfrac{x_3}{0.8}, \tfrac{x_4}{0.9}\}), ((a_{12}, a_{22}, a_{31}), \{\tfrac{x_1}{0.5}, \tfrac{x_4}{0.4}\})\}.$$

and

$$(G, \mathbf{B}) = \{((a_{11}, a_{21}, a_{31}), \{\tfrac{x_2}{0.2}, \tfrac{x_3}{0.9}\}), ((a_{11}, a_{22}, a_{31}), \{\tfrac{x_2}{0.6}\}),$$
$$((a_{11}, a_{21}, a_{32}), \{\tfrac{x_1}{0.4}, \tfrac{x_4}{0.7}\}), ((a_{11}, a_{22}, a_{32}), \{\tfrac{x_3}{0.2}, \tfrac{x_4}{0.8}\})\}.$$

We have excluded those $\boldsymbol{\alpha} \in \mathbf{A}$ for which $F(\boldsymbol{\alpha})$ is a null fuzzy set (similarly for those $\boldsymbol{\beta} \in \mathbf{B}$ for which $G(\boldsymbol{\beta})$ is a null fuzzy set).

Then the union and intersections of $(F, \mathbf{A})$ and $(G, \mathbf{B})$ are given by:

$(F,\mathbf{A})\,\tilde{\cup}\,(G,\mathbf{B}) = \{((a_{11},a_{21},a_{31}),\{\frac{x_1}{0.5},\frac{x_2}{0.7},\frac{x_3}{0.9}\}),((a_{11},a_{22},a_{31}),\{\frac{x_2}{0.6}\}),$
$((a_{12},a_{21},a_{31}),\{\frac{x_3}{0.8},\frac{x_4}{0.9}\}),((a_{12},a_{22},a_{31}),\{\frac{x_1}{0.5},\frac{x_4}{0.4}\}),$
$((a_{11},a_{21},a_{32}),\{\frac{x_1}{0.4},\frac{x_4}{0.7}\}),((a_{11},a_{22},a_{32}),\{\frac{x_3}{0.2},\frac{x_4}{0.8}\}),$
$((a_{12},a_{21},a_{32}),0_F),((a_{12},a_{22},a_{32}),0_F)\};$

and

$$(F,\mathbf{A})\,\tilde{\cap}\,(G,\mathbf{B}) = \{((a_{11},a_{21},a_{31}),\{\frac{x_2}{0.2}\}),((a_{11},a_{22},a_{31}),\{\frac{x_2}{0.3}\})\}.$$

The differences $(F,\mathbf{A})\tilde{\setminus}(G,\mathbf{B})$ and $(G,\mathbf{B})\tilde{\setminus}(F,\mathbf{A})$ are the following
$(F,\mathbf{A})\tilde{\setminus}(G,\mathbf{B}) = \{((a_{11},a_{21},a_{31}),\{\frac{x_1}{0.5},\frac{x_2}{0.5}\}),((a_{11},a_{22},a_{31}),0_F)\};$
$(G,\mathbf{B})\tilde{\setminus}(F,\mathbf{A}) = \{((a_{11},a_{21},a_{31}),\{\frac{x_3}{0.9}\}),((a_{11},a_{22},a_{31}),\{\frac{x_2}{0.3}\})\}.$

Example 3 Let $\mathrm{U} = \{\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3,\mathrm{x}_4\}$. Define the attributes sets by:

$$E_1 = \{a_{11},a_{12}\}, E_2 = \{a_{21},a_{22}\}, E_3 = \{a_{31},a_{32}\}.$$

Suppose that

$A_1 = \{a_{11},a_{12}\}, A_2 = \{a_{21},a_{22}\}, A_3 = \{a_{31}\}$, and
$B_1 = \{a_{11}\}, B_2 = \{a_{21},a_{22}\}, B_3 = \{a_{31},a_{32}\}$

that is,. $A_i, B_i \subseteq E_i$ for each $i = 1,2,3$.

Let the intuitionistic fuzzy hypersoft sets $(F,\mathbf{A})$ and $(G,\mathbf{B})$ be defined by

$(F,\mathbf{A}) = \{((a_{11},a_{21},a_{31}),\{\frac{x_1}{<0.5,0.3>},\frac{x_2}{<0.7,0.2>}\}),((a_{11},a_{22},a_{31}),\{\frac{x_2}{<0.3,0.5>}\}),$

$((a_{12},a_{21},a_{31}),\{\frac{x_3}{<0.8,0.1>},\frac{x_4}{<0.1,0.5>}\}),((a_{12},a_{22},a_{31}),\{\frac{x_1}{<0.5,0.3>},\frac{x_4}{<0.4,0.2>}\})\}.$

and

$(G,\mathbf{B}) = \{((a_{11},a_{21},a_{31}),\{\frac{x_2}{<0.2,0.6>},\frac{x_3}{<0.8,0.1>}\}),((a_{11},a_{22},a_{31}),\{\frac{x_2}{<0.6,0.3>}\}),$

$((a_{11},a_{21},a_{32}),\{\frac{x_1}{<0.4,0.5>},\frac{x_4}{<0.7,0.2>}\}),((a_{11},a_{22},a_{32}),\{\frac{x_3}{<0.4,0.2>},\frac{x_4}{<0.1,0.8>}\})\}.$

We have excluded all those $\boldsymbol{\alpha} \in \mathbf{A}$ for which $F(\boldsymbol{\alpha})$ is a null intuitionistic fuzzy set (similarly for those $\boldsymbol{\beta} \in \mathbf{B}$ for which $G(\boldsymbol{\beta})$ is a null intuitionistic fuzzy set).

The union and intersections of $(F,\mathbf{A})$ and $(G,\mathbf{B})$ are given by:

$(F,\mathbf{A})\,\tilde{\cup}\,(G,\mathbf{B})$

$= \{((a_{11},a_{21},a_{31}),\{\frac{x_1}{<0.5,0.3>},\frac{x_2}{<0.7,0.2>},\frac{x_3}{<0.8,0.1>}\}),((a_{11},a_{22},a_{31}),\{\frac{x_2}{<0.6,0.3>}\}),$

$((a_{12},a_{21},a_{31}),\{\frac{x_3}{<0.8,0.1>},\frac{x_4}{<0.1,0.5>}\}),((a_{12},a_{22},a_{31}),\{\frac{x_1}{<0.5,0.3>},\frac{x_4}{<0.4,0.2>}\}),$

$((a_{11},a_{21},a_{32}),\{\frac{x_1}{<0.4,0.5>},\frac{x_4}{<0.7,0.2>}\}),((a_{11},a_{22},a_{32}),\{\frac{x_3}{<0.4,0.2>},\frac{x_4}{<0.1,0.8>}\}),$

$((a_{12},a_{21},a_{32}),0_{IF}),((a_{12},a_{22},a_{32}),0_{IF})\};$

and

$$(F,\mathbf{A})\,\tilde{\cap}\,(G,\mathbf{B}) = \{((a_{11},a_{21},a_{31}),\{\frac{x_2}{<0.2,0.6>}\}),((a_{11},a_{22},a_{31}),\{\frac{x_2}{<0.3,0.5>}\})\}.$$

The differences $(F,\mathbf{A})\tilde{\setminus}(G,\mathbf{B})$ and $(G,\mathbf{B})\tilde{\setminus}(F,\mathbf{A})$ are the following

$$(F,\mathbf{A})\tilde{\setminus}(G,\mathbf{B}) = \{((a_{11},a_{21},a_{31}),\{\frac{x_1}{<0.5,0.3>},\frac{x_2}{<0.6,0.2>}\}),((a_{11},a_{22},a_{31}),\{\frac{x_2}{<0.3,0.6>}\})\};$$

$$(G,\mathbf{B})\tilde{\setminus}(F,\mathbf{A}) = \{((a_{11},a_{21},a_{31}),\{\frac{x_2}{<0.2,0.7>},\frac{x_3}{<0.8,0.1>}\}),((a_{11},a_{22},a_{31}),\{\frac{x_2}{<0.5,0.3>}\})\}.$$

Example 4 Let $\mathrm{U} = \{\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3,\mathrm{x}_4\}$. Define the attributes sets by:

$$E_1 = \{a_{11},a_{12}\}, E_2 = \{a_{21},a_{22}\}, E_3 = \{a_{31},a_{32}\}.$$

Suppose that

$A_1 = \{a_{11},a_{12}\}, A_2 = \{a_{21},a_{22}\}, A_3 = \{a_{31}\}$, and

$B_1 = \{a_{11}\}, B_2 = \{a_{21},a_{22}\}, B_3 = \{a_{31},a_{32}\}$

that is,. $A_i, B_i \subseteq E_i$ for each $i = 1,2,3$.

Let the neutrosophic hypersoft sets $(F,\mathbf{A})$ and $(G,\mathbf{B})$ be defined by

$$(F,\mathbf{A}) = \{((a_{11},a_{21},a_{31}),\{\frac{x_1}{<0.5,0.2,0.3>},\frac{x_2}{<0.7,0.3,0.2>}\}),((a_{11},a_{22},a_{31}),\{\frac{x_2}{<0.3,0.2,0.5>}\}),$$

$$((a_{12},a_{21},a_{31}),\{\frac{x_3}{<0.8,0.4,0.1>},\frac{x_4}{<0.1,0.5,0.5>}\}),((a_{12},a_{22},a_{31}),\{\frac{x_1}{<0.5,0.2,0.3>},\frac{x_4}{<0.4,0.3,0.2>}\})\}.$$

and

$$(G,\mathbf{B}) = \{((a_{11},a_{21},a_{31}),\{\frac{x_2}{<0.2,0.5,0.6>},\frac{x_3}{<0.8,0.6,0.1>}\}),((a_{11},a_{22},a_{31}),\{\frac{x_2}{<0.6,0.2,0.3>}\}),$$

$$((a_{11},a_{21},a_{32}),\{\frac{x_1}{<0.4,0.3,0.5>},\frac{x_4}{<0.7,0.3,0.2>}\}),((a_{11},a_{22},a_{32}),\{\frac{x_3}{<0.4,0.4,0.2>},\frac{x_4}{<0.1,0.3,0.8>}\})\}.$$

We have excluded those $\boldsymbol{\alpha} \in \mathbf{A}$ for which $F(\boldsymbol{\alpha})$ is a null intuitionistic fuzzy set (similarly for those $\boldsymbol{\beta} \in \mathbf{B}$ for which $G(\boldsymbol{\beta})$ is a null intuitionistic fuzzy set).

The union and intersections of $(F,\mathbf{A})$ and $(G,\mathbf{B})$ are given by:

$$(F,\mathbf{A}) \tilde{\cup} (G,\mathbf{B})$$

$$= \{((a_{11},a_{21},a_{31}),\{\frac{x_1}{<0.5,0.2,0.3>},\frac{x_2}{<0.7,0.3,0.2>},\frac{x_3}{<0.8,0.6,0.1>}\}),((a_{11},a_{22},a_{31}),\{\frac{x_2}{<0.6,0.2,0.3>}\}),$$

$$((a_{12},a_{21},a_{31}),\{\frac{x_3}{<0.8,0.4,0.1>},\frac{x_4}{<0.1,0.5,0.5>}\}),((a_{12},a_{22},a_{31}),\{\frac{x_1}{<0.5,0.2,0.3>},\frac{x_4}{<0.4,0.3,0.2>}\}),$$

$$((a_{11},a_{21},a_{32}),\{\frac{x_1}{<0.4,0.3,0.5>},\frac{x_4}{<0.7,0.3,0.2>}\}),((a_{11},a_{22},a_{32}),\{\frac{x_3}{<0.4,0.4,0.2>},\frac{x_4}{<0.1,0.3,0.8>}\}),$$

$$((a_{12},a_{21},a_{32}),0_N),((a_{12},a_{22},a_{32}),0_N)\};$$

and

$$(F,\mathbf{A}) \tilde{\cap} (G,\mathbf{B})$$

$$= \{((a_{11},a_{21},a_{31}),\{\frac{x_2}{<0.2,0.5,0.6>}\}),((a_{11},a_{22},a_{31}),\{\frac{x_2}{<0.3,0.2,0.5>}\})\}.$$

The differences $(F,\mathbf{A})\tilde{\setminus}(G,\mathbf{B})$ and $(G,\mathbf{B})\tilde{\setminus}(F,\mathbf{A})$ are the following

$$(F,\mathbf{A})\tilde{\setminus}(G,\mathbf{B})$$

$= \{((a_{11}, a_{21}, a_{31}), \{\frac{x_1}{<0.5,0.2,0.3>}, \frac{x_2}{<0.6,0.15,0.2>}\}), ((a_{11}, a_{22}, a_{31}), \{\frac{x_2}{<0.3,0.4,0.6>}\})\};$

$(G, \mathbf{B}) \tilde{\setminus} (F, \mathbf{A})$

$= \{((a_{11}, a_{21}, a_{31}), \{\frac{x_2}{<0.2,0.15,0.7>}, \frac{x_3}{<0.8,0.6,0.1>}\}), ((a_{11}, a_{22}, a_{31}), \{\frac{x_2}{<0.5,0.4,0.3>}\})\}.$

Remark 3 *There are four types of plithogenic hypersoft sets namely: plithogenic crisp hypersoft set, plithogenic fuzzy hypersoft set, plithogenic intuitionistic fuzzy hypersoft set, plithogenic neutrosophic hypersoft set. Here we discuss only plithogenic crisp hypersoft point whereas examples for other types of sets can be constructed in the similar way.*

Example 5 Let $\mathrm{U} = \{\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4\}$. Define the attributes sets by:

$$E_1 = \{a_{11}, a_{12}\}, E_2 = \{a_{21}, a_{22}\}, E_3 = \{a_{31}, a_{32}\}.$$

Suppose that

$A_1 = \{a_{11}, a_{12}\}, A_2 = \{a_{21}, a_{22}\}, A_3 = \{a_{31}\}$, and

$B_1 = \{a_{11}\}, B_2 = \{a_{21}, a_{22}\}, B_3 = \{a_{31}, a_{32}\}$

that is,. $A_i, B_i \subseteq E_i$ for each $i = 1,2,3$.

Let the plithogenic crisp hypersoft sets $(F, \mathbf{A})$ and $(G, \mathbf{B})$ be defined by

$(F, \mathbf{A}) = \{((a_{11}, a_{21}, a_{31}), \{x_1(1,0,1), x_2(1,1,1)\}), ((a_{11}, a_{22}, a_{31}), \{x_2(0,0,1)\}),$

$((a_{12}, a_{21}, a_{31}), \{x_3(1,1,0), x_4(1,1,1)\}), ((a_{12}, a_{22}, a_{31}), \{x_1(1,0,1), x_4(0,1,0)\})\}.$

and

$(G, \mathbf{B}) = \{((a_{11}, a_{21}, a_{31}), \{x_2(1,1,1), x_3(1,1,0)\}), ((a_{11}, a_{22}, a_{31}), \{x_2(0,1,0)\}),$

$((a_{11}, a_{21}, a_{32}), \{x_1(0,1,1), x_4(1,1,1)\}), ((a_{11}, a_{22}, a_{32}), \{x_3(1,1,1), x_4(1,1,1)\})\}.$

We have excluded all those $\boldsymbol{\alpha} \in \mathbf{A}$ for which $F(\boldsymbol{\alpha})$ is a null plithogenic crisp set (similarly for those $\boldsymbol{\beta} \in \mathbf{B}$ for which $G(\boldsymbol{\beta})$ is a null plithogenic crisp set).

The union and intersections of $(F, \mathbf{A})$ and $(G, \mathbf{B})$ are given by:

$(F, \mathbf{A}) \tilde{\cup} (G, \mathbf{B}) = \{((a_{11}, a_{21}, a_{31}), \{x_1(1,0,1), x_2(1,1,1), x_3(1,1,0)\}),$

$((a_{11}, a_{22}, a_{31}), \{x_2(0,1,1)\}), ((a_{12}, a_{21}, a_{31}), \{x_3(1,1,0), x_4(1,1,1)\}),$

$((a_{12}, a_{22}, a_{31}), \{x_1(1,0,1), x_4(0,1,0)\}),$

$((a_{11}, a_{21}, a_{32}), \{x_1(0,1,1), x_4(1,1,1)\}), ((a_{11}, a_{22}, a_{32}), \{x_3(1,1,1), x_4(1,1,1)\}),$

$((a_{12}, a_{21}, a_{32}), 0_{PC}), ((a_{12}, a_{22}, a_{32}), 0_{PC})\};$

and

$$(F, \mathbf{A}) \tilde{\cap} (G, \mathbf{B}) = \{((a_{11}, a_{21}, a_{31}), \{x_2(1,1,1)\}), ((a_{11}, a_{22}, a_{31}), 0_{PC})\}.$$

The differences $(F, \mathbf{A}) \tilde{\setminus} (G, \mathbf{B})$ and $(G, \mathbf{B}) \tilde{\setminus} (F, \mathbf{A})$ are the following

$(F, \mathbf{A}) \tilde{\setminus} (G, \mathbf{B}) = \{((a_{11}, a_{21}, a_{31}), \{x_1(1,0,1)\}), ((a_{11}, a_{22}, a_{31}), \{x_2(0,0,1)\})\};$

$(G, \mathbf{B}) \tilde{\setminus} (F, \mathbf{A}) = \{((a_{11}, a_{21}, a_{31}), \{x_3(1,1,0)\}), ((a_{11}, a_{22}, a_{31}), \{x_2(0,1,0)\})\}.$

Proposition 1 *Let* $(F, \boldsymbol{A})$ *be a *-hypersoft set over* U_*. *Then the following holds;*

1. $(F, \mathbf{A}) \tilde{\cup} \Phi_{\mathbf{A}} = (F, \mathbf{A});$

2. $(F, \mathbf{A}) \tilde{\cap} \Phi_{\mathbf{A}} = \Phi_{\mathbf{A}}$;
3. $(F, \mathbf{A}) \tilde{\cup} \tilde{U}_{\mathbf{A}} = \tilde{U}_{\mathbf{A}}$;
4. $(F, \mathbf{A}) \tilde{\cap} \tilde{U}_{\mathbf{A}} = (F, \mathbf{A})$;
5. $\tilde{U}_{\mathbf{A}} \tilde{\setminus} (F, \mathbf{A}) = (F, \mathbf{A})^c$;

6. $(F, \mathbf{A}) \tilde{\cup} (F, \mathbf{A})^c = \tilde{U}_{\mathbf{A}}$;
7. $(F, \mathbf{A}) \tilde{\cap} (F, \mathbf{A})^c = \Phi_{\mathbf{A}}$.

Proof. We will prove only (i), (ii) and (v) and proofs of remaining are similar.

(i) By the definition of union, we have

$$(F, \mathbf{A}) \tilde{\cup} \Phi_{\mathbf{A}} = (H, \mathbf{C}),$$

where $\mathbf{C} = \mathbf{A}$ and $H(\boldsymbol{\alpha}) = F(\boldsymbol{\alpha}) \cup_* 0_* = F(\boldsymbol{\alpha})$ for all $\boldsymbol{\alpha} \in \mathbf{C}$. Hence $(H, \mathbf{C}) = (F, \mathbf{A})$.

(ii) By the definition of intersection, we obtain that

$$(F, \mathbf{A}) \tilde{\cap} \Phi_{\mathbf{A}} = (H, \mathbf{C}),$$

where $\mathbf{C} = \mathbf{A}$ and $H(\boldsymbol{\alpha}) = F(\boldsymbol{\alpha}) \cap_* 0_* = 0_*$ for all $\boldsymbol{\alpha} \in \mathbf{C}$. Hence $(H, \mathbf{C}) = \Phi_{\mathbf{A}}$.

(v) By the definition of difference, we get

$$\tilde{U}_{\mathbf{A}} \tilde{\setminus} (F, \mathbf{A}) = (H, \mathbf{C}),$$

where $\mathbf{C} = \mathbf{A}$ and $H(\boldsymbol{\alpha}) = U \setminus_* F(\boldsymbol{\alpha}) = F^c(\boldsymbol{\alpha})$ for all $\boldsymbol{\alpha} \in \mathbf{C}$. Hence $(H, \mathbf{C}) = (F, \mathbf{A})^c$.

3. Hypersoft point

In this section, we define hypersoft point in different frameworks and study some basic properties of such points in each setup.

3.1 Crisp hypersoft point

Definition 9 *Let $\boldsymbol{A} \subseteq \boldsymbol{E}$, $\boldsymbol{\alpha} \in \boldsymbol{A}$, and $x \in U$. A hypersoft set $(F, \boldsymbol{A})$ is said to be a hypersoft point if $F(\boldsymbol{\alpha}')$ is an empty set for every $\boldsymbol{\alpha}' \in \boldsymbol{A} \setminus \{\boldsymbol{\alpha}\}$ and $F(\boldsymbol{\alpha})$ is a singleton set. We will denote hypersoft point $(F, \boldsymbol{A})$ simply by $P^{(\boldsymbol{\alpha},x)}$.*

Definition 10 *A hypersoft set $(F, \boldsymbol{A})$ is said to be an empty hypersoft point if $F(\boldsymbol{\alpha})$ is an empty set for each $\boldsymbol{\alpha} \in \boldsymbol{A}$. We will denote an empty hypersoft set, corresponding to $\boldsymbol{\alpha}$, by $P^{(\boldsymbol{\alpha},\varnothing)}$.*

As a matter of fact if $(F, \mathbf{A})$ is a null hypersoft set then for every $\boldsymbol{\alpha} \in \mathbf{A}$ it may be regarded as empty hypersoft set $P^{(\boldsymbol{\alpha},\varnothing)}$.

Definition 11 *A hypersoft point $P^{(\boldsymbol{\alpha},x)}$ is said to belong to a hypersoft set $(G, \boldsymbol{A})$ if $P^{(\boldsymbol{\alpha},x)} \tilde{\subseteq} (G, \boldsymbol{A})$. We write it as $P^{(\boldsymbol{\alpha},x)} \tilde{\in} (G, \boldsymbol{A})$.*

It is straightforward to check that the hypersoft union of hypersoft points of a hypersoft set $(G, \mathbf{A})$ returns the hypersoft set $(G, \mathbf{A})$, that is,

$$(G, \mathbf{A}) = \tilde{\cup} \{P^{(\boldsymbol{\alpha},x)} : P^{(\boldsymbol{\alpha},x)} \tilde{\in} (G, \mathbf{A})\}.$$

We illustrate the above observation through the following example.

Example 6 Let $\mathrm{U} = \{\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4\}$, and $(\mathrm{F}, \mathbf{A})$ be as given in the example 1. Then the hypersoft points of $(\mathrm{F}, \mathbf{A})$ are the following:

$$P_1^{((a_{11},a_{21},a_{31}),x_1)} = \{((a_{11}, a_{21}, a_{31}), \{x_1\})\};$$
$$P_2^{((a_{11},a_{21},a_{31}),x_2)} = \{((a_{11}, a_{21}, a_{31}), \{x_2\})\};$$
$$P_3^{((a_{11},a_{22},a_{31}),x_2)} = \{((a_{11}, a_{22}, a_{31}), \{x_2\})\};$$
$$P_4^{((a_{12},a_{21},a_{31}),x_3)} = \{((a_{12}, a_{21}, a_{31}), \{x_3\})\};$$
$$P_5^{((a_{12},a_{21},a_{31}),x_4)} = \{((a_{12}, a_{21}, a_{31}), \{x_4\})\};$$
$$P_6^{((a_{12},a_{22},a_{31}),x_1)} = \{((a_{12}, a_{22}, a_{31}), \{x_1\})\};$$
$$P_7^{((a_{12},a_{22},a_{31}),x_4)} = \{((a_{12}, a_{22}, a_{31}), \{x_4\})\}.$$

Moreover

$$(F, \mathbf{A}) = P_1^{((a_{11},a_{21},a_{31}),x_1)} \tilde{\cup} P_2^{((a_{11},a_{21},a_{31}),x_2)} \tilde{\cup} P_3^{((a_{11},a_{22},a_{31}),x_2)}$$
$$\tilde{\cup} P_4^{((a_{12},a_{21},a_{31}),x_3)} \tilde{\cup} P_5^{((a_{12},a_{21},a_{31}),x_4)} \tilde{\cup} P_6^{((a_{12},a_{22},a_{31}),x_1)} \tilde{\cup} P_7^{((a_{12},a_{22},a_{31}),x_4)}.$$

Proposition 2 *Let $(F, \boldsymbol{A}), (F_1, \boldsymbol{A})$ and $(F_2, \boldsymbol{A})$ be hypersoft sets over U. Then the following hold:*

1. *If $(F, \boldsymbol{A})$ is not a null hypersoft set, then $(F, \boldsymbol{A})$ contains at least one nonempty hypersoft point.*
2. *$(F_1, \boldsymbol{A}) \tilde{\subseteq} (F_2, \boldsymbol{A})$ if and only if $P^{(\alpha,x)} \tilde{\in} (F_1, \boldsymbol{A})$ implies that $P^{(\alpha,x)} \tilde{\in} (F_2, \boldsymbol{A})$.*
3. *$P^{(\alpha,x)} \tilde{\in} (F_1, \boldsymbol{A}) \tilde{\cup} (F_2, \boldsymbol{A})$ if and only if $P^{(\alpha,x)} \tilde{\in} (F_1, \boldsymbol{A})$ or $P^{(\alpha,x)} \tilde{\in} (F_2, \boldsymbol{A})$.*
4. *$P^{(\alpha,x)} \tilde{\in} (F_1, \boldsymbol{A}) \tilde{\cap} (F_2, \boldsymbol{A})$ if and only if $P^{(\alpha,x)} \tilde{\in} (F_1, \boldsymbol{A})$ and $P^{(\alpha,x)} \tilde{\in} (F_2, \boldsymbol{A})$.*
5. *$P^{(\alpha,x)} \tilde{\in} (F_1, \boldsymbol{A}) \tilde{\setminus} (F_2, \boldsymbol{A})$ if and only if $P^{(\alpha,x)} \tilde{\in} (F_1, \boldsymbol{A})$ and $P^{(\alpha,x)} \tilde{\notin} (F_2, \boldsymbol{A})$.*

Proof. We will prove (1), (2) and (3). Proofs of (4) and (5) are similar to that of (3).

(1) Suppose that $(F, \mathbf{A})$ is not a null hypersoft set, that is, $F(\boldsymbol{\alpha}) \neq \varnothing$ for some $\boldsymbol{\alpha} \in \mathbf{A}$. Now if $\boldsymbol{\alpha}_0 \in \mathbf{A}$ is such that $F(\boldsymbol{\alpha}_0) \neq \varnothing$, then for $x \in F(\boldsymbol{\alpha}_0)$, there will be a hypersoft point $P^{(\boldsymbol{\alpha}_0,x)}$ such that $P^{(\boldsymbol{\alpha}_0,x)} \tilde{\in} (F, \mathbf{A})$.

(2) Suppose that $(F_1, \mathbf{A}) \tilde{\subseteq} (F_2, \mathbf{A})$ and $P^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_1, \mathbf{A})$. By the definition 11, we have

$$P^{(\boldsymbol{\alpha},x)} \tilde{\subseteq} (F_1, \mathbf{A}).$$

Thus

$$P^{(\boldsymbol{\alpha},x)} \tilde{\subseteq} (F_1, \mathbf{A}) \tilde{\subseteq} (F_2, \mathbf{A})$$

implies that $P^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_2, \mathbf{A})$.

Conversely suppose that $P^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_1, \mathbf{A})$ which implies that $P^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_2, \mathbf{A})$. By the definition 11, we obtain that

$$P^{(\boldsymbol{\alpha},x)} \tilde{\subseteq} (F_2, \mathbf{A}) \text{forall} P^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_1, \mathbf{A}).$$

Thus we have

$$(F_1, \mathbf{A}) = \tilde{\cup} \{P^{(\boldsymbol{\alpha},x)}: P^{(\boldsymbol{\alpha},x)} \tilde{\in} (G, \mathbf{A})\} \tilde{\subseteq} (F_2, \mathbf{A}).$$

(3) Suppose that $P^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_1, \mathbf{A}) \tilde{\cup} (F_2, \mathbf{A})$. It follows from the definition 11 that

$$P^{(\boldsymbol{\alpha},x)} \tilde{\subseteq} (F_1, \mathbf{A}) \tilde{\cup} (F_2, \mathbf{A}),$$

which implies that $x \in F_1(\boldsymbol{\alpha}) \cup_C F_2(\boldsymbol{\alpha})$. Thus $x \in F_1(\boldsymbol{\alpha})$ or $F_2(\boldsymbol{\alpha})$. Hence we have

$$P^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_1, \mathbf{A}) \text{or} P^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_2, \mathbf{A}).$$

Conversely suppose that $P^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_1, \mathbf{A})$ or $P^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_2, \mathbf{A})$. This implies that $x \in F_1(\boldsymbol{\alpha})$ or $F_2(\boldsymbol{\alpha})$. Thus $x \in F_1(\boldsymbol{\alpha}) \cup_C F_2(\boldsymbol{\alpha})$ and so we have

$$P^{(\boldsymbol{\alpha},x)} \tilde{\subseteq} (F_1, \mathbf{A}) \tilde{\cup} (F_2, \mathbf{A}).$$

3.2 Fuzzy hypersoft point

Definition 12 *Let $\boldsymbol{A} \subseteq \boldsymbol{E}$, $\boldsymbol{\alpha} \in \boldsymbol{A}$, and $x \in U_F$. A fuzzy hypersoft set $(F, \boldsymbol{A})$ is said to be a fuzzy hypersoft point if $F(\boldsymbol{\alpha}')$ is a null fuzzy set for every $\boldsymbol{\alpha}' \in \boldsymbol{A} \backslash \{\boldsymbol{\alpha}\}$ and $F(\boldsymbol{\alpha})(y) = 0$ for all $y \neq x$. We will denote $(F, \boldsymbol{A})$ simply by $FP^{(\boldsymbol{\alpha},x)}$.*

Definition 13 *A fuzzy hypersoft set $(F, \boldsymbol{A})$ is said to be a null fuzzy hypersoft point if $F(\boldsymbol{\alpha})$ is a null fuzzy set for each $\boldsymbol{\alpha} \in \boldsymbol{A}$. We denote a null fuzzy hypersoft set, corresponding to $\boldsymbol{\alpha}$, by $FP^{(\boldsymbol{\alpha},0_F)}$.*

Note that if $(F, \mathbf{A})$ is a null fuzzy hypersoft set then for every $\boldsymbol{\alpha} \in \mathbf{A}$, it can be regarded as null fuzzy hypersoft set $FP^{(\boldsymbol{\alpha},0_F)}$.

Definition 14 *A fuzzy hypersoft point $FP^{(\boldsymbol{\alpha},x)}$ is said to belong to a fuzzy hypersoft set $(G, \boldsymbol{A})$ if $FP^{(\boldsymbol{\alpha},x)} \mathrel{\tilde{\subseteq}} (G, \boldsymbol{A})$. We write it as $FP^{(\boldsymbol{\alpha},x)} \mathrel{\tilde{\in}} (G, \boldsymbol{A})$.*

It is straightforward to check that the fuzzy hypersoft union of fuzzy hypersoft points of a fuzzy hypersoft set $(G, \mathbf{A})$ returns the fuzzy hypersoft set $(G, \mathbf{A})$, that is,

$$(G, \mathbf{A}) = \tilde{\cup} \{FP^{(\boldsymbol{\alpha},x)} : FP^{(\boldsymbol{\alpha},x)} \mathrel{\tilde{\in}} (G, \mathbf{A})\}.$$

We illustrate this observation through the following example.

Example 7 Let $\mathrm{U} = \{\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4\}$, and $(\mathrm{F}, \mathbf{A})$ be as given in the example 2. Then some of the fuzzy hypersoft points of $(\mathrm{F}, \mathbf{A})$ are given as:

$$FP_1^{((a_{11},a_{21},a_{31}),x_1)} = \left\{((a_{11}, a_{21}, a_{31}), \{\tfrac{x_1}{0.5}\})\right\};$$

$$FP_2^{((a_{11},a_{21},a_{31}),x_1)} = \left\{((a_{11}, a_{21}, a_{31}), \{\tfrac{x_1}{0.2}\})\right\};$$

$$FP_3^{((a_{11},a_{21},a_{31}),x_2)} = \left\{((a_{11}, a_{21}, a_{31}), \{\tfrac{x_2}{0.7}\})\right\};$$

$$FP_4^{((a_{11},a_{22},a_{31}),x_2)} = \left\{((a_{11}, a_{22}, a_{31}), \{\tfrac{x_2}{0.3}\})\right\};$$

$$FP_5^{((a_{12},a_{21},a_{31}),x_3)} = \left\{((a_{12}, a_{21}, a_{31}), \{\tfrac{x_3}{0.8}\})\right\};$$

$$FP_6^{((a_{12},a_{21},a_{31}),x_4)} = \left\{((a_{12}, a_{21}, a_{31}), \{\tfrac{x_4}{0.6}\})\right\};$$

$$FP_7^{((a_{12},a_{21},a_{31}),x_4)} = \left\{((a_{12}, a_{21}, a_{31}), \{\tfrac{x_4}{0.9}\})\right\};$$

$$FP_8^{((a_{12},a_{22},a_{31}),x_1)} = \left\{((a_{12}, a_{22}, a_{31}), \{\tfrac{x_1}{0.5}\})\right\};$$

$$FP_9^{((a_{12},a_{22},a_{31}),x_4)} = \left\{((a_{12}, a_{22}, a_{31}), \{\tfrac{x_4}{0.4}\})\right\}.$$

Moreover we have

$$\begin{aligned}(F, \mathbf{A}) &= FP_1^{((a_{11},a_{21},a_{31}),x_1)} \,\tilde{\cup}\, FP_2^{((a_{11},a_{21},a_{31}),x_1)} \,\tilde{\cup}\, FP_3^{((a_{11},a_{21},a_{31}),x_2)} \\ &\tilde{\cup}\, FP_4^{((a_{11},a_{22},a_{31}),x_2)} \,\tilde{\cup}\, FP_5^{((a_{12},a_{21},a_{31}),x_3)} \,\tilde{\cup}\, FP_6^{((a_{12},a_{21},a_{31}),x_4)} \\ &\tilde{\cup}\, FP_7^{((a_{12},a_{21},a_{31}),x_4)} \,\tilde{\cup}\, FP_8^{((a_{12},a_{22},a_{31}),x_1)} \,\tilde{\cup}\, FP_9^{((a_{12},a_{22},a_{31}),x_4)}.\end{aligned}$$

Proposition 3 *Let $(F, \boldsymbol{A}), (F_1, \boldsymbol{A})$ and $(F_2, \boldsymbol{A})$ be fuzzy hypersoft sets over U. Then the following hold:*

1. If $(F, \mathbf{A})$ is not a null fuzzy hypersoft set, then $(F, \mathbf{A})$ contains at least one nonnull fuzzy hypersoft point.
2. $(F_1, \mathbf{A}) \mathrel{\tilde{\subseteq}} (F_2, \mathbf{A})$ if and only if $FP^{(\boldsymbol{\alpha},x)} \mathrel{\tilde{\in}} (F_1, \mathbf{A})$ implies that $FP^{(\boldsymbol{\alpha},x)} \mathrel{\tilde{\in}} (F_2, \mathbf{A})$.

3. $FP^{(\mathbf{\alpha},x)} \tilde{\in} (F_1, \mathbf{A}) \tilde{\cup} (F_2, \mathbf{A})$ if and only if $FP^{(\mathbf{\alpha},x)} \tilde{\in} (F_1, \mathbf{A})$ or $FP^{(\mathbf{\alpha},x)} \tilde{\in} (F_2, \mathbf{A})$.
4. $FP^{(\mathbf{\alpha},x)} \tilde{\in} (F_1, \mathbf{A}) \tilde{\cap} (F_2, \mathbf{A})$ if and only if $FP^{(\mathbf{\alpha},x)} \tilde{\in} (F_1, \mathbf{A})$ and $FP^{(\mathbf{\alpha},x)} \tilde{\in} (F_2, \mathbf{A})$.

5. $FP^{(\mathbf{\alpha},x)} \tilde{\in} (F_1, \mathbf{A}) \tilde{\setminus} (F_2, \mathbf{A})$ if and only if $FP^{(\mathbf{\alpha},x)} \tilde{\in} (F_1, \mathbf{A})$ and $FP^{(\mathbf{\alpha},x)} \tilde{\notin} (F_2, \mathbf{A})$.

The proof of above proposition is similar as in the case of crisp hypersoft point.

3.3 Intuitionistic fuzzy hypersoft point

Definition 15 *Let $\boldsymbol{A} \subseteq \boldsymbol{E}$, $\boldsymbol{\alpha} \in \boldsymbol{A}$, and $x \in U_{IF}$. An intuitionistic fuzzy hypersoft set $(F, \boldsymbol{A})$ is said to be an intuitionistic fuzzy hypersoft point if $F(\boldsymbol{\alpha}')$ is a null intuitionistic fuzzy set for every $\boldsymbol{\alpha}' \in \boldsymbol{A}\setminus\{\boldsymbol{\alpha}\}$ and $F(\boldsymbol{\alpha})(y) =< 0,1 >$ for all $y \neq x$. We will denote $(F, \boldsymbol{A})$ simply by $IFP^{(\boldsymbol{\alpha},x)}$.*

Definition 16 *An intuitionistic fuzzy hypersoft set $(F, \boldsymbol{A})$ is said to be a null intuitionistic fuzzy hypersoft point if $F(\boldsymbol{\alpha})$ is a null intuitionistic fuzzy set for each $\boldsymbol{\alpha} \in \boldsymbol{A}$. We will denote a null intuitionistic fuzzy hypersoft set, corresponding to $\boldsymbol{\alpha}$, by $IFP^{(\boldsymbol{\alpha},0_{IF})}$.*

If $(F, \mathbf{A})$ is a null intuitionistic fuzzy hypersoft set, then for every $\mathbf{\alpha} \in \mathbf{A}$ it can be regarded as null intuitionistic fuzzy hypersoft set $IFP^{(\mathbf{\alpha},0_{IF})}$.

Definition 17 *An intuitionistic fuzzy hypersoft point $IFP^{(\boldsymbol{\alpha},x)}$ is said to belong to an intuitionistic fuzzy hypersoft set $(G, \boldsymbol{A})$ if $IFP^{(\boldsymbol{\alpha},x)} \tilde{\subseteq} (G, \boldsymbol{A})$. We write it as $IFP^{(\boldsymbol{\alpha},x)} \tilde{\in} (G, \boldsymbol{A})$.*

It is straightforward to check that the intuitionistic fuzzy hypersoft union of intuitionistic fuzzy hypersoft points of an intuitionistic fuzzy hypersoft set $(G, \mathbf{A})$ gives the intuitionistic fuzzy hypersoft set $(G, \mathbf{A})$, that is,

$$(G, \mathbf{A}) = \tilde{\cup} \{IFP^{(\mathbf{\alpha},x)} : IFP^{(\mathbf{\alpha},x)} \tilde{\in} (G, \mathbf{A})\}.$$

We illustrate this observation through the following example.

Example 8 Let $\mathrm{U} = \{\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4\}$, and $(\mathrm{F}, \mathbf{A})$ be as given in the example 3. Then some of the intuitionistic fuzzy hypersoft points of $(\mathrm{F}, \mathbf{A})$ are the following:

$$IFP_1^{((a_{11},a_{21},a_{31}),x_1)} = \left\{((a_{11}, a_{21}, a_{31}), \{\frac{x_1}{<0.5,0.3>}\})\right\};$$

$$IFP_2^{((a_{11},a_{21},a_{31}),x_1)} = \left\{((a_{11}, a_{21}, a_{31}), \{\frac{x_1}{<0.2,0.3>}\})\right\};$$

$$IFP_3^{((a_{11},a_{21},a_{31}),x_2)} = \left\{((a_{11}, a_{21}, a_{31}), \{\frac{x_2}{<0.7,0.2>}\})\right\};$$

$$IFP_4^{((a_{11},a_{22},a_{31}),x_2)} = \left\{((a_{11}, a_{22}, a_{31}), \{\frac{x_2}{<0.3,0.5>}\})\right\};$$

$$IFP_5^{((a_{12},a_{21},a_{31}),x_3)} = \left\{((a_{12}, a_{21}, a_{31}), \{\frac{x_3}{<0.8,0.1>}\})\right\};$$

$$IFP_6^{((a_{12},a_{21},a_{31}),x_4)} = \left\{((a_{12}, a_{21}, a_{31}), \{\frac{x_4}{<0.1,0.6>}\})\right\};$$

$$IFP_7^{(a_{12},a_{21},a_{31}),x_4)} = \left\{((a_{12}, a_{21}, a_{31}), \{\frac{x_4}{<0.1,0.5>}\})\right\};$$

$$IFP_8^{((a_{12},a_{22},a_{31}),x_1)} = \left\{((a_{12}, a_{22}, a_{31}), \{\frac{x_1}{<0.5,0.3>}\})\right\};$$

$$IFP_9^{((a_{12},a_{22},a_{31}),x_4)} = \left\{((a_{12}, a_{22}, a_{31}), \{\frac{x_4}{<0.4,0.2>}\})\right\}.$$

Moreover we have

$$(F,\mathbf{A}) = IFP_1^{((a_{11},a_{21},a_{31}),x_1)} \tilde{\cup}\, IFP_2^{((a_{11},a_{21},a_{31}),x_1)} \tilde{\cup}\, IFP_3^{((a_{11},a_{21},a_{31}),x_2)}$$
$$\tilde{\cup}\, IFP_4^{((a_{11},a_{22},a_{31}),x_2)} \tilde{\cup}\, IFP_5^{((a_{12},a_{21},a_{31}),x_3)} \tilde{\cup}\, IFP_6^{((a_{12},a_{21},a_{31}),x_4)}$$
$$\tilde{\cup}\, IFP_7^{((a_{12},a_{21},a_{31}),x_4)} \tilde{\cup}\, IFP_8^{((a_{12},a_{22},a_{31}),x_1)} \tilde{\cup}\, IFP_9^{((a_{12},a_{22},a_{31}),x_4)}.$$

Proposition 4 *Let* $(F,\boldsymbol{A}),(F_1,\boldsymbol{A})$ *and* $(F_2,\boldsymbol{A})$ *be intuitionistic fuzzy hypersoft sets over* U. *Then the following hold:*

1. If $(F,\mathbf{A})$ is not a null intuitionistic fuzzy hypersoft set then $(F,\mathbf{A})$ contains at least one nonnull intuitionistic fuzzy hypersoft point.
2. $(F_1,\boldsymbol{A}) \tilde{\subseteq} (F_2,\boldsymbol{A})$ *if and only if* $IFP^{(\alpha,x)} \tilde{\in} (F_1,\boldsymbol{A})$ *implies that* $IFP^{(\alpha,x)} \tilde{\in} (F_2,\boldsymbol{A})$.
3. $IFP^{(\alpha,x)} \tilde{\in} (F_1,\boldsymbol{A}) \tilde{\cup} (F_2,\boldsymbol{A})$ *if and only if* $IFP^{(\alpha,x)} \tilde{\in} (F_1,\boldsymbol{A})$ *or* $IFP^{(\alpha,x)} \tilde{\in} (F_2,\boldsymbol{A})$.
4. $IFP^{(\alpha,x)} \tilde{\in} (F_1,\boldsymbol{A}) \tilde{\cap} (F_2,\boldsymbol{A})$ *if and only if* $IFP^{(\alpha,x)} \tilde{\in} (F_1,\boldsymbol{A})$ *and* $IFP^{(\alpha,x)} \tilde{\in} (F_2,\boldsymbol{A})$.
5. $IFP^{(\alpha,x)} \tilde{\in} (F_1,\boldsymbol{A}) \tilde{\setminus} (F_2,\boldsymbol{A})$ *if and only if* $IFP^{(\alpha,x)} \tilde{\in} (F_1,\boldsymbol{A})$ *and* $IFP^{(\alpha,x)} \tilde{\notin} (F_2,\boldsymbol{A})$.

The proof of above proposition is similar as in the case of crisp hypersoft point.

3.4 Neutrosophic hypersoft point

Definition 18 *Let* $\boldsymbol{A} \subseteq \boldsymbol{E}$ *and* $\boldsymbol{\alpha} \in \boldsymbol{A}$, $x \in U_N$. *A neutrosophic hypersoft set* $(F,\boldsymbol{A})$ *is said to be a neutrosophic fuzzy hypersoft point if* $F(\boldsymbol{\alpha}')$ *is a null neutrosophic set for every* $\boldsymbol{\alpha}' \in \boldsymbol{A}\setminus\{\boldsymbol{\alpha}\}$ *and* $F(\boldsymbol{\alpha})(y) =< 0,1,1 >$ *for all* $y \neq x$. *We will denote* $(F,\boldsymbol{A})$ *simply by* $NP^{(\alpha,x)}$.

Definition 19 *A neutrosophic hypersoft set* $(F,\boldsymbol{A})$ *is said to be a null neutrosophic hypersoft point if* $F(\boldsymbol{\alpha})$ *is a null neutrosophic set for each* $\boldsymbol{\alpha} \in \boldsymbol{A}$. *We will denote a null neutrosophic hypersoft set, corresponding to* $\boldsymbol{\alpha}$, *by* $NP^{(\alpha,0_N)}$.

Its a matter of fact that if $(F,\mathbf{A})$ is a null neutrosophic hypersoft set then for every $\boldsymbol{\alpha} \in \mathbf{A}$ it can be regarded as null neutrosophic hypersoft set $NP^{(\alpha,0_N)}$.

Definition 20 *A neutrosophic hypersoft point* $NP^{(\alpha,x)}$ *is said to belong to a neutrosophic hypersoft set* $(G,\boldsymbol{A})$ *if* $NP^{(\alpha,x)} \tilde{\subseteq} (G,\boldsymbol{A})$. *We write it as* $NP^{(\alpha,x)} \tilde{\in} (G,\boldsymbol{A})$.

It is straightforward to check that the neutrosophic hypersoft union of neutrosophic hypersoft points of a neutrosophic hypersoft set $(G,\mathbf{A})$ returns the neutrosophic hypersoft set $(G,\mathbf{A})$, that is,

$$(G,\mathbf{A}) = \tilde{\cup}\, \{NP^{(\alpha,x)} : NP^{(\alpha,x)} \tilde{\in} (G,\mathbf{A})\}.$$

We illustrate this observation through the following example.

Example 9 Let $U = \{x_1, x_2, x_3, x_4\}$, and $(F,\mathbf{A})$ be as given in the example 4. Some of the neutrosophic hypersoft points of $(F,\mathbf{A})$ are the following:

$$NP_1^{((a_{11},a_{21},a_{31}),x_1)} = \left\{((a_{11},a_{21},a_{31}),\{\frac{x_1}{<0.5,0.2,0.3>}\})\right\};$$

$$NP_2^{((a_{11},a_{21},a_{31}),x_1)} = \left\{((a_{11},a_{21},a_{31}),\{\frac{x_1}{<0.2,0.2,0.3>}\})\right\};$$

$$NP_3^{((a_{11},a_{21},a_{31}),x_2)} = \left\{((a_{11},a_{21},a_{31}),\{\frac{x_2}{<0.7,0.3,0.2>}\})\right\};$$

$$NP_4^{((a_{11},a_{22},a_{31}),x_2)} = \left\{((a_{11},a_{22},a_{31}),\{\frac{x_2}{<0.3,0.2,0.5>}\})\right\};$$

$$NP_5^{((a_{12},a_{21},a_{31}),x_3)} = \left\{((a_{12},a_{21},a_{31}),\{\frac{x_3}{<0.8,0.4,0.1>}\})\right\};$$

$$NP_6^{((a_{12},a_{21},a_{31}),x_4)} = \left\{((a_{12},a_{21},a_{31}),\{\frac{x_4}{<0.1,0.5,0.5>}\})\right\};$$

$$NP_7^{((a_{12},a_{22},a_{31}),x_1)} = \left\{((a_{12},a_{22},a_{31}),\{\frac{x_1}{<0.5,0.2,0.3>}\})\right\};$$

$$NP_8^{((a_{12},a_{22},a_{31}),x_4)} = \left\{((a_{12},a_{22},a_{31}),\{\frac{x_4}{<0.4,0.3,0.2>}\})\right\}.$$

Moreover we have

$$(F,\mathbf{A}) = NP_1^{((a_{11},a_{21},a_{31}),x_1)} \tilde{\cup} NP_2^{((a_{11},a_{21},a_{31}),x_1)} \tilde{\cup} NP_3^{((a_{11},a_{21},a_{31}),x_2)}$$
$$\tilde{\cup} NP_4^{((a_{11},a_{22},a_{31}),x_2)} \tilde{\cup} NP_5^{((a_{12},a_{21},a_{31}),x_3)} \tilde{\cup} NP_6^{((a_{12},a_{21},a_{31}),x_4)}$$
$$\tilde{\cup} NP_7^{((a_{12},a_{22},a_{31}),x_1)} \tilde{\cup} NP_8^{((a_{12},a_{22},a_{31}),x_4)}.$$

Proposition 5 *Let $(F,\boldsymbol{A}),(F_1,\boldsymbol{A})$ and $(F_2,\boldsymbol{A})$ be neutrosophic hypersoft sets over U. Then the following hold:*

1. *If $(F,\boldsymbol{A})$ is not a null neutrosophic hypersoft set then $(F,\boldsymbol{A})$ contains at least one nonnull neutrosophic hypersoft point.*
2. *$(F_1,\boldsymbol{A}) \tilde{\subseteq} (F_2,\boldsymbol{A})$ if and only if $NP^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_1,\boldsymbol{A})$ implies that $NP^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_2,\boldsymbol{A})$.*
3. *$NP^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_1,\boldsymbol{A}) \tilde{\cup} (F_2,\boldsymbol{A})$ if and only if $NP^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_1,\boldsymbol{A})$ or $NP^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_2,\boldsymbol{A})$.*
4. *$NP^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_1,\boldsymbol{A}) \tilde{\cap} (F_2,\boldsymbol{A})$ if and only if $NP^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_1,\boldsymbol{A})$ and $NP^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_2,\boldsymbol{A})$.*
5. *$NP^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_1,\boldsymbol{A}) \tilde{\setminus} (F_2,\boldsymbol{A})$ if and only if $NP^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_1,\boldsymbol{A})$ and $NP^{(\boldsymbol{\alpha},x)} \tilde{\notin} (F_2,\boldsymbol{A})$.*

The proof of above proposition is similar as in the case of crisp hypersoft point.

3.5 Plithogenic hypersoft point

There may be four types of plithogenic hypersoft points namely: plithogenic crisp hypersoft point, plithogenic fuzzy hypersoft point, plithogenic intuitionistic fuzzy hypersoft point, plithogenic neutrosophic hypersoft point. But in this section we discuss only plithogenic crisp hypersoft point whereas other concepts and examples can be given in the similar way.

Definition 21 *Let $\boldsymbol{A} \subseteq \boldsymbol{E}$, $\boldsymbol{\alpha} \in \boldsymbol{A}$, and $x \in U_P$. A plithogenic crisp hypersoft set $(F,\boldsymbol{A})$ is said to be a plithogenic crisp hypersoft point if $F(\boldsymbol{\alpha}')$ is a null plithogenic crisp set for every $\boldsymbol{\alpha}' \in \boldsymbol{A}\setminus\{\boldsymbol{\alpha}\}$ and $F(\boldsymbol{\alpha})(y)(\boldsymbol{0})$ for all $y \neq x$. We will denote $(F,\boldsymbol{A})$ simply by $P_cP^{(\boldsymbol{\alpha},x)}$.*

Definition 22 *A plithogenic crisp hypersoft set $(F,\boldsymbol{A})$ is said to be a null plithogenic crisp hypersoft point if $F(\boldsymbol{\alpha})$ is a null plithogenic crisp set for each $\boldsymbol{\alpha} \in \boldsymbol{A}$. We will denote a null plithogenic crisp hypersoft set, corresponding to $\boldsymbol{\alpha}$, by $P_cP^{(\boldsymbol{\alpha},0_{PC})}$.*

Note that if $(F,\mathbf{A})$ is a null plithogenic crisp hypersoft set, then for every $\boldsymbol{\alpha} \in \mathbf{A}$ it can be regarded as a null plithogenic crisp hypersoft set $P_cP^{(\boldsymbol{\alpha},0_{PC})}$.

Definition 23 *A plithogenic crisp hypersoft point $P_cP^{(\boldsymbol{\alpha},x)}$ is said to belong to a plithogenic crisp hypersoft set $(G,\boldsymbol{A})$ if $P_cP^{(\boldsymbol{\alpha},x)} \tilde{\subseteq} (G,\boldsymbol{A})$. We write it as $P_cP^{(\boldsymbol{\alpha},x)} \tilde{\in} (G,\boldsymbol{A})$.*

It is straightforward to check that the plithogenic crisp hypersoft union of plithogenic crisp hypersoft points of a plithogenic crisp hypersoft set $(G,\mathbf{A})$ gives back the plithogenic crisp hypersoft set $(G,\mathbf{A})$, that is,

$$(G,\mathbf{A}) = \tilde{\cup}\{P_cP^{(\boldsymbol{\alpha},x)} : P_cP^{(\boldsymbol{\alpha},x)} \tilde{\in} (G,\mathbf{A})\}.$$

We illustrate this observation through the following example.

Example 10 Let $\mathrm{U} = \{\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3,\mathrm{x}_4\}$, and $(\mathrm{F},\mathbf{A})$ be as given in the example 5. Then some of the plithogenic crisp hypersoft points of $(\mathrm{F},\mathbf{A})$ are the following:

$$P_cP_1^{((a_{11},a_{21},a_{31}),x_1)} = \{((a_{11},a_{21},a_{31}),\{x_1(1,0,1)\})\};$$
$$P_cP_2^{((a_{11},a_{21},a_{31}),x_2)} = \{((a_{11},a_{21},a_{31}),\{x_2(1,1,1)\})\};$$
$$P_cP_3^{((a_{11},a_{22},a_{31}),x_2)} = \{((a_{11},a_{22},a_{31}),\{x_2(0,0,1)\})\};$$
$$P_cP_4^{((a_{12},a_{21},a_{31}),x_3)} = \{((a_{12},a_{21},a_{31}),\{x_3(1,1,0)\})\};$$
$$P_cP_5^{((a_{12},a_{21},a_{31}),x_4)} = \{((a_{12},a_{21},a_{31}),\{x_4(1,1,1)\})\};$$
$$P_cP_6^{((a_{12},a_{22},a_{31}),x_1)} = \{((a_{12},a_{22},a_{31}),\{x_1(1,0,1)\})\};$$
$$P_cP_7^{((a_{12},a_{22},a_{31}),x_4)} = \{((a_{12},a_{22},a_{31}),\{x_4(0,1,0)\})\}.$$

Moreover we have

$$(F,\mathbf{A}) = P_cP_1^{((a_{11},a_{21},a_{31}),x_1)} \tilde{\cup} P_cP_2^{((a_{11},a_{21},a_{31}),x_2)}$$
$$\tilde{\cup} P_cP_3^{((a_{11},a_{22},a_{31}),x_2)} \tilde{\cup} P_cP_4^{((a_{12},a_{21},a_{31}),x_3)} \tilde{\cup} P_cP_5^{((a_{12},a_{21},a_{31}),x_4)}$$
$$\tilde{\cup} P_cP_6^{((a_{12},a_{22},a_{31}),x_1)} \tilde{\cup} P_cP_7^{((a_{12},a_{22},a_{31}),x_4)}.$$

Proposition 6 *Let $(F,\boldsymbol{A}),(F_1,\boldsymbol{A})$ and $(F_2,\boldsymbol{A})$ be plithogenic crisp hypersoft sets over U. Then the following hold:*

1. *If $(F,\boldsymbol{A})$ is not a null plithogenic crisp hypersoft set then $(F,\boldsymbol{A})$ contains at least one nonnull plithogenic crisp hypersoft point.*
2. $(F_1,\boldsymbol{A}) \tilde{\subseteq} (F_2,\boldsymbol{A})$ *if and only if* $P_cP^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_1,\boldsymbol{A})$ *implies that* $P_cP^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_2,\boldsymbol{A})$.
3. $P_cP^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_1,\boldsymbol{A}) \tilde{\cup} (F_2,\boldsymbol{A})$ *if and only if* $P_cP^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_1,\boldsymbol{A})$ *or* $P_cP^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_2,\boldsymbol{A})$.
4. $P_cP^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_1,\boldsymbol{A}) \tilde{\cap} (F_2,\boldsymbol{A})$ *if and only if* $P_cP^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_1,\boldsymbol{A})$ *and* $P_cP^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_2,\boldsymbol{A})$.
5. $P_cP^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_1,\boldsymbol{A}) \tilde{\setminus} (F_2,\boldsymbol{A})$ *if and only if* $P_cP^{(\boldsymbol{\alpha},x)} \tilde{\in} (F_1,\boldsymbol{A})$ *and* $P_cP^{(\boldsymbol{\alpha},x)} \tilde{\notin} (F_2,\boldsymbol{A})$.

The proof of above proposition is similar as in the case of crisp hypersoft point.

4. Conclusions

In this paper, we have initiated the concept of hypersoft point that will lead to define Cartesian product and then function on *-hypersoft sets. As a future work, one may carry out the study of *-hypersoft topological spaces. Once the functions on *-hypersoft sets are defined, this may lead to the study of fixed point results in this new framework.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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Received: Apr 15, 2020. Accepted: July 5 2020

Neutrosophic Sets and Systems, Vol. 35, 2020
University of New MeSico

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Abstract: In this paper, we introduce the notion of neutrosophic ℵ-bi-ideal for a semigroup. We infer different semigroups using neutrosophic ℵ-bi-ideal structures. Moreover, for regular semigroups, neutrosophic ℵ-product and intersection of neutrosophic ℵ-ideals are identical.

Keywords: Semigroup, ideal, bi-ideal, neutrosophic ℵ− ideals, neutrosophic ℵ -bi-ideals, neutrosophic ℵ −product.

1. Introduction

In 1965, Zadeh [16] introduced the idea of fuzzy sets for modeling the ambiguous theories in the globe. In 1986, Atanassov [1] generalized fuzzy set and named as intuitionistic fuzzy set, and discussed it. Also from his view point, there are two degrees for any object in the world. They are degree of membership to a vague subset and degree of non-membership to that given subset.

Smarandache generalized fuzzy and intuitionistic fuzzy set, and referred as Neutrosophic set (see [2, 3, 6, 13-15]). It is identified by a truth, a falsity and an indeterminacy membership function. These sets are applied to many branches of mathematics to overcome the complexities arising from uncertain data. Neutrosophic set can distinguish between absolute membership and relative membership. Smarandache used this in non-standard analysis such as result of sport games (winning/defeating/tie), decision making and control theory, etc. This area has been studied by several authors (see [5, 10-12]).

In [8], M. Khan et al. presented and discussed the concepts of neutrosophic ℵ −subsemigroup of semigroup. In [5], Gulistan et al. have studied the idea of complex neutrosophic subsemigroups. They have introduced the notion of characteristic function of complex neutrosophic sets, direct product of complex neutrosophic sets.

In [4], B. Elavarasan et al. introduced the concepts of neutrosophic ℵ −ideal of semigroup and explored its properties. Also, the conditions are given for neutrosophic ℵ −structure becomes neutrosophic ℵ −ideal. Further, presented the notion of characteristic neutrosophic ℵ −structure over semigroup.

Throughout this article, X denotes a semigroup. Recall that for any subsets A and B of X, $AB = \{uw | u \in A \text{ and } w \in B\}$, the multiplication of A and B.

For a semigroup X,

(i) $\emptyset \neq U \subseteq X$ is a subsemigroup of X if $U^2 \subseteq U$.

(ii) A subsemigroup U of X is left (resp., right) ideal if $XU \subseteq U$ (resp., $UX \subseteq U$). U is an ideal of X if U is both left and right ideal of X.

(iii) X is left (resp., right) regular if for each $s \in X$, there exists $x \in X$ such that $s = xs^2$(resp., $s = s^2x$) [7].

(iv) X is regular if for each $s \in X$, there exists $x \in X$ such that $s = sxs$ [9].

(v) X is intra-regular if for every $s \in X$, there exist $x, y \in X$ such that $s = xs^2y$ [9].

(vi) A subsemigroup Y of X is bi-ideal if $YXY \subseteq Y$. For any $r' \in X$, $B(r') = \{r', r'^2, r'Xr'\}$ is the principal bi-ideal of X generated by r'.

2. Basics of neutrosophic ℵ – structures

In this section, we present the required basic definitions of neutrosophic ℵ −structures of X that we need in the sequel.

The collection of functions from a set X to $[-1, 0]$ is denoted by $\Im(X, [-1, 0])$. Note that $f \in \Im(X, [-1, 0])$ is a negative-valued function from X to $[-1, 0]$ (briefly, ℵ −function on X). Here ℵ −structure means (X, f) of X.

Definition 2.1. [8] A neutrosophic ℵ − structure of X is defined to be the structure:

$$X_N := \frac{X}{(T_N, I_N, F_N)} = \left\{ \frac{x}{T_N(x), I_N(x), F_N(x)} \mid x \in X \right\}$$

where T_N is the negative truth membership function on X, I_N is the negative indeterminacy membership function on X and F_N is the negative falsity membership function on X.

Note that for any $x \in X$, X_N satisfies the condition $-3 \leq T_N(x) + I_N(x) + F_N(x) \leq 0$.

Definition 2.2. [8] A neutrosophic ℵ −structure X_N of X is called a neutrosophic ℵ −subsemigroup of X if the below condition is valid:

$$(\forall\, g_i, h_j \in X) \begin{pmatrix} T_N(g_ih_j) \leq T_N(g_i) \vee T_N(h_j) \\ I_N(g_ih_j) \geq I_N(g_i) \wedge I_N(h_j) \\ F_N(g_ih_j) \leq F_N(g_i) \vee F_N(h_j) \end{pmatrix}.$$

Let X_N be a neutrosophic ℵ − structure of X and let $\lambda, \delta, \varepsilon \in [-1, 0]$ with $-3 \leq \lambda + \delta + \varepsilon \leq 0$. Then the set $X_N(\lambda, \delta, \varepsilon) := \{x \in X | T_N(x) \leq \lambda,\ I_N(x) \geq \delta,\ F_N(x) \leq \varepsilon\}$ is called a *(λ, δ, ε)* – level set of X_N.

Definition 2.3. [4] A neutrosophic ℵ −structure X_N of X is called a neutrosophic ℵ −left (resp., right) ideal of X if it satisfies:

$$(\forall\, g_i, h_j \in X) \begin{pmatrix} T_N(g_ih_j) \leq T_N(h_j)\ (resp., T_N(g_ih_j) \leq T_N(g_i)) \\ I_N(g_ih_j) \geq I_N(h_j)\ (resp., I_N(g_ih_j) \geq I_N(g_i)) \\ F_N(g_ih_j) \leq F_N(h_j)\ (resp., F_N(g_ih_j) \leq F_N(g_i)) \end{pmatrix}.$$

If X_N is both neutrosophic ℵ −left and neutrosophic ℵ −right ideal of X, then it is called a neutrosophic ℵ −ideal of X.

Definition 2.4. A neutrosophic ℵ −subsemigroup X_N of X is a neutrosophic ℵ −bi-ideal of X if the following condition is valid:

$$(\forall\, r,s,t \in X)\begin{pmatrix} T_N(rst) \le T_N(r) \vee T_N(t) \\ I_N(rst) \ge I_N(r) \wedge I_N(t) \\ F_N(rst) \le F_N(r) \vee F_N(t) \end{pmatrix}.$$

Clearly any neutrosophic $\aleph-$left (resp., right) ideal is neutrosophic $\aleph-$bi-ideal, but the neutrosophic $\aleph-$bi-ideal is not necessary to be a neutrosophic $\aleph-$left (resp., right) ideal.

Example 2.5. Consider the semigroup $X = \{0, a, b, c\}$ with binary operation as follows:

.	0	a	b	c
0	0	0	0	0
a	0	0	0	b
b	0	0	0	b
c	b	b	b	c

Then $X_N = \left\{\frac{0}{(-0.9,-0.1,-0.7)}, \frac{a}{(-0.8,-0.2,-0.5)}, \frac{b}{(-0.7,-0.3,-0.3)}, \frac{c}{(-0.5,-0.4,-0.1)}\right\}$ is a neutrosophic $\aleph-$bi-ideal of X, but X_N is not neutrosophic $\aleph-$left ideal as well as neutrosophic $\aleph-$right ideal of X. □

Definition 2.6. [8] For $\Phi \neq A \subseteq X$, the characteristic neutrosophic $\aleph-$structure of X is denoted by $\chi_A(X_N)$ and is defined to be neutrosophic $\aleph-$structure

$$\chi_A(X_N) = \frac{X}{(\chi_A(T)_N, \chi_A(I)_N, \chi_A(F)_N)}$$

where

$$\chi_A(T)_N : X \to [-1, 0],\ x \to \begin{cases} -1 \ if\ x \in A \\ 0 \ \ otherwise, \end{cases}$$

$$\chi_A(I)_N : X \to [-1, 0],\ x \to \begin{cases} 0 \ if\ x \in A \\ -1\ otherwise, \end{cases}$$

$$\chi_A(F)_N : X \to [-1, 0],\ x \to \begin{cases} -1\ \ if\ x \in A \\ 0\ \ otherwise. \end{cases}$$

Definition 2.7. [8] Let $X_N := \frac{X}{(T_N, I_N, F_N)}$ and $X_M := \frac{X}{(T_M, I_M, F_M)}$.

(i) X_M is called a neutrosophic $\aleph-$ substructure of X_N over X, denoted by $X_N \subseteq X_M$, if
$T_N(t) \ge T_M(t),\ I_N(t) \le I_M(t),\ F_N(t) \ge F_M(t)\ \forall t \in X.$
If $X_N \subseteq X_M$ and $X_M \subseteq X_N$, then we say that $X_N = X_M$.

(ii) The neutrosophic $\aleph-$ product of X_N and X_M is defined to be a neutrosophic $\aleph-$structure of X,

$$X_N \odot X_M := \frac{X}{(T_{N\circ M}, I_{N\circ M}, F_{N\circ M})} = \left\{\frac{h}{T_{N\circ M}(h),\ I_{N\circ M}(h),\ F_{N\circ M}(h)} \mid h \in X\right\},$$

where

$$(T_N \circ T_M)(h) = T_{N\circ M}(h) = \begin{cases} \bigwedge\limits_{h=rs} \{T_N(r) \vee T_M(s)\} & if\ \exists\ r,s \in X\ such\ that\ h = rs \\ 0 & otherwise, \end{cases}$$

$$(I_N \circ I_M)(h) = I_{N\circ M}(h) = \begin{cases} \bigvee\limits_{h=rs} \{I_N(r) \wedge I_M(s)\} & if\ \exists\ r,s \in X\ such\ that\ h = rs \\ -1 & otherwise, \end{cases}$$

$$(F_N \circ F_M)(h) = F_{N\circ M}(h) = \begin{cases} \bigwedge\limits_{h=rs} \{F_N(r) \vee F_M(s)\} & if\ \exists\ r,s \in X\ such\ that\ h = rs \\ 0 & otherwise. \end{cases}$$

(iii) For $t \in X$, the element $\frac{t}{(T_{N\circ M}(t),\ I_{N\circ M}(t),\ F_{N\circ M}(t))}$ is simply denoted by

$(X_N \odot X_M)(t) = (T_{N\circ M}(t),\ I_{N\circ M}(t),\ F_{N\circ M}(t))$ for the sake of convenience.

(iv) The union of X_N and X_M is a neutrosophic $\aleph$ −structure over X is defined as

$$X_N \cup X_M = X_{N\cup M} = \left(X; T_{N\cup M},\ I_{N\cup M},\ F_{N\cup M}\right),$$

where

$$(T_N \cup T_M)(h_i) = T_{N\cup M}(h_i) = T_N(h_i) \wedge T_M(h_i),$$
$$(I_N \cup I_M)(h_i) = I_{N\cup M}(h_i) = I_N(h_i) \vee I_M(h_i),$$
$$(F_N \cup F_M)(h_i) = F_{N\cup M}(h_i) = F_N(h_i) \wedge F_M(h_i)\ \ \forall h_i \in X.$$

(v) The intersection of X_N and X_M is a neutrosophic $\aleph$ −structure over X is defined as

$$X_N \cap X_M = X_{N\cap M} = \left(X; T_{N\cap M},\ I_{N\cap M},\ F_{N\cap M}\right),$$

where

$$(T_N \cap T_M)(h_i) = T_{N\cap M}(h_i) = T_N(h_i) \vee T_M(h_i),$$
$$(I_N \cap I_M)(h_i) = I_{N\cap M}(h_i) = I_N(h_i) \wedge I_M(h_i),$$
$$(F_N \cap F_M)(h_i) = F_{N\cap M}(h_i) = F_N(h_i) \vee F_M(h_i)\ \ \forall\ \ h_i \in X.$$

3. Neutrosophic $\aleph$ −bi-ideals of semigroups

In this section, we examine different properties of neutrosophic $\aleph$ −bi-ideals of X.

Theorem 3.1. For $\Phi \neq B \subseteq X$, the following assertions are equivalent:

(i) $\chi_B(X_N)$ is a neutrosophic $\aleph$ −bi-ideal of X,

(ii) B is a bi-ideal of X.

Proof: Suppose $\chi_B(X_N)$ is a neutrosophic $\aleph$ −bi-ideal of X. Let $r, t \in B$ and $s \in X$. Then

$$\chi_B(T)_N(rst) \leq \chi_B(T)_N(r) \vee \chi_B(T)_N(t) = -1,$$
$$\chi_B(I)_N(rst) \geq \chi_B(I)_N(r) \wedge \chi_B(I)_N(t) = 0,$$
$$\chi_B(F)_N(rst) \leq \chi_B(F)_N(r) \vee \chi_B(F)_N(t) = -1.$$

Thus $rst \in B$ and hence B is a bi-ideal of X,

Conversely, assume B is a bi-ideal of X. Let $r, s, t \in X$.

If $r \in B$ and $t \in B$, then $rst \in B$. Now

$$\chi_B(T)_N(rst) = -1 = \chi_B(T)_N(r) \vee \chi_B(T)_N(t),$$
$$\chi_B(I)_N(rst) = 0 = \chi_B(I)_N(r) \wedge \chi_B(I)_N(t),$$
$$\chi_B(F)_N(rst) = -1 = \chi_B(F)_N(r) \vee \chi_B(F)_N(t).$$

If $r \notin B$ or $t \notin B$, then

$$\chi_B(T)_N(rst) \leq 0 = \chi_B(T)_N(r) \vee \chi_B(T)_N(t),$$
$$\chi_B(I)_N(rst) \geq -1 = \chi_B(I)_N(r) \wedge \chi_B(I)_N(t)$$
$$\chi_B(F)_N(rst) \leq 0 = \chi_B(F)_N(r) \vee \chi_B(F)_N(t).$$

Therefore $\chi_B(X_N)$ is a neutrosophic $\aleph$ −bi-ideal of X. □

Theorem 3.2. Let $\lambda, \delta, \varepsilon \in [-1, 0]$ be such that $-3 \leq \lambda + \delta + \varepsilon \leq 0$. If X_N is a neutrosophic $\aleph$ −bi-ideal, then $(\lambda, \delta, \varepsilon)$ −level set of X_N is a neutrosophic bi- ideal of X whenever $X_N(\lambda, \delta, \varepsilon) \neq \emptyset$.

Proof: Suppose $X_N(\lambda, \delta, \varepsilon) \neq \emptyset$ for $\lambda, \delta, \varepsilon \in [-1, 0]$ with $-3 \leq \lambda + \delta + \varepsilon \leq 0$. Let X_N be a neutrosophic $\aleph$ −bi-ideal and let $x, y, z \in X_N(\lambda, \delta, \varepsilon)$. Then

$$T_N(xyz) \leq T_N(x) \vee T_N(z) \leq \lambda,$$

$$I_N(xyz) \geq I_N(x) \wedge I_N(z) \geq \delta,$$

$$F_N(xyz) \leq F_N(x) \vee F_N(z) \leq \varepsilon$$

which imply $xyz \in X_N(\lambda, \delta, \varepsilon)$. Therefore $X_N(\lambda, \delta, \varepsilon)$ is a neutrosophic $\aleph$ −bi-ideal of X. □

Theorem 3.3. Let X_M be a neutrosophic $\aleph -$ structure of X. Then the equivalent assertions are:

(i) $X_M \odot X_M \subseteq X_M$ and $X_M \odot \chi_X(X_N) \odot X_M \subseteq X_M$ for any neutrosophic $\aleph -$ structure X_N,

(ii) X_M is a neutrosophic $\aleph$ −bi-ideal of X.

Proof: Suppose (i) holds. Then X_M is neutrosophic $\aleph -$ subsemigroup of X by Theorem 4.6 of [8]. Let $r, s, t \in X$ and let $a = rst$. Then

$$(T_M)(rst) \leq (T_M \circ \chi_X(T)_N \circ T_M)(rst) = \bigwedge_{a=rst} \{(T_M \circ \chi_X(T)_N)(rs) \vee T_M(t)\}$$

$$= \bigwedge_{a=bt} \{ \bigwedge_{b=rs} \{(T_M(r) \vee \chi_X(T)_N(s)\} \vee T_M(t)\}$$

$$\leq \bigwedge \{T_M(r) \vee T_M(t)\} \leq T_M(r) \vee T_M(t),$$

$$I_M(rst) \geq (I_M \circ \chi_X(I)_N \circ I_M)(rst) = \bigvee_{a=rst} \{(I_M \circ \chi_X(I)_N)(rs) \wedge I_M(t)\}$$

$$= \bigvee_{a=bt} \{ \bigvee_{b=rs} \{ I_M(r) \wedge \chi_X(I)_N(s)\} \wedge I_M(t)\}$$

$$\geq \bigvee_{a=rst} \{ I_M(r) \wedge I_M(t)\} \geq I_M(r) \wedge I_M(t),$$

$$(F_M)(rst) \leq (F_M \circ \chi_X(F)_N \circ F_M)(rst) = \bigwedge_{a=rst} \{(F_M \circ \chi_X(F)_N)(rs) \vee F_M(t)\}$$

$$= \bigwedge_{a=bt} \{ \bigwedge_{b=rs} \{(F_M(r) \vee \chi_X(F)_N(s)\} \vee F_M(t)\}$$

$$\leq \bigwedge_{a=rst} \{F_M(r) \vee F_M(t)\} \leq F_M(r) \vee F_M(t).$$

Therefore X_M is a neutrosophic $\aleph -$ bi-ideal of X.

For converse, suppose (ii) holds. Then $X_M \odot X_M \subseteq X_M$ by Theorem 4.6 of [8].

Let $x \in X$. If $x = rb$ and r$= st$ for some r$, b, s, t \in X$, then

$$(T_M \circ \chi_X(T)_N \circ T_M)(x) = \bigwedge_{x=rb} \{(T_M \circ \chi_X(T)_N)(r) \vee T_M(b)\}$$

$$= \bigwedge_{x=rb} \{ \bigwedge_{r=st} \{T_M(s) \vee \chi_X(T)_N(t)\} \vee T_M(b)\}$$

$$= \bigwedge_{x=rb} \{ \bigwedge_{r=st} \{(T_M(s)\} \vee T_M(b)\}$$

$= \wedge_{x=rb}\{T_M(s_i) \vee T_M(b)\}$ for some $s_i \in X$ and r$= s_i t_i$

$$\geq \bigwedge_{x=s_i t_i b} T_M(s_i t_i b) = T_M(x),$$

$$(I_M \circ \chi_X(I)_N \circ I_M)(x) = \bigvee_{x=rb} \{(I_M \circ \chi_X(I)_N)(r) \wedge I_M(b)\}$$

$$= \bigvee_{x=rb} \{ \bigvee_{r=pq} \{ I_M(s) \wedge \chi_X(I)_N(t)\} \wedge I_M(b)\}$$
$$= \bigvee_{x=rb} \{ \bigvee_{r=st} \{ I_M(s)\} \wedge I_M(b)\}$$

$= \vee_{x=ab}\{I_M(s_i) \wedge I_M(b)\}$, for some $s_i \in X$ and $r = s_i t_i$

$$\leq \bigvee_{x=s_i t_i b} I_M(s_i t_i b) = I_M(x),$$

$$(F_M \circ \chi_X(F)_N \circ F_M)(x) = \bigwedge_{x=rb} \{(F_M \circ \chi_X(F)_N)(r) \vee F_M(b)\}$$

$$= \bigwedge_{x=rb} \{ \bigwedge_{a=st} \{(F_M(s) \vee \chi_X(F)_N(t)\} \vee F_M(b)\}$$

$$= \bigwedge_{x=rb} \{ \bigwedge_{r=st} \{(F_M(s)\} \vee F_M(b)\}$$
$= \wedge_{x=rb}\{F_M(s_i) \vee F_M(b)\}$ for some $s_i \in X$ and $a = s_i t_i$

$$\geq \bigwedge_{x=s_i t_i b} F_M(s_i t_i b) = F_M(x).$$

Otherwise $x \neq rb$ or $a \neq st$ for all r, $b, s, t \in X$. Then

$$(T_M \circ \chi_X(T)_N \circ T_M)(x) = 0 \geq T_M(x),$$
$$(I_M \circ \chi_X(I)_N \circ I_M)(x) = -1 \leq I_M(x),$$
$$(F_M \circ \chi_X(F)_N \circ F_M)(x) = 0 \geq F_M(x).$$

Therefore $X_M \odot \chi_X(X_N) \odot X_M \subseteq X_M$ for any neutrosophic $\aleph -$ structure X_N over X. □

Definition 3.4. A semigroup X is called neutrosophic $\aleph -$left (resp., right) duo if every neutrosophic $\aleph -$left (resp., right) ideal is neutrosophic $\aleph -$ideal of X.

If X is both neutrosophic $\aleph -$left duo and neutrosophic $\aleph -$right duo, then X is called neutrosophic $\aleph -$duo

Theorem 3.5. If X is regular left duo (resp., duo, right duo), then the equivalent assertions are:

(i) X_M in X is neutrosophic $\aleph -$bi- ideal,

(ii) X_M in X is neutrosophic $\aleph -$right ideal (resp., ideal, left ideal).

Proof: $(\boldsymbol{i}) \Longrightarrow (\boldsymbol{ii})$ Suppose X_M is a neutrosophic $\aleph -$bi- ideal and $g, h \in X$. As X is regular, we get $g = gtg \in gX \cap Xg$ for some $t \in X$ which gives $gh \in (gX \cap Xg)X \subseteq gX \cap Xg$ as X is left duo. So $gh = gs$ and $gh = s'g$ for some $s, s' \in X$. As X is regular, $\exists r \in X$: $gh = ghrgh = gsrs'g = g(srs')g$. Since X_M is neutrosophic $\aleph -$bi- ideal, we have

$$T_M(gh) = T_M(g(srs')g) \leq T_M(g) \vee T_M(g) = T_M(g),$$

$$I_M(gh) = I_M(g(srs')g) \geq I_M(g) \wedge I_M(g) = I_M(g),$$

$$F_M(gh) = F_M(g(srs')g) \leq F_M(g) \vee F_M(g) = F_M(g).$$

Therefore X_M is neutrosophic $\aleph -$right ideal.

$(\boldsymbol{ii}) \Longrightarrow (\boldsymbol{i})$ Suppose X_M is neutrosophic $\aleph -$right ideal and let $x, y, z \in X$. Then

$$T_M(xyz) \leq T_M(x) \leq T_M(x) \vee T_M(z),$$

$$I_M(xyz) \geq I_M(x) \geq I_M(x) \wedge I_M(z),$$

$$F_M(xyz) \leq F_M(x) \leq F_M(x) \vee F_M(z).$$

Therefore X_M is a neutrosophic $\aleph$ −bi-ideal. □

Theorem 3.6. If X is regular, then the equivalent assertions are:

(i) X is left duo (resp., right duo, duo),

(ii) X is neutrosophic $\aleph$ −left duo (resp., right duo, duo).

Proof: $(\boldsymbol{i}) \Longrightarrow (\boldsymbol{ii})$ Let r, s $\in X$, we have $rs \in (rXr)s \subseteq r(Xr)X \subseteq Xr$ as Xr is left ideal. Since X is regular, we have $rs = tr$ for some $t \in X$.

If X_M is neutrosophic $\aleph$ −left ideal, then $T_M(rs) = T_M(tr) \leq T_M(r), I_M(rs) = I_M(tr) \geq I_M(r)$ and $F_M(rs) = F_M(tr) \leq F_M(r)$. Thus X_M is neutrosophic $\aleph$ −right ideal and therefore X is neutrosophic $\aleph$ −left duo.

$(\boldsymbol{ii}) \Longrightarrow (\boldsymbol{i})$ Let A be a left ideal of X. Then $\chi_A(X_M)$ is a neutrosophic $\aleph$ −left ideal by Theorem 3.5 of [4]. By assumption, $\chi_A(X_M)$ is neutrosophic $\aleph$ −ideal. Thus A is a right ideal of X. □

Theorem 3.7. If X is regular, then the equivalent assertions are:

(i) Every neutrosophic $\aleph$ −bi-ideal is a neutrosophic $\aleph$ −right (resp., left ideal, ideal) ideal,

(ii) Every bi-ideal of X is a right ideal (resp., left ideal, ideal).

Proof: $(\boldsymbol{i}) \Longrightarrow (\boldsymbol{ii})$ Let A be a bi-ideal of X. Then *by* Theorem 3.1 $\chi_A(X_M)$ is neutrosophic $\aleph$ −bi-ideal for a neutrosophic $\aleph$ −structure X_M. Now by assumption, $\chi_A(X_M)$ is neutrosophic $\aleph$ −right ideal. So by Theorem 3.5 of [4], A is right ideal.

$(\boldsymbol{ii}) \Longrightarrow (\boldsymbol{i})$ Let X_M be a neutrosophic $\aleph$ −bi-ideal and let $r, s \in X$. Then we get rXr is a bi-ideal of X. By hypothesis, we can have rXr is right ideal. Since X is regular, we can get r$\in rXr$. So $rs \in (rXr)X \subseteq rXr$ implies $rs = rxr$ for some $x \in X$. Now,

$$T_M(rs) = T_M(rxr) \leq T_M(r) \vee T_M(r) = T_M(r),$$
$$I_M(rs) = I_M(rxr) \geq I_M(r) \wedge I_M(r) = I_M(r)$$
$$F_M(rs) = F_M(rxr) \leq F_M(r) \vee F_M(r) = F_M(r).$$

Thus X_M is a neutrosophic $\aleph$ −right ideal of X. □

Theorem 3.8. For any X, the equivalent conditions are:

(i) X is regular,

(ii) $X_M \cap X_N = X_M \odot X_N \odot X_M$ for every neutrosophic $\aleph$ − bi-ideal X_M and neutrosophic $\aleph$ − ideal X_N of X.

Proof: $(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$ Suppose $X$is regular, X_M is a neutrosophic $\aleph$ − bi-ideal and X_N is a neutrosophic $\aleph$ − ideal of X. Then by Theorem 3.3, we have $X_M \odot X_N \odot X_M \subseteq X_M$ and $X_M \odot X_N \odot X_M \subseteq X_N$. So $X_M \odot X_N \odot X_M \subseteq X_M \cap X_N$.

Let $r' \in X$. As X is regular, there is $p \in X$ such that $r' = r'pr' = r'pr'pr'$. Now

$$T_{M \circ N \circ M}(r') = \bigwedge_{r'=de} \{T_M(d) \vee T_{N \circ M}(e)\}$$

$$= \bigwedge_{r'=r'e} \{T_M(r') \vee \{ \bigwedge_{v=pr'pr'} \{T_N(pr'p) \vee T_M(r')\}\}$$

$$\leq \bigwedge_{r'=r'e} \{T_M(r') \vee T_N(r')\} \leq T_M(r') \vee T_N(r') = T_{M \cap N}(r'),$$

$$I_{M \circ N \circ M}(r') = \vee_{r'=de} \{I_M(d) \wedge I_{N \circ M}(e)\}$$

$$= \bigvee_{r'=r'e} \{I_M(r') \wedge \{ \bigvee_{v=pr'pr'} \{ I_N(pr'p) \wedge I_M(r')\}\}$$

$$\geq \bigvee_{r'=r'e} \{I_M(r') \wedge I_N(r')\} \geq I_M(r') \wedge I_N(r') = I_{M\cap N}(r'),$$

$$F_{M\circ N\circ M}(r') = \bigwedge_{r'=de} \{F_M(d) \vee F_{N\circ M}(e)\}$$

$$= \bigwedge_{r'=r'e} \{F_M(r') \vee \{ \bigwedge_{v=pr'pr'} \{F_N(pr'p) \vee F_M(r')\}\}$$

$$\leq \bigwedge_{r'=r'e} \{F_M(r') \vee F_N(r')\} \leq F_M(r') \vee F_N(r') = F_{M\cap N}(r').$$

Thus $X_{M\cap N} \subseteq X_M \odot X_N \odot X_M$ and hence $X_{M\cap N} = X_M \odot X_N \odot X_M$.

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{i})$ Suppose (ii) holds. Then $X_M \cap \chi_X(X_N) = X_M \odot \chi_X(X_N) \odot X_M$. But $X_M \cap \chi_X(X_N) = X_M$, so $X_M = X_M \odot \chi_X(X_N) \odot X_M$ for every neutrosophic $\aleph -$ bi-ideal X_M of X.

Let $u' \in X$. Then $\chi_{B(u')}(X_M)$ is neutrosophic $\aleph -$ bi-ideal by Theorem 3.1.

By assumption, we have

$$\chi_{B(u')}(T)_M = \chi_{B(u')}(T)_M \circ \chi_X(T)_N \circ \chi_{B(u')}(T)_M = \chi_{B(u')XB(u')}(T)_M,$$
$$\chi_{B(u')}(I)_M = \chi_{B(u')}(I)_M \circ \chi_X(I)_N \circ \chi_{B(u')}(I)_M = \chi_{B(u')XB(u')}(I)_M,$$
$$\chi_{B(u')}(F)_M = \chi_{B(u')}(F)_M \circ \chi_X(F)_N \circ \chi_{B(u')}(F)_M = \chi_{B(u')XB(u')}(F)_M.$$

Since $u' \in B(u')$, we have

$$\chi_{B(u')XB(u')}(T)_M(u') = \chi_{B(u')}(T)_M(u') = -1,$$
$$\chi_{B(u')XB(u')}(I)_M(u') = \chi_{B(u')}(I)_M(u') = 0,$$
$$\chi_{B(u')XB(u')}(F)_M(u') = \chi_{B(u')}(F)_M(u') = -1$$

Thus u'$\in B(u')XB(u')$ and hence X is regular. □

Theorem 3.9. For any X, the below statements are equivalent:

(i) X is regular,

(ii) $X_M \cap X_N = X_M \odot X_N$ for every neutrosophic $\aleph -$ bi-ideal X_M and neutrosophic $\aleph -$ left ideal X_N of X.

Proof:$(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$ Let X_M and X_N be neutrosophic $\aleph -$ bi-ideal and neutrosophic $\aleph -$left ideal of X respectively. Let r $\in X$. Then $\exists x \in X$: r$= rxr$. Now

$$T_{M\circ N}(r) = \bigwedge_{r=uv} \{T_M(u) \vee T_N(v)\} \leq T_M(r) \vee T_N(xr) \leq T_M(r) \vee T_N(r) = T_{M\cap N}(r),$$
$$I_{M\circ N}(r) = \bigvee_{r=uv} \{I_M(u) \wedge I_N(v)\} \geq I_M(r) \wedge I_N(xr) \geq I_M(r) \wedge I_N(r) = I_{M\cap N}(r),$$
$$F_{M\circ N}(r) = \bigwedge_{r=uv} \{F_M(u) \vee F_N(v)\} \leq F_M(r) \vee F_N(xr) \leq F_M(r) \vee F_N(r) = F_{M\cap N}(r).$$

Therefore $X_{M\cap N} \subseteq X_M \odot X_N$.

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{i})$ Suppose (ii) holds, and let X_M and X_N be neutrosophic $\aleph -$ right ideal and neutrosophic $\aleph -$ left ideal of X respectively. Since every neutrosophic $\aleph -$ right ideal is neutrosophic $\aleph -$ bi-ideal, X_M is neutrosophic $\aleph -$ bi-ideal. Then by assumption, $X_{M\cap N} \subseteq X_M \odot X_N$. By Theorem 3.8 and Theorem 3.9 of [4], we can get $X_M \odot X_N \subseteq X_N$ and $X_M \odot X_N \subseteq X_M$ and so $X_M \odot X_N \subseteq X_M \cap X_N = X_{M\cap N}$. Therefore $X_M \odot X_N = X_{M\cap N}$.

Let K and L be right and left ideals of X respectively, and r $\in K \cap L$. Then $\chi_K(X_M) \odot \chi_L(X_M) = \chi_K(X_M) \cap \chi_L(X_M)$ which implies $\chi_{KL}(X_M) = \chi_{K\cap L}(X_M)$. Since r$\in K \cap L$, we have

$\chi_{K\cap L}(T)_M(r) = -1 = \chi_{KL}(T)_M(r), \chi_{K\cap L}(I)_M(r) = 0 = \chi_{KL}(I)_M(r)$ and $\chi_{K\cap L}(F)_M(r) = -1 = \chi_{KL}(F)_M(r)$ which imply r $\in KL$.Thus $K \cap L \subseteq KL \subseteq K \cap L$. So $K \cap L = KL$. Thus X is regular. □

Theorem 3.10. For any X, the equivalent conditions are:

(i) X is regular,

(ii) $X_M \cap X_N \subseteq X_M \odot X_N$ for every neutrosophic $\aleph-$ right ideal X_N and neutrosophic $\aleph-$ bi-ideal X_M of X.

Proof: It is same as Theorem 3.9. □

Theorem 3.11. For any X, the equivalent assertions are:

(i) X is regular,

(ii) $X_L \cap X_M \cap X_N \subseteq X_L \odot X_M \odot X_N$ for every neutrosophic$\aleph-$ right ideal $X_{L,}$ neutrosophic $\aleph-$ bi-ideal X_M and neutrosophic $\aleph-$ left ideal X_N of X.

Proof: $(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$ Suppose X is regular, and let X_L, X_M, X_N be neutrosophic $\aleph-$ right, bi-ideal, left ideals of X respectively. Let r $\in X$. Then there is $x \in X$ with r= $rxr = rxrxr$. Now

$$
\begin{aligned}
T_{L\circ M\circ N}(r) &= \bigwedge_{r=uv}\{T_L(u) \vee T_{M\circ N}(v)\} \le T_L(rx) \vee T_{M\circ N}(rxr) \le T_L(r) \vee \{T_M(r) \vee T_N(xr)\}\\
&\le T_L(r) \vee T_M(r) \vee T_N(r) = T_{L\cap M\cap N}(r),\\
I_{L\circ M\circ N}(r) &= \bigvee_{r=uv}\{I_L(u) \wedge I_{M\circ N}(v)\} \ge I_L(rx) \wedge I_{M\circ N}(rxr) \ge I_L(r) \wedge \{I_M(r) \wedge I_N(xr)\}\\
&\ge I_L(r) \wedge I_M(r) \wedge I_N(r) = I_{L\cap M\cap N}(r),\\
F_{L\circ M\circ N}(r) &= \bigwedge_{r=uv}\{F_L(u) \vee F_{M\circ N}(v)\} \le F_L(rx) \vee F_{M\circ N}(rxr) \le F_L(r) \vee F_M(r) \vee F_N(xr)\\
&\le F_L(r) \vee F_M(r) \vee F_N(r) = F_{L\cap M\cap N}(r).
\end{aligned}
$$

Therefore $X_{L\cap M\cap N} \subseteq X_L \odot X_M \odot X_N$.

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{i})$ Suppose (ii) holds, and let X_L and X_N be neutrosophic $\aleph-$ right and neutrosophic $\aleph-$ left ideal of X respectively, and X_M a neutrosophic $\aleph-$bi-ideal of X. Then $\chi_X(X_M)$ is a neutrosophic $\aleph-$ bi-ideal by Theorem 3.1. Now $X_L \cap X_N = X_L \cap \chi_X(X_M) \cap X_N \subseteq X_L \odot \chi_X(X_M) \odot X_N \subseteq X_L \odot X_N$. Again by Theorem 3.8 and Theorem 3.9 of [4], we can get $X_L \odot X_N \subseteq X_L \cap X_N$and so $X_L \odot X_N = X_L \cap X_N$.

Let K and L be right and left ideals of X respectively. Then $\chi_K(X_M) \odot \chi_L(X_M) = \chi_K(X_M) \cap \chi_L(X_M)$. By Theorem 3.6 of [4], we have $\chi_{KL}(X_M) = \chi_{K\cap L}(X_M)$. Let r $\in K \cap L$. Then

$$\chi_{KL}(T)_M(r) = \chi_{K\cap L}(T)_M(r) = -1,$$

$$\chi_{KL}(I)_M(r) = \chi_{K\cap L}(I)_M(r) = 0,$$

$$\chi_{KL}(F)_M(r) = \chi_{K\cap L}(F)_M(r) = -1.$$

So r $\in KL$. Thus $K \cap L \subseteq KL \subseteq K \cap L$. Hence $K \cap L = KL$. Therefore X is regular. □

Theorem 3.12. For any X, the equivalent conditions are:

(i) X is regular and intra- regular,

(ii) $X_M \cap X_N \subseteq X_M \odot X_N$ for every neutrosophic $\aleph-$ bi-ideals X_M, X_N of X.

Proof: $(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$ Let X_M and X_N be neutrosophic $\aleph-$ bi-ideals. Let $h \in X$. Then by regularity of X , h $= hxh = hxhxh$ for some x $\in X$. Since X is intra-regular, $\exists y, z \in X$: h $= yh^2z$. Then $h = hxyhhzxh$. Now

$$T_{M\circ N}(h) = \bigwedge_{h=rt}\{T_M(r) \vee T_N(t)\} \leq T_M(hxyh) \vee T_N(hzxh) \leq T_M(h) \vee T_N(h) \quad = T_{M\cap N}(h),$$
$$I_{M\circ N}(h) = \bigvee_{h=rt}\{I_M(r) \wedge I_N(t)\} \geq I_M(hxyh) \wedge I_N(hzxh) \geq I_M(h) \wedge I_N(h) = I_{M\cap N}(h),$$
$$F_{M\circ N}(h) = \bigwedge_{h=rt}\{F_M(r) \vee F_N(t)\} \leq F_M(hxyh) \vee F_N(hzxh) \leq F_M(h) \vee F_N(h) = F_{M\cap N}(h).$$

Therefore $X_M \cap X_N \subseteq X_M \odot X_N$ for every neutrosophic $\aleph-$ bi-ideals X_M and X_N.

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{i})$ Suppose (ii) holds, and let X_M and X_N be neutrosophic $\aleph-$ right and left ideal of X respectively. Then X_M and X_N are neutrosophic $\aleph-$ bi-ideals. By assumption, $X_{M\cap N} \subseteq X_M \odot X_N$. By Theorem 3.8 and Theorem 3.9 of [4], we can get $X_M \odot X_N \subseteq X_N$ and $X_M \odot X_N \subseteq X_M$ and so $X_M \odot X_N \subseteq X_M \cap X_N = X_{M\cap N}$. Therefore $X_M \odot X_N = X_{M\cap N}$.

Let K, L be right, left ideals of X respectively. Then $\chi_K(X_M) \odot \chi_L(X_M) = \chi_K(X_M) \cap \chi_L(X_M)$.

By Theorem 3.6 of [4], $\chi_{KL}(X_M) = \chi_{K\cap L}(X_M)$. Let $r \in K \cap L$. Then $\chi_{K\cap L}(T)_M(r) = -1 = \chi_{KL}(T)_M(r), \chi_{K\cap L}(I)_M(r) = 0 = \chi_{KL}(I)_M(r)$ and $\chi_{K\cap L}(F)_M(r) = -1 = \chi_{KL}(F)_M(r)$ which imply $r \in KL$. Thus $K \cap L \subseteq KL \subseteq K \cap L$ and hence $K \cap L = KL$. Therefore X is regular.

Also, for $r \in X, \chi_{B(r)}(X_M) \cap \chi_{B(r)}(X_M) = \chi_{B(r)}(X_M) \odot \chi_{B(r)}(X_M)$. By Theorem 3.8 and Theorem 3.9 of [4], we get $\chi_{B(r)}(X_M) = \chi_{B(r)B(r)}(X_M)$.since$\chi_{B(r)}(T)_M(r) = -1 = \chi_{B(r)}(F)_M(r)$and $\chi_{B(r)}(I)_M(r) = 0$, we get $\chi_{B(r)B(r)}(T)_M(r) = -1 = \chi_{B(r)B(r)}(F)_M(r)$ and $\chi_{B(r)B(r)}(I)_M(r) = 0$ which imply $r \in B(r)B(r)$. Thus X is intra-regular. □

Theorem 3.13. For any X, the equivalent conditions are:

(i) X is intra-regular and regular,

(ii) $X_M \cap X_N \subseteq (X_M \odot X_N) \cap (X_N \odot X_M)$ for every neutrosophic $\aleph-$ bi-ideals X_M and X_N of X.

Proof:$(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$ Suppose X is regular and intra- regular, and let X_M and X_N be neutrosophic $\aleph-$ bi-ideals of X. Then by Theorem 3.12, $X_M \odot X_N \supseteq X_M \cap X_N$. Similarly we can prove that $X_N \odot X_M \supseteq X_N \cap X_M$.Therefore $(X_M \odot X_N) \cap (X_N \odot X_M) \supseteq X_M \cap X_N$ for every neutrosophic $\aleph-$ bi-ideals X_M and X_N of X.

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{i})$ Let X_M and X_N be neutrosophic $\aleph-$ bi-ideals of X. Then $X_M \cap X_N \subseteq X_M \odot X_N$ gives X is intra-regular and regular by Theorem 3.12. □

Theorem 3.14. For any X, the equivalent assertions are:

(i) X is intra-regular and regular,

(ii) $X_M \cap X_N \subseteq X_M \odot X_N \odot X_M$ for every neutrosophic $\aleph-$ bi-ideals X_M and X_N of X.

Proof:$(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$ Let X_M and X_N be neutrosophic $\aleph-$ bi-ideals, and $a \in X$. As X is regular, $a = axa = axaxaxa$ for some $x \in X$. Since X is intra-regular, $a = ya^2z$ for some $y, z \in X$. Then $a = (axya)(azxya)(azxa)$. Now

$$T_{M\circ N\circ M}(a) = \bigwedge_{a=km}\{T_M(k) \vee T_{N\circ M}(m)\}$$
$$= \bigwedge_{a=(axya)v}\{T_M(axya) \vee \{\bigwedge_{v=rt}\{T_N(r) \vee T_M(t)\}\}$$
$$\leq T_M(axya) \vee T_N(azxya) \vee T_M(azxa)$$
$$\leq T_M(a) \vee T_N(a) \vee T_M(a) \quad = T_{M\cap N}(a),$$
$$I_{M\circ N\circ M}(a) = \bigvee_{a=km}\{I_M(k) \wedge I_{N\circ M}(m)\}$$
$$= \bigvee_{a=(axya)v}\{I_M(axya) \wedge \{\bigvee_{v=rt}\{I_N(r) \wedge I_M(t)\}\}$$

$$\geq I_M(axya) \wedge I_N(azxya) \wedge I_M(azxa)$$
$$\geq I_M(a) \wedge I_N(a) \wedge I_M(a) = I_{M\cap N}(a),$$

and

$$F_{M\circ N\circ M}(a) = \bigwedge_{a=km} \{F_M(k) \vee F_{N\circ M}(m)\}$$
$$= \bigwedge_{a=(axya)v} \{F_M(axya) \vee \{\bigwedge_{v=rt} \{F_N(r) \vee F_M(t)\}\}$$
$$\leq F_M(axya) \vee F_N(azxya) \vee F_M(azxa)$$
$$\leq F_M(a) \vee F_N(a) \vee F_M(a) = F_{M\cap N}(a).$$

Therefore $X_M \cap X_N \subseteq X_M \odot X_N \odot X_M$ for every neutrosophic $\aleph-$ bi-ideals X_M and X_N of X.

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{i})$ Let $h_j \in X$. Then

$$\chi_{B(h_j)}(X_M) \subseteq \chi_{B(h_j)}(X_M) \cap \chi_{B(h_j)}(X_M) \subseteq \chi_{B(h_j)}(X_M) \odot \chi_{B(h_j)}(X_M) \odot \chi_{B(h_j)}(X_M).$$

So

$$\chi_{B(h_j)}(T)_M(h_j) \geq \chi_{B(h_j)B(h_j)B(h_j)}(T)_M(h_j),$$
$$\chi_{B(h_j)}(I)_M(h_j) \leq \chi_{B(h_j)B(h_j)B(h_j)}(I)_M(h_j),$$
$$\chi_{B(h_j)}(F)_M(h_j) \geq \chi_{B(h_j)B(h_j)B(h_j)}(F)_M(h_j).$$

Since $\chi_{B(h_j)}(T)_M(h_j) = -1 = \chi_{B(h_j)}(F)_M(h_j)$ and $\chi_{B(h_j)}(I)_M(h_j) = 0$, we get $\chi_{B(h_j)B(h_j)B(h_j)}(T)_M(h_j) = -1 = \chi_{B(h_j)B(h_j)B(h_j)}(F)_M(h_j)$ and $\chi_{B(h_j)B(h_j)B(h_j)}(I)_M(h_j) = 0$ which imply $h_j \in B(h_j)B(h_j)B(h_j)$. Therefore X is intra-regular and regular. □

Theorem 3.15. For any X, the equivalent assertions are:

(i) X is intra-regular,

(ii) For each neutrosophic $\aleph-$ideal X_M of X, $X_M(a) = X_M(a^2)$ $\forall a \in X$.

Proof: $(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$ Let $a \in X$. Then $a = ya^2z$ for some $y, z \in X$. For a neutrosophic $\aleph-$ideal X_M, we have

$$T_M(a) = T_M(ya^2z) \leq T_M(a^2z) \leq T_M(a^2) \leq T_M(a),$$
$$I_M(a) = I_M(ya^2z) \geq I_M(a^2z) \geq I_M(a^2) \geq I_M(a),$$
$$F_M(a) = F_M(ya^2z) \leq F_M(a^2z) \leq F_M(a^2) \leq F_M(a),$$

so $T_M(a) = T_M(a^2)$; $I_M(a) = I_M(a^2)$ and $F_M(a) = F_M(a^2)$ for all $a \in X$. Therefore $X_M(a) = X_M(a^2)$

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{i})$ Let $a \in X$. Then $I(a^2)$ is an ideal of X. Thus $\chi_{I(a^2)}(X_M)$ is neutrosophic $\aleph-$ideal by Theorem 3.5 of [4]. By assumption, $\chi_{I(a^2)}(X_M)(a) = \chi_{I(a^2)}(X_M)(a^2)$. Since $\chi_{I(a^2)}(T)_M(a^2) = -1 = \chi_{I(a^2)}(F)_M(a^2)$ and $\chi_{I(a^2)}(I)_M(a^2) = 0$, we get $\chi_{I(a^2)}(T)_M(a) = -1 = \chi_{I(a^2)}(F)_M(a)$ and $\chi_{I(a^2)}(I)_M(a) = 0$ imply $a \in I(a^2)$. Thus X is intra-regular. □

Theorem 3.16. For any X, the equivalent assertions are:

(i) X is left (resp., right) regular,

(ii) For each neutrosophic $\aleph-$left (resp., right) ideal X_M of X, $X_M(a) = X_M(a^2)$ $\forall a \in X$.

Proof: $(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$ Suppose X is left regular. Then $a = ya^2$ for some $y \in X$ Let X_M be neutrosophic $\aleph-$ left ideal. Then $T_M(a) = T_M(ya^2) \leq T_M(a^2)$ and so $T_M(a) = T_M(a^2), I_M(a) =$

$I_M(ya^2) \geq I_M(a)$ and so $I_M(a) = I_M(a^2)$, and $F_M(a) = F_M(ya^2) \leq F_M(a)$ and so $F_M(a) = F_M(a^2)$. Therefore $X_M(a) = X_M(a^2)$ for all $a \in X$.

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{i})$ Let X_M be neutrosophic $\aleph$ −left ideal. Then for any $a \in X$, we have $\chi_{L(a^2)}(T)_M(a) = \chi_{L(a^2)}(T)_M(a^2) = -1, \chi_{L(a^2)}(I)_M(a) = \chi_{L(a^2)}(I)_M(a^2) = 0$ and $\chi_{L(a^2)}(F)_M(a) = \chi_{L(a^2)}(F)_M(a^2) = -1$ imply $a \in L(a^2)$. Thus X is left regular. □

Corollary 3.17. Let X be a regular right duo (resp., left duo). Then the equivalent conditions are:

(i) X is left regular,

(ii) For each neutrosophic $\aleph$ −bi- ideal X_M of X, we have $X_M(a) = X_M(a^2)$ for all $a \in X$.

Proof: It is evident from Theorem 3.5 and Theorem 3.16. □

Conclusions

In this paper, we have presented the concept of neutrosophic $\aleph$ − bi −ideals of semigroups and explored their properties, and characterized regular semigroups, intra-regular semigroups and semigroups using neutrosophic $\aleph$-bi-ideal structures. We have also shown that the neutrosophic $\aleph$-product of ideals and the intersection of neutrosophic $\aleph$-ideals are identical for a regular semigroup. In future, we will focus on the idea of neutrosophic $\aleph$ −prime ideals of semigroups and its properties.

Acknowledgments: The authors express their gratitude to the referees for valuable comments and suggestions which improve the article a lot.

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Received: Apr 28, 2020. Accepted: July 17, 2020

Neutrosophic Sets and Systems, Vol. 35, 2020
University of New Mexico

A Novel Approach for Pairwise Separation Axioms on Bi-Soft Topology Using Neutrosophic Sets and An Output Validation in Real Life Application

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Abstract: The set that lightens the vagueness stage more energetically than fuzzy sets are neutrosophic sets. Bi-soft topological space is a space which goes for two different topologies with certain parameters. This work carries out, construction of such type of topology on neutrosophic. Besides by means of this, separation axioms are extended to pairwise separation axioms by using neutrosophic and to analyze the relationship among the class of such spaces. Here some of their properties are discussed with illustrative examples. In addition to it, we initiate the matrix form of neutrosophic soft sets in such space. Here problems deal to take a decision in life by the choice of two different groups. The aim of this decision making problem is to determine the unique thing or person from the universe by giving marks depending on parameters. Step by step process of solving the problem is explained in algorithm, also formulae given to determine their values with illustrative examples.

Keywords: Neutrosophic sets (NSs); neutrosophic soft sets (NSSs); neutrosophic soft topological spaces (NSTSs); neutrosophic soft $T_{i=0,1,2,3,4}$ -spaces (NS $T_{i=0,1,2,3,4}$ -spaces); neutrosophic bitopological spaces (NBTSs); neutrosophic bi-soft topological spaces (NBSTSs); pairwise neutrosophic soft $T_{i=0,1,2,3,4}$ -spaces (pairwise NS $T_{i=0,1,2,3,4}$ -spaces); decision making (DM).

1. Introduction

Zadeh [54], evaluated a fuzzy set (1965) to explore the situations like risky, unclear, erratic and distortion occurs in our life cycle. Fuzzy sets simplify classical sets and are unique cases of the membership functions. It has been used in a spacious collection of domains. This set extended to develop intuitionistic fuzzy set (IFS) theory (1986) by Atanassov [47]. Smarandache [7] originated a set which forecast the indeterminancy part along with truth and false statements, called NS (1998), such as blending of network arises to unpredictable states. It is a dynamic structure which postulates the concept of all other sets introduced before. Later, he generalized the NS on IFS [8] and recently proposed his work on attributes valued set, plithogenic set (PS) [9]. In day-to-day life decisions taken to diagnostic the problems either positive or negative even not both. Such types of problems are key role in all fields and so most of the researchers studied DM problem. In recent times various works have been done on these sets by Salma and Alblowi [33] and on extension of neutrosophic analysis on DM by Abdel et al. [1-6].

Soft set (1999) is a broad mathematical gadget which accord with a group of objects based on fairly accurate descriptions with orientation to elements of a parameter set was projected by Molodtsov [46]. Topological structure on this set explored by Shabir & Naz [38] as soft topological spaces (2011). Anon this thought were developed by Ali et al. [35, 40], Bayramov and Gunduz [22, 29], Cagman et al. [37], Chen [43], Feng et al. [41], Hussain and Ahmad [36], Maji et al. [44, 45], Min [39], Nazmul and Samanta [32], Pie and Miao [42], Tantawy et al. [26], Varol and Aygun [31], Zorlutuna et al. [34]. Maji [30] presented the binding of neutrosophic with soft set termed as NSSs (2013). Bera & Mahapatra [23] defined such type of set on topological structure, named as NSTSs (2017). Using these concepts, Deli & Broumi [27], Bera & Mahapatra [10, 24], on separation axioms by Cigdem et al. [20, 21] have done some research works. Mostly DM applied on these sets related to fuzzy with multicriteria by Chinnadurai et al. [13, 14 & 19], Abishek et al. [12], Muhammad et al. [16], Mehmood et al. [17], Evanzalin Ebenanjar et al. [18] and Faruk [25].

Kelly [55] imported the approach of a set equipped with two topologies, named as bitopological space (BTS) (1963), which is the generic system of topological space. Also it was carried out by Lane [53], Patty [52], Kalaiselvi and Sindhu [15] and pairwise concepts by Kim [51], Singal and Asha [50], Lal [48], Reilly [49]. Naz, Shabir and Ali [28] introduced bi-soft topological spaces (BSTSs) (2015) and studied the separation axioms on it. Taha and Alkan [11] presented BTS on neutrosophic structure as NBTSs (2019) which is engaged with two neutrosophic topologies (NTs).

The intension of this paper is to initiate the idea of NS on BSTS and to study some essential properties of such spaces. Also, the pairwise concept on separation axioms implemented in NBSTS. In addition, the NSSs referred as matrix form on NBSTS. As real life application, decisions made to select the one among the universe based on its parameters by considering two different groups as neutrosophic soft topologies (NSTs).

The arrangements made in this paper are as follows. Some basic definitions related to NS are in segment 2. The results of NBSTS are proved and disproved by counter examples in segment 3. The bonding among the pairwise separation axioms on NBSTS are stated with illustrative examples in segment 4. In segment 5, the method of evaluating DM problems are described in algorithm and formula specified to calculate the scores of universe set, to choose the best among them with illustrative examples. Finally, concluded with future work in segment 6.

2. Preamble

In this segment, we evoke few primary definitions associated to NSS, NSTS, BSTS and NBSTS.

Definition 2.1 [30] Let V be the set of universe and E be a set of parameters. Let $NS(V)$ denote the set of all NSs of V. Then a estimated function of NSS K over V is a set defined by a mapping $f_K : E \to NS(V)$. The NSS is a parameterized family of the set $NS(V)$ which can be written as a set of ordered pairs,

$$K = \left\{\left(e, \left\{\langle v, T_{f_K(e)}(v), I_{f_K(e)}(v), F_{f_K(e)}(v)\rangle : v \in V\right\}\right) : e \in E\right\}$$

where $T_{f_K(e)}(v), I_{f_K(e)}(v), F_{f_K(e)}(v) \in [0,1]$ respectively called the truth-membership, indeterminacy-membership and false-membership functions of $f_K(e)$ and the inequality $0 \leq T_{f_K(e)}(v) + I_{f_K(e)}(v) + F_{f_K(e)}(v) \leq 3$ is obvious.

Definition 2.2 [23] Let $NSS(V)$ denote the set of all NSSs of V through all $e \in E$ and $\tau_u \subset NSS(V,E)$. Then τ_u is called NST on (*V*, *E*) if the following conditions are satisfied.

(i) $\phi_u, 1_u \in \tau_u$, where null NSS $\phi_u = \{(e, \{\langle v, 0, 0, 1 \rangle\} : v \in V\}) : e \in E\}$ and absolute NSS $1_u = \{(e, \{\langle v, 1, 1, 0 \rangle\} : v \in V\}) : e \in E\}$.

(ii) the intersection of any finite number of members of τ_u belongs to τ_u.

(iii) the union of any collection of members of τ_u belongs to τ_u.

Then the triplet (*V*, *E*, τ_u) is called a NSTS.
Every member of τ_u is called τ_u-open NSS, whose complement is τ_u-closed NSS.

Definition 2.3 [21] Let *NSS*(*V*, *E*) denote the family of all NSSs over *V*. The NSS $u_e^{(\alpha,\beta,\gamma)}$ is called a NSP, for every $u \in V, 0 < \alpha, \beta, \gamma \leq 1, e \in E$ and is defined as follows:

$$u_{(e)}^{(\alpha,\beta,\gamma)}(e')(v) = \begin{cases} (\alpha,\beta,\gamma), & if\ e' = e\ and\ v = u \\ (0,0,1), & if\ e' \neq e\ and\ v \neq u \end{cases}$$

Obviously, every NSS is the union of its NSPs.

Definition 2.4 [11] Let (*V*, τ_{u1}) and (*V*, τ_{u2}) be the two different NTs on *V*. Then (*V*, τ_{u1}, τ_{u2}) is called a NBTS.

3. NBSTS

In this segment, the conception of NBSTS is defined and some key resources of topology are studied on it. The theoretical results are supported by some significant descriptive examples.

Definition 3.1 The quadruple (*V*, *E*, τ_{u1}, τ_{u2}) is called a NBSTS over (*V*, *E*), where τ_{u1} and τ_{u2} are NSTs independently satisfy the axioms of NSTS.
The elements of τ_{u1} are τ_{u1}-neutrosophic soft open sets (τ_{u1}-NSOSs) and the complement of it are τ_{u1}-neutrosophic soft closed sets (τ_{u1}-NSOSs).

Example 3.2 Let $V = \{v_1, v_2\}$, $E = \{e_1, e_2\}$ and $\tau_{u1} = \{\phi_u, 1_u, K_1\}$ and $\tau_{u2} = \{\phi_u, 1_u, L_1, L_2\}$ where K_1, L_1, L_2 are NSSs over (*V*, *E*), defined as follows

$f_{K_1}(e_1) = \{< v_1, (1,1,0) >, < v_2, (0,0,1) >\}$,

$f_{K_1}(e_2) = \{< v_1, (0,0,1) >, < v_2, (1,0,1) >\}$

and

$f_{L_1}(e_1) = \{< v_1, (1,0,1) >, < v_2, (0,0,1) >\}$,

$f_{L_1}(e_2) = \{< v_1, (0,1,0) >, < v_2, (1,1,0) >\}$;

$f_{L_2}(e_1) = \{< v_1, (0,1,0) >, < v_2, (1,1,0) >\}$,

$f_{L_2}(e_2) = \{< v_1, (1,0,1) >, < v_2, (0,0,1) >\}$.

Thus τ_{u1} and τ_{u2} are NSTs on (*V*, *E*) and so (*V*, *E*, τ_{u1}, τ_{u2}) is a NBSTS over (*V*, *E*).

Example 3.3 Let the neutrosophic soft indiscrete (trivial) topology $\tau_{u1} = \{\phi_u, 1_u\}$ and neutrosophic soft discrete topology $\tau_{u2} = NSS(V, E)$.
Then (*V*, *E*, τ_{u1}, τ_{u2}) is a NBSTS over (*V*, *E*).

Definition 3.4 Let (*V*, *E*, τ_{u1}, τ_{u2}) be a NBSTS over (*V*, *E*), where τ_{u1} and τ_{u2} are NSTs on (*V*, *E*) and $P, Q \in NSS(V, E)$ be any two arbitrary NSSs. Suppose $\tau_{v1} = \{P \cap K_i / K_i \in \tau_{u1}\}$ and $\tau_{v2} = \{Q \cap L_i / L_i \in \tau_{u2}\}$. Then τ_{v1} and τ_{v2} are also NSTs on (*V*, *E*). Thus (*V*, *E*, τ_{v1}, τ_{v2}) is a NBST subspace of (*V*, *E*, τ_{u1}, τ_{u2}).

Theorem 3.5 Let (*V*, *E*, τ_{u1}, τ_{u2}) be a NBSTS over (*V*, *E*), where $\tau_{u1}(e)$ and $\tau_{u2}(e)$ are defined as

$\tau_{u1}(e) = \{f_K(e) / K \in \tau_{u1}\}$

$\tau_{u2}(e) = \{f_L(e) / L \in \tau_{u2}\}$ for each $e \in E$.

Then (*V*, *E*, $\tau_{u1}(e)$, $\tau_{u2}(e)$) is a NBTS over (*V*, *E*).
Proof. Follows from the fact that τ_{u1} and τ_{u2} are NTs on *V*.

Example 3.6 Let $V = \{v_1, v_2, v_3\}$, $E = \{e_1, e_2\}$ and $\tau_{u1} = \{\phi_u, 1_u, K_1, K_2\}$ and $\tau_{u2} = \{\phi_u, 1_u, L_1, L_2, L_3, L_4\}$ where $K_1, K_2, L_1, L_2, L_3, L_4$ are NSSs over (*V*, *E*), defined as follows

$f_{K_1}(e_1) = \{< v_1, (1,.5,.4) >, < v_2, (.6,.6,.6) >, < v_3, (.3,.4,.9) >\}$,

$f_{K_1}(e_2) = \{< v_1, (.8,.4,.5) >, < v_2, (.7,.7,.3) >, < v_3, (.7,.5,.6) >\}$;

$f_{K_2}(e_1) = \{< v_1, (.8,.5,.1) >, < v_2, (.8,.6,.5) >, < v_3, (.5,.6,.4) >\}$,

$f_{K_2}(e_2) = \{< v_1, (.9,.7,.1) >, < v_2, (.9,.9,.2) >, < v_3, (.8,.6,.3) >\}$

and

$f_{L_1}(e_1) = \{< v_1, (.3,.7,.6) >, < v_2, (.4,.3,.8) >, < v_3, (.6,.4,.5) >\}$,

$f_{L_1}(e_2) = \{< v_1, (.4,.6,.8) >, < v_2, (.3,.7,.2) >, < v_3, (.3,.3,.7) >\}$;

$f_{L_2}(e_1) = \{< v_1, (.6,.6,.8) >, < v_2, (.2,.9,.3) >, < v_3, (.1,.2,.4) >\}$,

$f_{L_2}(e_2) = \{< v_1,(.7,.9,.5) >,< v_2,(.4,.2,.3) >,< v_3,(.5,.5,.4) >\}$;

$f_{L_3}(e_1) = \{< v_1,(.6,.7,.6) >,< v_2,(.4,.9,.3) >,< v_3,(.6,.4,.4) >\}$,

$f_{L_3}(e_2) = \{< v_1,(.7,.9,.5) >,< v_2,(.4,.7,.2) >,< v_3,(.5,.5,.4) >\}$;

$f_{L_4}(e_1) = \{< v_1,(.3,.6,.8) >,< v_2,(.2,.3,.8) >,< v_3,(.1,.2,.5) >\}$,

$f_{L_4}(e_2) = \{< v_1,(.4,.6,.8) >,< v_2,(.3,.2,.3) >,< v_3,(.3,.3,.7) >\}$

Thus τ_{u1} and τ_{u2} are NSTs on (*V, E*) and so (*V, E,* τ_{u1}, τ_{u2}) is a NBSTS over (*V, E*).

Now,

$$\tau_{u1}(e_1) = \begin{Bmatrix} \phi_u, 1_u, \{< v_1,(1,.5,.4) >,< v_2,(.6,.6,.6) >,< v_3,(.3,.4,.9) >\}, \\ \{< v_1,(.8,.5,.1) >,< v_2,(.8,.6,.5) >,< v_3,(.5,.6,.4) >\} \end{Bmatrix},$$

$$\tau_{u2}(e_1) = \begin{Bmatrix} \phi_u, 1_u, \{< v_1,(.3,.7,.6) >,< v_2,(.4,.3,.8) >,< v_3,(.6,.4,.5) >\}, \\ \{< v_1,(.6,.6,.8) >,< v_2,(.2,.9,.3) >,< v_3,(.1,.2,.4) >\}, \\ \{< v_1,(.6,.7,.6) >,< v_2,(.4,.9,.3) >,< v_3,(.6,.4,.4) >\}, \\ \{< v_1,(.3,.6,.8) >,< v_2,(.2,.3,.8) >,< v_3,(.1,.2,.5) >\} \end{Bmatrix}$$

and

$$\tau_{u1}(e_2) = \begin{Bmatrix} \phi_u, 1_u, \{< v_1,(.8,.4,.5) >,< v_2,(.7,.7,.3) >,< v_3,(.7,.5,.6) >\}, \\ \{< v_1,(.9,.7,.1) >,< v_2,(.9,.9,.2) >,< v_3,(.8,.6,.3) >\} \end{Bmatrix},$$

$$\tau_{u2}(e_2) = \begin{Bmatrix} \phi_u, 1_u, \{< v_1,(.4,.6,.8) >,< v_2,(.3,.7,.2) >,< v_3,(.3,.3,.7) >\}, \\ \{< v_1,(.7,.9,.5) >,< v_2,(.4,.2,.3) >,< v_3,(.5,.5,.4) >\}, \\ \{< v_1,(.7,.9,.5) >,< v_2,(.4,.7,.2) >,< v_3,(.5,.5,.4) >\}, \\ \{< v_1,(.4,.6,.8) >,< v_2,(.3,.2,.3) >,< v_3,(.3,.3,.7) >\} \end{Bmatrix}$$

are NTs on *V*.

Thus (*V, E,* $\tau_{u1}(e), \tau_{u2}(e)$) is a NBTS over (*V, E*).

Definition 3.7 Let (*V, E,* τ_{u1}, τ_{u2}) be a NBSTS over (*V, E*). Then the supremum NST is $\tau_{u1} \vee \tau_{u2}$, which is the smallest NST on *V* that contains $\tau_{u1} \cup \tau_{u2}$.

Example 3.8 Let us consider 3.5 example, where τ_{u1} and τ_{u2} are NSTs on (*V, E*).

Then,

$$K_1 \cup L_1 = P = \begin{cases} f_P(e_1) = \{< v_1,(.3,.7,.4) >,< v_2,(.6,.6,.6) >,< v_3,(.6,.4,.5) >\} \\ f_P(e_2) = \{< v_1,(.8,.6,.5) >,< v_2,(.7,.7,.2) >,< v_3,(.7,.5,.6) >\} \end{cases}$$

and

$$K_1 \vee L_1 = Q = \begin{cases} f_Q(e_1,e_1) = \{< v_1,(.3,.7,.4) >,< v_2,(.6,.6,.6) >,< v_3,(.6,.4,.5) >\} \\ f_Q(e_1,e_2) = \{< v_1,(.8,.4,.5) >,< v_2,(.7,.7,.3) >,< v_3,(.7,.5,.4) >\} \\ f_Q(e_2,e_1) = \{< v_1,(1,.6,.4) >,< v_2,(.6,.7,.2) >,< v_3,(.3,.4,.7) >\} \\ f_Q(e_2,e_2) = \{< v_1,(.8,.6,.5) >,< v_2,(.7,.7,.2) >,< v_3,(.7,.5,.6) >\} \end{cases}$$

Thus $K_1 \vee L_1$ is the smallest NSS on *V* that contains $K_1 \cup L_1$.

Theorem 3.9 If (V, E, τ_{u1} , τ_{u2}) is a NBSTS over (V, E), then $\tau_{u1} \cap \tau_{u2}$ is a NST over (V, E).
Proof. Let (V, E, τ_{u1} , τ_{u2}) be a NBSTS over (V, E).

(i) Since $\phi_u, 1_u \in \tau_{u1}$ and $\phi_u, 1_u \in \tau_{u1}$, it follows that $\phi_u, 1_u \in \tau_{u1} \cap \tau_{u2}$.

(ii) Suppose that $\{K_i / i \in I\}$ is a family of NSSs in $\tau_{u1} \cap \tau_{u2}$.
Then $K_i \in \tau_{u1}$ and $K_i \in \tau_{u2}$ for all $i \in I$.

Thus $\bigcup_{i \in I} K_i \in \tau_{u1}$ and $\bigcup_{i \in I} K_i \in \tau_{u2}$.

Therefore $\bigcup_{i \in I} K_i \in \tau_{u1} \cap \tau_{u2}$.

(iii) Let $K, L \in \tau_{u1} \cap \tau_{u2}$.
Then $K, L \in \tau_{u1}$ and $K, L \in \tau_{u2}$.
Since $K \cap L \in \tau_{u1}$ and $K \cap L \in \tau_{u2}$, we have $K \cap L \in \tau_{u1} \cap \tau_{u2}$.
Hence $\tau_{u1} \cap \tau_{u2}$ is a NST over (V, E).

Remark 3.10 If (V, E, τ_{u1} , τ_{u2}) is a NBSTS over (V, E), then $\tau_{u1} \cup \tau_{u2}$ need not be a NST over (V, E).

Example 3.11 Let us consider 3.5 example where τ_{u1} and τ_{u2} are NSTs on (V, E).
Then,

$$K_1 \cup L_1 = P = \begin{cases} f_P(e_1) = \{< v_1, (0.3, 0.7, 0.4) >, < v_2, (0.6, 0.6, 0.6) >, < v_3, (0.6, 0.4, 0.5) >\} \\ f_P(e_2) = \{< v_1, (0.8, 0.6, 0.5) >, < v_2, (0.7, 0.7, 0.2) >, < v_3, (0.7, 0.5, 0.6) >\} \end{cases}$$

Thus $K_1 \cup L_1 \notin \tau_{u1} \cup \tau_{u2}$.
Hence $\tau_{u1} \cup \tau_{u2}$ is not a NST over (V, E).

4. Neutrosophic bi-soft separation axioms

In this segment, the separation of NBSTS is explored. The pairwise NS $T_{i=0,1,2,3,4}$-spaces on NBSTS are introduced and the relationships among them are examined with relevant examples.

Definition 4.1 A NBSTS (V, E, τ_{u1} , τ_{u2}) over (V, E) is called a pairwise NS T_0-space, if $u_{(e)}^{(\alpha,\beta,\gamma)}$ and $v_{(e')}^{(\alpha,\beta,\gamma)}$ are distinct NSPs then there exist τ_{u1}-NSOS K and τ_{u2}-NSOS L such that

$$u_{(e)}^{(\alpha,\beta,\gamma)} \in K \;\; ; \;\; u_{(e)}^{(\alpha,\beta,\gamma)} \cap L = \phi_u$$

$$\text{or} \;\; v_{(e')}^{(\alpha,\beta,\gamma)} \in L \;\; ; \;\; v_{(e')}^{(\alpha,\beta,\gamma)} \cap K = \phi_u .$$

Example 4.2 Consider neutrosophic soft indiscrete (trivial) topology $\tau_{u1} = \{\phi_u, 1_u\}$ and neutrosophic soft discrete topology $\tau_{u2} = NSS(U,E)$.
Thus (*V, E,* τ_{u1}, τ_{u2}) is a pairwise NS T_0-space.

Theorem 4.3 Let (*V, E,* τ_{u1}, τ_{u2}) be a NBSTS over (*V, E*). If (*V, E,* τ_{u1}, τ_{u2}) is a pairwise NS T_0-space then (*V, E,* $\tau_{u1} \vee \tau_{u2}$) is a NS T_0-space.
Proof. Let (*V, E,* τ_{u1}, τ_{u2}) be a NBSTS over (*V, E*).
Suppose that (*V, E,* τ_{u1}, τ_{u2}) is a pairwise NS T_0-space.

Let $u_{(e)}^{(\alpha,\beta,\gamma)}$ and $v_{(e')}^{(\alpha,\beta,\gamma)}$ be any two distinct NSPs.

Then there exist τ_{u1}-NSOS *K* and τ_{u2}-NSOS *L* such that

$$u_{(e)}^{(\alpha,\beta,\gamma)} \in K \quad ; \quad u_{(e)}^{(\alpha,\beta,\gamma)} \cap L = \phi_u$$

$$\text{or} \quad v_{(e')}^{(\alpha,\beta,\gamma)} \in L \quad ; \quad v_{(e')}^{(\alpha,\beta,\gamma)} \cap K = \phi_u$$

In either case $K, L \in \tau_{u1} \vee \tau_{u2}$.
Hence (*V, E,* $\tau_{u1} \vee \tau_{u2}$) is a NS T_0-space.

Theorem 4.4 Let (*V, E,* τ_{u1}, τ_{u2}) be a NBSTS over (*V, E*). If (*V, E,* τ_{u1}, τ_{u2}) is a pairwise NS T_0-space then (*V, E,* τ_{v1}, τ_{v2}) is also a pairwise NS T_0-space.
Proof. Let (*V, E,* τ_{u1}, τ_{u2}) be a NBSTS over (*V, E*).

Let $u_{(e)}^{(\alpha,\beta,\gamma)}$ and $v_{(e')}^{(\alpha,\beta,\gamma)}$ be any two distinct NSPs and $P, Q \in NSS(U,E)$.

Suppose that (*V, E,* τ_{u1}, τ_{u2}) is a pairwise NS T_0-space.
Then there exist τ_{u1}-NSOS *K* and τ_{u2}-NSOS *L* such that

$$u_{(e)}^{(\alpha,\beta,\gamma)} \in K \quad ; \quad u_{(e)}^{(\alpha,\beta,\gamma)} \cap L = \phi_u$$

$$\text{or} \quad v_{(e')}^{(\alpha,\beta,\gamma)} \in L \quad ; \quad v_{(e')}^{(\alpha,\beta,\gamma)} \cap K = \phi_u$$

Now $u_{(e)}^{(\alpha,\beta,\gamma)} \in P$ and $u_{(e)}^{(\alpha,\beta,\gamma)} \in K$.

Then $u_{(e)}^{(\alpha,\beta,\gamma)} \in P \cap K$, where $K \in \tau_{u1}$.

Consider $u_{(e)}^{(\alpha,\beta,\gamma)} \cap L = \phi_u$.

$$\Rightarrow u_{(e)}^{(\alpha,\beta,\gamma)} \cap L \cap Q = \phi_u \cap Q.$$

$$\Rightarrow u_{(e)}^{(\alpha,\beta,\gamma)} \cap (Q \cap L) = \phi_u.$$

Thus $u_{(e)}^{(\alpha,\beta,\gamma)} \in P \cap K$; $u_{(e)}^{(\alpha,\beta,\gamma)} \cap (Q \cap L) = \phi_u$, where $P \cap K \in \tau_{v1}$, $Q \cap L \in \tau_{v2}$.

Or if $v_{(e')}^{(\alpha,\beta,\gamma)} \in L$; $v_{(e')}^{(\alpha,\beta,\gamma)} \bigcap K = \phi_u$, it can be proved that

$v_{(e')}^{(\alpha,\beta,\gamma)} \in Q \bigcap L$; $v_{(e')}^{(\alpha,\beta,\gamma)} \bigcap (Q \bigcap L) = \phi_u$, where $P \bigcap K \in \tau_{v1}$, $Q \bigcap L \in \tau_{v2}$.

Hence (*V*, *E*, τ_{v1}, τ_{v2}) is also a pairwise NS T_0 -space.

Definition 4.5 A NBSTS (*V*, *E*, τ_{u1}, τ_{u2}) over (*V*, *E*) is called pairwise NS T_1 -space, if $u_{(e)}^{(\alpha,\beta,\gamma)}$ and $v_{(e')}^{(\alpha,\beta,\gamma)}$ are distinct NSPs then there exist τ_{u1} -NSOS *K* and τ_{u2} -NSOS *L* such that

$$u_{(e)}^{(\alpha,\beta,\gamma)} \in K \quad ; \quad u_{(e)}^{(\alpha,\beta,\gamma)} \bigcap L = \phi_u$$

and $v_{(e')}^{(\alpha,\beta,\gamma)} \in L$; $v_{(e')}^{(\alpha,\beta,\gamma)} \bigcap K = \phi_u$.

Example 4.6 Let $V = \{v_1, v_2\}$, E = {*e*}, and $v_{1(e)}^{(.2,.3,.7)}$ and $v_{2(e)}^{(.9,.4,.1)}$ be NSPs. Let $\tau_{u1} = \{\phi_u, 1_u, K\}$ and $\tau_{u2} = \{\phi_u, 1_u, L\}$ where *K* and *L* are NSSs over (*V*, *E*), defined as

$$K = v_{1(e)}^{(.2,.3,.7)} = f_K(e) = \{< v_1, (.2,.3,.7) >, < v_2, (0,0,1) >\}$$

and

$$L = v_{2(e)}^{(.9,.4,.1)} = f_L(e) = \{< v_1, (0,0,1) >, < v_2, (.9,.4,.1) >\} .$$

Thus (*V*, *E*, τ_{u1}, τ_{u2}) is a NBSTS over (*V*, *E*).
Hence (*V*, *E*, τ_{u1}, τ_{u2}) is a pairwise NS T_1 -space, also a pairwise NS T_0 -space.

Theorem 4.7 Every pairwise NS T_1 -space is also a pairwise NS T_0 -space.
Proof. Follows from the Definitions 4.1 and 4.3.

Remark 4.8 The converse of the 4.7 theorem is not true, which is shown in the following example.

Example 4.9 Let *V* = $\{v_1, v_2\}$, *E* = $\{e_1, e_2\}$, and $v_{1(e_1)}^{(.2,.5,.7)}$, $v_{1(e_2)}^{(.2,.8,.2)}$, $v_{2(e_1)}^{(.2,.7,.5)}$ and $v_{2(e_2)}^{(.1,.1,.9)}$ be NSPs. Let $\tau_{u1} = \{\phi_u, 1_u, K_1, K_2, K_3\}$ and $\tau_{u2} = \{\phi_u, 1_u, L_1, L_2\}$ where K_1, K_2, K_3, L_1, L_2 are NSSs over (*V*, *E*), defined as

$$K_1 = v_{1(e_1)}^{(.2,.5,.7)} = \begin{cases} f_{K_1}(e_1) = \{< v_1, (.2,.5,.7) >, < v_2, (0,0,1) >\} \\ f_{K_1}(e_2) = \{< v_1, (0,0,1) >, < v_2, (0,0,1) >\} \end{cases} ;$$

$$K_2 = v_{2(e_2)}^{(.1,.1,.9)} = \begin{cases} f_{K_2}(e_1) = \{< v_1,(0,0,1) >, < v_2,(0,0,1) >\} \\ f_{K_2}(e_2) = \{< v_1,(0,0,1) >, < v_2,(.1,.1,.9) >\} \end{cases} ;$$

$$K_3 = K_1 \cup K_2$$

and

$$L_1 = v_{2(e_1)}^{(.2,.7,.5)} = \begin{cases} f_{L_1}(e_1) = \{< v_1,(0,0,1) >, < v_2,(.2,.7,.5) >\} \\ f_{L_1}(e_2) = \{< v_1,(0,0,1) >, < v_2,(0,0,1) >\} \end{cases} ;$$

$$L_2 = \left\{ v_{1(e_1)}^{(.2,.5,.7)}, v_{1(e_2)}^{1(.2,.8,.2)}, v_{2(e_1)}^{(.2,.7,.5)}, v_{2(e_2)}^{(.1,.1,.9)} \right\} = \begin{cases} f_{L_2}(e_1) = \{< v_1,(.2,.5,.7) >, < v_2,(.2,.7,.5) >\} \\ f_{L_2}(e_2) = \{< v_1,(.2,.8,.2) >, < v_2,(.1,.1,.9) >\} \end{cases}$$

Thus $(V, E, \tau_{u1}, \tau_{u2})$ is a NBSTS over (V, E).
Hence $(V, E, \tau_{u1}, \tau_{u2})$ is a pairwise NS T_0-space, but not a pairwise NS T_1-space since for NSPs $v_{1(e_1)}^{(.2,.5,.7)}$ and $v_{2(e_2)}^{(.1,.1,.9)}$, $(V, E, \tau_{u1}, \tau_{u2})$ is not a pairwise NS T_1-space.

Theorem 4.10 Let $(V, E, \tau_{u1}, \tau_{u2})$ be a NBSTS over (V, E). If (V, E, τ_{u1}) or (V, E, τ_{u2}) is not a NS T_0-space, then $(V, E, \tau_{u1}, \tau_{u2})$ is a pairwise NS T_0-space but not a pairwise NS T_1-space.

Proof. Let $K \in \tau_{u1}$ and $L \in \tau_{u2}$, also $u_{(e)}^{(\alpha,\beta,\gamma)}$ and $v_{(e')}^{(\alpha,\beta,\gamma)}$ be any two distinct NSPs.

Suppose (V, E, τ_{u1}) is a NS T_0-space and (V, E, τ_{u2}) is not a NS T_0-space.

Then, $u_{(e)}^{(\alpha,\beta,\gamma)} \in K \;\; ; \;\; u_{(e)}^{(\alpha,\beta,\gamma)} \cap L = \phi_u$

and $v_{(e')}^{(\alpha,\beta,\gamma)} \in L \; ; \;\; v_{(e')}^{(\alpha,\beta,\gamma)} \cap K \neq \phi_u$

Thus by Definitions 4.1 and 4.3,
$(V, E, \tau_{u1}, \tau_{u2})$ is a pairwise NS T_0-space but not a pairwise NS T_1-space.

Theorem 4.11 Let $(V, E, \tau_{u1}, \tau_{u2})$ be a NBSTS over (V, E). Then $(V, E, \tau_{u1}, \tau_{u2})$ is a pairwise NS T_1-space if and only if (V, E, τ_{u1}) and (V, E, τ_{u2}) are NS T_1-spaces.
Proof. Let $(V, E, \tau_{u1}, \tau_{u2})$ be a NBSTS over (V, E).

Let $u_{(e)}^{(\alpha,\beta,\gamma)}$ and $v_{(e')}^{(\alpha,\beta,\gamma)}$ be any two distinct NSPs.

Suppose that (V, E, τ_{u1}) and (V, E, τ_{u2}) are NS T_1-spaces.
Then there exist τ_{u1}-NSOS K and τ_{u2}-NSOS L such that

$$u_{(e)}^{(\alpha,\beta,\gamma)} \in K \;\; ; \;\; u_{(e)}^{(\alpha,\beta,\gamma)} \cap L = \phi_u$$

and $v_{(e')}^{(\alpha,\beta,\gamma)} \in L \;\; ; \;\; v_{(e')}^{(\alpha,\beta,\gamma)} \cap K = \phi_u$

In either case the result follows immediately.
Thus $(V, E, \tau_{u1}, \tau_{u2})$ is a pairwise NS T_1-space.
Conversely, assume that $(V, E, \tau_{u1}, \tau_{u2})$ is a pairwise NS T_1-space.

Then there exist some τ_{u1}-NSOS K_1 and τ_{u2}-NSOS L_1 such that

$$u_{(e)}^{(\alpha,\beta,\gamma)} \in K_1 \;\; ; \;\; u_{(e)}^{(\alpha,\beta,\gamma)} \bigcap L_1 = \phi_u$$

and $v_{(e')}^{(\alpha,\beta,\gamma)} \in L_1 \;\; ; \;\; v_{(e')}^{(\alpha,\beta,\gamma)} \bigcap K_1 = \phi_u$

Also there exist some τ_{u1}-NSOS K_2 and τ_{u2}-NSOS L_2 such that

$$u_{(e)}^{(\alpha,\beta,\gamma)} \in K_2 \;\; ; \;\; u_{(e)}^{(\alpha,\beta,\gamma)} \bigcap L_2 = \phi_u$$

and $v_{(e')}^{(\alpha,\beta,\gamma)} \in L_2 \;\; ; \;\; v_{(e')}^{(\alpha,\beta,\gamma)} \bigcap K_2 = \phi_u$

Hence (*V*, *E*, τ_{u1}) and (*V*, *E*, τ_{u2}) are NS T_1-spaces.

Theorem 4.12 Let (*V*, *E*, τ_{u1}, τ_{u2}) be a NBSTS over (*V*, *E*). If (*V*, *E*, τ_{u1}, τ_{u2}) is a pairwise NS T_1-space then (*V*, *E*, $\tau_{u1} \vee \tau_{u2}$) is a NS T_1-space.

Proof. Let (*V*, *E*, τ_{u1}, τ_{u2}) be a NBSTS over (*V*, *E*).

Suppose that (*V*, *E*, τ_{u1}, τ_{u2}) is a pairwise NS T_1-space.

Let $u_{(e)}^{(\alpha,\beta,\gamma)}$ and $v_{(e')}^{(\alpha,\beta,\gamma)}$ be any two distinct NSPs.

Then there exist τ_{u1}-NSOS *K* and τ_{u2}-NSOS *L* such that

$$u_{(e)}^{(\alpha,\beta,\gamma)} \in K \;\; ; \;\; u_{(e)}^{(\alpha,\beta,\gamma)} \bigcap L = \phi_u$$

and $v_{(e')}^{(\alpha,\beta,\gamma)} \in L \;\; ; \;\; v_{(e')}^{(\alpha,\beta,\gamma)} \bigcap K = \phi_u$

In either case $K, L \in \tau_{u1} \vee \tau_{u2}$.

Hence (*V*, *E*, $\tau_{u1} \vee \tau_{u2}$) is a NS T_1-space.

Theorem 4.13 Let (*V*, *E*, τ_{u1}, τ_{u2}) be a NBSTS over (*V*, *E*). If (*V*, *E*, τ_{u1}, τ_{u2}) is a pairwise NS T_1-space then (*V*, *E*, τ_{v1}, τ_{v2}) is also a pairwise NS T_1-space.

Proof. Let (*V*, *E*, τ_{u1}, τ_{u2}) be a NBSTS over (*V*, *E*).

Let $u_{(e)}^{(\alpha,\beta,\gamma)}$ and $v_{(e')}^{(\alpha,\beta,\gamma)}$ be any two distinct NSPs and $P, Q \in NSS(U, E)$.

Suppose that (*V*, *E*, τ_{u1}, τ_{u2}) is a pairwise NS T_1-space.

Then there exist τ_{u1}-NSOS *K* and τ_{u2}-NSOS *L* such that

$$u_{(e)}^{(\alpha,\beta,\gamma)} \in K \;\; ; \;\; u_{(e)}^{(\alpha,\beta,\gamma)} \bigcap L = \phi_u$$

and $v_{(e')}^{(\alpha,\beta,\gamma)} \in L \;\; ; \;\; v_{(e')}^{(\alpha,\beta,\gamma)} \bigcap K = \phi_u$

Now $u_{(e)}^{(\alpha,\beta,\gamma)} \in P$ and $u_{(e)}^{(\alpha,\beta,\gamma)} \in K$.

Then $u_{(e)}^{(\alpha,\beta,\gamma)} \in P \bigcap K$, where $K \in \tau_{u1}$.

Consider $u_{(e)}^{(\alpha,\beta,\gamma)} \bigcap L = \phi_u$.

$\Rightarrow u_{(e)}^{(\alpha,\beta,\gamma)} \bigcap L \bigcap Q = \phi_u \bigcap Q$.

$\Rightarrow u_{(e)}^{(\alpha,\beta,\gamma)} \bigcap (Q \bigcap L) = \phi_u$.

Thus $u_{(e)}^{(\alpha,\beta,\gamma)} \in P \bigcap K$; $u_{(e)}^{(\alpha,\beta,\gamma)} \bigcap (Q \bigcap L) = \phi_u$, where $P \bigcap K \in \tau_{v1}$, $Q \bigcap L \in \tau_{v2}$.

Further if $v_{(e')}^{(\alpha,\beta,\gamma)} \in L$; $v_{(e')}^{(\alpha,\beta,\gamma)} \bigcap K = \phi_u$, it can be proved that

$v_{(e')}^{(\alpha,\beta,\gamma)} \in Q \bigcap L$; $v_{(e')}^{(\alpha,\beta,\gamma)} \bigcap (Q \bigcap L) = \phi_u$, where $P \bigcap K \in \tau_{v1}$, $Q \bigcap L \in \tau_{v2}$.

Hence (*V*, *E*, τ_{v1}, τ_{v2}) is also a pairwise NS T_1-space.

Theorem 4.14 Let (*V*, *E*, τ_{u1}, τ_{u2}) be a NBSTS over (*V*, *E*). For each pair of distinct NSPs $u_{(e)}^{(\alpha,\beta,\gamma)}$ and $v_{(e')}^{(\alpha,\beta,\gamma)}$, $u_{(e)}^{(\alpha,\beta,\gamma)}$ is a τ_{u2}-NSCS and $v_{(e')}^{(\alpha,\beta,\gamma)}$ is a τ_{u1}-NSCS, then (*V*, *E*, τ_{u1}, τ_{u2}) is a pairwise NS T_1-space.

Proof. Let (*V*, *E*, τ_{u1}, τ_{u2}) be a NBSTS over (*V*, *E*).

Suppose that for each pair of distinct NSPs $u_{(e)}^{(\alpha,\beta,\gamma)}$ and $v_{(e')}^{(\alpha,\beta,\gamma)}$, $u_{(e)}^{(\alpha,\beta,\gamma)}$ is a τ_{u2}-NSCS.

Then $\left(u_{(e)}^{(\alpha,\beta,\gamma)}\right)^c$ is a τ_{u2}-NSOS.

Let $u_{(e)}^{(\alpha,\beta,\gamma)}$ and $v_{(e')}^{(\alpha,\beta,\gamma)}$ be any two distinct NSPs.

(i.e.,) $u_{(e)}^{(\alpha,\beta,\gamma)} \bigcap v_{(e')}^{(\alpha,\beta,\gamma)} = \phi_u$.

Thus

$$v_{(e')}^{(\alpha,\beta,\gamma)} \in \left(u_{(e)}^{(\alpha,\beta,\gamma)}\right)^c \text{ and } u_{(e)}^{(\alpha,\beta,\gamma)} \bigcap \left(u_{(e)}^{(\alpha,\beta,\gamma)}\right)^c = \phi_u \tag{1}$$

Similarly assume that for each NSP, $v_{(e')}^{(\alpha,\beta,\gamma)}$ is a τ_{u1}-NSCS.

Then $\left(v_{(e')}^{(\alpha,\beta,\gamma)}\right)^c$ is a τ_{u1}-NSOS such that

$$u_{(e)}^{(\alpha,\beta,\gamma)} \in \left(v_{(e')}^{(\alpha,\beta,\gamma)}\right)^c \text{ and } v_{(e')}^{(\alpha,\beta,\gamma)} \bigcap \left(v_{(e')}^{(\alpha,\beta,\gamma)}\right)^c = \phi_u \tag{2}$$

From (1) and (2),

$u_{(e)}^{(\alpha,\beta,\gamma)} \in \left(v_{(e')}^{(\alpha,\beta,\gamma)}\right)^c$; $u_{(e)}^{(\alpha,\beta,\gamma)} \bigcap \left(u_{(e)}^{(\alpha,\beta,\gamma)}\right)^c = \phi_u$

and $v_{(e')}^{(\alpha,\beta,\gamma)} \in \left(u_{(e)}^{(\alpha,\beta,\gamma)}\right)^c$; $v_{(e')}^{(\alpha,\beta,\gamma)} \bigcap \left(v_{(e')}^{(\alpha,\beta,\gamma)}\right)^c = \phi_u$

Hence (V, E, τ_{u1}, τ_{u2}) is a pairwise NS T_1-space.

Definition 4.15 A NBSTS (V, E, τ_{u1}, τ_{u2}) over (V, E) is called pairwise NS T_2-space or pairwise NS Hausdorff space, if $u_{(e)}^{(\alpha,\beta,\gamma)}$ and $v_{(e')}^{(\alpha,\beta,\gamma)}$ are distinct NSPs then there exist τ_{u1}-NSOS K and τ_{u2}-NSOS L such that $u_{(e)}^{(\alpha,\beta,\gamma)} \in K$, $v_{(e')}^{(\alpha,\beta,\gamma)} \in L$ and $K \cap L = \phi_u$.

Example 4.16 Let $V = \{v_1, v_2\}$, $E = \{e_1, e_2\}$, and $v_{1(e_1)}^{(.2,.5,.7)}$, $v_{1(e_2)}^{(.2,.8,.2)}$, $v_{2(e_1)}^{(.2,.7,.5)}$ and $v_{2(e_2)}^{(.1,.1,.9)}$ be NSPs. Let $\tau_{u1} = \{\phi_u, 1_u, K_1, K_2, K_3\}$ and $\tau_{u2} = \{\phi_u, 1_u, L_1, L_2, L_3\}$ where $K_1, K_2, K_3, L_1, L_2, L_3$ are NSSs over (V, E), defined as follows

$$K_1 = v_{1(e_1)}^{(.2,.5,.7)} = \begin{cases} f_{K_1}(e_1) = \{< v_1, (.2,.5,.7) >, < v_2, (0,0,1) >\} \\ f_{K_1}(e_2) = \{< v_1, (0,0,1) >, < v_2, (0,0,1) >\} \end{cases};$$

$$K_2 = v_{2(e_2)}^{(.1,.1,.9)} = \begin{cases} f_{K_2}(e_1) = \{< v_1, (0,0,1) >, < v_2, (0,0,1) >\} \\ f_{K_2}(e_2) = \{< v_1, (0,0,1) >, < v_2, (.1,.1,.9) >\} \end{cases};$$

$$K_3 = K_1 \cup K_2$$

and

$$L_1 = v_{2(e_1)}^{(.2,.7,.5)} = \begin{cases} f_{L_1}(e_1) = \{< v_1, (0,0,1) >, < v_2, (.2,.7,.5) >\} \\ f_{L_1}(e_2) = \{< v_1, (0,0,1) >, < v_2, (0,0,1) >\} \end{cases};$$

$$L_2 = v_{1(e_2)}^{(.2,.8,.2)} = \begin{cases} f_{L_2}(e_1) = \{< v_1, (0,0,1) >, < v_2, (0,0,1) >\} \\ f_{L_2}(e_2) = \{< v_1, (.2,.8,.2) >, < v_2, (0,0,1) >\} \end{cases};$$

$$L_3 = L_1 \cup L_2$$

Then (V, E, τ_{u1}, τ_{u2}) is a NBSTS over (V, E).
Hence (V, E, τ_{u1}, τ_{u2}) is a pairwise NS T_2-space.

Theorem 4.17 Let (V, E, τ_{u1}, τ_{u2}) be a NBSTS over (V, E). If (V, E, τ_{u1}, τ_{u2}) is a pairwise NS T_2-space then (V, E, $\tau_{u1} \vee \tau_{u2}$) is a NS T_2-space.
Proof. Let (V, E, τ_{u1}, τ_{u2}) be a NBSTS over (V, E).
Suppose that (V, E, τ_{u1}, τ_{u2}) is a pairwise NS T_2-space.

Let $u_{(e)}^{(\alpha,\beta,\gamma)}$ and $v_{(e')}^{(\alpha,\beta,\gamma)}$ be any two distinct NSPs.

Then there exist τ_{u1}-NSOS K and τ_{u2}-NSOS L such that

$$u_{(e)}^{(\alpha,\beta,\gamma)} \in K, \ v_{(e')}^{(\alpha,\beta,\gamma)} \in L \text{ and } K \cap L = \phi_u.$$

In either case $K, L \in \tau_{u1} \vee \tau_{u2}$.
Hence (V, E, $\tau_{u1} \vee \tau_{u2}$) is a NS T_2-space.

Theorem 4.18 Let (*V*, *E*, τ_{u1}, τ_{u2}) be a NBSTS over (*V*, *E*). If (*V*, *E*, τ_{u1}, τ_{u2}) is a pairwise NS T_2-space then (*V*, *E*, τ_{v1}, τ_{v2}) is also a pairwise NS T_2-space.

Proof. Let (*V*, *E*, τ_{u1}, τ_{u2}) be a NBSTS over (*V*, *E*).

Let $u_{(e)}^{(\alpha,\beta,\gamma)}$ and $v_{(e')}^{(\alpha,\beta,\gamma)}$ be any two distinct NSPs and $P, Q \in NSS(U,E)$.

Suppose that (*V*, *E*, τ_{u1}, τ_{u2}) is a pairwise NS T_2-space.

Then there exist τ_{u1}-NSOS *K* and τ_{u2}-NSOS *L* such that

$u_{(e)}^{(\alpha,\beta,\gamma)} \in K$, $v_{(e')}^{(\alpha,\beta,\gamma)} \in L$ and $K \cap L = \phi_u$.

Now $u_{(e)}^{(\alpha,\beta,\gamma)} \in P$ and $u_{(e)}^{(\alpha,\beta,\gamma)} \in K$; $v_{(e')}^{(\alpha,\beta,\gamma)} \in Q$ and $v_{(e')}^{(\alpha,\beta,\gamma)} \in L$

Then $u_{(e)}^{(\alpha,\beta,\gamma)} \in P \cap K$, $v_{(e')}^{(\alpha,\beta,\gamma)} \in Q \cap L$ where $K \in \tau_{u1}$, $L \in \tau_{u2}$.

Consider $K \cap L = \phi_u$.

$$\Rightarrow (P \cap K) \cap (L \cap Q) = P \cap \phi_u \cap Q.$$
$$\Rightarrow (P \cap K) \cap (Q \cap L) = \phi_u..$$

Thus $u_{(e)}^{(\alpha,\beta,\gamma)} \in P \cap K$, $v_{(e')}^{(\alpha,\beta,\gamma)} \in Q \cap L$ and $(P \cap K) \cap (Q \cap L) = \phi_u$.

Hence (*V*, *E*, τ_{v1}, τ_{v2}) is also a pairwise NS T_2-space.

Theorem 4.19 Every pairwise NS T_2-space is also a pairwise NS T_1-space.

Proof. Follows from Definitions 4.3 and 4.15.

Theorem 4.20 Let (*V*, *E*, τ_{u1}, τ_{u2}) be a NBSTS over (*V*, *E*). (*V*, *E*, τ_{u1}, τ_{u2}) is a pairwise NS T_2-space if and only if for any two distinct NSPs $u_{(e)}^{(\alpha,\beta,\gamma)}$ and $v_{(e')}^{(\alpha,\beta,\gamma)}$, there exist τ_{u1}-NSOS *K* containing $u_{(e)}^{(\alpha,\beta,\gamma)}$ but not $v_{(e')}^{(\alpha,\beta,\gamma)}$ such that $v_{(e')}^{(\alpha,\beta,\gamma)} \notin \overline{K}$.

Proof. Let (*V*, *E*, τ_{u1}, τ_{u2}) be a NBSTS over (*V*, *E*).

Let $u_{(e)}^{(\alpha,\beta,\gamma)}$ and $v_{(e')}^{(\alpha,\beta,\gamma)}$ be any two distinct NSPs.

Suppose that (*V*, *E*, τ_{u1}, τ_{u2}) is a pairwise NS T_2-space.

Then there exist τ_{u1}-NSOS *K* and τ_{u2}-NSOS *L* such that

$u_{(e)}^{(\alpha,\beta,\gamma)} \in K$, $v_{(e')}^{(\alpha,\beta,\gamma)} \in L$ and $K \cap L = \phi_u$.

Since $u_{(e)}^{(\alpha,\beta,\gamma)} \cap v_{(e')}^{(\alpha,\beta,\gamma)} = \phi_u$ and $K \cap L = \phi_u$, $v_{(e')}^{(\alpha,\beta,\gamma)} \notin K$.

Thus $v_{(e')}^{(\alpha,\beta,\gamma)} \notin \overline{K}$.

Conversely, assume that for any two distinct NSPs $u_{(e)}^{(\alpha,\beta,\gamma)}$ and $v_{(e')}^{(\alpha,\beta,\gamma)}$, there exist τ_{u1}-NSOS K containing $u_{(e)}^{(\alpha,\beta,\gamma)}$ but not $v_{(e')}^{(\alpha,\beta,\gamma)}$ such that $v_{(e')}^{(\alpha,\beta,\gamma)} \notin \overline{K}$.

Then $v_{(e')}^{(\alpha,\beta,\gamma)} \notin \left(\overline{K}\right)^c$.

Thus $\overline{K}$ and $\left(\overline{K}\right)^c$ are disjoint τ_{u1}-NSOS and τ_{u1}-NSOS containing $u_{(e)}^{(\alpha,\beta,\gamma)}$ and $v_{(e')}^{(\alpha,\beta,\gamma)}$ respectively.

Theorem 4.21 Let (V, E, τ_{u1}, τ_{u2}) be a NBSTS over (V, E) and (V, E, τ_{u1}, τ_{u2}) be a pairwise NS T_1-space for every NSP $u_{(e)}^{(\alpha,\beta,\gamma)} \in K \in \tau_{u1}$. If there exist τ_{u2}-NSOS L such that $u_{(e)}^{(\alpha,\beta,\gamma)} \in L \subseteq \overline{L} \subseteq K$, then ($V$, E, τ_{u1}, τ_{u2}) is a pairwise NS T_2-space.
Proof. Let (V, E, τ_{u1}, τ_{u2}) be a NBSTS over (V, E) and let it be a pairwise NS T_1-space
Suppose that $u_{(e)}^{(\alpha,\beta,\gamma)} \cap v_{(e')}^{(\alpha,\beta,\gamma)} = \phi_u$.

Let $u_{(e)}^{(\alpha,\beta,\gamma)}$ be a τ_{u1}-NSCS and $v_{(e')}^{(\alpha,\beta,\gamma)}$ be a τ_{u2}-NSCS.

Then $\left(v_{(e')}^{(\alpha,\beta,\gamma)}\right)^c$ is a τ_{u2}-NSOS such that

$$u_{(e)}^{(\alpha,\beta,\gamma)} \in \left(v_{(e')}^{(\alpha,\beta,\gamma)}\right)^c \in \tau_{u2}$$

Then there exist a τ_{u2}-NSOS L such that

$$u_{(e)}^{(\alpha,\beta,\gamma)} \in L \subseteq \overline{L} \subseteq \left(v_{(e')}^{(\alpha,\beta,\gamma)}\right)^c.$$

Thus $\left(v_{(e')}^{(\alpha,\beta,\gamma)}\right)^c \in \left(\overline{L}\right)^c$, $u_{(e)}^{(\alpha,\beta,\gamma)} \in L$ and $L \cap \left(\overline{L}\right)^c = \phi_u$.

Hence (V, E, τ_{u1}, τ_{u2}) is a pairwise NS T_2-space.

Remark 4.22 Let (V, E, τ_{u1}, τ_{u2}) be a NBSTS over (V, E). For any NSS K over (V, E), $\left(\overline{K}\right)^{\tau_{u2}}$ denotes the NS closure of K with respect to τ_{u2}-NST over (V, E).

Theorem 4.23 Let (V, E, τ_{u1}, τ_{u2}) be a NBSTS over (V, E). Then the following are equilavent:

(1) (V, E, τ_{u1}, τ_{u2}) is a pairwise NS Hausdorff space over (V, E).

(2) If $u_{(e)}^{(\alpha,\beta,\gamma)}$ and $v_{(e')}^{(\alpha,\beta,\gamma)}$ are distinct NSPs, there exist τ_{u1}-NSOS K such that

$$u_{(e)}^{(\alpha,\beta,\gamma)} \in K \quad \text{and} \quad v_{(e')}^{(\alpha,\beta,\gamma)} \in \left(\left(\overline{K}\right)^{\tau_{u2}}\right)^c.$$

Proof. (1) $\Rightarrow$ (2). Suppose that (V, E, τ_{u1}, τ_{u2}) is a pairwise NS Hausdorff space over (V, E).
Then there exist τ_{u1}-NSOS K and τ_{u2}-NSOS L such that

$u_{(e)}^{(\alpha,\beta,\gamma)} \in K$, $v_{(e')}^{(\alpha,\beta,\gamma)} \in L$ and $K \cap L = \phi_u$.

So that $K \subseteq L^c$.

Since $\left(\overline{K}\right)^{\tau_{u2}}$ is the smallest τ_{u2}-NSCS that contains K and L^c is a τ_{u2}-NSCS, then $\left(\overline{K}\right)^{\tau_{u2}} \subseteq L^c$

$$\Rightarrow \; L \subseteq \left(\left(\overline{K}\right)^{\tau_{u2}}\right)^c .$$

Thus $v_{(e')}^{(\alpha,\beta,\gamma)} \in L \subseteq \left(\left(\overline{K}\right)^{\tau_{u2}}\right)^c$.

Hence $v_{(e')}^{(\alpha,\beta,\gamma)} \in \left(\left(\overline{K}\right)^{\tau_{u2}}\right)^c$.

(2) $\Rightarrow$ (1). Let $u_{(e)}^{(\alpha,\beta,\gamma)}$ and $v_{(e')}^{(\alpha,\beta,\gamma)}$ be any two distinct NSPs.

By assumption, there exist τ_{u1}-NSOS K such that $u_{(e)}^{(\alpha,\beta,\gamma)} \in K$ and $v_{(e')}^{(\alpha,\beta,\gamma)} \in \left(\left(\overline{K}\right)^{\tau_{u2}}\right)^c$.

As $\left(\overline{K}\right)^{\tau_{u2}}$ is a τ_{u2}-NSCS so $L = \left(\left(\overline{K}\right)^{\tau_{u2}}\right)^c \in \tau_{u2}$.

Now $u_{(e)}^{(\alpha,\beta,\gamma)} \in K$, $v_{(e')}^{(\alpha,\beta,\gamma)} \in L$ and

$$K \cap L = K \cap \left(\overline{K}\right)^{\tau_{u2}} .$$

$$\supseteq K \cap \left(\left(\overline{K}\right)^{\tau_{u2}}\right)^c \qquad (\because K \subseteq \left(\overline{K}\right)^{\tau_{u2}})$$

$$= \phi_u .$$

Thus $K \cap L = \phi_u$.

Hence (V, E, τ_{u1} , τ_{u2}) is a pairwise NS Hausdorff space over (V, E).

Definition 4.24 Let $NSS(V, E)$ be the family of all NSSs over the universe V and $u \in V$. Then $u_E^{(\alpha,\beta,\gamma)}$ denotes the NSS over (V, E) for which $u_{(e)}^{(\alpha,\beta,\gamma)} = \left\{u^{(\alpha,\beta,\gamma)}\right\}$, for all $e \in E$.

Corollary 4.25 Let (V, E, τ_{u1} , τ_{u2}) be a pairwise NS T_2-space over (V, E). Then for each NSP $u_{(e)}^{(\alpha,\beta,\gamma)}$,

$$u_E^{(\alpha,\beta,\gamma)} = \bigcap \left\{\left(\overline{K}\right)^{\tau_{u2}} : u_{(e)}^{(\alpha,\beta,\gamma)} \in K \in \tau_{u1}\right\}.$$

Proof. Let (V, E, τ_{u1} , τ_{u2}) be a pairwise NS T_2-space over (V, E) and $u_{(e)}^{(\alpha,\beta,\gamma)}$ be a NSP.

Then there exist a NSOS $u_{(e)}^{(\alpha,\beta,\gamma)} \in K \in \tau_{u1}$.

If $u_{(e)}^{(\alpha,\beta,\gamma)}$ and $v_{(e')}^{(\alpha,\beta,\gamma)}$ are distinct NSPs, by 4.24 theorem, there exist τ_{u1}-NSOS K such that

$u_{(e)}^{(\alpha,\beta,\gamma)} \in K$ and $v_{(e')}^{(\alpha,\beta,\gamma)} \in \left(\left(\overline{K}\right)^{\tau_{u2}}\right)^{c}$.

$$\Rightarrow v_{(e')}^{(\alpha,\beta,\gamma)} \notin f_{\left(\overline{K}\right)^{\tau_{u2}}}(e').$$

$$\Rightarrow v_{(e')}^{(\alpha,\beta,\gamma)} \notin \bigcap_{u_{(e)}^{(\alpha,\beta,\gamma)} \in K \in \tau_{u1}} \left(f_{\left(\overline{K}\right)^{\tau_{u2}}}(e')\right) \quad \text{for all } e' \in E.$$

Thus

$$\bigcap\left\{\left(\overline{K}\right)^{\tau_{u2}} : u_{(e)}^{(\alpha,\beta,\gamma)} \in K \in \tau_{u1}\right\} \subseteq u_{E}^{(\alpha,\beta,\gamma)} \tag{1}$$

Also it is obvious that $u_{(e)}^{(\alpha,\beta,\gamma)} \in K \subseteq \left(\overline{K}\right)^{\tau_{u2}}$.

Thus

$$u_{E}^{(\alpha,\beta,\gamma)} \subseteq \bigcap\left\{\left(\overline{K}\right)^{\tau_{u2}} : u_{(e)}^{(\alpha,\beta,\gamma)} \in K \in \tau_{u1}\right\} \tag{2}$$

Hence from (1) and (2),

$$u_{E}^{(\alpha,\beta,\gamma)} = \bigcap\left\{\left(\overline{K}\right)^{\tau_{u2}} : u_{(e)}^{(\alpha,\beta,\gamma)} \in K \in \tau_{u1}\right\}.$$

Corollary 4.26 Let (V, E, τ_{u1}, τ_{u2}) be a pairwise NS T_2-space over (V, E). Then for each NSP $u_{(e)}^{(\alpha,\beta,\gamma)}$, $\left(u_{E}^{(\alpha,\beta,\gamma)}\right)^{c} \in \tau_{ui}$ for i = 1, 2.

Proof. Let (V, E, τ_{u1}, τ_{u2}) be a pairwise NS T_2-space over (V, E) and $u_{(e)}^{(\alpha,\beta,\gamma)}$ be a NSP.

By 4.25 corollary,

$$\left(u_{E}^{(\alpha,\beta,\gamma)}\right)^{c} = \bigcup\left\{\left(\left(\overline{K}\right)^{\tau_{u2}}\right)^{c} : u_{(e)}^{(\alpha,\beta,\gamma)} \in K \in \tau_{u1}\right\}$$

Since $\left(\overline{K}\right)^{\tau_{u2}}$ is a τ_{u2}-NSCS, then $\left(\left(\overline{K}\right)^{\tau_{u2}}\right)^{c} \in \tau_{u2}$.

By the axioms of a NS topological space,

$$\bigcup\left\{\left(\left(\overline{K}\right)^{\tau_{u2}}\right)^{c} : u_{(e)}^{(\alpha,\beta,\gamma)} \in K \in \tau_{u1}\right\} \in \tau_{u2}.$$

Thus $\left(u_{E}^{(\alpha,\beta,\gamma)}\right)^{c} \in \tau_{u2}$.

Similarly it can be proved that, $\left(u_{E}^{(\alpha,\beta,\gamma)}\right)^{c} \in \tau_{u1}$.

Hence $\left(u_{E}^{(\alpha,\beta,\gamma)}\right)^{c} \in \tau_{ui}$ for i = 1, 2.

Definition 4.27 A NBSTS (*V*, *E*, τ_{u1}, τ_{u2}) over (*V*, *E*) is called pairwise NS regular space, if *K* is a τ_{u1}-NSCS and $u_{(e)}^{(\alpha,\beta,\gamma)} \cap K = \phi_u$ then there exist τ_{u2}-NSOSs L_1 and L_2 such that $u_{(e)}^{(\alpha,\beta,\gamma)} \in L_1$, $K \subseteq L_2$ and $L_1 \cap L_2 = \phi_u$.

A NBSTS (*V*, *E*, τ_{u1}, τ_{u2}) over (*V*, *E*) is called pairwise NS T_3-space, if it is both a pairwise NS regular space and a pairwise NS T_1-space.

Theorem 4.28 Let (*V*, *E*, τ_{u1}, τ_{u2}) be a NBSTS over (*V*, *E*). Then (*V*, *E*, τ_{u1}, τ_{u2}) is a pairwise NS T_3-space if and only if for every $u_{(e)}^{(\alpha,\beta,\gamma)} \in K \in \tau_{u1}$, there exists $L \in \tau_{u2}$ such that $u_{(e)}^{(\alpha,\beta,\gamma)} \in L \subseteq \overline{L} \subseteq K$.

Proof. Let (*V*, *E*, τ_{u1}, τ_{u2}) be a NBSTS over (*V*, *E*).

Suppose that (*V*, *E*, τ_{u1}, τ_{u2}) is a pairwise NS T_3-space and $u_{(e)}^{(\alpha,\beta,\gamma)} \in K \in \tau_{u1}$.

Since (*V*, *E*, τ_{u1}, τ_{u2}) is a pairwise NS T_3-space for the NSP $u_{(e)}^{(\alpha,\beta,\gamma)}$ and τ_{u1}-NSCS K^c, there exist τ_{u2}-NSOSs L_1 and L_2 such that $u_{(e)}^{(\alpha,\beta,\gamma)} \in L_1$, $K^c \subseteq L_2$ and $L_1 \cap L_2 = \phi_u$.

Thus $u_{(e)}^{(\alpha,\beta,\gamma)} \in L_1 \subseteq (L_2)^c \subseteq K$.

Since $(L_2)^c$ is a τ_{u2}-NSCS, $\overline{L_1} \subseteq (L_2)^c$.

Hence $u_{(e)}^{(\alpha,\beta,\gamma)} \in L_1 \subseteq \overline{L_1} \subseteq K$.

Conversely, let $u_{(e)}^{(\alpha,\beta,\gamma)} \cap K = \phi_u$ and *K* be a τ_{u1}-NSCS.

Thus $u_{(e)}^{(\alpha,\beta,\gamma)} \in K^c$.

From the condition of the theorem,

$$u_{(e)}^{(\alpha,\beta,\gamma)} \in L \subseteq \overline{L} \subseteq K^c.$$

Then $u_{(e)}^{(\alpha,\beta,\gamma)} \in L$, $K \subseteq \overline{L}^c$ and $L \cap \overline{L}^c = \phi_u$.

Hence (*V*, *E*, τ_{u1}, τ_{u2}) is a pairwise NS T_3-space.

Definition 4.29 A NBSTS (*V*, *E*, τ_{u1}, τ_{u2}) over (*V*, *E*) is called pairwise NS normal space, if for every pair of disjoint τ_{u1}-NSCSs K_1 and K_2, there exists disjoint τ_{u2}-NSOSs L_1 and L_2 such that $K_1 \subseteq L_1$ and $K_2 \subseteq L_2$.

A NBSTS (*V*, *E*, τ_{u1}, τ_{u2}) over (*V*, *E*) is called pairwise NS T_4-space, if it is both a pairwise NS normal space and a pairwise NS T_1-space.

Theorem 4.30 Let (V, E, τ_{u1}, τ_{u2}) be a NBSTS over (V, E). Then (V, E, τ_{u1}, τ_{u2}) is a pairwise NS T_4-space if and only if for each τ_{u1}-NSCS K and τ_{u1}-NSOS L with $K \subseteq L$, there exists a τ_{u2}-NSOS P such that $K \subseteq P \subseteq \overline{P} \subseteq L$.

Proof. Let (V, E, τ_{u1}, τ_{u2}) be a NBSTS over (V, E).

Suppose that (V, E, τ_{u1}, τ_{u2}) is a pairwise NS T_4-space and K be τ_{u1}-NSCS and $K \subseteq L \in \tau_{u1}$.

Then L^c is a τ_{u1}-NSCS and $K \bigcap L^c = \phi_u$.

Since (V, E, τ_{u1}, τ_{u2}) is a pairwise NS T_4-space, there exist τ_{u2}-NSOSs P_1 and P_2 such that $K \subseteq P_1$, $M^c \subseteq P_2$ and $P_1 \bigcap P_2 = \phi_u$.

Thus $K \subseteq P_1 \subseteq (P_2)^c \subseteq L$.

Since $(P_2)^c$ is a τ_{u2}-NSCS, $\overline{P_1} \subseteq (P_2)^c$.

Hence $K \subseteq P_1 \subseteq \overline{P_1} \subseteq L$.

Conversely, let K_1 and K_2 be any two disjoint τ_{u1}-NSCSs.

Then $K_1 \subseteq (K_2)^c$.

From the condition of the theorem, there exists a τ_{u2}-NSOS P such that $K_1 \subseteq P \subseteq \overline{P} \subseteq (K_2)^c$.

Thus P and $(\overline{P})^c$ are τ_{u2}-NSOSs.

Then $K_1 \subseteq P$, $K_2 \subseteq (\overline{P})^c$ and $P \bigcap (\overline{P})^c = \phi_u$.

Hence (V, E, τ_{u1}, τ_{u2}) is a pairwise NS T_4-space.

Example 4.31 Let $V = \{v_1, v_2\}$, $E = \{e_1, e_2, e_3\}$, and $v_{1(e_1)}^{(.2,.4,.3)}$, $v_{1(e_2)}^{(.2,.8,.2)}$, $v_{1(e_3)}^{(.2,.5,.7)}$, $v_{2(e_1)}^{(.1,.2,.5)}$, $v_{2(e_2)}^{(.2,.7,.5)}$ and $v_{2(e_3)}^{(.1,.1,.9)}$ be NSPs.

Then $\tau_{u1} = \{\phi_u, 1_u, K_1, K_2, K_3, K_4, K_5, K_6, K_7\}$ and $\tau_{u2} = \{\phi_u, 1_u, L_1, L_2, L_3, L_4, L_5, L_6, L_7\}$ where $K_1, K_2, K_3, K_4, K_5, K_6, K_7, L_1, L_2, L_3, L_4, L_5, L_6, L_7$ are NSSs over (V, E), defined as follows

$$K_1 = \begin{cases} f_{K_1}(e_1) = \{< v_1, (.2,.4,.3) >, < v_2, (1,1,0) >\} \\ f_{K_1}(e_2) = \{< v_1, (1,1,0) >, < v_2, (1,1,0) >\} \\ f_{K_1}(e_3) = \{< v_1, (1,1,0) >, < v_2, (1,1,0) >\} \end{cases};$$

$$K_2 = \begin{cases} f_{K_2}(e_1) = \{< v_1, (1,1,0) >, < v_2, (1,1,0) >\} \\ f_{K_2}(e_2) = \{< v_1, (1,1,0) >, < v_2, (.2,.7,.5) >\} \\ f_{K_2}(e_3) = \{< v_1, (1,1,0) >, < v_2, (1,1,0) >\} \end{cases};$$

$$K_3 = \begin{cases} f_{K_3}(e_1) = \{< v_1, (1,1,0) >, < v_2, (1,1,0) >\} \\ f_{K_3}(e_2) = \{< v_1, (1,1,0) >, < v_2, (1,1,0) >\} \\ f_{K_3}(e_3) = \{< v_1, (.2,.5,.7) >, < v_2, (1,1,0) >\} \end{cases};$$

$K_4 = K_1 \cap K_2$;

$K_5 = K_1 \cap K_3$;

$K_6 = K_2 \cap K_3$;

$K_7 = K_1 \cap K_2 \cap K_3$

and

$$L_1 = \begin{cases} f_{L_1}(e_1) = \{< v_1, (.5,.7,.1) >, < v_2, (0,0,1) >\} \\ f_{L_1}(e_2) = \{< v_1, (0,0,1) >, < v_2, (0,0,1) >\} \\ f_{L_1}(e_3) = \{< v_1, (0,0,1) >, < v_2, (0,0,1) >\} \end{cases} ;$$

$$L_2 = \begin{cases} f_{L_2}(e_1) = \{< v_1, (0,0,1) >, < v_2, (0,0,1) >\} \\ f_{L_2}(e_2) = \{< v_1, (0.0,1) >, < v_2, (.8,.6,.1) >\} \\ f_{L_2}(e_3) = \{< v_1, (0,0,1) >, < v_2, (0,0,1) >\} \end{cases} ;$$

$$L_3 = \begin{cases} f_{L_3}(e_1) = \{< v_1, (0,0,1) >, < v_2, (0,0,1) >\} \\ f_{L_3}(e_2) = \{< v_1, (0,0,1) >, < v_2, (0,0,1) >\} \\ f_{L_3}(e_3) = \{< v_1, (.7,.5,.2) >, < v_2, (0,0,1) >\} \end{cases} ;$$

$L_4 = L_1 \cup L_2$;

$L_5 = L_1 \cup L_3$;

$L_6 = L_2 \cup L_3$;

$L_7 = L_1 \cup L_2 \cup L_3$;

Thus (*V*, *E*, τ_{u1}, τ_{u2}) is a NBSTS over (*V*, *E*).

Consider $(\tau_{u1})^c = \{\phi_u, 1_u, (K_1)^c, (K_2)^c, (K_3)^c, (K_4)^c, (K_5)^c, (K_6)^c, (K_7)^c\}$

where $(K_1)^c, (K_2)^c, (K_3)^c, (K_4)^c, (K_5)^c, (K_6)^c, (K_7)^c$ are τ_{u1}-NSCSs over (*V*, *E*), defined as follows

$$(K_1)^c = \begin{cases} f_{(K_1)^c}(e_1) = \{< v_1, (.3,.6,.2) >, < v_2, (0,0,1) >\} \\ f_{(K_1)^c}(e_2) = \{< v_1, (0,0,1) >, < v_2, (0,0,1) >\} \\ f_{(K_1)^c}(e_3) = \{< v_1, (0,0,1) >, < v_2, (0,0,1) >\} \end{cases} ;$$

$$(K_2)^c = \begin{cases} f_{(K_2)^c}(e_1) = \{< v_1, (0,0,1) >, < v_2, (0,0,1) >\} \\ f_{(K_2)^c}(e_2) = \{< v_1, (0,0,1) >, < v_2, (.5,.3,.2) >\} \\ f_{(K_2)^c}(e_3) = \{< v_1, (0,0,1) >, < v_2, (0,0,1) >\} \end{cases} ;$$

$$(K_3)^c = \begin{cases} f_{(K_3)^c}(e_1) = \{< v_1, (0,0,1) >, < v_2, (0,0,1) >\} \\ f_{(K_3)^c}(e_2) = \{< v_1, (0,0,1) >, < v_2, (0,0,1) >\} \\ f_{(K_3)^c}(e_3) = \{< v_1, (.7,.5,.2) >, < v_2, (0,0,1) >\} \end{cases} ;$$

$(K_4)^c = (K_1)^c \cup (K_2)^c$;

$(K_5)^c = (K_1)^c \cup (K_3)^c$;

$(K_6)^c = (K_2)^c \cup (K_3)^c$;

$(K_7)^c = (K_1)^c \cup (K_2)^c \cup (K_3)^c$

Hence (*V, E,* τ_{u1} , τ_{u2}) is a pairwise NS T_4 -space, also a pairwise NS T_3 -space.

5. DM Problem in NBSTS

In this segment, measured the output of problem and evaluated the decision on NBSTS.

Definition 5.1 Let *V* be the set of universal set, *E* be its parameter and $\tau_{u1} = \{\phi_u, 1_u, P\}$ and $\tau_{u2} = \{\phi_u, 1_u, Q\}$ be two NSTs. Then NSSs *P* and *Q* in NBSTS (*V, E,* τ_{u1} , τ_{u2}) over (*V, E*) are defined by $k \times l$ matrix where every entries are marks of v_k based on each parameters e_l .

$$[P]_{k\times l} = \begin{bmatrix} <T_{f_P(e_1)}(v_1), F_{f_P(e_1)}(v_1), I_{f_P(e_1)}(v_1)> & <T_{f_P(e_2)}(v_1), F_{f_P(e_2)}(v_1), I_{f_P(e_2)}(v_1)> & \cdots & <T_{f_P(e_l)}(v_1), F_{f_P(e_l)}(v_1), I_{f_P(e_l)}(v_1)> \\ <T_{f_P(e_1)}(v_2), F_{f_P(e_1)}(v_2), I_{f_P(e_1)}(2)> & <T_{f_P(e_2)}(v_2), F_{f_P(e_2)}(v_2), I_{f_P(e_2)}(v_2)> & \cdots & <T_{f_P(e_l)}(v_2), F_{f_P(e_l)}(v_2), I_{f_P(e_l)}(v_2)> \\ \vdots & \vdots & & \vdots \\ <T_{f_P(e_1)}(v_k), F_{f_P(e_1)}(v_k), I_{f_P(e_1)}(v_k)> & <T_{f_P(e_2)}(v_k), F_{f_P(e_2)}(v_k), I_{f_P(e_2)}(v_k)> & \cdots & <T_{f_P(e_l)}(v_k), F_{f_P(e_l)}(v_k), I_{f_P(e_l)}(v_k)> \end{bmatrix}$$

and

$$[Q]_{k\times l} = \begin{bmatrix} <T_{f_Q(e_1)}(v_1), F_{f_Q(e_1)}(v_1), I_{f_Q(e_1)}(v_1)> & <T_{f_Q(e_2)}(v_1), F_{f_Q(e_2)}(v_1), I_{f_Q(e_2)}(v_1)> & \cdots & <T_{f_Q(e_l)}(v_1), F_{f_Q(e_l)}(v_1), I_{f_Q(e_l)}(v_1)> \\ <T_{f_Q(e_1)}(v_2), F_{f_Q(e_1)}(v_2), I_{f_Q(e_1)}(2)> & <T_{f_Q(e_2)}(v_2), F_{f_Q(e_2)}(v_2), I_{f_Q(e_2)}(v_2)> & \cdots & <T_{f_Q(e_l)}(v_2), F_{f_Q(e_l)}(v_2), I_{f_Q(e_l)}(v_2)> \\ \vdots & \vdots & & \vdots \\ <T_{f_Q(e_1)}(v_k), F_{f_Q(e_1)}(v_k), I_{f_Q(e_1)}(v_k)> & <T_{f_Q(e_2)}(v_k), F_{f_Q(e_2)}(v_k), I_{f_Q(e_2)}(v_k)> & \cdots & <T_{f_Q(e_l)}(v_k), F_{f_Q(e_l)}(v_k), I_{f_Q(e_l)}(v_k)> \end{bmatrix}$$

where $v_1, v_2, \ldots, v_k \in V$ and $e_1, e_2, \ldots, e_l \in E$.

Clearly $\tau_{u1} = \{\phi_u, 1_u, [P]_{k\times l}\}$ and $\tau_{u2} = \{\phi_u, 1_u, [Q]_{k\times l}\}$ are also NSTs in NBSTS (*V, E,* τ_{u1} , τ_{u2}) over (*V, E*).

Thus the outcome result (OR) of $v \in V$ is given by the formula

$$OR(v)^e = \left[\frac{\left(T_{f_P(e)}(v) - F_{f_Q(e)}(v)\right) + \left(T_{f_Q(e)}(v) - F_{f_P(e)}(v)\right)}{2}\right]\left[1 - \frac{\left(I_{f_P(e)}(v) + I_{f_Q(e)}(v)\right)}{2}\right] \tag{5.1.1}$$

where $e \in E$.

The Net Result (NR) of each $v_1, v_2, \ldots, v_k \in V$ is

$$NR(v_i)^{e_j} = \sum_{j=1}^{l}\left(R(v_i)^{e_j}\right) \tag{5.1.2}$$

for all *i* = 1 to *k*.

Example 5.2 Let $V = \{v_1, v_2\}$, $E = \{e_1, e_2\}$ and $\tau_{u1} = \{\phi_u, 1_u, [K_1]_{2\times 2}, [K_2]_{2\times 2}, [K_3]_{2\times 2}, [K_4]_{2\times 2}\}$ and

$\tau_{u2} = \{\phi_u, 1_u, [L_1]_{2\times 2}, [L_2]_{2\times 2}\}$ where $[K_1]_{2\times 2}, [K_2]_{2\times 2}, [K_3]_{2\times 2}, [K_4]_{2\times 2}, [L_1]_{2\times 2}, [L_2]_{2\times 2}$ are NSSs over (*V, E*), defined as follows

$$[K_1]_{2\times 2} = \begin{bmatrix} <.1,.2,.3> & <.4,.5,.6> \\ <.9,.4,.7> & <.2,.6.3> \end{bmatrix},$$

$$[K_2]_{2\times 2} = \begin{bmatrix} <.2,.3,.1> & <.4,.7,.2> \\ <.5,.7,.6> & <.3,.5.2> \end{bmatrix},$$

$$[K_3]_{2\times 2} = \begin{bmatrix} <.2,.3,.1> & <.4,.7,.2> \\ <.9,.7,.6> & <.3,.6.2> \end{bmatrix},$$

$$[K_4]_{2\times 2} = \begin{bmatrix} <.1,.2,.3> & <.4,.5,.6> \\ <.5,.4,.7> & <.2,.5.3> \end{bmatrix}.$$

and

$$[L_1]_{2\times 2} = \begin{bmatrix} <.5,.8,.1> & <.3,.4,.2> \\ <.3,.1,.2> & <.6,.7.2> \end{bmatrix},$$

$$[L_2]_{2\times 2} = \begin{bmatrix} <.4,.7,.2> & <.2,.3,.5> \\ <.2,.1,.6> & <.3,.6.9> \end{bmatrix}.$$

Thus (V, E, τ_{u1}, τ_{u2}) is a NBSTS over (V, E).

Algorithm

Step 1: List the set of things or person $v \in V$ with their parameters $e \in E$.

Step 2: Go through the records of the particulars.

Step 3: Collect the data for each $v \in V$ according to all $e \in E$.

Step 4: Define NSSs.

Step 5: Define two different topologies τ_{u1} and τ_{u2} where each satisfies the condition of NST and so (V, E, τ_{u1}, τ_{u2}) is a NBSTS over (V, E).

Step 6: Form $NSSs \in \tau_{ui=u1,u2}$ matrix with collected data where v_k as rows and e_l as columns.

Step 7: Calculate the OR for all $v \in V$.

Step 8: Calculate the NR for all $v \in V$.

Step 9: Select a highest value among all the calculated NR.

Step 10: If two or more NR are identical, add one more parameter and repeat the process.

Step 11: End the process while we acquire the unique NR of v_k.

Problem 5.3 Let us suppose that there are two groups of women. First group consists of young age women (YAW, aging 20-25), say τ_{u1}, and second group consists of middle age women (MAW, aging 30-35), say τ_{u2}. Our aim is to insist both groups of women to select a saree together according to their desire and choice.

1. Let $V = \{sr_1, sr_2, sr_3, sr_4, sr_5\}$ be the set of sample sarees and selection done by the set of parameters let it be $E = \{c, q, d, p\}$ where is c = colour, q = quality, d = design and p = price.
2. Both groups are analyzing the sarees collections.
3. Data are collected for each sarees according to its paramaters given.
4. Convert these data as NSSs, say YAW and MAW.
5. Let $\tau_{u1} = \{\phi_u, 1_u, YAW\}$ and $\tau_{u2} = \{\phi_u, 1_u, MAW\}$ be two NSTs and so (V, E, τ_{u1}, τ_{u2}) is a NBSTS over (V, E).
6. The matrix form of NSSs YAW and MAW are as follows:

$$[YAW]_{5\times 4} = \begin{bmatrix} <.5,.7,.2> & <.7,.2,.2> & <.4,.2,.7> & <.8,.4,.5> \\ <.9,.3,.4> & <.3,.4,.2> & <.8,.7,.5> & <.9,.3,.6> \\ <.6,.3,.9> & <.7,.4,.8> & <.2,.7,.1> & <.3,.2,.5> \\ <.2,.4,.6> & <.4,.6,.8> & <.5,.3,.1> & <.7,.1,.3> \\ <.7,.5,.2> & <.1,.2,.3> & <.3,.6,.9> & <.1,.3,.4> \end{bmatrix}$$

and

$$[MAW]_{5\times 4} = \begin{bmatrix} <.6,.4,.3> & <.7,.3,.2> & <.3,.6,.2> & <.4,.7,.3> \\ <.8,.4,.2> & <.8,.4,.2> & <.5,.9,.4> & <.4,.4,.4> \\ <.1,.9,.2> & <.6,.7,.3> & <.8,.2,.1> & <.5,.1,.2> \\ <.2,.6,.7> & <.3,.4,.5> & <.7,.2,.1> & <.2,.9,.1> \\ <.3,.1,.2> & <.2,.1,.2> & <.3,.5,.2> & <.2,.4,.8> \end{bmatrix}$$

7. The **Table 5.3.1** is obtained by using the formula (5.1.1),

Table 5.3.1. OR table.

	sr1	**sr2**	**sr3**	**sr4**	**sr5**
c	.105	.3575	−.08	−.225	.21
q	.375	.21	.045	.15	−.085
d	.06	.04	.22	.45	−.1125
p	.09	.0975	.0425	.125	−.2925

8. The **Table 5.3.2** is obtained by using the formula (5.1.2),

Table 5.3.2. NR table.

	sr1	**sr2**	**sr3**	**sr4**	**sr5**
c	.105	.3575	−.08	−.225	.21
q	.375	.21	.045	.15	−.085
d	.06	.04	.22	.45	−.1125
p	.09	.0975	.0425	.125	−.2925
NR	.63	**.705**	.2275	.2	−.28

Thus the second saree has selected by both the categories of women.

Problem 5.4 Consider the situation of problem 5.3.

1. Let $V = \{sr_1, sr_2, sr_3, sr_4, sr_5\}$ be the set of sample sarees and selection done by the set of parameters let it be $E = \{c, q, d, p\}$ where is c = colour, q = quality, d = design and p = price.
2. Both groups are analyzing the sarees collections.
3. Data are collected for each sarees according to its paramaters given.
4. Convert these data as NSSs, say YAW and MAW.
5. Let $\tau_{u1} = \{\phi_u, 1_u, YAW\}$ and $\tau_{u2} = \{\phi_u, 1_u, MAW\}$ be two NSTs and so $(V, E, \tau_{u1}, \tau_{u2})$ is a NBSTS over (V, E).
6. The matrix form of NSSs YAW and MAW are as follows:

$$[YAW]_{5\times4} = \begin{bmatrix} <.5,.7,.2> & <.5,.3,.1> & <.4,.2,.7> & <.8,.4,.5> \\ <.9,.3,.4> & <.3,.4,.2> & <.8,.7,.5> & <.9,.3,.6> \\ <.1,.3,.4> & <.7,.4,.8> & <.2,.7,.1> & <.3,.2,.5> \\ <.2,.4,.6> & <.4,.6,.8> & <.7,.2,.2> & <.7,.1,.3> \\ <.7,.5,.2> & <.1,.2,.3> & <.3,.6,.9> & <.6,.3,.9> \end{bmatrix}$$

and

$$[MAW]_{5\times4} = \begin{bmatrix} <.6,.4,.3> & <.7,.2,.1> & <.3,.6,.2> & <.4,.7,.3> \\ <.8,.4,.2> & <.8,.4,.2> & <.5,.9,.4> & <.4,.4,.4> \\ <.2,.4,.8> & <.6,.7,.3> & <.8,.2,.1> & <.5,.1,.2> \\ <.2,.6,.7> & <.3,.4,.5> & <.7,.3,.2> & <.2,.9,.1> \\ <.3,.1,.2> & <.2,.1,.2> & <.3,.5,.2> & <.1,.9,.2> \end{bmatrix}$$

7. The **Table 5.4.1** is obtained by using the formula (5.1.1),

Table 5.4.1. OR table.

	sr1	sr2	sr3	sr4	sr5
c	.105	.3575	−.2925	−.225	.21
q	.45	.21	.045	.15	−.085
d	.06	.04	.22	.375	−.1125
p	.09	.0975	.0425	.125	−.08

8. The **Table 5.4.2** is obtained by using the formula (5.1.2),

Table 5.4.2. NR table.

	sr1	sr2	sr3	sr4	sr5
c	.105	.3575	−.2925	−.225	.21
q	.45	.21	.045	.15	−.085
d	.06	.04	.22	.375	−.1125
p	.09	.0975	.0425	.125	−.08
NR	**.705**	**.705**	.015	.125	−.0675

Thus first and second sarees have selected by both categories of women.

In this situation, we just add a parameter *f* = fabric in *E* and repeat the process.

4. After adding one more parameter, convert these data as NSSs, say YAW^* and MAW^*.

5. Let $\tau_{u1} = \{\phi_u, 1_u, YAW^*\}$ and $\tau_{u2} = \{\phi_u, 1_u, MAW^*\}$ be two NSTs and so (*V*, *E*, τ_{u1}, τ_{u2}) is a NBSTS over (*V*, *E*).

6. The matrix form of NSSs YAW^* and MAW^* are as follows:

$$\left[YAW^*\right]_{5\times5} = \begin{bmatrix} <.5,7,2> & <.5,3,1> & <.4,2,7> & <.8,4,5> & <.6,7,2> \\ <.9,3,4> & <.3,4,2> & <.8,7,5> & <.9,3,6> & <.5,1,3> \\ <.1,3,4> & <.7,4,8> & <.2,7,1> & <.3,2,5> & <.4,5,2> \\ <.2,4,6> & <.4,6,8> & <.7,2,2> & <.7,1,3> & <.7,8,4> \\ <.7,5,2> & <.1,2,3> & <.3,6,9> & <.6,3,9> & <.1,3,6> \end{bmatrix}$$

and

$$\left[MAW^*\right]_{5\times5} = \begin{bmatrix} <.6,4,3> & <.7,2,1> & <.3,6,2> & <.4,7,3> & <.9,6,3> \\ <.8,4,2> & <.8,4,2> & <.5,9,4> & <.4,4,4> & <.7,8,1> \\ <.2,4,8> & <.6,7,3> & <.8,2,1> & <.5,1,2> & <.6,5,4> \\ <.2,6,7> & <.3,4,5> & <.7,3,2> & <.2,9,1> & <.2,3,4> \\ <.3,1,2> & <.2,1,2> & <.3,5,2> & <.1,9,2> & <.6,2,7> \end{bmatrix}$$

7. The **Table 5.4.3** is obtained by using the formula (5.1.1),

Table 5.4.3. OR table after adding a parameter.

	sr_1	sr_2	sr_3	sr_4	sr_5
c	.105	.3575	−.2925	−.225	.21
q	.45	.21	.045	.15	−.085
d	.06	.04	.22	.375	−.1125
p	.09	.0975	.0425	.125	−.08
f	.175	.22	.1	.0225	−.225

8. The **Table 5.4.4** is obtained by using the formula (5.1.2),

Table 5.4.4. NR table after adding a parameter.

	sr_1	sr_2	sr_3	sr_4	sr_5
c	.105	.3575	−.2925	−.225	.21
q	.45	.21	.045	.15	−.085
d	.06	.04	.22	.375	−.1125
p	.09	.0975	.0425	.125	−.08
f	.175	.22	.1	.0225	−.225
NR	.88	**.925**	.115	.1475	−.2925

Thus the second saree has selected by both categories of women.

Problem 5.5 Consider the situation that there are six students on the main stage for Quiz Finale. There are two teams, each team consists of three students, one is Winner (*W*) and other is Runner (*R*). Let *FA*1 and *FA*2 be two final authorities to judge the event. Our problem is to find the best player in the winning team whose teammates are not mentioned here.

1. Let $V = \{st_1, st_2, st_3, st_4, st_5, st_6\}$ be the set of students and judgement is based on the set of parameters let it be *E* = {*ra*, *eff*, *ca*, *mr*, *gp*} where *ra* = right answers, *eff* = effectiveness, *ca* = complex analysis, *mr* = memory, *gp* = grasping power.

2. First of all these final authorities will go through the records of the students.

3. They will collect student's data according to their paramaters given.

4. These data are converted into two different NSSs, say $FA1$ and $FA2$.

5. Let $\tau_{u1} = \{\phi_u, 1_u, FA1\}$ and $\tau_{u2} = \{\phi_u, 1_u, FA2\}$ be two NSTs and so $(V, E, \tau_{u1}, \tau_{u2})$ is a NBSTS over (V, E).

6. The matrix form of NSSs $FA1$ and $FA2$ are as follows:

$$[FA1]_{6\times5} = \begin{bmatrix} <.4,.2,.7> & <.6,.3,.1> & <.2,.4,.8> & <.2,.9,.1> & <.6,.5,.3> \\ <.7,.3,.2> & <.8,.6,.1> & <.5,.4,.3> & <.9,.7,.2> & <.2,.7,.5> \\ <.3,.6,.6> & <.3,.5,.4> & <.6,.4,.2> & <.1,.2,.3> & <.5,.4,.6> \\ <.2,.6,.3> & <.7,.5,.4> & <.8,.6,.1> & <.4,.2,.7> & <.7,.3,.4> \\ <.6,.5,.4> & <.9,.2,.1> & <.7,.3,.4> & <.3,.5,.4> & <.4,.1,.4> \\ <.7,.3,.4> & <.6,.7,.2> & <.8,.9,.6> & <.3,.5,.4> & <.3,.5,.2> \end{bmatrix}$$

and

$$[FA2]_{6\times5} = \begin{bmatrix} <.4,.7,.3> & <.2,.3,.4> & <.5,.7,.2> & <.3,.4,.2> & <.2,.7,.1> \\ <.5,.1,.2> & <.6,.7,.3> & <.9,.3,.4> & <.7,.4,.8> & <.7,.2,.2> \\ <.7,.8,.1> & <.9,.3,.6> & <.1,.3,.4> & <.4,.6,.8> & <.3,.6,.9> \\ <.2,.6,.7> & <.7,.4,.8> & <.2,.4,.6> & <.1,.2,.3> & <.8,.4,.5> \\ <.5,.9,.4> & <.1,.2,.3> & <.7,.5,.2> & <.4,.2,.7> & <.3,.2,.5> \\ <.8,.4,.5> & <.9,.6,.3> & <.5,.3,.1> & <.1,.3,.4> & <.7,.3,.2> \end{bmatrix}$$

7. The **Table 5.5.1** is obtained by using the formula (5.1.1),

Table 5.5.1. OR table.

	st$_1$	**st$_2$**	**st$_3$**	**st$_4$**	**st$_5$**	**st$_6$**
ra	.11	.32	.045	$-.12$	.045	.195
eff	.105	.175	.24	.055	.24	.175
ca	.0675	.2275	.0325	.075	.24	.12
mr	.035	.135	$-.018$	$-.02$	$-.13$	.24
gp	.08	.055	$-.175$	.195	$-.085$	.18

8. The **Table 5.5.2** is obtained by using the formula (5.1.2),

Table 5.5.2. NR table.

	st$_1$	**st$_2$**	**st$_3$**	**st$_4$**	**st$_5$**	**st$_6$**
ra	.11	.32	.045	$-.12$	.045	.195
eff	.105	.175	.24	.055	.24	.175
ca	.0675	.2275	.0325	.075	.24	.12
mr	.035	.135	$-.018$	$-.02$	$-.13$	.24
gp	.08	.055	$-.175$	.195	$-.085$	.18
NR	.3975	**.9125**	$-.0375$	.005	.31	**.91**

Here both st_2 and st_6 got high score from judges, so they both does not belongs to R.

Case (i). If $st_2 \in W$ and $st_6 \in R$, then the best player award goes to st_2.

Case (ii). If $st_2 \in R$ and $st_6 \in W$, then the best player award goes to st_6.

Case (ii). If $st_2 \in W$ and $st_6 \in W$, then we just add a parameter ld = leadership.

4. After adding one more parameter, convert these data as NSSs, say $FA1^*$ and $FA2^*$.

5. Let $\tau_{u1} = \{\phi_u, 1_u, FA1^*\}$ and $\tau_{u2} = \{\phi_u, 1_u, FA2^*\}$ be two NSTs and so ($V, E, \tau_{u1}, \tau_{u2}$) is a NBSTS over ($V, E$).

6. The matrix form of NSSs $FA1^*$ and $FA2^*$ are as follows:

$$[FA1^*]_{6\times 6} = \begin{bmatrix} <.4,.2,.7> & <.6,.3,.1> & <.2,.4,.8> & <.2,.9,.1> & <.6,.5,.3> & <.3,.2,.4> \\ <.7,.3,.2> & <.8,.6,.1> & <.5,.4,.3> & <.9,.7,.2> & <.2,.7,.5> & <.9,.1,.1> \\ <.3,.6,.6> & <.3,.5,.4> & <.6,.4,.2> & <.1,.2,.3> & <.5,.4,.6> & <.7,.5,.3> \\ <.2,.6,.3> & <.7,.5,.4> & <.8,.6,.1> & <.4,.2,.7> & <.7,.3,.4> & <.8,.2,.1> \\ <.6,.5,.4> & <.9,.2,.1> & <.7,.3,.4> & <.3,.5,.4> & <.4,.1,.4> & <.3,.1,.5> \\ <.7,.3,.4> & <.6,.7,.2> & <.8,.9,.6> & <.3,.5,.4> & <.3,.5,.2> & <.6,.2,.6> \end{bmatrix}$$

and

$$[FA2^*]_{6\times 6} = \begin{bmatrix} <.4,.7,.3> & <.2,.3,.4> & <.5,.7,.2> & <.3,.4,.2> & <.2,.7,.1> & <.7,.5,.4> \\ <.5,.1,.2> & <.6,.7,.3> & <.9,.3,.4> & <.7,.4,.8> & <.7,.2,.2> & <.6,.9,.1> \\ <.7,.8,.1> & <.9,.3,.6> & <.1,.3,.4> & <.4,.6,.8> & <.3,.6,.9> & <.8,.4,.4> \\ <.2,.6,.7> & <.7,.4,.8> & <.2,.4,.6> & <.1,.2,.3> & <.8,.4,.5> & <.3,.4,.5> \\ <.5,.9,.4> & <.1,.2,.3> & <.7,.5,.2> & <.4,.2,.7> & <.3,.2,.5> & <.7,.3,.2> \\ <.8,.4,.5> & <.9,.6,.3> & <.5,.3,.1> & <.1,.3,.4> & <.7,.3,.2> & <.5,.9,.4> \end{bmatrix}$$

7. The **Table 5.5.3** is obtained by using the formula (5.1.1),

Table 5.5.3. OR table after adding a parameter.

	st1	st2	st3	st4	st5	st6
ra	.11	.32	.045	−.12	.045	.195
eff	.105	.175	.24	.055	.24	.175
ca	.0675	.2275	.0325	.075	.24	.12
mr	.035	.135	−.018	−.02	−.13	.24
gp	.08	.055	−.175	.195	−.085	.18
ld	.065	.352	−.055	.175	.12	.0675

8. The **Table 5.5.4** is obtained by using the formula (5.1.2),

Table 5.5.4. NR table after adding a parameter.

	st1	st2	st3	st4	st5	st6
ra	.11	.32	.045	−.12	.045	.195
eff	.105	.175	.24	.055	.24	.175
ca	.0675	.2275	.0325	.075	.24	.12
mr	.035	.135	−.018	−.02	−.13	.24
gp	.08	.055	−.175	.195	−.085	.18

Id	.065	.352	$-.055$	.175	.12	.0675
NR	.4625	**1.2375**	$-.0925$	.18	.43	.9775

Thus the best player award goes to st_2.

6. Conclusion

The main involvement of this paper is to preface the definition of NBSTSs and the study of some important properties of such spaces including separation axioms and the relationship between $T_{i=0,1,2,3,4}$-spaces. The key of this paper is to apply NBSTS in real life problems to take a decision, which might be positive or negative. In our problems two different types of NSTs are combined together to choose a unique decision according to the algorithm and calculation made by the formulae given here. Subsequently, NBSTS can be built up to pairwise NS separated sets, pairwise NS connected spaces, pairwise NS connected sets, pairwise NS disconnected spaces, pairwise NS disconnected sets and so on. We look forward to encourage this type of NBSTS will find a way to other types of topological structures. In future, some case studies which we mention in this paper need to develop on multicriteria DM also.

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Received: Apr 22, 2020. Accepted: July 12 2020

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Ranking of Pentagonal Neutrosophic Numbers and its Applications to solve Assignment Problem

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Abstract: Introduction of Neutrosophic sets and Neutrosophic numbers paves a way to handle uncertainty more effectively. In this paper we propose a new approach for ranking neutrosophic number by using its magnitude. We develop an algorithm for the solution of neutrosophic assignment problems involving pentagonal neutrosophic number. The proposed method is easy to understand and to apply for finding solution of neutrosophic assignment problems occurring in real life situations. To show the proposed strategy numerical models are given and the acquired results are analyzed.

Keywords: Neutrosophic sets, Neutrosophic number, Pentagonal neutrosophic number, Neutrosophic Assignment Problem, Optimal Solution.

1. Introduction

This section gives a survey of research work carried out so far to handle uncertainity. The novelty of present work, motivation behind it and structure of the remaining sections were also provided.

1.1: Literature survey

Smarandache [1] introduced neutrosophic sets having three components truthiness, indeterminacies, and falseness. Wang et al [3] introduced a single valued neutrosophic set, which is a subclass of a neutrosophic set presented by Smarandache [1]. Introduction of neutrosophic measure, neutrosophic integral, and neutrosophic probability by Smarandache [2,4] gave notation and many examples for neutrosophic measure, and consequently, the neutrosophic integral and neutrosophic probability are also defined. Many researchers have applied the neutrosophic logic in various fields.

To develop an optimization problem and its solution procedure in uncertain environment, the study of fuzzy number, intuitionistic fuzzy number, neutrosophic number and their ranking is necessary. Several researchers paid attention to fuzzy and intuitionistic fuzzy optimization methods by adopting various ranking techniques. But ranking of neutrosophic number is a risk task. To handle optimization problems having indeterminacy, ranking of neutrosophic numbers plays a vital role. S.Subasri and K.Selvakumari [5] ranked triangular neutrosophic number and applied the same to solve travelling salesman problems. Avishek Chakraborty [6], [7] gave a new ranking method to rank pentagonal neutrosophic number. Chakraborty A, Mondal SP, Ahmadian A, Senu N, Alam S,Salahshour S in 2018 [8] formatted Different forms of triangular neutrosophic numbers, and introduced de-neutrosophication techniques, and applied in critical path analysis. Smarandache [9] in 2019 approached TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number. Nabeeh NA, Abdel-Basset M, El-Ghareeb HA, Aboelfetouh A in 2019[10] developed multi-criteria decision making approach for IoT-based enterprises using neutrosophic numbers. Neutrosophic functions and neutrosophic calculus, was defined by Florentin Smarandache [11].Neutrosophic ordinary differential equation of first order via neutrosophic numbers is epitomized by Sumathi IR, MohanaPriya V [12], Differential equations in neutrosophic environment are explored, and solution of second-order linear differential equation with trapezoidal neutrosophic numbers as boundary conditions is discussed by R. Sumathi [13].Minimal spanning tree is one of the important fact in the field of graph theory. Single valued neutrosophic minimal spanning tree and clustering method was solved by Ye [14] in 2014. Mandal & Basu [15] solved similarity measure to find spanning tree related with neutrosophic arena. Mullai et.al [16] formulated minimum spanning tree problem in bipolar neutrosophic number. Broumi et.al [17] formulated shortest path problem on single valued neutrosophic graphs. Kandasamy [18] developed double-valued neutrosophic sets and their application in minimum spanning tree problems. Broumi et.al [19] formulated neutrosophic shortest path for solving Dijkstra's algorithm in graph theory. Mohamed Abdel-Basset [20] introduced bipolar neutrosophic number and applied in decission making problems he also proposed a model in [21] to evaluate the supply chain sustainability metrics based on a combination of quality function deployment and plithogenic aggregation operations. Assignment problems plays an important role in optimization. Many researchers have handled assignment problems in fuzzy and intuitionistic fuzzy environment but in neutrosophic environment, only few articles were published, that too involving other forms neutrosophic numbers. This was the first attempt to discuss assignment problems in neutrosophic environments involving pentagonal neutrosophic numbers.

1.2. Motivation

For the past few years the ambiguous data were handled by fuzzy sets, intuitionistic fuzzy sets, interval valued fuzzy sets and many such structures. Recently, the introduction of neutrosophic sets proves to be more suited to handle vagueness than existing set theoretical structure. Fuzzy number can measure only uncertainty, intuitionistic and interval valued intuitionistic fuzzy number can measure uncertainty and vagueness not hesitation. Only neutrosophic number can measure all the three parameters effectively. Thus pentagonal neutrosophic number attracts more attention and paves path for new research.

1.3. Novelties

From its inception, a few research articles had just distributed in various journals in neutrosophic field. Only a countable amount of articles had dealt with pentagonal neutrosophic number in that other types neutrosophic number can be generalized from pentagonal neutrosophic number. Neutrosophic assignment problem is an area in which focus on the de-neutrosophication technique applied to solve neutrosophic assignment problem.

1.4. Contribution

In this research article, symmetric pentagonal neutrosophic fuzzy numbers are considered. These numbers are converted into crisp values by means of ranking approach by magnitude. There are many ranking procedures which rank uncertainty and vagueness separately. Here our ranking procedure converts all the three parts of pentagonal neutrosophic number into crisp number. Lastly, the proposed ranking was applied to solve neutrosophic assignment problem. Section-1 throws an introduction to neutrosophic number and literature survey in the field. Section-2 gives the preliminaries Section-3 covers representation, definition and ranking of pentagonal neutrosophic number. Section-4 provides mathematical formation of neutrosophic assignment problem, algorithm to solve it and numerical example illustrating the procedure. The last section gives the conclusion and scope of the future work.

2. Preliminaries

Definition 2.1: [1] Let X be a universe set. A neutrosophic set A on X is defined as $A=\{T_A(x), I_A(x), F_A(x): x \in X\}$, where $T_A(x), I_A(x), F_A(x): X \to -]0,1[+$ represents the degree of membership, degree of indeterministic, and degree of non-membership respectively of the element $x \in X$, such that $-0 \le T_A(x)+I_A(x)+F_A(x) \le 3+$.

Definition 2.2: [12] $(\alpha,\beta,\gamma)-$cut: The $(\alpha,\beta,\gamma)-$cut neutrosophic set is denoted by $F(\alpha,\beta,\gamma)$, where $\alpha,\beta,\gamma \in [0,1]$ and are fixed numbers, such that $\alpha+\beta+\gamma \le 3$ is defined as $F(\alpha,\beta,\gamma)=\{T_A(x), I_A(x), F_A(x): x \in X, T_A(x) \ge \alpha, I_A(x) \le \beta, F_A(x) \le \gamma\}$.

Definition 2.3: [12] A neutrosophic set A defined on the universal set of real numbers R is said to be neutrosophic number if it has the following properties. (i) A is normal if there exist $x_0 \in R$, , such that $T_A(x_0)=1, I_A(x_0)=F_A(x_0)=0$. (ii) A is convex set for the truth function $T_A(x)$, i.e., $T_A(\mu x_1+(1-\mu)x_2) \ge \min(T_A(x_1), T_A(x_2)), \forall x_1, x_2 \in R, \mu \in [0,1]$. (iii) A is concave set for the indeterministic function and false fuunction $I_A(x)$ and $F_A(x)$, i.e., $I_A(\mu x_1+(1-\mu)x_2) \ge \max(I_A(x_1), I_A(x_2)), \forall x_1, x_2 \in R, \mu \in [0,1]$, $F_A(\mu x_1+(1-\mu)x_2) \ge \max(F_A(x_1), F_A(x_2)), \forall x_1, x_2 \in R, \mu \in [0,1]$.

3 Ranking of pentagonal Neutrosophic number

This section gives the definition of symmetric pentagonal neutrosphic number and a method of ranking it by means of magnitude. Numerical examples were illustrated to explain the proposed ranking procedure.

Definition: Symmetric Pentagonal neutrosophic number: A is a subset of neutrosophic number in R with the following truth function, indeterministic function, and falsity function which is given by the following:

$$A=\{(a_1,a_2,a_3,a_4,a_5),(b_1,b_2,b_3,b_4,b_5),(c_1,c_2,c_3,c_4,c_5);p,q,r\}, \text{ where } p,q,r \in [0,1].$$

The accuracy membership function $A_\mu(x): \mathbb{R} \to [0,1]$, the indeterminacy membership function $A_\nu(x): \mathbb{R} \to [0,1]$ and the falsity membership function $A_\varphi(x): \mathbb{R} \to [0,1]$ are defined as follows:

$$A_\mu(x) = \begin{cases} 0, x \le a_1 \\ p\left(\dfrac{x-a_1}{a_2-a_1}\right), a_1 \le x \le a_2 \\ 1+\dfrac{(p-1)(x-a_3)}{a_3-a_2}, a_2 \le x \le a_3 \\ 1,\ x = a_3 \\ 1+\dfrac{(p-1)(x-a_3)}{a_3-a_2}, a_2 \le x \le a_3 \\ p\left(\dfrac{a_5-x}{a_5-a_4}\right), a_4 \le x \le a_5 \\ 0, x \ge a_5 \end{cases}$$

$$A_\nu(x) = \begin{cases} 1, x \le b_1 \\ 1+\dfrac{(q-1)(x-b_1)}{b_2-b_1}, b_1 \le x \le b_2 \\ q\left(\dfrac{b_3-x}{b_3-b_2}\right), b_2 \le x \le b_3 \\ 0, x = b_3 \\ q\left(\dfrac{x-b_3}{b_4-b_3}\right), b_3 \le x \le b_4 \\ q+\dfrac{(1-q)(x-b_4)}{b_5-b_4}, b_4 \le x \le b_5 \\ 1, x \ge b_5 \end{cases}$$

$$A_\varphi(x) = \begin{cases} 1, x \le c_1 \\ 1+\dfrac{(r-1)(x-c_1)}{c_2-c_1}, c_1 \le x \le c_2 \\ r\left(\dfrac{c_3-x}{c_3-c_2}\right), c_2 \le x \le c_3 \\ 0, x = c_3 \\ r\left(\dfrac{x-c_3}{c_4-c_3}\right), c_3 \le x \le c_4 \\ r+\dfrac{(1-r)(x-c_4)}{c_5-c_4}, c_4 \le x \le c_5 \\ 1, x \ge c_5 \end{cases}$$

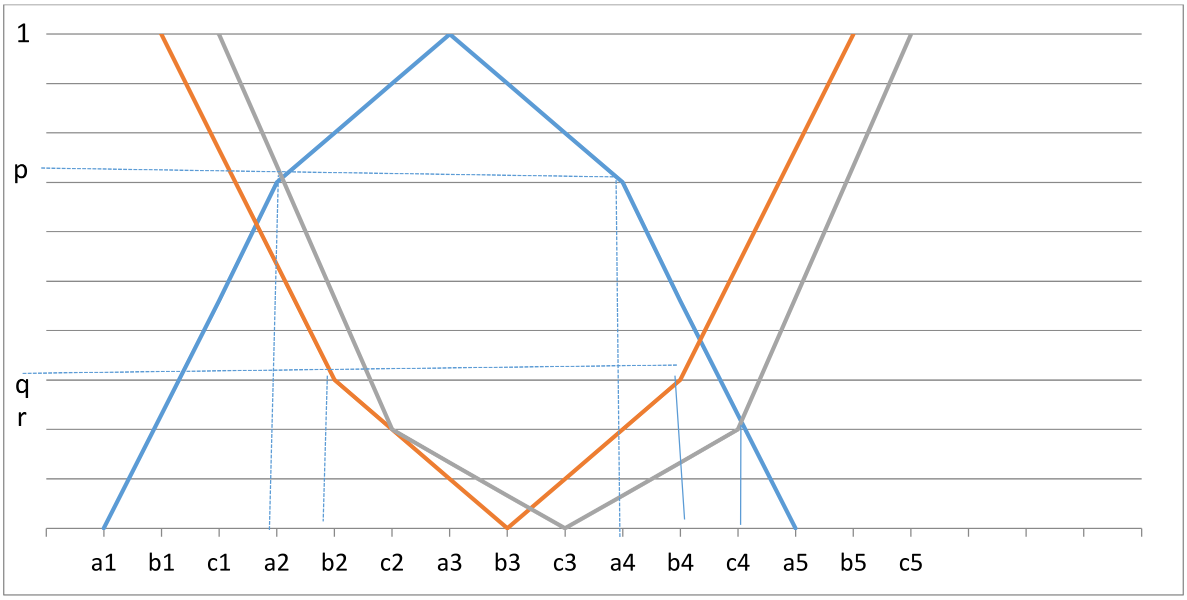

Figure 1.Representation of Symmetric Pentagonal neutrosophic number

3.1 Magnitude of a Pentagonal Neutrosophic Number

Let $A=\{(a_1,a_2,a_3,a_4,a_5),(b_1,b_2,b_3,b_4,b_5),(c_1,c_2,c_3,c_4,c_5);p,q,r\}$, where $p,q,r\in[0,1]$ be a symmetric pentagonal neutrosophic number whose accuracy membership function is given by

$$A_\mu(x)=\begin{cases}0, & x\le a_1\\ z_A^{L1}, & a_1\le x\le a_2\\ z_A^{L2}, & a_2\le x\le a_3\\ 1, & x=a_3\\ z_A^{R1}, & a_3\le x\le a_4\\ z_A^{R2}, & a_4\le x\le a_5\\ 0, & x\ge a_5\end{cases}$$

indeterminacy membership function is given by

$$A_\nu(x)=\begin{cases}1, & x\le b_1\\ k_A^{L1}, & b_1\le x\le b_2\\ k_A^{L2}, & b_2\le x\le b_3\\ 0, & x=b_3\\ k_A^{R1}, & b_3\le x\le b_4\\ k_A^{R2}, & b_4\le x\le b_5\\ 1, & x=b_5\end{cases}$$ and

and falsity membership function is given by

$$A_\phi(x)=\begin{cases}1, & x\le c_1\\ m_A^{L1}, & c_1\le x\le c_2\\ m_A^{L2}, & c_2\le x\le c_3\\ 0, & x=c_3\\ m_A^{R1}, & c_3\le x\le c_4\\ m_A^{R2}, & c_4\le x\le c_5\\ 1, & x\ge c_5\end{cases}$$

Here $z_A^{L1}(x):[a_1,a_2]\to[0,\mathrm{p}], z_A^{L2}(x):[a_2,a_3]\to[\mathrm{p},1], z_A^{R1}(x):[a_3,a_4]\to[\mathrm{p},1], \mathrm{z}_A^{R2}(\mathrm{x}):[\mathrm{a}_4,\mathrm{a}_5]\to[0,p]$,

where $z_A^{L1}(x), z_A^{L2}(\mathrm{x})$ are non-decreasing left accuracy functions and $z_A^{R1}(x),\ z_A^{R2}(x)$ are non-increasing right accuracy functions of symmetric Pentagonal neutrosophic number. Also $k_A^{L1}(x):[b_1,b_2]\to[q,1], k_A^{L2}(x):[b_2,b_3]\to[0,\mathrm{q}], k_A^{R1}(x):[b_3,b_4]\to[0,\mathrm{q}], k_A^{R2}(x):[b_4,b_5]\to[\mathrm{q},1]$, where $k_A^{L1}(x),\ \mathrm{k}_A^{L2}(\mathrm{x})$ are non-increasing left indeterminacy membership functions and $k_A^{R1}(x),\ \mathrm{k}_A^{R2}(x)$ are non-decreasing right indeterminacy membership functions of symmetric pentagonal neutrosophic number. Similarly the functions that occur in falsity membership function were defined as follows:

$m_A^{L1}(x):[\mathrm{c}_1, c_2]\to[\mathrm{r},1], \mathrm{m}_A^{L2}(x):[\mathrm{c}_2, c_3]\to[0,\mathrm{r}], \mathrm{m}_A^{R1}(x):[\mathrm{c}_3, c_4]\to[0,\mathrm{r}], \mathrm{m}_A^{R2}(x):[\mathrm{c}_4, c_5]\to[\mathrm{r},1]$, where $m_A^{L1}(x),\ \mathrm{m}_A^{L2}(\mathrm{x})$ are non-increasing left falsity membership function and $m_A^{R1}(x),\ \mathrm{m}_A^{R2}(x)$ are non-decreasing right falsity membership function of symmetric Pentagonal neutrosophic number. It is clear that $z_A^{L1}(x), z_A^{L2}(\mathrm{x})$ $z_A^{R1}(x), z_A^{R2}(x)$, $k_A^{L1}(x),\mathrm{k}_A^{L2}(\mathrm{x})$, $k_A^{R1}(x),\mathrm{k}_A^{R2}(x)$, $m_A^{L1}(x),\mathrm{m}_A^{L2}(\mathrm{x})$, $m_A^{R1}(x),\ \mathrm{m}_A^{R2}(x)$ are one to one and inverse exist.

The inverse functions of left and right accuracy, indeterminacy and falsity functions are defined as follows:

$f_A^{L1}:[0,p]\to R, f_A^{L2}:[p,1]\to R, f_A^{R1}:[p,1]\to R, f_A^{R2}:[0,p]\to R, g_A^{L1}:[1,q]\to R, g_A^{L2}:[0,q]\to R,$
$g_A^{R1}:[0,q]\to R, g_A^{R2}:[1,q]\to R, h_A^{L1}:[1,q]\to R, h_A^{L2}:[0,q]\to R, h_A^{R1}:[0,q]\to R, h_A^{R2}:[1,q]\to R,$ where

$$f_A^{L1}(y)=a_1+y\frac{(a_2-a_1)}{p},\quad 0\le y\le p$$
$$f_A^{L2}(y)=a_2+\frac{(a_3-a_2)(y-p)}{1-p},\quad p\le y\le 1$$
$$f_A^{R1}(y)=a_3+\frac{(a_4-a_3)(y-1)}{p-1},\quad p\le y\le 1$$
$$f_A^{R2}(y)=a_5+y\frac{(a_4-a_5)}{p},\quad 0\le y\le p$$

$$g_A^{L1}(y) = b_1 + \frac{(b_2 - b_1)(y-1)}{q-1}, \quad q \le y \le 1$$
$$g_A^{L2}(y) = b_3 - \frac{(b_3 - b_2)y}{q}, \quad 0 \le y \le q$$
$$g_A^{R1}(y) = b_3 + \frac{(b_4 - b_3)y}{q}, \quad 0 \le y \le q$$
$$g_A^{R2}(y) = b_4 + \frac{(b_5 - b_4)(y-q)}{1-q}, \quad q \le y \le 1$$
$$h_A^{L1}(y) = c_1 + \frac{(c_2 - c_1)(y-1)}{r-1}, \quad r \le y \le 1$$
$$h_A^{L2}(y) = c_3 - \frac{(c_3 - c_2)y}{r}, \quad 0 \le y \le r$$
$$h_A^{R1}(y) = c_3 + \frac{(c_4 - c_3)y}{r}, \quad 0 \le y \le r$$
$$h_A^{R2}(y) = c_4 + \frac{(c_5 - c_4)(y-r)}{1-r}, \quad r \le y \le 1$$

The magnitude denoted by Mag(A) of a symmetric pentagonal neutrosophic number $A = \{(a_1, a_2, a_3, a_4, a_5), (b_1, b_2, b_3, b_4, b_5), (c_1, c_2, c_3, c_4, c_5); p, q, r\}$, is determined as follows:

$$Mag(A) = \frac{1}{2}\int (f_A^{L1}(\alpha) + f_A^{L2}(\alpha) + f_A^{R1}(\alpha) + f_A^{R2}(\alpha) + 2\,\mathrm{a}_3 + g_A^{L1}(\alpha) + g_A^{L2}(\alpha) + g_A^{R1}(\alpha) + g_A^{R2}(\alpha) + 2\,\mathrm{b}_3 + h_A^{L1}(\alpha) + h_A^{L2}(\alpha) + h_A^{R1}(\alpha) + h_A^{R2}(\alpha) + 2\,\mathrm{c}_3)\, t(\alpha)\, d\alpha.$$
$$= \frac{1}{12}[p^2(a_1 + a_5) + (1+p)(a_2 + a_4) - (2p^2 + 2p - 10)a_3 - (q^2 + q - 2)(b_1 + b_5) + (1+q)(b_2 + b_4) + (2q^2 + 6)b_3 - (r^2 + r - 2)(c_1 + c_5) + (1+r)(c_2 + c_4) + (2r^2 + 6)c_3]. \tag{1}$$

where the function t(α)is a weighted function and is a non-negative and increasing function on [0,1] with t(0)=0,t(1)=1and $\int_0^1 t(\alpha)d\alpha = \frac{1}{2}$ we choose t(α) $= \alpha$.The scalar value Mag(A) is used to rank Pentagonal neutrosophic number.

Remark:

When $p = 0, q = 1, r = 1$ pentagonal neutrosophic number becomes triangular neutrosophic number. Then the magnitude of A defined in equation (1) will be transformed into

$$Mag(A) = \frac{1}{12}[(a_2 + a_4) + 10a_3 + 2(b_2 + b_4) + 8b_3 + 2(c_2 + c_4) + 8c_3]$$

3.2 Ranking Procedure

Using the magnitude of symmetric pentagonal neutrosophic number defined above, the ordering of pentagonal neutrosophic numbers is explained in this section.

Let $A=\{(a_1,a_2,a_3,a_4,a_5),(b_1,b_2,b_3,b_4,b_5),(c_1,c_2,c_3,c_4,c_5);p,q,r\}$, and

$B=\{(d_1,d_2,d_3,d_4,d_5),(e_1,e_2,e_3,e_4,e_5),(i_1,i_2,i_3,i_4,i_5);u,v,w\}$ be any two arbitrary Pentagonal neutrosophic numbers. Then the ranking procedure is as follows:

Step 1: Compute Mag(A) , Mag(B),any one of the following cases prevail.

Step 2:

(i) If $Mag(A) > \text{Mag(B)}$, then A > B

(ii) If $Mag(A) < \text{Mag(B)}$, then A < B

(iii) If $Mag(A) = \text{Mag(B)}$, then A = B

3.3 Numerical examples

The ordering procedure in the previous section is illustrated by numerical examples. Consider the following sets of Pentagonal neutrosophic numbers.

Set 1:A={(.5,1.5,2.5,3.5,4.5)(0.3,1.3,2.3,3.3,4.3)(1.8,2.8,3.8,4.8,5.8);0.5,0.5,0.5}

B={(.7,1.7,2.5,3.5,4.7)(.5,1.5,2.2,3.2,4)(1.7,2.7,3.,4.7,5.7);0.5,0.5,0.5}

C={(1,4,7,10,13)(0.5,3.5,6.5,9.5,12.5)(4.5,7.5,9,12,14.5);0.5,0.5,0.5}

Set 2:A={(10,15,20,25,30)(0,3,5,7,10)(0,1,2,3,4,);0.5,0.5,0.5}

B={(5,10,15,20,25)(1,2,3,4,5)(1,1.5,2,2.5,3);0.5,0.5,0.5}

C={(10,20,30,40,50)(1,4,7,8,10)(1,1.5,2,2.5,3);0.5,0.5,0.5}

The table.1 gives the comparison of proposed ranking of Pentagonal neutrosophic numbers with the existing methods.

Author name and method	Set 1	Set 2
Proposed method	A=8.6 B=8.4 C=22.79 Result: C > A > B	A=27 B=20 C=38.5 Result: C>A>B
Avishek Chakraborty's De-Neutrosophication value [6]	A=2.86 B=2.66 C=7.66 Result: C > A = B	A=9 B=6.66 C=12.70 Result: C>A>B
Avishek Chakraborty's accuracy function value [7]	A= -.533 B= -.45 C= -2.33 Result: B > A >C	A=5 B=4 C=8 Result: C>A>B

Table.1 Comparison table for ranking Pentagonal neutrosophic numbers

The table 2 gives the numerical example of the De-Neutrosophication value of Triangular neutrosophic numbers.

Sl.No	Triangular neutrosophic numbers	proposed method of ranking
1	A={(1,2,3)(0.5,1.5,2.5)(1.2,2.7,3.5)}	6.083
	B={(.5,1.5,2.5)(.3,1.3,2.2)(.7,1.7,2.2}	3.723
	Result	A > B

Table.2 De-Neutrosophication value Triangular neutrosophic numbers

4. Application of Ranking of Pentagonal Neutrosophic Number in solving Neutosphic Assignment Problem

In this section neutrosophic assignment problem with pentagonal neutrosophic numbers as parameters was formulated, algorithm for identifying the optimal solution to neutrosophic assignment problem was stated. Finally a numerical example was produced to explain the proposed algorithm.

Need for Pentagonal neutrosophic numbers

Suppose there are n facilities and n jobs it is clear that in this case, there will be n assignments. Each facility or say worker can perform each job, one at a time. But there should be certain procedure by which assignment should be made so that the profit is maximized or the cost or time is minimized. But in our real life applications the times taken to complete the job undergo uncertainty, hesitation and vagueness. In such cases we cannot have the parameter as a real value. So we have to use some other representation of the parameter with which the uncertainty, hesitation and vagueness can be measured. The below discussion justify the need for selecting the cost parameter in the terms of Pentagonal neutrosophic number.

- **If the parameter is a real value -** uncertainty hesitation and vagueness cannot be handled
- **If the parameter is a fuzzy value-** uncertainty but hesitation and vagueness cannot be handled
- **If the parameter is an Intuitionistic Fuzzy value -** uncertainty and hesitation can be handled but vagueness cannot be handled.
- **If the parameter is a Pentagonal neutrosophic value -** uncertainty, hesitation and vagueness (i.e) all the components can be handled.

From the above discussion, it is clear that only pentagonal neutrosophic environment can tackle the impreciseness, hesitation and truthiness in a membership function of an uncertain number, which is more reliable, logical and realistic for a decision maker. Pentagonal neutrosophic numbers enabled to meet the imprecise parameters as well, which is approvingly the advantageous for the decision makers to analyze the result in a more precise manner. Moreover Pentagonal neutrosophic numbers generalize other types of neutrosophic numbers.

Pentagonal neutrosophic assignment problem may be formulated as follows:

Consider the assignment problem with cost function as Pentagonal neutrosophic number.

Minimize $z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$, subject to

$\sum_{i=1}^{n} x_{ij} = 1, j = 1,2,3,......n$ and $\sum_{j=1}^{n} x_{ij} = 1, j = 1,2,3,......n,$ $x_{ij} = 1$ or 0 for al $i, j,$ where c_{ij} is a

Pentagonal neutrosophic number and the total cost for performing all the activity is given by $\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$.

Fundamental Theorems of a Pentagonal neutrosophic Assignment Problem

The solution of a Pentagonal neutrosophic assignment problem is fundamentally based on the following two theorems:

Theorem 1:

In a Pentagonal neutrosophic assignment problem, if we add or subtract an Pentagonal neutrosophic number to every element of any row (or column) of the Pentagonal neutrosophic parameter matrix $[c_{ij}]$, then an assignment that minimizes the total Pentagonal neutrosophic parameter on one matrix also minimizes the total Pentagonal neutrosophic parameter on the other matrix.

Minimize $Z = \sum^{n}\sum^{n} c_{ij}x_{ij}$ with $\sum_{n}^{n} x_{ij} = 1, j = 12....n,$ $\sum^{n} x_{ij} = 1, i = 1,2..n$ $x_{ij} = 0$ or 1 for every i, j then x_{ij}^{*} also minimize $Z^{*} = \sum\sum c_{ij}^{*}x_{ij}$ where $c_{ij}^{*} = c_{ij} - u_i - v_j$ *for all* $i, j = 1,2...n$ for all i, j=1, 2,……n are some real valued Pentagonal neutrosophic number.

Proof:

$$Z^{*} = \sum\sum c_{ij}^{*}x_{ij}$$

$$= \sum_{j}^{n}\sum_{i}^{n}(c_{ij} - u_i - v_j)x_{ij}$$

$$= \sum_{i}^{n}\sum_{j}^{n} c_{ij}x_{ij} - \sum_{i}^{n} u_i \sum_{j}^{n} x_{ij} - \sum_{j}^{n} v_j - \sum_{i}^{n} x_{ij}$$

$$= Z - \sum_{i}^{n} u_i - \sum_{j}^{n} v_j \quad \text{since} \sum_{i}^{n} x_{ij} = 1 \, and \sum_{j}^{n} x_{ij} = 1$$

This shows that the minimization of the new objective function Z^{*}yields the same solution as the minimization of original objective function Z

Theorem 2:

In a pentagonal neutrosophic assignment problem with parameter function c_{ij} if all $c_{ij} \geq 0$ the feasible solution which x_{ij} satisfies $\sum_{i}^{n}\sum_{j}^{n} c_{ij}x_{ij} = 0$ is an optimal solution.

Proof: Since all $c_{ij} \geq 0$ all $x_{ij} \geq 0$. The objective function $Z = \sum^{n}\sum^{n} c_{ij}x_{ij}$ cannot be negative the minimum possible that Z can have is 0.Therefore any feasible solution x_{ij} obtained that satisfies $Z = \sum^{n}\sum^{n} c_{ij}x_{ij} = 0$ will be optimal.

Algorithm to solve Pentagonal neutrosophic assignment problem:

We, now introduce a new algorithm called the Pentagonal neutrosophic Hungarian method for finding a Pentagonal neutrosophic optimal assignment for Pentagonal neutrosophic assignment problem.

Step 1: Determine the Pentagonal neutrosophic parameter table from the given problem.

Step 2: Convert the given Pentagonal neutrosophic assignment matrix to crisp by using the magnitude method.

Step 3: Subtract the row minimum from each row entry of that row. Subtract the column minimum of the resulting matrix from each column entry of that column. Each column and row now has at least one zero.

Step 4: In the modified assignment table obtained in step 3, search for optimal assignment as follows.

Examine the rows successively until a row with a single zero is found. Assign the zero and cross off all other zeros in its column. Continue this for all the rows. Repeat the procedure for each column of reduced assignment table. If a row and / or column have two or more zeros assign arbitrary any one of these zeros and cross off all other zeros of that row/column. Repeat the above process successively until the chain of assigning or cross ends.

Step 5: If the number of assignments is equal to n, the order of the parameter matrix, optimal solution is reached. If the number of assignments is less than n, parameter matrix, go to the step 6.

Step 6: Draw the minimum number of horizontal and / or vertical lines to cover all the zeros of the reduced assignment matrix. This can be done by using the following:

(i)Mark rows that do not have any assigned zero. (ii)Mark columns that have zeros in the marked rows. (iii)Mark rows that do have zeros in the marked columns. Repeat (ii) and (iii) of the above until the chain of marking is completed. Draw lines through all the unmarked rows and marked columns. This gives the desired minimum number of lines.

Step 7: Develop the new revised reduced parameter matrix as follows: Find the smallest entry of the reduced matrix not covered by any of the lines. Subtract this entry from all the uncovered entries and add the same to all the entries lying at the intersection of any two lines.

Step 8: Repeat step 5 to step 7 until optimal solution to the given assignment problem is attained.

Numerical example:

Suppose we want to assign jobs A, B, C, D to machine M1, M2, M3, and M4. Our aim is to find the minimum time so that the job is completed so that each machine is assigned only one job, The time parameter may not be a real value since the time taken to complete a job depend on the facts such as (i) working condition of the machine (ii) climatic condition and so on. So we represent the time parameter as Pentagonal neutrosophic number. The problem can be considered as follows.

Minimize $Z = \sum_{j}^{4}\sum_{i}^{4} c_{ij}x_{ij}$

Where

$c_{11} = \{(8,13,19,24,30)(7,10,15,22,27)(10,16,23,25,32);0.5,0.5,0.5\}$

$c_{12} = \{(7,12,18,24,30)(6,10,14,20,25)(10,15,20,25,35);0.6,0.4,0.3\}$

$c_{13} = \{(3,8,14,20,26)(2,7,12,18,22)(5,10,15,24,30);0.4,0.3,0.4\}$

$c_{14} = \{(10,15,20,26,32)(7,12,18,22,26)(12,16,22,28,35);0.6,0.4,0.3\}$

$c_{21} = \{(8,14,20,26,32)(6,12,18,22,28)(10,18,24,28,35);0.4,0.5,0.4\}$

$c_{22} = \{(6,10,15,20,25)(4,8,12,18,22)(8,14,20,24,30);0.6,0.6,0.5\}$

$c_{23} = \{(9,14,20,25,30)(6,12,16,21,24)(12,15,23,28,35);0.5,0.4,0.3\}$

$c_{24} = \{(11,15,19,23,27)(8,12,17,21,24)(14,18,22,26,30);0.6,0.6,0.4\}$

$c_{31} = \{(7,10,13,16,20)(6,8,12,15,18)(10,14,18,22,25);0.7,0.4,0.5\}$

$c_{32} = \{(12,15,18,24,26)(5,9,13,17,21)(10,14,18,22,26);0.8,0.6,0.2\}$

$c_{33} = \{(6,11,15,18,24,26)(9,13,17,20,23)(14,18,22,25,29);0.6,0.4,0.3\}$

$c_{34} = \{(7,11,14,17,26)(4,8,13,19,23)(9,14,19,24,28);0.6,0.5,0.4\}$

$c_{41} = \{(4,9,15,21,27)(3,8,13,19,23)(6,11,16,25,31);0.6,0.6,0.4\}$

$c_{42} = \{(11,14,17,23,25)(7,11,16,20,23)(13,17,21,25,29);0.5,0.3,0.4\}$

$c_{43} = \{(6,11,15,18,24,26)(7,11,15,18,20)(13,17,21,24.27);0.7,0.3,0.4\}$

$c_{44} = \{(10,14,18,22,26)(5,9,13,19,23)(9,15,21,25,31);0.5,0.4,0.3\}$

Subject to $\sum_{j=1}^{4} x_{ij} = 1,\ i = 1,2,3,4.\ \sum_{i}^{4} x_{ij} = 1,\ j = 1,2,3,4.,\ x_{ij} = 1 \text{ or } 0,\ \text{for all } i, j.$

Pentagonal neutrosophic assignment matrix in the crisp form

$$C_{ij} = \begin{array}{ccccc} & A & B & C & D \\ M1 & 56.5 & 53.52 & 42.18 & 60.0 \\ M2 & 60.9 & 47.0 & 58.54 & 57.69 \\ M3 & 42.82 & 49.55 & 54.12 & 46.25 \\ M4 & 45.23 & 54.98 & 49.99 & 51.98 \end{array}$$

Applying step 3, 4 the following time parameter matrix is obtained is

$$C_{ij} = \begin{array}{ccccc} & A & B & C & D \\ M1 & 14.32 & 11.34 & 0 & 17.82 \\ M2 & 13.9 & 0 & 11.54 & 10.69 \\ M3 & 0 & 6.73 & 11.3 & 3.43 \\ M4 & 0 & 9.75 & 4.76 & 6.75 \end{array}$$

Applying step 5 we the following result

$$C_{ij} = \begin{array}{ccccc} & A & B & C & D \\ M1 & 14.32 & 11.34 & 0 & 14.39 \\ M2 & 13.9 & 0 & 11.54 & 6.26 \\ M3 & 0 & 6.73 & 11.3 & 0 \\ M4 & 0 & 8.81 & 4.76 & 3.32 \end{array}$$

Number of assignment is equal to the order of the matrix. Therefore the optimal assignment is $A \rightarrow M4, B \rightarrow M2, C \rightarrow M1, D \rightarrow M3$ the minimum time to complete the job is $45.23 + 47 + 42.18 + 46.25 = 180.66.$.

5. Conclusions

In this research article the de-Neutrosophication Pentagonal neutrosophic number into a real number has been introduced by means of magnitude approach. The resulted ranking has been applied to solve neutrosophic assignment problems. The algorithm stated in this paper is simple to use and applicable to solve neutrosophic assignment problems in short time. Also it produces accurate result. There is much scope for future work in this field. This ranking can be applied to solve linear, non-linear and transportation problems involving pentagonal neutrosophic number. Further image processing multi-criteria decision making problems can also make use of this ranking method for smart computation.

Funding: "This research received no external funding"

Acknowledgments: In this section we acknowledge the Chief Editor and reviewers for their valuable suggestions.

Conflicts of Interest: "The authors declare no conflict of interest."

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Received: Apr 10, 2020. Accepted: July 13 2020

Neutrosophic Sets and Systems, Vol. 35, 2020
University of New Mexico

A new distance measure for trapezoidal fuzzy neutrosophic numbers based on the centroids

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Abstract: Distance measure is a numerical measurement of the distance between any two objects. The aim of this paper is to propose a new distance measure for trapezoidal fuzzy neutrosophic numbers based on the centroids with graphical representation. In addition, the metric properties of the proposed measure are examined in detail. A decision making problem also has been solved using the proposed distance measure for a software selection process. comparative analysis has been done with the existing methods to show the potential of the proposed distance measure and various forms of trapezoidal fuzzy neutrosophic number have been listed out to show the uniqueness of the proposed graphical representation. Further, advantages of the proposed distance measure have been given.

Keywords: trapezoidal fuzzy neutrosophic numbers; centroids; distance measure

1-Introduction

Zadeh introduced a mathematical frame work called fuzzy set [43] which plays a very significant role in many aspects of science. Intuitionistic fuzzy set is the generalization of the Zadeh's fuzzy set which was presented by Atanassov [3]. Later, triangular intuitionistic fuzzy sets was developed by Liu and Yuan [22] which is based on the combination of triangular fuzzy numbers and intuitionistic fuzzy sets. The fundamental characteristic of the triangular intuitionistic fuzzy set is that the values of its membership function and non-membership function are triangular fuzzy numbers rather than exact numbers. Furthermore, Ye [38] extended the triangular intuitionistic fuzzy set to the trapezoidal intuitionistic fuzzy set, where its fundamental characteristic is that the values of its membership function and non-membership function are trapezoidal fuzzy numbers rather than triangular fuzzy numbers, and proposed the trapezoidal intuitionistic fuzzy prioritized weighted averaging (TIFPWA) operator and trapezoidal intuitionistic fuzzy prioritized weighted geometric (TIFPWG) operator and their multi-criteria decision-making method, in which the criteria are in different priority level. Recently, Wang et al. [35] introduced a single-valued neutrosophic set, which is a subclass of a neutrosophic set presented by Smarandache [30], as a generalization of the classic set, fuzzy set and intuitionistic fuzzy set. The single-valued neutrosophic set can independently express truth-membership degree, indeterminacy-membership degree and falsity-membership degree and deal with incomplete, indeterminate and inconsistent information. All the factors described by the single-valued neutrosophic set are very suitable for human thinking due to the imperfection of knowledge that human receives or observes from the external world. For example, for a given proposition ''Movie X would be hit,'' in this situation human brain certainly cannot generate precise answers in terms of yes or no, as indeterminacy is the sector of unawareness of a proposition's value between truth and falsehood. Obviously, the neutrosophic components are best fit in the representation of indeterminacy and inconsistent information, while the intuitionistic fuzzy set cannot represent and handle indeterminacy and inconsistent information. Hence, the single-valued neutrosophic set has been a rapid development and a wide range of applications [39, 40]. Ye [42] introduced the trapezoidal neutrosophic set and its application to multiple attribute decision-making. Cui and Ye [10], Donghai et al. [16], Ebadi et al. [17], Guha and Chakraborty [18], Hajjari [19], Nayagam et al. [25], Rouhparvar et al. [29], Wu [37], Ye [40], Zou et al. [45] and more researchers have shown interest on decision making problem using distance measures. Weighted projection measure, the combination of angle cosine and weighted projection measure,similarity measure, hybrid vector similarity measure of single valued neutrosophic set and interval valued neutrosophic set, outranking strategy, complete ranking, new ranking function have been introduced so far under fuzzy, intuitionistic fuzzy and neutrosophic environments and applied in decision making problem. The rest of the paper is organized as follows. In section 2, literature review is given. In section 3, basic concepts are presented for better understanding. In section 4, proposed a new distnace measure and its graphical representation, and derived its properties in detail. In section 5, new methodology is described for a decision making process using the proposed measure. In section 6, a numerical example is using the proposed methodology to choose the best software system. In section 7, comparative analysis has been done with the existing methods and various forms of trapezoidal fuzzy neutrosophic numbers have been listed out to ahow the uniqueness of the proposed graphical representation. In section 8, advantages of the proposed measure are given. In section 9, conclusion of the present work is given with the future direction.

2-Literature Review

The authors of, Ahmad et al. [1] proposed a similarity measure based on the distance and set theory for generalized trapezoidal fuzzy numbers. Allahviranloo et al. [2] contributed a new distance measure and ranking method for generalized trapezoidal fuzzy numbers. Atanassov [3] introduced intuitionistic fuzzy sets. Azman and Abdullah [4] proposed a novel centroid method for trapezoidal fuzzy numbers for ranking. Biswas et al. [6] solved a decision making problem using expected value of neutrosophic trapezoidal numbers. Biswas et al. [6] solved a decision making problem using distance measure under interval trapezoidal neutrosophic numbers. Bolos et al. [7] designed the performance indicators of financial assets using neutrosophic fuzzy numbers. Bora and Gupta [8] studied the reaction of distance measure on the work of K-Means algorithm Matlab. Chakraborty et al. [9] presented different forms of trapezoidal neutrosophic number and deneutrosophication techniques. Cui and Ye [10] proposed logarithmic similarity measure and applied in medical diagnosis under dynamic neutrosophic cubic setting. Darehmiraki [11] introduced a new ranking methodology to solve linear programming problem. Das and De [12] introduced a new distance measure for the ranking IFNs. Das and Guha [13] introduced a ranking method for IFN using the point of centroid. Deli and Oztaurk [14] introduced a defuzzification method and applied in a decision-making problem for single valued trapezoidal neutrosophic numbers. Dhar et al. [15] indicated square neutrosophic fuzzy matrices. Donghai et al. [16] proposed a new similarity measure and distance measure between hesitant linguisticterm sets and applied the proposed concepts in a decision making problem. Ebadi et al. [17] proposed a novel distance measure for trapezoidal fuzzy numbers. Guha and Chakraborty [18] contributed a theoretical development of distance measure for intuitionistic fuzzy numbers (IFNs). Hajjari [19] conferred a new distance measure for Trapezoidal fuzzy numbers. Huang and Wu [20] presented equivalent forms of the triangle inequalities in fuzzy metric spaces. Liang et al. [21] proposed an integrated approach under a single valued trapezoidal neutrosophic environment. Liu and Yuan [22] prospected fuzzy number of intuitionistic fuzzy set. Llopis and Micheli [23] rectified a state of conflict in the sequence of input images. Minculete and Paltanea [24] introduced an enhanced estimates for the triangle inequality. Nayagam et al. [25] contributed a complete ranking of IFNs. Pardha Saradhi et al. [26] presented ordering of IFNs using centroids of centroids. Ravi Shankar et al. [27] developed a new ranking formula using centroid of centroids for fuzzy numbers and applied in a fuzzy critical path method. Rezvani [28] proposed a new ranking exponential formula using median value for trapezoidal fuzzy numbers. Rouhparvar et al. [29] introduced a novel fuzzy distance measure. Uppada [31] examined clustering algorithm using centroid clearly. Varghese and Kuriakose [32] proposed a formula to find the centroid of the fuzzy number. Wang [33] introduced geometric aggregation operator and applied in a decision making problem under intuitionistic fuzzy environment. Wang [34] proposed arithmetic aggregation operators. Wang et al. [35] introduced single valued neutrosophic sets. Wei et al. [36] introduced some persuaded aggregation operators under intuinistic fuzzy setting and applied in a group decision making problem. Wu [37] explained about distance metrics and their role in data transformations.Ye [38] proposed prioritized aggregation operators based on trapezoidal intuitionistic fuzzy concept and applied in a multi-criteria decision making problem. Ye [39] solved minimum spanning tree problem under single valued neutrosophic setting and its clustering method. Ye [40] proposed single valued neutrosophic cross entropy measure and applied in a decision making problem. Ye [41] introduced the expected Dice similarity measure and applied in a decision making problem. Ye [42] projected trapezoidal neutrosophic set and applied in a multiple attribute decision making. Zhang et al. [44] introduced interval neutrosophic sets and used in multi criteria decision making problem. Zou et al. [45] introduced a distance measure between neutrosophic sets as an evidential approach. From the literature, it is found that distance measure for trapezoidal neutrosophic numbers using centroids with its properties has not yet been studied so far. Hence the motivation of the present study.

Hence, in this paper a new distance measure for trapezoidal fuzzy neutrosophic numbers based on centroids has been proposed with its metric properties in detail. Also the graphical representation is presented for trapezoidal fuzzy neutrosophic number. Comparative study also have been made with the existing cases for both proposed distance measure and proposed graphical representation. Further advantages of the proposed distance measure are presented.

3-Preliminaries

Definition 1. [38] Let X be a space of discourse, a trapezoidal intuitionistic fuzzy set B in X is defined as: $B=\{\langle y,\alpha_B(y),\beta_B(y)\rangle | y\in X\}$, where $\alpha_B(y)\subset[0,1]$ and $\beta_B(y)\subset[0,1]$ are two trapezoidal fuzzy numbers $\alpha_B(y)=(\alpha_B^1(y),\alpha_B^2(y),\alpha_B^3(y),\alpha_B^4(y)):Y\to[0,1]$ and $\beta_B(y)=(\beta_B^1(y),\beta_B^2(y),\beta_B^3(y),\beta_B^4(y)):Y\to[0,1]$ with the condition that $0\le\alpha_B^4(y)+\beta_B^4(y)\le 1,\ \forall\, y\in Y$.

For Convenience, let $\alpha_B(y)=(a,b,c,d)$ and $\beta_B(y)=(e,f,g,h)$ be two trapezoidal fuzzy numbers, thus a trapezoidal intuitionistic fuzzy number (TrIFN) can be denoted by $j=\langle(a,b,c,d),(e,f,g,h)\rangle$, which is basic element in a trapezoidal intuitionistic fuzzy set.

If $b=c$ and $f=g$ hold in a TrIFN j, which is a special case of the TrIFN.

Definition 2. [38] Let $j_1=\langle(a_1,b_1,c_1,d_1),(e_1,f_1,g_1,h_1)\rangle$ and

$j_2=\langle(a_2,b_2,c_2,d_2),(e_2,f_2,g_2,h_2)\rangle$, be two TrIFNs. Then there are the following operational rules:

1. $$j_1\oplus j_2=\left\langle\begin{array}{l}(a_1+a_2-a_1a_2,b_1+b_2-b_1b_2,c_1+c_2-c_1c_2,d_1+d_2-d_1d_2),\\(e_1e_2,f_1f_2,g_1g_2,h_1h_2)\end{array}\right\rangle$$
2. $$j_1\otimes j_2=\left\langle\begin{array}{l}(a_1a_2,b_1b_2,c_1c_2,d_1d_2),\\(e_1+e_2-e_1e_2,f_1+f_2-f_1f_2,g_1+g_2-g_1g_2,h_1+h_2-h_1h_2)\end{array}\right\rangle$$
3. $$\lambda j_1=\left\langle\begin{array}{l}\left(1-(1-a_1)^\lambda,1-(1-b_1)^\lambda,1-(1-c_1)^\lambda,1-(1-d_1)^\lambda\right),\\(e_1^\lambda,f_1^\lambda,g_1^\lambda,h_1^\lambda)\end{array}\right\rangle,\lambda>0;$$
4. $$m_1^\lambda=\left\langle\begin{array}{l}(a_1^\lambda,b_1^\lambda,c_1^\lambda,d_1^\lambda),\left(1-(1-e_1)^\lambda,1-(1-f_1)^\lambda,1-(1-g_1)^\lambda,1-(1-h_1)^\lambda\right),\\\left(1-(1-i_1)^\lambda,1-(1-j_1)^\lambda,1-(1-k_1)^\lambda,1-(1-l_1)^\lambda\right)\end{array}\right\rangle,\lambda\ge 0$$

Definition 3. [30] From philosophical point of view, Smarandache [30] originally presented the concept of a neutrosophic set B in a universal set Y, which is characterized independently by a truth-membership function $T_B(y)$, an indeterminacy membership function $I_B(y)$ and a falsity-membership function $F_B(y)$. The function $T_B(y)$, $I_B(y)$ and $F_B(y)$ in Y are real standard or nonstandard subsets of $]^-0,1^+[$, such that $T_B(y):Y\to]^-0,1^+[$, $I_B(y):Y\to]^-0,1^+[$, and $F_B(y):Y\to]^-0,1^+[$ .Then, the sum of $T_B(y),I_B(y)$ and $F_B(y)$ satisfies the condition $^-0\le\sup T_B(y)+\sup I_B(y)+\sup F_B(y)\le 3^+$. Obviously, it is difficult to apply the neutrosophic set to practical problems. To easily apply it in science and engineering fields, Wang et al. [35] introduced the concept of a single-valued neutrosophic set as a subclass of the neutrosophic set and gave the following definition.

Definition 4. [35] A single-valued neutrosophic set B in a universal set Y is characterized by a truth-membership function $T_B(y)$, an indeterminacy-membership function $I_B(y)$ and a falsity-membership function $F_B(y)$. Then, a single-valued neutrosophic set B can be denoted by

$$B=\left\{\langle y,T_B(y),I_B(y),F_B(y)\rangle \middle| y\in Y\right\}$$

where, $T_B(y),I_B(y),F_B(y)\in[0,1]$ for each $y\in Y$. Therefore, the sum of $T_B(y),I_B(y)$ and $F_B(y)$ satisfies $0\le T_B(y)+I_B(y)+F_B(y)\le 3$.

Let $M=\left\{\langle y,T_M(y),I_M(y),F_M(y)\rangle \middle| y\in Y\right\}$ and $N=\left\{\langle y,T_N(y),I_N(y),F_N(y)\rangle \middle| y\in Y\right\}$ be two single-valued neutrosophic sets, then we the following relations [8,11]:

1. Complement: $M^C=\left\{\langle y,F_M(y),1-I_M(y),T_M(y)\rangle \middle| y\in Y\right\}$;
2. Inclusion: $M\subseteq N$ if and only if $T_M(y)\le T_N(y)$, $I_M(y)\ge I_N(y)$ and $F_M(y)\ge F_N(y)$ for any $y\in Y$;
3. Equality: $M=N$ if and only if $M\subseteq N$ and $N\subseteq M$;
4. Union: $M\cup N=\left\{\langle y,T_M(y)\vee T_N(y),I_M(y)\wedge I_N(y),F_M(y)\wedge F_N(y)\rangle \middle| y\in Y\right\}$;
5. Intersection: $M\cap N=\left\{\langle y,T_M(y)\wedge T_N(y),I_M(y)\vee I_N(y),F_M(y)\vee F_N(y)\rangle \middle| y\in Y\right\}$;

6. Addition: $M \oplus N = \left\{ \left\langle \begin{array}{l} y, T_M(y) + T_N(y) - T_M(y)T_N(y), I_M(y)I_N(y), \\ F_M(y)F_N(y) \end{array} \right\rangle \middle| y \in Y \right\}$;

7. Multiplication: $M \otimes N = \left\{ \left\langle \begin{array}{l} y, T_M(y)T_N(y), I_M(y) + I_N(y) - I_M(y)I_N(y), \\ F_M(y) + F_N(y) - F_M(y)F_N(y) \end{array} \right\rangle \middle| y \in Y \right\}$.

Definition 5. [42] Let Y be a space of discourse, a trapezoidal neutrosophic set H in Y is defined as follow:

$H = \{\langle y, T_H(y), I_H(y), F_H(y)\rangle | y \in Y\}$, where $T_H(y) \subset [0,1]$, $I_H(y) \subset [0,1]$ and $F_H(y) \subset [0,1]$ are three trapezoidal fuzzy numbers $T_H(y) = (t_H^1(y), t_H^2(y), t_H^3(y), t_H^4(y)) : Y \to [0,1]$,

$I_H(y) = (i_H^1(y), i_H^2(y), i_H^3(y), i_H^4(y)) : Y \to [0,1]$ and

$F_H(y) = (f_H^1(y), f_H^2(y), f_H^3(y), f_H^4(y)) : Y \to [0,1]$ with the condition

$0 \le t_H^4(y) + i_H^4(y) + f_H^4(y) \le 3,\ y \in Y$.

For convenience, the three trapezoidal fuzzy numbers are denoted by $T_H(y) = (a,b,c,d)$, $I_H(y) = (e,f,g,h)$ and $F_H(y) = (i,j,k,l)$. Thus, a trapezoidal neutrosophic numbers is denoted by $m = \langle (a,b,c,d),(e,f,g,h),(i,j,k,l)\rangle$, which is a basic element in the trapezoidal neutrosophic set.

If $b = c$, $f = g$ and $j = k$ hold in a trapezoidal neutrosophic number j_1, it reduces to the triangular neutrosophic number, which is considered as a special case of the trapezoidal neutrosophic number.

Definition 6. [42] Let $m_1 = \langle (a_1,b_1,c_1,d_1),(e_1,f_1,g_1,h_1),(i_1,j_1,k_1,l_1)\rangle$, and

$m_2 = \langle (a_2,b_2,c_2,d_2),(e_2,f_2,g_2,h_2),(i_2,j_2,k_2,l_2)\rangle$ be two trapezoidal neutrosophic numbers. Then there are the following operational rules:

1. $m_1 \oplus m_2 = \left\langle \begin{array}{l} (a_1 + a_2 - a_1a_2, b_1 + b_2 - b_1b_2, c_1 + c_2 - c_1c_2, d_1 + d_2 - d_1d_2), \\ (e_1e_2, f_1f_2, g_1g_2, h_1h_2), (i_1i_2, j_1j_2, k_1k_2, l_1l_2) \end{array} \right\rangle$,

2. $$m_1 \otimes m_2 = \left\langle \begin{array}{l} (a_1a_2, b_1b_2, c_1c_2, d_1d_2), \\ (e_1+e_2-e_1e_2, f_1+f_2-f_1f_2, g_1+g_2-g_1g_2, h_1+h_2-h_1h_2), \\ (i_1+i_2-i_1i_2, j_1+j_2-j_1j_2, k_1+k_2-k_1k_2, l_1+l_2-l_1l_2) \end{array} \right\rangle;$$

3. $$\lambda m_1 = \left\langle \begin{array}{l} \left(1-(1-a_1)^\lambda, 1-(1-b_1)^\lambda, 1-(1-c_1)^\lambda, 1-(1-d_1)^\lambda\right), \\ \left(e_1^\lambda, f_1^\lambda, g_1^\lambda, h_1^\lambda\right), \left(i_1^\lambda, j_1^\lambda, k_1^\lambda, l_1^\lambda\right) \end{array} \right\rangle, \lambda > 0;$$

4. $$m_1^\lambda = \left\langle \begin{array}{l} \left(a_1^\lambda, b_1^\lambda, c_1^\lambda, d_1^\lambda\right), \\ \left(1-(1-e_1)^\lambda, 1-(1-f_1)^\lambda, 1-(1-g_1)^\lambda, 1-(1-h_1)^\lambda\right), \\ \left(1-(1-i_1)^\lambda, 1-(1-j_1)^\lambda, 1-(1-k_1)^\lambda, 1-(1-l_1)^\lambda\right) \end{array} \right\rangle, \lambda \geq 0$$

Definition 7. [18] Let P and Q be the intuitionistic fuzzy sets with membership functions $\mu_P(x), \mu_Q(x)$, non-membership functions $\nu_P(x), \nu_Q(x)$ and hesitation degree $\pi_P(x), \pi_Q(x)$. Then the normalized Hamming distance is

$$D(P,Q) = \frac{1}{2n}\sum_{i=1}^{n}\left[\left|\mu_P(x_i)-\mu_Q(x_i)\right| + \left|\nu_P(x_i)-\nu_Q(x_i)\right| + \left|\pi_P(x_i)-\pi_Q(x_i)\right|\right]$$

And the normalized Euclidean distance is

$$D_E(P,Q) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left[\left(\mu_P(x_i)-\mu_Q(x_i)\right)^2 + \left(\nu_P(x_i)-\nu_Q(x_i)\right)^2 + \left(\pi_P(x_i)-\pi_Q(x_i)\right)^2\right]}$$

Definition 8. [17] Consider the real values r_i, $i = 1,2,3,\ldots,6$ and if $r_1 \leq r_2, r_3 \leq r_4, r_5 \leq r_6$ then the following results are true.

1. $\max\{r_1, r_3, r_5\} \leq \max\{r_2, r_4, r_6\}$

2. $\max\{r_1+r_2, r_3+r_4, r_5+r_6\} \leq \max\{r_1, r_3, r_5\} + \max\{r_2, r_4, r_6\}$

Definition 9. [34] For any real numbers $r_i, s_i \geq 0, i = 1,2,\ldots,d$, the Euclidean distance is defined as, $D(r,s) = \sqrt{\sum_{i=1}^{d}(r_i - s_i)^2}$ and satisfies the condition that $\left[\sum_{i=1}^{d}(r_i+s_i)^p\right]^{1/p} \leq \left(\sum_{i=1}^{d}(r_i)^p\right)^{1/p} + \left(\sum_{i=1}^{d}(s_i)^p\right)^{1/p}$.

Definition 10. [42] Let $m_p = \langle (a_p, b_p, c_p, d_p), (e_p, f_p, g_p, h_p), (i_p, j_p, k_p, l_p) \rangle, p = 1,2,3,\ldots,n$ be the trapezoidal fuzzy neutrosophic numbers then the trapezoidal fuzzy neurosophic weighted geometric operator is defined by

$$TFNWG(m_1, m_2, \ldots, m_n) = m_1^{\omega_1} \otimes m_2^{\omega_2} \otimes m_3^{\omega_3} \otimes \ldots \otimes m_n^{\omega_n}$$

$$= \left\langle \left(\prod_{p=1}^{n} a_p^{\omega_p}, \prod_{p=1}^{n} b_p^{\omega_p}, \prod_{p=1}^{n} c_p^{\omega_p}, \prod_{p=1}^{n} d_p^{\omega_p} \right), \left(1-\prod_{p=1}^{n}(1-e_p)^{\omega_p}, 1-\prod_{p=1}^{n}(1-f_p)^{\omega_p}, 1-\prod_{p=1}^{n}(1-g_p)^{\omega_p}, 1-\prod_{p=1}^{n}(1-h_p)^{\omega_p} \right), \right.$$

$$\left. \left(1-\prod_{p=1}^{n}(1-i_p)^{\omega_p}, 1-\prod_{p=1}^{n}(1-j_p)^{\omega_p}, 1-\prod_{p=1}^{n}(1-k_p)^{\omega_p}, 1-\prod_{p=1}^{n}(1-l_p)^{\omega_p} \right) \right\rangle$$

where, $\omega_1, \omega_2, \ldots, \omega_n$ are the weight vectors and the sum of the weight vectors is 1.

Definition 11. [9] Graphical representation of trapezoidal neutrosophic number

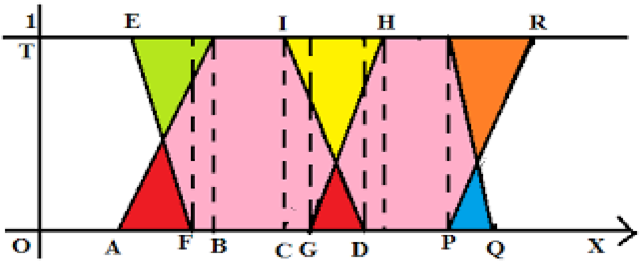


Figure 1. Graphical representation of Trapezoidal neutrosophic number

Figure 1 shows that graphical representation of trapezoidal fuzzy neutrosophic number can be done in different ways. It is a linear trapezoidal neutrosophic number.

4-Proposed Distance Measure for Trapezoidal Fuzzy Neutrosophic Number

Here we propose a new distance measure for trapezoidal fuzzy neutrosophic number based on centroids. Firstly, individual graphical representation proposed measure is presented here with the individual representation of truth, indeterminacy, falsity membership functions and trapezoidal fuzzy neutrosophic fuzzy number described by Figure 2-Figure 6.

Centre point of the object is called centroid. It should lie inside the object. At this point, the three medians of the triangle intersect and is termed point of intersection. Centroid is the average of coordinate points in X axis and Y axis of each vertex of the triangle. Centroid is the fixed point of all linear transformation which maintains length in translation, rotation, glides and reflection.

The centroid of the truth, indeterminacy and falsity trapezoid is treated as a balance point for the trapezoid. The centroid of each part are estimated using the calculation of centroid and the simple area and this combination will generate a triangle. Also the distance is measured from the centroid of all the parts to X axis and Y axis. Here the area of all the parts are multiplied by the distance and find their sum to get the total value. And the sum of the products of the area and distances is divided by the total area and obtain the centroid of circumcentre described by x and y point. Since centroid based distance measure may be derived using Euclidean measure, here it is obtained from the circumcentre of the centroids and the authentic point for the trapezoidal fuzzy neutrosophic number.

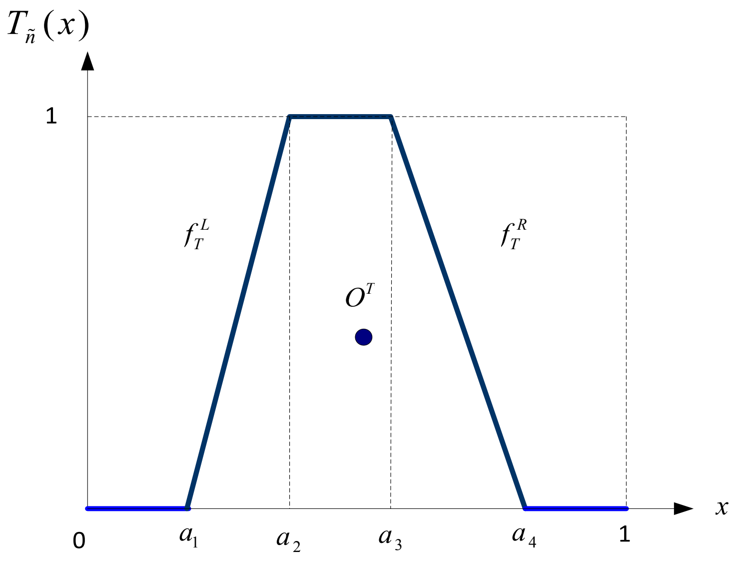


Figure 2. Truth membership function of trapezoidal fuzzy neutrosophic set with centroid

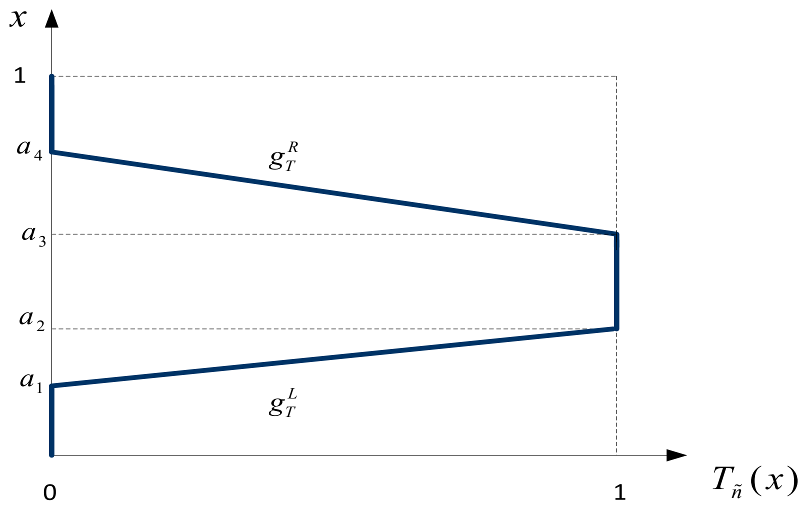


Figure 3. Truth membership function of trapezoidal fuzzy neutrosophic set

Suppose $\tilde{n}=\langle(a_1,a_2,a_3,a_4),(b_1,b_2,b_3,b_4),(c_1,c_2,c_3,c_4)\rangle$ be a trapezoidal fuzzy neutrosophic number. Based on the literature (Y. M. Wang et al. On the centroids of fuzzy numbers), we can get the centroid point $O^T=(x_o^T(\tilde{n}),y_o^T(\tilde{n}))$ of the truth membership function of trapezoidal fuzzy neutrosophic number $\tilde{n}$.

$$x_o^T(\tilde{n}) = \frac{\int_{a_1}^{a_2} x f_T^L dx + \int_{a_2}^{a_3} x \cdot 1 dx + \int_{a_3}^{a_4} x f_T^R dx}{\int_{a_1}^{a_2} f_T^L dx + \int_{a_2}^{a_3} 1 dx + \int_{a_3}^{a_4} f_T^R dx} = \frac{1}{3}[a_1 + a_2 + a_3 + a_4 - \frac{a_4 a_3 - a_1 a_2}{(a_4 + a_3) - (a_1 + a_2)}],$$

$$y_o^T(\tilde{n}) = \frac{\int_0^1 y(g_T^L - g_T^R)dy}{\int_0^1 (g_T^L - g_T^R)dy} = \frac{1}{3}[1 + \frac{a_3 - a_2}{(a_4 + a_3) - (a_1 + a_2)}].$$

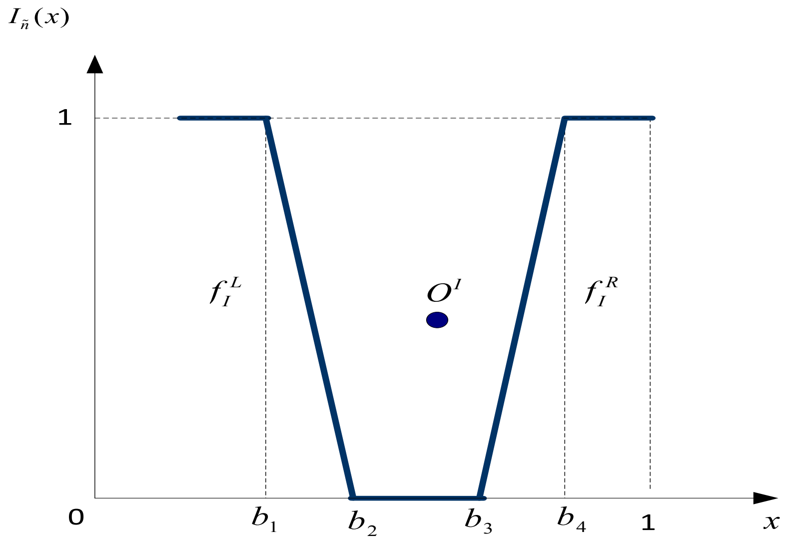


Figure 4. Indeterminate membership function of trapezoidal fuzzy neutrosophic set with centroid

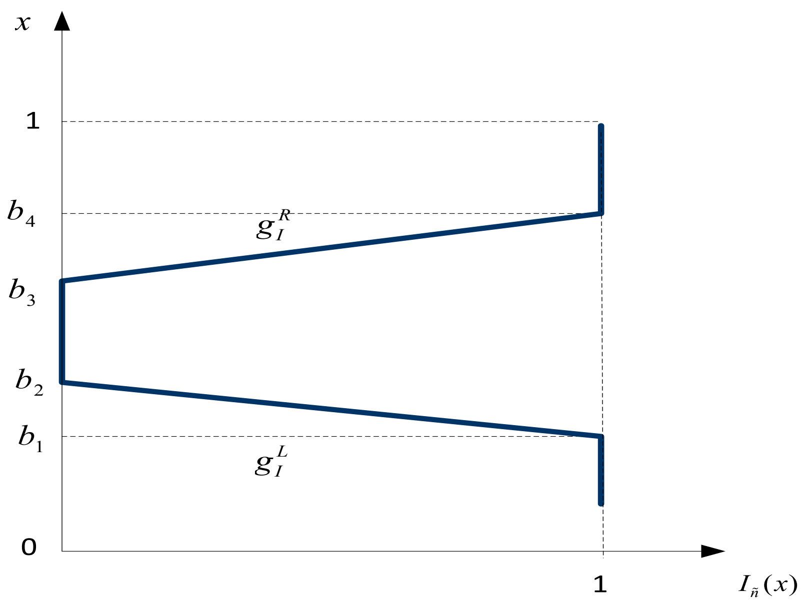


Figure 5. Indeterminate membership function of trapezoidal fuzzy neutrosophic number

we can get the centroid point $O^I = (x_o^I(\tilde{n}), y_o^I(\tilde{n}))$ of indeterminacy membership function of trapezoidal fuzzy neutrosophic number $\tilde{n}$.

$$x_o^I(\tilde{n}) = \frac{\int_{b_1}^{b_2} x f_I^L dx + \int_{b_2}^{b_3} x \cdot 1 dx + \int_{b_3}^{b_4} x f_I^R dx}{\int_{b_1}^{b_2} f_I^L dx + \int_{b_2}^{b_3} 1 dx + \int_{b_3}^{b_4} f_I^R dx} = \frac{1}{3}[b_1 + b_2 + b_3 + b_4 - \frac{b_4 b_3 - b_1 b_2}{(b_4 + b_3) - (b_1 + b_2)}],$$

$$y_o^I(\tilde{n}) = \frac{\int_0^1 y(g_I^L - g_I^R) dy}{\int_0^1 (g_I^L - g_I^R) dy} = \frac{1}{3}[1 + \frac{b_3 - b_2}{(b_4 + b_3) - (b_1 + b_2)}].$$

Similarly, we can get the centroid point $O^F = (x_o^F(\tilde{n}), y_o^F(\tilde{n}))$ of falsity membership function of trapezoidal fuzzy neutrosophic number $\tilde{n}$.

$$x_o^F(\tilde{n}) = \frac{\int_{c_1}^{c_2} x f_F^L dx + \int_{c_2}^{c_3} x \cdot 1 dx + \int_{c_3}^{c_4} x f_F^R dx}{\int_{c_1}^{c_2} f_F^L dx + \int_{c_2}^{c_3} 1 dx + \int_{c_3}^{c_4} f_F^R dx} = \frac{1}{3}[c_1 + c_2 + c_3 + c_4 - \frac{c_4 c_3 - c_1 c_2}{(c_4 + c_3) - (c_1 + c_2)}],$$

$$y_o^F(\tilde{n}) = \frac{\int_0^1 y(g_F^L - g_F^R) dy}{\int_0^1 (g_F^L - g_F^R) dy} = \frac{1}{3}[1 + \frac{c_3 - c_2}{(c_4 + c_3) - (c_1 + c_2)}].$$

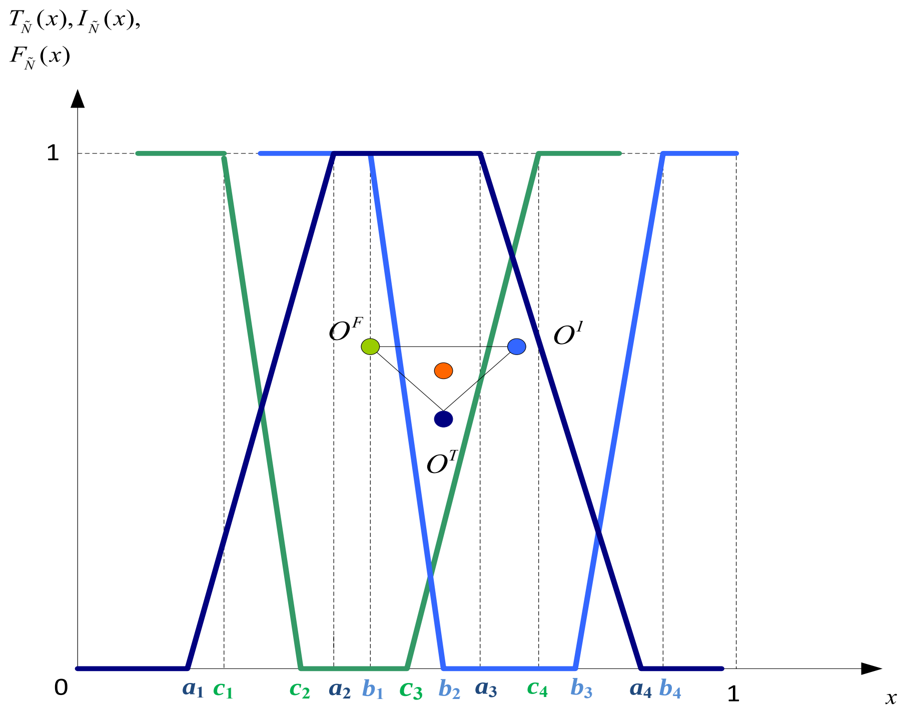


Figure 6. Trapezoidal fuzzy neutrosophic number with circumcentre of Centroids

In the above figure 5, the red dot represents the center of gravity of the triangle consisting of O^T, O^I, and O^F. According to the coordinate formula of the center of gravity of the triangle, we can get the coordinates of red dots $O=(x(\tilde{n}),y(\tilde{n}))$.

$$x(\tilde{n})=\frac{x_o^T(\tilde{n})+x_o^I(\tilde{n})+x_o^F(\tilde{n})}{3}$$

$$=\frac{1}{3}\left\{\begin{array}{l}\frac{1}{3}[a_1+a_2+a_3+a_4-\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}]\\ +\frac{1}{3}[b_1+b_2+b_3+b_4-\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}]+\frac{1}{3}[c_1+c_2+c_3+c_4-\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}]\end{array}\right\}$$

$$=\frac{1}{9}[\sum_{i=1}^4 a_i+\sum_{i=1}^4 b_i+\sum_{i=1}^4 c_i-\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}]$$

$$y(\tilde{n})=\frac{y_o^T(\tilde{n})+y_o^I(\tilde{n})+y_o^F(\tilde{n})}{3}$$

$$=\frac{\frac{1}{3}[1+\frac{a_3-a_2}{(a_4+a_3)-(a_1+a_2)}]+\frac{1}{3}[1+\frac{b_3-b_2}{(b_4+b_3)-(b_1+b_2)}]+\frac{1}{3}[1+\frac{c_3-c_2}{(c_4+c_3)-(c_1+c_2)}]}{3}$$

$$=\frac{1}{9}[3+\frac{a_3-a_2}{(a_4+a_3)-(a_1+a_2)}+\frac{b_3-b_2}{(b_4+b_3)-(b_1+b_2)}+\frac{c_3-c_2}{(c_4+c_3)-(c_1+c_2)}]$$

Definition1: Let $\tilde{n}_1=\langle(a_1,a_2,a_3,a_4),(b_1,b_2,b_3,b_4),(c_1,c_2,c_3,c_4)\rangle$ and $\tilde{n}_2=\langle(e_1,e_2,e_3,e_4),(f_1,f_2,f_3,f_4),(g_1,g_2,g_3,g_4)\rangle$ be two trapezoidal fuzzy neutrosophic numbers, and their centroids are $O_1=(x(\tilde{n}_1),y(\tilde{n}_1))$, $O_2=(x(\tilde{n}_2),y(\tilde{n}_2))$ respectively, then the distance between $\tilde{n}_1$ and $\tilde{n}_2$ is

$$D(\tilde{n}_1,\tilde{n}_2)=\frac{1}{9}\sqrt{\begin{array}{l}[\sum_{i=1}^4 a_i+\sum_{i=1}^4 b_i+\sum_{i=1}^4 c_i-\sum_{i=1}^4 e_i-\sum_{i=1}^4 f_i-\sum_{i=1}^4 g_i-(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})\\ -(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})-(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})]^2\\ +[(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})+(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})\\ +(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})]^2\end{array}}$$

Theorem 1: This distance $D(\tilde{n}_1,\tilde{n}_2)$ of $\tilde{n}_1$ and $\tilde{n}_2$ fulfills the following properties:

1. $0 \le D(\tilde{n}_1, \tilde{n}_2) \le 1$;

2. $D(\tilde{n}_1, \tilde{n}_2) = 0$ if and only if $\tilde{n}_1 = \tilde{n}_2$, i.e., $a_i = e_i$, $b_i = f_i$ and $c_i = g_i$ hold for $i = 1, 2, 3, 4$;

3. $D(\tilde{n}_1, \tilde{n}_2) = D(\tilde{n}_2, \tilde{n}_1)$.

4. If $\tilde{n}_1, \tilde{n}_2 \,\&\, \tilde{n}_3$ are the trapezoidal fuzzy neutrosophic numbers then

$$D(\tilde{n}_1, \tilde{n}_3) \le D(\tilde{n}_1, \tilde{n}_2) + D(\tilde{n}_2, \tilde{n}_3)$$

Proof

1. It is easy to prove $0 \le D(\tilde{n}_1, \tilde{n}_2)$. In addition, it can be seen from figure 1, the maximum distance is the distance between the point $(0,0)$ and the point $(1,1)$, or the point $(0,1)$ and the point $(1,0)$, assume the coordinates of centroids of $\tilde{n}_1$ and $\tilde{n}_2$ are O_1 and O_2, and $O_1 = (0,1)$ and $O_2 = (1,0)$, or $O_1 = (1,0)$ and $O_2 = (0,1)$, or $O_1 = (0,0)$ and $O_2 = (1,1)$, or $O_1 = (1,1)$ and $O_2 = (0,0)$, then the $D(\tilde{n}_1, \tilde{n}_2) = 1$, otherwise, $D(\tilde{n}_1, \tilde{n}_2) < 1$, thus $0 \le D(\tilde{n}_1, \tilde{n}_2) \le 1$.

2. if $\tilde{n}_1 = \tilde{n}_2$, i.e., $a_i = e_i$, $b_i = f_i$ and $c_i = g_i$, then

$$D(\tilde{n}_1,\tilde{n}_2)=\frac{1}{9}\sqrt{\begin{aligned}&[\sum_{i=1}^{4}a_i+\sum_{i=1}^{4}b_i+\sum_{i=1}^{4}c_i-\sum_{i=1}^{4}a_i-\sum_{i=1}^{4}b_i-\sum_{i=1}^{4}c_i-(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)})\\&-(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)})-(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)})]^2\\&+[(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)})+(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)})\\&+(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)})]^2\end{aligned}}$$

$$=0.$$

if $D\left(\tilde{n}_1,\tilde{n}_2\right)=0$, then

$$\begin{aligned}&[\sum_{i=1}^{4}a_i+\sum_{i=1}^{4}b_i+\sum_{i=1}^{4}c_i-\sum_{i=1}^{4}e_i-\sum_{i=1}^{4}f_i-\sum_{i=1}^{4}g_i-(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})\\&-(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})-(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})]^2\\&+[(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})+(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})\\&+(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})]^2=0,\end{aligned}$$

Thus,

$$\begin{aligned}&\sum_{i=1}^{4}a_i+\sum_{i=1}^{4}b_i+\sum_{i=1}^{4}c_i-\sum_{i=1}^{4}e_i-\sum_{i=1}^{4}f_i-\sum_{i=1}^{4}g_i-(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})\\&-(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})-(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})\\&=0,\end{aligned}$$

$$\begin{aligned}&(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})+(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})\\&+(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})\\&=0,\end{aligned}$$

thus

$$\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)}=0,$$
$$\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)}=0,$$
$$\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)}=0,$$

thus

$a_i=e_i,\ b_i=f_i,\ c_i=g_i$, that is $\tilde{n}_1=\tilde{n}_2$.

3. Since,

$$[\sum_{i=1}^{4}a_i+\sum_{i=1}^{4}b_i+\sum_{i=1}^{4}c_i-\sum_{i=1}^{4}e_i-\sum_{i=1}^{4}f_i-\sum_{i=1}^{4}g_i-(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})$$
$$-(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})-(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})]^2$$
$$+[(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})+(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})$$
$$+(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})]^2$$

$$=[\sum_{i=1}^{4}e_i+\sum_{i=1}^{4}f_i+\sum_{i=1}^{4}g_i-\sum_{i=1}^{4}a_i-\sum_{i=1}^{4}b_i-\sum_{i=1}^{4}c_i-(\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)}-\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)})$$
$$-(\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)}-\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)})-(\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)}-\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)})]^2$$
$$+[(\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)}-\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)})+(\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)}-\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)})$$
$$+(\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)}-\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)})]^2$$

then $D(\tilde{n}_1,\tilde{n}_2)=D(\tilde{n}_2,\tilde{n}_1)$.

4. Using Def. 8, we can prove (4).

Let $\tilde{n}_1=\langle(a_1,a_2,a_3,a_4),(b_1,b_2,b_3,b_4),(c_1,c_2,c_3,c_4)\rangle$,

$\tilde{n}_2=\langle(e_1,e_2,e_3,e_4),(f_1,f_2,f_3,f_4),(g_1,g_2,g_3,g_4)\rangle$ and

$\tilde{n}_3 = \langle (j_1, j_2, j_3, j_4), (k_1, k_2, k_3, k_4), (l_1, l_2, l_3, l_4) \rangle$ are the three trapezoidal fuzzy neutrosophic numbers then $D(\tilde{n}_1, \tilde{n}_3) \leq D(\tilde{n}_1, \tilde{n}_2) + D(\tilde{n}_2, \tilde{n}_3)$

Using the results we have,

$$D(\tilde{n}_1, \tilde{n}_3)$$

$$= \frac{1}{9}\sqrt{\begin{array}{l} [\sum_{i=1}^{4} a_i + \sum_{i=1}^{4} b_i + \sum_{i=1}^{4} c_i - \sum_{i=1}^{4} j_i - \sum_{i=1}^{4} k_i - \sum_{i=1}^{4} l_i - (\frac{a_4 a_3 - a_1 a_2}{(a_4 + a_3) - (a_1 + a_2)} - \frac{j_4 j_3 - j_1 j_2}{(j_4 + j_3) - (j_1 + j_2)}) \\ -(\frac{b_4 b_3 - b_1 b_2}{(b_4 + b_3) - (b_1 + b_2)} - \frac{k_4 k_3 - k_1 k_2}{(k_4 + k_3) - (k_1 + k_2)}) - (\frac{c_4 c_3 - c_1 c_2}{(c_4 + c_3) - (c_1 + c_2)} - \frac{l_4 l_3 - l_1 l_2}{(l_4 + l_3) - (l_1 + l_2)})]^2 \\ +[(\frac{a_4 a_3 - a_1 a_2}{(a_4 + a_3) - (a_1 + a_2)} - \frac{j_4 j_3 - j_1 j_2}{(j_4 + j_3) - (j_1 + j_2)}) + (\frac{b_4 b_3 - b_1 b_2}{(b_4 + b_3) - (b_1 + b_2)} - \frac{k_4 k_3 - k_1 k_2}{(k_4 + k_3) - (k_1 + k_2)}) \\ +(\frac{c_4 c_3 - c_1 c_2}{(c_4 + c_3) - (c_1 + c_2)} - \frac{l_4 l_3 - l_1 l_2}{(l_4 + l_3) - (l_1 + l_2)})]^2 \end{array}}$$

$$
\leq \frac{1}{9}\sqrt{\begin{aligned}
&\{\sum_{i=1}^{4}a_i+\sum_{i=1}^{4}b_i+\sum_{i=1}^{4}c_i-\sum_{i=1}^{4}e_i-\sum_{i=1}^{4}f_i-\sum_{i=1}^{4}g_i+\sum_{i=1}^{4}e_i+\sum_{i=1}^{4}f_i+\sum_{i=1}^{4}g_i-\sum_{i=1}^{4}j_i-\sum_{i=1}^{4}k_i-\sum_{i=1}^{4}l_i\\
&-[(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})-(\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)}-\frac{j_4j_3-j_1j_2}{(j_4+j_3)-(j_1+j_2)})]\\
&-[(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})-(\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)}-\frac{k_4k_3-k_1k_2}{(k_4+k_3)-(k_1+k_2)})]\\
&-[(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})-(\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)}-\frac{l_4l_3-l_1l_2}{(l_4+l_3)-(l_1+l_2)})]\}^2\\
&+\{[(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})+(\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)}-\frac{j_4j_3-j_1j_2}{(j_4+j_3)-(j_1+j_2)})]\\
&-[(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})+(\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)}-\frac{k_4k_3-k_1k_2}{(k_4+k_3)-(k_1+k_2)})]\\
&-[(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})+(\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)}-\frac{l_4l_3-l_1l_2}{(l_4+l_3)-(l_1+l_2)})]\}^2
\end{aligned}}
$$

$$
\leq \frac{1}{9}\left[\sqrt{\begin{aligned}
&\{\sum_{i=1}^{4}a_i+\sum_{i=1}^{4}b_i+\sum_{i=1}^{4}c_i-\sum_{i=1}^{4}e_i-\sum_{i=1}^{4}f_i-\sum_{i=1}^{4}g_i+\sum_{i=1}^{4}e_i+\sum_{i=1}^{4}f_i+\sum_{i=1}^{4}g_i-\sum_{i=1}^{4}j_i-\sum_{i=1}^{4}k_i-\sum_{i=1}^{4}l_i\\
&-[(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})-(\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)}-\frac{j_4j_3-j_1j_2}{(j_4+j_3)-(j_1+j_2)})]\\
&-[(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})-(\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)}-\frac{k_4k_3-k_1k_2}{(k_4+k_3)-(k_1+k_2)})]\\
&-[(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})-(\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)}-\frac{l_4l_3-l_1l_2}{(l_4+l_3)-(l_1+l_2)})]\}^2\\
&+\{[(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})+(\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)}-\frac{j_4j_3-j_1j_2}{(j_4+j_3)-(j_1+j_2)})]\\
&-[(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})+(\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)}-\frac{k_4k_3-k_1k_2}{(k_4+k_3)-(k_1+k_2)})]\\
&-[(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})+(\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)}-\frac{l_4l_3-l_1l_2}{(l_4+l_3)-(l_1+l_2)})]\}^2
\end{aligned}}\right]^2
$$

$$
\leq \frac{1}{9}\left(\begin{aligned}
&[\sum_{i=1}^{4}a_i+\sum_{i=1}^{4}b_i+\sum_{i=1}^{4}c_i-\sum_{i=1}^{4}e_i-\sum_{i=1}^{4}f_i-\sum_{i=1}^{4}g_i-(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})\\
&-(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})-(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})]^2\\
&+[(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})+(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})\\
&+(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})]^2
\end{aligned}\right)^{1/2}
$$

$$+\frac{1}{9}\left(\begin{array}{l}
[\sum_{i=1}^{4} e_i+\sum_{i=1}^{4} f_i+\sum_{i=1}^{4} g_i-\sum_{i=1}^{4} j_i-\sum_{i=1}^{4} k_i-\sum_{i=1}^{4} l_i-(\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)}-\frac{j_4j_3-j_1j_2}{(j_4+j_3)-(j_1+j_2)}) \\
-(\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)}-\frac{k_4k_3-k_1k_2}{(k_4+k_3)-(k_1+k_2)})-(\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)}-\frac{l_4l_3-l_1l_2}{(l_4+l_3)-(l_1+l_2)})]^2 \\
+[(\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)}-\frac{j_4j_3-j_1j_2}{(j_4+j_3)-(j_1+j_2)})+(\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)}-\frac{k_4k_3-k_1k_2}{(k_4+k_3)-(k_1+k_2)}) \\
+(\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)}-\frac{l_4l_3-l_1l_2}{(l_4+l_3)-(l_1+l_2)})]^2
\end{array}\right)^{1/2}$$

Using Def.9 we have,

$$\leq\frac{1}{9}\sqrt{\begin{array}{l}
[\sum_{i=1}^{4} a_i+\sum_{i=1}^{4} b_i+\sum_{i=1}^{4} c_i-\sum_{i=1}^{4} e_i-\sum_{i=1}^{4} f_i-\sum_{i=1}^{4} g_i-(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)}) \\
-(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})-(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})]^2 \\
+[(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})+(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)}) \\
+(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})]^2
\end{array}}$$

$$+\frac{1}{9}\sqrt{\begin{array}{l}
[\sum_{i=1}^{4} e_i+\sum_{i=1}^{4} f_i+\sum_{i=1}^{4} g_i-\sum_{i=1}^{4} j_i-\sum_{i=1}^{4} k_i-\sum_{i=1}^{4} l_i-(\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)}-\frac{j_4j_3-j_1j_2}{(j_4+j_3)-(j_1+j_2)}) \\
-(\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)}-\frac{k_4k_3-k_1k_2}{(k_4+k_3)-(k_1+k_2)})-(\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)}-\frac{l_4l_3-l_1l_2}{(l_4+l_3)-(l_1+l_2)})]^2 \\
+[(\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)}-\frac{j_4j_3-j_1j_2}{(j_4+j_3)-(j_1+j_2)})+(\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)}-\frac{k_4k_3-k_1k_2}{(k_4+k_3)-(k_1+k_2)}) \\
+(\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)}-\frac{l_4l_3-l_1l_2}{(l_4+l_3)-(l_1+l_2)})]^2
\end{array}}$$

$\leq D\left(\tilde{n}_1,\tilde{n}_2\right)+D\left(\tilde{n}_2,\tilde{n}_3\right)$ and hence the result (4).

5- Decision Making method based on new distance measure based on centroids

In this section, we establish an approach based an trapezoidal fuzzy neutrosophic number weighted geometric arithmetic operator and a new distance measure based on centroid to deal with trapezoidal fuzzy neutrosophic information. The proposed approach is described as follows.

Step 1: Apply trapezoidal fuzzy neutrosophic number weighted geometric arithmetic operator [39] to find the aggregated trapezoidal fuzzy neutrosophic numbers for all the alternatives.

Step 2: Use the proposed distance measure, find the distances between all the alternatives and the ideal trapezoidal fuzzy neutrooshic number

Step 3: Rank the alternatives in which smaller value of distance indicate the best one.
Step 4: End

6- Numerical Example for the application of the proposed distance measure

In this section, a numerical example of a software selection problem and the aggregation operator called trapezoidal neutrosophic number weighted geometric averaging operator are get used from Ye [39] for a multiple attribute decision making problem is contributed to exhibit the application and effectiveness of the proposed distance measure under trapezoidal fuzzy neutrosophic environment. For a software selection process, consider candidate software systems are given as the set of five alternatives S_1, S_2, S_3, S_4, S_5 and the investment company need to take a decision according to four criteria: (i). the contribution to organization performance, (ii). The effort totranform from current system, (iii). The costs of hardware/software investment, (iv). The outsourcing software deneloper reliability denoted by C_1, C_2, C_3, C_4 respectively with the weight vector $\omega = (0.25, 0.25, 0.3, 0.2)^T$. The experts evaluate the five alternatives with repect to the four criteions under trapezoidal fuzzy neutrosophic environment and thus we can form the trapezoidal fuzzy neutrosophic decision matrix:

Table 1: Decision matrix using trapezoidal fuzzy neutrosophic numbers

$$D = \begin{bmatrix} \langle(0.4,0.5,0.6,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)\rangle & \langle(0.0,0.1,0.2,0.3),(0.0,0.1,0.2,0.3),(0.2,0.3,0.4,0.5)\rangle & \langle(0.3,0.4,0.5,0.6),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)\rangle & \langle(0.3,0.4,0.5,0.6),(0.1,0.1,0.1,0.1),(0.1,0.2,0.3,0.4)\rangle \\ \langle(0.3,0.4,0.5,0.5),(0.1,0.2,0.3,0.4),(0.0,0.1,0.1,0.1)\rangle & \langle(0.2,0.3,0.4,0.5),(0.0,0.1,0.2,0.3),(0.0,0.1,0.2,0.3)\rangle & \langle(0.0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.5,0.6,0.7,0.8)\rangle & \langle(0.3,0.4,0.5,0.6),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.2)\rangle \\ \langle(0.1,0.1,0.1,0.1),(0.1,1.1,0.1,0.1),(0.6,0.7,0.8,0.9)\rangle & \langle(0.0,0.1,0.1,0.2),(0.0,0.1,0.2,0.3),(0.3,0.4,0.5,0.6)\rangle & \langle(0.2,0.3,0.4,0.5),(0.0,0.1,0.2,0.3),(0.1,0.2,0.2,0.3)\rangle & \langle(0.1,0.2,0.3,0.4),(0.1,0.1,0.1,0.1),(0.3,0.4,0.5,0.6)\rangle \\ \langle(0.7,0.7,0.7,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)\rangle & \langle(0.4,0.5,0.6,0.7),(0.1,0.1,0.1,0.1),(0.0,0.1,0.2,0.2)\rangle & \langle(0.2,0.3,0.4,0.5),(0.0,0.1,0.2,0.3),(0.1,0.2,0.3,0.3)\rangle & \langle(0.1,0.2,0.3,0.4),(0.1,0.1,0.1,0.1),(0.1,0.1,0.1,0.1)\rangle \\ \langle(0.0,0.1,0.2,0.2),(0.1,0.1,0.1,0.1),(0.5,0.6,0.7,0.8)\rangle & \langle(0.4,0.4,0.4,0.4),(0.0,0.1,0.2,0.3),(0.0,0.1,0.2,0.3)\rangle & \langle(0.6,0.7,0.7,0.8),(0.1,0.1,0.1,0.1),(0.0,0.1,0.1,0.2)\rangle & \langle(0.1,0.2,0.3,0.3),(0.1,0.2,0.3,0.4),(0.2,0.3,0.4,0.5)\rangle \end{bmatrix}$$

Here we used the developed method to obtain the best software system(s) and it is described as follows:

Step 1: Using trapezoidal fuzzy neutrosophic weighted geometric operator in Definition 10, get the aggregated trapezoidal fuzzy neutrosophic numbers of $n_i,\ i = 1,2,3,4,5$ for the software system $S_i,\ i = 1,2,3,4,5$ as follows:

$$n_1 = \langle (0.0000, 0.2985, 0.4162, 0.5244), (0.0209, 0.1003, 0.1809, 0.2639), (0.1261, 0.1745, 0.2266, 0.2836) \rangle$$
$$n_2 = \langle (0.0000, 0.2458, 0.2919, 0.3798), (0.0563, 0.1262, 0.1984, 0.2739), (0.1879, 0.2944, 0.3717, 0.4743) \rangle$$
$$n_3 = \langle (0.0000, 0.1599, 0.1888, 0.2545), (0.0464, 0.1000, 0.1566, 0.2162), (0.3437, 0.4502, 0.5424, 0.6655) \rangle$$
$$n_4 = \langle (0.2833, 0.3885, 0.4807, 0.5658), (0.0464, 0.1000, 0.1566, 0.2162), (0.1480, 0.2276, 0.3109, 0.3109) \rangle$$
$$n_5 = \langle (0.0000, 0.2912, 0.3756, 0.3910), (0.0760, 0.1210, 0.1690, 0.2208), (0.1958, 0.3012, 0.3877, 0.5020) \rangle$$

Step 2: Use the proposed distance measure and find the distance between all $n_i,\ i = 1,2,3,4,5$ and the ideal trapezoidal fuzzy neutrosophic number $n_{Ideal} = \langle (1,1,1,1), (0,0,0,0), (0,0,0,0) \rangle$.

The obtained distances are as follows:

$$D(n_1, I) = 0.1712 = D_1$$
$$D(n_2, I) = 0.1276 = D_2$$
$$D(n_3, I) = 0.1000 = D_3$$
$$D(n_4, I) = 0.1280 = D_4$$
$$D(n_5, I) = 0.1246 = D_5$$

Step 3: Find the best alternative by considering the smaller value of the distance as the smaller value of distance indicates the best one.

Using step 2 it is found that, $D_3 > D_5 > D_2 > D_4 > D_1$ and from the ranking order, S_3 is the best is the best software system.

7- Comparative analysis for the proposed distance measure and graphical representation

In this section, a comparative study is made to show the effectiveness of the proposed distance measure with the existing methods and to show the uniqueness of the proposed graphical representation.

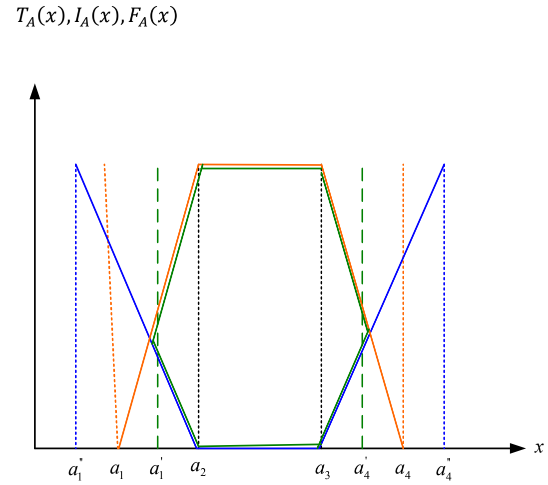
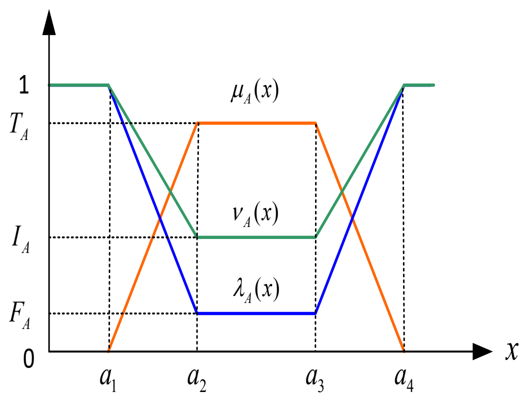
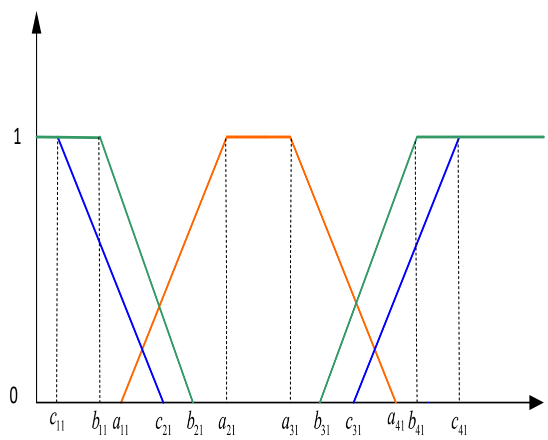
Table 2: Comparative analysis with the existing methods

Existing Methods	Score/ distance values					Ranking
	D_1	D_2	D_3	D_4	D_5	
[6]	0.6092	0.4512	0.6039	0.6121	0.6321	$S_2 > S_3 > S_1 > S_4 > S_5$
[16]	0.2788	0.6790	0.9394	0.6564	0.4014	$S_3 > S_2 > S_4 > S_5 > S_1$
[42]	0.6553	0.5779	0.5069	0.6835	0.5904	$S_4 > S_1 > S_5 > S_2 > S_3$
[45]	0.7716	0.7798	0.7349	0.8124	0.8201	$S_3 > S_1 > S_2 > S_4 > S_5$

From the Table 2, it is found that, the third software system is the best one among the five alternatives. The results in the existing methods overlaps the proposed result. Theresore the proposed methodology using the proposed under trapezoidal fuzzy neutrosophic environment to solve the decision making problem suitably in comparision with the existing methods.

Table 3 represents the various forms of trapezoidal fuzzy neutrosophic numbers (TrFNN) have been listed out and it shows the uniqueness of the proposed graphical representation among the existing graphical representations.

Table 3: Comparative analysis with the existing graphical representation

Trapezoidal fuzzy neutrosophic forms	Graphical representation
Darehmiraki [11]; A is a TrFNN, $a_1'', a_1, a_1', a_2, a_3, a_4', a_4, a_4'' \in R$ such that $a_1'' \le a_1 \le a_1' \le a_2 \le a_3 \le a_4' \le a_4 \le a_4''$ $A = \left\langle \left(a_1'', a_1, a_1', a_2, a_3, a_4', a_4, a_4''\right), T_A, I_A, F_A \right\rangle$	
Liang [21]; A is a TrFNN, $a_1, a_2, a_3, a_4 \in [0,1]$ such that $0 \le a_1 \le a_2 \le a_3 \le a_4 \le 1$ $A = \left\langle [a_1, a_2, a_3, a_4], (T_A, I_A, F_A) \right\rangle$	
Biswas [5]; A is a TpFNN, $(a_{41}, a_{21}, a_{31}, a_{41}), (b_{41}, b_{21}, b_{31}, b_{41}),$ $(c_{41}, c_{21}, c_{31}, c_{41}) \in R$ such that $c_{11} \le b_{11} \le a_{11} \le c_{21} \le b_{21} \le a_{21}$ $\le a_{31} \le b_{31} \le c_{31} \le a_{41} \le b_{41} \le c_{41}$ and $A = \left\langle (a_{11}, a_{21}, a_{31}, a_{41}), (b_{11}, b_{21}, b_{31}, b_{41}),\right.$ $\left.(c_{11}, c_{21}, c_{31}, c_{41}) \right\rangle$	

8-Advantages of the proposed measure

An efficient distance measure boosts the performance of task analysis or clustering. Also centroid method is specific and location based one and acquire the best geographical location in consideration of the distance between all the competences. Though the existing methods namely Euclidean measure, Manhattan measure Minkowski measure and Hamming distance measure have been applied in many real time problems they could not provide good results for the indeterminate data. Hence in this paper, we proposed a new distance measure for trapezoidal neutrosophic fuzzy numbers based on centroids and the significant advantages of the proposed measure are given as follows.

(i). Trapezoidal fuzzy neutrosophic number is a simple design of arithmetic operations and easy and perceptive interpretation as well. Therefore the proposed measure is an easy and effective one under neutrosophic environment.

(ii). Distance measure can be estimated with simple algorithm and significant level of accuracy can be acquired as well.

(iii). While taking the important decision of choosing the method to measure a distance it can be used due its simplicity.

(iv). The proposed distance measure is based on centroid and hence estimation of the distance between all objects of the data set is possible and indeterminacy also can be addressed.

(v). It is derived using Euclidean distance and hence it is very useful in correlation analysis.

(vi). Also it can be applied in location planning, operations management, Neutrosophic Statistics, clustering, medical diagnosis, Optimization and image processing to get more accurate results without any computational complexity.

9-Conclusion and Future Research

The concept of distance measure of trapezoidal fuzzy neutrosophic number has sufficient scope of utilization in different studies in various domain. In this paper, we proposed a new distance measure for the trapezoidal fuzzy neutrosophic number based on centroid with the graphical representation. Also, the properties of the proposed measure have been derived in detail. In addition, a decision making problem has been solved using the proposed measure as a numerical example. Further, comparative analysis has been done with the existing methods to show the potential of the proposed distance measure and various forms of trapezoidal fuzzy neutrosophic number have been listed and shown the uniqueness of the proposed graphical representation. Furthermore, advantages of the proposed measure are given. In future, the present work may be extended to other special types of neutrosophic set like pentagonal neutrosophic set, neutrosophic rough set, interval valued neutrosophic set and plithogeneic environments.

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Received: Apr 17, 2020. Accepted: July 3 2020

Neutrosophic Sets and Systems, Vol. 35, 2020
University of New Mexico

Introduction to Combined Plithogenic Hypersoft Sets

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Abstract: Plithogenic Hypersoft sets was introduced by Florentin Smarandache, who has extended crisp sets, fuzzy sets, intuitionistic sets, neutrosophic sets to plithogenic sets. The plithogenic sets considers the degree of appurtenance of the elements with respect to the attribute system. Smarandache has presented the classification of the plithogenic hypersoft sets and the applications of plithogenic fuzzy whole hypersoft sets in multi attribute decision making. Inspired by these research works, the concept of combined plithogenic hypersoft sets is introduced in this article. The representations of the degree of appurtenance of the elements determines the type of plithogenic hypersoft set, if it takes a combination of values then the new archetype of plithogenic hypersoft sets gets emerged into decision making scenario. This research work is put forth to project the realistic perception of the experts in the construction process of optimal conclusions.

Keywords: Plithogenic hypersoft set, combined plithogenic hypersoft set, decision making, multi attribute system.

1. Introduction

Classical set theory deals with the sets consisting of elements with membership values 0 or 1. The degree of belongingness of an element in a set has been extended to [0,1] by Zadeh [1] in the name of fuzzy sets, which is gaining momentum since its introduction. Sets comprising of quantitative elements can be defined by conventional concepts of sets, but the elements of qualitative nature can be treated only by fuzzy concepts and its membership value states the degree of confidence of its presence in the set. Atanassov [2] investigated on the degree of its absence in the set, by defining non-membership values. This paved way for the intuitionistic fuzzy sets, which consists of degree of membership, non-membership and hesitation. Fuzzy sets and intuitionistic fuzzy sets are extensively applied in decision making process. But still the human perception was not completely reflected in these two kinds of sets. This gap was filled by Florentine Smarandache [3-5] who introduced neutrosophic fuzzy sets, comprising of degree of truth membership, indeterminacy and degree of false membership. Smarandache has represented each of the three function in a more generalized and independent manner. Neutrosophic sets have extensive application in decision making at recent times. Abdel- Basset et al [6-7] has developed neutrosophic decision making models to solve transition difficulties of IoT-based enterprises and to evaluate green supply chain management practices.

Smarandache also extended the classical sets, fuzzy sets, intuitionistic fuzzy sets and neutrosophic fuzzy sets to plithogenic sets which is a quintuple (P, a, V, d, c) consisting of a set P, the attribute a, the range of attribute values V, degree of appurtenance d, and the degree of contradiction c. The nature of d determines the type of plithogenic sets. Smarandache presented an elaborate discussion on the genesis of plithogenic sets in his research work [8]. Abdel-Basset et al [9-11] has developed decision making models with incorporation of plithogenic sets to evaluate green supply chain management practices and intelligent Medical Decision Support Model Based on Soft Computing and IoT was also built; a hybrid plithogenic decision-making approach with quality function deployment for selecting supply chain sustainability metrics was also framed. These plithogenic decision making models are highly robust and feasible.

Molodtsov [12] introduced and applied soft sets in decision making which was extended to fuzzy soft sets predominantly by Maji [13]. The comprehensive outlook of hypersoft sets was made by Smarandache [14] which took the different forms of fuzzy sets in the course of time. Shazia Rana et al [15] in their recent work on application of plithogenic fuzzy whole hypersoft set in multi attribute decision making introduced the matrix representation of plithogenic hypersoft set and plithogenic fuzzy whole hypersoft set which adds to the compatibility of this decision making technique. The validation of the proposed decision making model with a numerical example in this work has inspired to introduce combined plithogenic hypersoft set.

The paper is organized as follows; section 2 presents a brief description of combined plithogenic hypersoft sets; section 3 comprises the application of combined plithogenic hypersoft sets in decision making based on the technique of Shazia Rana et al [15]; section 4 discusses the results and the last section concludes with the future extension of the proposed concept.

2. Combined plithogenic hypersoft sets

This section comprises of the direct narration and representation of the combined plithogenic hypersoft sets based on the preliminaries discussed by Smarandache [14] and Shazia Rana et al [15] to avoid the repetition of the elementary definitions. Smarandache presented the classification of plithogenic hypersoft sets and the categorization was purely based on the nature of degree of appurtenance. Based on his discussion, let us consider the following example to explain the need of combined plithogenic hypersoft sets

Let U be the universe of discourse that consists of pollution mitigation methods say

$U = \{M_1, M_2, M_3, M_4, M_5\}$ and the set $\mathscr{M} = \{M1, M4\} \subset U$.

The attributes are a_1 = Cost efficiency, a_2 = Eco-compatibility, a_3 = Time efficacy, a_4 = Profit yield. If the pollution abatement methods are supposed to fulfill these attributes, then in realistic perspective the possible attribute values are taken as follows,

Cost efficiency = A_1 = {low, medium, high}, Eco-compatibility = A_2 = {very high, high}, Time efficacy = A_3 = {less, more}, Profit yield = A_4 ={maximum, minimum}.

Suppose a manufacturing firm has decided to implement a pollution control method, then the above attributes and its values are considered for making optimal decision with the possible range of values of attributes. By usual consideration,

Let the function be: $G: A_1 \times A_2 \times A_3 \times A_4 \longrightarrow P(U)$

Let's assume: G ({low, high, more, maximum}) = $\{M_1, M_4\}$.

The degree of appurtenance of an element x to the set $\mathcal{M}$, with respect to each attribute value a is d_x^0(a) that is the deciding factor of the nature of plithogenic hypersoft set.

In the context of decision making with the expert's opinion, then d_x^0(a) is the resultant of the expert's perception. Sometimes the expert's outlook may be a combination of certain, fuzzy, intuitionistic and neutrosophic, which is expressed as follows

G({low, high, more, maximum}) = { M_1 (1,0.8,0.7,(0.4,0.5)),

M_4 (1,0.9,(0.8,0.1,0.1),(0.5,0.6)) }.

This is the pragmatic reflection of the person's perception in decision making process and this is the point of origin of combined plithogenic hypersoft sets. Thus a combined plithogenic hypersoft sets is a plithogenic hypersoft set in which the degree of appurtenance of an element x to the set $\mathcal{M}$, with respect to each attribute value is a combination of either crisp, fuzzy, intuitionistic or neutrosophic.

Combined plithogenic hypersoft sets can be classified into completely combined plithogenic hypersoft sets and partially combined plithogenic hypersoft sets based on the nature and combination of values taken by d_x^0(a). In the above stated example G({low, high, more, maximum}) = { M_1 (1,0.8,0.7,(0.4,0.5)), M_4 (1,0.9,(0.8,0.1,0.1),(0.5,0.6))} is a completely combined plithogenic hypersoft sets as $dx0$(a) takes all possible types of values. Suppose G({low, high, more, maximum}) = { M_1 (0.9,0.8,0.7,(0.4,0.5)), M_4 (0.8,0.9,0.6,(0.5,0.6))} then this combined plithogenic hypersoft set is partial in nature as d_x^0(a) takes only a combination of two types of values. Thus a combined plithogenic hypersoft set which is not complete is partial in its nature.

It is very evident that combined plithogenic hypersoft sets are highly rational in nature and it will certainly play a vital role in receiving the expert's opinion, which is very significant in any multi attribute decision making process. Also the need of such new types of plithogenic hypersoft sets are very essential, because in the manufacturing firms and in business sectors the implementation of certain methods and installation of certain mechanisms and machinery may not be characterized by only crisp or fuzzy values with regard to the degree of appurtenance as the possibility aspect has some extent of participation in it. To handle such situations the combined plithogenic hypersoft sets may lend a helping hand to the decision makers.

3. Application of Combined Plithogenic Hypersoft set in Multi Attribute Decision Making

The previous section presented an elaborate portrayal of combined plithogenic hypersoft set, the significant feature is the realistic representation, but it has certain difficulties in computations as the degree of appurtenance varies for each attribute. To handle such crisis, all the values of d_x^0(a) must be similar in nature, i.e. either all the values must be fuzzy values which is the lower level of fuzzy representation or it must be neutrosophic values, the higher level of fuzzy representation.

A manufacturing sector has decided to enhance its production rate by installing new kinds of machinery. The ultimate aim of incorporating such a change in the production mechanism is quality production and customer satisfaction. The market is flooded with several varieties of well equipped, modern machines and since the manufacturing sector makes huge investment, the decision making process takes place in two phases based on the expert's opinion and advice. In the first phase, ten machines were selected by the manufacturing sector and in the next phase five were selected based on the feedback of the users. The decision making problem begins here, as the company has to purchase only three out of five based on the extent of satisfaction of the attributes by these machines.

Let U = { M_1, M_2, M_3, M_4, M_5, M_6, M_7,M_8, M_9, M_{10}} be the university of discourse and set

T = {M_1, M_3, M_6, M_7, M_9} $\subset$ U.

The attribute system is represented as follows A = { (A_1)Maintenance Cost {Maximum in the initial years of utility(A_1^1), Maximum in the latter years of utility(A_1^2)}, (A_2)Reliability {High with additional expenditure(A_2^1), Moderate with no extra expense(A_2^2)}, (A_3)Flexibility {Single task oriented(A_3^1), Multi task oriented(A_3^2)}, (A_4)Durability {Very high in the beginning years of service(A_4^1), High in the latter years of service(A_4^2), }, (A_5)Profitability {Moderate in the initial years(A_5^1), Maximum in the latter years(A_5^2)}}.

The attributes are quite common, but the attribute values are more realistic as it mirror the actual aspects involved in making decision.

Let the function be: G: $A_1^1 \times A_2^2 \times A_3^2 \times A_4^1 \times A_5^2 \longrightarrow P(U)$. Based on the Expert's opinion, the degree of appurtenance of the elements with respect to the attribute values is represented as follows

G(A_1^1, A_2^2 , A_3^2 , A_4^1, A_5^2) =

{M1(0.9,(0.7,0.1),0.8,(0.6,0.2),0.5),M3((0.6,0.3),0.5,(0.4,0.1,0.3),0.8,0.7), M6(0.8,0.7,0.6,(0.5,0.2),(0.6,0.1,0.1)),M7((0.7,0.2,0.1),(0.7,0.1),0.9,(0.7,0.2),0.8),M9(1,0.9,0.5,0.8,(0.6,0.1,0.2))}.

The modified lower and higher fuzzy values of the degree of appurtenance of the elements with respect to the attribute values are denoted as G_L(A_1^1, A_2^2 , A_3^2 , A_4^1, A_5^2) and G_H(A_1^1, A_2^2 , A_3^2 , A_4^1, A_5^2)

G_L(A_1^1, A_2^2 , A_3^2 , A_4^1, A_5^2) = {M1(0.9,0.875,0.8,0.75,0.5),M3(0.67,0.5,0.4,0.8,0.7),M6(0.8,0.7,0.6,0.7,0.5), M7(0.67,0.875,0.9,0.78,0.8), M9(1,0.9,0.5,0.8,0.47)}

G_H(A_1^1, A_2^2 , A_3^2 , A_4^1, A_5^2) =

{M1(0.9,0.1,0.1),(0.7,0.2,0.1),(0.8,0.1,0.1),(0.6,0.3,0.2),(0.5,0.2,0.7)),M3((0.6,0.3,0.3), (0.5,0.2,0.7),(0.4,0.1,0.3),(0.8,0.1,0.1),(0.7,0.2,0.1)),M6((0.8,0.1,0.1),(0.7,0.2,0.1),(0.6,0.2,0.3),(0.5,0.3,0.2),(0.6,0.1,0.1)),M7((0.7,0.2,0.1),(0.7,0.1,0.1),(0.9,0.1,0.1),(0.7,0.1,0.2),(0.8,0.1,0.1)),M9((1,0,0),(0.9,0.1,0.1),(0.5,0.2,0.7),(0.8,0.1,0.1),(0.6,0.1,0.2))}

The lower and higher fuzzy values of the degree of appurtenance correspond to single fuzzy value and neutrosophic values. The matrix representation C of the degree of appurtenance of the elements with respect to the attribute values in combined plithogenic hypersoft sets is

$$\begin{array}{c|ccccc} & A_1^1 & A_2^2 & A_3^2 & A_4^1 & A_5^2 \\ \hline M_1 & 0.9 & (0.7,0.1) & 0.8 & (0.6,0.2) & 0.5 \\ M_3 & (0.6,0.3) & 0.5 & (0.4,0.1,0.3) & 0.8 & 0.7 \\ M_6 & 0.8 & 0.7 & 0.6 & (0.5,0.2) & (0.6,0.1,0.1)), \\ M_7 & (0.7,0.2,0.1) & (0.7,0.1) & 0.9 & (0.7,0.2) & 0.8 \\ M_9 & 1 & 0.9 & 0.5 & 0.8 & (0.6,0.1,0.2) \end{array}$$

The intuitionistic and neutrosophic values are transformed to the above fuzzy values by the methods of imprecision and Defuzzification [16]

Method I (Imprecision membership): Any neutrosophic fuzzy set $N_A = (T_A, I_A, F_A)$ including neutrosophic fuzzy values are transformed into intuitionistic fuzzy values or vague values as η(A) = $(T_A,\ f_A)$ where f_A is estimated the formula stated below which is called as Impression membership method.

$$f_A = \begin{cases} F_A + \dfrac{[1-F_A-I_A][1-F_A]}{[F_A+I_A]} & \text{if } F_A = 0 \\ F_A + \dfrac{[1-F_A-I_A][F_A]}{[F_A+I_A]} & \text{if } 0 < F_A \le 0.5 \\ F_A + [1-F_A-I_A]\left[0.5 + \dfrac{F_A-0.5}{F_A+I_A}\right] & \text{if } 0.5 < F_A \le 1 \end{cases}$$

Method II (Defuzzification): After Method I (Median membership), intuitionistic (vague),fuzzy values of the form η(A)= $(T_A,\ f_A)$ are transformed into fuzzy set including fuzzy values as $<\Delta(A)> = <\frac{T_A}{[T_A+f_A]}>$.

The matrix representation C_L of the lower fuzzy values of the degree of appurtenance of the elements with respect to the attribute values in combined plithogenic hypersoft sets is

$$\begin{array}{c|ccccc|} & A_1^1 & A_2^2 & A_3^2 & A_4^1 & A_5^2 \\ M_1 & 0.9 & 0.875 & 0.8 & 0.75 & 0.5 \\ M_3 & 0.67 & 0.5 & 0.4 & 0.8 & 0.7 \\ M_6 & 0.8 & 0.7 & 0.6 & 0.7 & 0.5 \\ M_7 & 0.67 & 0.875 & 0.9 & 0.78 & 0.8 \\ M_9 & 1 & 0.9 & 0.5 & 0.8 & 0.47 \end{array}$$

By using the procedure of ranking as discussed by Shazia Rana et. al [15] the machines are ranked by considering the values of C_L.

The frequency matrix F_L representing the ranking of the machines is

$$\begin{array}{c|ccccc|} & R_1 & R_2 & R_3 & R_4 & R_5 \\ M_1 & 1 & 2 & 0 & 0 & 0 \\ M_3 & 0 & 0 & 0 & 1 & 2 \\ M_6 & 0 & 1 & 0 & 2 & 0 \\ M_7 & 2 & 0 & 1 & 0 & 0 \\ M_9 & 1 & 1 & 1 & 0 & 0 \end{array}$$

The percentage measure of authenticity of ranking is presented below in Table 3.1

Table 3.1

R_1	M_7	50%
R_2	M_1	50%
R_3	M_9	50%
R_4	M_6	67%
R_5	M_3	100%

The matrix representation C_H of higher fuzzy values (neutrosophic representations) of the degree of appurtenance of the elements with respect to the attribute values in combined plithogenic hypersoft sets is

$$\begin{array}{c} \\ M_1 \\ M_3 \\ M_6 \\ M_7 \\ M_9 \end{array} \begin{pmatrix} A_1^1 & A_2^2 & A_3^2 & A_4^1 & A_5^2 \\ (0.9,0.1,0.1) & (0.7,0.2,0.1) & (0.8,0.1,0.1) & (0.6,0.3,0.2) & (0.5,0.2,0.7) \\ (0.6,0.3,0.3) & (0.5,0.2,0.7) & (0.4,0.1,0.3) & (0.8,0.1,0.1) & (0.7,0.2,0.1) \\ (0.8,0.1,0.1) & (0.7,0.2,0.1) & (0.6,0.2,0.3) & (0.5,0.3,0.2) & (0.6,0.1,0.1) \\ (0.7,0.2,0.1) & (0.7,0.1,0.1) & (0.9,0.1,0.1) & (0.7,0.1,0.2) & (0.8,0.1,0.1) \\ (1,0,0) & (0.9,0.1,0.1) & (0.5,0.2,0.7) & (0.8,0.1,0.1) & (0.6,0.1,0.2) \end{pmatrix}$$

To make the ranking of the machines based on the higher values in C_H the score values K(A) of the single valued neutrosophic representations [say A = (a,b,c)] are determined by $K(A) = \frac{1+a-2b-c}{2}$ [17]

$$\begin{array}{c} \\ M_1 \\ M_3 \\ M_6 \\ M_7 \\ M_9 \end{array} \begin{pmatrix} A_1^1 & A_2^2 & A_3^2 & A_4^1 & A_5^2 \\ 0.8 & 0.6 & 0.75 & 0.4 & 0.2 \\ 0.35 & 0.2 & 0.45 & 0.75 & 0.6 \\ 0.75 & 0.6 & 0.45 & 0.35 & 0.65 \\ 0.6 & 0.7 & 0.8 & 0.65 & 0.75 \\ 1 & 0.8 & 0.2 & 0.75 & 0.6 \end{pmatrix}$$

The frequency matrix F_H representing the ranking of machines is

$$\begin{array}{c|c} & \begin{array}{ccccc} R_1 & R_2 & R_3 & R_4 & R_5 \end{array} \\ \begin{array}{c} M_1 \\ M_3 \\ M_6 \\ M_7 \\ M_9 \end{array} & \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

The percentage measure of authenticity of ranking is presented below in Table 3.2

Table 3.2

R_1	M_7	60%
R_2	M_9	50%
R_3	M_6	25%
R_4	M_1	33%
R_5	M_3	100%

4. Discussion

The combined plithogenic hypersoft set representations are so deliberate in nature. The resultant of computations in making decisions in two ways is represented in Table 3.1 and 3.2. The machines M_7 and M_3 occupy first and fifth rank respectively in both kinds of representation of degree of appurtenance. Also by making inferences from the table values M_1, M_3 and M_6 can be ranked in second ,third and fourth positions respectively. It is very evident that the transformation of combined attribute values to lower order fuzzy values yields best results in ranking the machines, but still the higher order values will also yield optimum results based on the selection of the score functions. The methods of converting combined attribute value to the values of similar fashion have to be constituted in the upcoming research works to attain feasible solutions to the decision making problems.

5. Conclusions

This research work encompasses the discussion of the new concept of combined plithogenic hypersoft set and its application in multi attribute decision making. Besides these types of appurtenance degrees, others can be used under the plithogenic umbrella, such as: Pythagorean, picture fuzzy, spherical fuzzy, spherical neutrosophic, etc. and even the most general one, refined neutrosophic type of appurtenance degree. The combined plithogenic hypersoft set can be extended to interval-valued combined plithogenic hypersoft sets and it can be converted to simple fuzzy values using score functions. The matrix representations of degree of appurtenance in combined plithogenic hypersoft set has induced the author to extend the proposed theoretical conceptualization to plithogenic concentric hypergraphs, most probably the scope and future research work.

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Received: Apr 22, 2020. Accepted: July 2 2020

Neutrosophic Sets and Systems, Vol. 35, 2020
University of New Mexico

On Neutrosophic Generalized Alpha Generalized Continuity

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Abstract: This article demonstrates a further class of neutrosophic closed sets named neutrosophic generalized αg-closed sets and discuss their essential characteristics in neutrosophic topological spaces. Moreover, we submit neutrosophic generalized αg-continuous functions with their elegant features.

Keywords: neutrosophic generalized αg-closed sets, neutrosophic generalized αg-continuous functions, and neutrosophic generalized αg-irresolute functions.

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1. Introduction

Smarandache [1,2] originally handed the theory of "neutrosophic set". Recently, Abdel-Basset et al. discussed a novel neutrosophic approach [3-8] in several fields, for a few names, information and communication technology. Salama et al. [9] gave the clue of neutrosophic topological space (or simply NTS). Arokiarani et al. [10] added the view of neutrosophic α-open subsets of neutrosophic topological spaces. Imran et al. [11] presented the idea of neutrosophic semi-α-open sets in neutrosophic topological spaces. Dhavaseelan et al. [12] presented the idea of neutrosophic α^{m}-continuity. Our aim is to introduce a new idea of neutrosophic generalized αg-closed sets and examine their vital merits in neutrosophic topological spaces. Additionally, we propose neutrosophic generalized αg-continuous functions by employing neutrosophic generalized αg-closed sets and emphasizing some of their primary characteristics.

2. Preliminaries

Everywhere of these following sections, we assume that NTSs $(\mathcal{U},\xi),(\mathcal{V},\varrho)$ and $(\mathcal{W},\mu)$ are briefly denoted as $\mathcal{U},\mathcal{V}$, and $\mathcal{W}$, respectively. Let $\mathcal{C}$ be a neutrosophic set in $\mathcal{U}$, and we are easily symbolized it by NS, then the complement of $\mathcal{C}$ is basically given by $\bar{\mathcal{C}}$. If $\mathcal{C}$ is a neutrosophic open set in $\mathcal{U}$ and shortly indicated by Ne-OS. Then, $\bar{\mathcal{C}}$ is termed a neutrosophic closed set in $\mathcal{U}$ and simply referred by Ne-CS. The neutrosophic closure and the neutrosophic interior of $\mathcal{C}$ are merely signified by Ne-$cl(\mathcal{C})$ and Ne-$int(\mathcal{C})$, correspondingly.

Definition 2.1 [10]: A NS $\mathcal{C}$ in a NTS $\mathcal{U}$ is named a neutrosophic α-open set and simply written as Ne-αOS if $\mathcal{C} \subseteq$Ne-int(Ne-cl(Ne-$int(\mathcal{C})$)). Besides, if Ne-cl(Ne-int(Ne-$cl(\mathcal{C})$)) $\subseteq \mathcal{C}$, then $\mathcal{C}$ is called a neutrosophic α-closed set, and we are shortly given it as Ne-αCS. The collection of all such these

Ne-αOSs (correspondently, Ne-αCSs) in $\mathcal{U}$ is referred to Ne-α$O(\mathcal{U})$ (correspondently, Ne-α$C(\mathcal{U})$). The intersection of all Ne-αCSs that contain $\mathcal{C}$ is called the neutrosophic α-closure of $\mathcal{C}$ in $\mathcal{U}$ and represented by Ne-$\alpha cl(\mathcal{C})$.

Definition 2.2 [13]: A NS $\mathcal{C}$ in NTS $\mathcal{U}$ is so-called a neutrosophic generalized closed set and denoted by Ne-gCS if for any Ne-OS $\mathcal{M}$ in $\mathcal{U}$ such that $\mathcal{C} \subseteq \mathcal{M}$, then Ne-$cl(\mathcal{C}) \subseteq \mathcal{M}$. Moreover, its complement is named a neutrosophic generalized open set and referred to Ne-gOS.

Definition 2.3 [14]: A NS $\mathcal{C}$ in NTS $\mathcal{U}$ is so-called a neutrosophic αg-closed set and indicated by Ne-αgCS if for any Ne-OS $\mathcal{M}$ in $\mathcal{U}$ such that $\mathcal{C} \subseteq \mathcal{M}$, then Ne-$\alpha cl(\mathcal{C}) \subseteq \mathcal{M}$. Furthermore, its complement is named a neutrosophic αg-open set and symbolized by Ne-αgOS.

Definition 2.4 [15]: A NS $\mathcal{C}$ in NTS $\mathcal{U}$ is so-called a neutrosophic gα-closed set and signified by Ne-gαCS if far any Ne-αOS $\mathcal{M}$ in $\mathcal{U}$ such that $\mathcal{C} \subseteq \mathcal{M}$, then Ne-$\alpha cl(\mathcal{C}) \subseteq \mathcal{M}$. Besides, its complement is named a neutrosophic gα-open set and briefly written as Ne-gαOS.

Theorem 2.5 [10,13-15]: For any NTS $\mathcal{U}$, the next declarations valid and but not vice versa:
(i) for all Ne-OSs (correspondingly, Ne-CSs) are Ne-αOSs (correspondingly, Ne-αCSs).
(ii) for all Ne-OSs (correspondingly, Ne-CSs) are Ne-gOSs (correspondingly, Ne-gCSs).
(iii) for all Ne-gOSs (correspondingly, Ne-gCSs) are Ne-αgOSs (correspondingly, Ne-αgCSs).
(iv) for all Ne-αOS (correspondingly, Ne-αCSs) are Ne-gαOSs (correspondingly, Ne-gαCSs).
(v) for all Ne-gαOSs (correspondingly, Ne-gαCSs) are Ne-αgOSs (correspondingly, Ne-αgCSs).

Definition 2.6: Let $(\mathcal{U}, \xi)$ and $(\mathcal{V}, \varrho)$ be NTSs and $\eta: (\mathcal{U}, \xi) \longrightarrow (\mathcal{V}, \varrho)$ be a mapping, we have
(i) if for each Ne-OS (correspondingly, Ne-CS) $\mathcal{K}$ in $\mathcal{V}$, $\eta^{-1}(\mathcal{K})$ is a Ne-OS (correspondingly, Ne-CS) in $\mathcal{U}$, then η is known as neutrosophic continuous and denoted by Ne-continuous. [16]
(ii) if for each Ne-OS (correspondingly, Ne-CS) $\mathcal{K}$ in $\mathcal{V}$, $\eta^{-1}(\mathcal{K})$ is a Ne-αOS (correspondingly, Ne-αCS) in $\mathcal{U}$, then η is known as neutrosophic α-continuous and referred to Ne-α-continuous. [10]
(iii) if for each Ne-OS (correspondingly, Ne-CS) $\mathcal{K}$ in $\mathcal{V}$, $\eta^{-1}(\mathcal{K})$ is a Ne-gOS (correspondingly, Ne-gCS) in $\mathcal{U}$, then $\eta$is known as neutrosophic g-continuous and signified by Ne-g-continuous. [17]

Remark 2.7 [17,10]: Let $\eta: (\mathcal{U}, \xi) \longrightarrow (\mathcal{V}, \varrho)$ be a map, the next declarations valid and but not vice versa:
(i) For all Ne-continuous functions are Ne-α-continuous.
(ii) For all Ne-continuous functions are Ne-g-continuous.

3. Neutrosophic Generalized αg-Closed Sets

The neutrosophic generalized **α**g-closed sets and their features are studied and discussed in this part of the paper.

Definition 3.1: Let $\mathcal{C}$ be a NS in $NTS\ \mathcal{U}$, then it called a neutrosophic generalized αg-closed set and denoted by Ne-gαgCS if for any Ne-αgOS $\mathcal{M}$ in $\mathcal{U}$ such that $\mathcal{C} \subseteq \mathcal{M}$, then Ne-$cl(\mathcal{C}) \subseteq \mathcal{M}$.We indicated the collection of all Ne-gαgCSs in $NTS\ \mathcal{U}$ by Ne-gαgC$(\mathcal{U})$.

Definition 3.2: Let $\mathcal{C}$ be a NS in $TS\ \mathcal{U}$, then its neutrosophic gαg-closure is the intersection of each Ne-gαgCS in $\mathcal{U}$ including $\mathcal{C}$, and we are shortly written it as Ne-gαg$cl(\mathcal{C})$. In other words, Ne-gαg$cl(\mathcal{C}) = \cap\{\mathcal{D}: \mathcal{C} \subseteq \mathcal{D}, \mathcal{D}$ is a Ne-gαgCS$\}$.

Theorem 3.3: The subsequent declarations are valid in any $TS\ \mathcal{U}$:

(i) for all Ne-CSs are Ne-gαgCSs.

(ii) for all Ne-gαgCSs are Ne-gCSs.

(iii) for all Ne-gαgCSs are Ne-αgCSs.

(iv) for all Ne-gαgCSs are Ne-gαCSs.

Proof:

(i) Suppose a Ne-CS $\mathcal{C}$ is in $TS\ \mathcal{U}$. For any Ne-αgOS $\mathcal{M}$, including $\mathcal{C}$, we have $\mathcal{M} \supseteq \mathcal{C} =$Ne-$cl(\mathcal{C})$. Therefore, $\mathcal{C}$ stands a Ne-gαgCS.

(ii) Suppose Ne-gαgCS $\mathcal{C}$ is in $TS\ \mathcal{U}$. For any Ne-OS $\mathcal{M}$, including $\mathcal{C}$, we have theorem (2.5) states that $\mathcal{M}$ stands a Ne-αgOS in $\mathcal{U}$. Because a Ne-gαgCS $\mathcal{C}$ satisfying this fact Ne-$cl(\mathcal{C}) \subseteq \mathcal{M}$. As a result, $\mathcal{C}$ considers a Ne-gCS.

(iii) Assume Ne-gαgCS $\mathcal{C}$ is in $TS\ \mathcal{U}$. For any Ne-OS $\mathcal{M}$, including $\mathcal{C}$, we have theorem (2.5) states that $\mathcal{M}$ remains a Ne-αgOS in $\mathcal{U}$. Subsequently, Ne-gαgCS $\mathcal{C}$ satisfying this statement Ne-α$cl(\mathcal{C}) \subseteq$Ne-$cl(\mathcal{C}) \subseteq \mathcal{M}$. Therefore, $\mathcal{C}$ becomes a Ne-αgCS.

(iv) Assume Ne-gαgCS $\mathcal{C}$ is in $TS\ \mathcal{U}$. For any Ne-αOS $\mathcal{M}$ including $\mathcal{C}$, we have theorem (2.5) states that $\mathcal{M}$ remains a Ne-αgOS in $\mathcal{U}$. Subsequently, Ne-gαgCS $\mathcal{C}$ satisfying this statement Ne-α$cl(\mathcal{C}) \subseteq$Ne-$cl(\mathcal{C}) \subseteq \mathcal{M}$. Therefore, $\mathcal{C}$ considers a Ne-gαCS.

The opposite conditions for this previous theorem do not look accurate by the next obvious examples.

Example 3.4: Suppose $\mathcal{U} = \{p, q\}$ and let $\xi = \{0_N, \mathcal{A}, \mathcal{B}, 1_N\}$, such that we have the sets $\mathcal{A} = \langle u, (0.6,0.7), (0.1,0.1), (0.4,0.2)\rangle$ and $\mathcal{B} = \langle u, (0.1,0.2), (0.1,0.1), (0.8,0.8)\rangle$, so that $(\mathcal{U}, \xi)$ is a NTS. However, the $NS\ \mathcal{C} = \langle u, (0.2,0.2), (0.1,0.1), (0.6,0.7)\rangle$ is a Ne-gαgCS but not a Ne-CS.

Example 3.5: Suppose $\mathcal{U} = \{p, q, r\}$ and let $\xi = \{0_N, \mathcal{A}, \mathcal{B}, 1_N\}$, where such that we have the sets $\mathcal{A} = \langle u, (0.5,0.5,0.4), (0.7,0.5,0.5), (0.4,0.5,0.5)\rangle$ and $\mathcal{B} = \langle u, (0.3,0.4,0.4), (0.4,0.5,0.5), (0.3,0.4,0.6)\rangle$, so that $(\mathcal{U}, \xi)$ is a NTS. However, the $NS\ \mathcal{C} = \langle u, (0.4,0.6,0.5), (0.4,0.3,0.5), (0.5,0.6,0.4)\rangle$ is a Ne-gCS but not a Ne-gαgCS.

Example 3.6: Suppose $\mathcal{U} = \{p, q\}$ and let $\xi = \{0_N, \mathcal{A}, \mathcal{B}, 1_N\}$, where such that we have the sets $\mathcal{A} = \langle u, (0.5,0.6), (0.3,0.2), (0.4,0.1)\rangle$ and $\mathcal{B} = \langle u, (0.4,0.4), (0.4,0.3), (0.5,0.4)\rangle$, so that $(\mathcal{U}, \xi)$ is a NTS.

However, the NS $\mathcal{C} = \langle u, (0.5,0.4), (0.4,0.4), (0.4,0.5)\rangle$ is a Ne-αgCS and hence Ne-gαCS but not a Ne-gαgCS.

Definition 3.7: Let $\mathcal{C}$ be any NS in TS $\mathcal{U}$, then it is called a neutrosophic generalized αg-open set and referred to by Ne-gαgOS iff the set $\mathcal{U} - \mathcal{C}$ is a Ne-gαgCS. The collection of the whole Ne-gαgOSs in NTS $\mathcal{U}$ indicated by Ne-gαgO$(\mathcal{U})$.

Definition 3.8: The union of the whole Ne-gαgOSs in NTS $\mathcal{U}$ included in NS $\mathcal{C}$ is termed neutrosophic gαg-interior of $\mathcal{C}$ and symbolized by Ne-gαg$int(\mathcal{C})$. In symbolic form, we have this thing Ne-gαg$int(\mathcal{C}) = \cup\{\mathcal{D}: \mathcal{C} \supseteq \mathcal{D}, \mathcal{D}$ is a Ne-gαgOS$\}$.

Proposition 3.9: For any NS $\mathcal{M}$ in TS $\mathcal{U}$, the subsequent features stand:
(i) Ne-gαg$int(\mathcal{M}) = \mathcal{M}$ iff $\mathcal{M}$ is a Ne-gαgOS.
(ii) Ne-gαg$cl(\mathcal{M}) = \mathcal{M}$ iff $\mathcal{M}$ is a Ne-gαgCS.
(iii) Ne-gαg$int(\mathcal{M})$ is the biggest Ne-gαgOS included in $\mathcal{M}$.
(iv) Ne-gαg$cl(\mathcal{M})$ is the littlest Ne-gαgCS, including $\mathcal{M}$.
Proof: the features (i-iv) are understandable.

Proposition 3.10: For any NS $\mathcal{M}$ in TS $\mathcal{U}$, the subsequent features stand:
(i) Ne-gαg$int(\overline{\mathcal{M}}) = \overline{(\text{Ne} - \text{g}\alpha\text{g}cl(\mathcal{M}))}$,
(ii) Ne-gαg$cl(\overline{\mathcal{M}}) = \overline{(\text{Ne} - \text{g}\alpha\text{g}int(\mathcal{M}))}$.
Proof:
(i) The proof will be evident by symbolic definition, Ne-gαg$cl(\mathcal{M}) = \cap\{\mathcal{D}: \mathcal{M} \subseteq \mathcal{D}, \mathcal{D}$ is a Ne-gαgCS$\}$

$$\begin{aligned}\overline{(\text{Ne} - \text{g}\alpha\text{g}cl(\mathcal{M}))} &= \cap\{\overline{D}: \overline{\mathcal{M}} \subseteq \overline{D}, \overline{D} \text{ is a Ne-g}\alpha\text{gCS}\}\\ &= \cup\{\overline{\mathcal{D}}: \overline{\mathcal{M}} \subseteq \overline{\mathcal{D}}, \overline{\mathcal{D}} \text{ is a Ne-g}\alpha\text{gCS}\}\\ &= \cup\{\mathcal{N}: \mathcal{M} \supseteq \mathcal{N}, \mathcal{N} \text{is a Ne-g}\alpha\text{gOS}\}\\ &= \text{Ne-g}\alpha\text{g}int(\overline{\mathcal{M}}).\end{aligned}$$

(ii) This feature has undeniable proof analogous to feature (i).

Theorem 3.11: For any Ne-OS $\mathcal{C}$ in TS $\mathcal{U}$, then this set is a Ne-gαgOS.
Proof: Suppose Ne-OS $\mathcal{C}$ in TS $\mathcal{U}$, so we obtain that $\bar{\mathcal{C}}$ is a Ne-CS. Therefore, $\bar{\mathcal{C}}$ is a Ne-gαgCS via the previous theorem (3.3), part (i). Consequently, $\mathcal{C}$ is a Ne-gαgOS.

Theorem 3.12: For any Ne-gαgOS $\mathcal{C}$ in TS $\mathcal{U}$, then this set is a Ne-gOS.
Proof: Suppose Ne-gαgOS $\mathcal{C}$ in TS $\mathcal{U}$, so we obtain that $\bar{\mathcal{C}}$ is a Ne-gαgCS. Therefore, $\bar{\mathcal{C}}$ is a Ne-gCS via the previous theorem (3.3), part (ii). Consequently, $\mathcal{C}$ is a Ne-gOS.

Lemma 3.13: For any Ne-gαgOS $\mathcal{C}$ in TS $\mathcal{U}$, then this set is Ne-αgOS (correspondingly, Ne-gαOS).
Proof: The proof of this lemma is similar to one of the previous theorem.

Proposition 3.14: For any two Ne-gαgCSs $\mathcal{C}$ and $\mathcal{D}$ in TS $\mathcal{U}$, then the set $\mathcal{C} \cup \mathcal{D}$ is a Ne-gαgCS.

Proof: Suppose any two Ne-gαgCSs $\mathcal{C}$ and $\mathcal{D}$ in $NTS\ \mathcal{U}$ and $\mathcal{M}$ is a Ne-αgOS, including $\mathcal{C}$ and $\mathcal{D}$. In other words, we have $\mathcal{C}\cup\mathcal{D} \subseteq \mathcal{M}$. So, $\mathcal{C},\mathcal{D} \subseteq \mathcal{M}$. Because $\mathcal{C}$ and $\mathcal{D}$ are Ne-gαgCSs, we get that Ne-$cl(\mathcal{C})$, Ne-$cl(\mathcal{D}) \subseteq \mathcal{M}$. Therefore, Ne-$cl(\mathcal{C}\cup\mathcal{D})$ = Ne-$cl(\mathcal{C})\cup$Ne-$cl(\mathcal{D}) \subseteq \mathcal{M}$. Then Ne-$cl(\mathcal{C}\cup\mathcal{D}) \subseteq \mathcal{M}$. Thus, $\mathcal{C}\cup\mathcal{D}$ stands a Ne-gαgCS.

Proposition 3.15: For any two Ne-gαgOSs $\mathcal{C}$ and $\mathcal{D}$ in $TS\ \mathcal{U}$, then the set $\mathcal{C}\cap\mathcal{D}$ is a Ne-gαgOS.
Proof: Suppose any two Ne-gαgOSs $\mathcal{C}$ and $\mathcal{D}$ in $TS\ \mathcal{U}$. Subsequently, we have that $\bar{\mathcal{C}}$ and $\bar{\mathcal{D}}$ are Ne-gαgCSs. So, we reach to this fact $\bar{\mathcal{C}}\cup\bar{\mathcal{D}}$ is a Ne-gαgCS by proposition (3.14). Because $\bar{\mathcal{C}}\cup\bar{\mathcal{D}} = \overline{(\mathcal{C}\cap\mathcal{D})}$, we obtain to our final result $\mathcal{C}\cap\mathcal{D}$ is a Ne-gαgOS.

Proposition 3.16: Let Ne-gαgCS $\mathcal{C}$ be in $TS\ \mathcal{U}$, then Ne-$cl(\mathcal{C}) - \mathcal{C}$ does not include non-empty Ne-CS in $\mathcal{U}$.
Proof: Assume we have Ne-gαgCS $\mathcal{C}$ and Ne-CS $\mathcal{F}$ in $NTS\ \mathcal{U}$ so as $\mathcal{F} \subseteq$ Ne-$cl(\mathcal{C}) - \mathcal{C}$. Because $\mathcal{C}$ stands a Ne-gαgCS, this gives us the fact Ne-$cl(\mathcal{C}) \subseteq \bar{\mathcal{F}}$. The latter means $\mathcal{F} \subseteq \overline{\mathrm{Ne} - cl(\mathcal{C})}$. Subsequently, we arrive to $\mathcal{F} \subseteq$ Ne-$cl(\mathcal{C})\cap(\overline{\mathrm{Ne} - cl(\mathcal{C})}) = 0_N$. Therefore, $\mathcal{F} = 0_N$ and so, we reach to our final result Ne-$cl(\mathcal{C}) - \mathcal{C}$ does not include non-empty Ne-CS.

Proposition 3.17: Let Ne-gαgCS $\mathcal{C}$ be in $NTS\ \mathcal{U}$ iff Ne-$cl(\mathcal{C}) - \mathcal{C}$ does not include non-empty Ne-αgCS in $\mathcal{U}$.
Proof: Assume we have Ne-gαgCS $\mathcal{C}$ and Ne-αgCS $\mathcal{G}$ in $NTS\ \mathcal{U}$ so as $\mathcal{G} \subseteq$ Ne-$cl(\mathcal{C}) - \mathcal{C}$. Because $\mathcal{C}$ considers a Ne-gαgCS, this gives us the fact Ne-$cl(\mathcal{C}) \subseteq \bar{\mathcal{G}}$. The latter means $\mathcal{G} \subseteq \overline{\mathrm{Ne} - cl(\mathcal{C})}$. Subsequently, we arrive to $\mathcal{G} \subseteq$ Ne-$cl(\mathcal{C})\cap(\overline{\mathrm{Ne} - cl(\mathcal{C})}) = 0_N$. Therefore, $\mathcal{G}$ is empty.
On The Other Hand, let us assume that Ne-$cl(\mathcal{C}) - \mathcal{C}$ does not include non-empty Ne-αgCS in $\mathcal{U}$. Suppose $\mathcal{M}$ is Ne-αgOS so as $\mathcal{C} \subseteq \mathcal{M}$. If we have this truth Ne-$cl(\mathcal{C}) \subseteq \mathcal{M}$ but then we get this fact Ne-$cl(\mathcal{C})\cap(\bar{\mathcal{M}})$ is non-empty. Meanwhile, we know that Ne-$cl(\mathcal{C})$ is Ne-CS and at the same time, we have $\bar{\mathcal{M}}$ is Ne-αgCS, so Ne-$cl(\mathcal{C})\cap(\bar{\mathcal{M}})$ is non-empty Ne-αgCS included Ne-$cl(\mathcal{C}) - \mathcal{C}$. This leads us to a contradiction. Consequently Ne-$cl(\mathcal{C}) \nsubseteq \mathcal{M}$. Therefore, $\mathcal{C}$ considers a Ne-gαgCS.

Theorem 3.18: Let Ne-αgOS and Ne-gαgCS $\mathcal{C}$ be in $TS\ \mathcal{U}$, then $\mathcal{C}$ considers a Ne-CS in $\mathcal{U}$.
Proof: Assume we have Ne-αgOS and Ne-gαgCS $\mathcal{C}$ is in $TS\ \mathcal{U}$, so we get that Ne-$cl(\mathcal{C}) \subseteq \mathcal{C}$ and subsequently, we reach to $\mathcal{C} \subseteq$ Ne-$cl(\mathcal{C})$. Consequently, Ne-$cl(\mathcal{C}) = \mathcal{C}$. Therefore, $\mathcal{C}$ stands a Ne-CS.

Theorem 3.19: Let Ne-gαgCS $\mathcal{C}$ be in $NTS\ \mathcal{U}$ so as $\mathcal{C} \subseteq \mathcal{D} \subseteq$ Ne-cl($\mathcal{C}$), but then again $\mathcal{D}$ considers a Ne-gαgCS in $\mathcal{U}$.
Proof: Assume we have Ne-gαgCS $\mathcal{C}$ and Ne-αgOS $\mathcal{M}$ are in $NTS\ \mathcal{U}$ so as $\mathcal{D} \subseteq \mathcal{M}$. Later, $\mathcal{C} \subseteq \mathcal{M}$. Subsequently, $\mathcal{C}$ stands a Ne-gαgCS; this fact pursues Ne-$cl(\mathcal{C}) \subseteq \mathcal{M}$. So, $\mathcal{D} \subseteq$ Ne-$cl(\mathcal{C})$ infers Ne-$cl(\mathcal{D}) \subseteq$ Ne-$cl($Ne-$cl(\mathcal{C}))$ = Ne-$cl(\mathcal{C})$. Consequently, Ne-$cl(\mathcal{D}) \subseteq \mathcal{M}$. Therefore, $\mathcal{D}$ exists a Ne-gαgCS.

Theorem 3.20: Let Ne-gαgOS $\mathcal{C}$ be in $NTS\ \mathcal{U}$ so as Ne-$int(\mathcal{C}) \subseteq \mathcal{D} \subseteq \mathcal{C}$, but then again $\mathcal{D}$ considers a Ne-gαgOS in $\mathcal{U}$.

Proof: Assume we have Ne-gαgOS $\mathcal{C}$ is in NTS $\mathcal{U}$ so as Ne-$int(\mathcal{C}) \subseteq \mathcal{D} \subseteq \mathcal{C}$. After that, $\mathcal{U} - \mathcal{C}$ stands a Ne-gαgCS as well as $\bar{\mathcal{C}} \subseteq \bar{\mathcal{D}} \subseteq$ Ne-$cl(\bar{\mathcal{C}})$. But then again, we depend on theorem (3.19) to get $\mathcal{U} - \mathcal{D}$ is a Ne-gαgCS. Therefore, $\mathcal{D}$ exists a Ne-gαgOS.

Theorem 3.21: A NS $\mathcal{C}$ is Ne-gαgOS iff $\mathcal{P} \subseteq$ Ne-$int(\mathcal{C})$ so as $\mathcal{P} \subseteq \mathcal{C}$ and $\mathcal{P}$ considers a Ne-gαgCS.
Proof: Assume we have that Ne-gαgCS $\mathcal{P}$ satisfying $\mathcal{P} \subseteq \mathcal{C}$ and $\mathcal{P} \subseteq$ Ne-$int(\mathcal{C})$. Afterward, $\overline{\mathcal{C}} \subseteq \bar{\mathcal{P}}$ and we have by lemma (3.13), $\bar{\mathcal{P}}$ remains a Ne-αgOS. Accordingly, Ne-$cl(\overline{\mathcal{C}}) = \overline{\text{Ne} - int(\mathcal{C})} \subseteq \bar{\mathcal{P}}$. Subsequently, $\bar{\mathcal{C}}$ stands a Ne-gαgCS. Therefore, $\mathcal{C}$ stands a Ne-gαgOS.
On the contrary, we assume Ne-gαgOS $\mathcal{C}$ and Ne-gαgCS $\mathcal{P}$ is so as $\mathcal{P} \subseteq \mathcal{C}$. Subsequently, $\bar{\mathcal{C}} \subseteq \bar{\mathcal{P}}$. While $\bar{\mathcal{C}}$ exists a Ne-gαgCS and $\bar{\mathcal{P}}$ remains a Ne-αgOS, we reach to that Ne-$cl(\overline{\mathcal{C}}) \subseteq \bar{\mathcal{P}}$. Therefore, $\mathcal{P} \subseteq$ Ne-$int(\mathcal{C})$.

Remark 3.22: The next illustration demonstrates the relative among the distinct kinds of Ne-CS:

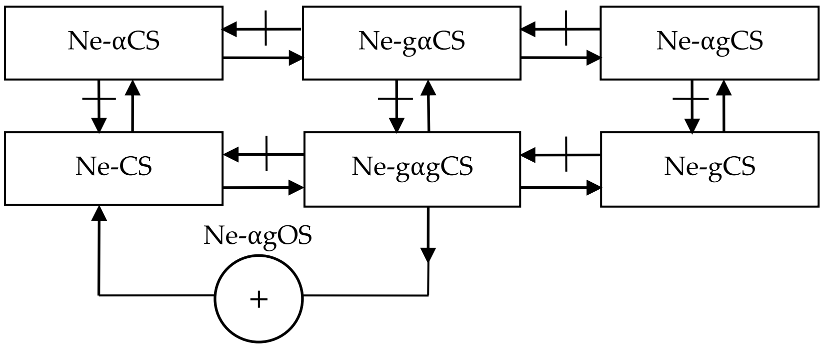

Fig. 3.1

4. Neutrosophic Generalized αg-Continuous Functions

In this part of this paper, the neutrosophic generalized **α**g-continuous functions are performed and examined their fundamental features.

Definition 4.1: Let $\eta: (\mathcal{U}, \xi) \longrightarrow (\mathcal{V}, \varrho)$ be a map so as $\mathcal{U}$ and $\mathcal{V}$ are NTSs, then:
(i) η is named a neutrosophic αg-continuous and signified by Ne-αg-continuous if for every Ne-OS (correspondingly, Ne-CS) $\mathcal{K}$ in $\mathcal{V}$, $\eta^{-1}(\mathcal{K})$ is a Ne-αgOS (correspondingly, Ne-αgCS) in $\mathcal{U}$.
(ii) η is named a neutrosophic gα-continuous and signified by Ne-gα-continuous if for every Ne-OS (correspondingly, Ne-CS) $\mathcal{K}$ in $\mathcal{V}$, $\eta^{-1}(\mathcal{K})$ is a Ne-gαOS (correspondingly, Ne-gαCS) in $\mathcal{U}$.

Theorem 4.2: Let η be a function on NTS $\mathcal{U}$ and valued in TS $\mathcal{V}$. So, we have the following:
(i) all Ne-g-continuous functions are Ne-αg-continuous.
(ii) all Ne-α-continuous functions are Ne-gα-continuous.
(iii) all Ne-gα-continuous functions are Ne-αg-continuous.
Proof:
(i) Let Ne-CS $\mathcal{K}$ be in NTS $\mathcal{V}$ and Ne-g-continuous function η defined on NTS $\mathcal{U}$ and valued in TS $\mathcal{V}$. By definition of Ne-g-continuous, $\eta^{-1}(\mathcal{K})$ remains a Ne-gCS in $\mathcal{U}$. So, we have $\eta^{-1}(\mathcal{K})$ is a Ne-αgCS in $\mathcal{U}$ because of theorem (2.5) part (iii). As a result, η stands a Ne-αg-continuous.

(ii) Let Ne-CS $\mathcal{K}$ be in $NTS\ \mathcal{V}$ and Ne-α-continuous function η defined on $NTS\ \mathcal{U}$ and valued in $NTS\ \mathcal{V}$. By definition of Ne-α-continuous, $\eta^{-1}(\mathcal{K})$ remains a Ne-αCS in $\mathcal{U}$. So, we have $\eta^{-1}(\mathcal{K})$ is a Ne-gαCS in $\mathcal{U}$ because of theorem (2.5) part (iv). As a result, η stands a Ne-gα-continuous.
(iii) Let Ne-CS $\mathcal{K}$ be in $NTS\ \mathcal{V}$ and Ne-gα-continuous function η defined on $NTS\ \mathcal{U}$ and valued in $TS\ \mathcal{V}$. So, we have $\eta^{-1}(\mathcal{K})$ is a Ne-gαCS and then $\eta^{-1}(\mathcal{K})$ is a Ne-αgCS in $\mathcal{U}$ because of theorem (2.5) part (v). Therefore, η stands a Ne-αg-continuous.
The reverse of the beyond proposition does not become valid as shown in the next examples.

Example 4.3: (i) Assume $\mathcal{U}=\{p,q\}$ and $\xi=\{0_N,\mathcal{A},\mathcal{B},1_N\}$ and $\varrho=\{0_N,\mathcal{B},\mathcal{C},1_N\}$, where $\mathcal{A}=\langle u,(0.6,0.7),(0.4,0.3),(0.5,0.2)\rangle$, $\mathcal{B}=\langle u,(0.5,0.5),(0.5,0.4),(0.6,0.5)\rangle$ and $\mathcal{C}=\langle u,(0.5,0.5),(0.6,0.4),(0.7,0.5)\rangle$ are the neutrosophic sets, then $(\mathcal{U},\xi)$ and $(\mathcal{U},\varrho)$ are NTSs. Define $\eta:(\mathcal{U},\xi)\longrightarrow(\mathcal{U},\varrho)$ as a $\eta(p)=q$ and $\eta(q)=p$. Then η is Ne- α g- continuous. But $\bar{\mathcal{C}}=\langle u,(0.7,0.5),(0.6,0.4),(0.5,0.5)\rangle$ is a Ne-CS in $(\mathcal{U},\varrho)$, $\eta^{-1}(\bar{\mathcal{C}})$ is not a Ne-gCS in $(\mathcal{U},\xi)$. Thus η is not a Ne-g-continuous.
(ii) Let $\mathcal{U}=\{p,q\}$ and let $\xi=\{0_N,\mathcal{A},\mathcal{B},1_N\}$ and $\varrho=\{0_N,\mathcal{B},\mathcal{C},1_N\}$, where $\mathcal{A}=\langle u,(0.6,0.7),(0.4,0.3),(0.5,0.2)\rangle$, $\mathcal{B}=\langle u,(0.5,0.5),(0.5,0.4),(0.6,0.5)\rangle$ and $\mathcal{C}=\langle u,(0.5,0.5),(0.5,0.5),(0.4,0.5)\rangle$ are the neutrosophic sets, then $(\mathcal{U},\xi)$ and $(\mathcal{U},\varrho)$ are NTSs. Define $\eta:(\mathcal{U},\xi)\longrightarrow(\mathcal{U},\varrho)$ as a $\eta(p)=p$ and $\eta(q)=q$. Then η is Ne-g α - continuous . But $\bar{\mathcal{C}}=\langle u,(0.4,0.5),(0.5,0.5),(0.5,0.5)\rangle$ is a Ne-CS in $(\mathcal{U},\varrho)$, $\eta^{-1}(\bar{\mathcal{C}})$ is not a Ne-αCS in $(\mathcal{U},\xi)$. Thus η is not a Ne-α-continuous.
(iii) Let $\mathcal{U}=\{p,q\}$ and let $\xi=\{0_N,\mathcal{A},\mathcal{B},1_N\}$ and $\varrho=\{0_N,\mathcal{B},\mathcal{C},1_N\}$, where $\mathcal{A}=\langle u,(0.6,0.7),(0.4,0.3),(0.5,0.2)\rangle$, $\mathcal{B}=\langle u,(0.5,0.5),(0.5,0.4),(0.6,0.5)\rangle$ and $\mathcal{C}=\langle u,(0.5,0.5),(0.6,0.4),(0.7,0.5)\rangle$ are the neutrosophic sets, then $(\mathcal{U},\xi)$ and $(\mathcal{U},\varrho)$ are NTSs. Define $\eta:(\mathcal{U},\xi)\longrightarrow(\mathcal{U},\varrho)$ as a $\eta(p)=q$ and $\eta(q)=p$. Then η is Ne- α g- continuous . But $\bar{\mathcal{C}}=\langle u,(0.5,0.5),(0.5,0.5),(0.6,0.4)\rangle$ is a Ne-CS in $(\mathcal{U},\varrho)$, $\eta^{-1}(\bar{\mathcal{C}})$ is not a Ne-$g\alpha$CS in $(\mathcal{U},\xi)$. Thus η is not a Ne-gα-continuous.

Definition 4.4: Let η be a function on $NTS\ \mathcal{U}$ and valued in $TS\ \mathcal{V}$. Then, we named η as neutrosophic generalized αg-continuous and shortly wrote it as Ne-gαg-continuous if for each Ne-CS $\mathcal{K}$ in $\mathcal{V}$, $\eta^{-1}(\mathcal{K})$ is a Ne-gαgCS in $\mathcal{U}$.

Theorem 4.5: Let η be a function on $NTS\ \mathcal{U}$ and valued in $TS\ \mathcal{V}$. Afterward, η remains a Ne-gαg-continuous function iff for each Ne-OS $\mathcal{K}$ in $\mathcal{V}$, $\eta^{-1}(\mathcal{K})$ is a Ne-gαgOS in $\mathcal{U}$.
Proof: Let Ne-OS $\mathcal{K}$ and Ne-CS $\bar{\mathcal{K}}$ are in $\mathcal{V}$. Therefore, $\eta^{-1}(\bar{\mathcal{K}})=\overline{(\eta^{-1}(\mathcal{K}))}$ remains a Ne-gαgCS in $\mathcal{U}$. Consequently, $\eta^{-1}(\mathcal{K})$ exists a Ne-gαgOS in $\mathcal{U}$. The reverse proof is evident.

Proposition 4.6: For all Ne-gαg-continuous functions are Ne-αg-continuous.
Proof: Let Ne-CS $\mathcal{K}$ be in $NTS\ \mathcal{V}$ and Ne-gαg-continuous function η defined on $NTS\ \mathcal{U}$ and valued in $TS\ \mathcal{V}$. By definition of Ne-gαg-continuous function, $\eta^{-1}(\mathcal{K})$ stands a Ne-gαgCS in $\mathcal{U}$. So, we have $\eta^{-1}(\mathcal{K})$ remains a Ne-αgCS in $\mathcal{U}$ because of theorem (3.3) part (iii). As a result, η exists a Ne-αg-continuous.

Proposition 4.7: For all Ne-gαg-continuous functions are Ne-gα-continuous.

Proof: Let Ne-CS $\mathcal{K}$ be in $NTS\ \mathcal{V}$ and Ne-gαg-continuous function η defined on $NTS\ \mathcal{U}$ and valued in $TS\ \mathcal{V}$. By definition of Ne-gαg-continuous function, $\eta^{-1}(\mathcal{K})$ stands a Ne-gαgCS in $\mathcal{U}$. So, we have $\eta^{-1}(\mathcal{K})$ remains a Ne-gαCS in $\mathcal{U}$ because of theorem (3.3) part (iv). As a result, η exists a Ne-gα-continuous.

The reverse of the beyond proposition does not become valid as shown in the next examples.

Example 4.8: Let $\mathcal{U} = \{p, q\}$ and let $\xi = \{0_N, \mathcal{A}, \mathcal{B}, 1_N\}$ and $\varrho = \{0_N, \mathcal{C}, 1_N\}$, where $\mathcal{A} = \langle u, (0.5,0.6), (0.3,0.2), (0.4,0.1)\rangle$, $\mathcal{B} = \langle u, (0.4,0.4), (0.4,0.3), (0.5,0.4)\rangle$ and $\mathcal{C} = \langle u, (0.5,0.4), (0.4,0.4), (0.4,0.5)\rangle$ are the neutrosophic sets, then $(\mathcal{U}, \xi)$ and $(\mathcal{U}, \varrho)$ are NTSs. Define $\eta: (\mathcal{U}, \xi) \longrightarrow (\mathcal{U}, \varrho)$ as a $\eta(p) = q$ and $\eta(q) = p$. Then η is Ne- α g- continuous . But $\mathcal{C} = \langle u, (0.4,0.5), (0.4,0.4), (0.5,0.4)\rangle$ is a Ne-CS in $(\mathcal{U}, \varrho), \eta^{-1}(\bar{\mathcal{C}})$ is a Ne-αgCS but not a Ne-gαgCS in $(\mathcal{U}, \xi)$. Thus η is not a Ne-gαg-continuous.

Example 4.9: Let $\mathcal{U} = \{p, q\}$ and let $\xi = \{0_N, \mathcal{A}, \mathcal{B}, 1_N\}$ and $\varrho = \{0_N, \mathcal{C}, 1_N\}$, where $\mathcal{A} = \langle u, (0.5,0.6), (0.3,0.2), (0.4,0.1)\rangle$, $\mathcal{B} = \langle u, (0.4,0.4), (0.4,0.3), (0.5,0.4)\rangle$ and $\mathcal{C} = \langle u, (0.5,0.4), (0.4,0.4), (0.4,0.5)\rangle$ are the neutrosophic sets, then $(\mathcal{U}, \xi)$ and $(\mathcal{U}, \varrho)$ are NTSs. Define $\eta: (\mathcal{U}, \xi) \longrightarrow (\mathcal{U}, \varrho)$ as a $\eta(p) = q$ and $\eta(q) = p$. Then η is Ne-g α - continuous . But $\mathcal{C} = \langle u, (0.4,0.5), (0.4,0.4), (0.5,0.4)\rangle$ is a Ne-CS in $(\mathcal{U}, \varrho), \eta^{-1}(\bar{\mathcal{C}})$ is a Ne-gαCS but not a Ne-gαgCS in $(\mathcal{U}, \xi)$. Thus η is not a Ne-gαg-continuous.

Definition 4.10: Let η be a function on $NTS\ \mathcal{U}$ and valued in $TS\ \mathcal{V}$. Then, we named η as neutrosophic generalized αg-irresolute and shortly wrote it as Ne-gαg-irresolute if for each Ne-gαgCS $\mathcal{K}$ in $\mathcal{V}$, $\eta^{-1}(\mathcal{K})$ is a Ne-gαgCS in $\mathcal{U}$.

Theorem 4.11: Let η be a function on $NTS\ \mathcal{U}$ and valued in $TS\ \mathcal{V}$. Afterward, η remains a Ne-gαg-irresolute function iff for each Ne-gαgOS $\mathcal{K}$ in $\mathcal{V}$, $\eta^{-1}(\mathcal{K})$ is a Ne-gαgOS in $\mathcal{U}$.

Proof: Let Ne-gαgOS $\mathcal{K}$ and Ne-gαgCS $\bar{\mathcal{K}}$ are in $\mathcal{V}$. Therefore, $\eta^{-1}(\bar{\mathcal{K}}) = \overline{(\eta^{-1}(\mathcal{K}))}$ remains a Ne-gαgCS in $\mathcal{U}$. Consequently, $\eta^{-1}(\mathcal{K})$ exists a Ne-gαgOS in $\mathcal{U}$. The reverse proof is evident.

Proposition 4.12: For all Ne-gαg-irresolute functions are Ne-gαg-continuous.

Proof: Let Ne-CS $\mathcal{K}$ be in $NTS\ \mathcal{V}$ and Ne-gαg-irresolute function η defined on $NTS\ \mathcal{U}$ and valued in $TS\ \mathcal{V}$. So, we have $\mathcal{K}$ stands a Ne-gαgCS in $\mathcal{V}$ by theorem (3.3) part (i). By definition of Ne-g α g-irresolute function, $\eta^{-1}(\mathcal{K})$ stands a Ne-g α gCS in $\mathcal{U}$. As a result, η exists a Ne-gαg-continuous.

The subsequent example explains that the inverse of the overhead proposition does not work.

Example 4.13: Suppose $\mathcal{U} = \{p, q\}$ and let $\xi = \{0_N, \mathcal{B}, 1_N\}$ and $\varrho = \{0_N, \mathcal{A}, \mathcal{B}, 1_N\}$, where $\mathcal{A} = \langle u, (0.6,0.7), (0.4,0.3), (0.5,0.2)\rangle$ and $\mathcal{B} = \langle u, (0.5,0.5), (0.5,0.4), (0.6,0.5)\rangle$ are the neutrosophic sets, then $(\mathcal{U}, \xi)$ and $(\mathcal{U}, \varrho)$ are NTSs. Define $\eta: (\mathcal{U}, \xi) \longrightarrow (\mathcal{U}, \varrho)$ as a $\eta(p) = q$ and $\eta(q) = p$. Then η is Ne-gαg-continuous. But $\mathcal{C} = \langle u, (0.5,0.5), (0.6,0.4), (0.5,0.7)\rangle$ is a Ne-gαgCS in $(\mathcal{U}, \varrho)$, $\eta^{-1}(\mathcal{C})$ is not a Ne-gαgCS in $(\mathcal{U}, \xi)$. Thus η is not a Ne-gαg-irresolute.

Definition 4.14: We called a $NTS\ \mathcal{U}$ with a neutrosophic $T_{\frac{1}{2}}$-space if for each Ne-gCS in $\mathcal{U}$ is a Ne-CS and we denoted it by Ne-$T_{\frac{1}{2}}$-space.

Definition 4.15: We called a $NTS\ \mathcal{U}$ with a neutrosophic $T_{g\alpha g}$-space if for each Ne-gαgCS in $\mathcal{U}$ is a Ne-CS and we denoted by Ne-$T_{g\alpha g}$-space.

Proposition 4.16: Every Ne-$T_{\frac{1}{2}}$-space stands a Ne-$T_{g\alpha g}$-space.

Proof: Let $\mathcal{C}$ be a Ne-gαgCS in Ne-$T_{\frac{1}{2}}$-space $\mathcal{U}$. By theorem (3.3) part (ii), we obtain $\mathcal{C}$ is a Ne-gCS. By definition of Ne-$T_{\frac{1}{2}}$-space, we reach to that $\mathcal{C}$ is a Ne-CS in $\mathcal{U}$. Therefore, $\mathcal{U}$ endures a Ne-$T_{g\alpha g}$-space.

Theorem 4.17: Let η_1 be a Ne-gαg-continuous function on $NTS\ \mathcal{U}$ and valued in $NTS\ \mathcal{V}$ and let η_2 be a Ne-g-continuous function on $NTS\ \mathcal{V}$ and valued in $TS\ \mathcal{W}$. If $\mathcal{V}$ is a Ne-$T_{\frac{1}{2}}$-space, then $\eta_2 \circ \eta_1$ is a Ne-gαg-continuous function.

Proof: Assume Ne-CS $\mathcal{K}$ is in $\mathcal{W}$. Meanwhile, we have a Ne-g-continuous function η_2 defined on a Ne-$T_{\frac{1}{2}}$-space $\mathcal{V}$, then ${\eta_2}^{-1}(\mathcal{K})$ stands a Ne-CS in $\mathcal{V}$. Subsequently, we also see a Ne-gαg-continuous function η_1 defined on $\mathcal{U}$, then ${\eta_1}^{-1}({\eta_2}^{-1}(\mathcal{K}))$ stands a Ne-gαgCS in $\mathcal{U}$. Therefore, $\eta_2 \circ \eta_1$ stands a Ne-gαg-continuous.

Theorem 4.18: Let η be a function on $NTS\ \mathcal{U}$ and valued in $TS\ \mathcal{V}$, we have the following results:

(i) If $NTS\ \mathcal{U}$ stands a Ne-$T_{\frac{1}{2}}$-space then the function η becomes a Ne-g-continuous iff it considers a a Ne-gαg-continuous.

(ii) If $NTS\ \mathcal{U}$ stands a Ne-$T_{g\alpha g}$-space then the function η becomes a Ne-continuous iff it considers a Ne-gαg-continuous.

Proof:

(i) Let Ne-CS $\mathcal{K}$ be in $\mathcal{V}$ and η be a Ne-g-continuous function. By definition of Ne-g-continuous, $\eta^{-1}(\mathcal{K})$ is a Ne-gCS in $\mathcal{U}$. Besides, the definition of Ne-$T_{\frac{1}{2}}$-space states $\eta^{-1}(\mathcal{K})$ is a Ne-CS. So, $\eta^{-1}(\mathcal{K})$ is a Ne-gαgCS in $\mathcal{U}$ by theorem (3.3) part (i). Therefore, η is a Ne-gαg-continuous.

On the contrary, let Ne-CS $\mathcal{K}$ be in $\mathcal{V}$ and let η be a Ne-gαg-continuous. By definition of Ne-gαg-continuous, $\eta^{-1}(\mathcal{K})$ is a Ne-gαgCS in $\mathcal{U}$. Besides, we have $\eta^{-1}(\mathcal{K})$ is a Ne-gCS in $\mathcal{U}$ by theorem (3.3) part (ii). Therefore, η is a Ne-g-continuous.

(ii) Let Ne-CS $\mathcal{K}$ be in $\mathcal{V}$ and let η be a Ne-continuous. By definition of Ne-continuous, $\eta^{-1}(\mathcal{K})$ is a Ne-CS in $\mathcal{U}$. So, we have $\eta^{-1}(\mathcal{K})$ is a Ne-gαgCS in $\mathcal{U}$ by theorem (3.3) part (i). Therefore, η is a Ne-gαg-continuous.

On the contrary, let Ne-CS $\mathcal{K}$ be in $\mathcal{V}$ and let η be a Ne-gαg-continuous. Besides, we have $\eta^{-1}(\mathcal{K})$ is a Ne-gαgCS in $\mathcal{U}$. Furthermore, the definition of Ne-$T_{g\alpha g}$-space gives $\eta^{-1}(\mathcal{K})$ is a Ne-CS in $\mathcal{U}$. Therefore, η is a Ne-continuous.

Remark 4.19: The subsequent illustration indicates the relative among the various kinds of Ne-continuous functions:

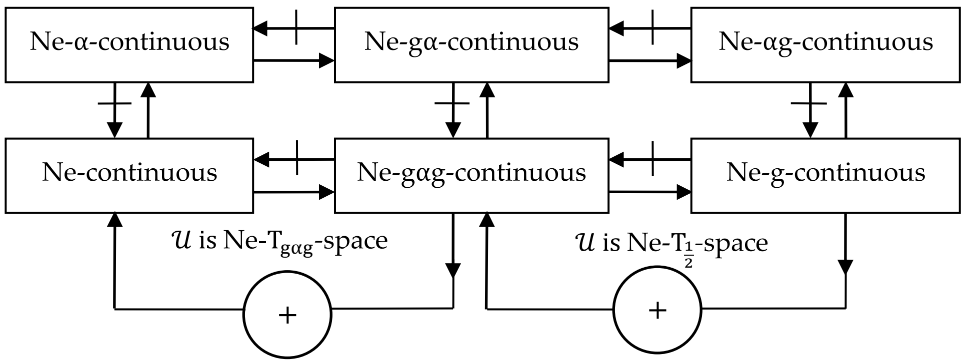

Fig. 4.1

5. Conclusion

The class of Ne-gαgCS described employing Ne-αgCS structures a neutrosophic topology and deceptions between the classes of Ne-CS and Ne-gCS. We as well illustration Ne-gαg-continuous functions by applying Ne-gαgCS. The Ne-gαgCS know how to be developed to establish another neutrosophic homeomorphism.

Funding: This work does not obtain any external grant.

Acknowledgments: The authors are highly grateful to the Referees for their constructive suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

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Received: Apr 25, 2020. Accepted: July 5 2020

Neutrosophic Sets and Systems, Vol. 35, 2020
University of New Mexico

Generalized neutrosophic b-open sets in neutrosophic topological space

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Abstract: The purpose of the study is to introduce the notion of generalized neutrosophic *b*-open set in neutrosophic topological space. We define generalized neutrosophic *b*-open set, generalized neutrosophic *b*-interior, generalized neutrosophic *b*-closure and investigate some of their properties. By defining generalized neutrosophic *b*-open set, we prove some theorems on neutrosophic topological spaces. We also furnish some suitable examples.

Keywords: Neutrosophic set; neutrosophic b-open set; generalized neutrosophic b-open set; generalized neutrosophic *b*-interior; generalized neutrosophic *b*-closure

1. Introduction

Smarandache (1998) grounded the Neutrosophic Set (NS) in 1998. From then it became very popular and attracted many researchers' attention for theoretical and practical researches (Broumi et al., 2018; Khalid, 2020; Peng & Dai, 2018; Pramanik, 2013; 2016a; 2016b; 2020; Pramanik & Mallick, 2018; 2019; Pramanik & Mondal, 2016; Pramanik & Roy, 2014; Smarandache & Pramanik, 2016; 2018, Biswas, Pramanik & Giri, 2014; 2016a; 2016b; Dalapati et al., 2017; Dey, Pramanik, & Giri, 2016a; 2016b; Pramanik, Mallick, & Dasgupta, 2018; Mondal & Pramanik, 2015; Pramanik & Dalapati, 2018, Pramanik, Dey, & Smarandache, 2018; Pramanik, Mondal, & Smarandache, 2016a; 2016b).

Salama and Alblowi (2012a) grounded the "Neutrosophic Topological Space" (NTS). Salama and Alblowi (2012b) also presented generalized NS and generalized NTSs. Salama, Smarandache, & Alblowi (2014) studied the concept of neutrosophic crisp topological space. Arokiarani, Dhavaseelan, Jafari, and Parimala (2017) defined neutrosophic semi-open functions and established relation between them. Iswaraya and Bageerathi (2016) studied neutrosophic semi-closed set and neutrosophic semi-open set. Rao and Srinivasa (2017) introduced neutrosophic pre-open set and pre-closed set. Dhavaseelan and Jafari (2018) studied generalized neutrosophic closed sets. Pushpalatha and Nandhini (2019) defined the neutrosophic generalized closed sets in NTSs. Shanthi, Chandrasekar, Safina, and Begam (2018) presented the neutrosophic generalized semi closed sets in

NTSs. Ebenanjar, Immaculate, and Wilfred (2018) studied neutrosophic *b* -open sets in NTSs. Maheswari, Sathyabama, and Chandrasekar (2018) studied the neutrosophic generalized b- closed sets in NTSs.

Research gap: No investigation on neutrosophic generalized *b*-open set has been reported in the recent literature.

Motivation: In order to fill the research gap, we introduce neutrosophic generalized b-open set.

Remaining of the paper is designed as follows:

Section 2 recalls of NTS, neutrosophic b- closed sets and a theorem. Section 3 introduces neutrosophic generalized *b*-open set and proofs of some theorems on neutrosophic *b*-open sets. Section 4 presents concluding remarks.

2. Preliminaries and some properties

Definition 2.1 Assume that (W,τ) is an NTS. Then χ, an *NS* over W is said to be a Neutrosophic b-Open (N-*b*-open) set (Ebenanjar, Immaculate, & Wilfred, 2018) if and only if (iff) $\chi \subseteq N_{int}(N_{cl}(\chi)) \cup N_{cl}(N_{int}(\chi))$.

Definition 2.2 In an NTS (W,τ), an *NS* χ is said to be a Neutrosophic b-Closed (N-*b*-closed) set (Ebenanjar, Immaculate, & Wilfred, 2018) iff $\chi \supseteq N_{int}(N_{cl}(\chi)) \cap N_{cl}(N_{int}(\chi))$.

Remark 2.1 An NS χ over W is said to be an N-*b*-closed set (Ebenanjar, Immaculate, & Wilfred, 2018) in (W,τ) iff χ^c is a N-*b*-open set in (W,τ).

In 2018, Ebenanjar, Immaculate, and Wilfred (2018) studied the concept of N-*b*-open set in NTS but they did not check whether the union or intersection of two N-*b*-open sets (N-*b*-closed sets) is again an N-*b*-open set (N-*b*-closed set) or not. In this paper we show some results on the intersection and union of neutrosophic *b*-closed sets.

Theorem 2.1 The intersection of any two N-*b*-closed sets is again an N-b-closed set.

Proof. Assume that *E*, *F* be any two N-*b*-closed sets in an *NTS* (W,τ). Then we have

$$E \supseteq N_{int}(N_{cl}(E)) \cap N_{cl}(N_{int}(E)) \quad (1)$$

and $F \supseteq N_{int}(N_{cl}(F)) \cap N_{cl}(N_{int}(F))$ (2)

For any two NSs *E* and *F* We know that $E\cap F \subseteq E$ and $E\cap F \subseteq F$.

Now $E\cap F \subseteq E \Rightarrow N_{int}(E\cap F) \subseteq N_{int}(E) \Rightarrow N_{cl}(N_{int}(E\cap F)) \subseteq N_{cl}(N_{int}(E))$ (3)

$E\cap F \subseteq E \Rightarrow N_{cl}(E\cap F) \subseteq N_{cl}(E) \Rightarrow N_{int}(N_{cl}(E\cap F)) \subseteq N_{int}(N_{cl}(E))$ (4)

$E\cap F \subseteq F \Rightarrow N_{int}(E\cap F) \subseteq N_{int}(F) \Rightarrow N_{cl}(N_{int}(E\cap F)) \subseteq N_{cl}(N_{int}(F))$ (5)

$E\cap F \subseteq F \Rightarrow N_{cl}(E\cap F) \subseteq N_{cl}(F) \Rightarrow N_{int}(N_{cl}(E\cap F)) \subseteq N_{int}(N_{cl}(F))$ (6)

From (1) and (2) we have,

$$E\cap F \supseteq N_{int}(N_{cl}(E)) \cap N_{cl}(N_{int}(E)) \cap N_{int}(N_{cl}(F)) \cap N_{cl}(N_{int}(F))$$

$$\supseteq N_{int}(N_{cl}(E\cap F)) \cap N_{cl}(N_{int}(E\cap F)) \cap N_{int}(N_{cl}(E\cap F)) \cap N_{cl}(N_{int}(E\cap F))$$

[by eqs (3), (4), (5) & (6)]

$$= N_{int}(N_{cl}(E\cap F)) \cap N_{cl}(N_{int}(E\cap F))$$

$\Rightarrow E\cap F \supseteq N_{cl}(N_{int}(E\cap F)) \cap N_{int}(N_{cl}(E\cap F))$.

Therefore $E\cap F$ is an N-*b*-closed set.

Hence the intersection of any two N-*b*-closed sets is again an N-*b*-closed set.

Remark 2.2: The union of any two N-*b*-closed sets may not be an N-*b*-closed set. This is proved as follows:

Example 2.1: Assume that $W=\{p_1, p_2\}$ and τ = {0_N, 1_N, {(p_1, 0.5, 0.2, 0.4), (p_2, 0.6, 0.1, 0.3)}, {(p_1, 0.3, 0.5, 0.6), (p_2, 0.4, 0.4, 0.5)}} be the family of some NSs over W. Then τ is an NT on W. Now it can be verified that *E*= {(*a*, 0.6, 0.5, 0.6), (*b*, 0.5, 0.6, 0.7)}, *F*={(*a*,1, 0, 1), (*b*, 0.9, 0.1, 0.1)} are two N-*b*-closed sets in (W,τ). But their union *E*∪*F* = {(*a*, 1, 0, 0.6), (*b*, 0.9, 0.1, 0.1)} is not an N-*b*-closed set.

Definition 2.3 Assume that (W,τ) is an NTS and χ is an *NS* over W. Then the Neutrosophic *b*-Closure (N_{bcl}) and Neutrosophic *b*-Interior (N_{bint}) (Ebenanjar, Immaculate & Wilfred, 2018) of χ are defined by

$N_{bcl}(\chi) = \cap\{\psi: \psi$ is an N-*b*-closed set in (W,τ) and $\chi\subseteq\psi\}$;

$N_{bint}(\chi) = \cup\{\xi: \xi$ is an N-*b*-open set in (W,τ) and $\xi\subseteq\chi\}$.

Remark 2.3 Clearly $N_{bint}(\chi)$ is the largest N-*b*-open set (Ebenanjar, Immaculate, & Wilfred, 2018) in (W,τ) which is contained in χ and $N_{bcl}(\chi)$ is the smallest N-*b*-closed set in (W,τ) which contains χ.

Definition 2.4 Assume that (W,τ) is an NTS. A neutrosophic subset *E* of (W,τ) is said to be a Neutrosophic Generalized Closed Set (NGCS) (Dhavaseelan & Jafari, 2018) if $N_{cl}(E)\subseteq F$ whenever $E\subseteq F$ and *F* is an NOS. A subset *K* of (W,τ) is called Neutrosophic Generalized Open Set (NGOS) iff K^c is an NGCS in (W,τ).

3. Generalized neutrosophic *b*-open set

Definition 3.1 Assume that (W,τ) is an NTS. An NS G over W is called a Generalized Neutrosophic *b*-Open (g-N-*b*-open) set if ∃ an N-*b*-closed set *H* (except 1_N) with $G\subseteq H$ such that $G\subseteq N_{int}(H)$. A *neutrosophic subset K* in (W,τ) is called a Generalized Neutrosophic *b*-Closed (g-N-*b*-closed) set iff K^c is a g-N-*b*-open set in (W,τ).

Example 3.1 Assume that $W=\{p_1, p_2\}$ and τ={0_N, 1_N, {(p_1, 0.5, 0.6, 0.7), (p_2,0.6, 0.7, 0.8)}, {(p_1,0.6, 0.5, 0.6), (p_2,0.7, 0.6, 0.7)}} are the collection of some *NSs* over W. Then (W,τ) is clearly an NTS. Here *K* = {(p_1, 0.6, 0.7, 0.8), (p_2, 0.5, 0.8, 0.8)} is a g-N-b-open set, because there exists an N-*b*-closed set *G* = { p_1, 0.7, 0.3, 0.4), (p_2, 0.8, 0.3, 0.4)} in (W,τ) with $K\subseteq G$ such that $K\subseteq N_{int}(G)$.

Proposition 3.1 In an NTS (W,τ), 0_N is a g-N-*b*-open set but 1_N is not a g-N-*b*-open set.

Proof. Assume that (W,τ) is an NTS. Since a Neutrosophic Open Set (NOS) is an N-*b*-open set, so 1_N is an N-*b*-open set. Therefore, 0_N is an N-*b*-closed set (since it is the complement of N-*b*-open set 1_N).

Now $0_N\subseteq 0_N$ and $0_N \subseteq N_{int}(0_N)= 0_N$.

Thus there exist an N-*b*-closed set 0_N (except 1_N) with $0_N \subseteq 0_N$ such that $0_N \subseteq N_{int}(0_N)$. Hence 0_N is a g-N-*b*-open set in (W,τ).

But in case of NS 1_N, we cannot find any neutrosophic *b*-closed set *H* (except 1_N) with $1_N\subseteq H$ such that $1_N\subseteq N_{int}(H)$. Hence 1_N is not a g-N-*b*-open set in (W,τ).

Proposition 3.2 Assume that ψ is a g-N-*b*-open set in an NTS (W,τ). Then, every NS contained in ψ is a g-N-*b*-open set.

Proof. Assume that ψ be a g-N-b-open set in an NTS (W,τ) and ξ be any arbitrary NS over W which is contained in ψ. Since ψ is a g-N-b-open set, so there exists an N-b-closed set η (except 1_N) with $\psi \subseteq \eta$ such that $\psi \subseteq N_{int}(\eta)$.

Now ξ is contained in *A*, so

$$\xi \subseteq \psi$$

$$\Rightarrow \xi \subseteq \psi \subseteq \eta \;\&\; \xi \subseteq \psi \subseteq N_{int}(\eta).$$

Therefore there exists an N-b-closed set η (except 1_N) with $\xi \subseteq \eta$ such that $\xi \subseteq N_{int}(\eta)$. Hence ξ is a g-N-b-open set. Thus each NS contained in ψ is again a g-N-b-open set in (W,τ).

Definition 3.2 Assume that (W,τ) is an NTS and ψ be an NS over W. Then the Generalized Neutrosophic b-Interior ($g\text{-}N_{bint}$) and Generalized Neutrosophic b- Closure ($g\text{-}N_{bcl}$) of ψ are defined by

$$g\text{-}N_{bint}(\psi) = \cup\{\xi : \xi \text{ is a g-N-}b\text{-open set and } \xi \subseteq \psi\};$$

$$g\text{-}N_{bcl}(\psi) = \cap\{\eta : \eta \text{ is a g-N-}b\text{-closed set and } \psi \subseteq \eta\}.$$

Theorem 3.1 Assume that (W,τ) is an NTS. Then each neutrosophic open subset of (W,τ) is a g-N-b-open set.

Proof. Assume that ψ be an arbitrary NOS in an NTS (W,τ). So $\psi = N_{int}(\psi)$. Since each neutrosophic closed set is an N-b-closed set so $N_{cl}(\psi)$ is an N-b-closed set. Also we know that $\psi \subseteq N_{cl}(\psi)$.

Now $\psi \subseteq N_{cl}(\psi)$

$\Rightarrow N_{int}(\psi) \subseteq N_{int}(N_{cl}(\psi))$

$\Rightarrow \psi = N_{int}(\psi) \subseteq N_{int}(N_{cl}(\psi))$

$\Rightarrow \psi \subseteq N_{int}(N_{cl}(\psi))$

Therefore there exists an N-b-closed set $N_{cl}(\psi)$ with $\psi \subseteq N_{cl}(\psi)$ such that $\psi \subseteq N_{int}(N_{cl}(\psi))$. Hence ψ is a g-N-b-open set in (W,τ). Thus each neutrosophic open subset of (W,τ) is again a g-N-b-open set.

Remark 3.1 The converse of the theorem 3.1 is not true. This can be shown by the example 3.2.

Example 3.2 In example 3.1, it can be easily seen that $K = \{(a, 0.6, 0.7, 0.8), (b, 0.5, 0.8, 0.8)\}$ is a g-N-b-open set in (W,τ) but it is not an NOS.

Theorem 3.2 Assume that (W,τ) is an NTS. Then each Neutrosophic Pre-Open Set (NPOS) in (W,τ) is a g-N-b-open set.

Proof. Assume that (W,τ) is an NTS and ψ is an NPOS. Then $\psi \subseteq N_{int}(N_{cl}(\psi))$. Since for any NS ψ, $N_{cl}(\psi)$ is an N-b-closed set and $\psi \subseteq N_{cl}(\psi)$. Therefore there exists an N-b-closed set $N_{cl}(\psi)$ with $\psi \subseteq N_{cl}(\psi)$ such that $\psi \subseteq N_{int}(N_{cl}(\psi))$. Hence ψ is a g-N-b-open set in (W,τ). Thus each NPOS in (W,τ) is again a g-N-b-open set.

Theorem 3.3 If ψ is both NOS and Neutrosophic Semi-Open Set (NSOS) in an NTS (W,τ) then it is a g-N-b-open set.

Proof. Assume that (W,τ) is an NTS and ψ is both NSOS and NOS. Since ψ is an NOS, so $\psi = N_{int}(\psi)$. Again since ψ is an NSOS, so $\psi \subseteq N_{cl}(N_{int}(\psi))$. It can be verified that $N_{cl}(N_{int}(\psi))$ is an N-*b*-closed set (since it is an NCS).

Now $\psi \subseteq N_{cl}(N_{int}(\psi))$

$\Rightarrow N_{int}(\psi) \subseteq N_{int}(N_{cl}(N_{int}(\psi)))$ [since $\psi \subseteq \delta \Rightarrow N_{int}(\psi) \subseteq N_{int}(\delta)$]

$\Rightarrow \psi = N_{int}(\psi) \subseteq N_{int}(N_{cl}(N_{int}(\psi)))$ [since $\psi = N_{int}(\psi)$]

$\Rightarrow \psi \subseteq N_{int}(N_{cl}(N_{int}(\psi)))$

Therefore there exists an N-*b*-closed set $N_{cl}(N_{int}(\psi))$ with $\psi \subseteq N_{cl}(N_{int}(\psi))$ in (W,τ) such that $\psi \subseteq N_{int}(N_{cl}(N_{int}(\psi)))$. Hence ψ is a g-N-*b*-open set.

Theorem 3.4 Assume that (W,τ) is an NTS and ψ is both neutrosophic α-open and neutrosophic open set. Then ψ is again a g-N-*b*-open set.

Proof. Assume that ψ is an arbitrary NS which is both neutrosophic α-open set and NOS. Since ψ is an NOS so $\psi = N_{int}(\psi)$. Again since ψ is a neutrosophic α-open set, so $\psi \subseteq N_{int}(N_{cl}(N_{int}(\psi)))$. Hence, it is clear that $N_{cl}(N_{int}(\psi))$ is an N-*b*-closed set (since it is an NCS) in (W,τ).

Now $\psi = N_{int}(\psi)$

$\Rightarrow \psi = N_{int}(\psi) \subseteq N_{cl}(N_{int}(\psi))$

$\Rightarrow \psi \subseteq N_{cl}(N_{int}(\psi))$

Therefore there exists an N-*b*-closed set $N_{cl}(N_{int}(\psi))$ with $\psi \subseteq N_{cl}(N_{int}(\psi))$ such that $\psi \subseteq N_{int}(N_{cl}(N_{int}(\psi)))$. Hence ψ is a generalized N-*b*-open set in (W,τ).

Theorem 3.5 The intersection of any two g-N-*b*-open sets in an NTS (W,τ) is again a g-N-*b*-open set.

Proof. Let ψ and ξ be any two g-N-*b*-open sets in an NTS (W,τ). Then there exist two N-*b*-closed sets *K, L* with $\psi \subseteq K$, $\xi \subseteq L$ such that $\psi \subseteq N_{int}(K)$ and $\xi \subseteq N_{int}(L)$.

Here $\psi \cap \xi \subseteq K \cap L$.

We know that the intersection of two N-*b*-closed sets is again an N-*b*-closed set. So $K \cap L$ is an N-*b*-closed set in (W,τ).

Now $\psi \cap \xi \subseteq N_{int}(K) \cap N_{int}(L)$ [since $\psi \subseteq N_{int}(K)$, $\xi \subseteq N_{int}(L)$]

$= N_{int}(K \cap L)$

$\Rightarrow \psi \cap \xi \subseteq N_{int}(K \cap L)$.

Therefore there exists an N-*b*-closed set $K \cap L$ with $\psi \cap \xi \subseteq K \cap L$ such that $\psi \cap \xi \subseteq N_{int}(K \cap L)$. Hence $\psi \cap \xi$ is a g-N-*b*-open set in (W,τ). Thus the intersection of any two g-N-*b*-open sets in (W,τ) is again a g-N-*b*-open set.

Theorem 3.6 The union of two g-N-*b*-open sets is a g-N-*b*-open set if one is contained in the other.

Proof. Let ψ, ξ are any two g-N-*b*-open sets in (W,τ) such that $\psi \subseteq \xi$. Since ψ and ξ are g-N-*b*-open sets, so there exist two N-*b*-closed sets G_1, G_2 with $\psi \subseteq G_1$ and $\xi \subseteq G_2$ such that $\psi \subseteq N_{int}(G_1)$ and $\xi \subseteq N_{int}(G_2)$.

Now $\psi \cup \xi \subseteq \xi$ [since $\psi \subseteq \xi$]

$\subseteq G_2$

$\Rightarrow \psi \cup \xi \subseteq G_2$

Again $\psi \cup \xi \subseteq \xi \subseteq N_{int}(G_2)$, where G_2 is an N-*b*-closed set in (X,τ).

Therefore there exists an N-b-closed set G_2 with $\psi \cup \xi \subseteq G_2$ in (X,τ) such that $\psi \cup \xi \subseteq N_{int}(G_2)$.

Hence the union of two g-N-b-open sets is again a g-N-open set if one is contained in the other.

Definition 3.3 An NS χ is called a g-N-b-open set relative to an NS ψ if there exists an N-b-closed set ξ with $\chi \subseteq \psi \cap \xi$ such that $\chi \subseteq N_{int}(\psi \cap \xi)$.

Theorem 3.7 Assume that (W,τ) is an NTS. If ξ is a g-N-b-open set relative to ψ and ψ is a g-N-b-open set relative to χ then ξ is a g-N-b-open set relative to χ.

Proof. Since ξ is a g-N-b-open set relative to ψ so there exists an N-b-closed set K with $\xi \subseteq \psi \cap K$ such that $\xi \subseteq N_{int}(\psi \cap K)$. Similarly, since ψ is a g-N-b-open set relative to χ then there exists an N-b-closed set L with $\psi \subseteq \chi \cap L$ such that $\psi \subseteq N_{int}(\chi \cap L)$.

We know that the intersection of two N-b-closed sets is again an N-b-closed set. So $K \cap L$ is an N-b-closed set.

Now $\xi \subseteq \psi \cap K \subseteq \chi \cap L \cap K$

$= \chi \cap (L \cap K)$

$= \chi \cap G$, where $G = K \cap L$ is an N-b-closed set.

Again $\xi \subseteq N_{int}(\psi \cap K)$

$\subseteq N_{int}(\chi \cap G)$.

Therefore there exists an N-b-closed set G with $\xi \subseteq \chi \cap G$ such that $\xi \subseteq N_{\text{int}}(\chi \cap G)$

Hence ξ is a g-N- b-open relative to χ.

4. Conclusion

In this article, we introduce generalized neutrosophic b-open set, generalized neutrosophic b-interior, generalized neutrosophic b-closure and investigate some of their properties. By defining generalized neutrosophic b-open set, we prove some theorems on NTSs and few illustrative examples are provided. In the future, we hope that based on these notions in NTSs, many new investigations can be carried out.

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Received: Apr 20, 2020. Accepted: July 15 2020

Neutrosophic Sets and Systems, Vol. 35, 2020
University of New Mexico

Neutrosophic Soft Fixed Points

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Abstract. In a wide spectrum of mathematical issues, the presence of a fixed point (FP) is equal to the presence of a appropriate map solution. Thus in several fields of math and science, the presence of a fixed point is important. Furthermore, an interesting field of mathematics has been the study of the existence and uniqueness of common fixed point (CFP) and coincidence points of mappings fulfilling the contractive conditions. Therefore, the existence of a FP is of significant importance in several fields of mathematics and science. Results of the FP, coincidence point (CP) contribute conditions under which maps have solutions. The aim of this paper is to explore these conditions (mappings) used to obtain the FP, CP and CFP of a neutrosophic soft set. We study some of these mappings (conditions) such as contraction map, L-lipschitz map, non-expansive map, compatible map, commuting map, weakly commuting map, increasing map, dominating map, dominated map of a neutrosophic soft set. Moreover we introduce some new points like a coincidence point, common fixed point and periodic point of neutrosophic soft mapping. We establish some basic results, particular examples on these mappings and points. In these results we show the link between FP and CP. Moreover we show the importance of mappings for obtaining the FP, CP and CFP of neutrosophic soft mapping.

Keyword. Neutrosophic set, fuzzy neutrosophic soft mapping, fixed point, coincidence point.

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1. Introduction

It is well known fact that fuzzy sets (FS) [1], complex fuzzy sets (CFS) [2], intuitionistic fuzzy sets (IFSs), the soft sets [3], fuzzy soft sets (FSS) and the fuzzy parameterized fuzzy soft sets (FPFS-sets) [4], [5] have been used to model the real life problems in various fields like in medical science, environments, economics, engineering, quantum physics and psychology etc.
In 1965, L. A. Zadeh [1] introduced a FS, which is the generalization of a crisp set. A grade value of a crisp set is either 1 or 0 but a grade value of fuzzy set has all the values in closed interval $[0,1]$. A FS plays a central role in modeling of real world problems. There are a lot of applications of FS theory in various branches of science such as in engineering, economics, medical science, mathematical chemistry, image processing, non-equilibrium thermodynamics etc. The concept for IFSs is provided in [3] which are generalizations of FS. An IFS P can be expressed as $P=\{\langle \nu,, \beta_P(\nu), \gamma_P(\nu)\rangle : \nu \in X\}$, where $\beta_P(\nu)$ represents the degree of membership, $\gamma_P(\nu)$ represents the degree of non-membership of the element $\nu \in X$. FPFS-sets is the extension of a FS and soft set proposed in [4], [5]. FPFS-sets maintain a proper degree of membership to both elements and parameters.
The notion of a complex CFS, the extension of the fuszy set, was introduced by Ramot et, al., [2]. A CFS membership function has all the values in the unit disk. A complex fuzzy set is used for representing two-dimensional phenomena and plays an important role in periodic phenomena. Complex fuzzy set is used in signals and systems to identify a reference signal out of large signals detected by a digital receiver. Moreover it is used for expressing complex fuzzy solar activity (solar maximum and solar minimum) through the average number

of sunspot.
Smarandache [6], [7] has given the notion of a neutrosophic set (NS). A NS is the extension of a crisp set, FS and IFS. In NS, truth membership (TM), falsity membership (FM) and indeterminacy membership (IM) are independent. In decision-making problems, the indeterminacy function is very significant. A NS and its extensions plays a vital role in many fields such as decision making problems, educational problems, image processing, medical diagnosis and conflict resolution. Moreover the field of neutrosophic probability, statistics, measures and logic have been developed in [8]. The generalization of fuzzy logic (FL) has been suggested by Smarandache in [8] and is termed as neutrosophic logic (NL). A proposition in NL is true (t), indeterminate (i) and false (f) are real values from the ranges T, I, F. T, I, F and also the sum of t, i, f are not restricted. In neutrosophic logic, there is indeterminacy term, which have no other logics, such as intuitionistic logic (IL), FL, boolean logic (BL) etc. Neutrosophic probability (NP) [8] is the extension of imprecise probability and classical probability. In NP, the chance occurs by an event is $t\%$ true, $i\%$ indeterminate and $f\%$ false where t, i, f varies in the subsets T, I and F respectively. Dynamically these subsets are functions based on parameters, but they are subsets on a static basis. In NP $n\_\sup \leq 3^{+}$, while in classical probability $n\_\sup \leq 1$. The extension of classical statistics is neutrosophic statistics [8] which is the analysis of events described by NP. There are twenty seven new definitions derived from NS, neutrosophic statistics and a neutrosophic probability. Each of these are independent. The sets derived from NS are intuitionistic set, paradoxist set, paraconsistent set, nihilist set, faillibilist set, trivialist set, and dialetheist set. Intuitionistic probability and statistics, faillibilist probability and statistics,tautological probability and statistics, dialetheist probabilityand statistics, paraconsistent probability and statistics, nihilist probability and statistics and trivialist probability and statistics are derived from neutosophic probability and statistics. N. A. Nabeeh [9] suggested a technique that would promote a personal selection process by integrating the neutrosophic analytical hierarchy process to show the ideal solution among distinct options with order preference tevhnique similar to an ideal solution (TOPSIS). M. A. Baset [10] introduced a new type of neutrosophy technique called type 2 neutrosophic numbers. By combining type 2 neutrosophic number and TOPSIS, they suggested a novel method T2NN-TOPSIS which is very useful in group decision making. They researched a multi criteria group decision making technique of the analytical network process method and Visekriterijusmska Optmzacija I Kommpromisno Resenje method under neutrosophic environment that deals high order imprecision and incomplete information [11]. M. A. Baet suggested a new strategy for estimating the smart medical device selecting process in a GDM in a vague decision environment. Neutrosophic with TOPSIS strategy is used in decision-making processes to deal with incomplete information, vagueness and uncertainty, taking into account the decision requirements in the information gathered by decision-makers [12]. They suggested the robust ranking method with NS to manage supply chain management (GSCM) performance and methods that have been widely employed to promote environmental efficiency and gain competitive benefits. The NS theory was used to manage imprecise understanding, linguistic imprecision, vague data and incomplete information [13]. Moreover M. A. Baset [14] et, al., used NS for assessment technique and decision-making to determine and evaluate the factors affecting supplier selection of supply chain management. T. Bera [15] et, al., defined a neutrosophic norm on a soft linear space known as neutrosophic soft linear space. They also modified the concept of neutrosophic soft (Ns) prime ideal over a ring. They presented the notion of Ns completely semi prime ideals, Ns completely prime ideals and Ns prime K-ideals [16]. Moreover T. Bera [17] introduced the concept of compactness and connectedness on Ns topological space along with their several characteristics. R. A. Cruz [18] et, al., discussed P-intersection, P- union, P-AND and P-OR of neutrosophic cubic sets and their related properties. N. Shah [19] et, al., studied neutrosophic soft graphs. They presented a link between neutorosophic soft sets and graphs. Moreover they also discussed the notion of strong neutosophic soft graphs.
Smarandache [20] discussed the idea of a single valued neutrosophic set (SVNS). A SVNS defined as for any

space of points set U' with u in U', a SVNS W in U', the truth membership, false memebership and indeterminac membership functions denoetd as T_A, F_A and I_A respectively with $T_A, F_A, I_A \in [0,1]$ for each u in U'. A SVNS W is expressed as $W = \int_X \langle T_W(\nu), I_W(\nu), F_W(\nu)\rangle / \nu, \nu \in X,$ when X is continous. For a discrete case, a SVNS can be expressed as $W = \sum_{i=1}^{n} \langle T(\nu i), I(\nu i), F(\nu i)\rangle / \nu i,\ \nu i \in X.$ Later, Maji [21] gave a new concept neutrosophic soft set (NSS). For any initial universal set W and any parameters set E with $A \subset E$ and $P(W)$ represents all the NS of W. The order set (ϕ, A) is said to be the soft NS over W where $\phi : A \to P(W)$. Arockiarani et al., [22] introduced fuzzy neutrosophic soft topological space and presents main results of fuzzy neutrosophic soft topological space. Later on the researchers linked the above theories with different field of sciences.

The purpose of this paper is to study the mappings such as contraction mapping, expansive mapping, non-expansive mapping, commuting mapping, and weakly commuting mapping used to attain the FP, CP and CFP of a neutrosophic soft set. We present some basic resultsnd particular examples of fixed points, coincidence points, common fixed points in which contraction mapping, expansive mapping, non-expansive mapping, commuting mapping, and weakly commuting mapping are used.

2. Preliminaries

We will discuss here the basic notions of NS and neutrosophic soft sets. We will also discuss some new neutrosophic soft mappings such as contraction mapping, increasing mapping, dominated mapping, dominating mapping, K-lipschitz mapping, non-expansive mapping, commuting mapping, weakly compatible mapping. Moreover we will study periodic point, common fixed point, coinciding point of neutrosophic soft-mapping. Here $\hat{N}S(U'_E)$ is the collection of all neutrosophic soft points.

Definition 2.1 [7] Let U be any universal set, with generic element $\nu \in U'$. A NS $\hat{N}$ is defined by $\hat{N} = \{\langle \nu, T_{\hat{N}}(\nu), I_{\hat{N}}(\nu), F_{\hat{N}}(\nu)\rangle, \nu \in U'\}$, where $T, I, F : U \to \,]^-0, 1^+[$ and

$$^-0 \leq T_{\hat{N}}(\nu) + I_{\hat{N}}(\nu) + F_{\hat{N}}(\nu) \leq 3^+.$$

$T_{\hat{N}}(\nu), I_{\hat{N}}(\nu)$ and $F_{\hat{N}}(\nu)$ denote TM, IM and FM functions respectively. In $]^-0, 1^+[, 1^+ = 1 + \varepsilon,$ where ε is it's non-standard part and 1 is it's standard part. Likely $^-0 = 0 - \varepsilon,$ ε is it's non-standard part and 0 is it's standard part. It is difficult to employ these values in real life applications. Hence we take all the values of neutrosophic set from subset $[0,1]$.

Definition 2.2 [23] Let E and W be the set of parameters and initial universal set respectively. Let the power set of W is denoted by $P(W)$. Then a pair (β, A) is called soft set (SS) over W, where $A \subseteq E$ and $\beta : A \to P(W)$.

Definition 2.3 [21] Let E and W be the set of parameters and initial universal set respectively. Suppose that the set of all neutrosophic soft set (NSS) is denoted as $\hat{N}S(W)$. Then for $\mathrm{P} \subseteq E$, a pair (β, P) is called a $\hat{N}SS$ over W, where $\beta : \mathrm{P} \to \hat{N}S(W)$ is a mapping.

Definition 2.4 [24] Let E and W be the set of parameters and initial universal set respectively. Suppose that the set of all NSS is denoted as $\hat{N}S(W)$. A NSS $\hat{N}$ over W is a set which defined by a set valued function $\mathbf{P}_{\hat{N}}$ representing a mapping $\mathbf{P}_{\hat{N}} : E \to \hat{N}S(W)$. $\mathbf{P}_{\hat{N}}$ is known as approximate function of the $\hat{N}S(W)$. The neutrosophic soft set can be written as:

$$N = \{(e, \{\left\langle \upsilon, T_{\mathrm{P}_{\hat{N}}(e)}(\upsilon), I_{\mathrm{P}_{\hat{N}}(e)}(\upsilon), F_{\mathrm{P}_{\hat{N}}(e)}(\upsilon) \right\rangle \ : \ \upsilon \in W\}) : \ e \in E\}$$

where $T_{N'}(\nu), I_{N'}(\nu), F_{N'}(\nu)$ represents the TM, IM and FM functions of $\mathrm{P}_{\hat{N}}(e)$ respectively and has values in $[0,1]$. Also

$$0 \le T_{\mathrm{P}_{\hat{N}}(e)}(\nu), I_{\mathrm{P}_{\hat{N}}(e)}(\nu), F_{\mathrm{P}_{\hat{N}}(e)}(\nu) \le 3.$$

Definition 2.5 [22] Let U' be any universal set. The fuzzy neutrosophic set (fn-s) N' is defined as

$$N' = \{\langle \alpha, T_{N'}(\alpha), I_{N'}(\alpha), F_{N'}(\alpha) \rangle, \alpha \in X\}$$

where $T_{N'}(\alpha), I_{N'}(\alpha), F_{N'}(\alpha)$ represents the TM, IM and FM functions respectively and $T, I, F \ : \ N' \to [0,1]$. Also $0 \le T_{N'}(\alpha) + I_{N'}(\alpha) + F_{N'}(\alpha) \le 3.$

Definition 2.6 [22] Let *E* and *W* be the set of parameters and initial universal set respectively. Suppose that the set of all fuzzy neutrosophic soft set (FNS-set) is denoted as $\hat{FN} S(U_E^{'})$. Then for $\mathrm{P} \subseteq E$, a pair (β, P) is said to be a FNS-set over W , where $\beta \ : \ \mathrm{P} \to \hat{N} S(W)$ is a mapping.

Definition 2.7 [25] Let $\Lambda_{A'}, \Lambda_{B'}$ be two fuzzy neutrosophic soft set. An fuzzy neutrosophic soft (FNS) relation ξ from $\Lambda_{A'}$ to $\Lambda_{B'}$ is known as FNS mapping if the two conditions are fulfilled.

(i) For every $\Lambda^{\alpha}_{A_1^{'}} \in \Lambda_{A'}$, there exists $\Lambda^{\alpha}_{B_1^{'}} \in \Lambda_{B'}$, where $\Omega^{\alpha}_{A_1^{'}}, \Omega^{\alpha}_{B_1^{'}}$ are FNS elements.

(ii) For empty fuzzy FNS element in $\Lambda_{A'}$, the $\xi(\Lambda_{A'})$ is also empty FNS element.

Definition 2.8 [25] Let $\Lambda_{A'} \in FNS(W, R)$ be a FNS-set and $\phi \ : \ \Lambda_{A'} \to \Lambda_{A'}$ an FNS-mapping. A fuzzy neutrosophic element $\Lambda^{\alpha}_{A'}$ is called a fixed point of ϕ if $\phi(\Lambda^{\alpha}_{A'}) = \Lambda^{\alpha}_{A'}$.

Criterion [26], [27] Let $\hat{N} S(W)$ be the set of all neutrosophic points over (W, E). Then the neutrosophic soft metric on based of neutrosophic points is defined as $d \ : \ \hat{N} S(W_E) \to \hat{N} S(W_E)$ having the following properties.

M_1). $d(\Lambda^{\alpha}_{A'}, \Lambda^{\alpha}_{B'}) \ge 0$ for all $\Lambda^{\alpha}_{A'}, \Lambda^{\alpha}_{B'} \in \hat{N} S(W_E)$.

M_2). $d(\Lambda^{\alpha}_{A'}, \Lambda^{\alpha}_{B'}) = 0 \ \Leftrightarrow \ \Lambda^{\alpha}_{A'} = \Lambda^{\alpha}_{B'}$.

M_3). $d(\Lambda^{\alpha}_{A'}, \Lambda^{\alpha}_{B'}) = d(\Lambda^{\alpha}_{B'}, \Lambda^{\alpha}_{A'})$.

M_4). $d(\Lambda^{\alpha}_{A'}, \Lambda^{\alpha}_{B'}) \le d(\Lambda^{\alpha}_{A'}, \Lambda^{\alpha}_{C'}) + d(\Lambda^{\alpha}_{C'}, \Lambda^{\alpha}_{B'})$.

Then $(\hat{N} S(U_E^{'}), d)$ is said to be neutrosophic soft metric space. Here $\Lambda^{\alpha}_{A'} = \Lambda^{\alpha}_{B'}$ implies $T_{\Lambda^{\alpha}_{A'}} = T_{\Lambda^{\alpha}_{B'}}, I_{\Lambda^{\alpha}_{A'}} = I_{\Lambda^{\alpha}_{B'}}$ and $F_{\Lambda^{\alpha}_{A'}} = F_{\Lambda^{\alpha}_{B'}}$.

3. Mappings on Neutrosophic Soft Set

Here, we introduced some new neutrosophic soft mappings such as contraction mapping, increasing mapping, dominated mapping, dominating mapping, K-lipschitz mapping, non-expansive mapping, commuting mapping, weakly compatible mapping. Also we introduced periodic point, common fixed point, coinciding point of neutrosophic soft-mapping. Here $\hat{N} S(U_E^{'})$ is the collection of all neutrosophic soft points.

Definition 3.1 Let ϕ be a mapping from $\hat{N}S(U'_E)$ to $\hat{N}S(U'_E)$. Then ϕ is called neutrosophic soft contraction if $d(\phi(\Lambda^{\alpha}_{A'}),\phi(\Lambda^{\alpha}_{B'})) \leq kd(\Lambda^{\alpha}_{A'},\Lambda^{\alpha}_{B'})$ for all $\Lambda^{\alpha}_{A'},\Lambda^{\alpha}_{B'} \in F\hat{N}S(U'_E)$ and $k \in [0,1)$. Where k is called contraction factor.

Example 3.1 Let $U' = \{\theta_1,\theta_1,\theta_3\}$ be any initial universal set and $R = A' = B' = \{\alpha_1,\alpha_2\}$. Define a NSS $\Lambda^{\alpha}_{A'}$ and $\Lambda^{\alpha}_{B'}$ as below:

$$\begin{aligned}\Lambda^{\alpha}_{A'} = \{&(\alpha_1,\{\langle\theta_1,0.8,0.1,0.3\rangle,\langle\theta_2,0.6,0.7,0.4\rangle,\langle\theta_3,1,0.2,0.4\rangle\}),\\ &(\alpha_2,\{\langle\theta_1,0.3,0.7,0.6\rangle,\langle\theta_2,0.1,0.9,0.3\rangle,\langle\theta_3,0.1,0.8,0.7\rangle\})\}\end{aligned}$$

and

$$\begin{aligned}\Lambda^{\alpha}_{B'} = \{&(\alpha_1,\{\langle\theta_1,0.9,0.7,0.1\rangle,\langle\theta_2,1,0.8,0.6\rangle,\langle\theta_3,1,0.2,0.4\rangle\}\}),\\ &(\alpha_2,\{\langle\theta_1,0.1,0.3,0.6\rangle,\langle\theta_2,0.2,0.3,0.9\rangle,\langle\theta_3,0.1,0.8,0.7\rangle\})\}.\end{aligned}$$

The distance defined [27] as

$$d(\phi(\Lambda^{\alpha_1}_{A'}),\phi(\Lambda^{\alpha_2}_{A'})) = \min_{\theta_i}\{(|T_{\Lambda^{\alpha_1}_{B'}}(\theta_i) - T_{\Lambda^{\alpha_2}_{B'}}(\theta_i)|^p + |I_{\Lambda^{\alpha_1}_{B'}}(\theta_i) - I_{\Lambda^{\alpha_2}_{B'}}(\theta_i)|^p + |T_{\Lambda^{\alpha_1}_{B'}}(\theta_i) - T_{\Lambda^{\alpha_2}_{B'}}(\theta_i)|^p)^{\frac{1}{p}}\}$$

$(p \geq 1)$.

In this example, we take $p = 1$, now

$$\begin{aligned}d(\phi(\Lambda^{\alpha_1}_{A'_1}),\phi((\Lambda^{\alpha_2}_{A'_1})) &= \min_{\theta_i}\{|T_{\Lambda^{\alpha_1}_{B'}}(\theta_i) - T_{\Lambda^{\alpha_2}_{B'}}(\theta_i)| + |I_{\Lambda^{\alpha_1}_{B'}}(\theta_i) - I_{\Lambda^{\alpha_2}_{B'}}(\theta_i)|\\ &\quad + |F_{\Lambda^{\alpha_1}_{B'}}(\theta_i) - F_{\Lambda^{\alpha_2}_{B'}}(\theta_i)|\}\\ &= |T_{\Lambda^{\alpha_1}_{B'}}(\theta_2) - T_{\Lambda^{\alpha_2}_{B'}}(\theta_2)| + |I_{\Lambda^{\alpha_1}_{B'}}(\theta_2) - I_{\Lambda^{\alpha_2}_{B'}}(\theta_2)|\\ &\quad + |F_{\Lambda^{\alpha_1}_{B'}}(\theta_2) - F_{\Lambda^{\alpha_2}_{B'}}(\theta_2)|\\ &= |1-0.2| + |0.8-0.3| + |0.6-0.9|\\ &= 0.8+0.5+0.3\\ &= 0.16\\ &= (0.2)(0.8)\\ &= 0.2d(\Lambda^{\alpha_1}_{A'_1},\Lambda^{\alpha_2}_{A'_1}).\end{aligned}$$

Here $k = 0.2$, so ϕ is a contraction.

Definition 3.2 Let ϕ be a mapping from $\hat{N}S(W_E)$ to $F\hat{N}S(W_E)$. Then ϕ is called neutrosophic soft non-expansive mapping if $d(\phi(\Lambda^{\alpha}_{A'}),\phi(\Lambda^{\alpha}_{B'})) \leq kd(\Lambda^{\alpha}_{A'},\Lambda^{\alpha}_{B'})$ for all $\Lambda^{\alpha}_{A'},\Lambda^{\alpha}_{B'} \in \hat{N}S(W_E)$ and $k = 1$.

Example 3.2 Let $W = \{\upsilon_1,\upsilon_2,\upsilon_3\}$ and $R = A' = B' = \{\alpha_1,\alpha_2\}$. Define a neutrosophic soft sets $\Lambda^{\alpha}_{A'}$ and $\Lambda^{\alpha}_{B'}$ as follows:

$$\begin{aligned}\Lambda^{\alpha}_{A'} = \{&(\alpha_1,\{\langle\upsilon_1,1,0.1,0.2\rangle,\langle\upsilon_2,0.6,0.7,0.4\rangle,\langle\upsilon_3,0.2,0.4,0.6\rangle\}),\\ &(\alpha_2,\{\langle\upsilon_1,0.3,0.7,0.6\rangle,\langle\upsilon_2,0.1,0.9,0.3\rangle,\langle\upsilon_3,0.4,0.6,0.7\rangle\})\}\end{aligned}$$

and

$$\Lambda^{\alpha}_{B'} = \{(\alpha_1,\{\langle \upsilon_1,1,0.5,0.2\rangle,\langle \upsilon_2,1,0.5,0.6\rangle,\langle \upsilon_3,0.2,0.4,0.6\rangle\}\}),$$
$$(\alpha_2,\{\langle \upsilon_1,0.1,0.3,0.6\rangle,\langle \upsilon_2,0.2,0.3,0.9\rangle,\langle \upsilon_3,0.4,0.6,0.7\rangle\})\}.$$

$$\begin{aligned} d(\phi(\Lambda^{\alpha_1}_{A_1'}),\phi((\Lambda^{\alpha_2}_{A_1'})) &= \min_{x_i}\{|\,T_{\Lambda^{\alpha_1}_{B'}}(\upsilon_i)-T_{\Lambda^{\alpha_2}_{B'}}(\upsilon_i)\,|+|\,I_{\Lambda^{\alpha_1}_{B'}}(\upsilon_i)-I_{\Lambda^{\alpha_2}_{B'}}(\upsilon_i)\,| \\ &\quad +|\,F_{\Lambda^{\alpha_1}_{B'}}(\upsilon_i)-F_{\Lambda^{\alpha_2}_{B'}}(\upsilon_i)\,|\} \\ &=|\,T_{\Lambda^{\alpha_1}_{B'}}(\upsilon_3)-T_{\Lambda^{\alpha_2}_{B'}}(\upsilon_3)\,|+|\,I_{\Lambda^{\alpha_1}_{A'}}(\upsilon_3)-I_{\Lambda^{\alpha_2}_{B'}}(\upsilon_3)\,| \\ &\quad +|\,F_{\Lambda^{\alpha_1}_{A'}}(\upsilon_3)-F_{\Lambda^{\alpha_2}_{B'}}(\upsilon_3)\,| \\ &=|\,0.2-0.4\,|+|\,0.4-0.6\,|+|\,0.6-0.7\,| \\ &= 0.2+0.2+0.1 \\ &= 0.5 \\ &= (1)(0.5) \\ &= 1d(\Lambda^{\alpha_1}_{A_1'},\Lambda^{\alpha_2}_{A_1'}). \end{aligned}$$

Here $k=1$, so ϕ is non-expansive.

Definition 3.3 Let ϕ be a mapping from $\tilde{N}S(W_E)$ to $\tilde{N}S(W_E)$. Then ϕ is called neutrosophic soft k-Lipschitz mapping if $d(\phi(\Lambda^{\alpha}_{A'}),\phi(\Lambda^{\alpha}_{B'})) \le kd(\Lambda^{\alpha}_{A'},\Lambda^{\alpha}_{B'})$ for all $\Lambda^{\alpha}_{A'},\Lambda^{\alpha}_{B'} \in F\tilde{N}S(W_E)$ and $k>0$.

Example 3.3 Let $W=\{\upsilon_1,\upsilon_2,\upsilon_3\}$ and $R=A'=B'=\{\alpha_1,\alpha_2\}$. Define a NSS $\Lambda^{\alpha}_{A'}$ and $\Lambda^{\alpha}_{B'}$ as below:

$$\Lambda^{\alpha}_{A'} = \{(\alpha_1,\{\langle \upsilon_1,0.3,0.4,0.3\rangle,\langle \upsilon_2,0.6,0.7,0.4\rangle,\langle \upsilon_3,0.2,0.4,0.6\rangle\}),$$
$$(\alpha_2,\{\langle \upsilon_1,0.5,0.6,0.4\rangle,\langle \upsilon_2,0.1,0.9,0.3\rangle,\langle \upsilon_3,0.4,0.6,0.7\rangle\})\}$$

and

$$\Lambda^{\alpha}_{B'} = \{(\alpha_1,\{\langle \upsilon_1,1,0.4,0.3\rangle,\langle \upsilon_2,1,0.6,0.3\rangle,\langle \upsilon_3,0.2,0.4,0.6\rangle\}\}),$$
$$(\alpha_2,\{\langle \upsilon_1,0.5,0.7,0.5\rangle,\langle \upsilon_2,0.3,0.2,0.9\rangle,\langle \upsilon_3,1,0.3,0.9\rangle\})\}.$$

$$
\begin{aligned}
d(\phi(\Lambda^{\alpha_1}_{A'_1}),\phi((\Lambda^{\alpha_2}_{A'_1})) &= \min_{?_i}\{|T_{\Lambda^{\alpha_1}_{B'}}(\upsilon_i)-T_{\Lambda^{\alpha_2}_{B'}}(\upsilon_i)|+|I_{\Lambda^{\alpha_1}_{B'}}(\upsilon_i)-I_{\Lambda^{\alpha_2}_{B'}}(\upsilon_i)| \\
&+|F_{\Lambda^{\alpha_1}_{B'}}(\upsilon_i)-F_{\Lambda^{\alpha_2}_{B'}}(\upsilon_i)|\} \\
&=|T_{\Lambda^{\alpha_1}_{B'}}(\upsilon_1)-T_{\Lambda^{\alpha_2}_{B'}}(\upsilon_1)|+|I_{\Lambda^{\alpha_1}_{A'}}(\upsilon_1)-I_{\Lambda^{\alpha_2}_{B'}}(\upsilon_1)| \\
&+|F_{\Lambda^{\alpha_1}_{A'}}(\upsilon_1)-F_{\Lambda^{\alpha_2}_{B'}}(\upsilon_1)| \\
&=|1-0.5|+|0.4-0.7|+|0.3-0.5| \\
&= 0.5+0.3+0.2 \\
&= 1 \\
&= (2)(0.5) \\
&= 2d(\Lambda^{\alpha_1}_{A'_1},\Lambda^{\alpha_2}_{A'_1}).
\end{aligned}
$$

Here $k=2,$ so ϕ is k-lipschitz.

Note: Every neutrosophic soft contraction mapping is neutrosophic soft K-lipschitz mapping but its converse does not hold.

Definition 3.4 Let ϕ be a mapping from $\hat{N}S(W_E)$ to $\hat{N}S(W_E)$. Then ϕ is said to be neutrosophic soft kanan contraction if $d(\phi(\Lambda^{\alpha}_{A'}),\phi(\Lambda^{\alpha}_{B'}))\le k[d(\Lambda^{\alpha}_{A'},\phi(\Lambda^{\alpha}_{A'}))+d(\Lambda^{\alpha}_{B'},\phi(\Lambda^{\alpha}_{B'}))]$ for all $\Lambda^{\alpha}_{A'},\Lambda^{\alpha}_{B'}\in\hat{N}S(W_E)$ and $k\in[0,\frac{1}{2})$. Where k is called contraction factor.

Definition 3.5 Let ϕ and ψ be two mappings from $\hat{N}S(U'_E)$ to $\hat{N}S(U'_E)$. Then ϕ and ψ are called neutrosophic soft commuting mapping if $\phi(\psi(\Omega^{\alpha}_{A'}))=\psi(\phi(\Omega^{\alpha}_{A'}))$ for all $\Omega^{\alpha}_{A'}\in\hat{N}S(U'_E)$.

Definition 3.6 Let ϕ and ψ be two mappings from $\hat{N}S(U'_E)$ to $\hat{N}S(U'_E)$. Then ϕ and ψ are called neutrosophic soft weakly commuting mapping if $d(\phi(\psi(\Lambda^{\alpha}_{A'})),\psi(\phi(\Lambda^{\alpha}_{A'})))\le d(\phi(\Lambda^{\alpha}_{A'}),\psi(\Lambda^{\alpha}_{A'}))$ for all $\Lambda^{\alpha}_{A'}\in\hat{N}S(U'_E)$.

Definition 3.7 Let ϕ and ψ be two mappings from $\hat{N}S(U'_E)$ to $\hat{N}S(U'_E)$. If for $\phi(\Omega^{\alpha}_{A'_n})\to\Omega^{\alpha}_{A'_0}$ and $\psi(\Omega^{\alpha}_{A'_n})\to\Omega^{\alpha}_{A'_0}$ as $n\to\infty$ and $\Omega^{\alpha}_{A'_n},\Omega^{\alpha}_{A'_0}\in\hat{N}S(U'_E)$. Then it is called neutrosophic soft compatible mapping if $\lim_{n\to\infty}d(\phi(\psi(\Omega^{\alpha}_{A'})),\psi(\phi(\Omega^{\alpha}_{A'})))\to 0$.

Definition 3.8 Let $\phi,\psi:\hat{N}S(U'_E)\to\hat{N}S(U'_E)$ be two mappings. If there is $\Omega^{\alpha}_{A'}\in\hat{N}S(U'_E)$ such that $\phi(\Omega^{\alpha}_{A'})=\psi(\Omega^{\alpha}_{A'})=\Omega^{\alpha}_{A'}$, then $\Omega^{\alpha}_{A'}\in\hat{N}S(U'_E)$ is called common fixed point neutrosophic soft mappings.

Definition 3.9 If $\Omega^{\alpha}_{A'}$ is a fixed point of $\phi:\hat{N}S(U'_E)\to\hat{N}S(U'_E)$, then $\Omega^{\alpha}_{A'}$ is also a fixed point ϕ^k that is $\phi^k(\Omega^{\alpha}_{A'})=\Omega^{\alpha}_{A'}$ for all $\Omega^{\alpha}_{A'}\in\hat{N}S(U'_E)$. So $\Omega^{\alpha}_{A'}$ is called periodic point of neutrosophic soft mapping ϕ and k is called period of ϕ.

Remark Every fixed point of neutrosophic soft mapping is a periodic point but every periodic point of neutrosophic soft mapping is not a fixed point.

Definition 3.9 Let ϕ, ψ be two mappings from $\hat{N}S(U'_E)$ to $\hat{N}S(U'_E)$. If $\phi(\Omega^{\alpha}_{A'})=\psi(\Omega^{\alpha}_{A'})=\Omega^{\alpha}_{B'}$, for all

$\Omega^{\alpha}_{A'}, \Omega^{\alpha}_{B'} \in F\hat{N}S(U'_E)$. Then $\Omega^{\alpha}_{A'}$ is called coincidence point of ϕ and ψ and $\Omega^{\alpha}_{B'}$ is called point of coincidence for ϕ and ψ.

Definition 3.10 Let $\phi : \hat{N}S(U'_E) \to \hat{N}S(U'_E)$ be a mapping. Then ϕ is said to be neutrosophic soft increasing map if for any $\Omega^{\alpha}_{A'} \le \Omega^{\alpha}_{B'}$ implies $\phi(\Omega^{\alpha}_{A'}) \le \phi(\Omega^{\alpha}_{B'})$ for all $\Omega^{\alpha}_{A'}, \Omega^{\alpha}_{B'} \in \hat{N}S(U'_E)$.

Definition 3.11 Let $\phi : \hat{N}S(U'_E) \to \hat{N}S(U'_E)$ be a mapping. Then ϕ is said to be neutrosophic soft dominated map if $\phi(\Omega^{\alpha}_{A'}) \le \Omega^{\alpha}_{A'}$ for all $\Omega^{\alpha}_{A'} \in \hat{N}S(U'_E)$.

Definition 3.12 Let $\phi : \hat{N}S(U'_E) \to \hat{N}S(U'_E)$ be a mapping. Then ϕ is said to be neutrosophic soft dominating map if $\Omega^{\alpha}_{A'} \le \phi(\Omega^{\alpha}_{A'})$ for all $\Omega^{\alpha}_{A'} \in \hat{N}S(U'_E)$.

4. Main Results

Banach Contraction Theorem

Proposition 1 Let $\hat{N}S(U'_E)$ be a non-empty set of neutrosophic points and $(\hat{N}S(U'_E), d)$ be a complete neutrosophic soft metric space. Suppose ϕ is a mapping from $\hat{N}S(U'_E)$ to $\hat{N}S(U'_E)$ be contraction. Then fixed point of ϕ exists and unique.

Proof Let $\Omega^{\alpha}_{A'_0} \in \hat{N}S(U'_E)$ be arbitrary. Define $\Omega^{\alpha}_{A'_1} = \phi(\Omega^{\alpha}_{A'_0})$ and by continuing we have a sequence in the form $\Omega^{\alpha}_{A'_{n+1}} = \phi(\Omega^{\alpha}_{A'_n})$. Now

$$\begin{aligned}
d(\Omega^{\alpha}_{A'_{n+1}}, \Omega^{\alpha}_{A'_n}) &= d(\phi(\Omega^{\alpha}_{A'_n}), \phi(\Omega^{\alpha}_{A'_{n-1}})) \\
&\le kd(\Omega^{\alpha}_{A'_n}, \Omega^{\alpha}_{A'_{n-1}}) \\
&= kd(\phi(\Omega^{\alpha}_{A'_{n-1}}), \phi(\Omega^{\alpha}_{A'_{n-2}})) \\
&\le k^2 d(\Omega^{\alpha}_{A'_{n-1}}, \Omega^{\alpha}_{A'_{n-2}}) \\
&= k^2 d(\phi(\Omega^{\alpha}_{A'_{n-2}}), \phi(\Omega^{\alpha}_{A'_{n-3}})) \\
&\le k^3 d(\Omega^{\alpha}_{A'_{n-2}}, \Omega^{\alpha}_{A'_{n-3}}) \\
&\quad . \\
&\quad . \\
&\quad . \\
&\le k^n d(\Omega^{\alpha}_{A'_1}, \Omega^{\alpha}_{A'_0}).
\end{aligned}$$

Now for $m, n > n_0$, we have

$$d(\Omega^{\alpha}_{A'_{n+1}},\Omega^{\alpha}_{A'_{n}}) \leq d(\Omega^{\alpha}_{A'_{n}},\Omega^{\alpha}_{A'_{n+1}}) + d(\Omega^{\alpha}_{A'_{n+1}},\Omega^{\alpha}_{A'_{n+2}}) + ... + d(\Omega^{\alpha}_{A'_{m-1}},\Omega^{\alpha}_{A'_{m}})$$
$$\leq k^{n} d(\Omega^{\alpha}_{A'_{1}},\Omega^{\alpha}_{A'_{0}}) + k^{n+1} d(\Omega^{\alpha}_{A'_{1}},\Omega^{\alpha}_{A'_{0}}) + ... + k^{m-1} d(\Omega^{\alpha}_{A'_{1}},\Omega^{\alpha}_{A'_{0}})$$
$$= k^{n}[1 + k + k^{2} + ... + k^{m-n-1}] d(\Omega^{\alpha}_{A'_{1}},\Omega^{\alpha}_{A'_{0}})$$
$$= \frac{k^{n}}{1-k} d(\Omega^{\alpha}_{A'_{1}},\Omega^{\alpha}_{A'_{0}})$$
$$d(\Omega^{\alpha}_{A'_{n+1}},\Omega^{\alpha}_{A'_{n}}) \to 0 \text{ as } n \to \infty.$$

So $\Omega^{\alpha}_{A'_{n}}$ is a cauchy sequence in $(\hat{N}S(U'_{E}),d)$, but $(\hat{N}S(U'_{E}),d)$ is complete, so there exists $\Omega^{\alpha *}_{A'} \in \hat{N}S(U'_{E})$ such that $d(\Omega^{\alpha}_{A'_{n}},\Omega^{\alpha *}_{A'}) \to 0$ as $n \to \infty$. Now

$$d(\Omega^{\alpha}_{A'_{n+1}},\phi(\Omega^{\alpha *}_{A'})) = d(\phi(\Omega^{\alpha}_{A'_{n}}),\phi(\Omega^{\alpha *}_{A'}))$$
$$\leq kd(\Omega^{\alpha}_{A'_{n}},\Omega^{\alpha *}_{A'}).$$

On taking limit as $n \to \infty$, we get

$$d(\phi(\Omega^{\alpha *}_{A'}),\Omega^{\alpha *}_{A'}) \leq 0.$$

But

$$d(\phi(\Omega^{\alpha *}_{A'}),\Omega^{\alpha *}_{A'}) \geq 0.$$

So

$$d(\phi(\Omega^{\alpha *}_{A'}),\Omega^{\alpha *}_{A'}) = 0$$
$$\phi(\Omega^{\alpha *}_{A'}) = \Omega^{\alpha *}_{A'}.$$

So $\Omega^{\alpha *}_{A'}$ is the FP of ⩯

Now we have to show that $\Omega^{\alpha *}_{A'}$ is unique. Suppose there exists another FP $\Omega^{\alpha *}_{B'} \in \hat{N}S(U'_{E})$ such that $\phi(\Omega^{\alpha *}_{B'}) = \Omega^{\alpha *}_{B'}$. Now

$$d(\Omega^{\alpha *}_{A'},\Omega^{\alpha *}_{B'}) = d(\phi(\Omega^{\alpha *}_{A'}),\phi(\Omega^{\alpha *}_{B'}))$$
$$\leq kd(\Omega^{\alpha *}_{A'},\Omega^{\alpha *}_{B'})$$
$$(1-k)d(\Omega^{\alpha *}_{A'},\Omega^{\alpha *}_{B'}) \leq 0.$$

Here $(1-k) \not\leq 0$, so

$$d(\Omega^{\alpha ?}_{A'},\Omega^{\alpha ?}_{B'}) \leq 0.$$

But

$$d(\Omega^{\alpha *}_{A'},\Omega^{\alpha *}_{B'}) \geq 0$$
$$d(\Omega^{\alpha *}_{A'},\Omega^{\alpha *}_{B'}) = 0.$$

Hence $\Omega^{\alpha *}_{A'} = \Omega^{\alpha *}_{B'}$, so the fixed point is unique.

Proposition 2 Let $(\hat{N}S(U'_{E}),d)$ be a complete neutrosophic soft metric space. Suppose ϕ be a mapping from $F\hat{N}S(U'_{E})$ to $F\hat{N}S(U'_{E})$ satisfies the contraction $d(\phi^{m}(\Omega^{\alpha}_{A'_{1}}),\phi^{m}(\Omega^{\alpha}_{B'_{1}})) \leq kd(\Omega^{\alpha}_{A'_{1}},\Omega^{\alpha}_{B'_{1}})$ for all $\Omega^{\alpha}_{A'_{1}},\Omega^{\alpha}_{B'_{1}} \in \hat{N}S(U'_{E})$, where $k \in [0,1)$ and m is any natural number. Then ϕ has a FP.

Proof It follows from banach contraction theorem that ϕ^m has unique a FP that is $\phi^m(\Omega^{\alpha}_{A_1'}) = \Omega^{\alpha}_{A_1'}$. Now

$$\begin{aligned} \phi^m(\phi(\Omega^{\alpha}_{A_1'})) &= \phi^{m+1}(\Omega^{\alpha}_{A_1'}) \\ &= \phi(\phi^m(\Omega^{\alpha}_{A_1'})) \\ &= \phi(\Omega^{\alpha}_{A_1'}). \end{aligned}$$

By the uniqueness of FP, we have $\phi(\Omega^{\alpha}_{A_1'}) = \Omega^{\alpha}_{A_1'}$.

Proposition 3 Let $(\hat{N}S(U_E'), d)$ be a complete neutrosophic soft metric space. Suppose ϕ , ψ satisfy $d(\phi(\Omega^{\alpha}_{A_1'}), \psi(\Omega^{\alpha}_{B_1'})) \leq \alpha d(\Omega^{\alpha}_{A_1'}, \phi(\Omega^{\alpha}_{A_1'})) + \beta d(\Omega^{\alpha}_{B_1'}, \psi(\Omega^{\alpha}_{B_1'})) + \gamma[d(\Omega^{\alpha}_{A_1'}, \psi(\Omega^{\alpha}_{B_1'})) + d(\Omega^{\alpha}_{B_1'}, \phi(\Omega^{\alpha}_{A_1'})]$ for all $\Omega^{\alpha}_{A_1'}, \Omega^{\alpha}_{B_1'} \in F\hat{N}S(U_E')$ with α, β, γ are non-negative and $\alpha + \beta + \gamma < 1$. Then ϕ and ψ have a unique FP.

Proof Let $\Omega^{\alpha}_{A_1'} \in \hat{N}S(U_E')$ be a fixed point of ϕ that is $\phi(\Omega^{\alpha}_{A_1'}) = \Omega^{\alpha}_{A_1'}$. We need to show that $\psi(\Omega^{\alpha}_{A_1'}) = \Omega^{\alpha}_{A_1'}$.
Now

$$\begin{aligned} d(\Omega^{\alpha}_{A_1'}, \psi(\Omega^{\alpha}_{A_1'})) &= d(\phi(\Omega^{\alpha}_{A_1'}), \psi(\Omega^{\alpha}_{A_1'})) \\ &\leq \alpha d(\Omega^{\alpha}_{A_1'}, \phi(\Omega^{\alpha}_{A_1'})) + \beta d(\Omega^{\alpha}_{A_1'}, \psi(\Omega^{\alpha}_{A_1'})) + \gamma[d(\Omega^{\alpha}_{A_1'}, \psi(\Omega^{\alpha}_{A_1'})) + d(\Omega^{\alpha}_{A_1'}, \phi(\Omega^{\alpha}_{A_1'}))] \\ &= \alpha d(\Omega^{\alpha}_{A_1'}, \Omega^{\alpha}_{A_1'}) + \beta d(\Omega^{\alpha}_{A_1'}, \psi(\Omega^{\alpha}_{A_1'})) + \gamma[d(\Omega^{\alpha}_{A_1'}, \psi(\Omega^{\alpha}_{A_1'})) + d(\Omega^{\alpha}_{A_1'}, \Omega^{\alpha}_{A_1'})] \\ &= \beta d(\Omega^{\alpha}_{A_1'}, \psi(\Omega^{\alpha}_{A_1'}) + \gamma d(\Omega^{\alpha}_{A_1'}, \psi(\Omega^{\alpha}_{A_1'})) \end{aligned}$$

$$(1-\beta-\gamma) d(\Omega^{\alpha}_{A_1'}, \psi(\Omega^{\alpha}_{A_1'})) \leq 0$$

Since $(1-\beta-\gamma) \not\leq 0$, so

$$d(\Omega^{\alpha}_{A_1'}, \psi(\Omega^{\alpha}_{A_1'})) \leq 0.$$

But

$$d(\Omega^{\alpha}_{A_1'}, \psi(\Omega^{\alpha}_{A_1'})) \geq 0$$

hence

$$d(\Omega^{\alpha}_{A_1'}, \psi(\Omega^{\alpha}_{A_1'})) = 0.$$

Thus $\psi(\Omega^{\alpha}_{A_1'}) = \Omega^{\alpha}_{A_1'}$.

Proposition 4 Let $\hat{N}S(U_E')$ be a non-empty set of neutrosophic points and $(\hat{N}S(U_E'), d)$ be a complete neutrosophic soft metric space. Suppose ϕ is a mapping from $\hat{N}S(U_E')$ to $\hat{N}S(U_E')$ be kanan contraction. Then fixed point of ϕ exists and unique.

Proof Let $\Omega^{\alpha}_{A_0'} \in \hat{N}S(U_E')$ be arbitrary. Define $\Omega^{\alpha}_{A_1'} = \phi(\Omega^{\alpha}_{A_0'})$ and by continuing we have a sequence in the form $\Omega^{\alpha}_{A_{n+1}'} = \phi(\Omega^{\alpha}_{A_n'})$. Now

$$
\begin{aligned}
d(\Omega^{\alpha}_{A'_{n+1}},\Omega^{\alpha}_{A'_{n}}) &= d(\phi(\Omega^{\alpha}_{A'_{n}}),\phi(\Omega^{\alpha}_{A'_{n-1}}))\\
&\leq k[d(\Omega^{\alpha}_{A'_{n}},\phi(\Omega^{\alpha}_{A'_{n}}))+d(\Omega^{\alpha}_{A'_{n-1}},\phi(\Omega^{\alpha}_{A'_{n-1}}))\\
&= k[d(\Omega^{\alpha}_{A'_{n}},\Omega^{\alpha}_{A'_{n+1}})+d(\Omega^{\alpha}_{A'_{n-1}},\Omega^{\alpha}_{A'_{n}})]\\
&= kd(\Omega^{\alpha}_{A'_{n}},\Omega^{\alpha}_{A'_{n+1}})+kd(\Omega^{\alpha}_{A'_{n-1}},\Omega^{\alpha}_{A'_{n}})\\
(1-k)d(\Omega^{\alpha}_{A'_{n+1}},\Omega^{\alpha}_{A'_{n}}) &\leq kd(\Omega^{\alpha}_{A'_{n}},\Omega^{\alpha}_{A'_{n-1}})\\
d(\Omega^{\alpha}_{A'_{n+1}},\Omega^{\alpha}_{A'_{n}}) &\leq \frac{k}{1-k}d(\Omega^{\alpha}_{A'_{n}},\Omega^{\alpha}_{A'_{n-1}})\\
&= hd(\Omega^{\alpha}_{A'_{n}},\Omega^{\alpha}_{A'_{n-1}})
\end{aligned}
$$

for $h=\frac{k}{1-k}$

$$
\begin{aligned}
d(\Omega^{\alpha}_{A'_{n+1}},\Omega^{\alpha}_{A'_{n}}) &\leq hd(\Omega^{\alpha}_{A'_{n}},\Omega^{\alpha}_{A'_{n-1}})\\
&\leq h^2 d(\Omega^{\alpha}_{A'_{n-1}},\Omega^{\alpha}_{A'_{n-2}})\\
&\leq h^3 d(\Omega^{\alpha}_{A'_{n-2}},\Omega^{\alpha}_{A'_{n-3}})\\
&\quad .\\
&\quad .\\
&\quad .\\
&\leq h^n d(\Omega^{\alpha}_{A'_{1}},\Omega^{\alpha}_{A'_{0}}).
\end{aligned}
$$

For $m>n$

$$
\begin{aligned}
d(\Omega^{\alpha}_{A'_{n}},\Omega^{\alpha}_{A'_{m}}) &\leq d(\Omega^{\alpha}_{A'_{n}},\Omega^{\alpha}_{A'_{n+1}})+d(\Omega^{\alpha}_{A'_{n+1}},\Omega^{\alpha}_{A'_{n+2}})+...+d(\Omega^{\alpha}_{A'_{m-1}},\Omega^{\alpha}_{A'_{m}})\\
&\leq h^n d(\Omega^{\alpha}_{A'_{1}},\Omega^{\alpha}_{A'_{0}})+h^{n+1}d(\Omega^{\alpha}_{A'_{1}},\Omega^{\alpha}_{A'_{0}})+...+h^{m-1}d(\Omega^{\alpha}_{A'_{1}},\Omega^{\alpha}_{A'_{0}})\\
&= h^n[1+h+h^2+...+h^{m-n-1}]d(\Omega^{\alpha}_{A'_{1}},\Omega^{\alpha}_{A'_{0}})\\
&= h^n(\frac{1}{1+h})d(\Omega^{\alpha}_{A'_{1}},\Omega^{\alpha}_{A'_{0}})\\
d(\Omega^{\alpha}_{A'_{n}},\Omega^{\alpha}_{A'_{m}}) &\to 0 \text{ as } n\to\infty.
\end{aligned}
$$

The sequence $\Omega^{\alpha}_{A'_{n}}$ is a cauchy sequence in $(\tilde{N}S(U'_E),d)$. Since $(\tilde{N}S(U'_E),d)$ is complete, so $\Omega^{\alpha}_{A'_{n}}$ converges to any $\Omega^{\alpha*}_{A'}\in\tilde{N}S(U'_E)$. Now

$$
\begin{aligned}
d(\phi(\Omega^{\alpha*}_{A'}),\Omega^{\alpha}_{A'_{n+1}}) &= d(\phi(\Omega^{\alpha*}_{A'}),\phi(\Omega^{\alpha}_{A'_{n}}))\\
&\leq h[d(\Omega^{\alpha*}_{A'},\phi(\Omega^{\alpha*}_{A'}))+d(\Omega^{\alpha}_{A'_{n}},\phi(\Omega^{\alpha}_{A'_{n}}))].
\end{aligned}
$$

Taking limit as $n\to\infty$, we have

$$
\begin{aligned}
d(\phi(\Omega^{\alpha*}_{A'}),\Omega^{\alpha*}_{A'}) &\leq h[d(\Omega^{\alpha*}_{A'},\phi(\Omega^{\alpha*}_{A'}))+d(\Omega^{\alpha*}_{A'},\phi(\Omega^{\alpha*}_{A'}))]\\
&= 2hd(\Omega^{\alpha*}_{A'},\phi(\Omega^{\alpha*}_{A'}))\\
(1-2h)d(\phi(\Omega^{\alpha*}_{A'}),\Omega^{\alpha*}_{A'}) &\leq 0
\end{aligned}
$$

As $(1-2h)\not\leq 0$, so

$$d(\phi(\Omega^{\alpha *}_{A'}),\Omega^{\alpha *}_{A'}) \leq 0$$

but

$$d(\phi(\Omega^{\alpha *}_{A'}),\Omega^{\alpha *}_{A'}) \geq 0$$

thus

$$d(\phi(\Omega^{\alpha *}_{A'}),\Omega^{\alpha *}_{A'}) = 0.$$

Hence $\Omega^{\alpha *}_{A'} \in \hat{N}S(U'_E)$ is a FP of ϕ.

Suppose $\Omega^{\alpha *}_{B'} \in \hat{N}S(U'_E)$ be another FP. Now

$$\begin{aligned} d(\Omega^{\alpha *}_{A'},\Omega^{\alpha *}_{B'}) &= d(\phi(\Omega^{\alpha *}_{A'}),\phi(\Omega^{\alpha *}_{B'})) \\ &\leq h[d(\Omega^{\alpha *}_{A'},\phi(\Omega^{\alpha *}_{A'})) + d(\Omega^{\alpha *}_{B'},\phi(\Omega^{\alpha *}_{B'}))] \\ &\leq h[d(\Omega^{\alpha *}_{A'},\Omega^{\alpha *}_{A'}) + d(\Omega^{\alpha *}_{B'},\Omega^{\alpha *}_{B'})] \end{aligned}$$

$$d(\Omega^{\alpha *}_{A'},\Omega^{\alpha *}_{B'}) \leq 0 \qquad (1)$$

but

$$d(\Omega^{\alpha *}_{A'},\Omega^{\alpha *}_{B'}) \geq 0. \qquad (2)$$

From (1) and (2) we have

$$d(\Omega^{\alpha *}_{A'},\Omega^{\alpha *}_{B'}) = 0.$$

Hence $\Omega^{\alpha *}_{A'} = \Omega^{\alpha *}_{B'}$.

Proposition 5 Let ϕ,ψ : $\hat{N}S(U'_E) \to \hat{N}S(U'_E)$ be weakly compatible maps. If ϕ and ψ have unique coincidence point. Then ϕ and ψ have unique common fixed point (CFP).

Proof Suppose there is $\Omega^{\alpha}_{A'_1} \in \hat{N}S(U'_E)$ such that $\phi(\Omega^{\alpha}_{A'_1}) = \psi(\Omega^{\alpha}_{A'_1}) = \Omega^{\alpha}_{B'_1}$. Since ϕ and ψ are weakly compatible, so $\phi(\psi(\Omega^{\alpha}_{A'_1})) = \psi(\phi(\Omega^{\alpha}_{A'_1}))$ for all $\Omega^{\alpha}_{A'_1} \in \hat{N}S(U'_E)$. Now

$$\phi(\Omega^{\alpha}_{B'_1}) = \phi(\Omega^{\alpha}_{B'_1}) = \phi(\psi(\Omega^{\alpha}_{A'_1})) = \psi(\phi(\Omega^{\alpha}_{A'_1})).$$

So $\Omega^{\alpha}_{B'_1}$ is also coincidence point (CP) of ϕ and ψ, but $\Omega^{\alpha}_{A'_1}$ is the unique CP of ϕ and ψ, so

$$\phi(\Omega^{\alpha}_{A'_1}) = \psi(\Omega^{\alpha}_{A'_1}) = \phi(\Omega^{\alpha}_{B'_1}) = \psi(\Omega^{\alpha}_{B'_1})$$

$$\Omega^{\alpha}_{B'_1} = \phi(\Omega^{\alpha}_{B'_1}) = \psi(\Omega^{\alpha}_{B'_1}).$$

So $\Omega^{\alpha}_{B'_1} \in \hat{N}S(U'_E)$ is CFP.

Proposition 6 Let $(\hat{N}S(U'_E),d)$ be a complete metric space and ϕ : $\hat{N}S(U'_E) \to \hat{N}S(U'_E)$ be a mapping satisfies $d(\phi^2(\Omega^{\alpha}_{A'_1}),\phi(\Omega^{\alpha}_{A'_1})) \leq kd(\phi(\Omega^{\alpha}_{A'_1}),\Omega^{\alpha}_{A'_1})$ for all $\Omega^{\alpha}_{A'_1} \in \hat{N}S(U'_E)$ and $k \in [0,1)$. Then fixed point of ϕ is singleton.

Proof Let $\Omega^{\alpha}_{A'_0} \in \hat{N}S(U'_E)$ be arbitrary and defines $\Omega^{\alpha}_{A'_{n+1}} = \phi^n(\Omega^{\alpha}_{A'_0}) = \phi(\Omega^{\alpha}_{A'_n})$. Now

$$\begin{aligned} d(\phi^{n+1}(\Omega^{\alpha}_{A_0'}),\phi^{n}(\Omega^{\alpha}_{A_0'})) &\leq kd(\phi^{n}(\Omega^{\alpha}_{A_0'}),\phi^{n-1}(\Omega^{\alpha}_{A_0'})) \\ &\leq k^2 d(\phi^{n-1}(\Omega^{\alpha}_{A_0'}),\phi^{n-2}(\Omega^{\alpha}_{A_0'})) \\ &\leq k^3 d(\phi^{n-3}(\Omega^{\alpha}_{A_0'}),\phi^{n-4}(\Omega^{\alpha}_{A_0'})) \\ &\;\;. \\ &\;\;. \\ &\;\;. \\ &\leq k^n d(\phi(\Omega^{\alpha}_{A_0'}),\Omega^{\alpha}_{A_0'}). \end{aligned}$$

Now for $m > n$

$$\begin{aligned} d(\phi^{m}(\Omega^{\alpha}_{A_0'}),\phi^{n}(\Omega^{\alpha}_{A_0'})) &\leq d(\phi^{n}(\Omega^{\alpha}_{A_0'}),\phi^{n+1}(\Omega^{\alpha}_{A_0'})) + d(\phi^{n+1}(\Omega^{\alpha}_{A_0'}),\phi^{n+2}(\Omega^{\alpha}_{A_0'})) \\ &\quad +\ldots+ d(\phi^{m-1}(\Omega^{\alpha}_{A_0'}),\phi^{m}(\Omega^{\alpha}_{A_0'})) \\ &\leq k^n d(\phi(\Omega^{\alpha}_{A_0'}),\Omega^{\alpha}_{A_0'}) + k^{n+1} d(\phi(\Omega^{\alpha}_{A_0'}),\Omega^{\alpha}_{A_0'}) \\ &\quad +\ldots+ k^{m-1} d(\phi(\Omega^{\alpha}_{A_0'}),\Omega^{\alpha}_{A_0'}) \\ &\leq k^n [1+k+k^2+\ldots+k^{m-n-1}] d(\phi(\Omega^{\alpha}_{A_0'}),\Omega^{\alpha}_{A_0'}) \\ &\leq \frac{k^n}{1-k} d(\phi(\Omega^{\alpha}_{A_0'}),\Omega^{\alpha}_{A_0'}) \end{aligned}$$

$$d(\phi^{m}(\Omega^{\alpha}_{A_0'}),\phi^{n}(\Omega^{\alpha}_{A_0'})) \to 0 \text{ as } n \to \infty.$$

So $\phi^{n}(\Omega^{\alpha}_{A_0'})$ is a cauchy sequence in $(\hat{N}S(U'_E),d)$, but $(\hat{N}S(U'_E),d)$ is complete, so every cauchy sequence is convergent that is $\phi^{n}(\Omega^{\alpha}_{A_0'}) \to \Omega^{\alpha *}_{A_0'}$ as $n \to \infty$. Now

$$d(\phi^{n+1}(\Omega^{\alpha}_{A_0'}),\phi(\Omega^{\alpha *}_{A_0'})) \leq kd(\phi^{n}(\Omega^{\alpha}_{A_0'}),\Omega^{\alpha *}_{A_0'})$$

taking limit as $n \to \infty$, we have

$$d(\Omega^{\alpha *}_{A_0'},\phi(\Omega^{\alpha *}_{A_0'})) \leq kd(\Omega^{\alpha *}_{A_0'},\Omega^{\alpha *}_{A_0'})$$

$$d(\Omega^{\alpha *}_{A_0'},\phi(\Omega^{\alpha *}_{A_0'})) \leq 0$$

but

$$d(\Omega^{\alpha *}_{A_0'},\phi(\Omega^{\alpha *}_{A_0'})) \geq 0$$

$$d(\Omega^{\alpha *}_{A_0'},\phi(\Omega^{\alpha *}_{A_0'})) = 0$$

$$\Rightarrow \phi(\Omega^{\alpha *}_{A_0'}) = \Omega^{\alpha *}_{A_0'}.$$

Hence $\Omega^{\alpha *}_{A_0'} \in \hat{N}S(U'_E)$ is the FP of ϕ.

Now suppose $\Omega^{\alpha *}_{B_0'} \in \hat{N}S(U'_E)$ is another FP of with $\phi(\Omega^{\alpha *}_{B_0'}) = \Omega^{\alpha *}_{B_0'}$, then

$$d(\Omega^{\alpha*}_{B_0'},\Omega^{\alpha*}_{A_0'}) = d(\phi(\Omega^{\alpha*}_{B_0'}),\phi(\Omega^{\alpha*}_{A_0'}))$$
$$= d(\phi(\phi(\Omega^{\alpha*}_{B_0'})),\phi(\Omega^{\alpha*}_{A_0'}))$$
$$\leq kd(\phi(\Omega^{\alpha*}_{B_0'})),\phi(\Omega^{\alpha*}_{A_0'}))$$
$$= kd(\Omega^{\alpha*}_{B_0'},\Omega^{\alpha*}_{A_0'})$$
$$(1-k)d(\Omega^{\alpha*}_{B_0'},\Omega^{\alpha*}_{A_0'}) \leq 0.$$

As $(1-k) \not\leq 0,$ so

$$d(\Omega^{\alpha ?}_{B_0'},\Omega^{\alpha ?}_{A_0'}) \leq 0 \qquad (1)$$

but

$$d(\Omega^{\alpha*}_{B_0'},\Omega^{\alpha*}_{A_0'}) \geq 0. \qquad (2)$$

From (1) and (2) we have

$$d(\Omega^{\alpha*}_{B_0'},\Omega^{\alpha*}_{A_0'}) = 0$$
$$\Rightarrow \Omega^{\alpha*}_{B_0'} = \Omega^{\alpha*}_{A_0'}.$$

Hence the FP is unique.

Proposition 7 Let ϕ,ψ : $\hat{N}S(U_E') \to \hat{N}S(U_E')$ be commuting maps. If ϕ and ψ have unique coincidence point. Then ϕ and ψ have unique common fixed point.

Proof Suppose there is $\Omega^{\alpha}_{A_1'} \in \hat{N}S(U_E')$ such that $\phi(\psi(\Omega^{\alpha}_{A_1'})) = \psi(\phi(\Omega^{\alpha}_{A_1'}))$. Since ϕ and ψ have unique coincidence point, so let $\phi(\Omega^{\alpha}_{A_1'}) = \psi(\Omega^{\alpha}_{A_1'}) = \Omega^{\alpha}_{B_1'}$. Now

$$\phi(\Omega^{\alpha}_{B_1'}) = \phi(\psi(\Omega^{\alpha}_{A_1'})) = \psi(\phi(\Omega^{\alpha}_{A_1'})) = \psi(\Omega^{\alpha}_{B_1'}).$$

Here $\Omega^{\alpha}_{B_1'} \in \hat{N}S(U_E')$ is also a coincidence point, but $\Omega^{\alpha}_{A_1'} \in \hat{N}S(U_E')$ is unique coincidence point, so

$$\phi(\Omega^{\alpha}_{B_1'}) = \psi(\Omega^{\alpha}_{B_1'}) = \psi(\Omega^{\alpha}_{A_1'}) = \phi(\Omega^{\alpha}_{A_1'}) = \Omega^{\alpha}_{B_1'}.$$

Hence $\Omega^{\alpha}_{B_1'} \in \hat{N}S(U_E')$ is also a fixed point.

Proposition 8 Every neutrosophic soft identity map is non-expansive.

Proof Suppose that I from $\hat{N}S(U_E')$ to $\hat{N}S(U_E')$ be a neutrosophic soft identity map such that $I(\Omega^{\alpha}_{A_1'}) = \Omega^{\alpha}_{A_1'}$ for all $\Omega^{\alpha}_{A_1'} \in \hat{N}S(U_E')$. Now

$$d(I(\Omega^{\alpha}_{A_1'}),I(\Omega^{\alpha}_{B_1'})) = d(\Omega^{\alpha}_{A_1'},\Omega^{\alpha}_{B_1'})$$

Here $k=1$, so I is non-expansive map.

5. Conclusion

In this paper, we have discussed some new mappings of NSS and some basic results and particular examples. Like fixed point, here also present some new concepts of points that is coincidence point, periodic point and CFP.

FP theory has a lot of applications in control and communicating system. FP theory is an important mathematical instrument used to demonstrate the existence of a solution in mathematical economics and game theory. So the notion of a neutrosophic soft fixed point can be used in these areas. For stabilization of dynamic systems, neutrosophic soft fixed point can be used. In addition, dynamic programming may employ the notion of presence and uniqueness of the common solution of neutrosophic soft set.

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Received: Apr 22, 2020. Accepted: July 10 2020

Neutrosophic Sets and Systems, Vol. 35, 2020

University of New Mexico

Selection of Alternative under the Framework of Single-Valued Neutrosophic Sets

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Abstract. The multiple criteria decision making (MCDM) problems indicate the alternatives which have more or less resemblance to each other. An important mathematical tool used by decision-makers (DMs) to quantify these resemblances is the similarity measure (SM). SM is a powerful tool that measures the resemblance more accurately. Mostly, fuzzy sets (FSs) and its extensions handle the vague and uncertain information by considering the membership, non-membership, and indeterminacy degrees whose sum always lies in the interval $[0, 1]$. However, single-valued neutrosophic sets (SVNSs) and interval-valued neutrosophic sets (IVNSs) have information whose sum is bounded in $[0, 3]$. In the present work, we extended the SM presented by William and Steel for SVNSs and IVNSs by using the concept of Euclidean distance. The weights of criteria indicate much influence for the selection of the best alternative, sometimes DMs feel hesitation to allocate the weights to the criteria. We applied the linear programming (LP) model to evaluate the weights of the criteria to reduce the hesitancy. Later on, SM is utilized to establish an MCDM model for the selection of the best option. Moreover, the Spearman's rank correlation coefficient is implemented to analyze the ranking order. Finally, a medical diagnosis example is illustrated for the feasibility and effectiveness of the proposed model.

Keywords: picture fuzzy sets; fuzzy sets; similarity measure; neutrosophic sets; linear programming model.

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1. Introduction

Most of the information provided to the experts or decision makers (DMs) are ambiguous and uncertain. DMs handle such information precisely by using the fuzzy sets (FSs) theory presented by Zadeh [31] in 1965. FSs contain a single value in its specification, called a membership degree (MDg) which is always bounded in the closed interval $[0, 1]$. FSs have been broadly used in different fields, for example, medical diagnosis, image processing, etc. [12, 17].

In various ambiguous decision making problems, the MD_g is assumed not exactly as a numerical value but as an interval. Therefore, Zadeh [32] introduced the interval-valued fuzzy sets (IVFSs), an augmentation of FSs. Though, the FSs and IVFSs only have the MD_g, and they cannot designate the non membership degree (NMDg) of the element belonging to the set. Consider that in a competition of university's postgraduate students, a board of seven experts evaluate the efficiency of a student. According to three experts a student can be accepted for admission, according to two experts he or she is rejected and the remaining two experts remained impartial. In such circumstances, FSs and IVFSs could not handle the vagueness and uncertainty precisely. Atanassov [6] further extended the notion of FSs into intuitionistic fuzzy sets (IFSs) to cope such problems which comprise both MD_g and NMD_g in its structure so that, $0 \leq MD_g + NMD_g \leq 1$. Most rapidly, IFSs become an important device to deal with the imprecise and ambiguous information than the FSs and IVFSs.

In spite of the fact that, IFSs have been successfully implemented in distinct fields, however, IFSs were not covering the human's attitude perfectly. Casting of vote is an excellent example of such type of attitude, we may divide the voters into four groups: vote for, vote against, neutral and refusal of voting. When a person refuses to vote, we can say that the person is not anxious about the general election. Cuong [11] focused such types of human's attitude by presenting the idea of picture fuzzy sets P_cFSs, the generalized form of IFSs. P_cFSs have three components in its formation called, MD_g, NMD_g and of degree refusal (D_gR) such that, $0 \leq MD_g + NMD_g + D_gR \leq 1$. But P_cFSs also have some limitations to express the decision information. For instance, three groups of decision makers (DMs) assess the advantages of a new business. First group predicts that the business will be profitable is 0.7, according to second group the possibility of loss is 0.2 and the third group is not sure whether the business will be profitable is 0.4. In this scenario, P_cFSs cannot handle the information because, $0.7 + 0.2 + 0.3 = 1.2 > 1$.

Therefore, to handle such situations Wang et al. [22] introduced an amazing concept of single-valued neutrosophic sets (SVNSs) that consists of three degrees, the truth-membership ($T_n(x)$) degree, indeterminacy-membership ($I_n(x)$) degree, and falsity membership ($F_n(x)$) degree in the closed interval $[0, 1]$ so that it satisfy the condition, $0 \leq T_n(x) + I_n(x) + F_n(x) \leq 3$. Later on, Wang [23] described these three degrees in the form of an interval, called an interval-valued neutrosophic sets (IVNSs). Nowadays, NSs have become the center of the eye of the researcher due to its innovation. Many researchers are trying to print it for example, Abdel-Basset et.al [1–4] used the score and accuracy functions of trapezoidal neutrosophic numbers to minimize the cost of projects under uncertain environmental conditions, in order to tackle the ambiguity and uncertainty present in the data for MCDM problems, utilized the plithogenic set, a generalization of NSs, a novel hybrid neutrosophic MCDM model is presented on the basis of TOPSIS by using bipolar neutrosophic numbers and resolve the supply chain issues with the help of best-worst method (evaluating weights) and plithogenic set, respectively.

SM is one of the vital and powerful tools that measures the level of resemblance among the objects. In order to show the preference strength among the alternatives, the similarity measures have achieved more attention from the DMs since the previous few decades. Various DMs have presented a number of similarity measures for MCDM problems to select the most favorable alternative from the various options having identical features under the certain criteria. For example, Beg and Ashraf discussed the various characteristic of similarity measures under the framework of FSs [7]. Ye [28–30] introduced the cosine similarity measures (vector similarity) and implemented it to pattern recognition and medical diagnosis under the environments of simplified neutrosophic sets, interval neutrosophic sets and IFSs. Intarapaiboon [14] applied two new similarity measures to pattern recognition in IFSs situations. Moreover, Song and Hu [20] established two measures of similarity between hesitant fuzzy linguistic term sets and used it for MCDM problems. Recently, Wei and Gao [26] developed the generalized Dice similarity measures for P_cFSs and implemented for pattern recognition. Consequently, Wang et al. [24] presented the generalized Dice similarity measures for Pythagorean fuzzy sets and used it in multiple attribute group decision making.

The linear programming (LP) model introduced by Vanderbei [21], permits some target function to be minimized or maximized inside the system of given situational limitations. LP is a computational technique that enables DMs to solve the problems which they face in decision-making model. It encourages the DMs to deal with constrained ideal conditions which they need to make the best of their resources. Various experts utilized LP model in MCDM for different extensions of FSs [5, 10, 13, 18, 25]. Recently, Sindhu et al. [19] implemented the LP methodology with extended TOPSIS (technique for order of preference by similarity to ideal solution) for picture fuzzy sets. The weights of criteria appear to specify that the DMs identify the significance of people views and its influence on attaining the objective. Sometimes DMs hesitate or confused to allocate the weights to criteria. Thereby, we applied TOPSIS to get the objective function and then find out the weights of criteria under some constraints by using LP model. The novelty of this article is concerned about proposing the SM to overcome the shortcoming present in the existing technique. The following are the major contributions of this study:

- William and Steel SM is extended on the basis of novel distance measure.
- Evaluate the objective function by using TOPSIS.

- Weights of criteria are calculated with the help of LP model.
- An MCDM model is developed on the basis of SM and implemented it for medical diagnosis under the framework of SVNSs and IVNSs.
- Spearman's rank-correlation coefficient and the critical value are applied to strength the proposed MCDM model.

Rest of the article is organized as: Section 2 encloses some preliminaries regarding SVNSs and IVNSs. Various pre-existing similarity measures of SVNSs, IVNSs and their shortcoming are elaborated in Section 3. The modified similarity measures for SVNSs and IVNSs are described in Section 4. An MCDM model is proposed in Section 5 and the developed model is then applied on an example of medical diagnosis in Section 6 to elaborate the validity and effectiveness. A comprehensive comparative analysis based on Spearman's rank correlation coefficient is penned in Section 7. Conclusions and future work are highlighted in Section 8.

2. **Preliminaries**

A brief introduction of the notions FSs, P_cFSs, $SVNS$ and $IVNS$ and the LP model is presented in this section.

Definition 2.1. [31] Let $X = \{x_1, x_2, ..., x_n\}$ be a discourse set. A fuzzy set (FS) A on X is represented in terms of a functions $m : X \to [0, 1]$ such that

$$A = \{\langle x_i, m_A(x_i)\rangle \,|x_i \in X\}.$$

Definition 2.2. [11] Let $X = \{x_1, x_2, ..., x_n\}$ be a fixed set. A picture fuzzy set P_c on X is defined as:

$$P_c = \{\langle x_i, \alpha_{P_c}(x_i), \gamma_{P_c}(x_i), \beta_{P_c}(x_i)\rangle \,|x_i \in X, i = 1, 2, ..., n\},$$

where $\alpha_{P_c}(x_i)$, $\beta_{P_c}(x_i)$, $\gamma_{P_c}(x_i) \in [0, 1]$ are called the acceptance membership, neutral and rejection membership degrees of $x_i \in X$ to the set P_c, respectively and $\alpha_{P_c}(x_i)$, $\gamma_{P_c}(x_i)$ and $\beta_{P_c}(x_i)$ fulfil the condition: $0 \leq \alpha_{P_c}(x_i) + \gamma_{P_c}(x_i) + \beta_{P_c}(x_i) \leq 1$, for all $x_i \in X$. Also $\zeta_{P_c}(x_i) = 1 - \alpha_{P_c}(x_i) - \gamma_{P_c}(x_i) - \beta_{P_c}(x_i)$, then $\zeta_{P_c}(x_i)$ is said to be a degree of refusal membership of $x_i \in X$ in P_c. For our convenience, we can write $p_i = (\alpha_{P_c}(x_i),\ \beta_{P_c}(x_i), \gamma_{P_c}(x_i))$ as the picture fuzzy numbers (P_cFNs) over a set P_c, where $i = 1, 2, ..., n$.

Definition 2.3. [22] Let $X = \{x_1, x_2, ..., x_n\}$ be a fixed set. A SVNS N_s on X is defined as:

$$N_s = \{\langle x_i, \alpha_{N_s}(x_i), \gamma_{N_s}(x_i), \beta_{N_s}(x_i)\rangle \,|x_i \in X, i = 1, 2, ..., n\},$$

where $\alpha_{N_s}(x_i)$, $\gamma_{N_s}(x_i)$, $\beta_{N_s}(x_i) \in [0, 1]$ are called the truth-membership, indeterminacy and falsity- membership degrees of $x_i \in X$ to the set N_s, respectively and $\alpha_{N_s}(x_i)$, $\gamma_{N_s}(x_i)$ and

$\beta_{N_s}(x_i)$ fulfil the condition:
for all $x_i \in X$ then, $0 \leq \alpha_{N_s}(x_i) + \gamma_{N_s}(x_i) + \beta_{N_s}(x_i) \leq 3$. Let N_s^1 and N_s^2 be two SVNS, then following conditions hold:

(1) $N_s^1 \subseteq N_s^2$ iff $\alpha_{N_s^1}(x_i) \leq \alpha_{N_s^2}(x_i)$, $\beta_{N_s^1}(x_i) \geq \beta_{N_s^2}(x_i)$ and $\gamma_{N_s^1}(x_i) \geq \gamma_{N_s^2}(x_i)$,
(2) $N_s^1 = N_s^2$ iff $N_s^1 \subseteq N_s^2$ and $N_s^2 \subseteq N_s^1$.

Definition 2.4. [23] Let $X = \{x_1, x_2, ..., x_n\}$ be a fixed set. An ISVNS $\tilde{N}_s$ on X is defined as:

$$\tilde{N}_s = \{\left\langle x_i, \alpha_{\tilde{N}_s}(x_i), \gamma_{\tilde{N}_s}(x_i), \beta_{\tilde{N}_s}(x_i)\right\rangle | x_i \in X, i = 1, 2, ..., n\},$$

where $\alpha_{\tilde{N}_s}(x_i) = [\alpha^l_{\tilde{N}_s}(x_i), \alpha^u_{\tilde{N}_s}(x_i)] \subseteq [0,1]$, $\gamma_{\tilde{N}_s}(x_i) = [\gamma^l_{\tilde{N}_s}(x_i), \gamma^u_{\tilde{N}_s}(x_i)] \subseteq [0,1]$, $\beta_{\tilde{N}_s}(x_i) = [\beta^l_{\tilde{N}_s}(x_i), \beta^u_{\tilde{N}_s}(x_i)] \subseteq [0,1]$ are called the truth-membership, indeterminacy and falsity- membership degrees of $x_i \in X$ to the set $\tilde{N}_s$, respectively and satisfy the condition:
for all $x_i \in X$ then, $0 \leq \alpha^u_{\tilde{N}_s}(x_i) + \gamma^u_{\tilde{N}_s}(x_i) + \beta^u_{\tilde{N}_s}(x_i) \leq 3$. Let $\tilde{N}_s^1$ and $\tilde{N}_s^2$ be two SVNS, then following conditions hold:

(1) $\tilde{N}_s^1 \subseteq \tilde{N}_s^2$ iff $\alpha^l_{N_s^1}(x_i) \leq \alpha^l_{N_s^2}(x_i), \alpha^u_{N_s^1}(x_i) \leq \alpha^u_{N_s^2}(x_i)$, $\beta^l_{N_s^1}(x_i) \geq \beta^l_{N_s^2}(x_i), \beta^u_{N_s^1}(x_i) \geq \beta^u_{N_s^2}(x_i)$, $\gamma^l_{N_s^1}(x_i) \geq \gamma^l_{N_s^2}(x_i)$ and $\gamma^u_{N_s^1}(x_i) \geq \gamma^u_{N_s^2}(x_i)$,
(2) $\tilde{N}_s^1 = \tilde{N}_s^2$ iff $\tilde{N}_s^1 \subseteq \tilde{N}_s^2$ and $\tilde{N}_s^2 \subseteq \tilde{N}_s^1$.

Definition 2.5. [21]. The linear programming model is constructed as:

$$\begin{aligned}
\text{Maximize:}\quad & Z = c_1t_1 + c_2t_2 + c_3t_3 + ... + c_nt_n \\
\text{Subject to:}\quad & a_{11}t_1 + a_{12}t_2 + a_{13}t_3 + ... + a_{1n}t_n \leq b_1 \\
& a_{21}t_1 + a_{22}t_2 + a_{23}t_3 + ... + a_{2n}t_n \leq b_2 \\
& \vdots \\
& a_{m1}t_1 + a_{m2}t_2 + a_{m3}t_3 + ... + a_{mn}t_n \leq b_m \\
& t_1, t_2, ..., t_n \geq 0,
\end{aligned}$$

where m and n denotes the cardinalities of the constraints and decision variables $t_1, t_2, ..., t_n$, respectively. A solution $(t_1, t_2, ..., t_n)$ is called feasible point if it fulfils all of the restrictions. LP model is used to find the optimal solution of the decision variables to maximize or minimize the linear function Z.

3. Some existing similarity measures for SVNSs and IVNSs

Similarity measure is a most widely used tool to evaluate the relationship between two sets. Two sets are said to be perfectly similar if similarity measure between them is exactly 1. The following are the compulsory axioms for the sets (SVNSs or IVNSs) to be perfectly similar:

Definition 3.1. Let $X = \{x_1, x_2, ..., x_n\}$ be a universal set and $N_s^1 = \{< x_i, \alpha_{N_s^1}(x_i), \gamma_{N_s^1}(x_i), \beta_{N_s^1}(x_i)\}$ and $N_s^2 = \{< x_i, \alpha_{N_s^1}(x_i), \gamma_{N_s^2}(x_i), \beta_{N_s^2}(x_i) >\}$ be two SVNS, where, $i = 1, 2, ..., n$. Then,

(1) $0 \leq S(N_s^1, N_s^2) \leq 1$,
(2) $S(N_s^1, N_s^2) = S(N_s^2, N_s^1)$,
(3) $S(N_s^1, N_s^2) = 1$ if and only if $N_s^1 = N_s^2$.

A cosine similarity measure $S(N_s^1, N_s^2)$ of SVNS presented by Ye [29] is given as:
$S(N_s^1, N_s^2) = \frac{(\alpha_{N_s^1}(x_i))(\alpha_{N_s^2}(x_i))+(\gamma_{N_s^1}(x_i))(\gamma_{N_s^2}(x_i))+(\beta_{N_s^1}(x_i))(\beta_{N_s^2}(x_i))}{[\sqrt{(\alpha_{N_s^1}(x_i))^2+(\gamma_{N_s^1}(x_i))^2+(\beta_{N_s^1}(x_i))^2}][\sqrt{(\alpha_{N_s^2}(x_i))^2+(\gamma_{N_s^2}(x_i))^2+(\beta_{N_s^2}(x_i))^2}]}$.
Suppose that $N_s^1 = (x, 0.4, 0.2, 0.6)$ and $N_s^2 = (x, 0.2, 0.1, 0.3)$ are two SVNSs, the Definition 2.3 shows that $N_s^1 \neq N_s^2$. However, by using cosine similarity measure presented by Ye [29], we see that, $S(N_s^1, N_s^2) = 1$, show the contradiction of the property 3 of Definition 3.1 which describe that $S(N_s^1, N_s^2) = 1$ if and only if $N_s^1 = N_s^2$. Similarly, if we take, $\alpha_{N_s^1}(x_i) = (k+1)\alpha_{N_s^2}(x_i)$, $\gamma_{N_s^1}(x_i) = (k+1)\gamma_{N_s^2}(x_i)$ and $\beta_{N_s^1}(x_i) = (k+1)\beta N_s^2(x_i)$, where $k \geq 1$, then according to cosine similarity measure, its value is:
$S(N_s^1, N_s^2) = \frac{(\alpha_{N_s^1}(x_i))(\alpha_{N_s^2}(x_i))+(\gamma_{N_s^1}(x_i))(\gamma_{N_s^2}(x_i))+(\beta_{N_s^1}(x_i))(\beta_{N_s^2}(x_i))}{[\sqrt{(\alpha_{N_s^1}(x_i))^2+(\gamma_{N_s^1}(x_i))^2+(\beta_{N_s^1}(x_i))^2}][\sqrt{(\alpha_{N_s^2}(x_i))^2+(\gamma_{N_s^2}(x_i))^2+(\beta_{N_s^2}(x_i))^2}]}$,
$S(N_s^1, N_s^2) = \frac{((k+1)\alpha_{N_s^2}(x_i))(\alpha_{N_s^2}(x_i))+((k+1)\gamma_{N_s^2}(x_i))(\gamma_{N_s^2}(x_i))+((k+1)\beta_{N_s^2}(x_i))(\beta_{N_s^2}(x_i))}{[\sqrt{((k+1)\alpha_{N_s^2}(x_i))^2+((k+1)\gamma_{N_s^2}(x_i))^2+((k+1)\beta_{N_s^2}(x_i))^2}][\sqrt{(\alpha_{N_s^2}(x_i))^2+(\gamma_{N_s^2}(x_i))^2+(\beta_{N_s^2}(x_i))^2}]}$,
$S(N_s^1, N_s^2) = \frac{(k+1)(\alpha_{N_s^2}(x_i))^2+(\gamma_{N_s^2}(x_i))^2+(\beta_{N_s^2}(x_i))^2)}{(k+1)((\alpha_{N_s^2}(x_i))^2+(\gamma_{N_s^2}(x_i))^2+(\beta_{N_s^2}(x_i))^2)}$ =1, which again opposes the property 3 of Definition 3.1.
Further, if $N_s^1 = (0, 0, 0)$ and $N_s^2 = (0, 0, 0)$ are two SVNS then according to Jaccrd and Dice similarity measures presented in [29] become undefined or meaningless.
Same as, if $\tilde{N}_s^1 = (y, [0.3, 0.4], [0.2, 0.3], [0.4, 0.5])$ and
$\tilde{N}_s^1 = (y, [0.6, 0.8], [0.4, 0.6], [0.8, 1])$ are two IVNSs, then according to Definition 2.4, $\tilde{N}_s^1 \neq \tilde{N}_s^2$, but the similarity measure presented by Ye [30] gives that, $S(\tilde{N}_s^1, \tilde{N}_s^2) = 1$, that is, $\tilde{N}_s^1 = \tilde{N}_s^2$ which again presents a contradiction with property 3 of Definition 3.1. Also for two IVNSs, $\tilde{N}_s^1 = [0, 0]$ and $\tilde{N}_s^2 = [0, 0]$, we get the meaningless or undefined results by using Equation 9 presented in [15]. So the similarity measures presented in [15, 29, 30] have a deficiency.
Hence, from the above discussion, it is clear that the existing similarity measures have some drawbacks and cannot be able to select the best alternative. Consequently, there is a need to improve the similarity measure which satisfy the axiom of Definition 3.1.

4. **Proposed similarity measures for SVNSs and IVNSs**

In order to overcome the deficiencies present in the above discussed similarity measures, we extend a similarity measure presented by William and Steel [27] for the SVNSs (IVNSs) based on the novel distance measure as:

$$D(N_s^1, N_s^2) = \frac{1}{3n}\sum_{i=1}^{n}\left(\begin{array}{c} \left[\left|\alpha_{N_s^1}(x_i) - \alpha_{N_s^2}(x_i)\right| + \left|\gamma_{N_s^1}(x_i) - \gamma_{N_s^2}(x_i)\right| + \left|\beta_{N_s^1}(x_i) - \beta_{N_s^2}(x_i)\right|\right] + \\ \max\left[\left|\alpha_{N_s^1}(x_i) - \alpha_{N_s^2}(x_i)\right|, \left|\gamma_{N_s^1}(x_i) - \gamma_{N_s^2}(x_i)\right|, \left|\beta_{N_s^1}(x_i) - \beta_{N_s^2}(x_i)\right|\right] \end{array}\right), \tag{1}$$

$$S_m^i(N_s^1, N_s^2) = e^{-\frac{1}{n}D(N_s^1, N_s^2)}, \tag{2}$$

where n is the number of alternatives and $1 \leq i \leq n$.

Similarly for the IVNSs the distance and similarity measures are:

$$\tilde{D}(\tilde{N}_s^1, \tilde{N}_s^2) = \frac{1}{3n}\sum_{i=1}^{n}\left(\begin{array}{c} [|\alpha^l_{\tilde{N}_s^1}(x_i) - \alpha^l_{\tilde{N}_s^2}(x_i)| + |\alpha^u_{\tilde{N}_s^1}(x_i) - \alpha^u_{\tilde{N}_s^2}(x_i)| + \\ |\gamma^l_{\tilde{N}_s^1}(x_i) - \gamma^l_{\tilde{N}_s^2}(x_i)| + |\gamma^u_{\tilde{N}_s^1}(x_i) - \gamma^u_{\tilde{N}_s^2}(x_i)| + \\ |\beta^l_{\tilde{N}_s^1}(x_i) - \beta^l_{\tilde{N}_s^2}(x_i)| + |\beta^u_{\tilde{N}_s^1}(x_i) - \beta^u_{\tilde{N}_s^2}(x_i)|] + \\ \max[|\alpha^l_{\tilde{N}_s^1}(x_i) - \alpha^l_{\tilde{N}_s^2}(x_i)|, |\alpha^u_{\tilde{N}_s^1}(x_i) - \alpha^u_{\tilde{N}_s^2}(x_i)|, \\ |\gamma^l_{\tilde{N}_s^1}(x_i) - \gamma^l_{\tilde{N}_s^2}(x_i)|, |\gamma^u_{\tilde{N}_s^1}(x_i) - \gamma^u_{\tilde{N}_s^2}(x_i)| \\ , |\beta^l_{\tilde{N}_s^1}(x_i) - \beta^l_{\tilde{N}_s^2}(x_i)|, |\beta^u_{\tilde{N}_s^1}(x_i) - \beta^u_{\tilde{N}_s^2}(x_i)|] \end{array}\right), \tag{3}$$

$$\tilde{S}_m^i(\tilde{N}_s^1, \tilde{N}_s^2) = e^{-\frac{1}{n}\tilde{D}(\tilde{N}_s^1, \tilde{N}_s^2)}. \tag{4}$$

Theorem 4.1. *The SM $S_m^i(N_s^1, N_s^2)$ defined in Equation (2) amongst $N_s^1 = \{\langle x_i, \alpha_{N_s^1}(x_i), \gamma_{N_s^1}(x_i), \beta_{N_s^1}(x_i)\rangle\}$ and $N_s^2 = \{\langle x_i, \alpha_{N_s^2}(x_i), \gamma_{N_s^2}(x_i), \beta_{N_s^2}(x_i)\rangle\}$ satisfies the given properties:*

(1) $S_m^i(N_s^1, N_s^2) = 1$ *if and only if* $N_s^1 = N_s^2$,
(2) $S_m^i(N_s^1, N_s^2) = S_m^i(N_s^2, N_s^1)$,
(3) $0 \leq S_m^i(N_s^1, N_s^2) \leq 1$.

Proof

(1) Suppose that, $N_s^1 = N_s^2$ that is, $\alpha_{N_s^1}(x_i) = \alpha_{N_s^2}(x_i)$, $\gamma_{N_s^1}(x_i) = \gamma_{N_s^2}(x_i)$ and $\beta_{N_s^1}(x_i) = \beta_{N_s^2}(x_i)$, then by using Equation (2), we have

$$S_m^i(N_s^1, N_s^2) = e^0 = 1.$$

(2) Consider $S_m^i(N_s^1, N_s^2) = e^{-\frac{1}{n}D(N_s^1, N_s^2)}$

$$= e^{-\frac{1}{3n^2}\sum_{i=1}^{n}\left(\begin{array}{c} \left[\left|\alpha_{N_s^1}(x_i) - \alpha_{N_s^2}(x_i)\right| + \left|\gamma_{N_s^1}(x_i) - \gamma_{N_s^2}(x_i)\right| + \left|\beta_{N_s^1}(x_i) - \beta_{N_s^2}(x_i)\right|\right] + \\ \max\left[\left|\alpha_{N_s^1}(x_i) - \alpha_{N_s^2}(x_i)\right|, \left|\gamma_{N_s^1}(x_i) - \gamma_{N_s^2}(x_i)\right|, \left|\beta_{N_s^1}(x_i) - \beta_{N_s^2}(x_i)\right|\right] \end{array}\right)},$$

$$= e^{-\frac{1}{3n^2}\sum_{i=1}^{n}\left(\begin{array}{c} \left[\left|\alpha_{N_s^2}(x_i) - \alpha_{N_s^1}(x_i)\right| + \left|\gamma_{N_s^2}(x_i) - \gamma_{N_s^1}(x_i)\right| + \left|\beta_{N_s^2}(x_i) - \beta_{N_s^1}(x_i)\right|\right] + \\ \max\left[\left|\alpha_{N_s^2}(x_i) - \alpha_{N_s^1}(x_i)\right|, \left|\gamma_{N_s^2}(x_i) - \gamma_{N_s^1}(x_i)\right|, \left|\beta_{N_s^2}(x_i) - \beta_{N_s^1}(x_i)\right|\right] \end{array}\right)},$$

$$= e^{-\frac{1}{n}D(N_s^2, N_s^1)} = S_m^i(N_s^2, N_s^1),$$

(3) From Equations (1) and (2), it is obvious that, $S^i_m(N^1_s, N^2_s) \leq 1$ and it become zero i.e., $S^i_m(N^1_s, N^2_s) = 0$ only when the distance between N^1_s and N^2_s is very large.

Example 4.2. Let $N^1_s = (x, 0.4, 0.2, 0.6)$ and $N^2_s = (x, 0.2, 0.1, 0.3)$ be two SVNSs, then by using Equations (1) and (2), the similarity measure is, $S^i_m(N^1_s, N^2_s) = 0.7408$.

Example 4.3. Let $\tilde{N}^1_s = (x, [0.3, 0.4], [0.2, 0.3], [0.4, 0.5])$ and $\tilde{N}^2_s = (x, [0.6, 0.8], [0.4, 0.6], [0.8, 1])$ be two IVNSs, then by using Equations (3) and (4), the similarity measure is, $S^i_m(\tilde{N}^1_s, \tilde{N}^2_s) = 0.3679$.

Theorem 4.4. *The SM $\tilde{S}^i_m(\tilde{N}^1_s, \tilde{N}^2_s)$ defined in Equation (4) amongst $\tilde{N}^1_s = \{\left\langle x_i, \alpha_{\tilde{N}^1_s}(x_i), \gamma_{\tilde{N}^1_s}(x_i), \beta_{\tilde{N}^1_s}(x_i)\right\rangle\}$ and $\tilde{N}^2_s = \{\left\langle x_i, \alpha_{\tilde{N}^2_s}(x_i), \gamma_{\tilde{N}^2_s}(x_i), \beta_{\tilde{N}^2_s}(x_i)\right\rangle\}$ satisfies the given properties:*

(1) $\tilde{S}^i_m(\tilde{N}^1_s, \tilde{N}^2_s) = 1$ *if and only if* $\tilde{N}^1_s = \tilde{N}^2_s$,
(2) $\tilde{S}^i_m(\tilde{N}^1_s, \tilde{N}^2_s) = \tilde{S}^i_m(\tilde{N}^2_s, \tilde{N}^1_s)$,
(3) $0 \leq \tilde{S}^i_m(\tilde{N}^1_s, \tilde{N}^2_s) \leq 1$.

Proof The proof of this Theorem is obvious.

4.1. ***Proposed weighted similarity measures (WSM) for SVNSs and IVNSs***

Since the weights of the criteria have a great impact in making decision process therefore we can further extend the proposed similarity measures into the WSM. Let $w = (w_1, w_2, ..., w_m)^T$ be a weight vector of the m criteria with $\sum_{j=1}^m w_j = 1$. In order to get WSM $S^{iw}_m(N^1_s, N^2_s)$ for SVNSs, we first define the weighted distance as:

$$D^w(N^1_s, N^2_s) = \sum_{i=1}^n \sum_{j=1}^m w_j \left(\begin{array}{c} \left[\left|\alpha_{N^1_s}(x_i) - \alpha_{N^2_s}(x_i)\right| + \left|\gamma_{N^1_s}(x_i) - \gamma_{N^2_s}(x_i)\right| + \left|\beta_{N^1_s}(x_i) - \beta_{N^2_s}(x_i)\right|\right] + \\ \max\left[\left|\alpha_{N^1_s}(x_i) - \alpha_{N^2_s}(x_i)\right|, \left|\gamma_{N^1_s}(x_i) - \gamma_{N^2_s}(x_i)\right|, \left|\beta_{N^1_s}(x_i) - \beta_{N^2_s}(x_i)\right|\right] \end{array} \right), \tag{5}$$

and

$$S^{iw}_m(N^1_s, N^2_s) = e^{-\frac{1}{n}D^w(N^1_s, N^2_s)}. \tag{6}$$

In the similar way, a WSM $\tilde{S}^{iw}_m(\tilde{N}^1_s, \tilde{N}^2_s)$ on the basis of weighted distance $\tilde{D}^w(\tilde{N}^1_s, \tilde{N}^2_s)$ for IVNSs is obtained as:

$$\tilde{D}^w(\tilde{N}^1_s, \tilde{N}^2_s) = \sum_{i=1}^n \sum_{j=1}^m w_j \left(\begin{array}{c} [|\alpha^l_{\tilde{N}^1_s}(x_i) - \alpha^l_{\tilde{N}^2_s}(x_i)| + |\alpha^u_{\tilde{N}^1_s}(x_i) - \alpha^u_{\tilde{N}^2_s}(x_i)| + \\ |\gamma^l_{\tilde{N}^1_s}(x_i) - \gamma^l_{\tilde{N}^2_s}(x_i)| + |\gamma^u_{\tilde{N}^1_s}(x_i) - \gamma^u_{\tilde{N}^2_s}(x_i)| + \\ |\beta^l_{\tilde{N}^1_s}(x_i) - \beta^l_{\tilde{N}^2_s}(x_i)| + |\beta^u_{\tilde{N}^1_s}(x_i) - \beta^u_{\tilde{N}^2_s}(x_i)|] + \\ \max[|\alpha^l_{\tilde{N}^1_s}(x_i) - \alpha^l_{\tilde{N}^2_s}(x_i)|, |\alpha^u_{\tilde{N}^1_s}(x_i) - \alpha^u_{\tilde{N}^2_s}(x_i)|, \\ |\gamma^l_{\tilde{N}^1_s}(x_i) - \gamma^l_{\tilde{N}^2_s}(x_i)|, |\gamma^u_{\tilde{N}^1_s}(x_i) - \gamma^u_{\tilde{N}^2_s}(x_i)| \\ , |\beta^l_{\tilde{N}^1_s}(x_i) - \beta^l_{\tilde{N}^2_s}(x_i)|, |\beta^u_{\tilde{N}^1_s}(x_i) - \beta^u_{\tilde{N}^2_s}(x_i)|] \end{array} \right), \tag{7}$$

and

$$\tilde{S}_m^{iw}(\tilde{N}_s^1, \tilde{N}_s^2) = e^{-\frac{1}{n}\tilde{D}^w(\tilde{N}_s^1, \tilde{N}_s^2)}. \tag{8}$$

Theorem 4.5. *Let $N_s^1 = \{< x_i,\ \alpha_{N_s^1}(x_i),\ \gamma_{N_s^1}(x_i),\ \beta_{N_s^1}(x_i) >\}$ and $N_s^2 = \{< x_i,\ \alpha_{N_s^2}(x_i),\ \gamma_{N_s^2}(x_i),\ \beta_{N_s^2}(x_i) >\}$ be two SVNSs (IVNSs) , then the WSM presented in Equation (6) (Equation (8)) between two SVNSs (IVNSs) satisfies the following properties:*

(1) $0 \leq S_m^{iw}(N_s^1, N_s^2) \leq 1$,

(2) $S_m^{iw}(N_s^1, N_s^2) = S_m^{iw}(N_s^2, N_s^1)$,

(3) $S_m^{iw}(N_s^1, N_s^2) = 1$ *if and only if* $N_s^1 = N_s^2$.

Proof It is obvious as Theorem 4.1.

Example 4.6. Let $N_s^1 = \{x, (0.3, 0.2, 0.5), (0.4, 0.6, 0.0)\}$ and $N_s^2 = \{x, (0.1, 0.1, 0.8), (0.2, 0.1, 0.7)\}$ be two SVNSs and $w = (0.7, 0.3)^T$ the weight vector, then the WSM for SVNSs is: $S_m^{iw}(N_s^1, N_s^2) = 0.9162$.

Example 4.7. Let $\tilde{N}_s^1 = \{x, ([0.4, 0.6], [0.2, 0.3], [0.3, 0.4]), ([0.5, 0.8], [0.1, 0.4], [0.1, 0.3])\}$ and $\tilde{N}_s^2 = \{x, ([0.7, 0.9], [0.1, 0.2], [0.1, 0.2]), ([0.3, 0.6], [0.1, 0.3], [0.4, 0.7])\}$ be two IVNSs and $w = (0.6, 0.4)^T$ the weight vector, then the weighted similarity measure for IVNSs is: $S_m^{iw}(N_s^1, N_s^2) = 0.8781$.

5. Decision making model under SVNSs (IVNSs)

The model for MCDM problems is presented on the basis of proposed weighted similarity measure in this section. Suppose that $Q = \{Q_1, Q_2, ..., Q_n\}$ is a discrete set of alternatives and $G = \{G_1, G_2, ..., G_m\}$ is another discrete set of criteria. If the DMs gave the various values for the alternative $Q_i (i = 1, 2, ..., n)$ under the criteria $G_j (j = 1, 2, ..., m)$, and form a neutrosofic decision matrix $N = [b_{ij}]_{n\times m}$. The concept of optimal solution assists the DMs to identify the best alternative from the decision set in MCDM framework. In spite of the fact that the perfect option does not exist in actual, it provides a valuable paradigm to appraise alternatives. Hence, we can find the ideal options $N^\star$ from the given information as $N^\star = \max([b_{ij}]_{n\times m})$. Since the weights of the criteria have an excessive impact, thereby a weighing vector of criteria is provided as $w = (w_1, w_2, w_3, ..., w_m)^T$, where $\sum_{j=1}^m w_j = 1$ and $w_j > 0$, can be evaluated by using the LP model presented in Definition 2.5. The model based on proposed weighted similarity measure described by Equation (6) (Equation (8)) has the following steps.

Step 1. Based on the information provided by DMs, form a single valued neutrosophic decision matrix (SVNDM) denoted by $N = [b_{ij}]_{n\times m}$.
Step 2. Find the optimal solution $N^\star$ from the SVNDM.
Step 3. On the basis of TOPSIS, an objective function is obtained and then calculate the

weights of criteria by using LP model as described in Definition 2.5.

Step 4. With the aid of weights evaluated in Step 3, calculate the similarity measures amongst the alternative $Q_i(i = 1, 2, ..., n)$ and the optimal alternative $N^\star$ by using Equation (6) (Equation (8)).

Step 5. Rank all the alternatives $Q_i(i = 1, 2, ..., n)$ from highest to lowest values of similarity measures obtained in Step 4 and choose the alternative having highest value of the similarity measure.

6. **Practical examples**

In this section, a medical diagnosis decision problem is considered to see the validity and effectiveness of the proposed MCDM model.

Example 1. For parents, it is significant to be aware of the most updated treatment process so you can be certain about your kids are getting the superlative care possible. According to the child specialist, some common childhood sicknesses and their appropriate symptoms are listed. Suppose a collection of diagnoses, chest infections (C), malaria (M), typhoid (T), sore throat (S) and bronchitis (B) are examined on the basis of some symptoms, fever (S_1), headache (S_2), breathlessness (S_3), cough (S_4) and chest pain (S_5). All the information is given in the form of neutrosophic decision matrix (NDM) $N = [b_{ij}]_{n\times m}$. Assume that patient $K_1 = N^\star$ has all the symptoms in the diagnosis process, all the information collected about the kids $K_i(i = 1, 2, ..., n)$ is provided in the form of SVNS in Table 1.

$$\begin{aligned}
\text{Maximize:}\quad & Z = 0.2175w_1 + 0.2350w_2 + 0.2200w_3 + 0.1950w_4 + 0.1850w_5\\
\text{Subject to:}\quad & 10w_1 + 8w_2 + 12w_3 + 10w_4 + 15w_5 \geq 10,\\
& 10w_1 + 8w_2 + 12w_3 + 10w_4 + 15w_5 \leq 10.5,\\
& 8w_1 + 11w_2 + 7w_3 + 10w_4 + 10w_5 \geq 8,\\
& 8w_1 + 11w_2 + 7w_3 + 10w_4 + 10w_5 \leq 8.5,\\
& 12w_1 + 15w_2 + 12w_3 + 10w_4 + 6w_5 \geq 12,\\
& 12w_1 + 15w_2 + 12w_3 + 10w_4 + 6w_5 \leq 12.5,\\
& w_1 + w_2 + w_3 + w_4 + w_5 = 1,\\
& w_1, w_2, ..., w_5 \geq 0.
\end{aligned}$$

Table 1. Neutrosophic decision matrix NDM

Daignosis	S_1	S_2	S_3	S_4	S_5
C	$< 0.4, 0.6, 0.0 >$	$< 0.3, 0.2, 0.5 >$	$< 0.1, 0.3, 0.7 >$	$< 0.4, 0.3, 0.3 >$	$< 0.1, 0.2, 0.7 >$
M	$< 0.7, 0.3, 0.0 >$	$< 0.2, 0.2, 0.6 >$	$< 0.0, 0.1, 0.9 >$	$< 0.7, 0.3, 0.0 >$	$< 0.1, 0.1, 0.8 >$
T	$< 0.3, 0.4, 0.3 >$	$< 0.6, 0.3, 0.1 >$	$< 0.2, 0.1, 0.7 >$	$< 0.2, 0.2, 0.6 >$	$< 0.1, 0.0, 0.9 >$
S	$< 0.1, 0.2, 0.7 >$	$< 0.2, 0.4, 0.4 >$	$< 0.8, 0.2, 0.0 >$	$< 0.2, 0.1, 0.7 >$	$< 0.2, 0.1, 0.7 >$
B	$< 0.1, 0.1, 0.8 >$	$< 0.0, 0.2, 0.8 >$	$< 0.2, 0.0, 0.8 >$	$< 0.2, 0.0, 0.8 >$	$< 0.8, 0.1, 0.1 >$

Step 1. Based on the information provided by the professional, form a SVNDM $N = [n_{ij}]_{5\times 5}$.

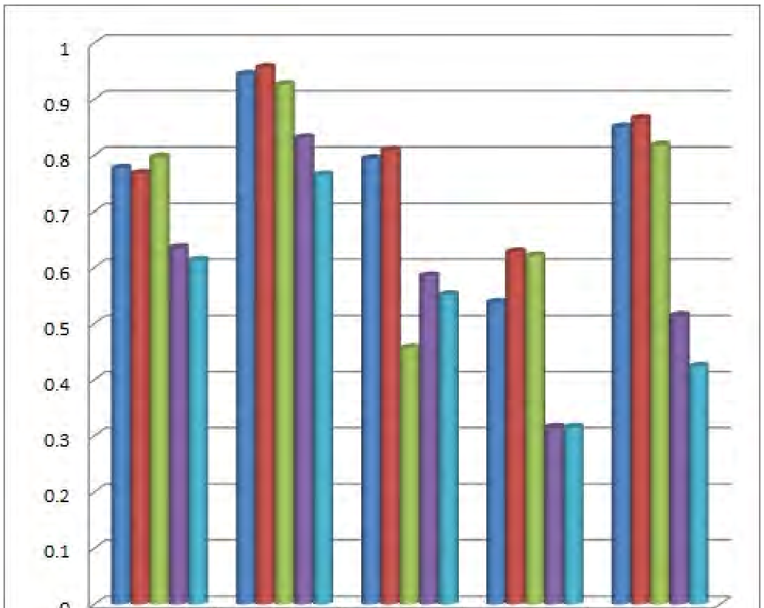
Step 2. Assume that a kid $K_1 = \{(0.8, 0.2, 0.1), (0.9, 0.3, 0.2), (0.2, 0.1, 0.8), (0.6, 0.5, 0.1), (0.1, 0.4, 0.6)\}$ has all the symptoms in the process of diagnosis.

Step 3. By using TOPSIS an objective function is obtained and then calculate the weights of criteria by applying the LP model as described in Definition 2.5.

Step 4. The values of the weighted similarity measure calculated with the help of Equation (6) amongst the diagnoses and the kid K_1 are: $S_m^{1w} = 0.7774$, $S_m^{2w} = 0.7675$, $S_m^{3w} = 0.7969$, $S_m^{4w} = 0.6353$ and $S_m^{5w} = 0.6127$.

Step 5. According to values obtained in Step 4, we get the ranking order as: $T \succ C \succ M \succ B \succ S$. Figure 1 indicates the ranking order presented in [8, 9, 16, 29] and the proposed model graphically.

Figure 1. Ranking order of alternatives



	Proposed	[4]	[5]	[12]	[24]
C	0.7774	0.9443	0.7941	0.5383	0.8505
M	0.7675	0.9571	0.8094	0.6282	0.8661
T	0.7969	0.9264	0.4568	0.6206	0.8185
S	0.6353	0.8314	0.5851	0.3154	0.5148
B	0.6127	0.765	0.5517	0.3154	0.4244

Example 2. Consider the same scenario as Example 1 with interval-valued data provided in Table 2. Assume that another Kid K_2 suffers from all the symptoms, which can be expressed by the following IVNS data.

Step 1. Based on the information given by the professional form an interval-valued neutrosofic decision matrix (INDM) denoted by $\tilde{N} = [\tilde{n}_{ij}]_{5\times5}$.

Step 2. Assume a kid $K_2 = \{([0.3, 0.5], [0.2, 0.3], [0.4, 0.5]), ([0.7, 0.9], [0.1, 0.2], [0.1, 0.2]), ([0.4, 0.6], [0.2, 0.3], [0.3, 0.4]), ([0.3, 0.6], [0.1, 0.3], [0.4, 0.7]), ([0.5, 0.8], [0.1, 0.4], [0.1, 0.3])\}$ has all the symptoms in the process of diagnosis.

Step 3. Use the same weights for the symptoms which are evaluated in Example 1.

Step 4. The values of the weighted similarity measure calculated with the help of Equation

Table 2. Neutrosofic decision matrix NDM

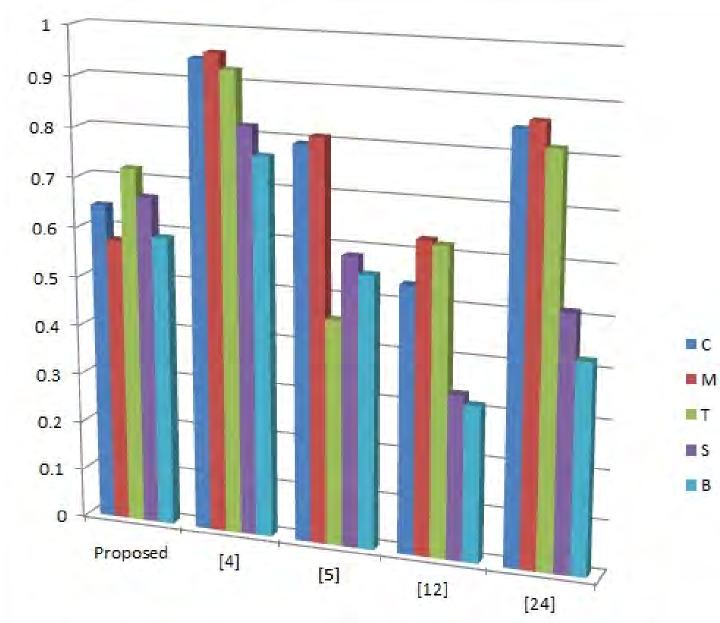
Daignosis	S_1	S_2	S_3
C	$([0.4, 0.4], [0.6, 0.6], [0.0, 0.0])$	$([0.3, 0.3], [0.2, 0.2], [0.5, 0.5])$	$([0.1, 0.1], [0.3, 0.3], [0.7, 0.7])$
M	$([0.7, 0.7], [0.3, 0.3], [0.0, 0.0])$	$([0.2, 0.2], [0.2, 0.2], [0.6, 0.6])$	$([0.0, 0.0], [0.1, 0.1], [0.9, 0.9])$
T	$([0.3, 0.3], [0.4, 0.4], [0.3, 0.3])$	$([0.6, 0.6], [0.3, 0.3], [0.1, 0.1])$	$([0.2, 0.2], [0.1, 0.1], [0.7, 0.7])$
S	$([0.1, 0.1], [0.2, 0.2], [0.7, 0.7])$	$([0.2, 0.2], [0.4, 0.4], [0.4, 0.4])$	$([0.8, 0.8], [0.2, 0.2], [0.0, 0.0])$
B	$([0.1, 0.1], [0.1, 0.1], [0.8, 0.8])$	$([0.0, 0.0], [0.2, 0.2], [0.8, 0.8])$	$([0.2, 0.2], [0.0, 0.0], [0.8, 0.8])$

Daignosis	S_4	S_5
C	$([0.4, 0.4], [0.3, 0.3], [0.3, 0.3])$	$([0.1, 0.1], [0.2, 0.2], [0.7, 0.7])$
M	$([0.7, 0.7], [0.3, 0.3], [0.0, 0.0])$	$([0.1, 0.1], [0.1, 0.1], [0.8, 0.8])$
T	$([0.2, 0.2], [0.2, 0.2], [0.6, 0.6])$	$([0.1, 0.1], [0.0, 0.0], [0.9, 0.9])$
S	$([0.2, 0.2], [0.1, 0.1], [0.7, 0.7])$	$([0.2, 0.2], [0.1, 0.1], [0.7, 0.7])$
B	$([0.2, 0.2], [0.0, 0.0], [0.8, 0.8])$	$([0.8, 0.8], [0.1, 0.1], [0.1, 0.1])$

(8) amongst the diagnoses and the kid K_1 are: $\tilde{S}_m^{1w} = 0.6445$, $\tilde{S}_m^{2w} = 0.5760$, $\tilde{S}_m^{3w} = 0.7222$, $\tilde{S}_m^{4w} = 0.6668$ and $\tilde{S}_m^{5w} = 0.5884$.

Step 5. The ranking order obtained by using the values calculated in Step 4 is: $T \succ C \succ M \succ B \succ S$. A graphical representation of ranking order presented in [8, 9, 16, 29] and the proposed model is shown in Figure 2.

Figure 2. Ranking order of alternatives



	Proposed	[4]	[5]	[12]	[24]
C	0.6445	0.9443	0.7941	0.5385	0.8505
M	0.576	0.9571	0.8094	0.6282	0.8661
T	0.7222	0.9264	0.4568	0.6206	0.8185
S	0.6668	0.8214	0.5851	0.3336	0.5148
B	0.5884	0.765	0.5517	0.3154	0.4244

7. Comparative analysis with the existing techniques

Various DMs have applied the SMs for medical diagnosis in the environment of SVNSs and IVNSs [8, 9, 16, 29]. In order to portray the usefulness and validation of the proposed SMs, we apply it for the same problem and the results are shown in the Tables 3 and 4. According to the results obtained by applying our proposed MCDM model, we see that the Kids K_1 and K_2 suffered in the disease typhoid (T) under the observations of five symptoms $S_j (j = 1, 2, ..., 5)$. The results obtained by proposed and existing methods are different because of assigning the weights to the criteria, These results are further analyzed by using Spearman's correlation coefficient.

Table 3. Results obtained by proposed SVNS's SM

SMs	C	M	T	S	B	Ranking
Proposed	0.7774	0.7675	0.7969	0.6353	0.6127	$T \succ C \succ M \succ B \succ S$
[8]	0.9443	0.9571	0.9264	0.8214	0.7650	$M \succ C \succ T \succ S \succ B$
[9]	0.7941	0.8094	0.4568	0.5851	0.5517	$M \succ C \succ S \succ B \succ T$
[16]	0.5385	0.6282	0.6206	0.3336	0.3154	$M \succ T \succ C \succ S \succ B$
[28]	0.8505	0.8661	0.8185	0.5148	0.4244	$M \succ C \succ T \succ S \succ B$

Table 4. Results obtained by proposed IVNS's SM

SMs	C	M	T	S	B	Ranking
Proposed	0.6445	0.5760	0.7222	0.6668	0.5884	$T \succ C \succ M \succ B \succ S$
[8]	0.9443	0.9571	0.9264	0.8214	0.7650	$M \succ C \succ T \succ S \succ B$
[9]	0.7941	0.8094	0.4568	0.5851	0.5517	$M \succ C \succ S \succ B \succ T$
[16]	0.5385	0.6282	0.6206	0.3336	0.3154	$M \succ T \succ C \succ S \succ B$
[28]	0.8505	0.8661	0.8185	0.5148	0.4244	$M \succ C \succ T \succ S \succ B$

7.1. ***Ranking analysis with Spearman's rank-correlation coefficient***

. The ranking preference of the diagnosis obtained by our and existing techniques are different and presented in Tables 3 and 4. In order to compare the diagnosis further, we use the Spearman's rank-correlation coefficient (ρ_s) and the critical value Z, where, ρ_s and Z can be calculated with the formulae given below:

$$\rho_s = 1 - 6 \sum_{l=1}^{i-1=k} \frac{(\triangle^l)^2}{n(n-1)},$$

and

$$Z = \rho_s \sqrt{n-1}.$$

Here, $\triangle^l$ is the difference between two sets of ranking. The values of ρ_s are always bounded in the closed interval $[-1,1]$. The values of ρ_s which are nearer to ± 1 show the perfect relationship amongst two ranking orders. Moreover,the critical value Z is compared with a pre-estimated degree of significance value η. The critical value Z corresponding to the degree of significance value $\eta = 0.05$ for the examples $(n = 5)$ is, $Z_{0.05} = 0.9$. If the critical value Z more than 0.9, it indicates that there exist a strong relationship between two rankings. On the other hand, the two rankings can be considered as dissimilar or have weaker relationship.

There are five collections of preference rankings obtained by the proposed method and [8, 9, 16, 28], represented by X, Y, V, T and U, respectively and their ranking order can be seen in Tables 3 and 4. In order to compare these ranking orders, ρ_s and Z evaluated in Table 5. The analysis of the results is summarized in Table 5 as follows:

The results obtained by the proposed model with those obtained in [8] and [28], the critical

Table 5. Comparison with existing methods

Daignosis	X	Y	V	T	U	X-Y	X-V	X-T	X-U
C	2	2	2	3	2	0	0	-1	0
M	3	1	1	1	1	2	2	2	2
T	1	3	5	2	3	-2	-4	-1	-2
S	5	4	3	4	4	1	2	1	1
B	4	5	4	5	5	-1	0	-1	-1
Spearman's rank-correlation coefficient ρ_s						0.5	-0.2	0.6	0.5
Critical value Z						1	-0.4	1.2	1

value $Z = 1 > 0.9$, shows that there is a positive relationship between the ranking of the proposed model (X), the ranking [8] (Y) and [28] (U). Also, the results obtained by the proposed model (X) with those obtained in [16] (T), the critical value $Z = 1.2 > 0.9$ indicates that there is a strongly positive relationship between the ranking X and T. However, the ranking X of the proposed model is significantly dissimilar to the ranking [9] (V) because the critical value $Z = -0.4$ is smaller than 0.9.

8. Conclusions

The similarity measures are extensively utilized in MCDM problems from the last few decades. This paper suggested a novel technique to develop the similarity measures on the basis of Euclidean distance measure for SVNSs and IVNSs, respectively. However, the similarity measures presented in [15, 29, 30] have some shortcoming. On the other hand the suggested similarity measures satisfy all the axioms of the similarity measure. Moreover, we used the suggested similarity measures to medical diagnosis decision problems. A practical example is used to exemplify the practicability and efficiency of the proposed similarity measure, which are then compared to other existing similarity measures. We will emphasize to apply the proposed similarity measure in pattern recognition and supply chain problems under the framework of SVNSs and IVNSs in future.

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Received: 23 March 2020 / Accepted: 22 May 2020

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Application of Single Valued Trapezoidal Neutrosophic Numbers in Transportation Problem

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Abstract: In the present paper, we introduced the concept of single valued trapezoidal neutrosophic number, which is generalization of single valued neutrosophic number. A generalization of crisp, fuzzy and intuitionistic fuzzy sets represents as neutrosophic sets, which have uncertainty, inconsistent, and incompleteness information in real world problem. De-neutrosophication is a process to convert neutrosophic number into a crisp number for practical applications. For unbalanced neutrosophic transportation problem, we also use here minimum row column method and set a comparison among crisp and neutrosophic optimal solutions. Here we use two models of transportation problems to understand the applications in neutrosophic environment.

Keywords: Fuzzy Number, Single valued trapezoidal neutrosophic number, De-neutrosophication, neutrosophic transportation problem.

1. Introduction

In present scenario the classical theory of mathematics can't be handling the different kind of uncertainties, vagueness or impreciseness of mathematical problems. Many researchers around the world define many approaches to understand or define it. In 1965, Zadeh [37] first time introduce the mathematical formulation of a fuzzy set (FS) as a set with its membership function or membership grade. Sometimes the membership function in FS was not suitable one to describe the ambiguity of a problem.

After development of FS theory in various fields of uncertainty, Atanassov [1] in 1986, believe about the belongingness and non-belongingness in fuzzy set and present it's extension as intuitionistic fuzzy set (IFS) theory, which included the degree of membership and degree of non-membership function of each element in the set. More development of IFS theory in decision problems plays key role in recent scenario [17, 20]. In real life decision making problems, the theory of FS and IFS is much applicable, IFS approach in the solution of transportation problems used by many researchers [15, 22, 23].

The basic theme of a transportation problem is to find a direct connection between source and destination in minimum time with minimum cost. Hitchcock [12] was first, who originally developed the basic results of transportation problem by simplex method, which was recognized as special mathematical module. Since in early stage the transportation parameters like transportation cost, demand and supply were on the crisp values. In present time the real life transportation problems have uncertain, uncontrolled factors as the transportation cost, supply and demand are in fuzzy values.

In that period many research problems related to fuzzy transportation problem (FTP) were solved, in which some are partial fuzzy and some are fully fuzzy. A FTP in which cost demand and supply are as fuzzy number is called fully FTP while in case of either cost, demand or supply are in fuzzy number, then it is FTP see [24, 7]. In a fuzzy solid transportation problem the parameters are trapezoidal fuzzy number (TrFN), introduced by Jim´enez and Verdegay [16] in 1999. For more research work about FTP, see [18, 19, 22, 25].

In current scenario, due to uncertainty, unawareness, vagueness, ambiguity in the constraints or some poor handling of data, the indeterminacy exists in transportation problems. The IFS theory can handle the problems of incomplete information but not the indeterminate and inconsistent information exists in transportation modal.

The problems with inconsistent information or indeterminate cannot be handled by any evocation of fuzzy set, so to overpower of such problems, Smarandache [27] introduced the neutrosophic set (NS) in 1988, which was an extension of classical set, FS and IFS. The well applicable fundamentals of NS, to represent the indeterminacy and inconsistent information are truth-membership degree, indeterminacy membership degree, and falsity-membership degree. The NS becomes the IFS, if indeterminacy membership degree I(x) of NS is equal to hesitancy degree h(x) of IFS. For practical applications and some technical references in NS, Wang et. al. [31] in 2010 introduced the idea of single valued neutrosophic set (SVNS). The notion of SVNS is more suitable and effective in solving many real life problems of decision making and supply chain management. For more applications of FS, IFS and NS in some different fields see [1- 10, 14, 21, 29- 32, 34, 36].

Since the study of transportation models with optimal and effective cost play a key role in every real life situation. Many researchers formulated efficient mathematical models in various uncertain environments. For practical application, two models of neutrosophic transportation problem (NTP) with all entries such as cost, demand, and supply are as single valued trapezoidal neutrosophic number (SVTrNN). Here we also use minimum row column method (MRCM) for balance the unbalance crisp transportation problem (CTP) and NTP with some existing method.

The main features of the paper are obtaining the optimal solutions of CTP and NTP after balancing with different methods and to compare the results. The paper is well organized in seven sections. In section first, introduction of the present paper with some earlier research are given. In second section, the basics concepts of FS, IFS and NS are discussed and reviewed. In third section, introduce the de-neutrosophication as score function to convert neutrosophic values into crisp values. Section fourth composes the classification and mathematical formulation of CTP & NTP of type-2 & 3. In fifth section, we introduce the procedures for solutions of CTP & NTP. In section six, seven and eight, we introduce two models of transportation with their solutions in different tables, their comparison, result and discussion. The conclusion and future aspects of research work exhibit in last section of the paper.

2. Preliminaries

2.1. Some basic definitions and examples

Definition 2.1.1. (FS [37]): A FS $\tilde{A}$ of a non empty set X is defined as $\tilde{A}=\{\langle x,\mu_{\tilde{A}}(x)\rangle / x\in X\}$ where $\mu_{\tilde{A}}(x)$ is called the membership function such that $\mu_{\tilde{A}}(x): X\rightarrow[0,1]$.

Definition 2.1.2. (FN)**:** A convex, normalized fuzzy set $\tilde{A}$ is called fuzzy number on the universal set of real numbers R, if the membership function $\mu_{\tilde{A}}$ of $\tilde{A}$ has the following belongingness:

(i) $\mu_{\tilde{A}} : X\rightarrow \left[0,1\right]$ is continuous

(ii) $\mu_{\tilde{A}}(x)=0,$ for all $x\in\left(-\infty,a\right]\cup\left[d,\infty\right)$

(iii) $\mu_{\tilde{A}}(x)$ is strictly increasing on $\left[a,\ b\right]$ and strictly decreasing on $\left[c,\ d\right]$

(iv) $\mu_{\tilde{A}}(x) = 1,$ for all $x \in [b,\ c]$, where $a \le b \le c \le d$.

Definition 2.1.3. (TrFN[19]): A trapezoidal fuzzy number (TrFN) denoted as $\tilde{A} = (a,b,c,d)$, with its membership function $\mu_{\tilde{A}}(x)$, on **R**, is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} (x-a)/(b-a), & \text{for } a \le x < b \\ 1, & \text{for } b \le x < c \\ (d-x)/(d-c), & \text{for } c < x \le d \\ 0, & \text{otherwise} \end{cases}$$

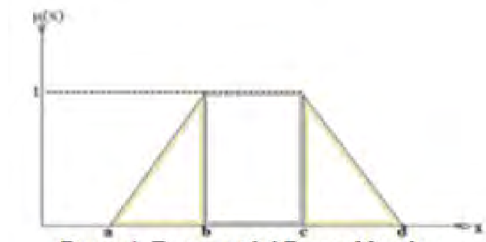

Figure1: Trapezoidal Fuzzy Number

If $b = c$ in TrFN $\tilde{A} = (a,b,c,d)$, then it becomes TFN $\tilde{A} = (a,b,d)$.

Definition 2.1.4. An IFS in a non-empty set X is denoted by $\tilde{A}^I$ and defined as $\tilde{A}^I = \left\{\left\langle x, \mu_{\tilde{A}^I}, \nu_{\tilde{A}^I} \right\rangle : x \in X\right\}$, where $\mu_{\tilde{A}^I}, \nu_{\tilde{A}^I} : X \to [0,1]$, such that $0 \le \mu_{\tilde{A}^I}, \nu_{\tilde{A}^I} \le 1, \forall x \in X$. The degree of membership $\mu_{\tilde{A}^I}$ and degree of non-membership $\nu_{\tilde{A}^I}$ are functions from X to [0,1] in $\tilde{A}^I$. The degree of hesitation is defined as $h(x) = 1 - \mu_{\tilde{A}^I} - \nu_{\tilde{A}^I} \le 1, \forall x \in X$ in $\tilde{A}^I$.

Definition 2.1.5. (ITrFN [20]): An Intutinistic trapezoidal fuzzy number (ITrFN) is denoted by $\tilde{A}^I = (a_1,a_2,a_3,a_4)(a_1',a_2,a_3,a_4')$ where $a_1' \le a_1 \le a_2 \le a_3 \le a_4 \le a_4'$ with membership function $\mu_{\tilde{A}^I}$ and non-membership function $\nu_{\tilde{A}^I}$ defined by

$$\mu_{\tilde{A}^I}(x) = \begin{cases} 0, & \text{for } x < a_1, \\ \dfrac{x - a_1}{a_2 - a_1}, & \text{for } a_1 \le x \le a_2, \\ 1, & \text{for } a_2 \le x \le a_3, \\ \dfrac{a_4 - x}{a_4 - a_3}, & \text{for } a_3 \le x \le a_4, \\ 0, & \text{for } x > a_4. \end{cases} \qquad \nu_{\tilde{A}^I}(x) = \begin{cases} 1, & \text{for } x < a_1', \\ \dfrac{x - a_1'}{a_2 - a_1'}, & \text{for } a_1' \le x \le a_2, \\ 0, & \text{for } a_2 \le x \le a_3, \\ \dfrac{a_4' - x}{a_4' - a_3}, & \text{for } a_3 \le x \le a_4', \\ 1, & \text{for } x > a_4'. \end{cases}$$

If $a_2 = a_3$ then ITrFN becomes ITFN denoted as $\tilde{A}^I = (a_1,a_2,a_3)(a_1',a_2',a_3')$ where $a_1' \le a_1 \le a_2 \le a_3 \le a_3'$.

Definition 2.1.6. ([4]): Let x be a generic element of a non empty set X. A neutrosophic number $\tilde{A}^N$ in X is defined as $\tilde{A}^N = \{\langle x, T_{\tilde{A}^N}(x), I_{\tilde{A}^N}(x), F_{\tilde{A}^N}(x)\rangle / x \in X\}$, $\forall\ T_{\tilde{A}^N}(x),\ I_{\tilde{A}^N}(x)$ and $F_{\tilde{A}^N}(x) \in]^-0,1^+[$ where $T_{\tilde{A}^N} : X \to]0^-,1^+[$, $I_{\tilde{A}^N} : X \to]0^-,1^+[$ and $F_{\tilde{A}^N} : X \to]0^-,1^+[$ are functions of truth-membership, indeterminacy membership and falsity-membership in $\tilde{A}^N$ respectively also there is no restrictions on the sum of $T_{\tilde{A}^N}(x)$, $I_{\tilde{A}^N}(x)$ and $F_{\tilde{A}^N}(x)$ so that $^-0 \le T_{\tilde{A}^N}(x) + I_{\tilde{A}^N}(x) + F_{\tilde{A}^N}(x) \le 3^+$.

For the practical applications it is difficult to apply directly NS theory, hence the notion of SVNS as well as single valued neutrosophic numbers [SVNN] introduced by Deli I., S¸uba Y[8] in 2014 .

Definition 2.1.7. (SVNS [8]): Let x be the generic point of a non-empty space X. A SVNS is denoted and defined as $\tilde{A}^N = \{\langle x, T_{\tilde{A}^N}(x), I_{\tilde{A}^N}(x), F_{\tilde{A}^N}(x)\rangle / x \in X\}$, where for each point x in X, $T_{\tilde{A}^N}(x)$, called truth membership $I_{\tilde{A}^N}(x)$ called indeterminacy membership and $F_{\tilde{A}^N}(x)$ called falsity membership function in [0, 1] and $0 \le T_{\tilde{A}^N}(x) + I_{\tilde{A}^N}(x) + F_{\tilde{A}^N}(x) \le 3$.

For continuous SVNS $\tilde{A}^N$ can be written as

$$\tilde{A}^N = \int_{\tilde{A}^N} \left\langle T_{\tilde{A}^N}(x), I_{\tilde{A}^N}(x), F_{\tilde{A}^N}(x) \right\rangle / x, \quad x \in X$$

When X is discrete, a SVNS $\tilde{A}^N$ can be written as

$$\tilde{A}^N = \sum_{i=1}^{n} \left\langle T_{\tilde{A}^N}(x_i), I_{\tilde{A}^N}(x_i), F_{\tilde{A}^N}(x_i) \right\rangle / x_i, \quad x_i \in X$$

Example 2.1.1. Let X be a space with capability x_1, trustworthiness x_2 and price x_3 in [0,1]. If expert wants "degree of good services", "degree of indeterminacy" and degree of poor services", then a SVNS $\tilde{A}^N$ of X is defined as

$$\tilde{A}^N = \langle 0.7, 0.1, 0.3\rangle / x_1 + \langle 0.4, 0.2, 0.7\rangle / x_2 + \langle 0.5, 0.1, 0.6\rangle / x_3.$$

Definition 2.1.8. An $(\alpha, \beta, \gamma)-$cut set of SVNS $\tilde{A}^N$, a crisp subset of R is defined by

$$\tilde{A}^N_{\alpha,\beta,\gamma} = \{x : T_{\tilde{A}}(x) \ge \alpha, I_{\tilde{A}}(x) \le \beta, F_{\tilde{A}}(x) \le \gamma\}$$

where $0 \le \alpha \le 1, 0 \le \beta \le 1, 0 \le \gamma \le 1$ and $0 \le \alpha + \beta + \gamma \le 3$.

Definition 2.1.9. A SVNS $\tilde{A}^N = \{\langle x, T_{\tilde{A}^N}(x), I_{\tilde{A}^N}(x), F_{\tilde{A}^N}(x)\rangle : x \in X\}$ is called neut-normal, if there exist at least three points $x_1, x_2, x_3 \in X$ such that $T_{\tilde{A}^N}(x_1) = 1, I_{\tilde{A}^N}(x_2) = 1, F_{\tilde{A}^N}(x_3) = 1$.

Definition 2.1.10. A SVNS $\tilde{A}^N = \{\langle x, T_{\tilde{A}^N}(x), I_{\tilde{A}^N}(x), F_{\tilde{A}^N}(x)\rangle : x \in X\}$ is called neut-convex set on the real line; if the following conditions are satisfied $\forall\ x_1, x_2, x_3 \in R$ and $\lambda \in [0,1]$

(i) $T_{\tilde{A}^N}(\lambda x_1 + (1-\lambda)x_2) \ge \min(T_{\tilde{A}^N}(x_1), T_{\tilde{A}^N}(x_2))$

(ii) $I_{\tilde{A}^N}(\lambda x_1 + (1-\lambda)x_2) \le \max(I_{\tilde{A}^N}(x_1), I_{\tilde{A}^N}(x_2))$

(iii) $F_{\tilde{A}^N}(\lambda x_1 + (1-\lambda)x_2) \le \max(F_{\tilde{A}^N}(x_1), F_{\tilde{A}^N}(x_2))$

Definition 2.1.11. (SVTrNN [8]): A single valued trapezoidal neutrosophic number (SVTrNN) $\tilde{a}^N = \langle (a_1, a_2, a_3, a_4); w_{\tilde{a}}, u_{\tilde{a}}, v_{\tilde{a}}\rangle$ is a special NS on the real line **R,** whose truth-membership $T_{\tilde{a}^N}(x)$, indeterminacy-membership $I_{\tilde{a}^N}(x)$, and a falsity-membership $F_{\tilde{a}^N}(x)$ are given as follows:

$$T_{\tilde{a}^N}(x)=\begin{cases}\dfrac{((x-a_1)w_{\tilde{a}^N}}{a_2-a_1}, & \text{for } a_1\le x\le a_2,\\ w_{\tilde{a}^N}, & \text{for } a_2\le x\le a_3,\\ \dfrac{(a_4-x)w_{\tilde{a}^N}}{a_4-a_3}, & \text{for } a_3\le x\le a_4,\\ 0, & \text{for } x>a_4 \text{ and } x<a_1\end{cases}$$

$$I_{\tilde{a}^N}(x)=\begin{cases}\dfrac{a_2-x+(x-a_1)u_{\tilde{a}^N}}{a_2-a_1}, & \text{for } a_1\le x\le a_2,\\ u_{\tilde{a}^N}, & \text{for } a_2\le x\le a_3,\\ \dfrac{x-a_3+(a_4-x)u_{\tilde{a}^N}}{a_4-a_3}, & \text{for } a_3\le x\le a_4,\\ 1, & \text{for } x>a_4 \text{ and } x<a_1\end{cases}$$

$$F_{\tilde{a}^N}(x)=\begin{cases}\dfrac{a_2-x+(x-a_1)v_{\tilde{a}^N}}{a_2-a_1}, & \text{for } a_1\le x\le a_2,\\ v_{\tilde{a}^N}, & \text{for } a_2\le x\le a_3,\\ \dfrac{x-a_3+(a_4-x)v_{\tilde{a}^N}}{a_4-a_3}, & \text{for } a_3\le x\le a_4,\\ 1, & \text{for } x>a_4 \text{ and } x<a_1\end{cases}$$

where $w_{\tilde{a}}$, $u_{\tilde{a}}$, and $v_{\tilde{a}}$ denotes the maximum truth-membership degree, minimum-indeterminacy membership degree and minimum falsity-membership degree in [0,1] respectively and $a_1, a_2, a_3, a_4 \in \mathbf{R}$ such that $a_1 \le a_2 \le a_3 \le a_4$. When $a_1 > 0, \tilde{a} = \langle (a_1, a_2, a_3, a_4); w_{\tilde{a}}, u_{\tilde{a}}, v_{\tilde{a}} \rangle$ is called positive SVTrNN, denoted by $\tilde{a} > 0$, and if $a_4 \le 0$, then $\tilde{a} = \langle (a_1, a_2, a_3, a_4); w_{\tilde{a}}, u_{\tilde{a}}, v_{\tilde{a}} \rangle$ becomes a negative SVTrNN, denoted by $\tilde{a} < 0$.

If $0 \le a_1 \le a_2 \le a_3 \le a_4 \le 1$, $w_{\tilde{a}}, u_{\tilde{a}}, v_{\tilde{a}} \in [0,1]$, then $\tilde{a}$ called a normalized SVTrNN. When $I_{\tilde{a}^N} = 1 - T_{\tilde{a}^N} - F_{\tilde{a}^N}$, then SVTrNN reduces as TIFN. If $a_2 = a_3$, then SVTrNN is reduces single valued triangular neutrosophic number (SVTNN), denoted as $\tilde{a} = \langle (a_1, a_3, a_4); w_{\tilde{a}}, u_{\tilde{a}}, v_{\tilde{a}} \rangle$.

Definition 2.1.12. A single valued trapezoidal neutrosophic number (SVTrNN) with twelve components is defined and denoted as:

$$\tilde{A}^N = \langle (p_1, p_2, p_3, p_4); (q_1, q_2, q_3, q_4); (r_1, r_2, r_3, r_4); w_{\tilde{A}^N}, u_{\tilde{A}^N}, v_{\tilde{A}^N} \rangle$$

where $r_1 \le q_1 \le p_1 \le r_2 \le q_2 \le p_2 \le r_3 \le q_3 \le p_3 \le r_4 \le q_4 \le p_4$ in which the quantity of the truth membership, indeterminacy membership and falsity membership are not dependent and is defined as follows:

$$T_{\tilde{A}^N}(x)=\begin{cases}\dfrac{(x-p_1)w_{\tilde{A}^N}}{p_2-p_1}, & \text{for } p_1\le x\le p_2,\\ w_{\tilde{A}^N}, & \text{for } p_2\le x\le p_3,\\ \dfrac{(p_4-x)w_{\tilde{A}^N}}{p_4-p_3}, & \text{for } p_3\le x\le p_4,\\ 0, & \text{for } x>p_4 \text{ and } x<p_1\end{cases}$$

$$I_{\tilde{A}^N}(x)=\begin{cases}\dfrac{q_2 - x+(x - q_1)u_{\tilde{A}^N}}{q_2 - q_1}, & \text{for } q_1 \le x \le q_2,\\ u_{\tilde{A}^N}, & \text{for } q_2 \le x \le q_3,\\ \dfrac{x-q_3+(q_4-x)u_{\tilde{A}^N}}{q_4 - q_3}, & \text{for } q_3 \le x \le q_4,\\ 1, & \text{for } x > q_4 \text{ and } x < q_1\end{cases}$$

$$F_{\tilde{A}^N}(x)=\begin{cases}\dfrac{r_2 - x+(x - r_1)\upsilon_{\tilde{A}^N}}{r_2 - r_1}, & \text{for } r_1 \le x \le r_2,\\ \upsilon_{\tilde{A}^N}, & \text{for } r_2 \le x \le r_3,\\ \dfrac{x-r_3+(r_4-x)\upsilon_{\tilde{A}^N}}{r_4 - r_3}, & \text{for } r_3 \le x \le r_4,\\ 1, & \text{for } x > r_4 \text{ and } x < r_1\end{cases}$$

where $0 \le T_{\tilde{A}^N}(x)+I_{\tilde{A}^N}(x)+F_{\tilde{A}^N}(x) \le 3,\ x \in \tilde{A}^N$.

Definition 2.1.13. The parametric form $\tilde{A}^N$ of SVTrNN for some $0 \le \alpha \le 1, 0 \le \beta \le 1, 0 \le \gamma \le 1$ and $0 \le \alpha+\beta+\gamma \le 3$ is defined as $(\tilde{A}^N)_{\alpha,\beta,\gamma} = [T'_{\tilde{A}^N}(\alpha), T''_{\tilde{A}^N}(\alpha), I'_{\tilde{A}^N}(\beta), I''_{\tilde{A}^N}(\beta), F'_{\tilde{A}^N}(\gamma), F''_{\tilde{A}^N}(\gamma)]$,

where $T'_{\tilde{A}^N}(\alpha)=p_1+\dfrac{\alpha}{w_{\tilde{A}^N}}(p_2-p_1)$, $\quad T''_{\tilde{A}^N}(\alpha)=p_4-\dfrac{\alpha}{w_{\tilde{A}^N}}(p_4-p_3)$

$$I'_{\tilde{A}^N}(\beta)=\frac{q_1u_{\tilde{A}^N}-q_2+\beta(q_2-q_1)}{u_{\tilde{A}^N}-1}, \quad I''_{\tilde{A}^N}(\beta)=\frac{q_3-q_4u_{\tilde{A}^N}+\beta(q_4-q_3)}{1-u_{\tilde{A}^N}}$$

$$F'_{\tilde{A}^N}(\gamma)=\frac{r_1\upsilon_{\tilde{A}^N}-r_2+\gamma(r_2-r_1)}{\upsilon_{\tilde{A}^N}-1}, \quad F''_{\tilde{A}^N}(\gamma)=\frac{r_3-r_4\upsilon_{\tilde{A}^N}+\gamma(r_4-r_3)}{1-\upsilon_{\tilde{A}^N}}$$

Example 2.1.2. let us take $\tilde{A}^N = \langle(7,12,16,22),(6,11,15,20),(5,10,14,19);0.4,0.6,0.6\rangle$. The parametric representation is $T'_{0.4}(\alpha)=7+12.5\alpha$, $T''_{0.4}(\alpha)=22-15\alpha$,

$I'_{0.6}(\beta)=18.5-12.5\beta$, $I''_{0.6}(\beta)=7.5+12.5\beta$,

$F'_{0.6}(\gamma)=17.5-12.5\gamma$, $F''_{0.6}(\gamma)=6.5+12.5\gamma$

For different values of α,β,γ the degree of truthfulness, degree of indeterminacy and degree of falsity shown in table 1 and their graphical representation in figure 2:

Table 1

α,β,γ	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**
$T'_{\tilde{A}^N}(\alpha)$	7.00	8.25	9.50	10.75	12.00	13.25	14.50	15.75	17.00	18.25	19.50
$T''_{\tilde{A}^N}(\alpha)$	22.00	20.50	19.5	17.50	16.00	14.50	13.00	11.50	10.00	8.50	7.00
$I'_{\tilde{A}^N}(\beta)$	18.5	17.25	16.00	14.75	13.50	12.25	11.00	9.75	8.50	7.25	6
$I''_{\tilde{A}^N}(\beta)$	7.50	8.75	10.00	11.25	12.50	13.75	15.00	16.25	17.50	18.75	20.00
$F'_{\tilde{A}^N}(\gamma)$	17.5	16.25	15.00	13.75	12.50	11.25	10.00	8.75	7.50	6.25	5.00
$F''_{\tilde{A}^N}(\gamma)$	6.50	7.75	9.00	10.25	11.50	12.75	14.00	15.25	16.50	17.75	19.50

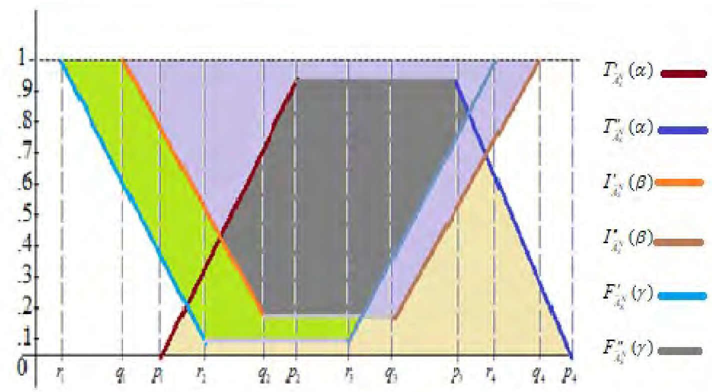

Figure 2: Graphical representation of
$\tilde{A}^N = \langle (p_1, p_2, p_3, p_4); (q_1, q_2, q_3, q_4); (r_1, r_2, r_3, r_4); w_{\tilde{A}^N}, u_{\tilde{A}^N}, v_{\tilde{A}^N} \rangle$
where $r_1 \le q_1 \le p_1 \le r_2 \le q_2 \le p_2 \le r_3 \le q_3 \le p_3 \le r_4 \le q_4 \le p_4$

2.2. Operational Laws on SVTrNN

Definition 2.2. 1. If $\tilde{A}^N$ and $\tilde{B}^N$ are two SVTrNN with twelve components having truth-membership $T'_{\tilde{A}^N}(x)$, $T''_{\tilde{B}^N}(x)$, indeterminacy-membership $I'_{\tilde{A}^N}(x), I''_{\tilde{B}^N}(x)$ and falsity-membership $F'_{\tilde{A}^N}(x), F''_{\tilde{B}^N}(x)$ respectively and three real numbers in [0.1], such as

$$\tilde{A}^N = \langle (p'_1, p'_2, p'_3, p'_4); (q'_1, q'_2, q'_3, q'_4); (r'_1, r'_2, r'_3, r'_4); w'_{\tilde{A}^N}, u'_{\tilde{A}^N}, v'_{\tilde{A}^N} \rangle$$
$$\tilde{B}^N = \langle (p''_1, p''_2, p''_3, p''_4); (q''_1, q''_2, q''_3, q''_4); (r''_1, r''_2, r''_3, r''_4); w''_{\tilde{B}^N}, u''_{\tilde{B}^N}, v''_{\tilde{B}^N} \rangle$$

Addition of SVTrNN:

$$\begin{aligned}\tilde{C}^N = \tilde{A}^N + \tilde{B}^N = \langle &(p'_1 + p''_1, p'_2 + p''_2, p'_3 + p''_3, p'_4 + p''_4)(q'_1 + q''_1, q'_2 + q''_2, q'_3 + q''_3, q'_4 + q''_4);\\ &(r'_1 + r''_1, r'_2 + r''_2, r'_3 + r''_3, r'_4 + r''_4); w'_{\tilde{A}^N} \wedge w''_{\tilde{B}^N}, u'_{\tilde{A}^N} \vee u''_{\tilde{B}^N}, v'_{\tilde{A}^N} \vee v''_{\tilde{B}^N} \rangle\end{aligned}$$

Negative of SVTrNN: If $\tilde{A}^N = \langle (p_1, p_2, p_3, p_4); (q_1, q_2, q_3, q_4); (r_1, r_2, r_3, r_4); w_{\tilde{A}^N}, u_{\tilde{A}^N}, v_{\tilde{A}^N} \rangle$, then

$$-\tilde{A}^N = \langle (-p_4, -p_3, -p_2, -p_1); (-q_4, -q_3, -q_2, -q_1); (-r_4, -r_3, -r_2, -r_1); w_{\tilde{A}^N}, u_{\tilde{A}^N}, v_{\tilde{A}^N} \rangle$$

Subtraction of SVTrNN:

$$\begin{aligned}\tilde{A}^N - \tilde{B}^N = \langle &(p'_1 - p''_4, p'_2 - p''_3, p'_3 - p''_2, p'_4 - p''_1)(q'_1 - q''_4, q'_2 - q''_3, q'_3 - q''_2, q'_4 - q''_1);\\ &(r'_1 - r''_4, r'_2 - r''_3, r'_3 - r''_2, r'_4 - r''_1); w'_{\tilde{A}^N} \wedge w''_{\tilde{B}^N}, u'_{\tilde{A}^N} \vee u''_{\tilde{B}^N}, v'_{\tilde{A}^N} \vee v''_{\tilde{B}^N} \rangle\end{aligned}$$

Multiplication of SVTrNN:

$$\tilde{A}^N \cdot \tilde{B}^N = \begin{cases} \langle (p'_1 \cdot p''_1, p'_2 \cdot p''_2, p'_3 \cdot p''_3, p'_4 \cdot p''_4); (q'_1 \cdot q''_1, q'_2 \cdot q''_2, q'_3 \cdot q''_3, q'_4 \cdot q''_4);\\ (r'_1 \cdot r''_1, r'_2 \cdot r''_2, r'_3 \cdot r''_3, r'_4 \cdot r''_4); w'_{\tilde{A}^N} \wedge w''_{\tilde{B}^N}, u'_{\tilde{A}^N} \vee u''_{\tilde{B}^N}, v'_{\tilde{A}^N} \vee v''_{\tilde{B}^N} \rangle \\ \qquad if \ p'_4 > 0, p''_4 > 0; q'_4 > 0, q''_4 > 0; r'_4 > 0, r''_4 > 0 \\ \langle (p'_1 \cdot p''_4, p'_2 \cdot p''_3, p'_3 \cdot p''_2, p'_4 \cdot p''_1); (q'_1 \cdot q''_4, q'_2 \cdot q''_3, q'_3 \cdot q''_2, q'_4 \cdot q''_1);\\ (r'_1 \cdot r''_4, r'_2 \cdot r''_3, r'_3 \cdot r''_2, r'_4 \cdot r''_1); w'_{\tilde{A}^N} \wedge w''_{\tilde{B}^N}, u'_{\tilde{A}^N} \vee u''_{\tilde{B}^N}, v'_{\tilde{A}^N} \vee v''_{\tilde{B}^N} \rangle \\ \qquad if \ p'_4 < 0, p''_4 > 0; q'_4 < 0, q''_4 > 0; r'_4 < 0, r''_4 > 0 \\ \langle (p'_4 \cdot p''_4, p'_3 \cdot p''_3, p'_2 \cdot p''_2, p'_1 \cdot p''_1); (q'_4 \cdot q''_4, q'_3 \cdot q''_3, q'_2 \cdot q''_2, q'_1 \cdot q''_1);\\ (r'_4 \cdot r''_4, r'_3 \cdot r''_3, r'_2 \cdot r''_2, r'_1 \cdot r''_1); w'_{\tilde{A}^N} \wedge w''_{\tilde{B}^N}, u'_{\tilde{A}^N} \vee u''_{\tilde{B}^N}, v'_{\tilde{A}^N} \vee v''_{\tilde{B}^N} \rangle \\ \qquad if \ p'_4 < 0, p''_4 < 0; q'_4 < 0, q''_4 < 0; r'_4 < 0, r''_4 < 0 \end{cases}$$

Scalar multiplication of SVTrNN:

$$k\tilde{A}^N = \begin{cases} \langle (kp_1, kp_2, kp_3, kp_4); (kq_1, kq_2, kq_3, kq_4); (kr_1, kr_2, kr_3, kr_4); w_{\tilde{A}^N}, u_{\tilde{A}^N}, v_{\tilde{A}^N} \rangle, if\ k > 0, \\ \langle (kp_4, kp_3, kp_2, kp_1); (kq_4, kq_3, kq_2, kq_1); (kr_4, kr_3, kr_2, kr_1); w_{\tilde{A}^N}, u_{\tilde{A}^N}, v_{\tilde{A}^N} \rangle, if\ k < 0. \end{cases}$$

Inverse of SVTrNN:

$$(\tilde{A}^N)^{-1} = \frac{1}{\tilde{A}^N} = \begin{cases} \langle (\frac{1}{p_4}, \frac{1}{p_3}, \frac{1}{p_2}, \frac{1}{p_1}); (\frac{1}{q_4}, \frac{1}{q_3}, \frac{1}{q_2}, \frac{1}{q_1}); (\frac{1}{r_4}, \frac{1}{r_3}, \frac{1}{r_2}, \frac{1}{r_1}); w_{\tilde{A}^N}, u_{\tilde{A}^N}, v_{\tilde{A}^N} \rangle, \\ \qquad if\ p_{'s} > 0, q_{'s} > 0, r_{'s} > 0, \\ \langle (\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}, \frac{1}{p_4}); (\frac{1}{q_1}, \frac{1}{q_2}, \frac{1}{q_3}, \frac{1}{q_4}); (\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}, \frac{1}{r_4}); w_{\tilde{A}^N}, u_{\tilde{A}^N}, v_{\tilde{A}^N} \rangle, \\ \qquad if\ p_{'s} < 0, q_{'s} < 0, r_{'s} < 0. \end{cases}$$

Division of SVTrNN:

$$\frac{\tilde{A}^N}{\tilde{B}^N} = \begin{cases} \langle (\frac{p_1'}{p_4''}, \frac{p_2'}{p_3''}, \frac{p_3'}{p_2''}, \frac{p_4'}{p_1''}); (\frac{q_1'}{q_4''}, \frac{q_2'}{q_3''}, \frac{q_3'}{q_2''}, \frac{q_4'}{q_1''}); (\frac{r_1'}{r_4''}, \frac{r_2'}{r_3''}, \frac{r_3'}{r_2''}, \frac{r_4'}{r_1''}); w_{\tilde{A}^N}' \wedge w_{\tilde{B}^N}'', u_{\tilde{A}^N}' \vee u_{\tilde{B}^N}'', v_{\tilde{A}^N}' \vee v_{\tilde{B}^N}'' \rangle \\ \qquad if\ p_4' > 0, p_4'' > 0; q_4' > 0, q_4'' > 0; r_4' > 0, r_4'' > 0 \\ \langle (\frac{p_4'}{p_4''}, \frac{p_3'}{p_3''}, \frac{p_2'}{p_2''}, \frac{p_1'}{p_1''}); (\frac{q_4'}{q_4''}, \frac{q_3'}{q_3''}, \frac{q_2'}{q_2''}, \frac{q_1'}{q_1''}); (\frac{r_4'}{r_4''}, \frac{r_3'}{r_3''}, \frac{r_2'}{r_2''}, \frac{r_1'}{r_1''}); w_{\tilde{A}^N}' \wedge w_{\tilde{B}^N}'', u_{\tilde{A}^N}' \vee u_{\tilde{B}^N}'', v_{\tilde{A}^N}' \vee v_{\tilde{B}^N}'' \rangle \\ \qquad if\ p_4' < 0, p_4'' > 0; q_4' < 0, q_4'' > 0; r_4' < 0, r_4'' > 0 \\ \langle (\frac{p_4'}{p_1''}, \frac{p_3'}{p_2''}, \frac{p_2'}{p_3''}, \frac{p_1'}{p_4''}); (\frac{q_4'}{q_1''}, \frac{q_3'}{q_2''}, \frac{q_2'}{q_3''}, \frac{q_1'}{q_4''}); (\frac{r_4'}{r_1''}, \frac{r_3'}{r_2''}, \frac{r_2'}{r_3''}, \frac{r_1'}{r_4''}); w_{\tilde{A}^N}' \wedge w_{\tilde{B}^N}'', u_{\tilde{A}^N}' \vee u_{\tilde{B}^N}'', v_{\tilde{A}^N}' \vee v_{\tilde{B}^N}'' \rangle \\ \qquad if\ p_4' < 0, p_4'' < 0; q_4' < 0, q_4'' < 0; r_4' < 0, r_4'' < 0 \end{cases}$$

Example 2.2.1. let $\tilde{A}^N = \langle (7,11,16,21), (6,10,15,20), (5,9,14,19); 0.4, 0.6, 0.6 \rangle$ and

$\tilde{B}^N = \langle (6,11,13,20), (5,10,12,18), (3,8,11,16); 0.3, 0.6, 0.7 \rangle$ be two SVTrNN, then

$\tilde{A}^N + \tilde{B}^N = \langle (13,22,29,41), (11,20,27,38), (8,17,25,35); 0.4, 0.6, 0.6 \rangle$

$\tilde{A}^N - \tilde{B}^N = \langle (-13,-2,5,15), (-12,-2,5,15), (-11,-2,6,16); 0.4, 0.6, 0.6 \rangle$

$\tilde{A}^N . \tilde{B}^N = \langle (42,121,208,420), (30,100,180,360), (15,72,154,304); 0.4, 0.6, 0.6 \rangle$

$\tilde{A}^N / \tilde{B}^N = \langle (0.35, 0.85, 1.45, 3.50), (0.33, 0.83, 1.50, 4.00, (0.31, 0.81, 1.75, 6.33); 0.4, 0.6, 0.6 \rangle$

$5\tilde{A}^N = \langle (35,55,80,105), (30,50,75,100), (25,45,70,95); 0.4, 0.6, 0.6 \rangle$

3. De-Neutrosophication by using score function

We use the score and accuracy functions of a SVTrNN, is defined by an expert [31] to compare any two SVTrNN. So that the score function is defined as

$$S(\tilde{A}^N) = \left(\frac{p_1 + p_2 + p_3 + p_4 - q_1 - q_2 - q_3 - q_4 - r_1 - r_2 - r_3 - r_4}{12} \right) \times \left(2 + w_{\tilde{A}^N} - u_{\tilde{A}^N} - v_{\tilde{A}^N} \right)$$

and accuracy function is

$$A(\tilde{A}^N) = \left(\frac{p_1 + p_2 + p_3 + p_4 - q_1 - q_2 - q_3 - q_4}{4} \right) \times \left(2 + w_{\tilde{A}^N} - u_{\tilde{A}^N} + v_{\tilde{A}^N} \right)$$

Example 3.1. Let $\tilde{A}^N = \langle (7,11,16,21), (6,10,15,20), (5,10,14,19); 0.4, 0.6, 0.6 \rangle$ then $S(\tilde{A}^N) = -4.4$ and $A(\tilde{A}^N) = -0.7$

Definition 3.1. (Comparison of SVTrNN). Let $\tilde{A}^N$ and $\tilde{B}^N$ be any two SVTrNN, then one has the following:

(a) $S(\tilde{A}^N) < S(\tilde{B}^N) \Rightarrow \tilde{A}^N < \tilde{B}^N$

(b) If $S(\tilde{A}^N_S) = S(\tilde{B}^N_S)$ and if

(i) $A(\tilde{A}^N) < A(\tilde{B}^N)$ then $\tilde{A}^N < \tilde{B}^N$

(ii) $A(\tilde{A}^N) > A(\tilde{B}^N)$ then $\tilde{A}^N > \tilde{B}^N$

(iii) $A(\tilde{A}^N) = A(\tilde{B}^N)$ then $\tilde{A}^N = \tilde{B}^N$

Example 3.2. Let $\tilde{A}^N = \langle(6,10,16,20),(5,9,14,19),(3,8,12,18);0.3,0.6,0.7\rangle$ and

$\tilde{B}^N = \langle(7,11,16,21),(6,15,14,20),(5,10,14,19);0.3,0.6,0.7\rangle$

$\tilde{C}^N = \langle(8,11,16,22),(6,15,14,21),(5,10,14,20);0.3,0.6,0.7\rangle$ be two SVTrNN, then

$S(\tilde{A}^N) = -3.00$, $A(\tilde{A}^N) = 1.25$, $S(\tilde{B}^N) = -0.4$, $A(\tilde{B}^N) = 0.0$, and $S(\tilde{C}^N) = -0.4$, $A(\tilde{C}^N) = 0.25$,

which implies that if $S(\tilde{A}^N) < S(\tilde{B}^N)$ then $\tilde{A}^N < \tilde{B}^N$

Also $S(\tilde{B}^N) = S(\tilde{C}^N)$ and $A(\tilde{B}^N) < A(\tilde{C}^N)$ then $\tilde{B}^N < \tilde{C}^N$.

4. Neutrosophic Transportation Problem (NTP) and its Mathematical formulation

4.1. Classification of NTP

Definition 4.1.1. In a TP, if atleast one parameter such as cost, demand or supply is in form of neutrosophic numbers, the TP is termed as NTP.

Definition 4.1.2. A NTP having neutrosophic availabilities and neutrosophic demand but crisp cost, is classified as NTP of type-1.

Definition 4.1.3. The NTP having crisp availabilities and crisp demand but neutrosophic cost, is classified as NTP of type-2.

Definition 4.1.4. If all the specifications of TP such as cost, demand and availabilities are combination of crisp, triangular or trapezoidal neutrosophic numbers, then NTP classified as NTP of type-3.

Definition 4.1.5. If all the specifications of TP must be in neutrosophic numbers form, then TP is said to be NTP of type-4 or fully NTP.

4.2. Mathematical Formulation of NTP

The TP is very important for transporting goods from one source to another destination. In TP if ambiguity occurs in cost, demand or supply then it is more difficult to solve it. To handle this type of impreciseness in cost to transferred product from i^{th} sources to j^{th} destination or uncertainty in supply and demand the decision maker introduce NTP of SVTrNN.

Here we consider two models in which the decision maker is unsettled about the specific values i.e. cost from i^{th} sources to j^{th} destination and also certainty or uncertainty in supply or demand of the product, so that a new type of TP is introduced namely NTP with parameters like cost, demand and supply as SVTrNN. The NTP with assumptions and constraints is defined as the number of unites x^N_{ij}

and the neutrosophic cost $\tilde{c}_{ij}^{N}$ are transported from i^{th} sources to j^{th} destination. For balance NTP $\sum_{i=0}^{m} a_i^N = \sum_{j=0}^{n} b_j^N$ i.e. total supply is equal to total demand.

For the formulation of NTP the following assumptions and constraints are required:

m	Total number of source point
n	Total number of destination point
i	Table of source (for all m)
j	Table of destination (for all n)
$\tilde{x}_{ij}^{N}$	Number of transported neutrosophic unites from i^{th} source to j^{th} destination
$\tilde{c}_{ij}^{N}$	Neutrosophic cost of one unit transported from i^{th} source to j^{th} destination
$\tilde{a}_{i}^{N}$	Available neutrosophic supply quantity from i^{th} source
$\tilde{b}_{j}^{N}$	Required neutrosophic demand quantity to j^{th} destination
c_{ij}	Crisp cost of one unit quantity
x_{ij}	Number of transported crisp unites from i^{th} source to j^{th} destination
a_i	Available crisp supply quantity from i^{th} source
b_j	Required crisp demand quantity to j^{th} destination

Modal I

In NTP the objective is to minimize the cost of transported product from source to destination. The mathematical formulation of NTP with uncertain transported units and transportation cost, demand and supply is:

$$(P_1)\quad \begin{aligned} &\text{Minimum } \tilde{Z}^N = \sum_{i=0}^{m}\sum_{j=0}^{n} \tilde{x}_{ij}^N \tilde{c}_{ij}^N \\ &\text{Subject to } \sum_{j=0}^{n} \tilde{x}_{ij}^N = \tilde{a}_i^N, \forall i(\text{sources}) = 1, 2, 3, \dots, m, \\ &\sum_{i=0}^{m} \tilde{x}_{ij}^N = \tilde{b}_j^N, \forall\ j(\text{destination}) = 1, 2, 3, \dots, n, \\ &\tilde{x}_{ij}^N \geq 0, \forall\ i = 1, 2, 3, \dots, m, j = 1, 2, 3, \dots, n. \end{aligned}$$

Modal II

The mathematical formulation of NTP with uncertain transported units and transportation cost but curtained about demand and supply is termed a NTP of type-2 is:

$$(P_2)\quad \begin{aligned} &\text{Minimum } \tilde{Z}^N = \sum_{i=0}^{m}\sum_{j=0}^{n} x_{ij} \tilde{c}_{ij}^N \\ &\text{Subject to } \sum_{j=0}^{n} x_{ij} = a_i, \forall i(\text{sources}) = 1, 2, 3, \dots, m, \\ &\sum_{i=0}^{m} x_{ij} = b_j, \forall\ j(\text{destination}) = 1, 2, 3, \dots, n, \\ &x_{ij} \geq 0, \forall\ i = 1, 2, 3, \dots, m, j = 1, 2, 3, \dots, n. \end{aligned}$$

5. Procedure for Proposed Algorithms for solution of CTP and NTP

5.1. Basic Assumptions of the Proposed Algorithms

The total transportation cost does not depends on the mode of transportation and distance, also the framework of the problem will be denoted by either crisp or SVTrNN.

If $\sum_{i=0}^{m} \tilde{a}_i^N$ < or > $\sum_{j=0}^{n} \tilde{b}_j^N$, $\forall i,j$, then first one can make sure to balance the TP as $\sum_{i=0}^{m} \tilde{a}_i^N = \sum_{j=0}^{n} \tilde{b}_j^N$, $\forall i,j$,

5.2. Steps for solution of CTP after balancing by existing method

Step5.2.1. To change the each neutrosophic cost $\tilde{c}_{ij}^N$, neutrosophic supply $\tilde{a}_i^N$ and neutrosophic demand $\tilde{b}_j^N$ of NTP in cost matrix to crisp values, we use here score function method i.e. we convert these by using $S(\tilde{A}^N)$.

Step5.2.2. For balance TP, verify that the sum of demands is equal to the sum of supply i.e. $\sum_{i=0}^{m} \tilde{a}_i = \sum_{j=0}^{n} \tilde{b}_j$, $\forall i,j$. If $\sum_{i=0}^{m} \tilde{a}_i$ < or > $\sum_{j=0}^{n} \tilde{b}_j$, $\forall i,j$, then first one can make sure to balance the TP by adding a row or column with zero entries in cost matrix [30].

Step5.2.3. After conversion of NTP into TP, choose the minimum entry in each row and subtract it to all other entries in that row. The same way is applicable in each column to find minimum one zero in each row and each column in TP matrix. For better (see table 4 and table 6).

Step5.2.4. Verify that the sum of demands is greater than the supply in each row and the sum of supplies are greater than the demand in each column, if ok go on step 5.2.6, otherwise go on step 5.2.5.

Step5.2.5. Draw the horizontal and vertical lines that cover all the zeros and equal to minimum number of order of matrix or reduced table. Now if number of lines is less than to the minimum number, revise table by choose the least element which is not under horizontal or vertical line and add it to the entry at the cross point of the lines. Again go to step 5.2.3 to check the condition.

Step5.2.6. To allot the maximum possible units of supply or demand in the cost cell, choose a cell of maximum cost in the reduced cost matrix. If the maximum cost exists more than one place, choose any one cell of maximum supply or demand.

Step5.2.7. If none cell occur for the maximum cost then go for next maximum. If such cell does not occur for any value, then choose any cell at random, whose reduced cost is zero.

Step5.2.8. From the reduced table omit the row which are fully exhausted or column which are fully satisfied, then repeats steps and again. Repeat the procedure until all the demand units and all the supply units are fully received respectively.

The procedure for the solution of NTP by using existing method is same as steps in 5.2, while the cost, demand, supply and solution vales are in SVTrNN.

5.3. Steps for solution of CTP after balancing by MRCM

For balance the unbalance CTP, we use MRCM which is generalization of method in [27]. We use the following steps for solution of CTP by MRCM:

Step5.3.1. Convert neutrosophic cost $\tilde{c}_{ij}^N$, neutrosophic supply $\tilde{a}_i^N$ and neutrosophic demand $\tilde{b}_j^N$ of NTP in cost matrix to crisp values by using score function $S(\tilde{A}^N)$ i.e NTP convert into CTP.

Step5.3.2 If CTP is unbalance then to make it balance one by applying the steps of MRCM that is if sum of supply is less then to the sum of demand, then add a row of minimum costs in each row with a supply equal to sum of supplies and add a column of minimum costs in each column with demand equal to the difference value from sum of all supplies differ to sum of demand. The same is applicable when sum of demand is less than the sum of supply. i.e.

$$\tilde{a}_{m+1} = \sum_{i=0}^{m} \tilde{a}_i \text{ and } \tilde{b}_{n+1} = \sum_{j=0}^{n} \tilde{a}_j \oplus \text{excess supply},$$

or

$$\tilde{b}_{n+1} = \sum_{j=0}^{n} \tilde{b}_j \text{ and } \tilde{a}_{m+1} = \sum_{i=1}^{m} \tilde{b}_i \oplus \text{excess demand}.$$

The unit transportation costs are taken as follows:

$$\tilde{c}_{i(n+1)} = \min_{1\le j\le n} \tilde{c}_{ij},\ 1\le i\le m, \qquad \tilde{c}_{(m+1)j} = \min_{1\le i\le m} \tilde{c}_{ij},\ 1\le j\le n,$$

$$\tilde{c}_{ij} = \tilde{c}_{ji},\ 1\le i\le m,\ 1\le j\le n, \text{ and } \tilde{c}_{(m+1)(n+1)} = 0.$$

Step5.3.3 Obtain optimal solution of converted CTP after balance it by existing method using excel solver. Let the crisp optimal solution be $\tilde{x}_{ij},\ 1\le i\le m+1,\ 1\le j\le n+1.$

Step5.3.4 By assuming $\tilde{\omega}'_{m+1}=0$ and using the relation $\tilde{\omega}'_i \oplus \tilde{v}'_j = \tilde{\sigma}'_{ij}$ for basic variables, find the values of all the dual variables $\tilde{\omega}'_i,\ 1\le i\le m$ and $\tilde{v}'_j,\ 1\le j\le n+1,$

Step5.3.5. According to MRCM, $\tilde{\omega}'_i = \tilde{\omega}_i$ and $\tilde{v}'_j = \tilde{v}_j$ for $1\le i\le m, 1\le j\le n$, obtain only central rank zero duals. After that in terms of original supply S_i and demand M_j find the neutrosophic optimal solution of the problem.

5.4. Steps for solution of NTP after balancing by MRCM

Step5.4.1. Convert neutrosophic cost $\tilde{c}_{ij}^N$, neutrosophic supply $\tilde{a}_i^N$ and neutrosophic demand $\tilde{b}_j^N$ of NTP in cost matrix to crisp values by using score function $S(\tilde{A}^N)$ to check either it is balance or unbalance.

Step5.4.2 If NTP is unbalance than same procedure as in 5.3 is applicable. i.e.

$$\tilde{a}^N_{m+1} = \sum_{i=0}^{m} \tilde{a}^N_i \text{ and } \tilde{b}^N_{n+1} = \sum_{j=0}^{n} \tilde{a}^N_j \oplus \text{excess supply},$$

or

$$\tilde{b}^N_{n+1} = \sum_{j=0}^{n} \tilde{b}^N_j \text{ and } \tilde{a}^N_{m+1} = \sum_{i=1}^{m} \tilde{b}^N_i \oplus \text{excess demand}.$$

The unit transportation costs are taken as follows:

$$\tilde{c}^N_{i(n+1)} = \min_{1\le j\le n} \tilde{c}^N_{ij},\ 1\le i\le m, \qquad \tilde{c}^N_{(m+1)j} = \min_{1\le i\le m} \tilde{c}^N_{ij},\ 1\le j\le n,$$

$$\tilde{c}^N_{ij} = \tilde{c}^N_{ji},\ 1\le i\le m,\ 1\le j\le n, \text{ and } \tilde{c}^N_{(m+1)(n+1)} = 0.$$

Step5.4.3 Obtain optimal solution of NTP by excel solver. Let the neutrosophic optimal solution obtained be $\tilde{x}^N_{ij},\ 1\le i\le m+1,\ 1\le j\le n+1.$

Step5.4.4 By assuming $\tilde{\omega}'^N_{m+1}=0$ and using the relation $\tilde{\omega}'^N_i \oplus \tilde{v}'^N_j = \tilde{\sigma}'^N_{ij}$ for basic variables, find the values of all the dual variables $\tilde{\omega}'^N_i,\ 1\le i\le m$ and $\tilde{v}'^N_j,\ 1\le j\le n+1,$

Step5.4.5. According to MRCM, $\tilde{\omega}_i^{'N} = \tilde{\omega}_i^N$ and $\tilde{v}_j^{'N} = \tilde{v}_j^N$ for $1 \le i \le m, 1 \le j \le n$, obtain only central rank zero duals.

6. Numerical Example

6.1. Modal I (NTP of type-3)

Let us consider a NTP with three sources say S_1, S_2, S_3 in which wheat are initially stored and ready to transport in three flour mills namely M_1, M_2, M_3 with unit transportation cost, demand and supply are as SVTrNN. The input data of SVTrNN -TP given in table 2 as follows:

Table 2

	M_1	M_2	M_3	***Supply***
S_1	$\left\langle\begin{pmatrix}(3,5,7.5,11)\\(2,4,7,10)\\(1,3.5,6,9)\end{pmatrix}\begin{matrix}0.2\\;0.4\\0.8\end{matrix}\right\rangle$	$\left\langle\begin{pmatrix}(2,4.5,10,15)\\(0.5,3.5,8,14)\\(0,2.5,6,12)\end{pmatrix}\begin{matrix}0.3\\;0.5\\0.8\end{matrix}\right\rangle$	$\left\langle\begin{pmatrix}(1,5,9,14.5)\\(-3,3.5,8,12)\\(-4,2,7,11)\end{pmatrix}\begin{matrix}0.2\\;0.5\\0.7\end{matrix}\right\rangle$	$\left\langle\begin{pmatrix}(9,17,26,36)\\(6,14,23,33)\\(3,11,20,30)\end{pmatrix}\begin{matrix}0.4\\;0.7\\0.7\end{matrix}\right\rangle$
S_2	$\left\langle\begin{pmatrix}(1,7,11.5,16)\\(-1,5,10,14)\\(-3,3,8.5,12)\end{pmatrix}\begin{matrix}0.4\\;0.5\\0.7\end{matrix}\right\rangle$	$\left\langle\begin{pmatrix}(1,6,9.5,12)\\(-1,4,8.5,11)\\(-2,2,8,10)\end{pmatrix}\begin{matrix}0.5\\;0.7\\0.8\end{matrix}\right\rangle$	$\left\langle\begin{pmatrix}(-1,4,8,15)\\(-2,3,6,12)\\(-3,2,5,11)\end{pmatrix}\begin{matrix}0.3\\;0.6\\0.6\end{matrix}\right\rangle$	$\left\langle\begin{pmatrix}(7,17,25,31)\\(3,12,22.5,29)\\(1,10,19.5,27)\end{pmatrix}\begin{matrix}0.3\\;0.6\\0.7\end{matrix}\right\rangle$
S_3	$\left\langle\begin{pmatrix}(3,6,9,12)\\(2,5,8,11)\\(1,4,7,10)\end{pmatrix}\begin{matrix}0.2\\;0.4\\0.8\end{matrix}\right\rangle$	$\left\langle\begin{pmatrix}(-1,3.5,9,12)\\(-2,2.5,7,11)\\(-4,1,5,10)\end{pmatrix}\begin{matrix}0.2\\;0.4\\0.8\end{matrix}\right\rangle$	$\left\langle\begin{pmatrix}(0,5,8,14)\\(-2,3,7,12)\\(-4,1,6,10)\end{pmatrix}\begin{matrix}0.2\\;0.6\\0.6\end{matrix}\right\rangle$	$\left\langle\begin{pmatrix}(9,16,22,31)\\(5,14,20,27)\\(1,12,18,23)\end{pmatrix}\begin{matrix}0.3\\;0.6\\0.7\end{matrix}\right\rangle$
Demand	$\left\langle\begin{pmatrix}(12,21,30,37)\\(9,19,28,34)\\(6,16,25,33)\end{pmatrix}\begin{matrix}0.3\\;0.6\\0.7\end{matrix}\right\rangle$	$\left\langle\begin{pmatrix}(10,16,22,27)\\(5,14,20,25)\\(0,12,18,23)\end{pmatrix}\begin{matrix}0.4\\;0.7\\0.7\end{matrix}\right\rangle$	$\left\langle\begin{pmatrix}(7,12,19,27)\\(4,11,18,24)\\(1,9,15,21)\end{pmatrix}\begin{matrix}0.2\\;0.6\\0.6\end{matrix}\right\rangle$	

6.2. Neutrosophic optimal solution with score function method

One can use score function to convert SVTrNN cost, demand and supply to obtain the crisp numbers in TP of table 2 as follows:

$$S(\tilde{A}^N) = \left(\frac{p_1 + p_2 + p_3 + p_4 - q_1 - q_2 - q_3 - q_4 - r_1 - r_2 - r_3 - r_4}{12}\right) \times \left(2 + w_{\tilde{A}^N} - u_{\tilde{A}^N} - v_{\tilde{A}^N}\right)$$

Here $$S\left(\tilde{c}_{11}^N\right) = \left\langle\begin{pmatrix}(3,5,7.5,11)\\(2,4,7,10)\\(1,3.5,6,9)\end{pmatrix}\begin{matrix}0.2\\;0.4\\0.8\end{matrix}\right\rangle = \left(\frac{3+5+7.5+11-2-4-7-10-1-3.5-6-9}{12}\right) \times (0.2+0.6+0.2) = -1.33$$

$S\left(\tilde{c}_{12}^N\right) = -1.25$, $S\left(\tilde{c}_{13}^N\right) = \text{-}0.58$, $S\left(\tilde{c}_{21}^N\right) = -1.08$, $S\left(\tilde{c}_{22}^N\right) = -1.00$, $S\left(\tilde{c}_{23}^N\right) = -0.67$, $S\left(\tilde{c}_{31}^N\right) = -1.50$, $S\left(\tilde{c}_{32}^N\right) = -0.50$, $S\left(\tilde{c}_{33}^N\right) = -0.50$, $S\left(\tilde{a}_1^N\right) = -4.33$, $S\left(\tilde{a}_2^N\right) = -3.67$, $S\left(\tilde{a}_3^N\right) = -3.50$, $S\left(\tilde{b}_1^N\right) = -5.83$, $S\left(\tilde{b}_2^N\right) = -3.50$, $S\left(\tilde{b}_3^N\right) = -3.17$.

After converting cost, demand and supply of NTP from SVTrNN to the crisp numbers by using score function method, the unbalance CTP cost matrix is given in table 3:

Table 3

	M_1	M_2	M_3	***Supply***
S_1	-1.33	-1.25	-0.58	***-4.33***
S_2	-1.08	-1.00	-0.67	***-3.67***
S_3	-1.50	-0.50	-0.50	***-3.50***
Demand	***-5.83***	***-3.50***	***-3.17***	

By using the steps in 5.2, the optimal crisp solution of CTP and their allotment of demand and supply in cost matrix shown in table 4:

Table 4

	M_1	M_2	M_3	***Supply***
S_1	-1.33**(-2.16)**	-1.25**(-2.17)**	-0.58	*-4.33*
S_2	-1.08**(-3.67)**	-1.00	-0.67	*-3.67*
S_3	-1.50	-0.50**(-0.33)**	-0.50**(-3.17)**	*-3.50*
S_4	0	0**(-1.00)**	0	*-1.00*
Demand	*-5.83*	*-3.50*	*-3.17*	

The complete solution of CTP is $x_{11} = -2.16$, $x_{12} = -2.17$, $x_{21} = -3.67$, $x_{32} = -0.33$, $x_{33} = -3.17$, $x_{42} = -1.00$, and $\tilde{Z} = 11.30$. The corresponding optimal solution of NTP with allotment of SVTrNN is shown in table 5 as follows:

Table 5

	M_1	M_2	M_3	***Supply***
S_1	$\left\langle \begin{pmatrix}(-19,-4,13,30)\\(-20,-3.5,16,31)\\(-21,-3.5,15,32)\end{pmatrix};\begin{matrix}0.3\\0.6\\0.7\end{matrix}\right\rangle$	$\left\langle \begin{pmatrix}(-21,4,30,55)\\(-25,-2,26.5,53)\\(-29,-4,23.5,51)\end{pmatrix};\begin{matrix}0.3\\0.6\\0.7\end{matrix}\right\rangle$	-	-
S_2	$\left\langle \begin{pmatrix}(7,17,25,31)\\(3,12,22.5,29)\\(1,10,19.5,27)\end{pmatrix};\begin{matrix}0.3\\0.6\\0.7\end{matrix}\right\rangle$	-	-	-
S_3	-	$\left\langle \begin{pmatrix}(-18,-3,10,24)\\(-19,-4,9,23)\\(-20,-3,9,22)\end{pmatrix};\begin{matrix}0.2\\0.6\\0.6\end{matrix}\right\rangle$	$\left\langle \begin{pmatrix}(7,12,19,27)\\(4,11,18,24)\\(1,9,15,21)\end{pmatrix};\begin{matrix}0.2\\0.6\\0.6\end{matrix}\right\rangle$	-
S_4	-	$\left\langle \begin{pmatrix}(-69,-24,21,66)\\(-71,-21.5,26,69)\\(-73,-20.5,25,72)\end{pmatrix};\begin{matrix}0.3\\0.6\\0.7\end{matrix}\right\rangle$	-	-
Demand	-	-	-	-

$$\tilde{Z}^N = \left\langle \begin{pmatrix}(3,5,7.5,11)\\(2,4,7,10)\\(1,3.5,6,9)\end{pmatrix};\begin{matrix}0.2\\0.4\\0.8\end{matrix}\right\rangle \left\langle \begin{pmatrix}(-19,-4,13,30)\\(-20,-3.5,16,31)\\(-21,-3.5,15,32)\end{pmatrix};\begin{matrix}0.3\\0.6\\0.7\end{matrix}\right\rangle + \left\langle \begin{pmatrix}(2,4.5,10,15)\\(0.5,3.5,8,14)\\(0,2.5,6,12)\end{pmatrix};\begin{matrix}0.3\\0.5\\0.8\end{matrix}\right\rangle \left\langle \begin{pmatrix}(-21,4,30,55)\\(-25,-2,26.5,53)\\(-29,-4,23.5,51)\end{pmatrix};\begin{matrix}0.3\\0.6\\0.7\end{matrix}\right\rangle$$

i.e.

$$+\left\langle \begin{pmatrix}(1,7,11.5,16)\\(-1,5,10,14)\\(-3,3,8.5,12)\end{pmatrix};\begin{matrix}0.4\\0.5\\0.7\end{matrix}\right\rangle \left\langle \begin{pmatrix}(7,17,25,31)\\(3,12,22.5,29)\\(1,10,19.5,27)\end{pmatrix};\begin{matrix}0.3\\0.6\\0.7\end{matrix}\right\rangle + \left\langle \begin{pmatrix}(-1,3.5,9,12)\\(-2,2.5,7,11)\\(-4,1,5,10)\end{pmatrix};\begin{matrix}0.2\\0.4\\0.8\end{matrix}\right\rangle \left\langle \begin{pmatrix}(-18,-3,10,24)\\(-19,-4,9,23)\\(-20,-3,9,22)\end{pmatrix};\begin{matrix}0.2\\0.6\\0.6\end{matrix}\right\rangle +$$

$$+\left\langle \begin{pmatrix}(0,5,8,14)\\(-2,3,7,12)\\(-4,1,6,10)\end{pmatrix};\begin{matrix}0.2\\0.6\\0.6\end{matrix}\right\rangle \left\langle \begin{pmatrix}(7,12,19,27)\\(4,11,18,24)\\(1,9,15,21)\end{pmatrix};\begin{matrix}0.2\\0.6\\0.6\end{matrix}\right\rangle + \left\langle \begin{pmatrix}(0,0,0,0)\\(0,0,0,0)\\(0,0,0,0)\end{pmatrix};\begin{matrix}0.2\\0.6\\0.6\end{matrix}\right\rangle \left\langle \begin{pmatrix}(-69,-24,21,66)\\(-71,-21.5,26,69)\\(-73,-20.5,25,72)\end{pmatrix};\begin{matrix}0.3\\0.6\\0.7\end{matrix}\right\rangle$$

$$\tilde{Z}^N = \left\langle \begin{pmatrix} (-74,187.5,927,2317) \\ (-25.5,62,738,1999) \\ (52,13.75,531.75,1654) \end{pmatrix} \begin{matrix} 0.4 \\ ;0.4 \\ 0.6 \end{matrix} \right\rangle = -194.54$$

Now for application of MRCM, we use steps in 5.3 to balance the unbalance CTP of table 2 as follows in table 6:

Table 6

	M_1	M_2	M_3	M_4	***Supply***
S_1	$\left\langle \begin{pmatrix} (3,5,7.5,11) \\ (2,4,7,10) \\ (1,3.5,6,9) \end{pmatrix} \begin{matrix} 0.2 \\ ;0.4 \\ 0.8 \end{matrix} \right\rangle$	$\left\langle \begin{pmatrix} (2,4.5,10,15) \\ (0.5,3.5,8,14) \\ (0,2.5,6,12) \end{pmatrix} \begin{matrix} 0.3 \\ ;0.5 \\ 0.8 \end{matrix} \right\rangle$	$\left\langle \begin{pmatrix} (1,5,9,14.5) \\ (-3,3.5,8,12) \\ (-4,2,7,11) \end{pmatrix} \begin{matrix} 0.2 \\ ;0.5 \\ 0.7 \end{matrix} \right\rangle$	$\left\langle \begin{pmatrix} (1,5,9,14.5) \\ (-3,3.5,8,12) \\ (-4,2,7,11) \end{pmatrix} \begin{matrix} 0.2 \\ ;0.5 \\ 0.7 \end{matrix} \right\rangle$	$\left\langle \begin{pmatrix} (9,17,26,36) \\ (6,14,23,33) \\ (3,11,20,30) \end{pmatrix} \begin{matrix} 0.4 \\ ;0.7 \\ 0.7 \end{matrix} \right\rangle$
S_2	$\left\langle \begin{pmatrix} (1,7,11.5,16) \\ (-1,5,10,14) \\ (-3,3,8.5,12) \end{pmatrix} \begin{matrix} 0.4 \\ ;0.5 \\ 0.7 \end{matrix} \right\rangle$	$\left\langle \begin{pmatrix} (1,6,9.5,12) \\ (-1,4,8.5,11) \\ (-2,2,8,10) \end{pmatrix} \begin{matrix} 0.5 \\ ;0.7 \\ 0.8 \end{matrix} \right\rangle$	$\left\langle \begin{pmatrix} (-1,4,8,15) \\ (-2,3,6,12) \\ (-3,2,5,11) \end{pmatrix} \begin{matrix} 0.3 \\ ;0.6 \\ 0.6 \end{matrix} \right\rangle$	$\left\langle \begin{pmatrix} (-1,4,8,15) \\ (-2,3,6,12) \\ (-3,2,5,11) \end{pmatrix} \begin{matrix} 0.3 \\ ;0.6 \\ 0.6 \end{matrix} \right\rangle$	$\left\langle \begin{pmatrix} (7,17,25,31) \\ (3,12,22.5,29) \\ (1,10,19.5,27) \end{pmatrix} \begin{matrix} 0.3 \\ ;0.6 \\ 0.7 \end{matrix} \right\rangle$
S_3	$\left\langle \begin{pmatrix} (3,6,9,12) \\ (2,5,8,11) \\ (1,4,7,10) \end{pmatrix} \begin{matrix} 0.2 \\ ;0.4 \\ 0.8 \end{matrix} \right\rangle$	$\left\langle \begin{pmatrix} (-1,3.5,9,12) \\ (-2,2.5,7,11) \\ (-4,1,5,10) \end{pmatrix} \begin{matrix} 0.2 \\ ;0.4 \\ 0.8 \end{matrix} \right\rangle$	$\left\langle \begin{pmatrix} (0,5,8,14) \\ (-2,3,7,12) \\ (-4,1,6,10) \end{pmatrix} \begin{matrix} 0.2 \\ ;0.6 \\ 0.6 \end{matrix} \right\rangle$	$\left\langle \begin{pmatrix} (0,5,8,14) \\ (-2,3,7,12) \\ (-4,1,6,10) \end{pmatrix} \begin{matrix} 0.2 \\ ;0.6 \\ 0.6 \end{matrix} \right\rangle$	$\left\langle \begin{pmatrix} (9,16,22,31) \\ (5,14,20,27) \\ (1,12,18,23) \end{pmatrix} \begin{matrix} 0.3 \\ ;0.6 \\ 0.7 \end{matrix} \right\rangle$
S_4	$\left\langle \begin{pmatrix} (1,7,11.5,16) \\ (-1,5,10,14) \\ (-3,3,8.5,12) \end{pmatrix} \begin{matrix} 0.4 \\ ;0.5 \\ 0.7 \end{matrix} \right\rangle$	$\left\langle \begin{pmatrix} (-1,3.5,9,12) \\ (-2,2.5,7,11) \\ (-4,1,5,10) \end{pmatrix} \begin{matrix} 0.2 \\ ;0.4 \\ 0.8 \end{matrix} \right\rangle$	$\left\langle \begin{pmatrix} (0,5,8,14) \\ (-2,3,7,12) \\ (-4,1,6,10) \end{pmatrix} \begin{matrix} 0.2 \\ ;0.6 \\ 0.6 \end{matrix} \right\rangle$	$\left\langle \begin{pmatrix} (0,0,0,0) \\ (0,0,0,0) \\ (0,0,0,0) \end{pmatrix} \begin{matrix} 0.2 \\ ;0.6 \\ 0.6 \end{matrix} \right\rangle$	$\left\langle \begin{pmatrix} (25,50,73,98) \\ (14,40,65.5,89) \\ (5,33,57.5,80) \end{pmatrix} \begin{matrix} 0.3 \\ ;0.6 \\ 0.7 \end{matrix} \right\rangle$
Demand	$\left\langle \begin{pmatrix} (12,21,30,37) \\ (9,19,28,34) \\ (6,16,25,33) \end{pmatrix} \begin{matrix} 0.3 \\ ;0.6 \\ 0.7 \end{matrix} \right\rangle$	$\left\langle \begin{pmatrix} (10,16,22,27) \\ (5,14,20,25) \\ (0,12,18,23) \end{pmatrix} \begin{matrix} 0.4 \\ ;0.7 \\ 0.7 \end{matrix} \right\rangle$	$\left\langle \begin{pmatrix} (7,12,19,27) \\ (4,11,18,24) \\ (1,9,15,21) \end{pmatrix} \begin{matrix} 0.2 \\ ;0.6 \\ 0.6 \end{matrix} \right\rangle$	$\left\langle \begin{pmatrix} (-41,29,97,167) \\ (-55,14,87,160) \\ (-67,8,78,153) \end{pmatrix} \begin{matrix} 0.3 \\ ;0.6 \\ 0.7 \end{matrix} \right\rangle$	

After converting cost, demand and supply of NTP in table 6 from SVTrNN to the crisp numbers by using score function method, the balance CTP cost matrix is given in table 7:

Table 7

	M_1	M_2	M_3	M_4	***Supply***
S_1	-1.33	-1.25	-0.58	-0.58	*-4.33*
S_2	-1.08	-1.00	-0.67	-0.67	*-3.67*
S_3	-1.50	-0.50	-0.50	-0.50	*-3.50*
S_4	-1.08	-0.50	-0.50	0	*-11.50*
Demand	*-5.83*	*-3.50*	*-3.17*	*-10.50*	

The complete allotment of demand and supply in cost matrix of CTP shown in table 8:

Table 8

	M_1	M_2	M_3	M_4	***Supply***
S_1	-1.33**(-1.16)**	-1.25	-0.58**(-3.17)**	-0.58	*-4.33*
S_2	-1.08**(-3.67)**	-1.00	-0.67	-0.67	*-3.67*
S_3	-1.50	-0.50**(-3.50)**	-0.50	-0.50	*-3.50*
S_4	-1.08**(-1.00)**	-0.50	-0.50	0**(-10.50)**	*-11.50*
Demand	*-5.83*	*-3.50*	*-3.17*	*-10.50*	

The optimal crisp solution and minimum cost of balance CTP of table 8 is $x_{11} = -1.16$, $x_{13} = -3.17$, $x_{21} = -3.67$, $x_{32} = -3.50$, $x_{41} = -1.00$, $x_{44} = -10.50$ and $Z = 10.18$.

Similarly after balance the unbalance NTP by MRCM, the corresponding optimal solution of balance NTP with allotment of SVTrNN is shown in table 9 as follows:

Table 9

	M_1	M_2	M_3	M_4	***Supply***
S_1	$\left\langle\begin{pmatrix}(-18,-2,14,29)\\(-18,-4,12,29)\\(-18,-4,11,29)\end{pmatrix};\begin{matrix}0.4\\0.6\\0.6\end{matrix}\right\rangle$	-	$\left\langle\begin{pmatrix}(7,12,19,27)\\(4,11,18,24)\\(1,9,15,21)\end{pmatrix};\begin{matrix}0.2\\0.6\\0.6\end{matrix}\right\rangle$	-	-
S_2	$\left\langle\begin{pmatrix}(7,17,25,31)\\(3,12,22.5,29)\\(1,10,19.5,27)\end{pmatrix};\begin{matrix}0.3\\0.6\\0.7\end{matrix}\right\rangle$	-	-	-	-
S_3	-	$\left\langle\begin{pmatrix}(9,16,22,31)\\(5,14,20,27)\\(1,12,18,23)\end{pmatrix};\begin{matrix}0.3\\0.6\\0.7\end{matrix}\right\rangle$	-	-	-
S_4	$\left\langle\begin{pmatrix}(-142,-47,44,139)\\(-146,-47,51.5,144)\\(-148,-45,49.5,147)\end{pmatrix};\begin{matrix}0.3\\0.6\\0.7\end{matrix}\right\rangle$	-	-	$\left\langle\begin{pmatrix}(-41,29,97,167)\\(-55,14,87,160)\\(-67,8,78,153)\end{pmatrix};\begin{matrix}0.3\\0.6\\0.7\end{matrix}\right\rangle$	-
Demand		-	-	-	-

$$\begin{aligned}\tilde{Z}^N = {} & \left\langle\begin{pmatrix}(3,5,7.5,11)\\(2,4,7,10)\\(1,3.5,6,9)\end{pmatrix};\begin{matrix}0.2\\0.4\\0.8\end{matrix}\right\rangle.\left\langle\begin{pmatrix}(-18,-2,14,29)\\(-18,-4,12,29)\\(-18,-4,11,29)\end{pmatrix};\begin{matrix}0.4\\0.6\\0.6\end{matrix}\right\rangle+\left\langle\begin{pmatrix}(1,5,9,14.5)\\(-3,3.5,8,12)\\(-4,2,7,11)\end{pmatrix};\begin{matrix}0.2\\0.5\\0.7\end{matrix}\right\rangle.\left\langle\begin{pmatrix}(7,12,19,27)\\(4,11,18,24)\\(1,9,15,21)\end{pmatrix};\begin{matrix}0.2\\0.6\\0.6\end{matrix}\right\rangle+\\ & +\left\langle\begin{pmatrix}(1,7,11.5,16)\\(-1,5,10,14)\\(-3,3,8.5,12)\end{pmatrix};\begin{matrix}0.4\\0.5\\0.7\end{matrix}\right\rangle.\left\langle\begin{pmatrix}(7,17,25,31)\\(3,12,22.5,29)\\(1,10,19.5,27)\end{pmatrix};\begin{matrix}0.3\\0.6\\0.7\end{matrix}\right\rangle+\left\langle\begin{pmatrix}(-1,3.5,9,12)\\(-2,2.5,7,11)\\(-4,1,5,10)\end{pmatrix};\begin{matrix}0.2\\0.4\\0.8\end{matrix}\right\rangle.\left\langle\begin{pmatrix}(9,16,22,31)\\(5,14,20,27)\\(1,12,18,23)\end{pmatrix};\begin{matrix}0.3\\0.6\\0.7\end{matrix}\right\rangle+\\ & +\left\langle\begin{pmatrix}(1,7,11.5,16)\\(-1,5,10,14)\\(-3,3,8.5,12)\end{pmatrix};\begin{matrix}0.4\\0.5\\0.7\end{matrix}\right\rangle.\left\langle\begin{pmatrix}(-142,-47,44,139)\\(-146,-47,51.5,144)\\(-148,-45,49.5,147)\end{pmatrix};\begin{matrix}0.3\\0.6\\0.7\end{matrix}\right\rangle+\left\langle\begin{pmatrix}(0,0,0,0)\\(0,0,0,0)\\(0,0,0,0)\end{pmatrix};\begin{matrix}0.2\\0.6\\0.6\end{matrix}\right\rangle.\left\langle\begin{pmatrix}(-41,29,97,167)\\(-55,14,87,160)\\(-67,8,78,153)\end{pmatrix};\begin{matrix}0.3\\0.6\\0.7\end{matrix}\right\rangle\end{aligned}$$

this implies $$\tilde{Z}^N = \left\langle\begin{pmatrix}(-191,-104,1267.5,3802.5)\\(85,-117.5,1108,3297)\\(415,-89,847.5,2810)\end{pmatrix};\begin{matrix}0.4\\0.4\\0.6\end{matrix}\right\rangle = -417.77$$

6.3. Model II (NTP of type-2)

For solution of NTP of type-2 i.e. a problem in which costs are in SVTrNN while demand and supply are given in crisp numbers. Here we are taking the problem in table 2 in which costs are in SVTrNN while demand and supply are as crisp numbers given as follows in table 10:

Table 10

	M_1	M_2	M_3	***Supply***
S_1	$\left\langle\begin{pmatrix}(3,5,7.5,11)\\(2,4,7,10)\\(1,3.5,6,9)\end{pmatrix};\begin{matrix}0.2\\0.4\\0.8\end{matrix}\right\rangle$	$\left\langle\begin{pmatrix}(2,4.5,10,15)\\(0.5,3.5,8,14)\\(0,2.5,6,12)\end{pmatrix};\begin{matrix}0.3\\0.5\\0.8\end{matrix}\right\rangle$	$\left\langle\begin{pmatrix}(1,5,9,14.5)\\(-3,3.5,8,12)\\(-4,2,7,11)\end{pmatrix};\begin{matrix}0.2\\0.5\\0.7\end{matrix}\right\rangle$	***-4.33***
S_2	$\left\langle\begin{pmatrix}(1,7,11.5,16)\\(-1,5,10,14)\\(-3,3,8.5,12)\end{pmatrix};\begin{matrix}0.4\\0.5\\0.7\end{matrix}\right\rangle$	$\left\langle\begin{pmatrix}(1,6,9.5,12)\\(-1,4,8.5,11)\\(-2,2,8,10)\end{pmatrix};\begin{matrix}0.5\\0.7\\0.8\end{matrix}\right\rangle$	$\left\langle\begin{pmatrix}(-1,4,8,15)\\(-2,3,6,12)\\(-3,2,5,11)\end{pmatrix};\begin{matrix}0.3\\0.6\\0.6\end{matrix}\right\rangle$	***-3.67***
S_3	$\left\langle\begin{pmatrix}(3,6,9,12)\\(2,5,8,11)\\(1,4,7,10)\end{pmatrix};\begin{matrix}0.2\\0.4\\0.8\end{matrix}\right\rangle$	$\left\langle\begin{pmatrix}(-1,3.5,9,12)\\(-2,2.5,7,11)\\(-4,1,5,10)\end{pmatrix};\begin{matrix}0.2\\0.4\\0.8\end{matrix}\right\rangle$	$\left\langle\begin{pmatrix}(0,5,8,14)\\(-2,3,7,12)\\(-4,1,6,10)\end{pmatrix};\begin{matrix}0.2\\0.6\\0.6\end{matrix}\right\rangle$	***-3.50***
Demand	***-5.83***	***-3.50***	***-3.17***	

The optimal crisp solution of NTP type-2 is shown in table 11 as follows:

Table 11

	M_1	M_2	M_3	***Supply***
S_1	-2.16	-2.17	-	***-4.33***
S_2	-3.67	-	-	***-3.67***
S_3	-	-0.33	-3.17	***-3.50***
S_4	-	-1.00	-	***-3.50***
Demand	***-5.83***	***-3.50***	***-3.17***	

The corresponding neutrosophic solution of NTP type-2 is:

$$\tilde{Z}^N_{t_2}=-2.16\times\left\langle\begin{pmatrix}(3,5,7.5,11)\\(2,4,7,10)\\(1,3.5,6,9)\end{pmatrix};\begin{matrix}0.2\\0.4\\0.8\end{matrix}\right\rangle-2.17\times\left\langle\begin{pmatrix}(2,4.5,10,15)\\(0.5,3.5,8,14)\\(0,2.5,6,12)\end{pmatrix};\begin{matrix}0.3\\0.5\\0.8\end{matrix}\right\rangle-3.67\times\left\langle\begin{pmatrix}(1,7,11.5,16)\\(-1,5,10,14)\\(-3,3,8.5,12)\end{pmatrix};\begin{matrix}0.4\\0.5\\0.7\end{matrix}\right\rangle$$

$$-0.33\times\left\langle\begin{pmatrix}(-1,3.5,9,12)\\(-2,2.5,7,11)\\(-4,1,5,10)\end{pmatrix};\begin{matrix}0.2\\0.4\\0.8\end{matrix}\right\rangle-3.17\times\left\langle\begin{pmatrix}(0,5,8,14)\\(-2,3,7,12)\\(-4,1,6,10)\end{pmatrix};\begin{matrix}0.2\\0.6\\0.6\end{matrix}\right\rangle=\left\langle\begin{pmatrix}(-14.16,-63.27,-108.44,-163.37)\\(5.26,-44.93,-93.68,-145.07)\\(22.85,-27.5,-77.87,-124.52)\end{pmatrix};\begin{matrix}0.2\\0.6\\0.6\end{matrix}\right\rangle=15.89$$

Similarly after balance the unbalance NTP of type-2 by MRCM, the corresponding optimal neutrosophic solution of balance NTP of type-2 with allotment is as follows:

$$\tilde{Z}^N_{t_2}=-1.16\times\left\langle\begin{pmatrix}(3,5,7.5,11)\\(2,4,7,10)\\(1,3.5,6,9)\end{pmatrix};\begin{matrix}0.2\\0.4\\0.8\end{matrix}\right\rangle-3.67\times\left\langle\begin{pmatrix}(1,5,9,14.5)\\(-3,3.5,8,12)\\(-4,2,7,11)\end{pmatrix};\begin{matrix}0.2\\0.5\\0.7\end{matrix}\right\rangle-3.17\left\langle\begin{pmatrix}(1,7,11.5,16)\\(-1,5,10,14)\\(-3,3,8.5,12)\end{pmatrix};\begin{matrix}0.4\\0.5\\0.7\end{matrix}\right\rangle-3.50\times\left\langle\begin{pmatrix}(-1,3.5,9,12)\\(-2,2.5,7,11)\\(-4,1,5,10)\end{pmatrix};\begin{matrix}0.2\\0.4\\0.8\end{matrix}\right\rangle$$

$$-1.00\times\left\langle\begin{pmatrix}(1,7,11.5,16)\\(-1,5,10,14)\\(-3,3,8.5,12)\end{pmatrix};\begin{matrix}0.4\\0.5\\0.7\end{matrix}\right\rangle-10.50\times\left\langle\begin{pmatrix}(0,0,0,0)\\(0,0,0,0)\\(0,0,0,0)\end{pmatrix};\begin{matrix}0.2\\0.6\\0.6\end{matrix}\right\rangle=\left\langle\begin{pmatrix}7.82,65.59,121.85,174.70\\-19.86,47.09,103.68,152.52\\-40.03,27.41,85.60,135.85\end{pmatrix};\begin{matrix}0.2\\0.6\\0.6\end{matrix}\right\rangle=14.35$$

7. Comparative Study

Real life application of single valued trapezoidal neutrosophic numbers in transportation problem have been solved by some existing and proposed MRCM methods. In present paper, the minimum cost obtained through proposed method with some existing method discussed in [30] have been summarized in table 12. From the table it is clear that minimum cost obtained by using MRCM is better than to the existing method in both either crisp or in neutrosophic environment. Figure 3 shows the graphical representation of the minimum crisp or neutrosophic cost degree of satisfaction by different approaches.

Comparison				
	Balance by existing method		Balance by MRCM	
Model I	Crisp cost of CTP $Z = 11.30$	The neutrosophic cost of NTP $\tilde{Z}^N=\left\langle\begin{pmatrix}(-74,187.5,927,2317)\\(-25.5,62,738,1999)\\(52,13.75,531.75,1654)\end{pmatrix};\begin{matrix}0.4\\0.4\\0.6\end{matrix}\right\rangle$ corresponding Crisp cost of NTP $\tilde{Z}^N = -194.54$	Crisp cost of CTP $Z = 10.18$	The neutrosophic cost of NTP $\tilde{Z}^N=\left\langle\begin{pmatrix}(-191,-104,1267.5,3802.5)\\(85,-117.5,1108,3297)\\(415,-89,847.5,2810)\end{pmatrix};\begin{matrix}0.4\\0.4\\0.6\end{matrix}\right\rangle$ corresponding Crisp cost of NTP $\tilde{Z}^N = -417.77$
Model II	The neutrosophic cost of NTP $\tilde{Z}^N_{t_2}=\left\langle\begin{pmatrix}(-14.16,-63.27,-108.44,-163.37)\\(5.26,-44.93,-93.68,-145.07)\\(22.85,-27.5,-77.87,-124.52)\end{pmatrix};\begin{matrix}0.2\\0.6\\0.6\end{matrix}\right\rangle$		The neutrosophic cost of NTP $\tilde{Z}^N_{t_2}=\left\langle\begin{pmatrix}7.82,65.59,121.85,174.70\\-19.86,47.09,103.68,152.52\\-40.03,27.41,85.60,135.85\end{pmatrix};\begin{matrix}0.2\\0.6\\0.6\end{matrix}\right\rangle$	
	corresponding Crisp cost of NTP $\tilde{Z}^N_{t_2} = 15.89$		corresponding Crisp cost of NTP $\tilde{Z}^N_{t_2} = 14.35$	

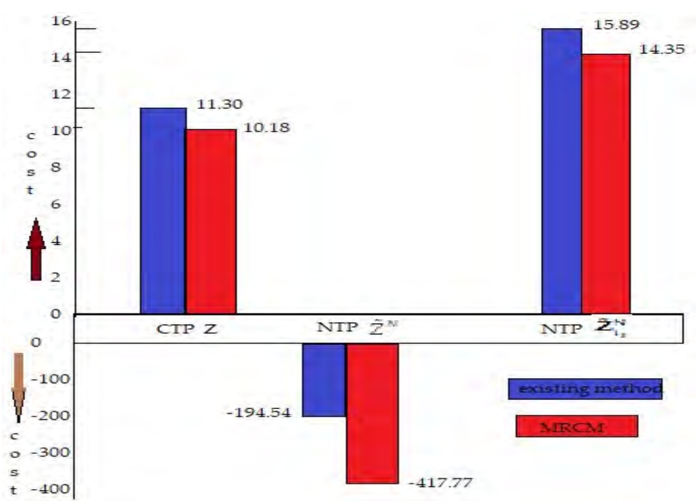

Figure 3: Comparision of results with proposed MRCM and existing method

8. Result and discussion

In this present study the optimal transportation crisp cost and optimal transportation neutrosophic cost of unbalanced NTP using MRCM is minimum than the existing method in [30]. It is also verified that in de-neutrosophication, the crisp values before and after conversion from neutrosophic to crisp and crisp to neutrosophic are different in score function method.

For the real life applications one can find the degree of result. The best of minimum neutrosophic cost of unbalanced NTP is $\tilde{Z}^N = \left\langle \begin{pmatrix} (-191,-104,1267.5,3802.5) \\ (85,-117.5,1108,3297) \\ (415,-89,847.5,2810) \end{pmatrix} ; \begin{matrix} 0.4 \\ 0.4 \\ 0.6 \end{matrix} \right\rangle$ i.e. total minimum transportation cost lies between -191 to 3802.5 for level of truthfulness or acceptance, 85 to 3297 for level of indeterminacy and 415 to 2810 for level of falsity. The degree of truthfulness or acceptance, degree of indeterminacy and degree of falsity is defined as $w_{T_{\tilde{Z}^N}}(x)\times 100$, $u_{I_{\tilde{Z}^N}}(x)\times 100$ and $v_{F_{\tilde{Z}^N}}(x)\times 100$ respectively, where x denotes the total cost and

$$w_{T_{\tilde{Z}^N}}(x) = \begin{cases} \dfrac{0.4(x+191)}{191-104}, & \text{for } -191 \le x \le -104, \\ 0.4, & \text{for } -104 \le x \le 1267.5, \\ \dfrac{0.4(3802.5-x)}{3802.5-1267.5}, & \text{for } 1267.5 \le x \le 3802.5, \\ 0, & \text{for otherwise.} \end{cases}$$

$$u_{I_{\tilde{Z}^N}}(x) = \begin{cases} \dfrac{(-117.5-x)+0.4(x-85)}{62+25.5}, & \text{for } -117.5 \le x \le 85, \\ 0.4, & \text{for } -117.5 \le x \le 1108, \\ \dfrac{(x-1108)+0.4(3297-x)}{3297-1108}, & \text{for } 1108 \le x \le 3297, \\ 0, & \text{for otherwise.} \end{cases}$$

$$v_{F_{\tilde{Z}^N}}(x) = \begin{cases} \dfrac{(-89-x)+0.6(x-415)}{13.75-52}, & \text{for } -89 \le x \le 415, \\ 0.6, & \text{for } -89 \le x \le 847, \\ \dfrac{(x-847)+0.6(2810-x)}{2810-847}, & \text{for } 847 \le x \le 2810, \\ 0, & \text{for otherwise.} \end{cases}$$

$x \rightarrow$ Degree $\downarrow$	-100	0	500	1000	2000	3000
$w_{T_{\tilde{Z}^N}} \times 100$	40.0	40.0	40.0	40.0	30.0	12.6
$u_{T_{\tilde{Z}^N}} \times 100$	40.0	40.0	40.0	40.0	64.4	91.8
$v_{T_{\tilde{Z}^N}} \times 100$	60.0	60.0	60.0	63.1	83.4	0

With the help of degree of truthfulness, degree of indeterminacy and degree of falsity, we can conclude the total neutrosophic cost from the range of -191 to 3802.5 for truthfulness, 85 to 3297 for indeterminacy and 415 to 2810 for falsity to scheduled the transportation and budget allocation.

9. Conclusions

In recent scenario the applied mathematical modeling with uncertainty or vagueness is necessity of the society. Nowadays the concept of neutrosophic number is very popular to handle such type of problems. In this research paper, we study of unbalance NTP and introduced a new approach MRCM for optimal solution with the concept of single valued trapezoidal neutrosophic number of twelve components from different viewpoints. Also the optimal neutrosophic solution and minimum cost obtained by using MRCM is better than by using some existing methods. The proposed method provides the more practical structure and considers the various characteristics of transportation problems in neutrosophic environment. In future the proposed MRCM is applied to the unbalance multi-attribute transportation problem, assignment problems and multilevel programming problem in neutrosophic environment. The present research will be a mile stone for transportation problems with generalization of the pick value of truth, indeterminacy and falsity functions by considering, which are very important for uncertainty theory.

Acknowledgement : Prof. Yanhui Guo, University of Illinois at Springfield, One University Plaza, Springfield, IL 62703, United States,

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Received: July 26, 2020. Accepted: Jun 24, 2020

Neutrosophic Sets and Systems (NSS) is an academic journal, published quarterly online and on paper, that has been created for publications of advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics etc. and their applications in any field.

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ISSN (print): 2331-6055, ISSN (online): 2331-608X
Impact Factor: 1.739

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