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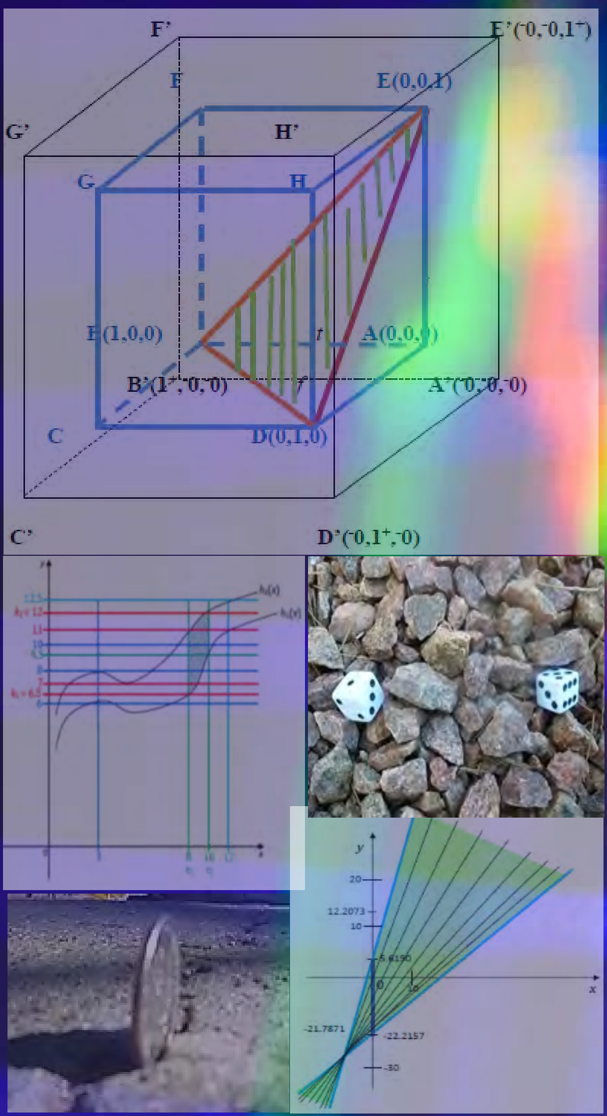

<A> <neutA> <antiA>

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The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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March 20, 2019

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n-Refined Neutrosophic Modules

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Abstract: This paper introduces the concept of n-refined neutrosophic module as a new generalization of neutrosophic modules and refined neutrosophic modules respectively and as a new algebraic application of n-refined neutrosophic set. It studies elementary properties of these modules. Also, This work discusses some corresponding concepts such as weak/strong n-refined neutrosophic modules, n-refined neutrosophic homomorphisms, and kernels.

Keywords: n-Refined weak neutrosophic module, n-Refined strong neutrosophic module, n-Refined neutrosophic homomorphism.

1. Introduction

In 1980s the international movement called paradoxism, based on contradictions in science and literature, was founded by Smarandache, who then extended it to neutrosophy, based on contradictions and their neutrals. [30]

Neutrosophy as a new branch of philosophy studies origin, nature, and indeterminacies, it was founded by F. Smarandache and became a useful tool in algebraic structures. Many neutrosophic algebraic structures were defined and studied such as neutrosophic groups, neutrosophic rings, neutrosophic vector spaces, and neutrosophic modules [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15]. In 2013 Smarandache proposed a new idea, when he extended the neutrosophic set to refined [n-valued] neutrosophic set, i.e. the truth value T is refined/split into types of sub-truths such as (T_1, T_2, …,) similarly indeterminacy I is refined/split into types of sub-indeterminacies (I_1, I_2, …,) and the falsehood F is refined/split into sub-falsehood (F_1, F_2,..,) [17,18].
Recently, there are increasing efforts to study the neutrosophic generalized structures and spaces such as refined neutrosophic modules, spaces, equations, and rings [5,14,21,22,23,24]. Smarandache et.al introduced the concept of n-refined neutrosophic ring [20], and n-refined neutrosophic vector space [19] by using n-refined neutrosophic set concept. Also, neutrosophic sets played an important role in applied science such as health care, industry, and optimization [25,26,27,28].
In this paper we give a new concept based on n-refined neutrosophic set, where we define and study the concept of n-refined neutrosophic modules, submodules, and homomorphisms as a generalization of similar concepts in the case of neutrosophic and refined neutrosophic modules [13,14]. Also, we discuss some elementary properties.

For our purpose we use multiplication operation (defined in [20]) between indeterminacies

$I_1, I_2, \dots, I_n$ as follows:

$I_m I_s = I_{\min(m,s)}$.

All rings considered through this paper are commutative.

2. Preliminaries

Definition 2.1: [20]

Let (R,+,×) be a ring and $I_k; 1 \leq k \leq n$ be n indeterminacies. We define

R_n(I)={$a_0 + a_1 I + \cdots + a_n I_n$; $a_i \in R$} to be n-refined neutrosophic ring.

Definition 2.2: [20]

(a) Let R_n(I) be an n-refined neutrosophic ring and P = $\sum_{i=0}^{n} P_i I_i$ = { $a_0 + a_1 I + \cdots + a_n I_n$: $a_i \in P_i$},

where P_i is a subset of R, we define P to be an AH-subring if P_i is a subring of R for all .

AHS-subring is defined by the condition $P_i = P_j$ for all i, j.

(b) P is an AH-ideal if P_i is an two sides ideal of R for all i, the AHS-ideal is defined by the condition

$P_i = P_j$ for all i, j.

(c) The AH-ideal P is said to be null if $P_i = R$ or $P_i = \{0\}$ for all i.

Definition 2.3 :[10]

Let (V , + , .) be a vector space over the field K then (V(I) , + ,.) is called a weak neutrosophic vector space over the field K , and it is called a strong neutrosophic vector space if it is a vector space over the neutrosophic field K(I).

Definition 2.4: [13]

Let (M,+,.) be a module over the ring R then (M(I),+,.) is called a weak neutrosophic module over the ring R, and it is called a strong neutrosophic module if it is a module over the neutrosophic ring R(I).

Elements of M(I) have the form $x + yI; x, y \in M$, i.e M(I) can be written as $M(I) = M + MI$.

Definition 2.5: [13]

Let M(I) be a strong neutrosophic module over the neutrosophic ring R(I) and W(I) be a non empty set of M(I), then W(I) is called a strong neutrosophic submodule if W(I) itself is a strong neutrosophic module.

Definition 2.6: [13]

Let U(I) and W(I) be two strong neutrosophic submodules of M(I) and let $f: U(I) \to W(I)$, we say that f is a neutrosophic vector space homomorphism if

(a) $f(I) = I$.

(b) f is a module homomorphism.

3. Main concepts and results

Definition 3.1 :

Let (M,+,.) be a module over the ring R, we say that

$M_n(I) = M + MI_1 + \cdots + MI_n = \{x_0 + x_1I_1 + \cdots + x_nI_n; x_i \in M\}$ is a weak n-refined neutrosophic module over the ring R. Elements of $M_n(I)$ are called n-refined neutrosophic vectors, elements of R are called scalars.

If we take scalars from the n-refined neutrosophic ring $R_n(I)$, we say that $M_n(I)$ is a strong n-refined neutrosophic module over the n-refined neutrosophic ring $M_n(I)$. Elements of $M_n(I)$ are called n-refined neutrosophic scalars.

Remark 3.2:

If we take n=1 we get the classical neutrosophic module.

Addition on $M_n(I)$ is defined as:

$\sum_{i=0}^{n} a_i I_i + \sum_{i=0}^{n} b_i I_i = \sum_{i=0}^{n} (a_i + b_i) I_i$.

Multiplication by a scalar $m \in R$ is defined as:

$m. \sum_{i=0}^{n} a_i I_i = \sum_{i=0}^{n} (m. a_i) I_i$.

Multiplication by an n-refined neutrosophic scalar $m = \sum_{i=0}^{n} m_i I_i \in R_n(I)$ is defined as:

$\sum_{i=0}^{n} m_i I_i . \sum_{i=0}^{n} a_i I_i = \sum_{i=0}^{n} (m_i. a_i) I_i I_j$.

Where $a_i \in M, m_i \in R, I_i I_j = I_{\min(i,j)}$.

Theorem 3.3 :

Let (M,+,.) be a module over the ring R. Then a weak n-refined neutrosophic module $M_n(I)$ is a module over the ring R. A strong n-refined neutrosophic module is a module over the n-refined neutrosophic ring R_n(I).

Proof:

It is similar to that of Theorem 5 in [9].

Example 3.4:

Let $M = Z_2$ be the finite module of integers modulo 2 over itself, we have:

(a) The corresponding weak 2-refined neutrosophic module over the ring Z_2 is

$M_n(I) = \{0,1,I_1,I_2,I_1+I_2,1+I_1+I_2,1+I_1,1+I_2\}$.

Definition 3.5:

Let $M_n(I)$ be a weak n-refined neutrosophic module over the ring R, a nonempty subset $W_n(I)$ is called a weak n-refined neutrosophic module of $M_n(I)$ if $W_n(I)$ is a submodule of $M_n(I)$ itself.

Definition 3.6:

Let $M_n(I)$ be a strong n-refined neutrosophic module over the n-refined neutrosophic ring $R_n(I)$, a nonempty subset $W_n(I)$ is called a strong n-refined neutrosophic submodule of $M_n(I)$ if $W_n(I)$ is a submodule of $M_n(I)$ itself.

Theorem 3.7:

Let $M_n(I)$ be a weak n-refined neutrosophic module over the ring R, $W_n(I)$ be a nonempty subset of $M_n(I)$. Then $W_n(I)$ is a weak n-refined neutrosophic submodule if and only if:

$x+y \in W_n(I), m.x \in W_n(I)$ for all $x,y \in W_n(I), m \in R$.

Proof:

It holds directly from the fact that is $W_n(I)$ is a submodule of $M_n(I)$.

Theorem 3.8:

Let $M_n(I)$ be a strong n-refined neutrosophic module over the n-refined neutrosophic ring $R_n(I)$, $W_n(I)$ be a nonempty subset of $M_n(I)$. Then $W_n(I)$ is a strong n-refined neutrosophic submodule if and only if:

$x + y \in W_n(I), m.x \in W_n(I)$ for all $x, y \in W_n(I), m \in R_n(I)$.

Proof:

It holds directly from the fact that is $W_n(I)$ is a submodule of $M_n(I)$ over the n-refined neutrosophic ring $R_n(I)$.

Example 3.9:

$M = R^2$ is a module over the ring R, $W = <(0,1)>$ is a submodule of M,

$R_2^2(I) = \{(a,b) + (m,s)I_1 + (k,t)I_2; a,b,m,s,k,t \in R\}$ is the corresponding weak/strong 2-refined neutrosophic module.

$W_2(I) = \{a_0 + a_1I_1 + a_2I_2\} = \{(0,x) + (0,y)I_1 + (0,z)I_2; x,y,z \in R\}$ is a weak 2-refined neutrosophic submodule of the weak 2-refined neutrosophic module $R_2^2(I)$ over the ring R.

$W_2(I) = \{a_0 + a_1I_1 + a_2I_2\} = \{(0,x) + (0,y)I_1 + (0,z)I_2; x,y,z \in R\}$ is a strong 2-refined neutrosophic submodule of the strong 2-refined neutrosophic module $R_2^2(I)$ over the n-refined neutrosophic ring $R_2(I)$.

Definition 3.10:

Let $M_n(I)$ be a weak n-refined neutrosophic module over the ring R, x be an arbitrary element of $M_n(I)$, we say that x is a linear combination of $\{x_1, x_2, \dots, x_m\} \leq M_n(I)$ is

$x = a_1x_1 + a_2x_2 + \dots + a_mx_m;\ a_i \in R, x_i \in M_n(I)$.

Example 3.11:

Consider the weak 2-refined neutrosophic module in Example 3.11,

$x = (0,2) + (1,3)I_1 \in \mathrm{R}_2^2(\mathrm{I})$, we have

$x = 2(0,1) + 1.(1,0)I_1 + 3(0,1)I_1$

i.e x is a linear combination of the set $\{(0,1), (1,0)I_1, (0,1)I_1\}$ over the ring R.

Definition 3.12:

Let $M_n(I)$ be a strong n-refined neutrosophic module over the n-refined neutrosophic ring $R_n(I)$, x be an arbitrary element of $M_n(I)$, we say that x is a linear combination of $\{x_1, x_2, \dots, x_m\} \leq M_n(I)$ is $x = a_1x_1 + a_2x_2 + \cdots + a_mx_m$: $a_i \in R_n(I), x_i \in M_n(I)$.

Example 3.12:

Consider the strong 2-refined neutrosophic module

$R_2^2(I) = \{(a, b) + (m, s)I_1 + (k, t)I_2; a, b, m, s, k, t \in R\}$ over the 2-refined neutrosophic ring $R_2(I)$,

$=(1 + I_1)(1,1)I_2 + I_2(2,1)I_1 = (1,1)I_1 + (1,1)I_2 + (2,1)I_1 = x$, hence x is a linear $x = (3,2)I_1 + (1,1)I_2$ combination of the set

$\{(2,1)I_1, (1,1)I_2\}$

over the 2-refined neutrosophic ring $R_2(I)$.

Definition 3.15:

Let $X = \{x_1, \dots, x_m\}$ be a subset of a weak n-refined neutrosophic module $M_n(\mathrm{I})$ over the ring R, X is a weak linearly independent set if $\sum_{i=0}^{m} a_i x_i = 0$ *implies* $a_i = 0$; $a_i \in R$.

Definition 3.16:

Let $X = \{x_1, \dots, x_m\}$ be a subset of a strong n-refined neutrosophic module $M_n(\mathrm{I})$ over the n-refined neutrosophic ring $R_n(I)$, X is a weak linearly independent set if

$\sum_{i=0}^{m} a_i x_i = 0$ *implies* $a_i = 0$; $a_i \in R_n(I)$.

Definition 3.17:

Let $M_n(I), W_n(I)$ be two strong n-refined neutrosophic modules over the n-refined neutrosophic ring $R_n(I)$, let $f: M_n(I) \to U_n(I)$ be a well defined map. It is called a strong n-refined neutrosophic homomorphism if:

$f(a.x + b.y) = a.f(x) + b.f(y)$ for all $x, y \in M_n(I), a, b \in R_n(I)$.

A weak n-refined neutrosophic homomorphism can be defined as the same.

Definition 3.18:

Let $f: M_n(I) \to U_n(I)$ be a weak/strong n-refined neutrosophic homomorphism, we define:

(a) $Ker(f) = \{x \in M_n(I); f(x) = 0\}$.

(b) $Im(f) = \{y \in U_n(I);\ \exists x \in M_n(I) and\ y = f(x)\}$.

Theorem 3.19:

Let $f: M_n(I) \to U_n(I)$ be a weak n-refined neutrosophic homomorphism. Then

(a) $Ker(f)$ is a weak n-refined neutrosophic submodule of M_n(I).

(b) $Im(f)$ is a weak n-refined neutrosophic submodule of $U_n(I)$.

Proof:

(a) f is a module homomorphism since $M_n(I), U_n(I)$ are modules, hence $Ker(f)$ is a submodule of the module $M_n(I)$, thus $Ker(f)$ is a weak n-refined neutrosophic submodule of $M_n(I)$.

(b) Holds by similar argument.

Theorem 3.20:

Let $f: M_n(I) \to U_n(I)$ be a strong n-refined neutrosophic homomorphism. Then

(a) $Ker(f)$ is a strong n-refined neutrosophic submodule of M_n(I).

(b) $Im(f)$ is a strong n-refined neutrosophic submodule of $U_n(I)$.

Proof:

(a) f is a module homomorphism since $M_n(I), U_n(I)$ are modules over the n-refined neutrosophic ring $R_n(I)$, hence $Ker(f)$ is a submodule of the module $M_n(I)$, thus $Ker(f)$ is a strong n-refined neutrosophic submodule of $M_n(I)$.

(b) Holds by similar argument.

Theorem 3.21:

Let $f: M_n(I) \to U_n(I)$ be a strong n-refined neutrosophic homomorphism. Then

(a) $Ker(f)$ is a strong n-refined neutrosophic submodule of M_n(I).

(b) $Im(f)$ is a strong n-refined neutrosophic submodule of $U_n(I)$.

Proof:

(a) f is a module homomorphism since $M_n(I), U_n(I)$ are modules over the n-refined neutrosophic ring $R_n(I)$, hence $Ker(f)$ is a submodule of the module $M_n(I)$, thus $Ker(f)$ is a strong n-refined neutrosophic submodule of $M_n(I)$.

(b) Holds by similar argument.

Example 3.22:

Let $R_2^2(I) = \{x_0 + x_1 I_1 + x_2 I_2;\ x_0, x_1, x_2 \in R^2\}$, $R_2^3(I) = \{y_0 + y_1 I_1 + y_2 I_2;\ y_0, y_1, y_2 \in R^3\}$ be two weak 2-refined neutrosophic modules over the ring of real numbers R. Consider $f: R_2^2(I) \to R_2^3(I)$, where $f[(a,b) + (m,n)I_1 + (k,s)I_2] = (a,0,0) + (m.0.0)I_1 + (k,0,0)I_2$, f is a weak 2-refined neutrosophic homomorphism over the ring R.

$Ker(f) = \{(0,b) + (0,n)I_1 + (0,s)I_2; b,n,s \in R\}$.

$Im(f) = \{(a,0,0) + (m,0,0)I_1 + (k,0,0)I_2; a,m,k \in R\}$.

Example 3.23:

Let $W_2(I) =< (0,0,1)I_1 >= \{q.(0,0,a)I_1; a \in R, q \in R_2(I)\}$,

$U_2(I) =< (0,1,0)I_1 >= \{q.(0,a,0)I_1; a \in R; q \in R_2(I)\}$ be two strong 2-refined neutrosophic modules of the strong 2-refined neutrosophic module $R_2^3(I)$ over 2-refined neutrosophic ring R_2(I). Define

$f: W_2(I) \to U_2(I); f[q(0,0,a)I_1] = q(0,a,0)I_1;\ q \in R_2(I)$.

f is a strong 2-refined neutrosophic homomorphism:

Let $A = q_1(0,0,a)I_1, B = q_2(0,0,b)I_1 \in W_2(I);\ q_1, q_2 \in R_2(I)$, we have

$A + B = (q_1 + q_2)(0,0,a+b)I_1, f(A+B) = (q_1+q_2).(0,a+b,0)I_1 = f(A) + f(B)$.

Let $m = c + dI_1 + eI_2 \in R_2(I)$ be a 2-refined neutrosophic scalar, we have

$m.A = c.q_1(0,0,a)I_1 + d.q_1(0,0,a)I_1I_1 + e.q_1(0,0,a)I_2I_1 = q_1(0,0,c.a + d.a + e.a)I_1$,

$f(m.A) = q_1(0,c.a + d.a + e.a, 0)I_1 = m.f(A)$, hence f is a strong 2-refined neutrosophic homomorphism.

$Ker(f) = (0,0,0) + (0,0,0)I_1 + (0,0,0)I_2$.

$.Im(f) = U_2(I)$

5. Conclusion

In this paper, we continuo the efforts about defining and studying n-refined neutrosophic algebraic structures, where we have introduced the concept of weak/strong n-refined neutrosophic module. Also, some related concepts such as weak/strong n-refined neutrosophic submodule, n-refined neutrosophic homomorphism have been presented and studied.

Future research

Authors hope that some corresponding notions will be studied in future such as weak/strong n-refined neutrosophic basis of n-refined neutrosophic modules, and AH-submodules.

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A Neutrosophic Approach to Digital Images

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Abstract: This research paper presents a neutrosophic mathematical representation of the elements of the digital image by dividing the points of the digital picture matrix into neutrosophic sets (PNS - Picture Neutrosophic Set), and studying the degree of connection between the points of the digital image for us to reach to the connected neutrosophic sets. We have also introduced many mathematical theories and results to calculate the difference and dissimilarity between the neutrosophic sets, which contributes practically in the comparison between digital images and their different uses. Our results help mainly to upgrade and create new neutrosophic algorithms for searching inside images and videos databases**.**

Keywords: Neutrosophic set; connected neutrosophic set; picture neutrosophic set (PNS); difference measure; dissimilarity measure.

1. Introduction

The neutrosophic logic, which resulted in a revolution in the mathematical logic world, was first introduced by Florentin in 1995 [1, 2]. It is a generalization of intuitionistic fuzzy logic. Several papers have been published in this field by Florentin and Salama et al [3-15]. It is necessary to take advantage of the features of this logic in various applied sciences.

Having studied researches related to digital image processing [16-18], we have noted that applied sciences researchers are interested in the use of fuzzy logic, first introduced by Lotfi Zadeh [19], for digital image processing because of its flexibility and appropriate features to deal with different forms of digital images. Moreover, the neutrosophic logic is a generalization and extension of fuzzy logic. It has provided many additional methods and tools, which we can be used to study digital images with greater accuracy and comprehensiveness than before.

Digital image processing is mainly based on mathematical concepts [20-26], such as mathematical logic, linear algebra (matrices), topology, statistics (especially Bayes' theory), Shannon information theory, and Fourier transform in different representations along with neural networks.

Several researchers have performed studies specifying methods to measure the dissimilarity, difference and distance between NSs. Salama, Smarandache, & Eisa, (2014) [27] have introduced image processing via neutrosophic techniques. Mohana & Mohanasundari (2019) [28] have studied some similarity measures of single valued neutrosophic rough sets. Sinha & Majumdar (2019) [29] have studied an approach to similarity measure between neutrosophic soft sets. Das, Samanta, Khan, Naseem & De (2020) [30] also have a study on discrete mathematics: sum distance in neutrosophic graphs with application.

We have organized this paper into 4 sections. In section 2, we discuss preliminaries about digital images and the neutrosophic set. In section 3, we have introduced new neutrosophic concepts, such as $K_S(\alpha)$ (the extent to which the series of points (α) belongs to the neutrosophic set S), and $C_S(p,q)$ (the connection strength between the points $p,q \in S$), based on which we have deduced connected neutrosophic sets. In addition, we have presented our vision in the field of distance and dissimilarity measures in neutrosophic sets. In section 4, we have concluded our paper.

2. Preliminaries

2.1. Digital Image:[31] It is a representation of a two-dimensional image in the form of a matrix of small squares, each image consists of thousands or millions of small squares, each of which is called the elements of the image or pixels.

When the computer starts drawing the image, it divides the screen or printed page into a grid of pixels. Then the computer uses the stored values of the digital image to give each pixel its color and brightness. The images posted on websites or by mobile phone are examples of digital images. For example, the small picture (Felix) can be represented in Figure 1:

Figure 1: Image of Cat Felix [31]

With an array (35 × 35), its elements are composed of numbers 0 and 1. Each element indicates the color of the pixel. It takes the value (0) for the black pixel and the value (1) for the white pixel. Note that digital images using two colors are called binary or Boolean images.

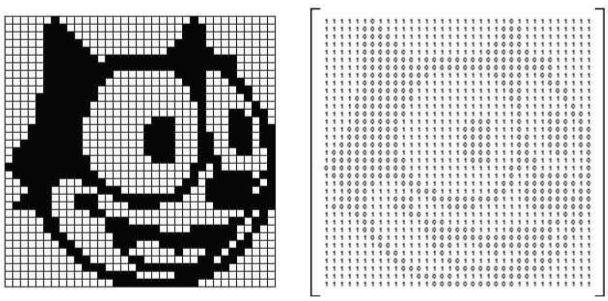

Figure 2: Matrix representing the image of Cat Felix [31]

The grayscale images are represented by a matrix, each element of which specifies the corresponding pixel intensity. For practical reasons, most of the current digital files use integers enclosed between zero-0 (for black pixels, very low color) and 255 (for the white pixel, the color is super hard).

2.2. Neutrosophic Logic:[2] Was created by professor Florentin Smarandache in 1995. It is a generalization of (fuzzy, intuitionistic, paraconsistent) logic. For any logical variable x in the neutrosophic logic A, it is described by (t, i, f), where:

$t = T_A(x)$: Truth membership function: a degree of membership function, for any x in the neutrosophic set A, and its values range in the open interval non-standard, where:

$T_A(x)$: $A \rightarrow \,]0^-, 1^+[$

$i = I_A(x)$: Indeterminacy membership function: a degree of indeterminacy, for any x in the neutrosophic set A, and its values range in the open interval non-standard, where:

$I_A(x)$: $A \rightarrow \,]0^-, 1^+[$

$f = F_A(x)$: Falsity membership function: a degree of non-membership, for any x in the neutrosophic set A, and its values range in the open interval non-standard, where:

$F_A(x)$: $A \rightarrow \,]0^-, 1^+[$

3. Neutrosophic Digital Image

Let M be the digital image matrix A, so any pixel (point) of image A that is expressed by the element $p\,(x, y)$ of the matrix M has four horizontal and vertical adjacent points $(x \pm 1, y)$ and $(x, y \pm 1)$ and four diagonal adjacent points $(x \pm 1, y \pm 1)$, so any point or pixel is surrounded by eight adjacent points (8-adjacent), noting the cases where the point P is present on the border of the matrix M.[18]

3.1. Connected Neutrosophic Sets:

Definition 3.1: Let S be a subset of M. For any p, q from S, they are connected in S if you find a path of points from S that connects p with q as follows:

$\alpha: p = p_0, p_1, p_2, \dots\dots, p_{n-1}, p_n = q$.

Where p_i is adjacent to p_{i-1} $(1 \leq i \leq n)$.

We denote the connection relationship between p, q by $p\rho q$.

Obviously, the relationship (ρ) represents an equivalence relationship:

$Reflexive: [p\rho p]$ & $Symmetric: [p\rho q \Rightarrow q\rho p]$ & $Transitive: [p\rho q \;\&\; q\rho z \Rightarrow p\rho z]$

Remark 3.1: By introducing the concept of non- member function and the function of indeterminacy to the neutrosophic logic, it has got more accuracy than fuzzy logic in different cases, such as an equal degree of membership. Thus, we can introduce the order relation $(\dddot{\leq})$ between any two elements in the neutrosophic set:

Definition 3.2: $\forall\, p, q \in S$, (S is neutrosophic set), then:

$$p \dddot{\leq} q \Leftrightarrow \begin{cases} \quad T_S(p) < T_S(q) \\ (or)\ F_S(p) > F_S(q) \ ;\ T_S(p) = T_S(q) \\ (or)\ I_S(p) \geq I_S(q) \ ;\ T_S(p) = T_S(q), \quad F_S(p) = F_S(q) \end{cases}$$

Remark 3.2: The order relation ($\ddot{\leqq}$) maintains its consistency with fuzzy logic in the case of $T_S(p) \neq T_S(q)$, and maintains consistency with intuitionistic fuzzy logic in the case of $F_S(p) \neq F_S(q)$ & $T_S(p) = T_S(q)$.

Example 3.1: $(0,1,1) \ \ddot{\leqq} \ (0,0,0)$, $(1,1,1) \ \ddot{\leqq} \ (1,0.5,1)$

$(0.9,1,0.8) \ \ddot{\leqq} \ (1,0.3,0.1)$, $(0.7,0,0.4) \ \ddot{\leqq} \ (0.7,1,0.3)$

Definition 3.3: [2] $\forall\, p,q \in S$, then:$[p \leq q] \Leftrightarrow [T_S(p) \leq T_S(q)\ ,\ I_S(p) \geq I_S(q) \quad and \quad F_S(p) \geq F_S(q)]$

Remark 3.3: $\forall\, p,q \in S \ \Rightarrow \begin{bmatrix} [p \leq (1,0,0) = 1_N & \& & 0_N = (0,1,1) \leq p] \\ [\, p \ \ddot{\leqq} \ (1,0,0) = 1_N & \& & 0_N = (0,1,1) \ \ddot{\leqq} \ p] \end{bmatrix}$ & $[p \leq q \ \Rightarrow \ p \ \ddot{\leqq} \ q]$

Definition 3.4: Let $(\alpha: p = p_0, p_1, p_2, \ldots, p_{n-1}, p_n = q)$, series of adjacent points, between the points $, q \ : \ p_i \in S$ (S neutrosophic set). The extent to which the series of points (α) belongs to the neutrosophic set S, denote by $K_S(\alpha)$:

$K_S(\alpha) = x \quad ; \quad (x \in \alpha) \ \ and \ (\, x \ \ddot{\leqq} \ \ p_i \ ; \ i = 0,1, \ldots, n\,)$

$\Rightarrow \ K_S(\alpha) = \min_{\ddot{\leqq}}(p_i)$

Definition 3.5: The connect strength between the points $p, q \in S$ (S neutrosophic set). denote by $C_S(p,q)$: $C_S(p,q) = K_S(\beta)$; $K_S(\alpha_i) \ \ddot{\leqq} \ K_S(\beta) \quad (\forall \alpha_i, \beta: p, \ldots, q)$

$\Rightarrow \ C_S(p,q) = \max_{\ddot{\leqq}}(K_S(\alpha_i))$

Theorem 3.1: S neutrosophic set and $\forall\, p,q \in S$, then:

1: $C_S(p,p) = p$

2: $C_S(p,q) = C_S(q,p)$

Proof:

1: α_i any path, from p to $p \quad \Rightarrow \quad K_S(\alpha_i) = \min_{\ddot{\leqq}}(p_i) \ \ddot{\leqq} \ p$

On the other hand:

The point p alone represents a series with a length of 0 from p to p, then:

$$\exists\ \alpha_i : \ K_S(\alpha_i) = \ p$$

Thus: $C_S(p,p) = \max_{\ddot{\leqq}}(K_S(\alpha_i)) = p$

2: Obviously. (by Definition 3.4)

Theorem 3.2: $\forall\, p,q \in S$ **(**S neutrosophic set**)**, then**:**

$$C_S(p,q) \leq \min_{\ddot{\leqq}}(p,q)$$

Proof:

α any path, from p to q: $(\alpha: p = p_0, p_1, \ldots, p_{n-1}, p_n = q)$, then:

$K_S(\alpha) = \min_{\ddot{\leqq}}(p_i) \ \leq \min_{\ddot{\leqq}}(p_0, p_n) = \min_{\ddot{\leqq}}(p,q) \quad ; i = 0,1, \ldots, n$

$\Rightarrow \ C_S(p,q) = \max_{\ddot{\leqq}}\big(K_S(\alpha_i)\big) \ \leq \min_{\ddot{\leqq}}(p,q)$

Definition 3.6: $\forall\, p,q \in S$, p and q is connected in S iff: $C_S(p,q) = \min_{\ddot{\leqq}}(p,q)$.

Theorem 3.3: S neutrosophic set and $\forall\, p,q \in S$, then**:**

p and q is connected in $S \ \Leftrightarrow \exists\ \alpha' : p = p_0, p_1, \ldots, p_{n-1}, p_n = q \ : \ P_i \in S$ & $P_i \geq \min_{\ddot{\leqq}}(p,q)$ (for all i)

Proof:

Let α' path from p to : $P_i \in S$ & $P_i \geq \min_{\ddot{\leqq}}(p,q)$, then:

$C_S(p,q) = \max_{\ddot{\leqq}}(K_S(\alpha)) \geq K_S(\alpha') = \min_{\ddot{\leqq}}(p_i) \geq \min_{\ddot{\leqq}}(p,q)$

And $C_S(p,q) \leq \min_{\ddot{\leqq}}(p,q)$ (Theorem 3.2)

Then: $C_S(p,q) = \min_{\ddot{\leqq}}(p,q) \Rightarrow p$ and q is connected in S

On the other hand: p and q is connected in , then:

$\exists\, \alpha'$ path from p to q : $K_S(\alpha') = \max_{\ddot{\leqq}}(K_S(\alpha)) = C_S(p,q) = \min_{\ddot{\leqq}}(p,q)$

Then for all P_i on α', we have: $P_i \geq K_S(\alpha') = \min_{\ddot{\leqq}}(p,q)$

Corollary 3.1: From the above we note that the relationship of the connection between two points is the relationship of: 1: reflexivity, 2: Symmetry, 3: not necessarily transitive.

Proof:

1: $C_S(p,p) = p = \min_{\ddot{\leqq}}(p,p)$

2: $\min_{\ddot{\leqq}}(p,q) = C_S(p,q) \Rightarrow C_S(q,p) = C_S(p,q) = \min_{\ddot{\leqq}}(p,q)$

3: Let p,q,z three points from neutrosophic set $S = [p,q,z]$ ($Matrix$ 1×3):

$q \ddot{\leqq} p = z$, then:

$C_S(p,q) = C_S(q,z) = q$ and $C_S(p,z) = q \neq \min_{\ddot{\leqq}}(p,z)$, thus:

(p and q is connected in S) and (q and z is connected in S), but (p and z is not connected in S)

Definition 3.7: S neutrosophic set, S is connected iff: $[\forall\, p,q \in S] \Rightarrow [C_S(p,q) = \min_{\ddot{\leqq}}(p,q)]$.

3.2. Operations on neutrosophic sets:

We will now in this section, we present our vision of distance and dissimilarity measures between two neutrosophic sets.

Definition 3.8: Let U be the set of points of the matrix M, a representative of the digital image I. denote by $PNS(U)$ for set of all neutrosophic sets in U (PNS - Picture Neutrosophic Set).

For $A,B \in PNS(U)$:

Union: $A \cup B = \{u: (T_{A\cup B}(u), I_{A\cup B}(u), F_{A\cup B}(u)); u \in U\}$, where:

$$T_{A\cup B}(u) = \max(T_A(u), T_B(u))$$

$$I_{A\cup B}(u) = \min(I_A(u), I_B(u))$$

$$F_{A\cup B}(u) = \min(F_A(u), F_B(u))$$

Intersection: $A \cap B = \{u: (T_{A\cap B}(u), I_{A\cap B}(u), F_{A\cap B}(u)); u \in U\}$, where:

$$T_{A\cap B}(u) = \min(T_A(u), T_B(u))$$

$$I_{A\cap B}(u) = \max(I_A(u), I_B(u))$$

$$F_{A\cap B}(u) = \max(F_A(u), F_B(u))$$

Example 3.2: Let us consider the following neutrosophic sets A and B in $U = \{u_1, u_2, u_3, u_4\}$, where:

$A = \{u_1: (0,1,1),\ u_2: (1,0.2,0),\ u_3: (0.7,0.3,1),\ u_4: (0.9,0.3,1)\}$

$B = \{u_1: (0.2,1,0.2),\ u_2: (1,0.5,0.3),\ u_3: (1,0.8,1),\ u_4: (0.9,0.8,0.2)\}$

Then:

$A \cup B = \{u_1: (0.2,1,0.2),\ u_2: (1,0.2,0),\ u_3: (1,0.3,1),\ u_4: (0.9,0.3,0.2)\}$

$A \cap B = \{u_1: (0,1,1),\ u_2: (1,0.5,0.3),\ u_3: (0.7,0.8,1),\ u_4: (0.9,0.8,1)\}$

Theorem 3.4: Let $A, B \in PNS\,(U)$, then: (for all $u \in U$)

$A \subseteq B \iff [T_A(u) \le T_B(u)\ ,\quad I_A(u) \ge I_B(u) \quad and \quad F_A(u) \ge F_B(u)]$

Proof:

$A \subseteq B \iff A \cup B = B$

$$\iff \begin{cases} \max\big(T_A(u), T_B(u)\big) = T_B(u) \\ \min\big(I_A(u), I_B(u)\big) = I_B(u) \\ \min\big(F_A(u), F_B(u)\big) = F_B(u) \end{cases} \iff \begin{cases} T_A(u) \le T_B(u) \\ I_A(u) \ge I_B(u) \\ F_A(u) \ge F_B(u) \end{cases}$$

Definition 3.9: An operator $\backslash: PNS(U) \times PNS(U) \to PNS(U)$

Is the difference, if it satisfies for all $A, B, C \in PNS(U)$, follow properties:

DIF1: $A\backslash B \subseteq A$

DIF2: $A\backslash\emptyset = A$

DIF3: $A \subseteq B \iff A\backslash B = \emptyset$

DIF4: $if\ B \subseteq C \implies B\backslash A \subseteq C\backslash A$

Theorem 3.5: The function $\backslash: PNS(U) \times PNS(U) \to PNS(U)$ given by:

$A\backslash B = \{u: \big(T_{A\backslash B}(u), I_{A\backslash B}(u), F_{A\backslash B}(u)\big); u \in U\}$, where:

$$T_{A\backslash B}(u) = \max(0, T_A(u) - T_B(u))$$
$$I_{A\backslash B}(u) = \min(1, 1 + (I_A(u) - I_B(u)))$$
$$F_{A\backslash B}(u) = \min(1, 1 + (F_A(u) - F_B(u)))$$

Is the difference between $PNS(U)$ sets.

Proof:

DIF1: $A\backslash B \subseteq A$

$$\forall\, u \in U \Rightarrow \begin{cases} T_{A\backslash B}(u) = \max\big(0, T_A(u) - T_B(u)\big) \\ I_{A\backslash B}(u) = \min\Big(1, 1 + \big(I_A(u) - I_B(u)\big)\Big) \\ F_{A\backslash B}(u) = \min\Big(1, 1 + \big(F_A(u) - F_B(u)\big)\Big) \end{cases}$$

$[0 \le T_A(u)] \quad and \quad [0 \le T_B(u) \Rightarrow T_A(u) - T_B(u) \le T_A(u)]$

Hence: $T_{A\backslash B}(u) = \max\big(0, T_A(u) - T_B(u)\big) \le T_A(u)$

$$I_A(u) \le 1 \quad and \quad \begin{bmatrix} I_B(u) \le 1 & \Rightarrow I_B(u) + I_A(u) \le 1 + I_A(u) \\ & \Rightarrow I_A(u) \le 1 + \big(I_A(u) - I_B(u)\big) \end{bmatrix}$$

Hence: $I_{A\backslash B}(u) = \min(1, 1 + (I_A(u) - I_B(u))) \ge I_A(u)$

Similarity: $F_{A\backslash B}(u) = \min(1, 1 + (F_A(u) - F_B(u))) \ge F_A(u)$

Thus: $A\backslash B \subseteq A$ (by Theorem 3.4)

DIF2: $A \backslash \emptyset \ = A$

$\emptyset = \{u: (0,1,1) \ ; \forall u \in U\} \ \Rightarrow$

$$\forall \, u \in U \Rightarrow \begin{cases} T_{A\backslash\emptyset}(u) = \max(0, T_A(u) - 0) \ = \ T_A(u) \\ I_{A\backslash\emptyset}(u) \ = \min\big(1, 1 + (I_A(u) - 1)\big) = \ I_A(u) \\ F_{A\backslash\emptyset}(u) = \min\big(1, 1 + (F_A(u) - 1)\big) = \ F_A(u) \end{cases}$$

Then: $A \backslash \emptyset \ = A$

DIF3: $A \subseteq B \ \Leftrightarrow \ A \backslash B = \emptyset$

$$A \subseteq B \ \Leftrightarrow \begin{cases} T_A(u) \ \le T_B(u) \\ I_A(u) \ \ge I_B(u) \\ F_A(u) \ge F_B(u) \end{cases} \Leftrightarrow \begin{cases} T_{A\backslash B}(u) = 0 \\ I_{A\backslash B}(u) = 1 \\ F_{A\backslash B}(u) = 1 \end{cases} \Leftrightarrow \ A \backslash B = \emptyset$$

DIF4: $if \ B \subseteq C \ \Longrightarrow B \backslash A \subseteq C \backslash A$

$$B \subseteq C \ \Longrightarrow \begin{cases} T_B(u) \ \le T_C(u) \\ I_B(u) \ \ge I_C(u) \\ F_B(u) \ge F_C(u) \end{cases} \Longrightarrow \begin{cases} T_B(u) - T_A(u) \le T_C(u) - T_A(u) \\ I_B(u) - I_A(u) \ge I_C(u) - I_A(u) \\ F_B(u) - F_A(u) \ge F_C(u) - F_A(u) \end{cases}$$

$$\Longrightarrow \begin{cases} T_{B\backslash A}(u) \le T_{C\backslash A}(u) \\ I_{B\backslash A}(u) \ \ge I_{C\backslash A}(u) \\ F_{B\backslash A}(u) \ge F_{C\backslash A}(u) \end{cases} \Longrightarrow \ B - A \subseteq C - A$$

Example 3.3: Let $U = \{u_1, u_2, u_3\}$, and $A, B \in PNS(U)$:

$A = \{u_1: (0.8, 0.1, 0.3), \ u_2: (0.9, 0.2, 0), \ u_3: (0.9, 0.8, 1)\}$

$B = \{u_1: (0.2, 1, 0.2), \ u_2: (1, 0.5, 0.3), \ u_3: (0.9, 0.8, 1)\}$

Then: $A \backslash B = \{u_1: (0.6, 0.1, 1), u_2: (0, 0.7, 0.7), u_3: (0, 1, 1)\}$

Definition 3.10: An operator $D: PNS(U) \times PNS(U) \ \rightarrow \ ]^{-}0, 1^{+}[$
Is the distance measure, if it satisfies for all $A, B, C \in \ PNS(U)$, follow properties:

DIS1: $D(A, B) = 0 \ \Leftrightarrow \ A = B$

DIS2: $D(A, B) = D(B, A)$

DIS3: $D(A, C) \le D(A, B) + D(B, C)$

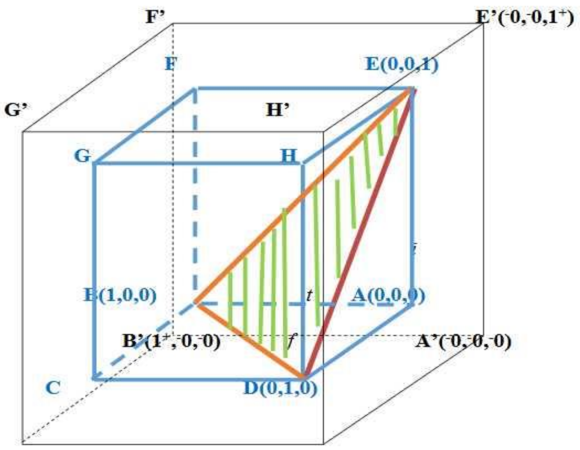

Figure 3: A three-dimension representation of a neutrosophic set [27]

Theorem 3.6: The function $D: PNS(U) \times PNS(U) \to]^{-}0,1^{+}[$ given by: [27]

$$D(A,B) = \sqrt{\frac{1}{3n}\sum_{i=1}^{n}[(T_A(u_i) - T_B(u_i))^2 + (I_A(u_i) - I_B(u_i))^2 + (F_A(u_i) - F_B(u_i))^2]}$$

Is the distance measure between $PNS(U)$ sets.

Proof: Obviously: $D(A,B)$ is generalization of the usually used to measure the distance of objects in Euclidean geometry

Example 3.4: Let $U = \{u_1, u_2, u_3\}$, and $A, B \in INS(U)$:

$$A = \{u_1: (0.8,0.1,0.3),\ u_2: (0.4,0.5,0),\ u_3: (0.7,0.8,1)\}$$
$$B = \{u_1: (0.7,1,0.5),\ u_2: (1,0.5,0.3),\ u_3: (0.9,0.8,0.7)\}$$

Then: $D_{NE}(A,B) = \sqrt{\frac{1}{9}(0.86 + 0.45 + 0.13)} = \sqrt{\frac{1}{9}1.44} = \sqrt{0.16} = 0.4$

Definition 3.11: An operator $DM: PNS(U) \times PNS(U) \to (DM_T, DM_I, DM_F)$, where:

DM_T: denote the degree of dissimilarity. $(^{-}0 \leq DM_T \leq 1^{+})$

DM_I: denote the degree of indeterminate dissimilarity. $(^{-}0 \leq DM_I \leq 1^{+})$

DM_F: denote the degree of non-dissimilarity. $(^{-}0 \leq DM_F \leq 1^{+})$

Is the dissimilarity measure, if it satisfies for all $A, B, C \in PNS(U)$, follow properties:

DISM1: $DM(A,A) = (0,1,1)$

DISM2: $DM(A,B) = DM(B,A)$

DISM3: $A \subseteq B \subseteq C \implies DM(A,B) \leq DM(A,C)$ & $DM(B,C) \leq DM(A,C)$

Remark 3.4: Let $A, B \in PNS(U)$, then:

$(A\backslash B) \cup (B\backslash A) = \{u: (T'(u), I'(u), F'(u)); u \in U\}$, where:

$$T'(u) = \max\big(\max\big(0, T_A(u) - T_B(u)\big), \max\big(0, T_B(u) - T_A(u)\big)\big) = |T_A(u) - T_B(u)|$$

$$I'(u) = \min\Big(\min\Big(1,1 + \big(I_A(u) - I_B(u)\big)\Big), \min\Big(1,1 + \big(I_B(u) - I_A(u)\big)\Big)\Big) = 1 - |I_A(u) - I_B(u)|$$

$$F'(u) = \min\Big(\min\Big(1,1 + \big(F_A(u) - F_B(u)\big)\Big), \min\Big(1,1 + \big(F_B(u) - F_A(u)\big)\Big)\Big) = 1 - |F_A(u) - F_B(u)|$$

Theorem 3.7: The function $DM: PNS(U) \times PNS(U) \to (DM_T, DM_I, DM_F)$

Given by, $\forall A, B \in PNS(U)$: $DM(A,B) = (DM_T(A,B), DM_I(A,B), DM_F(A,B))$, where:

$$DM_T(A,B) = \frac{1}{n}\sum_{i=1}^{n}[|T_A(u_i) - T_B(u_i)|]$$

$$DM_I(A,B) = \frac{1}{n}\sum_{i=1}^{n}[1 - |I_A(u_i) - I_B(u_i)|]$$

$$DM_F(A,B) = \frac{1}{n}\sum_{i=1}^{n}[1 - |F_A(u_i) - F_B(u_i)|]$$

Is the dissimilarity measure between $PNS(U)$ sets.

Proof:

DISM1: $DM(A,A) = (0,1,1)$

$$DM_T(A,A) = \frac{1}{n}\sum_{i=1}^{n}[|T_A(u_i) - T_A(u_i)|] = \frac{1}{n}\sum_{i=1}^{n}[0] = 0$$

$$DM_I(A,A) = \frac{1}{n}\sum_{i=1}^{n}[1 - |I_A(u_i) - I_A(u_i)|] = \frac{1}{n}\sum_{i=1}^{n}[1 - 0] = 1$$

$$DM_F(A,A) = \frac{1}{n}\sum_{i=1}^{n}[1 - |F_A(u_i) - F_A(u_i)|] = \frac{1}{n}\sum_{i=1}^{n}[1 - 0] = 1$$

DISM2: $DM(A,B) = DM(B,A)$

$$DM_T(A,B) = \frac{1}{n}\sum_{i=1}^{n}[|T_A(u_i) - T_B(u_i)|] = \frac{1}{n}\sum_{i=1}^{n}[|T_B(u_i) - T_A(u_i)|] = DM_T(B,A)$$

$$DM_I(A,B) = \frac{1}{n}\sum_{i=1}^{n}[1 - |I_A(u_i) - I_B(u_i)|] = \frac{1}{n}\sum_{i=1}^{n}[1 - |I_B(u_i) - I_A(u_i)|] = DM_I(B,A)$$

$$DM_F(A,B) = \frac{1}{n}\sum_{i=1}^{n}[1 - |F_A(u_i) - F_B(u_i)|] = \frac{1}{n}\sum_{i=1}^{n}[1 - |F_A(u_i) - F_B(u_i)|] = DM_F(B,A)$$

DISM3: $A \subseteq B \subseteq C \implies DM(A,B) \leq DM(A,C)$ & $DM(B,C) \leq DM(A,C)$

$A \subseteq B \subseteq C \implies T_A(u) \leq T_B(u) \leq T_C(u)$

$$\implies |T_A(u) - T_B(u)| + |T_B(u) - T_C(u)| = |T_A(u) - T_C(u)|$$

$$\implies |T_A(u) - T_B(u)| \leq |T_A(u) - T_C(u)| \ \& \ |T_B(u) - T_C(u)| \leq |T_A(u) - T_C(u)|$$

$$\implies DM_T(A,B) \leq DM_T(A,C) \ \& \ DM_T(B,C) \leq DM_T(A,C)$$

$A \subseteq B \subseteq C \implies I_A(u) \geq I_B(u) \geq I_c(u)$

$$\implies |I_A(u) - I_B(u)| + |I_B(u) - I_c(u)| = |I_A(u) - I_C(u)|$$

$$\implies |I_A(u) - I_B(u)| \leq |I_A(u) - I_C(u)| \ \& \ |I_B(u) - I_c(u)| \leq |I_A(u) - I_C(u)|$$

$$\implies 1 - |I_A(u) - I_B(u)| \geq 1 - |I_A(u) - I_C(u)| \ \& \ 1 - |I_B(u) - I_c(u)| \geq 1 - |I_A(u) - I_C(u)|$$

$$\implies DM_I(A,B) \geq DM_I(A,C) \ \& \ DM_I(B,C) \geq DM_I(A,C)$$

Similarity, $DM_F(A,B) \geq DM_F(A,C)$ & $DM_F(B,C) \geq DM_F(A,C)$.

Then: $DM(A,B) \leq DM(A,C)$ & $DM(B,C) \leq DM(A,C)$

Example 3.5: Let $U=\{u_1,u_2,u_3\}$, and $A,B\in PNS(U)$:

$$A=\{u_1:(0.1,0.1,1),\ u_2:(0,0.2,0.8),\ u_3:(1,1,0)\}$$
$$B=\{u_1:(0.5,1,0.5),\ u_2:(1,0.5,0.2),\ u_3:(0,1,1)\}$$

Then:

$$DM_T(A,B)=\frac{1}{n}\sum_{i=1}^{n}[|T_A(u_i)-T_B(u_i)|]=\frac{1}{3}(0.4+1+1)=\frac{2.4}{3}=0.8$$

$$DM_I(A,B)=\frac{1}{n}\sum_{i=1}^{n}[1-|I_A(u_i)-I_B(u_i)|]=\frac{1}{3}(0.1+0.7+1)=\frac{1.8}{3}=0.6$$

$$DM_F(A,B)=\frac{1}{n}\sum_{i=1}^{n}[1-|F_A(u_i)-F_B(u_i)|]=\frac{1}{3}(0.5+0.4+0)=\frac{0.9}{3}=0.3$$

Thus: $DM(A,B)=(0.8,0.6,0.3)$

4. Conclusion

By combining the concepts of algebraic with the neutrosophic sets, we introduce the neutrosophic order relation ($\ddot{\leqq}$), the connected points, the connection strength between the points inside the neutrosophic set and the connected neutrosophic sets. Thus, it became a new and interesting research topic on which researchers can do further studies. In addition, in this paper, we have defined the basic operations (union, intersection, difference) on the picture neutrosophic set $PNS(U)$. We have proposed a new method for dissimilarity measure between $PNS(U)$ sets. These measures and operations are used basically in image processing and comparison. In the future, we will study the properties of these measures and their applications in practical problems.

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A Novel Method for Neutrosophic Assignment Problem by using Interval-Valued Trapezoidal Neutrosophic Number

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Abstract: Assignment problem (AP) is well- studied and important area in optimization. In this research manuscript, an assignment problem in neutrosophic environment, called as neutrosophic assignment problem (NAP), is introduced. The problem is proposed by using the interval-valued trapezoidal neutrosophic numbers in the elements of cost matrix. As per the concept of score function, the interval-valued trapezoidal neutrosophic assignment problem (IVTNAP) is transformed to the corresponding an interval-valued AP. To optimize the objective function in interval form, we use the order relations. These relations are the representations of choices of decision maker. The maximization (or minimization) model with objective function in interval form is changed to multi- objective based on order relations introduced by the decision makers' preference in case of interval profits (or costs). In the last, we solve a numerical example to support the proposed solution methodology.

Keywords: Assignment problem; Interval-valued trapezoidal neutrosophic numbers; Score function; Interval-valued assignment problem; Multi-objective assignment problem; Weighting Tchebycheff program; Decision Making.

Glossary	
AP: Assignment problem.	LP : Linear programming.
DM: Decision makers.	MOLP: Multi-objective linear programming
FN-LPP: Fuzzy neutrosophic LPP.	MOAP: Multi-objective assignment problem
GAMS: General Algebraic Modeling System.	MOOP: Multi-objective optimization problem.
IVN : Interval-valued neutrosophic.	NAP: Neutrosophic assignment problem.

IVTNAP: Interval- valued trapezoidal neutrosophic assignment.	

1. Introduction

In important real-life applications, an AP appears such as production planning, telecommunication, resource scheduling, vehicle routing and distribution, economics, plant location and flexible manufacturing systems, and attracts more and more researchers' attention [10, 13, 37], where it deals with the question how to set n number of people or machines to m number of works in such a way that an optimal assignment can be obtained to minimize the cost (or maximize the profit).

Following these research objectives, the DM has to make an attempt for the optimization of models starting from linear AP to nonlinear AP. In view of this, the linear AP is a special kind of linear programming problem (LPP) where the people or machines are being assigned to various works as one to one rule so that the assignment profit (or cost) is optimized. An optimal assignee for the work is a good description of the AP, where number of rows is equal to the number of columns as explained in Ehrgott et al. [14]. A new approach was developed to study the assignment problem with several objectives, by Bao et al. [4], which was followed with applications to determine the cost-time AP problem as multiple criteria decision making problem by Geetha and Nair [16].

Few decades ago, a large number of authors and policy makers around the world have investigated the basic idea of fuzzy sets. The theory of fuzzy sets was, first, originated by Zadeh [45], which has been intensely applied to study several practical problems, including financial risk management. Then the fuzzy concept is also represented by fuzzy constraints and / or fuzzy quantities. Dubois and Prade [13] suggested the implementation of algebraic operations on crisp numbers to fuzzy numbers with the help of fuzzification method. However, AP representing real-life scenario consists of a set of parameters. The values of these parameters are set by decision makers. DMs required fixing exact values to the parameters that in the conventional approach. In that case, DMs do not precisely estimate the exact value of parameters, therefore the model parameters are generally defined in an uncertain manner. Zimmermann [46] was the first solved LP model having many objectives through suitable membership functions. Bellmann and Zadeh [6] implemented fuzzy set notion to the decision-making problem consisting of imprecision as well as uncertainty.

Sakawa and Yano [39] suggested the idea of fuzzy multiobjective linear programming (MOLP) problems. Hamadameen [18] derived an approach for getting the optimal solution of fuzzy MOLP model considering the coefficients of objective function as triangular fuzzy numbers. The fuzzy MOLP problem was reduced to crisp MOLP with the help of ranking function as explained by Wang [42]. Thereafter, the problem was solved with the help of the fuzzy programming method. Leberling [28] solved vector maximum LP problem using a particular kind of nonlinear membership functions. Bit et al. [7] applied fuzzy methodology for multiple objective transportation model. Belacela and Boulasselb [5] studied a multiple criteria fuzzy AP. Lin and Wen [29] designed an algorithm for the solution of fuzzy AP problem. Kagade and Bajaj [22] discussed interval numbers cost coefficients MOAP problem. Yang and Liu [44] developed a Tabu search method with the help of fuzzy simulation to determine an optimal solution to the fuzzy AP. Moreover, De and Yadav [11] proposed a solution approach to MOAP with the implementation of fuzzy goal programming technique. Mukherjee and Basu [32] solved fuzzy cost AP problem using the ranking method introduced by Yager [43]. Pramanik and Biswas [36] studied multi-objective AP with imprecise costs, time and ineffectiveness. Haddad et al. [17] investigated some generalized AP models in imprecise environment. Emrouznejad et al. An alternative development was suggested for the fuzzy AP with fuzzy profits or fuzzy costs for all possible assignments as explained by Emrouznejad et al. [15]. Kumar and Gupta [26] investigated a methodology to solve fuzzy AP as well as fuzzy travelling salesman problem under various membership functions and ranking index introduced by Yager [43]. Medvedeva and Medvedev [31] applied the concept of the primal and dual for getting the optimal solution to a MOAP. Hamou and Mohamed [19] applied the branch & bound based method to generate the set of each efficient solution to MOAP. Jayalakshmi and Sujatha [21] investigated a novel procedure, referred as optimal flowing method providing the ideal and set of all efficient solutions. Pandian and Anuradha [34] investigated a novel methodology to determine the optimal solution of the problem consisting of zero-point method which was introduced by Pandian and Natarajan [33].

Khalifa and Al- Shabi [23] studied the multi-objective assignment problem with trapezoidal fuzzy numbers. They introduced an interactive approach for solving it and then determined the stability set of the first kind corresponding the solution. Khalifa [25] introduced an approach based on the Weighting Tchebycheff program to solve the multi- objective assignment problem in neutrosophic environment.

The extension of intuitionistic fuzzy set is the neutrosophic set. The neutrosophic set consists of three defining functions. These functions are the membership function, the non-membership function, and the indeterminacy function. All these functions are entirely independent to each other. A new solution approach for the FN-LPP was proposed with real life application by Abdel *et al.* [3]. Kumar et al. [27] investigated a novel solution procedure for the computation of fuzzy pythagorean transportation problem, where they extended the interval basic feasible solution, then existing optimality method to obtain the cost of transportation. Khalifa et al. [24] studied the complex programming problem with neutrosophic concept. They applied the lexicographic order to determine the optimal solution of neutrosophic complex programming. Vidhya et al. [41] studied neutrosophic MOLP problem. Pramanik and Banerjee [35] proposed a goal programming methodology to MOLP problem under neutrosophic numbers. Broumi and Smarandaache [8] introduced some novel operations for interval neutrosophic sets in terms of arithmetic, geometrical, and harmonic means. Rizk-Allah et al. [38] suggested a novel compromise approach for many objective transportation problem, which was further studied by Zimmermann's fuzzy programming approach as well as the neutrosophic set terminology. Abdel- Basset et al. [1] introduced a plithogenic multi- criteria decision- making model based on neutrosophic analytic hierarchy process in order of performance by similarity to the ideal solution of financial performance. Abdel- Basset et al. [2] evaluated a set of measurements for providing sustainable supply chain finance in the gas industry in the uncertain environment. Abdel- Basset et al. [3] proposed an integrated method based on neutrosophic set to evaluate innovation value for smart product- service systems.

In this paper, the assignment problem having interval- valued trapezoidal neutrosophic numbers in all the parameters is introduced. This problem is converted into two objectives assignment problem, then the Weighting Tchebycheff program with the ideal targets are applied for solving it.

The outlay of the proposed research article is organized as follows: In the next Section, we present some sort of preliminaries, which is essential for the present study. Section 3 formulate interval-valued trapezoidal neutrosophic assignment problem. Section 4 proposes solution approach for the determination of preferred solution. A numerical example is solved, in Section 5, to support the efficiency of the solution approach. In the last, some concluding remarks as well as the further research directions are summarized in Section 6.

2. Preliminaries

This section introduces some of basic concepts and results related to fuzzy numbers, neutrosophic set, and their arithmetic operations.

Definition 1. A fuzzy set $\widetilde{P}$ defined on the set of real numbers $\mathbb{R}$ is called fuzzy number when the membership function

$\mu_{\widetilde{P}}(x)\colon \mathbb{R} \to [0,1]$, have the following properties:

1. $\mu_{\widetilde{P}}(x)$ is an upper semi-continuous membership function;
2. $\widetilde{P}$ is convex fuzzy set, i.e., $\mu_{\widetilde{P}}(\delta\, x + (1-\delta)\, y) \geq \min\{\mu_{\widetilde{P}}(x), \mu_{\widetilde{P}}(y)\}$ for all $x, y \in \mathbb{R}; 0 \leq \delta \leq 1$;
3. $\widetilde{P}$ is normal, i.e., $\exists\, x_0 \in \mathbb{R}$ for which $\mu_{\widetilde{P}}(x_0) = 1$;
4. $\mathrm{Supp}\,(\widetilde{P}) = \{x \in \mathbb{R}\colon \mu_{\widetilde{P}}(x) > 0\}$ is the support of $\widetilde{P}$, and the closure $\mathrm{cl}(\mathrm{Supp}(\widetilde{P}))$ is compact set.

Definition 2. (Ishibuchi and Tanaka [20). An interval on $\mathbb{R}$ is defined as

$A = [a^L, a^R] = \{a\colon, a^L \leq a \leq a^R, a \in \mathbb{R}\}$, where a^L is left limit and a^R is right limit of A.

Definition 3. (Ishibuchi and Tanaka [20]).The interval is also defined by

$A = \langle a_C, a_W \rangle = \{\mathbf{a}\colon a_C - a_W \leq \mathbf{a} \leq a_C + a_W, \mathbf{a} \in \mathbb{R}\}$, where $a_C = \frac{1}{2}(a^R + a^L)$ is center and $a_W = \frac{1}{2}(a^R - a^L)$ is width of A.

Definition 4. (Neutrosophic set, Wang et al. [42]). Let X be a nonempty set. Then a neutrosophic set $\overline{P}_N$ of nonempty set X is defined as

$$\overline{P}_N = \left\{\langle x; , T_{\overline{P}_N}, I_{\overline{P}_N}, F_{\overline{P}_N} \rangle\colon x \in X\right\},$$

where $T_{\overline{P}_N}, I_{\overline{P}_N}, F_{\overline{P}_N} : X \to]0_-, 1^+[$ define respectively the degree of membership function, the degree of indeterminacy , and the degree of non-membership of element $x \in X$ to the set $\overline{P}_N$ with the condition:

$$0_- \le T_{\overline{P}_N} + I_{\overline{P}_N} + F_{\overline{P}_N} \le 3^+. \tag{1}$$

Definition 5. (Interval-valued neutrosophic set, Broumi and Smarandache [8]). Let X be a nonempty set. Then an interval valued neutrosophic (IVN) set $\overline{P}_N^{IV}$ of X is defined as:

$$\overline{P}_N^{IV} = \left\{ \langle x;\ \left[T^L_{\overline{P}_N}, T^U_{\overline{P}_N}\right], \left[I^L_{\overline{P}_N}, I^U_{\overline{P}_N}\right], \left[F^L_{\overline{P}_N}, F^U_{\overline{P}_N}\right] \rangle : x \in X \right\},$$

$$\text{where } \left[T^L_{\overline{P}_N}, T^U_{\overline{P}_N}\right], \left[I^L_{\overline{P}_N}, I^U_{\overline{P}_N}\right], \text{ and } \left[F^L_{\overline{P}_N}, F^U_{\overline{P}_N}\right] \subset [0,1] \text{ for each } x \in X .$$

Definition 6. (Broumi and Smarandache [8]). Let

$$\overline{P}_N^{IV} = \left\{ \langle x;\ \left[T^L_{\overline{P}_N}, T^U_{\overline{P}_N}\right], \left[I^L_{\overline{P}_N}, I^U_{\overline{P}_N}\right], \left[F^L_{\overline{P}_N}, F^U_{\overline{P}_N}\right] \rangle : x \in X \right\} \text{ be IVNS, then}$$

(i) $\overline{P}_N^{IV}$ is empty if $T^L_{\overline{P}_N} = T^U_{\overline{P}_N} = 0, I^L_{\overline{P}_N} = I^U_{\overline{P}_N} = 1, F^L_{\overline{P}_N} = F^U_{\overline{P}_N} = 1$, for all $x \in \overline{P}_N$,

(ii) Let $\underline{0} = \langle x;\ 0,1,1 \rangle$, and $\underline{1} = \langle x;\ 1,0,0 \rangle$.

Definition 7. (Interval-valued trapezoidal neutrosophic number). Let $u_{\tilde{a}}, v_{\tilde{a}}, w_{\tilde{a}} \subset [0,1]$, and $a_1, a_2, a_3, a_4 \in \mathbb{R}$ such that $a_1 \le a_2 \le a_3 \le a_4$. Then an interval-valued trapezoidal fuzzy neutrosophic number,

$$\tilde{a} = \langle (a_1, a_2, a_3, a_4);\ [u^L_{\tilde{a}}, u^U_{\tilde{a}}], [v^L_{\tilde{a}}, v^U_{\tilde{a}}], [w^L_{\tilde{a}}, w^U_{\tilde{a}}] \rangle,$$

whose degrees of membership function, the degrees of indeterminacy, and the degrees of non-membership are

$$\vartheta_{\tilde{a}}(x) = \begin{cases} u_{\tilde{a}}\left(\frac{x-a_1}{a_2-a_1}\right), & \text{for } a_1 \le x \le a_2, \\ u_{\tilde{a}}, & \text{for } a_2 \le x \le a_3, \\ u_{\tilde{a}}\left(\frac{a_4-x}{a_4-a_3}\right), & \text{for } a_3 \le x \le a_4, \\ 0, & \text{Otherwise,} \end{cases}$$

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{a_2-x+v_{\tilde{a}}(x-a_1)}{a_2-a_1}, & \text{for } a_1 \le x \le a_2, \\ v_{\tilde{a}}, & \text{for } a_2 \le x \le a_3, \\ \frac{x-a_3+v_{\tilde{a}}(a_4-x)}{a_4-a_3}, & \text{for } a_3 \le x \le a_4, \\ 1, & \text{Otherwise,} \end{cases} \tag{2}$$

$$\varphi_{\tilde{a}}(x) = \begin{cases} \frac{a_2-x+w_{\tilde{a}}(x-a_1)}{a_2-a_1}, & \text{for } a_1 \le x \le a_2, \\ w_{\tilde{a}}, & \text{for } a_2 \le x \le a_3, \\ \frac{x-a_3+w_{\tilde{a}}(a_4-x)}{a_4-a_3}, & \text{for } a_3 \le x \le a_4, \\ 1, & \text{Otherwise.} \end{cases}$$

Where, $u_{\tilde{a}}$, $v_{\tilde{a}}$, and $w_{\tilde{a}}$ are the upper bound of membership degree, lower bound of indeterminacy degree, and lower bound of non-membership degree, respectively.

Definition 8. (Arithmetic operations). Let $\tilde{a} = \langle (a_1, a_2, a_3, a_4);\ [u^L_{\tilde{a}}, u^U_{\tilde{a}}], [v^L_{\tilde{a}}, v^U_{\tilde{a}}], [w^L_{\tilde{a}}, w^U_{\tilde{a}}] \rangle$, and

$\tilde{b} = \langle (b_1, b_2, b_3, b_4);\ [u^L_{\tilde{b}}, u^U_{\tilde{b}}], [v^L_{\tilde{b}}, v^U_{\tilde{b}}], [w^L_{\tilde{b}}, w^U_{\tilde{b}}] \rangle$ be two IVN numbers. Then,

1. $\tilde{a} \oplus \tilde{b} = \langle (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4): A, B, C \rangle,$
2. $\tilde{a} \ominus \tilde{b} = \langle (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1): A, B, C \rangle,$
3. $\tilde{a} \odot \tilde{b} = \begin{cases} \langle (a_1b_1, a_2b_2, a_3b_3, a_4b_4): A, B, C \rangle, \text{if } a_4 > 0, b_4 > 0, \\ \langle (a_1b_4, a_2b_3, a_3b_2, a_4b_1): A, B, C \rangle, \text{if } a_4 < 0, b_4 > 0\ , \\ \langle (a_4b_4, a_3b_3, a_2b_2, a_1b_1): A, B, C \rangle, \text{if } a_4 < 0, b_4 < 0. \end{cases}$
4. $\tilde{a} \oslash \tilde{b} = \begin{cases} \langle (a_1/b_4, a_2/b_3, a_3/b_2, a_4/b_1): A, B, C \rangle, \text{if } a_4 > 0, b_4 > 0, \\ \langle (a_4/b_4, a_3/b_3, a_2/b_2, a_1/b_1): A, B, C \rangle, \text{if } a_4 < 0, b_4 > 0 \\ \langle (a_4/b_1, a_3/b_2, a_2/b_3, a_1/b_4): A, B, C \rangle, \text{if } a_4 < 0, b_4 < 0. \end{cases}$
5. $k\,\tilde{a} = \begin{cases} \langle (ka_1, k\,a_2, ka_3, ka_4);\ [u^L_{\tilde{a}}, u^U_{\tilde{a}}], [v^L_{\tilde{a}}, v^U_{\tilde{a}}], [w^L_{\tilde{a}}, w^U_{\tilde{a}}] \rangle, \text{if } k > 0 \\ \langle (ka_4, ka_3, ka_2, ka_1);\ [u^L_{\tilde{a}}, u^U_{\tilde{a}}], [v^L_{\tilde{a}}, v^U_{\tilde{a}}], [w^L_{\tilde{a}}, w^U_{\tilde{a}}] \rangle, k < 0. \end{cases}$
6. $\tilde{a}^{-1} = \langle (1/a_4, 1/a_3, 1/a_2, 1/a_1);\ [u^L_{\tilde{a}}, u^U_{\tilde{a}}], [v^L_{\tilde{a}}, v^U_{\tilde{a}}], [w^L_{\tilde{a}}, w^U_{\tilde{a}}] \rangle, \tilde{a} \neq 0.$

Where, $A = [\min(u^L_{\tilde{a}}, u^L_{\tilde{b}},), \min(u^U_{\tilde{a}}, u^U_{\tilde{b}})], B = [\max(v^L_{\tilde{a}}, v^L_{\tilde{b}},), \max(v^U_{\tilde{a}}, v^U_{\tilde{b}})]$, and

$C = [\max(w^L_{\tilde{a}}, w^L_{\tilde{b}},), \max(w^U_{\tilde{a}}, w^U_{\tilde{b}})].$

Definition 9. (Score function, Tharmaraiselvi and Santhi [40]). The score function for the IVN number $\tilde{a} = \langle (a_1, a_2, a_3, a_4);\ [u^L_{\tilde{a}}, u^U_{\tilde{a}}], [v^L_{\tilde{a}}, v^U_{\tilde{a}}], [w^L_{\tilde{a}}, w^U_{\tilde{a}}] \rangle$ is defined as

$$S(\tilde{a}) = \frac{1}{16}(a_1 + a_2 + a_3 + a_4) \times [\vartheta_{\tilde{a}} + (1 - \mu_{\tilde{a}}) + (1 - \varphi_{\tilde{a}})].$$

3. Problem statement and solution concepts

3.1 Assumptions, Index and notation

3.1.1. Assumption

We assume that there are n number of jobs, which must be performed by and n persons, where the costs are based on the specific assignments. Each job must be assigned to exactly one person and each person has to perform exactly one job.

3.1.2. Index

i: Persons.

j: Jobs.

3.1.3. Notation

$(\tilde{c}_{ij})^{IV}_{N}$: Interval-valued trapezoidal neutrosophic cost of ith person assigned to jth job.

x_{ij}: Number of jth jobs assigned to ith person.

Consider the following interval-valued trapezoidal neutrosophic assignment problem (IVTNAP)

$$\text{(IVTNAP)} \quad \text{Min} \quad \tilde{Z}_N^{IV} = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\tilde{c}_{ij}\right)_N^{IV} x_{ij}$$

Subject to

$\sum_{i=1}^{n} x_{ij} = 1, j = 1, 2, \dots, n$ (only one person would be assigned the jth job)

$\sum_{j=1}^{n} x_{ij} = 1, i = 1, 2, \dots, n$ (only one job selected by ith person)

$x_{ij} = 0$ or 1.

It obvious that $\left(\tilde{c}_{ij}\right)_N^{IV}$ ($i = j = 1, 2, 3, \dots, n; 1, 2, 3, \dots, K$) are interval-valued trapezoidal neutrosophic numbers.

Based on score function defined in Definition 9, the IVTNAP in converted into the following interval-valued assignment problem (IVAP)

$$\text{(IVAP)} \quad \text{Min} \quad Z^{IV} = \sum_{i=1}^{n} \sum_{j=1}^{n} \left[c_{ij}^{L}, c_{ij}^{U}\right] x_{ij}$$

Subject to

$x \in X' = \left\{ \sum_{i=1}^{n} x_{ij} = 1, j = 1, 2, \dots, n;\ \sum_{j=1}^{n} x_{ij} = 1, i = 1, 2, \dots, n;\ x_{ij} = 0 \text{ or } 1 \right\}.$

Definition 10. $x \in X'$ is solution of problem IVAP if and only if there is no $\hat{x} \in X'$ satisfies $Z(\hat{x}) \leq_{LR} Z(x)$, or $Z(\hat{x}) <_{CW} Z$.

Or equivalently,

Definition 11. $x \in X'$ is solution of problem IVAP if and only if there is no $\hat{x} \in X'$ satisfies that $Z(\hat{x}) \leq_{RC} Z(x)$.

The solution set of problem IVAP can be obtained as the efficient solution of the following MOAP:

$$\text{Min } (Z^R, Z^C)$$
$$\text{Subject to} \quad x \in X'. \tag{3}$$

Using the Weighting Tchebycheff problem, the Problem (3) is described in the following form

$$\begin{aligned} &\text{Min } \psi \\ &\text{Subject to} \\ &\quad w_1\left[Z^R - \hat{Z}^R\right] \leq \psi, \\ &\quad w_2\left[Z^C - \hat{Z}^C\right] \leq \psi, \\ &\quad x \in X'. \end{aligned} \tag{4}$$

Where $w_1, w_2 \geq 0$; $\hat{Z}^R$, and $\hat{Z}^C$ are defined as the ideal targets.

4. Solution procedure

The steps of the solution procedure to solve the IVTNAP can be summarized as:

Step 1: Formulate the IVTNAP

Step 2: Convert the IVTNAP using the score function (Definition 9) into the IVAP.

Step 3: Estimate the ideal points $\hat{Z}^R$ and $\hat{Z}^C$ for the IVAP from the following relation

$$\hat{Z}^R = \text{Min } Z^R,$$

Subject to $x \in X'$, and

$$\hat{Z}^C = \text{Min}\, Z^C,$$

Subject to $x \in X'$.

Step 4: Determine the value of individual maximum and minimum for every objective function subject to given constraints.

Step 5: Compute the weights from the relation

$$w_1 = \frac{\overline{Z}^R - \underline{Z}^R}{\left(\overline{Z}^R - \underline{Z}^R\right) + \left(\overline{Z}^C - \underline{Z}^C\right)},\quad w_2 = \frac{\overline{Z}^C - \underline{Z}^C}{\left(\overline{Z}^R - \underline{Z}^R\right) + \left(\overline{Z}^C - \underline{Z}^C\right)} \qquad (5)$$

Here $\overline{Z}^R$, $\overline{Z}^C$ and $\underline{Z}^R$, $\underline{Z}^C$ are the value of individual maximum and minimum of the Z^R, and Z^C, respectively.

Step 6: Applying the GAMS software to problem (5) to obtain the optimum compromise solution, and hence the fuzzy cost.

Step 7: Stop.

The flowchart of the proposed method is presented in Figure 1, below.

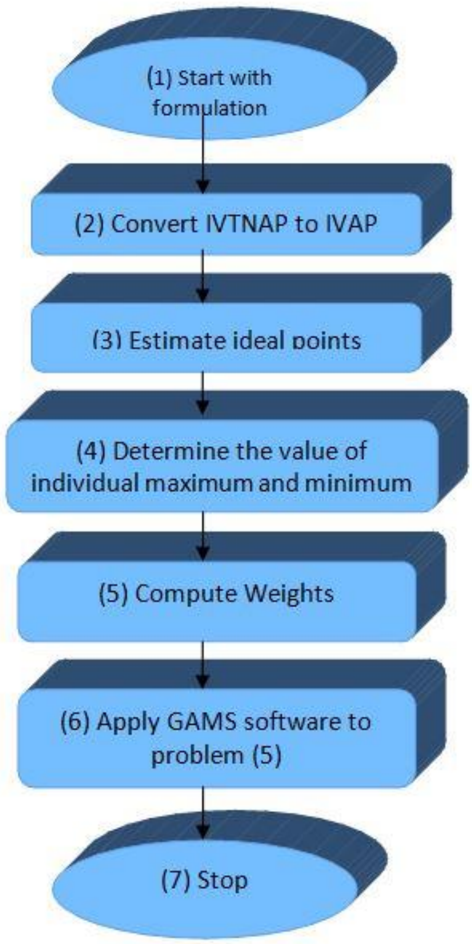

Figure 1: Flowchart of the proposed method

5. Numerical example

Consider the following IVTNAP

$$\text{Min } Z(x)_N^{IV} = \begin{pmatrix} \langle(14,17,21,28);[0.7,0.9],[0.1,0.3],[0.5,0.7]\rangle x_{11} \\ \oplus \langle((13,18,20,24));[0.5,0.7],[0.3,0.5],[0.4,0.6]\rangle x_{12} \\ \oplus \langle(20,25,30,35);[0.8,1.0],[0.2,0.4],[0.1,0.3]\rangle x_{13} \\ \oplus \langle(15,18,23,30);[0.7,1.0],[0.2,0.3],[0.2,0.5]\rangle x_{21} \\ \oplus \langle(6,10,13,15);[0.6,0.8],0.1,0.4],[0.2,0.6]\rangle x_{22} \\ \oplus \langle(15,18,23,30);[0.7,0.9],[0.1,0.4],[0.3,0.5]\rangle x_{23} \\ \langle(13,18,20,24);[0.3,0.7],[0.1,0.4],[0.3,0.7]\rangle x_{31} \\ \oplus \langle(13,18,20,24);[0.2,0.7],0.2,0.5],[0.3,0.6]\rangle x_{32} \\ \oplus \langle(14,16,21,23);[0.6,0.8],[0.3,0.6],[0.2,0.4]\rangle x_{33} \end{pmatrix}$$

Subject to

$$\sum_{i=1}^{3} x_{ij} = 1,\ j = 1,2,3;\ \sum_{j=1}^{3} x_{ij} = 1,\ i = 1,2,3,$$
$$x_{ij} = 0 \text{ or } 1.$$

Step 2:

$$\text{Min } Z(x)^{IV} = \begin{pmatrix} [8.5,11.5]x_{11} + [6.5625,9.375]x_{12} + [14.4375,18.5625]x_{13} \\ [10.2125,13.975]x_{21} + [4.4,6.875]x_{22} + [9.675,13.4375]x_{23} \\ [5.625,10.78125]x_{31} + [5.15625,10.3125]x_{32} + [7.4,10.6375]x_{33} \end{pmatrix}$$

Subject to

$$\sum_{i=1}^{3} x_{ij} = 1,\ j = 1,2,3;\ \sum_{j=1}^{3} x_{ij} = 1,\ i = 1,2,3,$$
$$x_{ij} = 0 \text{ or } 1.$$

Step 4: We determine optimal solution for the following problems individually with respect to the given constraints:

$$\hat{Z}^R = \text{Min } Z^R = \begin{pmatrix} 11.5x_{11} + 9.375x_{12} + 18.5625x_{13} + 13.975x_{21} + 6.875x_{22} \\ 13.4375x_{23} + 10.78125x_{31} + 10.3125x_{32} + 10.6375x_{33} \end{pmatrix}$$

$$\hat{Z}^C = \text{Min } Z^C = \begin{pmatrix} 10x_{11} + 7.96875x_{12} + 16.5x_{13} + 12.09375x_{21} + 5.6375x_{22} \\ 11.55625x_{23} + 8.203125x_{31} + 7.734375x_{32} + 9.01875x_{33} \end{pmatrix}$$

$$\text{Max } Z^R = \begin{pmatrix} 11.5x_{11} + 9.375x_{12} + 18.5625x_{13} + 13.975x_{21} + 6.875x_{22} \\ 13.4375x_{23} + 10.78125x_{31} + 10.3125x_{32} + 10.6375x_{33} \end{pmatrix}$$

$$\text{Max } Z^C = \begin{pmatrix} 10x_{11} + 7.96875x_{12} + 16.5x_{13} + 12.09375x_{21} + 5.6375x_{22} \\ 11.55625x_{23} + 8.203125x_{31} + 7.734375x_{32} + 9.01875x_{33} \end{pmatrix}$$

Subject to

$$\sum_{i=1}^{3} x_{ij} = 1,\ j = 1,2,3;\ \sum_{j=1}^{3} x_{ij} = 1,\ i = 1,2,3,$$
$$X_{ij} = 0 \text{ or } 1.$$

$$\hat{Z}^R = \text{Min } Z^R = 29.01, \quad \hat{Z}^C = \text{Min } Z^C = 24.66,\ \text{Max } Z^R = 42.85,\ \text{Max } Z^C = 36.33$$

Step 5: Calculate the weights

$$w_1 = \frac{13.84}{25.15} = 0.542532, \quad w_2 = \frac{11.67}{25.51} = 0.45747$$

Step6: Determine the optimal solution of the problem:

$$\text{Min } \psi$$

Subject to

$$\begin{pmatrix} 11.5x_{11} + 9.375x_{12} + 18.5625x_{13} + 13.975x_{21} + 6.875x_{22} \\ 13.4375x_{23} + 10.78125x_{31} + 10.3125x_{32} + 10.6375x_{33} - 1.84321\psi \end{pmatrix} \le 29.01,$$

$$\begin{pmatrix} 10x_{11} + 7.96875x_{12} + 16.5x_{13} + 12.09375x_{21} + 5.6375x_{22} \\ 11.55625x_{23} + 8.203125x_{31} + 7.734375x_{32} + 9.01875x_{33} - 2.18595\psi, \end{pmatrix} \le 24.66,$$

$$x \in X'.$$

The optimal compromise solution is $x_{11} = 1, x_{22} = 1,\ x_{33} = 1$,

$x_{12} = x_{13} = \ x_{21} = x_{23} = x_{31} = x_{32} = 0$, and $\psi = 0.0014$.

So, the interval-valued trapezoidal neutrosophic optimum value is

$Z(x)_N^{IV} = \langle (34, 43, 55, 66); [0.6, 0.8], [0.3, 0.6], [0.5, 0.7] \rangle$.

It is evident that the total minimum assigned cost will be greater than 34 and less than 66 . The total minimum assigned cost lies in between 43 and 55, the overall satisfaction lies in between 60% and 80%. Then, for the remaining of total minimum assigned cost, the truthfulness degree is

$$\vartheta_{\tilde{a}}(x) \times 100 = \begin{cases} [0.6, 0.8] \left(\frac{x-34}{43-34}\right), & \text{for } 34 \le x \le 43, \\ [0.6, 0.8], & \text{for } 43 \le x \le 55, \\ [0.6, 0.8] \left(\frac{66-x}{66-55}\right), & \text{for } 55 \le x \le 66, \\ 0, & \text{Otherwise,} \end{cases}$$

Also, the indeterminacy and falsity degrees for the assigned cost are

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{43-x+[0.3,0.6](x-34)}{43-34}, & \text{for } 34 \le x \le 43, \\ v_{\tilde{a}}, & \text{for } 43 \le x \le 55, \\ \frac{x-55+[0.3,0.6](66-x)}{66-55}, & \text{for } 55 \le x \le 66, \\ 1, & \text{Otherwise,} \end{cases}$$

$$\varphi_{\tilde{a}}(x) = \begin{cases} \frac{43-x+[0.5,0.7](x-34)}{43-34}, & \text{for } 34 \le x \le 43, \\ w_{\tilde{a}}, & \text{for } 43 \le x \le 55, \\ \frac{x-55+[0.5,0.7](66-x)}{66-55}, & \text{for } 55 \le x \le 66 \\ 1, & \text{Otherwise.} \end{cases}$$

Thus, the DM concludes that the total interval-valued trapezoidal neutrosophic assigned cost lies in between 34 and 66 with truth, indeterminacy, and falsity degrees lies in between $[0.6, 0.8], [0.3, 0.6]$, and $[0.5, 0.7]$, respectively, and also he is able to schedule the assignment and constraints under budgetary.

6. Concluding remarks and further research directions

The present research article addressed a novel solution methodology to the assignment problem with objective function coefficients characterized by interval-valued trapezoidal neutrosophic numbers. The problem is transformed to the corresponding interval–valued problem, and hence into the multi-objective optimization problem (MOOP). Then, the so obtained MOOP is undertaken for the solution by using the Weighting Tchebycheff problem beside the GAMS software. The advantage of this approach is more flexible than the standard assignment problem, where it allows the DM to choose the targets he is willing.

For further research, one may incorporate this concept in transportation model. Also, one may consider the stochastic nature in assignment problem and develop the same methodology to solve the problem. Additionally, one possible extension might be explored by considering the fuzzy-random, fuzzy-stochastic, etc. In addition, the proposed solution methodology may be applied in different branches (viz. management science, financial management and decision science) where the assignment problems occur in neutrosophic environment.

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Conflicts and Interest

The author declares no conflict of interest**.**

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Connectivity index in neutrosophic trees and the algorithm to find its maximum spanning tree

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Abstract: In this paper, we first define the Neutrosophic tree using the concept of the strong cycle. We then define a strong spanning Neutrosophic tree. In the following, we propose an algorithm for detecting the maximum spanning tree in Neutrosophic graphs. Next, we discuss the Connectivity index and related theorems for Neutrosophic trees.

Keywords: Neutrosophic trees; totally and partial Connectivity indices; maximum spanning tree; strong spanning tree; strong cycle; strong edge

1. Introduction

In recent years, neutrosophic graphs as one of the new branches of graph theory has been welcomed by many researchers and a lot of work has been done on the features and applications of this particular type of graph [1, 2, 4-6, 17-25]. One of these is finding the spanning tree in neutrosophic graphs. In an article by S.Broumi et al. [7], an algorithm for finding the minimum spanning tree is presented. Using the score function, they calculated a rank for each edge, then constructed a minimum spanning tree based on the lowest score. Other people, including I.Kandasamy [13], also provided algorithms for the minimum spanning tree in the Double-Valued neutrosophic graph.

What we present here is an algorithm for finding the maximum spanning tree in neutrosophic graphs. Our proposed algorithm is similar in appearance to the algorithm presented in [7] but differs from it. First, the algorithm is presented for graphs that have weighted edges, while our algorithm includes the general state of the neutrosophic graphs. The second difference is in how you choose to build the tree. In [7], the score function is used and we use the strength function. The strength function has the advantage of having a more realistic view of indeterminacy-membership (I). In fact, in this function, we have improved the effect of effect indeterminacy-membership (I). In [7, 16], the effect of falsity-membership (F) and indeterminacy-membership (I) was the same, which does not seem very appropriate due to the different nature of falsity-membership (F) and indeterminacy-membership (I).

The definition of a neutrosophic tree used in this paper is similar in structure to the definition given in [12]. The difference between the two definitions stems from the difference in the definition of the strength of connectivity between the two vertices.

2. Preliminaries

In this section, some of the important and basic concepts required are given by mentioning the source.

Definition 1. [3] A single-valued neutrosophic graph on a nonempty V is a pair $G=(N,M)$. Where N is single-valued neutrosophic set in V and M single-valued neutrosophic relation on V such that

$$T_M(uv) \leq \min\{T_N(u), T_N(v)\},$$

$$I_M(uv) \leq \min\{I_N(u), I_N(v)\},$$
$$F_M(uv) \leq \max\{F_N(u), F_N(v)\},$$

For all $u, v \in V$. N is called single-valued neutrosophic vertex set of G and, M is called single-valued neutrosophic edge set of G, respectively.

Definition 2. [12] A connected SVN-graph $G = (N, M)$ is said to be a SVN-tree if it has a SVN spanning subgraph $H = (N, B)$ which is a tree, where for all edges uv not in H satisfying

$$T_M(uv) < T_B^{\infty}(uv), \qquad I_M(uv) > I_B^{\infty}(uv), \qquad F_M(uv) > F_B^{\infty}(uv).$$

3. Neutrosophic tree

In this section, the types of edges are first classified and defined in terms of edge strength. Then we will provide some other definitions depending on the type of edges. Based on the strength of connectivity between the end vertices of an edge, edges of neutrosophic graphs can be divided into two categories as given below.

Definition 3. An edge uv in a neutrosophic graph $G = (N, M)$ is called

a. A ***weak*** edge if $CONN_{(G-uv)}(u, v) = CONN_G(u, v)$ *and* $CONN_G(u, v) \neq M(uv)$,

b. A ***neutral*** edge if $CONN_{(G-uv)}(u, v) = CONN_G(u, v)$ *and* $CONN_G(u, v) = M(uv)$,

c. A $\mathbf{I}-$***strong edge*** if $CONN_{(G-uv)}(u, v) < CONN_G(u, v)$ *and,*
$$CONN_G(u, v) = \big(T_M(uv), I_M(uv), F_M(uv)\big) = M(uv),$$

d. A $\mathbf{II}-$***strong edge*** if $CONN_{(G-uv)}(u, v) < CONN_G(u, v)$ *and,* $CONN_G(u, v) \neq M(uv)$.

Example 1. Consider the neutrosophic graph $G = (N, M)$ on $V = \{a, b, c, d, e, f\}$ as shown in figure 1.

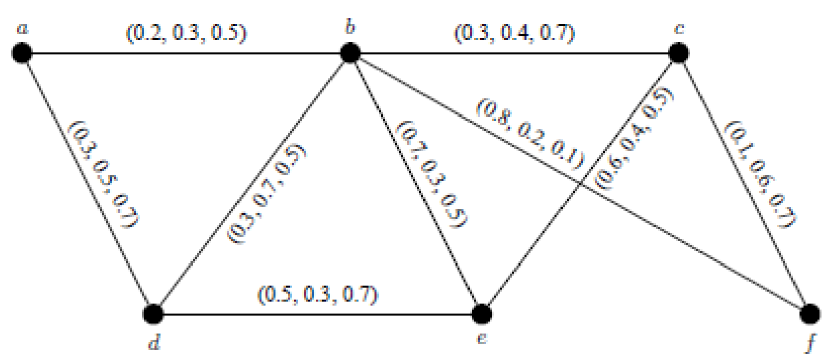


Figure 1. A neutrosophic graph

Table 1. The strength of connectedness between each pair of vertices u and v.

	$\boldsymbol{CONN_G(u,v)}$	$\boldsymbol{CONN_{G-uv}(u,v)}$	$\boldsymbol{M(uv)}$
a,b	(0.3, 0.3, 0.5)	(0.3, 0.5, 0.7)	(0.2, 0.3, 0.5)
a,d	(0.3, 0.3, 0.5)	(0.2, 0.3, 0.5)	(0.3, 0.5, 0.7)
b,c	(0.6, 0.4, 0.5)	(0.6, 0.4, 0.5)	(0.3, 0.4, 0.7)
b,d	(0.5, 0.3, 0.5)	(0.5, 0.3, 0.7)	(0.3, 0.7, 0.5)
b,e	(0.7, 0.3, 0.5)	(0.3, 0.4, 0.7)	(0.7, 0.3, 0.5)
b,f	(0.8, 0.2, 0.1)	(0.1, 0.6, 0.7)	(0.8, 0.2, 0.1)
c,e	(0.6, 0.4, 0.5)	(0.3, 0.4, 0.7)	(0.6, 0.4, 0.5)
c,f	(0.6, 0.4, 0.5)	(0.6, 0.4, 0.5)	(0.1, 0.6, 0.7)
d,e	(0.5, 0.3, 0.5)	(0.3, 0.5, 0.5)	(0.5, 0.3, 0.7)

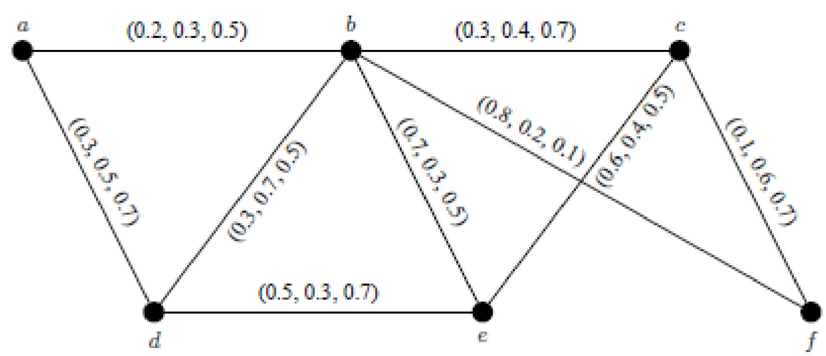
As can be seen in Table 1, edge bc and cf are weak, be, bf and ce are $\mathrm{I}-strong\ edges$, and ac, ad, bd and de are $\mathrm{II}-strong\ edge$.

Definition 4. A path in a neutrosophic graph is called a $\mathrm{I}-strong\ path$ if all its edges are $\mathrm{I}-strong$ and called a $\mathrm{II}-strong\ path$ if all its edges are $\mathrm{II}-strong$. Also is said to be a $strong\ path$ if all its edges are either $\mathrm{I}-strong\ edge$ or $\mathrm{II}-strong\ edge$.

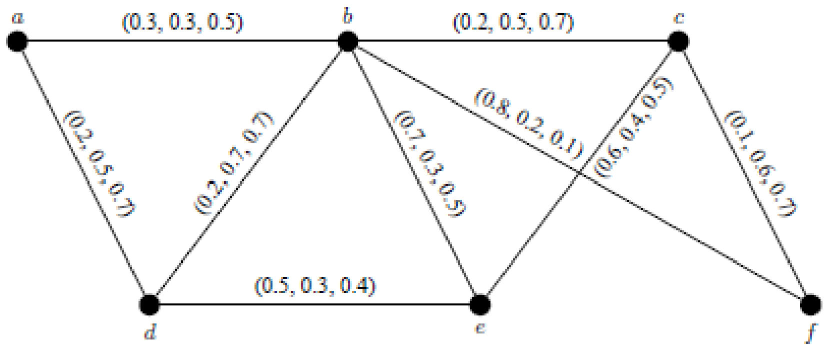
Definition 5. Let $G=(N,M)$ be a neutrosophic graph and C be a cycle in G. C called strong cycle if all its edges are either $\mathrm{I}-strong\ edge$ or $\mathrm{II}-strong\ edge$.

Definition 6. Let $G=(N,M)$ be a neutrosophic graph. G called a neutrosophic tree if it has no strong cycle.

Example 1. Consider a neutrosophic graph $G=(N,M)$ and $H=(A,B)$ as shown in figure 2.



a. G is not a neutrosophic tree



b. H is a neutrosophic tree

Figure 2. a. G is not a neutrosophic tree and b. H is a neutrosophic tree

It is clear from fig 1 that G is not a neutrosophic tree. Since G contains strong neutrosophic cycles. Cycles such as $abda$, $abeda$, $abceda$, ect. are strong neutrosophic cycles in G. But H is a neutrosophic tree, H has no strong neutrosophic cycle.

Definition 7. Let $G = (N, M)$ be a connected neutrosophic graph and T, is a neutrosophic spanning subgraph of G that T spanned by the vertex set of G and T^* is a tree. If the edges of T are selected from G such that for each edge uv of T, uv is either $\mathrm{I} - strong\ edge$ or $\mathrm{II} - strong\ edge$. Then T called a strong spanning tree and denoted by (SST).

Definition 8. Let $G = (N, M)$ be a connected neutrosophic graph with at least one strong spanning tree. Then the strength of strong spanning tree in G is defined and denoted by

$$S(T) = \sum_{uv\in T} S(uv) = \sum_{uv\in T} \frac{4 + 2T_M(uv) - 2F_M(uv) - I_M(uv)}{6}.$$

Also, F called maximum spanning tree if $S(F) \geq S(T)$ for any strong spanning tree T.

Theorem 1. Let $G = (N, M)$ be a connected neutrosophic graph. Then G is a neutrosophic tree if and only if the following conditions are equivalent for any $u, v \in V$.

a. uv is a $\mathrm{I} - strong\ edge$
b. $\left(CONN_{TG}(u, v), CONN_{IG}(u, v), CONN_{FG}(u, v)\right) = (T_M(uv), I_M(uv), F_M(uv))$.

Proof. This theorem can be easily proved by defining a strong edge.
□

Definition 9. Let $G = (N, M)$ be the Neutrosophic Graph. The ***partial connectivity index*** of G is defined as

$$PCI_T(G) = \sum_{u,v\in N} T_N(u)T_N(v)CONN_{T_G}(u, v),$$
$$PCI_I(G) = \sum_{u,v\in N} I_N(u)I_N(v)CONN_{I_G}(u, v),$$
$$PCI_F(G) = \sum_{u,v\in N} F_N(u)F_N(v)CONN_{F_G}(u, v),$$

Where $CONN_{T_G}(u,v)$ is the strength of truth, $CONN_{I_G}(u,v)$ is the strength of indeterminacy and $CONN_{F_G}(u,v)$ is the strength of falsity between two vertices u and v. we have

$$CONN_{T_G}(u,v) = \max\{\min T_M(e) \mid e \in P \text{ and } P \text{ is a path between } u \text{ and } v\},$$
$$CONN_{I_G}(u,v) = \min\{\max I_M(e) \mid e \in P \text{ and } P \text{ is a path between } u \text{ and } v\},$$
$$CONN_{F_G}(u,v) = \min\{\max F_M(e) \mid e \in P \text{ and } P \text{ is a path between } u \text{ and } v\},$$

Also, the ***totally connectivity index*** of G is defined as

$$TCI(G) = \frac{4 + 2PCI_T(G) - 2PCI_F(G) - PCI_I(G)}{6}.$$

3.1. Maximum spanning tree

In this section, a version of the maximum spanning tree discussed on a graph by strength of edges. In the following, we propose a neutrosophic maximum spanning tree algorithm, whose computing steps are described below. Note that the strength function $S(uv) = \frac{4+2T_M(uv)-2F_M(uv)-I_M(uv)}{6}$ is used to label here.

The algorithm for finding the maximum spanning tree (MST)

Here, the input is adjacency matrix $M = \left[\left(T_M(u_iu_j), I_M(u_iu_j), F_M(u_iu_j)\right)\right]_{n\times n}$ of the neutrosophic graph $G = (N, M)$, and output is a tree F with weighted edges.

Step 1. Input matrix M;

Step 2. Using the strength function $S(u_iu_j) = \frac{4+2T_M(u_iu_j)-2F_M(u_iu_j)-I_M(u_iu_j)}{6}$, convert the neutrosophic matrix into a strength matrix $S = \left[S(u_iu_j)\right]_{n\times n}$;

Step 3. Iterate steps 4 and 5 until all $n-1$ elements of S are either labeled to 0 or all the nonzero elements of the matrix are labeled;

Step 4. Find the M either column or row to compute the unlabeled maximum element $S(u_iu_j)$, which is the value of the corresponding are $e(u_iu_j) \in M$;

Step 5. If the corresponding edge $e(u_iu_j) \in M$ of chosen S produce a cycle whit the previous labeled entries of the strength matrix S than set $S(u_iu_j) = 0$ else label $S(u_iu_j)$;

Step 6. Design the tree F including only the labeled elements from the S which will be computed MST of G;

Step 6. Stop (end algorithm).

Example 3. Consider a neutrosophic graph $G = (N, M)$ on $V = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ as shown in Figure 3.

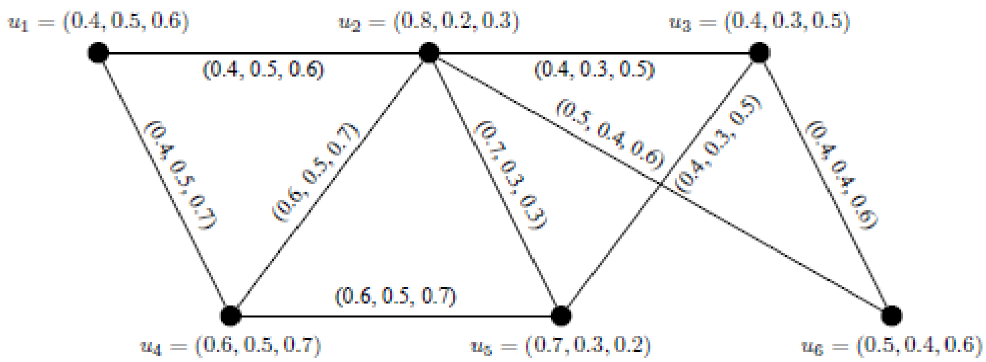


Figure 3. a neutrosophic graph G on $V = \{u_1, u_2, u_3, u_4, u_5, u_6\}$

And

$$M = \begin{bmatrix} 0 & (0.4, 0.5, 0.6) & 0 & (0.4, 0.5, 0.7) & 0 & 0 \\ (0.4, 0.5, 0.6) & 0 & (0.4, 0.3, 0.5) & (0.6,0.5, 0.7) & (0.7,0.3, 0.3) & (0.5, 0.4, 0.6) \\ 0 & (0.4, 0.3, 0.5) & 0 & 0 & (0.4, 0.3, 0.5) & (0.4, 0.4, 0.6) \\ (0.4, 0.5, 0.7) & (0.6,0.5, 0.7) & 0 & 0 & (0.7, 0.3, 0.2) & 0 \\ 0 & (0.7, 0.3, 0.3) & (04, 0.3, 0.5) & (0.6, 0.5, 0.7) & 0 & 0 \\ 0 & (0.5, 0.4, 0.6) & (0.4, 0.4, 0.6) & 0 & 0 & 0 \end{bmatrix}.$$

Using the strength function $S(u_i u_j) = \frac{4+2T_M(u_i u_j)-2F_M(u_i u_j)-I_M(u_i u_j)}{6}$ we have

$$S(u_i u_j) = \begin{bmatrix} 0 & 0.517 & 0 & 0.483 & 0 & 0 \\ 0.517 & 0 & 0.583 & 0.550 & 0.750 & 0.567 \\ 0 & 0.583 & 0 & 0 & 0.583 & 0.533 \\ 0.483 & 0.550 & 0 & 0 & 0.550 & 0 \\ 0 & 0.750 & 0.583 & 0.550 & 0 & 0 \\ 0 & 0.567 & 0.533 & 0 & 0 & 0 \end{bmatrix},$$

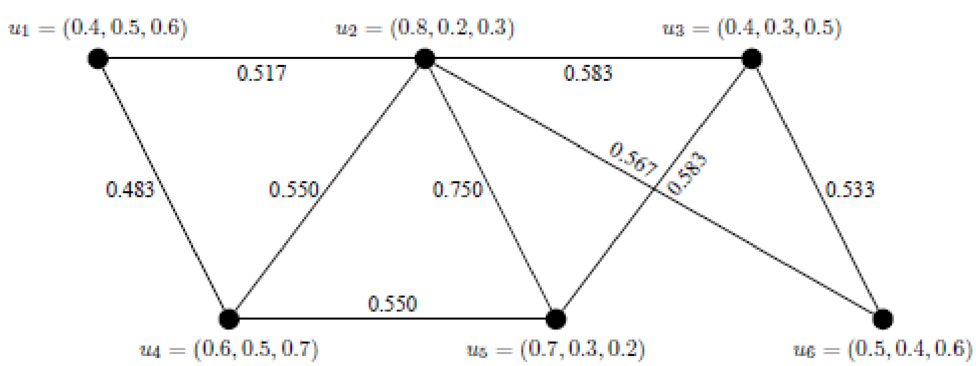


Figure 4. A neutrosophic graph G whit strength of edges

Now search the matrix S to find the maximum value and select the edge corresponding to the row and column of that element. The following figure edge u_2u_5 is highlighted.

$$S(u_iu_j)=\begin{bmatrix} 0 & 0.517 & 0 & 0.483 & 0 & 0 \\ 0.517 & 0 & 0.583 & 0.550 & 0.750 & 0.567 \\ 0 & 0.583 & 0 & 0 & 0.583 & 0.533 \\ 0.483 & 0.550 & 0 & 0 & 0.550 & 0 \\ 0 & 0.750 & 0.583 & 0.550 & 0 & 0 \\ 0 & 0.567 & 0.533 & 0 & 0 & 0 \end{bmatrix},$$

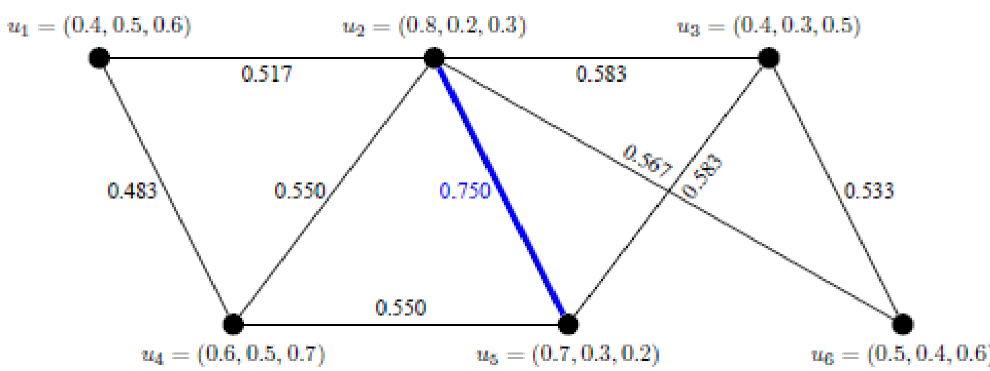


Figure 5. An edge u_2u_5 is highlighted

The next maximum element 0.583 is marked and corresponding edges u_2u_3 and u_3u_5, but the simultaneous selection of these two edges causes the formation of a cycle, so we choose one of these two edges arbitrarily and ignore the other.

$$S(u_iu_j)=\begin{bmatrix} 0 & 0.517 & 0 & 0.483 & 0 & 0 \\ 0.517 & 0 & 0.583 & 0.550 & 0.750 & 0.567 \\ 0 & 0.583 & 0 & 0 & 0.583 & 0.533 \\ 0.483 & 0.550 & 0 & 0 & 0.550 & 0 \\ 0 & 0.750 & 0.583 & 0.550 & 0 & 0 \\ 0 & 0.567 & 0.533 & 0 & 0 & 0 \end{bmatrix},$$

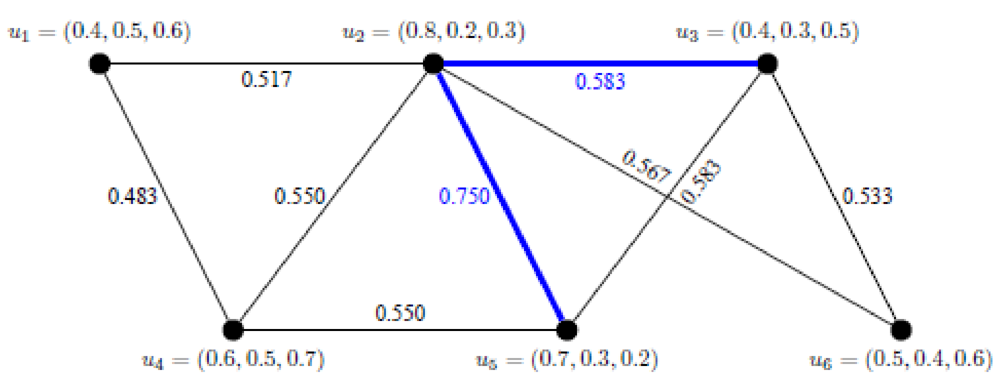


Figure 6. An edge u_2u_3 is highlighted

Continuing this process, edges u_2u_6, u_2u_4, and u_2u_1 are selected, respectively. The maximum spanning tree is obtained as figure 8.

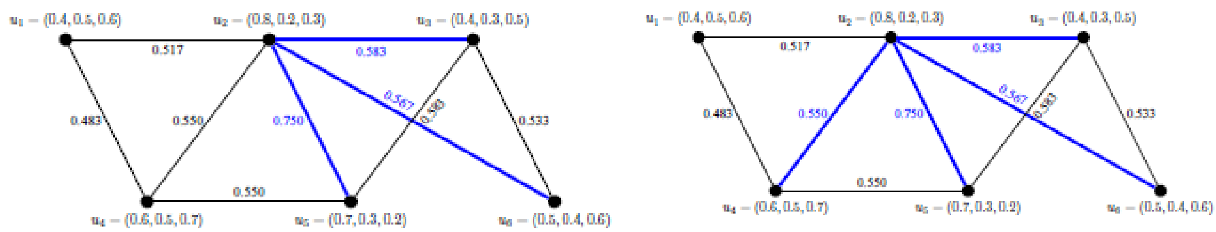


Figure 7. The edges u_2u_6 and u_2u_4 are highlighted

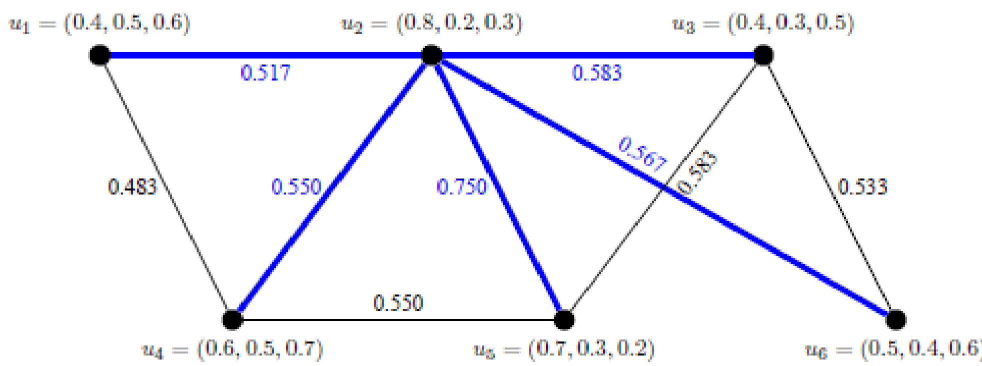


Figure 8. Maximum spanning tree (MST)

As it was observed, the selection of the maximum spanning tree was not unique, so neutrosophic graph $G = (N, M)$ is not a neutrosophic tree, also G contains a strong neutrosophic cycle.

Note. Obviously, if $G = (N, M)$ has a unique strong spanning tree, it will also have a unique maximum spanning tree, but the conversely is not necessarily true.

3.2. Partial connectivity index in the neutrosophic tree

In this section, the results of examining the Partial connectivity index and totally connectivity index on the neutrosophic trees are presented and proved.

Theorem 2. Let $G = (N, M)$ be a neutrosophic graph. Then $TCI(G - uv) = TCI(G)$ if and only if either uv is a weak edge or neutral edge.

Proof. The proof of this theorem is clear using definition 8.
□

Corollary 1. Let $G = (N, M)$ be a neutrosophic graph and, uv is an edge in G, uv is a bridge if and only if uv is either $\text{I} - strong\ edge$ or $\text{II} - strong\ edge$.

Corollary 2. Let $G = (N, M)$ be a neutrosophic graph. Then for any uv, $TCI(G - uv) \neq TCI(G)$ if G^* is a tree.

Theorem 3. Let $G = (N, M)$ be a connected neutrosophic graph whit strong spanning tree (SST) T. for any $uv \in M$, where uv is an edge of T, then either

$$PCI_T(G - uv) < PCI_T(G)$$
$$or$$
$$[(PCI_I(G - uv) > PCI_I(G)) \vee (PCI_F(G - uv) > PCI_F(G))]$$

Hence we have $TCI(G - uv) < TCI(G)$.

Proof. Suppose $G = (N, M)$ be a connected neutrosophic graph whit strong spanning tree (SST) T. Since T is SST then any edge of T is either I $- strong\ edge$ or II $- strong\ edge$. By Corollary 1, for each $uv \in M$, uv is a bridge. Then $PCI_T(G - uv) < PCI_T(G)\ or\ [(PCI_I(G - uv) > PCI_I(G)) \vee (PCI_F(G - uv) > PCI_F(G))]$.

□

Theorem 4. Let $G = (N, M)$ be a connected neutrosophic tree and G^* is not a tree. Then there exists at least one edge $uv \in M^*$ such that $TCI(G - uv) = TCI(G)$.

Proof. Let $G = (N, M)$ be a neutrosophic tree and G^* is not a tree. Hence there is at least one cycle in G^*. As respects a tree is a connected forest, there exist $uv \in M^*$ so that at least one of the following

$$T_M(uv) < CONN_{T(G-uv)}(u, v),$$
$$I_M(uv) > CONN_{I(G-uv)}(u, v), \qquad F_M(uv) > CONN_{F(G-uv)}(u, v))$$

Then

$$PCI_T(G - uv) = PCI_T(G)\ \ and\ \ PCI_I(G - uv) = PCI_I(G)\ \ and\ \ PCI_F(G - uv) = PCI_F(G)$$

Therefore, $TCI(G - uv) = TCI(G)$.

□

Theorem 5. Let $G = (N, M)$ be a connected neutrosophic graph then G is a neutrosophic tree if and only if G has a unique strong spanning tree.

Proof. Suppose $G = (N, M)$ is a connected neutrosophic graph with only one strong spanning tree T. Then G has no strong edges except the edges of T. hence G has no strong cycle. Therefore by definition 6, G is a neutrosophic tree. Conversely, assume that G is a neutrosophic tree. Again according to definition 6, G lacks a strong circle. Therefore, there is only one strong path between the two arbitrary vertices of G. then the strong spanning tree of G is unique.

□

Theorem 6. Let $G = (N, M)$ be a connected neutrosophic graph and T the corresponding SST of G. Then $TCI(T) = TCI(G)$ if and only if T is the unique strong spanning tree of G.

Proof. Suppose $G = (N, M)$ is a connected neutrosophic graph and T the corresponding SST of G. And $TCI(T) = TCI(G)$. Now, shown that T is a unique strong spanning tree of G. Proof of this is easily possible using Theorem 5. Conversely, assume that T is the unique strong spanning tree of G. It is clear that to obtain the connectivity index of G, only the strong paths will be the same paths of T. then $TCI(T) = TCI(G)$

□

Corollary 3. Let $G = (N, M)$ be a neutrosophic tree with the unique strong spanning tree (T) and the unique maximum spanning tree (F). Then $TCI(T) = TCI(G) = TCI(F)$.

Theorem 7. Let $G=(N,M)$ be a connected neutrosophic graph and $uv \in M^*$. Then $TCI(G-uv)<TCI(G)$ for any uv and $\left(CONN_{TG}(u,v), CONN_{IG}(u,v), CONN_{FG}(u,v)\right)=(T_M(uv), I_M(uv), F_M(uv))$ if and only if G^* is a tree.

Proof. Suppose $G=(N,M)$ is a connected neutrosophic graph and G^* is a tree. It is clear $TCI(G-uv)<TCI(G)$. Since G^* is a tree, for any $uv \in M^*$, $G-uv$ is not connected. Also for any $uv \in G$ we have $\left(CONN_{TG}(u,v), CONN_{IG}(u,v), CONN_{FG}(u,v)\right)=(T_M(uv), I_M(uv), F_M(uv))$. Conversely assume that for each uv, $TCI(G-uv)<TCI(G)$ and $\left(CONN_{TG}(u,v), CONN_{IG}(u,v), CONN_{FG}(u,v)\right)=(T_M(uv), I_M(uv), F_M(uv))$, then both uv is a neutrosophic bridge and a $\mathrm{I}-strong\ edge$. By theorem 1, G is a tree. Since, for each uv, $TCI(G-uv)<TCI(G)$, G^* is a tree.

□

Theorem 8. Let $G=(N,M)$ be a connected neutrosophic graph such that G^* is a star graph. If v_1 is the center vertex and for any $uv \in M^*$,

$$T_M(uv)=\min\{T_N(u),T_N(v)\},\ \ I_M(uv)=\min\{I_N(u),I_N(v)\},\ F_M(uv)=\max\{F_N(u),F_N(v)\}.$$

Also $\forall\, j\geq 2$, $t_1 \leq t_j$, $i_1 \leq i_j$ and $f_1 \geq f_j$ where $t_j=T_N\left(v_j\right)$, $i_j=I_N\left(v_j\right)$ and $f_j=F_N\left(v_j\right)$ for $j=1,2,\dots,n$. Then

$$PCI_T(G)=t_1\sum_{j=1}^{n-1} t_j \sum_{k=j+1}^{n} t_k,$$
$$PCI_I(G)=i_1\sum_{j=1}^{n-1} i_j \sum_{k=j+1}^{n} i_k,$$
$$PCI_F(G)=f_1\sum_{j=1}^{n-1} f_j \sum_{k=j+1}^{n} f_k.$$

Proof. Let $G=(N,M)$ be a neutrosophic graph such that G^* is a star graph and v_1 is the center vertex. Therefore for any vertex v_j, we have

$$CONN_{TG}\left(v_1,v_j\right)=T_M\left(v_1v_j\right)=\min\left\{T_N(v_1),T_N\left(v_j\right)\right\}=\ T_N(v_1),$$
$$CONN_{IG}\left(v_1,v_j\right)=I_M\left(v_1v_j\right)=\min\left\{I_N(v_1),I_N\left(v_j\right)\right\}=\ I_N(v_1),$$
$$CONN_{FG}\left(v_1,v_j\right)=F_M\left(v_1v_j\right)=\max\left\{F_N(v_1),F_N\left(v_j\right)\right\}=\ F_N(v_1).$$

Then

$$\sum_{k=2}^{n} T_N(v_1)T_N(v_k)CONN_{TG}(v_1,v_k)=(T_N(v_1))^2\sum_{k=2}^{n} T_N(v_k)=t_1^2\sum_{k=2}^{n} t_k,$$

Too for any $j,k\neq 1$, we have $CONN_{TG}\left(v_j,v_k\right)=T_N(v_1)=t_1$. Hence

$$PCI_T(G) = \sum_{u,v\in N} T_N(u)T_N(v)CONN_{T_G}(u,v)$$
$$= \sum_{k=2}^{n} T_N(v_1)T_N(v_k)CONN_{TG}(v_1,v_k) + \sum_{k=3}^{n} T_N(v_2)T_N(v_k)CONN_{TG}(v_2,v_k) + \cdots$$
$$+ T_N(v_{n-1})T_N(v_n)CONN_{TG}(v_{n-1},v_n)$$
$$= \left(T_N(v_1)\right)^2 \sum_{k=2}^{n} T_N(v_k) + T_N(v_1)\sum_{k=3}^{n} T_N(v_2)T_N(v_k) + \cdots + T_N(v_1)T_N(v_{n-1})T_N(v_n)$$
$$= \left(T_N(v_1)\right)^2 \sum_{k=2}^{n} T_N(v_k) + T_N(v_1)\sum_{j=n}^{n-1} T_N(v_j) \sum_{k=j+1}^{n} T_N(v_k) = t_1 \sum_{j=1}^{n-1} t_j \sum_{k=j+1}^{n} t_k.$$

Using a similar proof, we can show that $PCI_I(G) = i_1 \sum_{j=1}^{n-1} i_j \sum_{k=j+1}^{n} i_k$ and $PCI_F(G) = f_1 \sum_{j=1}^{n-1} f_j \sum_{k=j+1}^{n} f_k$.
□

Theorem 9. Let $G = (N, M)$ be a connected neutrosophic graph such that $G^* = C_n$. Then the following are equivalent.

a. $TCI(G - uv) = TCI(G)$ for any uv.
b. M is a constant function.
c. G has n strong spanning tree whit $S(T) = \gamma$ that γ is a constant value.

Proof. Suppose $G = (N, M)$ be a neutrosophic graph with $G^* = C_n$.

a → b Assume that $TCI(G - uv) = TCI(G)$ for any uv. This means that deleting each edge will not change the value of the connectivity index. Therefore, the membership function will be the same for all edges.

b → c Assume that M is a constant function. Hence all the edges of G are $I - strong\ edge$. Since removing each edge from the cycle will result a new tree of G. then the number of strong spanning trees of G will be n and strength of any strong spanning tree is a constant value.

c → a Assume that G has n strong spanning tree whit $S(T) = \gamma$ that γ is a constant value. It is clear for each edge of G we have $TCI(G - uv) = TCI(G)$.
□

4. Conclusion

In the paper, deals with a maximum spanning tree (MST) and a strong spanning tree (SST) problem under the neutrosophic graphs. Also, the Partial connectivity index and totally connectivity index in neutrosophic trees was presented here and some results obtained from the study of this index in trees were presented and proved. It should be noted that the results obtained in this article can be generalized to directed neutrosophic graphs, bipolar neutrosophic graphs and interval-valued neutrosophic graph, in general.

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An Intelligent Dual Simplex Method to Solve Triangular Neutrosophic Linear Fractional Programming Problem

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Abstract: This paper develops a general form of neutrosophic linear fractional programming (NLFP) problem and proposed a novel model to solve it. In this method the NLFP problem is decomposed into two neutrosophic linear programming (NLP) problem. Furthermore, the problem has been solved by combination of dual simplex method and a special ranking function. In addition, the model is compared with an existing method. An illustrative example is shown for better understanding of the proposed method. The results show that the method is computationally very simple and comprehensible.

Keywords: Triangular neutrosophic numbers; dual simplex method, ranking function, linear fractional programming, linear programming

1. Introduction

Liner fractional programming (LFP) problem is a special type of linear programming(LP) problem where the constraints are in linear form and the objective functions must be a ratio of two linear functions. Last few years, many researchers have been developed various methods to solve LFP problem in both classical logic and fuzzy logic [1-8]. These methods are interesting, however, in daily life circumstances, due to ambiguous information supplied by decision makers, the parameters are often illusory and it is very hard challenge for decision maker to make a decision. In such a case, it is more appropriate to interpret the ambiguous coefficients and the vague aspirations parameters by means of the fuzzy set theory.

In this manuscript a real life problem was presented, having vague parameters. Perhaps the best task given to mankind is to control the earth inside which they live. Anyway, some guidance's have been given and, limits have been set with the end goal that some law of nature ought not to be abused. During the time spent controlling nature, humanity has developed extremely incredible instruments that permit them to have control of some significant spots like the ocean, air, and ground. For instance, and as a genuine circumstance, an infection called Covid-19 was recognized first in the city of Wuhan China, which is the capital of Hubei territory on December 31, 2019. In the wake of appearing the pneumonia without an unmistakable reason and for which the antibodies or medicines were not found. Further, it is indicated that the transmission of the infection differs from Human to Human. The case spread Wuhan city as well as to different urban areas of China. Moreover, the disease spread to other area of the world, for example, Europe, North America, and Asia. It is obscure to all whether the infection will be spread all world or constrained to some nation. In this point, what is the amount of the affected related to the number of the individuals? It's absolutely blind in regards to everybody and the information's are uncertain. Whether or not it was impacted to each age social occasion or some specific get-together? Everything is questionable and uncertain. Hence, from the above real-life conditions, the values are incomplete and ambiguous. This type of problem can be handled by way of fuzzy sets.

The thought of fuzzy logic was setup by Zadeh [16] and from that point forward it has discovered enormous applications in different fields. When applied the LFP problem with fuzzy numbers, it is termed as fuzzy LFP (FLFP) problem. As yet, exceptional sorts of FLFP problem have already been interpreted within many articles to resolve such kind of problems. Li and Chen [9] developed an approach for solving FLFP problem via triangular fuzzy numbers, inspired by them, a multi-objective LFP problem with the fuzzy strategy is viewed via Luhandjula [10]. Meher et al. [20] proposed an idea to compute an (α, β) optimal solution for finding FLFP problem. Subsequently, Veeramani and Sumathi [15] examined a FLFP problem with triangular fuzzy number by way of multi-objective LFP problem and changed into a single objective linear programming problem. A goal programming approach used to be delivered to solve FLFP problem via Veeramani and

Sumathi [6]. However, Das et al. [11] solved the FLFP problem using the concepts of simple ranking approach between two triangular fuzzy numbers. Das and Mandal [14] introduced a ranking method for solving the FLFP problem. A method was introduced by Pop and Minasian [2] for solving fully FLFP (FFLFP) problem where the cost of objective, constraints and the variables are triangular fuzzy numbers. Later on, some of the mathematicians [3-4,17-19,21-23] proposed a different methods for solving FFLFP problem. A new method of lexicographic optimal solution was proposed by Das et al. [12].

The drawback of fuzzy sets, whence its incapacity to successfully symbolize facts as its only take into consideration the truth membership function. To conquer this trouble, Atanassov [24]presented the concept of intuitionistic fuzzy sets (IFS) which is a hybrid of fuzzy sets, he took into consideration both truth and falsity membership functions. However, in real-life situations it's still facing some difficulty in case of decision making. Therefore, new set theory was introduced which dealt with incomplete, inconsistency and indeterminate informations called neutrosophic set (NS).

Neutrosophic logic was introduced by Smarandache [27] as a new generalization of fuzzy logic and IFSs. Neutrosophic set may be characterized by three independent components i.e. (i) truth-membership component *(T),* (ii) indeterminacy membership component *(I),* (iii) falsity membership component *(F).*

The decision makers in neutrosophic set want to increase the degree of truth membership and decrease the degree of both indeterminacy and falsity memberships. The truth membership function is exactly the inverse or in the opposite side of the falsity membership function, while the indeterminate membership function took some of its values from the truth membership function and other values of indeterminacy are took from the falsity membership function, that is mean the indeterminate membership is in the middle position between truth and falsity. For solving practical problems, a single value neutrosophic set (SVNS) was introduced by Wang et al. [45]. Some authors [46-48] considered the problem of SVNS in practical applications like educational sector, social sector. The basic definitions and notions of neutrosophic number (NN) were set up by Samarandache [37]. Recently, Abdel-Basset et al. [38] presented a novel technique for neutrosophic

LP problem by considering trapezoidal neutrosophic numbers. Edalatpanah [34] proposed a direct model for solving LP by considering triangular neutrosophic numbers. A new method to find the optimal solution of LP problem in NNs environment was proposed by Ye et al. [43]. The field of solving LP problem with single objective in NNs environment with the help of goal programming introduced by Banerjee and Pramanik [42]. Again, Pramanik and Dey [41] have solved the problem of linear bi-level-LP problem under NNs. Maiti et al. [39] introduced a strategy for multi-level multi-objective LP problem with NNs. Huda E. Khalid [54] established a new branch in neutrosophic theory named as neutrosophic geometric programming problems with their newly algorithms , and novel definitions and theorems.

Here, we consider NLFP problem in which all the parameters, except crisp decision variables are considered as triangular neutrosophic numbers. We emphasize that there are few manuscripts have used triangular neutrosophic numbers in LFP problems. Recently, an interesting method was proposed by Abdel-Basset et al. [28] for solving neutrosophic LFP (NLFP). The NLFP problem is transformed into an equivalent crisp multi-objective linear fractional programming (MOLFP) problem, where the authors have transformed the crisp MOLFP problem is reduced to a single objective LP problem which can be solved easily by suitable LP technique. However, the above mentioned method has a drawback where the solutions are obtained does not satisfy the constraints, more constraints arise step by step.

In this paper, the NLFP problem is decomposed in two NLP problem. The NLP problem is transformed into crisp LP problem by using ranking function. By using dual simplex method, the crisp LP problem was solved. Consequently, the adequacy of the applied procedure is shown through a numerical example.

The remain parts of this paper were orchestrated as follow: some basic definitions and arithmetical operation with respect to the neutrosophic numbers are introduced in Section 2. The strategy of the proposed technique was contained in Section 3. In section 4, the proposed system applied with representation numerical guide to explain its appropriateness. The article reaches a conclusion containing the finishing up comments introduced in Section 5.

2. Preliminaries

In this section, we present the basic notations and definitions, which are used throughout this paper.

Definition 1. [28]

Assume X is a universal set and x∈X. A neutrosophic set N may be defined via three membership functions for truth, indeterminacy along with falsity and denoted by $T_N(x)$, $I_N(x)$ and $F_N(x)$ .These are real standard or real nonstandard subsets of $]0^-,1^+[$. That is $T_N(x): X \to \left]^-0,1^+\right[$, $I_N(x): X \to \left]^-0,1^+\right[$ and $F_N(x): X \to \left]^-0,1^+\right[$. There is no restriction on the sum of $T_N(x)$, $I_N(x)$ and $F_N(x)$,so $0^- \le \sup T_N(x) + \sup I_N(x) + \sup F_N(x) \le 3^+$.

Definition 2. [38]

A single-valued neutrosophic set (SVNS) N over X is an object having the form $N = \{x, T_N(x), I_N(x), F_N(x)\}$, where X be a space of discourse, $T_N(x): X \to [0,1],\ I_N(x): X \to [0,1]\ and\ F_N(x): X \to [0,1]$ with $0 \le T_N(x) + I_N(x) + F_N(x) \le 3, \forall x \in X$.

Definition 3 [34]. A triangular neutrosophic number (TNNs) is signified via $N =< (a_1^l, a_2^l, a_3^l); (q_1^l, q_2, q_3^l) >$ is an extended version of the three membership functions for the truth, indeterminacy, and falsity of x can be defined as follows:

$$T_N(x) = \begin{cases} \dfrac{(x - a_1^l)}{(a_2^l - a_1^l)} & a_1^l \le x < a_2^l, \\ 1 & \text{x} = a_2^l, \\ \dfrac{(a_3^l - x)}{(a_3^l - a_2^l)} & a_2^l \le x < a_3^l, \\ 0 & \text{something else.} \end{cases}$$

$$I_N(x)=\begin{cases}\dfrac{\left(a_2^l-x\right)}{\left(a_2^l-a_1^l\right)} & a_1^l\le x<a_2^l,\\ 1 & \mathrm{x}=a_2^l,\\ \dfrac{\left(x-a_2^l\right)}{\left(a_3^l-a_2^l\right)} & a_2^l\le x<a_3^l,\\ 0 & \text{something else.}\end{cases}$$

$$F_N(x)=\begin{cases}\dfrac{\left(a_2^l-x\right)}{\left(a_2^l-a_1^l\right)} & a_1^l\le x<a_2^l,\\ 0 & \mathrm{x}=a_2^l,\\ \dfrac{\left(x-a_2^l\right)}{\left(a_3^l-a_2^l\right)} & a_2^l\le x<a_3^l,\\ 1 & \text{something else.}\end{cases}$$

Where, $0\le T_N(x)+I_N(x)+F_N(x)\le 3, x\in N$ Additionally, when $a_1^l\ge 0$, N is called a non negative TNN. Similarly, when $a_1^l<0$, N becomes a negative TNN.

Definition 4 [28].(Arithmetic Operations)

Suppose $N_1=<(a_1^l,a_2^l,a_3^l);(q_1^l,q_2,q_3^l)>$ and $N_2=<(b_1^l,b_2^l,b_3^l);(r_1^l,r_2,r_3^l)>$ be two TNNs. Then the arithmetic relations are defined as:

$(i) N_1\oplus N_2=<(a^l{}_1+b^l{}_1,a^l{}_2+b^l{}_2,a^l{}_3+b^l{}_3);(q_1^l\wedge r_1^l,q_2\vee r_2,q_3^l\vee r_3^l)>$

$(ii) N_1-N_2=<(a^l{}_1-b^l{}_3,a^l{}_2-b^l{}_2,a^l{}_3-b^l{}_1);(q_1^l\wedge r_1^l,q_2\vee r_2,q_3^l\vee r_3^l)>$

$(iii) N_1\otimes N_2=<(a^l{}_1b^l{}_1,a^l{}_2b^l{}_2,a^l{}_3b^l{}_3);(q_1^l\wedge r_1^l,q_2\vee s_2,q_3^l\wedge r_3^l)>,\ \ if\ a^l{}_1>0,b^l{}_1>0,$

$$(iv)\lambda N_1=\begin{cases}<(\lambda a^l{}_1,\lambda a^l{}_2,\lambda a^l{}_3);(q_1^l,q_2,q_3^l)>, & if\ \ \lambda>0\\ <(\lambda a^l{}_3,\lambda a^l{}_2,\lambda a^l{}_1);(q_1^l,q_2,q_3^l)>, & if\ \ \lambda<0\end{cases}$$

$$(v)\,\frac{N_1}{N_2}=\begin{cases}\left\langle(\frac{a^l{}_1}{b^l{}_3},\frac{a^l{}_2}{b^l{}_2},\frac{a^l{}_3}{b^l{}_1});q_1^l\wedge r^l{}_1,q_2\vee r_2,q_3^l\vee r_3^l\right\rangle(a^l{}_3>0,b^l{}_3>0)\\ \left\langle(\frac{a^l{}_3}{b^l{}_3},\frac{a^l{}_2}{b^l{}_2},\frac{a^l{}_1}{b^l{}_1});q_1^l\wedge r^l{}_1,q_2\vee r_2,q_3^l\vee r_3^l\right\rangle(a^l{}_3<0,b^l{}_3>0)\\ \left\langle(\frac{a^l{}_3}{b^l{}_1},\frac{a^l{}_2}{b^l{}_2},\frac{a^l{}_1}{b^l{}_3});q_1^l\wedge r^l{}_1,q_2\vee r_2,q_3^l\vee r_3^l\right\rangle(a^l{}_3<0,b^l{}_3<0)\end{cases}$$

Definition 5 [28].Suppose N_1 and N_2 be two TNNs. Then:

(i) $N_1\le N_2$ if and only if $\Re(N_1)\le\Re(N_2)$.

(ii) $N_1 < N_2$ if and only if $\Re(N_1) < \Re(N_2)$.

Where $\Re(.)$ is a ranking function.

Definition 6 [32].The ranking function for triangular neutrosophic number $N =< (a_1^l, a_2^l, a_3^l); (q_1^l, q_2, q_3^l) >$ is defined as:

$$\Re(N) = \frac{a_1^l + a_2^l + a_3^l}{9}(q_1^l + (1 - q_2) + (1 - q_3^l)).$$

3. Proposed Method

One of the main aims of this paper is to extend the linear fractional problem into neutrosophic linear fractional programming problem.

The crisp LFP problem can be presented in the following way:

$$Max\ (or\ Min)\ z(x) = \frac{N(x)}{D(x)}$$

Subject to (1)

$$x \in S.$$

where $N(x)\ and\ D(x)$ are linear functions and the set S is defined as $S = \{x / Ax \leq b, x > 0\}$.

Here A is a fuzzy $m\,x\,n$ matrix.

The problem (1) can be written as:

$$\begin{array}{ll} \max\,(or \min)\, z(x) & N(x) \\ s.t. & x \in S \\ \min\,(or \max)\, z(x) & D(x) \\ s.t. & x \in S \end{array} \qquad (2)$$

Now, we consider the neutrosophic linear fractional programming (NLFP) problem with m constraints and n variables:

$$\text{Max (or min)}\ \tilde{Z} = \frac{\tilde{n}^T \otimes x + \tilde{r}}{\tilde{d}^T \otimes x + \tilde{s}} \qquad (3)$$

Subject to

$$\tilde{A} \otimes x \leq \tilde{b}$$

$$x \geq 0,\ j = 1, 2, \ldots, n.$$

where $\tilde{n}^T = [\tilde{n}_{1\times i}{}^T]_{1\times n}$, $\tilde{d}^T = [\tilde{d}_{1\times i}{}^T]_{1\times n}$ and rank $(\tilde{A},\tilde{b}) = rank(\tilde{A}) = m$. $\tilde{r}$ and $\tilde{s} \in \mathbb{R}$ are constants.

Due to some challenges exists in the crisp methods, these methods cannot be tackling above NLFP problem, and to overcome the challenges, another strategy is proposed. The means of the proposed technique are described in the following algorithm:

Step-1:substituting $\tilde{n}^T = [\tilde{n}_j]_{1\times n}, \tilde{d}^T = [\tilde{d}_j]_{1\times n}, \tilde{A} = [a_{ij}]_{m\times n}, \tilde{r} = [\tilde{r}_j]_{1\times n}, \tilde{s} = [\tilde{s}_j]_{1\times n}$ and $\tilde{b} = [\tilde{b}_i]_{m\times 1}$

the above NLFP problem may be rewritten as :

$$\text{Max (or min)} \quad \tilde{Z}(x) = \frac{\sum_{j=1}^{n} \tilde{n}_j x_j + \tilde{r}_j}{\sum_{j=1}^{n} \tilde{d}_j x_j + \tilde{s}_j}$$

Subject to (4)

$$\sum_{i=1}^{m} \tilde{a}_{ij} x_j \leq \tilde{b}_i$$

$$x_j \geq 0, i = 1, 2, .m$$

Step-2: By considering the following triangular neutrosophic numbers:

$$\tilde{n}_j = (l_j, m_j, n_j); (\mu_j, v_j, w_j), \tilde{d}_j = (c_j, d_j, f_j); (\mu_d, v_d, w_d), \tilde{a}_{ij} = (a_{ij}, b_{ij}, c_{ij}); (\mu_a, v_a, w_a),$$
$$\tilde{r}_j = (r_{j1}, r_{j2}, r_{j3}; \mu_{jr}, v_{jr}, w_{jr}), \tilde{s}_j = (s_{j1}, s_{j2}, s_{j3}; \mu_{js}, v_{js}, w_{js}) \, and \, \tilde{b}_i = (p_i, q_i, r_i); (\mu_b, v_b, w_b).$$

the NLFP problem may be written as :

$$\text{max (or min)} \quad \tilde{Z}(x) = \frac{\sum_{j=1}^{n} (l_j, m_j, n_j; \mu_j, v_j, w_j) x_j + (r_{j1}, r_{j2}, r_{j3}; \mu_{jr}, v_{jr}, w_{jr})}{\sum_{j=1}^{n} (c_j, d_j, f_j; \mu_d, v_d, w_d) x_j + (s_{j1}, s_{j2}, s_{j3}; \mu_{js}, v_{js}, w_{js})}$$

subject to

$$\sum_{j=1}^{n} (a_{ij}, b_{ij}, c_{ij}; \mu_j, v_j, w_j) x_j \leq \left(p_i, q_i, r_i; \mu_j, v_j, w_j\right)$$

$$x_j \geq 0, i = 1, 2, ..., m.$$

Step-3. To determine the optimal value of the above problem, we take transform the objective function into two neutrosophic linear programming problem and the problem may be written as follows:

(E-1) $$\text{max (or min)} \quad \tilde{Z}(x) = \sum_{j=1}^{n} (l_j, m_j, n_j; \mu_j, v_j, w_j) x_j + (r_{j1}, r_{j2}, r_{j3}; \mu_j, v_j, w_j)$$

subject to

$$\sum_{j=1}^{n}(a_{ij},b_{ij},c_{ij};\mu_j,v_j,w_j)x_j \le \left(p_i,q_i,r_i;\mu_j,v_j,w_j\right)$$

$$x_j \ge 0, i = 1,2,...,m.$$

(E-2) min (or max) $\tilde{Z}(x) = \sum_{j=1}^{n}(c_j,d_j,f_j;\mu_j,v_j,w_j)x_j + (s_{j1},s_{j2},s_{j3};\mu_j,v_j,w_j)$

subject to

$$\sum_{j=1}^{n}(a_{ij},b_{ij},c_{ij};\mu_j,v_j,w_j)x_j \le \left(p_i,q_i,r_i;\mu_j,v_j,w_j\right)$$

$$x_j \ge 0, i = 1,2,...,m.$$

Step-4: Using arithmetic operations, defined in definition 5 and 7, the above NLP problems (E-1) and (E-2) are converted into crisp linear programming problems, separately.

(E-3) max (or min) $\tilde{Z}(x) = \Re(\sum_{j=1}^{n}(l_j,m_j,n_j;\mu_j,v_j,w_j)x_j + (r_{j1},r_{j2},r_{j3};\mu_j,v_j,w_j))$

subject to

$$\Re(\sum_{j=1}^{n}(a_{ij},b_{ij},c_{ij};\mu_j,v_j,w_j)x_j \le \left(p_i,q_i,r_i;\mu_j,v_j,w_j\right))$$

$$x_j \ge 0, i = 1,2,...,m.$$

(E-4) min (or max) $\tilde{Z}(x) = \Re(\sum_{j=1}^{n}(c_j,d_j,f_j;\mu_j,v_j,w_j)x_j + (s_{j1},s_{j2},s_{j3};\mu_j,v_j,w_j))$

subject to

$$\Re(\sum_{j=1}^{n}(a_{ij},b_{ij},c_{ij};\mu_j,v_j,w_j)x_j \le \left(p_i,q_i,r_i;\mu_j,v_j,w_j\right))$$

$$x_j \ge 0, i = 1,2,...,m.$$

Step-5: Now solve the above crisp LP problem (E-3) and (E-4) by using the dual simplex method.

Step-6: Find the optimal solution of x_j by solving crisp LP problem obtained in Step-5.

Step-7: Find the fuzzy optimal value by putting x_j in both (E-3) and (E-4) and get crisp linear fractional programming problem.

4. Numerical Example

Here, we select a case of [28] to represent the model alongside correlation of existing technique.

Example-1

In Jamshedpur City, India, A Wooden company is the producer of three kinds of products A, B and C with profit around 8, 7 and 9 dollar per unit, respectively. However the cost for each one unit of the above products is around 8, 9 and 6 dollars respectively. It is assume that a fixed cost of around 1.5 dollar is added to the cost function due to expected duration through the process of production. Suppose the raw material needed for manufacturing product A, B and C is about 4, 3 and 5 units per dollar respectively, the supply for this raw material is restricted to about 28 dollar. Man-hours per unit for the product A is about 5 hour, product B is about 3 hour and C is about 3 hour per unit for manufacturing but total Man-hour available is about 20 hour daily. Determine how many products A, B and C should be manufactured in order to maximize the total profit.

Let x_1, x_2 and x_3 component be the measure of A, B and C, individually to be created. Afterprediction of evaluated parameters, the above issue can be defined as the followingNLFPP:

$$Max\ Z = \frac{8^l x_1 + 7^l x_2 + 9^l x_3}{8^l x_1 + 9^l x_2 + 6^l x_3 + 1.5^l} \tag{5}$$

Subject to

$$4^l x_1 + 3^l x_2 + 5^l x_3 \leq 28^l$$

$$5^l x_1 + 3^l x_2 + 3^l x_3 \leq 20^l$$

$$x_1, x_2, x_3 \geq 0.$$

Here we consider,

$$8^l = (7,8,9;0.5,0.8,0.3), 7^l = (6,7,8;0.2,0.6,0.5), 9^l = (8,9,100.8,0.1,0.4), 6^l = (4,6,8;0.75,0.25,0.1),$$
$$1.5^l = (1,1.5,2;0.75,0.5,0.25), 4^l = (3,4,5;0.4,0.6,0.5), 3^l = (2,3,4;1,0.25,0.3), 5^l = (4,5,6;0.3,0.4,0.8)$$
$$28^l = (25,28,30;0.4,0.25,0.6), 20^l = (18,20,22;0.9,0.2,0.6)$$

Presently the problem is modified as follows:

$$Max\ Z = \frac{(7,8,9;0.5,0.8,0.3)x_1 + (6,7,8;0.2,0.6,0.5)x_2 + (8,9,10.8,0.1,0.4)x_3}{(7,8,9;0.5,0.8,0.3)x_1 + (8,9,10.8,0.1,0.4)x_2 + (4,6,8;0.75,0.25,0.1)x_3 + (1,1.5,2;0.75,0.5,0.25)}$$

Subject to

$$(3,4,5;0.4,0.6,0.5)x_1 + (2,3,4;1,0.25,0.3)x_2 + (4,5,6;0.3,0.4,0.8)x_3 \le (25,28,30;0.4,0.25,0.6)$$
$$(4,5,6;0.3,0.4,0.8)x_1 + (2,3,4;1,0.25,0.3)x_2 + (2,3,4;1,0.25,0.3)x_3 \le (18,20,22;0.9,0.2,0.6)$$

$$x_1, x_2, x_3 \ge 0.$$

Utilizing Step 2 the NFP problem can be transformed into two NLP problem as:

$$Max\ Z = (7,8,9;0.5,0.8,0.3)x_1 + (6,7,8;0.2,0.6,0.5)x_2 + (8,9,10.8,0.1,0.4)x_3$$

(E-1) Subject to

$$(3,4,5;0.4,0.6,0.5)x_1 + (2,3,4;1,0.25,0.3)x_2 + (4,5,6;0.3,0.4,0.8)x_3 \le (25,28,30;0.4,0.25,0.6)$$
$$(4,5,6;0.3,0.4,0.8)x_1 + (2,3,4;1,0.25,0.3)x_2 + (2,3,4;1,0.25,0.3)x_3 \le (18,20,22;0.9,0.2,0.6)$$

$$x_1, x_2, x_3 \ge 0.$$

$$Max\ Z = (7,8,9;0.5,0.8,0.3)x_1 + (8,9,10.8,0.1,0.4)x_2 + (4,6,8;0.75,0.25,0.1)x_3 + (1,1.5,2;0.75,0.5,0.25)$$

(E-2) Subject to

$$(3,4,5;0.4,0.6,0.5)x_1 + (2,3,4;1,0.25,0.3)x_2 + (4,5,6;0.3,0.4,0.8)x_3 \le (25,28,30;0.4,0.25,0.6)$$
$$(4,5,6;0.3,0.4,0.8)x_1 + (2,3,4;1,0.25,0.3)x_2 + (2,3,4;1,0.25,0.3)x_3 \le (18,20,22;0.9,0.2,0.6)$$

$$x_1, x_2, x_3 \ge 0.$$

Using Step-3, the ranking function the problem (E-1) and (E-2) can be written as follows:

$$Max\ Z = 3.73x_1 + 2.56x_2 + 6.9x_3$$

(E-1) Subject to

$$1.73x_1 + 2.45x_2 + 1.83x_3 \le 14.29$$
$$1.83x_1 + 2.45x_2 + 2.45x_3 \le 14$$

$$x_1, x_2, x_3 \geq 0.$$

$$Max\ Z = 3.73x_1 + 6.9x_2 + 4.8x_3 + 1$$

(E-2) Subject to

$$1.73x_1 + 2.45x_2 + 1.83x_3 \leq 14.29$$
$$1.83x_1 + 2.45x_2 + 2.45x_3 \leq 14$$

$$x_1, x_2, x_3 \geq 0.$$

Now solve the problem (E-1) by dual simplex method

$$Max\ Z = 3.73x_1 + 2.56x_2 + 6.9x_3$$

(E-3) Subject to

$$1.73x_1 + 2.45x_2 + 1.83x_3 \leq 14.29$$
$$1.83x_1 + 2.45x_2 + 2.45x_3 \leq 14$$

$$x_1, x_2, x_3 \geq 0.$$

Now the problem (E-3) is solved by dual simplex method and get the optimal solution is as:

$x_1 = 0, x_2 = 0, x_3 = 5.71$ and the objective solution is $z = 39.42$.

$$Max\ Z = 3.73x_1 + 6.9x_2 + 4.8x_3 + 1$$

(E-4) Subject to

$$1.73x_1 + 2.45x_2 + 1.83x_3 \leq 14.29$$
$$1.83x_1 + 2.45x_2 + 2.45x_3 \leq 14$$

$$x_1, x_2, x_3 \geq 0.$$

Now the problem (E-4) is solved by dual simplex method and get the optimal solution is as:

$x_1 = 0, x_2 = 2.65, x_3 = 3.05$ and the objective solution is $z = 33.92$.

Finally, the optimum solution of crisp linear fractional programming problem is obtained.
Thus,

$$\max z = \frac{39.42}{33.92} = 1.16$$

By comparing the results of objective solutions, we can conclude that our solution is more maximize the cost.

$$1.16 = z_{(Proposed\ Method)} > z_{(Existing\ Method[28])} = 1$$

By comparing the results of the proposed method with existing method [28] based on ranking function and ordering by using Definition 6, we can conclude that our
result is more effective, because:

$$0.208 = z_{(Proposed\ Method)} > z_{(Existing\ Method[28])} = 0.069\ .$$

This example has been solved by the proposed method to show that one can overcome the limitations of the existing method [28] by using the proposed method. Earlier this problem was also solved by Abel-Basset [28]. Obtained result of the present method has been compared with the results of existing method [28]. It is worth mentioning that one may check that the results obtained by the existing method may not satisfy the constraints properly where the results obtained by the present method satisfied those constraints exactly. Based on the ranking function the proposed method is higher optimized the value as compare to the existing method. In the proposed methodology the FFLP problem turns into a crisp linear programming problem and that problem is solved by using LINGO Version 13.0.

Result Analysis:

In this segment, we give an outcome examination of the proposed strategy with existing technique. In the above writing perusing, we infer that there is exceptionally less exploration paper for taking care of neutrosophic LFP issue. In this manner, we consider the traditional LFP issue and fuzzy LFP issue for correlation with our proposed strategy.

- Our proposed outcomes are better than traditional LFP [22] and fluffy LFP [56] model. The objective solution of our proposed technique is 1.16, anyway in the current strategy [22,56] the objective solution is 1.09. Obviously our target arrangement is maximized.

- In real-life problem, the leaders faces numerous issue to take choice as truth, not truth and bogus. In any case, in Das et al. [56] the fluffy model the leaders consider just truth work. This is the fundamental downside of Das et al. fluffy model. Taking these points of interest, we proposed new technique.

- Our model is applied in any genuine issue.
- In the above conversation, we reason that our model is another approach to deal with the vulnerability and indeterminacy in genuine issue.

5. Conclusions

This paper introduced a novel method for solving NLFP problem where all the parameters are triangular neutrosophic numbers except decision variables. In our proposed method, NLFP problem is transformed into two equivalent NLP problems and the resultant problem is converted into crisp LP problem by using ranking function. Dual simplex method is used for solving the crisp LP problem. From the computational discussion, we conclude that with respect to the existing method [28], proposed method has less computational steps and the optimum solution is maximize the values. The proposed method NLFP problem has successfully overcome the drawbacks of the existing work [28]. Finally, from the procured results, it might be derived that the model is capable and supportive.

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Neutrosophic $\aleph$ −interior ideals in semigroups

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Abstract: We define the concepts of neutrosophic $\aleph$ -interior ideal and neutrosophic $\aleph$ −characteristic interior ideal structures of a semigroup. We infer different types of semigroups using neutrosophic $\aleph$-interior ideal structures. We also show that the intersection of neutrosophic $\aleph$-interior ideals and the union of neutrosophic $\aleph$-interior ideals is also a neutrosophic $\aleph$-interior ideal.

Keywords: Semi group, neutrosophic $\aleph$ − ideals, neutrosophic $\aleph$ -interior ideals, neutrosophic $\aleph$ −product.

1. Introduction

Nowadays, the theory of uncertainty plays a vital role to manage different issues relating to modelling engineering problems, networking, real-life problem relating to decision making and so on. In 1965, Zadeh[24] introduced the idea of fuzzy sets for modelling vague concepts in the globe. In 1986, Atanassov [1] generalized fuzzy set and named as Intuitionistic fuzzy set. Also, from his viewpoint, there are two degrees of freedom in the real world, one a degree of membership to a vague subset and the other is a degree of non-membership to that given subset.

Smarandache generalized fuzzy set and intuitionistic fuzzy set, and named as neutrosophic set (see [4, 7, 8, 14, 19, 22-23]). These sets are characterized by a truth membership function, an indeterminacy membership function and a falsity membership function. These sets are applied to many branches of mathematics to overcome the complexities arising from uncertain data. A Neutrosophic set can distinguish between absolute membership and relative membership. Smarandache used this in non-standard analysis such as the result of sports games (winning/defeating/tie), decision making and control theory, etc. This area has been studied by several authors (see [3, 11, 12, 16-18]).

For more details on neutrosophic set theory, the readers visit the website http://fs.gallup.unm.edu/FlorentinSmarandache.htm

In [2], Abdel Basset et al. designed a framework to manage scheduling problems using neutrosophic theory. As the concept of time-cost tradeoffs and deterministic project scheduling disagree with the real situation, some data were changed during the implementation process. Here fuzzy scheduling and time-cost tradeoffs models assumed only truth-membership functions dealing with uncertainties of the project and their activities duration which were unable to treat indeterminacy and inconsistency.

In [6], Abdel Basset et al. evaluated the performance of smart disaster response systems under uncertainty. In [5], Abdel Basset et al. introduced different hybrid neutrosophic multi-criteria decision-making framework for professional selection that employed a collection of neutrosophic analytical network process and order preference by similarity to the ideal solution under bipolar neutrosophic numbers.

In [21], Prakasam Muralikrishna1 et al. presented the characterization of MBJ – Neutrosophic β – Ideal of β – algebra. They analyzed homomorphic image, pre–image, cartesian product and related results, and these concepts were explored to other substructures of a β – algebra. In [9], Chalapathi et al. constructed certain Neutrosophic Boolean rings, introduced Neutrosophic complement elements and mainly obtained some properties satisfied by the Neutrosophic complement elements of Neutrosophic Boolean rings.

In [14], M. Khan et al. presented the notion of neutrosophic ℵ-subsemigroup in semigroup and explored several properties. In [11], Gulistan et al. have studied the idea of complex neutrosophic subsemigroups and introduced the concept of the characteristic function of complex neutrosophic sets, direct product of complex neutrosophic sets.

In [10], B. Elavarasan et al. introduced the notion of neutrosophic ℵ-ideal in semigroup and explored its properties. Also, the conditions for neutrosophic ℵ-structure to be neutrosophic ℵ-ideal are given, and discussed the idea of characteristic neutrosophic ℵ-structure in semigroups and obtained several properties. In [20], we have introduced and discussed several properties of neutrosophic ℵ-bi-ideal in the semigroup. We have proved that neutrosophic ℵ-product and the intersection of neutrosophic ℵ-ideals were identical for regular semigroups. In this paper, we define and discuss the concepts of neutrosophic ℵ-interior ideal and neutrosophic ℵ-characteristic interior ideal structures of a semigroup.

Throughout this paper, X denotes a semigroup. Now, we present the important definitions of semigroup that we need in sequel.

Recall that for any $X_1, X_2 \subseteq X$, $X_1X_2 = \{ab|a \in X_1 \; and \; b \in X_2\}$, multiplication of X_1 and X_2.

Let X be a semigroup and $\emptyset \neq X_1 \subseteq X$. Then

(i) X_1 is known as subsemigroup if ${X_1}^2 \subseteq X_1$.

(ii) A subsemigroup X_1 is known as left (resp., right) ideal if $X_1X \subseteq X_1$(resp., $XX_1 \subseteq X_1$).

(iii) X_1 is known as ideal if X_1 is both a left and a right ideal.

(iv) X is known as left (resp., right) regular if for each $r \in X$, there exists $i \in X$ such that $r = ir^2$(resp., $r = r^2i$) [13].

(v) X is known as regular if for each $b_1 \in X$, there exists $i \in X$ such that $b_1 = b_1ib_1$

(vi) X is known as intra-regular if for each $x_1 \in X$, there exist $i, j \in X$ such that $x_1 = ix_1^2j$ [15].

2. Definitions of neutrosophic ℵ - structures

We present definitions of neutrosophic $\aleph$ −structures namely neutrosophic $\aleph$ −subsemigroup, neutrosophic $\aleph$ −ideal, neutrosophic $\aleph$ −interior ideal of a semigroup X

The set of all the functions from X to $[-1,0]$ is denoted by $\Im(X,[-1,0])$. We call that an element of $\Im(X,[-1,0])$ is $\aleph$ −function on X. A $\aleph$ −structure means an ordered pair (X,g) of X and an $\aleph$ −function g on X.

Definition 2.1.[14] A neutrosophic $\aleph-$ structure of X is defined to be the structure:

$$X_M := \frac{X}{(T_M,\ I_M,\ F_M)} = \left\{\frac{r}{T_M(r), I_M(r),\ F_M(r)} \mid r \in X\right\},$$

where T_M, I_M and F_M are the negative truth, negative indeterminacy and negative falsity membership function on X ($\aleph-$ functions).

It is evident that $-3 \le T_M(r) + I_M(r) + F_M(r) \le 0$ for all $r \in X$.

Definition 2.2.[14] A neutrosophic $\aleph-$ structure X_M of X is called a neutrosophic $\aleph$ −subsemigroup of X if the following assertion is valid:

$$(\forall\, g_i, h_j \in X)\begin{pmatrix} T_M(g_ih_j) \le T_M(g_i) \vee T_M(h_j) \\ I_M(g_ih_j) \ge I_M(g_i) \wedge I_M(h_j) \\ F_M(g_ih_j) \le F_M(g_i) \vee F_M(h_j) \end{pmatrix}.$$

.Let X_M be a neutrosophic $\aleph$ −structure and $\gamma,\delta,\varepsilon \in [-1,0]$ with $-3 \le \gamma+\delta+\varepsilon \le 0.$ Consider the sets:

$$T_M^{\gamma} = \{r_i \in X | T_M(r_i) \le \gamma\}$$
$$I_M^{\delta} = \{r_i \in X | I_M(r_i) \ge \delta\}$$
$$F_M^{\varepsilon} = \{r_i \in X | F_M(r_i) \le \varepsilon\}.$$

The set $X_M(\gamma,\delta,\varepsilon) \coloneqq \{r_i \in X\,|T_M(r_i) \le \gamma,\ I_M(r_i) \ge \delta,\ F_M(r_i) \le \epsilon\}$ is known as $(\gamma,\delta,\varepsilon)$-level set of X_M. It is easy to observe that $X_M(\gamma,\delta,\varepsilon) = T_M^{\gamma} \cap I_M^{\delta} \cap F_M^{\varepsilon}$.

Definition 2.3.[10] A neutrosophic $\aleph$ −structure X_M of X is called a neutrosophic $\aleph$ −left (resp., right) ideal of X if

$$(\forall\, g_i, h_j \in X)\begin{pmatrix} T_M(g_ih_j) \le T_M(h_j)\ (resp., T_M(g_ih_j) \le T_M(g_i)) \\ I_M(g_ih_j) \ge I_M(h_j)\ (resp.,\ I_M(g_ih_j) \ge I_M(g_i)) \\ F_M(g_ih_j) \le F_M(h_j)\ (resp.,\ F_M(g_ih_j) \le F_M(g_i)) \end{pmatrix}.$$

X_M is neutrosophic $\aleph$ −ideal of X if X_M is neutrosophic $\aleph$ −left and $\aleph$ −right ideal of X.

Definition 2.4. A neutrosophic $\aleph$ −subsemigroup X_M of X is known as neutrosophic $\aleph$ −interior ideal if

$$(\forall\, x,a,y \in X)\begin{pmatrix} T_M(xay) \le T_M(a) \\ I_M(xay) \ge I_M(a) \\ F_M(xay) \le F_M(a) \end{pmatrix}.$$

It is easy to observe that every neutrosophic $\aleph$ −ideal is neutrosophic $\aleph$ −interior ideal, but neutrosophic $\aleph$ −interior ideal need not be a neutrosophic $\aleph-$ ideal, as shown by an example.

Example 2.5. Let X be the set of all non-negative integers except 1. Then X is a semigroup with usual multiplication.

Let $X_M = \left\{\frac{0}{(-0.9,-0.1,-0.7)}, \frac{2}{(-0.4\ -0.6,-0.5)}, \frac{5}{(-0.3,-0.8,-0.3)}, \frac{10}{(-0.3,-0.8,-0.1)}, \frac{otherwise}{(-0.7,-0.4,-0.6)}\right\}$. Then X_M is neutrosophic $\aleph$ −interior ideal, but not neutrosophic $\aleph-$ ideal with $T_N(2.5) = -0.3 \nleq T_N(2)$.

Definition 2.6.[14] For any $E \subseteq X$, the characteristic neutrosophic $\aleph$ −structure is defined as

$$\chi_E(X_M) = \frac{X}{(\chi_E(T)_M,\ \chi_E(I)_M,\ \chi_E(F)_M)}$$

where

$$\chi_E(T)_M: X \to [-1,0],\ r \to \begin{cases} -1 \ if\ r \in E \\ 0\ \ otherwise, \end{cases}$$

$$\chi_E(I)_M: X \to [-1,0],\ r \to \begin{cases} 0 \ if\ r \in E \\ -1\ otherwise, \end{cases}$$

$$\chi_E(F)_M: X \to [-1,0],\ r \to \begin{cases} -1 \ if\ r \in E \\ 0\ \ otherwise. \end{cases}$$

Definition 2.7.[14] Let $X_N := \frac{X}{(T_N,\ I_N,\ F_N)}$ and $X_M := \frac{X}{(T_M,\ I_M,\ F_M)}$ be neutrosophic $\aleph-$structures of X. Then

(i) X_N is called a neutrosophic $\aleph-$ substructure of X_M, denote by $X_M \subseteq X_N$, if $T_M(r) \geq T_N(r),\ I_M(r) \leq I_N(r),\ F_M(r) \geq F_N(r)$ for all $r \in X$.

(ii) If $X_N \subseteq X_M$ and $X_M \subseteq X_N$, then we say that $X_N = X_M$.

(iii) The neutrosophic $\aleph-$ product of X_N and X_M is defined to be a neutrosophic $\aleph-$structure of X,

$$X_N \odot X_M := \frac{X}{(T_{N\circ M},\ I_{N\circ M},\ F_{N\circ M})} = \left\{\frac{h}{T_{N\circ M}(h),\ I_{N\circ M}(h),\ F_{N\circ M}(h)} \mid h \in X\right\},$$

where

$$(T_N \circ T_M)(h) = T_{N\circ M}(h) = \begin{cases} \bigwedge_{h=rs}\{T_N(r) \vee T_M(s)\} & if\ \exists\ r,s \in X\ such\ that\ h = rs \\ 0 & otherwise, \end{cases}$$

$$(I_N \circ I_M)(h) = I_{N\circ M}(h) = \begin{cases} \bigvee_{h=rs}\{I_N(r) \wedge I_M(s)\} & if\ \exists\ u,v \in X\ such\ that\ h = rs \\ -1 & otherwise, \end{cases}$$

$$(F_N \circ F_M)(h) = F_{N\circ M}(h) = \begin{cases} \bigwedge_{h=rs}\{F_N(r) \vee F_M(s)\} & if\ \exists\ u,v \in X\ such\ that\ h = rs \\ 0 & otherwise. \end{cases}$$

For $i \in X$, the element $\frac{\mathbf{i}}{(\mathbf{T_{N\circ M}(i)},\ \mathbf{I_{N\circ M}(i)},\ \mathbf{F_{N\circ M}(i)})}$ is simply denoted by $(\mathbf{X_N} \odot \mathbf{X_M})(\mathbf{i}) = (\mathbf{T_{N\circ M}(i)},\ \mathbf{I_{N\circ M}(i)},\ \mathbf{F_{N\circ M}(i)})$.

(iii) The union of X_N and X_M, a neutrosophic $\aleph-$structure over X is defined as

$$X_N \cup X_M = X_{N\cup M} = (X; T_{N\cup M},\ I_{N\cup M},\ F_{N\cup M}),$$

where

$$(T_N \cup T_M)(h_i) = T_{N\cup M}(h_i) = T_N(h_i) \wedge T_M(h_i),$$
$$(I_N \cup I_M)(h_i) = I_{N\cup M}(h_i) = I_N(h_i) \vee I_M(h_i),$$
$$(F_N \cup F_M)(h_i) = F_{N\cup M}(h_i) = F_N(h_i) \wedge F_M(h_i)\ \ \forall h_i \in X.$$

(iv) The intersection of X_N and X_M, a neutrosophic $\aleph-$structure over X is defined as

$$X_N \cap X_M = X_{N\cap M} = (X; T_{N\cap M},\ I_{N\cap M},\ F_{N\cap M}),$$

where

$$(T_N \cap T_M)(h_i) = T_{N\cap M}(h_i) = T_N(h_i) \vee T_M(h_i),$$
$$(I_N \cap I_M)(h_i) = I_{N\cap M}(h_i) = I_N(h_i) \wedge I_M(h_i),$$
$$(F_N \cap F_M)(h_i) = F_{N\cap M}(h_i) = F_N(h_i) \vee F_M(h_i)\ \ \forall\ h_i \in X.$$

3. Neutrosophic $\aleph-$interior ideals

We study different properties of neutrosophic $\aleph-$interior ideals of X. It is evident that neutrosophic $\aleph-$ ideal is a neutrosophic $\aleph-$interior ideal of X, but not the converse. Further, for a regular and for an intra-regular semigroup, every neutrosophic $\aleph-$interior ideal is neutrosophic $\aleph-$ideal.

All throughout this part, we consider X_M and X_N are neutrosophic $\aleph-$structures of X.

Theorem 3.1. For any $L \subseteq X$, the equivalent assertions are:

(i) L is an interior ideal,

(ii) The characteristic neutrosophic $\aleph-$structure $\chi_L(X_N)$ is a neutrosophic $\aleph-$interior ideal.

Proof: Suppose L is an interior ideal and let $x, a, y \in X$.

If $a \in L$, then $xay \in L$, so $\chi_L(T)_N(xay) = -1 = \chi_L(T)_N(a)$, $\chi_L(I)_N(xay) = 0 = \chi_L(I)_N(a)$ and $\chi_L(F)_N(xay) = -1 = \chi_L(F)_N(a)$.

If $a \notin L$, then $\chi_L(T)_N(xay) \leq 0 = \chi_L(T)_N(a)$, $\chi_L(I)_N(xay) \geq -1 = \chi_L(I)_N(a)$ and $\chi_L(F)_N(xay) \leq 0 = \chi_L(F)_N(a)$.

Therefore $\chi_L(X_N)$ is a neutrosophic $\aleph-$interior ideal.

Conversely, assume that $\chi_L(X_N)$ is a neutrosophic $\aleph-$ interior ideal. Let $u \in L$ and $x, y \in X$. Then

$$\chi_L(T)_N(xuy) \leq \chi_L(T)_N(u) = -1,$$
$$\chi_L(I)_N(xuy) \geq \chi_L(I)_N(u) = 0,$$
$$\chi_L(F)_N(xuy) \leq \chi_L(F)_N(u) = -1\ .$$

So $xuy \in L$. □

Theorem 3.2. If X_M and X_N are neutrosophic $\aleph-$ interior ideals, then $X_{M\cap N}$ is neutrosophic $\aleph-$ interior ideal.

Proof: Let X_M and X_N be neutrosophic $\aleph-$ interior ideals. For any $r, s, t \in X$, we have

$$T_{M\cap N}(rst) = T_M(rst) \vee T_N(rst) \leq T_M(s) \vee T_N(s) = T_{M\cap N}(s),$$
$$I_{M\cap N}(rst) = I_M(rst) \wedge I_N(rst) \geq I_M(s) \wedge I_N(s) = I_{M\cap N}(s),$$
$$F_{M\cap N}(rst) = F_M(rst) \vee F_N(rst) \leq F_M(s) \vee F_N(s) = F_{M\cap N}(s).$$

Therefore $X_{M\cap N}$ is neutrosophic $\aleph-$ interior ideal. □

Corollary 3.3. The arbitrary intersection of neutrosophic $\aleph-$ interior ideals is a neutrosophic $\aleph-$ interior ideal.

Theorem 3.4. If X_M and X_N are neutrosophic $\aleph-$ interior ideals, then $X_{M\cup N}$ is neutrosophic $\aleph-$ interior ideal.

Proof: Let X_M and X_N be neutrosophic $\aleph-$ interior ideals. For any $r, s, t \in X$, we have

$$T_{M\cup N}(rst) = T_M(rst) \wedge T_N(rst) \leq T_M(s) \wedge T_N(s) = T_{M\cup N}(s),$$
$$I_{M\cup N}(rst) = I_M(rst) \vee I_N(rst) \geq I_M(s) \vee I_N(s) = I_{M\cup N}(s),$$
$$F_{M\cup N}(rst) = F_M(rst) \wedge F_N(rst) \leq F_M(s) \wedge F_N(s) = F_{M\cup N}(s).$$

Therefore $X_{M\cup N}$ is neutrosophic $\aleph-$ interior ideal. □

Corollary 3.5. The arbitrary union of neutrosophic $\aleph-$ interior ideals is neutrosophic $\aleph-$ interior ideal.

Theorem 3.6. Let X be a regular semigroup. If X_M is neutrosophic $\aleph-$ interior ideal, then X_M is neutrosophic $\aleph-$ ideal.

Proof: Assume that X_M is an interior ideal, and let $u, v \in X$. As X is regular and $u \in X$, there exists $r \in X$ such that $u = uru$. Now, $T_M(uv) = T_M(uruv) \leq T_M(u)$, $I_M(uv) = I_M(uruv) \geq I_M(u)$ and $F_M(uv) = F_M(uruv) \leq F_M(u)$. Therefore X_M is neutrosophic $\aleph -$ right ideal.

Similarly, we can show that X_M is neutrosophic $\aleph -$ left ideal and hence X_M is neutrosophic $\aleph -$ ideal. □

Theorem 3.7. Let X be an intra-regular semigroup. If X_M is neutrosophic $\aleph -$ interior ideal, then X_M is neutrosophic $\aleph -$ ideal.

Proof: Suppose that X_M is neutrosophic $\aleph -$ interior ideal, and let $u, v \in X$. As X is intra regular and $u \in X$, there exist $s, t \in S$ such that $u = su^2t$. Now,

$$T_M(uv) = T_M(su^2tv) \leq T_M(u),$$
$$I_M(uv) = I_M(su^2tv) \geq I_M(u)$$
$$F_M(uv) = F_M(su^2tv) \leq F_M(u).$$

Therefore X_M is neutrosophic $\aleph -$ right ideal. similarly, we can show that X_M is neutrosophic $\aleph -$ left ideal and hence X_M is neutrosophic $\aleph -$ ideal. □

Definition 3.8. A semigroup X is left simple (resp., right simple) if it does not contain any proper left ideal (resp., right ideal) of X. A semigroup X is simple if it does not contain any proper ideal of X.

Definition 3.9. A semigroup X is said to be neutrosophic $\aleph -$simple if every neutrosophic $\aleph -$ ideal is a constant function

i.e., for every neutrosophic $\aleph -$ideal X_M of X, we have $T_M(i) = T_M(j)$, $I_M(i) = I_M(j)$ and $F_M(i) = F_M(j)$ for all $i, j \in X$.

Notation 3.10. If X is a semigroup and $s \in X$, we define a subset, denoted by I_s as follows:

$$I_s \coloneqq \{i \in X \mid T_N(i) \leq T_N(s),\ I_N(i) \geq I_N(s) \ \ and \ \ F_N(i) \leq F_N(s)\}.$$

Proposition 3.11. If X_N is neutrosophic $\aleph -$ right (resp., $\aleph -$ left, $\aleph -$ ideal) ideal, then I_s is right (resp., left, ideal) ideal for every $s \in X$.

Proof: Let $s \in X$. Then it is clear that $\varphi \neq \mathrm{I_s} \subseteq X$. Let $u \in \mathrm{I_s}$ and $x \in X$. Then $ux \in \mathrm{I_s}$. Indeed; Since X_N is neutrosophic $\aleph -$ right ideal and $u, x \in X$, we get $T_N(ux) \leq T_N(u)$, $I_N(ux) \geq I_N(u)$ and $F_N(ux) \leq F_N(t)$. Since $u \in \mathrm{I_s}$, we get $T_N(u) \leq T_N(s)$, $I_N(u) \geq I_N(s)$ and $F_N(u) \leq F_N(s)$ which imply $ux \in \mathrm{I_s}$. Therefore I_s is a right ideal for every $s \in X$. □

Theorem 3.12.[4] For any $L \subseteq X$, the equivalent assertions are:

(i) L is left (resp., right) ideal,

(ii) Characteristic neutrosophic $\aleph -$structure $\chi_{\mathrm{L}}(\mathrm{X_N})$ is neutrosophic $\aleph -$left (resp., right) ideal.

Theorem 3.13. Let X be a semigroup. Then X is simple if and only if X is neutrosophic $\aleph -$simple.

Proof: Suppose X is simple. Let X_M be a neutrosophic $\aleph-$ ideal and $u,v \in X$. Then by Proposition 3.11, I_u is an ideal of X. As X is simple, we have $I_u = X$. Since $v \in I_u$, we have $T_M(v) \leq T_M(u)$, $I_M(v) \geq I_M(u)$ and $F_M(v) \leq F_M(u)$.

Similarly, we can prove that $T_M(u) \leq T_M(v)$, $I_M(u) \geq I_M(v)$ and $F_M(u) \leq F_M(v)$. So $T_M(u) = T_M(v)$, $I_M(u) = I_M(v)$ and $F_M(u) = F_M(v)$. Hence X is neutrosophic $\aleph-$ simple.

Conversely, assume that X is neutrosophic $\aleph-$ simple and I is an ideal of X. Then by Theorem 3.12, $\chi_I(X_N)$ is a neutrosophic $\aleph-$ ideal. We now claim that $X = I$. Let $w \in X$. Since X is neutrosophic $\aleph-$ simple, we have $\chi_I(X_N)$ is a constant function and $\chi_I(X_N)(w) = \chi_I(X_N)(y)$ for every $y \in X$. In particular, we have $\chi_I(T_N)(w) = \chi_I(T_N)(d) = -1$, $\chi_I(I_N)(w) = \chi_I(I_N)(d) = 0$ and $\chi_I(F_N)(w) = \chi_I(F_N)(d) = -1$ for any $d \in I$ which implies $w \in I$. Thus $X \subseteq I$ and hence $X = I$. □

Lemma 3.14. Let X be a semigroup. Then X is simple if and only for every $t \in X$, we have $X = XtX$.

Proof: Suppose X is simple and let $t \in X$. Then $X(XtX) \subseteq XtX$ and $(XtX)X \subseteq XtX$ imply that XtX is an ideal. Since X is simple, we have $XtX = X$.

Conversely, let P be an ideal and let $a \in P$. Then $X = XaX$, $XaX \subseteq XPX \subseteq P$ which implies $P = X$. Therefore X is simple. □

Theorem 3.15. Suppose X is a semigroup. Then X is simple if and only every neutrosophic $\aleph-$ interior ideal of X is a constant function.

Proof: Suppose X is simple and $s,t \in X$. Let X_N be neutrosophic $\aleph-$ interior ideal. Then by Lemma 3.14, we get $X = XsX = XtX$. As $s \in XsX$, we have $s = atb$ for $a,b \in X$. Since X_N is neutrosophic $\aleph-$ interior ideal, we have $T_N(s) = T_N(atb) \leq T_N(t)$, $I_N(s) = I_N(atb) \geq I_N(t)$ and $F_N(s) = F_N(atb) \leq F_N(t)$. Similarly, we can prove that $T_N(t) \leq T_N(s)$, $I_N(t) \geq I_N(s)$ and $F_N(t) \leq F_N(s)$. So X_N is a constant function.

Conversely, suppose X_N is neutrosophic $\aleph-$ ideal. Then X_N is neutrosophic $\aleph-$ interior ideal. By hypothesis, X_N is a constant function and so X_N is neutrosophic $\aleph-$simple. By Theorem 3.13, X is simple. □

Theorem 3.16. Let X_M be neutrosophic $\aleph-$ structure and let $\gamma,\delta,\varepsilon \in [-1,0]$ with $-3 \leq \gamma+\delta+\varepsilon \leq 0$. If X_M is neutrosophic $\aleph-$interior ideal, then $(\gamma,\delta,\varepsilon)$-level set of X_M is neutrosophic $\aleph-$interior ideal whenever $X_M(\gamma,\delta,\varepsilon) \neq \emptyset$.

Proof: Suppose $\boldsymbol{X_M(\gamma,\delta,\varepsilon)} \neq \emptyset$ for $\boldsymbol{\gamma,\delta,\varepsilon} \in [\boldsymbol{-1,0}]$ with $\boldsymbol{-3 \leq \gamma+\delta+\varepsilon \leq 0}$.

Let $\boldsymbol{X_M}$ be a neutrosophic $\aleph-$interior ideal and let $\boldsymbol{u,v,w} \in \boldsymbol{X_M(\gamma,\delta,\varepsilon)}$. Then $\boldsymbol{T_M(uvw)} \leq \boldsymbol{T_M(v)} \leq \boldsymbol{\alpha}$; $\boldsymbol{I_M(uvw)} \geq \boldsymbol{I_M(v)} \geq \boldsymbol{\beta}$ and $\boldsymbol{F_M(uvw)} \leq \boldsymbol{F_M(v)} \leq \boldsymbol{\gamma}$ which imply $\boldsymbol{uvw} \in \boldsymbol{X_M(\alpha,\beta,\gamma)}$. Therefore $\boldsymbol{X_M(\gamma,\delta,\varepsilon)}$ is a neutrosophic $\aleph-$interior ideal of $\boldsymbol{X}$. □

Theorem 3.17. Let X_N be neutrosophic $\aleph-$ structure with $\alpha,\beta,\gamma \in [-1,0]$ such that $-3 \leq \alpha+\beta+\gamma \leq 0$. If T_N^{α}, I_N^{β} and F_N^{γ} are interior ideals, then X_N is neutrosophic $\aleph-$ interior ideal of X whenever it is non-empty.

Proof: Suppose that for $\boldsymbol{a,b,c} \in \boldsymbol{X}$ with $\boldsymbol{T_N(abc)} > \boldsymbol{T_N(b)}$. Then $\boldsymbol{T_N(abc)} > \boldsymbol{t_\alpha} \geq \boldsymbol{T_N(b)}$ for some $\boldsymbol{t_\alpha} \in [\boldsymbol{-1,0})$. So $\boldsymbol{b} \in \boldsymbol{T_N^{t_\alpha}(b)}$ but $\boldsymbol{abc} \notin \boldsymbol{T_N^{t_\alpha}(b)}$, a contradiction. Thus $\boldsymbol{T_N(abc)} \leq \boldsymbol{T_N(b)}$.

Suppose that for $a, b, c \in X$ with $I_N(abc) < I_N(b)$. Then $I_N(abc) < t_\alpha \leq I_N(b)$ for some $t_\alpha \in [-1, 0)$. So $b \in I_N^{t_\alpha}(b)$ but $abc \notin I_N^{t_\alpha}(b)$, a contradiction. Thus $I_N(abc) \geq I_N(b)$.

Suppose that for $a, b, c \in X$ with $F_N(abc) > F_N(b)$. Then $F_N(abc) > t_\alpha \geq F_N(b)$ for some $t_\alpha \in [-1, 0)$. So $b \in F_N^{t_\alpha}(b)$ but $abc \notin F_N^{t_\alpha}(b)$, a contradiction. Thus $F_N(abc) \leq F_N(b)$.

Thus X_N is neutrosophic $\aleph-$ interior ideal. □

Theorem 3.18. Let X_M be neutrosophic $\aleph-$ structure over X. Then the equivalent assertions are:

(i) X_M is neutrosophic $\aleph-$interior ideal,

(ii) $X_N \odot X_M \odot X_N \subseteq X_M$ for any neutrosophic $\aleph-$ structure X_N.

Proof: Suppose X_M is neutrosophic $\aleph-$ interior ideal. Let $x \in X$. For any $u, v, w \in X$ such that $x = uvw$. Then $T_M(x) = T_M(uvw) \leq T_M(v) \leq T_N(u) \vee T_M(v) \vee T_N(w)$ which implies $T_M(x) \leq T_{N\circ M\circ N}(x)$. Otherwise $x \neq uvw$. Then $T_M(x) \leq 0 = T_{N\circ M\circ N}(x)$. Similarly, we can prove that $I_M(x) \geq I_{N\circ M\circ N}(x)$ and $F_M(x) \leq F_{N\circ M\circ N}(x)$. Thus $X_N \odot X_M \odot X_N \subseteq X_M$.

Conversely, assume that $X_N \odot X_M \odot X_N \subseteq X_M$ for any neutrosophic $\aleph-$structure X_N.

Let $u, v, w \in X$. If $x = uvw$, then

$$T_M(uvw) = T_M(x) \leq (\chi_X(T)_N \circ T_M \circ \chi_X(T)_N)(x) = \bigwedge_{x=rw} \{\chi_X(T)_N \circ T_M)(r) \vee \chi_X(T)_N(w)\}$$

$$= \bigwedge_{x=rc} \{ \bigwedge_{r=uv} \{\chi_X(T)_N(u) \vee (T)_M(v)\} \vee \chi_X(T)_N(w)\}$$

$$\leq \chi_X(T)_N(u) \vee (T)_M(v) \vee \chi_X(T)_N(w) = T_M(v),$$

$$I_M(uvw) = I_M(x) \leq (\chi_X(I)_N \circ I_M \circ \chi_X(I)_N)(x) = \bigvee_{x=rw} \{\chi_X(I)_N \circ I_M)(r) \wedge \chi_X(I)_N(w)\}$$

$$= \bigvee_{x=rc} \{ \bigvee_{r=uv} \{\chi_X(I)_N(u) \wedge (I)_M(v)\} \wedge \chi_X(I)_N(w)\}$$

$$\geq \chi_X(I)_N(u) \wedge (I)_M(v) \wedge \chi_X(I)_N(w) = (I)_M(v),$$

and

$$F_M(uvw) = F_M(x) \leq (\chi_X(F)_N \circ F_M \circ \chi_X(F)_N)(x) = \bigwedge_{x=rw} \{\chi_X(F)_N \circ F_M)(r) \vee \chi_X(F)_N(w)\}$$

$$= \bigwedge_{x=rc} \{ \bigwedge_{r=uv} \{\chi_X(F)_N(u) \vee (F)_M(v)\} \vee \chi_X(F)_N(w)\}$$

$$\leq \chi_X(F)_N(u) \vee (F)_M(v) \vee \chi_X(F)_N(w) = F_M(v).$$

Therefore X_M is neutrosophic $\aleph-$interior ideal. □

Notation 3.19. Let X and Z be semigroups. A mapping $g: X \rightarrow Z$ is said to be a homomorphism if $g(uv) = g(u)g(v)$ for all $u, v \in X$. Throughout this remaining section, we denote $Aut(X)$, the set of all automorphisms of X.

Definition 3.20. An interior ideal J of a semigroup X is called a characteristic interior ideal if $h(J) = J$ for all $h \in Aut(X)$.

Definition 3.21. Let $\boldsymbol{X}$ be a semigroup. A neutrosophic $\aleph-$ interior ideal $\boldsymbol{X_N}$ is called neutrosophic $\aleph-$ characteristic interior ideal if $\boldsymbol{T_N(h(u)) = T_N(u)}$, $\boldsymbol{I_N(h(u)) = I_N(u)}$ and $\boldsymbol{F_N(h(u)) = F_N(u)}$ for all $\boldsymbol{u \in X}$ and all $\boldsymbol{h \in Aut(X)}$.

Theorem 3.22. For any $\mathrm{L} \subseteq \mathrm{X}$, the equivalent assertions are:

(i) L is characteristic interior ideal,

(ii) The characteristic neutrosophic $\aleph-$structure $\chi_{\mathrm{L}}(\mathrm{X_M})$ is neutrosophic $\aleph-$characteristic interior ideal.

Proof: Suppose L is characteristic interior ideal and let $x \in X$. Then by Theorem 3.1, $\chi_{\mathrm{L}}(\mathrm{X_M})$ is neutrosophic $\aleph-$interior ideal. If $x \in L$, then $\chi_L(T)_M(x) = -1$, $\chi_L(I)_M(x) = 0$, and $\chi_L(F)_M(x) = -1$. Now, for any $h \in Aut(X)$, $h(x) \in h(L) = L$ which implies $\chi_L(T)_M(h(x)) = -1, \chi_L(I)_M(h(x)) = 0$, and $\chi_L(F)_M(h(x)) = -1$. If $x \notin L$, then $\chi_L(T)_M(x) = 0, \chi_L(I)_M(x) = -1$, and $\chi_L(F)_M(x) = 0$. Now, for any $h \in Aut(X)$, $h(x) \notin h(L)$ which implies $\chi_L(T)_M(h(x)) = 0, \chi_L(I)_M(h(x)) = -1$, and $\chi_L(F)_M(h(x)) = 0$. Thus $\chi_L(T)_M(h(x)) = \chi_L(T)_M(x)$, $\chi_L(I)_M(h(x)) = \chi_L(I)_M(x)$, and $\chi_L(F)_M(h(x)) = \chi_L(F)_M(x)$ for all $x \in X$ and hence $\chi_{\mathrm{L}}(\mathrm{X_M})$ is neutrosophic $\aleph-$ characteristic interior ideal.

Conversely, assume that $\chi_{\mathrm{L}}(\mathrm{X_M})$ is neutrosophic $\aleph-$characteristic interior ideal. Then by Theorem 3.1, L is an interior ideal. Now, let $h \in Aut(X)$ and $x \in L$. Then $\chi_L(T)_M(x) = -1$, $\chi_L(I)_M(x) = 0$ and $\chi_L(F)_M(x) = -1$. Since $\chi_{\mathrm{L}}(\mathrm{X_M})$ is neutrosophic $\aleph-$characteristic interior ideal , we have $\chi_L(T)_M(h(x)) = \chi_L(T)_M(x)$, $\chi_L(I)_M(h(x)) = \chi_L(I)_M(x)$ and $\chi_L(F)_M(h(x)) = \chi_L(T)_M(x)$ which imply $h(x) \in L$. So $h(L) \subseteq L$ for all $h \in Aut(X)$. Again, since $h \in Aut(X)$ and $x \in L$, there exists $y \in L$ such that $h(y) = x$.

Suppose that $y \notin L$. Then $\chi_L(T)_M(y) = 0$, $\chi_L(I)_M(y) = -1$ and $\chi_L(F)_M(y) = 0$. Since $\chi_L(T)_M(h(y)) = \chi_L(T)_M(y)$, $\chi_L(I)_M(h(y)) = \chi_L(I)_M(y)$ and $\chi_L(F)_M(h(y)) = \chi_L(T)_M(y)$, we get $\chi_L(T)_M(h(y)) = 0$, $\chi_L(I)_M(h(y)) = -1$ and $\chi_L(F)_M(h(y)) = 0$ which imply $h(y) \notin L$, a contradiction. So $y \in L$ i.e., $h(y) \in L$. Thus $L \subseteq h(L)$ for all $h \in Aut(X)$ and hence L is characteristic interior ideal. □

Theorem 3.23. For a semigroup X, the equivalent statements are:

(i) X is intra-regular,

(ii) For any neutrosophic $\aleph-$interior ideal X_M, we have $X_M(w) = X_M(w^2)$ for all $w \in X$.

Proof: $(i) \Rightarrow (ii)$ Suppose X is intra-regular, and X_M is neutrosophic $\aleph-$ interior ideal and $w \in X$. Then there exist $r, s \in X$ such that $w = rw^2s$. Now $T_M(w) = T_M(rw^2s) \leq T_M(w^2) \leq T_M(w)$ and so $T_M(w) = T_M(w^2)$, $I_M(w) = I_M(rw^2s) \geq I_M(w^2) \geq I_M(w)$ and so $I_M(w) = I_M(w^2)$, and $F_M(w) = F_M(rw^2s) \leq F_M(w^2) \leq F_M(w)$ and so $F_M(w) = F_M(w^2)$. Therefore $X_M(w) = X_M(w^2)$ for all $w \in X$.

$(ii) \Rightarrow (i)$ Let (ii) holds and $s \in X$. Then $I(s^2)$ is an ideal of X. By Theorem 3.5 of [4], $\chi_{I(s^2)}(X_M)$ is neutrosophic $\aleph-$ideal. By assumption, $\chi_{I(s^2)}(X_M)(s) = \chi_{I(s^2)}(X_M)(s^2)$. Since $\chi_{I(s^2)}(T)_M(s^2) = -1 = \chi_{I(s^2)}(F)_M(s^2)$ and $\chi_{I(s^2)}(I)_M(s^2) = 0$, we get $\chi_{I(s^2)}(T)_M(s) = -1 = \chi_{I(s^2)}(F)_M(s)$ and $\chi_{I(s^2)}(I)_M(s^2) = 0$ which imply $s \in I(s^2)$. Hence X is intra-regular. □

Theorem 3.24. For a semigroup X, the equivalent statements are:

(i) X is left (resp., right) regular,

(ii) For any neutrosophic $\aleph$ −interior ideal X_M, we have $X_M(w) = X_M(w^2)$ for all $w \in X$.

Proof: $(i) \Rightarrow (ii)$ Let X be left regular. Then there exists $y \in X$ such that $w = yw^2$. Let X_M be a neutrosophic $\aleph$ −interior ideal. Then $T_M(w) = T_M(yw^2) \leq T_M(w)$ and so $T_M(w) = T_M(w^2)$, $I_M(w) = I_M(yw^2) \geq I_M(w)$ and so $I_M(w) = I_M(w^2)$, and $F_M(w) = F_M(yw^2) \leq F_M(w)$ and so $F_M(w) = F_M(w^2)$. Therefore $X_M(w) = X_M(w^2)$ for all $w \in X$.

$(ii) \Rightarrow (i)$ Suppose (ii) holds and let X_M be neutrosophic $\aleph$ −interior ideal. Then for any $w \in X$, $\chi_{L(w^2)}(T)_M(w) = \chi_{L(w^2)}(T)_M(w^2) = -1$, $\chi_{L(w^2)}(I)_M(w) = \chi_{L(w^2)}(I)_M(w^2) = 0$ and $\chi_{L(w^2)}(F)_M(w) = \chi_{L(w^2)}(F)_M(w^2) = -1$ which imply $w \in L(w^2)$. Thus X is left regular. □

Conclusions

In this paper, we have introduced the concepts of neutrosophic $\aleph -$ interior ideals and neutrosophic $\aleph -$ characteristic interior ideals in semigroups and studied their properties, and characterized regular and intra-regular semigroups using neutrosophic $\aleph$-interior ideal structures. We have also shown that R is a characteristic interior ideal if and only if the characteristic neutrosophic $\aleph$ −structure $\chi_R(X_N)$ is neutrosophic $\aleph$ −characteristic interior ideal. In future, we will define neutrosophic $\aleph$ −prime ideals in semigroups and study their properties.

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Introduction Totally and Partial Connectivity Indices in Neutrosophic graphs with Application in Behavioral Sciences

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Abstract: Connectivity is one of the most important concepts in graph theory. Since Neutrosophic Graphs are a branch of graphs, connectivity will be very important in this branch as well. In this paper, we will define the connectivity in Neutrosophic graphs using the strength of connectedness between each pair of its vertices. Also in this article, we define two new concepts of Partial connectivity index and totally connectivity index. We present several theorems related to these concepts and prove the theorems.

Keywords: neutrosophic graphs; partial connectivity index; totally connectivity index; m-barbell graph; connected neutrosophic graph

1. Introduction

Neutrosophic graphs are a new branch of graphs that has been very popular among graph theorists in recent decades. Neutrosophic graphs are a generalized form of fuzzy graph theory. One of the features that have been considered in fuzzy graphs is connectivity and types of connectivity indices in fuzzy graphs [7]. The connectivity index is a numerical quantity that can be used to calculate some of the properties of the studied graph in more detail. Many researchers have pointed to different uses of neutrosophic Graphs, such as the use of neutrosophic sets and graphs in medicine [3], social media [4], decision-making problem [9], Economics Theorizing [11] and so on. In this article, after introducing the partial connectivity index and totally connectivity index in neutrosophic graphs, we will point out some applications of it.

In our previous article [8], we also presented the correlation index in neutrosophic graphs and gave an example of its applications. In the following works, we will compare and examine the strengths and weaknesses of each.

2. Preliminaries

In this section, some of the important and basic concepts required are given by mentioning the source.

Definition 1. [4] A single-valued neutrosophic graph on a nonempty V is a pair $G=(N,M)$. Where N is single-valued neutrosophic set in V and M single-valued neutrosophic relation on V such that

$$T_M(uv) \leq \min\{T_N(u),T_N(v)\},$$
$$I_M(uv) \leq \min\{I_N(u),I_N(v)\},$$
$$F_M(uv) \leq \max\{F_N(u),F_N(v)\},$$

For all $u,v \in V$. N is called single-valued neutrosophic vertex set of G and, M is called single-valued neutrosophic edge set of G, respectively.

Definition 2. [4] Let $G = (N, M)$ be the Neutrosophic Graph of G^*. If $H = (N', M')$ is a neutrosophic graph of G^* such that

$$T'(u) \leq T(u), \quad I'(u) \geq I(u), \quad F'(u) \geq F(u), \quad \forall u \in X,$$
$$T_{M'}(uv) \leq T_M(uv), \quad I_{M'}(uv) \geq I_M(uv), \quad F_{M'}(uv) \geq F_M(uv), \quad \forall uv \in E,$$

Then H is called a Neutrosophic subgraph of the Neutrosophic graph G.

Definition 3. [4] A neutrosophic graph $G = (N, M)$ is called complete if the following conditions are satisfied:

$$T_M(uv) = \min\{T_N(u), T_N(v)\},$$
$$I_M(uv) = \min\{I_N(u), I_N(v)\},$$
$$F_M(uv) = \max\{F_N(u), F_N(v)\},$$

For all $u, v \in V$.

Definition 4. [4] A neutrosophic graph $G_1 = (N_1, M_1)$ of the graph $G_1^* = (V_1, E_1)$ is isomorphic with neutrosophic graph $G_2 = (N_2, M_2)$ of the graph $G_2^* = (V_2, E_2)$ if we have f where $f: V_1 \to V_2$ is a bijection and following relations are satisfied

$$T_{N_1}(u) = T_{N_2}(f(u)), \quad I_{N_1}(u) = I_{N_2}(f(u)), \quad F_{N_1}(u) = F_{N_2}(f(u)),$$

For all $u \in V_1$ and

$$T_{M_1}(uv) = T_{M_2}(f(u)f(v)), \quad I_{M_1}(uv) = I_{M_2}(f(u)f(v)), \quad F_{M_1}(uv) = F_{M_2}(f(u)f(v)),$$

For all $uv \in E_1$.

Definition 5. [4] the m-barbell graph $B_{(m,m)}$ is the simple graph obtained by connecting two copies of a complete graph K_m by abridge.

3. Totally and Partial connectivity index

In this section, which is the main part of the article, we first define the connected neutrosophic graph and connectivity index in the neutrosophic graphs. Note that definitions are provided for a connected neutrosophic graph in some references [5, 6], but the definition we use here will be based on connectivity. After providing some examples, the theorems related to the connectivity index are expressed and proved in neutrosophic graphs.

3.1. Partial connectivity index in neutrosophic graphs

Here we first define the Partial and totally connectivity indices in neutrosophic graphs and provide examples to better understand it. And then in the next part we will present the boundaries for the Partial and totally connectivity indices in neutrosophic graphs.

Definition 6. Let $G = (N, M)$ be the connected Neutrosophic Graph. The **partial connectivity index** of G is defined as

$$PCI_T(G) = \sum_{u,v \in N} T_N(u)T_N(v)CONN_{T_G}(u, v),$$
$$PCI_I(G) = \sum_{u,v \in N} I_N(u)I_N(v)CONN_{I_G}(u, v),$$
$$PCI_F(G) = \sum_{u,v \in N} F_N(u)F_N(v)CONN_{F_G}(u, v),$$

Where $CONN_{T_G}(u,v)$ is the strength of truth, $CONN_{I_G}(u,v)$ the strength of indeterminacy and $CONN_{F_G}(u,v)$ the strength of falsity between two vertices u and v. We have

$$CONN_{T_G}(u,v) = \max\{\min T_M(e) \mid e \in P \text{ } and \text{ } P \text{ } is \text{ } a \text{ } path \text{ } between \text{ } u \text{ } and \text{ } v\},$$
$$CONN_{I_G}(u,v) = \min\{\max I_M(e) \mid e \in P \text{ } and \text{ } P \text{ } is \text{ } a \text{ } path \text{ } between \text{ } u \text{ } and \text{ } v\},$$
$$CONN_{F_G}(u,v) = \min\{\max F_M(e) \mid e \in P \text{ } and \text{ } P \text{ } is \text{ } a \text{ } path \text{ } between \text{ } u \text{ } and \text{ } v\}.$$

Also, the **totally connectivity index** of G is defined as

$$TCI(G) = \frac{4 + 2PCI_T(G) - 2PCI_F(G) - PCI_I(G)}{6}.$$

Definition 7. Let $G = (N, M)$ be the Neutrosophic graph. G called a **connected** neutrosophic graph if for any two vertices $u, v \in N$, $CONN_{T_G}(u,v) > 0$, $CONN_{I_G}(u,v) > 0$, and $CONN_{F_G}(u,v) > 0$.

Example 1. Consider the Neutrosophic graph $G = (N, M)$ with $V = \{a, b, c, d\}$, that shown in figure 1. As can be seen, $(T_N, I_N, F_N)(a) = (0.4, 0.6, 0.5)$, $(T_N, I_N, F_N)(b) = (0.7, 0.5, 0.4)$, $(T_N, I_N, F_N)(c) = (0.7, 0.4, 0.3)$, and $(T_N, I_N, F_N)(d) = (0.5, 0.4, 0.5)$, The edge set contains $(T_M, I_M, F_M)(a,b) = (0.4, 0.5, 0.5)$, $(T_M, I_M, F_M)(b,c) = (0.7, 0.4, 0.4)$, $(T_M, I_M, F_M)(c,d) = (0.5, 0.4, 0.5)$, $(T_M, I_M, F_M)(a,d) = (0.4, 0.4, 0.5)$ and $(T_M, I_M, F_M)(b,d) = (0.3, 0.5, 0.7)$.
By direct calculations, we have

Table 1. The strength of connectedness between each pair of vertices u and v.

	$CONN_{T_G}(u,v)$	$CONN_{I_G}(u,v)$	$CONN_{F_G}(u,v)$
a, b	0.4	0.5	0.5
a, c	0.4	0.4	0.5
a, d	0.4	0.4	0.5
b, c	0.7	0.4	0.4
b, d	0.5	0.4	0.5
c, d	0.5	0.4	0.5

Then the partial connectivity index of G is,

$$\begin{aligned} PCI_T(G) = \sum_{u,v \in N} T_N(u)T_N(v)CONN_{T_G}(u,v) \\ = (0.4)(0.7)(0.4) + (0.4)(0.7)(0.4) + (0.4)(0.5)(0.4) + (0.7)(0.7)(0.7) + (0.7)(0.5)(0.5) \\ + (0.7)(0.5)(0.5) = 0.112 + 0.112 + 0.080 + 0.147 + 0.245 + 0.245 = 0.941, \end{aligned}$$

$$\begin{aligned} PCI_I(G) = \sum_{u,v \in N} I_N(u)I_N(v)CONN_{I_G}(u,v) \\ = (0.6)(0.5)(0.5) + (0.6)(0.4)(0.4) + (0.6)(0.4)(0.4) + (0.5)(0.4)(0.4) + (0.5)(0.4)(0.4) \\ + (0.4)(0.4)(0.4) = 0.180 + 0.096 + 0.096 + 0.080 + 0.080 + 0.064 = 0.596, \end{aligned}$$

$$\begin{aligned} PCI_F(G) = \sum_{u,v \in N} F_N(u)F_N(v)CONN_{F_G}(u,v) \\ = (0.5)(0.4)(0.5) + (0.5)(0.3)(0.5) + (0.5)(0.5)(0.5) + (0.4)(0.3)(0.4) + (0.4)(0.5)(0.5) \\ + (0.3)(0.5)(0.5) = 0.1 + 0.075 + 0.125 + 0.048 + 0.1 + 0.075 = 0.523. \end{aligned}$$

Also by definition 1, we have

$$TCI(G) = \frac{4 + 2PCI_T(G) - 2PCI_F(G) - PCI_I(G)}{6} = \frac{4 + 2(0.941) - 2(0.523) - 0.596}{6} = 0.707.$$

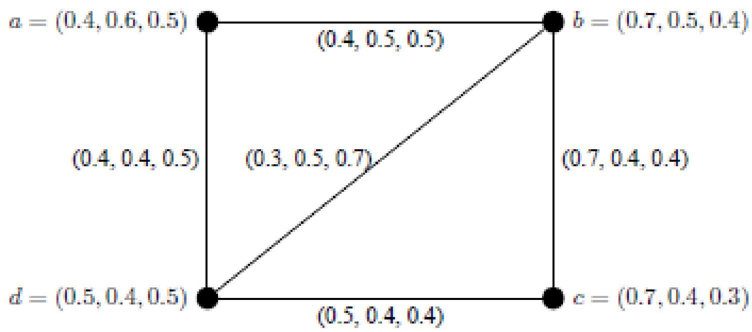

Figure 1. A neutrosophic graph with $V = \{a, b, c, d\}$

Theorem 1. Let $G = (N, M)$ be a connected neutrosophic graph and $H = (N', M')$ is a partial neutrosophic subgraph of G. then

$$PCI_T(H) \leq PCI_T(G),$$
$$PCI_I(H) \geq PCI_I(G),$$
$$PCI_F(H) \geq PCI_F(G),$$

Moreover, we have $TCI(H) \leq TCI(G)$.

Proof. Let $H = (N', M')$ is a partial neutrosophic subgraph of G, and $T_{N'}(u) \leq T_N(u)$ for $u \in V$. Since $T_{M'}(uv) \leq T_M(uv)$ for uv, then $CONN_{TH}(u, v) \leq CONN_{TG}(u, v)$ thus we get

$$PCI_T(H) = \sum_{u,v \in X} T_{N'}(u)T_{N'}(v)CONN_{TH}(u, v) \leq \sum_{u,v \in X} T_N(u)T_N(v)CONN_{TG}(u, v) = PCI_T(G).$$

Using a similar proof, we can show that

$$PCI_I(H) = \sum_{u,v \in X} I_{N'}(u)I_{N'}(v)CONN_{IH}(u, v) \geq \sum_{u,v \in X} I_N(u)I_N(v)CONN_{IG}(u, v) = PCI_I(G),$$

And

$$PCI_F(H) = \sum_{u,v \in X} F_{N'}(u)F_{N'}(v)CONN_{FH}(u, v) \geq \sum_{u,v \in X} F_N(u)F_N(v)CONN_{FG}(u, v) = PCI_F(G).$$

Now, we show that

$$TCI(H) \leq TCI(G).$$

By definition totally connectivity index, and since $PCI_T(H) \leq PCI_T(G), PCI_I(H) \geq PCI_I(G), PCI_F(H) \geq PCI_F(G)$, we have

$$TCI(H) = \frac{4 + 2PCI_T(H) - 2PCI_F(H) - PCI_I(H)}{6} \leq \frac{4 + 2PCI_T(G) - 2PCI_F(H) - PCI_I(H)}{6}$$
$$\leq \frac{4 + 2PCI_T(G) - 2PCI_F(G) - PCI_I(G)}{6} = TCI(G),$$

And, hence $TCI(H) \leq TCI(G)$.
□

Example 2. Consider the neutrosophic graph $G = (N,\ M)$ whit

$$N = \{(a, 0.7, 0.3, 0.4), (b, 0.5, 0.2, 0.3), (c, 0.7, 0.3, 0.6), (d, 0.4, 0.3, 0.5)\},$$

And

$$M = \{(ab, 0.5, 0.2, 0.4), (ac, 0.7, 0.3, 0.6), (bc, 0.5, 0.2, 0.6), (cd, 0.4, 0.3, 0.6)\}.$$

Also, let $H = (N', M')$ be a neutrosophic subgraph of G, whit

$$N' = \{(a, 0.6, 0.3, 0.5), (b, 0.4, 0.2, 0.4), (c, 0.6, 0.3, 0.7), (d, 0.3, 0.3, 0.6)\},$$

And

$$M' = \{(ab, 0.4, 0.2, 0.5), (ac, 0.5, 0.3, 0.7), (bc, 0.4, 0.2, 0.7), (cd, 0.3, 0.3, 0.7)\}.$$

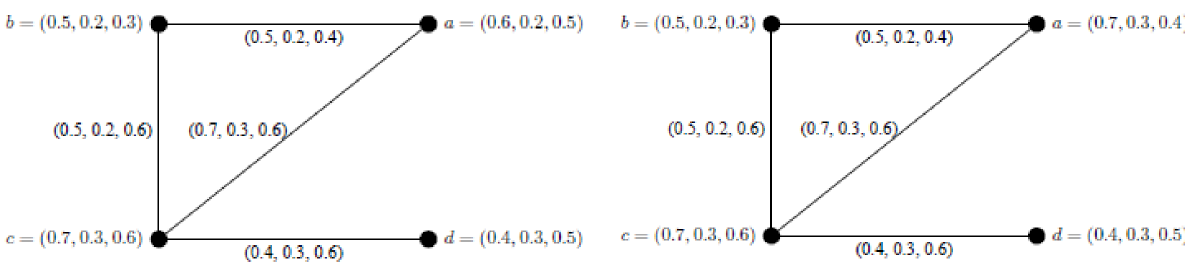

Figure 2. The neutrosophic graph G and the neutrosophic subgraph of G

By direct calculations, we have

$$PCI_T(G) = 0.997, \qquad PCI_I(G) = 0.120, \qquad PCI_F(G) = 0.690,$$

And

$$PCI_T(H) = 0.516, \qquad PCI_I(H) = 0.120, \qquad PCI_F(H) = 1.213.$$

Moreover

$$TCI(G) = \frac{4 + 2PCI_T(G) - 2PCI_F(G) - PCI_I(G)}{6} = \frac{4 + 2(0.997) - 2(0.690) - 0.120}{6} = 0.749.$$

$$TCI(H) = \frac{4 + 2PCI_T(H) - 2PCI_F(H) - PCI_I(H)}{6} = \frac{4 + 2(0.516) - 2(1.213) - 0.120}{6} = 0.622.$$

It is easy to see that $TCI(H) = 0.622 \leq TCI(G) = 0.749$.

Note 1. Note that if $H = (N', M')$ is a partial neutrosophic subgraph of $G = (N, M)$ such that $N' = N \backslash \{v\}$ then $PCI_T(H) < PCI_T(G),\ PCI_I(H) < PCI_{I|}(G),\ PCI_F(H) < PCI_F(G)$.

Theorem 2. Let $G_1 = (N_1, M_1)$ be isomorphic with $G_2 = (N_2, M_2)$. Then all of the following equation are established.

$$PCI_T(G_1) = PCI_T(G_2),$$
$$PCI_I(G_1) = PCI_I(G_2),$$
$$PCI_F(G_1) = PCI_F(G_2),$$

Also, we have $TCI(G_1) = TCI(G_2)$.

Proof. Let $G_1 = (N_1, M_1)$ be isomorphic with $G_2 = (N_2, M_2)$, and $f: V_1 \rightarrow V_2$ be the bijection from V_1 to V_2 such that

$$T_{N_1}(u) = T_{N_2}\big(f(u)\big), \qquad I_{N_1}(u) = I_{N_2}\big(f(u)\big), \qquad F_{N_1}(u) = F_{N_2}\big(f(u)\big),$$

For all $u \in V_1$, and

$$T_{M_1}(uv) = T_{M_2}\big(f(u)f(v)\big), \quad I_{M_1}(uv) = I_{M_2}\big(f(u)f(v)\big), \quad F_{M_1}(uv) = F_{M_2}\big(f(u)f(v)\big),$$

For all $uv \in E_1$. Since G_1 isomorphic with G_2, the strength of any strongest path between u and v in G_1 is equal to that between $f(u)$ and $f(v)$ in G_2. Hence

$$CONN_{T_{G_1}}(u,v) = CONN_{T_{G_2}}\big(f(u), f(v)\big), \qquad CONN_{I_{G_1}}(u,v) = CONN_{I_{G_2}}\big(f(u), f(v)\big),$$
$$CONN_{F_{G_1}}(u,v) = CONN_{F_{G_2}}(f(u), f(v)),$$

For $u, v \in N_1^*$. Therefore

$$PCI_T(G_1) = PCI_T(G_2), \qquad PCI_I(G_1) = PCI_I(G_2), \qquad PCI_F(G_1) = PCI_F(G_2),$$

And

$$TCI(G_1) = \frac{4 + 2PCI_T(G_1) - 2PCI_F(G_1) - PCI_I(G_1)}{6} = \frac{4 + 2PCI_T(G_2) - 2PCI_F(G_2) - PCI_I(G_2)}{6} = TCI(G_2).$$

□

Theorem 3. Let $G = (N, M)$ be a complete neutrosophic graph whit $V = \{v_1, v_2, \dots, v_n\}$ such that $t_1 \leq t_2 \leq \cdots \leq t_n$, $i_1 \leq i_2 \leq \cdots \leq i_n$ and $f_1 \geq f_2 \geq \cdots \geq f_n$ where $t_j = T_N\big(v_j\big)$, $i_j = I_N\big(v_j\big)$ and $f_j = F_N\big(v_j\big)$ for $j = 1, 2, \dots, n$. Then

$$PCI_T(G) = \sum_{j=1}^{n-1} t_j^2 \sum_{k=j+1}^{n} t_k,$$
$$PCI_I(G) = \sum_{j=1}^{n-1} i_j^2 \sum_{k=j+1}^{n} i_k,$$
$$PCI_F(G) = \sum_{j=1}^{n-1} f_j^2 \sum_{k=j+1}^{n} f_k.$$

Proof. Suppose v_1 is a vertex with the least Truth-membership value t_1. In a complete neutrosophic graph, $CONN_{TG}(u,v) = T_M(u,v)$ for all $u, v \in V$. Therefore $T_M(v_1 v_k) = t_1$ for $k = 2, 3, \dots, n$ and hence $T_N(v_1)T_N(v_k)CONN_{TG}(v_1, v_k) = t_1^2 t_k$ for $k = 2, 3, \dots, n$. Then for v_1, we have

$$\sum_{k=2}^{n} T_N(v_1)T_N(v_k)CONN_{TG}(v_1, v_k) = \sum_{k=2}^{n} t_1^2 t_k.$$

For v_2, $T_N(v_2)T_N(v_k)CONN_{TG}(v_2, v_k) = t_2^2 t_k$ for $k = 3, 4, \dots, n$.

$$\sum_{k=3}^{n} T_N(v_2)T_N(v_k)CONN_{TG}(v_2, v_k) = \sum_{k=3}^{n} t_2^2 t_k,$$

For v_{n-2}, $T_N(v_{n-2})T_N(v_k)CONN_{TG}(v_{n-2}, v_k) = t_{n-2}^2 t_k$ for $k = n-1, n$.

For v_{n-1}, $T_N(v_{n-1})T_N(v_k)CONN_{TG}(v_{n-1}, v_k) = t_{n-1}^2 t_j$ for $k = n$.

Thus, by summing over v_j, $j = 1, 2, 3, \dots, n-1$, we get

$$PCI_T(G) = \sum_{k=2}^{n} t_1^2 t_k + \sum_{k=3}^{n} t_2^2 t_k + \cdots + \sum_{k=n-1}^{n} t_{n-2}^2 t_k + \sum_{k=n}^{n} t_{n-1}^2 t_k = \sum_{j=1}^{n-1} t_j^2 \sum_{k=j+1}^{n} t_k.$$

Using the same argument, we can prove the other two cases.
□

Theorem 4. Let $G = (N, M)$ be a neutrosophic graph whit $V = \{v_1, v_2, \dots, v_n\}$ such that $G^* = (V, E)$ is a complete bipartite graph and $T_M(uv) = \min\{T_N(u), T_N(v)\}$, $I_M(uv) = \min\{I_N(u), I_N(v)\}$, $F_M(uv) = \max\{F_N(u), F_N(v)\}$ For all $u, v \in V$. Also, $V_1 = \{v_1, v_2, \dots, v_m\}$, and $V_2 = \{v_{m+1}, v_{m+2}, \dots, v_n\}$ whit $t_1 \leq t_2 \leq \cdots \leq t_n$, $i_1 \leq i_2 \leq \cdots \leq i_n$, and $f_1 \geq f_2 \geq \cdots \geq f_n$ where $t_j = T_N(v_j)$, $i_j = I_N(v_j)$ and $f_j = F_N(v_j)$ for $j = 1, 2, \dots, n$. Then

$$PCI_T(G) = \sum_{j=1}^{m} t_j^2 \sum_{k=j+1}^{n} t_k + t_m \sum_{j=m+1}^{n-1} t_j \sum_{k=j+1}^{n} t_k,$$
$$PCI_I(G) = \sum_{j=1}^{m} i_j^2 \sum_{k=j+1}^{n} i_k + i_m \sum_{j=m+1}^{n-1} i_j \sum_{k=j+1}^{n} i_k$$
$$PCI_F(G) = \sum_{j=1}^{m} f_j^2 \sum_{k=j+1}^{n} f_k + f_m \sum_{j=m+1}^{n-1} f_j \sum_{k=j+1}^{n} f_k.$$

Proof. Let $G = (N, M)$ be a neutrosophic graph whit $V = \{v_1, v_2, \dots, v_n\}$ and $G^* = K_{m,n}$, such that $t_1 \leq t_2 \leq \cdots \leq t_n$, $i_1 \leq i_2 \leq \cdots \leq i_n$ and $f_1 \geq f_2 \geq \cdots \geq f_n$.
Here we prove $PCI_F(G)$, states $PCI_T(G)$ and $PCI_I(G)$ are similarly proved.
Using definition, we have

$$PCI_F(G) = \sum_{v_j, v_k \in V} F_N(v_j)F_N(v_k)CONN_{FG}(v_j, v_k).$$

Too, for $v_1, v_k \in V$, we have

$$CONN_{FG}(v_1, v_k) = \min\{\max\{f_1\}, \max\{f_1, f_2\}, \dots, \max\{f_1, f_m\}\} = \min\{f_1, f_1, \dots, f_1\} = f_1.$$

Accordingly for $v_1, v_k \in V$

$$\sum_{\substack{v_k \neq v_1 \\ v_k \in V}} F_N(v_1)F_N(v_k)CONN_{FG}(v_1, v_k) = f_1 f_1 \sum_{k=2}^{n} f_k.$$

Similarly, for $v_j, v_k \in V$ $j = 2, 3, \dots, m$

$$\sum_{k=j+1} F_N(v_j)F_N(v_k)CONN_{FG}(v_j, v_k) = f_j f_j \sum_{k=j+1}^{n} f_k.$$

On the other hand, we have for $m < j < n$

$$\sum_{k=j+1} F_N(v_j)F_N(v_k)CONN_{FG}(v_j, v_k) = f_m f_j \sum_{k=j+1}^{n} f_k,$$

Then

$$PCI_F(G) = \sum_{v_j,v_k\in V} F_N(v_j)F_N(v_k)CONN_{FG}(v_j,v_k)$$
$$= f_1f_1\sum_{k=2}^{n} f_k + f_2f_2\sum_{k=3}^{n} f_k + \cdots + f_mf_m\sum_{k=m+1}^{n} f_k + f_mf_{m+1}\sum_{k=m+2}^{n} f_k + \cdots + f_mf_{n-1}f_n$$
$$= \sum_{j=1}^{m} f_j^2 \sum_{k=j+1}^{n} f_k + f_m \sum_{j=m+1}^{n-1} f_j \sum_{k=j+1}^{n} f_k.$$

□

Note 2. Clearly, in the above theorem it is enough to have

$$\forall\, v \in V_1\ \forall u \in V_2, \qquad T_N(v) \le T_N(u), \qquad I_N(v) \ge I_N(u), \qquad F_N(v) \ge F_N(u).$$

Then the case will be established. In the following example you can see the correctness of this claim.

Example 3. Consider the neutrosophic graph $G = (N, M)$ whit

$$N = \{(a, 0.2, 0.6, 0.7), (b, 0.4, 0.6, 0.5), (c, 0.7, 0.5, 0.4), (d, 0.5, 0.3, 0.5), (e, 0.6, 0.4, 0.5)\},$$

And

$$M = \{(ac, 0.2, 0.6, 0.7), (ad, 0.2, 0.6, 0.7), (ae, 0.2, 0.6, 0.7),$$
$$(bc, 0.4, 0.6, 0.5), (bd, 0.4, 0.6, 0.5), (be, 0.4, 0.6, 0.5)\}.$$

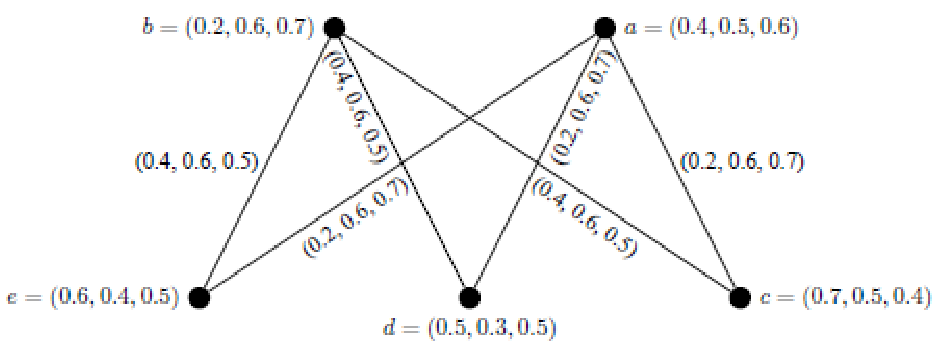

Figure 3. A complete bipartite neutrosophic graph whit $G^* = K_{2,3}$

By direct calculation, we have

$$CONN_{T_G}(a,b) = CONN_{T_G}(a,c) = CONN_{T_G}(a,d) = CONN_{T_G}(a,e) = 0.2 = T_N(a),$$
$$CONN_{T_G}(b,c) = CONN_{T_G}(b,d) = CONN_{T_G}(b,e) = 0.4 = T_N(b),$$
$$CONN_{T_G}(c,d) = CONN_{T_G}(c,e) = 0.4 = T_N(b),$$
$$CONN_{T_G}(d,e) = 0.4 = T_N(b),$$

$$PCI_T(G) = \sum_{u,v \in N} T_N(u)T_N(v)CONN_{T_G}(u,v)$$
$$= (0.2)(0.4)(0.2) + (0.2)(0.7)(0.2) + (0.2)(0.5)(0.2) + (0.2)(0.6)(0.2) + (0.4)(0.4)(0.7) + (0.4)(0.4)(0.5) + (0.4)(0.4)(0.6) + (0.7)(0.5)(0.4) + (0.7)(0.6)(0.4) + (0.5)(0.4)(0.6) = 0.804,$$

Using Theorem 4,

$$PCI_T(G) = \sum_{j=1}^{m} t_j^2 \sum_{k=j+1}^{n} t_k + t_m \sum_{j=m+1}^{n-1} t_j \sum_{k=j+1}^{n} t_k = \sum_{j=1}^{2} t_j^2 \sum_{k=j+1}^{5} t_k + t_m \sum_{3}^{4} t_j \sum_{k=j+1}^{5} t_k$$
$$= (0.2)(0.2)(0.4 + 0.7 + 0.5 + 0.6) + (0.4)(0.4)(0.7 + 0.5 + 0.6) + (0.4)(0.7)(0.5 + 0.6) + (0.4)(0.5)(0.6) = 0.804.$$

As observed, the value of truth- partial connectivity index $PCI_T(G)$ is obtained from both methods equally.

Theorem 5. Let $G = (N, M)$ be a wheel neutrosophic graph whit $V = \{v_1, v_2, \dots, v_n\}$ such that G^* is a wheel graph and for any $uv \in M^*$,

$$T_M(uv) = \min\{T_N(u), T_N(v)\}, \quad I_M(uv) = \min\{I_N(u), I_N(v)\}, \quad F_M(uv) = \max\{F_N(u), F_N(v)\}.$$

If $t_1 \le t_2 \le \cdots \le t_n$, $i_1 \le i_2 \le \cdots \le i_n$ and $f_1 \ge f_2 \ge \cdots \ge f_n$ where $t_j = T_N(v_j)$, $i_j = I_N(v_j)$ and $f_j = F_N(v_j)$ for $j = 1, 2, \dots, n$ and v_1 is the center vertex. Then

$$PCI_T(G) = \sum_{j=1}^{n-1} t_j^2 \sum_{k=j+1}^{n} t_k,$$
$$PCI_I(G) = \sum_{j=1}^{n-1} i_j^2 \sum_{k=j+1}^{n} i_k,$$
$$PCI_F(G) = \sum_{j=1}^{n-1} f_j^2 \sum_{k=j+1}^{n} f_k.$$

Proof. Let $G = (N, M)$ be a wheel neutrosophic graph whit the conditions stated in the theorem. Here we prove $PCI_I(G)$, states $PCI_T(G)$ and $PCI_F(G)$ are similarly proved. Then

Suppose v_1 is the center vertex. Using definition,

$$PCI_I(G) = \sum_{v_j, v_k \in V} I_N(v_j)I_N(v_k)CONN_{IG}(v_j, v_k).$$

Now, for $v_1, v_k \in V$ we have

$$CONN_{IG}(v_1, v_2) = \min\{\max\{i_1\}, \max\{i_1, i_2\}, \max\{i_1, i_2, i_3\}, \dots, \max\{i, i_m\}\} = \min\{i_1, i_2, \dots, i_n\} = i_1,$$

Hence

$$\sum_{k=2}^{n} I_N(v_1)I_N(v_k)CONN_{IG}(v_1, v_k) = i_1 i_1 i_2 + i_1 i_1 i_3 + \cdots + i_1 i_1 i_{n-1} + i_1 i_1 i_n = \sum_{k=2}^{n} i_1^2 i_k.$$

Similarly for $v_j, v_k \in V$ $j = 2, 3, \dots, n-1$

$$CONN_{IG}(v_j,v_k) = \sum_{k=j+1}^{n} I_N(v_j)I_N(v_k)CONN_{IG}(v_j,v_k) = \sum_{k=j+1}^{n} i_j^2 i_k,$$

This shows that

$$PCI_I(G) = \sum_{v_j,v_k\in V} I_N(v_j)I_N(v_k)CONN_{IG}(v_j,v_k) = \sum_{k=2}^{n} i_1^2 i_k + \sum_{k=3}^{n} i_2^2 i_k + \cdots + \sum_{k=j+1}^{n} i_j^2 i_k + \cdots + i_{n-1}i_{n-1}i_n$$
$$= \sum_{j=1}^{n-1} i_j^2 \sum_{k=j+1}^{n} i_k.$$

□

Theorem 6. Let $G = (N, M)$ be a complete neutrosophic graph of $G^* = (V, E)$, and $B_{(m,m)}$ is a m-barbell graph of G. if $t_1 \le t_2 \le \cdots \le t_n$, $i_1 \ge i_2 \ge \cdots \ge i_n$ and $f_1 \ge f_2 \ge \cdots \ge f_n$ where $t_j = T_N(v_j)$, $i_j = I_N(v_j)$ and $f_j = F_N(v_j)$ for $j = 1, 2, \dots, n$. And uv is a $\mathrm{I} - strong\ edge$ whit $M(uv) = (T_M(uv), I_M(uv), F_M(uv))$, where $T_M(uv) \le t_1, I_M(uv) \le i_1, F_M(uv) \ge f_1$, and uv connecting two copies of complete neutrosophic graphs G. Then

$$PCI_T(B_{(m,m)}) = 2\sum_{j=1}^{n-1} t_j^2 \sum_{k=j+1}^{n} t_k + T_M(uv)\sum_{j=1}^{n} t_j \sum_{k=j}^{n} t_k,$$
$$PCI_I(B_{(m,m)}) = 2\sum_{j=1}^{n-1} i_j^2 \sum_{k=j+1}^{n} i_k + I_M(uv)\sum_{j=1}^{n} i_j \sum_{k=j}^{n} i_k,$$
$$PCI_F(B_{(m,m)}) = 2\sum_{j=1}^{n-1} f_j^2 \sum_{k=j+1}^{n} f_k + F_M(uv)\sum_{j=1}^{n} f_j \sum_{k=j}^{n} f_k.$$

Proof. Let $G = (N, M)$ be a wheel neutrosophic graph whit the conditions stated in the theorem. By definition 5, here we have two copies of the complete graph K_m. Also using Theorem 3, for a complete neutrosophic graph

$$PCI_T(G) = \sum_{j=1}^{n-1} t_j^2 \sum_{k=j+1}^{n} t_k,$$
$$PCI_I(G) = \sum_{j=1}^{n-1} i_j^2 \sum_{k=j+1}^{n} i_k,$$
$$PCI_F(G) = \sum_{j=1}^{n-1} f_j^2 \sum_{k=j+1}^{n} f_k.$$

Now it suffices to obtain the connectivity between two vertices from two copies of K_m. Suppose vertex v_j is from one of the two copies of K_m and vertex v_k is from another copy, in which case we have

$$CONN_{TG}(v_j,v_k) = \max\{min\{T_M(uv) \wedge \min\{t_k \mid t_k \in P(v_j\_v_k)\}\} = T_M(uv),$$

Then

$$PCI_T\big(B_{(m,m)}\big) = \sum_{v_j,v_k \in V} I_N(v_j) I_N(v_k) CONN_{IG}(v_j, v_k)$$
$$= \sum_{j=1}^{n-1} t_j^2 \sum_{k=j+1}^{n} t_k + \sum_{j=1}^{n-1} t_j^2 \sum_{k=j+1}^{n} t_k + v_1 v_1 T_M(uv) + v_1 v_2 T_M(uv) + \cdots + v_n v_n T_M(uv)$$
$$= 2\sum_{j=1}^{n-1} t_j^2 \sum_{k=j+1}^{n} t_k + T_M(uv) \sum_{j=1}^{n} t_j \sum_{k=j}^{n} t_k.$$

The proof will be the same for the other two cases.

□

Example 4. Consider the neutrosophic graph $G = K_4 = (N, M)$ whit

$$N = \{(a, 0.2, 0.6, 0.8), (b, 0.3, 0.5, 0.7), (c, 0.3, 0.4, 0.7), (d, 0.4, 0.4, 0.5)\},$$

And

$$M = \{(ab, 0.2, 0.6, 0.8), (ac, 0.2, 0.6, 0.8), (ad, 0.2, 0.6, 0.8),$$
$$(bc, 0.3, 0.5, 0.7), (bd, 0.3, 0.4, 0.7), (cd, 0.3, 0.4, 0.7)\}.$$

Now suppose that the edge that connects the two complete graphs does not hold true. As shown in figure 4, for example, if we want to go from vertex b in the right graph to vertex a in the left graph, there are paths with different connectivity.

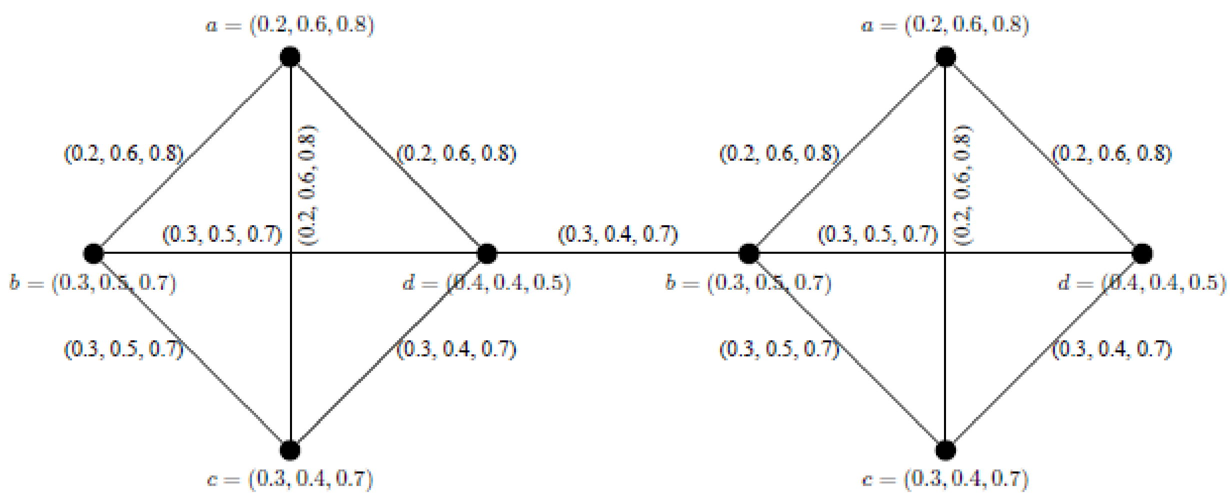

Figure 4. A m-barbell neutrosophic graph whit $G^* = K_4$

3.2. Bounds for connectivity index

In this section, we discuss bunds for partial connectivity index (PCI) and totally connectivity index (TCI). We show that, among all neutrosophic graphs whit a same support, the complete neutrosophic graph will have maximum totally connectivity index.

Theorem 7. Let $G = (N, M)$ be a neutrosophic graph whit $|N| = n$, and $G' = (N', M')$ is the complete neutrosophic graph spanned by the vertex set of G. Then,

$$0 \le PCI_T(G) \le PCI_T(G'),$$

$$0 \leq PCI_I(G) \leq PCI_I(G'),$$
$$0 \leq PCI_F(G) \leq PCI_F(G').$$

Also if $I_M(uv) = I_{M'}(uv)$, and $F_M(uv) = F_{M'}(uv)$, for all $uv \in E$ then $0 \leq TCI_F(G) \leq TCI_F(G')$.

proof. Consider the neutrosophic graph $G = (N, M)$ whit $|N| = n$. If $|E| = 0$ clearly, $PCI_T(G) = PCI_I(G) = PCI_F(G) = TCI(G) = 0$. Let $|E| > 0$ and $G' = (N', M')$ is the complete neutrosophic graph whit $|N'| = n$. Suppose $\left(T_N(u), I_N(u), F_N(u)\right) = (T_{N'}(u), I_{N'}(u), F_{N'}(u))$ for all $u \in X$. Since

$$T_M(uv) \leq T_{M'}; \quad I_M(uv) \leq I_{M'}(uv); \quad F_M(uv) \leq F_{M'}(uv); \quad \forall uv \in E.$$

Therefore, we have $CONN_{TG}(u,v) \leq CONN_{TG'}(u,v)$, $CONN_{IG}(u,v) \leq CONN_{IG'}(u,v)$ and $CONN_{FG}(u,v) \leq CONN_{FG'}(u,v)$. Then

$$0 \leq PCI_T(G) = \sum_{u,v \in X} T_N(u) T_N(v) CONN_{TG}(u,v) \leq \sum_{u,v \in X} T_{N'}(u) T_{N'}(v) CONN_{TG'}(u,v) = PCI_T(G').$$

Using a similar proof we can show that

$$0 \leq PCI_I(G) \leq PCI_I(G'), \qquad and \qquad 0 \leq PCI_F(G) \leq PCI_F(G').$$

Also, according to definition $TCI(G)$, if $I_M(uv) = I_{M'}(uv)$, and $F_M(uv) = F_{M'}(uv)$, for all $uv \in E$, then

$$TCI(G) = \frac{4 + 2PCI_T(G) - 2PCI_F(G) - PCI_I(G)}{6} \leq \frac{4 + 2PCI_T(G') - 2PCI_F(G') - PCI_I(G')}{6} = TCI(G').$$

□

Note 3. Note that the above theorem for case $TCI(G) \leq TCI(G')$ may not always be true.

4. Applications

Neutrosophic graphs are one of the most practical branches of graph theory. Different applications of it have been studied to date [1-3, 12-20]. Here we will mention another application.

Behavioral sciences, which is one of the branches of humanities, is one of the most extensive sciences in our time. Every day, many theorists in this field create new theories and cause them to expand more and more. So every day they are faced with a lot of new data and information.

Mathematics has always been one of the best tools for modeling and categorizing this data and information. Among these, graphic models are among the most appropriate models that come with the help of behavioral sciences and with proper modeling, provide the conditions for a more accurate analysis of these complex problems. What is very important in behavioral sciences is the existence of a relationship, the relationship between individuals, groups, communities, organizations and institutions, and, so on. Studying and discovering these relationships, categorizing them, and then examining and studying the extent and impact of these relationships on each other is a complex task. Neutrosophic graph models can help with these problems and help answer some of the questions. Questions such as: Which relationship is most effective? Which relationship should end? Which person is more influential in a relationship? And many other questions

Here we are dealing with the relationship between several families. Information related to this problem is data from a real study obtained from a behavioral science study clinic. Of course, given the limitations we had, we have provided a small sample of that data in this article.

In this problem, we studied 5 families that are related. First, each family was studied separately and the behavior of each family member was studied by experts, and then we obtained an average of the behaviors and traits studied in family members. These features were classified into three categories. Good qualities include the ability to communicate, cooperate, be honest, etc; Bad traits include jealousy, misconceptions, lack of anger control, personal aggression, etc; Neutral behaviors include behaviors that do not involve any behavioral actions. The experts then assigned a numerical value to each of these behaviors, which we named T, F, and I, respectively. Experts then studied the relationships between families and the extent of each family's impact on another family and the type of impact of each family. The effect of each family on other families was evaluated using behavioral science criteria. The experts coded these relationships into three categories: good, neutral, and bad, and obtained a numerical quantity for each category based on the coding results.

Here we present a neutrosophic graph model related to 5 families from 137 families surveyed.

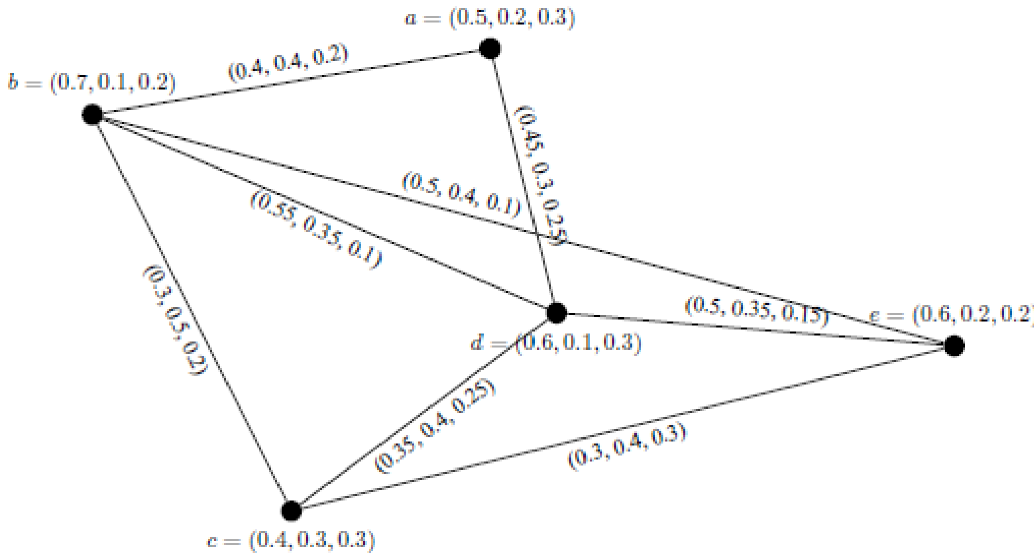

Figure 5. A neutrosophic graph model corresponding to 5 families

By direct calculations

Table 2. The strength of connectedness between each pair of vertices u and v.

	$CONN_{T_G}(u,v)$	$CONN_{I_G}(u,v)$	$CONN_{F_G}(u,v)$
a,b	0.45	0.35	0.2
a,c	0.35	0.4	0.2
a,d	0.45	0.3	0.2
a,e	0.45	0.35	0.2
b,c	0.35	0.4	0.2
b,d	0.55	0.35	0.1
b,e	0.5	0.35	0.1
c,d	0.35	0.4	0.2
c,e	0.35	0.4	0.2
d,e	0.5	0.35	0.1

Then

$$PCI_T(G) = \sum_{u,v \in N} T_N(u)T_N(v)CONN_{T_G}(u,v) = 1.3845,$$
$$PCI_I(G) = \sum_{u,v \in N} I_N(u)I_N(v)CONN_{I_G}(u,v) = 0.519,$$
$$PCI_F(G) = \sum_{u,v \in N} F_N(u)F_N(v)CONN_{F_G}(u,v) = 0.118.$$

Also, we have

$$TCI(G) = \frac{4 + 2PCI_T(G) - 2PCI_F(G) - PCI_I(G)}{6} = \frac{4 + 2(1.3845) - 2(0.118) - 0.519}{6} = 1.002.$$

The connectivity index is used as a numerical index in evaluating the interactions of these five families. Note that the analysis of this problem will be done by behavioral science experts and the results will be presented in detail in another article.

5. Conclusion

Connectivity is one of the major parameters associated with a neutrosophic network and a neutrosophic graph. In this paper, two concepts of partial connectivity index and totally connectivity index were studied. In a neutrosophic graph, according to the parameters of the problem, we can obtain the partial connectivity index and totally connectivity for it. The higher the Truth-partial connectivity index and the lower the Falsity-partial correlation index, the more complete our information is and the more reliable the problem will be.

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Neutrosophic g*-Closed Sets and its maps

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Abstract: Topology is one of the classical subjects in Mathematics. A lot of researchers have published their ideas.As a generalization of topological concepts many new kind of closed and open sets are published continuously. Salama presented Neutrosophic topological spaces by using Smarandache 's Neutrosophic sets. Many Researchers introduced so many closed sets in Neutrosophic topological spaces. Purpose of this research paper is we introduce Neutrosophic g*-Closed sets and Neutrosophicg*-open sets in Neutrosophic topological spaces. Also we study about study about mappings of Neutrosophic g*-Closed sets

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Keywords: Neutrosophic g-Closed sets Neutrosophic g*-Closed sets, Neutrosophicg*-open sets, Neutrosophic g*-continuous.

1. Introduction

Smarandache [10,11] characterized the Neutrosophic set on three segment Neutrosophic sets(T Truth, I-Indeterminacy, F-Falsehood). Neutrosophic topological spaces(NS-T-S) presented by Salama [19,20]et al. Neutrosophic have wide scope of constant applications for the fields of Electrical & Electronic, Artificial Intelligence, Mechanics, Computer Science, Information Systems, Applied Mathematics , basic leadership. Prescription and Management Science and so on.

Neutrosophic semi closed, α- closed, pre closed and regular closed sets are introduced by I. Arokiarani[6] et al.,R.Dhavaseelan[8] et al. introduced Neutrosophic g closed sets and gα closed sets .Point of this paper is R .Dhavaseelan[9] and S.Jafari, are introduced Generalized Neutrosophic Closed sets . D.Jayanthi [13]presented αG Closed Sets in Neutrosophic Topological Spaces, V.K.Shanthi [22] developed Neutrosophic gs and sg closed set. C.Mahesawri[14,15] et al introduced Neutrosophic gb closed sets.

Aim of this present paper is, we introduce and study the concepts of Neutrosophic g*-Closed sets and Neutrosophic g*-open sets in Neutrosophic topological spaces. Also we study about mappings of Neutrosophicg*-Closed sets

2. Preliminaries

In this section, we recall required and necessary definition and results of Neutrosophic sets

Definition 2.1 [16,17] Let Nu_X^* be a non-empty fixed set. A Neutrosophic set W_1^* is a object having the form $W_1^* = \{< w, \mu_{W_1^*}(w), \sigma_{W_1^*}(w), \gamma_{W_1^*}(w) >: w \in Nu_X^*\}$,

$\mu_{W_1^*}(w)$- membership function

$\sigma_{W_1^*}(w)$- Indeterminacy function

$\gamma_{W_1^*}(w)$- Non-Membership function

Definition 2.2 [16,17]. Neutrosophic set $W_1^* = \{< w, \mu_{W_1^*}(w), \sigma_{W_1^*}(w), \gamma_{W_1^*}(w) >: w \in Nu_X^*\}$, on Nu_X^* and $\forall w \in Nu_X^*$ then complement of W_1^* is

$W_1^{*C} = \{< w, \gamma_{W_1^*}((w)), 1 - \sigma_{W_1^*}(w), \mu_{W_1^*}(w) >: w \in Nu_X^*\}$

Definition 2.3 [16,17]. Let W_1^* and W_2^* are two Neutrosophic sets, $\forall w \in Nu_X^*$

$W_1^* = \{< w, \mu_{W_1^*}(w), \sigma_{W_1^*}(w), \gamma_{W_1^*}(w) >: w \in Nu_X^*\}$

$W_2^* = \{< w, \mu_{W_2^*}(w), \sigma_{W_2^*}(w), \gamma_{W_2^*}(w) >: w \in Nu_X^*\}$

Then $W_1^* \subseteq W_2^* \Leftrightarrow \mu_{W_1^*}(w) \leq \mu_{W_2^*}(w), \sigma_{W_1^*}(w) \leq \sigma_{W_2^*}(w) \quad \& \gamma_{W_1^*}(w) \geq \gamma_{W_2^*}(w)\}$

Definition 2.4[16,17]. Let Nu_X^* be a non-empty set, and Let $W_1^*$and W_2^* be two Neutrosophic sets are

$W_1^* = \{< w, \mu_{W_1^*}(w), \sigma_{W_1^*}(w), \gamma_{W_1^*}(w) >: w \in Nu_X^*\}$, $W_2^* = \{< w, \mu_{W_2^*}(w), \sigma_{W_2^*}(w), \gamma_{W_2^*}(w) >: w \in Nu_X^*\}$

Then $W_1^* \cap W_2^* = \{< w, \mu_{W_1^*}(w) \cap \mu_{W_2^*}(w), \sigma_{W_1^*}(w) \cap \sigma_{W_2^*}(w), \gamma_{W_1^*}(w) \cup \gamma_{W_2^*}(w) >: w \in Nu_X^*\}$

$W_1^* \cup W_2^* = \{< w, \mu_{W_1^*}(w) \cup \mu_{W_2^*}(w), \sigma_{W_1^*}(w) \cup \sigma_{W_2^*}(w), \gamma_{W_1^*}(w) \cap \gamma_{W_2^*}(w) >: w \in Nu_X^*\}$

Definition 2.5 [19,20]. Let Nu_X^* be non-empty set and Nu_τ be the collection of Neutrosophic subsets of Nu_X^*satisfying the accompanying properties:

1.$0_{Nu}, 1_{Nu} \in Nu_\tau$

2. $Nu_{T_1} \cap Nu_{T_2} \in Nu_\tau$ for any $Nu_{T_1}, Nu_{T_2} \in Nu_\tau$

3. $\cup Nu_{T_i} \in Nu_\tau$ for every $\{Nu_{T_i}: i \in j\} \subseteq Nu_\tau$

Then the space (Nu_X^*, Nu_τ), is called a Neutrosophic topological space(NS-T-S) The component of Nu_τ are called Nu-OS (Neutrosophic open set)and its complement is Nu-CS(Neutrosophic closed set)

Example 2.6. Let Nu_X^* ={w} and $\forall w \in Nu_X^*$,$W_1^* = \langle w, \frac{6}{10}, \frac{6}{10}, \frac{5}{10} \rangle$, $W_2^* = \langle w, \frac{5}{10}, \frac{7}{10}, \frac{9}{10} \rangle$

$W_3^* = \langle w, \frac{6}{10}, \frac{7}{10}, \frac{5}{10} \rangle$,$W_4^* = \langle w, \frac{5}{10}, \frac{6}{10}, \frac{9}{10} \rangle$ Then the collection $Nu_\tau = \{0_{Nu}, W_1^*, W_2^*, W_3^*, W_4^* 1_{Nu}\}$ is called a NS-T-S on Nu_X^*.

Definition 2.7. Let (Nu_X^*, Nu_τ), be a NS-T-S

and $W_1^* = \{< w, \mu_{W_1^*}(w), \sigma_{W_1^*}(w), \gamma_{W_1^*}(w) >: w \in Nu_X^*\}$ bea Neutrosophic set in Nu_X^*. Then W_1^* is said to be

[1] Neutrosophic α-closed set [6] (Nu $-$ αCS in short) Nu-cl(Nu-in(Nu-cl(W_1^*)))$\subseteq$ W_1^*,

[2] Neutrosophic pre-closed set [22] (Nu-PCS in short) Nu -cl(Nu -in(W_1^*))$\subseteq$ W_1^*,

[3] Neutrosophic regular closed set [6] (Nu -RCS in short) Nu-cl(Nu-in(W_1^*))=W_1^*,

[4] Neutrosophic semi closed set [7] (Nu-SCS in short) Nu -in(Nu-cl(W_1^*))$\subseteq$ W_1^*,

[5] Neutrosophic generalized closed set [4] (Nu -GCS in short) Nu-cl(W_1^* $\subseteq$ $\mathcal{H}$ whenever W_1^* $\subseteq$ $\mathcal{H}$ and $\mathcal{H}$ is a Nu -OS, in Nu_X^*

[6] Neutrosophic α generalized closed set [13] (Nu-(αG)CS in short) Nuαcl(W_1^*)$\subseteq$ $\mathcal{H}$ whenever W_1^* $\subseteq$ $\mathcal{H}$ And $\mathcal{H}$ is a Nu-OS, in Nu_X^*

[7] Neutrosophic generalized semi closed set [21](Nu-GSCS in short) Nu-Scl(W_1^*)$\subseteq$ $\mathcal{H}$ whenever W_1^* $\subseteq$ $\mathcal{H}$ and $\mathcal{H}$ is a Nu-OS in Nu_X^*

[8] Neutrosophic semi generalized closed set[21](Nu-SGCS in short) if Nu scl(W_1^*) $\subseteq$ $\mathcal{H}$ whenever W_1^* $\subseteq$ $\mathcal{H}$ and $\mathcal{H}$ is a Nu-SOS in Nu_X^* ,

[9] Neutrosophic generalized alpha closed set[9]. (Nu-GαCS in short) if Nu-αcl(W_1^*) $\subseteq$ $\mathcal{H}$ whenever $W_1^*$$\subseteq$ $\mathcal{H}$ and $\mathcal{H}$ is a Nu-αOS in Nu_X^*

[10] Neutrosophic generalized b closed set[14](Nu-GbCS in short) if Nu-bcl(W_1^*) $\subseteq$$\mathcal{H}$ whenever $W_1^*$$\subseteq$$\mathcal{H}$ and $\mathcal{H}$ is a Nu-OS in Nu_X^*

Definition 2.8.[13] An (NS)S W_1^* in an (NS)TS (Nu_X^*, Nu_τ), is said to be aNeutrosophic weakly generalized closed set ((Nu-WG)CS) Nu-cl(Nu-in(W_1^*))$\subseteq$ $\mathcal{K}$ whenever W_1^* $\subseteq$ $\mathcal{K}$, $\mathcal{K}$ is (Nu)OS in Nu_X^*.

Definition 2.9. (Nu_X^*, Nu_τ), be a NS-T-S and $W_1^* = \{< w, \mu_{W_1^*}(w), \sigma_{W_1^*}(w), \gamma_{W_1^*}(w) >: w \in Nu_X^*\}$ Nu_X^*.

Then Neutrosophic closure of W_1^* is Nu-Cl(W_1^*)= $\cap$ $\{\mathcal{H}: \mathcal{H}$ is a Nu-CS in Nu_X^* and W_1^* $\subseteq$ $\mathcal{H}$ $\}$
Neutrosophic interior of W_1^* is Nu-Int(W_1^*)= $\cup$\{M:M is a Nu-OS in Nu_X^* and M$\subseteq$ W_1^*\}.

Definition 2.10.[2] Let (Nu_X^*, Nu_τ), be a NS-T-S and
$W_1^* = \{< w, \mu_{W_1^*}(w), \sigma_{W_1^*}(w), \gamma_{W_1^*}(w) >: w \in Nu_X^*\}$
Nu-Sint(W_1^*)= $\cup$\{ $\mathcal{H}/\mathcal{H}$ is a Nu-SOS in Nu_X^* and $\mathcal{H}$ $\subseteq$ W_1^*\},
Nu -Scl(W_1^*)= $\cap$\{ $\mathcal{K}$ / $\mathcal{K}$ is a Nu -SCS in Nu_X^* and W_1^* $\subseteq$ $\mathcal{K}$ \}.
Nu-αint(W_1^*)= $\cup$\{ $\mathcal{H}$ / $\mathcal{H}$ is a Nu-αOS in Nu_X^* and $\mathcal{H}$ $\subseteq$ W_1^*\},
Nu-αcl(W_1^*)= $\cap$\{ $\mathcal{K}$ / $\mathcal{K}$ is a Nu-αCS in Nu_X^* and W_1^* $\subseteq$ $\mathcal{K}$ \}.

3. NEUTROSOPHIC G* CLOSED SETS

In this section we introduce Neutrosophic G*-Closed sets and studied some of its basic properties.

Definition 3.1: An NS W_1^* in (Nu_X^*, Nu_τ) is said to be a NeutrosophicG*-Closed set (Nu-G*CS in short) if Nu-cl(W_1^*) $\subseteq$ $\mathcal{K}$ whenever W_1^* $\subseteq$ $\mathcal{K}$ and $\mathcal{K}$ is Nu-GOS in (Nu_X^*, Nu_τ) .

The family of all Nu-G*CS's of A NTS (Nu_X^*, Nu_τ) is denoted by Nu-G*C(Nu_X^*).

Example 3.2: Let Nu_X^* = \{ w_1 , w_2 \} and let Nu_τ =\{0_{Nu}, $\mathcal{K}$, 1_{Nu}\}is NT on Nu_X^* ,where $\mathcal{K}$ = $\langle w, \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right)\rangle$.Then the NS W_1^*=$\langle w, \left(\frac{7}{10}, \frac{5}{10}, \frac{1}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{0}{10}\right)\rangle$ is Nu-G*CS in (Nu_X^*, Nu_τ)

Theorem 3.3: Every Nu-CS is Nu-G*CS .

Proof: Let W_1^* be aNu-CS in (Nu_X^*, Nu_τ) . Then Nu-cl(W_1^*) = W_1^*. Let $W_1^*$$\subseteq$$\mathcal{K}$ and $\mathcal{K}$ is Nu-GOS in (Nu_X^*, Nu_τ) . Therefore Nu-cl(W_1^*) = $W_1^*$$\subseteq$$\mathcal{K}$. Thus W_1^* is Nu-G*CS inNu_X^*.

Example 3.4: Let Nu_X^*=\{w_1,w_2\} and let Nu_τ=\{0_{Nu}, $\mathcal{K}$,1_{Nu}\}is NT on Nu_X^*, where

$\mathcal{K}$ = $\langle w, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{2}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$Then the NS W_1^* =$\langle w, \left(\frac{6}{10}, \frac{5}{10}, \frac{1}{10}\right), \left(\frac{7}{10}, \frac{5}{10}, \frac{1}{10}\right)\rangle$ is Nu-G*CS but not an

Nu-CS in Nu_X^*.

Theorem 3.5: Every Nu-G*CS is Nu-GCS.

Proof: Let W_1^* be aNu-G*CS in $(Nu_X^*,\ Nu_\tau)$. Let $W_1^* \subseteq \mathcal{K}$ and $\mathcal{K}$ is Nu-OS in $(Nu_X^*,\ Nu_\tau)$. Since every Nu-OS is Nu-GOS and since W_1^* is Nu-G*CS in Nu_X^*. Therefore Nu-cl(W_1^*)$\subseteq \mathcal{K}$ whenever $W_1^* \subseteq \mathcal{K}$, $\mathcal{K}$ is Nu-OS in Nu_X^*. Thus W_1^* is Nu-GCS in Nu_X^*.

Example 3.6: Let Nu_X^*={w_1,w_2, w_3} and

let Nu_τ={0_{Nu}, $\mathcal{K}$, 1_{Nu}}is NT on Nu_X^*, where $\mathcal{K}$ = $\langle w, \left(\frac{2}{10},\frac{5}{10},\frac{7}{10}\right), \left(\frac{5}{10},\frac{5}{10},\frac{4}{10}\right), \left(\frac{3}{10},\frac{5}{10},\frac{5}{10}\right)\rangle$.

Then the NS W_1^* =$\langle w, \left(\frac{5}{10},\frac{5}{10},\frac{5}{10}\right), \left(\frac{2}{10},\frac{5}{10},\frac{7}{10}\right), \left(\frac{3}{10},\frac{5}{10},\frac{6}{10}\right)\rangle$ is Nu-GCS but not an Nu-G*CS in Nu_X^*.

Theorem 3.7: Every Nu-G*CS is Nu-αGCS .

Proof: Let W_1^* be aNu-G*CS in $(Nu_X^*,\ Nu_\tau)$. By Theorem 3.6 W_1^* is Nu-GCS in Nu_X^*. Since Nuα-cl(W_1^*) $\subseteq$ Nu-cl(W_1^*) and W_1^* is a Nu-GCS in Nu_X^*. Therefore Nuα -cl(W_1^*) $\subseteq$ Nu-cl(W_1^*)$\subseteq$ $\mathcal{K}$ whenever $W_1^* \subseteq \mathcal{K}$, $\mathcal{K}$ is Nu-OS in Nu_X^*. Thus W_1^* is Nu-αGCS in Nu_X^*.

Example 3.8: Let Nu_X^* = {w_1, w_2} and let Nu_τ ={0_{Nu}, $\mathcal{K}$, 1_{Nu}}is NT on Nu_X^*,

where $\mathcal{K}$ =$\langle w, \left(\frac{1}{10},\frac{5}{10},\frac{6}{10}\right), \left(\frac{4}{10},\frac{5}{10},\frac{5}{10}\right)\rangle$. Then the NS W_1^* =$\langle w, \left(\frac{3}{10},\frac{5}{10},\frac{4}{10}\right), \left(\frac{5}{10},\frac{5}{10},\frac{5}{10}\right)\rangle$

is Nu-αGCS but not an Nu-G*CS in Nu_X^*.

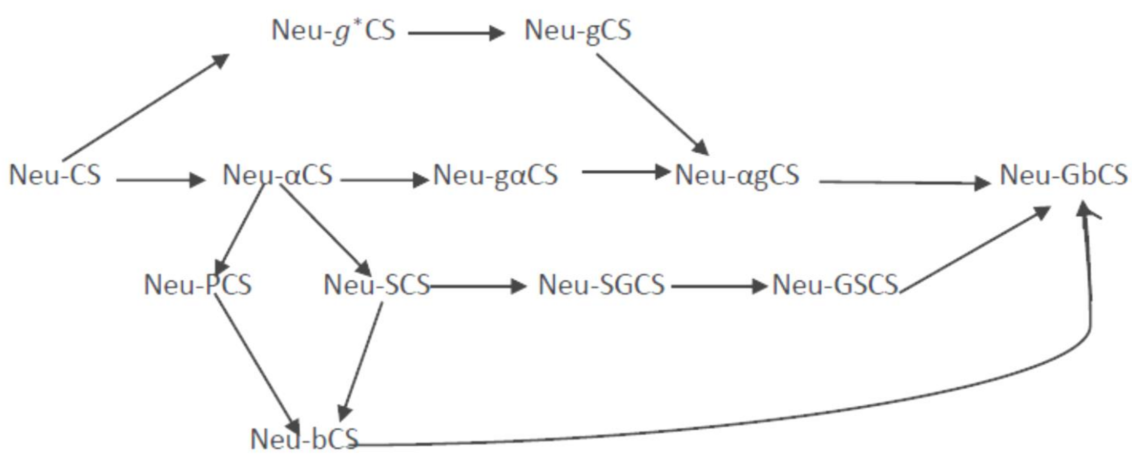
Theorem 3.9: Every Nu-RCS is Nu-G*CS .

Proof: Let W_1^* be a Nu-RCS in $(Nu_X^*,\ Nu_\tau)$. Then W_1^* = Nu-cl(Nu-int(W_1^*)). Let $W_1^* \subseteq \mathcal{K}$ and $\mathcal{K}$ is Nu-GOS in(Nu_X^*, Nu_τ) .Therefore Nu-cl(W_1^*)$\subseteq$Nu-cl(Nu-int(W_1^*)). This implies Nu-cl(W_1^*)$\subseteq$ $W_1^* \subseteq \mathcal{K}$. Thus W_1^* is Nu-G*CS in Nu_X^*.

Example 3.10: Let Nu_X^* = {w_1, w_2} and let Nu_τ = {0_{Nu}, $\mathcal{K}$, 1_{Nu}}is NT on Nu_X^*,Where

$\mathcal{K}$ =$\langle w, \left(\frac{3}{10},\frac{5}{10},\frac{6}{10}\right), \left(\frac{7}{10},\frac{5}{10},\frac{3}{10}\right)\rangle$Then NS W_1^* =$\langle w, \left(\frac{7}{10},\frac{5}{10},\frac{3}{10}\right), \left(\frac{6}{10},\frac{5}{10},\frac{4}{10}\right)\rangle$ is Nu-G*CS but not an Nu-RCS in Nu_X^*.

Diagram:I



Remark 3.11:

Nu-G*CS is independent from Nu-αCS, Nu-SCS, Nu-PCS, and Nu-bCS as seen from the following example.

Example 3.12: Let Nu^*_X = {w_1, w_2} and let Nu_τ = {0_{Nu}, $\mathcal{K}$, 1_{Nu}}is NT on Nu^*_X, where

$\mathcal{K}$ =⟨w $\left(\frac{2}{10},\frac{5}{10},\frac{6}{10}\right),\left(\frac{4}{10},\frac{5}{10},\frac{5}{10}\right)$ ⟩. Then NS W^*_1=⟨w, $\left(\frac{3}{10},\frac{5}{10},\frac{5}{10}\right),\left(\frac{4}{10},\frac{5}{10},\frac{5}{10}\right)$⟩

is NuSCS, Nu-bCS, but not an Nu-G*CS in Nu^*_X.

Example 3.13: Let Nu^*_X = {w_1, w_2} and let Nu_τ = {0_{Nu}, $\mathcal{K}$, 1_{Nu}}is NT on Nu^*_X, where

$\mathcal{K}$ =⟨w $\left(\frac{6}{10},\frac{5}{10},\frac{2}{10}\right),\left(\frac{5}{10},\frac{5}{10},\frac{4}{10}\right)$ ⟩. Then NS W^*_1=⟨w, $\left(\frac{1}{10},\frac{5}{10},\frac{7}{10}\right),\left(\frac{4}{10},\frac{5}{10},\frac{5}{10}\right)$⟩

is Nu-PCS, Nu-αCS, but not an Nu-G*CS in Nu^*_X.

Example 3.14: Let Nu^*_X = {w_1,w_2} and let Nu_τ={0_{Nu}, $\mathcal{K}$, 1_{Nu}}is NT on Nu^*_X, where

$\mathcal{K}$ =⟨w, $\left(\frac{6}{10},\frac{5}{10},\frac{2}{10}\right),\left(\frac{5}{10},\frac{5}{10},\frac{2}{10}\right)$⟩ Then the NS W^*_1=⟨w, $\left(\frac{1}{10},\frac{5}{10},\frac{3}{10}\right),\left(\frac{4}{10},\frac{5}{10},\frac{2}{10}\right)$⟩

is Nu-G*CS but not NuSCS, Nu-bCS Nu^*_X.

Example 3.15: Let Nu^*_X = {w_1,w_2} and let Nu_τ={0_{Nu}, $\mathcal{K}$, 1_{Nu}}is NT on Nu^*_X, where

$\mathcal{K}$ =⟨w, $\left(\frac{2}{10},\frac{5}{10},\frac{6}{10}\right),\left(\frac{3}{10},\frac{5}{10},\frac{5}{10}\right)$⟩ Then the NS W^*_1=⟨w, $\left(\frac{3}{10},\frac{5}{10},\frac{1}{10}\right),\left(\frac{3}{10},\frac{5}{10},\frac{3}{10}\right)$⟩

is Nu-G*CS but not NuαCS, Nu-PCS Nu^*_X.

Theorem 3.16: The union of two Nu-G*CS's is Nu-G*CS

Proof: Let W^*_1 and W^*_2 be the two Nu-G*CS's in Nu^*_X and let $W^*_1 \cup W^*_2 \subseteq \mathcal{K}$, where $\mathcal{K}$ is a Nu-GOS in Nu^*_X. Therefore $W^*_1 \subseteq \mathcal{K}$ or $W^*_2 \subseteq \mathcal{K}$ or both contained $\mathcal{K}$. Since W^*_1 and W^*_2 are Nu-G*CS, Nu-cl(W^*_1)$\subseteq \mathcal{K}$ and Nu-cl(W^*_2) $\subseteq \mathcal{K}$. Therefore Nu-cl($W^*_1 \cup W^*_2$)$\subseteq \mathcal{K}$. Thus $W^*_1 \cup W^*_2$ is Nu-G*CS.

Remark 3.17: The intersection of any two Nu-G*CSs is not an Nu-G*CS in general as seen in the following example.

Example 3.18: Let Nu^*_X = {w_1, w_2} and let Nu_τ={0_{Nu}, $\mathcal{K}$, 1_{Nu}}is NT on Nu^*_X,

where $\mathcal{K}$ =⟨w, $\left(\frac{5}{10},\frac{5}{10},\frac{1}{10}\right),\left(\frac{1}{10},\frac{5}{10},\frac{8}{10}\right)$⟩.

Then NS's W^*_1 = ⟨w, $\left(\frac{2}{10},\frac{5}{10},\frac{5}{10}\right),\left(\frac{7}{10},\frac{5}{10},\frac{0}{10}\right)$⟩ W^*_2 =⟨w, $\left(\frac{6}{10},\frac{5}{10},\frac{0}{10}\right),\left(\frac{3}{10},\frac{5}{10},\frac{7}{10}\right)$⟩

are Nu-G*CS's in Nu^*_X but $W^*_1 \cap W^*_2$ is not a Nu-G*CS in Nu^*_X.

Theorem 3.19: If W^*_1 is Nu-G*CS in (Nu^*_X,Nu_τ), such that $W^*_1 \subseteq W^*_2 \subseteq$ Nu-cl(W^*_1). Then W^*_2 is also a Nu-G*CS of (Nu^*_X,Nu_τ)

Proof: Let $\mathcal{K}$ be a Nu-GOS in (Nu^*_X,Nu_τ) such that $W^*_2 \subseteq \mathcal{K}$, Since $W^*_1 \subseteq W^*_2$, $W^*_1 \subseteq \mathcal{K}$ and $\mathcal{K}$ be a Nu-GOS. Also since W^*_1 is Nu-G*CS, Nu-cl(W^*_1)$\subseteq \mathcal{K}$. By hypothesis $W^*_2 \subseteq$Nu-cl(W^*_1). This implies Nu-cl(W^*_2)$\subseteq$Nu-cl(Nu-cl(W^*_1))$\subseteq \mathcal{K}$. Therefore Nu-cl(W^*_2)$\subseteq \mathcal{K}$. Hence W^*_2 is Nu-G*CS of Nu^*_X.

Theorem 3.20: If W^*_1 is both Nu-GOS and Nu-G*CS of (Nu^*_X,Nu_τ), then W^*_1 is Nu-CS in Nu^*_X.

Proof: Let W^*_1 is Nu-GOS in Nu^*_X. Since $W^*_1 \subseteq W^*_1$, by hypothesis Nu-cl(W^*_1)$\subseteq W^*_1$. But from the Definition, $W^*_1 \subseteq$Nu-cl(W^*_1). Therefore Nu-cl(W^*_1)=W^*_1. Hence W^*_1 is Nu-CS of Nu^*_X.

Theorem 3.21: Let (Nu^*_X,Nu_τ) be a NTS. Then Nu-GO(Nu^*_X)=Nu-GC(Nu^*_X) iff every NS in (Nu^*_X, Nu_τ) is Nu-G*CS in Nu^*_X.

Proof:

Necessity: Suppose that Nu-GO(Nu^*_X)=Nu-GC(Nu^*_X). Let $W^*_1 \subseteq \mathcal{K}$ and $\mathcal{K}$ is Nu-GOS in Nu^*_X. This implies Nu-cl(W^*_1)$\subseteq$Nu-cl($\mathcal{K}$). Since $\mathcal{K}$ is Nu-GOS in Nu^*_X. Since by hypothesis $\mathcal{K}$ is Nu-GCS in Nu^*_X, Nu-cl($\mathcal{K}$)$\subseteq \mathcal{K}$. This implies Nu-cl(W^*_1)$\subseteq \mathcal{K}$. Therefore W^*_1 is Nu-G*CS in Nu^*_X.

Sufficiency: Suppose that every NS in (Nu^*_X, Nu_τ) is Nu-G*CS in Nu^*_X. Let $\mathcal{K}$ $\subseteq$Nu-O(Nu^*_X), then

$\mathcal{K}$ ⊆Nu-GO(Nu_X^*). Since $\mathcal{K} \subseteq \mathcal{K}$ and $\mathcal{K}$ is Nu-OS in Nu_X^*, by hypothesis Nu-cl($\mathcal{K}$)⊆ $\mathcal{K}$. I.e., $\mathcal{K}$ ⊆Nu-GC(Nu_X^*). Hence Nu-GO(Nu_X^*)⊆Nu-GC(Nu_X^*).Let W_1^*⊆Nu-GC(Nu_X^*) then ${W_1^*}^C$ is an Nu-GOS in Nu_X^*. But Nu-GO(Nu_X^*)⊆Nu-GC(Nu_X^*). Therefore ${W_1^*}^C$⊆Nu-GC(Nu_X^*). I.e., W_1^*⊆Nu-GO(Nu_X^*). Hence Nu-GC(Nu_X^*)⊆Nu-GO(Nu_X^*). Thus Nu-GO(Nu_X^*)⊆Nu-GC(Nu_X^*).

Theorem 3.22: If W_1^* is Nu-OS and an Nu-G*CS in (Nu_X^*,Nu_τ) , then

W_1^* is Nu-ROS in Nu_X^*

W_1^* is Nu-RCS in Nu_X^*

Proof: (i) Let W_1^* be a Nu-OS and a Nu-G*CS in Nu_X^* .Then Nu-cl(W_1^*)⊆ W_1^* . I.e., Nu int(Nu-clW_1^*))⊆W_1^*.Since W_1^* is a Nu-OS, W_1^* is Nu-POS in Nu_X^*. Hence W_1^*⊆ Nu-int(Nu-cl(W_1^*)). Therefore W_1^*= Nu-int(Nu-cl(W_1^*)). Hence W_1^* is Nu-ROS in Nu_X^*.

(ii): Let W_1^* be a Nu-OS and an Nu-G*CS inNu_X^*. Then Nu-cl(W_1^*)⊆W_1^*. I.e., Nu-cl(Nu-int(W_1^*))⊆W_1^*. Since W_1^* is a Nu-OS, W_1^* is Nu-OS in Nu_X^*. Hence W_1^*⊆Nu-cl(Nu-int(W_1^*)). Therefore W_1^* = Nu-int(Nu-cl(W_1^*)). Hence W_1^* is Nu-RCS in Nu_X^*.

4. NEUTROSOPHIC g*-OPEN SETS

In this section we introduce Neutrosophic g*-open sets and studied some of its properties.

Definition 4.1: An NS W_1^* is said to be a Neutrosophic g*-open set (Nu-G*OS in short) in (Nu_X^*, Nu_τ) if the complement ${W_1^*}^C$ is Nu-G*CS in Nu_X^*.The family of all Nu-G*OS's of A NTS (Nu_X^*, Nu_τ) is denoted by Nu-G*O(Nu_X^*).

Theorem 4.2:A subset W_1^* of (Nu_X^*, Nu_τ) is Nu-G*OS iff W_2^*⊆Nu-int(W_1^*) whenever W_2^* is Nu-GCS in Nu_X^* and W_2^*⊆ W_1^*.

Proof: Necessity: Let W_1^* is Nu-G*OS in Nu_X^*. Let W_2^* be a Nu-GCS in Nu_X^* and W_2^*⊆ W_1^*. Then ${W_2^*}^C$ is Nu-GOS in Nu_X^* such that ${W_1^*}^C$⊆ ${W_2^*}^C$. Since ${W_1^*}^C$ is Nu-G*CS, we have Nu-cl(${W_1^*}^C$)⊆ ${W_2^*}^C$. Hence Nu-int(W_1^*))^C⊆ ${W_2^*}^C$. Therefore W_2^*⊆Nu-int(W_1^*).

Sufficiency: Let W_2^*⊆ Nu-int(W_1^*) whenever W_2^* is Nu-GCS in Nu_X^* and W_2^*⊆ W_1^*. Then ${W_1^*}^C$⊆ ${W_2^*}^C$ and ${W_2^*}^C$ is Nu-GOS. By hypothesis, (Nu-int(W_1^*))^C⊆ ${W_2^*}^C$, which implies Nu-cl(${W_1^*}^C$)⊆ ${W_2^*}^C$. Therefore ${W_1^*}^C$ is Nu-G*CS of Nu_X^*. Hence W_1^* is Nu-G*OS in Nu_X^*.

Theorem 4.3: Every Nu-OS is Nu-G*OS .

Proof: Let W_1^* be a Nu-OS. Then ${W_1^*}^C$ is Nu-CS. By Theorem 3.3, every Nu-CS is Nu-G*CS. Therefore ${W_1^*}^C$ is Nu-G*CS. Hence W_1^* is Nu-G*OS.

Example 4.4: Let Nu_X^* = {w_1, w_2} and let Nu_τ={0_{Nu}, $\mathcal{K}$, 1_{Nu}}is NT on Nu_X^*, where

$\mathcal{K}$ =⟨w, $\left(\frac{2}{10},\frac{5}{10},\frac{2}{10}\right)$, $\left(\frac{5}{10},\frac{5}{10},\frac{8}{10}\right)$⟩. Then NS W_1^* =⟨ w, $\left(\frac{2}{10},\frac{5}{10},\frac{3}{10}\right)$, $\left(\frac{7}{10},\frac{5}{10},\frac{8}{10}\right)$⟩

is Nu-G*OS but not an Nu-OS in Nu_X^*.

Theorem 4.5: Every Nu-ROS is Nu- G*OS .

Proof: Let W_1^* be aNu-WS. Then ${W_1^*}^C$ is Nu-RCS. By Theorem 3.15, every Nu-RCS is Nu-G*CS. Therefore ${W_1^*}^C$ is Nu-G*CS. Hence W_1^* is Nu-G*OS.

Example 4.6: Let Nu_X^* = {w_1, w_2} and let Nu_τ={0_{Nu}, $\mathcal{K}$, 1_{Nu}}is NT on Nu_X^*, where

$\mathcal{K}$ =⟨w, $\left(\frac{3}{10},\frac{5}{10},\frac{6}{10}\right)$, $\left(\frac{7}{10},\frac{5}{10},\frac{3}{10}\right)$⟩ Then NS W_1^*=⟨w, $\left(\frac{3}{10},\frac{5}{10},\frac{7}{10}\right)$, $\left(\frac{4}{10},\frac{5}{10},\frac{6}{10}\right)$⟩is Nu-G*OS but not an Nu-ROS in Nu_X^*.

Theorem 4.7: Every Nu-G*OS is Nu-GOS .

Proof: Let W_1^* be a Nu-G*OS in (Nu_X^*, Nu_τ) . Then ${W_1^*}^C$ is Nu-G*CS. By Theorem 3.6, every Nu-G*CS is Nu-GCS. Therefore ${W_1^*}^C$ is Nu-GCS. Hence W_1^* is Nu-GOS.

Example 4.8: Let Nu_X^* ={w_1,w_2} and let Nu_τ ={0_{Nu}, $\mathcal{K}$, 1_{Nu}}is NT on Nu_X^*, where

$\mathcal{K}$ =$\langle w, \left(\frac{5}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{2}{10}, \frac{5}{10}, \frac{2}{10}\right)\rangle$. Then NS W_1^* =$\langle w, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{2}{10}, \frac{5}{10}, \frac{6}{10}\right)\rangle$is Nu-GOS but not an Nu-G*OS inNu_X^*.

Theorem 4.9: Every Nu-G*OS is Nu-αGOS .

Proof: Let W_1^* be aNu-G*OS in (Nu_X^*,Nu_τ). Then ${W_1^*}^C$ is Nu-G*CS. By Theorem 3.9, every Nu-G*CS is Nu-αGCS. Therefore ${W_1^*}^C$ is Nu-αGCS. Hence W_1^* is Nu-αGOS.

Example 4.10: Let Nu_X^* ={ w_1 , w_2 } and let Nu_τ ={0_{Nu}, $\mathcal{K}$,1_{Nu}}is NT on Nu_X^*, where $\mathcal{K}$ = $\langle w, \left(\frac{4}{10}, \frac{5}{10}, \frac{2}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$Then the NS W_1^*=$\langle w, \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{3}{10}\right)\rangle$ is Nu-αGOS but not an Nu-G*OS in Nu_X^*.

Theorem 4.11: The intersection of two Nu-G*OS's is Nu-G*OS.

Proof: Let W_1^* and W_2^* be the two Nu-G*OS's inNu_X^*, ${W_1^*}^C$ and ${W_2^*}^C$ are Nu-G*CS. By Theorem 3.28 ${W_1^*}^C \cup {W_2^*}^C$ is Nu-G*CS in Nu_X^*. Therefore ($W_1^* \cap W_2^*$) is Nu-G*CS. Thus $W_1^* \cap W_2^*$ is Nu-G*OS in Nu_X^*.

Theorem 4.12: Let (Nu_X^*, Nu_τ) be a NTS. If W_1^* is NS of Nu_X^*. Then for every $W_1^* \in$ Nu-G*O(Nu_X^*) and every $W_2^* \in$(Nu_X^*), Nu-int(W_1^*)$\subseteq W_2^* \subseteq W_1^*$ implies $W_2^* \in$ Nu-G*O(Nu_X^*).

Proof: By hypothesis Nu-int(W_1^*)$\subseteq W_2^* \subseteq W_1^*$. Taking complement on both sides, we get ${W_1^*}^C \subseteq {W_2^*}^C \subseteq$Nu-cl(${W_1^*}^C$). Let ${W_2^*}^C \subseteq \mathcal{K}$ and $\mathcal{K}$ is Nu-GOS in Nu_X^*. Since${W_1^*}^C \subseteq {W_2^*}^C$, ${W_1^*}^C \subseteq \mathcal{K}$. Since ${W_1^*}^C$ is Nu-G*CS, Nu-cl(${W_1^*}^C$)$\subseteq \mathcal{K}$. Therefore Nu-cl(${W_2^*}^C$)$\subseteq$Nu-cl(${W_1^*}^C$)$\subseteq \mathcal{K}$. Hence ${W_2^*}^C$ is Nu-G*CS in Nu_X^*. Therefore W_2^* is Nu-G*OS in Nu_X^*. I.e., W_2^* $\subseteq$Nu-G*O(Nu_X^*)

Definition: 4.13: For any Nu. set W_1^* in any NSTS,

Nu-g*cl(W_1^*) =∩{ $\mathcal{U}$: $\mathcal{U}$ is Nu-g*CS Nu. set and $W_1^* \subseteq \mathcal{U}$ }

Nu-g*int (W_1^*) = ∪{ : $\mathcal{V}$ is Nu-g* OS and $W_1^* \supseteq \mathcal{V}$ }

Theorem: 4.14: In a Its (Nu_X^*, Nu_τ) a Nu. set W_1^* is Nu-g*- CS iff W_1^* = Nu-g* cl(W_1^*).

Proof: Let W_1^* be a Nu-g*CS Nu. set in NSTS (Nu_X^*, Nu_τ). Since $W_1^* \subseteq W_1^*$ and W_1^* is Nu-g*CS , $W_1^* \in${$\mathcal{K}$: $\mathcal{K}$ is a Nu-g*CS Nu. set and $W_1^* \subseteq \mathcal{K}$ } and $W_1^* \subseteq \mathcal{K} \Rightarrow W_1^*$ = ∩{ $\mathcal{K}$: $\mathcal{K}$ is Nu-g*CS and $W_1^* \subseteq \mathcal{K}$ } that is W_1^* = Nu-g* cl(W_1^*).

Conversely, suppose that W_1^* = Nu-g*cl(W_1^*),that is W_1^*= ∩{ $\mathcal{K}$: $\mathcal{K}$ is a Nu-g*- CS Nu. set and $W_1^* \subseteq$ $\mathcal{K}$ }. This denotes that $W_1^* \in${ $\mathcal{K}$: $\mathcal{K}$ is a Nu-g*CS Nu. set and $W_1^* \subseteq \mathcal{K}$ }. From now W_1^* is Nu-g*CS Nu. set.

Theorem: 4.15 In a NSTS Nu_X^* the subsequent results hold for Nu-g* - closure.

1) Nu-g*cl (0_{Nu}) = 0_{Nu}.

2) Nu-g*cl (W_1^*) is Nu-g*CS Nu. set in Nu_X^*.

3) Nu-g*cl (W_1^*) $\subseteq$Nu-g*cl (W_2^*) if $W_1^* \subseteq W_2^*$.

4) Nu-g* cl (Nu-g*cl(W_1^*)) = Nu-g* cl(W_1^*).

5) Nu-g* cl ($W_1^* \cup W_2^*$)$\supseteq$Nu-g* cl(W_1^*)∪Nu-g* cl(W_2^*).

6) Nu-g* cl ($W_1^* \cap W_2^*$)$\subseteq$Nu-g* cl(W_1^*)∩Nu-g* cl(W_2^*).

Proof: easy

Theorem: 4.16 In a NSTS Nu_X^*, a Nu. set W_1^* is Nu-g* OS iff W_1^* = Nu-g*int (W_1^*).

Proof: Let W_1^* be Nu-g*OS Nu. set in Nu_X^*. Since $W_1^* \subseteq W_1^*$ and W_1^* is Nu-g* OS and $W_1^* \in${ $\mathcal{K}$: $\mathcal{K}$ is a Nu-g* OS Nu. set and $W_1^* \supseteq \mathcal{K}$ } and $W_1^* \supseteq \mathcal{K} \Rightarrow W_1^*$ = ∪{ $\mathcal{K}$: $\mathcal{K}$ is Nu-g*OS and $W_1^* \supseteq \mathcal{K}$ }. That is W_1^*=Nu-g*int (W_1^*).

Conversely, suppose that W_1^* = Nu-g*int (W_1^*), that is W_1^*=∪($\mathcal{K}$: $\mathcal{K}$ is Nu-g*OS and $W_1^* \supseteq \mathcal{K}$ }. This implies that W_1^* ∈{ $\mathcal{K}$: $\mathcal{K}$ is Nu-g*OS and $W_1^* \supseteq \mathcal{K}$ }.Hence W_1^* is Nu-g*OS Nu. set.

Theorem: 4.17 In a NSTS Nu_X^*, the following hold for Nu-g* -interior.

1) Nu-g*int (0_{Nu}) = 0_{Nu}

2) Nu-g*int(W_1^*) ⊆Nu-g*int (W_2^*) if $W_1^* \subseteq W_2^*$.

3) Nu-g*int (W_1^*) is Nu-g*OS in Nu_X^*.

4) Nu-g*int (Nu-g*int (W_1^*)) = Nu-g*int (W_1^*).

5) Nu-g*int ($W_1^* \cup W_2^*$)⊇Nu-g*int(W_1^*)∪Nu-g*int (W_2^*).

6) Nu-g*int ($W_1^* \cap W_2^*$)⊆Nu-g*int(W_1^*)∩Nu-g*int (W_2^*).

Proof: proof is as usual.

5. NEUTROSOPHIC g*- CONTINUOUS

In this section we introduce Neutrosophic g*-continuous and studied some properties of neutrosophic g* - open map and closed map.

Definition:5.1 Let Nu_X^* and Nu_Y^* be two NTS. A function $f: Nu_X^* \longrightarrow Nu_Y^*$ is said to be neutrosophic g* - continuous (Nu-g* - continuous) if the inverse image of every neutrosophic open set in Nu_Y^* is g* - open in Nu_X^*.

Theorem:5.2 A function $f: Nu_X^* \longrightarrow Nu_Y^*$ is Nu-g* - continuous iff the inverse image of every Nu-closed set in Nu_Y^* is g* - closed set in Nu_X^*.

Proof: Suppose the function $f: Nu_X^* \longrightarrow Nu_Y^*$is Nu-g* - continuous. Let $\mathcal{F}$ be Nu-closed set in Nu_Y^*. Then $\mathcal{F}^c$ is Nu- open set in Nu_Y^*. Since f is Nu-g* - continuous, $f^{-1}(\mathcal{F}^c)$ is Nu- g* - open in Nu_X^*. But $f^{-1}(\mathcal{F}^c)$= $(f^{-1}(\mathcal{F}^c))^c$and so $f^{-1}(\mathcal{F})$is Nu-g* - closed in Nu_X^*.

Conversely, assume that the inverse image of every Nu-closed set in Nu_Y^* is Nu-g* - closed in Nu_X^*. Let $\mathcal{V}$ be neutrosophic open set in Nu_Y^*. Then$\mathcal{V}^c$is Nu-closed in Nu_Y^*. By hypothesis, $f^{-1}(\mathcal{V}^c)$ is Nu-g*-closed set in Nu_X^*. But $f^{-1}(\mathcal{V}^c)$ = $(f^{-1}(\mathcal{V}^c))^c$ and so $f^{-1}(\mathcal{V})$is Nu-g* - open set in Nu_X^*. Hence f is Nu-g*-continuous.

Theorem:5.3 Every Nu- continuous function is Nu-g* - continuous.

Proof: Let $f: Nu_X^* \longrightarrow Nu_Y^*$be Nu-continuous. Let $\mathcal{F}$ be Nu-closed set in Nu_Y^*. Then $f^{-1}(\mathcal{F})$ is Nu-closed set in Nu_X^* since f is neutrosophic continuous. And therefore $f^{-1}(\mathcal{F})$is Nu-g* - closed in Nu_X^*. Hence f is Nu-g* - continuous.

Theorem:5.4 Every Nu-g*-continuous function isNu-g-continuous.

Proof: Let $f: Nu_X^* \longrightarrow Nu_Y^*$ be Nu-g* - continuous. Let $\mathcal{F}$ be a Nu-closed set inNu_Y^*. Since f is Nu-g* - continuous, $f^{-1}(\mathcal{F})$ is Nu-g* - closed in Nu_X^*. And therefore $f^{-1}(\mathcal{F})$ is Nu-g - closed in Nu_X^* as every Nu-g*-closed set is Nu-g - closed. Hence f is Nu-g- continuous.

The converse of the above theorem need not be true as seen from the following example.

Theorem:5.5 If $f: Nu_X^* \longrightarrow Nu_Y^*$is Nu-g* - continuous and Nu_X^* is neutrosophic –$T^*_{1/2}$ NTS. Then f is neutrosophic -continuous.

Proof: Let $f: Nu_X^* \longrightarrow Nu_Y^*$be Nu-g*- continuous . Let $\mathcal{F}$ be Nu-closed set in Nu_Y^*. Then $f^{-1}(\mathcal{F})$is $f^{-1}(\mathcal{F})$ Nu- g* - closed in Nu_X^* since f is Nu-g* - continuous. Also since Nu_X^* is neutrosophic - $T^*_{1/2}$, $f^{-1}(\mathcal{F})$is closccl in Nu_X^*. Hence f is Nu-continuous.

Theorem:5.6 If $f: Nu_X^* \longrightarrow Nu_Y^*$is Nu-g - continuous and Nu_X^* is neutrosophic - $T^*_{1/2}$ NTS. Then f is Nu-g* - continuous.

Proof: Let $f: Nu_X^* \longrightarrow Nu_Y^*$ be Nu-g - continuous. Let $\mathcal{F}$ be Nu-closed set in Nu_Y^*, then $f^{-1}(\mathcal{F})$ is g-closed in Nu_X^*. Since X is neutrosophic - $T^*_{1/2}$, $f^{-1}(\mathcal{F})$is Nu- g* - closed in Nu_X^*. Hence f is Nu-g* -

continuous.

Theorem:5.7 If $f: \mathrm{Nu}_{\mathrm{X}}^{*} \longrightarrow \mathrm{Nu}_{\mathrm{Y}}^{*}$is Nu-g* - continuous and g : $\mathrm{Nu}_{\mathrm{Y}}^{*} \longrightarrow \mathrm{Nu}_{\mathrm{Z}}^{*}$ is Nu-continuous then gof : $\mathrm{Nu}_{\mathrm{X}}^{*} \longrightarrow \mathrm{Nu}_{\mathrm{Z}}^{*}$ is Nu-g * - continuous.

Proof: Let $\mathcal{F}$ be Nu-closed set in $\mathrm{Nu}_{\mathrm{Z}}^{*}$. Then $g^{-1}(\mathcal{F})$ is closed in $\mathrm{Nu}_{\mathrm{Y}}^{*}$ since g is Nu-continuous. And then $f^{-1}(g^{-1}(\mathcal{F}))$ is Nu-g* - closed in $\mathrm{Nu}_{\mathrm{X}}^{*}$ since f is Nu-g* - continuous.

Now $(g \circ f)^{-1}(\mathcal{F}) = f^{-1}(g^{-1}(\mathcal{F}))$ is Nu- g* - closed in $\mathrm{Nu}_{\mathrm{X}}^{*}$. Hence $g \circ f$: $\mathrm{Nu}_{\mathrm{X}}^{*} \longrightarrow \mathrm{Nu}_{\mathrm{Z}}^{*}$ is Nu-g*-continuous.

Theorem:5.8 If $f: \mathrm{Nu}_{\mathrm{X}}^{*} \longrightarrow \mathrm{Nu}_{\mathrm{Y}}^{*}$is Nu-g* - continuous and g : $\mathrm{Nu}_{\mathrm{Y}}^{*} \longrightarrow \mathrm{Nu}_{\mathrm{Z}}^{*}$ is Nu-g* - continuous and $\mathrm{Nu}_{\mathrm{Y}}^{*}$ is neutrosophic $-T^{*}_{1/2}$ space. Then gof : $\mathrm{Nu}_{\mathrm{X}}^{*} \longrightarrow \mathrm{Nu}_{\mathrm{Z}}^{*}$ is Nu-g*- continuous.

Proof: Let $\mathcal{F}$ be Nu-closed set in $\mathrm{Nu}_{\mathrm{Z}}^{*}$. Then $g^{-1}(\mathcal{F})$ is Nu-g*CS in $\mathrm{Nu}_{\mathrm{Y}}^{*}$ since g is Nu-g*-continuous. Since $\mathrm{Nu}_{\mathrm{Y}}^{*}$ is neutrosophic $-T^{*}_{1/2}$, $g^{-1}(\mathcal{F})$ is Nu-closed in $\mathrm{Nu}_{\mathrm{Y}}^{*}$. And then $f^{-1}(g^{-1}(\mathcal{F}))$ is Nu-g*CS in $\mathrm{Nu}_{\mathrm{X}}^{*}$ as f is Nu-g* - continuous. Now $(g \circ f)^{-1}(\mathcal{F}) = f^{-1}(g^{-1}(\mathcal{F}))$is Nu-g*CS in $\mathrm{Nu}_{\mathrm{X}}^{*}$. Hence $g \circ f$ is Nu-g* - continuous.

Definition:5.9A map $f: \mathrm{Nu}_{\mathrm{X}}^{*} \longrightarrow \mathrm{Nu}_{\mathrm{Y}}^{*}$ is said to be neutrosophic g* - open if the image of every neutrosophic open set in $\mathrm{Nu}_{\mathrm{X}}^{*}$ is Nu-g*-open set in $\mathrm{Nu}_{\mathrm{Y}}^{*}$.

Definition:5.10 A map $f: \mathrm{Nu}_{\mathrm{X}}^{*} \longrightarrow \mathrm{Nu}_{\mathrm{Y}}^{*}$is said to be neutrosophic g* - closed if the image of every Nu-closed set in $\mathrm{Nu}_{\mathrm{X}}^{*}$ is Nu-g*-closed set in $\mathrm{Nu}_{\mathrm{Y}}^{*}$.

Theorem: 5.11 Every neutrosophic open map is neutrosophic g* - open.

Proof: Let $f: \mathrm{Nu}_{\mathrm{X}}^{*} \longrightarrow \mathrm{Nu}_{\mathrm{Y}}^{*}$ be a neutrosophic open map let $\mathcal{V}$ be an neutrosophic open set in $\mathrm{Nu}_{\mathrm{X}}^{*}$ then f $(\mathcal{V})$ is Nu-open in $\mathrm{Nu}_{\mathrm{Y}}^{*}$ since f is neutrosophic open map. And therefore $f(\mathcal{V})$ is Nu-g* - open in $\mathrm{Nu}_{\mathrm{Y}}^{*}$. Hence f is neutrosophic g* open map.

Theorem :5.12 If $f: \mathrm{Nu}_{\mathrm{X}}^{*} \longrightarrow \mathrm{Nu}_{\mathrm{Y}}^{*}$is Nu-g*-open map and $\mathrm{Nu}_{\mathrm{Y}}^{*}$ is neutrosophic $-T^{*}_{1/2}$, then f is a Nu-open map.

Proof : Let $f: \mathrm{Nu}_{\mathrm{X}}^{*} \longrightarrow \mathrm{Nu}_{\mathrm{Y}}^{*}$is neutrosophic g*- open map. Let $\mathcal{V}$ be neutrosophic open set in $\mathrm{Nu}_{\mathrm{X}}^{*}$. Then f $(\mathcal{V})$ is Nu- g* - open in $\mathrm{Nu}_{\mathrm{Y}}^{*}$. Since $\mathrm{Nu}_{\mathrm{Y}}^{*}$ is neutrosophic $-T^{*}_{1/2}$, $f(\mathcal{V})$ is neutrosophic open set in $\mathrm{Nu}_{\mathrm{Y}}^{*}$. Hence f is Nu- open map.

Theorem:5.13 Every Nu-g* - open map is neutrosophic g - open.

Proof: Let $f: \mathrm{Nu}_{\mathrm{X}}^{*} \longrightarrow \mathrm{Nu}_{\mathrm{Y}}^{*}$ be a Nu-g*- open map. Let $\mathcal{V}$ be neutrosophic open set in $\mathrm{Nu}_{\mathrm{X}}^{*}$. Then f $(\mathcal{V})$ is Nu-g* - open in $\mathrm{Nu}_{\mathrm{Y}}^{*}$ since f is Nu-g* - open map. And therefore $f(\mathcal{V})$ is Nu-g- open set in $\mathrm{Nu}_{\mathrm{Y}}^{*}$. Hence f is neutrosophic g - open map.

Theorem : 5.14 If $f: \mathrm{Nu}_{\mathrm{X}}^{*} \longrightarrow \mathrm{Nu}_{\mathrm{Y}}^{*}$is neutrosophic g - open and $\mathrm{Nu}_{\mathrm{Y}}^{*}$ is neutrosophic - $^{*}T_{1/2}$ space, then f in Nu-g* - open map.

Proof: Let $\mathcal{V}$ be neutrosophic open set in $\mathrm{Nu}_{\mathrm{X}}^{*}$. Then $f(\mathcal{V})$ is Nu-g - open in $\mathrm{Nu}_{\mathrm{Y}}^{*}$. Since $\mathrm{Nu}_{\mathrm{Y}}^{*}$ is neutrosophic -$^{*}T_{1/2}$, f $(\mathcal{V})$ is Nu-g* - open in $\mathrm{Nu}_{\mathrm{Y}}^{*}$. And hence f is Nu-g* - open map.

Theorem : 5.15 Every Nu-closed map is Nu-g* - closed map.

Proof: Let $f: \mathrm{Nu}_{\mathrm{X}}^{*} \longrightarrow \mathrm{Nu}_{\mathrm{Y}}^{*}$ be Nu-closed map. Let $\mathcal{F}$ be Nu-closed set in $\mathrm{Nu}_{\mathrm{X}}^{*}$.Then $f(\mathcal{F})$ is closed in $\mathrm{Nu}_{\mathrm{Y}}^{*}$. And therefore $f(\mathcal{F})$ is Nu- g* - closed in $\mathrm{Nu}_{\mathrm{Y}}^{*}$. And hence f is Nu-g* - closed map.

Theorem :5.16 If $f: \mathrm{Nu}_{\mathrm{X}}^{*} \longrightarrow \mathrm{Nu}_{\mathrm{Y}}^{*}$is Nu-g* - closed and $\mathrm{Nu}_{\mathrm{Y}}^{*}$ is neutrosophic $-T^{*}_{1/2}$. Then f is Nu-closed map.

Proof: Let $f: \mathrm{Nu}_{\mathrm{X}}^{*} \longrightarrow \mathrm{Nu}_{\mathrm{Y}}^{*}$ be Nu-g* - closed map. Let $\mathcal{F}$ be Nu-closed set in $\mathrm{Nu}_{\mathrm{X}}^{*}$.Then f^{-1} $(\mathcal{F})$ is Nu- g* - closed in $\mathrm{Nu}_{\mathrm{Y}}^{*}$. Since $\mathrm{Nu}_{\mathrm{Y}}^{*}$ is neutrosophic $-T^{*}_{1/2}$, $f(\mathcal{F})$ is Nu-closed in $\mathrm{Nu}_{\mathrm{Y}}^{*}$. Hence f is neutrosophic closcd map.

Theorem: 5.17 A map $f: \mathrm{Nu}_{\mathrm{X}}^{*} \longrightarrow \mathrm{Nu}_{\mathrm{Y}}^{*}$is Nu-g* - closed iff for each neutrosophic set $\mathcal{S}$ of $\mathrm{Nu}_{\mathrm{Y}}^{*}$and for each neutrosophic open set $\mathcal{U}$ such that $f^{-1}(\mathcal{S})\subseteq \mathcal{U}$ there is a Nu-g*-open set $\mathcal{V}$ of $\mathrm{Nu}_{\mathrm{Y}}^{*}$ such that $\mathcal{S} \subseteq \mathcal{V}$ and $f^{-1}(\mathcal{V}) \subseteq \mathcal{U}$.

Proof: Suppose f is Nu-g* - closed map. Let $\mathcal{S}$ be a neutrosophic set of $\mathrm{Nu}_{\mathrm{Y}}^{*}$ and $\mathcal{U}$ be a neutrosophic open set of $\mathrm{Nu}_{\mathrm{X}}^{*}$ such that $f^{-1}(\mathcal{U})\subseteq \mathcal{U}$. Then $\mathcal{V} = \mathrm{Nu}_{\mathrm{Y}}^{*} - f\ (\mathcal{U}^{C})$ is a Nu-g*-open set in $\mathrm{Nu}_{\mathrm{Y}}^{*}$ such that $\mathcal{S} \subseteq \mathcal{V}$ and $f^{-1}(\mathcal{V})\subseteq \mathcal{U}$.

Conversely, suppose that $\mathcal{F}$ is a Nu-closed set of $\mathrm{Nu}_{\mathrm{X}}^{*}$. Then $f^{-1}\ (f(\mathcal{F}^{C}))\subseteq \mathcal{F}^{C}$and $\mathcal{F}^{C}$is Nu-open. By hypothesis, there is a Nu-g*-open set $\mathcal{V}$ of $\mathrm{Nu}_{\mathrm{Y}}^{*}$ such that $f(\mathcal{F})^{c}\subseteq \mathcal{V}$ and $f^{-1}(\mathcal{V})\subseteq \mathcal{F}^{C}$ Therefore $\mathcal{F} \subseteq f^{-1}(\mathcal{V})^{C}$. Hence $\mathcal{V}^{c}\subseteq f(\mathcal{F})\subseteq f(f^{-1}(\mathcal{V})^{c})\subseteq \mathcal{V}^{c}$ which implies $f(\mathcal{F}) = \mathcal{V}^{c}$. Since $\mathcal{V}^{c}$is Nu-g* - closed, $f(\mathcal{F})$ is Nu-g*CS and thus f is a Nu-g*- closed m ap.

Conclusion

In this paper, we have defined the neutrosophic g* closed sets and open sets.then discussed about neutrosophic g* continuity Then,we have presented some properties of these operations. We have also investigated neutrosophic topological structures of neutrosophic sets. Hence, we hope that the findings in this paper will help researchers enhance and promote the further study on neutrosophic topology.

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Neutrosophic bg-closed Sets and its Continuity

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Abstract: Smarandache introduced and developed the new concept of Neutrosophic set from the Intuitionistic fuzzy sets. A.A. Salama introduced Neutrosophic topological spaces by using the Neutrosophic crisp sets. Aim of this paper is we introduce and study the concepts Neutrosophic b generalized closed sets and Neutrosophic b generalized continuity in Neutrosophic topological spaces and its Properties are discussed details.

Keywords: Neutrosophic bg closed sets, Neutrosophic bg open sets, Neutrosophic bg continuity, Neutrosophic bg maps.

1. Introduction

Neutrosophic system plays important role in the fields of Information Systems, Computer Science, Artificial Intelligence, Applied Mathematics, Mechanics, decision making, Medicine, Management Science, and Electrical & Electronic, etc,. Topology is a classical subject, as a generalization topological spaces many type of topological spaces introduced over the year. T Truth, F -Falsehood, I- Indeterminacy are three component of Neutrosophic sets. Neutrosophic topological spaces(N-T-S) introduced by Salama [22,23]etal., R.Dhavaseelan[10], Saied Jafari are introduced Neutrosophic generalized closed sets. Neutrosophic b closed sets are introduced by C.Maheswari[17] et al.Aim of this paper is we introduce and study about Neutrosophic b generalized closed sets and Neutrosophic b generalized continuity in Neutrosophic topological spaces and its properties and Characterization are discussed details.

2. Preliminaries

In this section, we recall needed basic definition and operation of Neutrosophic sets and its fundamental Results

Definition 2.1 [13] Let $\mathfrak{X}$ be a non-empty fixed set. A Neutrosophic set $\mathcal{J}_1^*$ is a object having the form

$$\mathcal{J}_1^* = \{< x, \mu_{\mathcal{J}_1^*}(x), \sigma_{\mathcal{J}_1^*}(x), \gamma_{\mathcal{J}_1^*}(x) >: x \in \mathfrak{X}\},$$

$\mu_{\mathcal{J}_1^*}(x)$-represents the degree of membership function

$\sigma_{\mathcal{J}_1^*}(x)$-represents degree indeterminacy and then

$\gamma_{\mathcal{J}_1^*}(x)$-represents the degree of non-membership function

Definition 2.2 [13]. Neutrosophic set $\mathcal{J}_1^* = \{< x, \mu_{\mathcal{J}_1^*}(x), \sigma_{\mathcal{J}_1^*}(x), \gamma_{\mathcal{J}_1^*}(x) >: x \in \mathfrak{X}\}$, on $\mathfrak{X}$ and $\forall x \in \mathfrak{X}$ then complement of $\mathcal{J}_1^*$ is ${\mathcal{J}_1^*}^C = \{< x, \gamma_{\mathcal{J}_1^*}(x), 1 - \sigma_{\mathcal{J}_1^*}(x), \mu_{\mathcal{J}_1^*}(x) >: x \in \mathfrak{X}\}$

Definition 2.3 [13]. Let $\mathcal{J}_1^*$ and $\mathcal{J}_2^*$ are two Neutrosophic sets, $\forall x \in \mathfrak{X}$

$\mathcal{J}_1^* = \{< x, \mu_{\mathcal{J}_1^*}(x), \sigma_{\mathcal{J}_1^*}(x), \gamma_{\mathcal{J}_1^*}(x) >: x \in \mathfrak{X}\}$

$\mathcal{J}_2^* = \{< x, \mu_{\mathcal{J}_2^*}(x), \sigma_{\mathcal{J}_2^*}(x), \gamma_{\mathcal{J}_2^*}(x) >: x \in \mathfrak{X}\}$

Then $\mathcal{J}_1^* \subseteq \mathcal{J}_2^* \Leftrightarrow \mu_{\mathcal{J}_1^*}(x) \leq \mu_{\mathcal{J}_2^*}(x), \sigma_{\mathcal{J}_1^*}(x) \leq \sigma_{\mathcal{J}_2^*}(x) \& \gamma_{\mathcal{J}_1^*}(x) \geq \gamma_{\mathcal{J}_2^*}(x)\}$

Definition 2.4 [13]. Let $\mathfrak{X}$ be a non-empty set, and Let $\mathcal{J}_1^*$ and $\mathcal{J}_2^*$ be two Neutrosophic sets are

$\mathcal{J}_1^* = \{< x, \mu_{\mathcal{J}_1^*}(x), \sigma_{\mathcal{J}_1^*}(x), \gamma_{\mathcal{J}_1^*}(x) >: x \in \mathfrak{X}\}$, $\mathcal{J}_2^* = \{< x, \mu_{\mathcal{J}_2^*}(x), \sigma_{\mathcal{J}_2^*}(x), \gamma_{\mathcal{J}_2^*}(x) >: x \in \mathfrak{X}\}$Then

1. $\mathcal{J}_1^* \cap \mathcal{J}_2^* = \{< x, \mu_{\mathcal{J}_1^*}(x) \cap \mu_{\mathcal{J}_2^*}(x), \sigma_{\mathcal{J}_1^*}(x) \cap \sigma_{\mathcal{J}_2^*}(x), \gamma_{\mathcal{J}_1^*}(x) \cup \gamma_{\mathcal{J}_2^*}(x) >: x \in \mathfrak{X}\}$

2. $\mathcal{J}_1^* \cup \mathcal{J}_2^* = \{< x, \mu_{\mathcal{J}_1^*}(x) \cup \mu_{\mathcal{J}_2^*}(x), \sigma_{\mathcal{J}_1^*}(x) \cup \sigma_{\mathcal{J}_2^*}(x), \gamma_{\mathcal{J}_1^*}(x) \cap \gamma_{\mathcal{J}_2^*}(x) >: x \in \mathfrak{X}\}$

Definition 2.5 [23]. Let $\mathfrak{X}$ be non-empty set and τ_N be the collection of Neutrosophic subsets of $\mathfrak{X}$ satisfying the following properties:

1.$0_N, 1_N \in \tau_N$

2.$T_1 \cap T_2 \in \tau_N$ for any $T_1, T_2 \in \tau_N$

3. $\cup T_i \in \tau_N$ for every $\{T_i : i \in j\} \subseteq \tau_N$

Then the space $(\mathfrak{X}, \tau_N)$ is called a Neutrosophic topological space(N-T-S).

The element of τ_N are called Ne.OS (Neutrosophic open set)

and its complement is Ne.CS(Neutrosophic closed set)

Example 2.6. Let $\mathfrak{X}$ ={x} and $\forall x \in \mathfrak{X}$

$A_1 = \langle x, \frac{6}{10}, \frac{6}{10}, \frac{5}{10} \rangle$, $A_2 = \langle x, \frac{5}{10}, \frac{7}{10}, \frac{9}{10} \rangle$

$A_3 = \langle x, \frac{6}{10}, \frac{7}{10}, \frac{5}{10} \rangle$, $A_4 = \langle x, \frac{5}{10}, \frac{6}{10}, \frac{9}{10} \rangle$

Then the collection $\tau_N = \{0_N, A_1, A_2, A_3, A_4, 1_N\}$ is called a N-T-S on $\mathfrak{X}$.

Definition 2.7. Let $(\mathfrak{X}, \tau_N)$be a N-T-S and $\mathcal{J}_1^* = \{< x, \mu_{\mathcal{J}_1^*}(x), \sigma_{\mathcal{J}_1^*}(x), \gamma_{\mathcal{J}_1^*}(x) >: x \in \mathfrak{X}\}$ be a Neutrosophic set in $\mathfrak{X}$. Then $\mathcal{J}_1^*$ is said to be

1. Neutrosophic b closed set [17] (Ne.bCS) if Ne.cl(Ne.int($\mathcal{J}_1^*$))∩Ne.int(Ne.cl($\mathcal{J}_1^*$))⊆ $\mathcal{J}_1^*$,
2. Neutrosophic α-closed set [7] (Ne. αCS) if Ne.cl(Ne.int(Ne.cl($\mathcal{J}_1^*$)))⊆ $\mathcal{J}_1^*$,
3. Neutrosophic pre-closed set [25] (Ne.Pre-CS) if Ne.cl(Ne.int($\mathcal{J}_1^*$))⊆ $\mathcal{J}_1^*$,
4. Neutrosophic regular closed set [7] (Ne.RCS) if Ne.cl(Ne.int($\mathcal{J}_1^*$)) = $\mathcal{J}_1^*$,
5. Neutrosophic semi closed set [7] (Ne.SCS) if Ne.int(Ne.cl($\mathcal{J}_1^*$))⊆ $\mathcal{J}_1^*$,
6. Neutrosophic generalized closed set [10] (Ne.GCS) if Ne.cl($\mathcal{J}_1^*$)⊆H whenever $\mathcal{J}_1^*$ ⊆H and H is aNe.OS,

7. Neutrosophic generalized pre closed set [17] (Ne.GPCS in short) if Ne.Pcl($\mathcal{J}_1^*$) $\subseteq$ H whenever $\mathcal{J}_1^*$ $\subseteq$ H and H is aNe.OS,
8. Neutrosophic α generalized closed set [15] (Ne. αGCS in short) if Neu α-cl($\mathcal{J}_1^*$)⊆H whenever $\mathcal{J}_1^*$ ⊆H and H is a Ne.OS,
9. Neutrosophic generalized semi closed set [24](Ne.GSCS in short) if Ne.Scl($\mathcal{J}_1^*$)⊆H whenever $\mathcal{J}_1^*$ ⊆H and H is a Ne.OS.
10. Neutrosophic generalized α closed set [11] (Ne. G α CS in short) if Neu α-cl($\mathcal{J}_1^*$)⊆H whenever $\mathcal{J}_1^*$ ⊆H and H is aNe. αOS.
11. Neutrosophic semi generalized closed set [24](Ne.SGCS in short) if Ne.Scl($\mathcal{J}_1^*$)⊆H whenever $\mathcal{J}_1^*$ ⊆H and H is a Ne.SOS.

Definition 2.8.[9] $(\mathfrak{X}, \tau_N)$be a N-T-S and $\mathcal{J}_1^* = \{< x, \mu_{\mathcal{J}_1^*}(x), \sigma_{\mathcal{J}_1^*}(x), \gamma_{\mathcal{J}_1^*}(x) >: x \in \mathfrak{X}\}$ be a

Neutrosophic set in $\mathfrak{X}$.Then

Neutrosophic closure of $\mathcal{J}_1^*$ is Ne.Cl($\mathcal{J}_1^*$)=∩{H:H is a Ne.CS in $\mathfrak{X}$ and $\mathcal{J}_1^*$ ⊆H}

Neutrosophic interior of $\mathcal{J}_1^*$ is Ne.Int($\mathcal{J}_1^*$)=∪{M:M is a Ne.OS in $\mathfrak{X}$ and M⊆ $\mathcal{J}_1^*$}.

Definition 2.9. Let $(\mathfrak{X}, \tau_N)$be a N-T-S and $\mathcal{J}_1^* = \{< x, \mu_{\mathcal{J}_1^*}(x), \sigma_{\mathcal{J}_1^*}(x), \gamma_{\mathcal{J}_1^*}(x) >: x \in X\}$ be a

Neutrosophic set in $\mathfrak{X}$. Then the Neutrosophic b closure of $\mathcal{J}_1^*$(Ne.bcl($\mathcal{J}_1^*$)in short) and Neutrosophic b interior of $\mathcal{J}_1^*$ (Ne.bint($\mathcal{J}_1^*$) in short) are defined as

Ne.bint($\mathcal{J}_1^*$)= ∪{ G/G is a Ne.bOS in $\mathfrak{X}$ and G⊆ $\mathcal{J}_1^*$},

Ne.bcl($\mathcal{J}_1^*$)= ∩{ K/K is a Ne.bCS in $\mathfrak{X}$ and $\mathcal{J}_1^*$ ⊆K }.

Proposition 2.10. Let $(\mathfrak{X}, \mathcal{N}_\tau)$ be any N-T-S. Let $\mathcal{J}_1^*$ and $\mathcal{J}_2^*$ be any two Neutrosophic sets in $(\mathfrak{X}, \tau_N)$.Then the Neutrosophic generalized b closure operator satisfy the following properties.

1. Ne.bcl(0_N)=0_N and Ne.bcl(1_N) = 1_N,
2. $\mathcal{J}_1^*$ ⊆Ne.bcl($\mathcal{J}_1^*$),
3. Ne.bint($\mathcal{J}_1^*$)⊆ $\mathcal{J}_1^*$,
4. If $\mathcal{J}_1^*$ is a Ne.bCS then $\mathcal{J}_1^*$=Ne.bcl(Ne.bcl($\mathcal{J}_1^*$)),
5. $\mathcal{J}_1^* \subseteq \mathcal{J}_2^*$ ⇒Ne.bcl($\mathcal{J}_1^*$) ⊆Ne.bcl($\mathcal{J}_2^*$),
6. $\mathcal{J}_1^* \subseteq \mathcal{J}_2^*$ ⇒Ne.bint($\mathcal{J}_1^*$)⊆Ne.bint($\mathcal{J}_2^*$).

NEUTROSOPHIC b GENERALIZED CLOSED SETS

In this part we introduce neutrosophicb bG closed sets its properties are discussed.

Definition 3.1.

A Ne. set $\mathcal{J}_1^*$ in an NSTS $(\mathfrak{X}, \mathcal{N}_\tau)$is called Neutrosophic b generalized CS (briefly Ne.(bG)CS) iff Ne.bCl($\mathcal{J}_1^*$)⊆$\mathcal{J}_2^*$, whenever $\mathcal{J}_1^*$⊆$\mathcal{J}_2^*$ and $\mathcal{J}_2^*$ is Ne. (b)OS in $\mathfrak{X}$.

Example 3.2.

Let $\mathfrak{X}$= {j_1, j_2}, $\mathcal{N}_\tau$= {0, $\mathcal{J}_1^*$, 1}, is a N.T.on $\mathfrak{X}$ where $\mathcal{J}_1^* = \langle x, \left(\frac{2}{10}, \frac{5}{10}, \frac{8}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right) \rangle$.

Then the Neutrosophic set $\mathcal{J}_2^* = \langle x, \left(\frac{7}{10}, \frac{5}{10}, \frac{3}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right) \rangle$is a Ne.bGCS in $\mathfrak{X}$.

Remark 3.3.

A Ne. set $\mathcal{J}_1^*$ in a NSTS $(\mathfrak{X}, \mathcal{N}_\tau)$ is called Ne.(b)generalized open (briefly Ne.(bG)OS) if its compliment ${\mathcal{J}_1^*}^c$is Ne.(bG)CS.

Theorem 3.4.

Every Ne.-CS in $(\mathfrak{X}, \mathcal{N}_\tau)$is Ne.(bG)CS.

Proof.

Let $\mathcal{J}_1^*$be a Ne.CS in NSTS $\mathfrak{X}$. Let $\mathcal{J}_1^* \subseteq \mathcal{J}_2^*$, where $\mathcal{J}_2^*$ is Ne.(b)OS in $\mathfrak{X}$. Since $\mathcal{J}_1^*$ is Ne.CS it is Ne.(b)CS and so NeuCl($\mathcal{J}_1^*$) =Ne.bCl($\mathcal{J}_1^*$)=$\mathcal{J}_1^* \subseteq \mathcal{J}_2^*$.Thus Ne.bCl($\mathcal{J}_1^*$)$\subseteq \mathcal{J}_2^*$. Hence $\mathcal{J}_1^*$ is Ne.(bG)CS.

Example 3.5

Let $\mathfrak{X}$= $\{j_1, j_2\}$, $\mathcal{N}_\tau$= $\{0, \mathcal{J}_1^*, 1\}$, is a N.T.on $\mathfrak{X}$ where$\mathcal{J}_1^* = \langle x, \left(\frac{3}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{2}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$. Then the

Neutrosophic set $\mathcal{J}_2^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{3}{10}\right)\rangle$ is a Ne.bGCS but not a Ne.CS in $\mathfrak{X}$

Theorem 3.6.

Every Ne.(b)CS in $(\mathfrak{X}, \mathcal{N}_\tau)$is Ne.(bG)CS.

Proof.

Let $\mathcal{J}_1^*$be a Ne.(b)CS in NSTS $\mathfrak{X}$. Let $\mathcal{J}_1^* \subseteq \mathcal{J}_2^*$. where$\mathcal{J}_2^*$is Ne.(b)OS in $\mathfrak{X}$. Since $\mathcal{J}_1^*$is Ne.(b)CS , Ne.bCl($\mathcal{J}_1^*$) =$\mathcal{J}_1^* \subseteq \mathcal{J}_2^*$. Thus Ne.bCl($\mathcal{J}_1^*$)$\subseteq \mathcal{J}_2^*$.Hence $\mathcal{J}_1^*$is Ne.(bG)CS.

Example 3.7. Let $\mathfrak{X}$= $\{j_1, j_2\}$, $\mathcal{N}_\tau$= $\{0, \mathcal{J}_1^*, 1\}$, is a N.T.on $\mathfrak{X}$

where $\mathcal{J}_1^* = \langle x, \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{8}{10}, \frac{5}{10}, \frac{2}{10}\right)\rangle$.Then the Neutrosophic set $\mathcal{J}_2^* = \langle x, \left(\frac{8}{10}, \frac{5}{10}, \frac{2}{10}\right), \left(\frac{9}{10}, \frac{5}{10}, \frac{1}{10}\right)\rangle$is a Ne.bGCS but not a Ne.bCS in $\mathfrak{X}$.

Remark 3.8.

(i).Every Ne. (bG)CS is Ne.(Gb)CS.

(ii). Every Ne.(sG)CS is Ne.(bG)CS.

(iii) Every Ne.(Gα)CS is Ne.(bG)CS.

Example 3.9.

Let $\mathfrak{X}$= $\{j_1, j_2\}$, $\mathcal{N}_\tau$= $\{0, \mathcal{J}_1^*, 1\}$, is a N.T.on $\mathfrak{X}$

where $\mathcal{J}_1^*, = \langle x, \left(\frac{3}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{2}{10}\right)\rangle$.Then the Neutrosophic set $\mathcal{J}_2^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{3}{10}\right)\rangle$

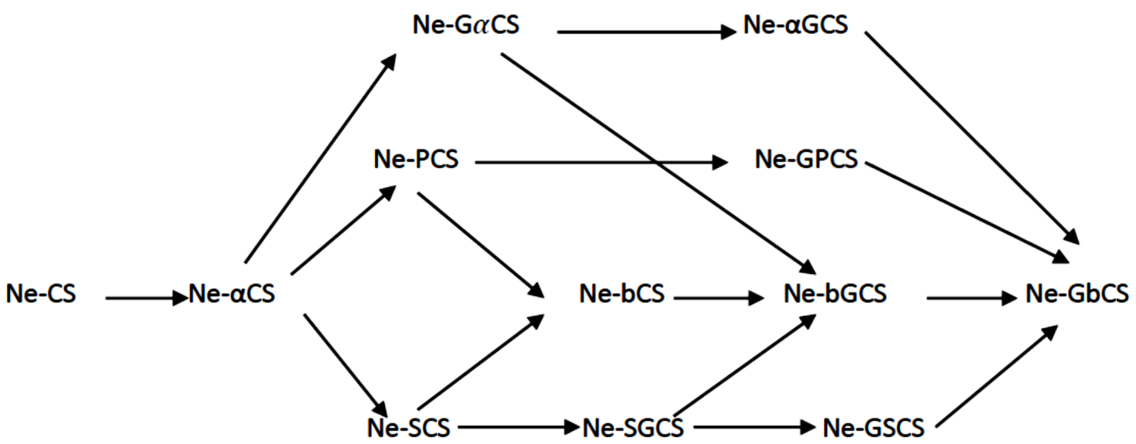
is a Ne.GbCS but not Ne.(bG)CS in $\mathfrak{X}$

Example 3.10.

Let $\mathfrak{X}$= $\{j_1, j_2\}$, $\mathcal{N}_\tau$= $\{0, \mathcal{J}_1^*, 1\}$, is a N.T.on $\mathfrak{X}$ where $\mathcal{J}_1^* = \langle x, \left(\frac{2}{10}, \frac{5}{10}, \frac{8}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$.

Then the Neutrosophic set $\mathcal{J}_2^* = \langle x, \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right)\rangle$is a Ne.bGCS in $\mathfrak{X}$ is not Ne.(sG)-CS

Diagram:1



Theorem 3.11.

A Ne. set $\mathcal{J}_1^*$ of a NSTS $(\mathfrak{X}, \mathcal{N}_\tau)$is called Ne.(bG)OSiff$\mathcal{J}_2^*$⊆Ne.bint $(\mathcal{J}_1^*)$, whenever $\mathcal{J}_2^*$ is Ne.(b)CS and $\mathcal{J}_2^*$⊆$\mathcal{J}_1^*$.

Proof.

Suppose $\mathcal{J}_1^*$is Ne.(bG)OS in $\mathfrak{X}$. Then ${\mathcal{J}_1^*}^c$is Ne.(bG)CS in $\mathfrak{X}$.Let $\mathcal{J}_2^*$ be a Ne.(b)CS in $\mathfrak{X}$ such that $\mathcal{J}_2^*$⊆$\mathcal{J}_1^*$. Then ${\mathcal{J}_1^*}^c$⊆${\mathcal{J}_2^*}^c$,${\mathcal{J}_2^*}^c$is Ne.(b)OS in $\mathfrak{X}$. Since ${\mathcal{J}_1^*}^c$is Ne.(bG)CS, Ne.bCl(${\mathcal{J}_1^*}^c$) ⊆${\mathcal{J}_2^*}^c$,which implies $(\mathrm{Ne.blnt}(\mathcal{J}_1^*))^c \subseteq {\mathcal{J}_2^*}^c$. Thus $\mathcal{J}_2^*$⊆Ne.blnt($\mathcal{J}_1^*$).

Conversely, assume that $\mathcal{J}_2^*$ ⊆Ne.bint($\mathcal{J}_1^*$), whenever $\mathcal{J}_2^* \subseteq \mathcal{J}_1^*$ and $\mathcal{J}_2^*$ isNe.(b)CS in $\mathfrak{X}$. Then $(\mathrm{Ne.blnt}(\mathcal{J}_1^*))^c \subseteq {\mathcal{J}_2^*}^c \subseteq \mathcal{J}_3^*$, where $\mathcal{J}_3^*$ is Ne.(b)OS in $\mathfrak{X}$. Hence Ne.bCl(${\mathcal{J}_1^*}^c$)⊆$\mathcal{J}_3^*$, which implies ${\mathcal{J}_1^*}^c$is Ne.(bG)CS.Therefore $\mathcal{J}_1^*$is Ne.(bG)OS.

Theorem 3.12.

If $\mathcal{J}_1^*$ is Ne.(bG)CS in $(\mathfrak{X}, \mathcal{N}_\tau)$ and $\mathcal{J}_1^*$⊆$\mathcal{J}_2^*$⊆Ne.bCl($\mathcal{J}_1^*$), then $\mathcal{J}_2^*$ is Ne.(bG)CS in $(\mathfrak{X}, \mathcal{N}_\tau)$.

Proof.

Let$\mathcal{J}_3^*$ be Ne.(b)-OS in $\mathfrak{X}$such that $\mathcal{J}_2^*$⊆$\mathcal{J}_3^*$., then $\mathcal{J}_1^*$⊆$\mathcal{J}_3^*$. Since $\mathcal{J}_1^*$is a Ne.(bG)CS in $\mathfrak{X}$, it follows that Ne.bCl($\mathcal{J}_1^*$)⊆$\mathcal{J}_3^*$. Now $\mathcal{J}_2^*$⊆Ne.bCl($\mathcal{J}_1^*$) implies Ne.bCl($\mathcal{J}_2^*$) ⊆Ne.bCl(Ne.bCl($\mathcal{J}_1^*$)) = Ne.bCl($\mathcal{J}_1^*$). Thus Ne.bCl($\mathcal{J}_2^*$)⊆$\mathcal{J}_3^*$. Hence $\mathcal{J}_2^*$is Ne.(bG)CS in $\mathfrak{X}$.

Theorem 3.13.

If $\mathcal{J}_1^*$ is Ne.(bG)OS in $(\mathfrak{X}, \mathcal{N}_\tau)$and Ne.blnt($\mathcal{J}_1^*$) ⊆$\mathcal{J}_2^*$⊆$\mathcal{J}_1^*$then $\mathcal{J}_2^*$ is Ne.(bG)-OS in $(\mathfrak{X}, \mathcal{N}_\tau)$.

Proof.

Let $\mathcal{J}_1^*$ be Ne.(bG)OS and $\mathcal{J}_2^*$be any Ne. set in $\mathfrak{X}$such that Ne.blnt($\mathcal{J}_1^*$)⊆$\mathcal{J}_2^*$⊆$\mathcal{J}_1^*$. Then ${\mathcal{J}_1^*}^c$is Ne.(bG)CS and ${\mathcal{J}_1^*}^c$⊆${\mathcal{J}_2^*}^c$⊆Ne.bCl(${\mathcal{J}_1^*}^c$). Then ${\mathcal{J}_2^*}^c$is Ne.(bG)CS.Hence $\mathcal{J}_2^*$is Ne.(bG)OS of $\mathfrak{X}$.

Theorem 3.14.

Finite intersection of Ne.(bG)CSs is a Ne.(bG)CS.

Proof.

Let$\mathcal{J}_1^*$and $\mathcal{J}_2^*$be Ne.(bG)CSs in $\mathfrak{X}$. Let $\mathfrak{F}$⊆$\mathcal{J}_1^* \cap \mathcal{J}_2^*$, where $\mathfrak{F}$is Ne.(b)CS in $\mathfrak{X}$. Then $\mathfrak{F}$⊆$\mathcal{J}_1^*$and $\mathfrak{F}$⊆$\mathcal{J}_2^*$.Since $\mathcal{J}_1^*$and $\mathcal{J}_2^*$are Ne.(bG)CSs, $\mathfrak{F}$⊆$\mathcal{J}_1^*$= Ne.blnt($\mathcal{J}_1^*$) and $\mathfrak{F}$⊆$\mathcal{J}_2^*$= Ne.bInt($\mathcal{J}_2^*$), which implies $\mathfrak{F}$⊆(Ne.blnt($\mathcal{J}_1^*$))∩(Ne.blnt($\mathcal{J}_2^*$)). Hence $\mathfrak{F}$⊆Ne.bInt($\mathcal{J}_1^* \cap \mathcal{J}_2^*$). Therefore $\mathcal{J}_1^* \cap \mathcal{J}_2^*$Ne.(bG)CS in $\mathfrak{X}$.

Theorem 3.15.

A finite union of Ne.(bG)OS is a Ne.(bG)OS.

Proof.

Let $\mathcal{J}_1^*$and $\mathcal{J}_2^*$be Ne.(bG)OS in $\mathfrak{X}$. Let $\mathcal{J}_1^* \cup \mathcal{J}_2^* \subseteq \mathfrak{F}$, where $\mathfrak{F}$is Ne.(b)OS in $\mathfrak{X}$. Then $\mathcal{J}_1^* \subseteq \mathfrak{F}$or $\mathcal{J}_2^* \subseteq \mathfrak{F}$.Since $\mathcal{J}_1^*$and $\mathcal{J}_2^*$are Ne.(bG)OS,Ne.bCl($\mathcal{J}_1^*$)=$\mathcal{J}_1^* \subseteq \mathfrak{F}$ or Ne.bCl($\mathcal{J}_2^*$)=$\mathcal{J}_2^* \subseteq \mathfrak{F}$, which implies Ne.bCl($\mathcal{J}_1^*$)∪Ne.bCl($\mathcal{J}_2^*$)⊆ $\mathfrak{F}$. Hence Ne.bCl($\mathcal{J}_1^* \cup \mathcal{J}_2^*$)⊆ $\mathfrak{F}$. Therefore $\mathcal{J}_1^* \cup \mathcal{J}_2^*$Ne.(bG)OS in $\mathfrak{X}$.

However, union of two Ne.(bG)CSs need not be a Ne.(bG)CSas shown in the following example.

Example 3.16.

Let $\mathfrak{X}$= $\{j_1, j_2\}$, $\mathcal{N}_\tau$= {0, $\mathcal{J}_1^*$, 1}, is a N.T.on $\mathfrak{X}$

Where $\mathcal{J}_1^* = \langle x, \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{8}{10}, \frac{5}{10}, \frac{2}{10}\right) \rangle$.

Then Neutrosophic set $\mathcal{J}_1^* = \langle x, \left(\frac{1}{10}, \frac{5}{10}, \frac{9}{10}\right), \left(\frac{8}{10}, \frac{5}{10}, \frac{2}{10}\right) \rangle$,

$\mathcal{J}_2^* = \langle x, \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{7}{10}, \frac{5}{10}, \frac{3}{10}\right) \rangle$is a are Ne.bGCSs but $\mathcal{J}_1^* \cup \mathcal{J}_2^*$ is not a Ne.bGCS in $\mathfrak{X}$,

Theorem 3.17.

If $\mathcal{J}_1^*$ is Ne.(b)OS in $(\mathfrak{X}, \mathcal{N}_\tau)$ and Ne.(bG)CS, then $\mathcal{J}_1^*$ is Ne.(b)CS in $(\mathfrak{X}, \mathcal{N}_\tau)$.

Proof.

Let $\mathcal{J}_1^*$be Ne.(b)OS and Ne.(bG)CS in $\mathfrak{X}$. For $\mathcal{J}_1^* \subseteq \mathcal{J}_1^*$, by definition Ne.bCl($\mathcal{J}_1^*$)⊆$\mathcal{J}_1^*$.

But $\mathcal{J}_1^*$⊆<u>Ne.bCl</u>($\mathcal{J}_1^*$),which implies $\mathcal{J}_1^*$=Ne.bCl($\mathcal{J}_1^*$).Hence $\mathcal{J}_1^*$is Ne.(b)CS in $\mathfrak{X}$.

Definition 3.18.

A NSTS $(\mathfrak{X}, \mathcal{N}_\tau)$ is called a Neutrosophic bT1/2 space (in short Ne.(b)T*1/2 space) if every Ne.(bG)CS in $\mathfrak{X}$ is Ne.-CS.

Definition 3.19.

A NSTS $(\mathfrak{X}, \mathcal{N}_\tau)$is called a Neutrosophic bT1/2 space (in short Ne.T1/2space) if every Ne.(bG)CS in $\mathfrak{X}$ is Ne.(b)CS.

Theorem 3.20.

A NSTS $(\mathfrak{X}, \mathcal{N}_\tau)$is Ne.(b)T1/2 space iff every Ne. set in $(\mathfrak{X}, \mathcal{N}_\tau)$is both Ne.(b)OS and Ne.(bG)OS.

Proof.

Let$\mathfrak{X}$be Ne.(b)T1/2space and let $\mathcal{J}_1^*$be Ne.(bG)OS in $\mathfrak{X}$. Then ${\mathcal{J}_1^*}^c$is Ne.(bG)CS $\mathfrak{X}$. By definition allNe.(bG)CS in $\mathfrak{X}$is Ne.(b)CS, so ${\mathcal{J}_1^*}^c$is Ne.(b)CS and hence $\mathcal{J}_1^*$is Ne.(b)OS in $\mathfrak{X}$.

Conversely, let$\mathcal{J}_1^*$be Ne.(bG)CS. Then ${\mathcal{J}_1^*}^c$is Ne.(bG)OS which implies ${\mathcal{J}_1^*}^c$isNe.(b)OS. Hence $\mathcal{J}_1^*$is Ne.(b)CS. Every Ne.(bG)CS in $\mathfrak{X}$is Ne.(b)CS.Therefore $\mathfrak{X}$is Ne.(b)T1/2 space.

Theorem 3.21.

A NSTS $(\mathfrak{X}, \mathcal{N}_\tau)$is Ne.(b)T1/2 space iff every Ne. set in $(\mathfrak{X}, \mathcal{N}_\tau)$is both Ne.OS and Ne.(bG)OS.

Remark 3.22.

A NSTS $(\mathfrak{X}, \mathcal{N}_\tau)$ is

(i) Ne.(b)T1/2space if every Ne.(bG)OS in $\mathfrak{X}$ is Ne.(b)OS.

(ii) Ne.(b)T*1/2space if ∀Ne.(Gb)OS in $\mathfrak{X}$ is Ne-open.

Remark 3.23.

In a NSTS ($(\mathfrak{X}, \mathcal{N}_\tau)$

(i) Every Ne.T1/2 space is Ne.(b)T1/2

(ii) Every Ne.(b)T1/2 space is Ne.(Gb)T1/2

(iii) Every Ne.(b)T1/2 space is Ne.(Gb)T1/2

4. Ne.(bG)-Continuous and Ne.(Gb)-closed mappings

In this section, Neutrosophic bg-CTS maps, Neutrosophic bg-irresolute maps, and Neutrosophic bg-homeomorphism in Neutrosophic topological spaces are introduced and studied.

Definition 4.1.

A mapping $\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$ is said to be Neutrosophic b generalized Continuous (Ne.(bG)-CTS), if $\varrho^{-1}(\mathcal{J}_1^*)$ is Ne.(bG)CS in $\mathfrak{X}$, for every Neutrosophic-CS $\mathcal{J}_1^*$ in $\mathfrak{Y}$.

Theorem 4.2.

$\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$isNe.(bG)-CTS iff the inverse image of each NSOS in $\mathfrak{Y}$ is Ne.(bG)OS in $\mathfrak{X}$.

Proof.

Let $\mathcal{J}_2^*$be a Ne.(bG)OS in $\mathfrak{Y}$. Then ${\mathcal{J}_2^*}^c$is Ne.(bG)CS in $\mathfrak{Y}$. Since ϱ is Ne.(bG)-CTS $\varrho^{-1}({\mathcal{J}_2^*}^c)$ = $(\varrho^{-1}(\mathcal{J}_2^*)\,)^c$is Ne.(bG)CS in $\mathfrak{X}$.Thus $\varrho^{-1}(\mathcal{J}_2^*)$ is Ne.(bG)OS in $\mathfrak{X}$.

Converse, is obvious.

Theorem 4.3.

Every Ne.-CTS map is Ne.(bG)-CTS.

Proof.

Let $\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$ be Ne.-CTS function. Let $\mathcal{J}_1^*$ be a Ne. OS in $\mathfrak{Y}$. Since ϱ is Ne.-CTS, ϱ^{-1} Ne. OS in$\mathfrak{X}$. Mean while each Ne.OS is Ne.(bG)OS, ϱ^{-1} is Ne.(bG)OS in $\mathfrak{X}$.Therefore ϱ is Ne.(bG)-CTS.

Example 4.4.

Let $\mathfrak{X}$= $\{j_1, j_2\}$= $\mathfrak{Y}$, $\mathcal{N}_\tau$= {0, $\mathcal{J}_1^*$, 1}, is a N.T.on $\mathfrak{X}$ $\mathcal{N}_\sigma$= {0, $\mathcal{J}_2^*$, 1} on $\mathfrak{Y}$,then Then the Neutrosophic sets

$\mathcal{J}_1^* = \langle \text{x}, \left(\frac{2}{10}, \frac{5}{10}, \frac{8}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$.

$\mathcal{J}_2^* = \langle \text{x}, \left(\frac{7}{10}, \frac{5}{10}, \frac{3}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right)\rangle$is a Ne.bGCS in $\mathfrak{X}$.

Identity mapping $\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$ ϱ is Ne.(Gb)-CTS but not Ne.-CTS

Definition 4.5

A mapping $\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$ is said to be Neutrosophic b-generalized irresolute (briefly Ne.(bG)-irresolute), if $\varrho^{-1}(\mathcal{J}_1^*)$ is Ne.(bG)CS set in $\mathfrak{X}$, for each Ne.(bG) CS $\mathcal{J}_1^*$ in $\mathfrak{Y}$.

Theorem 4.6.

Every Ne.(bG)-irresolute map is Ne.(bG)-CTS.

Proof.

Let $\varrho: \mathfrak{X} \longrightarrow \mathfrak{Y}$ be Ne.(bG)-irresolute and let $\mathcal{J}_1^*$be Ne.-CS in $\mathfrak{Y}$. Since every Ne.-CS is Also Ne.(bG)CS, $\mathcal{J}_1^*$ is Ne.(bG)CS in $\mathfrak{Y}$. Since $\varrho: \mathfrak{X} \longrightarrow \mathfrak{Y}$ is Ne.(bG)-irresolute, $\varrho^{-1}(\mathcal{J}_1^*)$ is Ne.(bG)CS. Thus inverse image of every Ne.CS in $\mathfrak{Y}$ is Ne.(bG)CS in $\mathfrak{X}$. Therefore the function $\varrho: \mathfrak{X} \longrightarrow \mathfrak{Y}$ is Ne.(bG)-CTS. The converse is not true.

Example 4.7.

Let $\mathfrak{X}$= $\{j_1, j_2\}$= $\mathfrak{Y}$, $\mathcal{N}_\tau$= {0, $\mathcal{J}_1^*$, 1}, is a N.T.on $\mathfrak{X}$ $\mathcal{N}_\sigma$= {0, $\mathcal{J}_2^*$, 1} on $\mathfrak{Y}$,then

Then the Neutrosophic sets

$\mathcal{J}_1^* = \langle \text{x}, \left(\frac{4}{10}, \frac{5}{10}, \frac{7}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{6}{10}\right)\rangle$, and$\mathcal{J}_2^* = \langle \text{x}, \left(\frac{8}{10}, \frac{5}{10}, \frac{3}{10}\right), \left(\frac{4}{10}, \frac{6}{10}, \frac{7}{10}\right)\rangle$.

Then Identity mapping $\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$

We have $\mathcal{J}_3^* = \langle \text{x}, \left(\frac{2}{10}, \frac{5}{10}, \frac{9}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{5}{10}\right)\rangle$ is a Ne.(bG)-CTS maps but not Ne.(bG)-irresolute maps.

Theorem 4.8.

Every Ne.(bG)- CTS map is Ne.(Gb) -CTS.

Proof.

Clear from the fact that Ne.(bG)CS is Ne.(Gb)CS.

Theorem 4.9.

Let $\varrho: \mathfrak{X} \longrightarrow \mathfrak{Y}$, $\zeta: \mathfrak{Y} \longrightarrow \mathfrak{Z}$be two mappings. Then

(i) ζ∘ϱ is Ne.(bG)-CTS, if ϱ is Ne.(bG)-CTS and ζ is Ne.-CTS.

(ii) ζ∘ϱ is Ne.(bG)- irresolute, if ϱ and ζ are Ne.(bG)- irresolute.

(iii) ζ∘ϱ is Ne.(bG)-CTS if ϱ is Ne.(bG)-irresolute and ζ is Ne.(bG)-CTS.

Proof.

(i)Let $\mathcal{J}_2^*$be Ne.CS in $\mathfrak{Z}$. Since ζ: $\mathfrak{Y} \longrightarrow \mathfrak{Z}$is Neutrosophic CTS, by definition $\zeta^{-1}(\mathcal{J}_2^*)$ is Ne.CS of $\mathfrak{Y}$. Now $\varrho: \mathfrak{X} \longrightarrow \mathfrak{Y}$is Ne.(bG)-CTS so $\varrho^{-1}(\zeta^{-1}(\mathcal{J}_2^*))$ =$(\zeta \circ \varrho)^{-1}(\mathcal{J}_2^*)$ is Ne.(bG)CS in $\mathfrak{X}$. Hence ζ∘ϱ: $\mathfrak{X} \longrightarrow \mathfrak{Z}$is Ne.(bG)-CTS.

(ii) Let ζ:$\mathfrak{Y} \longrightarrow \mathfrak{Z}$be Ne.(bG)-irresolute and let $\mathcal{J}_2^*$be Ne.(bG)CS subset in $\mathfrak{Z}$. Since ζis Ne.(bG)-irresolute by definition , $\zeta^{-1}(\mathcal{J}_2^*)$ is Ne.(bG)CS in $\mathfrak{Y}$. Also $\varrho: \mathfrak{X} \longrightarrow \mathfrak{Y}$is Ne.(bG)-irresolute, so $\varrho^{-1}(\zeta^{-1}(\mathcal{J}_2^*))$ =$(\zeta \circ \varrho)^{-1}(\mathcal{J}_2^*)$ is Ne.(bG)CS. Thus $\zeta \circ \varrho$:$\mathfrak{X} \longrightarrow \mathfrak{Z}$is Ne.(bG)-irresolute.

(iii) Let $\mathcal{J}_2^*$be Ne.(b)-CS in $\mathfrak{Z}$. Since ζ:$\mathfrak{Y} \longrightarrow \mathfrak{Z}$is Ne.(bG)-CTS, $\zeta^{-1}(\mathcal{J}_2^*)$ is Ne.(bG)CS in $\mathfrak{Y}$. Also $\varrho: \mathfrak{X} \longrightarrow \mathfrak{Y}$is Ne.(bG)-irresolute, so every Ne.(bG)CS in $\mathfrak{Y}$ is Ne.(bG)CS in $\mathfrak{X}$. Hence $\varrho^{-1}\zeta^{-1}(\mathcal{J}_2^*))$ = $(\zeta \circ \varrho)^{-1}$(H) is Ne.(bG)CS in $\mathfrak{X}$. Thus $\zeta \circ \varrho$: $\mathfrak{X} \longrightarrow \mathfrak{Z}$ is Ne.(bG)-irresolute.

Theorem 4.11.

If $\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$ is Ne.(b)*-CTS and ζ :($\mathfrak{Y}$, $\mathcal{N}_\sigma$)⟶ ($\mathfrak{Z}$, $\mathcal{N}_\delta$) is Ne.(bG)-CTS then $\zeta \circ \varrho$: $(\mathfrak{X}, \mathcal{N}_\tau)$ ⟶($\mathfrak{Z}$, $\mathcal{N}_\delta$)is Ne.(bG)CTS if $\mathfrak{Y}$ is Ne.(b)$T_{1/2}$-space.

Proof.

Suppose $\mathcal{J}_1^*$is Ne.(b)-CS subset of $\mathfrak{Z}$. Since ζ:$\mathfrak{Y} \longrightarrow \mathfrak{Z}$ is Ne.(bG)CTS $\zeta^{-1}(\mathcal{J}_2^*)$ is Ne.(bG)CS subset of $\mathfrak{Y}$. Now since $\mathfrak{Y}$ is Ne.(b)$T_{1/2}$-space, $\zeta^{-1}(\mathcal{J}_2^*)$ is Ne.(b)-CS subset of $\mathfrak{Y}$. Also since $\varrho: \mathfrak{X} \longrightarrow \mathfrak{Y}$ is Ne.(b)*-CTS $\varrho^{-1}(\zeta^{-1}(\mathcal{J}_2^*))$ =$(\zeta \circ \varrho)^{-1}(\mathcal{J}_2^*)$ is Ne.(b)-CS. Thus $\zeta \circ \varrho$: $\mathfrak{X} \longrightarrow \mathfrak{Z}$ is Ne.(bG)-CTS.

Theorem 4.12.

Let $\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$be Ne.(bG)-CTS. Then ϱ is Ne.(b)-CTS if $\mathfrak{X}$ is Ne.(b)$T_{1/2}$space.

Proof.

Let $\mathcal{J}_2^*$ be Ne.-CS in $\mathfrak{Y}$. Since $\varrho: \mathfrak{X} \longrightarrow \mathfrak{Y}$is Ne.(bG)CTS, $\varrho^{-1}(\mathcal{J}_2^*)$ is Ne.(bG)CS subset in $\mathfrak{X}$. Since $\mathfrak{X}$ is Ne.(b)$T_{1/2}$ space, by hypothesis every Ne.(bG)CS is Ne.(b)-CS . Hence $\varrho^{-1}(\mathcal{J}_1^*)$ is Ne.(b)CS subset in $\mathfrak{X}$. Therefore $\varrho: \mathfrak{X} \longrightarrow \mathfrak{Y}$ is Ne.(b)-CTS.

Theorem 4.13.

Let $\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$ be onto Ne.(bG)-irresolute and Ne. b*CS. If $\mathfrak{X}$ is Ne.(b)$T_{1/2}$-space, then $(\mathfrak{Y}, \mathcal{N}_\sigma)$ is Ne.(b)$T_{1/2}$-space.

Proof.

Let $\mathcal{J}_1^*$ be a Ne.(bG)CS in $\mathfrak{Y}$. Since $\varrho: \mathfrak{X} \longrightarrow \mathfrak{Y}$is Ne.(bG)irresolute,$\varrho^{-1}(\mathcal{J}_1^*)$ is Ne.(bG)CS in $\mathfrak{X}$. As $\mathfrak{X}$ is Ne.(b)$T_{1/2}$-space, $\varrho^{-1}(\mathcal{J}_1^*)$ is Ne.(b)CS in $\mathfrak{X}$. Also $\varrho: \mathfrak{X} \longrightarrow \mathfrak{Y}$ is Ne. b*CS, $\varrho(\varrho^{-1}(\mathcal{J}_1^*))$ is Ne.(b)CS in $\mathfrak{Y}$. Since $\varrho: \mathfrak{X} \longrightarrow \mathfrak{Y}$is onto, $\varrho(\varrho^{-1}(\mathcal{J}_1^*))$ =$\mathcal{J}_1^*$. Thus $\mathcal{J}_1^*$is Ne.(b)CS in $\mathfrak{Y}$. Hence$(\mathfrak{Y}, \mathcal{N}_\sigma)$is also Ne.(b)$T_{1/2}$-space.

Theorem 4.14.

Let $\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$ be Ne.(bG)-CTS and ζ :$(\mathfrak{Y}, \mathcal{N}_\sigma)$ ⟶$(\mathfrak{Z},\mathcal{N}_\delta)$ be Ne.g-CTS. Then $\zeta \circ \varrho$ is Ne.(bG)- CTS if $\mathfrak{Y}$ is Ne.$T_{1/2}$ space.

Proof.

Let $\mathcal{J}_1^*$ be Ne.-CS in $\mathfrak{Z}$. Since ζ is Ne.g-CTS, $\zeta^{-1}(\mathcal{J}_1^*)$ is Ne.g-CS in $\mathfrak{Y}$. But $\mathfrak{Y}$is Ne.T1/2 space and so $\zeta^{-1}(\mathcal{J}_1^*)$ is Ne.-CS in $\mathfrak{Y}$. Since ϱ is Ne.(bG)-CTS $\varrho^{-1}(\zeta^{-1}(\mathcal{J}_1^*)) = (\zeta \circ \varrho)^{-1}(\mathcal{J}_1^*)$ is Ne.(bG)CS in $\mathfrak{X}$. Hence $\zeta \circ \varrho$ Ne.(bG)-CTS.

Theorem 4.15.

If the bijective map $\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$ is Ne.(b)*-open and Ne.(b)-irresolute, then $\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$ is Ne.(bG)-irresolute.

Proof.

Let $\mathcal{J}_1^*$ be a Ne.(bG)CS in $\mathfrak{Y}$ and let $\varrho^{-1}(\mathcal{J}_1^*) \subseteq \mathcal{J}_2^*$where $\mathcal{J}_2^*$is a Ne.(b)OS in $\mathfrak{X}$. Clearly, $\mathcal{J}_1^* \subseteq \varrho(\mathcal{J}_2^*)$. Since $\varrho: \mathfrak{X} \longrightarrow \mathfrak{Y}$is Ne.(b)*-open map, $\varrho(\mathcal{J}_2^*)$ is Ne.(b)-open in $\mathfrak{Y}$ and $\mathcal{J}_1^*$ is Ne.(bG)CS in $\mathfrak{Y}$.Then Ne.bCl($\mathcal{J}_1^*$)⊆$\varrho(\mathcal{J}_2^*)$, and thus ϱ^{-1}(Ne.bCl($\mathcal{J}_1^*$))⊆$\mathcal{J}_2^*$. Also $\varrho: \mathfrak{X} \longrightarrow \mathfrak{Y}$ irresolute and Ne.bCl($\mathcal{J}_1^*$) is a Ne.(b)-CS in $\mathfrak{Y}$, then ϱ^{-1}(Ne.bCl($\mathcal{J}_1^*$)) is Ne.(b)CS in $\mathfrak{X}$. Thus Ne.bCl($\varrho^{-1}(\mathcal{J}_1^*)$)⊆Ne.bCl($\varrho^{-1}$Ne. bCl($\mathcal{J}_1^*$)))⊆$\mathcal{J}_2^*$. So $\varrho^{-1}(\mathcal{J}_1^*)$ is Ne.(bG)CS in $\mathfrak{X}$. Hence $\varrho: \mathfrak{X} \longrightarrow \mathfrak{Y}$is Ne.(bG)-irresolute map.

Definition 4.16.

A mapping $\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$is said to be Neutrosophic bg-open (briefly Ne.(bG)OS) if the image of every Ne.-OS in $\mathfrak{X}$, is Ne.(bG)OS in $\mathfrak{Y}$.

Definition 4.17.

A mapping $\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$is said to be Neutrosophic bg-CS (briefly Ne.(bG)CS) if the image of every Ne.CS in $\mathfrak{X}$ is Ne.(bG)CS in $\mathfrak{Y}$.

Definition 4.18.

A mapping $\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$ is said to be Neutrosophic bg*-open (briefly Ne.(bG)*-OS) if the image of every Ne.(bG)OS in $\mathfrak{X}$ is Ne.(bG)OS in $\mathfrak{Y}$.

Definition 4.19.

A mapping $\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$is said to be Neutrosophic bg-CS (briefly Ne.(bG)*-CS) if the image of every Ne.(bG)CS in $\mathfrak{X}$ is Ne.(bG)CS in $\mathfrak{Y}$.

Remark 4.20.

(i)Every Ne.(bG)*-CS mapping is Ne.(bG)CS.

(ii)Every Ne.(bG)*-CS mapping is Ne.(Gb)* -CS.

Theorem 4.23.

If $\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$) is Ne.CS and $\zeta :(\mathfrak{Y}, \mathcal{N}_\sigma) \longrightarrow (\mathfrak{Z},\mathcal{N}_\delta)$ is Ne.(bG)CS, then $\zeta \circ \varrho$ is Ne.(bG)CS.

Proof.

Let $\mathcal{J}_1^*$ be a Ne.CS in $\mathfrak{X}$. Then $\varrho(\mathcal{J}_1^*)$ is Ne.CSin $\mathfrak{Y}$. Since $\zeta :(\mathfrak{Y}, \mathcal{N}_\sigma) \longrightarrow (\mathfrak{Z},\mathcal{N}_\delta)$is Ne.(bG)CS, $\zeta(\varrho(\mathcal{J}_1^*)) = (\zeta \circ \varrho)(\mathcal{J}_1^*)$ is Ne.(bG)CS in $\mathfrak{Z}$. Therefore $\zeta \circ \varrho$ is Ne.(bG)CS.

Theorem 4.24.

If $\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$ is a Ne.(bG)CS map and $\mathfrak{Y}$ is Ne.(b)T1/2 space, then ϱ is a Ne.-CS.

Proof.

Let $\mathcal{J}_1^*$ be a Ne.CS in $\mathfrak{X}$. Then $\varrho(\mathcal{J}_1^*)$ is Ne.(Gb)-CS in $\mathfrak{Y}$,since ϱ is Ne.(Gb)CS. Again since $\mathfrak{Y}$ is Ne.(b)T1/2space, $\varrho(\mathcal{J}_1^*)$ is Ne.-CS in $\mathfrak{Y}$. Hence $\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$is a ϱ-CS.

Theorem 4.25.

If $\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$ is a Ne.(bG)CS map and $\mathfrak{Y}$ is Ne.(b)T1/2 space, then ϱ is a Ne.(b)-CS map.

Theorem 4.26.

A mapping $\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$ is Ne.(bG)CS iff for each Ne. set $\mathcal{J}_1^*$ in $\mathfrak{Y}$ and Ne.OS $\mathcal{J}_2^*$ such that $\varrho^{-1}(\mathcal{J}_1^*) \subseteq \mathcal{J}_2^*$, there is a Ne.(bG)OS$\mathcal{J}_3^*$ of $\mathfrak{Y}$ such that $\mathcal{J}_1^* \subseteq \mathcal{J}_3^*$ and $\varrho^{-1}(\mathcal{J}_3^*) \subseteq \mathcal{J}_2^*$.

Proof.

Suppose ϱ is Ne.(bG)CS map. Let $\mathcal{J}_1^*$ be a Ne. set of $\mathfrak{F}$,and $\mathcal{J}_2^*$be an Ne.OS of $\mathfrak{X}$, such that $\varrho^{-1}(\mathcal{J}_1^*)\subseteq\mathcal{J}_2^*$. Then $\mathcal{J}_3^*=(\mathcal{J}_2^{*C})^c$is a Ne.(bG)OS in $\mathfrak{F}$ such that $\mathcal{J}_1^*\subseteq\mathcal{J}_3^*$and $\varrho^{-1}(\mathcal{J}_3^*)\subseteq\mathcal{J}_2^*$.

Conversely, suppose that $\mathfrak{F}$ is a Ne.CS of$\mathfrak{X}$. Then $\varrho^{-1}((\varrho(\mathfrak{F}))^C)\subseteq\mathfrak{F}^C$, and $\mathfrak{F}^C$, is Ne.OS. By hypothesis, there is a Ne.(bG)OS $\mathcal{J}_3^*$ of $\mathfrak{Y}$ such that $(\varrho(\mathfrak{F}))^C\subseteq\mathcal{J}_3^*$and $\varrho^{-1}(\mathcal{J}_3^*)\subseteq\mathfrak{F}^C$. Therefore $\mathfrak{F}\subseteq\left(\varrho^{-1}\left(\mathcal{J}_3^*\right)\right)^C$ Hence $\mathcal{J}_3^{*C}\subseteq\varrho(\mathcal{J}_3^*)\subseteq\varrho\left(\varrho^{-1}\left(\mathcal{J}_3^*\right)\right)^C\subseteq\mathcal{J}_3^{*C}$, which implies $\varrho(\mathfrak{F})=\mathcal{J}_3^{*C}$. Since $\mathcal{J}_3^{*C}$is Ne.(bG)CS, ϱ($\mathfrak{F}$) is Ne.(bG)CS and thus ϱ is a Ne.(bG)CS map.

Theorem 4.27.

If $\varrho:(\mathfrak{X},\mathcal{N}_\tau)\longrightarrow(\mathfrak{Y},\mathcal{N}_\sigma)$ and $\zeta:(\mathfrak{Y},\mathcal{N}_\sigma)\longrightarrow(\mathfrak{Z},\mathcal{N}_\delta)$ are Ne.(bG)CS maps and $\mathfrak{Y}$ is Ne.(b)$T_{1/2}$ space, then $\zeta\circ\varrho:\mathfrak{X}\longrightarrow\mathfrak{Z}$ is Ne.(bG)CS.

Proof. Let $\mathcal{J}_1^*$ be a Ne.-CS in $\mathfrak{X}$. Since $\varrho:(\mathfrak{X},\mathcal{N}_\tau)\longrightarrow(\mathfrak{Y},\mathcal{N}_\sigma)$is Ne.(bG)CS, $\varrho(\mathcal{J}_1^*)$ is Ne.(bG)CS in $\mathfrak{F}$. Now $\mathfrak{F}$ is Ne.(b)$T_{1/2}$space, so $\varrho(\mathcal{J}_1^*)$ is Ne.-CS in $\mathfrak{F}$. Also $\zeta:\mathfrak{Y}\longrightarrow\mathfrak{Z}$ is Ne.(bG)CS, $\zeta(\varrho(\mathcal{J}_1^*))=(\zeta\circ\varrho)((\mathcal{J}_1^*)$ is Ne.(bG)CS in $\mathfrak{Z}$. Therefore $\zeta\circ\varrho$ is Ne.(bG)CS.

Theorem 4.28. If $\mathcal{J}_1^*$ is Ne.(bG)CS in $\mathfrak{X}$ and $\varrho:\mathfrak{X}\longrightarrow\mathfrak{Y}$ is bijective, Ne.(b)-irresolute and Ne.(bG)CS, then $\varrho(\mathcal{J}_1^*)$ is Ne.(bG)CS in $\mathfrak{Y}$.

Proof.

Let $\varrho(\mathcal{J}_1^*)\subseteq\mathcal{J}_2^*$ where $\mathcal{J}_2^*$is Ne.(b)OS in $\mathfrak{F}$. Since ϱ is Ne.(b)irresolute,$\varrho^{-1}(\mathcal{J}_2^*)$ is Ne.(b)OS containing $\mathcal{J}_1^*$. Hence Ne.bCl($\mathcal{J}_1^*$)$\subseteq\varrho^{-1}(\mathcal{J}_2^*)$ as $\mathcal{J}_1^*$ is Ne.(bG)CS. Since ϱ is Ne.(bG)CS, ϱ(Ne.bCl($\mathcal{J}_1^*$)) is Ne.(bG)CS contained in the Ne.(b)OS $\mathcal{J}_2^*$, which implies Ne.bCl(ϱ(Ne.bCl($\mathcal{J}_1^*$))) $\subseteq\mathcal{J}_2^*$and hence Ne.bCl(ϱ($\mathcal{J}_1^*$))$\subseteq\mathcal{J}_2^*$. So ϱ($\mathcal{J}_1^*$) is Ne.(bG)CS in $\mathfrak{F}$.

Theorem 4.29.

If $\varrho:(\mathfrak{X},\mathcal{N}_\tau)\longrightarrow(\mathfrak{Y},\mathcal{N}_\sigma)$ is Ne.(bG)CS and $\zeta:(\mathfrak{Y},\mathcal{N}_\sigma)\longrightarrow(\mathfrak{Z},\mathcal{N}_\delta)$is Ne.(bG)*-CS, then $\zeta\circ\varrho$ is Ne.(bG) -CS.

Proof. Let $\mathcal{J}_1^*$ be Ne.CS in $\mathfrak{X}$. Then ϱ($\mathcal{J}_1^*$) is Ne.(bG)CS in $\mathfrak{Y}$.Since $\zeta:(\mathfrak{Y},\mathcal{N}_\sigma)\longrightarrow(\mathfrak{Z},\mathcal{N}_\delta)$is Ne.(bG)*-CS. Thus $\zeta(\varrho(\mathcal{J}_1^*))=(\zeta\circ\varrho)(\mathcal{J}_1^*)$ is Ne.(bG)CS in $\mathfrak{Z}$. Therefore $\zeta\circ\varrho$ is Ne.(bG)CS.If $\varrho:(\mathfrak{X},\mathcal{N}_\tau)\longrightarrow(\mathfrak{Y},\mathcal{N}_\sigma)$and $\zeta:(\mathfrak{Y},\mathcal{N}_\sigma)\longrightarrow(\mathfrak{Z},\mathcal{N}_\delta)$ are Ne.(bG)*CS maps, then $\zeta\circ\varrho:\mathfrak{X}\longrightarrow\mathfrak{Z}$ is Ne.(bG)* CS.

Theorem 4.30.

Let $\varrho:(\mathfrak{X},\mathcal{N}_\tau)\longrightarrow(\mathfrak{Y},\mathcal{N}_\sigma)$, $\zeta:(\mathfrak{Y},\mathcal{N}_\sigma)\longrightarrow(\mathfrak{Z},\mathcal{N}_\delta)$be two maps such that $\zeta\circ\varrho:\mathfrak{X}\longrightarrow\mathfrak{Z}$ is Ne.(bG)CS.

(i) If ϱ is Ne.-CTS and surjective, then ζ is Ne.(bG)CS.

(ii) If ζ is Ne.(bG)-irresolute and injective, then ϱ is Ne.(bG)CS.

Proof.

(i). Let $\mathfrak{F}$ be Ne.CS in $\mathfrak{Y}$. Then $\varrho^{-1}(\mathfrak{F})$is Ne.CS in $\mathfrak{X}$,as ϱ is Ne.-CTS. Since $\zeta\circ\varrho$ is Ne.(bG)CS map and ϱ is surjective,$(\zeta\circ\varrho)(\varrho^{-1}(\mathfrak{F}))=\zeta(\mathfrak{F})$ is Ne.(bG)CS in $\mathfrak{Z}$. Hence $\zeta:\mathfrak{Y}\longrightarrow\mathfrak{Z}$ is Ne.(bG)CS.

(ii).Let $\mathfrak{F}$ be a Ne. CS in $\mathfrak{X}$. Then $(\zeta\circ\varrho)(\mathfrak{F})$ is Ne.(bG)CS in $\mathfrak{Z}$. Since ζ is Ne.(bG)-irresolute and injective $\zeta^{-1}(\zeta\circ\varrho)(\mathfrak{F})=\varrho(\mathfrak{F})$ is Ne.(bG)CS in $\mathfrak{Y}$. Hence ϱ is a Ne.(bG)CS.

Theorem 4.31.

Let $\varrho:(\mathfrak{X},\mathcal{N}_\tau)\longrightarrow(\mathfrak{Y},\mathcal{N}_\sigma)$, $\zeta:(\mathfrak{Y},\mathcal{N}_\sigma)\longrightarrow(\mathfrak{Z},\mathcal{N}_\delta)$be two maps such that $\zeta\circ\varrho:\mathfrak{X}\longrightarrow\mathfrak{Z}$ is Ne.(bG)*CS map.

(i) If ϱ is Ne.(bG)-CTS and surjective, then ζ is Ne.(bG)CS.

(ii) If ζ is Ne.(bG)-irresolute and injective, then ϱ is Ne.(bG)*-CS.

Theorem 4.32. Let $\varrho:(\mathfrak{X},\mathcal{N}_\tau)\longrightarrow(\mathfrak{Y},\mathcal{N}_\sigma)$ then the following statements are equivalent

(i) ϱ is Ne.(bG)-irresolute.

(ii) for every Ne.(bG)CS $\mathcal{J}_1^*$ in $\mathfrak{Y}$,$\varrho^{-1}(\mathcal{J}_1^*)$ is Ne.(bG)CS in $\mathfrak{X}$.

Proof.(i)⇒ (ii)Obvious.

(ii)⇒(i) Let $\mathcal{J}_1^*$is a Ne.(bG)CS in $\mathfrak{Y}$ which implies $\mathcal{J}_1^{*c}$, is Ne.(bG)OS in $\mathfrak{Y}$. $\varrho^{-1}(\mathcal{J}_1^{*c})$ is Ne.(bG)-open in $\mathfrak{X}$ implies $\varrho^{-1}(\mathcal{J}_1^*)$is Ne.(bG)CS in $\mathfrak{X}$.Hence ϱ is Ne.(bG)-irresolute.

Neutrosophic bg- homeomorphism and Neutrosophic bg*-homeomorphism are defined as follows.

Definition 4.33.

A mapping $\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$is called Neutrosophic bg-homeomorphism (briefly Ne.(b)-homeomorphism) if ϱ and ϱ^{-1} are Ne.(bG)CTS.

Definition 4.34.

A mapping $\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$is called Neutrosophic bg*-homeomorphism (briefly Ne.(bG)*-homeomorphism) if ϱ and $\varrho^{-1}$are Ne.(bG)irresolute.

Theorem 4.35.

Every Ne.-homeomorphism is Ne.(bG)-homeomorphism.

The converse of the above theorem need not be true as seen from the following example.

Example 4.36.

Let $\mathfrak{X}$= {j_1, j_2}= $\mathfrak{Y}$, $\mathcal{N}_\tau$= {0, $\mathcal{J}_1^*$, 1}, is a N.T.on $\mathfrak{X}$ $\mathcal{N}_\sigma$= {0, $\mathcal{J}_2^*$, 1} on $\mathfrak{Y}$,

Then Neutrosophic sets

$$\mathcal{J}_1^* = \langle x, \left(\frac{10}{10}, \frac{5}{10}, \frac{0}{10}\right), \left(\frac{8}{10}, \frac{5}{10}, \frac{2}{10}\right) \rangle,$$

$$\mathcal{J}_2^* = \langle x, \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right), \left(\frac{6}{10}, \frac{6}{10}, \frac{4}{10}\right) \rangle \text{ . and } \mathcal{J}_3^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right) \rangle$$

Define mapping $\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$ by ϱ(j_1)= j_1 and ϱ(j_2) = j_2

Then ϱ is Ne.(bG)-homeomorphism but not Ne.-homeomorphism

Theorem 4.37.

Every Ne.(bG)*-homeomorphism is Ne.(bG)- homeomorphism.

Proof.

Let $\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$ be Ne.(Gb)*-homeomorphism. Then ϱ and ϱ^{-1} are Ne.(bG)-irresolute mappings. By theorem 4.7 ϱ and ϱ^{-1} are Ne.(bG)-CTS. Hence $\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$ is Ne.(bG)-homeomorphism.

Theorem 4.38.

If $\varrho: (\mathfrak{X}, \mathcal{N}_\tau) \longrightarrow (\mathfrak{Y}, \mathcal{N}_\sigma)$is Ne.(bG)-homeomorphism and

ζ :$(\mathfrak{Y}, \mathcal{N}_\sigma)$ ⟶$(\mathfrak{Z}, \mathcal{N}_\delta)$ is Ne.(bG)-homeomorphism and $\mathfrak{Y}$ is Ne.(b)T1/2 space,

then ζ ∘ ϱ: $\mathfrak{X} \longrightarrow \mathfrak{Z}$ is Ne.(bG)-homeomorphism.

Proof.

To show that ζ ∘ ϱ and $(ζ ∘ ϱ)^{-1}$are Ne.(bG)- CTS. Let $\mathcal{J}_1^*$be a Ne.OS in $\mathfrak{Z}$. Since ζ: $\mathfrak{Y} \longrightarrow \mathfrak{Z}$is Ne.(bG)- CTS, $\zeta^{-1}(\mathcal{J}_1^*)$ is Ne.(bG)open in $\mathfrak{Y}$. Then $\zeta^{-1}(\mathcal{J}_1^*)$ is a Ne.-open in $\mathfrak{Y}$as $\mathfrak{Y}$ is Ne.(b)T1/2 space. Also since ϱ :$\mathfrak{X} \longrightarrow \mathfrak{Y}$ is Ne.(bG)- CTS, $\varrho^{-1}(\zeta^{-1}(\mathcal{J}_1^*))$ = $(ζ ∘ ϱ)^{-1}(\mathcal{J}_1^*)$ is Ne.(bG)-open in $\mathfrak{X}$. Therefore ζ ∘ ϱ is Ne.(bG) CTS. Again, let $\mathcal{J}_1^*$ be a Ne.OS in $\mathfrak{X}$. Since ϱ^{-1} : $\mathfrak{Y} \longrightarrow \mathfrak{X}$ is Ne.(bG)- CTS, $(\varrho^{-1})^{-1}(\mathcal{J}_1^*)$) = ϱ($\mathcal{J}_1^*$) is Ne.(bG)OS in $\mathfrak{Y}$. And so ϱ($\mathcal{J}_1^*$) is Ne.-open in $\mathfrak{Y}$ since $\mathfrak{Y}$ is Ne.(b)T1/2 space. Also since ζ^{-1}:$\mathfrak{Z} \longrightarrow \mathfrak{Y}$ is Ne.(bG)-CTS, $(\zeta^{-1})^{-1}$(ϱ($\mathcal{J}_1^*$)) = ζ(ϱ($\mathcal{J}_1^*$)) = (ζ ∘ ϱ)($\mathcal{J}_1^*$) is Ne.(bG)-open in $\mathfrak{Z}$. Therefore $((ζ ∘ ϱ)^{-1})^{-1}(\mathcal{J}_1^*)$ = (ζ ∘ ϱ)($\mathcal{J}_1^*$) is Ne.(bG)OS in $\mathfrak{Z}$. Hence $(ζ ∘ ϱ)^{-1}$ is Ne.(bG) - CTS. Thus ζ ∘ ϱ is Ne.(bG) - homeomorphism.

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Neutrosophic Vague Line Graphs

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Abstract: Neutrosophic graphs are employed as a mathematical key to hold an imprecise and unspecified data. Vague sets gives more intuitive graphical notation of vague information, that delicates crucially better analysis in data relationships, incompleteness and similarity measures. In this paper, the neutrosophic vague line graphs are introduced. The necessary and sufficient condition for a line graph to be neutrosophic vague line graph is provided. Further, homomorphism, weak vertex and weak line isomorphism are discussed. The given results are illustrated with suitable example.

Keywords: Neutrosophic vague line graph, Weak isomorphism of neutrosophic vague line graph, Homomorphism.

1. Introduction

The line graph, $L(G)$, of a graph G is the intersection graph of the set of lines of G. Hence the vertices of $L(G)$ are the lines of G with two vertices of $L(G)$ adjacent whenever the corresponding lines of G are adjacent [20]. Vague sets are denoted as a higher-order fuzzy sets which develops the solution process are more complex to obtain the results more accurate than fuzzy but not affecting the complexity on computation time/volume and memory space. Can we see an example, suppose there are 10 patients to check a pandemic during testing. In which, there are four patients having positive, five will have negative and one is undecided or yet to come. In the view of neutrosophic concepts, the mathematical form is represented as $x(0.4,0.1,0.5)$. Thus it is clear that, the neutrosophic field arises to hold the indeterminacy data. It generalizes the fuzzy sets and intuitionistic sets from the philosophical viewpoint. The single-valued neutrosophic set is the generalisation of intuitionistic fuzzy sets and is used expediently to deal with real-world problems, especially in decision support [1, 2, 3]. The computation of believe in that element (truth), the disbelieve in that element (falsehood) and the indeterminacy part of that element with the sum of these three components are strictly less than 1. The neutrosophic set is introduced by the author Smarandache in order to use the inconsistent and indeterminate information, and has been studied extensively (see [28]-[33]). In the definition of neutrosophic set, the indeterminacy value is quantified explicitly and truth-membership, indeterminacy-membership and false-membership are defined completely independent with the sum of these values lies between 0 and 3. Neutrosophic set and related notions paid attention by the researchers in many weird domains. The combination of neutrosophic set and vague set are introduced by Alkhazaleh in 2015 [6]. Single valued neutrosophic graph are established in [11, 12].

The neutrosophic graph is efficiently model the inconsistent information about any real-life problem. Some types of neutrosophic graphs and co-neutrosophic graphs are discussed in [16]. Intuitionistic bipolar neutrosophic set and its application to graphs are established in [25]. Al-Quran and Hassan in [5] introduced a combination of neutrosophic vague set and soft expert set to improving the reason-ability of decision making in real life application. Neutrosophic vague graphs are investigated in [24]. Comparative study of regular and (highly) irregular vague graphs with applications are obtained in [13]. Furthermore, some properties of degree of vague graphs, domination number and regularity properties of vague graphs are established by the author Borzooei in [7, 8, 9]. Neutrosophic vague set theory was extensively studied in [6]. The concept of a single-valued neutrosophic line graph of a single-valued neutrosophic graph is introduced by the authors in [21]. In which, a necessary and sufficient condition for a single-valued neutrosophic graph to be isomorphic to its corresponding single-valued neutrosophic line graph. Further, some remarkable properties of strong neutrosophic vague graphs, complete neutrosophic vague graphs and self-complementary neutrosophic vague graphs are investigated in [24]. Moreover, Cartesian product, lexicographic product, cross product, strong product and composition of neutrosophic vague graphs are investigated in [22]. As far, there exists no research work on the concept of neutrosophic vague line graphs until now. In order to fill this gap in the literature and motivated by papers [6, 21, 24], we put forward a new idea concerning the neutrosophic vague line graphs. The main contributions of this paper are as follows:

- Neutrosophic Vague Line Graphs (NVLGs) are introduced and explained with an example. The obtained neutrosophic vague line graph $L(\mathbb{G})$ is a strong neutrosophic vague graph.
- The necessary and sufficient condition for a line graph to be NVLG is formulated with supporting proofs.
- Furthermore, the results of homomorphism, weak vertex and weak line isomorphism are developed.

The manuscript is organised as follows: The basic definitions and example which are essential for the main results are given in Section 2. The necessary and sufficient condition of NVLG are provided and also the definition of NVLGs, homomorphism and weak isomorphism are given in Section 3. Finally, a conclusion is provided.

2 Preliminaries

In this section, basic definitions and example are given.

Definition 2.1 *[34] A vague set $\mathbb{A}$ on a non empty set $\mathbb{X}$ is a pair $(T_{\mathbb{A}}, F_{\mathbb{A}})$, where $T_{\mathbb{A}}: \mathbb{X} \to [0,1]$ and $F_{\mathbb{A}}: \mathbb{X} \to [0,1]$ are true membership and false membership functions, respectively, such that*
$0 \le T_{\mathbb{A}}(x) + F_{\mathbb{A}}(x) \le 1$ for any $x \in \mathbb{X}$.

Let $\mathbb{X}$ and $\mathbb{Y}$ be two non-empty sets. A vague relation $\mathbb{R}$ of $\mathbb{X}$ to $\mathbb{Y}$ is a vague set $\mathbb{R}$ on $\mathbb{X} \times \mathbb{Y}$ that is $\mathbb{R} = (T_{\mathbb{R}}, F_{\mathbb{R}})$, where $T_{\mathbb{R}}: \mathbb{X} \times \mathbb{Y} \to [0,1], F_{\mathbb{R}}: \mathbb{X} \times \mathbb{Y} \to [0,1]$ and satisfy the condition: $0 \leq T_{\mathbb{R}}(x,y) + F_{\mathbb{R}}(x,y) \leq 1$ for any $x, y \in \mathbb{X}$.

Definition 2.2 *[7] Let $\mathbb{G}^* = (\mathbb{V}, \mathbb{E})$ be a graph. A pair $\mathbb{G} = (\mathbb{J}, \mathbb{K})$ is called a vague graph on $\mathbb{G}^*$, where $\mathbb{J} = (T_{\mathbb{J}}, F_{\mathbb{J}})$ is a vague set on $\mathbb{V}$ and $\mathbb{K} = (T_{\mathbb{K}}, F_{\mathbb{K}})$ is a vague set on $\mathbb{E} \subseteq \mathbb{V} \times \mathbb{V}$ such that for each $xy \in \mathbb{E}$,*
$T_{\mathbb{K}}(xy) \leq \min(T_{\mathbb{J}}(x), T_{\mathbb{J}}(y))$ and $F_{\mathbb{K}}(xy) \geq \max(F_{\mathbb{J}}(x), F_{\mathbb{J}}(y))$.

Definition 2.3 *[28] A Neutrosophic set $\mathbb{A}$ is contained in another neutrosophic set $\mathbb{B}$, (i.e) $\mathbb{A} \subseteq \mathbb{B}$ if $\forall x \in \mathbb{X}, T_{\mathbb{A}}(x) \leq T_{\mathbb{B}}(x), I_{\mathbb{A}}(x) \geq I_{\mathbb{B}}(x)$and $F_{\mathbb{A}}(x) \geq F_{\mathbb{B}}(x)$.*

Definition 2.4 *[14, 28] Let $\mathbb{X}$ be a space of points (objects), with generic elements in $\mathbb{X}$ denoted by x. A single valued neutrosophic set $\mathbb{A}$ in $\mathbb{X}$ is characterised by truth-membership function $T_{\mathbb{A}}(x)$, indeterminacy-membership function $I_{\mathbb{A}}(x)$ and falsity-membership-function $F_{\mathbb{A}}(x)$,*
For each point x in $\mathbb{X}$, $T_{\mathbb{A}}(x), F_{\mathbb{A}}(x), I_{\mathbb{A}}(x) \in [0,1]$. Also $A = \{x, T_{\mathbb{A}}(x), F_{\mathbb{A}}(x), I_{\mathbb{A}}(x)\}$ and

$$0 \leq T_{\mathbb{A}}(x) + I_{\mathbb{A}}(x) + F_{\mathbb{A}}(x) \leq 3.$$

Definition 2.5 *[4, 12] A neutrosophic graph is defined as a pair $\mathbb{G}^* = (\mathbb{V}, \mathbb{E})$ where*

(i) $\mathbb{V} = \{v_1, v_2, .., v_n\}$ such that $T_1: \mathbb{V} \to [0,1]$, $I_1: \mathbb{V} \to [0,1]$ and $F_1: \mathbb{V} \to [0,1]$ denote the degree of truth-membership function, indeterminacy-function and falsity-membership function, respectively, and

$$0 \leq T_1(v) + I_1(v) + F_1(v) \leq 3,$$

(ii) $\mathbb{E} \subseteq \mathbb{V} \times \mathbb{V}$ where $T_2: \mathbb{E} \to [0,1]$, $I_2: \mathbb{E} \to [0,1]$ and $F_2: \mathbb{E} \to [0,1]$ are such that

$$T_2(uv) \leq \min\{T_1(u), T_1(v)\},$$
$$I_2(uv) \leq \min\{I_1(u), I_1(v)\},$$
$$F_2(uv) \leq \max\{F_1(u), F_1(v)\},$$
$$\text{and } 0 \leq T_2(uv) + I_2(uv) + F_2(uv) \leq 3, \ \forall uv \in \mathbb{E}.$$

Definition 2.6 *[6] A neutrosophic vague set $\mathbb{A}_{NV}$ (NVS in short) on the universe of discourse $\mathbb{X}$ be written as* $\mathbb{A}_{NV} = \{\langle x, \hat{T}_{\mathbb{A}_{NV}}(x), \hat{I}_{\mathbb{A}_{NV}}(x), \hat{F}_{\mathbb{A}_{NV}}(x)\rangle, x \in \mathbb{X}\}$, whose truth-membership, indeterminacy-membership and falsity-membership function is defined as

$$\hat{T}_{\mathbb{A}_{NV}}(x) = [T^-(x), T^+(x)], \hat{I}_{\mathbb{A}_{NV}}(x) = [I^-(x), I^+(x)] \text{and} \hat{F}_{\mathbb{A}_{NV}}(x) = [F^-(x), F^+(x)],$$

where $T^+(x) = 1 - F^-(x), F^+(x) = 1 - T^-(x)$, and $0 \leq T^-(x) + I^-(x) + F^-(x) \leq 2$.

Definition 2.7 *[6] The complement of NVS $\mathbb{A}_{NV}$ is denoted by $\mathbb{A}_{NV}^c$ and it is given by*

$$\hat{T}_{\mathbb{A}_{NV}}^c(x) = [1 - T^+(x), 1 - T^-(x)],$$
$$\hat{I}_{\mathbb{A}_{NV}}^c(x) = [1 - I^+(x), 1 - I^-(x)],$$
$$\hat{F}_{\mathbb{A}_{NV}}^c(x) = [1 - F^+(x), 1 - F^-(x)].$$

Definition 2.8 *[6] Let $\mathbb{A}_{NV}$ and $\mathbb{B}_{NV}$ be two NVSs of the universe $\mathbb{U}$. If for all $u_i \in \mathbb{U}$,*

$$\hat{T}_{\mathbb{A}_{NV}}(u_i) \leq \hat{T}_{\mathbb{B}_{NV}}(u_i), \hat{I}_{\mathbb{A}_{NV}}(u_i) \geq \hat{I}_{\mathbb{B}_{NV}}(u_i), \hat{F}_{\mathbb{A}_{NV}}(u_i) \geq \hat{F}_{\mathbb{B}_{NV}}(u_i),$$

then the NVS, $\mathbb{A}_{NV}$ are included in $\mathbb{B}_{NV}$, denoted by $\mathbb{A}_{NV} \subseteq \mathbb{B}_{NV}$ where $1 \leq i \leq n$.

Definition 2.9 *[6] The union of two NVSs $\mathbb{A}_{NV}$ and $\mathbb{B}_{NV}$ is a NVSs, $\mathbb{C}_{NV}$, written as $\mathbb{C}_{NV} = \mathbb{A}_{NV} \cup \mathbb{B}_{NV}$, whose truth-membership function, indeterminacy-membership function and false-membership function are related to those of $\mathbb{A}_{NV}$ and $\mathbb{B}_{NV}$ by*

$$\hat{T}_{\mathbb{C}_{NV}}(x) = [\max(T_{\mathbb{A}_{NV}}^-(x), T_{\mathbb{B}_{NV}}^-(x)), \max(T_{\mathbb{A}_{NV}}^+(x), T_{\mathbb{B}_{NV}}^+(x))]$$

$$\hat{I}_{\mathbb{C}_{NV}}(x) = [\min(I^-_{\mathbb{A}_{NV}}(x), I^-_{\mathbb{B}_{NV}}(x)), \min(I^+_{\mathbb{A}_{NV}}(x), I^+_{\mathbb{B}_{NV}}(x))]$$
$$\hat{F}_{\mathbb{C}_{NV}}(x) = [\min(F^-_{\mathbb{A}_{NV}}(x), F^-_{\mathbb{B}_{NV}}(x)), \min(F^+_{\mathbb{A}_{NV}}(x), F^+_{\mathbb{B}_{NV}}(x))].$$

Definition 2.10 *[6] The intersection of two NVSs, A_{NV} and B_{NV} is a NVSs C_{NV}, written as $C_{NV} = A_{NV} \cap B_{NV}$, whose truth-membership function, indeterminacy-membership function and false-membership function are related to those of A_{NV} and B_{NV} by*

$$\hat{T}_{\mathbb{C}_{NV}}(x) = [\min(T^-_{\mathbb{A}_{NV}}(x), T^-_{\mathbb{B}_{NV}}(x)), \min(T^+_{\mathbb{A}_{NV}}(x), T^+_{\mathbb{B}_{NV}}(x))]$$
$$\hat{I}_{\mathbb{C}_{NV}}(x) = [\max(I^-_{\mathbb{A}_{NV}}(x), I^-_{\mathbb{B}_{NV}}(x)), \max(I^+_{\mathbb{A}_{NV}}(x), I^+_{\mathbb{B}_{NV}}(x))]$$
$$\hat{F}_{\mathbb{C}_{NV}}(x) = [\max(F^-_{\mathbb{A}_{NV}}(x), F^-_{\mathbb{B}_{NV}}(x)), \max(F^+_{\mathbb{A}_{NV}}(x), F^+_{\mathbb{B}_{NV}}(x))].$$

Definition 2.11 *[24] Let $\mathbb{G}^* = (\mathbb{R}, \mathbb{S})$ be a graph. A pair $\mathbb{G} = (\mathbb{A}, \mathbb{B})$ is called a neutrosophic vague graph (NVG) on $\mathbb{G}^*$ or a neutrosophic vague graph where $\mathbb{A} = (\hat{T}_{\mathbb{A}}, \hat{I}_{\mathbb{A}}, \hat{F}_{\mathbb{A}})$ is a neutrosophic vague set on $\mathbb{R}$ and $\mathbb{B} = (\hat{T}_{\mathbb{B}}, \hat{I}_{\mathbb{B}}, \hat{F}_{\mathbb{B}})$ is a neutrosophic vague set $\mathbb{S} \subseteq \mathbb{R} \times \mathbb{R}$ where*

(1) $\mathbb{R} = \{v_1, v_2, \ldots, v_n\}$ such that $T^-_{\mathbb{A}}: \mathbb{R} \to [0,1], I^-_{\mathbb{A}}: \mathbb{R} \to [0,1], F^-_{\mathbb{A}}: \mathbb{R} \to [0,1]$ which satisfies the condition $F^-_{\mathbb{A}} = [1 - T^+_{\mathbb{A}}]$

$T^+_{\mathbb{A}}: \mathbb{R} \to [0,1], I^+_{\mathbb{A}}: \mathbb{R} \to [0,1], F^+_{\mathbb{A}}: \mathbb{R} \to [0,1]$ which satisfying the condition $F^+_{\mathbb{A}} = [1 - T^-_{\mathbb{A}}]$

denotes the degree of truth membership function, indeterminacy membership and falsity membership of the element $v_i \in \mathbb{R}$, and

$$0 \le T^-_{\mathbb{A}}(v_i) + I^-_{\mathbb{A}}(v_i) + F^-_{\mathbb{A}}(v_i) \le 2$$
$$0 \le T^+_{\mathbb{A}}(v_i) + I^+_{\mathbb{A}}(v_i) + F^+_{\mathbb{A}}(v_i) \le 2.$$

(2) $\mathbb{S} \subseteq \mathbb{R} \times \mathbb{R}$ where

$$T^-_{\mathbb{B}}: \mathbb{R} \times \mathbb{R} \to [0,1],\ I^-_{\mathbb{B}}: \mathbb{R} \times \mathbb{R} \to [0,1],\ F^-_{\mathbb{B}}: \mathbb{R} \times \mathbb{R} \to [0,1]$$
$$T^+_{\mathbb{B}}: \mathbb{R} \times \mathbb{R} \to [0,1],\ I^+_{\mathbb{B}}: \mathbb{R} \times \mathbb{R} \to [0,1],\ F^+_{\mathbb{B}}: \mathbb{R} \times \mathbb{R} \to [0,1]$$

represents the degree of truth membership function, indeterminacy membership and falsity membership of the element $v_i, v_j \in \mathbb{S}$, respectively and such that,

$$0 \le T^-_{\mathbb{B}}(v_i v_j) + I^-_{\mathbb{B}}(v_i v_j) + F^-_{\mathbb{B}}(v_i v_j) \le 2$$
$$0 \le T^+_{\mathbb{B}}(v_i v_j) + I^+_{\mathbb{B}}(v_i v_j) + F^+_{\mathbb{B}}(v_i v_j) \le 2,$$

such that

$$T^-_{\mathbb{B}}(v_i v_j) \le \min\{T^-_{\mathbb{A}}(v_i), T^-_{\mathbb{A}}(v_j)\}$$
$$I^-_{\mathbb{B}}(v_i v_j) \le \min\{I^-_{\mathbb{A}}(v_i), I^-_{\mathbb{A}}(v_j)\}$$
$$F^-_{\mathbb{B}}(v_i v_j) \le \max\{F^-_{\mathbb{A}}(v_i), F^-_{\mathbb{A}}(v_j)\},$$

and similarly

$$T^+_{\mathbb{B}}(v_i v_j) \le \min\{T^+_{\mathbb{A}}(v_i), T^+_{\mathbb{A}}(v_j)\}$$
$$I^+_{\mathbb{B}}(v_i v_j) \le \min\{I^+_{\mathbb{A}}(v_i), I^+_{\mathbb{A}}(v_j)\}$$
$$F^+_{\mathbb{B}}(v_i v_j) \le \max\{F^+_{\mathbb{A}}(v_i), F^+_{\mathbb{A}}(v_j)\}.$$

3 Neutrosophic Vague Line Graphs

In this section, the necessary and sufficient condition of NVLGs are provided. The definition of NVLGs, homomorphism and weak isomorphism are given.

Definition 3.1 *Let $\Lambda(D) = (D, S)$ be an intersection graph $G = (V, E)$ and let $\mathbb{G} = (H_1, K_1)$ be a NVG with underlying set V. A NVG of $\Lambda(D)$ is a pair (H_2, K_2), where $H_2 = (T^+_{H_2}, I^+_{H_2}, F^+_{H_2}, T^-_{H_2}, I^-_{H_2}, F^-_{H_2})$ and $K_2 = (T^+_{K_2}, I^+_{K_2}, F^+_{K_2}, T^-_{K_2}, I^-_{K_2}, F^-_{K_2})$ are NVSs of D and S, respectively, such that*

$$T^+_{H_2}(D_i) = T^+_{H_1}(v_i), I^+_{H_2}(D_i) = I^+_{H_1}(v_i), F^+_{H_2}(D_i) = F^+_{H_1}(v_i),$$

$$T^-_{H_2}(D_i) = T^-_{H_1}(v_i), I^-_{H_2}(D_i) = I^-_{H_1}(v_i), F^-_{H_2}(D_i) = F^-_{H_1}(v_i),$$

for all $D_i, D_j \in D$.

$$T^+_{K_2}(D_iD_j) = T^+_{K_1}(v_iv_j), I^+_{K_2}(D_iD_j) = I^+_{K_1}(v_iv_j), F^+_{K_2}(D_iD_j) = F^+_{K_1}(v_iv_j),$$
$$T^-_{K_2}(D_iD_j) = T^-_{K_1}(v_iv_j), I^-_{K_2}(D_iD_j) = I^-_{K_1}(v_iv_j), F^-_{K_2}(D_iD_j) = F^-_{K_1}(v_iv_j)$$

for all $D_iD_j \in S$.

That is, any NVG of intersection graph $\Lambda(D)$ is also a neutrosophic vague intersection graph of $\mathbb{G}$.

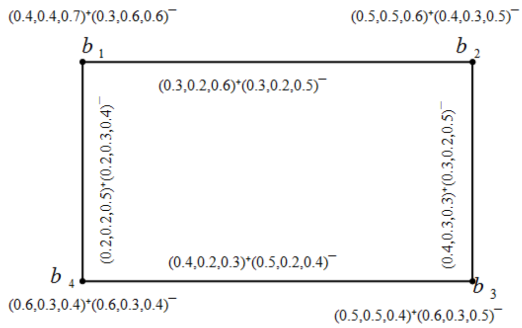
Definition 3.2 *Let $L(G) = (M, N)$ be a line graph of a graph $G = (V, E)$. A NVLG of a NVG $\mathbb{G} = (H_1, K_1)$ (with underlying set V) is a pair $L(\mathbb{G}) = (H_2, K_2)$, where $H_2 = (T^+_{H_2}, I^+_{H_2}, F^+_{H_2}, T^-_{H_2}, I^-_{H_2}, F^-_{H_2})$ and $K_2 = (T^+_{K_2}, I^+_{K_2}, F^+_{K_2}, T^-_{K_2}, I^-_{K_2}, F^-_{K_2})$ are NVSs of M and N, respectively such that,*

$$T^+_{H_2}(D_x) = T^+_{K_1}(x) = T^+_{K_1}(u_xv_x)$$
$$I^+_{H_2}(D_x) = I^+_{K_1}(x) = I^+_{K_1}(u_xv_x)$$
$$F^+_{H_2}(D_x) = F^+_{K_1}(x) = F^+_{K_1}(u_xv_x)$$
$$T^-_{H_2}(D_x) = T^-_{K_1}(x) = T^-_{K_1}(u_xv_x)$$
$$I^-_{H_2}(D_x) = I^-_{K_1}(x) = I^-_{K_1}(u_xv_x)$$
$$F^-_{H_2}(D_x) = F^-_{K_1}(x) = F^-_{K_1}(u_xv_x).$$

for all $D_x \in M, u_xv_x \in N$.

$$T^+_{K_2}(D_xD_y) = \min\{T^+_{K_1}(x), T^+_{K_1}(y)\}$$
$$I^+_{K_2}(D_xD_y) = \min\{I^+_{K_1}(x), I^+_{K_1}(y)\}$$
$$F^+_{K_2}(D_xD_y) = \max\{F^+_{K_1}(x), F^+_{K_1}(y)\}$$
$$T^-_{K_2}(D_xD_y) = \min\{T^-_{K_1}(x), T^-_{K_1}(y)\}$$
$$I^-_{K_2}(D_xD_y) = \min\{I^-_{K_1}(x), I^-_{K_1}(y)\}$$
$$F^-_{K_2}(D_xD_y) = \max\{F^-_{K_1}(x), F^-_{K_1}(y)\} \quad \text{for all } D_xD_y \in N.$$

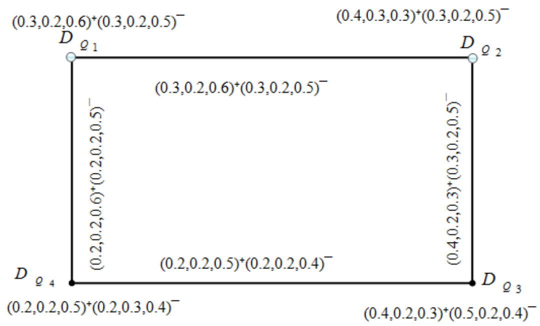
Example 3.3 *Consider $G = (V, E)$, where $V = \{b_1, b_2, b_3, b_4\}$ and $E = \{Q_1 = b_1b_2, Q_2 = b_2b_3, Q_3 = b_3b_4, Q_4 = b_4b_1\}$. Let $\mathbb{G} = (H_1, K_1)$ be a NVG of G as shown in figure 1, defined by*



$Figure$ 1

Neutrosophic Vague Graph

consider a line graph $L(G) = (M, N)$ where $M = (D_{Q_1}, D_{Q_2}, D_{Q_3}, D_{Q_4})$ and $N = (D_{Q_1}D_{Q_2}, D_{Q_2}D_{Q_3}, D_{Q_3}D_{Q_4}, D_{Q_4}D_{Q_1})$. Let $L(\mathbb{G})$ be the NVLG, as shown in figure 2.



$Figure\ 2$

Neutrosophic Vague Line Graph

$L(\mathbb{G})$

Proposition 3.4 *A NVLG is always a strong NVG.*

Proof. It is obvious from the definition, therefore it is omitted.

Proposition 3.5 *If* $L(\mathbb{G})$ *is NVLG of NVG* $\mathbb{G}$. *Then* $L(G)$ *is the line graph of* G.

Proof. Given $\mathbb{G} = (H_1, K_1)$ is NVLG of G and $L(\mathbb{G}) = (H_2, K_2)$ is a NVG of $L(G)$

$$T_{H_2}^{+}(D_x) = T_{K_1}^{+}(x)$$
$$I_{H_2}^{+}(D_x) = I_{K_1}^{+}(x)$$
$$F_{H_2}^{+}(D_x) = F_{K_1}^{+}(x)$$
$$T_{H_2}^{-}(D_x) = T_{K_1}^{-}(x)$$
$$I_{H_2}^{-}(D_x) = I_{K_1}^{-}(x)$$
$$F_{H_2}^{-}(D_x) = F_{K_1}^{-}(x),$$

$\forall x \in E$ and so $D_x \in M$ if and only if for $x \in E$,

$$T_{K_2}^{+}(D_x D_y) = \min\{T_{K_1}^{+}(x), T_{K_1}^{+}(y)\}$$
$$I_{K_2}^{+}(D_x D_y) = \min\{I_{K_1}^{+}(x), I_{K_1}^{+}(y)\}$$
$$F_{K_2}^{+}(D_x D_y) = \max\{F_{K_1}^{+}(x), F_{K_1}^{+}(y)\}$$
$$T_{K_2}^{-}(D_x D_y) = \min\{T_{K_1}^{-}(x), T_{K_1}^{-}(y)\}$$
$$I_{K_2}^{-}(D_x D_y) = \min\{I_{K_1}^{-}(x), I_{K_1}^{-}(y)\}$$
$$F_{K_2}^{-}(D_x D_y) = \max\{F_{K_1}^{-}(x), F_{K_1}^{-}(y)\},$$

for all $D_x D_y \in N$, and so $M = \{D_x D_y | D_x \cup D_y \neq \emptyset, x, y \in E, x \neq y\}$. Hence proved.

Proposition 3.6 *Let* $L(\mathbb{G}) = (H_2, K_2)$ *be a NVG of* $\mathbb{G}$. *Then* $L(\mathbb{G})$ *is a NVG of some NVG of* G *if and only if*

$$T_{K_2}^{+}(D_x D_y) = \min\{T_{H_2}^{+}(D_x), T_{H_2}^{+}(D_y)\}$$
$$T_{K_2}^{-}(D_x D_y) = \min\{T_{H_2}^{-}(D_x), T_{H_2}^{-}(D_y)\}$$
$$I_{K_2}^{+}(D_x D_y) = \min\{I_{H_2}^{+}(D_x), I_{H_2}^{+}(D_y)\}$$
$$I_{K_2}^{-}(D_x D_y) = \min\{I_{H_2}^{-}(D_x), I_{H_2}^{-}(D_y)\}$$

$F^{+}_{K_2}(D_xD_y) = \max\{F^{+}_{H_2}(D_x), F^{+}_{H_2}(D_y)\}$

$F^{-}_{K_2}(D_xD_y) = \max\{F^{-}_{H_2}(D_x), F^{-}_{H_2}(D_y)\}$,

for all $D_xD_y \in N$.

Proof. Suppose that $T^{P}_{K_2}(D_xD_y) = \min\{T^{+}_{H_2}(D_x), T^{+}_{H_2}(D_y)\}$,

$I^{+}_{K_2}(D_xD_y) = \min\{I^{+}_{H_2}(D_x), I^{+}_{H_2}(D_y)\}$, $F^{+}_{K_2}(D_xD_y) = \max\{F^{+}_{H_2}(D_x), F^{+}_{H_2}(D_y)\}$ for all $D_xD_y \in N$.

Define, $T^{+}_{H_2}(D_x) = T^{+}_{K_1}(x), I^{+}_{H_2}(D_x) = I^{+}_{K_1}(x), F^{+}_{H_2}(D_x) = F^{+}_{K_1}(x)$ for all $x \in E$, then

$T^{+}_{K_2}(D_xD_y) = \min\{T^{+}_{H_2}(D_x), T^{+}_{H_2}(D_y)\} = \min\{T^{+}_{K_1}(x), T^{+}_{K_1}(x)\}$,

$I^{+}_{K_2}(D_xD_y) = \min\{I^{+}_{H_2}(D_x), I^{+}_{H_2}(D_y)\} = \min\{I^{+}_{K_1}(x), I^{+}_{K_1}(x)\}$,

$F^{+}_{K_2}(D_xD_y) = \max\{F^{+}_{H_2}(D_x), F^{+}_{H_2}(D_y)\} = \max\{F^{+}_{K_1}(x), F^{+}_{K_1}(x)\}$,

for all $D_xD_y \in M$.

We know that NVG H_1 yields the properties,

$$T^{+}_{K_1}(uv) \leq \min\{T^{+}_{H_1}(u), T^{+}_{H_1}(v)\}$$

$$I^{+}_{K_1}(uv) \leq \min\{I^{+}_{H_1}(u), I^{+}_{H_1}(v)\}$$

$$F^{+}_{K_1}(uv) \leq \max\{F^{+}_{H_1}(u), F^{+}_{H_1}(v)\}.$$

In the similar way, we prove for the similar part also, The converse part of this theorem is obvious by using the definition of $L(\mathbb{G})$.

Theorem 3.7 *$L(\mathbb{G})$ is a NVLG if and only if $L(G)$ is a line graph and*

$T^{+}_{K_2}(uv) = \min\{T^{+}_{H_2}(u), T^{+}_{H_2}(v)\}$

$I^{+}_{K_2}(uv) = \min\{I^{+}_{H_2}(u), I^{+}_{H_2}(v)\}$

$F^{+}_{K_2}(uv) = \max\{F^{+}_{H_2}(u), F^{+}_{H_2}(v)\}$

$T^{-}_{K_2}(uv) = \min\{T^{-}_{H_2}(u), T^{-}_{H_2}(v)\}$

$I^{-}_{K_2}(uv) = \min\{I^{-}_{H_2}(u), I^{-}_{H_2}(v)\}$

$F^{-}_{K_2}(uv) = \max\{F^{-}_{H_2}(u), F^{-}_{H_2}(v)\} \quad \forall uv \in M.$

Proof. The proof follows from the above Proposition 3.5 and Proposition 3.6.

Definition 3.8 *A homomorphism $\chi: \mathbb{G}_1 \to \mathbb{G}_2$ of two NVGs $\mathbb{G}_1 = (H_1, K_1)$ and $\mathbb{G}_2 = (H_2, K_2)$ is mapping $\chi: V_1 \to V_2$ such that*

$(A) T^{+}_{H_1}(x_1) \leq T^{+}_{H_2}(\chi(x_1)), T^{-}_{H_1}(x_1) \leq T^{-}_{H_2}(\chi(x_1))$,

$I^{+}_{H_1}(x_1) \leq I^{+}_{H_2}(\chi(x_1)), I^{-}_{H_1}(x_1) \leq I^{-}_{H_2}(\chi(x_1))$,

$F^{+}_{H_1}(x_1) \leq F^{+}_{H_2}(\chi(x_1)), F^{-}_{H_1}(x_1) \leq F^{-}_{H_2}(\chi(x_1)), \quad \forall x_1 \in V_1$.

$(B) T^{+}_{K_1}(x_1y_1) \leq T^{+}_{K_2}(\chi(x_1)\chi(y_1)), T^{-}_{K_1}(x_1y_1) \leq T^{-}_{K_2}(\chi(x_1)\chi(y_1))$,

$I^{+}_{K_1}(x_1y_1) \leq I^{+}_{K_2}(\chi(x_1)\chi(y_1)), I^{-}_{K_1}(x_1y_1) \leq I^{-}_{K_2}(\chi(x_1)\chi(y_1))$,

$F^{+}_{K_1}(x_1y_1) \leq F^{+}_{K_2}(\chi(x_1)\chi(y_1)), F^{-}_{K_1}(x_1y_1) \leq F^{-}_{K_2}(\chi(x_1)\chi(y_1)), \quad \forall x_1y_1 \in E_1$.

Definition 3.9 *A (weak) vertex-isomorphism is a bijective homomorphism* $\chi: \mathbb{G}_1 \to \mathbb{G}_2$ *such that*

$$(A) T^{+}_{H_1}(x_1) = T^{+}_{H_2}(\chi(x_1)),$$
$$T^{-}_{H_1}(x_1) = T^{-}_{H_2}(\chi(x_1)),$$
$$I^{+}_{H_1}(x_1) = I^{+}_{H_2}(\chi(x_1)),$$
$$I^{-}_{H_1}(x_1) = I^{-}_{H_2}(\chi(x_1)),$$
$$F^{+}_{H_1}(x_1) = F^{+}_{H_2}(\chi(x_1)),$$
$$F^{-}_{H_1}(x_1) = F^{-}_{H_2}(\chi(x_1)), \quad \forall x_1 \in V_1.$$

A (weak) line-isomorphism is bijective homomorphism $\chi: \mathbb{G}_1 \to \mathbb{G}_2$ such that

$$(B) T^{+}_{K_1}(x_1 y_1) = T^{+}_{K_2}(\chi(x_1)\chi(y_1)),$$
$$T^{-}_{K_1}(x_1 y_1) = T^{-}_{K_2}(\chi(x_1)\chi(y_1)),$$
$$I^{+}_{K_1}(x_1 y_1) = I^{+}_{K_2}(\chi(x_1)\chi(y_1)),$$
$$I^{-}_{K_1}(x_1 y_1) = I^{-}_{K_2}(\chi(x_1)\chi(y_1)),$$
$$F^{+}_{K_1}(x_1 y_1) = F^{+}_{K_2}(\chi(x_1)\chi(y_1)),$$
$$F^{-}_{K_1}(x_1 y_1) = F^{-}_{K_2}(\chi(x_1)\chi(y_1)), \quad \forall x_1 y_1 \in E_1.$$

If $\chi: \mathbb{G}_1 \to \mathbb{G}_2$ is a weak-vertex isomorphism and a (weak) line-isomorphism, then χ is called a (weak) isomorphism.

Proposition 3.10 *Let* $\mathbb{G} = (H_1, K_1)$ *be a NVG with underlying set* V. *Then* (H_2, K_2) is a NVG of $\Lambda(D)$ and $(H_1, K_1) \cong (H_2, K_2)$

Proposition 3.11 *Let* $\mathbb{G}$ *and* $\mathbb{G}'$ *be NVGs of* G *and* G' *respectively, if* $\chi: \mathbb{G} \to \mathbb{G}'$ *is a weak isomorphism then* $\chi: \mathbb{G} \to \mathbb{G}'$ *is an isomorphism.*

Proof. Let $\chi: \mathbb{G} \to \mathbb{G}'$ be a weak isomorphism, then $u \in V$ if and only if $\chi(u) \in V'$ and $uv \in E$ if and only if $\chi(u)\chi(v) \in E'$. Hence proved.

Conclusion

A neutrosophic graph is very useful to interpret the real-life situations and it is regarded as a generalisation of intuitionistic fuzzy graph. Neutrosophic vague graphs are represented as a context-dependent generalized fuzzy graphs which holds the indeterminate and inconsistent information. This paper dealt with the necessary and sufficient condition for NVLG to be a line graph are also derived. The properties of homomorphism, weak vertex and weak line isomorphism are established. Further we are able to extend by investigating the regular and isomorphic properties of the interval valued neutrosophic vague line graph.

Conflict of Interest: The authors declare that they have no conflict of interest.

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Neutrosophic Nano Semi-Frontier

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Abstract: Smarandache presented and built up the new idea of Neutrosophic set from the Intuitionistic fuzzy sets. A.A. Salama presented Neutrosophic topological spaces by utilizing the Neutrosophic sets. M.L.Thivagar et al., created Nano topological spaces and Neutrosophic nano topological spaces. Point of this paper is we present and study the properties of Neutrosophic Nano semi frontier in Neutrosophic nano topological spaces and its portrayal are talked about subtleties.

Keywords: Neutrosophic Nano semi open set, Neutrosophic Nano semi closed set, Neutrosophic Nano frontier, Neutrosophic Nano semi frontier, Neutrosophic nano topology.

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1. Introduction

Nano topology explored by M.L.Thivagar [15]et.al can be communicated as an assortment of nano approximations, Neutrosophic sets set up by F.Smarandache[14]. Neutrosophic set is illustrate by three functions: a membership, indeterminacy and nonmembership functions that are independently related. Neutrosophic set have wide scope of uses, all things considered. M.L.Thivagar et al., created Neutrosophic nano topological spaces .Neutrosophic nano semi closed, neutrosophic nano αclosed, neutrosophic nano pre closed, neutrosophic nano semi pre closed and neutrosophic nano regular closed are presented by M.Parimala[17] et al. Point of the current paper is we learned about properties of Neutrosophic Nano frontier, Neutrosophic Nano semi frontier in Neutrosophic nano topological spaces

2. PRELIMINARIES

In this section, we recall needed basic definition and operation of Neutrosophic sets

Definition 2.1 : [15]

Let U be a non-empty set and R be an equivalence relation on U. Let F be a neutrosophic set in U with the membership function μ_F , the indeterminacy function σ_F and the non-membership functionν_F . The neutrosophic nano lower,neutrosophic nano upper approximation and neutrosophic nano boundary of F in the approximation (U,R) denoted by $\underline{\mathrm{N(F)}}$, $\overline{N(\mathrm{F})}$and $\mathrm{B_N(F)}$are respectively defined as follows:

(i) $\underline{\mathrm{N(F)}} = \{< u, \mu_{\underline{\mathrm{R}}}(M_1^*)(\mathrm{u}), \sigma_{\underline{\mathrm{R}}}(M_1^*)(\mathrm{u}), \nu_{\underline{\mathrm{R}}}(M_1^*)(\mathrm{u}) > /y \in [\mathrm{u}]_{\mathrm{R}}, \mathrm{u} \in \mathrm{U}\}$.

(ii) $\overline{N(\mathrm{F})} = \{< u, \mu_{\overline{\mathrm{R}}}(M_1^*)(\mathrm{u}), \sigma_{\overline{\mathrm{R}}}(M_1^*)(\mathrm{u}), \nu_{\overline{\mathrm{R}}}(M_1^*)(\mathrm{u}) > /y \in [\mathrm{u}]_{\mathrm{R}}, \mathrm{u} \in \mathrm{U}\}$.

(iii) $\mathrm{B_N(F)}=\overline{N(\mathrm{F})} - \underline{\mathrm{N(F)}}$

Definition 2.2 : [15]

Let U be an universe, R be an equivalence relation on U and F be a neutrosophic set in U and if the collection $N_N(\tau) = \{0_{N_N}, 1_{N_N}, \underline{\mathrm{N(F)}}, \overline{N(\mathrm{F})}, \mathrm{B_N(F)}\}$

forms a topology then it is said to be a neutrosophic nano topology. We call (U, $N_N(\tau)$) as the neutrosophic nano topological space. The elements of $N_N(\tau)$ are called neutrosophic nano open sets.

Definition 2.3 : [15]

Let U be a nonempty set and the Neutrosophic sets M_1^* and M_2^* in the form M_1^*={< u:$\mu_{M_1^*}(u)$, $\sigma_{M_1^*}(u)$, $\nu_{M_1^*}(u)$>, u∈U},

M_2^*={< u:$\mu_{M_2^*}(u)$, $\sigma_{M_2^*}(u)$, $\nu_{M_2^*}(u)$>, u∈U}}.

Then the following statements hold:

(i) 0_{N_N} = {< u, 0, 0, 1 >: u∈U} and 1_{N_N} = {< u, 1, 1, 0 >: u∈U}.

(ii) $M_1^* \subseteq M_2^*$ iff { $\mu_{M_1^*}(u) \leq \mu_{M_2^*}(u)$, $\sigma_{M_1^*}(u) \leq \sigma_{M_2^*}(u)$, $\nu_{M_1^*}(u) \geq \nu_{M_2^*}(u)$, ∀u∈U}.

(iii) $M_1^* = M_2^*$ iff $M_1^* \subseteq M_2^*$ and $M_2^* \subseteq M_1^*$.

(iv) M_1^{*C} ={< u, $\nu_{M_1^*}(u)$, $1-\sigma_{M_1^*}(u)$, $\mu_{M_1^*}(u)$ >: u ∈ U}.

(v) $M_1^* \cap M_2^*$ ={u, $\mu_{M_1^*}(u) \wedge \mu_{M_2^*}(u)$, $\sigma_{M_1^*}(u) \wedge \sigma_{M_2^*}(u)$, $\nu_{M_1^*}(u) \vee \nu_{M_2^*}(u)$, ∀u∈U}.

(vi) $M_1^* \cup M_2^*$ ={u, $\mu_{M_1^*}(u) \vee \mu_{M_2^*}(u)$, $\sigma_{M_1^*}(u) \vee \sigma_{M_2^*}(u)$, $\nu_{M_1^*}(u) \wedge \nu_{M_2^*}(u)$, ∀u∈U}.

(vii) $\cup M_j^* = \langle u, \vee, \vee, \wedge \rangle$

(viii) $\cap M_j^* = \langle u, \wedge, \wedge, \vee \rangle$

(ix) $M_1^* - M_2^* = M_1^* \cap \circledcirc {M_2^*}^C$

Proposition 2.4 [15]

For any Neutrosophic Nano set M_1^* in (U, $N_N(\tau)$)) we have

(1) N^NCl ((M_1^*)^C) = (N^NInt (M_1^*))^C,

(2) N^NInt ((M_1^*)^C) =(N^NCl (M_1^*))^C.

(3) $M_1^* \subseteq M_2^* \Rightarrow N^N$Int ($M_1^*$) ⊆ N^NInt (M_2^*),

(4) $M_1^* \subseteq M_2^* \Rightarrow N^N$Cl($M_1^*$) ⊆ N^NCl(M_2^*),

(5) N^NInt (N^NInt (M_1^*)) = N^NInt(M_1^*),

(6) N^NCl (N^NCl (M_1^*)) = N^NCl (M_1^*),

(7) N^NInt ($M_1^* \cap M_2^*$)) = N^NInt (M_1^*) ∩ N^NInt (M_2^*),

(8) N^NCl ($M_1^* \cup M_2^*$) = N^NCl(M_1^*) ∪ N^NCl(M_2^*),

(9) N^NInt (0_{N_N}) = 0_{N_N} ,

(10) N^NInt (1_{N_N}) = 1_{N_N} ,

(11) N^NCl (0_{N_N}) = 0_{N_N} ,

(12) N^NCl (1_{N_N}) = 1_{N_N} ,

(13) $M_1^* \subseteq M_2^* \Rightarrow ({M_2^*}^C \subseteq {M_1^*}^C$,

(14) N^NCl ($M_1^* \cap M_2^*$)⊆N^NCl (M_1^*) ∩ N^NCl (M_2^*),

(15) N^NInt ($M_1^* \cup M_2^*$)⊇ N^NInt (M_1^*) ∪ N^NInt (M_2^*).

3. NEUTROSOPHIC NANO FRONTIER

In this section, the concepts of the Neutrosophic Nano frontier in Neutrosophic Nano topological space are introduced and also discussed their characterizations with some related examples.

Definition 3.1.

Let U be a *N-N-T-S* and let M_1^* ⊚⊚*NNS* (U). Neutrosophic Nano frontier of M_1^* and is denoted by *NFr* (M_1^*). i.e., $N^N Fr\ (M_1^*) = N^N Cl(M_1^*) \cap \circledcirc N^N Cl(M_1^*)^C$.

Proposition 3.2. For each M_1^* ⊚⊚*NNS*(U), $M_1^* \cup \circledcirc N^N Fr\ (M_1^*) \subseteq N^N Cl\ (M_1^*)$.

Proof : Let M_1^* be the *NNS* in the N-N-T-S U. Using Definition 3.1.,

$M_1^* \cup \circledcirc N^N Fr\ (M_1^*) = M_1^* \cup (\ N^N Cl\ (M_1^*) \cap \circledcirc N^N Cl\ (({M_1^*}^C))$

$= (M_1^* \cup \circledcirc N^N Cl\ (M_1^*)\ ) \cap \circledcirc (\ M_1^* \cup \circledcirc N^N Cl\ (({M_1^*}^C))$

⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚$\subseteq \circledcirc N^N Cl\ (M_1^*) \cap \circledcirc N^N Cl\ ({M_1^*}^C)$

⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚⊚$\subseteq \circledcirc N^N Cl\ (M_1^*)$

Hence $M_1^* \cup \circledcirc N^N Fr\ (M_1^*) \subseteq \circledcirc N^N Cl\ (M_1^*)$.

Example 3.3.

Let U and $\mathcal{A}$ be two non-empty finite sets,

where U is the universe and $\mathcal{A}$ the set of attributes
The members of U= {P_1, P_2, P_3,P_4}are pressure patients
Let U/R ={{P_1, P_2, P_3}, {P_4}} be an equivalence relation
$\mathcal{A}$ = {Salt food , colostreal food } are two attributes

P_1= $\langle x, \left(\frac{5}{10},\frac{10}{10},\frac{1}{10}\right),\left(\frac{9}{10},\frac{2}{10},\frac{5}{10}\right)\rangle$

P_2= $\langle x, \left(\frac{2}{10},\frac{5}{10},\frac{9}{10}\right),\left(\frac{0}{10},\frac{5}{10},\frac{10}{10}\right)\rangle$

P_3= $\langle x, \left(\frac{5}{10},\frac{10}{10},\frac{1}{10}\right),\left(\frac{9}{10},\frac{5}{10},\frac{5}{10}\right)\rangle$

P_4= $\langle x, \left(\frac{2}{10},\frac{5}{10},\frac{9}{10}\right),\left(\frac{0}{10},\frac{2}{10},\frac{10}{10}\right)\rangle$

$N_N(\tau)$ = {0_{N_N}, 1_{N_N}, $\underline{N(F)}$, $\overline{N(F)}$,B_N(F)}

$\underline{N(F)} = \langle x, \left(\frac{2}{10},\frac{5}{10},\frac{9}{10}\right),\left(\frac{0}{10},\frac{5}{10},\frac{10}{10}\right)\rangle$

$\overline{N(F)} = \langle x, \left(\frac{5}{10},\frac{10}{10},\frac{1}{10}\right),\left(\frac{9}{10},\frac{5}{10},\frac{5}{10}\right)\rangle$

BN(F) = $\langle \left(\frac{5}{10},\frac{5}{10},\frac{2}{10}\right),\left(\frac{9}{10},\frac{5}{10},\frac{5}{10}\right)\rangle$

$N_N(\tau)$= {0_{N_N}, 1_{N_N}, $\langle x, \left(\frac{2}{10},\frac{5}{10},\frac{9}{10}\right),\left(\frac{0}{10},\frac{5}{10},\frac{10}{10}\right)\rangle$, $\langle \left(\frac{5}{10},\frac{10}{10},\frac{1}{10}\right),\left(\frac{9}{10},\frac{5}{10},\frac{5}{10}\right)\rangle$,
$\langle \left(\frac{5}{10},\frac{5}{10},\frac{2}{10}\right),\left(\frac{9}{10},\frac{5}{10},\frac{5}{10}\right)\rangle$, $\langle \left(\frac{2}{10},\frac{5}{10},\frac{9}{10}\right),\left(\frac{0}{10},\frac{2}{10},\frac{10}{10}\right)\rangle$}

Here N^NCl (P_3) = 1_{N_N} and N^NCl (P_3^c) = $\langle \left(\frac{2}{10},\frac{5}{10},\frac{5}{10}\right),\left(\frac{5}{10},\frac{5}{10},\frac{9}{10}\right)\rangle$.

Using Definition 2.1, N^NFr (M_1^*) = $\langle \left(\frac{2}{10},\frac{5}{10},\frac{5}{10}\right),\left(\frac{5}{10},\frac{5}{10},\frac{9}{10}\right)\rangle$.

Also M_1^* ∪◎N^NFr (M_1^*) = $\langle \left(\frac{5}{10},\frac{10}{10},\frac{1}{10}\right),\left(\frac{9}{10},\frac{5}{10},\frac{5}{10}\right)\rangle \subseteq 1_{N_N}$.

Therefore N^NCl (M_1^*) = $1_{N_N} \nsubseteq \langle \left(\frac{5}{10},\frac{10}{10},\frac{1}{10}\right),\left(\frac{9}{10},\frac{5}{10},\frac{5}{10}\right)\rangle$.

Theorem 3.5.
For a NNS M_1^* in the N-N-T-S U, N^NFr (M_1^*) =N^NFr (${M_1^*}^C$).

Proof : Let M_1^* be the *NNS* in the N-N-T-S U. Using Definition 3.1.,
N^NFr (M_1^*) = N^NCl (M_1^*) ∩◎N^NCl (${M_1^*}^C$)
= N^NCl (${M_1^*}^C$) ∩◎N^NCl (M_1^*)
= N^NCl (${M_1^*}^C$) ∩◎$N^NCl\left({M_1^*}^C\right)^C$
Again by Definition 3.1,= N^NFr (${M_1^*}^C$)
Hence N^NFr (M_1^*) = N^NFr (${M_1^*}^C$).

Theorem 3.6.
If a N^NS M_1^* is a *NCS*, then N^NFr (M_1^*) ⊆◎M_1^*.

Proof :
Let M_1^* be the N^NS in the Neutrosophic Nano topological space U. Using Definition 3.1.,
N^NFr (M_1^*) = N^NCl (M_1^*) ∩ N^NCl (${M_1^*}^C$) ⊆ N^NCl (M_1^*)
By Propositon (2.4) ,= M_1^*
Hence N^NFr (M_1^*) ⊆◎M_1^*, if M_1^* is N^NCS in U.
The converse of the above theorem needs not be true as shown by the following example.

Theorem 3.7.
If a *NNS* M_1^* is N^NOS, then N^NFr (M_1^*) ⊆◎◎${M_1^*}^C$.

Proof :
Let M_1^* be the *NNS* in the N-N-T-S U. Using Definition 3.1 ,
M_1^* is N^NOS implies ${M_1^*}^C$is N^NCS in U. By Theorem 3.6, N^NFr (${M_1^*}^C$) ⊆ ${M_1^*}^C$and by Theorem 3.5, we get N^NFr (M_1^*) ⊆ ${M_1^*}^C$

Theorem 3.8.
For a *NNS* M_1^* in the N^NTS U, (N^NFr (M_1^*))^C= N^NInt (M_1^*) ∪◎N^NInt (${M_1^*}^C$).

Proof :
Let M_1^* be the N^NS in the N-N-T-S U. Using Definition 3.1.,
(N^NFr (M_1^*))^C = (N^NCl (M_1^*))^C(∩◎N^NCl (${M_1^*}^C$)))

By Propositon (2.4) ,= $(N^N Cl\ (M_1^*)\ )^C \cup\circledcirc \left(N^N Cl\ (M_1^{*C}))\right)^C$
By Propositon (2.4),= $N^N Int\ (M_1^{*C}) \cup\circledcirc N^N Int\ (M_1^*)$
Hence $(N^N Fr\ (M_1^*))^C = N^N Int\ (M_1^*) \cup\circledcirc N^N Int\ (M_1^{*C})$.

Theorem 3.9

Let $M_1^* \subseteq\circledcirc M_2^*$ and $M_2^* \circledcirc\circledcirc N^N C$ (U) (resp.,$M_2^* \circledcirc\circledcirc N^N O$ (U)). Then $N^N Fr\ (M_1^*) \subseteq\circledcirc M_2^*$ (resp., $N^N Fr\ (M_1^*) \subseteq (N^N Cl\ (M_2^*)\ )^C$), where $N^N C$ (U) (resp., $N^N O$ (U)) denotes the class of Neutrosophic Nano closed (resp., Neutrosophic Nano open) sets in U.

Proof : Use Prop.,2.4 , $M_1^* \subseteq\circledcirc M_2^*$,
$N^N Cl\ (M_1^*) \subseteq\circledcirc N^N Cl\ (M_2^*)$ -------------------- (1).
By Definition 3.1.,
$N^N Fr\ (M_1^*) = N^N Cl\ (M_1^*) \cap\circledcirc N^N Cl\ (M_1^{*C})$
$\subseteq\circledcirc N^N Cl\ (M_2^*) \cap\circledcirc N^N Cl\ (M_1^{*C})$ by (1)
$\subseteq\circledcirc N^N Cl\ (M_2^*) = M_2^*$
Hence $N^N Fr\ (M_1^*) \subseteq\circledcirc M_2^*$.

Theorem 3.10

Let M_1^* be the *NNS* in the *N-N-T-S* U. Then
$N^N Fr\ (M_1^*) = N^N Cl\ (M_1^*) - N^N Int\ (M_1^*)$.

Proof : Let M_1^* be the *NNS* in the N-N-T-S U. By Propositon (2.4),
$\left((N^N Cl\ (M_1^{*C}))\right)^C = N^N Int\ (M_1^*)$ and by Definition 3.1.,
$N^N Fr\ (M_1^*) = N^N Cl\ (M_1^*) \cap\circledcirc N^N Cl\ (M_1^{*C})$
$= N^N Cl\ (M_1^*) - (NCl\ (M_1^{*C}))^C$
by using $M_1^* - M_2^* = M_1^* \cap\circledcirc M_2^{*C}$
By Propositon (2.4),
$= N^N Cl\ (M_1^*) - N^N Int\ (M_1^*)$
Hence $N^N Fr\ (M_1^*) = N^N Cl\ (M_1^*) - N^N Int\ (M_1^*)$.

Theorem 3.11.

For a *NNS* M_1^* in the $N^N TS$ U, $N^N Fr\ (N^N Int\ (M_1^*)) \subseteq\circledcirc N^N Fr\ (M_1^*)$.

Proof :

Let M_1^* be the NNS in the N-N-T-S U. Using Definition 3.1.,
$N^N Fr\ (N^N Int\ (M_1^*)) = N^N Cl\ (N^N Int\ (M_1^*)) \cap N^N Cl\ (N^N Int\ (M_1^*))\ )^C$
By Propositon (2.4),
$= N^N Cl\ (N^N Int\ (M_1^*)) \cap N^N Cl\ (N^N Cl\ (M_1^{*C})))$
By Propositon (2.4),
$= N^N Cl\ (N^N Int\ (M_1^*)) \cap N^N Cl\ (M_1^{*C})\ )$
. By Propositon (2.4), ,
$\subseteq\circledcirc N^N Cl\ (M_1^*) \cap N^N Cl\ (M_1^{*C}))$
Again by Definition 3.1.,
$= N^N Fr\ (M_1^*)$
Hence $N^N Fr\ (N^N Int\ (M_1^*)) \subseteq\circledcirc N^N Fr\ (M_1^*)$.
.

Example 3.12.

Let U and $\mathcal{A}$ be two non-empty finite sets,
where U is the universe and $\mathcal{A}$ the set of attributes
The members of U= {P_1, P_2, P_3,P_4}are pressure patients
Let U/R ={{P_1, P_2, P_3}, {P_4}} be an equivalence relation
$\mathcal{A} = \{\text{Head ache, Temperature}\}$ are two attributes
$P_1 = \langle x, \left(\frac{5}{10}, \frac{6}{10}, \frac{7}{10}\right), \left(\frac{10}{10}, \frac{9}{10}, \frac{4}{10}\right)\rangle$
$P_2 = \langle x, \left(\frac{3}{10}, \frac{9}{10}, \frac{2}{10}\right), \left(\frac{4}{10}, \frac{1}{10}, \frac{6}{10}\right)\rangle$

P3= ⟨x, $\left(\frac{5}{10},\frac{9}{10},\frac{2}{10}\right),\left(\frac{4}{10},\frac{9}{10},\frac{4}{10}\right)$⟩
P4= ⟨x, $\left(\frac{3}{10},\frac{6}{10},\frac{7}{10}\right),\left(\frac{1}{10},\frac{1}{10},\frac{6}{10}\right)$⟩
$N_N(\tau)$ = {0_{N_N}, 1_{N_N}, $\underline{N(F)}$, $\overline{N(F)}$, BN(F)}

$\underline{N(F)}$ = ⟨x, $\left(\frac{3}{10},\frac{6}{10},\frac{7}{10}\right),\left(\frac{4}{10},\frac{1}{10},\frac{6}{10}\right)$⟩
$\overline{N(F)}$ = ⟨x, $\left(\frac{5}{10},\frac{9}{10},\frac{2}{10}\right),\left(\frac{10}{10},\frac{9}{10},\frac{4}{10}\right)$⟩
BN(F) = ⟨x, $\left(\frac{5}{10},\frac{4}{10},\frac{3}{10}\right),\left(\frac{6}{10},\frac{9}{10},\frac{4}{10}\right)$⟩

$N_N(\tau)$= {0_{N_N}, 1_{N_N}, ⟨x, $\left(\frac{3}{10},\frac{6}{10},\frac{7}{10}\right),\left(\frac{4}{10},\frac{1}{10},\frac{6}{10}\right)$⟩, ⟨ $\left(\frac{5}{10},\frac{9}{10},\frac{2}{10}\right),\left(\frac{10}{10},\frac{9}{10},\frac{4}{10}\right)$⟩, ⟨$\left(\frac{5}{10},\frac{4}{10},\frac{3}{10}\right),\left(\frac{6}{10},\frac{9}{10},\frac{4}{10}\right)$⟩, $\left(\frac{3}{10},\frac{6}{10},\frac{7}{10}\right),\left(\frac{1}{10},\frac{1}{10},\frac{6}{10}\right)$}

$M_3^* = \langle\left(\frac{5}{10},\frac{4}{10},\frac{3}{10}\right),\left(\frac{6}{10},\frac{9}{10},\frac{4}{10}\right)\rangle$

Therefore by Definition 3.1., $N^N Fr\ (M_3^*) = \nsubseteq N^N Fr\ (N^N Int\ (M_3^*))$.

Theorem 3.13.

For a *NNS* M_1^* in the *N-N-T-S* U, $N^N Fr\ (N^N Cl\ (M_1^*)) \subseteq N^N Fr\ (M_1^*)$.

Proof : Let M_1^* be the *NNS* in the N-N-T-S U. Using Definition 3.1.,
$N^N Fr\ (N^N Cl\ (M_1^*)) = N^N Cl\ (N^N Cl\ (M_1^*)) \cap N^N Cl\ ((N^N Cl\ (M_1^*)))\ )^C$ By Propositon (2.4),
$= N^N Cl\ (M_1^*) \cap N^N Cl\ (N^N Int\ ({M_1^*}^C))$By Propositon (2.4),
$\subseteq N^N Cl\ (M_1^*) \cap N^N Cl\ ({M_1^*}^C)$Again by Definition 3.1.,
$= N^N Fr\ (M_1^*)$
Hence $N^N Fr\ (N^N Cl\ (M_1^*)) \subseteq N^N Fr\ (M_1^*)$.

Theorem 3.14.

Let M_1^* be the *NNS* in the *N-N-T-S* U. Then $N^N Int\ (M_1^*) \subseteq⊚ M_1^* - N^N Fr\ (M_1^*)$.

Proof : Let M_1^* be the *NNS* in the N-N-T-S U. Now by Definition 3.1.,
$M_1^* - N^N Fr\ (M_1^*) = M_1^* - (N^N Cl\ (M_1^*) \cap⊚ N^N Cl\ ({M_1^*}^C))$
$= (M_1^* - N^N Cl\ (M_1^*)) \cup (M_1^* - N^N Cl\ ({M_1^*}^C))$
$= M_1^* - N^N Cl\ ({M_1^*}^C)$
$N^N Int\ (M_1^*)$.
Hence $N^N Int\ (M_1^*) \subseteq M_1^* - N^N Fr\ (M_1^*)$.

Remark 3.15.

In general topology, the following conditions are hold :
$N^N Fr\ (M_1^*) \cap⊚ N^N Int\ (M_1^*) = 0N$,
$N^N Int\ (M_1^*) \cup⊚ N^N Fr\ (M_1^*) = N^N Cl\ (M_1^*)$,
$N^N Int\ (M_1^*) \cup⊚ N^N Int\ ({M_1^*}^C) \cup⊚ N^N Fr\ (M_1^*) = 1_{N_N}$.

Theorem 3.16.

Let M_1^* and M_2^* be the two *NNSs* in the *N-N-T-S*.
Then $N^N Fr\ (M_1^* \cup⊚ M_2^*) \subseteq⊚ N^N Fr\ (M_1^*) \cup⊚ N^N Fr\ (M_2^*)$.

Proof : Let M_1^* and M_2^* be the two *NNSs* in the *N-N-T-S* U.
Using Definition 3.1.,
$N^N Fr\ (M_1^* \cup⊚ M_2^*) = N^N Cl\ (M_1^* \cup⊚ M_2^*) \cap⊚ N^N Cl\ ((M_1^* \cup M_2^*)^C$
. By Propositon (2.4),
$= N^N Cl\ (M_1^* \cup⊚ M_2^*) \cap⊚ N^N Cl\ (\ {M_1^*}^C \cap⊚ {M_2^*}^C)$
$\subseteq⊚ (N^N Cl\ (M_1^*) \cup⊚ N^N Cl\ (M_2^*)) \cap⊚ (N^N Cl\ ({M_1^*}^C) \cap⊚ N^N Cl({M_2^*}^C))$
$= [(N^N Cl\ (M_1^*) \cup⊚ N^N Cl\ (M_2^*)\ ) \cap⊚ N^N Cl\ ({M_1^*}^C)\ ] \cap⊚ [\ (N^N Cl\ (M_1^*) \cup⊚ N^N Cl\ (M_2^*)\ ) \cap⊚ N^N Cl\ ({M_2^*}^C)\ ]$
$= [(N^N Cl\ (M_1^*) \cap⊚ N^N Cl\ ({M_1^*}^C)) \cup (\ N^N Cl\ (M_2^*) \cap⊚ N^N Cl({M_1^*}^C))] \cap⊚ [(\ N^N Cl\ (M_1^*) \cap⊚ N^N Cl\ ({M_2^*}^C)) \cup (\ N^N Cl(M_2^*) \cap⊚ N^N Cl({M_2^*}^C))]$
Again by Definition 3.1.,
$= [N^N Fr\ (M_1^*) \cup⊚ (\ N^N Cl\ (M_2^*) \cap⊚ N^N Cl\ ({M_1^*}^C))\ ] \cap⊚ [\ (N^N Cl\ (M_1^*) \cap⊚ N^N Cl\ ({M_2^*}^C)\ ) \cup⊚ N^N Fr(M_2^*)\ ]$
$= (N^N Fr\ (M_1^*) \cup⊚ N^N Fr\ (M_2^*)) \cap⊚ [\ (N^N Cl\ (M_2^*) \cap⊚ N^N Cl\ ({M_1^*}^C)\ )$

$\cup⊚(N^N Cl\ (M_1^*) \cap⊚ N^N Cl\ ({M_2^*}^C))]$
$\subseteq⊚ N^N Fr\ (M_1^*) \cup⊚ N^N Fr\ (M_2^*)$.
Hence $N^N Fr\ (M_1^* \cup⊚ M_2^*) \subseteq⊚ N^N Fr\ (M_1^*) \cup⊚ N^N Fr\ (M_2^*)$.

Note 3.17.

$N^N Fr\ (M_1^* \cap⊚ M_2^*) \nsubseteq N^N Fr\ (M_1^*) \cap⊚ N^N Fr\ (M_2^*)$ and
$N^N Fr\ (M_1^*) \cap⊚ N^N Fr\ (M_2^*) \nsubseteq N^N Fr\ (M_1^* \cap⊚ M_2^*)$.

Theorem 3.18.

For any *NNSs* M_1^* and M_2^* in the *N-N-T-S* U,
$N^N Fr\ (M_1^* \cap⊚ M_2^*) \subseteq⊚(N^N Fr\ (M_1^*) \cap⊚ N^N Cl\ (M_2^*)) \cup⊚(N^N Fr\ (M_2^*) \cap N^N Cl\ (M_1^*))$.

Proof : Let M_1^* and M_2^* be the two *NNSs* in the *N-N-T-S* U.
Using Definition 3.1.,
$N^N Fr\ (M_1^* \cap⊚ M_2^*) = N^N Cl\ (M_1^* \cap⊚ M_2^*) \cap⊚ N^N Cl\ ((M_1^* \ \cap M_2^*)^C$
Use Prop., 3.2 (1) [18] ,
$= N^N Cl\ (M_1^* \cap⊚ M_2^*) \cap⊚ N^N Cl\ {M_1^*}^C \cup⊚ {M_2^*}^C)$
. By Propositon (2.4),
$\subseteq⊚(N^N Cl\ (M_1^*) \cap⊚ N^N Cl\ (M_2^*)) \cap⊚(N^N Cl\ ({M_1^*}^C) \cup⊚ N^N Cl\ ({M_2^*}^C))$
$= [(N^N Cl\ (M_1^*) \cap⊚ N^N Cl\ (M_2^*)) \cap⊚ N^N Cl\ ({M_1^*}^C)] \cup⊚[(N^N Cl\ (M_1^*) \cap⊚ N^N Cl\ (M_2^*)) \cap⊚ N^N Cl\ ({M_2^*}^C)]$
Again by Definition 3.1.,
$= (N^N Fr\ (M_1^*) \cap⊚ N^N Cl\ (M_2^*)) \cup⊚(N^N Fr\ (M_2^*) \cap⊚ N^N Cl\ (M_1^*))$
Hence $N^N Fr\ (M_1^* \cap⊚ M_2^*) \subseteq⊚(N^N Fr\ (M_1^*) \cap⊚ N^N Cl\ (M_2^*)) \cup$
$(N^N Fr\ (M_2^*) \cap⊚ N^N Cl\ (M_1^*))$.

Corollary 3.19.

For any *NNSs* M_1^* and M_2^* in the *N-N-T-S* U,
$N^N Fr\ (M_1^* \cap⊚ M_2^*) \subseteq⊚ N^N Fr\ (M_1^*) \cup⊚ N^N Fr\ (M_2^*)$.

Proof :

Let M_1^* and M_2^* be the two *NNSs* in the *N-N-T-S* U. Using Definition 3.1.,
$N^N Fr\ (M_1^* \cap⊚ M_2^*) = N^N Cl\ (M_1^* \cap⊚ M_2^*) \cap⊚ N^N Cl\ ((M_1^* \ \cap M_2^*)^C$
. By Propositon (2.4),,
$= N^N Cl\ (M_1^* \cap⊚ M_2^*) \cap⊚ N^N Cl\ ({M_1^*}^C \cup⊚ {M_2^*}^C)$
. By Propositon (2.4),,
$\subseteq⊚(N^N Cl\ (M_1^*) \cap⊚ N^N Cl\ (M_2^*)) \cap⊚(N^N Cl\ ({M_1^*}^C) \cup⊚ N^N Cl\ ({M_2^*}^C))$
$= (N^N Cl\ (M_1^*) \cap⊚ N^N Cl\ (M_2^*) \cap⊚ N^N Cl\ ({M_1^*}^C)) \cup⊚(N^N Cl\ (M_1^*) \cap⊚ N^N Cl\ (M_2^*) \cap⊚ N^N Cl\ ({M_2^*}^C))$
Again by Definition 3.1.,
$= (N^N Fr\ (M_1^*) \cap⊚ N^N Cl\ (M_2^*)) \cup⊚(N^N Cl\ (M_1^*) \cap⊚ N^N Fr\ (M_2^*))$
$\subseteq⊚ N^N Fr\ (M_1^*) \cup⊚ N^N Fr\ (M_2^*)$
Hence $N^N Fr\ (M_1^* \cap⊚ M_2^*) \subseteq⊚ N^N Fr\ (M_1^*) \cup⊚ N^N Fr\ (M_2^*)$.

Theorem 3.20

For any *NNS* M_1^* in the *N-N-T-S* U,
(1) $N^N Fr\ (N^N Fr\ (M_1^*)) \subseteq⊚ N^N Fr\ (M_1^*)$,
(2) $N^N Fr\ (N^N Fr\ (N^N Fr\ (M_1^*))) \subseteq⊚ N^N Fr\ (N^N Fr\ (M_1^*))$.

Proof : (1) Let M_1^* be the *NNS* in the Neutrosophic Nano topological space U. Using Definition 3.1.,
$N^N Fr\ (N^N Fr\ (M_1^*)) = N^N Cl\ (N^N Fr\ (M_1^*)) \cap⊚ N^N Cl\ ((N^N Fr\ (M_1^*))^C$ Again by Definition 3.1.,
$= N^N Cl\ (N^N Cl\ (M_1^*) \cap⊚ N^N Cl\ ({M_1^*}^C)) \cap N^N Cl\ (\left((N^N Cl\ (M_1^*) \ \cap N^N Cl\ ({M_1^*}^C) \right)^C$
By Propositon (2.4), and by By Propositon (2.4),
$\subseteq⊚(N^N Cl\ (N^N Cl\ (M_1^*)) \cap⊚ N^N Cl\ (N^N Cl\ ({M_1^*}^C))) \cap⊚ N^N Cl\ (N^N Int\ ({M_1^*}^C) \cup⊚ N^N Int\ (M_1^*))$
Use Prop., 1.18 (f) [18] ,
$= (N^N Cl\ (M_1^*) \cap⊚ N^N Cl\ ({M_1^*}^C)) \cap⊚(N^N Cl\ (N^N Int\ ({M_1^*}^C)) \cup⊚ N^N Cl\ (N^N Int\ (M_1^*))$
$\subseteq⊚ N^N Cl\ (M_1^*) \cap⊚ N^N Cl\ ({M_1^*}^C)$
By Definition 3.1.,

= $N^N Fr\ (M_1^*)$
Therefore $N^N Fr\ (N^N Fr\ (M_1^*)) \subseteq_{\circledcirc} N^N Fr\ (M_1^*)$.
(2) By Definition 3.1.,
$N^N Fr\ (N^N Fr\ (N^N Fr\ (M_1^*))) = N^N Cl\ (N^N Fr\ (N^N Fr\ (M_1^*))) \cap N^N Cl\ ((N^N Fr\ (N^N Fr\ (M_1^*)))^C)$
Use Prop., 1.18 (f) [18] ,
$\subseteq_{\circledcirc}(N^N Fr\ (N^N Fr\ (M_1^*))) \cap_{\circledcirc} N^N Cl((N^N Fr\ (N^N Fr\ (M_1^*))))^C \subseteq_{\circledcirc} N^N Fr\ (N^N Fr\ (M_1^*))$.
Hence $N^N Fr\ (N^N Fr\ (N^N Fr\ (M_1^*))) \subseteq_{\circledcirc} N^N Fr\ (N^N Fr\ (M_1^*))$.

Example 3.21.
Let U and $\mathcal{A}$ be two non-empty finite sets,
where U is the universe and $\mathcal{A}$ the set of attributes
The members of U= {P_1, P_2, P_3,P_4}are patients
Let U/R ={{P_1, P_2, P_3}, {P_4}} be an equivalence relation
$\mathcal{A} = \{\text{Head ache, Temperature}\}$ are two attributes
$P_1 = \langle x, \left(\frac{8}{10},\frac{4}{10},\frac{5}{10}\right),\left(\frac{4}{10},\frac{6}{10},\frac{7}{10}\right)\rangle$
$P_2 = \langle x, \left(\frac{4}{10},\frac{2}{10},\frac{9}{10}\right),\left(\frac{1}{10},\frac{4}{10},\frac{9}{10}\right)\rangle$
$P_3 = \langle x, \left(\frac{5}{10},\frac{6}{10},\frac{8}{10}\right),\left(\frac{7}{10},\frac{4}{10},\frac{4}{10}\right)\rangle$
$P_4 = \langle x, \left(\frac{9}{10},\frac{8}{10},\frac{4}{10}\right),\left(\frac{9}{10},\frac{6}{10},\frac{1}{10}\right)\rangle$
$N_N(\tau) = \{0_{N_N}, 1_{N_N}, \underline{N(F)}, \overline{N(F)}, B_N(F)\}$

$\underline{N(F)} = \langle x, \left(\frac{4}{10},\frac{2}{10},\frac{9}{10}\right),\left(\frac{1}{10},\frac{4}{10},\frac{9}{10}\right)\rangle$
$\overline{N(F)} = \langle x, \left(\frac{8}{10},\frac{6}{10},\frac{5}{10}\right),\left(\frac{7}{10},\frac{6}{10},\frac{4}{10}\right)\rangle$
$BN(F) = \langle x, \left(\frac{9}{10},\frac{6}{10},\frac{4}{10}\right),\left(\frac{7}{10},\frac{6}{10},\frac{4}{10}\right)\rangle$

$N_N(\tau) = \{0_{N_N}, 1_{N_N}, \langle x, \left(\frac{4}{10},\frac{2}{10},\frac{9}{10}\right),\left(\frac{1}{10},\frac{4}{10},\frac{9}{10}\right)\rangle, \langle\left(\frac{8}{10},\frac{6}{10},\frac{5}{10}\right),\left(\frac{7}{10},\frac{6}{10},\frac{4}{10}\right)\rangle,$
$\langle\left(\frac{9}{10},\frac{6}{10},\frac{4}{10}\right),\left(\frac{7}{10},\frac{6}{10},\frac{4}{10}\right)\rangle, \left(\frac{9}{10},\frac{8}{10},\frac{4}{10}\right),\left(\frac{9}{10},\frac{6}{10},\frac{1}{10}\right)\}$
$M_1^* = \langle x, \left(\frac{6}{10},\frac{7}{10},\frac{8}{10}\right),\left(\frac{5}{10},\frac{4}{10},\frac{5}{10}\right)\rangle$
Then $N^N Fr\ (M_1^*) = \langle x, \left(\frac{9}{10},\frac{8}{10},\frac{4}{10}\right),\left(\frac{9}{10},\frac{6}{10},\frac{1}{10}\right)\rangle$
$N^N Fr\ (N^N Fr\ (M_1^*)) = \langle x, \left(\frac{4}{10},\frac{2}{10},\frac{9}{10}\right),\left(\frac{1}{10},\frac{4}{10},\frac{9}{10}\right)\rangle$.
$N^N Fr\ (M_1^*) \nsubseteq N^N Fr\ (N^N Fr\ (M_1^*))$

III. NEUTROSOPHIC NANO SEMI-FRONTIER

In this section, we introduce the Neutrosophic Nano semi-frontier and their properties in N-N-T-S s.

Definition 4.1.
Let M_1^* be a *NNS* in the *N-N-T-S* U. Then the Neutrosophic Nano semi-frontier of M_1^* is defined as
$NN(S)Fr\ (M_1^*) = N^N(S)Cl\ (M_1^*) \cap N^N(S)Cl({M_1^*}^C)$.
Obviously $N^N(S)Fr\ (M_1^*)$ is a *NN(S)C* set in U.

Theorem 4.2.
Let M_1^* be a *NNS* in the *N-N-T-S* U. Then the following conditions are holds :
(i) $N^N(S)Cl\ (M_1^*) = M_1^* \cup_{\circledcirc} N^N Int\ (N^N Cl\ (M_1^*))$,
(ii) $N^N(S)Int\ (M_1^*) = M_1^* \cap_{\circledcirc} N^N Cl\ (N^N Int\ (M_1^*))$.

Proof : (i) Let M_1^* be a *NNS* in U. Consider
$N^N Int\ (N^N Cl\ (M_1^* \cup_{\circledcirc} N^N Int\ (N^N Cl\ (M_1^*))\ )\ )$
$= N^N Int\ (N^N Cl\ (M_1^*) \cup_{\circledcirc} N^N Cl\ (N^N Int\ (N^N Cl\ (M_1^*))\ )\ )$
$= N^N Int\ (N^N Cl\ (M_1^*))$
$\subseteq_{\circledcirc} M_1^* \cup_{\circledcirc} N^N Int\ (N^N Cl\ (M_1^*))$
It follows that $M_1^* \cup_{\circledcirc} N^N Int\ (N^N Cl\ (M_1^*))$ is a *NN(S)C* set in U.
Hence $N^N(S)Cl\ (M_1^*) \subseteq_{\circledcirc} M_1^* \cup_{\circledcirc} N^N Int\ (N^N Cl\ (M_1^*))$ ... (1)
Use Prop $N^N(S)Cl\ (M_1^*)$ is $N^N(S)C$ set in

U. We have $N^N Int\ (N^N Cl\ (M_1^*)) \subseteq⊚ N^N Int\ (N^N Cl\ (N^N (S)Cl\ \ (M_1^*))) \subseteq⊚ N^N (S)Cl\ \ (M_1^*)$.
Thus $M_1^* \cup⊚ N^N Int\ (N^N Cl\ (M_1^*)) \subseteq⊚ N^N (S)Cl\ \ (M_1^*)$ ... (2).
From (1) and (2), $N^N (S)Cl\ \ (M_1^*) = M_1^* \cup⊚ N^N Int\ (N^N Cl\ (M_1^*))$.
(ii) This can be proved in a similar manner as (i).

Theorem 4.3.
For a *NNS* M_1^* in the *N-N-T-S* U, $N^N (S)Fr\ (M_1^*) = N^N (S)Fr\ ({M_1^*}^C)$.
Proof : Let M_1^* be the *NNS* in the N-N-T-S U. Using Definition 4.1,
$N^N (S)Fr\ (M_1^*) = N^N (S)Cl\ \ (M_1^*) \cap⊚ N^N (S)Cl\ \ ({M_1^*}^C)$
$\quad = N^N (S)Cl\ \ ({M_1^*}^C) \cap⊚ N^N (S)Cl\ \ (M_1^*)$
$\quad = N^N (S)Cl\ \ ({M_1^*}^C) \cap⊚ N^N (S)Cl\left({M_1^*}^C\right)^C$
Again by Definition 4.1,
$\quad = N^N (S)Fr\ ({M_1^*}^C)$
Hence $N^N (S)Fr\ (M_1^*) = N^N (S)Fr\ ({M_1^*}^C)$.

Theorem 4.4.
If M_1^* is *NN(S)C* set in U, then$N^N (S)Fr\ (M_1^*) \subseteq⊚ M_1^*$.
Proof : Let M_1^* be the *NNS* in the N-N-T-S U. Using Definition 4.1,
$N^N (S)Fr\ (M_1^*) = N^N (S)Cl\ \ (M_1^*) \cap⊚ N^N (S)Cl\ \ ({M_1^*}^C)$
$\quad \subseteq⊚\ N^N (S)Cl(M_1^*) = M_1^*$
Hence $N^N (S)Fr\ (M_1^*) \subseteq⊚ M_1^*$, if M_1^* is *NN(S)C* in U.
The converse of the above theorem is not true as shown by the following example.

Example 4.5.
Let U and $\mathcal{A}$ be two non-empty finite sets,
where U is the universe and $\mathcal{A}$ the set of attributes
U={F_1,F_2,F_3, F_4}are Fruits
Let U/R ={{ F_1,F_2,F_3}, { F_4}} be an equivalence relation
$\mathcal{A} = \{\text{Proteins, minerals, vitamins}\}$ are three attributes ,its Neutrosphic values are given below
$F_1 = \langle\left(\frac{4}{10},\frac{5}{10},\frac{3}{10}\right),\left(\frac{3}{10},\frac{2}{10},\frac{2}{10}\right),\left(\frac{9}{10},\frac{5}{10},\frac{8}{10}\right)\rangle$
$F_2 = \langle\left(\frac{2}{10},\frac{4}{10},\frac{5}{10}\right),\left(\frac{1}{10},\frac{1}{10},\frac{2}{10}\right),\left(\frac{6}{10},\frac{5}{10},\frac{8}{10}\right)\rangle$
$F_3 = \langle\left(\frac{5}{10},\frac{5}{10},\frac{3}{10}\right),\left(\frac{4}{10},\frac{2}{10},\frac{2}{10}\right),\left(\frac{9}{10},\frac{5}{10},\frac{8}{10}\right)\rangle$
$F_4 = \langle\left(\frac{4}{10},\frac{4}{10},\frac{5}{10}\right),\left(\frac{2}{10},\frac{2}{10},\frac{2}{10}\right),\left(\frac{6}{10},\frac{5}{10},\frac{8}{10}\right)\rangle$
$N_N(\tau) = \{0_{N_N}, 1_{N_N}, \underline{\mathrm{N(M)}}, \overline{N}(\mathrm{M}), B_N(\mathrm{M})\}$
$\underline{\mathrm{N(F)}} = \langle\left(\frac{2}{10},\frac{4}{10},\frac{5}{10}\right),\left(\frac{1}{10},\frac{1}{10},\frac{2}{10}\right),\left(\frac{6}{10},\frac{5}{10},\frac{8}{10}\right)\rangle$
$\overline{N}(\mathrm{F}) = \langle\left(\frac{5}{10},\frac{5}{10},\frac{3}{10}\right),\left(\frac{4}{10},\frac{2}{10},\frac{2}{10}\right),\left(\frac{9}{10},\frac{5}{10},\frac{8}{10}\right)\rangle$
$B_N(\mathrm{F}) = \langle\left(\frac{5}{10},\frac{5}{10},\frac{3}{10}\right),\left(\frac{2}{10},\frac{2}{10},\frac{2}{10}\right),\left(\frac{8}{10},\frac{5}{10},\frac{8}{10}\right)\rangle$

$$N_N(\tau) = \{0_{N_N}, 1_{N_N}, \langle\left(\frac{2}{10},\frac{4}{10},\frac{5}{10}\right),\left(\frac{1}{10},\frac{1}{10},\frac{2}{10}\right),\left(\frac{6}{10},\frac{5}{10},\frac{8}{10}\right)\rangle,$$
$$\langle\left(\frac{5}{10},\frac{5}{10},\frac{3}{10}\right),\left(\frac{4}{10},\frac{2}{10},\frac{2}{10}\right),\left(\frac{9}{10},\frac{5}{10},\frac{8}{10}\right)\rangle$$
$$\langle\left(\frac{5}{10},\frac{5}{10},\frac{3}{10}\right),\left(\frac{2}{10},\frac{2}{10},\frac{2}{10}\right),\left(\frac{8}{10},\frac{5}{10},\frac{8}{10}\right)\rangle,$$
$$\langle\left(\frac{4}{10},\frac{4}{10},\frac{5}{10}\right),\left(\frac{2}{10},\frac{2}{10},\frac{2}{10}\right),\left(\frac{6}{10},\frac{5}{10},\frac{8}{10}\right)\rangle\}$$

$M_1^*\ \langle\left(\frac{2}{10},\frac{5}{10},\frac{5}{10}\right),\left(\frac{2}{10},\frac{7}{10},\frac{2}{10}\right),\left(\frac{7}{10},\frac{5}{10},\frac{5}{10}\right)\rangle$, is Neutrosophic Nano semi-closed set
Then $N^N (S)Fr\ (M_1^*) \subseteq M_1^*$

Theorem 4.6.
If M_1^* is NNSO set in U, then $N^N(S)Fr\ (M_1^*) \subseteq {M_1^*}^C$
Proof : Let M_1^* be the NNS in the N-N-T-S U. Using Proposition 4.3 [18] ,

M_1^* is NNSO set implies ${M_1^*}^C$ is NN(S)C set in U. By Theorem 3.4, N^N(S)Fr (${M_1^*}^C$) ⊆ ${M_1^*}^C$ and by Theorem 3.3, we get N^N(S)Fr (M_1^*) ⊆ ${M_1^*}^C$

Theorem 4.7.

Let $M_1^* \subseteq M_2^*$ and $M_2^* \in N^N(S)C$ (U) (resp., $M_2^* \in N^N SO$ (U)). Then $N^N(S)Fr\ (M_1^*) \subseteq M_2^*$ (resp., $N^N(S)Fr\ (M_1^*) \subseteq {M_2^*}^C$, where $N^N(S)C$ (U) (resp., $N^N SO$ (U)) denotes the class of Neutrosophic Nano semi-closed (resp.,Neutrosophic Nano semi-open) sets in U.

Proof : Use Prop., 6.3 (iv) [18] , $M_1^* \subseteq M_2^*$,

$N^N(S)Cl\ (M_1^*) \subseteq N^N(S)Cl\ (M_2^*)$ -------------------- (1).

By Definition 4.1,

$N^N(S)Fr\ (M_1^*) = N^N(S)Cl(M_1^*) \cap N^N(S)Cl\ ({M_1^*}^C)$

$\subseteq N^N(S)Cl(M_2^*) \cap N^N(S)Cl\ ({M_1^*}^C)$ by (1)

$\subseteq N^N(S)Cl\ (M_2^*)$Use Prop., 6.3 (ii) [18] ,= M_2^*

Hence $N^N(S)Fr\ (M_1^*) \subseteq M_2^*$.

Theorem 4.8.

Let M_1^* be the NNS in the N-N-T-S U. Then $(N^N(S)Fr\ (M_1^*)\ )^C = N^N(S)Int\ (M_1^*) \cup N^N(S)Int\ ({M_1^*}^C\ )$.

Proof : Let M_1^* be the NNS in the N-N-T-S U. Using Definition 4.1,

$(N^N(S)Fr\ (M_1^*)\ )^C = \left((N^N(S)Cl\ (M_1^*)\ \cap N^N(S)Cl\ ({M_1^*}^C\ ))\right)^C$Use Prop., 3.2 (1) [18] ,

$= (N^N(S)Cl\ (M_1^*))^C \cup (N^N(S)Cl\ ({M_1^*}^C))^C$Use Prop., 6.2 (ii) [18] ,

$= N^N(S)Int\ ({M_1^*}^C) \cup N^N(S)Int\ (M_1^*)$

Hence $(N^N(S)Fr\ (M_1^*)\ )^C\ = N^N(S)Int\ (M_1^*) \cup N^N(S)Int\ ({M_1^*}^C)$.

Theorem 4.9.

For a NNS M_1^* in the N-N-T-S U, then $N^N(S)Fr\ (M_1^*) \subseteq N^N Fr\ (M_1^*)$.

Proof : Let M_1^* be the *NNS* in the N-N-T-S U. Using Proposition 6.4 [18] ,

$N^N(S)Cl\ (M_1^*) \subseteq N^N Cl\ (M_1^*)$ and $N^N(S)Cl\ ({M_1^*}^C) \subseteq N^N Cl\ ({M_1^*}^C)$.Now by Definition 4.1,

$N^N(S)Fr\ (M_1^*) = N^N(S)Cl\ (M_1^*) \cap N^N(S)Cl\ ({M_1^*}^C) \subseteq N^N Cl\ (M_1^*) \cap N^N Cl\ ({M_1^*}^C)$By Definition 3.1.,

$= N^N Fr\ (M_1^*)$

Hence $N^N(S)Fr\ (M_1^*) \subseteq N^N Fr\ (M_1^*)$.

Theorem 4.10.

For a *NNS* M_1^* in the *N-N-T-S* U, then $N^N(S)Cl\ (N^N(S)Fr\ (M_1^*)) \subseteq N^N Fr\ (M_1^*)$.

Proof : Let M_1^* be the *NNS* in the N-N-T-S U. Using Definition 4.1,

$N^N(S)Cl\ (N^N(S)Fr\ (M_1^*)) = N^N(S)Cl\ (N^N(S)Cl\ (M_1^*) \cap N^N(S)Cl\ (({M_1^*}^C)))$

$\subseteq N^N(S)Cl\ (N^N(S)Cl\ (M_1^*)) \cap N^N(S)Cl\ (N^N(S)Cl\ (({M_1^*}^C)))$Use Prop., 6.3 (iii) [18] ,

$= N^N(S)Cl\ (M_1^*) \cap N^N(S)Cl\ (({M_1^*}^C))$By Definition 4.1,

$= N^N(S)Fr\ (M_1^*)$By Theorem 3.10,

$\subseteq N^N Fr\ (M_1^*)$

Hence $N^N(S)Cl\ (N^N(S)Fr\ (M_1^*)) \subseteq N^N Fr\ (M_1^*)$.

Theorem 4.11

Let M_1^* be a $N^N S$ in the *N-N-T-S* U. Then $N^N(S)Fr\ (M_1^*) = N^N(S)Cl\ (M_1^*) - N^N(S)Int\ (M_1^*)$.

Proof :

Let M_1^* be the *NNS* in the N-N-T-S U. Use Prop., 6.2 (ii) [18] ,

$(N^N(S)Cl\ ({M_1^*}^C)))^C = N^N(S)Int\ (M_1^*)$ and by Definition 4.1,

$N^N(S)Fr\ (M_1^*) = N^N(S)Cl\ (M_1^*) \cap N^N(S)Cl\ (\ ({M_1^*}^C))$

$= N^N(S)Cl\ (M_1^*) - (N^N(S)Cl\ (({M_1^*}^C)))^C$by using $M_1^* - M_2^* = M_1^* \cap ({M_2^*}^C)$Use Prop., 6.2 (ii) [18] ,

$= N^N(S)Cl\ (M_1^*) - N^N(S)Int\ (M_1^*)$

Hence $N^N(S)Fr\ (M_1^*) = N^N(S)Cl\ (M_1^*) - N^N(S)Int\ (M_1^*)$.

Theorem 4.12.

For a *NNS* M_1^* in the *N-N-T-S* U, then $N^N(S)Fr\ (N^N(S)Int\ (M_1^*)) \subseteq N^N(S)Fr\ (M_1^*)$.

Proof : Let M_1^* be the *NNS* in the N-N-T-S U. Using Definition 4.1,

$N^N(S)Fr\ (N^N Int\ (M_1^*)) = N^N(S)Cl\ (N^N Int(M_1^*)) \cap\ N^N(S)Cl\ (\ (N^N(S)Int\ (M_1^*))^C)$ Use Prop., 6.2 (i) [18] ,

$= N^N(S)Cl\ (N^N(S)Int\ (M_1^*)) \cap\ N^N(S)Cl\ (N^N(S)Cl\ ((M_1^{*C})))$ Use Prop., 6.3 (iii) [18] ,

$= N^N(S)Cl\ (N^N(S)Int\ (M_1^*)) \cap N^N(S)Cl\ (M_1^{*C})$ Use Prop., 5.2 (ii) [18] ,

$\subseteq N^N(S)Cl\ (M_1^*) \cap N^N(S)Cl\ (M_1^{*C}))$ By Definition 4.1,

$= N^N(S)Fr\ (M_1^*)$

Hence $N^N(S)Fr\ (N^N(S)Int\ (M_1^*)) \subseteq N^N(S)Fr\ (M_1^*)$.

Example 4.13.

Let U and $\mathcal{A}$ be two non-empty finite sets,

where U is the universe and $\mathcal{A}$ the set of attributes

U={P_1,P_2,P_3, P_4}are Patients

Let U/R ={{ P_1,P_2,P_3, P_4 }, { P_5}} be an equivalence relation

$\mathcal{A} = \{\text{Head ache}, \text{Temperature}, \text{Cold}\}$ are three attributes

its Neutrosphic values are given below

$P_1 = \langle\left(\frac{3}{10},\frac{4}{10},\frac{2}{10}\right),\left(\frac{5}{10},\frac{6}{10},\frac{7}{10}\right),\left(\frac{9}{10},\frac{5}{10},\frac{2}{10}\right)\rangle$

$P_2 = \langle\left(\frac{3}{10},\frac{5}{10},\frac{1}{10}\right),\left(\frac{4}{10},\frac{6}{10},\frac{2}{10}\right),\left(\frac{8}{10},\frac{4}{10},\frac{6}{10}\right)\rangle$

$P_3 = \langle\left(\frac{3}{10},\frac{5}{10},\frac{1}{10}\right),\left(\frac{4}{10},\frac{6}{10},\frac{2}{10}\right),\left(\frac{8}{10},\frac{4}{10},\frac{6}{10}\right)\rangle$

$P_4 = \langle\left(\frac{3}{10},\frac{4}{10},\frac{2}{10}\right),\left(\frac{4}{10},\frac{6}{10},\frac{7}{10}\right),\left(\frac{8}{10},\frac{4}{10},\frac{6}{10}\right)\rangle$

$P_5 = \langle\left(\frac{5}{10},\frac{6}{10},\frac{1}{10}\right),\left(\frac{6}{10},\frac{7}{10},\frac{1}{10}\right),\left(\frac{9}{10},\frac{5}{10},\frac{2}{10}\right)\rangle$

$N_N(\tau) = \{0_{N_N}, 1_{N_N}, \underline{N(M)}, \overline{N(M)}, B_N(M)\}$

$\underline{N(F)} = \langle\left(\frac{3}{10},\frac{4}{10},\frac{2}{10}\right),\left(\frac{4}{10},\frac{6}{10},\frac{7}{10}\right),\left(\frac{8}{10},\frac{4}{10},\frac{6}{10}\right)\rangle$

$\overline{N(F)} = \langle\left(\frac{3}{10},\frac{5}{10},\frac{1}{10}\right),\left(\frac{5}{10},\frac{6}{10},\frac{2}{10}\right),\left(\frac{9}{10},\frac{5}{10},\frac{2}{10}\right)\rangle$

$B_N(F) = \langle\left(\frac{2}{10},\frac{5}{10},\frac{3}{10}\right),\left(\frac{5}{10},\frac{6}{10},\frac{4}{10}\right),\left(\frac{6}{10},\frac{5}{10},\frac{6}{10}\right)\rangle$

$N_N(\tau) = \{0_{N_N}, 1_{N_N}, \langle\left(\frac{3}{10},\frac{4}{10},\frac{2}{10}\right),\left(\frac{4}{10},\frac{6}{10},\frac{7}{10}\right),\left(\frac{8}{10},\frac{4}{10},\frac{6}{10}\right)\rangle, \langle\left(\frac{3}{10},\frac{5}{10},\frac{1}{10}\right),\left(\frac{5}{10},\frac{6}{10},\frac{2}{10}\right),\left(\frac{9}{10},\frac{5}{10},\frac{2}{10}\right)\rangle$
$\langle\left(\frac{2}{10},\frac{5}{10},\frac{3}{10}\right),\left(\frac{5}{10},\frac{4}{10},\frac{4}{10}\right),\left(\frac{6}{10},\frac{5}{10},\frac{6}{10}\right)\rangle, \langle\left(\frac{5}{10},\frac{6}{10},\frac{1}{10}\right),\left(\frac{6}{10},\frac{7}{10},\frac{1}{10}\right),\left(\frac{9}{10},\frac{5}{10},\frac{2}{10}\right)\rangle\}$

$M_1^* = \langle\left(\frac{2}{10},\frac{6}{10},\frac{5}{10}\right),\left(\frac{5}{10},\frac{4}{10},\frac{5}{10}\right),\left(\frac{6}{10},\frac{5}{10},\frac{7}{10}\right)\rangle$ is a *NNSO*

Therefore $N^N(S)Fr\ (M_1^*) \nsubseteq N^N(S)Fr\ (N^N(S)Int\ (M_1^*))$.

Theorem 4.14.

For a *NNS* M_1^* in the *N-N-T-S* U, then $N^N(S)Fr\ (N^N(S)Cl\ (M_1^*)) \subseteq N^N(S)Fr\ (M_1^*)$.

Proof :

Let M_1^* be the N^NS in the N-N-T-S U. Using Definition 4.1,

$N^N(S)Fr(N^N(S)Cl\ (M_1^*)) = N^N(S)Cl\ (N^N(S)Cl\ (M_1^*)) \cap\ N^N(S)Cl((N^N(S)Cl\ (M_1^*)))^C$

Use Prop., 6.3 (iii) and Proposition 6.2 (ii) [18] ,

$= N^N(S)Cl\ (M_1^*) \cap N^N(S)Cl\ (N^N(S)Int\ (M_1^{*C}))$ Use Prop., 5.2 (i) [18] ,

$\subseteq N^N(S)Cl\ (M_1^*) \cap N^N(S)Cl\ (M_1^{*C})$ By Definition 4.1,

$= N^N(S)Fr\ (M_1^*)$

Hence $N^N(S)Fr\ (N^N(S)Cl\ (M_1^*)) \subseteq N^N(S)Fr\ (M_1^*)$.

Remark 4.15.

In general topology, the following conditions conditions are hold :

$N^N(S)Fr\ (M_1^*) \cap\!\circledcirc N^N(S)Int\ (M_1^*) = 0N$,

$N^N(S)Int\ (M_1^*) \cup\!\circledcirc N^N(S)Fr\ (M_1^*) = N^N(S)Cl\ (M_1^*)$,

$N^N(S)Int\ (M_1^*) \cup\!\circledcirc N^N(S)Int\ (M_1^{*C}) \cup\!\circledcirc N^N(S)Fr\ (M_1^*) = 1_{N_N}$.

Theorem 4.16.

Let M_1^* and M_2^* be *NNSs* in the *N-N-T-S* U.

Then $N^N(S)Fr\ (M_1^* \cup\!\circledcirc M_2^*) \subseteq\!\circledcirc N^N(S)Fr\ (M_1^*) \cup\!\circledcirc N^N(S)Fr\ (M_2^*)$.

Proof : Let M_1^* and M_2^* be *NNSs* in the *N-N-T-S* U. Using Definition 4.1,

$N^N(S)Fr\ (M_1^* \cup⊚ M_2^*) = N^N(S)Cl\ (M_1^* \cup⊚ M_2^*) \cap⊚ N^N(S)Cl\ (M_1^* \cup M_2^*)^C$. By Propositon (2.4),

$= N^N(S)Cl\ (M_1^* \cup⊚ M_2^*) \cap⊚ N^N(S)Cl\ ((M_1^{*C}) \cap⊚ (M_2^{*C}))$Use Prop., 6.5 (i) and (ii) [18] ,

⊚⊚$\subseteq⊚(N^N(S)Cl\ (M_1^*) \cup⊚ N^N(S)Cl\ (M_2^*)) \cap (N^N(S)Cl\ (M_1^{*C}) \cap\ N^N(S)Cl\ (M_2^{*C}))$

$= [\ (N^N(S)Cl(M_1^*) \cup⊚ N^N(S)Cl\ (M_2^*)\ ) \cap⊚ N^N(S)Cl\ ((M_1^{*C}] \cap [\ (N^N(S)Cl\ (M_1^*) \cup⊚ N^N(S)Cl\ (M_2^*)\ ) \cap⊚ N^N(S)Cl\ (M_2^{*C})\ ]$

$= [(N^N(S)Cl\ (M_1^*) \cap\ N^N(S)Cl\ (M_1^{*C})) \cup (N^N(S)Cl\ (M_2^*) \cap\ N^N(S)Cl\ (M_1^{*C}))]$

$\cap [(N^N(S)Cl\ (M_1^*) \cap\ N^N(S)Cl\ (M_2^{*C})) \cup (N^N(S)Cl\ (M_2^*) \cap\ N^N(S)Cl\ (M_2^{*C}))]$By Definition 4.1,

$= [N^N(S)Fr\ (M_1^*) \cup⊚(N^N(S)Cl\ (M_2^*) \cap⊚ N^N(S)Cl\ (M_1^{*C})\ )\ ] \cap [\ (N^N(S)Cl\ (M_1^*) \cap⊚ N^N(S)Cl\ (M_2^{*C})\ ) \cup⊚ N^N(S)Fr(M_2^*)\ ]$

$= (N^N(S)Fr\ (M_1^*) \cup⊚ N^N(S)Fr\ (M_2^*)) \cap⊚[\ (N^N(S)Cl\ (M_2^*) \cap N^N(S)Cl\ (M_1^{*C})\ ) \cup⊚(N^N(S)Cl\ (M_1^*) \cap⊚ N^N(S)Cl\ (M_2^{*C})\ )\ ]$

$\subseteq⊚ N^N(S)Fr\ (M_1^*) \cup⊚ N^N(S)Fr\ (M_2^*)$.

Hence $N^N(S)Fr\ (M_1^* \cup⊚ M_2^*) \subseteq⊚ N^N(S)Fr\ (M_1^*) \cup⊚ N^N(S)Fr\ (M_2^*)$.

Example 4.17.

Let U and $\mathcal{A}$ be two non-empty finite sets,

where U is the universe and $\mathcal{A}$ the set of attributes

U={P_1,P_2 ,P_3, P_4}are Patients

Let U/R ={{ P_1,P_2 ,P_3}, { P_4}} be an equivalence relation

$\mathcal{A} =$ {Temperature} are one attributes

U/R = {P_1}{ P_2,P_3, P_4}

$P_1 = \langle\left(\frac{6}{10},\frac{5}{10},\frac{4}{10}\right)\rangle$

$P_2 = \langle\left(\frac{4}{10},\frac{5}{10},\frac{7}{10}\right)\rangle$

$P_3 = \langle\left(\frac{6}{10},\frac{5}{10},\frac{4}{10}\right)\rangle$

$P_4 = \langle\left(\frac{3}{10},\frac{5}{10},\frac{6}{10}\right)\rangle$.

Then $N_N(\tau)$ = {0_{N_N}, 1_{N_N}, $\underline{N}(M)$, $\overline{N}(M)$,$B_N(M)$}

$\underline{N}(F) = \langle\left(\frac{4}{10},\frac{5}{10},\frac{7}{10}\right)\rangle$

$\overline{N}(F) = \langle\left(\frac{6}{10},\frac{5}{10},\frac{4}{10}\right)\rangle$

$B_N(F) = \langle\left(\frac{6}{10},\frac{5}{10},\frac{4}{10}\right)\rangle$

$N_N(\tau) = \{0_{N_N}, 1_{N_N}, \left(\frac{4}{10},\frac{5}{10},\frac{7}{10}\right), \left(\frac{6}{10},\frac{5}{10},\frac{4}{10}\right), \left(\frac{6}{10},\frac{5}{10},\frac{4}{10}\right), \left(\frac{3}{10},\frac{8}{10},\frac{6}{10}\right)\}$

$N_N(\tau) = \{0_{N_N}, 1_{N_N}, \left(\frac{4}{10},\frac{5}{10},\frac{7}{10}\right), \left(\frac{6}{10},\frac{5}{10},\frac{4}{10}\right), \left(\frac{3}{10},\frac{5}{10},\frac{6}{10}\right)\}$

$M_1^* = \left(\frac{4}{10},\frac{5}{10},\frac{6}{10}\right), M_2^* = \left(\frac{6}{10},\frac{5}{10},\frac{4}{10}\right)$are NN(S)C

$M_1^* \cup⊚ M_2^* = \left(\frac{6}{10},\frac{5}{10},\frac{4}{10}\right)$

$N^N(S)Fr\ (M_1^* \cup⊚ M_2^*) \subseteq⊚ N^N(S)Fr\ (M_1^*) \cup⊚ N^N(S)Fr\ (M_2^*)$.

Theorem 4.18.

For any *NNSs* M_1^* and M_2^* in the *N-N-T-S* U,

$N^N(S)Fr\ (M_1^* \cap⊚ M_2^*) \subseteq⊚(N^N(S)Fr\ (M_1^*) \cap⊚ N^N(S)Cl\ (M_2^*)\ ) \cup (N^N(S)Fr\ (M_2^*) \cap⊚ N^N(S)Cl\ (M_1^*)\ )$.

Proof : Let M_1^* and M_2^* be $N^N Ss$ in the $N^N TS$ U. Using Definition 4.1,

$N^N(S)Fr\ (M_1^* \cap⊚ M_2^*) = N^N(S)Cl\ (M_1^* \cap⊚ M_2^*) \cap⊚ N^N(S)Cl\ (M_1^* \cap M_2^*)^C$Use Prop., 3.2 (1) [18] ,

$= N^N(S)Cl\ (M_1^* \cap⊚ M_2^*) \cap⊚ N^N(S)Cl\ ((M_1^{*C}) \cup⊚(M_2^{*C}))$ Use Prop., 6.5 (ii) and (i) [18] ,

$\subseteq⊚(N^N(S)Cl\ (M_1^*) \cap\ N^N(S)Cl\ (M_2^*)) \cap (N^N(S)Cl\ ((M_1^{*C})) \cup\ N^N(S)Cl\ ((M_2^{*C})))$

$= [\ (N^N(S)Cl\ (M_1^*) \cap⊚ N^N(S)Cl\ (M_2^*)\ ) \cap⊚ N^N(S)Cl\ ((M_1^{*C}] \cup [\ (N^N(S)Cl\ (M_1^*) \cap⊚ N^N(S)Cl\ (M_2^*)\ ) \cap⊚ N^N(S)Cl\ (M_2^{*C})\ ]$ By Definition 4.1,

$= (N^N(S)Fr\ (M_1^*) \cap⊚ N^N(S)Cl\ (M_2^*)\ ) \cup⊚(N^N(S)Fr\ (M_2^*) \cap N^N(S)Cl\ (M_1^*))$

Hence $N^N(S)Fr\ (M_1^* \cap⊚ M_2^*) \subseteq⊚(N^N(S)Fr\ (M_1^*) \cap⊚ N^N(S)Cl\ (M_2^*)\ ) \cup (N^N(S)Fr\ (M_2^*) \cap⊚ N^N(S)Cl\ (M_1^*)\ )$.

Corollary 4.19

For any *NNSs* M_1^* and M_2^* in the *N-N-T-S* U,

$N^N(S)Fr\ (M_1^* \cap⊚ M_2^*) \subseteq⊚ N^N(S)Fr\ (M_1^*) \cup⊚ N^N(S)Fr\ (M_2^*)$.

Proof : Let M_1^* and M_2^* be *NNSs* in the *N-N-T-S* U. Using Definition 4.1,

$N^N(S)Fr\ (M_1^* \cap⊚ M_2^*) = N^N(S)Cl\ (M_1^* \cap⊚ M_2^*) \cap⊚ N^N(S)Cl\ (M_1^* \cap M_2^*)^C$ Use Prop., 3.2 (1) [18] ,

$= N^N(S)Cl\ (M_1^* \cap⊚ M_2^*) \cap⊚ N^N(S)Cl\ (({M_1^*}^C) \cup⊚ ({M_2^*}^C))$,

$\subseteq⊚ (N^N(S)Cl\ (M_1^*) \cap\ N^N(S)Cl\ (M_2^*)) \cap (N^N(S)Cl\ (({M_1^*}^C \cup⊚ N^N(S)Cl\ (({M_2^*}^C)))$

$= (N^N(S)Cl\ (M_1^*) \cap⊚ N^N(S)Cl\ (M_2^*) \cap⊚ N^N(S)Cl\ (({M_1^*}^C) \cup⊚ (N^N(S)Cl(M_1^*) \cap⊚ N^N(S)Cl\ (M_2^*) \cap⊚ N^N(S)Cl$ $(({M_2^*}^C)))$

By Definition 4.1,

$= (N^N(S)Fr\ (M_1^*) \cap⊚ N^N(S)Cl\ (M_2^*)\ ) \cup⊚ (N^N(S)Cl\ (M_1^*) \cap⊚ N^N(S)Fr\ (M_2^*)) \subseteq⊚ N^N(S)Fr\ (M_1^*) \cup⊚ N^N(S)Fr$ (M_2^*).

Hence $N^N(S)Fr\ (M_1^* \cap⊚ M_2^*) \subseteq⊚ N^N(S)Fr\ (M_1^*) \cup⊚ N^N(S)Fr\ (M_2^*)$.

Theorem 4.20

For any *NNS* M_1^* in the *N-N-T-S* U,

(1) $N^N(S)Fr\ (N^N(S)Fr\ (M_1^*)) \subseteq⊚ N^N(S)Fr\ (M_1^*)$,

(2) $N^N(S)Fr\ (N^N(S)Fr\ (N^N(S)Fr\ (M_1^*))) \subseteq⊚ N^N(S)Fr\ (N^N(S)Fr\ (M_1^*))$.

Proof : (1) Let M_1^* be the *NNS* in the N-N-T-S U. Using Definition 4.1,

$N^N(S)Fr\ (N^N(S)Fr\ (M_1^*)) = N^N(S)Cl\ (N^N(S)Fr\ (M_1^*)) \cap⊚ N^N(S)Cl\ ((N^N(S)Fr\ (M_1^*)\ )^C\ (N^N(S)Fr\ (M_1^*)\ ))$

By Definition 4.1,

$= N^N(S)Cl\ (N^N(S)Cl\ (M_1^*) \cap⊚ N^N(S)Cl\ ({M_1^*}^C)) \cap N^N(S)Cl\ (\left((N^N(S)Cl\ (M_1^*)\ \cap N^N(S)Cl\ (({M_1^*}^C))\right)^C))$

Use Prop., 6.3 (iii) and 6.2 (ii) [18] ,

$\subseteq⊚ (N^N(S)Cl\ (N^N(S)Cl\ (M_1^*)) \cap⊚ N^N(S)Cl\ (N^N(S)Cl\ (({M_1^*}^C)))\ ) \cap N^N(S)Cl\ (N^N(S)Int\ (({M_1^*}^C))$ $\cup⊚ N^N(S)Int\ (M_1^*)\ )$Use Prop., 6.3 (iii) [18] ,

$= (N^N(S)Cl\ (M_1^*) \cap⊚ N^N(S)Cl\ ({M_1^*}^C))) \cap⊚ (N^N(S)Cl\ (N^N(S)Int\ ({M_1^*}^C)) \cup⊚ N^N(S)Cl\ (N^N(S)Int\ (M_1^*))$

$\subseteq⊚ N^N(S)Cl\ (M_1^*) \cap⊚ N^N(S)Cl\ ({M_1^*}^C))$ By Definition 4.1,

$= N^N(S)Fr\ (M_1^*)$

Therefore $N^N(S)Fr\ (N^N(S)Fr\ (M_1^*)) \subseteq⊚ N^N(S)Fr\ (M_1^*)$.(2) By Definition 4.1,

$N^N(S)Fr\ (N^N(S)Fr\ (N^N(S)Fr\ (M_1^*))) = N^N(S)Cl\ (N^N(S)Fr\ (N^N(S)Fr\ (M_1^*))) \cap⊚ N^N(S)Cl\ (\ (N^N(S)Fr$ $(N^N(S)Fr\ (M_1^*)))^C)$

Use Prop., 6.3 (iii) [18] ,

$\subseteq⊚ (N^N(S)Fr\ (N^N(S)Fr\ (M_1^*))) \cap⊚ N^N(S)Cl\ ((N^N(S)Fr\ (N^N(S)Fr\ (M_1^*)))^C)\ (N^N(S)Fr\ (N^N(S)Fr\ (M_1^*))))$

$\subseteq⊚ N^N(S)Fr\ (N^N(S)Fr\ (M_1^*))$.

Hence $N^N(S)Fr\ (N^N(S)Fr\ (N^N(S)Fr\ (M_1^*))) \subseteq⊚ N^N(S)Fr\ (N^N(S)Fr\ (M_1^*))$.

Conclusion

This research article shared some fundmental properties of introduce the Neutrosophic Nano semi-frontier .This concepts for further research will be on elaborating the structure of Neutrosophic Nano topology to more new classes of weak and strong forms of nano-open sets, new classes of generalized sets and new classes of continuous functions.There is further scope of launching into wider applications of Neutrosophic nano topology in different branches of Sciences and Humanities.

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A Kind of Non-associative Groupoids and Quasi Neutrosophic Extended Triplet Groupoids (QNET-Groupoids)

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Abstract: The various generalized associative laws can be considered as generalizations of traditional symmetry. Based on the theories of CA-groupoid, TA-groupoid and neutrosophic extended triplet (NET), this paper first proposes a new concept, which is type-2 cyclic associative groupoid (shortly by T2CA-groupoid), and gives some examples and basic properties. Furthermore, as a combination of neutrosophic extended triplet group (NETG) and T2CA-groupoid, the notion of type-2 cyclic associative neutrosophic extended triplet groupoid (T2CA-NET-groupoid) is introduced, and a decomposition theorem of T2CA-NET-groupoid is proved. Finally, as a generalization of neutrosophic extended triplet group (NETG), the concept of quasi neutrosophic extended triplet groupoid (QNET-groupoid) is introduced, and the relationships among T2CA-QNET-groupoid, T2CA-NET-groupoid and CA-NET-groupoid are discussed.

Keywords: Semigroup; Type-2 cyclic associative groupoid (T2CA-groupoid); neutrosophic extended triplet group (NETG); decomposition theorem; quasi neutrosophic extended triplet groupoid (QNET-groupoid)

1. Introduction

Groups and semigroups ([1–5, 7]) are essential branches of algebra, with the development of semigroup, the study of generalized semigroup has become an important topic. As far as we know the term groupoid (also called a magma) consists of a set *G* equipped with a binary operation. Despite the lack of further axioms, interesting results about groupoids exist [6].

The theory of non-associative algebras has seen new impetuous developments in recent years. Starting from algebraic topology, geometry and physics, new non-associative structures have emerged, such as triple systems, pairs, coalgebras and superalgebras. From a purely algebraic point of view, these structures are interesting. They have produced innovative ideas and methods that can help solve some algebraic problems. In fact, various generalized association identities are studied in many branches, for examples, functional equations [8-9], non-associative algebras [10], image processing [11] theory and networks [12].

The term "cyclic associative law" first appeared in the paper [13] published in 1954, which means an equation in the axiomatic system of Boolean algebra obtained in the literature [14] in 1946, namely $(ab)c=(bc)a$. Later, references [15-18] studied the relevant algebraic structures satisfying the cyclic binding law, however, the cyclic associative law in these references is actually a dual form of the cyclic associative law in [13-14], which is, $x(yz)=z(xy)$. In [19], we introduce the notion of formal cyclic associative groupoid (CA-groupoid), and systematically study its properties and the relationship between CA-groupoid and neutrosophic extended triplet group (NETG).

Moreover, in some literatures ([7-9, 20]), the cyclic associative law is also used to refer to the following equation:

$$x(yz)=(zx)y.$$

This meaning first appeared in Hosszü's study of function equation [20]. In this way, the term "cyclic associative law" has at least two different meanings in historical documents.

In order to avoid confusion, the equation $x(yz)=(zx)y$ is called type-2 cyclic associative law in this paper, and we focus on the basic properties and structures of the groupoid satisfying type-2 cyclic associative law, calling it type-2 cyclic associative groupoid.

In addition, Smarandache first proposed the new notion of neutrosophic extended triplet group (NETG) in [21], and many other significant results on NETGs and related algebraic systems can be found in [22-25]. In this paper, we analyze the structure of type-2 cyclic associative neutrosophic extended triplet groupoid (T2CA-NET-Groupoid) and study the relationship with the commutative regular semigroup.

This paper is organized as follows. In Section 2, we show some significant concepts and basic properties of groupoid, CA-groupoid and neutrosophic extended triplet groupoid (NETG). In Section 3, we put forward the concept of type-2 cyclic associative groupoid (T2CA-groupoid), and show some typical examples. In Section 4, we discuss the basic properties of the T2CA-groupoid and show some important results on cancellative T2CA-Groupoids. In Section 5, we introduce an important class of T2CA-groupoids for the first time, and we call it a type-2 cyclic associative neutrosophic extended triplet groupoid (T2CA-NET-groupoid). We first study its basic properties, and then gets its decomposition theorem, and finally, we study the relationship between T2CA-NET-groupoid and commutative regular semigroup. In Section 6, we introduce another significant class of groupoids. We call it a quasi neutrosophic extended triplet groupoid (QNET-groupoid) and further discuss the relationship between T2CA-QNET-groupoid, QNETG, T2CA-NET-groupoid, and CA-NET-groupoid. In Section 7, we present the summary and plans for future work.

2. Preliminaries

We give some notions and results about groupoids in this section.

A groupoid refers to an algebraic structure composed of non-empty sets, on which binary operations * are acted. Traditionally, when the * operator is omitted, it will not be confused. Assume $(S, *)$ is a groupoid, we show some concepts as follows:

(1) An element $x \in S$ is called idempotent if $x^2=x$.

(2) An element $x \in S$ is right cancellative (respectively left cancellative), if for all $y, z \in S$, $y*x = z*x \Rightarrow y = z$ ($x*y = x*z \Rightarrow y = z$). If an element both right and left cancellative, then it is cancellative. S is called right cancellative (left cancellative, cancellative), if each element of S is right cancellative (left cancellative, cancellative).

(3) If for any $x, y, z \in S$, $x*(y*z)=(x*y)*z$, S is called semigroup. A semigroup $(S, *)$ is commutative, if for all $x, y \in S$, $x*y=y*x$.

(4) If $\forall x \in S$, $x^2=x$, we call the semigroup $(S, *)$ as a band.

Definition 1. *([18, 26])* Let $(S, *)$ be a groupoid, for any $x, y, z \in S$.

(1) If $x*(y*z) = z*(x*y)$, then S is called a cyclic associative groupoid (or shortly CA-groupoid).
(2) If $(x*y)*z = (z*y)*x$, then S is called a CA-AG-groupoid.

Proposition 1. *[19]* If $(S, *)$ is a CA-groupoid ($\forall r, s, t, u, v, w \in S$), then*:*

(1) $(r*s)*(t*u) = (u*r)*(t*s)$;

(2) $(r*s)*((t*u)*(v*w)) = (u*r)*((t*s)*(v*w))$.

Definition 2. ([21,27]) Let S be a non-empty set, and * is a binary operation on S. S is called a neutrosophic extended triplet set, if for each $x \in S$, there is a neutral "x" (denote by neut(x)), and the opposite of "x" (denote by $anti(x)$), such that $x*neut(x) = neut(x)*x = x$, $x*anti(x) = anti(x)*x = neut(x)$.

The set of $neut(a)$ and $anti(a)$ is represented by the notations $\{neut(a)\}$ and $\{anti(a)\}$; any certain one of $neut(a)$ and $anti(a)$ is represented by us with $neut(a)$ and $anti(a)$.

Definition 3. ([21, 27]) Let $(S, *)$ be a neutrosophic extended triplet set. If
(1) $(S, *)$ is well-defined, *i.e.*, $x*y \in S$ ($\forall$ $x, y \in S$).
(2) $(S, *)$ is associative, *i.e.*, $(x * y) * z = x * (y * z)$ ($\forall$ $x, y, z \in S$).
Then, $(S, *)$ is called a neutrosophic extended triplet group (NETG). If $x*y=y*x$ ($\forall$ $x, y \in S$), S is called a commutative NETG.

Proposition 2. ([23, 24]) If $(S, *)$ is a NETG, then ($\forall x \in S$) $neut(x)$ is unique.

Theorem 1. ([19]) Let $(S, *)$ be a TA-NET-groupoid. Denote the set of all different neutral element in S by $N(S)$. Put $S(e) = \{x \in S \mid neut(x) = e\}$ ($\forall e \in N(S)$), then $S(e)$ is a subgroup of S.

Theorem 2. ([28]) Assume that (S, *) is a CA-groupoid, the following statements are equivalent:
(1) S is a CA-NET-groupoid;
(2) S is a CA- (r, l)-NET-groupoid;
(3) S is a CA- (r, r)-NET-groupoid;
(4) S is a CA- (l, r)-NET-groupoid;
(5) S is a CA- (l, l)-NET-groupoid;
(6) S is a commutative regular semigroup.

3. Type-2 Cyclic Associative Groupoids (T2CA-Groupoids)

Definition 4. Let $(S, *)$ be a groupoid, for any $r, s, t \in S$. If

$$r*(s*t) = (t*r)*s,$$

then $(S, *)$ is called a type-2 cyclic associative groupoid (shortly, T2CA-groupoid).

The following example shows that there is T2CA-groupoid, which is not a CA-groupoid, not a semigroup, not an AG-groupoid. Obviously, it is not a CA-AG-groupoid.

Example 1. Put $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$, and define the operations * on S as shown in Table 1. Then (S, *) is a T2CA-groupoid. We can verify that (S, *) is not a semigroup, due to the fact that $(6*7)*7 = 2 \neq 1 = 6*(7*7)$; ($S$, *) is not a CA-groupoid, because $6*(6*7) = 1 \neq 2 = 7*(6*6)$; (S, *) is not an AG-groupoid, since $(6*7)*7 = 2 \neq 1 = (7*7)*6$. Obviously, ($S$, *) is not a CA-AG-groupoid.

Table 1. The operation * on S

*	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1
3	1	1	1	2	1	1	2	1
4	1	1	1	1	1	1	1	1
5	1	1	1	1	1	2	2	1
6	1	1	5	1	1	4	5	1
7	1	1	2	2	1	4	2	2
8	1	1	2	1	1	1	2	2

From the following example, we know that there is T2CA-groupoid which is a semigroup but not commutative.

Example 2. Table 2 shows the non-commutative T2CA-groupoid of order 6, and (*S*, *) is a semigroup.

Table 2. Cayley table on *S = {r, s, t, u, v, w}*

*	***r***	***S***	***t***	***u***	***v***	***w***
r	*r*	*R*	*t*	*t*	*t*	*t*
s	*r*	*R*	*t*	*t*	*t*	*t*
t	*t*	*T*	*r*	*r*	*r*	*r*
u	*t*	*t*	*r*	*r*	*r*	*s*
v	*t*	*t*	*r*	*s*	*s*	*r*
w	*t*	*t*	*r*	*r*	*s*	*r*

Proposition 3. (1) Each commutative semigroup is a T2CA-groupoid. (2) Let (*S*, *) be a T2CA-groupoid. If *S* is commutative, then *S* is a commutative semigroup.

Proof. By Definition 4, this is obvious. □

Definition 5. Let (*S*, *) is a T2CA-groupoid, then an element *e* in S is called the quasi left identity element if for all *a* in *S*, $e*a = a$ $(a \neq e)$; and it is called the quasi right identity element for all *a* in *S*, $a*e = a$ $(a \neq e)$. If *e* is both quasi left and right identity element**,** it is called quasi identity element**.**

Example 3. As shown in Table 3, put *S* = {*f, g, h, j, k*}, and define the operations * on *S*. Then we can verify through MATLAB that (*S*, *) is a T2CA-groupoid, and *f* is the quasi identity element in *S*, due to the fact that $f*g = g*f = g$, $f*h = h*f = h$, $f*j = j*f = j$, $f*k = k*f = k$.

Table 3. The operation * on *S*

*	***f***	***g***	***h***	***j***	***k***
f	*G*	*g*	*h*	*j*	*k*
g	*G*	*g*	*h*	*j*	*k*
h	*H*	*h*	*g*	*j*	*k*
j	*J*	*j*	*j*	*j*	*j*
k	*k*	*k*	*k*	*j*	*k*

Theorem 3. Let (*S*, *) be a T2CA-groupoid with quasi identity element *e*, that is $\forall x \in S$, $e*x = x*e = x$ $(x \neq e)$. Then *S* is commutative.

Proof. For any $x, y \in S$, when $x = y$, obviously $x*y = y*x$. Suppose $x \neq y$, we have:

(1) Assume that $y = e$, due to $x \neq y$, then $x \neq e$. Therefore, $e*x = x = x*e$. That is, $x*y = y*x$.

(2) Suppose $x \neq e$, and $y \neq e$, there are:

Case 1, if $x*y \neq e$, by Definition 4, we can get that $x*y = e*(x*y) = (y*e)*x = y*x$.

Case 2, if $x*y = e$, we have $y*x = e$. Otherwise, suppose $y*x \neq e$, by Definition 4 we have $y*x = e*(y*x) = (x*e)*y = x*y = e$. This contradicts $y*x \neq e$.

Hence**,** *S* is commutative. □

Theorem 4. Let (*S*, *) be a T2CA-groupoid, $e \in S$.

(1) If *e* is the quasi left identity element of *S*, that is, $\forall x \in S$, $e*x = x$ $(x \neq e)$, then *e* is the quasi right identity element.

(2) If *e* is the quasi right identity element of *S*, that is, $\forall x \in S$, $x*e = x$ $(x \neq e)$, then *e* is the quasi left identity element.

Proof. (1) If *e* is the quasi left identity element of *S.* For each $x \in S$, $e*x = x$ $(x \neq e)$, we have $x*e = (e*x)*e = x*(e*e)$, and

$$x = e*x = e*(e*x) = (x*e)*e = (x*(e*e))*e = (e*e)*(e*x) = (e*e)*x = e*(x*e).$$

Case 1, when $x*e \neq e$, from $x = e*(x*e)$ and definition of quasi left identity element, we can get that $x = x*e$.

Case 2, when $x*e = e$, $x = e*(x*e) = e*e$, then $e*e \neq e$. Otherwise, if $e*e = e$, we have $x = e*e = e$. This contradicts $x \neq e$. Due to that e is the quasi left identity element of S, therefore, $e*(e*e) = e*e$. And,

$$x*e = x*(e*e) = x*(e*(e*e)) = (e*e)*(x*e) = (e*e)*e = e*(e*e) = e*e = x.$$

Hence, e is the quasi right identity element.

(2) Suppose that e is the quasi right identity element of S. For any $x \in S$, $x*e = x\ (x \neq e)$, we have $ex = e*(x*e) = (e*e)*x$, and

$$x = x*e = (x*e)*e = e*(e*x) = e*((e*e)*x) = (x*e)*(e*e) = x*(e*e) = (e*x)*e.$$

Case 1, when $e*x \neq e$, from definition of quasi right identity element, we can get that $(e*x)*x = e*x$, so $x = (e*x)*e = e*x$.

Case 2, when $e*x = e$, $x = (e*x)*e = e*e$, then $e*e \neq e$. Otherwise, if $e*e = e$, we have $x = e*e = e$. This contradicts $x \neq e$. Due to that e is the quasi right identity element of S, therefore, $e*(e*e) = e*e$. Moreover,

$$\begin{aligned} e*x &= (e*e)*x = x*x \quad \text{(Applying } x = e*e\text{)} \\ &= x*(e*e) = x. \end{aligned}$$

Hence, e is the quasi left identity element. □

4. Some Properties of Type-2 Cyclic Associative Groupoids (T2CA-Groupoids)

Proposition 4. Let $(S, *)$ be a T2CA-groupoid. Then,

(1) $\forall a, b, c, d \in S$, $(a*b)*(c*d) = (b*a)*(d*c)$;

(2) $\forall a, b, c, d, e, f \in S$, $(a*b)*[(c*d)*(e*f)] = [(b*f)*(c*a)]*(e*d)$.

Proof. (1) Suppose $(S, *)$ is a T2CA-groupoid, then for any $a, b, c, d, e, f \in S$, by Definition 4 we have

$$(a*b)*(c*d) = [d*(a*b)]*c = [(b*d)*a]*c = a*[c*(b*d)] = a*[(d*c)*b] = (b*a)*(d*c).$$

(2) For any $a, b, c, d, e, f \in S$, by Definition 4 we have

$$\begin{aligned} (a*b)*[(c*d)*(e*f)] &= (a*b)*[(d*c)*(f*e)] \qquad \text{(By } (a*b)*(c*d) = (b*a)*(d*c)\text{)} \\ &= b*[((d*c)*(f*e))*a] = b*[(f*e)*(a*(d*c))] = b*[(f*e)*((c*a)*d)] \\ &= b*[(d*(f*e))*(c*a)] = [(c*a)*b]*(d*(f*e)) = [a*(b*c)]*((e*d)*f) \\ &= [f*(a*(b*c))]*(e*d) = [f*((c*a)*b)]*(e*d) = [(b*f)*(c*a)]*(e*d). \ \square \end{aligned}$$

Theorem 5. Suppose $(S, *)$ is a T2CA-groupoid.

(1) If $\forall k \in S$, $\exists\ e \in S$ such that $e*k = k$, that is, S have a left identity element, then S is a commutative semigroup.

(2) If $\forall k \in S$, $\exists e \in S$ such that $k*e = k$, that is, S have a right identity element, then S is a commutative semigroup.

(3) If $e \in S$ is a left identity element, then e is an identity element.

(4) If $e \in S$ is a right identity element, then e is an identity element.

Proof. (1) Suppose $(S, *)$ is a T2CA-groupoid. $\forall k, w \in S$, we have

$$k*w = [e*(e*k)]*w = [(k*e)*e]*w = e*[w*(k*e)] = e*[(e*w)*k] = e*(w*k) = w*k.$$

Therefore, $(S, *)$ is a commutative T2CA-groupoid. Applying Proposition 3 (2), we get that $(S, *)$ is a commutative semigroup.

(2) Suppose $(S, *)$ is a T2CA-groupoid. $\forall\ k, w \in S$, there are:

$$\begin{aligned} k*w &= [e*(e*k)]*w = k*[(w*e)*e] = (e*k)*(w*e) \\ &= (k*e)*(e*w) \qquad \text{(By Proposition 4 (1))} \\ &= [w*(k*e)]*e = (w*k)*e = w*k. \end{aligned}$$

Therefore, $(S, *)$ is a commutative T2CA-groupoid. Applying Proposition 3 (2), we get that $(S, *)$ is a commutative semigroup.

(3) If e is a left identity element in S. $\forall\ k \in S$, applying Proposition 4 (1) there are:

$$k = e*k = e*(e*k) = (k*e)*e = (k*e)*(e*e) = (e*k)*(e*e) = k*e.$$

Thus, $e \in S$ is an identity element.

(4) If e is a right identity element in S, $\forall\ k \in S$, applying Proposition 4 (1) we get

$$k = k*e = (k*e)*e = e*(e*k) = (e*e)*(e*k) = (e*e)*(k*e) = e*k.$$

Therefore, $e \in S$ is an identity element. □

Definition 6. Suppose S is a T2CA-groupoid. S is called a left cancellative (right cancellative, cancellative) T2CA-groupoid, if each element of S is left cancellative (right cancellative, cancellative).

Theorem 6. Suppose (S, *) is a T2CA-groupoid, $\forall p, q \in S$:

(1) if p is right cancellative or left cancellative, then p is cancellative;

(2) if p is right cancellative and q is left cancellative, then $p*q$ is cancellative;

(3) if $p*q$ is right cancellative, then $p*q = q*p$;

(4) if $p*q$ is cancellative, then p and q are cancellative;

(5) if p and $p*q$ are right cancellative, then $p*q$ is cancellative.

Proof. Let (S, *) be a T2CA-groupoid, $p, q \in S$.

(1) If p is a right cancellative element, $p*k = p*w$ ($\forall\ k, w \in S$), using type-2 cyclic association:

$$(k*p)*p = p*(p*k) = p*(p*w) = (w*p)*p.$$

Applying right cancellation property of p two times, then $k = w$. Therefore, $p \in S$ is a left cancellative element, so p is a cancellative element in S.

Similarly, if p is a left cancellative element, $k*p = w*p$ ($\forall k, w \in S$), using type-2 cyclic association:

$$p*(p*k) = (k*p)*p = (w*p)*p = p*(p*w).$$

Using left cancellation property of p two times, then $k = w$. Therefore, $p \in S$ is a right cancellative element, so p is a cancellative element in S.

(2) If p is right cancellative, q is left cancellative, $k*(p*q) = w*(p*q)$ ($\forall k, w \in S$), using type-2 cyclic association:

$$(q*k)*p = k*(p*q) = k*(p*q) = w*(p*q) = w*(p*q) = (q*w)*p.$$

Since p is right cancellative, q is left cancellative, we get $k = w$. Therefore, $p*q$ is a right cancellative.

Moreover, if $(p*q)*k = (p*q)*w$ ($\forall k, w \in S$), we have:

$$q*(k*p) = (p*q)*k = (p*q)*k = (p*q)*w = q*(w*p).$$

Since p is right cancellative, q is left cancellative, we get $k = w$. Therefore, $p*q$ is a left cancellative. Hence, $p*q$ is cancellative.

(3) Suppose $p*q$ is right cancellative. By Proposition 4 (2), we have:

$$[(p*q)*(q*p)] * (p*q) = (q*p)*[(p*q)*(p*q)] = [(p*q)*(p*q)]*(p*q).$$

Since $p*q$ is right cancellative, then $(p*q)*(q*p) = (p*q)*(p*q)$. Applying Proposition 4 (1), we get that $(q*p)*(p*q) = (p*q)*(p*q)$.Moreover, since $p*q$ is right cancellative, then $q*p = p*q$.

(4) Suppose $p*q$ is cancellative. If $q*k = q*w$ ($\forall k, w \in S$), there are:

$$k*(p*q) = (q*k)*p = (q*w)*p = w*(p*q).$$

Since $p*q$ is cancellative, so k=w. This means that q is left cancellative. According to (1), we know q is cancellative.

And, since $p*q$ is cancellative, then $p*q$ is right cancellative, according to (5) we get $q*p = p*q$. So, $q*p$ is cancellative, p is cancellative. Therefore, if $p*q$ is cancellative, then p and q are cancellative.

(5) Assume that p and $p*q$ are right cancellative. If $(p*q) * k = (p*q) * w$ ($\forall k, w \in S$), using type-2 cyclic association:

$$(k*p)*(p*q) = p*((p*q)*k) = p*((p*q)*w) = (w*p)*(p*q).$$

Since $p*q$ is right cancellative, so $k*p = w*p$. Moreover, p is right cancellative, so $k = w$. Thus, $p*q$ is left cancellative, according to (1), we know $p*q$ is cancellative. □

According to Theorem 6, we have the following corollary.

Corollary 1. Suppose (S, *) is a T2CA-groupoid, then the following asserts are equivalent:

(i) S is a left cancellative T2CA-groupoid;

(ii) S is a right cancellative T2CA-groupoid;

(iii) S is a cancellative and commutative semigroup;

Proof. (i) ⇒ (ii): Follow Theorem 6 (1).

(ii) ⇒ (iii): Assume that S is right cancellative, by using Theorem 6 (1), we get S is cancellative. For any $p, q \in S$, according to Theorem 6 (3), we have $p*q = q*p$, then S is commutative. When applying Proposition 3 (2), we get that S is a commutative semigroup. Therefore, S is a cancellative and commutative semigroup.

(iii) ⇒ (i): Obviously. □

Corollary 2. Let (S, *) be a T2CA-groupoid. If there exists a cancellative element in S, then the set $M = \{p \in S: p \text{ is cancellative}\}$ is a sub T2CA-groupoid of S.

Proof. Through the existence of a condition for cancellative elements in S, we get that M is not empty. $\forall p, q \in M$, p and q are right and left cancellative. By Theorem 6 (2), we get $p*q$ is cancellative. Thus $p*q \in M$. Therefore, M is a sub T2CA-groupoid of S. □

Corollary 3. Let (S, *) be a T2CA-groupoid. If there exists a non-cancellative element in S, then the set $N = \{p \in S: p \text{ is non-cancellative}\}$ is a sub T2CA-groupoid of S.

Proof. Obviously, N is non-empty. $\forall\ p, q \in N$, p and q are non-cancellative. Through Theorem 6 (4), we know that $p*q$ is non-cancellative. Thus, $p*q \in N$. Therefore, N is a sub T2CA-groupoid of S.□

Theorem 7. Suppose (S, *) is a T2CA-groupoid, $r, s, t \in S$. Define on S the relation ~ as:

$$r \sim s \Leftrightarrow r \text{ and } s \text{ are both cancellative or non-cancellative.}$$

Then ~ is an equivalence relation.

Proof. Obviously, ~ is reflexive and symmetric.

Next, Assume $r \sim s$ and $s \sim t$. If r and s are non-cancellative, from $s \sim t$ we get t is non-cancellative, thus r and t are non-cancellative, i.e., $r \sim t$; if r and s are cancellative, from $s \sim t$ we get t is cancellative, thus r and t are cancellative, i.e., $r \sim t$. Thus ~ is transitive.

Therefore, ~ is an equivalence relation. □

Definition 7. Let $(S_1, *_1)$, $(S_2, *_2)$ be two T2CA-groupoids, $S_1 \times S_2 = \{(p, q) \mid p \in S_1, q \in S_2\}$. Define binary operation * on $S_1 \times S_2$ as following:

$$(p_1, p_2) * (q_1, q_2) = (p_1 *_1 q_1, p_2 *_2 q_2), \text{ for any } (p_1, p_2), (q_1, q_2) \in S_1 \times S_2.$$

S_1 and S_2 are called the direct factors of $S_1 \times S_2$, and $(S_1 \times S_2, *)$ is called the direct product of $(S_1, *_1)$ and $(S_2, *_2)$.

Theorem 8. Let $(S_1, *_1)$, $(S_2, *_2)$ be two T2CA-groupoids. Then the direct product $(S_1 \times S_2, *)$ is a T2CA-groupoid.

Proof. Let (r_1, r_2), (s_1, s_2), $(t_1, t_2) \in S_1 \times S_2$. We have:

$$\begin{aligned}(r_1, r_2) * ((s_1, s_2) * (t_1, t_2)) &= (r_1, r_2) * (s_1 *_1 t_1, s_2 *_2 t_2) \\ &= (r_1 *_1 (s_1 *_1 t_1), r_2 *_2 (s_2 *_2 t_2)) = ((t_1 *_1 r_1) *_1 s_1), (t_2 *_2 r_2) *_2 s_2)) \\ &= (t_1 *_1 r_1, t_2 *_2 r_2) * (s_1, s_2) = ((t_1, t_2) * (r_1, r_2)) * (s_1, s_2).\end{aligned}$$

Hence, $(S_1 \times S_2, *)$ is a T2CA-groupoid. □

Theorem 9. Let S_1, S_2 are T2CA-groupoids, if a and b are cancellative, then $(p, q) \in S_1 \times S_2$ is cancellative $(p \in S_1, q \in S_2)$.

Proof. Applying Theorem 8, we know $S_1 \times S_2$ is a T2CA-groupoid. Assume p and q are cancellative $(p \in S_1, q \in S_2)$, for any $(x_1, x_2), (y_1, y_2) \in S_1 \times S_2$, $(p, q)*(x_1, x_2) = (p, q)*(y_1, y_2)$. Then

$$(px_1, qx_2) = (py_1, qy_2);\ px_1 = py_1,\ qx_2 = qy_2.$$

And, according to p and q are cancellative, we get that $x_1 = y_1$, $x_2 = y_2$. That is $(x_1, x_2) = (y_1, y_2)$. Hence, (p, q) is cancellative. □

5. Type-2 Cyclic Associative Neutrosophic Extended Triplet Groupoids (T2CA-NET-Groupoids)

In this section, we first proposed an important class of T2CA-groupoids, namely T2CA-NET-groupoids. After giving the basic definitions and properties, this section focuses on the structure of T2CA-NET-groupoids, and the relationship between T2CA-NET-groupoids and commutative regular semigroups. Fortunately, we got very exciting results.

Definition 8. Let $(S, *)$ be a neutrosophic extended triplet set. $(S, *)$ is called a type-2 cyclic associative neutrosophic extended triplet groupoid (shortly, T2CA-NET-groupoid), if the following conditions are satisfied:
(1) $(S, *)$ is well-defined, i.e., $\forall a, b \in S$, one has $a*b \in S$.
(2) $(S, *)$ is type-2 cyclic associative, i.e., $a*(b*c) = (c*a)*b$, $\forall a, b, c \in S$.
S is called a commutative T2CA-NET-groupoid if $a*b = b*a$, $\forall a, b \in S$.

Theorem 10. Let $(S, *)$ be a T2CA-NET-groupoid, $\forall x \in S$, then $neut(x)$ is unique.

Proof. We assume that local unit element $neut(x)$ is not unique in S. Then, there is $s, t \in \{neut(x)\}$ such that $(p, q \in S)$

$$x*s = s*x = x \text{ and } x*p = p*x = s;\ x*t = t*x = x \text{ and } x*q = q*x = t.$$

(1) To prove $s = s*t$. Due to the fact

$$s = p*x = p*(t*x) = (x*p)*t = s*t.$$

(2) To prove $t = t*s$. Due to the fact

$$t = q*x = q*(s*x) = (x*q)*s = t*s.$$

(3) To prove $s = s*s$. Due to the fact

$$s = p*x = p*(s*x) = (x*p)*s = s*s.$$

(4) To prove $t*s = s*t$. Due to the fact

$$t*s = (t*s)*s = s*(s*t) = s*s = s = s*t.$$

Hence $s = t$, and $neut(x)$ is unique in S. □

Remark 1. In a T2CA-NET-groupoid (S, *), we know from Example 4 that *anti(x)* may be not unique.

Example 4. Let $S = \{g, k, u, v, w\}$. The operate * on S is defined as Table 4. Then, $(S, *)$ is T2CA-NET-groupoid. Moreover, $neut(g) = g$, and $\{anti(g)\} = \{g, k, u, v, w\}$.

Table 4. The operation * on S

*	*g*	*k*	*u*	*v*	*w*
g	*g*	*g*	*g*	*g*	*g*
k	*g*	*k*	*u*	*v*	*w*
u	*g*	*u*	*w*	*k*	*v*
v	*g*	*v*	*k*	*w*	*u*
w	*g*	*w*	*v*	*u*	*k*

Proposition 5. Suppose (S, *) is a T2CA-NET-groupoid. Then, for any $t \in S$,
(1) $neut(t)*neut(t) = neut(t)$;
(2) $neut(neut(t)) = neut(t)$;
(3) $anti(neut(t))*t = t$.

Proof. (1) Using $anti(t)*t = t*anti(t) = neut(t)$, we get

$$neut(t)*neut(t) = neut(t)*[anti(t)*t] = [t*neut(t)]*anti(t) = t*anti(t) = neut(t).$$

(2) According to the definition of $neut(neut(t))$ we can get:

$$neut(t)*neut(neut(t)) = neut(neut(t))*neut(t) = neut(t).$$

By the definition of $anti(neut(t))$ we can get:

$$neut(t)*anti(neut(t)) = anti(neut(t))*neut(t) = neut(neut(t)).$$

Applying (1) and Theorem 10, we get $neut(neut(t)) = neut(t)$.

(3) By Definition 4, Definition 8 and Proposition 5 (2), there are:

$$anti(neut(t))*t = anti(neut(t))*[t*neut(t)] = [neut(t)*anti(neut(t))]*t$$
$$= neut(neut(t))*t = neut(t)*t = t.$$

Therefore, $anti(neut(t))*t = t$. □

Remark 2. In a T2CA-NET-groupoid (S, *), we know from Example 5 that *neut(anti(t))* may be not equal to *neut(t).*

Example 5. *Let $S = \{g, u, v, w\}$. The operate * on S is defined as Table 5. Then, (S, *) is T2CA-NET-groupoid. And,*

$$neut(g) = g,\ neut(u) = u,\ \{anti(g)\} = \{g, u, v, w\}.$$

While $anti(g) = u$, $neut(anti(g)) \neq neut(g)$, because $neut(anti(g)) = neut(u) = u \neq g = neut(g)$.

Table 5. The operation * on S

*	*g*	*u*	*v*	*w*
G	*g*	*g*	*g*	*g*
U	*g*	*u*	*g*	*w*
V	*g*	*g*	*v*	*g*
W	*g*	*w*	*g*	*u*

Theorem 11. Suppose (S, *) is a T2CA-NET-groupoid, then its idempotents are commutative.

Proof. If k, w an idempotent in S, then

$$(k*w)*(k*w) = (w*k)*(w*k) \qquad \text{(Using Proposition 4 (1))}$$
$$= [(w*k)*w]*k = [k*(w*w)]*k = (w*w)*(k*k) = w*k.$$

Moreover,

$$(k*w)*(k*w) = [w*(k*w)]*k = [(w*w)*k]*k = (w*k)*k$$
$$= k*(k*w) = (neut(k)*k)*(k*w)$$
$$= (k*neut(k))*(w*k) \qquad \text{(Using Proposition 4 (1))}$$
$$= k*(w*k) = (k*k)*w = k*w.$$

Hence, we get $k*w = w*k$. That is, in a T2CA-NET-groupoid, its idempotents are commutative. □

Corollary 4. Every T2CA-NET-groupoid is commutative.

Proof. Suppose $(S, *)$ is a T2CA-NET-groupoid. Applying Theorem 11, $neut(x)$ $(\forall x\in S)$ is idempotent. $\forall\ k, w\in S$, we have

$$neut(k) * neut(w) = neut(w) * neut(k).$$

Further, for any $k, w \in S$, we have:

$$k*w = [k*neut(k)]*[w*neut(w)] = [neut(w)*(k*neut(k))]*w$$
$$= [(neut(k)*neut(w))*k]*w = k*[w*(neut(k)*neut(w))]$$
$$= k*[(neut(w)*w)*neut(k)] = k*[w*neut(k)]$$
$$= [neut(k)*k]*[w*neut(k)] = k*[(w*neut(k))*neut(k)]$$
$$= k*[neut(k)*(neut(k)*w)] = [(neut(k)*w)*k]*neut(k)$$
$$= [w*(k*neut(k))]*neut(k) = (w*k)*neut(k)$$
$$= (w*k)*[neut(k)*neut(k)]$$
$$= (k*w)*[neut(k)*neut(k)] \qquad \text{(Applying Proposition 4 (1))}$$
$$= (k*w)*neut(k) \qquad \text{(Applying Proposition 5 (1))}$$
$$= w*[neut(k)*k] = w*k.$$

Hence, every T2CA-NET-groupoid is commutative. □

Example 6. T2CA-NET-groupoid of order 5, given in Table 6, and

$$neut(a) = a, \{anti(a)\} = \{a, e\}; neut(b) = b, \{anti(b)\} = \{a, b, c, d, e\};$$
$$neut(c) = c, \{anti(c)\} = \{c, e\}; neut(d) = d, \{anti(d)\} = \{a, c, d, e\}; neut(e) = e, anti(e) = e.$$

Obviously, $(S, *)$ is a commutative.

Table 6. Cayley table on $S = \{a, b, c, d, e\}$.

*	a	b	c	d	e
a	a	b	d	d	a
b	b	b	b	b	b
c	d	b	c	d	c
d	d	b	d	d	d
e	a	b	c	d	e

Proposition 6. Let $(S, *)$ be a T2CA-NET-groupoid. Then for any $k \in S$, for all $t, u\in\{anti(k)\}$,

(1) $t*neut(k) = u*neut(k)$;
(2) $neut(u)*neut(k) = neut(k)*neut(u) = neut(k)$.
(3) $u*neut(k) \in \{anti(k)\}$;
(4) $u*neut(k) = (neut(k)*u)*neut(k)$;
(5) $u*neut(k) = neut(k)*u$;
(6) $neut(u*neut(k)) = neut(k)$.

Proof. (1) $\forall t, u\in\{anti(k)\}$, by the definition of opposite and neutral element, using Theorem 10, we get

$$k*t = t*k = neut(k), k*u = u*k =neut(k).$$
$$t*neut(k) = t*(u*k) = (k*t)*u = neut(k)*u = (k*u)*u = u*(u*k) = u*neut(k).$$

(2) $\forall u \in\{anti(k)\}$, by $k*u = u*k = neut(k)$, we have

$$neut(u)*neut(k) = neut(u)*(k*u) = [u*neut(u)]*k = u*k = neut(k);$$
$$neut(k)*neut(u) = (u*k)*neut(u) = k*[neut(u)*u] = k*u = neut(k).$$

That is, $neut(u)*neut(k) = neut(k)*neut(u) = neut(k)$ is true for all $k \in S$.

(3) $\forall k\in S$ and $u\in\{anti(k)\}$, $u*k=k*u=neut(k)$. Then, by Definition 4 and Proposition 5 (1), we have

$$k*[u*neut(k)] = [neut(k)*k]*u = k*u = neut(k);$$
$$[u*neut(k)]*k = neut(k)*(k*u) = neut(k)*neut(k) = neut(k).$$

This means that $u*neut(k)\in\{anti(k)\}$.

(4) $\forall k\in S$ and $u\in\{anti(k)\}$, $u*k = k*u = neut(k)$. Applying (1) and (3), we get

$$u*neut(k) = (u*neut(k))*neut(k).$$

On the other hand, by using Proposition 4 (1) and Proposition 5 (1), we get

$$[u*neut(k)]*neut(k) = [u*neut(k)]*[neut(k)*neut(k)]$$
$$= [neut(k)*u]*[neut(k)*neut(k)]= [neut(k)*u]*neut(k).$$

Combining two equations above, we get $u*neut(k) = [neut(k)*u]*neut(k)$.

(5) Assume that $u\in\{anti(k)\}$, then $k*u = u*k = neut(k)$, and $u*neut(u) = neut(u)*u = u$. By Proposition 5 (1) and (2), applying (2) and (3), there are

$$\begin{aligned}
neut(k)*u &= [neut(k)*neut(k)]*[u*neut(u)] = [neut(u)*(neut(k)*neut(k))]*u\\
&= [(neut(k)*neut(u))*neut(k)]*u = neut(k)*[u*(neut(k)*neut(u))]\\
&= neut(k)*[u*(neut(u)*neut(k))] = neut(k)*[(neut(k)*u)*neut(u)]\\
&= [neut(u)*neut(k)]*[neut(k)*u] = neut(k)*[neut(k)*u]\\
&= [u*neut(k)]*neut(k)\\
&= [neut(k)*u]*neut(k) \qquad (\text{By (3)}, [u*neut(k)]*neut(k) = [neut(k)*u]*neut(k))\\
&= u*neut(k). \qquad (\text{By (3)}, u*neut(k) = [neut(k)*u]*neut(k))
\end{aligned}$$

(6) Assume $u\in\{anti(k)\}$, denote $d = u*neut(k)$. We prove the following equations:

$$d*neut(k) = neut(k)*d = d;\ d*k = k*d = neut(k).$$

By Proposition 4 (1), Proposition 5 (1), and above (5), we get

$$\begin{aligned}
d*neut(k) &= [u*neut(k)]*neut(k) = [u*neut(k)]*[neut(k)*neut(k)]\\
&= [neut(k)*u]*[neut(k)*neut(k)] = [neut(k)*u]*neut(k)\\
&= u*[neut(k)*neut(k)] = u*neut(k) = d.
\end{aligned}$$

Using Definition 4 and (5), we have

$$neut(k)*d = neut(k)*[u*neut(k)] = neut(k)*[neut(k)*u] = [u*neut(k)]*neut(k) = d*neut(k) = d.$$

Moreover, using Proposition 5 (1), Definition 4, there are:

$$d*k = [u*neut(k)]*k = neut(k)*(k*u) = neut(k)*neut(k) = neut(k).$$
$$k*d = k*[u*neut(k)] = [neut(k)*k]*u = k*u = neut(k).$$

Thus,

$$d*neut(k) = neut(k)*d = d;\ d*k = k*d = neut(k).$$

According to the definition of neutral element and Theorem 10, we get $neut(k)$ is the neutral element of $d = u*neut(k)$. Hence, $neut(u*neut(k)) = neut(k)$. □

Theorem 12. Let $(S, *)$ be a T2CA-NET-groupoid. Put the set of all different neutral elements in S by $N(S)$, and $S(n) = \{a\in S \mid neut(a) = n\}$ $(\forall n\in N(S))$. Then:

(1) $S(n)$ is a subgroup of S;

(2) for any $n_1, n_2\in N(S)$, $n_1\neq n_2 \Rightarrow S(n_1) \cap S(n_2) = \varnothing$;

(3) $S = \bigcup_{n\in N(S)} S(n)$.

Proof. (1) For every $k\in S(n)$, $neut(k) = n$, we get that n is an identity element in $S(n)$. Applying Proposition 5 (1), there are $n*n = n$.

Assume $k, w \in S(n)$, then $neut(k) = neut(w) = n$. Next, we are going to prove that $neut(k*w) = n$.

Applying Definition 4, and Corollary 4, we have

$$(k*w)*n = y*(n*k) = w*k = k*w;$$
$$n*(k*w) = (w*n)*k = w*k = k*w.$$

Moreover, for any $anti(k)\in\{anti(k)\}$, $anti(w)\in\{anti(w)\}$. By using Definition 4 and Definition 8, we have

$$\begin{aligned}(k*w)*[anti(k)*anti(w)] &= w*[(anti(k)*anti(w))*k] = w*[anti(w)*(k*anti(k))]\\ &= w*(anti(w)*neut(k)) = [neut(k)*w]*anti(w) = (n*w)*anti(w)\\ &= w*[anti(w)*n] = w*anti(w) = neut(w) = n.\end{aligned}$$

$$\begin{aligned}[anti(k)*anti(w)]*(k*w) &= anti(w)*[(k*w)*anti(k)] = anti(w)*[w*(anti(k)*k)]\\ &= anti(w)*[w*neut(k)] = anti(w)*(w*n) = anti(w)*w\\ &= neut(w) = n.\end{aligned}$$

Thus, according to Theorem 10 and the definition of neutral elements, we get that $neut(k*w) = n$. Therefore, $k*w\in S(n)$, that is, $S(n)$ is closed under operation *.

Furthermore, $\forall k\in S(n)$, $\exists u\in S$ such that $u\in\{anti(k)\}$. Using Proposition 6(3), $u*neut(k)\in\{anti(k)\}$; and applying Proposition 6 (6), $neut(u*neut(k)) = neut(k)$.

Put $d = u*neut(k)$, we have

$$d = u*neut(k)\in\{anti(k)\},\ neut(d) = neut(u*neut(k)) = neut(k) = n.$$

Thus $d\in\{anti(k)\}$, $neut(d) = n$, *i.e.*, $d\in S(n)$ and d is the inverse element of k in $S(n)$.

Hence, $(S(n), *)$ is a subgroup of S.

(2) Suppose $k\in S(n_1)\cap S(n_2)$ and $n_1, n_2\in N(S)$. There are $neut(k)=n_1$, $neut(k)=n_2$. Applying Theorem 10, we get $n_1 = n_2$. Hence, $n_1\neq n_2 \Rightarrow S(n_1)\cap S(n_2) = \varnothing$.

(3) $\forall k\in S$, $\exists neut(k)\in S$. Put $n=neut(k)$, then $k\in S(n)$, $n\in N(S)$. This means that $S = \bigcup_{n\in N(S)} S(n)$. □

Example 7. T2CA-NET-groupoid of order 5, given in Table 7, and

$$neut(a) = a,\ anti(a) = a;\ neut(s) = a,\ anti(s) = s;$$
$$neut(d) = d,\ anti(d) = \{a, d, g\};\ neut(f) = d,\ anti(f) = \{s, f\};\ neut(g) = g,\ anti(g) = g.$$

Denote $S_1 = \{a, s\}$, $S_2 = \{d, f\}$, $S_3 = \{g\}$, then S_1, S_2 and S_3 are subgroup of S, and $S=S_1\cup S_2\cup S_3$, $S_1\cap S_2=\varnothing$, $S_1\cap S_3=\varnothing$, $S_2\cap S_3=\varnothing$.

Table 7. Cayley table on $S = \{a, s, d, f, g\}$.

*	*a*	*s*	*d*	*f*	*g*
a	*a*	*s*	*d*	*f*	*d*
s	*s*	*a*	*f*	*d*	*f*
d	*d*	*f*	*d*	*f*	*d*
f	*f*	*d*	*f*	*d*	*f*
g	*d*	*f*	*d*	*f*	*g*

Theorem 13. Suppose $(S, *)$ is a groupoid, then S is a T2CA-NET-groupoid if and only if it is a commutative regular semigroup.

Proof. If S is a T2CA-NET-groupoid. By Corollary 4 and Proposition 3 (2), we know that S is a commutative semigroup. By Definition 8, there are:

$$k*anti(k)*k = neut(k)*k = k.\ (\forall k\in S)$$

Therefore, element k is a regular element and S is a commutative regular semigroup.

Next, if S is a commutative regular semigroup. Applying Proposition 3 (1), we get S is a T2CA-groupoid. $\forall k\in S$, $\exists w\in S$ we have

$$k*(w*k) = k.$$

Also,

$$(w*k)*k = (w*k)*[k*(w*k)] = [(w*k)*(w*k)]*k = [(k*(w*k))*w]*k = (k*w)*k = k.$$

Therefore, there exists $(w*k)\in S$, such that $k*(w*k) = (w*k)*k = k$.

Moreover, since

$$w*k = w*[k*(w*k)] = [(w*k)*w]*k = k*[(w*k)*w].$$

Then,

$$\begin{aligned}
&[(w*k)*w]*k \\
&= [(w*k)*w]*[k*(w*k)] = [(w*k)*((w*k)*w)]*k \\
&= [(w*k)*(w*(w*k))]*k = [((w*k)*(w*k))*w]*k \\
&= [(w * k) * w] * k \qquad \text{(by } (w*k)*(w*k) = (w*k)\text{)} \\
&= w*[k*(w*k)] = w*k.
\end{aligned}$$

Thus, there exists $[(w*k)*w]\in S$, such that $k*[(w*k)*w] = [(w*k)*w]*k = w*k$. Then S is a T2CA-NET-groupoid. □

Example 8. T2CA-NET-groupoid of order 5, given in Table 8, and

$$neut(a)=a,\ \{anti(a)\} = \{a, w, r, t\};\ neut(q) = a,\ anti(q) = q;$$
$$neut(w) = r,\ anti(w) = w;\ neut(r) = r,\ anti(r) = r;\ neut(t) = t,\ anti(t) = t.$$

Also $(S, *)$ is a regular semigroup, due to the fact that $a = a*a*a$, $q = q*q*q$, $w = w*w*w$, $r = r*r*r$, $t = t*t*t$. Obviously, $(S, *)$ is a commutative.

Table 8. Cayley table on $S = \{a, q, w, r, t\}$.

*	*a*	*q*	*w*	*r*	*t*
a	*a*	*q*	*a*	*a*	*a*
q	*q*	*a*	*q*	*q*	*q*
w	*a*	*q*	*r*	*w*	*a*
r	*a*	*q*	*w*	*r*	*a*
t	*a*	*q*	*a*	*a*	*t*

Definition 9. Let $(S, *)$ be a T2CA-groupoid. (1) If $\forall\ k\in S$, $\exists\ s, t\in S$ such that $k*s = k$, and $t*k = s$. Then, S is called a T2CA-(r, l)-NET-groupoid.

(2) If $\forall\ k\in S$, $\exists\ s, t\in S$ such that $k*s = k$, and $k*t = s$. Then, S is called a T2CA-(r, r)-NET-groupoid.

(3) If $\forall\ k\in S$, $\exists\ s, t\in S$ such that $s*k = k$, and $k*t = s$. Then, S is called a T2CA-(l, r)-NET-groupoid.

(4) If $\forall\ k\in S$, $\exists\ s, t\in S$ such that $s*k = k$, and $t*k = s$. Then, S is called a T2CA-(l, l)-NET-groupoid.

Theorem 14. Suppose $(S, *)$ is a groupoid, then S is a T2CA- (r, l)-NET-groupoid if and only if it is a commutative regular semigroup.

Proof. If S is a T2CA-(r, l)-NET-groupoid. $\forall\ k\in S$, by Definition 8, Definition 9 (1) there are:

$$k*neut(k) = k,\ anti(k)*k = neut(k).$$

Moreover, we have

$$\begin{aligned}
k*anti(k) &= [k*neut(k)]*anti(k) = [anti(k)*k]*neut(k) = neut(k)*neut(k), \\
neut(k)*k &= (anti(k)*k)*k = k*(k*anti(k)) = k*[(k*neut(k))*anti(k)] \\
&= (anti(k)*k)*(k*neut(k)) = neut(k)*(k*neut(k)) = [neut(k)*neut(k)]*k \\
&= (k*anti(k))*k \qquad \text{(By } k*anti(k) = neut(k)*neut(k)\text{)} \\
&= anti(k)*(k*k) = anti(k)*[(k*neut(k))*k] = (k*anti(k))*(k*neut(k)) \\
&= (anti(k)*k)*(neut(k)*k) \qquad \text{(Using Proposition 4 (1))} \\
&= neut(k)*(neut(k)*k) = (k*neut(k))*neut(k) \\
&= k*neut(k) = k.
\end{aligned}$$

Thus, $neut(k)*k = k*neut(k) = k$.

Further, we have

$$\begin{aligned}
k*anti(k) &= neut(k)*neut(k) = (anti(k)*k)*(anti(k)*k) \\
&= (k*anti(k))*(k*anti(k)) \qquad \text{(Using Proposition 4 (1))}
\end{aligned}$$

$= anti(k)*[(k*anti(k))*k] = anti(k)*[anti(k)*(k*k)]$
$= anti(k)*[neut(k)*k]$ (By $anti(k)*(k*k) = neut(k)*k$)
$= [k*anti(k)]*neut(k) = anti(k)*[neut(k)*k]$
$= anti(k)*k = neut(k).$

Thus, $anti(k)*k = k*anti(k) = neut(k).$

Therefore, we prove that S is a T2CA-NET-groupoid. By Theorem 13, we get that T2CA-(r, l)-NET-groupoid is equivalent to commutative regular semigroup. □

Theorem 15. Suppose (S, *) is a groupoid, then S is a T2CA- (r, r)-NET-groupoid if and only if it is a commutative regular semigroup.

Proof. If S is a T2CA-(r, r)-NET-groupoid. $\forall k\in S$, by Definition 8, Definition 9 (2) there are:

$$k * neut(k) = k,\ k * anti(k) = neut(k).$$

Moreover, we have

$$\begin{aligned} neut(k)*k &= [k*anti(k)]*k = anti(k)*(k*k) = anti(k)*[k*(k*neut(k))] \\ &= [(k*neut(k))*anti(k)]*k = [neut(k)*(anti(k)*k)]*k \\ &= (anti(k)*k)*(k*neut(k)) = (anti(k)*k)*k = k*(k*anti(k)) \\ &= k*neut(k) = k. \end{aligned}$$

Thus, $neut(k)*k = k*neut(k) = k.$

Further, we have

$$\begin{aligned} anti(k)*k &= anti(k)*(neut(k)*k) = anti(k)*[neut(k)*(k*neut(k))] \\ &= anti(k)*[(neut(k)*neut(k))*k] = (k*anti(k))*[neut(k)*neut(k)] \\ &= neut(k)*[neut(k)*neut(k)] = neut(k)*[(k*anti(k))*neut(k)] \\ &= neut(k)*[anti(k)*(neut(k)*k)] = neut(k)*[anti(k)*k] \\ &= [k*neut(k)]*anti(k) \\ &= k*anti(k) = neut(k). \end{aligned}$$

That is, $anti(k)*k = k*anti(k) = neut(k).$

Therefore, we prove that S is a T2CA-NET-groupoid. By Theorem 13, we get that T2CA-(r, r)-NET-groupoid is equivalent to commutative regular semigroup. □

Theorem 16. Suppose (S, *) is a groupoid, then S is a T2CA-(l, r)-NET-groupoid if and only if it is a commutative regular semigroup.

Proof. If S is a T2CA-(l, r)-NET-groupoid. $\forall k\in S$, by Definition 8, Definition 9(3) we have

$$neut(k)*k = k,\ k*anti(k) = neut(k).$$

Moreover,

$$\begin{aligned} anti(k)*k &= anti(k)*(neut(k)*k) = (k*anti(k))*neut(k) = neut(k)*neut(k) \\ &= neut(k)*[k*anti(k)] = [anti(k)*neut(k)]*k = [anti(k)*(k*anti(k))]*k \\ &= [(anti(k)*anti(k))*k]*k = k*[k*(anti(k)*anti(k))] \\ &= k*[(anti(k)*k)*anti(k)] = [anti(k)*k]*[anti(k)*k]. \end{aligned}$$

$$\begin{aligned} neut(k)*neut(k) &= [neut(k)*neut(k)]*[neut(k)*neut(k)] \\ &= [anti(k)*k]*[neut(k)*neut(k)] \qquad \text{(By } anti(k)*k = neut(k)*neut(k)\text{)} \\ &= k*[(neut(k)*neut(k))*anti(k)] = k*[neut(k)*(anti(k)*neut(k))] \\ &= [(anti(k)*neut(k))*k]*neut(k) = [neut(k)*(k*anti(k))]*neut(k) \\ &= [neut(k)*neut(k)]*neut(k). \end{aligned}$$

Then,

$$\begin{aligned} k*neut(k) &= k*(k*anti(k)) = [anti(k)*k]*k \\ &= [neut(k)*neut(k)]*k \qquad \text{(By } anti(k)*k = neut(k)*neut(k)\text{)} \\ &= [(neut(k)*neut(k))*neut(k)]*k \\ &\qquad \text{(By } [neut(k)*neut(k)]*neut(k)=neut(k)*neut(k)\text{)} \\ &= [neut(k)*(neut(k)*neut(k))]*k = [neut(k)*neut(k)]*[k*neut(k)] \\ &= neut(k)*[(k*neut(k))*neut(k)] = neut(k)*[neut(k)*(neut(k)*k)] \end{aligned}$$

$= neut(k)*[neut(k)*k] = neut(k)*k = k.$

Thus, $neut(k)*k = k*neut(k) = k$.

Further, we have

$[neut(k)*neut(k)]*neut(k)= neut(k)*[neut(k)*neut(k)]$

$= neut(k)*[anti(k)*k]$ (By $anti(k)*k = neut(k)*neut(k)$)

$= [k*neut(k)]*anti(k)$

$= k*anti(k) = neut(k).$

Thus, $[neut(k)*neut(k)]*neut(k) = neut(k)*neut(k) = anti(k)*k = neut(k) = k*anti(k).$

Therefore, we prove that S is a T2CA-NET-groupoid. By Theorem 13, we get that T2CA-(l, r) -NET-groupoid is equivalent to commutative regular semigroup. □

Theorem 17. Suppose (S, *) is a groupoid, then S is a T2CA-(l, l)-NET-groupoid if and only if it is a commutative regular semigroup**.**

Proof. If S is a T2CA-(l, l)-NET-groupoid. $\forall k\in S$, by Definition 8, Definition 9 (4) we have $neut(k)*k = k$, $anti(k)*k = neut(k)$. Moreover,

$k*neut(k) = k*[anti(k)*k]$

$= (k*k)*anti(k) = [(neut(k)*k)*k]*anti(k)$

$= [(k*(k*neut(k)))]*anti(k) = [k*neut(k)]*[anti(k)*k]$

$= [k*neut(k)]*neut(k) = neut(k)*[neut(k)*k]$

$= neut(k)*k = k.$

Thus, $k*neut(k) = neut(k)*k = k.$

Further, we have

$k*anti(k)$

$= [k*neut(k)]*anti(k) = neut(k)*[anti(k)*k] = neut(k)*neut(k)$

$= [anti(k)*k]*neut(k) = k*[neut(k)*anti(k)] = k*[(anti(k)*k)*anti(k)]$

$= k*[k*(anti(k)*anti(k))] = [(anti(k)*anti(k))*k]*k = [anti(k)*(k*anti(k))]*k$

$= [k*anti(k)]*[k*anti(k)] = anti(k)*[(k*anti(k))*k] = anti(k)*[anti(k)*(k*k)]$

$= [(k*k)*anti(k)]*anti(k) = [k*(anti(k)*k)]*anti(k) = [k*neut(k)]*anti(k)$

$= [neut(k)*k]*anti(k) = k*[anti(k)*neut(k)] = k*[anti(k)*(anti(k)*k)]$

$= k*[(k*anti(k))*anti(k)] = [anti(k)*k]*[k*anti(k)] = neut(k)*(k*anti(k))$

$= neut(k)*(neut(k)*neut(k))$ (By $neut(k)*neut(k)= k*anti(k)$)

$= [neut(k)*neut(k)]*neut(k))$

$= [k*anti(k)]*neut(k)$ (By $neut(k)*neut(k)= k*anti(k)$)

$= anti(k)*[neut(k)*k] = anti(k)*k = neut(k).$

Thus, $anti(k)*k = k*anti(k) = neut(k).$

Therefore, we prove that S is a T2CA-NET-groupoid. By Theorem 13, we get that T2CA-(l, l)-NET-groupoid is equivalent to commutative regular semigroup. □

Example 9. T2CA-**(**r, l)**-**NET-groupoid of order 4, given in Table 9, and

$$neut_{(r,\,l)}(c) = c,\ \{anti_{(r,\,l)}(c)\} = \{c, v, b, n\};\ neut_{(r,\,l)}(v) = n,\ \{anti_{(r,\,l)}(v)\} = v;$$
$$neut_{(r,\,l)}(b) = b,\ anti_{(r,\,l)}(b) = b;\ neut_{(r,\,l)}(n) = n,\ anti_{(r,\,l)}(n) = n.$$

It is easy to verify that (S, *) is also a T2CA-(r, r)-NET-groupoid, T2CA-(l, r)-NET-groupoid, T2CA-(l, l)-NET-groupoid. Moreover, (S, *) is a regular semigroup**,** due to the fact that $c = c*c*c$, $v = v * v * v$, $b = b * b * b$, $n = n* n * n$. Obviously, (S, *) is a commutative.

Table 9. Cayley table on $S = \{c, v, b, n\}$.

*	*c*	*v*	*b*	*n*
c	*c*	*c*	*c*	*c*
v	*c*	*n*	*c*	*v*
b	*c*	*c*	*b*	*c*
n	*c*	*v*	*c*	*n*

6. Quasi Neutrosophic Extended Triplet (QNET) Groupoids and T2CA-QNET-Groupoids

Definition 10. Let $(S, *)$ be a groupoid. If for any $x \in S$, there exists $y, z \in S$ such that

$$(x*y = x, \text{ or } y*x = x), \text{ and } (z*x = y \text{ or } x*z = y).$$

then $(S, *)$ is called a quasi neutrosophic extended triplet groupoid (shortly, QNET-groupoid). Suppose $(S, *)$ is a semigroup and QNET-groupoid, then $(S, *)$ is called a quasi neutrosophic triplet group (shortly, QNETG). Suppose $(S, *)$ is a T2CA-groupoid and QNET-groupoid, then $(S, *)$ is called a T2CA-QNET-groupoid.

Let $(S, *)$ be a QNET-groupoid and $x \in S$. We introduce the following concepts:

(1) If $\exists y, z \in S$, s.t. $x * y = x$ and $x * z = y$, then x is called an QNET-element with (r-r)- property;
(2) If $\exists y, z \in S$, s.t. $x * y = x$ and $z * x = y$, then x is called an QNET-element with (r-l)- property;
(3) If $\exists y, z \in S$, s.t. $y * x = x$ and $z * x = y$, then x is called an QNET-element with (l-l)- property;
(4) If $\exists y, z \in S$, s.t. $y * x = x$ and $x * z = y$, then x is called an QNET-element with (l-r)- property;
(5) If $\exists y, z \in S$, s.t. $x * y = y * x = x$ and $z* x=y$, then x is called an QNET-element with (lr-l)-property;
(6) If $\exists y, z \in S$, s.t. $x * y = y * x = x$ and $x* z= y$, then x is called an QNET-element with (lr-r)-property;
(7) If $\exists y, z \in S$, s.t. $y * x = x$ and $x * z= z* x = y$, then x is called an QNET-element with (l-lr)-property;
(8) If $\exists y, z \in S$, s.t. $x * y=x$ and $x * z = z * x = y$, then x is called an QNET-element with (r-lr)-property;
(9) If $\exists y, z \in S$, s.t. $x * y = y * x = x$ and $x * z = z * x = y$, then x is called an QNET-element with (lr-lr)-property.

Easy to verify: (i) if x is an QNET-element with (r-lr)-property, then x is an QNET-element with (r-r)-property and (r-l)-property; if x is an QNET-element with (lr-r)-property, then x is an QNET-element with (l-r)-property and (r-r)-property; and soon; (ii) if * is commutative, then the above properties coincide.

Example 10. Denote $S = \{1, 2, 3, 4\}$, define the operation * on S in Table 10. Then $(S, *)$ is QNET-groupoid, and 1 is an QNET-element with (lr-lr)-property; 2 is an QNET-element with (lr-r)-property; 3 is an QNET-element with (r-r)-property; and 4 is an QNET-element with (l-lr)-property. Obviously, $(S, *)$ is not a NET-groupoid.

Table 10. The operation * on S

*	1	2	3	4
1	1	1	1	4
2	3	1	2	3
3	3	2	4	1
4	1	4	2	1

Example 11. Denote $S = \{1, 2, 3, 4, 5\}$, define the operation * on S in Table 11. Then $(S, *)$ is QNETG, and 1 is an QNET-element with (lr-lr)-property; 2 is an QNET-element with (lr-lr)-property; 3 is an QNET-element with (lr-lr)-property; 4 is an QNET-element with (lr-r)-property; and 5 is an QNET-element with (lr-r)-property. Obviously, $(S, *)$ is not a NETG. Moreover, since $5 * (5 * 4) = 5 \neq 1 = (4 * 5) * 5$, $(S, *)$ is not a T2CA-groupoid.

Table 11. The operation * on S

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	4	1
3	1	1	3	1	5
4	1	4	1	1	2
5	1	1	5	3	1

Theorem 18. Suppose $(S, *)$ is a groupoid, then S is a T2CA-QNET-groupoid if and only if it is a T2CA-NET-groupoid.

Proof. Let $(S, *)$ be a T2CA-QNET-groupoid. In particular, we consider the local unit element of each element. By Definition 10, we know that for any $a, b \in S$, there are four cases of their local unit.

Case 1: There exists $y, z \in S$, such that $a*y = a$, $b*z = b$. Then a is an QNET-element with (r-l, or r-r, or r-lr)-property, b is an QNET-element with (r-l, or r-r, or r-lr)-property. We have

$$
\begin{aligned}
a*b &= (a*y)*b = y*(b*a) = y*[b*(a*y)] = y*[(y*b)*a] \\
&= (a*y)*(y*b) = a*(y*b) = (b*a)*y = [(b*z)*a]*y \\
&= [z*(a*b)]*y = (a*b)*(y*z) \\
&= (b*a)*(z*y) && \text{(Applying Proposition 4 (1))} \\
&= [y*(b*a)]*z = [(a*y)*b]*z = (a*b)*z = b*(z*a) \\
&= z*[(z*a)*b] = z*[a*(b*z)] = z*(a*b) = (b*z)*a \\
&= b*a.
\end{aligned}
$$

Case 2: There exists $y, z \in S$, such that $y*a = a$, $z*b = b$. Then a is an QNET-element with (l-l, or l-r, or l-lr)-property, b is an QNET-element with (l-l, or l-r, or l-lr)-property. According to case 1, we can similarly get

$$a*b = (y*a)*(z*b) = (a*y)*(b*z) = b*a.$$

Case 3: There exists $y, z \in S$, such that $y*a = a$, $b*z = b$. Then a is an QNET-element with (l-l, or l-r, or l-lr)-property, b is an QNET-element with (r-l, or r-r, or r-lr)-property. We have

$$
\begin{aligned}
z*a &= z*(y*a) = z*[y*(y*a)] = [(y*a)*z]*y \\
&= [a*(z*y)]*y = (z*y)*(y*a) \\
&= (z*y)*a;
\end{aligned}
$$

$$
\begin{aligned}
b*a &= b*(y*a) = (a*b)*y = [(y*a)*b]*y = [a*(b*y)]*y = (b*y)*(y*a) \\
&= (b*y)*a = y*(a* b) && \text{(By } a*b = a*(b*y)\text{)} \\
&= y*[a*(b*y)] = [(b*y)*y]*a = [y*(y*b)]*a = (y*b)*(a*y) \\
&= (b*y)*(y*a) && \text{(Applying Proposition 4 (1))} \\
&= (b*y)*a \\
&= y*(a*b);
\end{aligned}
$$

$$
\begin{aligned}
a*b &= a*(b*z) = (z*a)*b = [(z*y)*a]*b && \text{(By } z*a = (z*y)*a\text{)} \\
&= a*[b*(z*y)] = a*[(y*b)*z] \\
&= (z*a)*(y*b) = (a*z)*(b*y) && \text{(Applying Proposition 4 (1))} \\
&= z*[(b*y)*a] = z*[y*(a*b)] \\
&= z*(b*a). && \text{(By } y*(a*b) = b*a\text{)}
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
b &= b*z = (b*z)*z = z*(z*b) = z*[z*(b*z)] = [(b*z)*z]*z = [z*(z*b)]*z \\
&= (z*b)*(z*z) = (b*z)*(z*z) && \text{(Applying Proposition 4 (1))} \\
&= b*(z*z).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
a*b &= a*(b*z) = a*[b*(z*z)] && \text{(By } b*z = b*(z*z)\text{)} \\
&= [(z*z)*a]*b = [z*(a*z)]*b= (a*z)*(b*z) \\
&= (z*a)*(z*b) && \text{(Applying Proposition 4 (1))} \\
&= [b*(z*a)]*z = [(a*b)*z]*z \\
&= (b*a)*z = [(b*a)*z]*z= z*[z*(b*a)] \\
&= z*(a*b) && \text{(By } a*b = z*(b*a)\text{)} \\
&= (b*z)*a = b*a.
\end{aligned}
$$

Case 4: There exists $y, z \in S$, such that $a*y = a$, $z*b =b$. Then a is an QNET-element with (r-l, or r-r, or r-lr)-property, b is an QNET-element with (l-l, or l-r, or l-lr)-property. According to case 3, we can similarly get $a*b = (a*y)*(z*b) = (y*a)*(b*z) = b*a$.

From Case 1, Case2, Case 3, and Case 4, we know that S is a commutative T2CA-QNET-groupoid. Then, for any $x \in S$, there exists y, $z \in S$ such that $x*y = x$, $y*x = x$, and $z*x = y$, $x*z = y$. Therefore, we prove that S is a T2CA-NET-groupoid.

Conversely, it is obvious. □

Corollary 5. Assume (S, *) is a T2CA-QNET-groupoid, then (S, *) is a QNETG.

Proof. Assume that S is a T2CA-QNET-groupoid. By Theorem 18 and 13, we get that S is a commutative regular semigroup. According to the Definition 10, we get that S is a QNETG. □

The inverse of Corollary 5 is not true, see Example 11.

Corollary 6. Let (S, *) be a T2CA-groupoid. Then, the following statements are equivalent:
(i) S is a T2CA-QNET-groupoid;
(ii) S is a T2CA-NET-groupoid;
(iii) S is a CA-NET-groupoid;
(iv) S is a commutative regular semigroup.

Proof. (i) Þ (ii). Suppose that S is a T2CA-QNET-groupoid. Applying Theorem 18, we get that S is a T2CA-NET-groupoid.

(ii) Þ (iii). Suppose that S is a T2CA-NET-groupoid. Applying Theorem 13, we get that S is a commutative regular semigroup. Then by Theorem 2 (1) and (6), we get S is a CA-NET-groupoid.

(iii) Þ (iv). Suppose that S is a CA-NET-groupoid. Applying Theorem 2 (1) and (6), we get that S is a commutative regular semigroup.

(iv) Þ (i). Suppose that S is a commutative regular semigroup. Applying Theorem 13, we get S is a T2CA-NET-groupoid. Then by Theorem 18, S is a T2CA-QNET-groupoid. □

7. Conclusions

In the paper, we introduced the new concepts of T2CA-groupoid, T2CA-NET-groupoid, and QNET-groupoid for the first time. We precisely discussed some fundamental characteristics of T2CA-groupoids and T2CA-NET-groupoids, then a decomposition theorem of T2CA-NET-groupoid is proved (see Theorem 8), and the relationship between T2CA-NET-groupoids and commutative regular semigroups is strictly proved. Furthermore, we investigated relationships among T2CA-QNET-groupoid, T2CA-NET-groupoid, CA-NET-groupoid and commutative regular semigroup**.** The results show that T2CA-groupoids, as a non-associative algebraic structure, are typically representative and closely related to a variety of algebraic structures.

For future research directions, we will discuss the integration of the related topics (such as algebraic systems related fuzzy logics and non-associative groupoids, see [29-34]).

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Fuzzy Neutrosophic Strongly Alpha Generalized Closed Sets in Fuzzy Neutrosophic Topological spaces

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Abstract: In this paper, we will define a new set called fuzzy neutrosophic strongly alpha generalized closed set, so we will prove some theorems related to this concept. After that, we will give some interesting properties were investigated and referred to some results related to the new definitions by theorems, propositions to get some relationships among fuzzy neutrosophic strongly alpha generalized closed sets, fuzzy neutrosophic closed sets, fuzzy neutrosophic regular closed sets, fuzzy neutrosophic alpha closed sets, fuzzy neutrosophic alpha generalized closed sets and fuzzy neutrosophic pre closed sets which are compared with necessary examples based of fuzzy neutrosophic topological spaces.

Keywords: Fuzzy neutrosophic set, fuzzy neutrosophic topological space, fuzzy neutrosophic strongly alpha generalized closed set.

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1. Introduction

The concept of fuzzy set "FS"was introduced by Lotfi Zadeh in 1965 [1], then Chang depended the fuzzy set to introduce the concept of fuzzy topological space "FTS" in 1968 [7]. After that the concept of fuzzy set was developed into the concept of intutionistic fuzzy set "IFS" by Atanassov in 1983 [4-6], the intutionistic fuzzy set gives a degree of membership and a degree of non-membership functions. Cokor in 1997 [7] relied on intutionistic fuzzy set to introduced the concept of intutionistic fuzzy topological space."IFTS". In 2005 Smaradache [23] study the concept of neutrosophic set. "NS". After that and as developed the term of neutrosophic set, Salama has studied neutrosophic topological space "NTS"and many of its applications [18-21]. In 2013 Arockiarani Sumathi and Martina Jency [2] introduced the concept of fuzzy neutrosophic set as generalizes the concept of fuzzy set and intutionistic fuzzy set. where each element had three associated defining functions on the universe of discourse X, namely the membership function (T), indeterminacy function (I), the non-membership function (F) that is added an indeterminacy degree between the degree of membership and the degree of non- membership. In 2012 Salama and Alblowi defined fuzzy neutrosophic topological space [18].

In the present work, we will generalized the concept of strongly alpha generalized closed set in fuzzy neutrosophic topological spaces which was studied by Santhi and Sakthivel in 2011 [22] via intuitinistic topological spaces and generalizing our works in 2018 [9,10], the new set will called fuzzy neutrosophic strongly alpha generalized closed set in fuzzy neutrosophic topological spaces.

Finally, there are many application of neutrosophic sets in many fields so we can enhance our work, we will try in the future to applied this work in different fields such as many authors applications see [11] and [13-17].

2. Preliminaries:

In this section, we will define some basic definitions and some operations which are useful in our present study.

Definition 2.1 [18]: Let X be a non-empty fixed set. The fuzzy neutrosophic set (FNS, for short), η_N is an object having the form $\eta_N = \{< x, \mu_{\eta N}(x), \sigma_{\eta N}(x), \nu_{\eta N}(x) >: x \in X\}$ where the functions $\mu_{\eta N}$, $\sigma_{\eta N}$, $\nu_{\eta N}$: $X \to [0, 1]$ denote the degree of membership function (namely $\mu_{\eta N}(x)$), the degree of indeterminacy function (namely $\sigma_{\eta N}(x)$) and the degree of non-membership (namely $\nu_{\eta N}(x)$) respectively of each element $x \in X$ to the set η_N and $0 \leq \mu_{\eta N}(x) + \sigma_{\eta N}(x) + \nu_{\eta N}(x) \leq 3$, for each $x \in X$.

Remark 2.2 [18]: FNS $\eta_N = \{< x, \mu_{\eta N}(x), \sigma_{\eta N}(x), \nu_{\eta N}(x) >: x \in X\}$ can be identified to an ordered triple $< x, \mu_{\eta N}, \sigma_{\eta N}, \nu_{\eta N} >$ in [0, 1] on X.

Definition 2.3 [18]: Let X be a non-empty set and the FNSs η_N and γ_N be in the form:
$\eta_N = \{< x, \mu_{\eta N}, \sigma_{\eta N}, \nu_{\eta N} >: x \in X\}$ and $\gamma_N = \{< x, \mu_{\gamma N}, \sigma_{\gamma N}, \nu_{\gamma N} >: x \in X\}$ on X then:

i. $\eta_N \subseteq \gamma_N$ iff $\mu_{\eta N} \leq \mu_{\gamma N}$, $\sigma_{\eta N} \leq \sigma_{\gamma N}$ and $\nu_{\eta N} \geq \nu_{\gamma N}$.

ii. $\eta_N = \gamma_N$ iff $\eta_N \subseteq \gamma_N$ and $\gamma_N \subseteq \eta_N$,

iii. $1_N - \eta_N = \{< x, \nu_{\eta N}, 1 - \sigma_{\eta N}, \mu_{\eta N} >: x \in X\}$,

iv. $\eta_N \cup \gamma_N = \{< x, Max(\mu_{\eta N}, \mu_{\gamma N}), Max(\sigma_{\eta N}, \sigma_{\gamma N}, Min(\nu_{\eta N}, \nu_{\gamma N}) >: x \in X\}$,

v. $\eta_N \cap \gamma_N = \{< x, Min(\mu_{\eta N}, \mu_{\gamma N}), Min(\sigma_{\eta N}, \sigma_{\gamma N}), Max(\nu_{\eta N}, \nu_{\gamma N}) >: x \in X\}$,

vi. $0_N = < x, 0, 0, 1>$ and $1_N = <x, 1, 1, 0>$.

Definition 2.4 [18]: "Fuzzy neutrosophic topology (FNT, for short) on a non-empty set X is a family τ_N of fuzzy neutrosophic subsets in X satisfying the following axioms.

i. $0_N, 1_N \in \tau_N$,

ii. $\eta_{N1} \cap \eta_{N2} \in \tau_N$ for any $\eta_{N1}, \eta_{N2} \in \tau_N$,

iii. $\cup\, \eta_{Ni} \in \tau_N$, $\forall \{\eta_{Ni}: i \in J\} \subseteq \tau_N$."

In this case the pair (X, τ_N) is called fuzzy neutrosophic topological space (FNTS, for short). The elements of τ_N are called fuzzy neutrosophic open set (FNOS, for short). The complement of FNOS in the FNTS (X, τ_N) is called fuzzy neutrosophic closed set (FNCS, for short).

Definition 2.5 [18]: Let (X, τ_N) be FNTS and η_N = < x, $\mu_{\eta N}$, $\sigma_{\eta N}$, $\nu_{\eta N}$ > be FNS in X. Then the fuzzy neutrosophic closure of η_N (FNCL, for short) and fuzzy neutrosophic interior of η_N (FNIn, for short) are defined by:

$$\text{FNCL}(\eta_N) = \cap \{C_N: C_N \text{ is FNCS in X and } \eta_N \subseteq C_N \},$$

$$\text{FNIn}(\eta_N) = \cup \{O_N: O_N \text{ is FNOS in X and } O_N \subseteq \eta_N \}.$$

We know, FNCL(η_N) is FNCS and FNIn (η_N) is FNOS in X. Further,

i. η_N is FNCS in X iff FNCL (η_N) = η_N,

ii. η_N is FNOS in X iff FNIn (η_N) = η_N.

Proposition 2.6 [25]: Let (X, τ_N) is FNTS and η_N, γ_N are FNSs in X. Then the following properties hold:

i. FNIn (η_N) $\subseteq$ η_N and $\eta_N \subseteq$ FNCL(η_N),

ii. $\eta_N \subseteq \gamma_N \Longrightarrow$ FNIn (η_N) $\subseteq$ FNIn (γ_N) and $\eta_N \subseteq \gamma_N \Longrightarrow$ FNCL(η_N) $\subseteq$ FNCL(γ_N),

iii. FNIn (FNIn (η_N)) = FNIn (η_N) and FNCL(FNCL(η_N)) = FNCL(η_N),

iv. FNIn ($\eta_N \cap \gamma_N$) = FNIn (η_N) $\cap$ FNIn (γ_N) and FNCL($\eta_N \cup \gamma_N$) = FNCL(η_N) $\cup$FNCL(γ_N),

v. FNIn (1_N) =1_N and FNCL(1_N) = 1_N,

vi. FNIn (0_N) = 0_N and FNCL(0_N) = 0_N.

Definition 2.7 [9]: FNS η_N in FNTS (X, τ_N) is called:

i. Fuzzy neutrosophic regular closed set (FNRCS, for short) if η_N = FNCL(FNIn (η_N)).

ii. Fuzzy neutrosophic pre closed set (FNPCS, for short) if FNCL(FNIn (η_N)) $\subseteq$ η_N.

iii. Fuzzy neutrosophic α closed set (FNαCS, for short) if FNCL(FNIn(FNCL(η_N))) $\subseteq$ η_N.

Definition 2.8 [10]: Let (X, τ_N) be FNTS and η_N = < x, $\mu_{\eta N}$, $\sigma_{\eta N}$, $\nu_{\eta N}$ > be FNS in X. Then the fuzzy neutrosophic alpha closure of η_N (FNαCL, for short) and fuzzy neutrosophic alpha interior of η_N (FNαIn,
for short) are defined by:

$$\text{FN}\alpha\text{CL}(\eta_N) = \cap \{C_N: C_N \text{ is FN}\alpha\text{CS in X and } \eta_N \subseteq C_N \},$$

$$\text{FN}\alpha\text{In}(\eta_N) = \cup \{O_N: O_N \text{ is FN}\alpha\text{OS in X and } O_N \subseteq \eta_N\}.$$

We know, FNαCL(η_N) is FNαCLS and FNαIn (η_N) is FNαOS in X. Further,

i. η_N is FNαCS in X iff FNαCL(η_N) = η_N,

ii. η_N is FNαOS in X iff FNαIn (η_N) = η_N.

Definition 2.9 [9,10]: Fuzzy neutrosophic sub set η_N of FNTS (X, τ_N) is called:

i. fuzzy neutrosophic generalized closed set (FNGCS, for short) if FNCL(η_N) $\subseteq$ U_N wherever, $\eta_N \subseteq U_N$ and U_N is FNOS in X. And η_N is said to be fuzzy neutrosophic generalized open set (FNGOS, for short) if the complement 1_N- η_N is FNGCS set in (X, τ_N).

ii. fuzzy neutrosophic alpha generalized closed set (FNαGCS, for short) if FNαCL(η_N) $\subseteq$ U_N wherever, $\eta_N \subseteq U_N$ and U_N is FNOS in X. And η_N is said to be fuzzy neutrosophic alpha generalized open set (FNαGOS, for short) if the complement 1_N- η_N is FNαGCS set in (X, τ_N).

3. Fuzzy Neutrosophic Strongly Alpha Generalized Closed Sets in Fuzzy Neutrosophic Topological Spaces.

Now, we will introduce the concept of fuzzy neutrosophic strongly alpha generalized closed set in fuzzy neutrosophic topological spaces.

Definition 3.1: Fuzzy neutrosophic subset η_N of FNTS (X, τ_N) is called fuzzy neutrosophic strongly alpha generalized closed set (FNSαGCS, for short) if FNαCL (η_N) $\subseteq U_N$ wherever, $\eta_N \subseteq U_N$ and U_N is FNGOS in X.

Example 3.2: Let X= {a, b} define FNS η_N in X as follows:

η_N =<x, ($0.2_{(a)}$, $0.3_{(b)}$), ($0.5_{(a)}$, $0.5_{(b)}$), ($0.8_{(a)}$,$0.7_{(b)}$) >, where the family τ_N ={0_N, 1_N, η_N }.

If we take, ψ_N = <x, ($0.8_{(a)}$, $0.7_{(b)}$), ($0.5_{(a)}$, $0.5_{(b)}$), ($0.1_{(a)}$,$0_{(b)}$) >.

And, $U_N = 1_N$ where U_N is FNGOS such that, $\psi_N \subseteq U_N$. Then, FNαCL(ψ_N) = 1_N. So, FNαCL(ψ_N) $\subseteq U_N$.

Hence, ψ_N is FNSαGCS.

Theorem 3.3: For any FNSs, the following statements are true in general:

i. Every FNOS is FNGOS.
ii. Every FNCS is FNαCS.
iii. Every FNCS is FNSαGCS.
iv. Every FNRCS is FNSαGCS.
v. Every FNαCS is FNSαGCS.
vi. Every FNαGCS is FNSαGCS.
vii. Every FNRCS is FNCS.
viii. Every FNαCS is FNαGCS.

Proof:.

i. Let η_N = <x, $\mu_{\eta N}$, $\sigma_{\eta N}$, $\nu_{\eta N}$ > be FNOS in the FNTS (X,τ_N).
Then **by Definition 2.5 ii** we get, FNIn (η_N) = η_N.
Now, let U_N is FNCS such that, $U_N \subseteq \eta_N$. Therefore, FNIn (η_N) = $\eta_N \supseteq U_N$.
Hence, η_N is FNGOS in (X,τ_N) .

ii. Let η_N = <x, $\mu_{\eta N}$, $\sigma_{\eta N}$, $\nu_{\eta N}$ > be FNCLS in the FNTS (X, τ_N).
Then by **Definition 2.5 (i)** we get, FN$\underline{CL}$(η_N) = η_N……(1).
And **by Proposition 2.6 i** we get, FNIn (η_N) $\subseteq \eta_N$.

So we get, FNIn (FN$\underline{CL}$(η_N)) $\subseteq \eta_N$

This implies FN$\underline{CL}$(FNIn (FNCL(η_N))) $\subseteq$ FNCL(η_N).

So **by (1)** we get, FNCL(FNIn (FNCL(η_N))) $\subseteq \eta_N$.

Hence, η_N is FNαCS in (X, τ_N).

iii. Let $\eta_N = <x, \mu_{\eta N}, \sigma_{\eta N}, \nu_{\eta N}>$ be FNCS in FNTS (X,τ_N).

Then by **Definition 2.5 (i)** we get, FNCL(η_N) = η_N. Now, let U_N be FNGOS such that, $\eta_N \subseteq U_N$.

Since, FNαCL(η_N) $\subseteq$FNCL(η_N) **by Definition 2.5 and Definition 2.8.**

So we get, FNαCL(η_N) $\subseteq$ FNCL(η_N) = $\eta_N \subseteq U_N$.

Hence, η_N is FNSαGCS in (X,τ_N) .

iv. Let $\eta_N = <x, \mu_{\eta N}, \sigma_{\eta N}, \nu_{\eta N}>$ be FNRCS in the FNTS (X, τ_N).

Then, FNCL(FNIn (η_N)) = η_N…….(1).

This implies, FNCL(FNIn (η_N)) = FNCL(η_N)…….(2).

Now, let U_N be FNGOS such that, $\eta_N \subseteq U_N$.

From **(1) and (2)** we get, FNCL(η_N) = η_N.

That η_N is FNCS in X.

So **by iii** we get, FNαCL(η_N) $\subseteq$ FNCL(η_N) = $\eta_N \subseteq U_N$.

Hence, η_N is FNSαGCS in (X,τ_N) .

v. Let $\eta_N = <x, \mu_{\eta N}, \sigma_{\eta N}, \nu_{\eta N}>$ be FNαCLOS in the FNTS (X, τ_N).

Then **by Definition 2.8 i** we get, FNαCL (η_N) = η_N.

Now, let U_N be FNGOS such that, $\eta_N \subseteq U_N$. So, FNαCL (η_N) = $\eta_N \subseteq U_N$.

Hence, η_N is FNSαGCS in (X, τ_N).

vi. Let $\eta_N = <x, \mu_{\eta N}, \sigma_{\eta N}, \nu_{\eta N}>$ be FNαGCS in the FNTS (X, τ_N).

Then, FNαCL (η_N) $\subseteq U_N$, $\eta_N \subseteq U_N$ and U_N be FNOS, so **by i** we get , FNOS be FNGOS in (X, τ_N).

Therefore, FNαCL (η_N) $\subseteq U_N$, $\eta_N \subseteq U_N$ and U_N be FNGOS. Hence, η_N is FNSαGCS in (X, τ_N).

vii. Let $\eta_N = <x, \mu_{\eta N}, \sigma_{\eta N}, \nu_{\eta N}>$ be FNαCS in the FNTS (X, τ_N). Then, FNαCL (η_N) = η_N.

Now, let U_N be FNOS such that, $\eta_N \subseteq U_N$, so, FNαCL (η_N) = $\eta_N \subseteq U_N$.

Hence, η_N is FNαGCS in (X, τ_N).

Remark 3.4: The convers of **Theorem 3.3** is not true and this can be clarified in the following examples.

Example 3.5:

i. Let X= {a, b} define FNS η_N in X as follows:

η_N = <x, ($0.5_{(a)}$, $0.7_{(b)}$), ($0.5_{(a)}$, $0.5_{(b)}$), ($0.5_{(a)}$,$0.2_{(b)}$) >.

The family τ_N = {0_N, 1_N, η_N } be FNT.

If we take, ψ_N = <x, ($0.1_{(a)}$, $0.6_{(b)}$), ($0.5_{(a)}$, $0.5_{(b)}$), ($0.9_{(a)}$,$0.3_{(b)}$) >.

And let, $U_N = 0_N$, where U_N be FNCS such that, $U_N \subseteq \psi_N$.

Then, FNIn (ψ_N) = <x, ($0_{(a)}$, $0_{(b)}$), ($0_{(a)}$, $0_{(b)}$), ($1_{(a)}$,$1_{(b)}$) > ⊆ <x, ($0.1_{(a)}$, $0.6_{(b)}$), ($0.5_{(a)}$, $0.5_{(b)}$), ($0.9_{(a)}$,$0.3_{(b)}$) > such that, ($0_{(a)}$, $0_{(b)}$) ≤ ($0.1_{(a)}$, $0.6_{(b)}$), ($0_{(a)}$, $0_{(b)}$) ≤ ($0.5_{(a)}$, $0.5_{(b)}$) and ($1_{(a)}$,$1_{(b)}$) ≥ ($0.9_{(a)}$,$0.3_{(b)}$) = 0_N. So, FNIn (ψ_N) ⊇ U_N. Hence, ψ_N is FNGOS but, not FNOS.

Since $\psi_N \notin \tau_N$.

ii. Let X={a } define the FNSs η_N and γ_N in X as follows:

η_N = <x, ($0.5_{(a)}$), ($0.4_{(a)}$), ($0.7_{(a)}$) >, γ_N = <x, ($0.4_{(a)}$), ($0.1_{(a)}$), ($0.8_{(a)}$) >.

The family τ_N ={0_N, 1_N, η_N, γ_N } be FNT.

If we take, ψ_N = <x, ($0.8_{(a)}$), ($0.6_{(a)}$), ($0.5_{(a)}$) >.

Then, FNCL(ψ_N) = <x, ($0.8_{(a)}$), ($0.9_{(a)}$), ($0.4_{(a)}$) >. And, FNIn (FNCL(ψ_N)) = <x, ($0.5_{(a)}$), ($0.4_{(a)}$), ($0.7_{(a)}$) >. So, FNCL(FNIn (FNCL(ψ_N))) = <x, ($0.7_{(a)}$), ($0.6_{(a)}$), ($0.5_{(a)}$) >.

Therefore, <x, ($0.7_{(a)}$), ($0.6_{(a)}$), ($0.5_{(a)}$) > ⊆ ψ_N.

Hence, ψ_N is FNαCS but not FNCS. Since $\psi_N \notin 1_N$-τ_N.

iii. Take **Example 3.2.** Then, ψ_N is FNSαGCS but, not FNCS.

Since, $\psi_N \notin 1_N$- τ_N.

iv. Take **Example 3.2**. Then ψ_N is FNSαGCS but, not FNRCS.

Since, FNIn (ψ_N) = <x, ($0.2_{(a)}$, $0.3_{(b)}$), ($0.5_{(a)}$, $0.5_{(b)}$), ($0.8_{(a)}$,$0.7_{(b)}$) > and

FNCL(FNIn (ψ_N)) = <x, ($0.8_{(a)}$, $0.7_{(b)}$), ($0.5_{(a)}$, $0.5_{(b)}$), ($0.2_{(a)}$,$0.3_{(b)}$) > ≠ ψ_N.

v. Let X={a, b } define the FNSs η_N and γ_N in X as follows:

η_N = <x, ($0.4_{(a)}$, $0.2_{(b)}$), ($0.5_{(a)}$, $0.5_{(b)}$), ($0.6_{(a)}$,$0.7_{(b)}$) >,

γ_N = <x, ($0.8_{(a)}$, $0.8_{(b)}$), ($0.5_{(a)}$, $0.5_{(b)}$), ($0.2_{(a)}$,$0.2_{(b)}$) >.

The family τ_N ={0_N, 1_N, η_N, γ_N } be FNT.

Now if, ψ_N = <x, ($0.6_{(a)}$, $0.7_{(b)}$), ($0.5_{(a)}$, $0.5_{(b)}$), ($0.4_{(a)}$,$0.3_{(b)}$) >.

By **Theorem 3.3** **i**. If U_N is FNOS then is FNGOS.

So, $U_N = \gamma_N$ where, U_N be FNGOS such that, $\psi_N \subseteq U_N$.

By **Theorem 3.3** **ii**. Every FNCS is FNαCS.

Then, FNαCL(ψ_N) = 1_N- η_N. Therefore FNαCL(ψ_N) $\subseteq$ U_N.
Hence, ψ_N is FNSαGCS but, not FNαCS.
Since, FNCL(ψ_N) = 1_N- η_N, FNIn (FNCL(ψ_N)) = η_N and
FNCL(FNIn (FNCL(ψ_N))) = 1_N- $\eta_N \nsubseteq \psi_N$.

vi. Let X={a} define the FNSs η_N and γ_N in X as follows:
η_N = <x, ($0.5_{(a)}$), ($0.5_{(a)}$), ($0.5_{(a)}$) >, γ_N = <x, ($0.5_{(a)}$), ($0_{(a)}$), ($1_{(a)}$) >.
The family τ_N ={0_N, 1_N, η_N, γ_N } be FNT.
Now if, ψ_N = <x, ($0.6_{(a)}$), ($0.6_{(a)}$), ($0.6_{(a)}$) >.
Let U_N = <x, ($1_{(a)}$), ($1_{(a)}$), ($0.4_{(a)}$) > be FNGOS such that, $\psi_N \subseteq U_N$.
Then, FNαCL(ψ_N) = 1_N- γ_N. So FNαCL(ψ_N) $\subseteq$ U_N.
Hence, ψ_N is FNSαGCS but, not FNαGCS.
Since, U_N is FNGOS but not FNOS.

vii. Let X={a } define the FNSs η_N and γ_N in X as follows:
η_N = <x, ($0.5_{(a)}$), ($0.5_{(a)}$), ($0.7_{(a)}$) >, γ_N = <x, ($0.4_{(a)}$), ($0_{(a)}$), ($1_{(a)}$) >.
The family τ_N ={0_N, 1_N, η_N, γ_N } be FNT.
Now if, ψ_N = <x, ($1_{(a)}$), ($1_{(a)}$), ($0.4_{(a)}$) >.
Then, ψ_N is FNCS. Since $\psi_N \in 1_N$- τ_N but, not FNRCS.
Since FNIn (ψ_N) = <x, ($0.5_{(a)}$), ($0.5_{(a)}$), ($0.7_{(a)}$) > and
FNCL(FNIn (ψ_N)) = <x, ($0.7_{(a)}$), ($0.5_{(a)}$), ($0.5_{(a)}$) > $\neq \psi_N$.

viii. Let X={a} define the FNSs η_N and γ_N in X as follows:
η_N = <x, ($0.5_{(a)}$), ($0.5_{(a)}$), ($0.6_{(a)}$) >, γ_N = <x, ($0.5_{(a)}$), ($0_{(a)}$), ($1_{(a)}$) >.
The family τ_N ={0_N, 1_N, η_N, γ_N } be FNT.
Now if, ψ_N = <x, ($0.6_{(a)}$), ($0.6_{(a)}$), ($0.6_{(a)}$) >.
Let, U_N = 1_N be FNOS such that, $\psi_N \subseteq U_N$.
Then, FNCL(ψ_N) = <x, ($1_{(a)}$), ($1_{(a)}$), ($0.5_{(a)}$) > and FNCL(ψ_N) $\subseteq$ U_N.
Hence, ψ_N is FNαGCS but, not FNαCS.
Since, FNCL(ψ_N) = <x, ($1_{(a)}$), ($1_{(a)}$), ($0.5_{(a)}$) >, FNIn (FNCL(ψ_N)) = <x, ($0.5_{(a)}$), ($0.5_{(a)}$), ($0.6_{(a)}$) > and
FNCL(FNIn (FNCL(ψ_N))) = <x, ($0.6_{(a)}$), ($0.5_{(a)}$), ($0.5_{(a)}$) > $\nsubseteq \psi_N$.

Remark 3.6: i. The relation between FNPCS and FNSαGCS is independent and this can be clarified in the next example.

ii. The intersection of two FNSαGCS is not FNSαGCS in general and we explained it in the next example.

Example 3.7:

i. (1) Let X= {a, b} define FNS η_N in X as follows:

η_N = <x, $(0.5_{(a)}, 0.5_{(b)}), (0.5_{(a)}, 0.5_{(b)}), (0.4_{(a)}, 0.5_{(b)})$ >.

The family $\tau_N = \{0_N, 1_N, \eta_N\}$ be FNT.

Now if, ψ_N = <x, $(0.5_{(a)}, 0.4_{(b)}), (0.5_{(a)}, 0.5_{(b)}), (0.6_{(a)}, 0.5_{(b)})$ >.

Then, FNIn (ψ_N) = 0_N and FNCL(FNIn (ψ_N)) = 0_N. So, FNCL(FNIn (ψ_N)) $\subseteq \psi_N$.

Hence, ψ_N is FNPCS but, not FNSαGCS. Since

Let, $U_N = \eta_N$, where U_N be FNGOS such that, $\psi_N \subseteq U_N$. Then, FNαCL(ψ_N) = 1_N. So FNαCL(ψ_N) $\nsubseteq$ U_N.

(2) Let X={a, b} define the FNSs η_N and γ_N in X as follows:

η_N = <x, $(0.5_{(a)}, 0.2_{(b)}), (0.5_{(a)}, 0.5_{(b)}), (0.5_{(a)}, 0.7_{(b)})$ >,

γ_N = <x, $(0.8_{(a)}, 0.8_{(b)}), (0.5_{(a)}, 0.5_{(b)}), (0.2_{(a)}, 0.2_{(b)})$ >.

The family $\tau_N = \{0_N, 1_N, \eta_N, \gamma_N\}$ be FNT.

Now if, ψ_N = <x, $(0.5_{(a)}, 0.7_{(b)}), (0.5_{(a)}, 0.5_{(b)}), (0.5_{(a)}, 0.3_{(b)})$ >.

Let, $U_N = \gamma_N$, where U_N be FNGOS such that, $\psi_N \subseteq U_N$.

Then, FNαCL(ψ_N) = <x, $(0.5_{(a)}, 0.7_{(b)}), (0.5_{(a)}, 0.5_{(b)}), (0.5_{(a)}, 0.2_{(b)})$ > $\subseteq U_N$.

Hence, ψ_N is FNSαGCS but, not FNPCS.

Since, FNIn (ψ_N) = η_N and FNCL(FNIn (ψ_N)) = <x, $(0.5_{(a)}, 0.7_{(b)}), (0.5_{(a)}, 0.5_{(b)}), (0.5_{(a)}, 0.2_{(b)})$ >.

So, FNCL(FNIn (ψ_N)) $\nsubseteq \psi_N$.

ii. Let X= {a, b} define FNS η_N in X as follows: η_N = <x, $(0.5_{(a)}, 0_{(b)}), (0.5_{(a)}, 0.5_{(b)}), (0.1_{(a)}, 1_{(b)})$ >.
The family $\tau_N = \{0_N, 1_N, \eta_N\}$ be FNT.
Now if, ψ_{N1} = <x, $(0.2_{(a)}, 1_{(b)}), (1_{(a)}, 1_{(b)}), (0.7_{(a)}, 0_{(b)})$ > and ψ_{N2} = <x, $(0.6_{(a)}, 0_{(b)}), (1_{(a)}, 1_{(b)}), (0.3_{(a)}, 1_{(b)})$ > are FNSαGCS. But, $\psi_{N1} \cap \psi_{N2}$ = <x, $(0.2_{(a)}, 0_{(b)}), (1_{(a)}, 1_{(b)}), (0.7_{(a)}, 1_{(b)})$>.
Now let, $U_N = \eta_N$, where U_N be FNGOS such that, $\psi_{N1} \cap \psi_{N2} \subseteq U_N$. Then, FNαCL($\psi_{N1} \cap \psi_{N2}$) = $1_N \nsubseteq$ U_N.
Hence, $\psi_{N1} \cap \psi_{N2}$ is not FNSαGCS.

Remark 3.8: The next diagram explains the relationships among different sets in the FNTS and the convers is not true in general.

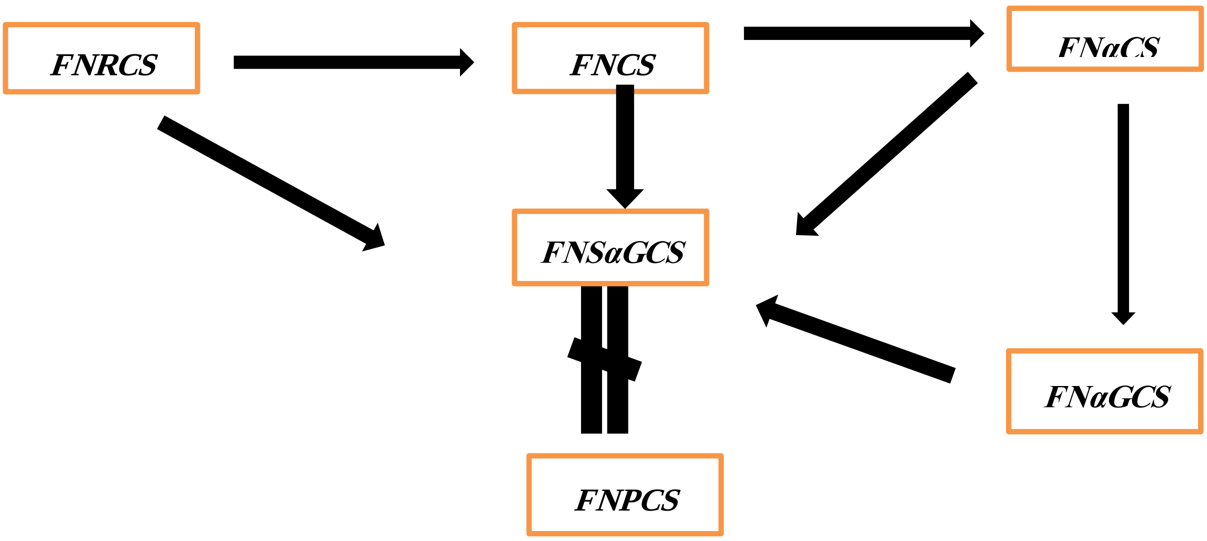

Diagram 3.1

5. Conclusions

In this present paper, we have defined new class of neutrosophic generalized closed sets called, fuzzy neutrosophic strongly alpha generalized closed set in fuzzy neutrosophic topological spaces. Many results have been discussed with some properties. Further, we giving some theorems, propositions and provided some useful examples where such properties failed to be preserved in order to get the relations between fuzzy neutrosophic strongly alpha generalized closed set and existing fuzzy neutrosophic closed sets in fuzzy neutrosophic topological spaces . We think, our studied class of sets belongs to the new class of fuzzy neutrosophic sets which is useful not only in the deepening of our understanding of some special features of the well-known notions of fuzzy neutrosophic topology but also useful in neutrosophic control theory.

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New Types of Neutrosophic Crisp Closed Sets

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Abstract: The neutrosophic sets were known since 1999, and because of their wide applications and their great flexibility to solve the problems, we used these the concepts to define a new types of neutrosophic crisp closed sets and limit points in neutrosophic crisp topological space, namly [neutrosophic crisp Gem sets and neutrosophic crisp Turig points] respactvely, we stady their properties in details and join it with topological concepts. Finally we used [neutrosophic crisp Gem sets and neutrosophic crisp Turig points] to introduce of topological concepts as : neutrosophic crisp closed (open) sets , neutrosophic crisp closure, neutrosophic crisp interior, neutrosophic crisp extrior and neutrosophic crisp boundary which are fundamental for further reserch on neutrosophic crisp topology and will setrengthen the foundations of theory of neutrosophic topological spaces.

Keywords: Neutrosophic crisp set, Neutrosophic crisp topology, Neutrosophic crisp closed set,

1. Introduction

In 1999, Smarandache firstly proposed the theory of neutrosophic set [1] which is the generalization of the class sets, conventional fuzz set [2] and intuitionistic set fuzzy [3]. After Smarandache, neutrosophic sets have been successfully applied to many fields such as; topology, control theory, databases, medical diagnosis problem, decision making problem and so on, [4-37] .

A.A. Salama, et, al.[38] proposed a new mathematical model called " Neutrosophic crisp sets and Neutrosophic crisp topological spaces " .

The idea of "Gem-Set", which is a characterization of the concept of closure is introduced by AL-Nafee ,Al-Swidi [39] . After AL-Nafee, the idea of "Gem-Set has been successfully using to many topological concepts such as; interior, exterior, boundary ,separation axioms, continuous functions , bitopological spaces, compactness, soft topological spaces, and so on, [40,41,42,43,44,45,46,47,48].

The idea of "controlling soft Gem-Set" and join it with topological concepts in soft topological space is introduced by [49]. The concept of the soft Turing point and used it with separation axioms in soft topological space is introduced by [50,51].

The goal of this research is to combine the concept of "Gem-Set" and Turing point with neutrosophic crisp set to define a new types of neutrosophic crisp closed sets and limit points in neutrosophic crisp topological space, namly [neutrosophic crisp Gem sets and neutrosophic crisp Turig points] respactvely, we stady their properties in details and we also use it to introduce the some of topological concepts as : neutrosophic crisp closed (open) sets , neutrosophic crisp closure, neutrosophic crisp interior, neutrosophic crisp extrior and neutrosophic crisp boundary which are fundamental for further reserch on neutrosophic crisp topology and will setrengthen the foundations of theory of neutrosophic topological spaces.

The paper is structured as follows; In section 2, we first recall the necessary background on neutrosophic and neutrosophic crisp points [NCP_N for short]. In section 3, a neutrosophic crisp Turing points properties are introduced with their properties. In section 4, the concept of neutrosophic crisp Gem sets are introduced and studied their properties.

Throughout this paper ,NCTS means a neutrosophic crisp topological space , also we write (H) by H (for short),the collection of all neutrosophic crisp sets on H will be denoted by N(H) .

2. Preliminaries

2.1. **Definition** [52]

Let H be a non-empty fixed set, a neutrosophic crisp set (for short NCS) D is an object having the form $D = <D_1,D_2,D_3>$ where D_1,D_2 and D_3 are .subsets of H .

We will exhibit the basic neutrosophic operations defnitions (union, intersection and complement) . Since there are different definitions of neutrosophic operations , we will organize the existing definitions into two types in each type these operations will be consistent and functional . In this work we will use one Type of neutrosophic crisp sets operations .

2.2. **Definition** [52]

A neutrosophic crisp topology (NCTS) on anon-empty set H is a family T of neutrosophic crisp susets in H satisfying the following conditions ;

$\emptyset_N, H_N \in T$.

$C \cap D \in T$, for $C,D \in T$

The union of any number of set in T belongs to T .

The pair (H,T) is said to be a neutrosophic crisp topological space(NCTS) in H . Moreover the elements in T are said to be neutrosophic crisp open sets. A neutrosophic crisp set F is closed iff its complement (F^C) is an open neutrosophic crisp set .

2.3. **Definition** [52]

Let NI be a non-null collection of neutrosophic crisp sets over a universe H . Then NI is called neutrosophic crisp ideal on H if ;

- $C \in NI$ and $D \in NI$ then $C \cup D \in NI$.
- $C \in NI$ and $D \subseteq C$ then $D \in NI$.

2.4. **Definition** [52]

Let (H,) be NCTS ,A be a neutrosophic crisp set then: The intersection of any neutrosophic crisp closed sets contained A is called neutrosophic crisp clusuer of A (for short NC-CL(A)) .

2.5. **Definition** [52]

((neutrosophic crisp sets operations of Type.I))

Let H be a non-empty set and $C = < C_1, C_2, C_3 >$, $D = < D_1, D_2, D_3 >$ be two neutrosophic crisp sets, where D_1, C_1,D_2,C_2 and D_3 C_1 are subsets of H ,such that $(D_1 \cap D_2)=\emptyset$, $(D_1 \cap D_3)=\emptyset$, $(D_2 \cap D_3)=\emptyset$, $(C_1 \cap C_2)=\emptyset$, $(C_1 \cap C_3)=\emptyset$, $(C_2 \cap C_3)=\emptyset$ then:

- $\emptyset_N = <\emptyset,\emptyset,H>$ (Neutrosophic empty set) .
- $H_N =<H,\emptyset,\emptyset>$ (Neutrosophic universal set) .
- $C \cap D = [C_1 \cap D_1] ,[C_2 \cap D_2]$ and $[C_3 \cup D_3]$.
- $C \cup D = [C_1 \cup D_1] ,[C_2 \cup D_2]$ and $[C_3 \cap D_3]$.
- $C \subseteq D \Leftrightarrow C_1 \subseteq D_1$,$C_2 \subseteq D_2$ and $D_3 \subseteq C_3$.

- The complement of a NCS (D) may be defined as: $D = < D_3, D_2, D_1 >$.
- $C = D \Leftrightarrow C \subseteq D, D \subseteq C$.

2.6. **Definition** [53]

((neutrosophic crisp sets operations of Type.2))

Let H be a non-empty set and $C = < C_1, C_2, C_3 >$, $D = < D_1, D_2, D_3 >$ be two neutrosophic crisp sets, where D_1, C_1, D_2 ,C_2 and D_3 ,C_1 are subsets of H then:

- $\emptyset_N = <\emptyset, \emptyset, \emptyset>$ (Neutrosophic empty set) .
- $H_N = <H,H,H>$ (Neutrosophic universal set) .
- $C \cap D = [C_1 \cap D_1], [C_2 \cap D_2]$ and $[C_3 \cap D_3]$.
- $C \cup D = [C_1 \cup D_1], [C_2 \cup D_2]$ and $[C_3 \cup D_3]$.
- $C \subseteq D \Leftrightarrow C_1 \subseteq D_1 , C_2 \subseteq D_2$ and $C_3 \subseteq D_3$.
- The complement of a NCS (D) may be defined as: $D^C = <D_1^c, D_2^c, D_3^c>$.
- $C = D \Leftrightarrow C \subseteq D, D \subseteq C$".

2.7. **Definition** [53]

For all a,b,c $\in$ H .Then the neutrosophic crisp points related to a,b,c are defined as follows ;

- $a_{N_1} = <\{a\},\emptyset,\emptyset>$ on H .
- $b_{N_2} = <\emptyset,\{b\},\emptyset>$ on H .
- $c_{N_3} = <\emptyset,\emptyset,\{c\}>$ on H .

(The set of all neutrosophic crisp points $(a_{N_1}, b_{N_2}, c_{N_3})$ is denoted by NCPN) .

3. **Neutrosophic crisp turing point**

In this work, we will use Type.2 of neutrosophic crisp sets operations , this was necessary to homogeneous suitable results for the upgrade of this research .

3.1. **Definition**

Let (H, T) be NCTS , $P \in NCPN$ in H, we define a neutrosophic crisp ideal NI with respect to a neutrosophic crisp point P, as follows :

$$NI(P) = \{D \in T : P \in (D)^C\}$$

3.2. **Definition**

Let (H, T) be NCTS, $P \in NCPN$ in (H, T), $Y \subseteq H$, we define a neutrosophic crisp ideal ${}^{Y}NI(P)$ respect to subspace (Y,T_Y), as follows:

$${}^{Y}NI(P) = \{D \in T_Y : P \in (H \setminus D)\} .$$

3.3. **Remark**

Let (H, T) be NCTS, $Y \subseteq H$, for each $D \neq \emptyset_N$ and $P \in NCPN$ in Y , then ;

$${}^{Y}NI(P) = \{ D \in T_Y : P \in (H \setminus D)\} = \{D \in T_Y : P \in (Y \setminus D) \} .$$

Proof

${}^{Y}NI(P) = \{D \in T_Y : P \in (H \setminus D)\} = \{ D \in T_Y : P \notin D,$ for each $P \in Y\} = \{ D \in T_Y : P \in (Y \setminus D)$, for each $P \in Y\}$.

3.4. **Remark**

Let (H, T) be NCTS ,$Y \subseteq H$, for each $D \neq \emptyset_N$ and $P \in NCPN$ in H,then ;

$${}^{Y}NI(P) = \{ D \in T_Y : P \in (H \setminus D)\} = \{D \cap Y : \text{for each } D \neq \emptyset_N \in NI(P)\}.$$

3.5. **Example**

Let (H, T) be NCTS , such that H={1} ,

$T = \{ \emptyset_N, H_N, A , B ,C ,D ,E ,F,G \}$, $P_1 = < \emptyset.\{1\},\emptyset >$, such that;

$A = < \{1\}, \emptyset, \emptyset >, B = < \emptyset, \{1\}, \emptyset >, C = < \{1\}, \{1\}, \emptyset >$, $D = < \{1\}, \emptyset, \{1\} >$,

$E = \langle \emptyset, \{1\}, \{1\} \rangle$, $F = \langle \emptyset, \emptyset, \{1\} \rangle$, $G = \langle \{1\}, \{1\}, \{1\} \rangle$.

Then, $NI(P_1) = \{\emptyset_N , A, D, F\}$.

3.6. **Definition**

Let (H, T) be NCTS , $P \in NCPN$ in H and NI be a neutrosophic crisp ideal on (H, T), we say that p is a neutrosophic crisp turing point of NI if $D^c \in NI$ for each $D \in T_P$,T_P is collection of all neutrosophic crisp open set of neutrosophic crisp point p.

3.7. **Remark**

Let (H, T) be NCTS, $P \in NCPN$ in H and $NI(P) = \{D \in T: P \in (D)^C\}$ be aneutrosophic crisp ideal on (H, T) .

Then, p is a neutrosophic crisp turing point of NI(P) .

3.8. **Example**

Let (H, T) be NCTS , such that H={1},

$T = \{ \emptyset_N, H_N, A, B, C, D, E, F, G \}$, $P_1 = \langle \emptyset, \{1\}, \emptyset \rangle$, $P_2 = \langle \{1\}, \emptyset, \emptyset \rangle$, such that;

$A = \langle \{1\}, \emptyset, \emptyset \rangle$, $B = \langle \emptyset, \{1\}, \emptyset \rangle$, $C = \langle \{1\}, \{1\}, \emptyset \rangle$, $D = \langle \{1\}, \emptyset, \{1\} \rangle$,

$E = \langle \emptyset, \{1\}, \{1\} \rangle$, $F = \langle \emptyset, \emptyset, \{1\} \rangle$, $G = \langle \{1\}, \{1\}, \{1\} \rangle$.

Then, P_1 is a neutrosophic crisp turing point of neutrosophic crisp ideal $NI(P_1)$,but not P_2 .

3.9. **Theorem**

Let (H, T) be NCTS, $a_{N_1} \neq b_{N_1} \in NCPN$ in H, then, $\langle \{b\}, \emptyset, \emptyset \rangle$ is a neutrosophic crisp closed set if and only if a_{N_1} is not a aneutrosophic crisp turing point of $NI(b_{N_1})$.

Proof

Let $a_{N_1} \neq b_{N_1} \in NCPN$ in H. Assume that $\langle \{b\}, \emptyset, \emptyset \rangle$ is a neutrosophic crisp closed set ,so that $\langle \{b\}, \emptyset, \emptyset \rangle = cl\ (\langle \{b\}, \emptyset, \emptyset \rangle)$. But $a_{N_1} \neq b_{N_1}$ get that $a_{N_1} \notin cl\ (\langle \{b\}, \emptyset, \emptyset \rangle)$. Therefore, there exists a neutrosophic crisp open set U such that, $a_{N_1} \in U$, $U \cap \langle \{b\}, \emptyset, \emptyset \rangle = \emptyset_N$. So that $a_{N_1} \in U$, $U^c \notin NI(b_{N_1})$,because if $U^c \in NI(b_{N_1})$, then $\langle \{b\}, \emptyset, \emptyset \rangle \in U$,that means $U \cap \langle \{b\}, \emptyset, \emptyset \rangle \neq \emptyset_N$,this a contradiction!. Hence a_{N_1} is not a neutrosophic crisp turing point of $NI(b_{N_1})$.

Conversely,

Let $a_{N_1} \neq b_{N_1} \in NCPN$ in H. Since a_{N_1} is not a neutrosophic crisp turing point of $NI(b_{N_1})$, then there exists a neutrosophic crisp open set U such that, $a_{N_1} \in U$, $U^c \notin NI(b_{N_1})$, so $\langle \{b\}, \emptyset, \emptyset \rangle \notin U$.Thus $a_{N_1} \in U$, $U \cap \langle \{b\}, \emptyset, \emptyset \rangle = \emptyset_N$ implies $a_{N_1} \notin cl(\langle \{b\}, \emptyset, \emptyset \rangle)$.

Hence $\langle \{b\}, \emptyset, \emptyset \rangle = cl(\langle \{b\}, \emptyset, \emptyset \rangle)$, thus $\langle \{b\}, \emptyset, \emptyset \rangle$ is a neutrosophic crisp closed set in H.

Proof by the same proof of 2.10. Theorem .

4. **Neutrosophic crisp Gem set**

4.1. **Definition**

Let (H, T) be NCTS, $P \in NCPN$ in H ,NI(P) be aneutrosophic crisp ideal on (H, T) and $D \subseteq (H, T)$, we defined the neutrosophic crisp set ND^{*P} with respect to space (H, T) as follows:

$ND^{*P} = \{P_1 \in NCPN$ in H ; $F \cap D \notin NI(P)$, for each $F \in T_{P_1}$,T_{P_1} is collection of all neutrosophic crisp open set of neutrosophic crisp point P_1.The neutrosophic crisp set ND^{*P} is called neutrosophic crisp Gem-Set .

4.2. **Example**

Let (H, T) be NCTS , such that H={1,2,3},

$T = \{ \emptyset\_N, H\_N, A, B, C, D, E, F, G \}$, $P = \langle \emptyset, \{1\}, \emptyset \rangle$, $D = \langle \emptyset, \{1,3\}, \emptyset \rangle$, such that;

$A = \langle \emptyset, \{1\}, \emptyset \rangle$, $B = \langle \emptyset, \{2\}, \emptyset \rangle$, $C = \langle \emptyset, \{3\}, \emptyset \rangle$, $D = \langle \emptyset, \{1,2\}, \emptyset \rangle$.

$E = <\emptyset,\{1,3\},\emptyset,\emptyset>$, $F = <\emptyset,\{2,3\},\emptyset>$, $G = <\emptyset,\{1,2,3\},\emptyset>$.

Then, $NI(P) = \{\emptyset, N, B, C, F\}$ and $ND^{*P} = <\{1\},\emptyset,\emptyset>$.

4.3. **Theorem**

Let (H,T) be NCTS, $P \in$ NCPN in H, and let D,C be subsets of (H,T).Then

1. $\emptyset_N{}^{*P} = \emptyset_N$
2. $H_N{}^{*P} = H_N$, whenever $NI(P) = \emptyset_N$.
3. $C \subseteq D \rightarrow NC^{*P} \subseteq ND^{*P}$.
4. For any points $P_1, P_2 \in$ NCPN in H ,with $NI(P_2) \supseteq NI(P_1)$ then $ND^{*P_2} \subseteq ND^{*P_1}$.
5. $P \in D$ if and only if $P \in ND^{*P}$.
6. If $P \in D$, then $(ND^{*P})^{*P} = ND^{*P}$.
7. If $P_1 \in D, P_2 \in C$ with $P_1 \neq, P_2$, $D \cap C = \emptyset_N$, then $ND^{*P_1} \cap NC^{*P_2} = \emptyset_N$.
8. If $a_{N_1}, b_{N_1} \in$ NCPN in Hwith $a_{N_1} \neq b_{N_1}$, then $b_{N_1} \in (a_{N_1})^c$ implies $a_{N_1} \notin (a_{N_1})^* b_{N_1}$ and $b_{N_1} \notin (b_{N_1})^* a_{N_1}$.

4.4 . **Remark**

The equality of theorem part (3),(4) does not necessarily hold as shown :

Let (H,T) be NCTS , such that $H=\{1,2\}$, $D = <\emptyset,\{2\},\emptyset>$, $C = <\emptyset,\{1\},\emptyset>$,

$T = \{\emptyset_N, H_N, A, B, G\}$, $P_1 = <\emptyset,\{2\},\emptyset>$, $P_2 = <\emptyset,\{1\},\emptyset>$,

$A = <\emptyset,\{1\},\emptyset>$, $B = <\emptyset,\{2\},\emptyset>$, $G = <\emptyset,\{1,2\},\emptyset>$,

Then, $NI(P_1) = \{\emptyset_N, A\}$, $NI(P_2) = \{\emptyset_N, B\}$ and $ND^{*P_1} = <\emptyset,\{2\},\emptyset>$, $ND^{*P_2} = \emptyset_N$,$NC^{*P_1} = \emptyset_N$

Note that,

1) $ND^{*P_2} \subseteq ND^{*P_1}$,but $NI(P_2) \not\supseteq NI(P_1)$.

2) $NC^{*P_1} \subseteq ND^{*P_1}$,but $C \not\subseteq D$.

4.5 .**Theorem**

Let(H,T) be NCTS, $P_1 \in$ NCPN in H and D,C be subsets of (H,T).Then $ND^{*P_1} \cup NC^{*P_1} = N(D \cup C)^{*P_1}$.

Proof

It is obviously known that $D \subset (D \cup C)$ and $C \subset (D \cup C)$, then from theorem 3.3 part(3) we get, $ND^{*P_1} \subset N(D \cup C)^{*P_1}$and $ND^{*P_1} \subset N(A \cup C)^{*P_1}$, for any $P_1 \in$ NCPN in H. Hence

$$ND^{*P_1} \cup NC^{*P_1} \subset N(D \cup C)^{*P_1} \text{ ----(1)}$$

For reverse inclusion, let $P_2 \notin ND^{*P_1}$.Then there exists neutrosophic crisp open set U containing p ,with $D \cap U \in NI(P_1)$.Similarly, if $P_2 \notin NC^{*P_1}$,then there exists neutrosophic crisp open set V containing P,with $C \cap V \in NI(P_1)$ .Then by hereditary property of neutrosophic crisp ideal, we get , $D \cap U \cap V \in NI(P_1)$ and $C \cap U \cap V \in NI(P_1)$. Again by the finite additivity condition of neutrosophic crisp ideal, we get $(D \cup C) \cap U \cap V \in NI(P_1)$.Hence $P_2 \notin N(D \cup C)^{*P_1}$.So,

$$N(D \cup C)^{*P_1} \subset ND^{*P_1} \cup NC^{*P_1} \text{ ----(2).}$$

From (1) and (2) we get, $ND^{*P_1} \cup NC^{*P_1} = N(D \cup C)^{*P_1}$.

4.6 .**Theorem**

Let (H,T) be NCTS, $P_1 \in NCPN$ in H and D,C be subsets of (H,T).Then $N(D \cap C)^{*P_1} \subset ND^{*P_1} \cap NC^{*P_1}$.

Proof

It is known that D∩C⊂D and D∩C⊂C, then from theorem part (3), $N(D \cap C)^{*P_1} \subset ND^{*P_1}$ and $N(D \cap C)^{*P_1} \subset NC^{*P_1}$.Hence $N(D \cap C)^{*P_1} \subset ND^{*P_1} \cap ND^{*P_1}$,for any $P_1 \in NCPN$ in H.

4.7 **.Theorem**

Let (H,T) be NCTS, $a_{N_1} \in NCPN$ in H, for each neutrosophic crisp open set U containinga_{N_1}, then $(a_{N_1})^* a_{N_1} \subseteq U$.

proof

Let $b_{N_1} \notin U$,so $a_{N_1} \neq b_{N_1}$,then we get that $U \cap (b_{N_1}) = \emptyset_N \in NI((a_{N_1})$.That means $(b_{N_1} \notin (a_{N_1})^* a_{N_1}$ .Thus$\{(a_{N_1})^* a_{N_1} \subseteq U$.

4.8 **.Theorem**

Let (H,T) be NCTS, $P_1 \in NCPN$ in H and D be subsets of (H,T).Then

$$D^{*P_1} = \begin{cases} \emptyset_N & \text{if } P_1 \notin D \\ cl(P_1) & \text{if } P_1 \in D \end{cases}$$

Proof

Case(1)

If $P_1 \notin D$,To prove $D^{*P_1} = \emptyset_N$ .Let $D^{*P_1} \neq \emptyset_N$, then there exists least one element. say $P_2 \in D^{*P_1}$ (by definition of D^{*P_1}), we have $C_{P_2} \cap D \notin NI(P_1)$. Hence$P_1 \in D \cap C_{P_2}$ So $P_1 \in D$ which contradiction!, then $D^{*P_1} = \emptyset_N$.

Case (2)

If $P_1 \in D$, to prove $D^{*P_1} = cl(P_1)$. Let $P_2 \in D^{*P_1}$ implies $P_1 \in D \cap V_{P_2}$ for each $V_{P_2} \in T_{P_2}$ implies that $P_1 \in V_{P_2}$ for each $V_{P_2} \in T_{P_2}$ it follows $P_2 \in cl(P_1)$ then $D^{*P_1} \subseteq cl(P_1)$ for each D be subsets of $(H.T)$. Let $P_2 \in cl(P_1)$ and $P_2 \notin D^{*P_1}$ then there exists neutrosophic crisp open set V_{P_2} containing P_2 such that $D \cap V_{P_2} \in NI(P_1)$, which implies that $P_1 \notin D \cap V_{P_2}$ then $P_1 \notin D$ or $P_1 \notin V_{P_2}$ which means that $P_1 \notin D$ or $P_2 \notin cl(P_1)$ which contradiction! in two case. Hence $P_2 \in D^{*P_1}$ implies that$cl(P_1) \subseteq D^{*P_1}$. Therefore, $D^{*P_1} = cl(P_1)$. if $P_1 \in D$.

4.9. **Definition**

Let (H,T) , (Y,δ) be NCTS. Then, the mapping $f:(H,T) \to (Y,\delta)$ is called NI^*- map ,if and only if, for every subset D of (H,T), $P_1 \in NCPN$ in H, $f(D^{*P_1}) = (f(D))^{*f(P_1)}$.

4.10. **Example**

Let (H,T) , (Y,δ) be NCTS, such that H={1,2,3}, Y={a,b,c},

T={ $\emptyset_N$, H_N, A , B }, δ={ $\emptyset_N$, Y_N, G }, such that .

$A < \{1\}, \emptyset, \emptyset >$, B= $< \{2,3\}, \emptyset, \emptyset >$, G $=< \{a\}, \emptyset, \emptyset >$.

Define f(2)=f(1)=c and f(3)=a, Put D={3} subset of (H,T) .

Then D^{*3} = B= $< \{2,3\}, \emptyset, \emptyset >$, so $f(D^{*3}) = (f(D))^{*f(3)=a} = (< \{a,c\}, \emptyset, \emptyset >)^{*\mathbf{a}} =< \{a,b,c\}, \emptyset, \emptyset >$.

4.11. **Definition**

Let (H,T) , (Y,δ) be NCTS.Then, the mapping $f:(H,T) \to (Y,\delta)$ is called NI^{**}-map if and only if, for every subset D of (Y,δ) ,p ∈ NCPN in Y, $f^{-1}(D^{*P}) = (f^{-1}(D))^{*f^{-1}(P)}$.

4.12. **Example**

Let (H,T) , (Y,δ) be NCTS, such that H={a,b,c}, Y={1,2,3}

T={ $\emptyset_N$, H_N, A , B }, δ={ $\emptyset_N$, Y_N, G }, such that .

$A = \langle \{a\}, \emptyset, \emptyset \rangle$, $B = \langle \{b, c\}, \emptyset, \emptyset \rangle$, $G = \langle \{1\}, \emptyset, \emptyset \rangle$.

Define f(b)=f(a)=3 and f(c)=1.Put D={3} subset of (Y, δ) .

Then $D^{*1} = B = \langle \{2,3\}, \emptyset, \emptyset \rangle$, so $f^{-1}(D^{*1}) = (f^{-1}(D))^{*c} = (\langle \{b, a\}, \emptyset, \emptyset \rangle)^{*c} = \langle \{b, c\}, \emptyset, \emptyset \rangle$.

Conclusion

We defined a new types of neutrosophic crisp closed sets and limit points in neutrosophic crisp topological space namly [neutrosophic crisp Gem sets and neutrosophic crisp Turig points] respactvely , we stady their properties in details and we also use it to introduce the some of topological concepts as : neutrosophic crisp closed (open) sets , neutrosophic crisp closure , neutrosophic crisp interior , neutrosophic crisp extrior and neutrosophic crisp boundary which are fundamental for further reserch on neutrosophic crisp topology and will setrengthen the foundations of theory of neutrosophic topological spaces .

We expect, this paper will promote the future study on neutrosophic crisp topological spaces and many other general frameworks .

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Pentapartitioned neutrosophic set and its properties

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Abstract: The main objective of this paper is to propose a new type of set which we call pentapartitioned neutrosophic set. We also prove some of its basic properties.

Keywords: Neutrosophic set, Single valued neutrosophic set, Pentapartitioned neutrosophic set.

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1. Introduction:

Smarandache [1] defined Neutrosophic Set (NS) to deal with uncertainty, indeterminacy and inconsistency involved in mathematical objects. It generalizes fuzzy set [2] and intuitionistic fuzzy set [3] by incorporating degrees of indeterminacy and rejection (falsity or non-membership) as independent components. Wang et al. [4] defined Single Valued Neutrosophic Set (SVNS) in 2010. Chatterjee et al. [5] defined Quadripartitioned SVNS (QSVNS) that involves truth, falsity, unknown and contradiction based on four valued logics [6, 7].

Smarandache [7] split indeterminacy into unknown, contradiction, ignorance and proposed Five Symbol Valued Neutrosophic Logic (FSVNL). In this paper we utilize FSVNL and propose pentapartitioned neutrosophic set. We also establish some basic properties of the proposed set. The proposed structure is generalization of existing theories of SVNS and QSVNS.

The organization of the paper is as follows: Section 1 provides a brief introduction; Section 2 is dedicated to recalling some preliminary results; Section 3 introduces the concept of a pentapartitioned neutrosophic set. Section 4 deals with some basic set-theoretic operations over pentapartitioned neutrosophic sets. Section 5 concludes the paper stating future scope of research.

1. Preliminary:

Definition 1: An NS [1] N on the universe of discourse Q is defined as:

$N=\{<q,T_N(q),I_N(q),F_N(q)>:q\in Q\}$ where $T,I,F:Q\to]^-0,1^+[$ and $^-0\le T_N(q)+I_N(q)+F_N(q)\le 3^+$.

2. Single Valued Pentapartitioned Neutrosophic Sets:

Based on Smarandache FSVNL [7], we define the concept of Pentapartitioned Neutrosophic Set (PNS). The term "pentapartitioned" means something that divided into five characteristic features. The indeterminacy is split into three parts signifying contradiction, ignorance and unknown respectively. We now defined a PNS as follows:

Definition 3: Let P be a non-empty set. A PNS A over P characterizes each element p in P by a truth-membership function T_A, a contradiction membership function C_A, an ignorance membership function G_A, unknown membership function U_A and a falsity membership function F_A such that for each $p \in P$, $T_A, C_A, G_A, U_A, F_A \in [0,1]$ and $0 \le T_A(p) + C_A(p) + G_A(p) + U_A(p) + F_A(p) \le 5$.

Example: Consider the statement: "Is Facebook good for society?".

Suppose, this statement is posed in front of a group of five people, say, $P = \{p_1, p_2, p_3 p_4, p_5\}$ (which constitute the universe under consideration) and they are requested to express their opinion regarding this statement. Now it may so happen that the opinion of the people may vary among the following possible options: "a degree of agreement with the statement", "a degree of both agreement as well as disagreement regarding the statement "a degree of neither agreement nor disagreement regarding the statement", "a degree of ignore agreement and disagreement" and "a degree of disagreement with respect to the statement". According to the response of the people, the available information can be represented in terms of a PNS as follows:

From the above PNS, it is seen that the person p_1 is to great extent, in agreement with the statement whereas, p_5 mostly disagrees with the statement while p_2 opines that the statement is both true as well as false, p_3 is mainly in ignorance regarding the truth of the statement and p_4 totally ignores the truth and false of the statement.

It is to be noted that when Indeterminacy (I) is refined into I1, I2, I3, and together T, I1, I2, I3, F form a pentapartitioned neutrosophic set. It is a special case of the n- valued refined neutrosophic set, introduced by Smarandache [7] in 2013.

Definition 4: A PNS A is said to be absolute PNS if and only if its truth-membership, contradiction membership, ignorance membership, unknown membership and falsity membership function values are defined as follow,

$$T_A(p) = 1, C_A(p) = 1, G_A(p) = 0, U_A(p) = 0, F_A(p) = 0.$$

Definition 5: A PNS is said to be null θ PNS if and only if its truth-membership, contradiction membership, ignorance membership, unknown membership and falsity membership function values are respectively defined as follows:

$$T_A(p) = 0, C_A(p) = 0, G_A(p) = 1, U_A(p) = 1, F_A(p) = 1.$$

3. Basic properties:

Definition 6: Consider two PNS R_1 and R_2 over P, R_1 is said to be contained in R_2, denoted by $R_1 \subseteq R_2$ iff $T_{R_1}(p) \le T_{R_2}(p), C_{R_1}(p) \le C_{R_2}(p), G_{R_1}(p) \ge G_{R_2}(p), U_{R_1}(p) \ge U_{R_2}(p)$ and $F_{R_1}(p) \ge F_{R_2}(p)$ where $p \in P$.

Definition 7: The complement of PNS R_1 is denoted by R_1^C and is defined as:

$R_1^C = \{(F_{R_1}(p), U_{R_1}(p), 1-G_{R_1}(p), C_{R_1}(p), T_{R_1}(p)) \mid p \in P\}$ i.e. $T_{R_1}(p) = F_{R_1}(p), C_{R_1}(p) = U_{R_1}(p),$

$G_{R_1}(p) = 1 - G_{R_1}(p), U_{R_1}(p) = C_{R_1}(p)$ and $F_{R_1}(p) = T_{R_1}(p), p \in P$

Definition 8: The union and intersection of any two PNSs R_1 and R_2 is denoted by $R_1 \cup R_2$ and $R_1 \cap R_2$ is defined as:

$$\begin{aligned} R_1 \cup R_2 &= \{(\max(T_{R_1}(p), T_{R_2}(p)), \max(C_{R_1}(p), C_{R_2}(p)), \min(G_{R_1}(p), G_{R_2}(p)), \min(U_{R_1}(p), U_{R_2}(p)), \min(F_{R_1}(p), F_{R_2}(p)) \mid p \in P\} \\ &= \{(T_{R_1}(p), C_{R_1}(p), G_{R_1}(p), U_{R_1}(p), F_{R_1}(p)) \vee (T_{R_2}(p), C_{R_2}(p), G_{R_2}(p), U_{R_2}(p), F_{R_2}(p)) \mid p \in P\} \end{aligned}$$

$$\begin{aligned} R_1 \cap R_2 &= \{(\min(T_{R_1}(p), T_{R_2}(p)), \min(C_{R_1}(p), C_{R_2}(p)), \max(G_{R_1}(p), G_{R_2}(p)), \max(U_{R_1}(p), U_{R_2}(p)), \max(F_{R_1}(p), F_{R_2}(p)) \mid p \in P\} \\ &= \{(T_{R_1}(p), C_{R_1}(p), G_{R_1}(p), U_{R_1}(p), F_{R_1}(p)) \wedge (T_{R_2}(p), C_{R_2}(p), G_{R_2}(p), U_{R_2}(p), F_{R_2}(p)) \mid p \in P\} \end{aligned}$$

Example: Consider any two PNSs defined over P, presented as:

$$E = \langle 0.6, 0.4, 0.3, 0.2, 0.3 \rangle / r_1 + \langle 0.5, 0.3, 0.4, 0.5, 0.4 \rangle / r_2 + \langle 0.3, 0.7, 0.5, 0.2, 0.4 \rangle / r_3$$
$$F = \langle 0.7, 0.2, 0.4, 0.3, 0.5 \rangle / r_1 + \langle 0.7, 0.4, 0.3, 0.4, 0.5 \rangle / r_2 + \langle 0.6, 0.5, 0.6, 0.4, 0.3 \rangle / r_3$$

Then we have,

$$E^C = \langle 0.3, 0.2, 0.7, 0.4, 0.6 \rangle / r_1 + \langle 0.4, 0.5, 0.6, 0.3, 0.5 \rangle / r_2 + \langle 0.4, 0.2, 0.5, 0.7, 0.3 \rangle / r_3$$
$$E \cup F = \langle 0.7, 0.4, 0.4, 0.3, 0.5 \rangle / r_1 + \langle 0.7, 0.4, 0.4, 0.5, 0.5 \rangle / r_2 + \langle 0.6, 0.7, 0.6, 0.4, 0.4 \rangle / r_3$$
$$E \cap F = \langle 0.6, 0.2, 0.3, 0.2, 0.3 \rangle / r_1 + \langle 0.5, 0.3, 0.3, 0.4, 0.4 \rangle / r_2 + \langle 0.3, 0.5, 0.5, 0.2, 0.3 \rangle / r_3$$

Proposition 1: PNSs satisfy the following properties under the aforementioned set theoretic operations:

i. Commutative law

$(a) R_1 \cup R_2 = R_2 \cup R_1$
$(b) R_1 \cap R_2 = R_2 \cap R_1$

ii. Associative law

$(c) R_1 \cup (R_2 \cup R_3) = (R_1 \cup R_2) \cup R_3$
$(d) R_1 \cap (R_2 \cap R_3) = (R_1 \cap R_2) \cap R_3$

iii. Distributive law

$(e) R_1 \cup (R_2 \cap R_3) = (R_1 \cup R_2) \cap (R_1 \cup R_3)$
$(f) R_1 \cap (R_2 \cup R_3) = (R_1 \cap R_2) \cup (R_1 \cap R_3)$

iv. Absorption law

$(g) R_1 \cup (R_1 \cap R_2) = R_1$
$(h) R_1 \cap (R_1 \cup R_2) = R_1$

v. Involution law

$(i)(R_1^C)^C = R_1$

vi. Law of contradiction

$(j) R_1 \cap R_1^C = \theta$

vii. De Morgan's law

$(k)(R_1 \cup R_2)^C = R_1^C \cap R_2^C$
$(l)(R_1 \cap R_2)^C = R_1^C \cup R_2^C$

Proof:

(a) $R_1 \cup R_2 = R_2 \cup R_1$

We know that,

$$R_1 \cup R_2 = \{(\max(T_{R_1}(p), T_{R_2}(p)), \max(C_{R_1}(p), C_{R_2}(p)), \min(G_{R_1}(p), G_{R_2}(p)), \min(U_{R_1}(p), U_{R_2}(p)), \min(F_{R_1}(p), F_{R_2}(p)) \mid p \in P\}$$
$$= \{(T_{R_1}(p), C_{R_1}(p), G_{R_1}(p), U_{R_1}(p), F_{R_1}(p)) \mathrm{V} (T_{R_2}(p), C_{R_2}(p), G_{R_2}(p), U_{R_2}(p), F_{R_2}(p)) \mid p \in P\}$$

$Let, x_i \in R_1 \cup R_2$
$\Rightarrow x_i \in \{\max(T_{R_1}, T_{R_2}), \max(C_{R_1}, C_{R_2}), \min(G_{R_1}, G_{R_2}), \min(U_{R_1}, U_{R_2}), \min(F_{R_1}, F_{R_2})\}$
$\Rightarrow x_i \in \{\max(T_{R_2}, T_{R_1}), \max(C_{R_2}, C_{R_1}), \min(G_{R_2}, G_{R_1}), \min(U_{R_2}, U_{R_1}), \min(F_{R_2}, F_{R_1})\}$
$\Rightarrow x_i \in R_2 \cup R_1$

$$\Rightarrow R_1 \cup R_2 \subset R_2 \cup R_1 \qquad (1)$$

$Let, y_i \in R_2 \cup R_1$
$\Rightarrow y_i \in \{\max(T_{R_2}, T_{R_1}), \max(C_{R_2}, C_{R_1}), \min(G_{R_2}, G_{R_1}), \min(U_{R_2}, U_{R_1}), \min(F_{R_2}, F_{R_1})\}$
$\Rightarrow y_i \in \{(\max(T_{R_1}, T_{R_2}), \max(C_{R_1}, C_{R_2}), \min(G_{R_1}, G_{R_2}), \min(U_{R_1}, U_{R_2}), \min(F_{R_1}, F_{R_2})\}$
$\Rightarrow y_i \in R_1 \cup R_2$

$$\Rightarrow R_2 \cup R_1 \subset R_1 \cup R_2 \qquad (2)$$

Therefore, from (1) and (2) we obtain,

$R_1 \cup R_2 = R_2 \cup R_1$

(b) Similarly, we can prove that

$R_1 \cap R_2 = R_2 \cap R_1$

(c) $R_1 \cup (R_2 \cup R_3) = (R_1 \cup R_2) \cup R_3$

$Assum\,that, x_i \in R_1 \cup (R_2 \cup R_3)$

$\Rightarrow x_i \in R_1 \cup \langle \max(T_{R_2}, T_{R_3}), \max(C_{R_2}, C_{R_3}), \min(G_{R_2}, G_{R_3}), \min(U_{R_2}, U_{R_3}), \min(F_{R_2}, F_{R_3}) \rangle$

$\Rightarrow x_i \in \langle \max(T_{R_1}, T_{R_2}, T_{R_3}), \max(C_{R_1}, C_{R_2}, C_{R_3}), \min(G_{R_1}, G_{R_2}, G_{R_3}), \min(U_{R_1}, U_{R_2}, U_{R_3}), \min(F_{R_2}, F_{R_2}, F_{R_3}) \rangle$

$\Rightarrow x_i \in \langle \max(T_{R_1}, T_{R_2}), \max(C_{R_1}, C_{R_2}), \min(G_{R_1}, G_{R_2}), \min(U_{R_1}, U_{R_2}), \min(F_{R_1}, F_{R_2}) \rangle \cup R_3$

$\Rightarrow x_i \in (R_1 \cup R_2) \cup R_3$

$R_1 \cup (R_2 \cup R_3) \subset (R_1 \cup R_2) \cup R_3 \qquad (3)$

$Assum\,that, y_i \in (R_1 \cup R_2) \cup R_3$

$\Rightarrow y_i \in \langle \max(T_{R_1}, T_{R_2}), \max(C_{R_1}, C_{R_2}), \min(G_{R_1}, G_{R_2}), \min(U_{R_1}, U_{R_2}), \min(F_{R_1}, F_{R_2}) \rangle \cup R$

$\Rightarrow y_i \in \langle \max(T_{R_1}, T_{R_2}, T_{R_3}), \max(C_{R_1}, C_{R_2}, C_{R_3}), \min(G_{R_1}, G_{R_2}, G_{R_3}), \min(U_{R_1}, U_{R_2}, U_{R_3}), \min(F_{R_1}, F_{R_2}, F_{R_3}) \rangle$

$\Rightarrow y_i \in R_1 \cup \langle \max(T_{R_2}, T_{R_3}), \max(C_{R_2}, C_{R_3}), \min(G_{R_2}, G_{R_3}), \min(U_{R_2}, U_{R_3}), \min(F_{R_2}, F_{R_3}) \rangle$

$\Rightarrow y_i \in R_1 \cup (R_2 \cup R_3)$

$(R_1 \cup R_2) \cup R_3 \subset R_1 \cup (R_2 \cup R_3) \qquad (4)$

From (3) and (4) we conclude that,

$R_1 \cup (R_2 \cup R_3) = (R_1 \cup R_2) \cup R_3$

(d) Similarly, we can prove that

$R_1 \cap (R_2 \cap R_3) = (R_1 \cap R_2) \cap R_3$

(e) $R_1 \cup (R_2 \cap R_3) = (R_1 \cup R_2) \cap (R_1 \cup R_3)$

$Assume\,that, x_i \in R_1 \cup (R_2 \cap R_3)$

$\Rightarrow x_i \in R_1 \cup \langle \min(T_{R_2}, T_{R_3}), \min(C_{R_2}, C_{R_3}), \max(G_{R_2}, G_{R_3}), \max(U_{R_2}, U_{R_3}) \max(F_{R_2}, F_{R_3}) \rangle$

$$\Rightarrow x_i \in \left\langle \begin{array}{l} \max(T_{R_1}, \min(T_{R_2}, T_{R_3})), \max(C_{R_1}, \min(C_{R_2}, C_{R_3})), \min(G_{R_1}, \max(G_{R_2}, G_{R_3})), \\ \min(U_{R_1}, \max(U_{R_2}, U_{R_3})), \min(F_{R_1}, \max(F_{R_2}, F_{R_3})) \end{array} \right\rangle$$

$\Rightarrow x_i \in \langle \max(T_{R_1}, T_{R}), \max(C_{R_1}, C_{R_2}), \min(G_{R_1}, G_{R_2}), \min(U_{R_1}, U_{R_2}), \min(F_{R_1}, F_{R_2}) \rangle$

$\cap \langle \max(T_{R_2}, T_{R_3}), \max(C_{R_2}, C_{R_3}), \min(G_{R_2}, G_{R_3}), \min(U_{R_2}, U_{R_3}), \min(F_{R_2}, F_{R_3}) \rangle$

$\Rightarrow x_i \in (R_1 \cup R_2) \cap (R_1 \cup R_3) \qquad (5)$

$Assume\,that, y_i \in (R_1 \cup R_2) \cap (R_1 \cup R_3)$

$\Rightarrow y_i \in \langle \max(T_{R_1}, T_{R}), \max(C_{R_1}, C_{R_2}), \min(G_{R_1}, G_{R_2}), \min(U_{R_1}, U_{R_2}), \min(F_{R_1}, F_{R_2}) \rangle$

$\cap \langle \max(T_{R_2}, T_{R_3}), \max(C_{R_2}, C_{R_3}), \min(G_{R_2}, G_{R_3}), \min(U_{R_2}, U_{R_3}), \min(F_{R_2}, F_{R_3}) \rangle$

$$\Rightarrow y_i \in \left\langle \begin{array}{l} \max(T_{R_1}, \min(T_{R_2}, T_{R_3})), \max(C_{R_1}, \min(C_{R_2}, C_{R_3})), \min(G_{R_1}, \max(G_{R_2}, G_{R_3})), \\ \min(U_{R_1}, \max(U_{R_2}, U_{R_3})), \min(F_{R_1}, \max(F_{R_2}, F_{R_3})) \end{array} \right\rangle$$

$\Rightarrow y_i \in < T_{R_1}, C_{R_1}, G_{R_1}, U_{R_1}, F_{R_1} > \cup \langle \min(T_{R_2}, T_{R_3}), \min(C_{R_2}, C_{R_3}), \max(G_{R_2}, G_{R_3}), \max(U_{R_2}, U_{R_3}) \max(F_{R_2}, F_{R_3}) \rangle$

$\Rightarrow y_i \in R_1 \cup \langle \min(T_{R_2}, T_{R_3}), \min(C_{R_2}, C_{R_3}), \max(G_{R_2}, G_{R_3}), \max(U_{R_2}, U_{R_3}) \max(F_{R_2}, F_{R_3}) \rangle$

$\Rightarrow y_i \in R_1 \cup (R_2 \cap R_3) \qquad (6)$

From (5) and (6), we conclude that

$R_1 \cup (R_2 \cap R_3) = (R_1 \cup R_2) \cap (R_1 \cup R_3)$

(g) $R_1 \cup (R_1 \cap R_2) = R_1$

$$Assume\,that, x_i \in R_1 \cup (R_1 \cap R_2)$$
$$=\succ x_i \in R_1 \cup \left\langle \min(T_{R_1}, T_{R_2}), \min(C_{R_1}, C_{R_2}), \max(G_{R_1}, G_{R_2}), \max(U_{R_1}, U_{R_2}) \max(F_{R_1}, F_{R_2}) \right\rangle$$
$$=\succ x_i \in \left\langle \begin{matrix} \max(T_{R_1}, \min(T_{R_1}, T_{R_2})), \max(C_{R_1}, \min(C_{R_1}, C_{R_2})), \min(G_{R_1}, \max(G_{R_1}, G_{R_2})), \\ \min(U_{R_1}, \max(U_{R_1}, U_{R_2})), \min(F_{R_1}, \max(F_{R_1}, F_{R_2})) \end{matrix} \right\rangle$$
$$=\succ x_i \in < T_{R_1}, C_{R_1}, G_{R_1}, U_{R_1}, F_{R_1} >$$
$$=\succ x_i \in R_1$$

$$\Rightarrow R_1 \cup (R_1 \cap R_2) \subset R_1 \qquad (7)$$
$$Assume\,that, x_i \in R_1$$
$$=\succ x_i \in < T_{R_1}, C_{R_1}, G_{R_1}, U_{R_1}, F_{R_1} >$$
$$=\succ x_i \in \left\langle \max(T_{R_1}, \min(T_{R_1}, T_{R_2})), \max(C_{R_1}, \min(C_{R_1}, C_{R_2})), \min(G_{R_1}, \max(G_{R_1}, G_{R_2})), \min(U_{R_1}, \max(U_{R_1}, U_{R_2})), \min(F_{R_1}, \max(F_{R_1}, F_{R_2})) \right\rangle$$
$$=\succ x_i \in R_1 \cup \left\langle \min(T_{R_1}, T_{R_2}), \min(C_{R_1}, C_{R_2}), \max(G_{R_1}, G_{R_2}), \max(U_{R_1}, U_{R_2}) \max(F_{R_1}, F_{R_2}) \right\rangle$$
$$=\succ x_i \in R_1 \cup (R_1 \cap R_2)$$
$$=\succ R_1 \subset R_1 \cup (R_1 \cap R_2) \qquad (8)$$

From (6) and (8), we conclude that

$$R_1 \cup (R_1 \cap R_2) = R_1$$

(h) Similarly, we can prove that

$$R_1 \cap (R_1 \cup R_2) = R_1$$

(i) $(R_1^{\,C})^C = R_1$

$$Assum\ that, x_i \in (R_1^{\,C})^C$$
$$=\succ x_i \in \left(F_{R_1}, U_{R_1}, 1 - G_{R_1}, C_{R_1}, T_{R_1}\right)^C$$
$$=\succ x_i \in (T_{R_1}, C_{R_1}, G_{R_1}, U_{R_1}, F_{R_1})$$
$$=\succ x_i \in R_1$$
$$=\succ (R_1^{\,C})^C \subset R_1 \qquad (9)$$

$$Assum\ that, y_i \in R_1$$
$$=\succ y_i \in (T_{R_1}, C_{R_1}, G_{R_1}, U_{R_1}, F_{R_1})$$
$$=\succ y_i \in \left(F_{R_1}, U_{R_1}, 1 - G_{R_1}, C_{R_1}, T_{R_1}\right)^C$$
$$=\succ y_i \in (R_1^{\,C})^C$$
$$=\succ R_1 \subset (R_1^{\,C})^C \qquad (10)$$

From (9) and (10), we obtain

$$(R_1^{\,C})^C = R_1$$

(j) $R_1 \cap R_1^{\,C} = \theta$

$$Assum\ that, x_i \in R_1 \cap R_1^C$$
$$\Rightarrow x_i \in (T_{R_1}, C_{R_1}, G_{R_1}, U_{R_1}, F_{R_1}) \cap (F_{R_1}, U_{R_1}, 1-G_{R_1}, C_{R_1}, T_{R_1})$$
$$\Rightarrow x_i \in < \min(T_{R_1}, F_{R_1}), \min(C_{R_1}, U_{R_1}), \max(G_{R_1}, 1-G_{R_1}), \max\left(U_{R_1}, C_{R_1}\right), \max(F_{R_1}, T_{R_1}) >$$
$$\Rightarrow x_i \in \theta$$
$$\Rightarrow R_1 \cap R_1^C \subset \theta \qquad (11)$$

$$Assum\ that, y_i \in \theta$$
$$\Rightarrow y_i \in < \min(T_{R_1}, F_{R_1}), \min(C_{R_1}, U_{R_1}), \max(G_{R_1}, 1-G_{R_1}), \max\left(U_{R_1}, C_{R_1}\right), \max(F_{R_1}, T_{R_1}) >$$
$$\Rightarrow y_i \in (T_{R_1}, C_{R_1}, G_{R_1}, U_{R_1}, F_{R_1}) \cap (F_{R_1}, U_{R_1}, 1-G_{R_1}, C_{R_1}, T_{R_1})$$
$$\Rightarrow y_i \in R_1 \cap R_1^C$$
$$\Rightarrow \theta \subset R_1 \cap R_1^C \qquad (12)$$

From (11) and (12), we obtain

$$R_1 \cap R_1^C = \theta$$

(k) $(R_1 \cup R_2)^C = R_1^C \cap R_2^C$

$$Assum\ that, x_i \in (R_1 \cup R_2)^C$$
$$\Rightarrow x_i \in (\max(T_{R_1}, T_{R_2}), \max(C_{R_1}, C_{R_2}), \min(G_{R_1}, G_{R_2}), \min(U_{R_1}, U_{R_2}), \min(F_{R_1}, F_{R_2}))^C$$
$$\Rightarrow x_i \in \left\langle \min(F_{R_1}, F_{R_2}), \min(U_{R_1}, U_{R_2}), 1-\min(G_{R_1}, G_{R_2}), \max(C_{R_1}, C_{R_2}), \max(T_{R_1}, T_{R_2}) \right\rangle$$
$$\Rightarrow x_i \in \left(F_{R_1}, U_{R_1}, 1-G_{R_1}, C_{R_1}, T_{R_1}\right) \cap \left(F_{R_2}, U_{R_2}, 1-G_{R_2}, C_{R_2}, T_{R_2}\right)$$
$$\Rightarrow x_i \in R_1^C \cap R_2^C$$
$$\Rightarrow (R_1 \cup R_2)^C \subset R_1^C \cap R_2^C \qquad (13)$$

$$Again,\ Assum\ that, y_i \in R_1^C \cap R_2^C$$
$$\Rightarrow y_i \in \left(F_{R_1}, U_{R_1}, 1-G_{R_1}, C_{R_1}, T_{R_1}\right) \cap \left(F_{R_2}, U_{R_2}, 1-G_{R_2}, C_{R_2}, T_{R_2}\right)$$
$$\Rightarrow y_i \in \left\langle \min(F_{R_1}, F_{R_2}), \min(U_{R_1}, U_{R_2}), 1-\min(G_{R_1}, G_{R_2}), \max(C_{R_1}, C_{R_2}), \max(T_{R_1}, T_{R_2}) \right\rangle$$
$$\Rightarrow y_i \in (\max(T_{R_1}, T_{R_2}), \max(C_{R_1}, C_{R_2}), \min(G_{R_1}, G_{R_2}), \min(U_{R_1}, U_{R_2}), \min(F_{R_1}, F_{R_2}))^C$$
$$\Rightarrow y_i \in (R_1 \cup R_2)^C$$
$$\Rightarrow R_1^C \cap R_2^C \subset (R_1 \cup R_2)^C \qquad (14)$$

From (13) and (14), we conclude that

$$(R_1 \cup R_2)^C = R_1^C \cap R_2^C$$

(l) $(R_1 \cap R_2)^C = R_1^C \cup R_2^C$

$$Assum\ that, x_i \in (R_1 \cap R_2)^C$$
$$\Rightarrow x_i \in (\min(T_{R_1}, T_{R_2}), \min(C_{R_1}, C_{R_2}), \max(G_{R_1}, G_{R_2}), \max(U_{R_1}, U_{R_2}), \max(F_{R_1}, F_{R_2}))^C$$
$$\Rightarrow x_i \in \left\langle \max(F_{R_1}, F_{R_2}), \max(U_{R_1}, U_{R_2}), 1-\max(G_{R_1}, G_{R_2}), \min(C_{R_1}, C_{R_2}), \min(T_{R_1}, T_{R_2}) \right\rangle$$
$$\Rightarrow x_i \in \left(F_{R_1}, U_{R_1}, 1-G_{R_1}, C_{R_1}, T_{R_1}\right) \cup \left(F_{R_1}, U_{R_1}, 1-G_{R_1}, C_{R_1}, T_{R_1}\right)$$
$$\Rightarrow x_i \in R_1^C \cup R_2^C$$

$$\Rightarrow (R_1 \cap R_2)^C \subset R_1^C \cup R_2^C \qquad (15)$$

$$Again,\ Assum\,that,\ y_i \in R_1^C \cup R_2^C$$
$$\Rightarrow y_i \in \left\langle F_{R_1}, U_{R_1}, 1-G_{R_1}, C_{R_1}, T_{R_1} \right\rangle \cup \left\langle F_{R_2}, U_{R_2}), 1-G_{R_2}, C_{R_2}, T_{R_2} \right\rangle$$
$$\Rightarrow y_i \in \left\langle \max(F_{R_1}, F_{R_2}), \max(U_{R_1}, U_{R_2}), 1-\max(G_{R_1}, G_{R_2}), \min(C_{R_1}, C_{R_2}), \min(T_{R_1}, T_{R_2}) \right\rangle$$
$$\Rightarrow y_i \in (\min(T_{R_1}, T_{R_2}), \min(C_{R_1}, C_{R_2}), \max(G_{R_1}, G_{R_2}), \max(U_{R_1}, U_{R_2}), \max(F_{R_1}, F_{R_2}))^C$$
$$\Rightarrow y_i \in (R_1 \cap R_2)^C$$
$$\Rightarrow R_1^C \cup R_2^C \subset (R_1 \cap R_2)^C \qquad (16)$$

From (15) and (16) we conclude that,

$$(R_1 \cap R_2)^C = R_1^C \cup R_2^C$$

4. Conclusion:

In this article we have develop pentapartitioned neutrosophic set. The pentapartitioned neutrosophic set is extension of SVNS and QSVNS. The concept of complement law, inclusion law, union law, intersection law, commutative law, etc. have been defined on pentapartitioned neutrosophic sets. Future works may comprise of the study of different types of operators on pentapartitioned neutrosophic sets dealing with actual problems and implementing them in decision-making problems [8-13].

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Soft Subring Theory Under Interval-valued Neutrosophic Environment

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Abstract. The primary goal of this article is to establish and investigate the idea of interval-valued neutrosophic soft subring. Again, we have introduced function under interval-valued neutrosophic soft environment and investigated some of its homomorphic attributes. Additionally, we have established product of two interval-valued neutrosophic soft subrings and analyzed some of its fundamental attributes. Furthermore, we have presented the notion of interval-valued neutrosophic normal soft subring and investigated some of its algebraic properties and homomorphic attributes.

Keywords: Neutrosophic set; Interval-valued neutrosophic soft set; Interval-valued neutrosophic soft subring; Interval-valued neutrosophic normal soft subring

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ABBREVIATIONS

TN indicates "T-norm".

SN indicates "S-norm".

IVTN indicates "Interval-valued T-norm".

IVSN indicates "Interval-valued S-norm".

CS indicates "Crisp set".

US indicates "Universal set".

FS indicates "Fuzzy set".

IFS indicates "Intuitionistic fuzzy set".

NS indicates "Neutrosophic set".

PS indicates "Plithogenic set".

SS indicates "Soft set".

IVFS indicates "Interval-valued fuzzy set".

IVIFS indicates "Interval-valued intuitionistic fuzzy set".

IVNS indicates "Interval-valued neutrosophic set".

NSSR indicates "Neutrosophic soft subring".

NNSSR indicates "Neutrosophic normal soft subring".

IVNSR indicates "Interval-valued neutrosophic subring".

IVNSSR indicates "Interval-valued neutrosophic soft subring".

IVNNSSR indicates "Interval-valued neutrosophic normal soft subring".

DMP indicates "Decision making problem".

$\boldsymbol{\phi(F)}$ indicates "Power set of F".

$\boldsymbol{K}$ indicates "The set $[0,1]$".

1. Introduction

Uncertainty plays a huge part in different economical, sociological, biological, as well as other scientific fields. It is not always possible to tackle ambiguous data using CS theory. To cope with its limitations Zadeh introduced the groundbreaking concept of FS [1] theory. Which was further generalized by Atanassov as IFS [2] theory. Later on, Smarandache extended these notions by introducing NS [3] theory, which became more reasonable for managing indeterminate situations. From the beginning, NS theory became very popular among various researchers. Nowadays, it is heavily utilized in numerous research domains. PS [4] theory is another innovative concept introduced by Smarandache, which is more general than all the previously mentioned notions. In NS and PS theory some of Smarandache's remarkable contributions are the notions of neutrosophic robotics [5], neutrosophic psychology [6], neutrosophic measure [7], neutrosophic calculus [8], neutrosophic statistics [9], neutrosophic probability [10], neutrosophic triplet group [11], plithogenic logic, probability [12], plithogenic subgroup [13], plithogenic aggregation operators [14], plithogenic hypersoft set [15], plithogenic fuzzy whole hypersoft set [16], plithogenic hypersoft subgroup [17], etc. Moreover, NS and PS theory has several contributions in various other scientific fields, for instance, in selection of suppliers [18], professional selection [19], fog and mobile-edge computing [20], fractional programming [21], linear programming [22], shortest path problem [23–30], supply chain problem [31], DMP [32–37], healthcare [38, 39], etc.

Interval-valued versions of FS [40], IFS [41], and NS [42] are further generalizations of their previously discussed counterparts. Since the beginning, various researchers have carried out this concepts and explored them in different research domains. For instance, nowadays in logic [42], abstract algebra [43–46], graph theory [47, 48], DMPs [49–51], etc., these concepts are widely used.

Another set theory of utmost importance is SS [52] theory. It was introduced by Molodtsov to deal with uncertainty more conveniently and easily. At present, it is extensively used in different scientific areas, like in DMPs [53–57], abstract algebra [58–61], stock treading [62], etc. Furthermore, to achieve higher uncertainty handling potentials researchers have implemented SS theory in different interval-valued environments. The following Table 1 comprises some momentous aspects of different interval-valued soft notions.

TABLE 1. Significance of different interval-valued soft notions in various fields.

Author & references	**Year**	**Contributions in various fields**
Yang et al. [63]	2009	Introduced soft IVFS and defined complement, "and" and "or" operations on them.
Jiang et al. [64]	2010	Proposed soft IVIFS and defined complement, "and", "or", union, intersection, necessity, and possibility operations on them.
Feng et al. [65]	2010	Introduced soft reduct fuzzy sets of soft IVFS and utilizing soft versions of reduct fuzzy sets and level sets, proposed flexible strategy for DMP.
Broumi et al. [66]	2014	Presented generalized soft IVNS, analyzed some set operations and further, applied it in DMP.
Mukherje et al. [67]	2014	Proposed relation on soft IVIFSs and presented a solution to a DMP.
Broumi et al. [68]	2014	Proposed relation on soft IVNSs and studied reflexivity, symmetry, transitivity of it.
Mukherje and Sarkar [69]	2015	Defined Euclidean and Hamming distances between two soft IVNSs and presented similarity measures according to distances within them.
Deli [70]	2017	Defined soft IVNS and introduced some operations. Further, implemented this in DMP.
Garg and Arora [71]	2018	Solved DMP with soft IVIFS information.

Group theory and ring theory are essential parts of abstract algebra, which have various applications in different research domains. But these were initially introduced under the crisp environment, which has certain limitations. From the year 1971, various mathematicians started implementing uncertainty theories to generalize these notions. Some noteworthy contributions in the field of group theory under uncertainty can be found on [72–76]. In ring theory under uncertainty, the following articles [77–80] are some important developments. Again, several researchers introduced these notions under soft environments. For instance, researchers have introduced the concepts of ring theory under soft fuzzy [81], soft intuitionistic fuzzy [82], and soft neutrosophic [83] environments. Also, some more articles which can be helpful to different researchers are [84–91], etc. Now, by mixing interval-valued environment with soft neutrosophic environment, we can introduce a more general version of NSSR, which will be called IVNSSR. Also, their homomorphic attributes can be studied. Again, their product and normal versions can be introduced and studied. Based on these perceptions, the followings are our primary objectives for this article:

- Introducing the concept of IVNSSR and a analyzing its homomorphic attributes.
- Introducing the product of IVNSSRs.
- Introducing subring of a IVNSSR.
- Introducing the concept of IVNNSSR and a analyzing its homomorphic properties.

The arrangement our article is: in Section 2, some desk researches of IVTN, IVSN, NS, IVNS, IVNSS, NSR, NSSR, etc., are discussed. In Section 3, the concept of IVNSSR has been introduced and some fundamental theories are provided. Also, their product and normal versions are defined and some theories are given to understand their different algebraic characteristics. Lastly, in Section 4, mentioning some future scopes, the concluding segment is given.

2. Literature Review

Definition 2.1. [92] A function $T : K \to K$ is known as a TN iff $\forall g, n, z \in K$, the followings can be concluded

(i) $T(g, 1) = g$
(ii) $T(g, n) = T(n, g)$
(iii) $T(g, n) \leq T(z, n)$ if $g \leq z$
(iv) $T(g, T(n, z)) = T(T(g, n), z)$

Definition 2.2. [93] A function $\bar{T} : \phi(K) \times \phi(K) \to \phi(K)$ defined as $\bar{T}(\bar{g}, \bar{n}) = [T(g^-, n^-), T(g^+, n^+)]$ (T is a TN) is known as an IVTN.

Definition 2.3. [92] A function $S : K \to K$ is known as SN iff $\forall g, n, z \in K$, the followings can be concluded

(i) $S(g, 0) = g$
(ii) $S(g, n) = S(n, g)$
(iii) $S(g, n) \leq S(z, n)$ if $g \leq z$
(iv) $S(g, S(n, z)) = S(S(g, n), z)$

Definition 2.4. [93] The function $\bar{S} : \phi(K) \times \phi(K) \to \phi(K)$ defined as $\bar{S}(\bar{g}, \bar{n}) = [S(g^-, n^-), S(g^+, n^+)]$ (S is a SN) is called an IVSN.

Definition 2.5. [3] A NS σ of a CS Q is denoted as $\sigma = \big\{\big(g, t_\sigma(g), i_\sigma(g), f_\sigma(g)\big) : g \in Q\big\}$. Here $\forall g \in Q$, $t_\sigma(g), i_\sigma(g),$ and $f_\sigma(g)$ are known as degree of truth, indeterminacy, and falsity which satisfy the inequality $^{-}0 \leq t_\sigma(g) + i_\sigma(g) + f_\sigma(g) \leq 3^{+}$.

The set of all NSs of Q will be expressed as NS(Q).

Definition 2.6. [52] Let Q be a US and A be a set of parameters. Also, let $L \subseteq A$. Then the ordered pair (f, L) is called a SS over Q, where $f : L \to \phi(Q)$ is a function.

Definition 2.7. [94] Let Q be a US and A be a set of parameters. Also, let $M \subseteq A$. Then a NSS over Q is denoted as (f, M) where $f : M \to \text{NS}(Q)$ is a function.

The following Definition 2.7 is a redefined version of NSS, which we have adopted in this article.

Definition 2.8. [56] Let Q be a US and A be a set of parameters. Then a NSS δ of Q is denoted as $\delta = \big\{\big(r, l_\delta(r)\big) : r \in A\big\}$ where $l_\delta : A \to \text{NS}(Q)$ is a function which is also known as an approximate function of NSS δ and $l_\delta(r) = \big\{\big(g, t_{l_{\delta(r)}}(g), i_{l_{\delta(r)}}(g), f_{l_{\delta(r)}}(g)\big) : g \in Q\big\}$. Here, $\forall g \in Q$, $t_{l_{\delta(r)}}(g)$, $i_{l_{\delta(r)}}(g)$, and $f_{l_{\delta(r)}}(g) \in [0,1]$ and they satisfy the inequality $3 \geq t_{l_{\delta(r)}}(g) + i_{l_{\delta(r)}}(g) + f_{l_{\delta(r)}}(g) \geq 0$.

The set of all NSSs of a set Q will be expressed as NSS(Q).

Definition 2.9. [42] An IVNS of Q is defined as the mapping $\bar{\sigma} : Q \to \phi(K) \times \phi(K) \times \phi(K)$, where $\bar{\sigma}(g) = \big\{\big(g, \bar{t}_{\bar{\sigma}}(g), \bar{i}_{\bar{\sigma}}(g), \bar{f}_{\bar{\sigma}}(g)\big) : g \in Q\big\}$, where $\forall g \in Q$, $\bar{t}_{\bar{\sigma}}(g)$, $\bar{i}_{\bar{\sigma}}(g)$, and $\bar{f}_{\bar{\sigma}}(g) \subseteq [0,1]$.

The set of all IVNSs of a set Q will be expressed as IVNS(Q).

Definition 2.10. [70] Let Q be a US and A be a set of parameters. Then a IVNSS Ψ of Q is denoted as $\Psi = \big\{\big(r, l_\Psi(r)\big) : r \in A\big\}$, where $l_\Psi : A \to \text{IVNS}(Q)$ is a function which is also known as an approximate function of IVNSS Ψ and $l_\Psi(r) = \big\{\big(g, \bar{t}_{l_{\Psi(r)}}(g), \bar{i}_{l_{\Psi(r)}}(g), \bar{f}_{l_{\Psi(r)}}(g)\big) : g \in Q\big\}$. Here, $\forall g \in Q$, $\bar{t}_{l_{\Psi(r)}}(g)$, $\bar{i}_{l_{\Psi(r)}}(g)$, and $\bar{f}_{l_{\Psi(r)}}(g) \subseteq [0,1]$.

The set of all IVNSSs of a set Q will be expressed as IVNSS(Q).

Definition 2.11. [70] $\Psi_1 = \big\{\big(r, l_{\Psi_1}(r)\big) : r \in A\big\}$ and $\Psi_2 = \big\{\big(r, l_{\Psi_2}(r)\big) : r \in A\big\}$ be two IVNSSs of Q. Then $\Psi = \Psi_1 \cup \Psi_2 = \big\{\big(r, l_\Psi(r)\big) : r \in A\big\}$ is defined as

$$\bar{t}_{l_{\Psi(r)}} = \big[\max\big\{\bar{t}^{-}_{l_{\Psi_1}(r)}, \bar{t}^{-}_{l_{\Psi_2}(r)}\big\}, \max\big\{\bar{t}^{+}_{l_{\Psi_1}(r)}, \bar{t}^{+}_{l_{\Psi_2}(r)}\big\}\big]$$
$$\bar{t}_{l_{\Psi(r)}} = \big[\min\big\{\bar{i}^{-}_{l_{\Psi_1}(r)}, \bar{i}^{-}_{l_{\Psi_2}(r)}\big\}, \min\big\{\bar{i}^{+}_{l_{\Psi_1}(r)}, \bar{i}^{+}_{l_{\Psi_2}(r)}\big\}\big]$$
$$\bar{t}_{l_{\Psi(r)}} = \big[\min\big\{\bar{f}^{-}_{l_{\Psi_1}(r)}, \bar{f}^{-}_{l_{\Psi_2}(r)}\big\}, \min\big\{\bar{f}^{+}_{l_{\Psi_1}(r)}, \bar{f}^{+}_{l_{\Psi_2}(r)}\big\}\big]$$

Definition 2.12. [70] $\Psi_1 = \{(r, l_{\Psi_1}(r)) : r \in A\}$ and $\Psi_2 = \{(r, l_{\Psi_2}(r)) : r \in A\}$ be two IVNSSs of Q. Then $\Psi = \Psi_1 \cap \Psi_2 = \{(r, l_\Psi(r)) : r \in A\}$ is defined as

$$\bar{t}_{l_\Psi(r)} = \left[\min\{\bar{t}^-_{l_{\Psi_1}(r)}, \bar{t}^-_{l_{\Psi_2}(r)}\}, \min\{\bar{t}^+_{l_{\Psi_1}(r)}, \bar{t}^+_{l_{\Psi_2}(r)}\}\right]$$
$$\bar{t}_{l_\Psi(r)} = \left[\max\{\bar{i}^-_{l_{\Psi_1}(r)}, \bar{i}^-_{l_{\Psi_2}(r)}\}, \max\{\bar{i}^+_{l_{\Psi_1}(r)}, \bar{i}^+_{l_{\Psi_2}(r)}\}\right]$$
$$\bar{t}_{l_\Psi(r)} = \left[\max\{\bar{f}^-_{l_{\Psi_1}(r)}, \bar{f}^-_{l_{\Psi_2}(r)}\}, \max\{\bar{f}^+_{l_{\Psi_1}(r)}, \bar{f}^+_{l_{\Psi_2}(r)}\}\right]$$

2.1. *Neutrosophic subring*

Definition 2.13. [80] Let $(Q, +, \cdot)$ be a crisp ring. A NS $\sigma = \{(g, t_\sigma(g), i_\sigma(g), f_\sigma(g)) : g \in Q\}$ is called a NSR of F, iff $\forall g, n \in Q$,

(i) $t_\sigma(g+n) \geq T(t_\sigma(g), t_\sigma(n)),\ i_\sigma(g+n) \geq I(i_\sigma(g), i_\sigma(n)),\ f_\sigma(g+n) \leq F(f_\sigma(g), f_\sigma(n))$

(ii) $t_\sigma(-g) \geq t_\sigma(g),\ i_\sigma(-g) \geq i_\sigma(g),\ f_\sigma(-g) \leq f_\sigma(g)$

(iii) $t_\sigma(g \cdot n) \geq T(t_\sigma(g), t_\sigma(n)),\ i_\sigma(g \cdot n) \geq I(i_\sigma(g), i_\sigma(n)),\ f_\sigma(g \cdot n) \leq S(f_\sigma(g), f_\sigma(n))$.

Here, T and I are two TNs and S is a SN.

The set of all NSR of a crisp ring $(Q, +, \cdot)$ will be expressed as NSR(Q).

Proposition 2.1. *[80] A NS $\sigma = \{(g, t_\sigma(g), i_\sigma(g), f_\sigma(g)) : g \in Q\}$ is called a NSR of Q, iff $\forall g, n \in Q$,*

(i) $t_\sigma(g-n) \geq T(t_\sigma(g), t_\sigma(n)),\ i_\sigma(g-n) \geq I(i_\sigma(g), i_\sigma(n)),\ f_\sigma(g-n) \leq F(f_\sigma(g), f_\sigma(n))$

(ii) $t_\sigma(g \cdot n) \geq T(t_\sigma(g), t_\sigma(n)),\ i_\sigma(g \cdot n) \geq I(i_\sigma(g), i_\sigma(n)),\ f_\sigma(g \cdot n) \leq S(f_\sigma(g), f_\sigma(n))$.

Here, T and I are two TNs and S is a SN.

Proposition 2.2. *[80] Let $\sigma_1, \sigma_2 \in NSR(Q)$. Then $\sigma_1 \cap \sigma_2 \in NSR(Q)$.*

Theorem 2.3. *[80] Let $(Q, +, \cdot)$ and $(Y, +, \cdot)$ be two crisp rings. Also, let $h : Q \to Y$ be a homomorphism. If σ is a NSR of Q then $h(\sigma)$ is a NSR of Y.*

Theorem 2.4. *[80] Let $(Q, +, \cdot)$ and $(Y, +, \cdot)$ be two crisp rings. Also, let $h : Q \to Y$ be a homomorphism. If σ' is a NSR of Y then $h^{-1}(\sigma')$ is a NSR of Q.*

Definition 2.14. [80] Let $\sigma = \{(g, t_\sigma(g), i_\sigma(g), f_\sigma(g)) : g \in Q\}$ be a NSR of Q. Then $\forall s \in [0, 1]$ the s-level sets of Q are defined as

(i) $(t_\sigma)_s = \{g \in Q : t_\sigma(g) \geq s\}$,

(ii) $(i_\sigma)_s = \{g \in Q : i_\sigma(g) \geq s\}$, and

(iii) $(f_\sigma)^s = \{g \in Q : f_\sigma(g) \leq s\}$.

Proposition 2.5. *[80] A NS $\sigma = \{(g, t_\sigma(g), i_\sigma(g), f_\sigma(g)) : g \in Q\}$ of a crisp ring $(Q, +, \cdot)$ is a NSR of Q iff $\forall s \in [0, 1]$ the s-level sets of Q, i.e. $(t_\sigma)_s$, $(i_\sigma)_s$, and $(f_\sigma)^s$ are crisp rings of Q.*

2.2. *Neutrosophic soft subring*

Definition 2.15. [83] Let $(Q,+,\cdot)$ be a crisp ring and A be a set of parameters. Then a NSS $\delta = \{(r, l_\delta(r)) : r \in A\}$ with $l_\delta : A \to \mathrm{NS}(Q)$ is called a NSSR if $\forall r \in A$, $l_\delta(r) \in \mathrm{NSR}(Q)$.

The set of all NSSR of a crisp ring $(Q,+,\cdot)$ will be expressed as NSSR(Q).

Proposition 2.6. *[83] A NSS* $\delta = \Big\{\Big(r, \big\{\big(g, t_{l_\delta(r)}(g), i_{l_\delta(r)}(g), f_{l_\delta(r)}(g)\big) : g \in Q\big\}\Big) : r \in A\Big\}$ *over a crisp ring* $(Q,+,\cdot)$ *is called a NSSR iff the following conditions hold:*

(i) $t_{l_\delta(r)}(g-n) \geq T\big(t_{l_\delta(r)}(g), t_{l_\delta(r)}(n)\big)$, $i_{l_\delta(r)}(g-n) \geq I\big(i_{l_\delta(r)}(g), i_{l_\delta(r)}(n)\big)$, $f_{l_\delta(r)}(g-n) \leq F\big(f_{l_\delta(r)}(g), f_{l_\delta(r)}(n)\big)$ *and*

(ii) $t_{l_\delta(r)}(g\cdot n) \geq T\big(t_{l_\delta(r)}(g), t_{l_\delta(r)}(n)\big)$, $i_{l_\delta(r)}(g\cdot n) \geq I\big(i_{l_\delta(r)}(g), i_{l_\delta(r)}(n)\big)$, $f_{l_\delta(r)}(g\cdot n) \leq S\big(f_{l_\delta(r)}(g), f_{l_\delta(r)}(n)\big)$.

Proposition 2.7. *[83] Let* $\delta_1, \delta_2 \in NSSR(Q)$*. Then* $\delta_1 \cap \delta_2 \in NSSR(Q)$*.*

Theorem 2.8. *[83] Let* $(Q,+,\cdot)$ *and* $(Y,+,\cdot)$ *be two crisp rings. Also, let* $h : Q \to Y$ *be an isomorphism. If* δ *is a NSSR of* Q *then* $h(\delta)$ *is a NSSR of* Y*.*

Theorem 2.9. *[83] Let* $(Q,+,\cdot)$ *and* $(Y,+,\cdot)$ *be two crisp rings. Also, let* $h : Q \to Y$ *be a homomorphism. If* δ' *is a NSSR of* Y *then* $h^{-1}(\delta')$ *is a NSSR of* Q*.*

Theorem 2.10. $\delta_1 \in NSSR(Q)$ *and* $\delta_2 \in NSSR(Y)$*, then their cartesian product* $\delta_1 \times \delta_2 \in NSSR(Q \times Y)$*.*

Definition 2.16. [83] A NSSR $\delta = \{(r, l_\delta(r)) : r \in A\}$ of a crisp ring $(Q,+,\cdot)$ is known as a NNSSR of Q iff $t_{l_\delta(r)}(g\cdot n) = t_{l_\delta(r)}(n\cdot g)$, $i_{l_\delta(r)}(g\cdot n) = i_{l_\delta(r)}(n\cdot g)$, and $f_{l_\delta(r)}(g\cdot n) = f_{l_\delta(r)}(n\cdot g)$.

The set of all NNSSR of Q will be expressed as NNSSR(Q).

Proposition 2.11. *[83] Let* $\delta_1, \delta_2 \in NNSSR(Q)$*. Then* $\delta_1 \cap \delta_2 \in NNSSR(Q)$*.*

Theorem 2.12. *[83] Let* $(Q,+,\cdot)$ *and* $(Y,+,\cdot)$ *be two crisp rings. Also, let* $h : Q \to Y$ *be an isomorphism. If* δ *is a NNSSR of* Q *then* $h(\delta)$ *is a NNSSR of* Y*.*

Theorem 2.13. *[83] Let* $(Q,+,\cdot)$ *and* $(Y,+,\cdot)$ *be two crisp rings. Also, let* $h : Q \to Y$ *be a ring homomorphism. If* δ' *is a NNSSR of* Y *then* $h^{-1}(\delta')$ *is a NNSSR of* Q*.*

3. **Proposed notion of interval-valued neutrosophic soft subring**

Definition 3.1. Let $(Q,+,\cdot)$ be a crisp ring and A be a set of parameters. An IVNSS $\Psi = \Big\{\Big(r, \big\{\big(g, \bar{t}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(g)\big) : g \in Q\big\}\Big) : r \in A\Big\}$ is called an IVNSSR of $(Q,+,\cdot)$ if $\forall g, n \in Q$, and $\forall r \in A$, the followings can be concluded:

(i) $\begin{cases} \bar{t}_{l_\Psi(r)}(g+n) \geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\big), \\ \bar{i}_{l_\Psi(r)}(g+n) \leq \bar{I}\big(\bar{i}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(n)\big), \\ \bar{f}_{l_\Psi(r)}(g+n) \leq \bar{F}\big(\bar{f}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(n)\big) \end{cases}$

(ii) $\begin{cases} \bar{t}_{l_\Psi(r)}(-r) \geq t_{l_\Psi(r)}(g), \\ \bar{i}_{l_\Psi(r)}(-r) \leq i_{l_\Psi(r)}(g), \\ \bar{f}_{l_\Psi(r)}(-r) \leq f_{l_\Psi(r)}(g) \end{cases}$

(iii) $\begin{cases} \bar{t}_{l_\Psi(r)}(g \cdot n) \geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\big), \\ \bar{i}_{l_\Psi(r)}(g \cdot n) \leq \bar{I}\big(\bar{i}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(n)\big), \\ \bar{f}_{l_\Psi(r)}(g \cdot n) \leq \bar{F}\big(\bar{f}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(n)\big), \end{cases}$

The set of all IVNSSR of a crisp ring $(Q, +, \cdot)$ will be expressed as IVNSSR(Q).

Example 3.2. Let $(\mathbb{Z}, +, \cdot)$ be the ring and $\mathbb{N}$ be a set of parameters. Also, let $\Psi = \Big\{\Big(r, \big\{\big(g, \bar{t}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(g)\big) \ : \ g \in \mathbb{Z}\big\}\Big) \ : \ e \in \mathbb{N}\Big\}$ be an IVNSS of $\mathbb{Z}$, where $l_\Psi : \mathbb{N} \to$ IVNS(Q) and $\forall g \in \mathbb{Z}$, $\forall r \in \mathbb{N}$ corresponding memberships are

$$\bar{t}_{l_\Psi(r)}(g) = \begin{cases} \Big[\dfrac{1}{r+1}, \dfrac{1}{r}\Big] & \text{if } g \in 2\mathbb{Z} \\ [0, 0] & \text{if } g \in 2\mathbb{Z}+1 \end{cases},$$

$$\bar{i}_{l_\Psi(r)}(g) = \begin{cases} [0, 0] & \text{if } g \in 2\mathbb{Z} \\ \Big[\dfrac{1}{2r+2}, \dfrac{1}{2r}\Big] & \text{if } g \in 2\mathbb{Z}+1 \end{cases}, \text{ and}$$

$$\bar{f}_{l_\Psi(r)}(g) = \begin{cases} [0, 0] & \text{if } g \in 2\mathbb{Z} \\ \Big[\dfrac{r-1}{r}, \dfrac{r}{r+1}\Big] & \text{if } g \in 2\mathbb{Z}+1 \end{cases}.$$

Here, considering minimum TN and maximum SNs $\forall r \in \mathbb{N}$, $\Psi \in$ IVNSSR$(\mathbb{Z})$.

Example 3.3. Let $(\mathbb{Z}_4, +, \cdot)$ be the ring of integers modulo 4 and $A = \{r_1, r_2, r_3\}$ be a set of parameters. Also, let $\Psi = \Big\{\Big(r, \big\{\big(r, \bar{t}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(g)\big) : g \in \mathbb{Z}_4\big\}\Big) : r \in A\Big\}$ be an IVNSS of $\mathbb{Z}_4$, where $l_\Psi : A \to$ IVNS(Q). Again, let the membership values of the elements belonging to Ψ are specified in Table 2, 3, and 4.

TABLE 2. Membership values of elements with respect to parameter r_1

$\Psi(r_1)$	$\bar{t}_{l_\Psi(r_1)}$	$\bar{i}_{l_\Psi(r_1)}$	$\bar{f}_{l_\Psi(r_1)}$
$\bar{0}$	$[0.64, 0.66]$	$[0.33, 0.35]$	$[0.13, 0.14]$
$\bar{1}$	$[0.7, 0.72]$	$[0.21, 0.23]$	$[0.77, 0.79]$
$\bar{2}$	$[0.74, 0.76]$	$[0.24, 0.26]$	$[0.51, 0.53]$
$\bar{3}$	$[0.66, 0.68]$	$[0.31, 0.33]$	$[0.28, 0.3]$

TABLE 3. Membership values of elements with respect to parameter r_2

$\Psi(r_2)$	$\bar{t}_{l_\Psi(r_2)}$	$\bar{i}_{l_\Psi(r_2)}$	$\bar{f}_{l_\Psi(r_2)}$
$\bar{0}$	$[0.68, 0.7]$	$[0.3, 0.32]$	$[0.31, 0.33]$
$\bar{1}$	$[0.61, 0.63]$	$[0.31, 0.33]$	$[0.41, 0.43]$
$\bar{2}$	$[0.57, 0.59]$	$[0.4, 0.42]$	$[0.65, 0.67]$
$\bar{3}$	$[0.7, 0.72]$	$[0.26, 0.28]$	$[0.52, 0.54]$

TABLE 4. Membership values of elements with respect to parameter r_3

$\Psi(r_3)$	$\bar{t}_{l_\Psi(r_3)}$	$\bar{i}_{l_\Psi(r_3)}$	$\bar{f}_{l_\Psi(r_3)}$
$\bar{0}$	$[0.71, 0.73]$	$[0.2, 0.23]$	$[0.15, 0.17]$
$\bar{1}$	$[0.83, 0.85]$	$[0.15, 0.17]$	$[0.24, 0.26]$
$\bar{2}$	$[0.68, 0.7]$	$[0.3, 0.32]$	$[0.38, 0.4]$
$\bar{3}$	$[0.78, 0.8]$	$[0.18, 0.2]$	$[0.4, 0.43]$

Here, considering the Łukasiewicz TN ($T(g,n) = \max\{0, g+n-1\}$) and bounded sum SNs ($S(g,n) = \min\{g+n, 1\}$), $\forall r \in A$, $\Psi \in$ IVNSSR($\mathbb{Z}_4$).

Proposition 3.1. *An IVNSS* $\Psi = \Big\{\Big(r, \big\{\big(g, \bar{t}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(g)\big) : g \in Q\big\}\Big) : r \in A\Big\}$ *of a crisp ring* $(Q, +, \cdot)$ *is an IVNSSR iff the following conditions hold (considering idempotent IVTN and IVSNs):*

(i) $\bar{t}_{l_\Psi(r)}(g-n) \geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\big)$, $\bar{i}_{l_\Psi(r)}(g-n) \leq \bar{I}\big(\bar{i}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(n)\big)$, $\bar{f}_{l_\Psi(r)}(g-n) \leq \bar{F}\big(\bar{f}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(n)\big)$ *and*

(ii) $\bar{t}_{l_\Psi(r)}(g \cdot n) \geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\big)$, $\bar{i}_{l_\Psi(r)}(g \cdot n) \leq \bar{I}\big(\bar{i}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(n)\big)$, $\bar{f}_{l_\Psi(r)}(g \cdot n) \leq \bar{F}\big(\bar{f}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(n)\big)$.

Proof. Let $\Psi \in$ IVNSSR(Q). Then

$$\begin{aligned}\bar{t}_{l_\Psi(r)}(g-n) &\geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(-n)\big) \text{ [by Definition 3.1]}\\ &\geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\big) \text{ [by Definition 3.1]}\end{aligned}$$

Similary, we will have

$$\begin{aligned}&\bar{i}_{l_\Psi(r)}(g-n) \leq \bar{I}\big(\bar{i}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(n)\big), \text{ and}\\ &\bar{f}_{l_\Psi(r)}(g-n) \leq \bar{F}\big(\bar{f}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(n)\big),\end{aligned}$$

Again, (ii) follows immediately from condition (iii) of Definition 3.1.

Conversely, let conditions (i) and (ii) of Proposition 3.1 hold. Assuming θ_Q as the additive neutral member of $(Q,+,\cdot)$, we have

$$\begin{aligned}\bar{t}_{l_\Psi(r)}(\theta_Q) &= \bar{t}_{l_\Psi(r)}(g-g)\\ &\geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g),\bar{t}_{l_\Psi(r)}(g)\big)\\ &= \bar{t}_{l_\Psi(r)}(g)\end{aligned} \tag{3.1}$$

Similaly,

$$\bar{i}_{l_\Psi(r)}(\theta_Q) \leq \bar{i}_{l_\Psi(r)}(g) \tag{3.2}$$

$$\bar{f}_{l_\Psi(r)}(\theta_Q) \leq \bar{f}_{l_\Psi(r)}(g) \tag{3.3}$$

Now,

$$\begin{aligned}\bar{t}_{l_\Psi(r)}(-g) &= \bar{t}_{l_\Psi(r)}(\theta_Q-g)\\ &\geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(\theta_Q),\bar{t}_{l_\Psi(r)}(g)\big)\\ &\geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g),\bar{t}_{l_\Psi(r)}(g)\big) \text{ [by 3.1]}\\ &= \bar{t}_{l_\Psi(r)}(g) \text{ [since } \bar{T} \text{ is idempotent]}\end{aligned} \tag{3.4}$$

Similarly,

$$\bar{i}_{l_\Psi(r)}(-g) \leq \bar{i}_{l_\Psi(r)}(g) \text{ [since } \bar{I} \text{ is idempotent]} \tag{3.5}$$

$$\bar{f}_{l_\Psi(r)}(-g) \leq \bar{f}_{l_\Psi(r)}(g) \text{ [since } \bar{F} \text{ is idempotent]} \tag{3.6}$$

Hence,

$$\begin{aligned}\bar{t}_{l_\Psi(r)}(g+n) &= \bar{t}_{l_\Psi(r)}(g-(-n))\\ &\geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g),\bar{t}_{l_\Psi(r)}(-n)\big)\\ &\geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g),\bar{t}_{l_\Psi(r)}(n)\big) \text{ [by 3.4]}\end{aligned} \tag{3.7}$$

Similarly,

$$\bar{i}_{l_\Psi(r)}(g+n) \leq \bar{I}\big(\bar{i}_{l_\Psi(r)}(g),\bar{i}_{l_\Psi(r)}(n)\big) \text{ [by 3.5]} \tag{3.8}$$

$$\bar{f}_{l_\Psi(r)}(g+n) \leq \bar{F}\big(\bar{f}_{l_\Psi(r)}(g),\bar{f}_{l_\Psi(r)}(n)\big) \text{ [by 3.6]} \tag{3.9}$$

Hence, Equations 3.7, 3.8, and 3.9 prove part (i) of Proposition 3.1. Again, part (ii) of Proposition 3.1 is similar to condition (iii) of Definition 3.1. So, $\Psi \in$ IVNSSR(Q). □

Theorem 3.2. *Let $(Q,+,\cdot)$ be a crisp ring. If $\Psi_1,\Psi_2 \in$ IVNSSR(Q), then $\Psi_1 \cap \Psi_2 \in$ IVNSSR(Q) (considering idempotent IVTN and IVSNs).*

Proof. Let $\Psi = \Psi_1 \cap \Psi_2$. Now, $\forall g, n \in Q$ and $\forall r \in A$

$$\begin{aligned}
\bar{t}_{l_{\Psi(r)}}(g+n) &= \bar{T}\big(\bar{t}_{l_{\Psi_1(r)}}(g+n), \bar{t}_{l_{\Psi_2(r)}}(g+n)\big) \\
&\geq \bar{T}\Big(\bar{T}\big(\bar{t}_{l_{\Psi_1(r)}}(g), \bar{t}_{l_{\Psi_1(r)}}(n)\big), \bar{T}\big(\bar{t}_{l_{\Psi_2(r)}}(g), \bar{t}_{l_{\Psi_2(r)}}(n)\big)\Big) \\
&= \bar{T}\Big(\bar{T}\big(\bar{t}_{l_{\Psi_1(r)}}(g), \bar{t}_{l_{\Psi_1(r)}}(n)\big), \bar{T}\big(\bar{t}_{l_{\Psi_2(r)}}(n), \bar{t}_{l_{\Psi_2(r)}}(g)\big)\Big) \text{ [as } \bar{T} \text{ is commutative]} \\
&= \bar{T}\Big(\bar{T}\big(\bar{t}_{\Psi_1}(g), \bar{t}_{l_{\Psi_2(r)}}(g)\big), \bar{T}\big(\bar{t}_{l_{\Psi_1(r)}}(n), \bar{t}_{l_{\Psi_2(r)}}(n)\big)\Big) \text{ [as } \bar{T} \text{ is associative]} \\
&= \bar{T}\big(\bar{t}_{l_{\Psi(r)}}(g), \bar{t}_{l_{\Psi(r)}}(n)\big) \qquad (3.10)
\end{aligned}$$

and

$$\begin{aligned}
\bar{t}_{l_{\Psi(r)}}(-g) &= \bar{T}\big(\bar{t}_{l_{\Psi_1(r)}}(-g), \bar{t}_{l_{\Psi_2(r)}}(-g)\big) \\
&\geq \bar{T}\big(\bar{t}_{l_{\Psi_1(r)}}(g), \bar{t}_{l_{\Psi_2(r)}}(g)\big) \text{ [by Definition 3.1]} \\
&= \bar{t}_{l_{\Psi(r)}}(g) \qquad (3.11)
\end{aligned}$$

Similarly, we can show

$$\bar{i}_{l_{\Psi(r)}}(g+n) \leq \bar{I}\big(\bar{i}_{l_{\Psi(r)}}(g), \bar{i}_{l_{\Psi(r)}}(n)\big) \qquad (3.12)$$

$$\bar{f}_{l_{\Psi(r)}}(g+n) \leq \bar{F}\big(\bar{f}_{l_{\Psi(r)}}(g), \bar{f}_{l_{\Psi(r)}}(n)\big) \qquad (3.13)$$

and

$$\bar{i}_{l_{\Psi(r)}}(-g) \leq \bar{i}_{l_{\Psi(r)}}(g) \qquad (3.14)$$

$$\bar{f}_{l_{\Psi(r)}}(-g) \leq \bar{f}_{l_{\Psi(r)}}(g) \qquad (3.15)$$

Also, we can show that

$$\bar{t}_{l_{\Psi(r)}}(g \cdot n) \geq \bar{T}\big(\bar{t}_{l_{\Psi(r)}}(g), \bar{t}_{l_{\Psi(r)}}(n)\big), \qquad (3.16)$$

$$\bar{i}_{l_{\Psi(r)}}(g \cdot n) \leq \bar{I}\big(\bar{i}_{l_{\Psi(r)}}(g), \bar{i}_{l_{\Psi(r)}}(n)\big), \text{ and} \qquad (3.17)$$

$$\bar{f}_{l_{\Psi(r)}}(g \cdot n) \leq \bar{F}\big(\bar{f}_{l_{\Psi(r)}}(g), \bar{f}_{l_{\Psi(r)}}(n)\big) \qquad (3.18)$$

So, from Equations 3.10–3.18 $\Psi \in$ IVNSSR(Q). □

Remark 3.3. *In general, if $\Psi_1, \Psi_2 \in IVNSSR(Q)$, then $\Psi_1 \cup \Psi_2$ may not always be an IVNSSR of $(Q, +, \cdot)$.*

The following Example 3.4 will prove Remark 3.3.

Example 3.4. Let $(\mathbb{Z}, +, \cdot)$ be the ring of integers and $\mathbb{N}$ be a set of parameters. Again, let $\Psi_1 = \Big\{\Big(r, \big\{\big(g, \bar{t}_{l_{\Psi_1(r)}}(g), \bar{i}_{l_{\Psi_1(r)}}(g), \bar{f}_{l_{\Psi_1(r)}}(g)\big) : g \in \mathbb{Z}\big\}\Big) : r \in \mathbb{N}\Big\}$ and $\Psi_2 =$

$\Big\{\Big(r, \big\{\big(g, \bar{t}_{l_{\Psi_2}(r)}(g), \bar{i}_{l_{\Psi_2}(r)}(g), \bar{f}_{l_{\Psi_2}(r)}(g)\big) : g \in \mathbb{Z}\big\}\Big) : r \in \mathbb{N} \setminus \{1\}\Big\}$ be two IVNSSs of $\mathbb{Z}$, where $l_{\Psi_1} : \mathbb{N} \to \mathrm{IVNSS}(Q)$ be defined as

$$\bar{t}_{l_{\Psi_1}(r)}(g) = \begin{cases} \Big[\dfrac{1}{r+1}, \dfrac{1}{r}\Big] & \text{if } g \in 2\mathbb{Z} \\ [0,0] & \text{if } g \in 2\mathbb{Z}+1 \end{cases},$$

$$\bar{i}_{l_{\Psi_1}(r)}(g) = \begin{cases} [0,0] & \text{if } g \in 2\mathbb{Z} \\ \Big[\dfrac{1}{2r+2}, \dfrac{1}{2r}\Big] & \text{if } g \in 2\mathbb{Z}+1 \end{cases}, \text{ and}$$

$$\bar{f}_{l_{\Psi_1}(r)}(g) = \begin{cases} [0,0] & \text{if } g \in 2\mathbb{Z} \\ \Big[\dfrac{r-1}{r}, \dfrac{r}{r+1}\Big] & \text{if } g \in 2\mathbb{Z}+1 \end{cases}.$$

and $l_{\Psi_2} : \mathbb{N} \setminus \{1\} \to \mathrm{IVNSS}(Q)$ be defined as

$$\bar{t}_{l_{\Psi_2}(r)}(g) = \begin{cases} \Big[\dfrac{1}{r}, \dfrac{1}{r-1}\Big] & \text{if } g \in 3\mathbb{Z} \\ [0,0] & \text{if } g \in 3\mathbb{Z}+1 \end{cases},$$

$$\bar{i}_{l_{\Psi_2}(r)}(g) = \begin{cases} [0,0] & \text{if } g \in 3\mathbb{Z} \\ \Big[\dfrac{1}{2r}, \dfrac{1}{2r-2}\Big] & \text{if } g \in 3\mathbb{Z}+1 \end{cases}, \text{ and}$$

$$\bar{f}_{l_{\Psi_2}(r)}(g) = \begin{cases} [0,0] & \text{if } g \in 3\mathbb{Z} \\ \Big[\dfrac{r-2}{r-1}, \dfrac{r-1}{r}\Big] & \text{if } g \in 3\mathbb{Z}+1 \end{cases}.$$

Here, considering minimum TN and maximum SNs $\Psi_1, \Psi_2 \in \mathrm{IVNSSR}(\mathbb{Z})$. Let $\Psi = \Psi_1 \cup \Psi_2$. Now considering $r = 3$ we will have

$$\bar{t}_{l_{\Psi_1}(3)}(g) = \begin{cases} \Big[\dfrac{1}{4}, \dfrac{1}{3}\Big] & \text{if } g \in 2\mathbb{Z} \\ [0,0] & \text{if } g \in 2\mathbb{Z}+1 \end{cases} \quad \text{and}$$

$$\bar{t}_{l_{\Psi_2}(3)}(g) = \begin{cases} \Big[\dfrac{1}{3}, \dfrac{1}{2}\Big] & \text{if } g \in 3\mathbb{Z} \\ [0,0] & \text{if } g \in 3\mathbb{Z}+1 \end{cases}$$

Now, taking $g = 10$ and $n = 15$, we will have

$$\begin{aligned} \bar{t}_{l_{\Psi}(3)}(g+n) &= \bar{t}_{l_{\Psi}(3)}(10+15) \\ &= \bar{t}_{l_{\Psi}(3)}(25) \\ &= \max\{\bar{t}_{l_{\Psi_1}(3)}(25), \bar{t}_{l_{\Psi_2}(3)}(25)\} \\ &= \max\{[0,0],[0,0]\} \\ &= [0,0] \end{aligned}$$

Again, if $\Psi \in$ IVNSSR(Q), then $\forall g, n \in Q$, $\bar{t}_{l_\Psi(3)}(g+n) \geq \min\{\bar{t}_{l_\Psi(3)}(g), \bar{t}_{l_\Psi(3)}(n)\}$. But, here for $g = 10$ and $n = 15$, $\min\{\bar{t}_{l_\Psi(3)}(10), \bar{t}_{l_\Psi(3)}(15)\} = \min\left\{\left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{1}{3}, \frac{1}{2}\right]\right\} = \left[\frac{1}{4}, \frac{1}{3}\right] \nleq [0,0] = \bar{t}_{l_\Psi(3)}(10+15)$. So, $\Psi \notin$ IVNSSR(Q).

Corollary 3.4. *If $\Psi_1, \Psi_2 \in IVNSSR(Q)$, then $\Psi_1 \cup \Psi_2 \in IVNSSR(Q)$ iff one is a subset of other.*

Definition 3.5. let $\Psi = \left\{\left(r, \left\{\left(g, \bar{t}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(g)\right) : g \in \mathbb{Z}_4\right\}\right) : r \in A\right\}$ be an IVNSS of a crisp ring $(Q, +, \cdot)$. Also, let $[g_1, n_1]$, $[g_2, n_2]$, and $[g_3, n_3] \in \phi(K)$. Then the CS $\Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$ is called a level set of IVNSSR Ψ, where for any $g \in \Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$ the following inequalities will hold: $\bar{t}_{l_\Psi(r)}(g) \geq [g_1, n_1]$, $\bar{i}_{l_\Psi(r)}(g) \leq [g_2, n_2]$, and $\bar{f}_{l_\Psi(r)}(g) \leq [g_3, n_3]$.

Theorem 3.5. *Let $(Q, +, \cdot)$ be a crisp ring. Then $\Psi \in IVNSSR(Q)$ iff $\forall [g_1, n_1], [g_2, n_2], [g_3, n_3] \in \phi(K)$ with $\bar{t}_{l_\Psi(r)}(\theta_Q) \geq [g_1, n_1]$, $\bar{i}_{l_\Psi(r)}(\theta_Q) \leq [g_2, n_2]$, and $\bar{f}_{l_\Psi(r)}(\theta_Q) \leq [g_3, n_3]$, $\Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$ is a crisp subring of $(Q, +, \cdot)$ (considering idempotent IVTN and IVSNs).*

Proof. Since, $\bar{t}_{l_\Psi(r)}(\theta_Q) \geq [g_1, n_1]$, $\bar{i}_{l_\Psi(r)}(\theta_Q) \leq [g_2, n_2]$, and $\bar{f}_{l_\Psi(r)}(\theta_Q) \leq [g_3, n_3]$, $\theta_Q \in \Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$, i.e., $\Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$ is non-empty. Now, let $\Psi \in$ IVNSSR(Q) and $g, n \in \Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$. To show that, $(g - n)$ and $g \cdot n \in \Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$. Here,

$$\begin{aligned}
\bar{t}_{l_\Psi(r)}(g-n) &\geq \bar{T}\left(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\right) \text{ [by Proposition 3.1]} \\
&\geq \bar{T}\left([g_1, n_1], [g_1, n_1]\right) \left[\text{as } g, n \in \Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}\right] \\
&\geq [g_1, n_1] \text{ [as } \bar{T} \text{ is idempotent]}
\end{aligned} \tag{3.19}$$

Again,

$$\begin{aligned}
\bar{t}_{l_\Psi(r)}(g \cdot n) &\geq \bar{T}\left(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\right) \text{ [by Proposition 3.1]} \\
&\geq \bar{T}\left([g_1, n_1], [g_1, n_1]\right) \left[\text{as } g, n \in \Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}\right] \\
&\geq [g_1, n_1] \text{ [as } \bar{T} \text{ is idempotent]}
\end{aligned} \tag{3.20}$$

Similarly, as $\bar{I}$ and $\bar{F}$ are idempotent, we can prove that

$$\bar{i}_{l_\Psi(r)}(g-n) \leq [g_2, n_2], \tag{3.21}$$

$$\bar{i}_{l_\Psi(r)}(g \cdot n) \leq [g_2, n_2], \tag{3.22}$$

$$\bar{f}_{l_\Psi(r)}(g-n) \leq [g_3, n_3], \text{ and} \tag{3.23}$$

$$\bar{f}_{l_\Psi(r)}(g \cdot n) \leq [g_3, n_3]. \tag{3.24}$$

So, from Equations 3.19–3.24 $(g-n)$ and $g\cdot n \in \Psi_{\big([g_1,n_1],[g_2,n_2],[g_3,n_3]\big)}$, i.e., $\Psi_{\big([g_1,n_1],[g_2,n_2],[g_3,n_3]\big)}$ is a crisp subring of $(Q,+,\cdot)$.

Conversely, let $\Psi_{\big([g_1,n_1],[g_2,n_2],[g_3,n_3]\big)}$ is a crisp subring of $(Q,+,\cdot)$. To show that, $\Psi \in$ IVNSSR(Q).

Let $g,n \in Q$, then there exists $[g_1,n_1] \in \phi(K)$ such that $\bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\big) = [g_1,n_1]$. Wherefrom $\bar{t}_{l_\Psi(r)}(g) \geq [g_1,n_1]$ and $\bar{t}_{l_\Psi(r)}(n) \geq [g_1,n_1]$. Also, let there exist $[g_2,n_2],[g_3,n_3] \in \phi(K)$ such that $\bar{I}\big(\bar{i}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(n)\big) = [g_2,n_2]$ and $\bar{F}\big(\bar{f}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(n)\big) = [g_3,n_3]$. Then $g,n \in \Psi_{\big([g_1,n_1],[g_2,n_2],[g_3,n_3]\big)}$.

Now, as $\Psi_{\big([g_1,n_1],[g_2,n_2],[g_3,n_3]\big)}$ is a crisp subring, $g-n \in \Psi_{\big([g_1,n_1],[g_2,n_2],[g_3,n_3]\big)}$ and $g\cdot n \in \Psi_{\big([g_1,n_1],[g_2,n_2],[g_3,n_3]\big)}$.

Hence,

$$\begin{aligned}\bar{t}_{l_\Psi(r)}(g-n) &\geq [k_1,s_1]\\ &= \bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\big) \text{ and} \qquad (3.25)\\ \bar{t}_{l_\Psi(r)}(g\cdot n) &\geq [k_1,s_1]\\ &= \bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\big) \qquad (3.26)\end{aligned}$$

Similarly, we can prove that

$$\begin{aligned}\bar{i}_{l_\Psi(r)}(g-n) &\leq [k_2,s_2]\\ &= \bar{I}\big(\bar{i}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(n)\big), \qquad (3.27)\\ \bar{i}_{l_\Psi(r)}(g\cdot n) &\leq [k_2,s_2]\\ &= \bar{T}\big(\bar{i}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(n)\big), \qquad (3.28)\\ \bar{f}_{l_\Psi(r)}(g-n) &\leq [k_3,s_3]\\ &= \bar{F}\big(\bar{f}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(n)\big), \text{ and} \qquad (3.29)\\ \bar{f}_{l_\Psi(r)}(g\cdot n) &\leq [k_3,s_3]\\ &= \bar{F}\big(\bar{f}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(n)\big) \qquad (3.30)\end{aligned}$$

Hence, from Equations 3.25–3.30 $\Psi \in$ IVNSSR(Q). □

Definition 3.6. Let Ψ and Ψ' be two IVNSSs of two CSs Q and Y, respectively. Also, let $h: Q \to Y$ be a function. Then

(i) image of Ψ under h will be
$h(\Psi) = \Big\{\Big(r, \big\{\big(n, \bar{t}_{h(l_\Psi(r))}(n), \bar{i}_{h(l_\Psi(r))}(n), \bar{f}_{h(l_\Psi(r))}(n)\big) : n \in Y\big\}\Big) : r \in A\Big\}$,
where $\bar{t}_{h(l_\Psi(r))}(n) = \underset{s\in h^{-1}(n)}{\vee} \bar{t}_{l_\Psi(r)}(s)$, $\bar{i}_{h(l_\Psi(r))}(n) = \underset{s\in h^{-1}(n)}{\wedge} \bar{i}_{l_\Psi(r)}(s)$, and $\bar{f}_{h(l_\Psi(r))}(v) =$

$\underset{s\in h^{-1}(n)}{\wedge}\bar{f}_{l_\Psi(r)}(s)$. Wherefrom, if h is injective then $\bar{t}_{h(l_\Psi(r))}(n) = \bar{t}_{l_\Psi(r)}\big(h^{-1}(n)\big)$, $\bar{i}_{h(l_\Psi(r))}(n) = \bar{i}_{l_\Psi(r)}\big(h^{-1}(n)\big)$, $\bar{f}_{h(l_\Psi(r))}(n) = \bar{f}_{l_\Psi(r)}\big(h^{-1}(n)\big)$.

(i) preimage of Ψ' under h will be
$h^{-1}(\Psi') = \Big\{\Big(r, \big\{\big(g, \bar{t}_{h^{-1}(l_{\Psi'}(r))}(g), \bar{i}_{h^{-1}(l_{\Psi'}(r))}(g), \bar{f}_{h^{-1}(l_{\Psi'}(r))}(g)\big) : g \in Q\big\}\Big) : r \in A\Big\}$,
where $\bar{t}_{h^{-1}(l_{\Psi'}(r))}(g) = \bar{t}_{l_{\Psi'}(r)}\big(h(g)\big)$, $\bar{i}_{h^{-1}(l_{\Psi'}(r))}(g) = \bar{i}_{l_{\Psi'}(r)}\big(h(g)\big)$, $\bar{f}_{h^{-1}(l_{\Psi'}(r))}(g) = \bar{f}_{l_{\Psi'}(r)}\big(h(g)\big)$.

Theorem 3.6. *Let $(Q, +, \cdot)$ and $(Y, +, \cdot)$ be two crisp rings. Also, let $h : Q \to Y$ be an isomorphism. If Ψ is an IVNSSR of Q then $h(\Psi)$ is an IVNSSR of Y.*

Proof. Let $n_1 = h(g_1)$ and $n_2 = h(g_2)$, where $g_1, g_2 \in Q$ and $n_1, n_2 \in Y$. Now,

$$\begin{aligned}
\bar{t}_{h(l_\Psi(r))}(n_1 - n_2) &= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_1 - n_2)\big) \text{ [as } h \text{ is injective]}\\
&= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_1) - h^{-1}(n_2)\big) \text{ [as } h^{-1} \text{ is a homomorphism]}\\
&= \bar{t}_{l_\Psi(r)}(g_1 - g_2)\\
&\geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g_1), \bar{t}_{l_\Psi(r)}(g_2)\big)\\
&= \bar{T}\Big(\bar{t}_{l_\Psi(r)}\big(h^{-1}(n_1)\big), \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_2)\big)\Big)\\
&= \bar{T}\big(\bar{t}_{h(l_\Psi(r))}(n_1), \bar{t}_{h(l_\Psi(r))}(n_2)\big) \qquad (3.31)
\end{aligned}$$

Again,

$$\begin{aligned}
\bar{t}_{h(l_\Psi(r))}(n_1 \cdot n_2) &= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_1 \cdot n_2)\big) \text{ [as } h \text{ is injective]}\\
&= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_1) \cdot h^{-1}(n_2)\big) \text{ [as } h^{-1} \text{ is a homomorphism]}\\
&= \bar{t}_{l_\Psi(r)}(g_1 \cdot g_2)\\
&\geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g_1), \bar{t}_{l_\Psi(r)}(g_2)\big)\\
&= \bar{T}\Big(\bar{t}_{l_\Psi(r)}\Big(h^{-1}(n_1)\big), \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_2)\big)\Big)\\
&= \bar{T}\big(\bar{t}_{h(l_\Psi(r))}(n_1), \bar{t}_{h(l_\Psi(r))}(n_2)\big) \qquad (3.32)
\end{aligned}$$

Similarly,

$$\bar{i}_{h(l_\Psi(r))}(n_1 - n_2) \leq \bar{I}\big(\bar{i}_{h(l_\Psi(r))}(n_1), \bar{i}_{h(l_\Psi(r))}(n_2)\big), \qquad (3.33)$$

$$\bar{i}_{h(l_\Psi(r))}(n_1 \cdot n_2) \leq \bar{I}\big(\bar{i}_{h(l_\Psi(r))}(n_1), \bar{i}_{h(l_\Psi(r))}(n_2)\big), \qquad (3.34)$$

$$\bar{f}_{h(l_\Psi(r))}(n_1 - n_2) \leq \bar{F}\big(\bar{f}_{h(l_\Psi(r))}(n_1), \bar{f}_{h(l_\Psi(r))}(n_2)\big), \text{ and} \qquad (3.35)$$

$$\bar{f}_{h(l_\Psi(r))}(n_1 \cdot n_2) \leq \bar{F}\big(\bar{f}_{h(l_\Psi(r))}(n_1), \bar{f}_{h(l_\Psi(r))}(n_2)\big) \qquad (3.36)$$

So, from Equations 3.31–3.36 $h(\Psi)$ is an IVNSSR of Y. $\square$

Theorem 3.7. *Let $(Q,+,\cdot)$ and $(Y,+,\cdot)$ be two crisp rings. Also, let $h : Q \rightarrow Y$ be a homomorphism. If Ψ' is an IVNSSR of Y then $h^{-1}(\Psi')$ is an IVNSSR of Q. (Note that, h^{-1} may not be an inverse function but $h^{-1}(\Psi')$ is an inverse image of Ψ').*

Proof. Let $n_1 = h(g_1)$ and $n_2 = h(g_2)$, where $g_1, g_2 \in Q$ and $n_1, n_2 \in Y$. Now,

$$\begin{aligned}
\bar{t}_{h^{-1}(l_{\Psi'}(r))}(g_1 - g_2) &= \bar{t}_{l_{\Psi'}(r)}\big(h(g_1 - g_2)\big) \\
&= \bar{t}_{l_{\Psi'}(r)}\big(h(g_1) - h(g_2)\big) \text{ [as } h \text{ is a homomorphism]} \\
&= \bar{t}_{l_{\Psi'}(r)}(n_1 - n_2) \\
&\geq \bar{T}\big(\bar{t}_{l_{\Psi'}(r)}(n_1), \bar{t}_{l_{\Psi'}(r)}(n_2)\big) \\
&= \bar{T}\Big(\bar{t}_{l_{\Psi'}(r)}\big(h(g_1)\big), \bar{t}_{l_{\Psi'}(r)}\big(h(g_2)\big)\Big) \\
&= \bar{T}\big(\bar{t}_{h^{-1}(l_{\Psi'}(r))}(g_1), \bar{t}_{h^{-1}(l_{\Psi'}(r))}(g_2)\big)
\end{aligned} \tag{3.37}$$

Again,

$$\begin{aligned}
\bar{t}_{h^{-1}(l_{\Psi'}(r))}(g_1 \cdot g_2) &= \bar{t}_{l_{\Psi'}(r)}\big(h(g_1 \cdot g_2)\big) \\
&= \bar{t}_{l_{\Psi'}(r)}\big(h(g_1) \cdot h(g_2)\big) \text{ [as } h \text{ is a homomorphism]} \\
&= \bar{t}_{l_{\Psi'}(r)}(n_1 \cdot n_2) \\
&\geq \bar{T}\big(\bar{t}_{l_{\Psi'}(r)}(n_1), \bar{t}_{l_{\Psi'}(r)}(n_2)\big) \\
&= \bar{T}\Big(\bar{t}_{l_{\Psi'}(r)}\big(h(g_1)\big), \bar{t}_{l_{\Psi'}(r)}\big(h(g_2)\big)\Big) \\
&= \bar{T}\big(\bar{t}_{h^{-1}(l_{\Psi'}(r))}(g_1), \bar{t}_{h^{-1}(l_{\Psi'}(r))}(g_2)\big)
\end{aligned} \tag{3.38}$$

Similarly,

$$\bar{i}_{h^{-1}(l_{\Psi'}(r))}(g_1 - g_2) \leq \bar{I}\big(\bar{i}_{h^{-1}(l_{\Psi'}(r))}(g_1), \bar{i}_{h^{-1}(l_{\Psi'}(r))}(g_2)\big) \tag{3.39}$$

$$\bar{i}_{h^{-1}(l_{\Psi'}(r))}(g_1 \cdot g_2) \leq \bar{I}\big(\bar{i}_{h^{-1}(l_{\Psi'}(r))}(g_1), \bar{i}_{h^{-1}(l_{\Psi'}(r))}(g_2)\big) \tag{3.40}$$

$$\bar{f}_{h^{-1}(l_{\Psi'}(r))}(g_1 - g_2) \leq \bar{F}\big(\bar{f}_{h^{-1}(l_{\Psi'}(r))}(g_1), \bar{f}_{h^{-1}(l_{\Psi'}(r))}(g_2)\big) \tag{3.41}$$

$$\bar{f}_{h^{-1}(l_{\Psi'}(r))}(g_1 \cdot g_2) \leq \bar{F}\big(\bar{f}_{h^{-1}(l_{\Psi'}(r))}(g_1), \bar{f}_{h^{-1}(l_{\Psi'}(r))}(g_2)\big) \tag{3.42}$$

So, from Equations 3.37–3.42 $h^{-1}(\Psi')$ is an IVNSSR of Q. □

Definition 3.7. Let $(Q,+,\cdot)$ be a crisp ring and $\Psi \in$ IVNSSR(Q). Again, let $\bar{\alpha} = [\alpha_1, \alpha_2], \bar{\nu} = [\nu_1, \nu_2], \bar{\chi} = [\chi_1, \chi_2] \in \phi(K)$. Then

(i) Ψ is called a $(\bar{\alpha},\bar{\nu},\bar{\chi})-$identity IVNSSR over Q, if $\forall g \in Q$

$$\bar{t}_{l_\Psi(r)}(g) = \begin{cases} \bar{\alpha} & \text{if } g = \theta_Q \\ [0,0] & \text{if } g \neq \theta_Q \end{cases},$$

$$\bar{i}_{l_\Psi(r)}(g) = \begin{cases} \bar{\nu} & \text{if } g = \theta_Q \\ [1,1] & \text{if } g \neq \theta_Q \end{cases}, \text{ and}$$

$$\bar{f}_{l_\Psi(r)}(g) = \begin{cases} \bar{\chi} & \text{if } g = \theta_Q \\ [1,1] & \text{if } g \neq \theta_Q \end{cases},$$

where θ_Q is the additive zero element of Q.

(ii) Ψ is called a $(\bar{\alpha},\bar{\nu},\bar{\chi})-$absolute IVNSSR over Q, if $\forall g \in Q$, $\bar{t}_{l_\Psi(r)}(g) = \bar{\alpha}$, $\bar{i}_{l_\Psi(r)}(g) = \bar{\nu}$, and $\bar{f}_{l_\Psi(r)}(g) = \bar{\chi}$.

Theorem 3.8. *Let $(Q,+,\cdot)$ and $(Y,+,\cdot)$ be two crisp rings and $\Psi \in$IVNSSR (Q). Again, let $h: Q \to Y$ be a homomorphism. Then*

(i) *$h(\Psi)$ will be a $(\bar{\alpha},\bar{\nu},\bar{\chi})-$identity IVNSSR over Y, if $\forall g \in Q$*

$$\bar{t}_{l_\Psi(r)}(g) = \begin{cases} \bar{\alpha} & \textit{if } g \in Ker(h) \\ [0,0] & \textit{otherwise} \end{cases},$$

$$\bar{i}_{l_\Psi(r)}(g) = \begin{cases} \bar{\nu} & \textit{if } g \in Ker(h) \\ [1,1] & \textit{otherwise} \end{cases}, \textit{ and}$$

$$\bar{f}_{l_\Psi(r)}(g) = \begin{cases} \bar{\chi} & \textit{if } g \in Ker(h) \\ [1,1] & \textit{otherwise} \end{cases},$$

(ii) *$h(\Psi)$ will be a $(\bar{\alpha},\bar{\nu},\bar{\chi})-$absolute IVNSSR over Y, if Ψ is a $(\bar{\alpha},\bar{\nu},\bar{\chi})-$absolute IVNSSR over Q.*

Proof. (i) Clearly, by Theorem 3.6 $h(\Psi) \in$ IVNSSR(Y). Let $g \in Ker(h)$, then $h(g) = \theta_Y$. So,

$$\begin{aligned} \bar{t}_{h(l_\Psi(r))}(\theta_Y) &= \bar{t}_{l_\Psi(r)}\big(h^{-1}(\theta_Y)\big) \\ &= \bar{t}_{l_\Psi(r)}(g) \\ &= \bar{\alpha} \end{aligned} \tag{3.43}$$

Similarly,

$$\bar{i}_{h(l_\Psi(r))}(\theta_Y) = \bar{\nu}, \text{ and} \tag{3.44}$$

$$\bar{f}_{h(l_\Psi(r))}(\theta_Y) = \bar{\chi} \tag{3.45}$$

Again, let $g \in Q \setminus Ker(h)$ and $h(g) = n$. Then

$$\begin{aligned}\bar{t}_{h(l_\Psi(r))}(n) &= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n)\big)\\ &= \bar{t}_{l_\Psi(r)}(g)\\ &= [0,0] \qquad (3.46)\end{aligned}$$

Similarly,

$$\bar{i}_{h(l_\Psi(r))}(n) = [1,1] \text{ and} \qquad (3.47)$$

$$\bar{f}_{h(l_\Psi(r))}(n) = [1,1] \qquad (3.48)$$

So, from the Equations 3.43–3.48 $h(\Psi)$ is a $(\bar{\alpha}, \bar{\nu}, \bar{\chi})-$identity IVNSSR over Y.

(ii) Let $h(g) = n$, for $g \in Q$ and $n \in Y$. Then

$$\begin{aligned}\bar{t}_{h(l_\Psi(r))}(n) &= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n)\big)\\ &= \bar{t}_{l_\Psi(r)}(g)\\ &= \bar{\alpha} \qquad (3.49)\end{aligned}$$

Similarly,

$$\bar{i}_{h(l_\Psi(r))}(n) = \bar{\nu} \text{ and} \qquad (3.50)$$

$$\bar{f}_{h(l_\Psi(r))}(n) = \bar{\chi} \qquad (3.51)$$

So, from the Equations 3.48–3.51 $h(\Psi)$ is a $(\bar{\alpha}, \bar{\nu}, \bar{\chi})-$absolute IVNSSR over Y. □

3.1. *Product of interval-valued neutrosophic subrings*

Definition 3.8. Let $(Q, +, \cdot)$ and $(Y, +, \cdot)$ be two crisp rings. Again, let $\Psi_1 \in$ IVNSSR(Q) and $\Psi_2 \in$ IVNSSR(Y), where $\Psi_1 = \Big\{\Big(r_1, \big\{\big(g, \bar{t}_{l_{\Psi_1}(r_1)}(g), \bar{i}_{l_{\Psi_1}(r_1)}(g), \bar{f}_{l_{\Psi_1}(r_1)}(g)\big) : g \in Q\big\}\Big) : r_1 \in A\Big\}$ and $\Psi_2 = \Big\{\Big(r_2, \big\{\big(v, \bar{t}_{l_{\Psi_2}(r_2)}(n), \bar{i}_{l_{\Psi_2}(r_2)}(n), \bar{f}_{l_{\Psi_2}(r_2)}(n)\big) : n \in Y\big\}\Big) : r_2 \in A\Big\}$. Then cartesian product of Ψ_1 and Ψ_2 will be

$$\begin{aligned}\Psi &= \Psi_1 \times \Psi_2\\ &= \big\{\big((r_1, r_2), l_{\Psi_1 \times \Psi_2}(r_1, r_2)\big) : (r_1, r_2) \in A \times A\big\}\end{aligned}$$

where the approximate function $l_{\Psi_1 \times \Psi_2} : A \times A \to$ IVNS$(Q \times Y)$ is defined as

$$\begin{aligned}\bar{t}_{l_{\Psi_1 \times \Psi_2}(r_1, r_2)}(g, n) &= \bar{T}\big(\bar{t}_{l_{\Psi_1}(r_1)}(g), \bar{t}_{l_{\Psi_2}(r_2)}(n)\big),\\ \bar{i}_{l_{\Psi_1 \times \Psi_2}(r_1, r_2)}(g, n) &= \bar{I}\big(\bar{i}_{l_{\Psi_1}(r_1)}(g), \bar{i}_{l_{\Psi_2}(r_2)}(n)\big), \text{ and}\\ \bar{f}_{l_{\Psi_1 \times \Psi_2}(r_1, r_2)}(g, n) &= \bar{F}\big(\bar{f}_{l_{\Psi_1}(r_1)}(g), \bar{f}_{l_{\Psi_2}(r_2)}(n)\end{aligned}$$

Similarly, product of 3 or more IVNSSRs can be defined.

Theorem 3.9. *Let $(Q,+,\cdot)$ and $(Y,+,\cdot)$ be two crisp rings with $\Psi_1 \in IVNSSR(Q)$ and $\Psi_2 \in IVNSSR(Y)$. Then $\Psi_1 \times \Psi_2 \in IVNSSR(Q \times Y)$.*

Proof. Let $\Psi = \Psi_1 \times \Psi_2$ and $(g_1,n_1),(g_2,n_2) \in Q \times R$. Then

$$\begin{aligned}
&\bar{t}_{l_{\Psi(r_1,r_2)}}\big((g_1,n_1)-(g_2,n_2)\big)\\
&\quad= \bar{t}_{l_{\Psi_1\times\Psi_2(r_1,r_2)}}\big((g_1-g_2,n_1-n_2)\big)\\
&\quad= \bar{T}\big(\bar{t}_{l_{\Psi_1(r_1)}}(g_1-g_2),\bar{t}_{l_{\Psi_2(r_2)}}(n_1-n_2)\big)\\
&\quad\geq \bar{T}\Big(\bar{T}\big(\bar{t}_{l_{\Psi_1(r_1)}}(g_1),\bar{t}_{l_{\Psi_1(r_1)}}(g_2)\big),\bar{T}\big(\bar{t}_{l_{\Psi_2(r_2)}}(n_1),\bar{t}_{l_{\Psi_2(r_2)}}(n_2)\big)\Big)\\
&\quad= \bar{T}\Big(\bar{T}\big(\bar{t}_{l_{\Psi_1(r_1)}}(g_1),\bar{t}_{l_{\Psi_2(r_2)}}(n_1)\big),\bar{T}\big(\bar{t}_{l_{\Psi_1(r_1)}}(g_2),\bar{t}_{l_{\Psi_2(r_2)}}(n_2)\big)\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{[as } \bar{T} \text{ is associative]}\\
&\quad= \bar{T}\big(\bar{t}_{l_{\Psi(r_1,r_2)}}(g_1,n_1),\bar{t}_{l_{\Psi(r_1,r_2)}}(g_2,n_2)\big) \qquad (3.52)
\end{aligned}$$

Again,

$$\begin{aligned}
&\bar{t}_{l_{\Psi(r_1,r_2)}}\big((g_1,n_1)\cdot(g_2,n_2)\big)\\
&\quad= \bar{t}_{l_{\Psi_1\times\Psi_2(r_1,r_2)}}\big((g_1\cdot g_2,n_1\cdot n_2)\big)\\
&\quad= \bar{T}\big(\bar{t}_{l_{\Psi_1(r_1)}}(g_1\cdot g_2),\bar{t}_{l_{\Psi_2(r_2)}}(n_1\cdot n_2)\big)\\
&\quad\geq \bar{T}\Big(\bar{T}\big(\bar{t}_{l_{\Psi_1(r_1)}}(g_1),\bar{t}_{l_{\Psi_1(r_1)}}(g_2)\big),\bar{T}\big(\bar{t}_{l_{\Psi_2(r_2)}}(n_1),\bar{t}_{l_{\Psi_2(r_2)}}(n_2)\big)\Big)\\
&\quad= \bar{T}\Big(\bar{T}\big(\bar{t}_{l_{\Psi_1(r_1)}}(g_1),\bar{t}_{l_{\Psi_2(r_2)}}(n_1)\big),\bar{T}\big(\bar{t}_{l_{\Psi_1(r_1)}}(g_2),\bar{t}_{l_{\Psi_2(r_2)}}(n_2)\big)\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{[as } \bar{T} \text{ is associative]}\\
&\quad= \bar{T}\big(\bar{t}_{l_{\Psi(r_1,r_2)}}(g_1,n_1),\bar{t}_{l_{\Psi(r_1,r_2)}}(g_2,n_2)\big) \qquad (3.53)
\end{aligned}$$

Similary,

$$\bar{i}_{l_{\Psi(r_1,r_2)}}\big((g_1,n_1)-(g_2,n_2)\big) \leq \bar{I}\big(\bar{i}_{l_{\Psi(r_1,r_2)}}(g_1,n_1),\bar{i}_{l_{\Psi(r_1,r_2)}}(g_2,n_2)\big), \qquad (3.54)$$

$$\bar{i}_{l_{\Psi(r_1,r_2)}}\big((g_1,n_1)\cdot(g_2,n_2)\big) \leq \bar{I}\big(\bar{i}_{l_{\Psi(r_1,r_2)}}(g_1,n_1),\bar{i}_{l_{\Psi(r_1,r_2)}}(g_2,n_2)\big), \qquad (3.55)$$

$$\bar{f}_{l_{\Psi(r_1,r_2)}}\big((g_1,n_1)-(g_2,n_2)\big) \leq \bar{F}\big(\bar{f}_{l_{\Psi(r_1,r_2)}}(g_1,n_1),\bar{f}_{l_{\Psi(r_1,r_2)}}(g_2,n_2)\big), \text{ and} \qquad (3.56)$$

$$\bar{f}_{l_{\Psi(r_1,r_2)}}\big((g_1,n_1)\cdot(g_2,n_2)\big) \leq \bar{F}\big(\bar{f}_{l_{\Psi(r_1,r_2)}}(g_1,n_1),\bar{f}_{l_{\Psi(r_1,r_2)}}(g_2,n_2)\big) \qquad (3.57)$$

So, by Proposition 3.1 and from Equations 3.52–3.57 $\Psi_1 \times \Psi_2 \in$ IVNSSR$(Q \times Y)$. □

Corollary 3.10. *Let $\forall i \in \{1,2,...,n\}$, $(Q_i,+,\cdot)$ are crisp rings and $\Psi_i \in IVNSSR(Q_i)$. Then $\Psi_1 \times \Psi_2 \times \cdots \times \Psi_n$ is a IVNSSR of $Q_1 \times Q_2 \times \cdots \times Q_n$, where $n \in \mathbb{N}$.*

3.2. *Subring of a interval-valued neutrosophic soft subgring*

Definition 3.9. Let $(Q,+,\cdot)$ be a crisp ring and $\Psi_1, \Psi_2 \in$ IVNSSR(Q), where $\Psi_1 = \Big\{\big(r, \{\big(g, \bar{t}_{l_{\Psi_1}(r)}(g), \bar{i}_{l_{\Psi_1}(r)}(g), \bar{f}_{l_{\Psi_1}(r)}(g)\big) : g \in Q\}\big) : r \in A\Big\}$ and $\Psi_2 = \Big\{\big(r, \{\big(g, \bar{t}_{l_{\Psi_2}(r)}(g), \bar{i}_{l_{\Psi_2}(r)}(g), \bar{f}_{l_{\Psi_2}(r)}(g)\big) : g \in Q\}\big) : r \in A\Big\}$. Then Ψ_1 is called a subring of Ψ_2 if $\forall g \in Q$, $\bar{t}_{l_{\Psi_1}(r)}(g) \leq \bar{t}_{l_{\Psi_2}(r)}(g)$, $\bar{i}_{l_{\Psi_1}(r)}(g) \geq \bar{i}_{l_{\Psi_2}(r)}(g)$, and $\bar{f}_{l_{\Psi_1}(r)}(g) \geq \bar{f}_{l_{\Psi_2}(r)}(g)$.

Theorem 3.11. *Let $(Q,+,\cdot)$ be a crisp ring and $\Psi \in$ IVNSSR(Q). Again, let Ψ_1 and Ψ_2 be two subrings of Ψ. Then $\Psi_1 \cap \Psi_2$ is also a subring of Ψ, considering all the IVTN and IVSNs as idempotent.*

Proof. Here, $\forall g \in Q$

$$\begin{aligned}\bar{t}_{l_{\Psi_1\cap\Psi_2}(r)}(g) &= \bar{T}\big(\bar{t}_{l_{\Psi_1}(r)}(g), \bar{t}_{l_{\Psi_2}(r)}(g)\big)\\ &\leq \bar{T}\big(\bar{t}_{l_{\Psi}(r)}(g), \bar{t}_{l_{\Psi}(r)}(g)\big)\\ &= \bar{t}_{l_{\Psi}(r)}(g) \text{ [as } \bar{T} \text{ is idempotent]}\end{aligned} \tag{3.58}$$

Similarly, since $\bar{I}$ and $\bar{F}$ are idempotent we have,

$$\bar{i}_{l_{\Psi_1\cap\Psi_2}(r)}(g) \geq \bar{i}_{l_{\Psi}(r)}(g) \text{ and} \tag{3.59}$$

$$\bar{f}_{l_{\Psi_1\cap\Psi_2}(r)}(g) \geq \bar{f}_{l_{\Psi}(r)}(g) \tag{3.60}$$

So, from Equations 3.58–3.60 $\Psi_1 \cap \Psi_2$ is a subring of Ψ. □

Theorem 3.12. *Let $(Q,+,\cdot)$ be a crisp ring and $\Psi_1, \Psi_2 \in$ IVNSSR(Q) such that Ψ_1 is a subring of Ψ_2. Let $(Y,+,\cdot)$ is another crisp ring and $h : Q \to Y$ be an isomorphism. Then*

(i) *$h(\Psi_1)$ and $h(\Psi_2)$ are two IVNSSRs over Y and*

(i) *$h(\Psi_1)$ is a subring of $h(\Psi_2)$.*

Proof. (i) can be proved by using Theorem 3.6.
(ii) Let $n = h(g)$, where $g \in Q$ and $n \in Y$. Then

$$\begin{aligned}&\bar{t}_{l_{\Psi_1}(r)}(g) \leq \bar{t}_{l_{\Psi_2}(r)}(g) \text{ [as } \Psi_1 \text{ is a subring of } \Psi_2]\\ \Rightarrow &\bar{t}_{l_{\Psi_1}(r)}\big(h^{-1}(n)\big) \leq \bar{t}_{l_{\Psi_2}(r)}\big(h^{-1}(n)\big)\\ \Rightarrow &\bar{t}_{h(l_{\Psi_1}(r))}(n) \leq \bar{t}_{h(l_{\Psi_2}(r))}(n)\end{aligned} \tag{3.61}$$

Similarly,

$$\bar{i}_{h(l_{\Psi_1}(r))}(n) \geq \bar{i}_{h(l_{\Psi_2}(r))}(n) \text{ and} \tag{3.62}$$

$$\bar{f}_{h(l_{\Psi_1}(r))}(n) \geq \bar{f}_{h(l_{\Psi_2}(r))}(n) \tag{3.63}$$

So, from Equations 3.61–3.63 $h(\Psi_1)$ is a subring of $h(\Psi_2)$. □

3.3. *Interval-valued neutrosophic normal soft subrings*

Definition 3.10. Let $(Q,+,\cdot)$ be a crisp ring and Ψ is an IVNSS of Q, where $\Psi = \Big\{\big(r, \big\{\big(g, \bar{t}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(g)\big) : g \in Q\big\}\big) : r \in A\Big\}$. Then Ψ is called an IVNNSSR over Q if

(i) Ψ is an IVNSSR of Q and

(ii) $\forall g, n \in Q$, $\bar{t}_{l_\Psi(r)}(g \cdot n) = \bar{t}_{l_\Psi(r)}(n \cdot g)$, $\bar{i}_{l_\Psi(r)}(g \cdot n) = \bar{i}_{l_\Psi(r)}(n \cdot g)$, and $\bar{f}_{l_\Psi(r)}(g \cdot n) = \bar{f}_{l_\Psi(r)}(n \cdot g)$.

The set of all IVNNSSR of $(Q,+,\cdot)$ will be expressed as IVNNSSR(Q).

Example 3.11. Let $(\mathbb{Z},+,\cdot)$ be the ring and $\mathbb{N}$ be the set of parameters. Also, let $\Psi = \Big\{\Big(r, \big\{\big(g, \bar{t}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(g)\big) : g \in \mathbb{Z}\big\}\Big) : r \in \mathbb{N}\Big\}$ be an IVNSS of $\mathbb{Z}$, where $l_\Psi(r) : \mathbb{N} \to$ IVNSS(Q) and $\forall g \in \mathbb{Z}$, $\forall r \in \mathbb{N}$ corresponding membership values are

$$\bar{t}_{l_\Psi(r)}(g) = \begin{cases} \left[\dfrac{1}{r+1}, \dfrac{1}{r-1}\right] & \text{if } g \in 2\mathbb{Z} \\ [0,0] & \text{if } g \in 2\mathbb{Z}+1 \end{cases},$$

$$\bar{i}_{l_\Psi(r)}(g) = \begin{cases} [0,0] & \text{if } g \in 2\mathbb{Z} \\ \left[\dfrac{1}{2r+2}, \dfrac{1}{2r-2}\right] & \text{if } g \in 2\mathbb{Z}+1 \end{cases}, \text{ and}$$

$$\bar{f}_{l_\Psi(r)}(g) = \begin{cases} [0,0] & \text{if } g \in 2\mathbb{Z} \\ \left[\dfrac{r-2}{r-1}, \dfrac{r}{r+1}\right] & \text{if } g \in 2\mathbb{Z}+1 \end{cases}.$$

Here, considering minimum TN and maximum SNs $\forall r \in \mathbb{N}$, $\Psi \in$ IVNNSSR$(\mathbb{Z})$.

Theorem 3.13. *Let $(Q,+,\cdot)$ be a crisp ring. If $\Psi_1, \Psi_2 \in$ IVNNSSR(Q), then $\Psi_1 \cap \Psi_2 \in$ IVNNSSR(Q).*

Proof. As $\Psi_1, \Psi_2 \in$ IVNSSR(Q) by Theorem 3.2 $\Psi_1 \cap \Psi_2 \in$ IVNSSR(Q). Again,

$$\begin{aligned} \bar{t}_{l_{\Psi_1 \cap \Psi_2}(r)}(g \cdot n) &= \bar{T}\big(\bar{t}_{l_{\Psi_1}(r)}(g \cdot n), \bar{t}_{l_{\Psi_2}(r)}(g \cdot n)\big) \\ &= \bar{T}\big(\bar{t}_{l_{\Psi_1}(r)}(n \cdot g), \bar{t}_{l_{\Psi_2}(r)}(n \cdot g)\big) \text{ [as } \Psi_1, \Psi_2 \in \text{IVNNSSR}(Q)] \\ &= \bar{t}_{\Psi_1 \cap \Psi_2}(n \cdot g) \end{aligned} \tag{3.64}$$

Similarly,

$$\bar{i}_{l_{\Psi_1 \cap \Psi_2}(r)}(g \cdot n) = \bar{i}_{l_{\Psi_1 \cap \Psi_2}(r)}(n \cdot g) \tag{3.65}$$

$$\bar{f}_{l_{\Psi_1 \cap \Psi_2}(r)}(g \cdot n) = \bar{f}_{l_{\Psi_1 \cap \Psi_2}(r)}(n \cdot g) \tag{3.66}$$

Hence, $\Psi_1 \cap \Psi_2 \in$ IVNNSSR(Q). □

Remark 3.14. *In general, if $\Psi_1, \Psi_2 \in IVNNSSR(Q)$, then $\Psi_1 \cup \Psi_2$ may not always be an IVNNSSR of $(Q, +, \cdot)$.*

Remark 3.14 can be shown by Example 3.4.

Theorem 3.15. *Let $(Q, +, \cdot)$ be a crisp ring. Then $\Psi \in IVNNSSR(Q)$ iff $\forall [g_1, n_1], [g_2, n_2], [g_3, n_3] \in \phi(K)$ with $\bar{t}_{l_\Psi(r)}(\theta_Q) \geq [g_1, n_1]$, $\bar{i}_{l_\Psi(r)}(\theta_Q) \leq [g_2, n_2]$, and $\bar{f}_{l_\Psi(r)}(\theta_Q) \leq [g_3, n_3]$, $\Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$ is a crisp normal subring of $(Q, +, \cdot)$ (considering idempotent IVTN and IVSNs).*

Proof. This can be proved using Theorem 3.5. □

Theorem 3.16. *Let $(Q, +, \cdot)$ and $(Y, +, \cdot)$ be two crisp rings. Also, let $h : Q \to Y$ be a ring isomorphism. If Ψ is an IVNNSSR of Q then $h(\Psi)$ is an IVNNSSR of Y.*

Proof. As Ψ is an IVNSSR of Q, by Theorem 3.6 $h(\Psi)$ is an IVNSSR of Y. Let $h(g_1) = n_1$ and $h(g_2) = n_2$, where $g_1, g_2 \in Q$ and $n_1, n_2 \in Y$. Then

$$
\begin{aligned}
\bar{t}_{h(l_\Psi(r))}(n_1 \cdot n_2) &= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_1 \cdot n_2)\big) \text{ [as } h \text{ is injective]} \\
&= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_1) \cdot h^{-1}(n_2)\big) \text{ [as } h^{-1} \text{ is a homomorphism]} \\
&= \bar{t}_{l_\Psi(r)}(g_1 \cdot g_2) \\
&= \bar{t}_{l_\Psi(r)}(g_2 \cdot g_1) \text{ [as } \Psi \text{ is an IVNNSSR of } Q\text{]} \\
&= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_2) \cdot h^{-1}(n_1)\big) \\
&= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_2 \cdot n_1)\big) \\
&= \bar{t}_{h(l_\Psi(r))}(n_2 \cdot n_1) \qquad (3.67)
\end{aligned}
$$

Similarly,

$$\bar{i}_{h(l_\Psi(r))}(n_1 \cdot n_2) = \bar{i}_{h(l_\Psi(r))}(n_2 \cdot n_1) \text{ and} \qquad (3.68)$$

$$\bar{f}_{h(l_\Psi(r))}(n_1 \cdot n_2) = \bar{f}_{h(l_\Psi(r))}(n_2 \cdot n_1) \qquad (3.69)$$

So, from Equations 3.67–3.69 $h(\Psi)$ is an IVNNSSR of Y. □

4. Conclusions

Interval-valued neutrosophic field is a dynamic research domain. Under soft environment, it becomes more general and productive. For this reason, we have adopted this mixed environment and defined the notions of interval-valued neutrosophic soft subring along with its normal version. Also, we have studied several homomorphic attributes of these newly introduced notions. Again, we have introduced the product of two interval-valued neutrosophic soft subrings. Furthermore, we have given several fundamental theories to understand some of its algebraic characteristics. These newly introduced notions have the potentials to become fruitful research domains. In future, for generalizing this concepts one can introduce them under the hypersoft set environment.

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Introduction to Interval-valued Neutrosophic Subring

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Abstract. The main purpose of this article is to develop and study the notion of interval-valued neutrosophic subring. Also, we have studied some homomorphic characteristics of interval-valued neutrosophic subring. Again, we have defined the concept of product of two interval-valued neutrosophic subrings and analyzed some of its important properties. Furthermore, we have developed the notion of interval-valued neutrosophic normal subring and studied some of its basic characteristics and homomorphic properties.

Keywords: Neutrosophic set; Interval-valued neutrosophic set; Interval-valued neutrosophic subring; Interval-valued neutrosophic normal subring

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ABBREVIATIONS

TN signifies “T-norm”.

SN signifies “S-norm”.

IVTN signifies “interval-valued T-norm”.

IVSN signifies “interval-valued S-norm”.

CS signifies “crisp set”.

FS signifies “fuzzy set”.

IFS signifies “intuitionistic fuzzy set”.

NS signifies “neutrosophic set”.

PS signifies “plithogenic set”.

FSG signifies “fuzzy subgroup”.

IFSG signifies “intuitionistic fuzzy subgroup”.

NSG signifies “neutrosophic subgroup”.

CR signifies “crisp ring”.

FSR signifies “fuzzy subring”.

IFSR signifies “intuitionistic fuzzy subring”.

NSR signifies "neutrosophic subring".

IVFSR signifies "interval-valued fuzzy subring".

IVIFSR signifies "interval-valued intuitionistic fuzzy subring".

IVNSR signifies "interval-valued neutrosophic subring".

IVNNSR signifies "interval-valued neutrosophic normal subring".

DMP signifies "decision making problem".

$\boldsymbol{\psi(P)}$ signifies "power set of P".

$\boldsymbol{L}$ signifies "the set $[0, 1]$".

1. Introduction

Zadeh's vision behind introducing the revolutionary concept of FS [1] theory was to tackle uncertainty in a better way than CS theory, which has certain drawbacks. Later on, following his vision Atanassov introduced a more general version of it, which is known as IFS [2] theory. These IFSs are a little step ahead in managing ambiguities and hence are welcomed by numerous researchers. Furthermore, following their footsteps Smarandache introduced NS [3] theory, which is more capable of handling vague situations. It is a significant generalization over CS, FS, and IFS theories. Smarandache has also initiated the concept of PS [4] theory which has broader aspects than those previously discussed concepts. In NS and PS theory, he has also developed the notions of neutrosophic calculus [5], neutrosophic probability [6], neutrosophic statistics [7], integral, measure [8], neutrosophic psychology [9], neutrosophic robotics [10], neutrosophic triplet group [11], plithogenic hypersoft set [12], plithogenic fuzzy whole hypersoft set [13], plithogenic logic, probability [14], plithogenic subgroup [15], plithogenic hypersoft subgroup [16], etc. Again, NS theory has various other contributions in different scientific researches, like in linear programming [17–20], decision making [21–27], healthcare [28,29], shortest path problem [30–37], neutrosophic forecasting [38], resource leveling [39], transportation problem [40,41], project scheduling [42], brain processing [43], etc.

Gradually, interval-valued versions of FS [44], IFS [45], and NS [46] were introduced, which are further generalizations of their CS, FS, IFS, and NS counterparts. Presently, these set theories are extensively used in different scientific domains. From the very start, various researchers have carried out this concepts and explored them in different dimensions. In the subsequent Table 1 we have referred some significant aspects of these notions.

Table 1. Importance of interval-valued notions in different domains.

Author & references	**Year**	**Contributions in various fields**
Biswas [47]	1994	Introduced interval-valued FSG.

continued . . .

Author & references	Year	Contributions in various fields
Atanassov [45]	1999	Studied basic definition and some properties of IVFS.
Mondal & Samanta [48]	2001	Defined and studied topology of IVIFSs.
Wang et al. [46]	2005	Proposed and studied IVNS and interval-valued neutrosophic logic.
Ye [49]	2009	Worked on multi-criteria DMP under IVIFSs.
Kang & Hur [50]	2010	Introduced and studied the notion of IVFSR.
Akram & Dudek [51]	2011	Defined some basic operations on interval-valued fuzzy graphs and studied some of their properties.
Aygünoğlu et al. [52]	2012	Introduced interval-valued IFSG and studied some homomorphic properties of it.
Moorthy & Arjunan [53]	2014	Introduced and studied some properties of IVIFSR.
Aiwu et al. [54]	2015	Worked on multi-attribute DMP under IVNSs.
Broumi et al. [56]	2016	Worked on interval-valued neutrosophic graph theory.
Deli [55]	2017	Applied soft version of IVNS in DMP.
Broumi et al. [56]	2019	Studied some properties of interval-valued neutrosophic graphs.

Group theory and ring theory are fundamental building blocks of abstract algebra, which are utilized in different scientific domains. But, initially, these concepts were introduced upon crisp environment. Gradually, from 1971 on-wards researchers started introducing these concepts under various uncertain environments. Some significant developments of these notions under uncertainty are the concepts of FSG [57], IFSG [58], NSG [59], FSR [60,61], IFSR [62], NSR [63], etc. Again some researchers have introduced these concepts under interval-valued environments and initiated the notions of interval-valued FSG [47], interval-valued IFSG [52], interval-valued NSG [64], interval-valued FSR [50], interval-valued IFSR [53], etc. Some more articles which can be helpful to different researchers are [65–71], etc. But, still, the notion of interval-valued NSR is undefined. Hence, by mixing interval-valued environment with neutrosophic environment, we can introduce a more general version of NSR, which will be called IVNSR. Also, their homomorphic properties can be studied. Again, their product and normal forms can be developed and analyzed. Based on these observations, the followings are some of our main objectives for this article:

- Introducing the notion of IVNSR and a analyzing its homomorphic properties.
- Introducing the product of IVNSRs.
- Introducing subring of a IVNSR.

- Introducing the notion of IVNNSR and a analyzing its homomorphic attributes.

The subsequent arrangement of this article is: in Section 2, some desk researches of FS, IFS, NS, IVFS, IVIFS, IVNS, FSR, IFSR, NSR, IVFSR, IVIFSR, etc., are discussed. In Section 3, the idea of IVNSR has been introduced and some basic theories are provided. Also, their product and normal versions are defined. Also, some theories are given to understand their algebraic attributes. Lastly, in Section 4, the concluding segment is given and also some opportunities for further studies are mentioned.

2. Literature Review

Definition 2.1. [1] A FS of a CS P is defined as the function $\nu : P \to L$.

Definition 2.2. [2] An IFS ρ of a CS P is defined as $\rho = \{(r, t_\rho(r), f_\gamma(r)) : r \in P\}$, where $\forall r \in P$, $t_\rho(r)$ and $f_\rho(r)$ known as the degree of membership and non-membership which satisfy the inequality $0 \leq t_\rho(r) + f_\rho(r) \leq 1$.

Definition 2.3. [3] A NS κ of a CS P is defined as $\kappa = \{(r, t_\kappa(r), i_\kappa(r), f_\kappa(r)) : r \in P\}$, where $\forall r \in P$, $t_\kappa(r), i_\kappa(r),$ and $f_\kappa(r)$ are known as degree of truth, indeterminacy, and falsity which satisfy the inequality $^{-}0 \leq t_\kappa(r) + i_\kappa(r) + f_\kappa(r) \leq 3^{+}$.

Definition 2.4. [52] An interval number of $L = [0, 1]$ is denoted as $\bar{k} = [k^-, k^+]$, where $1 \geq k^+ \geq k^- \geq 0$.

Definition 2.5. [44] An IVFS of P is defined as the mapping $\nu : P \to \psi(L)$.

Definition 2.6. [45] An IVIFS of P is defined as the mapping $\bar{\rho} : P \to \psi(L) \times \psi(L)$, It is denoted as $\bar{\rho} = \{(r, \bar{t}_{\bar{\rho}}(r), \bar{f}_{\bar{\rho}}(r)) : r \in P\}$, where $\bar{t}_{\bar{\rho}}(r),\ \bar{f}_{\bar{\rho}}(r) \subseteq [0, 1]$.

Definition 2.7. [46] An IVNS of P is defined as the mapping $\bar{\kappa} : P \to \psi(L) \times \psi(L) \times \psi(L)$, It is denoted as $\bar{\kappa} = \{(r, \bar{t}_{\bar{\kappa}}(r), \bar{i}_{\bar{\kappa}}(r), \bar{f}_{\bar{\kappa}}(r)) : r \in P\}$ where $\forall r \in P$, $\bar{t}_{\bar{\kappa}}(r),\ \bar{i}_{\bar{\kappa}}(r),$ and $\bar{f}_{\bar{\kappa}}(r) \subseteq L$.

Definition 2.8. [46] Let $\bar{\kappa_1} = \{(r, \bar{t}_{\bar{\kappa_1}}(r), \bar{i}_{\bar{\kappa_1}}(r), \bar{f}_{\bar{\kappa_1}}(r)) : r \in P\}$ and $\bar{\kappa_2} = \{(r, \bar{t}_{\bar{\kappa_2}}(r), \bar{i}_{\bar{\kappa_2}}(r), \bar{f}_{\bar{\kappa_2}}(r)) : r \in P\}$ be two IVNSs of P. Then union of $\bar{\kappa_1}$ and $\bar{\kappa_2}$ is defined as

$$\bar{t}_{\bar{\kappa_1} \cup \bar{\kappa_2}} = \left[\max\{\bar{t}^-_{\bar{\kappa_1}}, \bar{t}^-_{\bar{\kappa_2}}\}, \max\{\bar{t}^+_{\bar{\kappa_1}}, \bar{t}^+_{\bar{\kappa_2}}\}\right]$$
$$\bar{t}_{\bar{\kappa_1} \cup \bar{\kappa_2}} = \left[\min\{\bar{i}^-_{\bar{\kappa_1}}, \bar{i}^-_{\bar{\kappa_2}}\}, \min\{\bar{i}^+_{\bar{\kappa_1}}, \bar{i}^+_{\bar{\kappa_2}}\}\right]$$
$$\bar{t}_{\bar{\kappa_1} \cup \bar{\kappa_2}} = \left[\min\{\bar{f}^-_{\bar{\kappa_1}}, \bar{f}^-_{\bar{\kappa_2}}\}, \min\{\bar{f}^+_{\bar{\kappa_1}}, \bar{f}^+_{\bar{\kappa_2}}\}\right]$$

Then intersection of $\bar{\kappa_1}$ and $\bar{\kappa_2}$ is defined as

$$\bar{t}_{\bar{\kappa_1} \cap \bar{\kappa_2}} = \left[\min\{\bar{t}^-_{\bar{\kappa_1}}, \bar{t}^-_{\bar{\kappa_2}}\}, \min\{\bar{t}^+_{\bar{\kappa_1}}, \bar{t}^+_{\bar{\kappa_2}}\}\right]$$
$$\bar{t}_{\bar{\kappa_1} \cap \bar{\kappa_2}} = \left[\max\{\bar{i}^-_{\bar{\kappa_1}}, \bar{i}^-_{\bar{\kappa_2}}\}, \max\{\bar{i}^+_{\bar{\kappa_1}}, \bar{i}^+_{\bar{\kappa_2}}\}\right]$$
$$\bar{t}_{\bar{\kappa_1} \cap \bar{\kappa_2}} = \left[\max\{\bar{f}^-_{\bar{\kappa_1}}, \bar{f}^-_{\bar{\kappa_2}}\}, \max\{\bar{f}^+_{\bar{\kappa_1}}, \bar{f}^+_{\bar{\kappa_2}}\}\right]$$

Definition 2.9. [72] A function $T : L \to L$ is called a TN iff $\forall r, v, z \in L$, the followings can be concluded

(i) $T(r, 1) = r$
(ii) $T(r, v) = T(v, r)$
(iii) $T(r, v) \leq T(z, v)$ if $r \leq z$
(iv) $T(r, T(v, z)) = T(T(r, v), z)$

Definition 2.10. [73] A function $\bar{T} : \psi(L) \times \psi(L) \to \psi(L)$ defined as $\bar{T}(\bar{k}, \bar{w}) = [T(k^-, w^-), T(k^+, w^+)]$, where T is a TN is known as an IVTN.

Definition 2.11. [72] A function $S : L \to L$ is called a SN iff $\forall r, v, z \in L$, the followings can be concluded

(i) $S(r, 0) = r$
(ii) $S(r, v) = S(v, r)$
(iii) $S(r, v) \leq S(z, v)$ if $r \leq z$
(iv) $S(r, S(v, z)) = S(S(r, v), z)$

Definition 2.12. [73] The function $\bar{S} : \psi(L) \times \psi(L) \to \psi(L)$ defined as $\bar{S}(\bar{k}, \bar{w}) = [S(k^-, w^-), S(k^+, w^+)]$, where S is a SN is called an IVSN.

2.1. *Fuzzy, Intuitionistic fuzzy & Neutrosophic subrings*

Definition 2.13. [60] Let $(P, +, \cdot)$ be a crisp ring. A FS λ is called a FSR of P, iff $\forall r, v \in P$,

(i) $\lambda(r - v) \geq \min\{\lambda(r), \lambda(v)\}$,
(ii) $\lambda(r \cdot v) \geq \min\{\lambda(r), \lambda(v)\}$

The set of all FSR of a crisp ring $(P, +, \cdot)$ will be denoted as FSR(P).

Theorem 2.1. *[61] Any FS λ of a ring $(P, +, \cdot)$ is a FSR of P iff the level sets λ_s ($\lambda(\theta_P) \geq s \geq 0$) are crisp subrings of P, where θ_P is the zero element of P.*

Definition 2.14. [61] Let λ be a FSR of $(P, +, \cdot)$ and $\lambda(\theta_P) \geq s \geq 0$, where θ_P is the zero element of P. Then λ_s is called a level subring of λ.

Proposition 2.2. *[61] Let $\lambda_1, \lambda_2 \in FSR(P)$. Then $\lambda_1 \cap \lambda_2 \in FSR(P)$.*

Theorem 2.3. *[61] Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings. Also, let $l : P \to R$ be a homomorphism. If λ is a FSR of P then $l(\lambda)$ is a FSR of R.*

Theorem 2.4. *[61] Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings. Also, let $l : P \to R$ be a homomorphism. If λ' is a FSR of R then $l^{-1}(\lambda')$ is a FSR of P.*

Definition 2.15. [62] Let $(P, +, \cdot)$ be a crisp ring. An IFS $\gamma = \{(r, t_\gamma(r), f_\gamma(r)) : r \in P\}$ is called an IFSR of P, iff $\forall r, v \in P$,

(i) $t_\gamma(r+v) \geq T(t_\gamma(r), t_\gamma(v)),\ f_\gamma(r+v) \leq S(f_\gamma(r), i_\gamma(v))$

(ii) $t_\gamma(-r) \geq t_\gamma(r),\ f_\gamma(-r) \leq f_\gamma(r)$

(iii) $t_\gamma(r \cdot v) \geq T(t_\gamma(r), t_\gamma(v)),\ f_\gamma(r \cdot v) \leq S(f_\gamma(r), i_\gamma(v))$.

Here, T is a TN and S is a SN.

The set of all IFSR of a crisp ring $(P, +, \cdot)$ will be denoted as IFSR(P).

Proposition 2.5. *[62] Let $\gamma \in$ IFSR(P). Then the followings will hold*

(i) $t_\gamma(-r) = t_\gamma(r),\ f_\gamma(-r) = f_\gamma(r)$ *and*

(ii) $t_\gamma(\theta_P) \geq t_\gamma(r),\ f_\gamma(\theta_P) \leq f_\gamma(r)$, *where θ_P is the zero element of P.*

Proposition 2.6. *[62] An IFS $\gamma = \{(r, t_\gamma(r), f_\gamma(r)) : r \in P\}$ is called an IFSR of P, iff $\forall r, v \in P$,*

(i) $t_\gamma(r-v) \geq T(t_\gamma(r), t_\gamma(v)),\ f_\gamma(r-v) \leq S(f_\gamma(r), f_\gamma(v))$

(ii) $t_\gamma(r \cdot v) \geq T(t_\gamma(r), t_\gamma(v)),\ f_\gamma(r \cdot v) \leq S(f_\gamma(r), f_\gamma(v))$

Proposition 2.7. *[62] Let $\gamma_1, \gamma_2 \in$ IFSR(P). Then $\gamma_1 \cap \gamma_2 \in$ IFSR(P).*

Theorem 2.8. *[62] Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings. Also, let $l : P \to R$ be a homomorphism. If γ is an IFSR of P then $l(\gamma)$ is an IFSR of R.*

Theorem 2.9. *[62] Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings. Also, let $l : P \to R$ be a homomorphism. If γ' is an IFSR of R then $l^{-1}(\gamma')$ is an IFSR of P.*

Definition 2.16. [63] Let $(P, +, \cdot)$ be a crisp ring. A NS $\omega = \{(r, t_\omega(r), i_\omega(r), f_\omega(r)) : r \in P\}$ is called a NSR of P, iff $\forall r, v \in P$,

(i) $t_\omega(r+v) \geq T(t_\omega(r), t_\omega(v)),\ i_\omega(r+v) \geq I(i_\omega(r), i_\omega(v)),\ f_\omega(r+v) \leq F(f_\omega(r), f_\omega(v))$

(ii) $t_\omega(-r) \geq t_\omega(r),\ i_\omega(-r) \geq i_\omega(r),\ f_\omega(-r) \leq f_\omega(r)$

(iii) $t_\omega(r \cdot v) \geq T(t_\omega(r), t_\omega(v)),\ i_\omega(r \cdot v) \geq I(i_\omega(r), i_\omega(v)),\ f_\omega(r \cdot v) \leq S(f_\omega(r), f_\omega(v))$.

Here, T and I are two TNs and S is a SN.

The set of all NSR of a crisp ring $(P, +, \cdot)$ will be denoted as NSR(P).

Proposition 2.10. *[63] A NS $\omega = \{(r, t_\omega(r), i_\omega(r), f_\omega(r)) : r \in P\}$ is called a NSR of P, iff $\forall r, v \in P$,*

(i) $t_\omega(r-v) \geq T(t_\omega(r), t_\omega(v)),\ i_\omega(r-v) \geq I(i_\omega(r), i_\omega(v)),\ f_\omega(r-v) \leq F(f_\omega(r), f_\omega(v))$

(ii) $t_\omega(r \cdot v) \geq T(t_\omega(r), t_\omega(v)),\ i_\omega(r \cdot v) \geq I(i_\omega(r), i_\omega(v)),\ f_\omega(r \cdot v) \leq S(f_\omega(r), f_\omega(v))$.

Here, T and I are two TNs and S is a SN.

Proposition 2.11. *[63] Let $\omega_1, \omega_2 \in NSR(P)$. Then $\omega_1 \cap \omega_2 \in NSR(P)$.*

Theorem 2.12. *[63] Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings. Also, let $l : P \to R$ be a homomorphism. If ω is a NSR of P then $l(\omega)$ is a NSR of R.*

Theorem 2.13. *[63] Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings. Also, let $l : P \to R$ be a homomorphism. If ω' is a NSR of R then $l^{-1}(\omega')$ is a NSR of P.*

Definition 2.17. [63] Let $\omega = \{(r, t_\omega(r), i_\omega(r), f_\omega(r)) : r \in P\}$ be a NSR of P. Then $\forall s \in [0, 1]$ the s-level sets of P are defined as

(i) $(t_\omega)_s = \{r \in P : t_\omega(r) \geq s\}$,
(ii) $(i_\omega)_s = \{r \in P : i_\omega(r) \geq s\}$, and
(iii) $(f_\omega)^s = \{r \in P : f_\omega(r) \leq s\}$.

Proposition 2.14. *[63] A NS $\omega = \{(r, t_\omega(r), i_\omega(r), f_\omega(r)) : r \in P\}$ of a crisp ring $(P, +, \cdot)$ is a NSR of P iff $\forall s \in [0, 1]$ the s-level sets of P, i.e. $(t_\omega)_s$, $(i_\omega)_s$, and $(f_\omega)^s$ are crisp rings of P.*

2.2. *Interval-valued Fuzzy and intuitionistic fuzzy subrings*

Definition 2.18. [50] Let $(P, +, \cdot)$ be a crisp ring. An IVFS $\Lambda = \{(r, \bar{t}_\Lambda(r)) : r \in P\}$ is called an IVFSR of $(P, +, \cdot)$ with respect to IVTN $\bar{T}$ if $\forall r, v \in P$, the followings can be concluded:

(i) $\bar{t}_\Lambda(r + v) \geq \bar{T}(\bar{t}_\Lambda(r), \bar{t}_\Lambda(v))$,
(ii) $\bar{t}_\Lambda(-r) \geq \Lambda(r)$, and
(iii) $\bar{t}_\Lambda(r \cdot v) \geq \bar{T}(\bar{t}_\Lambda(r), \bar{t}_\Lambda(v))$,

The set of all IVFSR of a crisp ring $(P, +, \cdot)$ with respect to an IVTN $\bar{T}$ will be denoted as IVFSR$(P, \bar{T})$.

Proposition 2.15. *[50] Let $\lambda = \{(r, t_\lambda(r)) : r \in P\}$ be a FSR of $(P, +, \cdot)$. Then $\Lambda = [t_\lambda, t_\lambda]$ is an IVFSR of P.*

Proposition 2.16. *[50] Let $\Lambda = \{(r, \bar{t}_\Lambda(r)) : r \in P\}$ be an IVFSR of $(P, +, \cdot)$. Then $\Lambda^- = \{(r, \bar{t}^-_\Lambda(r)) : r \in P\}$ and $\Lambda^+ = \{(r, \bar{t}^+_\Lambda(r)) : r \in P\}$ are FSRs of P.*

Definition 2.19. [53] Let $(P, +, \cdot)$ be a crisp ring. An IVIFS $\Gamma = \{(r, \bar{t}_\Gamma(r), \bar{f}_\Gamma(r)) : r \in P\}$ is called an IVIFSR of $(P, +, \cdot)$ if $\forall r, v \in P$, the followings can be concluded:

(i) $\bar{t}_\Gamma(r + v) \geq \bar{T}(\bar{t}_\Gamma(r), \bar{t}_\Gamma(v))$, $\bar{f}_\Gamma(r + v) \leq \bar{F}(\bar{f}_\Gamma(r), \bar{f}_\Gamma(v))$,
(ii) $\bar{t}_\Gamma(-r) \geq \bar{t}_\Gamma(r)$, $\bar{f}_\Gamma(-r) \leq \bar{f}_\Gamma(r)$, and
(iii) $\bar{t}_\Gamma(r \cdot v) \geq \bar{T}(\bar{t}_\Gamma(r), \bar{t}_\Gamma(v))$, $\bar{f}_\Gamma(r \cdot v) \leq \bar{F}(\bar{f}_\Gamma(r), \bar{f}_\Gamma(v))$.

The set of all IVIFSR of a crisp ring $(P, +, \cdot)$ will be denoted as IVIFSR(P).

Theorem 2.17. *[53] If* $\Gamma = \{(r, \bar{t}_\Gamma(r), \bar{f}_\Gamma(r)) : r \in P\} \in IVIFSR(P)$*, then* $\bar{t}_\Gamma(r) \leq \bar{t}_\Gamma(\theta_P)$ *and* $\bar{f}_\Gamma(r) \geq \bar{f}_\Gamma(\theta_P)$*.*

Theorem 2.18. *[53] If* Γ_1 *and* $\Gamma_2 \in IVIFSR(P)$*, then* $\Gamma_1 \cap \Gamma_2 \in IVIFSR(P)$*.*

Theorem 2.19. *[53] Let* $\Gamma = \{(r, \bar{t}_\Gamma(r), \bar{f}_\Gamma(r)) : r \in P\} \in IVIFSR(P)$*, then* $\forall r, v \in P$

(i) $\bar{t}_\Gamma(r - v) = \bar{t}_\Gamma(\theta_P)$ *implies that* $\bar{t}_\Gamma(r) = \bar{t}_\Gamma(v)$*.*
(ii) $\bar{f}_\Gamma(r - v) = \bar{f}_\Gamma(\theta_P)$ *implies that* $\bar{f}_\Gamma(r) = \bar{f}_\Gamma(v)$*.*

3. Proposed notion of interval-valued neutrosophic subring

Definition 3.1. Let $(P, +, \cdot)$ be a crisp ring. An IVNS $\Omega = \{(r, \bar{t}_\Omega(r), \bar{i}_\Omega(r), \bar{f}_\Omega(r)) : r \in P\}$ is called an IVNSR of $(P, +, \cdot)$ if $\forall r, v \in P$, the followings can be concluded:

$$\text{(i)} \begin{cases} \bar{t}_\Omega(r+v) \geq \bar{T}(\bar{t}_\Omega(r), \bar{t}_\Omega(v)), \\ \bar{i}_\Omega(r+v) \leq \bar{I}(\bar{i}_\Omega(r), \bar{i}_\Omega(v)), \\ \bar{f}_\Omega(r+v) \leq \bar{F}(\bar{f}_\Omega(r), \bar{f}_\Omega(v)) \end{cases}$$

$$\text{(ii)} \begin{cases} \bar{t}_\Omega(-r) \geq t_\Omega(r), \\ \bar{i}_\Omega(-r) \leq i_\Omega(r), \\ \bar{f}_\Omega(-r) \leq f_\Omega(r) \end{cases}$$

$$\text{(iii)} \begin{cases} \bar{t}_\Omega(r \cdot v) \geq \bar{T}(\bar{t}_\Omega(r), \bar{t}_\Omega(v)), \\ \bar{i}_\Omega(r \cdot v) \leq \bar{I}(\bar{i}_\Omega(r), \bar{i}_\Omega(v)), \\ \bar{f}_\Omega(r \cdot v) \leq \bar{F}(\bar{f}_\Omega(r), \bar{f}_\Omega(v)), \end{cases}$$

where $\bar{T}$ is an IVTN, $\bar{I}$ and $\bar{F}$ are two IVSNs.

The set of all IVNSR of a crisp ring $(P, +, \cdot)$ will be denoted as IVNSR(P).

Example 3.2. Let $(\mathbb{Z}, +, \cdot)$ be the ring of integers with respect to usual addition and multiplication. Let $\Omega = \{(r, \bar{t}_\Omega(r), \bar{i}_\Omega(r), \bar{f}_\Omega(r)) : r \in \mathbb{Z}\}$ be an IVNS of $\mathbb{Z}$, where $\forall r \in \mathbb{Z}$

$$\bar{t}_\Omega(r) = \begin{cases} [0.2, 0.25] & \text{if } r \in 2\mathbb{Z} \\ [0, 0] & \text{if } r \in 2\mathbb{Z}+1 \end{cases},$$

$$\bar{i}_\Omega(r) = \begin{cases} [0, 0] & \text{if } r \in 2\mathbb{Z} \\ [0.1, 0.12] & \text{if } r \in 2\mathbb{Z}+1 \end{cases}, \text{ and}$$

$$\bar{f}_\Omega(r) = \begin{cases} [0, 0] & \text{if } r \in 2\mathbb{Z} \\ [0.75, 0.8] & \text{if } r \in 2\mathbb{Z}+1 \end{cases}.$$

Now, if we consider minimum TN and maximum SNs, then $\Omega \in$ IVNSR$(\mathbb{Z})$.

Example 3.3. Let $(\mathbb{Z}_4, +, \cdot)$ be the ring of integers modulo 4 with usual addition and multiplication. Let $\Omega = \{(r, \bar{t}_\Omega(r), \bar{i}_\Omega(r), \bar{f}_\Omega(r)) : r \in \mathbb{Z}_4\}$ be an IVNS of $\mathbb{Z}_4$, where interval-valued memberships of elements belonging to Ω are mentioned in Table 2.

TABLE 2. Membership values of elements belonging to Ω

Ω	$\bar{t}_\Omega$	$\bar{i}_\Omega$	$\bar{f}_\Omega$
$\bar{0}$	$[0.6, 0.7]$	$[0.33, 0.35]$	$[0.2, 0.3]$
$\bar{1}$	$[0.7, 0.8]$	$[0.21, 0.23]$	$[0.5, 0.6]$
$\bar{2}$	$[0.75, 0.85]$	$[0.24, 0.26]$	$[0.3, 0.7]$
$\bar{3}$	$[0.75, 0.9]$	$[0.31, 0.33]$	$[0.5, 0.7]$

Now, if we consider the Łukasiewicz T-norm ($T(r, v) = \max\{0, r + v - 1\}$) and bounded sum S-norms ($S(r, v) = \min\{r + v, 1\}$), then $\Omega \in \text{IVNSR}(\mathbb{Z}_4)$.

Proposition 3.1. *An IVNS* $\Omega = \{(r, \bar{t}_\Omega(r), \bar{i}_\Omega(r), \bar{f}_\Omega(r)) : r \in P\}$ *of a crisp ring* $(P, +, \cdot)$ *is an IVNSR iff the followings can be concluded (assuming that all the IVTN and IVSNs are idempotent):*

$$\text{(i)} \begin{cases} \bar{t}_\Omega(r - v) \geq \bar{T}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(v)\big), \\ \bar{i}_\Omega(r - v) \leq \bar{I}\big(\bar{i}_\Omega(r), \bar{i}_\Omega(v)\big), \\ \bar{f}_\Omega(r - v) \leq \bar{F}\big(\bar{f}_\Omega(r), \bar{f}_\Omega(v)\big) \end{cases}$$

$$\text{(ii)} \begin{cases} \bar{t}_\Omega(r \cdot v) \geq \bar{T}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(v)\big), \\ \bar{i}_\Omega(r \cdot v) \leq \bar{I}\big(\bar{i}_\Omega(r), \bar{i}_\Omega(v)\big), \\ \bar{f}_\Omega(r \cdot v) \leq \bar{F}\big(\bar{f}_\Omega(r), \bar{f}_\Omega(v)\big). \end{cases}$$

Proof. Let $\Omega \in$ IVNSR(P). Then we have

$$\begin{aligned} \bar{t}_\Omega(r - v) &\geq \bar{T}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(-v)\big) \text{ [by condition (i) of Definition 3.1]} \\ &\geq \bar{T}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(v)\big) \text{ [by condition (ii) of Definition 3.1]} \end{aligned}$$

Similary, we will have

$$\begin{aligned} \bar{i}_\Omega(r - v) &\leq \bar{I}\big(\bar{i}_\Omega(r), \bar{i}_\Omega(v)\big), \text{ and} \\ \bar{f}_\Omega(r - v) &\leq \bar{F}\big(\bar{f}_\Omega(r), \bar{f}_\Omega(v)\big), \end{aligned}$$

which proves (i).

Again, (ii) follows immediately from condition (iii) of Definition 3.1.

Conversely, let (i) and (ii) of Proposition 3.1 hold. Also, let θ_P be the additive neutral element in $(P, +, \cdot)$. Then

$$\begin{aligned}\bar{t}_{\Omega}(\theta_P) &= \bar{t}_{\Omega}(r - r)\\ &\geq \bar{T}\big(\bar{t}_{\Omega}(r), \bar{t}_{\Omega}(r)\big)\\ &= \bar{t}_{\Omega}(r) \quad (3.1)\end{aligned}$$

Similaly, we can show that

$$\bar{i}_{\Omega}(\theta_P) \leq \bar{i}_{\Omega}(r) \quad (3.2)$$

$$\bar{f}_{\Omega}(\theta_P) \leq \bar{f}_{\Omega}(r) \quad (3.3)$$

Now,

$$\begin{aligned}\bar{t}_{\Omega}(-r) &= \bar{t}_{\Omega}(\theta_P - r)\\ &\geq \bar{T}\big(\bar{t}_{\Omega}(\theta_P), \bar{t}_{\Omega}(r)\big)\\ &\geq \bar{T}\big(\bar{t}_{\Omega}(r), \bar{t}_{\Omega}(r)\big) \text{ [by 3.1]}\\ &= \bar{t}_{\Omega}(r) \text{ [since } \bar{T} \text{ is idempotent]} \quad (3.4)\end{aligned}$$

Similarly, we can prove

$$\bar{i}_{\Omega}(-r) \leq \bar{i}_{\Omega}(r) \text{ [since } \bar{I} \text{ is idempotent]} \quad (3.5)$$

$$\bar{f}_{\Omega}(-r) \leq \bar{f}_{\Omega}(r) \text{ [since } \bar{F} \text{ is idempotent]} \quad (3.6)$$

Hence,

$$\begin{aligned}\bar{t}_{\Omega}(r + v) &= \bar{t}_{\Omega}(r - (-v))\\ &\geq \bar{T}\big(\bar{t}_{\Omega}(r), \bar{t}_{\Omega}(-v)\big)\\ &\geq \bar{T}\big(\bar{t}_{\Omega}(r), \bar{t}_{\Omega}(v)\big) \text{ [by 3.4]} \quad (3.7)\end{aligned}$$

Similarly,

$$\bar{i}_{\Omega}(r + v) \leq \bar{I}\big(\bar{t}_{\Omega}(r), \bar{t}_{\Omega}(v)\big) \text{ [by 3.5]} \quad (3.8)$$

$$\bar{f}_{\Omega}(r + v) \leq \bar{F}\big(\bar{t}_{\Omega}(r), \bar{t}_{\Omega}(v)\big) \text{ [by 3.6]} \quad (3.9)$$

So, by Equations 3.7, 3.8, and 3.9 condition (i) of Proposition 3.1 has been proved. Also, condition (ii) of Proposition 3.1 is same as condition (iii) of Definition 3.1. Hence, $\Omega \in$ IVNSR(P). □

Theorem 3.2. *Let $(P, +, \cdot)$ be a crisp ring. If $\Omega_1, \Omega_2 \in IVNSR(P)$, then $\Omega_1 \cap \Omega_2 \in IVNSR(P)$ (assuming all the IVTN and IVSNs are idempotent).*

Proof. Let $\Omega = \Omega_1 \cap \Omega_2$. Now, $\forall r, v \in P$

$$\begin{aligned}
\bar{t}_{\Omega}(r+v) &= \bar{T}\big(\bar{t}_{\Omega_1}(r+v), \bar{t}_{\Omega_2}(r+v)\big) \\
&\geq \bar{T}\Big(\bar{T}\big(\bar{t}_{\Omega_1}(r), \bar{t}_{\Omega_1}(v)\big), \bar{T}\big(\bar{t}_{\Omega_2}(r), \bar{t}_{\Omega_2}(v)\big)\Big) \\
&= \bar{T}\Big(\bar{T}\big(\bar{t}_{\Omega_1}(r), \bar{t}_{\Omega_1}(v)\big), \bar{T}\big(\bar{t}_{\Omega_2}(v), \bar{t}_{\Omega_2}(r)\big)\Big) \text{ [as } \bar{T} \text{ is commutative]} \\
&= \bar{T}\Big(\bar{T}\big(\bar{t}_{\Omega_1}(r), \bar{t}_{\Omega_2}(r)\big), \bar{T}\big(\bar{t}_{\Omega_1}(v), \bar{t}_{\Omega_2}(v)\big)\Big) \text{ [as } \bar{T} \text{ is associative]} \\
&= \bar{T}\big(\bar{t}_{\Omega}(r), \bar{t}_{\Omega}(v)\big) \qquad (3.10)
\end{aligned}$$

Similarly, as both $\bar{I}$ and $\bar{S}$ are commutative as well as associative, we will have

$$\bar{i}_{\Omega}(r+v) \leq \bar{I}\big(\bar{i}_{\Omega}(r), \bar{i}_{\Omega}(v)\big) \qquad (3.11)$$

$$\bar{f}_{\Omega}(r+v) \leq \bar{F}\big(\bar{f}_{\Omega}(r), \bar{f}_{\Omega}(v)\big) \qquad (3.12)$$

Again,

$$\begin{aligned}
\bar{t}_{\Omega}(-r) &= \bar{T}\big(\bar{t}_{\Omega_1}(-r), \bar{t}_{\Omega_2}(-r)\big) \\
&\geq \bar{T}\big(\bar{t}_{\Omega_1}(r), \bar{t}_{\Omega_2}(r)\big) \text{ [by Definition 3.1]} \\
&= \bar{t}_{\Omega}(r) \qquad (3.13)
\end{aligned}$$

Also,

$$\bar{i}_{\Omega}(-r) \leq \bar{i}_{\Omega}(r) \qquad (3.14)$$

$$\bar{f}_{\Omega}(-r) \leq \bar{f}_{\Omega}(r) \qquad (3.15)$$

Similarly, we can show that

$$\bar{t}_{\Omega}(r \cdot v) \geq \bar{T}\big(\bar{t}_{\Omega}(r), \bar{t}_{\Omega}(v)\big), \qquad (3.16)$$

$$\bar{i}_{\Omega}(r \cdot v) \leq \bar{I}\big(\bar{i}_{\Omega}(r), \bar{i}_{\Omega}(v)\big), \text{ and} \qquad (3.17)$$

$$\bar{f}_{\Omega}(r \cdot v) \leq \bar{F}\big(\bar{f}_{\Omega}(r), \bar{f}_{\Omega}(v)\big) \qquad (3.18)$$

Hence, by Equations 3.10–3.18 $\Omega = \Omega_1 \cap \Omega_2 \in \text{IVNSR}(P)$. □

Remark 3.3. *In general, if $\Omega_1, \Omega_2 \in IVNSR(P)$, then $\Omega_1 \cup \Omega_2$ may not always be an IVNSR of $(P, +, \cdot)$.*

The following Example 3.4 will prove our claim.

Example 3.4. Let $(\mathbb{Z}, +, \cdot)$ be the ring of integers with respect to usual addition and multiplication. Let $\Omega_1 = \big\{\big(r, \bar{t}_{\Omega_1}(r), \bar{i}_{\Omega_1}(r), \bar{f}_{\Omega_1}(r)\big) : r \in \mathbb{Z}\big\}$ and $\Omega_2 = \big\{\big(r, \bar{t}_{\Omega_2}(r), \bar{i}_{\Omega_2}(r), \bar{f}_{\Omega_2}(r)\big) :$

$r \in \mathbb{Z}\big\}$ be two IVNSs of $\mathbb{Z}$, where $\forall r \in \mathbb{Z}$

$$\bar{t}_{\Omega_1}(r) = \begin{cases} [0.25, 0.4] & \text{if } r \in 2\mathbb{Z} \\ [0, 0] & \text{if } r \in 2\mathbb{Z}+1 \end{cases},$$

$$\bar{i}_{\Omega_1}(r) = \begin{cases} [0, 0] & \text{if } r \in 2\mathbb{Z} \\ [0.17, 0.2] & \text{if } r \in 2\mathbb{Z}+1 \end{cases}, \text{ and}$$

$$\bar{f}_{\Omega_1}(r) = \begin{cases} [0, 0] & \text{if } r \in 2\mathbb{Z} \\ [0.33, 0.4] & \text{if } r \in 2\mathbb{Z}+1 \end{cases}.$$

and

$$\bar{t}_{\Omega_2}(r) = \begin{cases} [0.5, 0.67] & \text{if } r \in 3\mathbb{Z} \\ [0, 0] & \text{if } r \in 3\mathbb{Z}+1 \end{cases},$$

$$\bar{i}_{\Omega_2}(r) = \begin{cases} [0, 0] & \text{if } r \in 3\mathbb{Z} \\ [0.2, 0.25] & \text{if } r \in 3\mathbb{Z}+1 \end{cases}, \text{ and}$$

$$\bar{f}_{\Omega_2}(r) = \begin{cases} [0, 0] & \text{if } r \in 3\mathbb{Z} \\ [0.33, 0.5] & \text{if } r \in 3\mathbb{Z}+1 \end{cases}.$$

Now, if we consider minimum TN and maximum SNs, then $\Omega_1, \Omega_2 \in \text{IVNSR}(\mathbb{Z})$.
Now let $\Omega = \Omega_1 \cup \Omega_2$. Then for $r = 4$ and $v = 9$

$$\begin{aligned} \bar{t}_{\Omega}(r+v) &= \bar{t}_{\Omega}(4+9) \\ &= \bar{t}_{\Omega}(13) \\ &= \max\{\bar{t}_{\Omega_1}(13), \bar{t}_{\Omega_2}(13)\} \\ &= \max\{[0,0], [0,0]\} \\ &= [0,0] \end{aligned}$$

Again, if $\Omega \in$ IVNSR(P), then $\forall r, v \in P$, $\bar{t}_{\Omega}(r+v) \geq \min\{\bar{t}_{\Omega}(r), \bar{t}_{\Omega}(v)\}$. But, here for $r = 4$ and $v = 9$, $\min\{\bar{t}_{\Omega}(4), \bar{t}_{\Omega}(9)\} = \min\{[0.25, 0.4], [0.5, 0.67]\} = [0.25, 0.4] \not\leq [0, 0] = \bar{t}_{\Omega}(4+9)$. Hence, $\Omega \notin$ IVNSR(P).

Corollary 3.4. *If $\Omega_1, \Omega_2 \in$ IVNSR(P), then $\Omega_1 \cup \Omega_2 \in$ IVNSR(P) iff one is contained in other.*

Definition 3.5. Let $\Omega = \big\{\big(r, \bar{t}_{\Omega}(r), \bar{i}_{\Omega}(r), \bar{f}_{\Omega}(r)\big) : r \in P\big\}$ be an IVNS of a crisp ring $(P, +, \cdot)$. Also, let $[k_1, s_1]$, $[k_2, s_2]$ and $[k_3, s_3] \in \Psi(L)$. Then the crisp set $\Omega_{\big([k_1,s_1],[k_2,s_2],[k_3,s_3]\big)}$ is called a level set of IVNSR Ω, where for any $r \in \Omega_{\big([k_1,s_1],[k_2,s_2],[k_3,s_3]\big)}$ the following inequalities will hold: $\bar{t}_{\Omega}(r) \geq [k_1, s_1]$, $\bar{i}_{\Omega}(r) \leq [k_2, s_2]$, and $\bar{f}_{\Omega}(r) \leq [k_3, s_3]$.

Theorem 3.5. *Let $(P,+,\cdot)$ be a crisp ring. Then $\Omega \in IVNSR(P)$ iff $\forall [k_1,s_1],[k_2,s_2],[k_3,s_3] \in \Psi(L)$ with $\bar{t}_\Omega(\theta_P) \geq [k_1,s_1]$, $\bar{i}_\Omega(\theta_P) \leq [k_2,s_2]$, and $\bar{f}_\Omega(\theta_P) \leq [k_3,s_3]$, $\Omega_{\big([k_1,s_1],[k_2,s_2],[k_3,s_3]\big)}$ is a crisp subring of $(P,+,\cdot)$ (assuming all the IVTN and IVSNs are idempotent).*

Proof. Since, $\bar{t}_\Omega(\theta_P) \geq [k_1,s_1]$, $\bar{i}_\Omega(\theta_P) \leq [k_2,s_2]$, and $\bar{f}_\Omega(\theta_P) \leq [k_3,s_3]$, $\theta_P \in \Omega_{\big([k_1,s_1],[k_2,s_2],[k_3,s_3]\big)}$, i.e., $\Omega_{\big([k_1,s_1],[k_2,s_2],[k_3,s_3]\big)}$ is non-empty.
Now, let $\Omega \in$ IVNSR(P) and $r,v \in \Omega_{\big([k_1,s_1],[k_2,s_2],[k_3,s_3]\big)}$. To show that, $(r-v)$ and $r\cdot v \in \Omega_{\big([k_1,s_1],[k_2,s_2],[k_3,s_3]\big)}$. Here,

$$\begin{aligned}
\bar{t}_\Omega(r-v) &\geq \bar{T}\big(\bar{t}_\Omega(r),\bar{t}_\Omega(v)\big) \text{ [by Proposition 3.1]}\\
&\geq \bar{T}\big([k_1,s_1],[k_1,s_1]\big) \Big[\text{as } r,v \in \Omega_{\big([k_1,s_1],[k_2,s_2],[k_3,s_3]\big)}\Big]\\
&\geq [k_1,s_1] \text{ [as } \bar{T} \text{ is idempotent]} \qquad (3.19)
\end{aligned}$$

Again,

$$\begin{aligned}
\bar{t}_\Omega(r\cdot v) &\geq \bar{T}\big(\bar{t}_\Omega(r),\bar{t}_\Omega(v)\big) \text{ [by Proposition 3.1]}\\
&\geq \bar{T}\big([k_1,s_1],[k_1,s_1]\big) \Big[\text{as } r,v \in \Omega_{\big([k_1,s_1],[k_2,s_2],[k_3,s_3]\big)}\Big]\\
&\geq [k_1,s_1] \text{ [as } \bar{T} \text{ is idempotent]} \qquad (3.20)
\end{aligned}$$

Similarly, we can show that

$$\bar{i}_\Omega(r-v) \leq [k_2,s_2], \qquad (3.21)$$

$$\bar{i}_\Omega(r\cdot v) \leq [k_2,s_2], \qquad (3.22)$$

$$\bar{f}_\Omega(r-v) \leq [k_3,s_3], \text{ and} \qquad (3.23)$$

$$\bar{f}_\Omega(r\cdot v) \leq [k_3,s_3] \qquad (3.24)$$

Hence, by Equations 3.19–3.24 $(r-v)$ and $r\cdot v \in \Omega_{\big([k_1,s_1],[k_2,s_2],[k_3,s_3]\big)}$, i.e., $\Omega_{\big([k_1,s_1],[k_2,s_2],[k_3,s_3]\big)}$ is a crisp subring of $(P,+,\cdot)$.
Conversely, let $\Omega_{\big([k_1,s_1],[k_2,s_2],[k_3,s_3]\big)}$ is a crisp subgroup of $(P,+,\cdot)$. To show that, $\Omega \in$ IVNSR(P).
Let $r,v \in P$, then there exists $[k_1,s_1] \in \Psi(L)$ such that $\bar{T}\big(\bar{t}_\Omega(r),\bar{t}_\Omega(v)\big) = [k_1,s_1]$. So, $\bar{t}_\Omega(r) \geq [k_1,s_1]$ and $\bar{t}_\Omega(v) \geq [k_1,s_1]$. Also, let there exist $[k_2,s_2],[k_3,s_3] \in \Psi(L)$ such that $\bar{I}\big(\bar{i}_\Omega(r),\bar{i}_\Omega(v)\big) = [k_2,s_2]$ and $\bar{F}\big(\bar{f}_\Omega(r),\bar{f}_\Omega(v)\big) = [k_3,s_3]$. Then $r,v \in \Omega_{\big([k_1,s_1],[k_2,s_2],[k_3,s_3]\big)}$.
Again, as $\Omega_{\big([k_1,s_1],[k_2,s_2],[k_3,s_3]\big)}$ is a crisp subring, $r-v \in \Omega_{\big([k_1,s_1],[k_2,s_2],[k_3,s_3]\big)}$ and $r\cdot v \in \Omega_{\big([k_1,s_1],[k_2,s_2],[k_3,s_3]\big)}$.

Hence,

$$\begin{aligned}\bar{t}_\Omega(r-v) &\geq [k_1, s_1] \\ &= \bar{T}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(v)\big) \text{ and} \qquad (3.25)\\ \bar{t}_\Omega(r\cdot v) &\geq [k_1, s_1] \\ &= \bar{T}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(v)\big) \qquad (3.26)\end{aligned}$$

Similarly, we can prove that

$$\begin{aligned}\bar{i}_\Omega(r-v) &\leq [k_2, s_2] \\ &= \bar{I}\big(\bar{i}_\Omega(r), \bar{i}_\Omega(v)\big), \qquad (3.27)\\ \bar{i}_\Omega(r\cdot v) &\leq [k_2, s_2] \\ &= \bar{I}\big(\bar{i}_\Omega(r), \bar{i}_\Omega(v)\big), \qquad (3.28)\\ \bar{f}_\Omega(r-v) &\leq [k_3, s_3] \\ &= \bar{F}\big(\bar{f}_\Omega(r), \bar{f}_\Omega(v)\big), \text{ and} \qquad (3.29)\\ \bar{f}_\Omega(r\cdot v) &\leq [k_3, s_3] \\ &= \bar{F}\big(\bar{f}_\Omega(r), \bar{f}_\Omega(v)\big) \qquad (3.30)\end{aligned}$$

So, Equations 3.25–3.30 imply that Ω follows Proposition 3.1, i.e., $\Omega \in \mathrm{IVNSR}(P)$. □

Definition 3.6. Let Ω and Ω' be two IVNSs of two CSs P and R, respectively. Also, let $l : P \to R$ be a function. Then

(i) image of Ω under l will be $l(\Omega) = \big\{\big(v, \bar{t}_{l(\Omega)}(v), \bar{i}_{l(\Omega)}(v), \bar{f}_{l(\Omega)}(v)\big) : v \in R\big\}$, where $\bar{t}_{l(\Omega)}(v) = \bigvee_{s\in l^{-1}(v)} \bar{t}_\Omega(s)$, $\bar{i}_{l(\Omega)}(v) = \bigwedge_{s\in l^{-1}(v)} \bar{i}_\Omega(s)$, $\bar{f}_{l(\Omega)}(v) = \bigwedge_{s\in l^{-1}(v)} \bar{f}_\Omega(s)$. Wherefrom, if l is injective then $\bar{t}_{l(\Omega)}(v) = \bar{t}_\Omega\big(l^{-1}(v)\big)$, $\bar{i}_{l(\Omega)}(v) = \bar{i}_\Omega\big(l^{-1}(v)\big)$, $\bar{f}_{l(\Omega)}(v) = \bar{f}_\Omega\big(l^{-1}(v)\big)$, and

(ii) preimage of Ω' under l will be $l^{-1}(\Omega') = \big\{\big(r, \bar{t}_{l^{-1}(\Omega')}(r), \bar{i}_{l^{-1}(\Omega')}(r), \bar{f}_{l^{-1}(\Omega')}(r)\big) : r \in R\big\}$, where $\bar{t}_{l^{-1}(\Omega')}(r) = \bar{t}_{\Omega'}\big(l(r)\big)$, $\bar{i}_{l^{-1}(\Omega')}(r) = \bar{i}_{\Omega'}\big(l(r)\big)$, $\bar{f}_{l^{-1}(\Omega')}(r) = \bar{f}_{\Omega'}\big(l(r)\big)$.

Theorem 3.6. *Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings. Also, let $l : P \to R$ be a ring isomorphism. If Ω is an IVNSR of P then $l(\Omega)$ is an IVNSR of R.*

Proof. Let $v_1 = l(r_1)$ and $v_2 = l(r_2)$, where $r_1, r_2 \in P$ and $v_1, v_2 \in R$. Now,

$$\begin{aligned}\bar{t}_{l(\Omega)}(v_1 - v_2) &= \bar{t}_{\Omega}\big(l^{-1}(v_1 - v_2)\big) \text{ [as } l \text{ is injective]}\\ &= \bar{t}_{\Omega}\big(l^{-1}(v_1) - l^{-1}(v_2)\big) \text{ [as } l^{-1} \text{ is a homomorphism]}\\ &= \bar{t}_{\Omega}(r_1 - r_2)\\ &\geq \bar{T}\big(\bar{t}_{\Omega}(r_1), \bar{t}_{\Omega}(r_2)\big)\\ &= \bar{T}\Big(\bar{t}_{\Omega}\big(l^{-1}(v_1)\big), \bar{t}_{\Omega}\big(l^{-1}(v_2)\big)\Big)\\ &= \bar{T}\big(\bar{t}_{l(\Omega)}(v_1), \bar{t}_{l(\Omega)}(v_2)\big) \qquad (3.31)\end{aligned}$$

Again,

$$\begin{aligned}\bar{t}_{l(\Omega)}(v_1 \cdot v_2) &= \bar{t}_{\Omega}\big(l^{-1}(v_1 \cdot v_2)\big) \text{ [as } l \text{ is injective]}\\ &= \bar{t}_{\Omega}\big(l^{-1}(v_1) \cdot l^{-1}(v_2)\big) \text{ [as } l^{-1} \text{ is a homomorphism]}\\ &= \bar{t}_{\Omega}(r_1 \cdot r_2)\\ &\geq \bar{T}\big(\bar{t}_{\Omega}(r_1), \bar{t}_{\Omega}(r_2)\big)\\ &= \bar{T}\Big(\bar{t}_{\Omega}\big(l^{-1}(v_1)\big), \bar{t}_{\Omega}\big(l^{-1}(v_2)\big)\Big)\\ &= \bar{T}\big(\bar{t}_{l(\Omega)}(v_1), \bar{t}_{l(\Omega)}(v_2)\big) \qquad (3.32)\end{aligned}$$

Similarly,

$$\bar{i}_{l(\Omega)}(v_1 - v_2) \leq \bar{I}\big(\bar{i}_{l(\Omega)}(v_1), \bar{i}_{l(\Omega)}(v_2)\big), \qquad (3.33)$$

$$\bar{i}_{l(\Omega)}(v_1 \cdot v_2) \leq \bar{I}\big(\bar{i}_{l(\Omega)}(v_1), \bar{i}_{l(\Omega)}(v_2)\big), \qquad (3.34)$$

$$\bar{f}_{l(\Omega)}(v_1 - v_2) \leq \bar{F}\big(\bar{f}_{l(\Omega)}(v_1), \bar{f}_{l(\Omega)}(v_2)\big), \text{ and} \qquad (3.35)$$

$$\bar{f}_{l(\Omega)}(v_1 \cdot v_2) \leq \bar{F}\big(\bar{f}_{l(\Omega)}(v_1), \bar{f}_{l(\Omega)}(v_2)\big) \qquad (3.36)$$

Hence, Equations 3.31–3.36 imply that $l(\Omega)$ follows Proposition 3.1, i.e., $l(\Omega)$ is an IVNSR of R. □

Theorem 3.7. *Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings. Also, let $l : P \to R$ be a ring homomorphism. If Ω' is an IVNSR of R then $l^{-1}(\Omega')$ is an IVNSR of P (Note that, l^{-1} may not be an inverse mapping but $l^{-1}(\Omega')$ is an inverse image of Ω').*

Proof. Let $v_1 = l(r_1)$ and $v_2 = l(r_2)$, where $r_1, r_2 \in P$ and $v_1, v_2 \in R$. Now,

$$\begin{aligned}\bar{t}_{l^{-1}(\Omega')}(r_1 - r_2) &= \bar{t}_{\Omega'}\big(l(r_1 - r_2)\big)\\ &= \bar{t}_{\Omega'}\big(l(r_1) - l(r_2)\big) \text{ [as } l \text{ is a homomorphism]}\\ &= \bar{t}_{\Omega'}(v_1 - v_2)\\ &\geq \bar{T}\big(\bar{t}_{\Omega'}(v_1), \bar{t}_{\Omega'}(v_2)\big)\\ &= \bar{T}\Big(\bar{t}_{\Omega'}\big(l(r_1)\big), \bar{t}_{\Omega'}\big(l(r_2)\big)\Big)\\ &= \bar{T}\big(\bar{t}_{l^{-1}(\Omega')}(r_1), \bar{t}_{l^{-1}(\Omega')}(r_2)\big) \qquad (3.37)\end{aligned}$$

Again,

$$\begin{aligned}\bar{t}_{l^{-1}(\Omega')}(r_1 \cdot r_2) &= \bar{t}_{\Omega'}\big(l(r_1 \cdot r_2)\big)\\ &= \bar{t}_{\Omega'}\big(l(r_1) \cdot l(r_2)\big) \text{ [as } l \text{ is a homomorphism]}\\ &= \bar{t}_{\Omega'}(v_1 \cdot v_2)\\ &\geq \bar{T}\big(\bar{t}_{\Omega'}(v_1), \bar{t}_{\Omega'}(v_2)\big)\\ &= \bar{T}\Big(\bar{t}_{\Omega'}\big(l(r_1)\big), \bar{t}_{\Omega'}\big(l(r_2)\big)\Big)\\ &= \bar{T}\big(\bar{t}_{l^{-1}(\Omega')}(r_1), \bar{t}_{l^{-1}(\Omega')}(r_2)\big) \qquad (3.38)\end{aligned}$$

Similarly,

$$\bar{i}_{l^{-1}(\Omega')}(r_1 - r_2) \leq \bar{I}\big(\bar{i}_{l^{-1}(\Omega')}(r_1), \bar{i}_{l^{-1}(\Omega')}(r_2)\big) \qquad (3.39)$$

$$\bar{i}_{l^{-1}(\Omega')}(r_1 \cdot r_2) \leq \bar{I}\big(\bar{i}_{l^{-1}(\Omega')}(r_1), \bar{i}_{l^{-1}(\Omega')}(r_2)\big) \qquad (3.40)$$

$$\bar{f}_{l^{-1}(\Omega')}(r_1 - r_2) \leq \bar{F}\big(\bar{f}_{l^{-1}(\Omega')}(r_1), \bar{f}_{l^{-1}(\Omega')}(r_2)\big) \qquad (3.41)$$

$$\bar{f}_{l^{-1}(\Omega')}(r_1 \cdot r_2) \leq \bar{F}\big(\bar{f}_{l^{-1}(\Omega')}(r_1), \bar{f}_{l^{-1}(\Omega')}(r_2)\big) \qquad (3.42)$$

Hence, Equations 3.37–3.42 imply that $l^{-1}(\Omega')$ follows Proposition 3.1, i.e., $l^{-1}(\Omega')$ is an IVNSR of P. □

Definition 3.7. Let $(P, +, \cdot)$ be a crisp ring and $\Omega \in \text{IVNSR}(P)$. Again, let $\bar{\sigma} = [\sigma_1, \sigma_2], \bar{\tau} = [\tau_1, \tau_2], \bar{\delta} = [\delta_1, \delta_2] \in \Psi(L)$. Then

(i) Ω is called a $(\bar{\sigma},\bar{\tau},\bar{\delta})-$identity IVNSR over P, if $\forall r\in P$

$$\bar{t}_\Omega(r)=\begin{cases}\bar{\sigma} & \text{if } r=\theta_P\\ [0,0] & \text{if } r\neq\theta_P\end{cases},$$

$$\bar{i}_\Omega(r)=\begin{cases}\bar{\tau} & \text{if } r=\theta_P\\ [1,1] & \text{if } r\neq\theta_P\end{cases}, \text{ and}$$

$$\bar{f}_\Omega(r)=\begin{cases}\bar{\delta} & \text{if } r=\theta_P\\ [1,1] & \text{if } r\neq\theta_P\end{cases},$$

where θ_P is the zero element of P.

(ii) Ω is called a $(\bar{\sigma},\bar{\tau},\bar{\delta})-$absolute IVNSR over P, if $\forall r\in P$, $\bar{t}_\Omega(r)=\bar{\sigma}$, $\bar{i}_\Omega(r)=\bar{\tau}$, and $\bar{f}_\Omega(r)=\bar{\delta}$.

Theorem 3.8. *Let $(P,+,\cdot)$ and $(R,+,\cdot)$ be two crisp rings and $\Omega\in$IVNSR (P). Again, let $l:P\to R$ be a ring homomorphism. Then*

(i) *$l(\Omega)$ will be a $(\bar{\sigma},\bar{\tau},\bar{\delta})-$identity IVNSR over R, if $\forall r\in P$*

$$\bar{t}_\Omega(r)=\begin{cases}\bar{\sigma} & \textit{if } r\in Ker(l)\\ [0,0] & \textit{otherwise}\end{cases},$$

$$\bar{i}_\Omega(r)=\begin{cases}\bar{\tau} & \textit{if } r\in Ker(l)\\ [1,1] & \textit{otherwise}\end{cases}, \textit{ and}$$

$$\bar{f}_\Omega(r)=\begin{cases}\bar{\delta} & \textit{if } r\in Ker(l)\\ [1,1] & \textit{otherwise}\end{cases},$$

(ii) *$l(\Omega)$ will be a $(\bar{\sigma},\bar{\tau},\bar{\delta})-$absolute IVNSR over R, if Ω is a $(\bar{\sigma},\bar{\tau},\bar{\delta})-$absolute IVNSR over P.*

Proof. (i) Clearly, by Theorem 3.6 $l(\Omega)\in$ IVNSR(R). Let $r\in Ker(l)$, then $l(r)=\theta_R$. So,

$$\begin{aligned}\bar{t}_{l(\Omega)}(\theta_R)&=\bar{t}_\Omega\big(l^{-1}(\theta_R)\big)\\&=\bar{t}_\Omega(r)\\&=\bar{\sigma}\end{aligned}\tag{3.43}$$

Similarly, we can show that

$$\bar{i}_{l(\Omega)}(\theta_R)=\bar{\tau}, \text{ and}\tag{3.44}$$

$$\bar{f}_{l(\Omega)}(\theta_R)=\bar{\delta}\tag{3.45}$$

Again, let $r \in P \setminus Ker(l)$ and $l(r) = v$. Then

$$\begin{aligned} \bar{t}_{l(\Omega)}(v) &= \bar{t}_{\Omega}\big(l^{-1}(v)\big) \\ &= \bar{t}_{\Omega}(r) \\ &= [0,0] \end{aligned} \tag{3.46}$$

Similarly, we can show that

$$\bar{i}_{l(\Omega)}(v) = [1,1] \text{ and} \tag{3.47}$$

$$\bar{f}_{l(\Omega)}(v) = [1,1] \tag{3.48}$$

Hence, by the Equations 3.43–3.48 $l(\Omega)$ is a $(\bar{\sigma}, \bar{\tau}, \bar{\delta})-$identity IVNSR over R.

(ii) Let $l(r) = v$, for $r \in P$ and $v \in R$. Then

$$\begin{aligned} \bar{t}_{l(\Omega)}(v) &= \bar{t}_{\Omega}\big(l^{-1}(v)\big) \\ &= \bar{t}_{\Omega}(r) \\ &= \bar{\sigma} \end{aligned} \tag{3.49}$$

Similarly, we can show that

$$\bar{i}_{l(\Omega)}(v) = \bar{\tau} \text{ and} \tag{3.50}$$

$$\bar{f}_{l(\Omega)}(v) = \bar{\delta} \tag{3.51}$$

Hence, by the Equations 3.48–3.51 $l(\Omega)$ is a $(\bar{\sigma}, \bar{\tau}, \bar{\delta})-$absolute IVNSR over R. □

3.1. *Product of interval-valued neutrosophic subrings*

Definition 3.8. Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings. Again, let $\Omega_1 = \big\{\big(r, \bar{t}_{\Omega_1}(r), \bar{i}_{\Omega_1}(r), \bar{f}_{\Omega_1}(r)\big) : r \in P\big\}$ and $\Omega_2 = \big\{\big(v, \bar{t}_{\Omega_2}(v), \bar{i}_{\Omega_2}(v), \bar{f}_{\Omega_2}(v)\big) : v \in R\big\}$ are IVNSRs of P and R respectively. Then Cartesian product of Ω_1 and Ω_2 will be

$$\begin{aligned} \Omega &= \Omega_1 \times \Omega_2 \\ &= \Big\{\Big((r,v), \bar{T}\big(\bar{t}_{\Omega_1}(r), \bar{t}_{\Omega_2}(v)\big), \bar{I}\big(\bar{i}_{\Omega_1}(r), \bar{i}_{\Omega_2}(v)\big), \bar{F}\big(\bar{f}_{\Omega_1}(r), \bar{f}_{\Omega_2}(v)\big)\Big) : (r,v) \in P \times R\Big\} \end{aligned}$$

Similarly, product of 3 or more IVNSRs can be defined.

Theorem 3.9. *Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings with $\Omega_1 \in$ IVNSR(P) and $\Omega_2 \in$ IVNSR(R). Then $\Omega_1 \times \Omega_2$ is a IVNSR of $P \times R$.*

Proof. Let $\Omega = \Omega_1 \times \Omega_2$ and $(r_1, v_1), (r_2, v_2) \in P \times R$. Then

$$\begin{aligned}
\bar{t}_\Omega\big((r_1, v_1) - (r_2, v_2)\big) &= \bar{t}_{\Omega_1 \times \Omega_2}\big((r_1 - r_2, v_1 - v_2)\big) \\
&= \bar{T}\big(\bar{t}_{\Omega_1}(r_1 - r_2), \bar{t}_{\Omega_2}(v_1 - v_2)\big) \text{ [by Definition 3.8]} \\
&\geq \bar{T}\Big(\bar{T}\big(\bar{t}_{\Omega_1}(r_1), \bar{t}_{\Omega_1}(r_2)\big), \bar{T}\big(\bar{t}_{\Omega_2}(v_1), \bar{t}_{\Omega_2}(v_2)\big)\Big) \text{ [by Proposition 3.1]} \\
&= \bar{T}\Big(\bar{T}\big(\bar{t}_{\Omega_1}(r_1), \bar{t}_{\Omega_2}(v_1)\big), \bar{T}\big(\bar{t}_{\Omega_1}(r_2), \bar{t}_{\Omega_2}(v_2)\big)\Big) \text{ [as } \bar{T} \text{ is associative]} \\
&= \bar{T}\big(\bar{t}_\Omega(r_1, v_1), \bar{t}_\Omega(r_2, v_2)\big) \qquad (3.52)
\end{aligned}$$

Again,

$$\begin{aligned}
\bar{t}_\Omega\big((r_1, v_1) \cdot (r_2, v_2)\big) &= \bar{t}_{\Omega_1 \times \Omega_2}\big((r_1 \cdot r_2, v_1 \cdot v_2)\big) \\
&= \bar{T}\big(\bar{t}_{\Omega_1}(r_1 \cdot r_2), \bar{t}_{\Omega_2}(v_1 \cdot v_2)\big) \text{ [by Definition 3.8]} \\
&\geq \bar{T}\Big(\bar{T}\big(\bar{t}_{\Omega_1}(r_1), \bar{t}_{\Omega_1}(r_2)\big), \bar{T}\big(\bar{t}_{\Omega_2}(v_1), \bar{t}_{\Omega_2}(v_2)\big)\Big) \text{ [by Proposition 3.1]} \\
&= \bar{T}\Big(\bar{T}\big(\bar{t}_{\Omega_1}(r_1), \bar{t}_{\Omega_2}(v_1)\big), \bar{T}\big(\bar{t}_{\Omega_1}(r_2), \bar{t}_{\Omega_2}(v_2)\big)\Big) \text{ [as } \bar{T} \text{ is associative]} \\
&= \bar{T}\big(\bar{t}_\Omega(r_1, v_1), \bar{t}_\Omega(r_2, v_2)\big) \qquad (3.53)
\end{aligned}$$

Similary, the followings can be shown

$$\bar{i}_\Omega\big((r_1, v_1) - (r_2, v_2)\big) \leq \bar{I}\big(\bar{i}_\Omega(r_1, v_1), \bar{i}_\Omega(r_2, v_2)\big), \qquad (3.54)$$

$$\bar{i}_\Omega\big((r_1, v_1) \cdot (r_2, v_2)\big) \leq \bar{I}\big(\bar{i}_\Omega(r_1, v_1), \bar{i}_\Omega(r_2, v_2)\big), \qquad (3.55)$$

$$\bar{f}_\Omega\big((r_1, v_1) - (r_2, v_2)\big) \leq \bar{F}\big(\bar{f}_\Omega(r_1, v_1), \bar{f}_\Omega(r_2, v_2)\big), \text{ and} \qquad (3.56)$$

$$\bar{f}_\Omega\big((r_1, v_1) \cdot (r_2, v_2)\big) \leq \bar{F}\big(\bar{f}_\Omega(r_1, v_1), \bar{f}_\Omega(r_2, v_2)\big) \qquad (3.57)$$

Hence, using Proposition 3.1 and by Equations 3.52–3.57 $\Omega_1 \times \Omega_2 \in$ IVNSR$(P \times R)$. □

Corollary 3.10. *Let $\forall i \in \{1, 2, ..., n\}$, $(P_i, +, \cdot)$ are crisp rings and $\Omega_i \in$ IVNSR(P_i). Then $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$ is a IVNSR of $P_1 \times P_2 \times \cdots \times P_n$, where $n \in \mathbb{N}$.*

3.2. *Subring of a interval-valued neutrosophic subgring*

Definition 3.9. Let $(P, +, \cdot)$ be a crisp ring and $\Omega_1, \Omega_2 \in$ IVNSR(P), where $\Omega_1 = \big\{\big(r, \bar{t}_{\Omega_1}(r), \bar{i}_{\Omega_1}(r), \bar{f}_{\Omega_1}(r)\big) : r \in P\big\}$ and $\Omega_2 = \big\{\big(r, \bar{t}_{\Omega_2}(r), \bar{i}_{\Omega_2}(r), \bar{f}_{\Omega_2}(r)\big) : r \in P\big\}$. Then Ω_1 is called a subring of Ω_2 if $\forall r \in P$, $\bar{t}_{\Omega_1}(r) \leq \bar{t}_{\Omega_2}(r)$, $\bar{i}_{\Omega_1}(r) \geq \bar{i}_{\Omega_2}(r)$, and $\bar{f}_{\Omega_1}(r) \geq \bar{f}_{\Omega_2}(r)$.

Theorem 3.11. *Let $(P, +, \cdot)$ be a crisp ring and $\Omega \in$ IVNSR(P). Again, let Ω_1 and Ω_2 be two subrings of Ω. Then $\Omega_1 \cap \Omega_2$ is also a subring of Ω, assuming that all the IVTN and IVSNs are idempotent.*

Proof. Here, $\forall r \in P$

$$\begin{aligned} \bar{t}_{\Omega_1 \cap \Omega_2}(r) &= \bar{T}\big(\bar{t}_{\Omega_1}(r), \bar{t}_{\Omega_2}(r)\big) \\ &\leq \bar{T}\big(\bar{t}_{\Omega}(r), \bar{t}_{\Omega}(r)\big) \\ &= \bar{t}_{\Omega}(r) \text{ [as } \bar{T} \text{ is idempotent]} \end{aligned} \tag{3.58}$$

Similarly, as $\bar{I}$ and $\bar{F}$ are idempotent we can show that,

$$\bar{i}_{\Omega_1 \cap \Omega_2}(r) \geq \bar{i}_{\Omega}(r) \text{ and} \tag{3.59}$$

$$\bar{f}_{\Omega_1 \cap \Omega_2}(r) \geq \bar{f}_{\Omega}(r) \tag{3.60}$$

Hence, by Equations 3.58–3.60 $\Omega_1 \cap \Omega_2$ is a subring of Ω. □

Theorem 3.12. *Let $(P, +, \cdot)$ be a crisp ring and $\Omega_1, \Omega_2 \in IVNSR(P)$ such that Ω_1 is a subring of Ω_2. Let $(R, +, \cdot)$ is another crisp ring and $l : P \to R$ be a ring isomorphism. Then*

(i) *$l(\Omega_1)$ and $l(\Omega_2)$ are two IVNSRs over R and*

(ii) *$l(\Omega_1)$ is a subring of $l(\Omega_2)$.*

Proof. (i) can be proved by using Theorem 3.6.
(ii) Let $v = l(r)$, where $r \in P$ and $v \in R$. Then

$$\begin{aligned} & \bar{t}_{\Omega_1}(r) \leq \bar{t}_{\Omega_2}(r) \text{ [as } \Omega_1 \text{ is a subring of } \Omega_2] \\ \Rightarrow & \bar{t}_{\Omega_1}\big(l^{-1}(v)\big) \leq \bar{t}_{\Omega_2}\big(l^{-1}(v)\big) \\ \Rightarrow & \bar{t}_{l(\Omega_1)}(v) \leq \bar{t}_{l(\Omega_2)}(v) \end{aligned} \tag{3.61}$$

Similarly, we can prove that

$$\bar{i}_{l(\Omega_1)}(v) \geq \bar{i}_{l(\Omega_2)}(v) \text{ and} \tag{3.62}$$

$$\bar{f}_{l(\Omega_1)}(v) \geq \bar{f}_{l(\Omega_2)}(v) \tag{3.63}$$

Hence, by Equations 3.61–3.63 $l(\Omega_1)$ is a subring of $l(\Omega_2)$. □

3.3. *Interval-valued neutrosophic normal subrings*

Definition 3.10. Let $(P, +, \cdot)$ be a crisp ring and Ω is an IVNS of P, where $\Omega = \big\{\big(r, \bar{t}_{\Omega}(r), \bar{i}_{\Omega}(r), \bar{f}_{\Omega}(r)\big) : r \in P\big\}$. Then Ω is called an IVNNSR over P if

(i) Ω is an IVNSR of P and

(ii) $\forall r, v \in P$, $\bar{t}_{\Omega}(r \cdot v) = \bar{t}_{\Omega}(v \cdot r)$, $\bar{i}_{\Omega}(r \cdot v) = \bar{i}_{\Omega}(v \cdot r)$, and $\bar{f}_{\Omega}(r \cdot v) = \bar{f}_{\Omega}(v \cdot r)$.

The set of all IVNNSR of a crisp ring $(P, +, \cdot)$ will be denoted as IVNNSR(P).

Example 3.11. Let $(\mathbb{Z}, +, \cdot)$ be the ring of integers with respect to usual addition and multiplication. Let $\Omega = \{(r, \bar{t}_\Omega(r), \bar{i}_\Omega(r), \bar{f}_\Omega(r)) : r \in \mathbb{Z}\}$ be an IVNS of $\mathbb{Z}$, where $\forall r \in \mathbb{Z}$

$$\bar{t}_\Omega(r) = \begin{cases} [0.67, 1] & \text{if } r \in 2\mathbb{Z} \\ [0, 0] & \text{if } r \in 2\mathbb{Z}+1 \end{cases},$$

$$\bar{i}_\Omega(r) = \begin{cases} [0, 0] & \text{if } r \in 2\mathbb{Z} \\ [0.33, 0.5] & \text{if } r \in 2\mathbb{Z}+1 \end{cases}, \text{ and}$$

$$\bar{f}_\Omega(r) = \begin{cases} [0, 0] & \text{if } r \in 2\mathbb{Z} \\ [0, 0.33] & \text{if } r \in 2\mathbb{Z}+1 \end{cases}.$$

Now, if we consider minimum TN and maximum SNs, then $\Omega \in \text{IVNNSR}(\mathbb{Z})$.

Theorem 3.13. *Let $(P, +, \cdot)$ be a crisp ring. If $\Omega_1, \Omega_2 \in IVNNSR(P)$, then $\Omega_1 \cap \Omega_2 \in IVNNSR(P)$.*

Proof. As $\Omega_1, \Omega_2 \in \text{IVNSR}(P)$ by Theorem 3.2 $\Omega_1 \cap \Omega_2 \in \text{IVNSR}(P)$. Again,

$$\begin{aligned} \bar{t}_{\Omega_1 \cap \Omega_2}(r \cdot v) &= \bar{T}\big(\bar{t}_{\Omega_1}(r \cdot v), \bar{t}_{\Omega_2}(r \cdot v)\big) \\ &= \bar{T}\big(\bar{t}_{\Omega_1}(v \cdot r), \bar{t}_{\Omega_2}(v \cdot r)\big) \text{ [as } \Omega_1, \Omega_2 \in \text{IVNNSR}(P)] \\ &= \bar{t}_{\Omega_1 \cap \Omega_2}(v \cdot r) \end{aligned} \tag{3.64}$$

Similarly,

$$\bar{i}_{\Omega_1 \cap \Omega_2}(r \cdot v) = \bar{i}_{\Omega_1 \cap \Omega_2}(v \cdot r) \tag{3.65}$$

$$\bar{f}_{\Omega_1 \cap \Omega_2}(r \cdot v) = \bar{f}_{\Omega_1 \cap \Omega_2}(v \cdot r) \tag{3.66}$$

Hence, $\Omega_1 \cap \Omega_2 \in \text{IVNNSR}(P)$. □

Remark 3.14. *In general, if $\Omega_1, \Omega_2 \in IVNNSR(P)$, then $\Omega_1 \cup \Omega_2$ may not always be an IVNNSR of $(P, +, \cdot)$.*

Remark 3.14 can be proved by Example 3.4.

Theorem 3.15. *Let $(P, +, \cdot)$ be a crisp ring. Then $\Omega \in IVNNSR(P)$ iff $\forall [k_1, s_1], [k_2, s_2], [k_3, s_3] \in \Psi(L)$ with $\bar{t}_\Omega(\theta_P) \geq [k_1, s_1]$, $\bar{i}_\Omega(\theta_P) \leq [k_2, s_2]$, and $\bar{f}_\Omega(\theta_P) \leq [k_3, s_3]$, $\Omega_{\big([k_1,s_1],[k_2,s_2],[k_3,s_3]\big)}$ is a crisp normal subring of $(P, +, \cdot)$ (assuming all the IVTN and IVSNs are idempotent).*

Proof. This can be proved using Theorem 3.5. □

Theorem 3.16. *Let $(P,+,\cdot)$ and $(R,+,\cdot)$ be two crisp rings. Also, let $l : P \to R$ be a ring isomorphism. If Ω is an IVNNSR of P then $l(\Omega)$ is an IVNNSR of R.*

Proof. As Ω is an IVNSR of P by Theorem 3.6 $l(\Omega)$ is an IVNSR of R. Let $l(r_1) = v_1$ and $l(r_2) = v_2$, where $r_1, r_2 \in P$ and $v_1, v_2 \in R$. Then

$$\begin{aligned}
\bar{t}_{l(\Omega)}(v_1 \cdot v_2) &= \bar{t}_{\Omega}\big(l^{-1}(v_1 \cdot v_2)\big) \text{ [as } l \text{ is injective]} \\
&= \bar{t}_{\Omega}\big(l^{-1}(v_1) \cdot l^{-1}(v_2)\big) \text{ [as } l^{-1} \text{ is a homomorphism]} \\
&= \bar{t}_{\Omega}(r_1 \cdot r_2) \\
&= \bar{t}_{\Omega}(r_2 \cdot r_1) \text{ [as } \Omega \text{ is an IVNNSR of } P\text{]} \\
&= \bar{t}_{\Omega}\big(l^{-1}(v_2) \cdot l^{-1}(v_1)\big) \\
&= \bar{t}_{\Omega}\big(l^{-1}(v_2 \cdot v_1)\big) \\
&= \bar{t}_{l(\Omega)}(v_2 \cdot v_1) \qquad (3.67)
\end{aligned}$$

Similarly,

$$\bar{i}_{l(\Omega)}(v_1 \cdot v_2) = \bar{i}_{l(\Omega)}(v_2 \cdot v_1) \text{ and} \qquad (3.68)$$

$$\bar{f}_{l(\Omega)}(v_1 \cdot v_2) = \bar{f}_{l(\Omega)}(v_2 \cdot v_1) \qquad (3.69)$$

Hence, by Equations 3.67–3.69 $l(\Omega)$ is an IVNNSR of R. □

4. Conclusions

As interval-valued neutrosophic environment is more general than regular one, we have adopted it and defined the notions of interval-valued neutrosophic subring and its normal version. Also, we have analyzed some homomorphic properties of these newly defined notions. Again, we have studied product of two interval-valued neutrosophic subrings. Furthermore, we have provided some essential theories to study some of their algebraic structures. These newly introduced notions have potentials to become fruitful research areas. For instance, soft set theory can be implemented and the notion of interval-valued neutrosophic soft subring can be defined.

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About Neutrosophic Countably Comapctness

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Abstract. We answer the following question: Are neutrosophic $\underline{\mu}$-compactness and neutrosophic $\underline{\mu}$-countably compactness equivalent? which posted in [10]. Since every neutrosophic topology is neutrosophic $\underline{\mu}$-topology, we answer the question for neutrosophic topological spaces, more precisely, we give an example of neutrosophic topology which is neutrosophic countably comapact but not neutrosophic compact

Keywords: Neutrosophic topological spaces; Neutrosophic compact; Neutrosophic Lindelöf; Neutrosophic countably compact space

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1. **Introduction**

Neutrosophic sets first introduced in [25, 27] as a generalization of intuitionistic fuzzy sets [14], where each element $x \in X$ has a degree of indeterminacy with the degree of membership and the degree of non-membership . Operations on neutrosophic sets are investigated after that. Neutrosophic topological spaces are studied by Smarandache [27], Lupianez [19,20] and Salama [23]. The interior , closure, exterior and boundary of neutrosophic sets can be found in [26]. Neutrosophic sets applied to generalize many notaions about soft topology and applications [18], [22], [15], generalized open and closed sets [28] , fixed point theorems [18] , graph theory [17]and rough topology and applications [21]. Neutrosophy has many applications specially in decision making, for more details about new trends of neutrosophic applications one can consult [1]- [7].

Generalized topology and continuity introduced in 2002 in [13], where many generalized open sets in general topology become examples in generalized topological spaces, and it become one of the most important generalization in topology which has different properties than general topology, see for example [9], [11] and [12]. There are a lot of studies about neutrosophic topological spaces that shows the importance of studying neutrosophic topology where it has possible applications, see for example [24], Neutrosophic μ-topological spaces first introduced in [10], and since Neutrosophic μ-topological space is a generalization of neutrosophic topological space it guarantees generalized results that are still true for neutrosophic topological spaces, see for example Theorem 2.30 in [10] which shows that neutrosophic μ-compactness and neutrosophic μ-countably compactness are equivalent, and this is not true in crisp topology, but it becomes true for neutrosophic topological spaces since every neutrosophic μ-topological space is neutrosophic topological space, another thing about the importance of neutrosophic μ-topological space is that some existing notations about neutrosophic topology can be considered as examples of neurosophic μ-topological spaces, see for example Theorem 2.9 in [10] which shows the relationship between μ-topological space and previous studies where we can consider all neutrosophic α-open sets over (X; τ) and all neutrosophic pre-closed sets in (X; τ) (introduced in [8]) as examples of strong neutrosophic μ-topology over X. The following question appeared in [10].

Definition 1.1. [25]: A set A is said *neutrosophic* on X if $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle; x \in X\}$; $\mu, \sigma, \nu : X \to]^-0, 1^+[$ and $^-0 \leq \mu(x) + \sigma(x) + \nu(x) \leq 3^+$.

The class of all neutrosophic set on the universe X is by $\mathcal{N}(X)$. We will exhibit the basic neutrosophic operations definitions (union, intersection and complement. Since there are different definitions of neutrosophic operations, we will organize the existing definitions into two types, in each type these operation will be consistent and functional.

Definition 1.2. [24][*Neutrosophic sets operations*] Let $A, A_\alpha, B \in \mathcal{N}(X)$ such that $\alpha \in \Delta$. Then we define the neutrsophic:

(1) (*Inclusion*): $A \sqsubseteq B$ If $\mu_A(x) \leq \mu_B(x)$, $\sigma_A(x) \geq \sigma_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.
(2) (*Equality*): $A = B \Leftrightarrow A \sqsubseteq B \wedge B \sqsubseteq A$.
(3) (*Intersection*) $\underset{\alpha\in\Delta}{\sqcap} A_\alpha(x) = \{\langle x, \underset{\alpha\in\Delta}{\wedge} \mu_{A_\alpha}(x), \underset{\alpha\in\Delta}{\vee} \sigma_A(x), \underset{\alpha\in\Delta}{\vee} \nu_A(x)\rangle; x \in X\}$.
(4) (*Union*) $\underset{\alpha\in\Delta}{\sqcup} A_\alpha(x) = \{\langle x, \underset{\alpha\in\Delta}{\vee} \mu_{A_\alpha}(x), \underset{\alpha\in\Delta}{\wedge} \sigma_A(x), \underset{\alpha\in\Delta}{\wedge} \nu_A(x)\rangle; x \in X\}$.
(5) (*Complement*) $A^c = \{\langle x, \nu_A(x), 1 - \sigma_A(x), \mu_A(x)\rangle; x \in X\}$
(6) (*Universal set*) $1_X = \{\langle x, 1, 0, 0\rangle; x \in X\}$; will be called the *neutrosophic universal set*.
(7) (*Empty set*) $0_X = \{\langle x, 0, 1, 1\rangle; x \in X\}$; will be called the *neutrosophic empty set*.

Proposition 1.3. *[24] For $A, A_\alpha \in \mathcal{N}(X)$ for every $\alpha \in \Delta$ we have:*

(1) $A \sqcap (\underset{\alpha\in\Delta}{\sqcup} A_\alpha) = \underset{\alpha\in\Delta}{\sqcup} (A \sqcap A_\alpha)$.
(2) $A \sqcup (\underset{\alpha\in\Delta}{\sqcap} A_\alpha) = \underset{\alpha\in\Delta}{\sqcap} (A \sqcup A_\alpha)$.

Definition 1.4. [24] [*Neutrosophic Topology*] $\tau \subset \mathcal{N}(X)$ is called a *neutrosophic topology* for X if

(1) $0_X, 1_X \in \tau$.
(2) If $A_\alpha \in \tau$ for every $\alpha \in \Delta$, then $\sqcup_{\alpha\in\Delta} A_\alpha \in \tau$,
(3) For every $A, B \in \tau$, we have $A, B \in \tau$.

The ordered pair (X, τ) will be said a *neutrosophic space* over X. The elements of τ will be called *neutrosophic open sets*. For any $A \in \mathcal{N}(X)$, If $A^c \in \tau$, then we say A is *neutrosophic closed*.

2. Neutrosophic Countably Compact Spaces

Definition 2.1. [10] Let X be nonempty, $0 < \alpha, \beta, \gamma < 1$. Then $A \in \mathcal{N}(X)$ is said a *neutrosophic point* iff there exists $x \in X$ such that $A = \{\langle x, \alpha, \beta, \gamma\rangle\} \cup \{\langle \acute{x}, 0, 1, 1\rangle; \acute{x} \neq x\}$. Neutrosophic points will be denoted by $x_{\alpha,\beta,\gamma}$.

Definition 2.2. [10] We say $x_{\alpha,\beta,\gamma}$ in the neutrosophic set A -in symbols $x_{\alpha,\beta,\gamma} \in A$- iff $\alpha < \mu_A(x), \beta > \sigma_A(x)$ and $\gamma > \nu_A(x)$.

Lemma 2.3. *[10] Let $A \in \mathcal{N}(X)$ and suppose that for every $x_{\alpha,\beta,\gamma} \in A$ there exists $B(x_{\alpha,\beta,\gamma}) \in \mathcal{N}(X)$ such that $x_{\alpha,\beta,\gamma} \in B(x_{\alpha,\beta,\gamma}) \sqsubseteq A$. Then $A = \sqcup\{B(x_{\alpha,\beta,\gamma}); x_{\alpha,\beta,\gamma} \in A\}$.*

Corollary 2.4. *[10] $A \in \mathcal{N}(X)$ is neutrosophic open in (X, τ) iff for every $x_{\alpha,\beta,\gamma} \in A$ there exists a neutrosophic set $B(x_{\alpha,\beta,\gamma}) \in \tau$; $x_{\alpha,\beta,\gamma} \in B(x_{\alpha,\beta,\gamma}) \sqsubseteq A$.*

Definition 2.5. [10] Let (X, τ) be a neutrosophic topology on X. A sub-collection $\mathcal{B} \subseteq \tau$ is called a neutrosophic base for τ if for any $U \in \tau$ there exists $\acute{\mathcal{B}} \subseteq \mathcal{B}$ such that $U = \sqcup\{B; B \in \acute{\mathcal{B}}\}$.

Definition 2.6. [10] Consider the neutrosophic space (X, τ) . We say the collection $\mathcal{U}$ from τ is a *neutrosophic open cover* of X, if $1_X = \sqcup\{U; U \in \mathcal{U}\}$.

Definition 2.7. [10] Consider the space (X, τ) and the neutrosophic open cover $\mathcal{U}$ of X. Then we say the sub-collection $\acute{\mathcal{U}} \subseteq \mathcal{N}(X)$ is a *neutrosophic subcover* of X from $\mathcal{U}$, if $\acute{\mathcal{U}}$ is neutrosophic covers X and $\acute{\mathcal{U}} \subseteq \mathcal{U}$.

The following is an immediate result of Corollary2.4.

Corollary 2.8. *[10] A sub-collection $\mathcal{U}$ from the neutrosophic space (X, τ) is an open cover of X iff for every $x_{\alpha,\beta,\gamma}$ in X there exists $U \in \mathcal{U}$ such that $x_{\alpha,\beta,\gamma} \in U$.*

Theorem 2.9. *Consider the collection $\mathcal{B}$ of neutrosophic sets on the universe X. Then $\mathcal{B}$ is a neutrosophic base for some neutrosophic topology on X iff*

(1) *For every $U \in \tau$ and every $x_{\alpha,\beta,\gamma} \in U$ there exists $B \in \mathcal{B}$ such that $x_{\alpha,\beta,\gamma} \in B \sqsubseteq U$.*
(2) *For every $A, B \in \mathcal{B}$ we have $A \sqcap B$ is a union of elements from $\mathcal{B}$.*

Proof. $\rightarrow$) Obvious!

$\leftarrow$) Suppose $\mathcal{B}$ satisfies the two conditions in the theorem. Let $\tau(\mathcal{B})$ be all possible neutrosophic unions of elements from $\mathcal{B}$ with 0_X. It suffices to show that $\tau(\mathcal{B})$ is a neutrosophic topology on X. From the first condition and the construction of $\tau(\mathcal{B})$ we have $0_X, 1_X \in \tau(\mathcal{B})$. Now let $H, K \in \tau(\mathcal{B})$. Then $H = \sqcup_i H_i$ and $K = \sqcup_j K_j$ where $H_i, K_j \in \mathcal{B}$ for every i, j. So we have (by parts (3) and (4) of Proposition1.3)

$H \sqcap K = (\sqcup_i H_i) \sqcap (\sqcup_j K_j) = \sqcup_j ((\sqcup_i H_i) \sqcap K_j) = \sqcup_j \sqcup_i (H_i \sqcap K_j)$

Since $H_i, K_j \in \mathcal{B}$ for every i, j, we have $H \sqcap K \in \tau(\mathcal{B})$. The proof that the union of elements from $\tau(\mathcal{B})$ is an element from $\tau(\mathcal{B})$ is easy! And we done. $\square$

$\tau(\mathcal{B})$ will be called the neutrosophic topology generated by the neutrosophic base $\mathcal{B}$ on X .

Definition 2.10. [10] (X, τ) is said to be *neutrosophic compact* if each neutrosophic open (in τ) cover of X has a finite neutrosophic subcover.

Theorem 2.11. *[10] Consider the space (X, τ), and let $\mathcal{B}$ be a neutrosophic base for τ. Then (X, τ) is a neutrosophic compact space iff every neutrosophic open cover of X from $\mathcal{B}$ has a finite neutrosophic subcover.*

Definition 2.12. [10] A neutrosophic space (X, τ) is said:

(1) A *neutrosophic* Lindelöf space if each neutrosophic open cover of X from τ has a countable neutrosophic subcover of X.

(2) A *neutrosophic* countably compact space if each neutrosophic open countable cover of X from τ has a finite neutrosophic subcover of X.

The following thee results have proofs similar to their correspondings about neutrosophic $\underline{\mu}$-topological spaces in [10].

Theorem 2.13. *Every neutrosophic space with a countable neutrosophic base is neutrosophic $Lindel\ddot{o}f$.*

Theorem 2.14. *Every neutrosophic Lindelöf and* countably compact space *is compact.*

Corollary 2.15. *Every neutrosophic* countably compact space *with a neutrosophic countable base is neutrosophic compact.*

The following example show that neutrosophic $Lindel\ddot{o}f$ spaces are not neutrosophic countably compact.

Example 2.16. Let $Y = \{a, b\}$ and let $\mathcal{B} = \{A_n; n = 1, 2, 3, ...\}$ where $A_n = \{\langle y, 1 - \frac{1}{2n}, \frac{1}{2n}, \frac{1}{2n}\rangle; y \in X\}$. We will show that $\mathcal{B}$ is a base for some neutrosophic topology on Y ;

i.e. we want to show $\mathcal{B}$ satisfies (1) and (2) in Theorem2.9.

First condtion: $\mathcal{B}$ neutrosophic covers Y, actually:

$\sqcup\mathcal{B} = \sqcup\{A_n; n = 1, 2, 3, ...\} = \{\langle y, \vee_1^\infty 1 - \frac{1}{2n}, \wedge_1^\infty \frac{1}{2n}, \wedge_1^\infty \frac{1}{2n}\rangle; y \in Y\} = \{\langle y, 1, 0, 0\rangle; y \in Y\} = 1_Y$.

Second condition: The neutrosophic intersection of two elements from $\mathcal{B}$ is the neutrosophic union of elements from $\mathcal{B}$, but is clear that for any A_n and A_m in $\mathcal{B}$ we have $A_n \sqcap A_m = A_t$ where $t = \max\{n, m\}$ which an element of $\mathcal{B}$, so that $\mathcal{B}$ is a neutrosophic base form some neutrosophic topology $\tau(\mathcal{B})$ on Y. Since $\tau(\mathcal{B})$ has a countable base, $\tau(\mathcal{B})$ is neutrosophic Lindelöf. Now, we will show that $\tau(\mathcal{B})$ is not neutrosophic countably paracompact (which implies it is not neutrosophic compact). By contrapositive, suppose Y is neutrosophic countably paracompact. Then $\mathcal{U} = \mathcal{B}$ is a countable neutrosophic open cover of Y. But Y is a neutrosophic countably paracompact space, so that we have $\mathcal{U}$ has a neutrosophic finite subcover, say $\mathcal{U}^* = \{A_{n1}, A_{n2}, ..., A_{nk}\}$. But $A_{n1} \sqcup A_{n2} \sqcup ... \sqcup A_{nk} = A_t$ where $t = \max\{n_1, n_2, ..., n_k\}$, and $A_t = \{\langle y, 1 - \frac{1}{2t}, \frac{1}{2t}, \frac{1}{2t}\rangle; y \in Y\} \neq 1_Y$, a contradiction. So Y is not neutrosophic countably paracompact and hence it is not neutrosophic compact.

The following theorem shows that neutrosophic compact spaces and neutrusophic countably compact spaces are equivalent if the universe of discourse is countable, which is not true for topological spaces.

Theorem 2.17. *For every countable neutrosophic topological space Y, the following two statements are equivalent :*

(1) *Y is neutrosophic compact.*
(2) *Y is neutrosophic countably compact.*

Proof. $\Rightarrow$) Obvious!

$\Leftarrow$) Suppose that Y is a countable neutrosophic countably compact space, and let $\mathcal{U}$ be a neutrosophic open cover of Y. For every $y \in Y$ we define the following three subsets of $[0, 1]$.

(1) $D_\mu^y = \{\mu_A(y); A \in \mathcal{U}\}$.
(2) $D_\sigma^y = \{\sigma_A(y); A \in \mathcal{U}\}$.
(3) $D_\nu^y = \{\nu_A(y); A \in \mathcal{U}\}$.

Let D_1^y, D_2^y and D_3^y be three countable dense subsets of D_μ^y, D_σ^y and D_ν^y respectively in the usual sense (the usual topology on the unit interval). Since $\mathcal{U}$ is a neutrosophic μ-open cover of Y, we have $\sup D_1^y = \sup D_\mu^y = 1$,$\inf D_2^y = \inf D_\sigma^y = 0$ and $\inf D_3^y = \inf D_\nu^y = 0$. Let $\mathcal{U}(y) = \{A \in \mathcal{U}; \mu_A(y) \in D_1^y, \sigma_A(y) \in D_2^y$ or $\nu_A(y) \in D_3^y\}$. It is clear that $\mathcal{U}(y)$ is countable. Let $\mathcal{U}^* = \cup\{\mathcal{U}(y); y \in Y\}$. Since Y is countable, $\mathcal{U}^*$ is a countable sub-collection from $\mathcal{U}$. We will show that $\mathcal{U}^*$ is a neutrosophic cover of Y. Set $B = \sqcup\mathcal{U}^*$. For every $y \in Y$ we have:

(1) $\mu_B(y) = \vee\{\mu_A(y); A \in B\} \geq \vee\{\mu_A(y); A \in D_1^y\} = \sup D_1^y = 1$.

(2) $\sigma_B(y) = \wedge\{\sigma_A(y); A \in B\} \geq \wedge\{\sigma_A(y); A \in D_1^y\} = \inf D_2^y = 0$.

(3) $\nu_B(y) = \wedge\{\nu_A(y); A \in B\} \geq \wedge\{\vee_A(y); A \in D_1^y\} = \inf D_3^y = 0$.

Which implies that $B = 1_Y$ and $\mathcal{U}^*$ is a neutrosophic countable open cover. Since Y is a neutrosophic μ-countably compact space, $\mathcal{U}^*$ has a finite subcover , that is Y is compact. □

The following example shows that neutrosophic compactness and neutrosophic countably compactness are not equivalent.

Example 2.18. Consider the set of all countable ordinals W_0 with the usual ordering. Let $\beta = \{[s,t), s, t < \omega_1(\text{the first uncountable ordinal})\}$. We know that β is a base for some topology τ on $Y = W_0$. For every $[s,t) \in \beta$ define the neutrosophic set

$$A_{[s,t)} = \begin{cases} (y,1,0,0) & \text{if } y \in [s,t) \\ (y,0,1,1) & \text{if } y \notin [s,t) \end{cases}$$

Set $\acute{\beta} = \{A_{[s,t)}; [s,t) \in \beta\}$. We will show that $\acute{\beta}$ is a base for some neutrosophic topology on Y. First we show it is a neutrosophic cover for Y. Let $A = \sqcup\beta$; it suffices to show that $A = 1_Y$. But for every $y \in Y$, we have $y \in [s,y)$ for some $s < y$, so that $\mu_A(y) = \vee\{\mu_C(y); C \in \acute{\beta}\} \geq \mu_{[s,y)} = 1$, $\sigma_A(y) = \wedge\{\sigma_C(y); C \in \acute{\beta}\} \leq \sigma_{[s,y)} = 0$, and $\nu_A(y) = \wedge\{\nu_C(y); C \in \acute{\beta}\} \leq \nu_{[s,y)} = 0$, that means $A = 1_Y$ and β covers Y. Now, we will show that the intersection of any two elements from β is empty or an element of β. Let $A_{[s_1,t_1)}$ and $A_{[s_2,t_2)}$ be two neutrosophic sets in β and set $C = A_{[s_1,t_1)} \sqcap A_{[s_2,t_2)}$, if $[s_1,t_1) \cap [s_2,t_2) = \emptyset$, then for every $y \in Y$ we have $y \notin [s_1,t_1)$ or $y \notin [s_2,t_2)$, which implies $\mu_C = \mu_{[s_1,t_1)} \wedge \mu_{[s_2,t_2)} = 0$, $\sigma_C = \sigma_{[s_1,t_1)} \vee \sigma_{[s_2,t_2)} = 1$ and $\nu_C = \nu_{[s_1,t_1)} \vee \nu_{[s_2,t_2)} = 1$ and that means $A_{[s_1,t_1)} \sqcap A_{[s_2,t_2)} = 0_Y$. Now, suppose that $[s_1,t_1) \cap [s_2,t_2) \neq \emptyset$. Then for every $y < \max\{s_1,s_2\}$ or $y \geq \min\{t_1,t_2\}$ we have $y \notin [s_1,t_1)$ or $y \notin [s_2,t_2)$, which means $\mu_C = 0$, $\sigma_C = 1$ and $\nu_C = 1$, and if $\max\{s_1,s_2\} \leq y < \min\{t_1,t_2\}$, then $y \in [s_1,t_1)$ and $y \in [s_2,t_2)$, that is $\mu_C = 1$, $\sigma_C = 0$ and $\nu_C = 0$, so that we have

$$A_{[s_1,t_1)} \sqcap A_{[s_2,t_2)} = A_{[s,t)} = \begin{cases} (y,1,0,0) & \text{if } y \in [s,t) \\ (y,0,1,1) & \text{if } y \notin [s,t) \end{cases} \in \beta$$

where $s = \max\{s_1,s_2\}$ and $t = \max\{t_1,t_2\}$. Let $\tau(\beta)$ be the neutrosophic topology generated on Y by β. Then $\tau(\beta)$ is a neutrosophic countably compact space: We will prove this by showing $\tau(\beta)$ has no countable cover form β. Let $\mathcal{C} = \{A_n = [s_n,t_n); n = 1,2,3,...\}$ be any countable subset from β, it suffices to show that $\mathcal{C}$ does not cover Y; by contapositive, suppose $\mathcal{C}$ covers Y, then $D = \sqcup\mathcal{C} = \bigsqcup_{i=1}^{\infty} A_n = 1_Y$. So that for every $y \in Y$ we have

$\mu_C = \overset{\infty}{\underset{i=1}{\vee}} \mu_{A_n} = 1$; since $\mu_{A_n} = 1$ or 0 for every $n = 1, 2, 3, ...$, there exist i such that $\mu_{A_i} = 1$, that is $y \in A_i = [s_i, t_i)$, which implies $Y = \overset{\infty}{\underset{i=1}{\cup}} [s_n, t_n)$, a contradiction, since Y is uncountable and $\overset{\infty}{\underset{i=1}{\cup}} [s_n, t_n)$ is countable, so β has no countable cover for Y, and so Y is neutrosophic countably compact. Now, to show that Y is not neutrosophic compact. But β is a neutrosophic open cover of Y and has no countable, and hence no finite, subcover, that means Y is not neutrosophic compact.

Corollary 2.19. *There is a neutrosophic $\underline{\mu}-$topological spaces which is neutrosophic countably compact but not neutrosophic compact.*

Proof. Since every neutrosophic space is $\underline{\mu}-$topological space, we have Example2.18 is an example of a neutrosophic $\underline{\mu}-$topological spaces which is neutrosophic countably compact but not neutrosophic compact. □

The approach we used in Example2.18 can be generalized to get more counterexample for neutrosophic topological spaces as follows.

Theorem 2.20. *Let (X, τ) be a topological space and for every $U \in \tau$ set*

$$A_U = \begin{cases} (x, 1, 0, 0) & \textit{if } x \in U \\ (x, 0, 1, 1) & \textit{if } x \notin U \end{cases}$$

and let $Neut(\tau) = \{A_U; U \in \tau\}$. Then $(X, Neut(\tau))$ is a neutrosophic topological space.

Proof. Since $\emptyset, X \in \tau$, we have $A_\emptyset, A_X \in Neut(\tau)$, but

$$A_\emptyset = \begin{cases} (x, 1, 0, 0) & \text{if } x \in \emptyset \\ (x, 0, 1, 1) & \text{if } x \notin \emptyset \end{cases} = \begin{cases} (x, 1, 0, 0) & \text{if } x \in \emptyset \\ (x, 0, 1, 1) & \text{if } x \in X \end{cases} = 0_X$$

$$A_X = \begin{cases} (x, 1, 0, 0) & \text{if } x \in X \\ (x, 0, 1, 1) & \text{if } x \notin X \end{cases} = \begin{cases} (x, 1, 0, 0) & \text{if } x \in X \\ (x, 0, 1, 1) & \text{if } x \in \emptyset \end{cases} = 1_X$$

So we have $0_X, 1_X \in Neut(\tau)$. Now, let $H = A_U \sqcap A_V$ where $A_U, A_V \in Neut(\tau)$. Then

$$\mu_H(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases} \wedge \begin{cases} 1 & \text{if } x \in V \\ 0 & \text{if } x \notin V \end{cases} = \begin{cases} 1 & \text{if } x \in U \cap V \\ 0 & \text{if } x \notin U \cap V \end{cases} = \mu_{A_{(U \cap V)(x)}}$$

$$\sigma_H(x) = \begin{cases} 0 & \text{if } x \in U \\ 1 & \text{if } x \notin U \end{cases} \vee \begin{cases} 0 & \text{if } x \in V \\ 1 & \text{if } x \notin V \end{cases} = \begin{cases} 0 & \text{if} x \in U \cap V \\ 1 & \text{if} x \notin U \cap V \end{cases} = \sigma_{A_{(U \cap V)}}(x)$$

$$\nu_H(x) = \begin{cases} 0 & \text{if } x \in U \\ 1 & \text{if } x \notin U \end{cases} \vee \begin{cases} 0 & \text{if } x \in V \\ 1 & \text{if } x \notin V \end{cases} = \begin{cases} 0 & \text{if} x \in U \cap V \\ 1 & \text{if} x \notin U \cap V \end{cases} = \nu_{A_{(U \cap V)}}(x)$$

So we have $A_U \sqcap A_V = A_{(U \cap H)} \in Neut(\tau)$. Similarly we show that $\underset{\alpha \in \Delta}{\sqcup} A_\alpha \in Neut(\tau)$ whenever $A_\alpha \in Neut(\tau)$ for every $\alpha \in \Delta$. □

3. Applications and further studies

This paper is a completion part of [10] and gives an answer for the following question: Are neutrosophic $\underline{\mu}$-compactness and neutrosophic $\underline{\mu}$-countably compactness equivalent? which posted in [10]. We give an example to show that the answer is no! the approach is used to give such example can be generalized to give many counter examples in neutrosophic topology using those existing in general topology. This paper, also, studied more advanced notations about neutrosophic topology such as neutrosophic comapactness and neutrosophic Lindelöf, which opens doors for more studies about neutrosophic topology, such as neutrosophic para-compactness, and other covering properties

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On Some Properties of Neutrosophic Quadruple H_v-rings

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Abstract. Hyperstructure theory, an 86 years old theory, has been of great interest for many algebraists where their researches were divided in to two categories: theory and applications. On the other hand, neutrosophic theory which is the study of neutralities, was introduced and developed by F. Smarandache in 1995 as an extension of dialectics. The purpose of this paper is to study some connections between the two theories: Neutrosophy and hyperstructures. In this regard, we define neutrosophic quadruple H_v-rings, neutrosophic quadruple H_v-subrings, and neutrosophic quadruple homomorphism and study their various properties.

Keywords: H_v-ring; neutrosophic quadruple number; neutrosophic quadruple H_v-ring; neutrosophic homomorphism.

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1. **Introduction**

The concept of neutrosophic quadruple numbers was introduced by Smarandache [14] in 2015. Where he defined and presented some arithmetic operations of these numbers such as addition, subtraction, multiplication, and scalar multiplication. Later in 2017, Akinleye et al. [2] considered the set of neutrosophic quadruple numbers and defined some operations on it and discussed neutrosophic quadruple algebraic structures. A generalization of the latter work was done in 2016 where Agboola et al. [1] considered the set of neutrosophic quadruple numbers and defined some hyperoperations on it and discussed neutrosophic quadruple hyperstructures. For more details about neutrosophy and its applications, we refer to [3–7, 10, 13, 15, 16].

A generalization of hyperstructures, known as H_v-structures was introduced by T. Vougiouklis [19, 20]. We refer to [19, 20] for basic definitions and results on H_v-rings. Al Tahan and Davvaz in [3] discussed neutrosophic H_v-groups and studied their properties. In this work, we extend the results to H_v-rings and it is constructed as follows: after an Introduction, in

Section 2, we present some basic definitions about hyperstructures that are used throughout the paper. In Section 3, we define neutrosophic quadruple H_v-rings and provide some examples on it. In Section 4, we define neutrosophic quadruple H_v-subrings and neutrosophic quadruple homomorphism and study their properties.

2. **Basic definitions about algebraic hyperstructures**

In this section, we present some definitions and theorems related to hyperstructure theory that are used throughout the paper. (See [8, 9, 19].)

Let H be a non-empty set and $\mathcal{P}^*(H)$ the set of all non-empty subsets of H. Then, a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a *binary hyperoperation* on H. The couple $(H, \circ)$ is called a *hypergroupoid*. In this definition, if X and Y are two non-empty subsets of H and $h \in H$, then we define:

$$X \circ Y = \bigcup_{\substack{x \in X \\ y \in Y}} x \circ y,\ h \circ X = \{h\} \circ X \text{ and } X \circ h = X \circ \{h\}.$$

Definition 2.1. A hypergroupoid $(H, \circ)$ is called a:

(1) *semihypergroup* if for every $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$;
(2) *quasi-hypergroup* if for every $x \in H$, $x \circ H = H = H \circ x$ (The latter condition is called the reproduction axiom);
(3) *hypergroup* if it is a semihypergroup and a quasi-hypergroup.

T. Vougiouklis [19, 20] introduced H_v-structures as a generalization of the well-known algebraic hyperstructures. The equality in some axioms of classical algebraic hyperstructures is replaced by non-empty intersection in H_v-structures. The majority of H_v-structures are applied in representation theory.

Definition 2.2. A hypergroupoid $(H, \circ)$ is called an H_v*-semigroup* if the weak associative axiom is satisfied. i.e., $(x \circ (y \circ z)) \cap ((x \circ y) \circ z) \neq \emptyset$ for all $x, y, z \in H$.

An element $0 \in H$ is called an *identity* if $x \in 0 \circ h \cap h \circ 0$ for all $h \in H$ and it is called a *scalar identity* if $h = 0 \circ h = h \circ 0$ for all $h \in H$. A scalar identity (if it exists) is unique. A hypergroupoid $(H, \circ)$ is called an H_v*-group* if it is a quasi-hypergroup and an H_v-semigroup. A non-empty subset M of an H_v-group $(H, \circ)$ is called H_v*-subgroup of* H if $(M, \circ)$ is an H_v-group.

Definition 2.3. Let R be a non-empty set and "+", "·" be hyperoperations. Then $(R, +, \cdot)$ is a *hyperring* if the following conditions hold. (1) $(R, +)$ is a hypergroup; (2) $(R, \cdot)$ is is a semihypergroup; (3) $\cdot$ is distributive with respect to $+$. And it is an H_v*-ring* if (1) $(R, +)$ is an H_v-group; (2) $(R, \cdot)$ is is an H_v-semigroup; (3) $\cdot$ is weak distributive with respect to $+$.

$(R, +, \cdot)$ is said to be commutative if $x + y = y + x$ and $x \cdot y = y \cdot x$ for all $x, y \in R$. An element $1 \in R$ is called a *unit* if $x \in 1 \cdot x \cap x \cdot 1$ for all $x \in R$ and it is called a *scalar unit* if $x = 1 \cdot x = x \cdot 1$ for all $x \in R$. If the scalar unit exists then it is unique. A subset M of an H_v-ring $(R, +, \cdot)$ is called an *H_v-subring* if $(M, +, \cdot)$ is an H_v-ring. To prove that $(M, +, \cdot)$ is an H_v-subring of $(M, +, \cdot)$, it suffices to show that $m + M = M + m = M$ and $M \cdot M \subseteq M$ for all $m \in M$.
Let $(R, +, \star)$ and $(R', +', \star')$ be two H_v-rings. Then $f : R \to R'$ is said to be *H_v-ring homomorphism* if $f(r + s) = f(r) +' f(s)$ and $f(r \star s) = f(r) \star' f(s)$ for all $r, s \in R$. $(R, +, \star)$ and $(S, +', \star')$ are called *isomorphic H_v-rings*, and written as $R \cong S$, if there exists a bijective homomorphism $f : R \to S$.

The concept of very thin hyperstructures was introduced and studied by Vougioklis [17,18].

An H_v-structure is called a *very thin H_v-structure*, denoted as VT-H_v-structure, if all hyperoperations are operations except one which has all hyperproducts singletons except only one. For example an H_v-ring $(H, \star, \circ)$ is said to be a VT-H_v-ring if there exists only one $(x, y) \in H^2$ with the property $|x \star y| > 1$ or $|x \circ y| > 1$.

3. Construction of neutrosophic quadruple H_v-rings

Symbolic (or Literal) neutrosophic theory is referring to the use of abstract symbols (i.e. the letters T, I, F, representing the neutrosophic components: truth, indeterminacy, and falsehood) in neutrosophics.

In [1,2], Agboola et al. and Akinleye et al. respectively based their study of neutrosophic quadruple algebraic structures (hyperstructures) on quadruple numbers based on the set of real numbers. In this section, we consider neutrosophic quadruple numbers based on a set instead of real or complex numbers and we use them to define neutrosophic quadruple H_v-rings.

Definition 3.1. [11] Let X be a nonempty set. A neutrosophic quadruple X-number is an ordered quadruple (a, bT, cI, dF) where $a, b, c, d \in X$ and T, I, F have their usual neutrosophic logical meanings.
The set of all neutrosophic quadruple X-numbers is denoted by $NQ(X)$, that is,

$$NQ(X) = \{(a, bT, cI, dF) : a, b, c, d \in X\}.$$

With respect to the preference law $T < I < F$, we define the Absorbance Law for the multiplications of T, I, and F, in the sense that the bigger one absorbs the smaller one (or the big fish eats the small fish); for example:
$FT = TF = F$ (because F is bigger), $TT = T$ (T absorbs itself), $TI = IT = I$ (because I is bigger), (because F is bigger), and $FI = IF = I$ (because F is bigger).

Let $(R,+,\cdot)$ be an H_v-ring with zero "0" and unit "1"and define "$\oplus$" and "$\odot$" on $NQ(R)$ as follows:

$$(x_1,x_2T,x_3I,x_4F)\oplus(y_1,y_2T,y_3I,y_4F)$$
$$=\{(a,bT,cI,dF):\ a\in x_1+y_1, b\in x_2+y_2, c\in x_3+y_3, d\in x_4+y_4\}.$$

and

$$(x_1,x_2T,x_3I,x_4F)\odot(y_1,y_2T,y_3I,y_4F)$$
$$=\{(a,bT,cI,dF):\ a\in x_1\cdot y_1, b\in x_1\cdot y_2\cup x_2\cdot y_1\cup x_2\cdot y_2,$$
$$c\in x_1\cdot y_3\cup x_2\cdot y_3\cup x_3\cdot y_1\cup x_3\cdot y_2\cup x_3\cdot y_3,$$
$$d\in x_1\cdot y_4\cup x_2\cdot y_4\cup x_3\cdot y_4\cup x_4\cdot y_1\cup x_4\cdot y_2\cup x_4\cdot y_3\cup x_4\cdot y_4\}.$$

Throughout this section, $T<I<F$ and $(R,+,\cdot)$ is an H_v-ring with identity "0", unit "1", $0+0=0$, $1\cdot 1=1$ and $x\cdot 0=0\cdot x=0$ for all $x\in R$ (i.e, 0 is an absorbing element).

Theorem 3.2. *[3] Let R be a set with $0\in R$. Then $(NQ(R),\oplus)$ is an H_v-group (called neutrosophic H_v-group) with identity $\overline{0}=(0,0T,0I,0F)$ if and only if $(R,+)$ is an H_v-group with identity "0" and $0+0=0$.*

Theorem 3.3. *[3] Let R be a set with $0\in R$. Then $(NQ(R),\oplus)$ is a hypergroup (called neutrosophic hypergroup) with identity $\overline{0}=(0,0T,0I,0F)$ if and only if $(R,+)$ is a hypergroup with identity "0" and $0+0=0$.*

In [1], Agboola et al. gave an example on a hypergroup of order 3 (Example 2.4) and said that it is a neutrosophic hypergroup which is an impossible case. We illustrate it by the following remark.

Remark 3.4. A neutrosophic H_v-group (hypergroup) $NQ(R)=\{(a,bT,cI,dF):a,b,c,d\in R\}$ is either infinite or of order $|R|^4$ where $|R|$ is the number of elements in R in case R is finite. This is clear by using Theorem 3.2 and Theorem 3.3 respectively.

Theorem 3.5. *[3] Let R be a set with $0\in R$. Then $(NQ(R),\oplus)$ is a commutative H_v-group with identity $\overline{0}=(0,0T,0I,0F)$ if and only if $(R,+)$ is a commutative H_v-group with identity "0" and $0+0=0$.*

Proposition 3.6. *Let R be a set containing "0" and "1" with a hyperoperation "$\cdot$". Then $(NQ(R),\odot)$ is a quadruple H_v-semigroup with unit 1 if and only if $(R,\cdot)$ is an H_v-semigroup with unit $\overline{1}=(1,0T,0I,0F)$.*

Proof. Let $(NQ(R),\odot)$ be a quadruple H_v-semigroup and let $a,b,c\in R$. Having $x=(a,0T,0I,0F)\in NQ(R)$, $y=(b,0T,0I,0F)\in NQ(R)$, $z=(c,0T,0I,0F)\in NQ(R)$ and $(x\odot(y\odot z))\cap((x\odot y)\odot z)\neq\emptyset$ implies that $(a\cdot(b\cdot c))\cap((a\cdot b)\cdot c)\neq\emptyset$.

Let $(R,\cdot)$ be an H_v-semigroup and let $x,y,z\in NQ(R)$. Then there exist $x_i,y_i,z_i\in R$ with $i=1,2,3,4$ such that $x=(x_1,x_2T,x_3I,x_4F)$, $y=(y_1,y_2T,y_3I,y_4F)$ and $z=$

(z_1, z_2T, z_3I, z_4F). We have $(x_i \cdot (y_i \cdot z_i)) \cap (x_i \cdot y_i) \cdot z_i) \neq \emptyset$ for $i = 1, 2, 3, 4$. Applying the latter with some computations on $x \odot (y \odot z)$ and on $(x \odot y) \odot z$, we get $(x \odot (y \odot z)) \cap ((x \odot y) \odot z) \neq \emptyset$. □

Proposition 3.7. *R be a set containing "0" and "1" with a hyperoperation "$\cdot$". Then $(NQ(R), \odot)$ is a quadruple semihypergroup with $\bar{1} = (1, 0T, 0I, 0F)$ as unit if and only if $(R, \cdot)$ is a semihypergroup with 1 as unit.*

Proof. The proof is the same as that of Proposition 3.6 but instead of nonempty intersection, we have equality. □

Proposition 3.8. *Let $(NQ(R), \oplus, \odot)$ be an H_v-ring with zero "$\bar{0}$" and unit "$\bar{1}$". Then for all $a, b, c \in R$, we have:*

$$(a \cdot (b + c)) \cap ((a \cdot b) + (a \cdot c)) \neq \emptyset.$$

Proof. Let $a, b, c \in R$. Then $x = (a, 0T, 0I, 0F), y = (b, 0T, 0I, 0F), z = (c, 0T, 0I, 0F) \in NQ(R)$. Since $(x \odot (y \oplus z)) \cap ((x \odot y) \oplus (x \odot z)) \neq \emptyset$, it follows that $(a \cdot (b + c)) \cap ((a \cdot b) + (a \cdot c)) \neq \emptyset$. □

Proposition 3.9. *Let $(R, +, \cdot)$ be an H_v-ring with identity "0" and unit "1". Then for all $x, y, z \in NQ(R)$, we have:*

$$(x \odot (y \oplus z)) \cap ((x \odot y) \oplus (y \odot z)) \neq \emptyset.$$

Proof. Let $x = (x_1, x_2T, x_3I, x_4F), y = (y_1, y_2T, y_3I, y_4F), z = (z_1, z_2T, z_3I, z_4F) \in NQ(R)$. We have:

$x \odot (y \oplus z) = \{(t_1, t_2, t_3, t_4) : t_1 \in x_1 \cdot (y_1 + z_1), t_2 \in x_1 \cdot (y_2 + z_2) \cup x_2 \cdot (y_1 + z_1) \cup x_2 \cdot (y_2 + z_2),$
$t_3 \in x_1 \cdot (y_3 + z_3) \cup x_2 \cdot (y_3 + z_3) \cup x_3 \cdot (y_1 + z_1) \cup x_3 \cdot (y_2 + z_2) \cup x_3 \cdot (y_3 + z_3),$
$t_4 \in x_1 \cdot (y_4 + z_4) \cup x_2 \cdot (y_4 + z_4) \cup x_3 \cdot (y_4 + z_4) \cup x_4 \cdot (y_1 + z_1) \cup x_4 \cdot (y_2 + z_2) \cup x_4 \cdot (y_3 + z_3) \cup x_4 \cdot (y_4 + z_4)\}$.

On the other hand, we have:

$(x \odot y) \oplus (x \odot z) = \{s = (s_1, s_2T, s_3I, s_4F) : q = (q_1, q_2T, q_3I, q_4F) \in x \odot y, r = (r_1, r_2T, r_3I, r_4F) \in x \odot z, s_i \in q_i + r_i$ for $i = 1, 2, 3, 4\}$.

Having $q = (q_1, q_2T, q_3I, q_4F) \in x \cdot y$ and $r = (r_1, r_2T, r_3I, r_4F) \in x \cdot z$ implies that $q_1 \in x_1 \cdot y_1$, $q_2 \in x_1 \cdot y_2 \cup x_2 \cdot y_1 \cup x_2 \cdot y_2$, $q_3 \in x_1 \cdot y_3 \cup x_2 \cdot y_3 \cup x_3 \cdot y_1 \cup x_3 \cdot y_2 \cup x_3 \cdot y_3$, $q_4 \in x_1 \cdot y_4 \cup x_2 \cdot y_4 \cup x_3 \cdot y_4 \cup x_4 \cdot y_1 \cup x_4 \cdot y_2 \cup x_4 \cdot y_3 \cup x_4 \cdot y_4$, $r_1 \in x_1 \cdot z_1$, $r_2 \in x_1 \cdot z_2 \cup x_2 \cdot z_1 \cup x_2 \cdot z_2$, $r_3 \in x_1 \cdot z_3 \cup x_2 \cdot z_3 \cup x_3 \cdot z_1 \cup x_3 \cdot z_2 \cup x_3 \cdot z_3$ and $r_4 \in x_1 \cdot z_4 \cup x_2 \cdot z_4 \cup x_3 \cdot z_4 \cup x_4 \cdot z_1 \cup x_4 \cdot z_2 \cup x_4 \cdot z_3 \cup x_4 \cdot z_4$. Since $x_i \cdot (y_i + z_i) \cap (x_i \cdot y_i + x_i \cdot z_i) \neq \emptyset$ for $i = 1, 2, 3, 4$, it follows that $(x \odot (y \oplus z)) \cap ((x \odot y) \oplus (y \odot z)) \neq \emptyset$. □

Proposition 3.10. *Let $(NQ(R), \oplus, \odot)$ be an hyperring with zero "$\bar{0}$" and unit "$\bar{1}$". Then for all $a, b, c \in R$, we have:*

$$(a \cdot (b + c)) = ((a \cdot b) + (a \cdot c)).$$

Proof. The proof is the same as that of Proposition 3.8 but instead of nonempty intersection, we have equality. □

Proposition 3.11. *Let $(R, +, \cdot)$ be a hyperring with identity "0" and unit "1". Then for all $x, y, z \in NQ(R)$, we have:*

$$x \odot (y \oplus z) \subseteq (x \odot y) \oplus (y \odot z).$$

Proof. The proof is straightforward. □

Remark 3.12. The equality in Proposition 3.11 may not hold. We illustrate it by the following example.

Example 3.13. Let $R = \mathbb{Z}_2$ be the ring of integers under standard addition and multiplication modulo 2 and let $x = (1, 1T, 0I, 0F)$, $y = (0, 1T, 0I, 0F)$ and $z = (1, 0T, 0I, 0F)$. Having $x \odot (y \oplus z) = (1, 1T, 0I, 0F)$ and $(x \odot y) \oplus (x \odot z) = \{(1, 0T, 0I, 0F), (1, 1T, 0I, 0F)\}$ implies that $x \odot (y \oplus z) \neq (x \odot y) \oplus (y \odot z)$.

In the proof of Theorem 2.11, [1], the proof of distributivity contains a gap. Our example, Example 3.13 can be used as an illustration.

Notation 1. *Let $(R, +, \cdot)$ be an H_v-ring with "0" and "1" as zero and unit respectively satisfying $0 + 0 = 0$, $1 \cdot 1 = 1$ and $0 \cdot x = x \cdot 0 = 0$ for all $x \in R$. Then $(NQ(R), \oplus, \odot)$ is called neutrosophic quadruple H_v-ring.*

Notation 2. *Let $(NQ(R), \oplus, \odot)$ be a hyperring. Then we call it a neutrosophic quadruple hyperring.*

Remark 3.14. Let $(R, +, \cdot)$ be a hyperring. Then $(NQ(R), \oplus, \odot)$ may fail to be a hyperring. One can easily see that $(NQ(R), \oplus, \odot)$ in Example 3.13 is not a hyperring (as the distributivity law does not hold.).

Theorem 3.15. *Let R be any set with two hyperoperations "+" and "$\cdot$". Then $(NQ(R), \oplus, \odot)$ is a neutrosophic H_v-ring with zero and unit $\overline{0} = (0, 0T, 0I, 0F)$ and $\overline{1} = (1, 0T, 0I, 0F)$ respectively if and only if $(R, +, \cdot)$ is an H_v-ring with zero and unit "0" and "1" respectively.*

Proof. The proof follows from Theorem 3.2, Proposition 3.6, Proposition 3.8 and Proposition 3.9. □

Corollary 3.16. *Let $(R, +, \cdot)$ be an H_v-ring containing an identity and absorbing element 0 and a unit 1 with the property that $0 + 0 = 0$, $1 \cdot 1 = 1$. Then we can construct infinite number of neutrosophic quadruple H_v-rings.*

Proof. Theorem 3.15 asserts that $(NQ(R), \oplus, \odot)$ is an H_v-ring with zero and unit $\overline{0} = (0, 0T, 0I, 0F)$ and $\overline{1} = (1, 0T, 0I, 0F)$ respectively. Applying Theorem 3.15 on $(NQ(R), \oplus, \odot)$, we get $NQ(NQ(R))$ is a quadruple H_v-ring. Continuing on this pattern, we can construct infinite number of quadruple H_v-rings. Particularly, we have $NQ(NQ(\ldots NQ(\ldots(R))\ldots)$ is a quadruple H_v-ring. □

Proposition 3.17. *Let $(R, +, \cdot)$ be any ring with unit. Then $(NQ(R), \oplus, \odot)$ is a neutrosophic H_v-ring. Moreover, $(NQ(R), \oplus, \odot)$ is not a ring.*

Proof. We can consider the ring $(R, +, \cdot)$ as an H_v-ring with zero and unit. Theorem 3.15 asserts that $(NQ(R), \oplus, \odot)$ is a neutrosophic H_v-ring.
Having $x = (1, 0T, 0I, 0F), y = (1, T, 0I, 0F) \in NQ(R)$ implies that $x \odot y \subseteq NQ(R)$. It is clear that $(1, 0T, 0I, 0F), (1, T, 0I, 0F) \in x \odot y$. Thus, $|x \odot y| > 1$. □

Example 3.18. Let $R_1 = \{0, 1\}$ and define $(R_1, +_1, \cdot_1)$ as follows:

$+_1$	0	1
0	0	1
1	1	R_1

$\cdot_1$	0	1
0	0	0
1	0	1

Then $(NQ(R_1), \oplus, \odot)$ is a quadruple H_v-ring with 16 elements.

By setting

$$\begin{array}{lll}
\overline{1} = (1, 0T, 0I, 0F), & a_6 = (0, 0T, I, F), & a_{11} = (1, 0T, 0I, F), \\
a_2 = (0, T, 0I, 0F), & a_7 = (0, T, I, 0F), & a_{12} = (1, T, 0I, F), \\
a_3 = (0, 0T, I, 0F), & a_8 = (0, T, 0I, F), & a_{13} = (1, 0T, I, F), \\
a_4 = (0, 0T, 0I, F), & a_9 = (1, T, 0I, 0F), & a_{14} = (1, T, I, 0F), \\
a_5 = (0, T, I, F), & a_{10} = (1, 0T, I, 0F), & a_{15} = (1, T, I, F),
\end{array}$$

we present some of the results for $a_i \oplus a_j = a_j \oplus a_i$, $i, j = 1, 2, \ldots, 15$ in the following table.

$\overline{0} \oplus x = \{x\}$ for all $x \in NQ(R_1)$	$\overline{1} \oplus \overline{1} = \{\overline{1}\}$
$\overline{1} \oplus a_2 = \{a_9\}$	$a_2 \oplus a_5 = \{a_5, a_6\}$
$a_3 \oplus a_4 = \{a_6\}$	$\overline{1} \oplus a_3 = \{a_{10}\}$
$a_5 \oplus a_5 = \{\overline{0}, \overline{1}, a_2, a_4, a_5, a_6, a_7, a_8, a_{10}\}$	$a_5 \oplus a_6 = \{a_5, a_7, a_8\}$
$\overline{1} \oplus a_4 = \{a_{11}\}$	$a_5 \oplus a_7 = \{a_4, a_5, a_6, a_8\}$
$a_5 \oplus a_8 = \{a_5, a_6, a_7, a_{10}\}$	$\overline{1} \oplus a_5 = \{a_{15}\}$
$a_5 \oplus a_9 = \{a_{13}, a_{14}, a_{15}\}$	$a_5 \oplus a_{10} = \{a_5, a_8\}$
$\overline{1} \oplus a_6 = \{a_{13}\}$	$a_5 \oplus a_{11} = \{a_{14}, a_{15}\}$
$a_5 \oplus a_{12} = \{a_{13}, a_{14}, a_{15}\}$	$\overline{1} \oplus a_7 = \{a_{14}\}$
$a_5 \oplus a_{13} = \{a_9, a_{12}, a_{14}, a_{15}\}$	$a_5 \oplus a_{14} = \{a_{11}, a_{13}, a_{15}\}$
$\overline{1} \oplus a_8 = \{a_{12}\}$	$a_4 \oplus a_{14} = \{a_{15}\}$
$a_4 \oplus a_{15} = \{a_{14}, a_{15}\}$	$\overline{1} \oplus a_9 = \{a_2, a_9\}$
$a_{14} \oplus a_{14} = \{\overline{1}, a_2, a_3, a_7, a_9, a_{10}, a_{14}\}$	$a_{14} \oplus a_{15} = \{a_4, a_5, a_6, a_8, a_{11}, a_{13}, a_{15}\}$
$\overline{1} \oplus a_{10} = \{a_3, a_{10}\}$	$a_{15} + a_{15} = NQ(R_1)$
$a_{15} \oplus a_3 = \{a_{12}, a_{15}\}$	$\overline{1} \oplus a_{11} = \{a_4, a_{10}\}$

and we present some of the results for $a_i \odot a_j = a_j \odot a_i$, $i, j = 1, 2, \ldots, 15$ in the following table.

$\overline{0} \odot x = \{\overline{0}\}$ for all $x \in NQ(R_1)$	$\overline{1} \odot \overline{1} = \{\overline{1}\}$
$\overline{1} \odot a_2 = \{\overline{0}, a_2\}$	$\overline{1} \odot a_3 = \{\overline{0}, a_3\}$
$\overline{1} \odot a_4 = \{\overline{0}, a_4\}$	$\overline{1} \odot a_5 = \{\overline{0}, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$
$\overline{1} \odot a_6 = \{\overline{0}, a_3, a_4, a_6\}$	$\overline{1} \odot a_7 = \{\overline{0}, a_2, a_3, a_7\}$
$\overline{1} \odot a_8 = \{\overline{0}, a_2, a_4, a_8\}$	$\overline{1} \odot a_9 = \{\overline{1}, a_9\}$
$\overline{1} \odot a_{10} = \{\overline{1}, a_{10}\}$	$\overline{1} \odot a_{11} = \{\overline{1}, a_{11}\}$
$\overline{1} \odot a_{12} = \{\overline{1}, a_9, a_{11}, a_{12}\}$	$\overline{1} \odot a_{13} = \{\overline{1}, a_{10}, a_{11}, a_{13}\}$
$\overline{1} \odot a_{14} = \{\overline{1}, a_9, a_{10}, a_{14}\}$	$\overline{1} \odot a_{15} = \{\overline{1}, a_9, a_{10}, a_1, a_{12}, a_{13}, a_{14}, a_{15}\}$
$a_2 \odot a_2 = \{\overline{0}, a_2\}$	$a_3 \odot a_3 = \{\overline{0}, a_3\}$

It is clear that $(NQ(R_1), \oplus, \odot)$ is a commutative quadruple H_v-ring.

Proposition 3.19. *Let $(R, +, \cdot)$ be an H_v-ring. Then "1" is the scalar unit of $(R, +, \cdot)$ if and only if $\overline{1} = (1, 0T, 0I, 0F)$ is the scalar unit of $(NQ(R), \oplus, \odot)$.*

Proof. The proof is straightforward by applying the uniqueness of the scalar unit. □

Proposition 3.20. *Let $(R, +, \cdot)$ be an H_v-ring. Then $(R, +, \cdot)$ is a commutative H_v-ring if and only if $(NQ(R), \oplus, \odot)$ is a commutative H_v-ring.*

Proof. Theorem 3.5 asserts that $(NQ(R), \oplus)$ is a commutative H_v-group if and only if $(R, +)$ is a commutative H_v-group. We need to show that $(NQ(R), \odot)$ is a commutative H_v-semigroup if and only if $(R, \cdot)$ is a commutative H_v-semigroup. Suppose that $(R, \cdot)$ is a commutative H_v-semigroup and let $x = (x_1, x_2T, x_3I, x_4F), y = (y_1, y_2T, y_3I, y_4F) \in NQ(R)$. Easy computations show that $x \odot y = y \odot x$. Thus, $(NQ(R), \odot)$ is a commutative H_v-semigroup.

Conversely, let $(NQ(R), \odot)$ be a commutative H_v-group and $a, b \in R$. Having $x = (a, 0T, 0I, 0F), y = (b, T, 0I, 0F) \in NQ(R)$ implies that $x \odot y = (a \cdot b, 0T, 0I, 0F) = y \odot x = (b \cdot a, 0T, 0I, 0F)$. Thus, $a \cdot b = b \cdot a$. Therefore, $(R, \cdot)$ is a commutative H_v-semigroup. □

Proposition 3.21. *If $(R, +, \cdot)$ is a VT-H_v-ring then $(NQ(R), \oplus, \odot)$ is not a VT-H_v-ring.*

Proof. Suppose that $(R, +)$ is a VT-H_v-ring. Then there exist $a, b \in R$ with either $|a + b| > 1$ or $|a \cdot b| > 1$.

- Case $|a + b| > 1$. Having $0 + 0 = 0$ implies that either $a \neq 0$ or $b \neq 0$ (or both are not equal to zero). Without loss of generality, we take $b \neq 0$. Let $x = (a, aI, 0T, 0F), y = (b, 0I, 0T, 0F), z = (0, bT, bI, bF) \in NQ(R)$. It is clear that $y \neq z$, $|x \oplus y| > 1$ and that $|x \oplus z| > 1$.
- Case $|a \cdot b| > 1$. Having $1 \cdot 1 = 1$ implies that either $a \neq 1$ or $b \neq 1$ (or both are not equal to 1). Without loss of generality, we take $b \neq 1$. Let $x = (a, 0I, 0T, 0F), y = (b, 0I, 0T, 0F), z = (0, bT, 0I, 0F) \in NQ(R)$. It is clear that $y \neq z$, $|x \odot y| > 1$ and that $|x \odot z| > 1$.

Therefore, $(NQ(R), \oplus, \odot)$ is not a VT-H_v-ring. □

4. Neutrosophic quadruple H_v-subrings and neutrosophic homomorphisms

In this section, we define neutrosophic quadruple H_v-subrings and neutrosophic homomorphisms and investigate some of their properties.

Definition 4.1. Let $(NQ(R), \oplus, \odot)$ be a neutrosophic quadruple H_v-ring and T be a non-empty subset of $NQ(R)$. Then $(T, \oplus, \odot)$ is called a *neutrosophic quadruple H_v-subring of* $NQ(R)$ if $(T, \oplus, \odot)$ is a neutrosophic quadruple H_v-ring.

Remark 4.2. Neutrosophic H_v-rings have no proper neutrosophic H_v-ideals. This is clear as if $NQ(J)$ is a neutrosophiv H_v-ideal of $NQ(R)$ then $(1, 0T, 0I, 0F) \in NQ(J)$. The latter implies that $(a, bT, cI, dF) = (a, bT, cI, dF) \odot (1, 0T, 0I, 0F) \in NQ(J)$ for all $(a, bT, cI, dF) \in NQ(R)$.

Theorem 4.3. *[3] Let $(R, +)$ be an H_v-group with identity "0", $S \subseteq R$ and $0 \in S$. Then $(NQ(S), \oplus)$ is an H_v-subgroup of $(NQ(R), \oplus)$ if and only if $(S, +)$ is an H_v-subgroup of $(R, +)$.*

Theorem 4.4. *Let $(R, +, \cdot)$ be an H_v-ring with identity "0" and unit 1, $S \subseteq R$ and $0, 1 \in S$. Then $(NQ(S), \oplus, \odot)$ is an H_v-subring of $(NQ(R), \oplus, \odot)$ if and only if $(S, +, \cdot)$ is an H_v-subring of $(R, +, \cdot)$.*

Proof. Theorem 4.3 asserts that $(NQ(S), \oplus)$ is an H_v-subgroup of $(NQ(R), \oplus)$ if and only if $(S, +)$ is an H_v-subgroup of $(R, +)$. We need to show that $(NQ(S), \odot)$ is an H_v-subsemigroup of $(NQ(R), \odot)$ if and only if $(S, \cdot)$ is an H_v-subsemigroup of $(R, \cdot)$. Suppose that $(S, \cdot)$ is an H_v-subsemigroup of $(R, \cdot)$. We need to show that $x \odot NQ(S) \cup NQ(S) \odot x \subseteq NQ(S)$ for all $x = (x_1, x_2T, x_3I, x_4F) \in NQ(S)$ which is clear.

Let $(NQ(S), \odot)$ be an H_v-subsemigroup of $(NQ(R), \odot)$ and let $x_1 \in S$. We need to show that $x_1 \cdot S \cup S \cdot x_1 \subseteq S$. For all $y_1 \in S$, we have $x = (x_1, 0T, 0I, 0F), y = (y_1, 0T, 0I, 0F) \in NQ(S)$. Since $x \odot y \subseteq NQ(S)$, it follows that $x_1 \cdot y_1 \subseteq S$. □

Example 4.5. Since $(R_1, +_1, \cdot_1)$ in Example 3.18 has only one H_v-subring (R_1) containing 0 and 1, it follows by applying Theorem 4.4 that $(NQ(R_1), \oplus, \odot)$ has only one neutrosophic H_v-subring: $(NQ(R_1), \oplus, \odot)$.

Example 4.6. Let $R_2 = \{0, 1, 2\}$ and define $(R_2, +_2, \cdot_2)$ as follows:

$+_2$	0	1	2
0	0	$\{0,1\}$	$\{0,2\}$
1	$\{0,1\}$	1	$\{1,2\}$
2	$\{0,2\}$	$\{1,2\}$	2

$\cdot_2$	0	1	2
0	0	0	0
1	0	1	$\{1,2\}$
2	0	$\{1,2\}$	2

It is clear that $(R_2, +_2, \cdot_2)$ is a commutative H_v-ring that has exactly two non-isomorphic H_v-subrings containing 0 and 1: $\{0, 1\}$ and R_2. We can deduce that $(NQ(R_2), \oplus, \odot)$ is a commutative neutrosophic quadruple H_v-ring and has two non-isomorphic neutrosophic quadruple H_v-subrings: $NQ(\{0, 1\}) = \{\overline{0}, \overline{1}\}$ and $NQ(R_2)$.

Proposition 4.7. *Let $n \geq 2$ be a natural number and $(\mathbb{Z}_n, +, \cdot)$ be the ring of integers under standard addition and multiplication modulo n. Then $(NQ(\mathbb{Z}_n), \oplus, \odot)$ has no proper neutrosophic H_v-subrings.*

Proof. Proposition 3.17 asserts that $(NQ(\mathbb{Z}_n), \oplus, \odot)$ is a neutrosophic H_v-ring. Let S be a subring of $\mathbb{Z}_n$. Then there exist $d \mid n$ with $1 \leq d \leq n$ such that $S = d\mathbb{Z}_n$. Since $1 \in S$ if and only if $d = 1$ and $(1, 0T, 0I, 0F) \in NQ(S)$, it follows that $NQ(S) = NQ(\mathbb{Z}_n)$. □

Proposition 4.8. *Let $(S, +, \cdot)$ be an H_v-subring of $(R, +, \cdot)$. Then $NQ(S) \oplus NQ(S) = NQ(S)$ and $NQ(S) \odot NQ(S) \subseteq NQ(S)$.*

Proof. The proof is straightforward. □

Definition 4.9. Let $(NQ(R), \oplus_1, \odot_1)$ and $(NQ(J), \oplus_2, \odot_2)$ be neutrosophic quadruple H_v-rings. A function $\phi : NQ(R) \to NQ(J)$ is called *neutosophic homomorphism* if

(1) $\phi(0_R, 0_RT, 0_RI, 0_RF) = (0_J, 0_JT, 0_JI, 0_JF)$;
(2) $\phi(1_R, 0_RT, 0_RI, 0_RF) = (1_J, 0_JT, 0_JI, 0_JF)$;
(3) $\phi(0_R, 1_RT, 0_RI, 0_RF) = (0_J, 1_JT, 0_JI, 0_JF)$;
(4) $\phi(0_R, 0_RT, 1_RI, 0_RF) = (0_J, 0_JT, 1_JI, 0_JF)$;
(5) $\phi(0_R, 0_RT, 0_RI, 1_RF) = (0_J, 0_JT, 0_JI, 1_JF)$;
(6) $\phi(x \oplus_1 y) = \phi(x) \oplus_2 \phi(y)$ for all $x, y \in NQ(R)$;
(7) $\phi(x \odot_1 y) = \phi(x) \odot_2 \phi(y)$ for all $x, y \in NQ(R)$.

If ϕ is a neutrosophic homomorphism and bijective then it is called *neutrosophic isomorphism* and we write $NQ(R) \cong NQ(J)$.

Example 4.10. Let $(R, +, \cdot)$ be an H_v-ring. Then $f : NQ(R) \to NQ(R)$ is an isomorphism, where $f(x) = x$ for all $x \in NQ(R)$.

Proposition 4.11. *Let $(R, +_1, \cdot_1)$ and $(J, +_2, \cdot_2)$ be H_v-rings. If there exist a homomorphism $f : R \to J$ with $f(0_R) = 0_J$ and $f(1_R) = 1_J$ then there exist a homomorphism from $(NQ(R), \oplus_1, \odot_1)$ to $(NQ(J), \oplus_2, \odot_2)$.*

Proof. Suppose that $f : R \to J$ is a homomorphism. We define $\phi : NQ(R) \to NQ(J)$ as follows: For $x = (x_1, x_2T, x_3I, x_4F) \in NQ(R)$

$$\phi((x_1, x_2T, x_3I, x_4F)) = (f(x_1), f(x_2)T, f(x_3)I, f(x_4)F).$$

It is clear that ϕ is well defined and that conditions 1. to 5. of Definition 4.9 are satisfied. Let $x = (x_1, x_2T, x_3I, x_4F), y = (y_1, y_2T, y_3I, y_4F) \in NQ(R)$. Since $f(x_i +_1 y_i) = f(x_i) +_2 f(y_i)$ for $i = 1, 2, 3, 4$, it follows that $\phi(x \oplus_1 y) = \phi(x) \oplus_2 \phi(y)$. Moreover, having $f(x_i \cdot_1 y_i) = f(x_i) \cdot_2 f(y_i)$ for $i = 1, 2, 3, 4$ implies that $\phi(x \odot_1 y) = \phi(x) \odot_2 \phi(y)$. □

Proposition 4.12. *Let $(R, +_1, \cdot_1)$ and $(J, +_2, \cdot_2)$ be isomorphic H_v-rings, $0_R, 1_R \in R$ with $0_R + 0_R = 0_R$, $1_R \cdot 1_R = 1_R$, $0_R \cdot x = 0_R$ for all $x \in R$ and $f : (R, +_1, \cdot_1) \to (J, +_2, \cdot_2)$ be an isomorphism. Then $f(0_R) = 0_J$ and $f(1_R) = 1_J$.*

Proof. let $f(0_R) = a, f(1_R) = b$. Since $a = f(0_R) = f(0_R +_1 0_R) = a +_2 a$ and $a +_2 y = f(0_R +_1 x) \ni f(x) = y$ for all $y \in J$, it follows that a is a zero of J satisfying $a +_2 a = a$. Moreover, having $b = f(1_R \cdot_1 1_R) = b \cdot_2 b$ and $b \cdot_2 y = f(1_R \cdot_1 x) \ni f(x) = y$ for all $y \in J$ implies that b is a unit of J satisfying $1_J \cdot_2 1_J = 1_J$. □

Corollary 4.13. *Let $(R, +_1, \cdot_1)$ and $(J, +_2, \cdot_2)$ be isomorphic H_v-rings. Then*

$$(NQ(R), \oplus_1, \odot_1) \cong (NQ(J), \oplus_2, \odot_2).$$

Proof. The proof is straightforward by using Proposition 4.11 and Proposition 4.12. □

Corollary 4.14. *Let $(R, +_1, \cdot_1)$ and $(J, +_2, \cdot_2)$ be H_v-rings and let $Hom(R, J) = \{f : R \to J : f$ is homomorphism, $f(0_R) = 0_J$ and $f(1_R) = 1_J\}$. If $|Hom(R, J)| < \infty$ then*

$$|Hom(R, J)| \leq |Hom(NQ(R), NQ(J)|.$$

Proof. The proof is straightforward using Proposition 4.11. □

Let $(R, +)$ be a commutative H_v-ring with identity "0" and unit "1" and $S \subseteq R$ be an H_v-subring of R. Then $(R/S, +', \cdot')$ is an H_v-ring with: S as a zero, "$1 + S$" as a unit and $S +' S = S$. Here "$+'$" and "$\cdot'$" are defined as follows: For all $x, y \in R$,

$$(x + S) +' (y + S) = (x + y) + S \text{ and } (x + S) \cdot' (y + S) = x \cdot y + S.$$

Proposition 4.15. *Let $(S, +, \cdot)$ be an H_v-subring of a commutative H_v-ring $(R, +, \cdot)$. Then $(NQ(R/S), \oplus, \odot)$ is an H_v-ring.*

Proof. Since $(R, +, \cdot)$ is commutative, it follows that "$+'$" and "$\cdot'$" are well defined. The proof follows from having $(R/S, +', \cdot')$ an H_v-ring with S as zero, $1 + S$ as unit, $S \cdot' (x + S) = (x + S) \cdot' S = S$ and from Theorem 3.15. □

Proposition 4.16. *Let $(S, +, \cdot)$ be an H_v-subring of a commutative H_v-ring $(R, +, \cdot)$. Then $(NQ(R/S), \oplus, \odot) \cong (NQ(R)/NQ(S), \oplus', \odot')$.*

Proof. Let $g : NQ(R)/NQ(S) \to NQ(R/S)$ be defined as follows:

$$g((x_1, x_2T, x_3I, x_4F) \oplus NQ(S)) = (x_1 + S, (x_2 + S)T, (x_3 + S)I, (x_4 + S)F).$$

We claim that g is a neutrosophic isomorphism, that is, g is well defined, one-to-one, onto and neutrosophic homomorphism.

(1) g is well defined. Let $x \oplus NQ(S) = y \oplus NQ(S) \in NQ(R)/NQ(S)$. Then there exist $x_i, y_i \in R$, $i = 1, 2, 3, 4$ such that $x = (x_1, x_2T, x_3I, x_4F), y = (y_1, y_2T, y_3I, y_4F)$. We need to show that $x_i + S = y_i + S$ for $i = 1, 2, 3, 4$, that is $x_i + S \subseteq y_i + S$ and $y_i + S \subseteq x_i + S$ for $i = 1, 2, 3, 4$. We show that $x_i + S \subseteq y_i + S$ and $y_i + S \subseteq x_i + S$ is done in a similar manner. Since $x \oplus NQ(S) = y \oplus NQ(S)$, it follows that $x \in x \oplus z \subseteq y \oplus NQ(S)$ for all $z = (z_1, z_2T, z_3I, z_4F) \in NQ(S)$. The latter implies that there exist $s = (s_1, s_2T, s_3I, s_4F) \in NQ(S)$ such that $x \oplus z \in y \oplus s$. We get $x_i + z_i \in y_i + s_i \subseteq y_i + S$ for $i = 1, 2, 3, 4$. The latter implies that $x_i + S \subseteq y_i + S$ for $i = 1, 2, 3, 4$.

(2) g is onto. The proof is straightforward.

(3) g is one-to-one. Let $x \oplus NQ(S) = (x_1, x_2T, x_3I, x_4F) \oplus NQ(S), y \oplus NQ(S) = (y_1, y_2T, y_3I, y_4F) \oplus NQ(S) \in NQ(R)/NQ(S)$ with $h(x \oplus NQ(S)) = h(y \oplus NQ(S))$. We need to show that $x \oplus NQ(S) = y \oplus NQ(S)$, that is, $x \oplus NQ(S) \subseteq y \oplus NQ(S)$ and $y \oplus NQ(S) \subseteq x \oplus NQ(S)$. We prove $x \oplus NQ(S) \subseteq y \oplus NQ(S)$ and $y \oplus NQ(S) \subseteq x \oplus NQ(S)$ is done in a similar manner.

Having $h(x \oplus NQ(S)) = h(y \oplus NQ(S))$ implies that $(x_1 + S, (x_2 + S)T, (x_3 + S)I, (x_4 + S)F) = (y_1 + S, (y_2 + S)T, (y_3 + S)I, (y_4 + S)F)$. The latter implies that $x_i + S = y_i + S$ for $i = 1, 2, 3, 4$. Let $z = (z_1, z_2T, z_3I, z_4F) \in NQ(S)$. Having $x_i + S = y_i + S$ for $i = 1, 2, 3, 4$ implies that there exist s_i, $i = 1, 2, 3, 4$, such that $x_i + z_i \subseteq y_i + s_i$ for $i = 1, 2, 3, 4$. The latter implies that $x \oplus NQ(S) \subseteq y \oplus s \subseteq y \oplus NQ(S)$.

(4) g is neutrosophic homomorphism.
- $g(0, 0T, 0I, 0F) = (S, ST, SI, SF)$,
- $g(1, 0T, 0I, 0F) = (1 + S, ST, SI, SF)$,
- $g(0, 1T, 0I, 0F) = (S, (1 + S)T, SI, SF)$,
- $g(0, 0T, 1I, 0F) = (S, ST, (1 + S)I, SF)$,
- $g(0, 0T, 0I, 1F) = (S, ST, SI, (1 + S)F)$,
- We have $g((x_1, x_2T, x_3I, x_4F) \oplus NQ(S) \oplus' (y_1, y_2T, y_3I, y_4F) \oplus NQ(S)) = g((x_1 + y_1, (x_2 + y_2)T, (x_3 + y_3)I, (x_4 + y_4)F) \oplus NQ(S)) = (x_1 + y_1 + S, (x_2 + y_2 + S)T, (x_3 + y_3 + S)I, (x_4 + y_4 + S)F)$. On the other hand, we have $g((x_1, x_2T, x_3I, x_4F) \oplus NQ(S)) \oplus g((y_1, y_2T, y_3I, y_4F) \oplus NQ(S)) = (x_1 + S, (x_2 + S)T, (x_3 + S)I, (x_4 + S)F) \oplus (y_1 + S, (y_2 + S)T, (y_3 + S)I, (y_4 + S)F)$.

- We have: $(x_1, x_2T, x_3I, x_4F) \oplus NQ(S) \odot' (y_1, y_2T, y_3I, y_4F) \oplus NQ(S) = (x_1, x_2T, x_3I, x_4F) \odot (y_1, y_2T, y_3I, y_4F) \oplus NQ(S)$ and $g((x_1, x_2T, x_3I, x_4F) \oplus NQ(S)) \odot g((y_1, y_2T, y_3I, y_4F) \oplus NQ(S)) = (x_1 + S, (x_2 + S)T, (x_3 + S)I, (x_4 + S)F) \odot (y_1 + S, (y_2 + S)T, (y_3 + S)I, (y_4 + S)F)$. Simple computations imply that $g((x_1, x_2T, x_3I, x_4F) \oplus NQ(S) \odot' (y_1, y_2T, y_3I, y_4F) \oplus NQ(S)) = g((x_1, x_2T, x_3I, x_4F) \oplus NQ(S)) \odot g((y_1, y_2T, y_3I, y_4F) \oplus NQ(S))$.

Therefore, $(NQ(R/S), \oplus, \odot) \cong (NQ(R)/NQ(S), \oplus', \odot')$. □

Example 4.17. Let $R_2 = \{0, 1, 2\}$ and $S = \{0, 1\}$ in Example 4.6. Then $NQ(R_2/S) \cong NQ(R_2)/NQ(S)$.

5. Conclusion

This paper contributed to the study of neutrosophic hyperstructures by introducing neutrosophic quadruple H_v-rings and studying their properties. For future work, it will be interesting to introduce and study other neutrosophic quadruple H_v-structures such as neutrosophic H_v-modules and neutrosophic H_v-vectorspaces.

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Generalized Aggregate Operators on Neutrosophic Hypersoft Set

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Abstract: Multi-criteria decision making (MCDM) is concerned about coordinating as well as looking after selection as well as planning problems which included multi-criteria. The neutrosophic soft set cannot handle the environment which involved more than one attribute. To overcome those hurdles neutrosophic hypersoft set (NHSS) is defined. In this paper, we proposed the generalized aggregate operators on NHSS such as extended union, extended intersection, OR-operation, AND-operation, etc. with their properties. Finally, the necessity and possibility operations on NHSS with suitable examples and properties are presented in the following research.

Keywords: Soft set; Neutrosophic Set; Neutrosophic soft set; Hypersoft set; Neutrosophic hypersoft set.

1. Introduction

Zadeh developed the notion of fuzzy sets [1] to solve those problems which contain uncertainty and vagueness. It is observed that in some cases circumstances cannot be handled by fuzzy sets, to overcome such types of situations Turksen [2] gave the idea of interval-valued fuzzy set. In some cases, we must deliberate membership unbiassed as the non- membership values for the suitable representation of an object in uncertain and indeterminate conditions that could not be handled by fuzzy sets nor interval-valued fuzzy sets. To overcome these difficulties Atanassov presented the notion of Intuitionistic fuzzy sets in [3]. The theory which was presented by Atanassov only deals the insufficient data considering both the membership and non-membership values, but the intuitionistic fuzzy set theory cannot handle the incompatible and imprecise information. To deal with such incompatible and imprecise data the idea of the neutrosophic set (NS) was developed by Smarandache [4].

A general mathematical tool was proposed by Molodtsov [5] to deal with indeterminate, fuzzy, and not clearly defined substances known as a soft set (SS). Maji et al. [6] extended the work on SS and defined some operations and their properties. In [7], they also used the SS theory for decision making. Ali et al. [8] revised the Maji approach to SS and developed some new operations with their properties. De Morgan's Law on SS theory was proved in [9] by using different operators. Cagman and Enginoglu [10] developed the concept of soft matrices with operations and discussed their properties, they also introduced a decision-making method to resolve those problems which contain uncertainty. In [11], they revised the operations proposed by Molodtsov's SS. In [12], the author's proposed some new operations on soft matrices such as soft difference product, soft restricted difference product, soft extended difference product, and soft weak-extended difference product with their properties.

Maji [13] offered the idea of a neutrosophic soft set (NSS) with necessary operations and properties. The idea of the possibility NSS was developed by Karaaslan [14] and introduced a possibility of neutrosophic soft decision-making method to solve those problems which contain uncertainty based on And-product. Broumi [15] developed the generalized NSS with some operations and properties and used the proposed concept for decision making. To solve MCDM problems with single-valued Neutrosophic numbers presented by Deli and Subas in [16], they constructed the concept of cut sets of single-valued Neutrosophic numbers. On the base of the correlation of intuitionistic fuzzy sets, the term correlation coefficient of SVNSs [17] was introduced. In [18], the idea of simplified NSs introduced with some operational laws and aggregation operators such as real-life Neutrosophic weighted arithmetic average operator and weighted geometric average operator. They constructed an MCDM method on the base of proposed aggregation operators.

Smarandache [19] generalized the SS to hypersoft set (HSS) by converting the function to a multi-attribute function to deal with uncertainty. Saqlain et al. [20] developed the generalization of TOPSIS for the NHSS, by using the accuracy function they transformed the fuzzy neutrosophic numbers to crisp form. In [21],s the author's proposed the fuzzy plithogenic hypersoft set in matrix form with some basic operations and properties. Martin and Smarandache developed the plithogenic hypersoft set by combining the plithogenic sets and hypersoft set in [22]. Saqlain et al. [23] proposed the aggregate operators and similarity measure [24] on NHSS. In [25], Abdel basset et al. applied TODIM and TOPSIS methods based on the best-worst method to increase the accuracy of evaluation under uncertainty according to the neutrosophic set. They also used the plithogenic set theory to solve the uncertain information and evaluate the financial performance of manufacturing industries, they used the AHP method to find the weight vector of the financial ratios to achieve this goal after that they used the VIKOR and TOPSIS methods to utilized the companies ranking in [26].

In the following paragraph, we explain some positive impacts of this research. The main focus of this study is too generalized the aggregate operators of the neutrosophic hypersoft set. We will use the proposed aggregate operators to solve multi-criteria decision-making problems after developing distance-based similarity measures. Saqlain et al. [23], developed the aggregate operators on NHSS but in some cases, we face some limitations such as in union and intersection. To overcome these limitations we develop the generalized version of aggregate operators on NHSS.

The following research is organized as follows: In section 2, we recall some basic definitions used in the following research such as SS, NS, NSS, HSS, and NHSS. We develop the generalized aggregate operators on NHSS such as extended union, extended intersection, And-operation, etc. in section 3 with properties. In section 4, the necessity and possibility of operations are presented with examples and properties.

2. Preliminaries

In this section, we recall some basic definitions such as SS, NSS, and NHSS which use in the following sequel.

Definition 2.1 [5] **Soft Set**

The soft set is a pair (F, Λ) over Ű if and only if F: Λ → P(Ű) is a mapping. That is the parameterized family of subsets of Ű known as a SS.

Definition 2.2 [4] **Neutrosophic Set**

Let Ű be a universe and Λ be an NS on Ű is defined as Λ = $\{< u, T_\Lambda(u), I_\Lambda(u), F_\Lambda(u) >: u \in$ Ű$\}$, where T, I, F: Ű $\to\ ]0^-,\ 1^+[$ and $0^- \leq T_\Lambda(u) + I_\Lambda(u) + F_\Lambda(u) \leq 3^+$.

Definition 2.3 [13] **Neutrosophic Soft Set**

Let Ű and Ḝ are universal set and set of attributes respectively. Let P(Ű) be the set of Neutrosophic values of Ű and Λ ⊆ Ḝ. A pair (F, Λ) is called an NSS over Ű and its mapping is given as

F: Λ → P(Ű)

Definition 2.4 [19] **Hypersoft Set**

Let Ű be a universal set and P(Ű) be a power set of Ű and for $n \geq 1$, there are n distinct attributes such as $k_1,\ k_2,\ k_3, \ldots,\ k_n$ and $K_1,\ K_2,\ K_3, \ldots,\ K_n$ are sets for corresponding values attributes respectively with following conditions such as $K_i \cap K_j = \emptyset\ (i \neq j)$ and $i, j \in \{1,2,3 \ldots n\}$. Then the pair (F, $K_1 \times K_2 \times K_3 \times \ldots \times K_n$) is said to be Hypersoft set over Ű where F is a mapping from $K_1 \times K_2 \times K_3 \times \ldots \times K_n$ to P(Ű).

Definition 2.5 [22] **Neutrosophic Hypersoft Set (NHSS)**

Let Ű be a universal set and P(Ű) be a power set of Ű and for $n \geq 1$, there are n distinct attributes such as $k_1,\ k_2,\ k_3, \ldots,\ k_n$ and $K_1,\ K_2,\ K_3, \ldots,\ K_n$ are sets for corresponding values attributes respectively with following conditions such as $K_i \cap K_j = \emptyset\ (i \neq j)$ and $i, j \in \{1,2,3 \ldots n\}$. Then the pair (F, Λ) is said to be NHSS over Ű if there exists a relation $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \Lambda$. F is a mapping from $K_1 \times K_2 \times K_3 \times \ldots \times K_n$ to P(Ű) and F($K_1 \times K_2 \times K_3 \times \ldots \times K_n$) = $\{< u, T_\Lambda(u), I_\Lambda(u), F_\Lambda(u) > : u \in$ Ű$\}$ where T, I, F are membership values for truthness, indeterminacy, and falsity respectively such that T, I, F: Ű $\rightarrow\ ]0^-,\ 1^+[$ and $0^- \leq T_\Lambda(u) + I_\Lambda(u) + F_\Lambda(u) \leq 3^+$.

Example 2.6 Assume that a person examines the attractiveness of a living house. Let Ű be a universe which consists of three choices Ű = $\{u_1,\ u_2\}$ and E = {έ$_1$, έ$_2$, έ$_3$} be a set of decision parameters. Then, the NHSS is given as

$$F_\Lambda = \{< \mathrm{u}_1, (\acute{\varepsilon}_1\{0.4, 0.7, 0.5\}, \acute{\varepsilon}_2\{0.8, 0.5, 0.3\}, \acute{\varepsilon}_3\{0.6, 0.5, 0.9\}) >$$
$$< \mathrm{u}_2, (\acute{\varepsilon}_1\{0.1,0.5,0.7\}, \acute{\varepsilon}_2\{0.5, 0.6, 0.2\}, \acute{\varepsilon}_3\{0.7, 0.4, 0.6\}) >\}$$

3. Generalized Aggregate Operators on Neutrosophic Hypersoft Set and Properties

In this section, we present the generalized aggregate operations on NHSS with examples. We prove commutative and associative laws by using proposed aggregate operators in the following section.

Definition 3.1

Let $F_\Lambda \in$ NHSS, then its complement, is written as $(F_\Lambda)^c = F^c(\Lambda)$ and defined as

$F^c(\Lambda) = \{< u, T(F^c(\Lambda)), I(F^c(\Lambda)), F(F^c(\Lambda)) > : u \in \mathrm{U}\}$ such that

$$T(F^c(\Lambda)) = 1 - T_\Lambda(u),$$
$$I(F^c(\Lambda)) = 1 - I_\Lambda(u),$$
$$F(F^c(\Lambda)) = 1 - F_\Lambda(u).$$

Example 3.2 Reconsider example 2.6

$$F^c(\Lambda) = \{< \mathrm{u}_1, (\acute{\varepsilon}_1\{0.6, 0.3, 0.5\}, \acute{\varepsilon}_2\{0.2, 0.5, 0.7\}, \acute{\varepsilon}_3\{0.4, 0.5, 0.1\}) >$$
$$< \mathrm{u}_2, (\acute{\varepsilon}_1\{0.9, 0.5, 0.3\}, \acute{\varepsilon}_2\{0.5, 0.4, 0.8\}, \acute{\varepsilon}_3\{0.3, 0.6, 0.4\}) >\}$$

Proposition 3.3

If $F_\Lambda \in$ NHSS, then $(F^c(\Lambda))^c = F_\Lambda$.

Proof

By using definition 3.1, we have

$$\begin{aligned} F^c(\Lambda) &= \{< u, T(F^c(\Lambda)), I(F^c(\Lambda)), F(F^c(\Lambda)) > : u \in \mathrm{U}\} \\ &= \{< u, 1 - T\ (F_\Lambda), 1 - I\ (F_\Lambda), 1 - F\ (F_\Lambda) > : u \in \mathrm{U}\}, \end{aligned}$$

Thus

$(F^c(\Lambda))^c = \{< u, 1 - (1 - T(F_\Lambda)), 1 - (1 - I(F_\Lambda)), 1 - (1 - F(F_\Lambda)) > : u \in \mathrm{U}\}$,

$(F^c(\Lambda))^c = \{< u, T(F_\Lambda), I(F_\Lambda), F(F_\Lambda) > : u \in \mathrm{U}\} = F_\Lambda$.

Which completes the proof.

Definition 3.4 Extended Union of Two Neutrosophic Hypersoft Set

Let $F_{\Lambda_1},\ F_{\Lambda_2} \in$ NHSS, then their extended union is

$$T\ (F_{\Lambda_1} \cup F_{\Lambda_2}) = \begin{cases} T(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ T(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Max\left(T(F_{\Lambda_1}), T(F_{\Lambda_2})\right) & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$I\ (F_{\Lambda_1} \cup F_{\Lambda_2}) = \begin{cases} I(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ I(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Min\left(I(F_{\Lambda_1}), I(F_{\Lambda_2})\right) & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$F\ (F_{\Lambda_1} \cup F_{\Lambda_2}) = \begin{cases} F(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ F(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Min\left(F(F_{\Lambda_1}), F(F_{\Lambda_2})\right) & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

Example 3.5 Let U = $\{u_1, u_2, u_3, u_4\}$ be a universal set and E = $\{\acute{\varepsilon}_1, \acute{\varepsilon}_2, \acute{\varepsilon}_3, \acute{\varepsilon}_4\}$ be a set of decision parameters and F_{Λ_1} = $\{u_1, u_4\}$ and F_{Λ_2} = $\{u_2, u_4\}$

$$F_{\Lambda_1} = \{< u_1, (\acute{\varepsilon}_1\{0.4, 0.7, 0.5\}, \acute{\varepsilon}_2\{0.8, 0.5, 0.3\}, \acute{\varepsilon}_3\{0.6, 0.5, 0.9\}, \acute{\varepsilon}_4\{0.3, 0.7, 0.2\}) >$$
$$< u_4, (\acute{\varepsilon}_1\{0.4, 0.7, 0.2\}, \acute{\varepsilon}_2\{0.6, 0.5, 0.3\}, \acute{\varepsilon}_3\{0.8, 0.4, 0.7\}, \acute{\varepsilon}_4\{0.6, 0.4, 0.3\}) >\}$$
$$F_{\Lambda_2} = \{< u_2, (\acute{\varepsilon}_1\{0.7, 0.4, 0.6\}, \acute{\varepsilon}_2\{0.4, 0.6, 0.9\}, \acute{\varepsilon}_3\{0.7, 0.4, 0.6\}, \acute{\varepsilon}_4\{0.7, 0.6, 0.3\}) >$$
$$< u_4, (\acute{\varepsilon}_1\{0.6, 0.2, 0.7\}, \acute{\varepsilon}_2\{0.5, 0.7, 0.3\}, \acute{\varepsilon}_3\{0.4, 0.8, 0.5\}, \acute{\varepsilon}_4\{0.5, 0.6, 0.4\}) >\}$$
$$F_{\Lambda_1} \cup F_{\Lambda_2} = \{< u_1, (\acute{\varepsilon}_1\{0.4, 0.7, 0.5\}, \acute{\varepsilon}_2\{0.8, 0.5, 0.3\}, \acute{\varepsilon}_3\{0.6, 0.5, 0.9\}, \acute{\varepsilon}_4\{0.3, 0.7, 0.2\}) >$$
$$< u_2, (\acute{\varepsilon}_1\{0.7, 0.4, 0.6\}, \acute{\varepsilon}_2\{0.4, 0.6, 0.9\}, \acute{\varepsilon}_3\{0.7, 0.4, 0.6\}, \acute{\varepsilon}_4\{0.7, 0.6, 0.3\}) >$$
$$< u_4, (\acute{\varepsilon}_1\{0.6, 0.7, 0.7\}, \acute{\varepsilon}_2\{0.6, 0.7, 0.3\}, \acute{\varepsilon}_3\{0.8, 0.8, 0.7\}, \acute{\varepsilon}_4\{0.6, 0.6, 0.4\}) >\}$$

Proposition 3.6

Let F_{Λ_1}, F_{Λ_2} and F_{Λ_3} are NHSSs than

1. $\left(F_{\Lambda_1} \cup F_{\Lambda_2}\right) = \left(F_{\Lambda_2} \cup F_{\Lambda_1}\right)$ (Commutative law)
2. $\left(F_{\Lambda_1} \cup F_{\Lambda_2}\right) \cup F_{\Lambda_3} = F_{\Lambda_1} \cup \left(F_{\Lambda_2} \cup F_{\Lambda_3}\right)$ (Associative law)

Proof 1. In the following proof first two cases are trivial, we consider only the third case in this proposition

$\left(F_{\Lambda_1} \cup F_{\Lambda_2}\right) = \{< u, \left(max\ \{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}, min\ \{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, min\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}\right) >\}$

$= \{< u, \left(max\{T(F_{\Lambda_2}), T(F_{\Lambda_1})\}, min\ \{I(F_{\Lambda_2}), I(F_{\Lambda_1})\}, min\{F(F_{\Lambda_2}), F(F_{\Lambda_1})\}\right) >\}$

$= \left(F_{\Lambda_2} \cup F_{\Lambda_1}\right)$

Proof 2: Let F_{Λ_1}, F_{Λ_2} and F_{Λ_3} are NHSSs than

$F_{\Lambda_1} \cup F_{\Lambda_2} = \{< u, \left(Max\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}, Min\ \{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, Min\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}\right) >\}$

$(F_{\Lambda_1} \cup F_{\Lambda_2}) \cup F_{\Lambda_3}$ =

$\{< u, \max\{\max\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}, T(F_{\Lambda_3})\}, \min\{\min\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, I(F_{\Lambda_3})\}, \min\{\min\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}, F(F_{\Lambda_3})\} >\}$

$= \left\{< u, \max\{T(F_{\Lambda_1}), T(F_{\Lambda_2}), T(F_{\Lambda_3})\}, \min\{\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, I(F_{\Lambda_3})\}, \min\left\{\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}, F(F_{\Lambda_3})\right\} >\right\}$

$= \left\{< u, \max\left\{T(F_{\Lambda_1}), \max\{T(F_{\Lambda_2}), T(F_{\Lambda_3})\}\right\}, \min\left\{I(F_{\Lambda_1}), \min\{I(F_{\Lambda_2}), I(F_{\Lambda_3})\}\right\}, \min\left\{F(F_{\Lambda_1}), \min\{F(F_{\Lambda_2}), F(F_{\Lambda_3})\}\right\} >\right\}$

$= F_{\Lambda_1} \cup \left(F_{\Lambda_2} \cup F_{\Lambda_3}\right)$

Definition 3.7 Extended Intersection of Two Neutrosophic Hypersoft Set

Let F_{Λ_1}, F_{Λ_2} ∈ NHSS, then their extended intersection is

$$T\ (F_{\Lambda_1} \cap F_{\Lambda_2}) = \begin{cases} T(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ T(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Min\left(T(F_{\Lambda_1}), T(F_{\Lambda_2})\right) & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$I\ (F_{\Lambda_1} \cap F_{\Lambda_2}) = \begin{cases} I(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ I(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Max\left(I(F_{\Lambda_1}), I(F_{\Lambda_2})\right) & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$F\ (F_{\Lambda_1}\ \cap\ F_{\Lambda_2}) = \begin{cases} F(F_{\Lambda_1}) & if\ u\ \in\ \Lambda_1 - \Lambda_2 \\ F(F_{\Lambda_2}) & if\ u\ \in\ \Lambda_2 - \Lambda_1 \\ Max\left(F(F_{\Lambda_1}), F(F_{\Lambda_2})\right) & if\ u\ \in\ \Lambda_1 \cap \Lambda_2 \end{cases}$$

Proposition 3.8 Let F_{Λ_1}, F_{Λ_2} and F_{Λ_3} are NHSSs than

1. $F_{\Lambda_1}\ \cap\ F_{\Lambda_2} = F_{\Lambda_2}\ \cap\ F_{\Lambda_1}$ (Commutative law)
2. $\left(F_{\Lambda_1}\ \cap\ F_{\Lambda_2}\right)\ \cap\ F_{\Lambda_3} = F_{\Lambda_1}\ \cap\ \left(F_{\Lambda_2}\ \cap\ F_{\Lambda_3}\right)$ (Associative law)

Proof 1. Similar to Proposition 3.6.

Proposition 3.9 Let F_{Λ_1}, F_{Λ_2} are NHSSs then

1. $(F_{\Lambda_1}\ \cup\ F_{\Lambda_2})^c = F^c(\Lambda_1)\ \cap\ F^c(\Lambda_2)$
2. $(F_{\Lambda_1}\ \cap\ F_{\Lambda_1})^c = F^c(\Lambda_1)\ \cup\ F^c(\Lambda_2)$

Proof 1. Let F_{Λ_1} and F_{Λ_1} ∈ NHSS, such as follows

$F_{\Lambda_1} = \{<\mathrm{u}, \{T(F_{\Lambda_1}), I(F_{\Lambda_1}), F(F_{\Lambda_1})\} >\}$ and $F_{\Lambda_2} = \{<\mathrm{u}, \{T(F_{\Lambda_2}), I(F_{\Lambda_2}), F(F_{\Lambda_2})\} >\}$

$$(F_{\Lambda_1}\ \cup\ F_{\Lambda_2})^c = \left\{<\mathrm{u}, \left(max\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}, min\ \{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, min\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\right) >\right\}^c$$

$$= \{<\mathrm{u}, \left(min\{1 - T(F_{\Lambda_1}), 1 - T(F_{\Lambda_2})\}, max\ \{1 - I(F_{\Lambda_1}), 1 -\ I(F_{\Lambda_2})\}, max\{1 - F(F_{\Lambda_1}), 1 - F(F_{\Lambda_2})\right) >\}$$

$$= \{<\mathrm{u}, (min\{T(F^c(\Lambda_1)), T(F^c(\Lambda_2))\}, max\ \{I(F^c(\Lambda_1)), I(F^c(\Lambda_2))\}, max\{F(F^c(\Lambda_1)), F(F^c(\Lambda_2))\}) >\}$$

$$= F^c(\Lambda_1)\ \cap\ F^c(\Lambda_2)$$

Proof 2. Similarly, we can prove 2.

Definition 3.10 OR-Operation of Two Neutrosophic Hypersoft Set

Let F_{Λ_1}, F_{Λ_2} ∈ NHSS. Consider k_1, k_2, k_3, …, k_n for $n \geq 1$, be n well-defined attributes, whose corresponding attributive values are respectively the set K_1, K_2, K_3, …, K_n with $K_i\ \cap\ K_j = \emptyset$, for $i \neq j$ and $i, j \epsilon \{1,2,3 \ldots n\}$ and their relation $K_1\ \times\ K_2\ \times\ K_3 \times \ldots \times\ K_n = \Lambda$, then $F_{\Lambda_1}\ \vee\ F_{\Lambda_2} = F_{\Lambda_1 \times \Lambda_2}$, then

$$T\ \ (F_{\Lambda_1 \times \Lambda_2}) = Max\left(T(F_{\Lambda_1}), T(F_{\Lambda_2})\right),$$

$$I\ \ (F_{\Lambda_1 \times \Lambda_2}) = Min\left(I(F_{\Lambda_1}), I(F_{\Lambda_2})\right),$$

$$F\ \ (F_{\Lambda_1 \times \Lambda_2}) = Min\left(F(F_{\Lambda_1}), F(F_{\Lambda_2})\right).$$

Example 3.11 Reconsider example 3.5

$$\begin{aligned} F_{\Lambda_1}\ \vee\ F_{\Lambda_2} &= F_{\Lambda_1 \times \Lambda_2} \\ &= \{< (\mathrm{u}_1, \mathrm{u}_2), (\acute{\varepsilon}_1\{0.7, 0.4, 0.5\}, \acute{\varepsilon}_2\{0.8, 0.5, 0.3\}, \acute{\varepsilon}_3\{0.7, 0.4, 0.6\}, \acute{\varepsilon}_4\{0.7, 0.6, 0.2\}) > \\ &\quad < (\mathrm{u}_1, \mathrm{u}_4), (\acute{\varepsilon}_1\{0.6, 0.2, 0.5\}, \acute{\varepsilon}_2\{0.8, 0.5, 0.3\}, \acute{\varepsilon}_3\{0.6, 0.5, 0.5\}, \acute{\varepsilon}_4\{0.5, 0.6, 0.2\}) > \\ &\quad < (\mathrm{u}_4, \mathrm{u}_2), (\acute{\varepsilon}_1\{0.7, 0.4, 0.2\}, \acute{\varepsilon}_2\{0.6, 0.5, 0.3\}, \acute{\varepsilon}_3\{0.8, 0.4, 0.6\}, \acute{\varepsilon}_4\{0.7, 0.4, 0.3\}) > \\ &\quad < (\mathrm{u}_4, \mathrm{u}_4), (\acute{\varepsilon}_1\{0.6, 0.2, 0.2\}, \acute{\varepsilon}_2\{0.6, 0.5, 0.3\}, \acute{\varepsilon}_3\{0.8, 0.4, 0.5\}, \acute{\varepsilon}_4\{0.6, 0.4, 0.3\}) >\} \end{aligned}$$

Definition 3.12 AND-Operation of Two Neutrosophic Hypersoft Set

Let F_{Λ_1}, F_{Λ_2} ∈ NHSS. Consider k_1, k_2, k_3, …, k_n for $n \geq 1$, be n well-defined attributes, whose corresponding attributive values are respectively the set K_1, K_2, K_3, …, K_n with $K_i\ \cap\ K_j = = \emptyset$, for $i \neq j$ and $i, j \epsilon \{1,2,3 \ldots n\}$ and their relation $K_1\ \times\ K_2\ \times\ K_3 \times \ldots \times\ K_n = \Lambda$ then $F_{\Lambda_1}\ \wedge\ F_{\Lambda_2} = F_{\Lambda_1 \times \Lambda_2}$, then

$$T\ (F_{\Lambda_1 \times \Lambda_2}) = Min\left(T(F_{\Lambda_1}), T(F_{\Lambda_2})\right),$$

$$I\ (F_{\Lambda_1 \times \Lambda_2}) = Max\left(I(F_{\Lambda_1}), I(F_{\Lambda_2})\right),$$

$$\boldsymbol{F}\ (\boldsymbol{F}_{\Lambda_1 \times \Lambda_2}) = \boldsymbol{Max}\left(\boldsymbol{F}(\boldsymbol{F}_{\Lambda_1}), \boldsymbol{F}(\boldsymbol{F}_{\Lambda_2})\right).$$

Proposition 3.13 Let F_{Λ_1}, F_{Λ_2} are NHSSs then

1. $\left(F_{\Lambda_1} \vee F_{\Lambda_2}\right)^c = F^c(\Lambda_1) \wedge F^c(\Lambda_2)$
2. $\left(F_{\Lambda_1} \wedge F_{\Lambda_2}\right)^c = F^c(\Lambda_1) \vee F^c(\Lambda_2)$

Proof 1. Let F_{Λ_1} and F_{Λ_1} $\in$ NHSS, such as follows

$F_{\Lambda_1} = \{< u_i, \{T(F_{\Lambda_1}), I(F_{\Lambda_1}), F(F_{\Lambda_1})\} >: u_i \in U\}$ and $F_{\Lambda_2} = \{< u_j, \{T(F_{\Lambda_2}), I(F_{\Lambda_2}), F(F_{\Lambda_2})\} >: u_j \in U\}$

By using definition 3.10 we get

$F_{\Lambda_1} \vee F_{\Lambda_2} = \{< (u_i, u_j), [e, \max\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}, \min\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, \min\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}] >\}$

$\left(F_{\Lambda_1} \vee F_{\Lambda_2}\right)^c = \{< (u_i, u_j), [e, 1 - \max\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}, 1 - \min\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, 1 - \min\{F(F_{\Lambda_1}), 1 - F(F_{\Lambda_2})\}] >\}$

$\left(F_{\Lambda_1} \vee F_{\Lambda_2}\right)^c = \{< (u_i, u_j), [e, \min\{1 - T(F_{\Lambda_1}), 1 - T(F_{\Lambda_2})\}, \max\{1 - I(F_{\Lambda_1}), 1 - I(F_{\Lambda_2})\}, \max\{1 - F(F_{\Lambda_1}), 1 - F(F_{\Lambda_2})\}] >\}$

$\left(F_{\Lambda_1} \vee F_{\Lambda_2}\right)^c = \{< (u_i, u_j), [e, \min\{T(F^c(\Lambda_1)), T(F^c(\Lambda_2))\}, \max\{I(F^c(\Lambda_1)), I(F^c(\Lambda_2))\}, \max\{F(F^c(\Lambda_1)), F(F^c(\Lambda_2))\}] >\}$

Since

$F^c(\Lambda_1) = \{< u_i, \{T(F^c(\Lambda_1)), I(F^c(\Lambda_1)), F(F^c(\Lambda_1))\} >: u_i \in U\}$ and

$F^c(\Lambda_2) = \{< u_j, \{T(F^c(\Lambda_2)), I(F^c(\Lambda_2)), F(F^c(\Lambda_2))\} >: u_j \in U\}$

By using definition 3.12, we get

$F^c(\Lambda_1) \wedge F^c(\Lambda_2) = \{< (u_i, u_j), [e, \min\{T(F^c(\Lambda_1)), T(F^c(\Lambda_2))\}, \max\{I(F^c(\Lambda_1)), I(F^c(\Lambda_2))\}, \max\{F(F^c(\Lambda_1)), F(F^c(\Lambda_2))\}] >\}$

So

$\left(F_{\Lambda_1} \vee F_{\Lambda_2}\right)^c = F^c(\Lambda_1) \wedge F^c(\Lambda_2)$.

Similarly, we can prove 2.

4. Necessity and Possibility Operations

The necessity and possibility operations on NHSS with some properties are presented in the following section.

Definition 4.1 Necessity operation

Let $F_\Lambda \in$ NHSS, then necessity operation on NHSS represented by $\oplus\ F_\Lambda$ and defined as follows

$\oplus\ F_\Lambda = \{< \mathrm{u}, \{T(F_\Lambda), I(F_\Lambda), 1 - T(F_\Lambda)\} >\}$ for all $u \in U$.

Example 4.2 Reconsider example 2.6

$$\oplus\ F_\Lambda = \{< \mathrm{u}_1, (\acute{\varepsilon}_1\{0.4, 0.7, 0.6\}, \acute{\varepsilon}_2\{0.8, 0.5, 0.2\}, \acute{\varepsilon}_3\{0.6, 0.5, 0.4\}) >$$
$$< \mathrm{u}_2, (\acute{\varepsilon}_1\{0.1, 0.5, 0.9\}, \acute{\varepsilon}_2\{0.5, 0.6, 0.5\}, \acute{\varepsilon}_3\{0.7, 0.4, 0.3\}) >\}$$

Proposition 4.3

1. $\oplus\ \left(F_{\Lambda_1} \cup F_{\Lambda_2}\right) = \oplus\ F_{\Lambda_2} \cup \oplus\ F_{\Lambda_1}$
2. $\oplus\ \left(F_{\Lambda_1} \cap F_{\Lambda_2}\right) = \oplus\ F_{\Lambda_2} \cap \oplus\ F_{\Lambda_1}$

Proof 1. Let $F_{\Lambda_1} \cup F_{\Lambda_2} = F_{\Lambda_3}$, then

$$T\ (F_{\Lambda_3}) = \begin{cases} T(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ T(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Max\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\} & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$I\ (F_{\Lambda_3}) = \begin{cases} I(F_{\Lambda_1}) & if\ u\ \in \Lambda_1 - \Lambda_2 \\ I(F_{\Lambda_2}) & if\ u\ \in \Lambda_2 - \Lambda_1 \\ Min\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\} & if\ u\ \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$F\ (F_{\Lambda_3}) = \begin{cases} F(F_{\Lambda_1}) & if\ u\ \in \Lambda_1 - \Lambda_2 \\ F(F_{\Lambda_2}) & if\ u\ \in \Lambda_2 - \Lambda_1 \\ Min\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\} & if\ u\ \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

By using the definition of necessity operation

$\oplus F_{\Lambda_3} = \{< u, \{\oplus T(F_{\Lambda_3}), \oplus I(F_{\Lambda_3}), \oplus F(F_{\Lambda_3})\} >: u \in U\}$, where

$$\oplus T\ (F_{\Lambda_3}) = \begin{cases} T(F_{\Lambda_1}) & if\ u\ \in \Lambda_1 - \Lambda_2 \\ T(F_{\Lambda_2}) & if\ u\ \in \Lambda_2 - \Lambda_1 \\ Max\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\} & if\ u\ \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$\oplus I\ (F_{\Lambda_3}) = \begin{cases} I(F_{\Lambda_1}) & if\ u\ \in \Lambda_1 - \Lambda_2 \\ I(F_{\Lambda_2}) & if\ u\ \in \Lambda_2 - \Lambda_1 \\ Min\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\} & if\ u\ \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$\oplus F\ (F_{\Lambda_3}) = \begin{cases} 1 - T(F_{\Lambda_1}) & if\ u\ \in \Lambda_1 - \Lambda_2 \\ 1 - T(F_{\Lambda_2}) & if\ u\ \in \Lambda_2 - \Lambda_1 \\ 1 - Max\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\} & if\ u\ \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

Assume

$\oplus F_{\Lambda_1} = \{< u, \{T(F_{\Lambda_1}), I(F_{\Lambda_1}), 1 - T(F_{\Lambda_1})\} >: u \in U\}$

$\oplus F_{\Lambda_2} = \{< u, \{T(F_{\Lambda_2}), I(F_{\Lambda_2}), 1 - T(F_{\Lambda_2})\} >: u \in U\}$

$\oplus F_{\Lambda_1} \cup \oplus F_{\Lambda_2} = F_{\delta}$, where

$F_{\delta} = \{< u, \{T(F_{\delta}), I(F_{\delta}), F(F_{\delta})\} >: u \in U\}$, such that

$$T\ (F_{\delta}) = \begin{cases} T(F_{\Lambda_1}) & if\ u\ \in \Lambda_1 - \Lambda_2 \\ T(F_{\Lambda_2}) & if\ u\ \in \Lambda_2 - \Lambda_1 \\ Max\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\} & if\ u\ \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$I\ (F_{\delta}) = \begin{cases} I(F_{\Lambda_1}) & if\ u\ \in \Lambda_1 - \Lambda_2 \\ I(F_{\Lambda_2}) & if\ u\ \in \Lambda_2 - \Lambda_1 \\ Min\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\} & if\ u\ \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$F\ (F_{\delta}) = \begin{cases} 1 - T(F_{\Lambda_1}) & if\ u\ \in \Lambda_1 - \Lambda_2 \\ 1 - T(F_{\Lambda_2}) & if\ u\ \in \Lambda_2 - \Lambda_1 \quad \text{OR} \\ Min\{1 - T(F_{\Lambda_1}), 1 - T(F_{\Lambda_2})\} & if\ u\ \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$F\ (F_{\delta}) = \begin{cases} 1 - T(F_{\Lambda_1}) & if\ u\ \in \Lambda_1 - \Lambda_2 \\ 1 - T(F_{\Lambda_2}) & if\ u\ \in \Lambda_2 - \Lambda_1 \\ 1 - Max\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\} & if\ u\ \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

Consequently $\oplus F_{\Lambda_3}$ and F_{δ} are same. So

$\oplus\ (F_{\Lambda_1} \cup F_{\Lambda_2}) = \oplus\ F_{\Lambda_2} \cup \oplus\ F_{\Lambda_1}$.

Similarly, we can prove 2.

Definition 4.4 Possibility operation

Let $F_{\Lambda} \in$ NHSS, then possibility operation on NHSS represented by $\otimes\ F_{\Lambda}$ and defined as follows

$\otimes\ F_{\Lambda} = \{< \mathrm{u}, \{1 - F(F_{\Lambda}), I(F_{\Lambda}), F(F_{\Lambda})\} >\}$ for all $u \in U$.

Example 4.5 Reconsider the example 2.6

$$\otimes\ F_{\Lambda} = \{< \mathrm{u}_1, (\acute{\varepsilon}_1\{0.5, 0.7, 0.5\}, \acute{\varepsilon}_2\{0.7, 0.5, 0.3\}, \acute{\varepsilon}_3\{0.1, 0.5, 0.9\}) >$$

$< \mathrm{u}_2, (\acute{\varepsilon}_1\{0.3, 0.5, 0.7\}, \acute{\varepsilon}_2\{0.8, 0.6, 0.2\}, \acute{\varepsilon}_3\{0.4, 0.4, 0.6\}) >\}$

Proposition 4.6

1. $\otimes\ (F_{\Lambda_1} \cup F_{\Lambda_2}) = \otimes\ F_{\Lambda_2} \cup \otimes\ F_{\Lambda_1}$
2. $\otimes\ (F_{\Lambda_1} \cap F_{\Lambda_2}) = \otimes\ F_{\Lambda_2} \cap \otimes\ F_{\Lambda_1}$

Proof 1. Let $F_{\Lambda_1} \cup F_{\Lambda_2} = F_{\Lambda_3}$, then

$$T\ (F_{\Lambda_3}) = \begin{cases} T(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ T(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Max\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\} & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$I\ (F_{\Lambda_3}) = \begin{cases} I(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ I(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Min\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\} & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$F\ (F_{\Lambda_3}) = \begin{cases} F(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ F(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Min\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\} & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

By using the definition of necessity operation

$\otimes F_{\Lambda_3} = \{< \mathrm{u}, \{\otimes T(F_{\Lambda_3}), \otimes I(F_{\Lambda_3}), \otimes F(F_{\Lambda_3})\} >: u \in U\}$, where

$$\otimes\ T\ (F_{\Lambda_3}) = \begin{cases} 1 - F(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ 1 - F(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ 1 - Max\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\} & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$= \begin{cases} 1 - F(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ 1 - F(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Min\{1 - F(F_{\Lambda_1}), 1 - F(F_{\Lambda_2})\} & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$\otimes\ I\ (F_{\Lambda_3}) = \begin{cases} I(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ I(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Min\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\} & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$\otimes\ F\ (F_{\Lambda_3}) = \begin{cases} F(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ F(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Min\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\} & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

Assume

$\otimes F_{\Lambda_1} = \{< \mathrm{u}, \{1 - F(F_{\Lambda_1}), I(F_{\Lambda_1}), F(F_{\Lambda_1})\} >: u \in U\}$

$\otimes F_{\Lambda_2} = \{< \mathrm{u}, \{1 - F(F_{\Lambda_2}), I(F_{\Lambda_2}), F(F_{\Lambda_2})\} >: u \in U\}$

$\otimes F_{\Lambda_1} \cup \oplus F_{\Lambda_2} = F_\delta$, where

$F_\delta = \{< \mathrm{u}, \{T(F_\delta), I(F_\delta), F(F_\delta)\} >: u \in U\}$, such that

$$\otimes\ T\ (F_\delta) = \begin{cases} 1 - F(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ 1 - F(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ 1 - Max\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\} & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$\otimes\ I\ (F_\delta) = \begin{cases} I(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ I(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Min\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\} & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$\otimes\ F\ (F_\delta) = \begin{cases} F(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ F(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Min\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\} & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

Consequently $\otimes F_{\Lambda_3}$ and F_δ are same. So

$\otimes\left(F_{\Lambda_1} \cup F_{\Lambda_2}\right) = \otimes F_{\Lambda_2} \cup \otimes F_{\Lambda_1}$

Similarly, we can prove 2.

Proposition 4.7 Let F_{Λ_1} and $F_{\Lambda_2} \in$ NHSS, than we have the following

1. $\oplus\left(F_{\Lambda_1} \wedge F_{\Lambda_2}\right) = \oplus F_{\Lambda_1} \wedge \oplus F_{\Lambda_2}$
2. $\oplus\left(F_{\Lambda_1} \vee F_{\Lambda_2}\right) = \oplus F_{\Lambda_1} \vee \oplus F_{\Lambda_2}$
3. $\otimes\left(F_{\Lambda_1} \wedge F_{\Lambda_2}\right) = \otimes F_{\Lambda_1} \wedge \otimes F_{\Lambda_2}$
4. $\otimes\left(F_{\Lambda_1} \vee F_{\Lambda_2}\right) = \otimes F_{\Lambda_1} \vee \otimes F_{\Lambda_2}$

Proof 1. Assume $F_{\Lambda_1} \wedge F_{\Lambda_2} = F_{\Lambda_1 \times \Lambda_2}$, where $(u_i, u_j) \in \Lambda_1 \times \Lambda_2$

$F_{\Lambda_1 \times \Lambda_2} = \left\{< (u_i, u_j), \left[e, \min\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}, \max\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, \max\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}\right] >\right\}$

By using definition 4.1, we have

$\oplus\left(F_{\Lambda_1} \wedge F_{\Lambda_2}\right) = \left\{< (u_i, u_j), \left[e, \min\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}, \max\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, 1 - \min\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}\right] >\right\}$

Since

$\oplus F_{\Lambda_1} = \left\{< \mathrm{u}, \{T(F_{\Lambda_1}), I(F_{\Lambda_1}), 1 - T(F_{\Lambda_1})\} >\right\}$, and

$\oplus F_{\Lambda_2} = \left\{< \mathrm{u}, \{T(F_{\Lambda_2}), I(F_{\Lambda_2}), 1 - T(F_{\Lambda_2})\} >\right\}$, then by using AND-operation, we get

$\oplus F_{\Lambda_1} \wedge \oplus F_{\Lambda_2} =$

$\left\{< (u_i, u_j), \left[e, \min\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}, \max\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, \max\{1 - T(F_{\Lambda_1}), 1 - T(F_{\Lambda_2})\}\right] >\right\}$

$= \left\{< (u_i, u_j), \left[e, \min\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}, \max\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, 1 - \min\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}\right] >\right\}$

$= \oplus\left(F_{\Lambda_1} \wedge F_{\Lambda_2}\right)$

Proof 2. Similar to Assertion 1.

Proof 3. Assume $F_{\Lambda_1} \wedge F_{\Lambda_2} = F_{\Lambda_1 \times \Lambda_2}$, where $(u_i, u_j) \in \Lambda_1 \times \Lambda_2$

$F_{\Lambda_1 \times \Lambda_2} = \left\{< (u_i, u_j), \left[e, \min\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}, \max\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, \max\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}\right] >\right\}$

By using definition 4.4, we have

$\otimes\left(F_{\Lambda_1} \wedge F_{\Lambda_2}\right) = \left\{< (u_i, u_j), \left[e, 1 - \max\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}, \max\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, \max\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}\right] >\right\}$

Since

$\otimes F_{\Lambda_1} = \left\{< \mathrm{u}, \{1 - F(F_{\Lambda_1}), I(F_{\Lambda_1}), F(F_{\Lambda_1})\} >\right\}$, and

$\otimes F_{\Lambda_2} = \left\{< \mathrm{u}, \{1 - F(F_{\Lambda_2}), I(F_{\Lambda_2}), F(F_{\Lambda_2})\} >\right\}$, then by using AND-operation, we get

$\otimes F_{\Lambda_1} \wedge \oplus F_{\Lambda_2} =$

$\left\{< (u_i, u_j), \left[e, \min\{1 - F(F_{\Lambda_1}), 1 - F(F_{\Lambda_2})\}, \max\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, \max\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}\right] >\right\}$

$= \left\{< (u_i, u_j), \left[e, 1 - \max\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}, \max\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, \max\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}\right] >\right\}$

$= \otimes\left(F_{\Lambda_1} \wedge F_{\Lambda_2}\right)$

Proof 4. Assume $F_{\Lambda_1} \vee F_{\Lambda_2} = F_{\Lambda_1 \times \Lambda_2}$, where $(u_i, u_j) \in \Lambda_1 \times \Lambda_2$

$F_{\Lambda_1 \times \Lambda_2} = \left\{< (u_i, u_j), \left[e, \max\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}, \min\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, \min\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}\right] >\right\}$

By using definition 4.4, we have

$\otimes\left(F_{\Lambda_1} \vee F_{\Lambda_2}\right) = \left\{< (u_i, u_j), \left[e, 1 - \min\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}, \min\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, \min\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}\right] >\right\}$

Since

$\otimes F_{\Lambda_1} = \left\{< \mathrm{u}, \{1 - F(F_{\Lambda_1}), I(F_{\Lambda_1}), F(F_{\Lambda_1})\} >\right\}$, and

$\otimes F_{\Lambda_2} = \left\{< \mathrm{u}, \{1 - F(F_{\Lambda_2}), I(F_{\Lambda_2}), F(F_{\Lambda_2})\} >\right\}$, then by using OR-operation, we get

$\otimes F_{\Lambda_1} \vee \oplus F_{\Lambda_2} =$

$\left\{< (u_i, u_j), \left[e, \max\{1 - F(F_{\Lambda_1}), 1 - F(F_{\Lambda_2})\}, \min\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, \min\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}\right] >\right\}$

$$= \{< (u_i, u_j), [e, 1 - \min\{F(F_{\wedge_1}), F(F_{\wedge_2})\}, \min\{I(F_{\wedge_1}), I(F_{\wedge_2})\}, \min\{F(F_{\wedge_1}), F(F_{\wedge_2})\}] >\}$$

$$= \otimes (\boldsymbol{F}_{\wedge_1} \wedge \boldsymbol{F}_{\wedge_2})$$

5. Conclusion

In this paper, we study neutrosophic hypersoft set with some basic definition. We proposed the generalized aggregate operators on neutrosophic hypersoft sets such as complement, extended union, extended intersection, And-operation, and Or-operation with their properties and proved the commutative and associative laws on NHSS by using extended union and extended intersection. Finally, the concept of necessity and possibility operations on NHSS with suitable numerical examples and properties are presented. For future trends, we can develop the distance-based similarity measure and will be used for decision making, medical diagnoses, pattern recognition, etc. We also develop the neutrosophic hypersoft matrices with its operations and properties by using proposed operations and use for decision making.

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Composite Neutrosophic Finite Automata

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Abstract. The idea behind the neutrosophic set is we can connect the concept by dynamics of opposite interacts and its neutral that are uncertain and get common parts. Automata theory is beneficial to solve computational complexity problem and also it is an influential mathematical modeling tool in computer science. Inspired by the concepts of neutrosophic sets and automata theory, here, we are introducing and discussing the algebraic concept of neutrosophic finite automata based on the paper [10]. Generally, composite machines can be achieved by the output of the one machine that will be used as input for another machines. This paper introduced the concept of composite automata under the environment of the neutrosophic set and also examined the box function between the composite neutrosophic finite automata.

Keywords: automata theory, stable, composite, box function, neutrosophic set

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1. Introduction

Smarandache [27, 28] has proposed an idea of neutrosophic sets which was extending from fuzzy sets. Neutrosophic sets have membership values lies in $]0^-, 1^+[$, the nonstandard unit interval [23] which includes the degree of truth, indeterminacy, and falsity. It is a device for handling the computational complexity of real-life and scientific problems whereas the fuzzy set has limited sources to depict it. The neutrosophic sets are different from intuitionistic fuzzy sets, it is because the neutrosophic set degree of indeterminacy can be defined independently since it is quantified explicitly. Aftermath, there are lots of research works done in various fields such as algebraic structures [5,21,29], topological structures [8,20,24], control theory [17,18,36], decision-making [2, 3, 14, 22, 34], medical [1, 25, 35] and smart product-service system [4].

Generally, computational complexity problems are solved by the automata theory. It has a wide application in computer science and discrete mathematics which is also used to study the behavior of dynamical discrete systems. Fuzzy automata emerge from the inclusion of fuzzy logic into automata theory. Fuzzy finite automata are beneficial to model uncertainties which inherent in many applications [6]. Wee [33] and Santos [26] first introduced the theory of fuzzy finite automata to deal with the notions frequently encountered in the study of natural languages such as vagueness and imprecision. Malik et al [16] introduced a considerably simpler notion of a fuzzy finite state machine that is almost identical to fuzzy finite automatons and greatly contributed to the algebraic study of the fuzzy automaton and fuzzy languages. In addition, several researchers contributed to the development of the theory of fuzzy automata ([11]). Fuzzy finite automata with output offer further inclination in providing output compare to one without outputs. For each assigning input, the machine will generate output and its value is a function of the current state and the current input. Verma and Tiwari [32] recently introduced and studied the concepts of state distinguishability, input-distinguishability, and output completeness of states of a crisp deterministic fuzzy automaton with output function based on [7].

In recent years neutrosophic sets and systems have become an area of interest for many researchers in different areas because it can provide a practical way to address real-world problems more efficiently along with indeterminacy naturally especially in the realm of decision-making. Neutrosophic automata is a newer model, which is extended from a fuzzy automata theory. The neutrosophic set idea was incorporated in automata theory by many researchers in different forms such as finite state machine and its switchboard machine was introduced by under the concept of interval neutrosophic sets [30] and single-valued neutrosophic sets [31]. Further, the finite automata theory has been extended by the concept of general fuzzy automata under the environment of neutrosophic sets, which is called as neutrosophic general finite automata [12]. In addition, the concept of distinguishability and inverse of neutrosophic finite automata was introduced by Kavikumar et al. in [10]. However, still, there are many algebraic structures of neutrosophic automata theory that haven't been studied yet especially automaton with output. Hence, it is important to study more algebraic structures on neutrosophic automata theory with outputs. Therefore, our motive is to study and introduce the concept of composite neutrosophic finite automata which we can obtain by using the outputs of one automaton as inputs to another automaton.

2. **Preliminaries**

Definition 2.1. Let X be a universe of discourse. The neutrosophic set is an object having the form $A = \{\prec x, \delta_1(x), \delta_2(x), \delta_3(x) \succ |\forall x \in X\}$ where the functions can be defined by $\delta_1, \delta_2, \delta_3 : X \rightarrow]0, 1[$ and δ_1 is the degree of membership or truth, δ_2 is the degree of indeterminancy and δ_3 is the degree of non-membership or false of the element $x \in X$ to the set A with the condition $\delta_1(x) + \delta_2(x) + \delta_3(x) \leq 3$.

Let X be a universe of discourse and λ is a neutrosophic subset of X. A map $\lambda : X \rightarrow L$, where L is a lattice-ordered monoid. The definition of lattice-ordered monoid is as follows:

Definition 2.2. An algebra $\mathbb{L} = (L, \leq, \wedge, \vee, \bullet, 0, 1)$ is called a lattice-ordered monoid if

(1) $\mathbb{L} = (L, \leq, \wedge, \vee, 0, 1)$ is a lattice with the least element 0 and the element element 1.
(2) $(L, \bullet, 1)$ is a monoid with 1 identity $1 \in L$ such that $a, b, c \in L$.
   (a) $a \bullet 0 = 0 \bullet a = 0$,
   (b) $a \leq b \Rightarrow a \bullet x \leq b \bullet b, \forall x \in L$,
   (c) $a \bullet (b \vee c) = (a \bullet b) \vee (b \bullet c)$ and $(b \vee c) \bullet a = (b \bullet a) \vee (c \bullet a)$.

Throughout, we work with a lattice-ordered monoid $\mathbb{L}$ so that the monoid $(L, \bullet, 1)$ satisfies the left cancellation law. A neutrosophic finite automaton with outputs (in short; neutrosophic finite automata (NFA)) has considered with neutrosophic transition function and neutrosophic output function.

Definition 2.3. A NFA is a five-tuple $\mathbb{M} = (Q, \Sigma, Z, \delta, \sigma)$, where Q is a finite non-empty set of states, Σ is a finite set of input alphabet, Z is a finite set of output alphabet, δ is a neutrosophic subset of $Q \times \Sigma \times Q$ which represents neutrosophic transition function, and σ is a neutrosophic subset of $Q \times \Sigma \times Z$ which represents neutrosophic output function.

Definition 2.4. Let $\mathbb{M} = (Q, \Sigma, Z, \delta, \sigma)$ be a NFA.

(1) $Q = \{q_1, q_2, \ldots, q_n\}$, is a finite set of states,
(2) $\Sigma = \{x_1, x_2, \ldots, x_n\}$, is a finite set of input symbols,
(3) $Z = \{y_1, y_2, \ldots, y_n\}$, is a finite set of output symbols,
(4) Let $\delta = \prec \delta_1, \delta_2, \delta_3 \succ$ is a neutrosophic subset of $Q \times \Sigma \times Q$ such that the neutrosophic transition function $\delta : A \times \Sigma \times Q \rightarrow L \times L \times L$ is defined as follows: $\forall q_i, q_j \in Q$ and $x_1, x_2 \in \Sigma$,

$$\delta_1(q_i, \Lambda, q_j) = \begin{cases} 1 & \text{if } q_i = q_j \\ 0 & \text{if } q_i \neq q_j \end{cases}$$
$$\delta_2(q_i, \Lambda, q_j) = \begin{cases} 0 & \text{if } q_i = q_j \\ 1 & \text{if } q_i \neq q_j \end{cases}$$
$$\delta_3(q_i, \Lambda, q_j) = \begin{cases} 0 & \text{if } q_i = q_j \\ 1 & \text{if } q_i \neq q_j \end{cases}$$

and

$$\delta_1(q_i, x_1x_2, q_j) = \bigvee_{r\in Q} \{\delta_1(q_i, x_1, r) \wedge \delta_1(r, x_2, q_j)\}$$
$$\delta_2(q_i, x_1x_2, q_j) = \bigwedge_{r\in Q} \{\delta_2(q_i, x_1, r) \vee \delta_2(r, x_2, q_j)\}$$
$$\delta_3(q_i, x_1x_2, q_j) = \bigwedge_{r\in Q} \{\delta_3(q_i, x_1, r) \vee \delta_3(r, x_2, q_j)\}$$

(5) Let $\sigma = \prec \sigma_1, \sigma_2, \sigma_3 \succ$ is a neutrosophic subset of $Q \times \Sigma \times Z$ such that the neutrosophic output function $\sigma : Q \times \Sigma \times Z \to L \times L \times L$ is defined as follows: $\forall q_i, q_j \in Q, x_1, x_2 \in \Sigma$ and $y_1, y_2 \in Z$,

$$\sigma_1(q_i, x_1, q_j) = \begin{cases} 1 & \text{if } x_1 = y_1 = \Lambda \\ 0 & \text{if } x_1 = \Lambda, y_1 \neq \Lambda \text{ or } x_1 \neq \Lambda, y_1 = \Lambda \end{cases}$$
$$\sigma_2(q_i, x_1, q_j) = \begin{cases} 0 & \text{if } x_1 = y_1 = \Lambda \\ 1 & \text{if } x_1 = \Lambda, y_1 \neq \Lambda \text{ or } x_1 \neq \Lambda, y_1 = \Lambda \end{cases}$$
$$\sigma_3(q_i, x_1, q_j) = \begin{cases} 0 & \text{if } x_1 = y_1 = \Lambda \\ 1 & \text{if } x_1 = \Lambda, y_1 \neq \Lambda \text{ or } x_1 \neq \Lambda, y_1 = \Lambda \end{cases}$$

and

$$\sigma_1(q_i, x_1x_2, y_1y_2) = \sigma_1(q_i, x_1, y_1) \bullet \bigvee_{r\in Q} \{\delta_1(q_i, x_1, r) \wedge \sigma_1(r, x_2, y_2)\}$$
$$\sigma_2(q_i, x_1x_2, y_1y_2) = \sigma_2(q_i, x_1, y_1) \bullet \bigwedge_{r\in Q} \{\delta_2(q_i, x_1, r) \vee \sigma_2(r, x_2, y_2)\}$$
$$\sigma_3(q_i, x_1x_2, y_1y_2) = \sigma_3(q_i, x_1, y_1) \bullet \bigwedge_{r\in Q} \{\delta_3(q_i, x_1, r) \vee \sigma_3(r, x_2, y_2)\}$$

3. Composite Neutrosophic Finite Automata

This section is interested in the concept of composite finite automata under the environment of neutrosophic sets.

Definition 3.1. For $i \leq n$, let $\mathbb{M}_i = (Q_i, \Sigma_i, Z_i, \delta^i, \sigma^i)$ be NFA's. Let $\mathbb{M}_T = \mathbb{M}_1 \to \mathbb{M}_2 \to \cdots \to \mathbb{M}_n$ be a composite NFA, where $(q_1, q_2, \ldots, q_n) = q_T \in Q_T$ and each $q_i \in Q_i$ if

(1) $Z_i \subseteq \Sigma_{i+1}$, for $i \leq n-1$.

(2) let $\{(x_T \in \Sigma_T \Rightarrow x_1 \in \Sigma_1)(y_T \in Z_T \Rightarrow y_n \in Z_n) | \sigma_1^1(q_1, x_T, y_1) > 0, \sigma_2^1(q_1, x_T, y_1) < 1, \sigma_3^1(q_1, x_T, y_1) < 1, \text{ for } i = 1\}$ then define

$$\delta_1^T \left[(q_1, q_2, \ldots, q_n), x_T, (q_1', q_2', \ldots, q_n')\right] = \begin{cases} \delta_1^1(q_1, x_1, q_1') > 0 & \text{for } i = 1, \\ \delta_1^i(q_i, (\sigma_1^i(q_i, y_{i-1}, y_i)), q_i') & \text{for } i > 1. \end{cases}$$

,

$$\delta_2^T \left[(q_1, q_2, \ldots, q_n), x_T, (q_1', q_2', \ldots, q_n')\right] = \begin{cases} \delta_2^1(q_1, x_1, q_1') < 1 & \text{for } i = 1, \\ \delta_2^i(q_i, (\sigma_2^i(q_i, y_{i-1}, y_i)), q_i') & \text{for } i > 1. \end{cases}$$

$$\delta_3^T\left[(q_1,q_2,\ldots,q_n),x_T,(q_1',q_2',\ldots,q_n')\right]=\begin{cases}\delta_3^1(q_1,x_1,q_1')<1 & \text{for } i=1,\\ \delta_3^i(q_i,(\sigma_3^i(q_i,y_{i-1},y_i)),q_i') & \text{for } i>1.\end{cases}$$

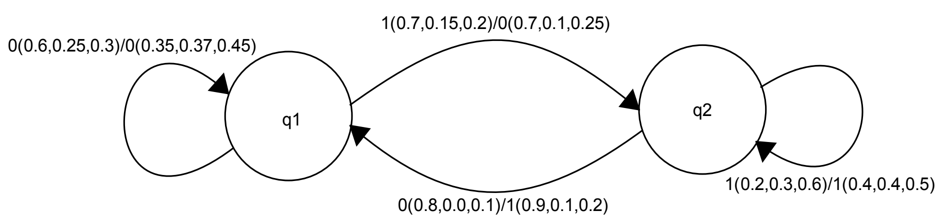
and

$$\sigma_1^T((q_1,q_2,\ldots,q_n),x_T,y_n)=\begin{cases}1 & \text{if } x_T=y_n=\Lambda\\ 0 & \text{if either } x_T\neq\Lambda \text{ and } y_n=\Lambda \text{ or } x_T=\Lambda \text{ and } y_n\neq\Lambda\end{cases}$$

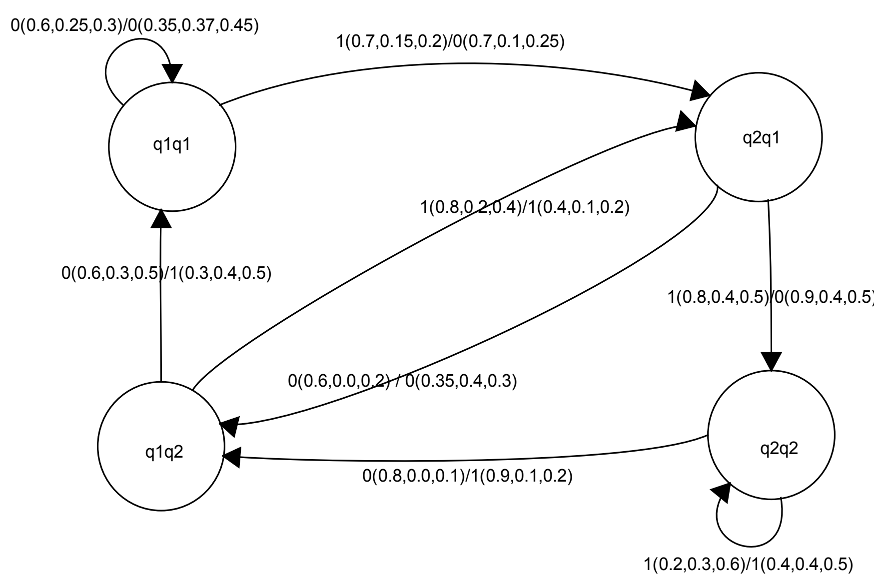
$$\sigma_2^T((q_1,q_2,\ldots,q_n),x_T,y_n)=\begin{cases}0 & \text{if } x_T=y_n=\Lambda\\ 1 & \text{if either } x_T\neq\Lambda \text{ and } y_n=\Lambda \text{ or } x_T=\Lambda \text{ and } y_n\neq\Lambda\end{cases}$$

$$\sigma_3^T((q_1,q_2,\ldots,q_n),x_T,y_n)=\begin{cases}0 & \text{if } x_T=y_n=\Lambda\\ 1 & \text{if either } x_T\neq\Lambda \text{ and } y_n=\Lambda \text{ or } x_T=\Lambda \text{ and } y_n\neq\Lambda\end{cases}$$

Example 3.2. Let $\mathbb{M}=(Q,\Sigma,Z,\delta,\sigma)$ is a NFA, where $Q=\{q_1,q_2\}$, $\Sigma=\{a,b\}$ and $Z=\{0,1\}$ and the transition diagram is given below:



Now, we define the composite NFA, $\mathbb{M}_T=\mathbb{M}\rightarrow\mathbb{M}$ and its transition diagram is given below:



Then the output for input $x_T=1001$ is $y_T=0010$.

Definition 3.3. Let $\mathbb{M} = (Q, \Sigma, Z, \delta, \sigma)$ be a NFA. A non-empty set of states $Q_A \subseteq \mathbb{M}$ is said to be stable if

$$\delta_1(q, x, p) > 0, \delta_2(q, x, p) < 1, \delta_3(q, x, p) < 1,$$

for all $q, p \in Q_A$ and $x \in \Sigma$.

Definition 3.4. Two NFA's $\mathbb{M}_1 = (Q_1, \Sigma_1, Z_1, \delta^1, \sigma^1)$ and $\mathbb{M}_2 = (Q_2, \Sigma_2, Z_2, \delta^2, \sigma^2)$ are said to be homomorphism if $\alpha\left[\delta^1(q, x, p)\right] = \delta^2(\alpha(q), \beta(x), \alpha(p))$ and $\sigma^1(q, x, y) \leq \sigma^2(\alpha(q), \beta(x), \gamma(y))$, $\forall q, p \in Q_1$, $x \in \Sigma_1$ and $y \in Z_1$, where the mapping $\alpha : Q_1 \to Q_2$, $\beta : \Sigma_1 \to \Sigma_2$ and $\gamma : Z_1 \to Z_2$ are monoid homomorphisms. Moreover, two NFA's are said to be isomorphism when the mapping α, β and γ are bijective.

Lemma 3.5. *Let $\mathbb{M}_1 = (Q_1, \Sigma_1, Z_1, \delta^1, \sigma^1)$, $\mathbb{M}_2 = (Q_2, \Sigma_2, Z_2, \delta^2, \sigma^2)$ and $\mathbb{M}_3 = (Q_3, \Sigma_3, Z_3, \delta^3, \sigma^3)$ be NFA's. Then $\mathbb{M}_1 \to (\mathbb{M}_2 \to \mathbb{M}_3)$ and $(\mathbb{M}_1 \to \mathbb{M}_2) \to \mathbb{M}_3$ are isomorphic.*

Proof. Since one neutrosophic finite automaton outputs are used as the another neutrosophic finite automaton inputs and omit the parentheses as follows $\mathbb{M}_1 \to \mathbb{M}_2 \to \mathbb{M}_3$. Now, we have an initial inputs for $\mathbb{M}_1$ and its outputs will become an input of $\mathbb{M}_2$. Then, the outputs of $\mathbb{M}_2$ will be an input of $\mathbb{M}_3$. In this manner, $\mathbb{M}_1 \to (\mathbb{M}_2 \to \mathbb{M}_3)$ and $(\mathbb{M}_1 \to \mathbb{M}_2) \to \mathbb{M}_3$ are isomorphic.

Remark 3.6. Lemma 3.5 can be easily extend to four or more NFA's.

Lemma 3.7. *Let $\mathbb{M}_i = (Q_i, \Sigma_i, Z_i, \delta^i, \sigma^i)$, where $i = 1, 2, \ldots, n$, be NFA's. If $\mathbb{M}_1 \to \mathbb{M}_2 \to \cdots \to \mathbb{M}_n$ is a composite NFA if and only if $\mathbb{M}_n$ is a NFA.*

Proof. Assume that $\mathbb{M}_1 \to \mathbb{M}_2 \to \cdots \to \mathbb{M}_n$ is a composite NFA. Then, by lemma 3.5, it is clear that $\mathbb{M}_n$ is a NFA. Conversely, since $\mathbb{M}_n$ is a NFA, the input of $\mathbb{M}_n$ is a output of the $\mathbb{M}_{n-1}$, so in this manner, $\mathbb{M}_1 \to \mathbb{M}_2 \to \cdots \to \mathbb{M}_n$ is a composite NFA.

Definition 3.8. A NFA $\mathbb{M} = (Q, \Sigma, Z, \delta, \sigma)$ is called free if $\forall q_i \in Q$, $x \in \Sigma$ $\exists$ $y \in Z$ such that

$$\sigma_1(q_i, x, y) > 0, \quad \sigma_2(q_i, x, y) < 1, \quad \text{and} \quad \sigma_3(q_i, x, y) < 1.$$

Theorem 3.9. *For each positive integer $i \leq n$, let $\mathbb{M}_i$ is a free NFA, then $\mathbb{M}_1 \to \mathbb{M}_2 \to \cdots \to \mathbb{M}_n$ is a composite NFA.*

Proof. Suppose $\mathbb{M}_i$, $i = 1, 2, \ldots, n$ is a NFA. Let $q, p \in Q_1$ and $x_1 \in \Sigma_1$ and $y_1 \in Z_1$. We prove the theorem by induction on $|i| = n$.

If $n = 1$, then $\mathbb{M}_1$ is a free NFA. Now, we have

$$\sigma_1^1(q_1, x_1, y_1) > 0, \quad \sigma_2^1(q_1, x_1, y_1) < 1, \quad \text{and} \quad \sigma_3^1(q_1 x_1, y_1) < 1,$$

since $\delta_1^1(q_1, x_1, p_1) > 0, \delta_2^1(q_1, x_1, p_1) < 1$ and $\delta_3^1(q_1, x_1, p_1) < 1$. This implies that $\mathbb{M}_1$ is a composite NFA. Hence, the theorem is true for $n = 1$.

Suppose the result is true for all $x_i \in \Sigma_i$ and $y_i \in Z_i$ such that $|i| = n-1$. Let $Z_i \subseteq \Sigma_{i+1}$ for $i \leq n-1$, $n > 1$, so that $\mathbb{M}_{n-1}$ is a free NFA. Now, we have,

$$\sigma_1^{n-1}(q_{n-1}, x_{n-1}, y_{n-1}) > 0, \quad \sigma_2^{n-1}(q_{n-1}, x_{n-1}, y_{n-1}) < 1, \quad \text{and} \quad \sigma_3^{n-1}(q_{n-1}, x_{n-1}, y_{n-1}) < 1.$$

Then by Definition 3.1, we have

$$\delta_1^n(q_n, y_{n-1}, p_n) > 0, \quad \delta_2^n(q_n, y_{n-1}, p_n) < 1 \quad \text{and} \quad \delta_3^n(q_n, y_{n-1}, p_n) < 1.$$

By the induction hypothesis and consider $y_{n-1} = x_n$, then we have

$$\delta_1^n(q_n, x_n, p_n) > 0, \quad \delta_2^n(q_n, x_n, p_n) < 1 \quad \text{and} \quad \delta_3^n(q_n, x_n, p_n) < 1.$$

This implies that, for $x_n \in \Sigma_n$ there exists $y_n \in Z_n$ such that

$$\sigma_1^n(q_n, x_n, y_n) > 0, \quad \sigma_2^n(q_n, x_n, y_n) < 1, \quad \text{and} \quad \sigma_3^n(q_n, x_n, y_n) < 1.$$

Hence, the theorem is true for induction.

Remark 3.10. The converse of Theorem 3.9 is not true since the outputs of composite NFA need not be satisfy the condition of free NFA.

Definition 3.11. Let $\mathbb{M}_1 = (Q_1, \Sigma_1, Z_1, \delta^1, \sigma^1)$ and $\mathbb{M}_2 = (Q_2, \Sigma_2, Z_2, \delta^2, \sigma^2)$ be NFA's. A box function β of $(\mathbb{M}_1, \mathbb{M}_2)$ is satisfy the following conditions, where $\beta : Q_1 \to Q_2$ such that

(1) $\Sigma_1 \subseteq Z_2$

(2) for all $q, p \in Q_1$ and $x \in \Sigma_1$ there exists $y \in Z_1$ such that

$$\beta\left[\delta^1(q, x, p)\right] = \delta^2\left[\beta(q), \sigma^1(q, x, y), \beta(p)\right].$$

Definition 3.12. Let $\mathbb{M}_i = (Q_i, \Sigma_i, Z_i, \delta^i, \sigma^i)$, i=1,2,...,n, be NFA's. To each box functions β_i of $(\mathbb{M}_i, \mathbb{M}_{i+1})$ for $1 \leq i \leq n-1$, there is a corresponding sub NFA $\mathbb{N}(\beta_1, \beta_2, \ldots, \beta_{n-1})$ of $\mathbb{M}_T = \mathbb{M}_1 \to \mathbb{M}_2 \to \cdots \to \mathbb{M}_n$.

Proposition 3.13. *Let $\mathbb{M}_T = (Q_T, \Sigma_T, Z_T, \delta^T, \sigma^T)$ be a composite NFA and $\mathbb{N} = (Q_N, \Sigma_N, Z_N, \delta^N, \sigma^N) \subseteq \mathbb{M}$, where $Q_N = \{(q_1, q_2, \ldots, q_n) | q_1 \in \mathbb{M}$ and $q_i = \beta_{i-1}(q_{i-1})$ for $i > 1\}$. If Q_T is stable, then $\mathbb{N}$ is a compositie NFA.*

Proof. Let $q = (q_1, \ldots, q_n), q' = (q'_1, \ldots, q'_n) \in Q_N, x_T \in \Sigma_T$ and $y_i \in Z_T$. Then, by definition 3.1 and $y_{i-1} = x_i$. Since $Q_N \subseteq Q_T$, it is enough to prove that Q_N is stable, for each $i > 1$. Then

$$\begin{aligned}
\delta_1^i(q_i, x_i, q'_i) &= \delta_1^i\left[\beta_{i-1}(q_{i-1}), (\sigma_1^{i-1}(q_{i-1}, y_{i-2}, y_{i-1})), \beta_{i-1}(q'_{i-1})\right] \\
&= \beta_{i-1}\left[\delta_1^{i-1}(q_{i-1}, x_{i-1}, q'_{i-1})\right], \text{since } \beta_{i-1} \text{ is a box function of } (\mathbb{M}_{i-1}, \mathbb{M}_i), \\
&= \delta_1^{i-1}\left[\beta_{i-1}(q_{i-1}), x_{i-1}, \beta_{i-1}(q'_{i-1})\right]
\end{aligned}$$

This implies that $\delta_1^{i-1}\left[\beta_{i-1}(q_{i-1}), x_{i-1}, \beta_{i-1}(q'_{i-1})\right]$ is stable, since $\delta_1^{i-1}(q_{i-1}, x_{i-1}, q'_{i-1})$ is stable. Hence, Q_N is stable. Therefore, $\mathbb{N}$ is a composite NFA.

Theorem 3.14. *Let $\mathbb{M}_1 = (Q_1, \Sigma_1, Z_1, \delta^1, \sigma^1)$ and $\mathbb{M}_2 = (Q_2, \Sigma_2, Z_2, \delta^2, \sigma^2)$ be two NFA's and let $\mathbb{H}$ be a NFA with inputs Σ_H which generating inputs set for Σ_1. Suppose $Z_1 \subseteq \Sigma_2$ and for all $p, q \in Q_1$, $x_1 \in \Sigma_H$, the map $\beta : Q_1 \to Q_2$ such that $\beta[\delta^1(q, x_1, p)] = \delta^2[\beta(q), \sigma^1(q, x_1, y_1), \beta(p)]$. Then β is a box function of $(\mathbb{M}_1, \mathbb{M}_2)$.*

Proof. We will prove the result by mathematical induction on the generated set of inputs Σ_H. For $n = 1$, let $x_1 \in \Sigma_H$ the result follows from 3.11.

For $n = 2$, let $x_1, x_2 \in \Sigma_H$ and $q, p \in Q_1$, then

$$\begin{aligned}
\beta\left[\delta^1(q, x_1x_2, p)\right] &= \beta\left[\bigvee_{r\in Q_1}\left\{\delta^1(q, x_1, r) \wedge \delta^1(r, x_2, p)\right\}\right] \\
&= \bigvee_{r\in Q_1}\left\{\beta(\delta^1(q, x_1, r)) \wedge \beta(\delta^1(r, x_2, p))\right\} \\
&= \bigvee_{\beta(r)\in Q_2}\left\{\delta^2(\beta(q), \sigma^1(q, x_1, y_1), \beta(r)) \wedge \delta^2(\beta(r), \sigma^1(q, x_2, y_2), \beta(p))\right\} \\
&= \delta^2\left[\beta(q), \sigma^1(q, x_1, y_1) \bullet \sigma^1(q, x_2, y_2), \beta(p)\right] \\
&= \delta^2\left[\beta(q), \sigma^1(q, x_1x_2, y_1y_2), \beta(p)\right]
\end{aligned}$$

If the induction continues for any finite sequence of inputs such as $n > 2$ for each $x_i \in \Sigma_H$, the results follows by induction. Hence β is a box function of $(\mathbb{M}_1, \mathbb{M}_2)$.

4. **Conclusions**

The main focus of this paper is to study the algebraic automata theory based on the concept of neutrosophic sets. Thus, this investigation contributes a small portion to algebraic automata theory such as composite neutrosophic finite automata which is established by outputs of one automaton as the inputs of another automaton. The future study will be concerned with similar concepts but the approaches are based on the combination of N-fuzzy structures [9,13] and type-2 fuzzy structures [15,19] under the environment of neutrosophic sets [27,28].

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Introduction to AntiRings

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Abstract. The objective of this paper is to introduce the concept of AntiRings. Several examples of AntiRings are presented. Specifically, certain types of AntiRings and their substructures are studied. It is shown that nonempty subsets of an AntiRing can be AntiRings with algebraic properties different from the algebraic properties of the parent AntiRing under the same binary operations. AntiIdeals, AntiQuotientRings and AntiRingHomomorphisms are studied with several examples. It is shown that the quotient of an AntiRing factored by an AntiIdeal can exhibit algebraic properties different from the algebraic properties of the AntiRing.

Keywords: NeutroRing; AntiRing; AntiSubring; QuasiAntiSubring; AntiIdeal; QuasiAntiIdeal; PseudoAntiIdeal; AntiQuotientRing; AntiRingHomomorphism.

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1. **Introduction**

Mathematical modeling of the real life space X requires that all the possible laws that can be defined on X as well as all the possible axioms that can be defined on X should all be considered. In order to be very close to reality, the laws as well as axioms on X should not be rigidly defined. The laws on X should be so flexibly defined to make provisions for both totally inner-defined, totally outer-defined, partially-defined and indeterminately-defined cases. Also, the axioms on X should be such that provisions are made for both totally inner-defined, totally outer-defined, partially-defined and indeterminately-defined cases. When the laws and axioms on X are totally inner-defined, they are called and referred to as ClassicalLaws and ClassicalAxioms respectively. When the laws and axioms on X are partially-defined, they are called and referred to as NeutroLaws and NeutroAxioms respectively. When the laws and axioms on X are totally outer-defined, they are called and referred to as AntiLaws and AntiAxioms respectively. Naturally, we have the neutrosophic triplets (Law, NeutroLaw, AntiLaw) and (Axiom, NeutroAxiom, AntiAxiom) where NonLaw $=$ NeutroLaw $\cup$ AntiLaw, NonAxiom $=$ NeutroAxiom $\cup$ AntiAxiom, NeutroLaw $\cap$ AntiLaw $= \emptyset$ and NeutroAxiom $\cap$ AntiAxiom $= \emptyset$. These concepts have several applications in sciences, engineering, technology, soft computing, social sciences, psychology, politics, sociology and humanities in general. For details on NeutroSociology the readers should see [14] and [7–11, 19] for more details on neutrosophy and applications.

Smarandache in [15–18] introduced and studied extensively the concepts of Neutro-Algebraic Structures and Anti-Algebraic Structures. Rezaei and Smarandache in [12] presented and studied Neutro-BE Algebras and Anti-BE Algebras. Agboola et al. in [4] studied NeutroAlgebras and AntiAlgebras, in [5] and [6], Agboola studied NeutroGroups and NeutroRings respectively. In [3], Agboola further studied NeutroGroups, in [2], he studied AntiGroups and in [1], he further studied NeutroRings. In the present paper, the concept of AntiRings is introduced. Several examples of AntiRings are presented. Specifically, certain types of AntiRings and their substructures are studied. It is shown that nonempty subsets of an AntiRing can be AntiRings with algebraic properties different from the algebraic properties of the parent AntiRing under the same binary operations. AntiIdeals, AntiQuotientRings and AntiRingHomomorphisms are studied with several examples. It is shown that the quotient of an AntiRing factored by an AntiIdeal can exhibit algebraic properties different from the algebraic properties of the AntiRing.

2. Preliminaries

In this section, some definitions and results that will be used later in the paper are presented.

Definition 2.1. [15]

A classical operation is an operation well defined for all the set's elements. A NeutroOperation is an operation partially well defined, partially indeterminate, and partially outer defined on the given set while an AntiOperation is an operation that is outer defined for all set's elements.

A classical law/axiom defined on a nonempty set is a law/axiom that is totally true (i.e. true for all set's elements). A NeutroLaw/NeutroAxiom (or Neutrosophic Law/Neutrosophic Axiom) defined on a nonempty set is a law/axiom that is true for some set's elements [degree of truth (T)], indeterminate for other set's elements [degree of indeterminacy (I)], or false for the other set's elements [degree of falsehood (F)], where $T, I, F \in [0, 1]$, with $(T, I, F) \neq (1, 0, 0)$ that represents the classical axiom, and $(T, I, F) \neq (0, 0, 1)$ that represents the AntiAxiom while an AntiLaw/AntiAxiom defined on a nonempty set is a law/axiom that is false for all set's elements.

A PartialOperation on a set is an operation that is well defined for some elements of the set and undefined for all the other elements of the set. A PartialAlgebra is an algebra that has at least one PartialOperation, and all its axioms are classical.

A NeutroAlgebra is an algebra that has at least one NeutroOperation or one NeutroAxiom (axiom that is true for some elements, indeterminate for other elements, and false for other elements), and no AntiOperation or AntiAxiom while an AntiAlgebra is an algebra endowed with at least one AntiOperation or at least one AntiAxiom. When a NeutroAlgebra has no NeutroAxiom, then it coincides with the PartialAlgebra.

Theorem 2.2. *[15] The NeutroAlgebra is a generalization of PartialAlgebra.*

Theorem 2.3. *[12] Let $\mathbb{U}$ be a nonempty finite or infinite universe of discourse and let S be a finite or infinite subset of $\mathbb{U}$. If n classical operations (laws and axioms) are defined on S where $n \geq 1$, then there will be $(2^n - 1)$ NeutroAlgebras and $(3^n - 2^n)$ AntiAlgebras.*

Definition 2.4. [Classical ring] [13]

Let R be a nonempty set and let $+,. : R \times R \to R$ be binary operations of the usual addition and multiplication respectively defined on R. The triple $(R, +, .)$ is called a classical ring if the following conditions $(R1 - R9)$ hold:

(R1) $x + y \in R\ \forall x, y \in R$ [closure law of addition].

(R2) $x + (y + z) = (x + y) + z\ \forall x, y, z \in R$ [axiom of associativity].

(R3) There exists $e \in R$ such that $x + e = e + x = x\ \forall x \in R$ [axiom of existence of neutral element].

(R4) There exists $-x \in R$ such that $x + (-x) = (-x) + x = e\ \forall x \in G$ [axiom of existence of inverse element]

(R5) $x + y = y + x\ \forall x, y \in R$ [axiom of commutativity].

(R6) $x.y \in R\ \forall x, y \in R$ [closure law of multiplication].

(R7) $x.(y.z) = (x.y).z\ \forall x, y, z \in R$ [axiom of associativity].

(R8) $x.(y + z) = (x.y) + (x.z)\ \forall x, y, z \in R$ [axiom of left distributivity].

(R9) $(y + z).x = (y.x) + (z.x)\ \forall x, y, z \in R$ [axiom of right distributivity].

If in addition we have,

(R10) $x.y = y.x\ \forall x, y \in R$ [axiom of commutativity],

then $(R, +, .)$ is called a commutative ring.

Definition 2.5. [1][NeutroSophication of the laws and axioms of the classical ring]

(NR1) There exist at least three duplets $(x, y), (u, v), (p, q) \in R$ such that $x+y \in R$ (inner-defined with degree of truth T) and [$u + v$ = indeterminate (with degree of indeterminacy I) or $p + q \notin R$ (outer-defined/falsehood with degree of falsehood F)] [NeutroClosure law of addition].

(NR2) There exist at least three triplets $(x, y, z), (p, q, r), (u, v, w) \in R$ such that $x+(y+z) = (x+y)+z$ (inner-defned with degree of truth T) and [$[p + (q + r)]$or$[(p + q) + r]$ = indeterminate (with degree of indeterminacy I) or $u + (v + w) \neq (u + v) + w$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of associativity (NeutroAssociativity)].

(NR3) There exists an element $e \in R$ such that $x + e = x + e = x$ (inner-defined with degree of truth T) and [$[x + e]$or$[e + x]$ = indeterminate (with degree of indeterminate I) or $x + e \neq x \neq e + x$ (outer-defined/falsehood with degree of falsehood F)] for at least one $x \in R$ [NeutroAxiom of existence of neutral element (NeutroNeutralElement)].

(NR4) There exists $-x \in R$ such that $x + (-x) = (-x) + x = e$ (inner-defined with degree of truth T) and [$[-x + x]$or$[x + (-x)]$ = indeterminate (with the degree of indeterminate I) or

$-x + x \neq e \neq x + (-x)$ (outer-defined/falsehood with degree of falsehood F)] for at least one $x \in R$ [NeutroAxiom of existence of inverse element (NeutroInverseElement)].

(NR5) There exist at least three duplets $(x, y), (u, v), (p, q) \in R$ such that $x + y = y + x$ (inner-defined with degree of truth T) and [$[p + q]$or$[q + p]$ = indeterminate (with degree of indeterminacy I) or $u + v \neq v + u$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of commutativity (NeutroCommutativity)].

(NR6) There exist at least three duplets $(x, y), (p, q), (u, v) \in R$ such that $x.y \in R$ (inner-defined with degree of truth T) and [$u.v$ = indeterminate (with degree of indeterminacy I) or $p.q \notin R$ (outer-defined/falsehood with degree of falsehood F)] NeutroClosure law of multiplication].

(NR7) There exist at least three triplets $(x, y, z), (p, q, r), (u, v, w) \in R$ such that $x.(y.z) = (x.y).z$ (inner-defined with degree of truth T) and [$[p.(q.r)]$or$[(p.q).r]$ = indeterminate (with degree of indeterminacy I) or $u.(v.w) \neq (u.v).w$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of associativity (NeutroAssociativity)].

(NR8) There exist at least three triplets $(x, y, z), (p, q, r), (u, v, w) \in R$ such that $x.(y + z) = (x.y) + (x.z)$ (inner-defined with degree of truth T) and [$[p.(q + r)]$or$[(p.q) + (p.r)]$ = indeterminate (with degree of indeterminacy I) or $u.(v + w) \neq (u.v) + (u.w)$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of left distributivity (NeutroLeftDistributivity)].

(NR9) There exist at least three triplets $(x, y, z), (p, q, r), (u, v, w) \in R$ such that $(y + z).x = (y.x) + (z.x)$ (inner-defined with degree of truth T) and [$[(v + w).u]$or$[(v.u) + (w.u)]$ = indeterminate (with degree of indeterminacy I) or $(v + w).u \neq (v.u) + (w.u)$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of right distributivity (NeutroRightDistributivity)].

(NR10) There exist at least three duplets $(x, y), (p, q), (u, v) \in R$ such that $x.y = y.x$ (inner-defined with degree of truth T) and [$[p.q]$or$[q.p]$ = indeterminate (with degree of indeterminacy I) or $u.v \neq v.u$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of commutativity (NeutroCommutativity)].

Definition 2.6. [1][AntiSophication of the laws and axioms of the classical ring]

(AR1) For all the duplets $(x, y) \in R$, $x + y \notin R$ [AntiClosure law of addition].

(AR2) For all the triplets $(x, y, z) \in R$, $x + (y + z) \neq (x + y) + z$ [AntiAxiom of associativity (AntiAssociativity)].

(AR3) There doest not exist an element $e \in R$ such that $x + e = x + e = x \ \forall x \in R$ [AntiAxiom of existence of neutral element (AntiNeutralElement)].

(AR4) There does not exist $-x \in R$ such that $x + (-x) = (-x) + x = e \ \forall x \in R$ [AntiAxiom of existence of inverse element (AntiInverseElement)].

(AR5) For all the duplets $(x, y) \in R$, $x+y \neq y+x$ [AntiAxiom of commutativity (AntiCommutativity)].

(AR6) For all the duplets $(x, y) \in R$, $x.y \notin R$ [AntiClosure law of multiplication].

(AR7) For all the triplets $(x, y, z) \in R$, $x.(y.z) \neq (x.y).z$ [AntiAxiom of associativity (AntiAssociativity)].

(AR8) For all the triplets $(x, y, z) \in R$, $x.(y + z) \neq (x.y) + (x.z)$ [AntiAxiom of left distributivity (AntiLeftDistributivity)].

(AR9) For all the triplets $(x, y, z) \in R$, $(y + z).x \neq (y.x) + (z.x)$ [AntiAxiom of right distributivity (AntiRightDistributivity)].

(AR10) For all the duplets $(x, y) \in R$, $x.y \neq y.x$ [AntiAxiom of commutativity (AntiCommutativity)].

Definition 2.7. [1][NeutroRing]
A NeutroRing NR is an alternative to the classical ring R that has at least one NeutroLaw or at least one of $\{NR1, NR2, NR3, NR4, NR5, NR6, NR7, NR8, NR9\}$ with no AntiLaw or AntiAxiom.

Definition 2.8. [[1]][AntiRing]
An AntiRing AR is an alternative to the classical ring R that has at least one AntiLaw or at least one of $\{AR1, AR2, AR3, AR4, AR5, AR6, AR7, AR8, AR9\}$.

Definition 2.9. [1][NeutroCommutativeRing]
A NeutroCommutativeRing NR is an alternative to the classical commutative ring R that has at least one NeutroLaw or at least one of $\{NR1, NR2, NR3, NR4, NR5, NR6, NR7, NR8, NR9\}$ and $NR10$ with no AntiLaw or AntiAxiom.

Definition 2.10. [1][AntiCommutativeRing]
An AntiCommutativeRing AR is an alternative to the classical commutative ring R that has at least one AntiLaw or at least one of $\{AR1, AR2, AR3, AR4, AR5, AR6, AR7, AR8, AR9\}$ and $AR10$.

Theorem 2.11. *[1] Let $(R, +, .)$ be a finite or infinite classical ring. Then:*

(i) *There are 511 types of NeutroRings.*
(ii) *There are 19171 types of AntiRings.*

Theorem 2.12. *[1] Let $(R, +, .)$ be a finite or infinite classical commutative ring. Then:*

(i) *There are 1023 types of NeutroCommutativeRings.*
(ii) *There are 58025 types of AntiCommutativeRings.*

Example 2.13. [1] Let $NR = \mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ and let $\oplus$ and $\odot$ be two binary operations defined on NR by

$$x \oplus y = x + y - 1,\ x \odot y = x + xy\ \forall\ x, y \in NR.$$

Then, $(NR, \oplus, \odot)$ is a NeutroCommutativeRing.

Example 2.14. [1] Let $NR = \{a, b, c, d\}$ and let $''+''$ and $''.''$ be binary operations defined on NR as shown in the Cayley tables below:

$+$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

$.$	a	b	c	d
a	a	b	c	d
b	a	c	b	c
c	c	d	c	d
d	d	a	d	a

Then, $(NR, +, .)$ is a NeutroCommutativeRing.

Example 2.15. Let $AR = \mathbb{Z}$ and let $\oplus$ and $\odot$ be two binary operations defined on AR such that $\oplus$ is the usual addition of integers and $\forall x, y \in AR$, $\odot$ is defined by

$$x \odot y = x^2 + x^2 y + 2.$$

Then $(AR, \oplus, \odot)$ is an AntiRing. To see this, we first note that $\oplus$ is well defined for all $x, y \in AR$ and that $R1 - R5$ are totally true. Hence, $(AR, \oplus)$ is an abelian group.

It is also noted that $\odot$ is well defined for all $x, y \in AR$ that is, $R6$ is totally true $\forall x, y \in AR$. Now let $x, y, z \in AR$. Then

$$\begin{aligned} x \odot (y \odot z) &= x^2 + x^2 y^2 + x^2 y^2 z + 2x^2 + 2, \\ (x \odot y) \odot z &= x^4 + 2x^4 y + x^4 y^2 + 4x^2 + 4x^2 y \\ &\quad + x^4 z + 2x^4 yz + x^4 y^2 z + 4x^2 z + 4x^2 yz + 4z + 6. \\ \therefore\ x \odot (y \odot z) &\neq (x \odot y) \odot z \ \forall x, y, z \in AR. \end{aligned}$$

It has just been shown that for all the elements of AR, $\odot$ is AntiAssociative over AR. Thus, $AR7$ is satisfied.

Also for all $x, y, z \in AR$, we have

$$\begin{aligned} x \odot (y \oplus z) &= x \odot (y + z) \\ &= x^2 + x^2 y + x^2 z + 2, \\ (x \odot y) \oplus (x \odot z) &= (x \odot y) + (x \odot z) \\ &= 2x^2 + x^2 y + x^2 z + 4. \\ \therefore\ x \odot (y \oplus z) &\neq (x \odot y) \oplus (x \odot y) \ \forall x, y, z \in AR. \end{aligned}$$

It has again been shown that over AR, $\odot$ is not left distributive over $\oplus$ for all $x, y, z \in AR$. Hence $AR8$ is satisfied.

Lastly for all $x, y, z \in AR$, we have

$$\begin{aligned}(y \oplus z) \odot x &= (y + z) \odot x \\ &= y^2 + z^2 + 2yz + xy^2 + 2xyz + xz^2 + 2, \\ (y \odot x) \oplus (z \odot x) &= (y \odot x) + (z \odot x) \\ &= y^2 + z^2 + y^2x + z^2x + 4. \\ \therefore\ (y \oplus z) \odot x &\neq (y \odot x) \oplus (y \odot x)\ \forall x, y, z \in AR.\end{aligned}$$

This also shows that over AR, $\odot$ is not right distributive over $\oplus$ for all $x, y, z \in AR$. Hence, $AR9$ is satisfied. It can easily be shown that $\odot$ is NeutroCommutative over AR. Accordingly by Definition 2.8, $(AR, \oplus, \odot)$ is an AntiRing.

Example 2.16. (i) Let $AR = M_{n\times n}[\mathbb{R}]$ be the set of all $n \times n$ matrices with real entries and let $\oplus$ and $\odot$ be two binary operations defined on AR such that $\oplus$ is the usual addition of matrices and $\forall X, Y \in AR$, $\odot$ is defined by

$$X \odot Y = X^2 + X^2Y + 2I$$

where I is the $n \times n$ unit matrix. Then, $(AR, \oplus, \odot)$ is an AntiRing.

(ii) Let M be an additive abelian group and let $AR = End(M)$ be the set of all endomorphisms of M into itself. Let $\oplus$ and $\odot$ be two binary operations defined on AR such that $\oplus$ is the usual addition of mappings and $\forall f, g \in AR$, $\odot$ is defined by

$$(f \odot g)(x) = f^2(x) + f^2(x)g(x) + 2i(x)$$

where i is the identity mapping. Then, $(AR, \oplus, \odot)$ is an AntiRing.

3. A Study of Certain Types of AntiRings

In this section, we are going to study certain types of AntiRings. Many examples and basic results will be presented. Since there are many types of AntiRings, then AntiRings in this section will be classified and named type-AR[,] according to which of $AR1 - AR10$ is(are) satisfied. If only $AR1$ is satisfied, the AntiRing will be called of type-AR[1], type-AR[3,4] if only $AR3$ and $AR4$ are satisfied and so on. AntiRings of type-AR[1,2,3,4-9] or of type-AR[1,2,3,4-10] will be called trivial AntiRings or trivial AntiCommutativeRings respectively.

Definition 3.1. Let $(AR, +, .)$ be an AntiRing.

(i) AR is called a finite AntiRing of order n if the cardinality of AR is n that is $o(AR) = n$. Otherwise, AR is called an infinite AntiRing and we write $o(AR) = \infty$.

(ii) AR is called an AntiRing with unity if there exists a multiplicative unit element $u \in AR$ such that $ux = xu = x$ for at least one $x \in R$.

(iii) If there exists a least positive integer n such that $nx = e$ for at least one $x \in AR$, then AR is called an AntiRing of characteristic n. If no such n exists, then AR is called an AntiRing of characteristic zero.

(iv) An element $x \in AR$ is called an idempotent element if $x^2 = x$.

(v) An element $x \in AR$ is called a nilpotent element if for the least positive integer n, we have $x^n = e$.

(vi) An element $e \neq x \in AR$ is called a zero divisor element if there exists an element $e \neq y \in AR$ such that $xy = e$ or $yx = e$.

(vii) An element $x \in AR$ is called a multiplicative inverse element if there exists at least one $y \in AR$ such that $xy = yx = u$ where u is the multiplicative unity element in AR.

Definition 3.2. Let $(AR, +, .)$ be an AntiCommutativeRing with unity. Then

(i) AR is called an AntiIntegralDomain if all the elements of AR are zero divisors.

(ii) AR is called an AntiField if all the elements of AR have no multiplicative inverse elements.

Definition 3.3. Let $(AR, +, .)$ be an AntiRing. A nonempty subset AS of AR is called an AntiSubring of AR if $(AS, +, .)$ is also an AntiRing of the same type as AR.

Definition 3.4. Let $(AR, +, .)$ be an AntiRing. A nonempty subset AS of AR is called a QuasiAntiSubring of AR if $(AS, +, .)$ is also an AntiRing not of the same type as AR.

Definition 3.5. Let $(AR, +, .)$ be an AntiRing. A nonempty subset AI of AR is called a left AntiIdeal of AR if the following conditions hold:

(i) AI is an AntiSubring of AR of the same type as AR.

(ii) $x \in AI$ and $r \in AR$ imply that $xr \notin AI$ for all $r \in AR$.

Definition 3.6. Let $(AR, +, .)$ be an AntiRing. A nonempty subset AI of AR is called a right AntiIdeal of AR if the following conditions hold:

(i) AI is an AntiSubring of AR of the same type as AR.

(ii) $x \in AI$ and $r \in AR$ imply that $rx \notin AI$ for all $r \in NR$.

Definition 3.7. Let $(AR, +, .)$ be an AntiRing. A nonempty subset AI of NR is called a two-sided AntiIdeal of AR if the following conditions hold:

(i) AI is an AntiSubring of AR of the same type as AR.

(ii) $x \in AI$ and $r \in AR$ imply that $xr \notin AI$ and $rx \notin AI$ for all $r \in AR$.

Definition 3.8. Let $(AR, +, .)$ be an AntiRing. A nonempty subset AI of AR is called a left QuasiAntiIdeal of AR if the following conditions hold:

(i) AI is a QuasiAntiSubring of AR.

(ii) $x \in AI$ and $r \in AR$ imply that $xr \notin AI$ for all $r \in AR$.

Definition 3.9. Let $(AR, +, .)$ be an AntiRing. A nonempty subset AI of AR is called a right QuasiAntiIdeal of AR if the following conditions hold:

(i) AI is a QuasiAntiSubring of AR.

(ii) $x \in AI$ and $r \in AR$ imply that $rx \notin AI$ for all $r \in NR$.

Definition 3.10. Let $(AR, +, .)$ be an AntiRing. A nonempty subset AI of NR is called a two-sided QuasiAntiIdeal of AR if the following conditions hold:

(i) AI is a QuasiAntiSubring of AR.

(ii) $x \in AI$ and $r \in AR$ imply that $xr \notin AI$ and $rx \notin AI$ for all $r \in AR$.

Definition 3.11. Let $(AR, +, .)$ be an AntiRing. A nonempty subset AI of AR is called a left PseudoAntiIdeal of AR if the following conditions hold:

(i) AI is an AntiSubring or a QuasiAntiSubring of AR.

(ii) For at least one $x \in AI$, $xr \notin AI$ for all $r \in AR$.

Definition 3.12. Let $(AR, +, .)$ be an AntiRing. A nonempty subset AI of AR is called a right PseudoAntiIdeal of AR if the following conditions hold:

(i) AI is an AntiSubring or a QuasiAntiSubring of AR.

(ii) For at least one $x \in AI$, $rx \notin AI$ for all $r \in AR$.

Definition 3.13. Let $(AR, +, .)$ be an AntiRing. A nonempty subset AI of NR is called a two-sided PseudoAntiIdeal of AR if the following conditions hold:

(i) AI is an AntiSubring or a QuasiAntiSubring of AR.

(ii) For at least one $x \in AI$, $xr \notin AI$ and $rx \notin AI$ for all $r \in AR$.

Example 3.14. Let $AR = \{a, b\}$ and let $''+''$ and $''.''$ be two binary operations defined on AR as shown in the Cayley tables below.

$+$	a	b
a	a	b
b	b	a

$.$	a	b
a	b	b
b	a	a

Since $x+y, xy \in AR\ \forall x, y \in AR$ and $(AR, +)$ is an abelian group, it follows that $R1 - R6$ of Definition 2.6 are totally true for all the elements of AR. Now consider the following:

(AR7) $a(aa) = b, (aa)a = a \neq b$, $a(ab) = b, (aa)b = a \neq b$, $a(ba) = b, (ab)a = a \neq b$, $b(aa) = a, (ba)a = b \neq a$,$a(bb) = b, (ab)b = a \neq b$, $b(ab) = a, (ba)b = b \neq a$, $b(ba) = a, (bb)a = b \neq a$, $b(bb) = a, (bb)b = b \neq a$. These show that the binary operation $''.''$ is totally AntiAssociative in AR.

(AR8) $a(a + a) = b$ while $aa + aa = a \neq b$. Also, $b(b + b) = a$ while $bb + bb = a$. These show that the binary operation $''.''$ is NeutroLeftDistributive over the binary operation $''+''$.

(AR9) $(a+a)a = b, aa+aa = a \neq b$, $(a+a)b = b, ab+ab = a \neq b$, $(a+b)a = a, aa+ba = b \neq a$, $(b+a)a = a, ba+aa = b \neq a$, $(a+b)b = a, ab+bb = b \neq a$, $(b+a)b = a, bb+ab = b \neq a$, $(b+b)a = b, ba+ba = a \neq b$, $(b+b)b = b, bb+bb = a \neq b$. These show that the binary operation $''.''$ is AntiRightDistributive over the binary operation $''+''$.

(AR10) $aa = b, bb = a$ but $ab = b, ba = a \neq b$. These show that the binary operation $''.''$ is NeutroCommutative in AR.

Since $AR7$ and $AR9$ are totally true for all the elements of AR, it follows from Definition 2.8 that $(AR, +, .)$ is an AntiRing which we call an AntiRing of type-AR[7,9].

Example 3.15. Let $(AR, +, .)$ be the AntiRing of Example 3.14. It is clear that $e = a$ is the additive identity element. The element b is idempotent since $bb = a$. Since $''.''$ is totally AntiAssociative, it follows that AR has no nilpotent elements. AR has no unity and consequently, none of the elements of AR is invertible. Since AR is not an AntiCommutativeRing, it follows that AR is neither an AntiIntegralDomain nor an AntiField.

Example 3.16. Let $AR = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$. Let $*$ and $\circ$ be two binary operations defined such that $*$ is the usual addition modulo 6 and for all $x, y \in AR$, $\circ$ is defined by

$$x \circ y = x + xy + 2.$$

It is clear that $x * y, x \circ y \in AR \ \forall x, y \in AR$. This shows that $R1 - R6$ of Definition 2.6 are totally true for all the elements of AR. Now consider the following:

(AR7) $x \circ (y \circ z) = 3x + xy + xyz + 2$ and $(x \circ y) \circ z = x + xy + xz + 2z + xyz + 4$. Equating the two expressions we obtain $2x = xz + 2z + 2$ from which we have that the triplet $(2, y, 2) \in AR$ can verify the associativity of $\circ$ in AR. Thus, $\circ$ is NeutroAssociative in AR.

(AR8) $x \circ (y * z) = x + xy + xz + 2$ and $(x \circ y) * (x \circ z) = 2x + xy + xz + 4$. Equating the two expressions we have $x = -2 \equiv 4$ modulo 6. Hence, only the triplets $(4, y, z) \in AR$ can verify the left distributivity of $\circ$ over $*$ in AR. Thus, $\circ$ is NeutroLeftDistributive in AR.

(R9) $(y * z) \circ x = y + z + xy + xz + 2$ and $(y \circ x) * (z \circ x) = y + z + xy + xz + 4$. Since $2 \neq 4$ modulo 6, it follows that $\circ$ is not right distributive over $*$ for all the triplets $(x, y, z) \in AR$. Hence $\circ$ is totally AntiRightDistributive over $*$ in AR.

(R10) $x \circ y = x + xy + 2$ and $y \circ x = y + yx + 2$. Equating the two expressions we have $x = y$ showing that only the duplets $(x, x) \in AR$ can verify the commutativity of $\circ$. Hence, $\circ$ is NeutroCommutative in AR.

According to Definition 2.8, we have that $(AR, *, \circ)$ is an AntiRing of type-AR[9].

Example 3.17. Let $AS = \{0, 3\}$ be a subset of AR where $(AR, *, \circ)$ is the AntiRing of Example 3.16. Consider the compositions of the elements of AS as shown in the Cayley tables below.

$*$	0	3
0	0	3
3	3	0

$\circ$	0	3
0	2	2
3	5	2

$R1 - R5$ are totally true since for all $x, y \in AS$, $x * y \in AS$ and $(AS, *)$ is an abelian group. Also for all the elements of AS, $R6 - R10$ are totally false. Accordingly, $(AS, *, \circ)$ is an AntiRing of the type-AR[6,7,8,9,10] which is different from the class of the parent AntiRing. Hence, AS is a QuasiAntiSubring of AR.

Example 3.18. Let $AT = \{0, 2, 4\}$ be a subset of AR where $(AR, *, \circ)$ is the AntiRing of Example 3.16. Consider the compositions of the elements of AT as shown in the Cayley tables below.

$*$	0	2	4
0	0	2	4
2	2	4	0
4	4	0	2

$\circ$	0	2	4
0	2	2	2
2	4	2	0
4	0	2	4

$R1 - R6$ are totally true since for all $x, y \in AS$, $x * y, x \circ y \in AT$ and $(AT, *)$ is an abelian group. Now consider the following:

(AR7) $2 \circ (4 \circ 2) = (2 \circ 4) \circ 2 = 2$ but $2 \circ (0 \circ 4) = 2, (2 \circ 0) \circ 4 = 4 \neq 2$. These show that the binary operation $\circ$ is NeutroAssociative over AT.

(AR8) $4 \circ (2 * 4) = (4 \circ 2) * (4 \circ 4) = 0$ but $2 \circ (4 * 0) = 0, (2 \circ 4) * (2 \circ 0) = 4 \neq 0$. These show that the binary operation $\circ$ is NeutroLeftDistributive over $*$ in AT.

(AR9) For all the triplets $(x, y, z) \in AS$, we have $(y * z) \circ x \neq (y \circ x) * (z \circ x)$. This shows that the binary operation $\circ$ is AntiRightDistributive over $*$ in AT.

(AR10) Since $0 \circ 0 = 2, 2 \circ 2 = 2, 4 \circ 4 = 4$ but $0 \circ 2 = 2, 2 \circ 0 = 4 \neq 2, 2 \circ 4 = 0, 4 \circ 2 = 2 \neq 0, 4 \circ 0 = 0, 0 \circ 4 = 2 \neq 0$, it follows that the binary operation $\circ$ is NeutroCommutative over AT.

Accordingly, $(AT, *, \circ)$ is an AntiRing of the type-AR[9] which is the same as the class of the parent AntiRing. Hence, AT is an AntiSubring of AR.

Example 3.19. Let $AR = \mathbb{Z}^+ = \{1, 2, 3, 4, \cdots,\}$ and let $AS = 2\mathbb{Z}^+ = \{2, 4, 6, 8, \cdots,\}$, $AT = 3\mathbb{Z}^+ = \{3, 6, 9, 12, \cdots,\}$. Suppose that $*$ and $\circ$ are binary operations respectively of the usual addition and multiplication of integers defined on AR, AS and AT. It can easily be shown that $(AR, *, \circ)$, $(AS, *, \circ)$ and $(AT, *, \circ)$ are AntiRings of type-AR[3,4] since $R1, R2$ and $R5 - R10$ are totally true but $R3$ and $R4$ are totally false. Since AS and AT are subsets of AR, it follows that AS and AT are AntiSubrings of AR. In general, $(n\mathbb{Z}^+, *, \circ)$ are AntiSubrings of the AntiRing $(\mathbb{Z}^+, *, \circ)$ for $n = 2, 3, 4, 5, \cdots$.

Remark 3.20. It is evident from Example 3.17 that an AntiRing of a particular type can have nonempty subsets which are AntiRings of types different from the type of the parent AntiRing under the same binary operations.

Example 3.21. Let AR be an AntiRing of Example 3.16 and let AS and AT be its QuasiAntiSubring and AntiSubring of Examples 3.17 and 3.18 respectively. Then $AS \cup AT = \{0, 2, 3, 4\}$ and $AS \cap AT = \{0\}$. It is clear that $AS \cap AT$ is neither an AntiSubring nor a QuasiAntiSubring of AR. However, it can be shown that $(AS \cup AT, *, \circ)$ is an AntiRing of type-AR[9]. Hence, $AS \cup AT$ is an AntiSubring of AR.

Example 3.22. Let AR be the AntiRing of Example 3.19 and let AS and AT be its AntiSubrings. Then $AS \cup AT = \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, \cdots\}$ and $AS \cap AT = \{6, 12, 18, 24, \cdots\} = 6\mathbb{Z}^+$. It can be shown that $AS \cup AT$ is a QuasiAntiSubring of AR and $AS \cap AT$ is an AntiSubring of AR.

Remark 3.23. The union of two AntiSubrings of an AntiRing can produce a QuasiAntiSubring of the AntiRing.

Example 3.24. Let AR be an AntiRing of Example 3.16 and let AS be its QuasiAntiSubring of Example 3.17. Then $[0 \circ 0 = 0 \circ 1 = 0 \circ 2 = 0 \circ 3 = 0 \circ 4 = 0 \circ 5 = 2, 3 \circ 0 = 3 \circ 2 = 5, 3 \circ 1 = 3 \circ 5 = 2, 3 \circ 3 = 1, 3 \circ 4 = 4] \notin AS$. These show that AS is a left QuasiAntiIdeal of AR.

However, $[0 \circ 0 = 2, 2 \circ 0 = 4, 3 \circ 0 = 5, 5 \circ 0 = 1, 0 \circ 3 = 2, 2 \circ 3 = 4, 3 \circ 3 = 1, 5 \circ 3 = 4] \notin AS$ but $[1 \circ 0 = 3, 4 \circ 0 = 1 \circ 3 = 4 \circ 3 = 0] \in AS$. These show that AS is a right PseudoAntiIdeal of AR.

Example 3.25. Let AR be an AntiRing of Example 3.16 and let AT be its AntiSubring of Example 3.18. Then $[0 \circ 0 = 0 \circ 1 = 0 \circ 2 = 0 \circ 3 = 0 \circ 4 = 0 \circ 5 = 2, 2 \circ 0 = 2 \circ 3 = 4, 2 \circ 1 = 2 \circ 4 = 0, 2 \circ 2 = 2 \circ 5 = 2, 4 \circ 0 = 4 \circ 3 = 0, 4 \circ 1 = 4 \circ 4 = 4, 4 \circ 2 = 4 \circ 5 = 2] \in AT$. These show that AT is neither a left QuasiAntiIdeal nor a left PseudoAntiIdeal of AR.

Also, $[0 \circ 0 = 0 \circ 2 = 2 \circ 2 = 4 \circ 2 = 0 \circ 4 = 2, 2 \circ 0 = 1 \circ 2 = 3 \circ 4 = 4 \circ 4 = 4, 4 \circ 0 = 2 \circ 4 = 0] \in AT$ but $[1 \circ 0 = 5 \circ 4 = 3, 3 \circ 0 = 3 \circ 2 = 5 \circ 2 = 5, 5 \circ 0 = 1 \circ 4 = 1] \notin AT$. These show that AT is a right PseudoAntiIdeal of AR.

Example 3.26. Let AR be the AntiRing of Example 3.19 and let AS and AT be its AntiSubrings. Then AS and AT are two-sided PseudoAntiIdeals of AR. To see this, let $x \in AS$ and $r \in AR$. Then

$$x \circ r = r \circ x = \begin{cases} a \in AS \text{ if } r = 2, 4, 6, 8, \cdots \\ b \notin AS \text{ if } r = 1, 3, 5, 7, \cdots \end{cases}$$

Also,

$$x \circ r = r \circ x = \begin{cases} c \in AT \text{ if } r = 3, 6, 9, 12, \cdots \\ d \notin AT \text{ if } r = 1, 2, 4, 8, \cdots \end{cases}$$

Definition 3.27. Let $(AR, +, .)$ be an AntiRing and let AI be a left(right)(two-sided) AntiIdeal or a left(right)(two-sided) QuasiAntiIdeal or a left(right)(two-sided) PseudoAntiIdeal of AR. The set AR/AI is defined by

$$AR/AI = \{x + AI : x \in AR\}.$$

For all $x + AI, y + AI \in AR/AI$, let $\oplus$ and $\odot$ be two binary operations on AR/AI defined as follows:

$$\begin{aligned}(x + AI) \oplus (y + AI) &= (x * y) + AI,\\ (x + AI) \odot (y + AI) &= (x \circ y) + AI.\end{aligned}$$

If $(AR/AI, \oplus, \odot)$ is an AntiRing, then AR/AI is called an AntiQuotientRing.

Example 3.28. Let AR be an AntiRing of Example 3.16 and let AS be its left QuasiAntiIdeal of Example 3.24. Then

$$AR/AS = \{AS, 1 + AS, 2 + AS\}.$$

Let $\oplus$ and $\odot$ be two binary operations defined on AR/AS as shown in the Cayley tables below.

$\oplus$	AS	$1+AS$	$2+AS$
AS	AS	$1+AS$	$2+AS$
$1+AS$	$1+AS$	$2+AS$	AS
$2+AS$	$2+AS$	AS	$1+AS$

$\odot$	AS	$1+AS$	$2+AS$
AS	$2+AS$	$2+AS$	$2+AS$
$1+AS$	AS	$1+AS$	$2+AS$
$2+AS$	$1+AS$	AS	$2+AS$

It can easily be shown that $R1-R6$ are totally true, $R7, R8$ and $R10$ are partially true and partially false and $R9$ is totally false. Hence, $(AR/AS, \oplus, \odot)$ is an AntiRing of type-AR[9].

Example 3.29. Let AR be an AntiRing of Example 3.16 and let AT be its right PseudoAntiIdeal of Example 3.25. Then

$$AR/AT = \{AT, 1 + AT\}.$$

Let $\oplus$ and $\odot$ be two binary operations defined on AR/AT as shown in the Cayley tables below.

$\oplus$	AT	$1+AT$
AT	AT	$1+AT$
$1+AT$	$1+AT$	AT

$\odot$	AT	$1+AT$
AT	AT	AT
$1+AT$	$1+AT$	AT

It can easily be shown that $R1-R6$ and $R9$ are totally true, $R7, R8$ and $R10$ are partially true and partially false. Hence, $(AR/AT, \oplus, \odot)$ is a NeutroRing.

Example 3.30. Let AR be the AntiRing of Example 3.19 and let AS be its PseudoAntiIdeal of Example 3.26. Then

$$AR/AS = \{1 + AS, 2 + AS, 3 + AS, 4 + AS, \cdots\}$$

If $\oplus$ and $\odot$ are two binary operations on AR/AS such that

$$\begin{aligned}(x + AS) \oplus (y + AS) &= (x * y) + AS,\\ (x + AS) \odot (y + AS) &= (x \circ y) + AS,\end{aligned}$$

It can be shown that $(AR/AS, \oplus, \odot)$ is an AntiRing of type-AR[3,4].

Remark 3.31. If $(AR, +, .)$ is an AntiRing and AI is a left(right)(two-sided) AntiIdeal or a left(right)(two-sided) QuasiAntiIdeal or a left(right)(two-sided) PseudoAntiIdeal of AR, then an AntiQuotientRing AR/AI can have algebraic properties different from the algebraic properties of AR.

Definition 3.32. Let $(AR, +, .)$ and $(AS, +', .')$ be any two AntiRings of the same type/class. The mapping $\phi : AR \to AS$ is called an AntiRingHomomorphism if ϕ anti-preserves the binary operations of AR and AS that is if for all the duplets $(x, y) \in AR$, we have:

$$\begin{aligned} \phi(x+y) &\neq \phi(x) +' \phi(y), \\ \phi(x.y) &\neq \phi(x).'\phi(y). \end{aligned}$$

The kernel of ϕ denoted by $Ker\phi$ is defined as

$$Ker\phi = \{x : \phi(x) = e_{AR}\}.$$

The image of ϕ denoted by $Im\phi$ is defined as

$$Im\phi = \{y \in AS : y = \phi(x) \text{ for at least one } y \in AS\}.$$

If in addition ϕ is an AntiBijection, then ϕ is called an AntiRingIsomorphism. AntiRingEpimorphism, AntiRingMonomorphism, AntiRingEndomorphism and AntiRingAutomorphism are similarly defined.

Example 3.33. Let AR be the AntiRing of Example 3.19 and let AR/AS be the AntiQuotientRing of Example 3.30. Then $\phi : AR \to AR/AS$ defined by

$$\phi(x) = x + AS \ \forall x \in AR$$

is a classical homomorphism and not an AntiRingHomomorphism. To see this, for all $m, n \in AR$, we have $\phi(m) = m + AS$ and $\phi(n) = n + AS$ so that

$$\begin{aligned} \phi(m) + \phi(n) &= (m + AS) \oplus (n + AS) \\ &= (m+n) + AS \\ &= \phi(m+n). \\ \phi(m)\phi(n) &= (m + AS) \odot (n + AS) \\ &= (mn) + AS \\ &= \phi(mn). \\ Ker\phi &= \emptyset. \\ Im\phi &= \{1 + AS, 2 + AS, 3 + AS, 4 + AS, \cdots\} = AR/AS. \end{aligned}$$

Remark 3.34. It is evident from Example 3.33 that the fundamental theorem of homomorphisms of the classical rings cannot hold in the classes of AntiRings.

4. **Conclusions**

We have in this paper introduced the concept of AntiRings with several examples. Specifically, certain types of AntiRings and their substructures were studied. It was shown that nonempty subsets of an AntiRing can be AntiRings with algebraic properties different from the parent AntiRing under the same binary operations. Also, we studied with several examples the concepts of AntiIdeals, AntiQuotientRings and AntiRingHomomorphisms. It was shown that an AntiQuotientRing of an AntiRing factored by an AntiIdeal can exhibit algebraic properties different from the algebraic properties of the AntiRing. We hope to study morphisms and AntiMorphisms of AntiSubrings and QuasiAntiSubrings of AntiRings and present further properties of different types of AntiRings in our future papers.

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Fixed Point Results for Contraction Theorems in Neutrosophic Metric Spaces

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Abstract. In this article, we present fixed and common fixed point results for Banach and Edelstein contraction theorems in neutrosophic metric spaces. Then some properties and examples are given for neutrosophic metric spaces. Thus, we added a new path in neutrosophic theory to obtain fixed point results. we investigate and prove some contraction theorems that are extended to neutrosophic metric space with the assistance of Grabiec.

Keywords: Fixed point; Neutrosophic Metric Space; Banach Contraction; Edelstein Contraction.

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1. Introduction

Fuzzy Sets was presented by Zadeh [20] as a class of elements with a grade of membership. Kramosil and Michalek [9] defined new notion called Fuzzy Metric Space (FMS). Later, many authors have examined the concept of fuzzy metric in various aspects. In 1984 Kaleva and Seikkala [8] have characterized the FMS, where separation between any two points to be positive number. In particular, George and Veeramani [4, 5] redefined the concept of fuzzy metric space with the assistance of continuous t-norm, and continuous t-co norm. FMS has utilized in applied science fields such as fixed point theory, decision making, medical imaging and signal processing. Heilpern [7] defined fuzzy contraction for Fixed point theorem. Park [14] defined Intuitionistic Fuzzy Metric Space (IFMS) from the concept of FMS and given some fixed point results. Fixed point theorems related to FMS and IFMS given by Alaca et al [2] and nemerous researchers [13, 19].In 1998, Smarandache [16] characterized the new concept called neutrosophic logic and neutrosophic set. In the idea of neutrosophic sets, there is T degree of membership, I degree of indeterminacy and F degree of non-membership. A neutrosophic value is appeared by (T, I, F). Hence, neutrosophic logic and neutrosophic set assists us to brief many uncertainties in our lives. In addition, several researchers have made significant development on this theory [26–30]. Recently, Baset et al. [22–25] explored the neutrosophic applications in different fields such as model for sustainable supply chain risk management, resource levelling problem in construction projects, Decision Making and financial performance evaluation of manufacturing industries. In fact, the idea of fuzzy sets deals with only a degree of membership. In addition, the concept of intuitionistic fuzzy set established while adding degree of non - membership with degree of membership. But these degrees are characterized relatively one another. Therefore, neutrosophic set is a generalized state of fuzzy and intuitionistic fuzzy set by incorporating degree of indeterminacy. In 2019, Kirisci et al [10, 11] defined neutrosophic metric space as a generalization of IFMS and brings about fixed point theorems in complete neutrosophic metric space.

In this paper, we investigate and prove some contraction theorems that are extended to neutrosophic metric space with the assistance of Grabiec [6].

2. **Preliminaries**

Definition 2.1 [17] Let Σ be a non-empty fixed set. A Neutrosophic Set (NS for short) N in Σ is an object having the form $N = \{\langle a, \xi_N(a), \varrho_N(a), \nu_N(a)\rangle : a \in \Sigma\}$ where the functions $\xi_N(a), \varrho_N(a)$ and $\nu_N(a)$ represent the degree of membership, degree of indeterminacy and the degree of non-membership respectively of each element a $\in$ N to the set Σ.

A neutrosophic set $N = \{\langle a, \xi_N(a), \varrho_N(a), \nu_N(a)\rangle : a \in \Sigma\}$ is expressed as an ordered triple $N = \langle a, \xi_N(a), \varrho_N(a), \nu_N(a)\rangle$ in Σ.

In NS, there is no restriction on $(\xi_N(a), \varrho_N(a), \nu_N(a))$ other than they are subsets of $]^-0, 1^+[$

Remark 2.2 [10] Neutrosophic Set N is included in another Neutrosophic set Γ ($N \subseteq \Gamma$) if and only if

$$inf\ \xi_N(a) \leq inf \xi_\Gamma(a) \quad sup\ \xi_N(a) \leq sup\ \xi_\Gamma(a)$$

$$inf\ \varrho_N(a) \geq inf \varrho_\Gamma(a) \quad sup\ \varrho_N(a) \geq sup\ \varrho_\Gamma(a)$$

$$inf\ \nu_N(a) \geq inf \nu_\Gamma(a) \quad sup\ \nu_N(a) \geq sup\ \nu_\Gamma(a)$$

Triangular Norms (TNs) were initiated by menger. Triangular co norms(TCs) knowns as dual operations of triangular norms (TNs).

Definition 2.3 [4] A binary operation $\star : [0, 1] \times [0, 1] \to [0, 1]$ is called continuous t - norm (CTN) if it satisfies the following conditions;

For all $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in$ [0, 1]

(i) $\varepsilon_1 \star 0 = \varepsilon_1$;

(ii) If $\varepsilon_1 \leq \varepsilon_3$ and $\varepsilon_2 \leq \varepsilon_4$ then $\varepsilon_1 \star \varepsilon_2 \leq \varepsilon_3 \star \varepsilon_4$;

(iii) $\star$ is continuous;

(iv) $\star$ is commutative and associative.

Definition 2.4 [4] A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is called continuous t - co norm (CTC) if it satisfies the following conditions;

For all $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in [0,\ 1]$

(i) $\varepsilon_1 \diamond 0 = \varepsilon_1$;

(ii) If $\varepsilon_1 \leq \varepsilon_3$ and $\varepsilon_2 \leq \varepsilon_4$ then $\varepsilon_1 \diamond \varepsilon_2 \leq \varepsilon_3 \diamond \varepsilon_4$;

(iii) $\diamond$ is continuous;

(iv) $\diamond$ is commutative and associative.

Remark 2.5 From the definitions of CTN and CTC, we note that if we take $0 < \varepsilon_1, \varepsilon_2 < 1$ for $\varepsilon_1 < \varepsilon_2$ then there exist $0 < \varepsilon_3, \varepsilon_4 < 1$ such that $\varepsilon_1 \star \varepsilon_3 \geq \varepsilon_2$ and $\varepsilon_1 \geq \varepsilon_2 \diamond \varepsilon_4$.

Further we choose $\varepsilon_5 \in (0,1)$ then there exists $\varepsilon_6, \varepsilon_7 \in (0,1)$ such that $\varepsilon_6 \star \varepsilon_6 \geq \varepsilon_5$ and $\varepsilon_7 \diamond \varepsilon_7 \leq \varepsilon_5$.

Definition 2.6 [13] A Sequence $\{t_n\}$ is called s - non-decreasing sequence if there exists $m_0 \in \mathbb{N}$ such that $t_m \leq t_{m+1}$ for all $m > m_0$.

3. **Neutrosophic Metric Space**

In this section, we apply neutrosophic theory to generalize the Intuitionistic fuzzy metric space. we also discuss some properties and examples in it.

Definition 3.1 A 6 - tuple $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$is called Neutrosophic Metric Space(NMS), if Σ is an arbitrary non empty set, $\star$ is a neutrosophic CTN and $\diamond$ is a neutrosophic CTC and Ξ, Θ, Υ are neutrosophic sets on $\Sigma^2 \times \mathbb{R}^+$ satisfying the following conditions:

For all $\zeta, \eta, \omega \in \Sigma, \lambda \in \mathbb{R}^+$

(i) $0 \leq \Xi(\zeta,\eta,\lambda) \leq 1;\ 0 \leq \Theta(\zeta,\eta,\lambda) \leq 1;\ 0 \leq \Upsilon(\zeta,\eta,\lambda) \leq 1$;

(ii) $\Xi(\zeta,\eta,\lambda) + \Theta(\zeta,\eta,\lambda) + \Upsilon(\zeta,\eta,\lambda) \leq 3$;

(iii) $\Xi(\zeta,\eta,\lambda) = 1$ if and only if $\zeta = \eta$;

(iv) $\Xi(\zeta,\eta,\lambda) = \Xi(\eta,\zeta,\lambda)$ for $\lambda > 0$;

(v) $\Xi(\zeta,\eta,\lambda) \star \Xi(\eta,\zeta,\mu) \leq \Xi(\zeta,\omega,\lambda+\mu)$, for all $\lambda, \mu > 0$;

(vi) $\Xi(\zeta,\eta,.) : [0,\infty) \to [0,1]$ is neutrosophic continuous ;

(vii) $lim_{\lambda\to\infty}\Xi(\zeta,\eta,\lambda) = 1$ for all $\lambda > 0$;

(viii) $\Theta(\zeta,\eta,\lambda) = 0$ if and only if $\zeta = \eta$;

(ix) $\Theta(\zeta,\eta,\lambda) = \Theta(\eta,\zeta,\lambda)$ for $\lambda > 0$;

(x) $\Theta(\zeta,\eta,\lambda) \diamond \Theta(\zeta,\omega,\mu) \geq \Theta(\zeta,\omega,\lambda+\mu)$, for all $\lambda, \mu > 0$;

(xi) $\Theta(\zeta,\eta,.) : [0,\infty) \to [0,1]$ is neutrosophic continuous ;

(xii) $lim_{\lambda\to\infty}\Theta(\zeta,\eta,\lambda) = 0$ for all $\lambda > 0$;

(xiii) $\Upsilon(\zeta,\eta,\lambda) = 0$ if and only if $\zeta = \eta$;

(xiv) $\Upsilon(\zeta, \eta, \lambda) = \Upsilon(\eta, \zeta, \lambda)$ for $\lambda > 0$;

(xv) $\Upsilon(\zeta, \eta, \lambda) \diamond \Upsilon(\zeta, \omega, \mu) \geq \Upsilon(\zeta, \omega, \lambda + \mu)$, for all $\lambda, \mu > 0$;

(xvi) $\Upsilon(\zeta, \eta, .) : [0, \infty) \to [0, 1]$ is neutrosophic continuous ;

(xvii) $lim_{\lambda\to\infty}\Upsilon(\zeta, \eta, \lambda) = 0$ for all $\lambda > 0$;

(xviii) If $\lambda > 0$ then $\Xi(\zeta, \eta, \lambda) = 0, \Theta(\zeta, \eta, \lambda) = 1, \Upsilon(\zeta, \eta, \lambda) = 1$.

Then (Ξ, Θ, Υ) is called Neutrosophic Metric on Σ. The functons Ξ, Θ and Υ denote degree of closedness, neturalness and non - closedness between ζ and η with respect to λ respectively.

Example 3.2 Let (Σ, d) be a metric space. Define $\zeta \star \eta = min\{\zeta, \eta\}$ and $\zeta \diamond \eta = max\{\zeta, \eta\}$, and $\Xi, \Theta, \Upsilon : \Sigma^2 \times \mathbb{R}^+ \to [0, 1]$ defined by , we define

$$\Xi(\zeta, \eta, \lambda) = \frac{\lambda}{\lambda + d(\zeta, \eta)}; \quad \Theta(\zeta, \eta, \lambda) = \frac{d(\zeta, \eta)}{\lambda + d(\zeta, \eta)}; \quad \Upsilon(\zeta, \eta, \lambda) = \frac{d(\zeta, \eta)}{\lambda}$$

for all $\zeta, \eta \in \Sigma$ and $\lambda > 0$. Then $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ is called neutrosophic metric space induced by a metric d the standard neutrosophic metric.

Example 3.3 If we take $\Sigma = \mathbb{N}$, consider the CTN, CTC are $\zeta \star \eta = min\{\zeta, \eta\}$ and $\zeta \diamond \eta = max\{\zeta, \eta\}$, $\Xi, \Theta, \Upsilon : \Sigma^2 \times \mathbb{R}^+ \to [0, 1]$ defined by

$$\Xi(\zeta, \eta, \lambda) = \begin{cases} \frac{\zeta}{\eta} & if \quad \zeta \leq \eta \\ \frac{\eta}{\zeta} & if \quad \eta \leq \zeta \end{cases}$$

$$\Theta(\zeta, \eta, \lambda) = \begin{cases} \frac{\eta-\zeta}{\eta} & if \quad \zeta \leq \eta \\ \frac{\zeta-\eta}{\zeta} & if \quad \eta \leq \zeta \end{cases}$$

$$\Upsilon(\zeta, \eta, \lambda) = \begin{cases} \eta - \zeta & if \quad \zeta \leq \eta \\ \zeta - \eta & if \quad \eta \leq \zeta \end{cases}$$

for all $\zeta, \eta \in \Sigma$ *and* $\lambda > 0$. Then $\Xi, \Theta, \Upsilon : \Sigma^2 \times \mathbb{R}^+ \to [0, 1]$ is a NMS.

Remark 3.4 In Neutrosophic Metric space Ξ is non - decreasing , Θ is a non - increasing , Υ is decreasing for all $\zeta, \eta \in \Sigma$.

Definition 3.5 Let $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ be neutrosophic metric space . Then

(a) a sequence $\{\zeta_n\}$ in Σ is converging to a point $\zeta \in \Sigma$ if for each $\lambda > 0$

$$lim_{\lambda\to\infty}\Xi(\zeta, \eta, \lambda) = 1; \quad lim_{\lambda\to\infty}\Theta(\zeta, \eta, \lambda) = 0; \quad lim_{\lambda\to\infty}\Upsilon(\zeta, \eta, \lambda) = 0.$$

(b) a sequence ζ_n in Σ is said to be Cauchy if for each $\epsilon > 0$ and $\lambda > 0$ there exist $N \in \mathbb{N}$ such that $\Xi(\zeta_n, \zeta_m, \lambda) > 1 - \epsilon$; $\Theta(\zeta_n, \zeta_m, \lambda) < \epsilon$; $\Upsilon(\zeta_n, \zeta_m, \lambda) < \epsilon$ for all n, m $\leq$ N.

(c) $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ is said to be complete neutrosophic metric space if every Cauchy sequence is convergent.

(d) $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ is called compact neutrosophic metric space if every sequence contains convergent sub sequence.

4. **Main Results**

Theorem 4.1 (Neutrosophic Banach Contraction Theorem) Let $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ be a complete neutrosophic metric space. Let $F : \Sigma \to \Sigma$ be a function satisfying

$$\Xi(F\zeta, F\eta, \lambda) \geq \Xi(\zeta, \eta, \lambda); \quad \Theta(F\zeta, F\eta, \lambda) \leq \Theta(\zeta, \eta, \lambda); \quad \Upsilon(F\zeta, F\eta, \lambda) \leq \Upsilon(\zeta, \eta, \lambda) \qquad (4.1.1)$$

for all $\zeta, \eta \in \Sigma$. $0 < k < 1$. Then F has unique fixed point.
Proof: Let $\zeta \in \Sigma$ and $\{\zeta_n\} = F^n(a)$ $(n \in \mathbb{N})$. By Mathematical induction, we obtain

$$\Xi(\zeta_n, \zeta_{n+1}, \lambda) \geq \Xi(\zeta, \zeta_1, \frac{\lambda}{k^n}); \Theta(\zeta_n, \zeta_{n+1}, \lambda) \leq \Theta(\zeta, \zeta_1, \frac{\lambda}{k^n}); \Upsilon(\zeta_n, \zeta_{n+1}, \lambda) \leq \Upsilon(\zeta, \zeta_1, \frac{\lambda}{k^n}) \text{ ....(4.1.2)}$$

for all $n > 0$ and $\lambda > 0$. Thus for any non-negative integer p, we have

$$\begin{aligned}
\Xi(\zeta_n, \zeta_{n+p}, \lambda) &\geq \Xi(\zeta, \zeta_{n+1}, \frac{\lambda}{p}) \star \cdots^{(p-times)} \cdots \star \Xi(\zeta_{n+p-1}, \zeta_{n+p}, \frac{\lambda}{p}) \\
&\geq \Xi(\zeta, \zeta_1, \frac{\lambda}{pk^n}) \star \cdots^{(p-times)} \cdots \star \Xi(\zeta, \zeta_1, \frac{\lambda}{pk^{n+p-1}}) \\
\Theta(\zeta_n, \zeta_{n+p}, \lambda) &\leq \Theta(\zeta, \zeta_{n+1}, \frac{\lambda}{p}) \diamond \cdots^{(p-times)} \cdots \diamond \Theta(\zeta_{n+p-1}, \zeta_{n+p}, \frac{\lambda}{p}) \\
&\leq \Theta(\zeta, \zeta_1, \frac{\lambda}{pk^n}) \diamond \cdots^{(p-times)} \cdots \diamond \Theta(\zeta, \zeta_1, \frac{\lambda}{pk^{n+p-1}}) \\
\Upsilon(\zeta_n, \zeta_{n+p}, \lambda) &\leq \Upsilon(\zeta, \zeta_{n+1}, \frac{\lambda}{p}) \diamond \cdots^{(p-times)} \cdots \diamond \Upsilon(\zeta_{n+p-1}, \zeta_{n+p}, \frac{\lambda}{p}) \\
&\leq \Upsilon(\zeta, \zeta_1, \frac{\lambda}{pk^n}) \diamond \cdots^{(p-times)} \cdots \diamond \Upsilon(\zeta, \zeta_1, \frac{\lambda}{pk^{n+p-1}})
\end{aligned}$$

by (4.1.2) and the definition of NMS conditions, we get

$$\begin{aligned}
lim_{n\to\infty}\Xi(\zeta_n, \zeta_{n+p}, \lambda) &\geq 1 \star \cdots^{(p-times)} \cdots \star 1 = 1 \\
lim_{n\to\infty}\Theta(\zeta_n, \zeta_{n+p}, \lambda) &\leq 0 \diamond \cdots^{(p-times)} \cdots \diamond 0 = 0 \\
lim_{n\to\infty}\Upsilon(\zeta_n, \zeta_{n+p}, \lambda) &\leq 0 \diamond \cdots^{(p-times)} \cdots \diamond 0 = 0.
\end{aligned}$$

Therefore, $\{\zeta_n\}$ is Cauchy sequence and it is convergent to a limit, let the limit point is η. Thus, we get

$$\begin{aligned}
\Xi(F\eta, \eta, t) &\geq \Xi(F\eta, F\zeta_n, \frac{\lambda}{2}) \star \Xi(\zeta_{n+1}, \eta, \frac{\lambda}{2}) \\
&\geq \Xi(\eta, \zeta_n, \frac{\lambda}{2k}) \star \Xi(\zeta_{n+1}, \eta, \frac{\lambda}{2}) \to 1 \star 1 = 1. \\
\Theta(F\eta, \eta, \lambda) &\leq \Theta(F\eta, F\zeta_n, \frac{\lambda}{2}) \diamond \Theta(\zeta_{n+1}, \eta, \frac{\lambda}{2}) \\
&\leq \Theta(\eta, \zeta_n, \frac{\lambda}{2k}) \diamond \Theta(\zeta_{n+1}, \eta, \frac{\lambda}{2}) \to 0 \diamond 0 = 0. \\
\Upsilon(F\eta, \eta, \lambda) &\leq \Upsilon(F\eta, F\zeta_n, \frac{\lambda}{2}) \diamond \Upsilon(\zeta_{n+1}, \eta, \frac{\lambda}{2})) \\
&\leq \Upsilon(\eta, \zeta_n, \frac{\lambda}{2k}) \diamond \Upsilon(\zeta_{n+1}, \eta, \frac{\lambda}{2}) \to 0 \diamond 0 = 0.
\end{aligned}$$

Since we see that

$$\Xi(\zeta,\eta,\lambda)=1 \;\; iff \;\; \zeta=\eta; \quad \Theta(\zeta,\eta,\lambda)=0 \;\; iff \;\; \zeta=\eta; \quad \Upsilon(\zeta,\eta,\lambda)=0 \;\; iff \;\; \zeta=\eta$$

we get $F\eta = \eta$, which is the fixed point of Neutrosophic metric space.
To show the uniqueness, let us assume that $F\omega = \omega$ for some $\omega \in \Sigma$

$$\begin{aligned}
&1 \geq \Xi(\zeta,\omega,\lambda) = \Xi(F\eta,F\omega,\lambda) \geq \Xi(\zeta,\omega,\frac{\lambda}{k}) = \Xi(F\eta,F\eta,\frac{\lambda}{k}) \geq \Xi(\zeta,\omega,\frac{\lambda}{k^2})\\
&\quad \geq \cdots \geq \Xi(\zeta,\omega,\frac{\lambda}{k^n}) \to 1 \; as \; n \to \infty\\
&0 \leq \Theta(\zeta,\omega,\lambda) = \Theta(F\eta,F\omega F\omega,\lambda) \leq \Theta(\zeta,\omega,\frac{\lambda}{k}) = \Theta(F\eta,F\omega,\frac{\lambda}{k}) \leq \Theta(\zeta,\omega,\frac{\lambda}{k^2})\\
&\quad \leq \cdots \leq \Theta(\zeta,\omega,\frac{\lambda}{k^n}) \to 0 \; as \; n \to \infty\\
&0 \leq \Upsilon(\zeta,\omega,\lambda) = \Upsilon(F\eta,F\omega,\lambda) \leq \Upsilon(\zeta,\omega,\frac{\lambda}{k}) = \Upsilon(F\eta,F\omega,\frac{\lambda}{k}) \leq \Upsilon(\zeta,\omega,\frac{\lambda}{k^2})\\
&\quad \leq \cdots \leq \Upsilon(\zeta,\omega,\frac{\lambda}{k^n}) \to 0 \; as \; n \to \infty.
\end{aligned}$$

From the definition of NMS, We get $\eta = \omega$. Therefor, F has a unique fixed point.
Lemma 4.2 (a) If $lim_{n\to\infty}\zeta_n = \zeta$ and $lim_{n\to\infty}\eta_n = \eta$, then

$$\begin{aligned}
\Xi(\zeta,\eta,\lambda-\epsilon) &\leq lim_{n\to\infty}inf \; \Xi(\zeta_n,\eta_n,\lambda)\\
\Theta(\zeta,\eta,\lambda-\epsilon) &\geq lim_{n\to\infty}sup \; \Theta(\zeta_n,\eta_n,\lambda)\\
\Upsilon(\zeta,\eta,\lambda-\epsilon) &\geq lim_{n\to\infty}sup \; \Upsilon(\zeta_n,\eta_n,\lambda)
\end{aligned}$$

(b) If $lim_{n\to\infty}\zeta_n = \zeta$ and $lim_{n\to\infty}\eta_n = \eta$, then

$$\begin{aligned}
\Xi(\zeta,\eta,\lambda+\epsilon) &\geq lim_{n\to\infty}sup \; \Xi(\zeta_n,\eta_n,\lambda)\\
\Theta(\zeta,\eta,\lambda+\epsilon) &\leq lim_{n\to\infty}inf \; \Theta(\zeta_n,\eta_n,\lambda)\\
\Upsilon(\zeta,\eta,\lambda+\epsilon) &\leq lim_{n\to\infty}inf \; \Upsilon(\zeta_n,\eta_n,\lambda)
\end{aligned}$$

for all $\lambda > 0$ and $0 < \epsilon < \lambda$.

Proof for(a): By the definition of NMS, conditions (v),(x) and (xv)

$$\Xi(\zeta_n, \eta_n, \lambda) \geq \Xi(\zeta_n, \zeta, \frac{\epsilon}{2}) \star \Xi(\zeta, \eta, \lambda - \epsilon) \star \Xi(\eta, \eta_n, \frac{\epsilon}{2})$$
$$lim_{n\to\infty}inf\ \Xi(\zeta_n, \eta_n, \lambda) \geq 1 \star \Xi(\zeta, \eta, \lambda - \epsilon) \star 1$$
$$Hence, \quad lim_{n\to\infty}inf\ \Xi(\zeta_n, \eta_n, \lambda) \geq \Xi(\zeta, \eta, \lambda - \epsilon)$$
$$\Theta(\zeta_n, \eta_n, \lambda) \leq \Theta(\zeta_n, \zeta, \frac{\epsilon}{2}) \diamond \Theta(\zeta, \eta, \lambda - \epsilon) \diamond \Theta(\eta, \eta_n, \frac{\epsilon}{2})$$
$$lim_{n\to\infty}sup\ \Theta(\zeta_n, \eta_n, \lambda) \leq 0 \diamond \Theta(\zeta, \eta, \lambda - \epsilon) \diamond 0$$
$$Hence, \quad lim_{n\to\infty}sup\ \Theta(\zeta_n, \eta_n, \lambda) \leq \Theta(\zeta, \eta, \lambda - \epsilon)$$
$$\Upsilon(\zeta_n, \eta_n, \lambda) \leq \Upsilon(\zeta_n, \zeta, \frac{\epsilon}{2}) \diamond \Upsilon(\zeta, \eta, \lambda - \epsilon) \diamond \Upsilon(\eta, \eta_n, \frac{\epsilon}{2})$$
$$lim_{n\to\infty}sup\ \Upsilon(\zeta_n, \eta_n, \lambda) \leq 0 \diamond \Upsilon(\zeta, \eta, \lambda - \epsilon) \diamond 0$$
$$Hence, \quad lim_{n\to\infty}sup\ \Upsilon(\zeta_n, \eta_n, \lambda) \leq \Theta(\zeta, \eta, \lambda - \epsilon)$$

Proof for (b):By the definition of NMS, conditions (v),(x) and (xv)

$$\Xi(\zeta, \eta, \lambda + \epsilon) \geq \Xi(\zeta, \zeta_n, \frac{\epsilon}{2}) \star \Xi(\zeta_n, \eta_n, \epsilon) \star \Xi(\eta_n, \eta, \frac{\epsilon}{2})$$
$$Hence, \quad \Xi(\zeta, \eta, \lambda + \epsilon) \geq lim_{n\to\infty}sup\Xi(\zeta_n, \eta_n, \epsilon)$$
$$\Theta(\zeta, \eta, \lambda + \epsilon) \leq \Xi(\zeta, \zeta_n, \frac{\epsilon}{2}) \diamond \Theta(\zeta_n, \eta_n, \epsilon) \diamond \Theta(\eta_n, \eta, \frac{\epsilon}{2})$$
$$Hence, \quad \Theta(\zeta, \eta, \lambda + \epsilon) \leq lim_{n\to\infty}inf\Theta(\zeta_n, \eta_n, \epsilon)$$
$$\Upsilon(\zeta, \eta, \lambda + \epsilon) \leq \Upsilon(\zeta, \zeta_n, \frac{\epsilon}{2}) \diamond \Upsilon(\zeta_n, \eta_n, \epsilon) \diamond \Upsilon(\eta_n, \eta, \frac{\epsilon}{2})$$
$$Hence, \quad \Upsilon(\zeta, \eta, \lambda + \epsilon) \leq lim_{n\to\infty}inf\Upsilon(\zeta_n, \eta_n, \epsilon)$$

Corollary 4.3 If $lim_{n\to\infty}\zeta_n = a$ and $lim_{n\to\infty}\eta_n = \eta$, then

$$(a) \quad \Xi(\zeta, \eta, \lambda) \leq lim_{n\to\infty}inf\ \Xi(\zeta_n, \eta_n, \lambda);$$
$$\Theta(\zeta, \eta, \lambda) \geq lim_{n\to\infty}sup\ \Theta(\zeta_n, \eta_n, \lambda);$$
$$\Upsilon(\zeta, \eta, \lambda) \geq lim_{n\to\infty}sup\ \Upsilon(\zeta_n, \eta_n, \lambda)....(4.3.1)$$

$$(b) \quad \Xi(\zeta, \eta, \lambda) \geq lim_{n\to\infty}sup\ \Xi(\zeta_n, \eta_n, \lambda)$$
$$\Theta(\zeta, \eta, \lambda) \leq lim_{n\to\infty}inf\ \Theta(\zeta_n, \eta_n, \lambda)$$
$$\Upsilon(\zeta, \eta, \lambda) \leq lim_{n\to\infty}inf\ \Upsilon\zeta_n, \eta_n, \lambda)....(4.3.2)$$

for all $\lambda > 0$ and $0 < \epsilon < \lambda$.

Theorem 4.4 (Neutrosophic Edelstein Contraction Theorem) Let $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ be compact neutrosophic metric space. Let $F : \Sigma \to \Sigma$ be a function satisfying

$$\Xi(F\zeta, F\eta, .) > \Xi(\zeta, \eta, .); \Theta(F\zeta, F\eta, .) < \Theta(\zeta, \eta, .); \Upsilon(F\zeta, F\eta, .) < \Upsilon(\zeta, \eta, .). ...(4.4.1)$$

Then $\digamma$ has fixed point.

Proof: Let $a \in \Sigma$ and $a_n = \digamma^n \zeta \quad (n \in \mathbb{N})$. Assume $\zeta_n \neq \zeta_{n+1}$ for each n (If not $\digamma \zeta_n = \zeta_n$) consequently $a_n \neq a_{n+1} \quad (n \neq m)$, For otherwise we get

$$\Xi(\zeta_n, \zeta_{n+1}, .) = \Xi(\zeta_m, \zeta_{m+1}, .) > \Xi(\zeta_{m-1}, \zeta_m, .) > \cdots > \Xi(\zeta_n, \zeta_{n+1}, .)$$
$$\Theta(\zeta_n, \zeta_{n+1}, .) = \Theta(\zeta_m, \zeta_{m+1}, .) < \Theta(\zeta_{m-1}, \zeta_m, .) < \cdots < \Theta(\zeta_n, \zeta_{n+1}, .)$$
$$\Upsilon(\zeta_n, \zeta_{n+1}, .) = \Upsilon(\zeta_m, \zeta_{m+1}, .) < \Upsilon(\zeta_{m-1}, \zeta_m, .) < \cdots < \Upsilon(\zeta_n, \zeta_{n+1}, .)$$

where $m > n$, which is a contradiction. Since Σ is compact set, $\{\zeta_n\}$ has convergent sub sequence $\{\zeta_{n_i}\}$. Let $\eta = lim_{i\to\infty}\zeta_{n_i}$, Also we assume that η such that $\digamma\eta \in \{\zeta_{n_i}; i \in \mathbb{N}\}$. According to the above assumption, we may now write,

$$\Xi(\digamma\zeta_{n_i}, \digamma\eta, .) > \Xi(\zeta_{n_i}, \eta, .); \quad \Theta(\digamma\zeta_{n_i}, \digamma\eta, .) < \Theta(\zeta_{n_i}, \eta, .); \quad \Upsilon(\digamma\zeta_{n_i}, \digamma\eta, .) < \Upsilon(\zeta_{n_i}, \eta, .)$$

for all $i \in \mathbb{N}$. Then by equation (4.3.1) we obtain

$$lim\ inf \Xi(\digamma\zeta_{n_i}, \digamma\eta, \lambda) \geq lim\ \Xi(\zeta_{n_i}, \eta, \lambda) = \Xi(\eta, \eta, \lambda) = 1$$
$$lim\ sup \Theta(\digamma\zeta_{n_i}, \digamma\eta, \lambda) \leq lim\ \Theta(\zeta_{n_i}, \eta, \lambda) = \Theta(\eta, \eta, \lambda) = 0$$
$$lim\ sup \Upsilon(\digamma\zeta_{n_i}, \digamma\eta, \lambda) \leq lim\ \Upsilon(\zeta_{n_i}, \eta, \lambda) = \Upsilon(\eta, \eta, \lambda) = 0$$

for each $\lambda > 0$. Hence

$$lim\ \digamma\zeta_{n_i} = \digamma\eta....(4.4.2)$$

Simillarly

$$lim\ \digamma^2\zeta_{n_i} = lim \digamma^2\eta...(4.4.3)$$

(we recall that $lim\ \digamma\zeta_{n_i} = \digamma\eta$ for all $(i \in \mathbb{N})$), Now observe that,

$$\begin{aligned}
\Xi(\zeta_{n_i}, \digamma\zeta_{n_i}, \lambda) &\leq \Xi(\digamma\zeta_{n_i}, \digamma^2\zeta_{n_i}, \lambda) \leq \cdots \leq \Xi(\zeta_{n_i}, \digamma\zeta_{n_i}, \lambda) \\
&\leq \Xi(\digamma\zeta_{n_i}, \digamma^2\zeta_{n_i}, \lambda) \leq \cdots \leq \Xi(\digamma\zeta_{n_{i+1}}, \digamma^2\zeta_{n_{i+1}}, \lambda) \\
&\leq \Xi(\digamma\zeta_{n_{i+1}}, \digamma^2\zeta_{n_{i+1}}, \lambda) \leq \cdots \leq 1. \\
\Theta(\zeta_{n_i}, \digamma\zeta_{n_i}, \lambda) &\geq \Theta(\digamma\zeta_{n_i}, \digamma^2\zeta_{n_i}, \lambda) \geq \cdots \geq \Theta(\zeta_{n_i}, \digamma\zeta_{n_i}, \lambda) \\
&\geq \Theta(\digamma\zeta_{n_i}, \digamma^2\zeta_{n_i}, \lambda) \geq \cdots \geq \Theta(\digamma\zeta_{n_{i+1}}, \digamma^2\zeta_{n_{i+1}}, \lambda) \\
&\geq \Theta(\digamma\zeta_{n_{i+1}}, \digamma^2\zeta_{n_{i+1}}, \lambda) \geq \cdots \geq 0. \\
\Upsilon(\zeta_{n_i}, \digamma\zeta_{n_i}, \lambda) &\geq \Upsilon(\digamma\zeta_{n_i}, \digamma^2\zeta_{n_i}, \lambda) \geq \cdots \geq \Upsilon(\zeta_{n_i}, \digamma\zeta_{n_i}, \lambda) \\
&\geq \Upsilon(\digamma\zeta_{n_i}, \digamma^2\zeta_{n_i}, \lambda) \geq \cdots \geq \Upsilon(\digamma\zeta_{n_{i+1}}, \digamma^2\zeta_{n_{i+1}}, \lambda) \\
&\geq \Upsilon(\digamma\zeta_{n_{i+1}}, \digamma^2\zeta_{n_{i+1}}, \lambda) \geq \cdots \geq 0.
\end{aligned}$$

for all $\lambda > 0$. Thus $\{\Xi(\zeta_{n_i}, F\zeta_{n_i}, \lambda)\}, \{\Theta(\zeta_{n_i}, F\zeta_{n_i}, \lambda)\}, \{\Upsilon(\zeta_{n_i}, F\zeta_{n_i}, \lambda)\}$ and $\{(F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda)\}$ $(\lambda > 0)$ are convergent to a common limit point . So by equations (4.3.1) , (4.3.2) and (4.4.1) and we get,

$$\begin{aligned}
\Xi(\eta, F\eta, \lambda) &\geq lim\ sup\ \Xi(\zeta_{n_i}, F\zeta_{n_i}, \lambda) = lim\ sup\ (F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda) \\
&\geq lim\ inf \Xi(F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda) \\
&\geq \Xi(F\eta, F^2\eta, \lambda) \\
\Theta(\eta, F\eta, \lambda) &\leq lim\ inf\ \Theta(\zeta_{n_i}, F\zeta_{n_i}, \lambda) = lim\ inf\ \Theta(F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda) \\
&\leq lim\ sup \Theta(F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda) \\
&\leq \Theta(F\eta, F^2\eta, \lambda) \\
\Upsilon(\eta, F\eta, \lambda) &\leq liminf\ \Upsilon(\zeta_{n_i}, F\zeta_{n_i}, \lambda) = lim\ inf\ \Upsilon(F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda) \\
&\leq lim\ sup \Upsilon(F\zeta F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda) \\
&\leq \Upsilon(F\eta, F^2\eta, \lambda)
\end{aligned}$$

for all $\lambda > 0$. Suppose $b \neq F\eta$, By equation (4.4.1)

$$\Xi(\eta, F\eta, .) < \Xi(F\eta, F^2\eta, .); \quad \Theta(\eta, F\eta, .) > \theta(F\eta, F^2\eta, .); \quad \Upsilon(\eta, F\eta, .) > \Upsilon(F\eta, F^2\eta, .).$$

which is a contradiction , because all the above functions are left continuous , non -decreasing and right continuous , non - increasing respectively. Hence $\eta = F\eta$ is a fixed point.
To prove the uniqueness of the fixed point, let us consider $F(\zeta) = \omega$ for some $\zeta \in \Sigma$.
Then

$$\begin{aligned}
1 &\geq \Xi(\zeta, \omega, \lambda) = \Xi(F\eta, F\omega, \lambda) \geq \Xi(\zeta, \omega, \frac{\lambda}{k}) = \Xi(F\eta, F\omega, \frac{\lambda}{k}) \geq \cdots \geq \Xi(\zeta, \omega, \frac{\lambda}{k^n}) \\
0 &\leq \Theta(\zeta, \omega, \lambda) = \Theta(F\eta, F\omega, \lambda) \leq \Theta(\zeta, \omega, \frac{\lambda}{k}) = \Theta(F\eta, F\omega, \frac{\lambda}{k}) \leq \cdots \leq \Theta(\zeta, \omega, \frac{\lambda}{k^n}) \\
0 &\leq \Upsilon(\zeta, \omega, \lambda) = \Upsilon(F\eta, F\omega, \lambda) \leq \Upsilon(\zeta, \omega, \frac{\lambda}{k}) = \Upsilon(F\omega, F\omega, \frac{\lambda}{k}) \leq \cdots \leq \Upsilon(\zeta, \omega, \frac{\lambda}{k^n})
\end{aligned}$$

Now , we easily verify that $\{\frac{\lambda}{k^n}\}$ is an s - increasing sequence, then by assumption for a given $\epsilon \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that

$$\Xi(\zeta, \omega, \frac{\lambda}{k^n}) \geq 1 - \epsilon; \quad \Theta(\zeta, \omega, \frac{\lambda}{k^n}) \leq \epsilon; \quad \Upsilon(\zeta, \omega, \frac{\lambda}{k^n}) \leq \epsilon.$$

Clearly

$$lim_{n\to\infty} \Xi(\zeta, \omega, \frac{\lambda}{k^n}) = 1; \ lim_{n\to\infty} \Theta(\zeta, \omega, \frac{\lambda}{k^n}) = 0; \ lim_{n\to\infty} \Upsilon(\zeta, \omega, \frac{\lambda}{k^n}) = 0.$$

Hence $\Xi(\zeta, \omega, \lambda) = 1$; $\Theta(\zeta, \omega, \lambda) = 0$; $\Upsilon(\zeta, \omega, \lambda) = 0$. Thus $\eta = \omega$. Hence proved.

Conclusion: In this study, we have investigated the concept of Neutrosophic Metric Space and its properties. We have proved fixed point results for contraction theorems in the setting of neutrosophic metric Space. There is a scope to establish many fixed point results in the areas such as fuzzy metric, generalized fuzzy metric, bipolar and partial fuzzy metric spaces by using the concept of Neutrosophic Set.

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An ideal decision making on Neutrosophic Q-fuzzy Setting

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Abstract: In this study a decision making model through Neutrosophic Q-fuzzy set has been designed. During Covid-19 – Pandemic situation, education sector is stabilizing its work through online mode. Information Communication Technology (ICT) platforms offer many opportunities for the academicians and Learners. This study intends to analyse the selection of best ICT tool by fixing important criteria. The selection of optimal ICT tool is scrutinized in this study using Significant Score of a Neutrosophic fuzzy number.

Keywords: Information Communication Technology, Neutrosophic Set, Neutrosophic Q- fuzzy set, Neutrosophic Q- fuzzy decision set, Neutrosophic fuzzy number, Significant Score of a NFN

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1. Introduction

Education Sector plays a vital role in the digital transformation and embraces the changes during Covid-19 Pandemic situation. In the 21st century, education sector slowly moves to the online education. Many educationists apply ICTs application in online education. Especially during lock down period, ICTs help the Academicians and Learners to balance the teaching – learning process. Yusuf M.O. [25] analyzed about the policy implications in Nigerian education system. The system offered maximum use of ICT potential in the schooling system itself. Neeti Roy [17] analyzed the ICT act as student centered - learning settings. It adopted the general component in teaching and learning process. It helps to enhance the quality and accessibility of education. It aims to learning motivation. Sivakumar Ramaraj [19] explored role of ICT has strong impact on teaching learning in 21st Century. Vibha Thakur et al. [24] studied about transmission of ICT in the field of teaching and learning system by implementing e-learning, virtual learning, e-meeting and e-collaboration. ICT tools integrate, enhance and interact with wide coverage of learning and teaching. It helps the learners to gain the knowledge in the wider range though they are in distant mode.

Many ICT platforms and research are developed and emerged into the market. The tools support the educators to transfer the ideas into implications. During this pandemic situation, the tools act as a bridge between the learners and teachers. It is inevitable to note the application of tools in the education sector effectively. For this purpose, the researchers intend to analyze the different characteristics of ICT tools which are very commonly used in the Academic platform. To identify the optimal ICT tool, this study wants to apply Neutrosophic Q-fuzzy set. Various properties enhance the education system using ICT. The major criteria have been selected for ICT tool which shows the higher ability of it. The Criteria helps to make decision on the application of ICT tools in teaching – learning process. To improve the accuracy in decision making, several types of fuzzy sets are applied in different situations. Muthumeenakshi et al.[15,16] applied the notions fuzzy soft set and bipolar valued Q-fuzzy set to design some multi criteria decision making models. Zhikang Lu[28] used intuitionistic fuzzy values for decision-making method. Smarandache[20.21] generalized the intuitionistic fuzzy set into Neutrosophic Set. After the invention of Neutrosophic settings, the notion is explored by the authors of [7,8,9,12,14,23] in various decision making problems. Later Mohseni et al.[11] introduced MBJ – Neutrosophic structure and applied it in BCK/BCI algebras. As an initiation, Surya et al.[22] applied MBJ – Neutrosophic structure in β-algebra. Recently in [10,13,18] also the concept of Neutrosophic set is applied to evaluate the management of internal control, applications to Multi-Criteria Decision-Making, solving the Fully Neutrosophic Linear Programming Problems.

With all these motivations, this paper incorporates the application of Neutrosophic Q- fuzzy set for the ideal selection of ICT tool in the education sector.

2. Preliminaries

This section discussed the essential notations for the construction of the model in this study.

2.1 Definition: [26, 27] A fuzzy set in an nonempty set Ψ is a mapping, $\omega : \Psi \rightarrow [0,1]$ for each x in Ψ, $\omega(x)$ is called the membership value of x .

2.2 Definition: [6] An intuitionistic fuzzy set in an non – empty set Ψ is defined by the structure $A = \{ < x, \omega_A(x), \lambda_A(x) > | \; x \in \Psi\}$, where $\omega_A : \Psi \rightarrow [0,1]$ is a membership function of A and $\lambda_A : \Psi \rightarrow [0,1]$ is a non – membership function of A with $0 \leq \omega_A + \lambda_A \leq 1$.

2.3 Definition: [20, 21] The term Neutrosophic Fuzzy Set N on a nonempty set Φ is is the structure of the form $\mathrm{N} = \{< x,\ \zeta_{\mathrm{N}}(x),\ \xi_{\mathrm{N}}(x),\ \eta_{\mathrm{N}}(x) > | \; x \in \Phi\}$ characterized by a truth – membership function ζ_{N} , an indeterminacy membership function ξ_{N}, and a falsity – membership function η_{N}, where $\zeta_{\mathrm{N}},\ \xi_{\mathrm{N}},\ \eta_{\mathrm{N}} : \Phi \rightarrow [0,1]$.

2.4 Definition: [16] A Q -fuzzy subset μ in a non-empty set X is a function $\mu: X \times Q \longrightarrow [0,1]$, where Q is any non-empty set.

2.5 Definition: [16] A Q-fuzzy decision (QFD) set of X denoted by QF_X^D and is defined by $QF_X^D = \left\{ \mu_{QF_X^D}(x) \;\middle|\; x \in X \right\}$ which is a fuzzy set over X and its membership function $\mu_{QF_X^D}$ is defined by $\mu_{QF_X^D}: X \longrightarrow [0,1]$, where $\mu_{QF_X^D}(x) = \frac{1}{|\mathrm{K}|}\sum_{j=1}^{n} \mu_X(x, q_j)$. Here $q_j \in Q$ and K is number of characteristics which influences the particular population.

3. Neutrosophic Q -Fuzzy Decision Set

3.1 Definition: A Neutrosophic-Q-Fuzzy Set (NQFS) Ω, in a non-empty set Γ is defined as an object of the form $\Omega = \{< (x,q),\ \zeta_\Omega(x,q),\ \xi_\Omega(x,q),\ \eta_\Omega(x,q) > | \; (x,q) \in \Gamma \times Q\}$, where $\zeta_\Omega,\ \xi_\Omega,\ \eta_\Omega : \Gamma \times Q \rightarrow [0,1]$ represents the truth membership function, intermediate membership function and false membership function of Ω respectively.

3.2 Definition: For the Neutrosophic Set $\mathrm{N} = \{< x,\ \zeta_{\mathrm{N}}(x),\ \xi_{\mathrm{N}}(x),\ \eta_{\mathrm{N}}(x) > |\ x \in \Phi\}$ in Φ, the triple $\langle \zeta_{\mathrm{N}},\ \xi_{\mathrm{N}},\ \eta_{\mathrm{N}} \rangle$ is called Neutrosophic Fuzzy Number(NFN) and is denoted by N_x.

3.3 Definition: The Significant Score of a NFN, $\mathrm{N}_x = \langle \zeta_{\mathrm{N}},\ \xi_{\mathrm{N}},\ \eta_{\mathrm{N}} \rangle$ is defined as $SS(\mathrm{N}_x) = \left(\zeta_{\mathrm{N}_x} - \eta_{\mathrm{N}_x} + \left(\frac{\xi_{\mathrm{N}_x}}{2}\right)\right)\left(1 - \left(\zeta_{N_x} - \eta_{N_x} + \left(\frac{\xi_{N_x}}{2}\right)\right)^2\right)$. This SS is used to identify an ideal solution from the various likewise objects of the given population.

3.4 Definition: A Neutrosophic Q-Fuzzy Decision (NQFD) set of Γ is defined by $NQF_\Gamma^D = \left\{ \left(\zeta_{\mathrm{N}_{QF_\Gamma^D}}(x),\ \xi_{\mathrm{N}_{QF_\Gamma^D}}(x),\ \eta_{\mathrm{N}_{QF_\Gamma^D}}(x) \right) \middle|\ x \in \Gamma \right\}$ which is a Neutrosophic fuzzy set over Γ, where $\zeta_{\mathrm{N}_{QF_\Gamma^D}} : \Gamma \longrightarrow [0,1]$, $\xi_{\mathrm{N}_{QF_\Gamma^D}} : \Gamma \longrightarrow [0,1]$ and $\eta_{\mathrm{N}_{QF_\Gamma^D}} : \Gamma \rightarrow [0,1]$ are the truth membership function, intermediate membership function and false membership function and respectively with $\zeta_{\mathrm{N}_{QF_\Gamma^D}}(x) = \frac{1}{|\mathrm{K}|}\sum_{j=1}^{n} \zeta_{\mathrm{N}_\Gamma}(x, q_j)$; $\xi_{\mathrm{N}_{QF_\Gamma^D}}(x) = \frac{1}{|\mathrm{K}|}\sum_{j=1}^{n} \xi_{\mathrm{N}_\Gamma}(x, q_j)$ and $\eta_{\mathrm{N}_{QF_\Gamma^D}}(x) = \frac{1}{|\mathrm{K}|}\sum_{j=1}^{n} \eta_{\mathrm{N}_\Gamma}(x, q_j)$. Here $q_j \in Q$ and K is number of characteristics which influences the particular population.

4. Ideal selection using Neutrosophic Q-Fuzzy Decision set

In this section, the responses from the Academicians and Learners are analyzed. The optimal selection of the ICT tool will be decided using Neutrosophic Q-Fuzzy Decision (NQFD) set. Based on the Experts' advice five major Criteria have been fixed for the ICT tool in E- Learning Process. The criteria are named as F1, F2, F3, F4 and F5 which are taken as the factor and the five different types of ICT tools are compared; E1, E2, E3, E4, E5. The commonly used ICT tools are selected based on the experts' opinion. These tools have different application strategy with wide range coverage. Here, the factors to be considered for the optimal selection process are Easy Access (F1), Advanced Features (F2), Consumption of Bytes (F3), Less Interruption (F4), and Allowable Participants (F5). For each factor, four questions were asked to the respondents. Totally twenty items were analyzed with the application of NQFD set. These twenty items directly or indirectly collate the opinion of the respondents in the education sector about the ICT application. The items are designed with the three point Likert Scale.

The Scales are Satisfied, Neutral and Dissatisfied. Satisfied referred to Truth membership value, Neutral denotes Intermediate membership value and Dissatisfied denotes False membership value.

The following procedure has been introduced for the purpose of selection.

1. Construct NQFS over X.
2. Build NQF_{Γ}^{D}.
3. Find SS (NQF_{Γ}^{D}) .
4. Interpretation.

Here $\Gamma = \{\mathrm{E1}, \mathrm{E2}, \mathrm{E3}, \mathrm{E4}, \mathrm{E5}\}$ and $Q = \{\mathrm{F1}, \mathrm{F2}, \mathrm{F3}, \mathrm{F4}, \mathrm{F5}\}$

Step 1: To apply NQFS for the selection of ICT tool in E-Learning process, the universal set Γ and the non-empty set Q of characteristics are designed as follows.

The responses are applied in the algorithm and values are calculated accordingly. Each characteristic is analyzed with four items in the form of statements. The google form has been structured and distributed to hundred respondents. The respondents are Academicians and Learners. The total satisfactory responses from the respondents for each statement are divided with number of respondents, i.e. 100. Likewise the total dissatisfactory and neutral responses are considered for the analysis.

Step 2: Truth, Intermediate and False Membership values have been assigned based on Step 1 Procedure. The following table shows the respective membership values for the optimal selection of ICT in E-Learning.

Table 1: Neutrosophic membership values

$\Gamma \rightarrow$ $Q \downarrow$	E_1	E_2	E_3	E_4	E_5
F_1	(0.43 ,0.23 , 0.34)	(0.36 , 0.32 , 0.32)	(0.57 , 0.21 , 0.22)	(0.70 , 0.14 , 0.16)	(0.63 , 0.21 , 0.16)
F_2	(0.45 , 0.30 , 0.25)	(0.47 , 0.32 , 0.21)	(0.52 , 0.29 , 0.19)	(0.61 , 0.21 , 0.18)	(0.53 , 0.23 , 0.24)
F_3	(0.36 , 0.42 , 0.22)	(0.41 , 0.20 , 0.39)	(0.49 , 0.24 , 0.27)	(0.69 , 0.21 , 0.10)	(0.30 , 0.50 , 0.20)
F_4	(0.38 , 0.32 , 0.30)	(0.40 , 0.21 , 0.39)	(0.60 , 0.20 , 0.20)	(0.72 , 0.21 , 0.07)	(0.50 , 0.32 , 0.18)
F_5	(0.34 , 0.31 , 0.35)	(0.31 , 0.32 , 0.37)	(0.43 , 0.32 , 0.25)	(0.82 , 0.12 , 0.06)	(0.62 , 0.21 , 0.17)

Step 3: The NQF_{Γ}^{D} has been attained using the definition 3.4.

$$NQF_{\Gamma}^{D}=\{(0.392\,,0.316, 0.292)/\,E_1\,,(0.390\,,0.274, 0.336)/\,E_2\,,$$
$$(0.522\,,0.252, 0.226)/\,E_3\,,(0.708\,,0.178, 0.114)/\,E_4$$
$$(0.516\,,0.294, 0.190)/\,E_5\,\}$$

Step 4: The Significant Score for all E_i's are identified as using the definition 3.3.

$SS(E_1) = 0.2376$

$SS(E_2) = 0.1840$

$SS(E_3) = 0.3468$

$SS(E_4) = 0.3644$

$SS(E_5) = 0.3672$

5. Conclusion

In Education Sector, ICT plays a vital role especially during Covid19 situation. Many ICT tools are in the education arena. Each ICT tool gives benefits with some unique characteristics. The very important and common usages of characters are considered as the criteria for the analysis. For the optimal selection of ICT tool, the Academicians and Learners are using different strategies in the Technology. In this study, Neutrosophic Q-Fuzzy Decision set has been used by considering the positive, intermediate and negative values of the responses from the Academicians and Learners opinions. The values are taken in the relative measures and applied in the Neutrosophic Q-Fuzzy Decision set. The result of the analysis revealed that the ICT (E5) is the best option which includes all the important characters of Tech tool for teaching and learning at the optimal level. This application enhances the opinion results and helps in decision making in the ICT tool selection and it can be explored in other such decision making scenarios.

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NeutroVectorSpaces I

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Abstract. Recently, the concept of NeutroAlgebraic and AntiAlgebraic Structures were introduced and analyzed by Florentin Smarandache. His new approach to the study of Neutrosophic Structures presents a more robust tool needed for managing uncertainty, incompleteness, indeterminate and imprecise information. In this paper, we introduce for the first time the concept of NeutroVectorSpaces. Specifically, we study a particular class of the NeutroVectorSpaces called of type $4S$ and their elementarily properties are presented. It is shown that the NeutroVectorSpaces of type $4S$ may contain NeutroSubspaces of other types and that the intersections of NeutroSubspaces of type $4S$ are not NeutroSubspaces. Also, it is shown that if NV is a NeutroVector Space of a particular type and NW is a NeutroSubspace of NV, the NeutroQuotientSpace NV/NW does not necessarily belong to the same type as NV.

Keywords: Neutrosophy; Vector Space; NeutroField; weak NeutroVectorSpace; strong NeutroVectorSpace; weak AntiVectorSpace; strong AntiVectorSpace; NeutroSubspace; weak NeutroQuotientSpace; strong NeutroQuotientSpace.

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1. Introduction

As an extension of his work in [15], Florentine Smarandache in [12] introduced a new way of handling uncertainty, incompleteness, indeterminate and imprecise information. He studied and presented the concept of NeutroAlgebraicStructures and AntiAlgebraicStructures, which can be generated from a classical algebraic structure by a process called neutro-sophication and anti-sophication respectively. The emergence of these processes has given birth to a new field of research in the theory of neutrosophic algebraic structures. More details on neutrosophic algebraic structures can be found in [4]- [10].

Smarandache in [13] recalled, improved and extended several definitions and properties of NeutroAlgebras and AntiAlgebras given in [12]. This new concept was examined by Agboola et al. in [1] viz-a-viz the classical number systems $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$. In [2], Agboola formally presented the notion of NeutroGroups by considering three NeutroAxioms (NeutroAssociativity, existence of NeutroNeutral element and existence of NeutroInverse element). In addition, he showed that generally, Langrange's theorem and 1st isomorphism theorem of the classical groups do not hold in the class of NeutroGroups considered. Also in [3], Agboola studied NeutroRing, NeutroSubring, NeutroIdeal, NeutroQuotientRings and he showed that the 1st isomorphism theorem of the classical rings holds in this class of NeutroRing. Recently, Rezaei and Smarandache [11] introduced the concept of Neutro-BE-algebras and Anti-BE-algebras and they showed that given any classical algebra S with n operations (laws and axioms) where $n \geq 1$ we can generate $(2^n - 1)$ NeutroStructures and $(3^n - 2^n)$ AntiStructures. For comprehensive review of new trends in neutrosophic theory readers should see [4–6, 8–10].

The present paper will be concerned with the introduction of NeutroVectorSpaces. Specifically in the paper, we will introduce and study a class of NeutroVectorSpaces called NeutroVectorSpaces of type $4S$ (i.e., 4 of its scalar multiplication axioms are NeutroAxioms) and we will present some of their elementarily properties. It will be shown that the NeutroVectorSpaces of type $4S$ may contain NeutroSubspaces of other types and that the intersections of NeutroSubspaces of type $4S$ are not NeutroSubspaces. Also, it will be shown that if NV is a NeutroVectorSpace of a particular type and NW is a NeutroSubspace of NV, then the NeutroQuotientSpace NV/NW does not necessarily belong to the same type as NV.

2. **Preliminaries**

In this section, we will give some definitions, examples and results that will be used in the sequel.

Definition 2.1. [14]

(i) A ClassicalOperation is an operation well-defined for all the set's elements while a NeutroOperation is an operation partially well-defined, partially indeterminate, and partially outer defined on the given set. An AntiOperation is an operation that is outer defined for all the set's elements.

(ii) A classicalLaw/Axiom defined on a nonempty set is a law/axiom that is totally true for all the set's elements while a NeutroLaw/Axiom defined on a nonempty set is a law/axiom that is true for some set's element, indeterminate for other set's elements, or false for the other set's elements. An AntiLaw/Axiom defined on a nonempty set is a law/axiom that is false for all set's elements.

(iii) A NeutroAlgebra is an algebra that has at least one NeutroOperation or one NeutroAxiom (axiom that is true for some elements, indeterminate for other elements, and false for other elements), and no AntiOperation or AntiAxiom while an AntiAlgebra is an algebra endowed with at least one AntiOperation or at least one AntiAxiom.

Theorem 2.2. *[11] Let $\mathbb{U}$ be a nonempty finite or infinite universe of discourse and let S be a finite or infinite subset of $\mathbb{U}$. If n classical operations (laws and axioms) are defined on S where $n \geq 1$, then there will be $(2^n - 1)$ NeutroAlgebraicStructures and $(3^n - 2^n)$ AntiAlgebraicStructures.*

Definition 2.3. [Classical group]

Let G be a nonempty set and let $* : G \times G \to G$ be a binary operation on G. The couple $(G, *)$ is called a classical group if the following conditions hold:

(G1) $x * y \in G \; \forall x, y \in G$ [closure law].

(G2) $x * (y * z) = (x * y) * z \; \forall x, y, z \in G$ [axiom of associativity].

(G3) There exists $e \in G$ such that $x * e = e * x = x \; \forall x \in G$ [axiom of existence of neutral element].

(G4) There exists $y \in G$ such that $x * y = y * x = e \; \forall x \in G$ [axiom of existence of inverse element] where e is the neutral element of G.

If in addition $\forall x, y \in G$, we have

(G5) $x * y = y * x$, then $(G, *)$ is called an abelian group.

Definition 2.4. [NeutroSophication of the law and axioms of the classical group]

(NG1) There exist at least three duplets $(x, y), (u, v), (p, q), \in G$ such that $x * y \in G$ (inner-defined with degree of truth T) and [$u * v =$ indeterminate (with degree of indeterminacy I) or $p * q \notin G$ (outer-defined/falsehood with degree of falsehood F)] [NeutroClosureLaw].

(NG2) There exist at least three triplets $(x, y, z), (p, q, r), (u, v, w) \in G$ such that $x * (y * z) = (x * y) * z$ (inner-defined with degree of truth T) and [[$p * (q * r)$]or [$(p * q) * r$] = indeterminate (with degree of indeterminacy I) or $u * (v * w) \neq (u * v) * w$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of associativity (NeutroAssociativity)].

(NG3) There exists an element $e \in G$ such that $x * e = e * x = x$ (inner-defined with degree of truth T) and [[$x * e$]or[$e * x$] = indeterminate (with degree of indeterminacy I) or $x * e \neq x \neq e * x$ (outer-defined/falsehood with degree of falsehood F)] for at least one $x \in G$ [NeutroAxiom of existence of neutral element (NeutroNeutralElement)].

(NG4) There exists an element $u \in G$ such that $x * u = u * x = e$ (inner-defined with degree of truth T) and [[$x * u$]or[$u * x$)] = indeterminate (with degree of indeterminacy I) or $x * u \neq e \neq u * x$ (outer-defined/falsehood with degre of falsehood F)] for at least one $x \in G$ [NeutroAxiom of existence of inverse element (NeutroInverseElement)] where e is a NeutroNeutralElement in G.

(NG5) There exist at least three duplets $(x, y), (u, v), (p, q) \in G$ such that $x * y = y * x$ (inner-defined with degree of truth T) and [[$u * v$]or[$v * u$] = indeterminate (with degree of indeterminacy I) or $p * q \neq q * p$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of commutativity (NeutroCommutativity)].

Definition 2.5. A NeutroGroup NG is an alternative to the classical group G that has at least one NeutroLaw or at least one of $\{NG1, NG2, NG3, NG4\}$ with no AntiLaw or AntiAxiom.

Definition 2.6. A NeutroAbelianGroup NG is an alternative to the classical abelian group G that has at least one NeutroLaw or at least one of $\{NG1, NG2, NG3, NG4\}$ and $NG5$ with no AntiLaw or AntiAxiom.

Example 2.7. Let $NG = \mathbb{N} = \{1, 2, 3, 4 \cdots, \}$. Then $(NG, .)$ is a finite NeutroGroup where $''.''$ is the binary operation of ordinary multiplication.

Definition 2.8. [Classical ring] Let R be a nonempty set and let $+, . : R \times R \to R$ be binary operations of the usual addition and multiplication respectively defined on R. The triple $(R, +, .)$ is called a classical ring if the following conditions $(R1 - R9)$ hold:

(R1) $x + y \in R \ \forall x, y \in R$ [closure law of addition].

(R2) $x + (y + z) = (x + y) + z \ \forall x, y, z \in R$ [axiom of associativity].

(R3) There exists $e \in R$ such that $x + e = e + x = x \ \forall x \in R$ [axiom of existence of neutral element].

(R4) There exists $-x \in R$ such that $x + (-x) = (-x) + x = e \ \forall x \in G$ [axiom of existence of inverse element]

(R5) $x + y = y + x \ \forall x, y \in R$ [axiom of commutativity].

(R6) $x.y \in R \ \forall x, y \in R$ [closure law of multiplication].

(R7) $x.(y.z) = (x.y).z \ \forall x, y, z \in R$ [axiom of associativity].

(R8) $x.(y + z) = (x.y) + (x.z) \ \forall x, y, z \in R$ [axiom of left distributivity].

(R9) $(y + z).x = (y.x) + (z.x) \ \forall x, y, z \in R$ [axiom of right distributivity].

If in addition we have,

(R10) $x.y = y.x \ \forall x, y \in R$ [axiom of commutativity],

then $(R, +, .)$ is called a commutative ring.

Definition 2.9. [NeutroSophication of the laws and axioms of the classical ring]

(NR1) There exist at least three duplets $(x, y), (u, v), (p, q) \in R$ such that $x + y \in R$ (inner-defined with degree of truth T) and [$u + v =$ indeterminate (with degree of indeterminacy I) or $p + q \notin R$ (outer-defined/falsehood with degree of falsehood F)] [NeutroClosure law of addition].

(NR2) There exist at least three triplets $(x, y, z), (p, q, r), (u, v, w) \in R$ such that $x + (y + z) = (x + y) + z$ (inner-defned with degree of truth T) and [[$p + (q + r)$]or[$(p + q) + r$] = indeterminate (with degree of indeterminacy I) or $u + (v + w) \neq (u + v) + w$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of associativity (NeutroAssociativity)].

(NR3) There exists an element $e \in R$ such that $x + e = x + e = x$ (inner-defined with degree of truth T) and [[$x + e$]or[$e + x$] = indeterminate (with degree of indeterminate I) or $x + e \neq x \neq e + x$ (outer-defined/falsehood with degree of falsehood F)] for at least one $x \in R$ [NeutroAxiom of existence of neutral element (NeutroNeutralElement)].

(NR4) There exists $-x \in R$ such that $x + (-x) = (-x) + x = e$ (inner-defined with degree of truth T) and [[$-x + x$]or[$x + (-x)$] = indeterminate (with the degree of indeterminate I) or $-x + x \neq e \neq x + (-x)$ (outer-defined/falsehood with degree of falsehood F)] for at least one $x \in R$ [NeutroAxiom of existence of inverse element (NeutroInverseElement)].

(NR5) There exist at least three duplets $(x,y),(u,v),(p,q) \in R$ such that $x+y=y+x$ (inner-defined with degree of truth T) and $[[p+q]\text{or}[q+p]$ = indeterminate (with degree of indeterminacy I) or $u+v \neq v+u$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of commutativity (NeutroCommutativity)].

(NR6) There exist at least three duplets $(x,y),(p,q),(u,v) \in R$ such that $x.y \in R$ (inner-defined with degree of truth T) and $[u.v$ = indeterminate (with degree of indeterminacy I) or $p.q \notin R$ (outer-defined/falsehood with degree of falsehood F)] NeutroClosure law of multiplication].

(NR7) There exist at least three triplets $(x,y,z),(p,q,r),(u,v,w) \in R$ such that $x.(y.z) = (x.y).z$ (inner-defined with degree of truth T) and $[[p.(q.r)]\text{or}[(p.q).r]$ = indeterminate (with degree of indeterminacy I) or $u.(v.w) \neq (u.v).w$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of associativity (NeutroAssociativity)].

(NR8) There exist at least three triplets $(x,y,z),(p,q,r),(u,v,w) \in R$ such that $x.(y+z) = (x.y) + (x.z)$ (inner-defined with degree of truth T) and $[[p.(q+r)]\text{or}[(p.q)+(p.r)]$ = indeterminate (with degree of indeterminacy I) or $u.(v+w) \neq (u.v)+(u.w)$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of left distributivity (NeutroLeftDistributivity)].

(NR9) There exist at least three triplets $(x,y,z),(p,q,r),(u,v,w) \in R$ such that $(y+z).x = (y.x) + (z.x)$ (inner-defined with degree of truth T) and $[[(v+w).u]\text{or}[(v.u)+(w.u)]$ = indeterminate (with degree of indeterminacy I) or $(v+w).u \neq (v.u)+(w.u)$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of right distributivity (NeutroRightDistributivity)].

(NR10) There exist at least three duplets $(x,y),(p,q),(u,v) \in R$ such that $x.y = y.x$ (inner-defined with degree of truth T) and $[[p.q]\text{or}[q.p]$ = indeterminate (with degree of indeterminacy I) or $u.v \neq v.u$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of commutativity (NeutroCommutativity)].

Definition 2.10. A NeutroRing NR is an alternative to the classical ring R that has at least one NeutroLaw or at least one of $\{NR1, NR2, NR3, NR4, NR5, NR6, NR7, NR8, NR9\}$ with no AntiLaw or AntiAxiom.

Definition 2.11. A NeutroNoncommutativeRing NR is an alternative to the classical noncommutative ring R that has at least one NeutroLaw or at least one of $\{NR1, NR2, NR3, NR4, NR5, NR6, NR7, NR8, NR9\}$ and $NR10$ with no AntiLaw or AntiAxiom.

Example 2.12. (i) Let $NR = \mathbb{Z}$ and let $\oplus$ be a binary operation of ordinary addition and for all $x,y \in NR$, let $\odot$ be a binary operation defined on NR as $x \odot y = \sqrt{xy}$. Then $(NR,\oplus,\odot)$ is a NeutroRing.

(ii) Let $NR = \mathbb{Q}$ and let $\oplus$ be a binary operation of ordinary addition and for all $x,y \in NR$, let $\odot$ be a binary operation defined on NR as $x \odot y = x/y$. Then $(NR,\oplus,\odot)$ is a NeutroRing.

3. Formulation of NeutroVectorSpaces

In this section, we present the concept of NeutroVectorSpaces and study their elementary properties.

Definition 3.1. [7] [Classical Vector Space]

A vector space consists of a nonempty set V of objects (called vectors) that can be added, that can be multiplied by a real or complex number (called a scalar in this context), and for which the following laws and axioms hold:

The Law and Axioms for vector addition

(A1) If u and v are in V, then $u+v$ is in V.

(A2) $u+(v+w)=(u+v)+w$ for all u,v, and w in V.

(A3) An element 0 in V exist such that $v+0=v=0+v$ for every v in V.

(A4) For each v in V, an element $-v$ in V exist such that $-v+v=0$ and $v+(-v)=0$.

(A5) $u+v=v+u$ for all u and v in V.

The Law and Axioms for scalar multiplication

(S1) If v is in V, then av is in V for all a in $\mathbb{R}$.

(S2) $a(v+w)=av+aw$ for all v and w in V and all $a\in\mathbb{R}$.

(S3) $(a+b)v=av+bv$ for all v in V and all a and $b\in\mathbb{R}$.

(S4) $a(bv)=(ab)v$ for all v in V and all a and b in $\mathbb{R}$.

(S5) $1v=v$ for all v in V.

Definition 3.2. [NeutroSophication of the law and axioms of the classical vector space]

NeutroSophication of the law and axioms for vector addition

(NA1) There exist at least three duplets $(u,v),(w,x),(y,z)\in V$ such that $u+v\in V$ (inner-defined with degree of truth T) and [$w+x=$ indeterminate (with degree of indeterminacy I) or $y+z\notin V$ (outer-defined/falsehood with degree of falsehood F)].

(NA2) There exist at least three triplets $(u,v,w),(x,y,z),(p,q,r)\in V$ such that $u+(v+w)=(u+v)+w$ (inner-defined with degree of truth T) and [$[x+(y+z)]$or$[(x+y)+z]=$ indeterminate (with degree of indeterminacy I) or $p+(q+r)\neq(p+q)+r$ (outer-defined/falsehood with degree of falsehood F)].

(NA3) There exists an element $e\in V$ such that $v+e=e+v=v$ (inner-defined with degree of truth T) and [$[v+e]$or$[e+v]=$ indeterminate (with degree of indeterminacy I) or $v+e\neq v\neq e+v$ (outer-defined/falsehood with degree of falsehood F)] for at least one $v\in V$.

(NA4) There exists $-v\in V$ such that $v+(-v)=(-v)+v=e$ (inner-defined with degree of truth T) and [$[-v+v]$or$[v+(-v)]=$ indeterminate (with degree of indeterminacy I) or $[-v+v\neq e\neq v+(-v)$ (outer-defined/falsehood with degree of falsehood F)]

(NA5) There exist at least three duplets $(u, v), (x, y), (w, z) \in V$ such that $u+v = v+u$ (inner-defined with degree of truth T) and $[[x + y]\text{or}[y + x]$ = indeterminate (with degree of indeterminacy I) or $w + z \neq z + w$ (outer-defined/falsehood with degree of falsehood F)].

NeutroSophication of the law and axioms for scalar multiplication

(NS1) There exist at least three duplets $(a, v), (b, u), (c, x)$ with $a, b, c \in K$ and $v, u, x \in V$ such that $av \in V$ (inner-defined with degree of truth T) and [bu = indeterminate (with degree of indeterminacy I) or $cx \notin V$ (degree of falsehood F)].

(NS2) There exist at least three triplets $(k, x, y), (m, u, v), (n, w, z)$ with $k, m, n \in K$ and $x, y, u, v, w, z \in X$ such that $k(x + y) = kx + ky$ (inner-defined with degree of truth T) and $[[m(u+v)]\text{or}[mu+mv]$ = indeterminate (with degree of indeterminacy I) or $n(w+z) \neq nw+nz$ (outer-defined/falsehood with degree of falsehood F)].

(NS3) There exist at least three triplets (k, m, x),(p,q,y),(r,s,z) with $k, m, p, q, r, s \in K$ and $x, y, z \in X$ such that $(k+m)x = kx+mx$ (inner-defined with degree of truth T) and $[[(p+q)y]\text{or}[py+qy]$ = indeterminate (with degree of indeterminacy I) or $(r + s)z \neq rz + sz$ (outer-defined/falsehood with degree of falsehood F)].

(NS4) There exist at least three triplets $(k, m, x), (p, q, y), (r, s, z)$ with $k, m, p, q, r, s \in K$ and $x, y, z \in X$ such that $k(mx) = (km)x = (mk)x$ (inner-defined with degree of truth T) and $[[p(qy)]\text{or}[q(py)]\text{or}[(pq)y]$ = indeterminate (with degree of indeterminacy I) or $r(sz) \neq (rs)z$ (outer-defined/falsehood with degree of falsehood F)].

(NS5) There exists an element $k \in K$ such that $kv = v$ (inner-defined with degree of truth T) and [kv = indeterminate (with degree of indeterminacy I) or $kv \neq v$ (outer-defined/falsehood with degree of falsehood F)] for at least one $v \in V$.

Definition 3.3. [AntiSophication of the law and axioms of the classical vector space]

AntiSophication of the law and axioms for vector addition

(AA1) For all the duplets $(u, v) \in V$, $u + v \notin V$.

(AA2) For all the triplets $(u, v, w) \in V$, $u + (v + w) \neq (u + v) + w$.

(AA3) There does not exist an element e in V such that $v + e = v = e + v$ for every v in V.

(AA4) There does not exist $-v$ in V such that $v + (-v) = (-v) + v = e$ for all $v \in V$ where e is a AntiNeutralElement in V.

(AA5) For all the duplets $(u, v) \in V$, $u + v \neq v + u$.

AntiSophication of the law and axioms for scalar multiplication

(AS1) For all $v \in V$ and $a \in \mathbb{R}$, $av \notin V$.

(AS2) For all $u, v \in V$ and $a \in \mathbb{R}$, $a(u + v) \neq au + av$.

(AS3) For all $v \in V$ and $a, b \in \mathbb{R}$, $(a + b)v \neq av + bv$.

(AS4) For all $v \in V$ and $a, b \in \mathbb{R}$, $a(bv) \neq (ab)v$.

(AS5) For all $v \in V$, $1v \neq v$.

Definition 3.4. Let $(K, +, \cdot)$ be a field. A NeutroField $(NK, +, \cdot)$ is an alternative to the classical field $(K, +, \cdot)$ that has at least one NeutroLaw or at least one NeutroAxiom with no Antilaw or AntiAxiom.

Definition 3.5. Let $(K, +, \cdot)$ be a field. An AntiField $(AK, +, \cdot)$ is an alternative to the classical field $(K, +, \cdot)$ that has at least one AntiLaw or at least one AntiAxiom.

Definition 3.6. Let " $+$ " be addition of vectors, " $\cdot$ " be multiplication of vector by scalars and let K be a Neutro/classical field. A NeutroVectorSpace $(NV, +, \cdot)$ is an alternative to the classical vector space $(V, +, \cdot)$ that has at least one NeutroLaw or at least one of $\{NA1 - NS5\}$ with no Antilaw or AntiAxiom.

If K is a classical field, then the quadruple $(NV, +, \cdot, K)$ is called a weak NeutroVectorSpace over K. And the quadruple $(NV, +, \cdot, K)$ is called a strong NeutroVectorSpace if K is a NeutroField (i.e., $K = NK$).

Definition 3.7. Let " $+$ " be addition of vectors, " $\cdot$ " be multiplication of vectors by scalars and let K be a Anti/classical field. An AntiVectorSpace $(AV, +, \cdot)$ is an alternative to the classical vector space $(V, +, \cdot)$ that has at least one AntiLaw or at least one of $\{AA1 - AS5\}$.

If K is a classical field, then the quadruple $(AV, +, \cdot, K)$ is called a weak AntiVectorSpace over K. And the quadruple $(AV, +, \cdot, K)$ is called a strong AntiVectorSpace if K is a AntiField (i.e., $K = AK$).

Theorem 3.8. *Let $(V, +, \cdot)$ be a classical vector space over a field K. Then,*

(1) *there are 1023 classes of NeutroVector Spaces.*

(2) *there are 58025 classes of AntiVector Spaces.*

Proof. The proof follows easily from Theorem 2.2. □

Theorem 3.8 shows that there are many classes of NeutroVector Spaces. The trivial cases from the 1023 classes are the cases where $NA1 - NS5$ hold. Examples of weak and strong NeutroVectorSpaces for the trivial cases are given in Example 3.9.

Example 3.9. Let $V = \mathbb{Z}_{12}$ and $K = \mathbb{R}$. Define addition and scalar multiplication by

$$x \oplus y = \frac{2x + 3y}{2} \text{ and } k \odot a = ka^2$$

where $\oplus$ is addition modulo 12. Then $(V, \oplus, \odot)$ is a weak NeutroVectorSpace over a field K.

To see this:

(1) We will show that $(V, \oplus)$ is a NeutroAbelianGroup.

(a) There exist at least $x, y \in V$ such that $x \oplus y \in V$ and at least $a, b \in V$ such that $a \oplus b \notin V$. For instance, if we take $(x, y) = (1, 2)$ and $(a, b) = (2, 1)$, we will see that $NV1$ holds. Therefore $\oplus$ is NeutroClosed.

(b) Let $x, y, z \in V$. Then
$x \oplus (y \oplus z) = \frac{4x+6y+9z}{4}$ and $(x \oplus y) \oplus z = \frac{4x+6y+6z}{4}$, equating these we have
$4x + 6y + 9z = 4x + 6y + 6z$ which gives $9z = 6z$,
$\therefore 3z = 0$ this implies that $z = 0, 4$ and 8.
Thus, only the triplets $(x, y, 0), (x, y, 4),$ and $(x, y, 8)$ can verify the associativity of $\oplus$ and therefore, $\oplus$ is NeutroAssociative.

(c) Let $e \in V$ such that $x \oplus e = \frac{2x+3e}{2} = x$ and $e \oplus x = \frac{2e+3x}{2} = x$.
Then $\frac{2x+3e}{2} = \frac{2e+3x}{2}$ from which we obtain $e = x$.
The elements of V that satisfy $x \oplus x = x$ are $0, 8$. This shows that V has NeutroNeutral element.

(d) Considering each NeutroNeutral element in (b) we can show that V has NeutroInverse element.

(e) Let $x, y \in V$, $x \oplus y = \frac{2x+3y}{2}$ and $y \oplus x = \frac{2y+3x}{2}$.
If $''\oplus''$ is commutative, we will have $\frac{2x+3y}{2} = \frac{2y+3x}{2}$ from which we obtain $x = y$. This shows that only the duplet (x, x) can verify commutativity of $\oplus$.
Thus, $\oplus$ is NeutroCommutative. Hence, $(V, \oplus)$ is a NeutroAbelianGroup.

(2) We wish to find at least a triplet (k, m, u) with $u \in V$ and $k, m \in K$, such that $k \odot (m \odot u) = (km) \odot u$.
Now, consider $(km) \odot u = (km)u^2 = kmu^2$ and $k \odot (m \odot u) = k \odot (mu^2) = k(mu^2)^2 = km^2u^4$.
Equating these we have

$$kmu^2 = km^2u^4,$$

which gives

$$mu^2 = 1.$$

Since we need at least a triplet, take $m = 1$, then elements of V that will satisfy $mu^2 = 1$ are $5, 7, 11$.
So, $k \odot (m \odot u) = (km) \odot u$ for at least the triplets $(k, 1, 5), (k, 1, 7)$ and $(k, 1, 11)$.

(3) We want to show that, there exist at least a triplet (k, m, u) with $u \in V$ and $k, m \in K$, such that $(k + m) \odot u = k \odot u \oplus m \odot u$.
Consider, $(k + m) \odot u = (k + m)u^2 = ku^2 + mu^2$ and
$k \odot u \oplus m \odot u = ku^2 \oplus mu^2 = \frac{2ku^2+3mu^2}{2}$.
Equating these we have

$$ku^2 + mu^2 = \frac{2ku^2 + 3mu^2}{2},$$

which gives

$$mu^2 = 0 \implies u^2 = 0$$

$$\therefore u = 0 \text{ and } 6.$$

This shows that only the triplets $(k, m, 0)$ and $(k, m, 6)$ can verify $(k + m) \odot u = k \odot u \oplus m \odot u$.

(4) We want to show that there exists at least a triplet (k, u, v) with $u, v \in V$ and $k \in K$, such that $k \odot (u \oplus v) = k \odot u \oplus k \odot v$.
Now, consider $k \odot (u \oplus v) = k \odot \frac{(2u+3v)}{2} = k\frac{(2u+3v)^2}{4} = \frac{4ku^2+12kuv+9kv^2}{4} = \frac{4ku^2+9kv^2}{4}$ and $k \odot u \oplus k \odot v = ku^2 \oplus kv^2 = \frac{2ku^2+3kv^2}{2}$.
Equating these we have

$$4ku^2 + 9kv^2 = 4ku^2 + 6kv^2,$$

which gives

$$9kv^2 = 6kv^2$$

$$3kv^2 = 0 \implies v^2 = 0.$$

So,

$$v = 0, 6.$$

This shows that only the triplets $(k, u, 0)$ and $(k, u, 6)$ can verify $k \odot (u \oplus v) = k \odot u \oplus k \odot v$.

(5) We want to show that there exists at least a $u \in V$ such that $1 \odot u = u$.
We have that the only elements of V that satisfy $1 \odot u = u^2 = u$ are 4 and 9.

Accordingly, $(V, \oplus, \odot)$ is a weak NeutroVectorSpace over a field $K = \mathbb{R}$.

Example 3.10. Let $X = \{a, b, c, d, e\}$ be a universe of discourse and let $\mathbb{K} = \{a, b, c, d\}$. Let $\oplus$ and $\odot$ be the binary operations defined on $\mathbb{K}$ as shown in the Cayley tables below.

TABLE 1. (a) Cayley table for the binary operation $''\oplus''$ and (b) Cayley table for the binary operation $''\odot''$

$\oplus$	a	b	c	d
a	a	c	a	c
b	b	d	b	d
c	c	a	c	a
d	d	b	d	b or d

(a)

$\odot$	a	b	c	d
a	a	c	a	c
b	b	d	b	d
c	a	c	a	c
d	b	d	b	d

(b)

(1) $(\mathbb{K}, \oplus, \odot)$ is a trivial NeutroField.
(2) $(\mathbb{K}, \oplus, \odot)$ taken over itself is a strong NeutroVector Space.

(1) To show that $(\mathbb{K}, \oplus, \odot)$ is a trivial NeutroField we proceed as follows:

(a) $(\mathbb{K}, \oplus)$ is a NeutroAbelianGroup. It is clear from the table that;

(i) $d \oplus d =$ b or d.

So, the composition $d \oplus d$ is indeterminate with 6.25% degree of indeterminacy and all other compositions are true with 93.75% degree of truth. Hence $''\oplus''$ is NeutroClosed.

(ii) $d \oplus (c \oplus a) = (d \oplus c) \oplus a = d$,

$a \oplus (c \oplus d) = a$, but $(a \oplus c) \oplus d = c \neq a$. Hence $''\oplus''$ is NeutroAssociative.

(iii) Since only the duplet $(x, x) \in \mathbb{K}$ verify commutativity, for $x = a, b, c \in \mathbb{K}$.

Hence $''\oplus''$ is NeutroCommutative.

(iv) Let N_x and I_x represent additive neutral and inverse element respectively with respect to any element $x \in \mathbb{K}$.

Then $N_a = a$, $N_c = c$ and N_b, N_d do not exist.

$I_a = a,\ I_c = c$ and $I_b,\ I_d$ do not exist.

Hence, $(\mathbb{K}, \oplus)$ is a NeutroAbelianGroup.

(b) $(\mathbb{K}, \odot)$ is a NeutroAbelianGroup. It is clear from the table that;

(i) $(a \odot b) \odot d = a \odot (b \odot d) = c$,

$(b \odot c) \odot d = d$ but $b \odot (c \odot d) = b \neq d$. Hence $''\odot''$ is NeutroAssociative.

(ii) $a \odot c = c \odot a = a$,

$a \odot b = c$ but $b \odot a = b$. Hence, $\odot$ is NeutroCommutative.

(iii) Let U_x and I_x represent multiplicative neutral and inverse element(s) respectively with respect to any element $x \in \mathbb{K}$.

Then, $U_a = a$ and c. $U_d = b$ and d. U_b and U_c do not exist.

$I_a = a$ and c. $I_d = b$ and d. I_c and I_b do not exist.

Hence, $(\mathbb{K}, \odot)$ is a NeutroAbelianGroup.

(c) Now, we show that $\odot$ is distributive over $\oplus$. It is clear from the table that ;

(i) $a \odot (b \oplus c) = a \odot b \oplus a \odot c = c$,

$b \odot (a \oplus b) = b$, but $b \odot a \oplus b \odot b = d \neq b$. So, $''\odot''$ is left NeutroDistributive over $''\oplus''$.

(ii) $(b \oplus c) \odot a = b \odot a \oplus c \odot a = b$,

$(c \oplus b) \odot d = c$, but $c \odot d \oplus b \odot d = a \neq c$. So, $''\odot''$ is right NeutroDistributive over $''\oplus''$. Hence, $''\odot''$ is NeutroDistributive over $''\oplus''$.

Accordingly, $(\mathbb{K}, \oplus, \odot)$ is a trivial NeutroField.

(2) That $(\mathbb{K}, \oplus, \odot)$ is a strong NeutroVector Space over itself, follows easily from all the properties established in solution of 1 above.

Proposition 3.11. *Every NeutroField taken over itself is a strong NeutroVectorSpace.*

Proof. The proof follows from Example 3.10 . □

4. **A Study of a Class of NeutroVectorSpaces**

In this section, we shall consider a particular class of NeutroVectorSpaces $(NV, +, \cdot)$ where

(1) $(NV, +)$ is a classical abelian group.

(2) $S1$ is totally true for all $v \in V$ and $a \in K$.

(3) $S2, S3, S4$ and $S5$ are either partially true or partially indeterminate or partially false for some elements of V and K.

We shall refer to this class of NeutroVectorSpace as NeutroVectorSpace of type $4S$ (i.e., 4 of its scalar multiplication axioms are NeutroAxioms).

Example 4.1. Let $K = \mathbb{Z}_p$ (where p is prime) and $NV = \mathbb{Z}_8$. Define $\oplus$ and $\odot$ by

$$a \oplus b = a + b \text{ and } k \odot a = a^2 + ka.$$

Where $''+''$ is addition modulo 8 .

Then $(NV, \oplus, \odot)$ is a weak NeutroVectorSpace of type $4S$ over the field $K = \mathbb{Z}_p$.

It is easy to show that $(NV, \oplus)$ is an abelian group. Also, it is easy to see that $S1$ holds.

Now it remains to show that $NS2, NS3, NS4$ and $NS5$ hold.

(1) We want to show that there exists at least a triplet (k, x, y) with $k \in K$ and $x, y \in NV$ such that

$$k \odot (x \oplus y) = k \odot x \oplus k \odot y.$$

Now, $k \odot (x \oplus y) = k \odot (x + y) = (x + y)^2 + k(x + y) = x^2 + y^2 + 2xy + kx + ky$.

And $k \odot x \oplus k \odot y = (x^2 + kx) \oplus (y^2 + ky) = x^2 + y^2 + kx + ky$.

$$\therefore x^2 + y^2 + 2xy + kx + ky = x^2 + y^2 + kx + ky$$

$$\Longrightarrow xy = 0.$$

Hence $x = 0$ or $y = 0, (x, y) = (2, 4), (x, y) = (4, 2), (x, y) = (4, 6)$ and $(x, y) = (6, 4)$.

This shows that only the triplets $(k, x, 0), (k, 0, y), (k, 2, 4), (k, 4, 2), (k, 4, 6)$ and $(k, 6, 4)$ can verify $NS2$.

(2) We want to show that there exists at least a triplet (k, m, u) with $k, m \in K$ and $u \in NV$ such that

$$(k + m) \odot u = k \odot u + m \odot u.$$

$(k + m) \odot u = u^2 + (k + m)u = u^2 + ku + mu$ and $k \odot u \oplus m \odot u = 2u^2 + ku + mu$.

Then, we have

$$u^2 + ku + mu = 2u^2 + ku + mu$$

$$\Longrightarrow u^2 = 0.$$

$$\therefore u = 0 \text{ and } 4.$$

Hence, only the triplet $(k, m, 0)$ and $(k, m, 4)$ can verify $NS3$.

(3) We want to show that there exists at least a triplet (k, m, u) with $u \in NV$ and $k, m \in K$, such that $k \odot (m \odot u) = (km) \odot u$.

Now, consider $(km) \odot u = u^2 + (km)u = u^2 + kmu$ and

$$k \odot (m \odot u) = k \odot (u^2 + mu) = (u^2 + mu)^2 + k(u^2 + mu) = u^4 + 2mu^3 + m^2u^2 + ku^2 + kmu.$$

Equating these we have

$$u^2 + 2mu + m^2 + k = 1.$$

Since we need at least a triplet, take $k = 1$, then we have $u^2 + 2um + m^2 = 0$ and this gives $u = -m$.

Hence, at least the triplet $(1, m, -m)$ satisfies $NS4$.

(4) We want to show that there exists at least $v \in NV$ such that $1 \odot v = v$.

From definition of $\odot$ we have that the only elements of NV that satisfy

$$1 \odot u = v^2 + v = v \text{ are } 0 \text{ and } 4.$$

Hence $(NV, \oplus, \odot)$ is a weak NeutroVectorSpace of type $4S$ over the field $K = \mathbb{Z}_p$.

Example 4.2. Let $X = \{a, b, c, d, e\}$ be a universe of discourse. Let $\mathbb{K} = \{a, b, c, d\}$ be the Neutrofield defined in Example 3.10 and let $NV = \left\{v_1 = \frac{a}{e}, v_2 = \frac{b}{e}, v_3 = \frac{c}{e}, v_4 = \frac{d}{e}\right\}$.

Define on NV the binary operation $+'$ as in the table below and scalar multiplication $\star$ by

$$a \star v = \frac{a \odot x}{e},$$

here $\odot$ is the multiplication in $\mathbb{K}$ defined in Table 1 (b) for all elements in $\mathbb{K}$.

Table 2. Cayley table for the binary operation $+'$

$+'$	v_1	v_2	v_3	v_4
v_1	v_1	v_2	v_3	v_4
v_2	v_2	v_3	v_4	v_1
v_3	v_3	v_4	v_1	v_2
v_4	v_4	v_1	v_2	v_3

Then $(NV, +', \star)$ is a strong NeutroVectorSpace of type $4S$ over $\mathbb{K}$.

It is clear from Table 2 that $(NV, +')$ is an abelian group. Also, it is easy to see that $S1$ holds. Now it remains to show that $NS2, NS3, NS4$ and $NS5$ hold.

It can be seen from Table 1 (a), Table 1 (b) and Table 2 that ;

(1) for $c \in \mathbb{K}$ and $v_2, v_3 \in NV$,

$$\begin{aligned} c \star (v_2 +' v_3) &= c \star (v_2 +' v_3) \\ &= c \star v_4 \text{ from Table 2} \\ &= \tfrac{c \odot d}{e} \\ &= \tfrac{c}{e} \because c \odot d = c, \text{ from Table 1 (b)} \\ &= v_3. \end{aligned}$$

$$\begin{aligned} c \star v_2 +' c \star v_3 &= \tfrac{c \odot b}{e} +' \tfrac{c \odot c}{e} \\ &= \tfrac{c}{e} +' \tfrac{a}{e} \text{ from Table1 (b)} \\ &= v_3 +' v_1 \\ &= v_3. \end{aligned}$$

$$\implies c \star (v_2 +' v_3) = c \star v_2 +' c \star v_3 = v_3,$$

and for $b \in \mathbb{K}$ and $v_3, v_4 \in NV$,

$$b \star (v_3 +' v_4) = v_4 \text{ but } b \star v_3 +' b \star v_4 = v_1 \neq v_4.$$

This shows that $NS2$ holds.

(2) for $a, c \in \mathbb{K}$ and $v_2 \in NV$,

$$(a \oplus c) \star v_2 = a \star v_2 +' c \star v_2 = v_3,$$

and for $a, b \in \mathbb{K}$ and $v_4 \in NV$,

$$(a \oplus b) \star v_4 = v_3 \text{ but } a \star v_4 +' b \star v_4 = v_2 \neq v_3.$$

This shows that $NS3$ holds.

(3) for $a, b \in \mathbb{K}$ and $v_4 \in NV$,

$$(a \odot b) \star v_4 = a \star (b \star v_4) = v_3,$$

and for $b, c \in \mathbb{K}$ and $v_4 \in NV$

$$(b \odot c) \star v_4 = v_4 \text{ but } b \star (c \star v_4) = v_2 \neq v_4.$$

This shows that $NS4$ holds.

(4) We know from Table 1 that NeutroUnityElements in K are $U_a = a, c$ and $U_d = b, d$.

Now, suppose we consider the NeutroUnityElement $U_d = b$ only.

We have that $b \star v_4 = v_4$ and $b \star v_3 = v_2 \neq v_3$.

This shows that $NS5$ holds.

Hence, we have that $(NV, +', \star)$ is a strong NeutroVectorSpace of type $4S$ over the NeutroField $\mathbb{K}$. From now on, every weak(strong) NeutroVectorSpaces of type $4S$ over $K(NK)$ will simply be called a weak(strong) NeutroVectorSpace over $K(NK)$.

Proposition 4.3. *Let $(NV, +'_1, \star_1)$ and $(NH, +'_2, \star_2)$ be two weak NeutroVectorSpace over the field K and let*

$$NV \times NH = \{(v, h) : v \in NV \text{ and } h \in NH\},$$

for $x = (v_1, h_1)$, $y = (v_2, h_2) \in NV \times NH$ and $k \in K$ define :

$$x \oplus y = ((v_1 +'_1 v_2), (h_1 +'_2 h_2),$$

$$k \odot x = (k \star_1 v_1, k \star_2 v_2).$$

Then $(NV \times NH, \oplus, \odot)$ *is a weak NeutroVectorSpace over the field* K.

Proof. Since $(NV, +_1')$ and $(NH, +_2')$ are classical abelian groups, then it can be shown that $(NV \times NH, \oplus)$ is a classical abelian group. Also, it is easy to see that $S1$ is true in $(NV \times NH)$.
Now, it remains to show that $NS2 - NS5$ hold in $NV \times NH$.

(1) There exists at least a triplet $(k, (v_1, h_1), (v_2, h_2))$ with $(v_1, h_1), (v_2, h_2) \in NV \times NH$ and $k \in K$, such that

$$\begin{aligned} k \odot ((v_1, h_1) \oplus (v_2, h_2)) &= k \odot (v_1 +_1' v_2,\ h_1 +_2' h_2) \\ &= (k \star_1 (v_1 +_1' v_2),\ k \star_2 (h_1 +_2' h_2)) \\ &= (k \star_1 v_1 +_1' k \star_1 v_2,\ k \star_2 h_1 +_2' k \star_2 h_2)\ \because NS2 \text{ holds in } NV \text{and } NH. \\ &= (k \star_1 v_1, k \star_2 h_1) \oplus\ (k \star_1 v_2, k \star_2 h_2) \\ &= k \odot (v_1, h_1) \oplus\ k \odot (v_2, h_2). \end{aligned}$$

Also, there exists at least a triplet $(m, (a_1, b_1), (a_2, b_2))$ with $(a_1, b_1), (a_2, b_2) \in NV \times NH$ and $m \in K$, such that

$$\begin{aligned} m \odot ((a_1, b_1) \oplus (a_2, b_2)) &= m \odot (a_1 +_1' a_2,\ b_1 +_2' b_2) \\ &= (m \star_1 (a_1 +_1' a_2),\ m \star_2 (b_1 +_2' b_2)) \\ &\neq (m \star_1 a_1 +_1' m \star_1 a_2,\ m \star_2 b_1 +_2' m \star_2 b_2)\ \because NS2 \text{ holds in } NV \text{and } NH. \\ &= (m \star_1 a_1, m \star_2 b_1) \oplus\ (m \star_1 a_2, m \star_2 b_2) \\ &= m \odot (a_1, b_1) \oplus\ m \odot (a_2, b_2). \end{aligned}$$

Hence, $NS2$ holds in $NV \times NH$.

(2) There exists at least a triplet $(k, m, (v, h))$ with $k, m \in K$ and $(v, h) \in NV \times NH$ such that

$$\begin{aligned} (k + m) \odot (v, h) &= ((k + m) \star_1 v,\ (k + m) \star_2 h) \\ &= ((k \star_1 v +_1' m \star_1 v),\ (k \star_2 h +_2' m \star_2 h))\ \because NS3 \text{ holds in } NV \text{and } NH. \\ &= ((k \star_1 v,\ k \star_2 h) \oplus (m \star_1 v, m \star_2 h)) \\ &= k \odot (v, h) \oplus m \odot (v, h). \end{aligned}$$

Also, there exists at least a triplet $(p, q, (a, b))$ with $p, q \in K$ and $(a, b) \in NV \times NH$ such that

$$\begin{aligned} (p + q) \odot (a, b) &= ((p + q) \star_1 a,\ (p + q) \star_2 b) \\ &\neq ((p \star_1 a +_1' q \star_1 a),\ (p \star_2 b +_2' q \star_2 b))\ \because NS3 \text{ holds in } NV \text{and } NH. \\ &= ((p \star_1 a,\ p \star_2 b) \oplus (q \star_1 a, q \star_2 b)) \\ &= p \odot (a, b) \oplus q \odot (a, b). \end{aligned}$$

Hence, $NS3$ holds in $NV \times NH$.

(3) There exists at least a triplet $(k, m, (v, h))$ with $k, m \in K$ and $(v, h) \in NV \times NH$ such that

$$\begin{aligned} (km) \odot (v, h) &= ((km) \star_1 v, (km) \star_2 h) \\ &= (k \star_1 (m \star_1 v), k \star_2 (m \star_2 h))\ \because NS4 \text{ holds in } NV \text{and } NH. \\ &= k \odot ((m \star_1 v), (m \star_2 h)) \\ &= k \odot (m \odot (v, h)). \end{aligned}$$

Also, there exists at least a triplet $(p, q, (a, b))$ with $p, q \in K$ and $(a, b) \in NV \times NH$ such that

$$\begin{aligned} (pq) \odot (a, b) &= ((pq) \star_1 a, (pq) \star_2 b) \\ &\neq (p \star_1 (q \star_1 a), p \star_2 (q \star_2 b)) \because NS4 \text{ holds in } NV \text{and } NH. \\ &= p \odot ((q \star_1 a), (p \star_2 b)) \\ &= p \odot (q \odot (v, h)). \end{aligned}$$

Hence, $NS4$ holds in $NV \times NH$.

(4) There exists $(v, h) \in NV \times NH$ such that

$$\begin{aligned} 1 \odot (v, h) &= (1 \star v, 1 \star h) \\ &= (v, h). \because NS5 \text{ holds in } NV \text{and } NH. \end{aligned}$$

Also, there exists $(a, b) \in NV \times NH$ such that

$$\begin{aligned} 1 \odot (a, b) &= (1 \star_1 a, 1 \star_2 b) \\ &\neq (a, b). \because NS5 \text{ holds in } NV \text{and } NH. \end{aligned}$$

Accordingly, $(NV \times NH, \oplus, \odot)$ is a weak NeutroVectorSpace over the field K. □

Proposition 4.4. *Let $(NV, +_1', \star_1)$ be a weak NeutroVectorSpace over the field K and let $(H, +, \cdot)$ be a classical vector space over the same field K and let*

$$NV \times H = \{(v, h) : v \in NV \text{ and } h \in H\}$$

and for $x = (v_1, h_1)$, $y = (v_2, h_2) \in NV \times H$ and $k \in K$ define :

$$x \oplus y = ((v_1 +_1' v_2), (h_1 + h_2) \text{ and } k \odot x = (k \star v_1, k \cdot v_2).$$

Then $(NV \times H, \oplus, \cdot)$ is a weak NeutroVectorSpace over the field K.

Proof. The proof is similar to the proof of Proposition 4.3 . □

Proposition 4.5. *Let $(NV, +_1', \star_1)$ and $(NH, +_2', \star_2)$ be two strong NeutroVectorSpaces over the NeutroField NK and let*

$$NV \times NH = \{(v, h) : v \in NV \text{ and } h \in NH\}$$

and for $x = (v_1, h_1)$, $y = (v_2, h_2) \in NV \times NH$ and $k \in NK$ define :

$$x \oplus y = ((v_1 +_1' v_2), (h_1 +_2' h_2) \text{ and } k \odot x = (k \star_1 v_1, k \star_2 v_2).$$

Then $(NV \times NH, \oplus, \odot)$ is a strong NeutroVectorSpace over the NeutroField NK.

Proof. The proof follows similar approach as the proof of Proposition 4.3. □

Definition 4.6. Let NV be a NeutroVectorSpace. Then NW is a NeutroSubspace of NV if and only if NW is a subset of NV, and NW is itself a NeutroVectorSpace with the same operations as in NV.

Example 4.7. Let $(NV, \oplus, \odot)$ be a weak NeutroVectorSpace of Example 4.1 and let $NW = 2\mathbb{Z}_8$ be a subset of NV. Following the approach in Example 4.1, it can be shown that $(NW, \oplus, \odot)$ is a weak NeutroVectorSpace over the field $\mathbb{Z}_p$. Hence NW is a weak NeutroSubspace of NV.

Example 4.8. Let $(NV, +', \star)$ be the strong NeutroVectorSpace of Example 4.2 , NV is the only strong NeutroSubspace of NV.

Remark 4.9. It should be noted that a NeutroVectorSpace NV of a particular class may contain a NeutroSubspace NW which belongs to another class.

We will illustrate Remark 4.9 with Example 4.10 .

Example 4.10. Let $(NV, +', \star)$ be the strong NeutroVectorSpace of Example 4.2 and let $NW = \{v_1, v_3\}$ be a subset of NV. Then $(NW, +', \star)$ is a NeutroVectorSpace of a class other than the class of NV.

We can see from Table 2 that $(NW, +')$ is an abelian group. Now, it can be seen from Table 1 (a), Table 1 (b) and Table 2 that ;

(1) $S1$ fails to hold. Since $\star$ is not true for all $a \in \mathbb{K}$ and $v \in NW$.
For instance, take $b \in \mathbb{K}$ and $v_3 \in NW$, then

$$b \star v_3 = \frac{b \cdot c}{e} = \frac{b}{e} = v_2 \notin V_3.$$

But if we take $a \in \mathbb{K}$ then for all $v \in NW$ we will have that $a \star v \in NW$.
Hence, $NS1$ holds in NW.

(2) for $a, b \in \mathbb{K}$ and $v_1, v_3 \in NW$, we have

$$a \star (v_1 +' v_3) = a \star v_1 +' a \star v_3 = v_1,$$

and $b \star (v_1 +' v_3) = v_2$ but $b \star v_1 +' b \star v_3 = v_4 \neq v_2$.
This shows that $NS2$ holds in NW.

(3) for $a, c \in \mathbb{K}$ and $v_3 \in NW$,

$$(a \oplus c) \star v_3 = a \star v_3 +' c \star v_3 = v_1,$$

and for $a, b \in \mathbb{K}$ and $v_3 \in NW$,

$$(a \oplus b) \star v_3 = v_1 \text{ but } a \star v_3 +' b \star v_3 = v_3 \neq v_1.$$

This shows that $NS3$ holds.

(4) for $a, c \in \mathbb{K}$ and $v_3 \in NW$,

$$(a \odot c) \star v_3 = a \star (c \star v_3) = v_1,$$

and for $a, b \in \mathbb{K}$ and $v_3 \in NW$,

$$(a \odot b) \star v_3 = v_1 \text{ but } a \star (b \star v_3) = v_3 \neq v_1.$$

This shows that $NS4$ holds.

(5) We know from Table 1 that NeutroUnityElements in K are $U_a = a, c$ and $U_d = b, d$.
Now, suppose we consider the NeutroUnityElement $U_a = c$ only.
We have that $c \star v_1 = v_1$ and $c \star v_3 = v_1 \neq v_3$.
This shows that $NS5$ holds.

Hence, we have that $(NW, +', \star)$ is a strong NeutroSubpace of type $5S$ over the NeutroField $\mathbb{K}$. This implies that the NeutroSubspace NW does not belong to the same class as NV.

Example 4.11. Let $NV = \mathbb{Z}_{12}$ and $K = \mathbb{Z}_p$. Define $\oplus$ and $\odot$ for all $u, v \in V$ and $k \in K$ by

$$u \oplus v = u + v \text{ and } k \odot v = v^2 + kv.$$

Where $''+''$ is addition mod 12.
Following the approach of Example 4.1 it can be shown that $(NV, \oplus, \odot)$ is a weak NeutroVectorSpace of type $4S$ over the field K.
Let $NW = 2\mathbb{Z}_{12}$ and $NH = 3\mathbb{Z}_{12}$ be two subsets of NV. Also, by following similar approach as in Example 4.1 it can be shown that $(NW, \oplus, \odot)$ and $(NH, \oplus, \odot)$ are weak NeutroSubspaces of NV.

Now consider the following :

(1) $NW + NH = \{0, 1, 2, \cdots, 11\} = NV$.
(2) $NW \cup NH = \{0, 2, 3, 4, 6, 8, 9, 10\}$.
(3) $NW \cap NH = \{0, 6\}$.

These show that $NW + NH$ is a NeutroSubspace of NV but $NW \cup NH$ and $NW \cap NH$ are not NeutroSubspaces of NV.
These observations are recorded in Proposition 4.12 .

Proposition 4.12. *Let NW and NH be any two weak NeutroSubspaces of a NeutroVectorSpace NV over a field K. Then*

(1) $NW + NH = \bigcup\{(w + h) : w \in NW \text{ and } h \in NU\}$ *is a NeutroSubspace of NV.*
(2) $NW \cap NU$ *is not necessarily a NeutroSubspace of NV.*
(3) $NW \cup NU$ *is not necessarily a NeutroSubspace of NV.*

Definition 4.13. Let NW be a weak(strong) NeutroSubspace of a weak(strong) NeutroVectorSpace NV over a field (NeutroField) $K(NK)$. The quotient NV/NW is defined by the set

$$\{v + NW : v \in NV\}.$$

Proposition 4.14. *Let $(NV, +', \star)$ be a weak NeutroVectorSpace and $(NW, +', \star)$ be a weak NeutroSubspace of NV. The quotient NV/NW is a weak NeutroVectorSpace over a field K if addition and multiplication are defined for all $\bar{u} = u + NW$, $\bar{v} = v + NW \in NV/NW$ and $k \in K$ as follows:*

$$\bar{u} \oplus \bar{v} = (u + NW) \oplus (v + NW) = (u +' v) + NW,$$

and

$$\alpha \odot \bar{u} = \alpha \odot (u + NW) = (\alpha \star u) + NW.$$

This weak NeutroVectorSpace $(NV/NW, \oplus, \odot)$ over a field K is called a weak NeutroQuotientSpace.

Proof. We can easily show that $\oplus$ and $\odot$ are well defined.
The proof that $(NV/NW, \oplus)$ is an abelian group follows similar approach as the proof in classical case.
Now it remains to show that $NS2, NS3, NS4$ and $NS5$ all hold.

(1) Since $NS2$ holds in NV, then there exist at least the triplets (k, u, v) and (m, a, b) with $u, v, a, b \in NV$ and $k, m \in K$ such that $k \star (u +' v) = k \star u +' k \star v$ and $m \star (a +' b) \neq m \star a +' m \star b$.
Let $\bar{u}, \bar{v}, \bar{a}, \bar{b} \in NV/NW$ and $k, m \in K(NK)$. Then

$$\begin{aligned} k \odot (\bar{u} \oplus \bar{v}) &= k \odot ((u +' v) + NW) \\ &= k \star (u +' v) + NW \\ &= (k \star u +' k \star v) + NW \\ &= (k \star u) + NW \oplus (k \star v) + NW \\ &= k \odot (u + NW) \oplus k \odot (v + NW) \\ &= k \odot \bar{u} \oplus k \odot \bar{v}. \end{aligned}$$

So, it implies $k \odot (\bar{u} \oplus \bar{v}) = k \odot \bar{u} \oplus k \odot \bar{v}$.
And also,

$$\begin{aligned} m \odot (\bar{a} \oplus \bar{b}) &= m \odot ((a +' b) + NW) \\ &= m \star (a +' b) + NW \\ &\neq (m \star a +' m \star b) + NW \\ &= (m \star a) + NW \oplus (m \star b) + NW \\ &= m \odot (a + NW) \oplus m \odot (b + NW) \\ &= m \odot \bar{a} \oplus m \odot \bar{b}. \end{aligned}$$

This implies $m \odot (\bar{a} \oplus \bar{b}) \neq m \odot \bar{a} \oplus m \odot \bar{b}$. Hence, we can conclude that $NS2$ holds in NV/NW.

(2) Since $NS3$ holds in NV, then there exist at least the triplets (k, m, u) and (p, q, v) with $u, v \in NV$ and $k, m, p, q \in K$ such that $(k + m) \star u = k \star u +' m \star u$ and $(p + q) \star v \neq p \star v +' q \star v$.
Let $\bar{u}, \bar{v} \in NV/NW$ and $k, m, p, q \in K(NK)$. Then

$$\begin{aligned} (k + m) \odot \bar{u} &= (k + m) \odot (u + NW) \\ &= (k + m) \star u + NW \\ &= (k \star u +' m \star u) + NW \\ &= (k \star u) + NW \oplus (m \star u) + NW \\ &= k \odot (u + NW) \oplus m \odot (u + NW) \\ &= k \odot \bar{u} \oplus m \odot \bar{u}. \end{aligned}$$

So, it implies $(k+m)\odot\bar{u} = k\odot\bar{u}\oplus m\odot\bar{u}$.

And also,

$$\begin{aligned}(p+q)\odot\bar{v} &= (p+q)\odot(v+NW)\\ &= (p+q)\star v + NW\\ &\neq (p\star v +' q\star v) + NW\\ &= (p\star v) + NW \oplus (q\star v) + NW\\ &= p\odot(v+NW)\oplus q\odot(v+NW)\\ &= p\odot\bar{v}\oplus q\odot\bar{v}.\end{aligned}$$

So, it implies $(p+q)\odot\bar{v} \neq p\odot\bar{v}\oplus q\odot\bar{v}$.

Hence, we can conclude that $NS3$ holds in NV/NW.

(3) Since $NS4$ holds in NV, then there exist at least the triplets (k,m,u) and (p,q,v) with $u,v\in NV$ and $k,m,p,q\in K$ such that $(km)\star u = k\star(m\star u)$ and $(pq)\star v \neq p\star(q\star v)$.

Let $\bar{u},\bar{v}\in NV/NW$ and $k,m,p,q\in K(NK)$. Then

$$\begin{aligned}(km)\odot\bar{u} &= (km)\odot(u+NW)\\ &= (km)\star u + NW\\ &= (k\star(m\star u)) + NW\\ &= k\odot((m\star u)+NW)\\ &= k\odot(m\odot(u+NW))\\ &= k\odot(m\odot\bar{u}).\end{aligned}$$

So, it implies $(km)\odot\bar{u} = k\odot(m\odot\bar{u})$.

And also,

$$\begin{aligned}(pq)\odot\bar{v} &= (pq)\odot(v+NW)\\ &= (pq)\star v + NW\\ &\neq (p\star(q\star v)) + NW\\ &= p\odot((q\star v)+NW)\\ &= p\odot(q\odot(v+NW))\\ &= p\odot(q\odot\bar{v}).\end{aligned}$$

So, it implies that $(pq)\odot\bar{v} \neq p\odot(q\odot\bar{v})$.

Hence, we can conclude that $NS4$ holds in NV/NW.

(4) In NV we have at least u and v such that $1\star u = u$ and $1\star v \neq v$.

So, in NV/NW there exist $\bar{u}$ and $\bar{v}$ such that

$$1\odot\bar{u} = 1\odot(u+NW) = (1\star u) + NW = u + NW = \bar{u}$$

and

$$1\odot\bar{v} = 1\odot(v+NW) = (1\star v) + NW \neq v + NW = \bar{v}.$$

So, it implies $1\odot\bar{u} = \bar{u}$ and $1\odot\bar{v} = \bar{v}$.

Hence, we can conclude that $NS5$ holds in NV/NW.

Accordingly, $(NV/NW,\oplus,\odot)$ is a weak NeutroVectorSpace over the field K. □

Remark 4.15. Let NV be weak(strong) NeutroVectorSpace of type $4S$ over a field(NeutroField) $K(NK)$ and let NW be a NeutroSubspace of NV. Then, the weak(strong) NeutroQuotient Space NV/NW over $K(NK)$ is not necessarily of type $4S$.

We illustrate Remark 4.15 by Example 4.16 .

Example 4.16. Let $(NV = \mathbb{Z}_{12}, +', \star)$ be a weak NeutroVectorSpace of type $4S$ over $K = \mathbb{Z}_p$ and let $(NW = 2\mathbb{Z}_{12}, +', \star)$ be a NeutroSubspace of NV. Where $+'$ is addition mod 12 and $\star$ is defined as

$$k \star v = v^2 +' kv$$

for all $v \in NV$ and $k \in K$.

Then for all $\bar{u}, \bar{v} \in NV/NW$ and $k \in K$ define the operation $\oplus$ and $\odot$ by

$$\bar{u} \oplus \bar{v} = (u +' v) + NW$$

and

$$k \odot \bar{u} = (k \star u) + NW.$$

Then $(NV/NW, \oplus, \odot)$ is a weak NeutroVectorSpace over K of type other than $4S$.

We know that $NV = \{0, 1, 2, \cdots, 11\}$ and $NW = \{0, 2, 4, 6, 8, 10\}$ then we have

$$NV/NW = \{NW, 1 + NW\}.$$

TABLE 3. Cayley table for the binary operation $\oplus$

$\oplus$	NW	$1 + NW$
NW	NW	$1 + NW$
$1 + NW$	$1 + NW$	NW

From Table 3 it is clear that $(NV/NW, \oplus)$ is an abelian group.

Now,

(1) $NS2$ fails to hold since for any triplet $(k, \bar{u}, \bar{v})$ we pick, with $k \in K$ and $\bar{u}, \bar{v} \in NV/NW$,

$$k \odot (\bar{u} \oplus \bar{v}) = k \odot \bar{u} \oplus k \odot \bar{v}$$

is always satisfied. This implies that $S2$ is totally true in NV/NW.

(2) There exists at least a triplet $(k, m, \bar{v})$ with $k, m \in K$ and $\bar{v} \in NV/NW$ such that

$$(k + m) \odot (\bar{v}) = k \odot \bar{v} \oplus m \odot \bar{v}.$$

Now,

$$(k + m) \odot (\bar{v}) = ((k + m) \star v) + NW = (v^2 +' kv +' mv) + NW$$

and

$$k \odot \bar{v} \oplus m \odot \bar{v} = ((k \star v) + NW) \oplus ((m \star v) + NW) = (2v^2 +' ku +' mv) + NW.$$

Equating these we have that $v^2 + NW = NW$ which implies $v^2 \in NW$.

So, only the triple (k, m, NW) satisfies $(k + m) \odot (\bar{v}) = k \odot \bar{v} \oplus m \odot \bar{v}$.

Hence, $NS3$ holds in NV/NW.

(3) There exists at least a triplet $(k, m, \bar{v})$ with $k, m \in K$ and $\bar{v} \in NV/NW$ such that

$$(km) \odot (\bar{v}) = k \odot (m \odot \bar{v}).$$

Now,

$$(km) \odot (\bar{v}) = ((km) \star v) + NW = (v^2 +' kmv) + NW$$

and

$$k \odot (m \odot \bar{v}) = k \odot ((m \star v) + NW) = (k \star (v^2 +' mv)) + NW = (v^4 +' 2v^3m +' m^2v^2 + kv^2 +' kmv) + NW.$$

Equating these we have that $(v^4 +' 2v^3m +' m^2v^2 + kv^2) + NW = v^2 + NW$ which implies $(v^2 +' 2vm +' m^2 +' k) + NW = 1 + NW$.

Since we needed at least a triplet, take $k = 1$, then we have $(v^2 +' 2vm +' m^2 +' 1) + NW = 1 + NW$ which gives $(v^2 +' 2vm +' m^2) + NW = NW$. So, we have that $(v^2 +' 2vm +' m^2) \in NW$. Then, at least the triplet $(1, m, NW)$ satisfies $(km) \odot (\bar{v}) = k \odot (m \odot \bar{v})$. Hence, $NS4$ holds in NV/NW.

(4) We can easily see that $1 \odot NW = NW$ and

$$1 \odot (1 + NW) = (1 \star 1) + NW = 2 + NW = NW \neq 1 + NW.$$

Hence, $NS5$ holds in NV/NW.

Accordingly, we have that $(NV/NW, \oplus, \odot)$ is a weak NeutroVectorSpace of type $3S$ over K.
This implies that the NeutroQuotient Space $(NV/NW, \oplus, \odot)$ does not belong to the class of NeutroVectorSpace NV.

5. **Conclusions**

In this paper, we have for the first time introduced the concept of NeutroVectorSpaces. Specifically, a class of NeutroVectorSpaces called of type $4S$ was investigated and some of their elementary properties and examples were presented. It was shown that NeutroVectorSpaces of type $4S$ contained NeutroSubspaces of other types and that the intersections of NeutroSubspaces of type $4S$ are not necessarily NeutroSubspaces. Also, it was shown that if NV is a NeutroVectorSpace of a particular type and NW is a NeutroSubspace of NV, the NeutroQuotientSpace NV/NW does not necessarily belong to the same type as NV. We hope to continue this work in our next paper to be titled "NeutroVectorSpaces II".

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Weak LA-hypergroups; Neutrosophy, Enumeration and Redox Reaction

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Abstract. The main motivation of this article is to introduce the theme of Neutrosophic triplet(NT) H_v-LA-Groups. This inspiration is recieved from the structure of weak non-associative Neutrosophic triplet(NT) structures. For it, firstly, we define that each element x have left neut(x) and left anti(x) , which may or may not unique. We further introduce the notion of neutrosophic triplet Hv-LA-subgroups and neutrosophic weak homomorphism on NT H_v-LA-Group. Secondly, presented NT H_v-LA-Group and develop two Mathematica Packages which help to check the left invertive law, weak left invertive law and reproductive axiom. Finally established a numerical example to validate the proposed approach in chemistry using redox reactions.

Keywords: H$_v$ LA-groups, NT sets, Neutro weak homomorphism, Mathematica Packages, Chemical applications,

1. **Introduction**

Neutrosophic logic: Neutrosophy is the new branch of philosophy that studies the origin and scope of neutralities, as well as their interaction with different ideational spectra. Smarandache used the idea of neutrosophic set. He defined the theme of t-membership, i- membership and f-membership, so neutrosophic logic generalize all previous versions, see [1], [2], [3]. Many researchers have studied neutrosophic cubic set, complex neutrosophic cubic set, N-cubic set and their applications in real life problems, see [52–55]. Further Abdel-Basset et. al., use neutrosophic set in different direction and discuss their use in real life probems [56–60] More about the neutrosophic algebraic structures we refer the reader [4–6] and [7–12]. For the NT groups see [13–18].

Hyperstructures theory: In 1934, Marty [19] introduced the theme of hyperstructures. More about the hyperstructures we refer the reader [20–22]. The idea of weak structure, which is known as H_v-structure is introduced by Vougiouklis [23], see also [24–31]. In 2007 Davvaz and Fotea mainly dedicated to the study of hyperring theory [32]. Davvaz and Vougiouklis [33], published recently a new book having title "A walk through weak hyperstructures, Hv-Structures" with some interesting applications of hyperstructures.

Left Invertive Structures: Kazim and Naseerudin [34] laid the idea of left almost semigroup (denoted by LA-semigroup). Afterwards, Mushtaq [35] and some other researcher, further worked in detail on the structure of LA-semigroup, see papers [36–42]. Hila and Dine [43] in 2011, furnished the idea of LA-semihypergroup. More detail can be seen in [44], [45], [46], [47], [48], [49], [50], [51].

Our Approach: This paper is the continuation of our published paper [18] and it consists of 6 sections. We arrange this work as: In section 2, we collected some of the relevant material after the introduction. In section 3, we give a new class of algebraic hyperstructure known as NT H_v-LA-Group, which is the main theme of LA-Group, LA-hypergroup, H_v-LA-Group. In NT H_v-LA-Group each element k have left neut(k) and left anti(k), which may or may not unique. We also define the neutro weak homomorphism on NT H_v-LA-Group. Moreover, we discuss many interesting properties of NT H_v-LA-Groups. In section 4, we provide the construction of NT H_v-LA-Groups with the two Mathematica Packages which help to check the left invertive law, weak left invertive law and reproductive axiom. In section 5, we present the application of propose structure in chemical reactions. In section 6, we end with the concluding remarks.

2. Preliminaries

In this section, we added some basic definition and result, which helped to prove the result of our proposed structure.

Definition 2.1. [44] "A hypergroupoid $(\aleph, \circ)$ is called LA-semihypergroup, if it satisfies the following law

$$(\flat_1 \circ \flat_2) \circ \flat_3 = (\flat_3 \circ \flat_2) \circ \flat_1 \text{ for all } \flat_1, \flat_2, \flat_3 \in \aleph.$$

"

Example 2.2. [44] "Let $\aleph = Z$ if we define $\flat_1 \circ \flat_2 = \flat_2 - \flat_1 + 3Z$, where $\flat_1, \flat_2 \in Z$. Then $(\aleph, \circ)$ become LA-semih ypergroup."

Definition 2.3. [24] "The hyperoperation $* : \aleph \times \aleph \longrightarrow P^*(\aleph)$ is called weakly associative hyperoperation (abbreviated as WASS) if for any $\flat_1, \flat_2, \flat_3 \in \aleph$

$$(\flat_1 * \flat_2) * \flat_3 \cap \flat_1 * (\flat_2 * \flat_3) \neq \phi$$

"

Definition 2.4. [24] "The hyperoperation is weakly commutative (abbreviated as COW) if for any $\flat_1, \flat_2 \in \aleph$

$$\flat_1 * \flat_2 \cap \flat_2 * \flat_1 \neq \phi$$

"

Definition 2.5. [47] "Let $\aleph$ be non-empty set and $*$ be hyperoperation on $\aleph$. Then $(\aleph, *)$ is called an $\aleph_v$-LA-semigroup, if it satisfies the weak left invertive law for all $\flat_1, \flat_2, \flat_3 \in \aleph$

$$(\flat_1 * \flat_2) * \flat_3 \cap (\flat_3 * \flat_2) * \flat_1 \neq \phi$$

"

Example 2.6. [47] "Let $\aleph = (0, \infty)$ we define $\flat_1 * \flat_2 = \left\{\frac{\flat_2}{\flat_1+1}, \frac{\flat_2}{\flat_1}\right\}$ where $\flat_1, \flat_2 \in \aleph$. Then for all $\flat_1, \flat_2, \flat_3 \in \aleph$. Then for all $\flat_1, \flat_2, \flat_3 \in \aleph$ satisfies $(\flat_1 * \flat_2) * \flat_3 \cap (\flat_3 * \flat_2) * \flat_1 \neq \phi$. Hence $(\aleph, *)$ is an H_v-LA-semigroup."

3. Neutrosophic Triplet(NT) H$_v$-LA-Groups

In this section, we define a new class of hyper algebraic structure known as NT H$_v$-LA-group and discuss some results on NT H$_v$-LA-group.

Definition 3.1. Let $(\aleph, *)$ be a left (resp., right, pure left, pure right) NT set. Then $\aleph$ is called left (resp., right, pure left , pure right) NT H$_v$-LA-group, if it satisfies the following axioms,

(1) $(\aleph, *)$ is well defined,

(2) $(\aleph, *)$ satisfies the weak left invertive law, i.e, $(\flat_1 * \flat_2) * \flat_3 \cap (\flat_3 * \flat_2) * \flat_1 \neq \phi$ for all $\flat_1, \flat_2, \flat_3 \in \aleph$,

(3) $\aleph * \flat_1 = \aleph = \aleph * \flat_1$ for all $\flat_1 \in \aleph$.

Example 3.2. Let $\aleph = \{\flat_1, \flat_2, \flat_3\}$ be a finite set. The hyperoperation $*$ is defined in Table-1

$*$	$\flat_1$	$\flat_2$	$\flat_3$
$\flat_1$	$\flat_1$	$\{\flat_1, \flat_2\}$	$\{\flat_1, \flat_3\}$
$\flat_2$	$\flat_3$	$\{\aleph\}$	$\{\flat_1, \flat_2\}$
$\flat_3$	$\flat_2$	$\{\flat_1, \flat_3\}$	$\{\aleph\}$

Table-1, neutrosophic triplet H$_v$**-LA-group**

Here all elements of $\aleph$ satisfy the weak left invertive law. Also left invertive law is not hold in $\aleph$, i.e.

$$\aleph = (\flat_1 * \flat_2) * \flat_3 \neq (\flat_3 * \flat_2) * \flat_1 = \{\flat_1, \flat_2\}.$$

Alike, associative law is not hold in $\aleph$ i.e.

$$\aleph = (\flat_3 * \flat_3) * \flat_1 \neq \flat_3 * (\flat_3 * \flat_1) = \{\flat_1, \flat_3\}.$$

Even, weak associative law is not valid here

$$\{\flat_2\} = (\flat_2 * \flat_1) * \flat_1 \cap \flat_2 * (\flat_1 * \flat_1) = \{\flat_3\} = \phi.$$

Here $(\flat_1, \flat_1, \flat_1)$, $(\flat_2, \flat_1, \flat_2)$, $(\flat_3, \flat_1, \flat_3)$ are left NT sets. Hence $(\aleph, *)$ is a NT H$_v$-LA-group.

Proposition 3.3. *Let* $(\aleph, *)$ *be a pure right NT* H_v*-LA-group. Then* $neut(\flat_1) * \flat_2 = neut(\flat_1) * \flat_3$ *if* $anti(\flat_1) * \flat_2 = anti(\flat_1) * \flat_3$ *for all* $\flat_1, \flat_2, \flat_3 \in \aleph$.

Proof. Suppose $(\aleph, *)$ is a pure right NT H$_v$-LA-group and $anti(\flat_1) * \flat_2 = anti(\flat_1) * \flat_3$ for $\flat_1, \flat_2, \flat_3 \in \aleph$. Multiply $\flat_1$ to the left side of $(\flat_1 * anti(\flat_1) * \flat_2 = (\flat_1 * anti(\flat_1)) * \flat_3$,

$$(\flat_1 * anti(\flat_1)) * \flat_2 = (\flat_1 * anti(\flat_1)) * \flat_3$$

$$neut(\flat_1) * \flat_2 = neut(\flat_1) * \flat_3 \text{ (because } neut(\flat_1) = \flat_1 * anti(\flat_1)\text{)}.$$

Therefore, $neut(\flat_1) * \flat_2 = neut(\flat_1) * \flat_3$. □

Theorem 3.4. *Let* $(\aleph, *)$ *be a pure right NT* H_v*-LA-group. Then* $neut(\flat_1) * neut(\flat_1) = neut(\flat_1)$.

Proof. Consider $neut(\flat_1) * neut(\flat_1) = neut(\flat_1)$. Multiply first with $\flat_1$ to the right, i.e.,

$$(\flat_1 * (neut(\flat_1)) * neut(\flat_1) = \flat_1 * neut(\flat_1)$$

$$((\flat_1 * neut(\flat_1)) * neut(\flat_1)) = \flat_1$$

$$\flat_1 * neut(\flat_1) = \flat_1$$

$$\flat_1 = \flat_1.$$

This shows that $neut(\flat_1) * neut(\flat_1) = neut(\flat_1)$. □

Theorem 3.5. *Let* $(\aleph, *)$ *be a pure right NT* H_v*-LA-group. Then* $neut(\flat_1) * anti(\flat_1) = anti(\flat_1)$.

Proof. Let $(\aleph, *)$ be a pure right NT H$_v$-LA-group. Multiply $\flat_1$ to the left of both side $neut(\flat_1) * anti(\flat_1) = anti(\flat_1)$, i.e.

$$(\flat_1 * (neut(\flat_1)) * anti(\flat_1) = \flat_1 * anti(\flat_1)$$
$$\flat_1 * anti(\flat_1) = neut(\flat_1)$$
$$neut(\flat_1) = neut(\flat_1)$$
$$neut(\flat_1) = neut(\flat_1)$$

This shows that $neut(\flat_1) * anti(\flat_1) = anti(\flat_1)$. □

Theorem 3.6. *Let $(\aleph, *)$ be a pure left NT H$_v$-LA-group. Then $neut(anti(\flat_1)) = neut(\flat_1)$.*

Proof. Let $neut(anti(\flat_1)) = neut(\flat_1)$. If we put $anti(\flat_1) = \flat_2$, then

$$neut(\flat_2) = neut(\flat_1). \text{ Post multiply by } \flat_2$$
$$neut(\flat_2) * \flat_2 = neut(\flat_1) * \flat_2$$
$$\flat_2 = neut(\flat_1) * \flat_2$$
$$anti(\flat_1) = neut(\flat_1) * anti(\flat_1), \text{ as } \flat_2 = anti(\flat_1)$$
$$anti(\flat_1) = anti(\flat_1), \text{ By Theorem 3.5 } neut(\flat_1) * anti(\flat_1) = anti(\flat_1).$$

Hence $neut(anti(\flat_1)) = neut(\flat_1)$. □

Definition 3.7. A non-empty subset B of a left NT H$_v$-LA-group $(\aleph, *)$ is called a left NT H$_v$-LA-subgroup of $\aleph$, if B itself form NT H$_v$-LA-group under same hyperoperation defined in $\aleph$.

Example 3.8. Let $\aleph = \{\flat_1, \flat_2, \flat_3, \flat_4\}$ and the hyperoperation is defined in the Table-2

$*$	$\flat_1$	$\flat_2$	$\flat_3$	$\flat_4$
$\flat_1$	$\flat_1$	$\flat_2$	$\flat_3$	$\flat_4$
$\flat_2$	$\flat_3$	$\{\flat_1, \flat_3\}$	$\{\flat_2, \flat_3\}$	$\flat_4$
$\flat_3$	$\flat_2$	$\{\flat_1, \flat_3\}$	$\{\flat_1, \flat_3\}$	$\flat_4$
$\flat_4$	$\flat_4$	$\flat_4$	$\flat_4$	$\{\flat_1, \flat_2, \flat_3\}$

Table-2, neutrosophic triplet H$_v$**-LA-group**

Here $(\flat_1, \flat_1, \flat_1), (\flat_2, \flat_1, \flat_2), (\flat_3, \flat_2, \flat_2)$ and $(\flat_4, \flat_3, \flat_4)$ are NT sets. As all elements of $\aleph$ satisfy the weak left invertive law but $\aleph$ do not satisfies the left invertive law, associative law and weak associative law i.e.

$$\begin{aligned} \{\flat_1,\flat_3\} &= (\flat_2 * \flat_2) * \flat_3 \neq (\flat_3 * \flat_2) * \flat_2 = \{\flat_1,\flat_2,\flat_3\} \\ \text{and } \{\flat_1,\flat_3\} &= (\flat_2 * \flat_2) * \flat_3 \neq \flat_2 * (\flat_2 * \flat_3) = \{\flat_1,\flat_2,\flat_3\}. \\ \text{Also } \{\flat_2\} &= (\flat_2 * \flat_1) * \flat_1 \cap \flat_2 * (\flat_1 * \flat_1) = \{\flat_3\} = \phi. \end{aligned}$$

So $(\aleph, *)$ is a NT H$_v$-LA-group. Here $\flat = \{\flat_1,\flat_2,\flat_3\}$ is a NT H$_v$-LA-subgroup of $\aleph$.

Lemma 3.9. *If* $(\aleph, *)$ *is a NT* H_v*-LA group, then*

$$(\flat_1 * \flat_2) * (\flat_3 * \flat_4) \cap (\flat_1 * \flat_3) * (\flat_2 * \flat_4) \neq \phi,$$
$$\textit{hold for all } \flat_1,\flat_2,\flat_3,\flat_4 \in \aleph.$$

Proof. Let

$$\begin{aligned} &(\flat_1 * \flat_2) * (\flat_3 * \flat_4) \\ &= (\flat_1 * \flat_2) * g, \text{ where } g = (\flat_3 * \flat_4) \\ &= (\flat_1 * \flat_2) * g \cap (g * \flat_2) * \flat_1 \text{ by the weak left invertive law} \\ &= (\flat_1 * \flat_2) * g \cap (g * \flat_2) * \flat_1 \text{ by the weak-left invertive law} \\ &= (\flat_1 * \flat_2) * g \cap \{(g * \flat_2) * \flat_1\} \text{ by the weak-left invertive law} \\ &= (\flat_1 * \flat_2) * (\flat_3 * \flat_4) \cap \{((\flat_3 * \flat_4) * \flat_2) * \flat_1\}, \text{ where } g = (\flat_3 * \flat_4) \\ &= (\flat_1 * \flat_2) * (\flat_3 * \flat_4) \cap \{\{((\flat_3 * \flat_4) * \flat_2) \cap (\flat_2 * \flat_4) * \flat_3\} * \flat_1\} \\ &= (\flat_1 * \flat_2) * (\flat_3 * \flat_4) \cap \{((\flat_3 * \flat_4) * \flat_2) * \flat_1\} \cap \{((\flat_2 * \flat_4) * \flat_3) * \flat_1\}\} \\ &= (\flat_1 * \flat_2) * (\flat_3 * \flat_4) \cap \left\{ \begin{array}{c} ((\flat_3 * \flat_4) * \flat_2) * \flat_1 \cap (\flat_1 * \flat_2) * (\flat_3 * \flat_4)\} \\ \cap\{((\flat_2 * \flat_4) * \flat_3) * \flat_1 \cap (\flat_1 * \flat_3) * (\flat_2 * \flat_4)\} \end{array} \right\} \to (1) \end{aligned}$$

Now

$$
\begin{aligned}
&(\flat_1 * \flat_3) * (\flat_2 * \flat_4) \\
&= (\flat_1 * \flat_3) * g, \text{ where } g = (\flat_2 * \flat_4) \\
&= (\flat_1 * \flat_3) * g \cap (g * \flat_3) * \flat_1 \text{ by the weak left invertive law} \\
&= (\flat_1 * \flat_3) * g \cap (g * \flat_3) * \flat_1 \text{ by the weak-left invertive law} \\
&= (\flat_1 * \flat_3) * g \cap \{(g * \flat_3) * \flat_1\} \text{ by the weak-left invertive law} \\
&= (\flat_1 * \flat_3) * (\flat_2 * \flat_4) \cap \{((\flat_2 * \flat_4) * \flat_3) * \flat_1\}, \text{ where } g = (\flat_2 * \flat_4) \\
&= (\flat_1 * \flat_3) * (\flat_2 * \flat_4) \cap \{\{((\flat_2 * \flat_4) * \flat_3) \cap (\flat_3 * \flat_4) * \flat_2\} * \flat_1\} \\
&= (\flat_1 * \flat_3) * (\flat_2 * \flat_4) \cap \{((\flat_2 * \flat_4) * \flat_3) * \flat_1\} \cap \{((\flat_3 * \flat_4) * \flat_2) * \flat_1\}\} \\
&= (\flat_1 * \flat_3) * (\flat_2 * \flat_4) \cap \left\{ \begin{array}{c} ((\flat_2 * \flat_4) * \flat_3) * \flat_1 \cap (\flat_1 * \flat_3) * (\flat_2 * \flat_4)\} \\ \cap\{((\flat_3 * \flat_4) * \flat_2) * \flat_1 \cap (\flat_1 * \flat_2) * (\flat_3 * \flat_4)\} \end{array} \right\} \rightarrow (2)
\end{aligned}
$$

From (1) and (2) we have $(\flat_1 * \flat_2) * (\flat_3 * \flat_4) \cap (\flat_1 * \flat_3) * (\flat_2 * \flat_4) \neq \phi$, hold for all $\flat_1, \flat_2, \flat_3, \flat_4 \in \aleph$. This law is known as weak medial law. □

Proposition 3.10. *Let $(\aleph, \circ)$ be a NT H_v-LA-group with left identity e and $\phi \neq A \subseteq \aleph$. If $(A \circ (A \circ \flat_1)) \circ \flat_2 \cap (A \circ (A \circ \flat_2)) \circ \flat_1 \neq \phi$ $\forall \flat_1, \flat_2 \in \aleph$ and we define a hyperoperation $A_R^{\otimes}$ on $\aleph$ as $\flat_1 A_R^{\otimes} \flat_2 = (\flat_1 \circ \flat_2) \circ A$, then $(\aleph, A_R^{\otimes})$ become a NT H_v-LA-group.*

Proof. Let $\flat_1, \flat_2, \flat_3 \in \aleph$, we have

$$
\begin{aligned}
(\flat_1 A_R^{\otimes} \flat_2) A_R^{\otimes} \flat_3 &= ((\flat_1 \circ \flat_2) \circ A) A_R^{\otimes} \flat_3 \\
&= (((\flat_1 \circ \flat_2) \circ A) \circ \flat_3) \circ A \\
&= ((\flat_3 \circ A) \circ (\flat_1 \circ \flat_2)) \circ A \\
&= (A \circ (A \circ \flat_3)) \circ (\flat_2 \circ \flat_1) \\
&= \flat_2 \circ ((A \circ (A \circ \flat_3)) \circ \flat_1)
\end{aligned}
$$

and on the other hand

$$
\begin{aligned}
(\flat_3 A_R^{\otimes} \flat_2) A_R^{\otimes} \flat_1 &= ((\flat_3 \circ \flat_2) \circ A) A_R^{\otimes} \flat_1 \\
&= (((\flat_3 \circ \flat_2) \circ A) \circ \flat_3) \circ A \\
&= ((\flat_1 \circ A) \circ (\flat_3 \circ \flat_2)) \circ A \\
&= (A \circ (A \circ \flat_1)) \circ (\flat_2 \circ \flat_3) \\
&= \flat_2 \circ ((A \circ (A \circ \flat_1)) \circ \flat_3)
\end{aligned}
$$

but

$$\flat_2 \circ ((A \circ (A \circ \flat_3)) \circ \flat_1) \cap \flat_2 \circ ((A \circ (A \circ \flat_1)) \circ \flat_3) \neq \phi$$

for all $\flat_1, \flat_2, \flat_3 \in \aleph$. It follows that

$$(\flat_1 A_R^{\otimes} \flat_2) A_R^{\otimes} \flat_3 \cap (\flat_3 A_R^{\otimes} \flat_2) A_R^{\otimes} \flat_1 \neq \phi.$$

Next, we have

$$\flat_1 A_R^{\otimes} \aleph = (\flat_1 \circ \aleph) \circ A = \aleph \text{ also } H A_R^{\otimes} \flat_1 = (\aleph \circ \flat_1) \circ A = \aleph.$$

Hence $(\aleph, A_R^{\otimes})$ become an H_v-LA-group. □

Definition 3.11. Let $(\aleph_1, \circ)$ and $(\aleph_2, *)$ be two NT H_v-LA-groups. The map $f : \aleph_1 \longrightarrow \aleph_2$ is called neutro homomorphism, if for all $\flat_1, \flat_2 \in \aleph_1$, the following conditions hold,

1. $f(\flat_1 \circ \flat_2) \cap f(\flat_1) * f(\flat_2) \neq \phi$,
2. $f\,(neut\,(\flat_1)) \cap neut\,(f\,(\flat_1)) \neq \phi$,
3. $f\,(anti\,(\flat_1)) \cap anti\,(f\,(\flat_1)) \neq \phi$.

Example 3.12. Let $\aleph_1 = \{v_1, v_2, v_3\}$ and $\aleph_2 = \{\flat_1, \flat_2, \flat_3\}$ are two finite sets, where $(\aleph_1, *)$ and $(\aleph_2, \circ)$ are NT H_v-LA-groups, the hyperoperation is defined in following tables 3,4:

$*$	v_1	v_2	v_3
v_1	$\{v_1\}$	$\{v_2\}$	$\{v_3\}$
v_2	$\{v_3\}$	$\{v_1, v_2\}$	$\{v_2\}$
v_3	$\{v_2\}$	$\{v_3\}$	$\{v_3, v_1\}$

Table-3, neutrosophic triplet H_v**-LA-group**

and

$\circ$	$\flat_1$	$\flat_2$	$\flat_3$
$\flat_1$	$\flat_1$	$\{\flat_1, \flat_2\}$	$\{\flat_1, \flat_3\}$
$\flat_2$	$\flat_3$	$\{\aleph\}$	$\{\flat_1, \flat_2\}$
$\flat_3$	$\flat_2$	$\{\flat_1, \flat_3\}$	$\{\aleph\}$

Table-4, neutrosophic triplet H_v**-LA-group**

The mapping $f : \aleph_1 \longrightarrow \aleph_2$ is defined by $f(v_1) = \flat_1$, $f(v_2) = \flat_2$, $f(v_3) = \flat_3$. Then clearly f is a neutro homomorphism.

4. Construction Of Neutrosophic triplet(NT) H$_v$-LA-groups

In this section we provide the construction of NT H$_v$-LA-groups and develop two Mathematica Packages which help us to check the left invertive law, weak left invertive law and reproductive axiom.

Consider a finite set $\aleph$, such that $|\aleph| > 2$. Define the hyperoperation $\circ$ on $\aleph$ as follows

$$\flat_i \circ \flat_j = \left\{ \begin{array}{cc} \flat_j & \text{for } i = 1 \\ \flat_\flat \text{ for } j = 1\text{and } \flat \equiv 2 - i \bmod |\aleph| \\ \aleph & \text{for } i = j, i \neq 1, j \neq 1 \\ \flat_i & \text{otherwise, for } i \prec j \text{ or } i \succ j \end{array} \right\}$$

and if $neut\,(\flat_i)$ and $anti\,(\flat_i)$ exist in $\aleph$. Then $\aleph$ under the hyperoperation $\circ$ forms a NT H$_v$-LA-group.

The above construction can be explained with the help of an example.

Example 4.1. Let $\aleph = \{\flat_1\flat_2, \flat_3\}$ under the binary hyperoperation $\circ$ defined in Table-5

$\circ$	$\flat_1$	$\flat_2$	$\flat_3$
$\flat_1$	$\flat_1$	$\flat_2$	$\flat_3$
$\flat_2$	$\flat_3$	$\aleph$	$\flat_2$
$\flat_3$	$\flat_2$	$\flat_3$	$\aleph$

Table-5, neutrosophic triplet H$_v$**-LA-group**

Here $(\flat_1, \flat_1, \flat_1), (\flat_2, \flat_1, \flat_2)$ and $(\flat_3, \flat_1, \flat_3)$ are NT set. One can see that $\circ$ satisfy the weak left invertive law, also $\circ$ is non-left invertive and non-associative i.e.

$$\aleph = (\flat_3 \circ \flat_3) \circ \flat_2 \neq (\flat_2 \circ \flat_3) \circ \flat_3 = \flat_2$$
$$\text{and } \aleph = (\flat_2 \circ \flat_2) \circ \flat_1 \neq \flat_2 \circ (\flat_2 \circ \flat_1) = \flat_2.$$

Also it is not WASS$(\flat_2 \circ \flat_1) \circ \flat_1 \cap \flat_2 \circ (\flat_1 \circ \flat_1) = \phi$. Hence $(\aleph, \circ)$ is a NT H$_v$-LA-group. The result of table can easily be generalized to n elements.

Remark 4.2. In NT H$_v$-LA-group, the property of H$_v$-LA-group can be checked by using the mathematica packages. The mathematica package(A) used to check the left invertive property and mathematica package(B) is used to check the weak non associative hypergroups. We paste the mathematica packages as under:

```
BeginPackage["LeftAlmostHyperGroupTest'"];
Clear["LeftAlmostHyperGroupTest'*"];
Begin["'Private'"]; Clear["LeftAlmostHyperGroupTest'Private'*"];
LeftAlmostHyperGroupTest[LookUpTable_List] :=
  Table[If[ReproductivityTest[LookUpTable[[j]]],
    If[LeftInvertiveTest[LookUpTable[[j]]], True, False],
    False], {j, 1, Length[LookUpTable]}];
LeftInvertiveTest[LookUpTable1_List] := Module[{i, j, k, len, test}, i = 1;
   j = 1;
   k = 1;
   test = True;
   len = Length[LookUpTable1];
   While[test && i ≤ len, test = Union[Flatten[Union[Extract[LookUpTable1,
          Distribute[{LookUpTable1[[i, j]], {k}}, List]]]]] == Union[Flatten[Union[
         Extract[LookUpTable1, Distribute[{LookUpTable1[[k, j]], {i}}, List]]]]];
    k = k + 1; If[k > len, k = 1; j = j + 1;
     If[j > len, i = i + 1; j = 1];];];
   Return[test]];
ReproductivityTest[LookUpTable1_List] :=
  Union[Apply[Union, LookUpTable1, 1]] == {Range[1, Length[LookUpTable1]]} &&
   Union[Apply[Union, Transpose[LookUpTable1], 1]] == {Range[1, Length[LookUpTable1]]};
End[];
EndPackage[];
```

Mathematica Package (A)

and

```
BeginPackage["WeakLeftAlmostHyperGroupTest'"];
Clear["WeakLeftAlmostHyperGroupTest'*"];
Begin["'Private'"]; Clear["WeakLeftAlmostHyperGroupTest'Private'*"];
WeakLeftAlmostHyperGroupTest[LookUpTable_List] :=
  Table[If[ReproductivityTest[LookUpTable[[j]]],
    If[WeakLeftInvertiveTest[LookUpTable[[j]]], True, False],
    False], {j, 1, Length[LookUpTable]}];
WeakLeftInvertiveTest[LookUpTable1_List] := Module[{i, j, k, len, test, ∅}, i = 1;
   j = 1; k = 1;
   test = True;
   len = Length[LookUpTable1];
   ∅ = NullSet;
   While[test && i ≤ len,
    test = Union[Flatten[Union[Extract[LookUpTable1, Distribute[{LookUpTable1[[i, j]],
          {k}}, List]]]]] ⋂ Union[Flatten[Union[Extract[LookUpTable1,
         Distribute[{LookUpTable1[[k, j]], {i}}, List]]]]] ≠ {};
    k = k + 1; If[k > len, k = 1; j = j + 1;
     If[j > len, i = i + 1; j = 1];];];
   Return[test]];
ReproductivityTest[LookUpTable1_List] :=
  Union[Apply[Union, LookUpTable1, 1]] == {Range[1, Length[LookUpTable1]]} &&
   Union[Apply[Union, Transpose[LookUpTable1], 1]] == {Range[1, Length[LookUpTable1]]};
End[];
EndPackage[];

WeakLeftAlmostHyperGroupTest.m;
```

Mathematica Package (B)

5. Application of Our proposed Structure

In the universe, the femininity, masculinity and neutrality exist. If we take the small particle, the small particle is an atom. The atom consists of three particle electrons, proton and neutron. So, from the above idea of the universe gave the concept of NT set. (Masculine, Neutral, feminine) and (Proton, Neutron, Electron) are the example of NT set.

There are three workers working in a factory. All three workers are disabled. The first worker has the right hand and no left hand. Factory made such a machine on which he can work with his right hand. The second worker has left hand but no right hand. Such a machine is made for him, on which he worked with his left hand. The third worker has an issue working with both of his hand. Such a machine is made for him, he works with his legs. All of these three worker's working performance is shown by the following Table-6.

$\circledast$	L	R	N
L	R	N	$\{L, N\}$
R	N	L	$\{R, N\}$
N	L	R	N

Table-6, neutrosophic triplet H_v**-LA-group**

In this table L represents the performance the worker, who work with his left hand. R represents the performance of the worker, who work with his right hand and N represents the performance of the worker, whose both hand are not functioning properly. Let $F = \{L, R, N\}$ be a finite set the hyperoperation is defined in the above table, and $(L, N, R), (R, N, L)$ and (N, L, L) are left NT set. $(F, \circledast)$ is a NT Hv-LA group.

5.1. *Chemical example of Neutrosophic Triplet(NT) H_v-LA- group*

The best example of NT H_v-LA-group in chemical reaction is a redox reaction.

Redox reaction: The chemical reaction in which one specie loss the electron and other specie gain the electron. Oxidation mean loss of electron. Reduction mean gain of electron. The redox reaction is a vital for biochemical reaction and industrial process. The electron transfer in cell and oxidation of glucose in the human body are the example of redox reaction. The reaction between hydrogen and fluorine is an example of redox reaction i.e.

$$\aleph_2 + F \longrightarrow 2\aleph F$$

$$\aleph_2 \longrightarrow 2\aleph^+ + 2e^- (\text{ Oxidation})$$

$$F_2 + 2e^- \longrightarrow 2F \text{ (Reduction)}$$

Each half reaction has standard reduction potential $\left(E^0\right)$ which is equal to the potential difference at equilibrium under the standard condition of an electrochemical cell in which the cathode reaction is half reaction considered and anode is a standard hydrogen electrode (SHE). For the redox reaction, the potential of cell is defined as

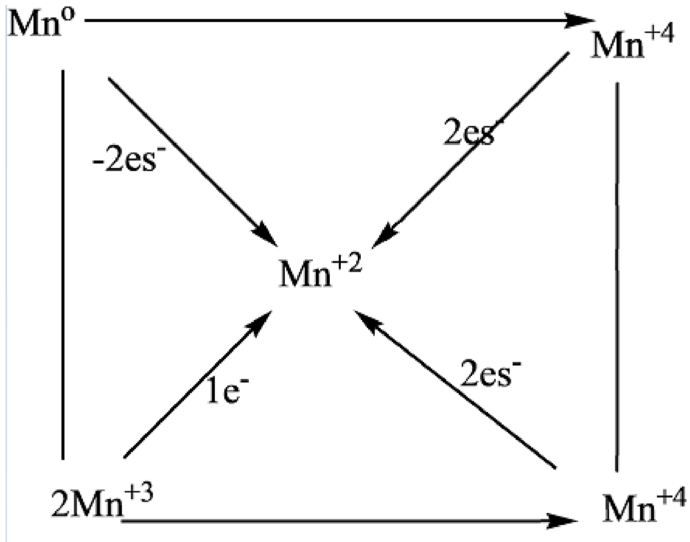
$$E^\circ cell = E^\circ{}_{cathode} - E^\circ{}_{anode}$$

where $E^\circ{}_{cathode}$ is the standard potential at the anode and $E^\circ{}_{cathode}$ is the standard potential at the cathode as given in the table of standard electrode potential. Now consider the redox reaction of Mn

$$Mn^0 \ + 2Mn^{+4} + 2Mn^{+3} \longrightarrow \ 3Mn^{+2} + 2Mn^{+4}$$

$$Mn^0 \ \longrightarrow Mn^{+2} + Mn^{+4} + 2e^- + 2Mn^{+3} + 2Mn^{+4}.$$

Manganese having a variable oxidation state of 0,+1, +2, +3, +4, +5, +6, +7. If we take $Mn^0, Mn^{+4}, Mn^{+3}, Mn^{+4}$ together we will get pure redox reaction. The flow chart is given as



Flow chart

Mn species with different oxidation state react with themselves. All possible reactions are presented in the following Table-7

$\oplus$	Mn^0	Mn^{+1}	Mn^{+2}	Mn^{+3}	Mn^{+4}
Mn^0	Mn^0	$\{Mn^0, Mn^{+1}\}$	$\{Mn^0, Mn^{+2}\}$	$\{Mn^0, Mn^{+3}\}$	$\{Mn^0, Mn^{+4}\}$
Mn^{+1}	$\{Mn^0, Mn^{+1}\}$	$\{Mn^0, Mn^{+2}\}$	$\{Mn^0, Mn^{+3}\}$	$\{Mn^{+2}\}$	$\{Mn^{+1}, Mn^{+4}\}$
Mn^{+2}	Mn^{+1}	$\{Mn^0, Mn^{+3}\}$	$\{Mn^{+1}, Mn^{+3}\}$	$\{Mn^{+1}, Mn^{+4}\}$	$\{Mn^{+2}, Mn^{+4}\}$
Mn^{+3}	$\{Mn^0, Mn^{+3}\}$	$\{Mn^{+1}, Mn^{+3}\}$	$\{Mn^{+2}, Mn^{+3}\}$	Mn^{+3}	$\{Mn^{+3}, Mn^{+4}\}$
Mn^{+4}	$\{Mn^0, Mn^{+4}\}$	$\{Mn^{+1}, Mn^{+4}\}$	$\{Mn^{+2}, Mn^{+4}\}$	$\{Mn^{+3}, Mn^{+4}\}$	Mn^{+4}

Table-7, All possible reactions

The standard reduction potentials (E^0) for conversion of each oxidation state to another are

$$E^0 \left(Mn^{+4}/Mn^{+3}\right) = +0.95,$$
$$E^0 \left(Mn^{+3}/Mn^{+2}\right) = +1.542,$$
$$E^0 \left(Mn^{+2}/Mn^{+1}\right) = -0.59,$$
$$E^0 \left(Mn^{+1}/Mn^{+0}\right) = 0.296.$$

If we replace

$$Mn^0 = \flat_1, Mn^{+1} = \flat_2, Mn^{+2} = \flat_3, Mn^{+3} = \flat_4, Mn^{+4} = \flat_5,$$

then we obtain the following Table-8

$\oplus$	$\flat_1$	$\flat_2$	$\flat_3$	$\flat_4$	$\flat_5$
$\flat_1$	$\{\flat_1\}$	$\{\flat_1,\flat_2\}$	$\{\flat_1,\flat_3\}$	$\{\flat_1,\flat_4\}$	$\{\flat_1,\flat_5\}$
$\flat_2$	$\{\flat_1,\flat_2\}$	$\{\flat_1,\flat_3\}$	$\{\flat_1,\flat_4\}$	$\{\flat_3\}$	$\{\flat_2,\flat_5\}$
$\flat_3$	$\{\flat_1,\flat_3\}$	$\{\flat_1,\flat_4\}$	$\{\flat_2,\flat_4\}$	$\{\flat_2,\flat_5\}$	$\{\flat_3,\flat_5\}$
$\flat_4$	$\{\flat_1,\flat_4\}$	$\{\flat_2,\flat_4\}$	$\{\flat_3,\flat_4\}$	$\{\flat_4\}$	$\{\flat_4,\flat_5\}$
$\flat_5$	$\{\flat_1,\flat_5\}$	$\{\flat_2,\flat_5\}$	$\{\flat_3,\flat_5\}$	$\{\flat_4,\flat_5\}$	$\{\flat_5\}$

Table-8, NT H$_v$-LA-group

As all elements of $\aleph$ satisfy the weak left invertive law but $\aleph$ do not satisfy the left invertive law, associative law and weak associative law

$$\{\flat_1,\flat_3\} = (\flat_2 \oplus \flat_2) \oplus \flat_1 \neq (\flat_1 \oplus \flat_2) \oplus \flat_2 = \{\flat_1,\flat_2,\flat_3\},$$
$$\{\flat_1,\flat_2,\flat_3,\flat_4\} = (\flat_2 \oplus \flat_2) \oplus \flat_3 \neq \flat_2 \oplus (\flat_2 \oplus \flat_3) = \{\flat_1,\flat_2,\flat_3\},$$
$$\text{and } (\flat_2 \oplus \flat_4) \oplus \flat_4 = \{\flat_2,\flat_5\} \cap \flat_3 = \flat_2 \oplus (\flat_4 \oplus \flat_4) = \phi$$

Here $(\flat_1,\flat_1,\flat_1), (\flat_2,\flat_4,\flat_3), (\flat_3,\flat_4,\flat_2), (\flat_4,\flat_5,\flat_3)$ and $(\flat_5,\flat_4,\flat_4)$ are NT sets. Hence $(\aleph,\oplus)$ is a NT H$_v$-LA-group.

Remark 5.1. NT set, which helps the chemist to take the state of M_n which react or not react easily with other state or themselves. M_n^{+0} plays the role of neuta with different oxidation state and themselves. If the M_n have the same neuta and anti, it means that Mn having equal chances of loss or gain of electron.

6. Difference between the proposed work and existing methods

Our proposed structure has two main purpose,

1) This structure generalize the structure of groups, LA-groups, semigroups, LA-semigroup and as well as the hyper versions of above mentioned structures.

2) As NT set has the abelity to capture indeterminacy in a much better way so our proposed stricture of NT LA-semigroups can handle the uncertanity in a better way as we have seen in the Redox reaction.

7. Conclusions

In this article, we have studied and introduced NT H$_v$ LA- groups. We presented some result on NT H$_v$ LA-groups and construction of NT H$_v$-LA groups. We defined the neutro homomorphism on NT H$_v$ LA groups. Also, we use the Mathematica packages to check the properties of left invertive and weak left invertive. Our defined structure have an interesting application in chemistry redox reaction.

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Some Properties of Q-Neutrosophic Ideals of Semirings

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Abstract. The intention of this paper is to introduce and study some properties of the ideals of semirings using the concept of Q-neutrosophic set.

Keywords: Semiring; Q-neutrosophic ideal; Cartesian Product; Composition.

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1. Introduction

The concept of fuzzy sets was introduced by Zadeh [21] in 1965 to overcome the uncertainties in various problems in environment, economics, engineering etc. As an extension of it, Atanassov [7] introduced intuitionistic fuzzy set in 1986, where a degree of non-membership was considered besides the degree of membership of each element with (membership value + non-membership value) ≤ 1.

After that several generalizations such as, rough sets, vague sets, interval-valued sets etc. are considered as mathematical tools for dealing with uncertainties. In 2005, F. Smarandache introduced Neutrosophic set [19] in which he introduced the indeterminacy to intuitionistic fuzzy sets. So, the resultant can be taken as a tri-component logic which can be applied to non-standard analysis such as decision making (for example, result of games (win/tie/defeat), votes, from no/yes/NA), control theory etc.. Since then several researchers has applied this concept in many practical fields such as multi-criteria decision making, signal processing, disaster management etc.. Some of its recent applications can be found in [1–5, 9, 18, 20].

In 2011, Majumder [13] introduced and studied the concept of Q-fuzzifcation of ideals of Γ-semigroup. Akram et al [6], Lekkoksung [11, 12], Mandal [14], Qamar et al [15, 16] extended this concept in case of Γ-semigroup, ordered semigroups [10], ordered Γ-semiring, soft fields, group theory and investigated some important properties.

Motivated by this idea and combining the concept with neutrosophic set, in the paper we have studied the ideal theory of semirings since it has several applications in graph theory, automata theory, mathematical modelling etc. [8].

2. Preliminaries

At first let us remember some definitions which will be used in the discussion of the paper.

Definition 2.1. A semiring is a nonempty set S on which two operations $+$ and $\cdot$ have been defined such that $(S,+)$ and $(S,\cdot)$ form monoid where $\cdot$ distributes over $+$ from any side.

Definition 2.2. A nonempty subset $X(\neq S)$ of semiring S is said to be an ideal if for all $x, y \in X$ and $p \in S$, $x+y \in X$, $px \in X$. Similarly we can define a right ideal also. An ideal of S is a nonempty subset which satisfies both properties of left ideal and right ideal.

Definition 2.3. A neutrosophic set N on the universe U is defined as $N = \{< u, A^T(u), A^I(u), A^F(u) >, u \in U\}$, where $A^T, A^I, A^F : U \rightarrow]^-0, 1^+[$ and $^-0 \leq A^T(u) + A^I(u) + A^F(u) \leq 3^+$. For practical purposes, it is difficult to consider $]^-0, 1^+[$. So, for studying neutrosophic set we consider the set which takes the value from the subset of [0, 1].

Definition 2.4. For a non-empty set Q, a mapping $\nu : S \times Q \longrightarrow [0,1]$ is said to be a Q-fuzzy subset of S and $\nu_l = \{(s,q) \in S \times Q | \nu(s,q) \geq l\}$ where $l \in [0,1]$ is its level subset.

3. Main Results

Definition 3.1. Let $\nu = (\nu^T, \nu^I, \nu^F)$ be a non empty neutrosophic subset of a semiring S. Then ν is called a Q-neutrosophic left ideal of S if

(i) $\nu^T(s_1+s_2,p) \geq \min\{\nu^T(s_1,p), \nu^T(s_2,p)\}$, $\nu^T(s_1s_2,p) \geq \nu^T(s_2,p)$
(ii) $\nu^I(s_1+s_2,p) \geq \frac{\nu^I(s_1,p)+\nu^I(s_2,p)}{2}$, $\nu^I(s_1s_2,p) \geq \nu^I(s_2,p)$
(iii) $\nu^F(s_1+s_2,p) \leq \max\{\nu^F(s_1,p), \nu^F(s_2,p)\}$, $\nu^F(s_1s_2,p) \leq \nu^F(s_2,p)$.

for all $s_1, s_2 \in S$ and $p \in Q$.

Theorem 3.2. *Any Q-neutrosophic set ν of a semiring S is a left ideal iff its level subsets $\nu^T_l := \{(x,p) \in S \times Q : \nu^T(x,p) \geq l,\ l \in [0,1],\ p \in Q\}$, $\nu^I_l := \{(x,p) \in S \times Q : \nu^I(x,p) \geq l,\ l \in [0,1]\}$ and $\nu^F_l := \{(x,p) \in S \times Q : \nu^F(x,p) \leq l,\ l \in [0,1]\}$ are left ideals of $S \times Q$.*

Proof. Suppose ν of S is a Q-neutrosophic left ideal of S. Then anyone of ν^T, ν^I or ν^F is not equal to zero for some $(s,p) \in S \times Q$. Without loss of generality we consider, all of them are not equal to zero.

Suppose $a, b \in \nu_l = (\nu_l^T,\ \nu_l^I,\ \nu_l^F)$, $s \in S$ and $p \in Q$. Then

$$\begin{aligned}
&\nu^T(a+b,p) \geq \min\{\nu^T(a,p), \nu^T(b,p)\} \geq \min\{l,l\} = l\\
&\nu^I(a+b,p) \geq \tfrac{\nu^I(a,p)+\nu^I(b,p)}{2} \geq \tfrac{l+l}{2} = l\\
&\nu^F(a+b,p) \leq \max\{\nu^F(a,p), \nu^F(b,p)\} \leq \max\{l,l\} = l
\end{aligned}$$

which implies $(a+b,p) \in \nu_l^T, \nu_l^I, \nu_l^F$ i.e., $(a+b,p) \in \nu_l$. Also

$$\begin{aligned}
&\nu^T(sa,p) \geq \nu^T(a,p) \geq l\\
&\nu^I(sa,p) \geq \nu^I(a,p) \geq l\\
&\nu^F(sa,p) \leq \nu^F(a,p) \leq l
\end{aligned}$$

Hence $(sa,p) \in \nu_l$.

Therefore ν_l is a left ideal of S.

Conversely, let $\nu_l(\neq \phi)$ is a left ideal of $S \times Q$ and ν is not a Q-neutrosophic left ideal of S. Then for $a, b \in S$ and $p \in Q$ anyone of the following inequality will hold.

$$\begin{aligned}
&\nu^T(a+b,p) < \min\{\nu^T(a,p), \nu^T(b,p)\}\\
&\nu^I(a+b,p) < \tfrac{\nu^I(a,p)+\nu^I(b,p)}{2}\\
&\nu^F(a+b,p) > \max\{\nu^F(a,p), \nu^F(b,p)\}
\end{aligned}$$

For the first inequality, choose $l_1 = \frac{1}{2}[\nu^T(a+b,p)+\min\{\nu^T(a,p), \nu^T(b,p)\}]$. Then $\nu^T(a+b,p) < l_1 < \min\{\nu^T(a,p), \nu^T(b,p)\} \Rightarrow (a,p),(b,p) \in \nu_{l_1}^T$. but $(a+b,p) \notin \nu_{l_1}^T$ - contradiction.

For the second inequality, choose $t_2 = \frac{1}{2}[\nu^I(a+b,p) + \min\{\nu^I(a,p), \nu^I(b,p)\}]$. Then $\nu^I(a+b,p) < l_2 < \frac{\nu^I(a,p)+\nu^I(b,p)}{2} \Rightarrow (a,p),(b,p) \in \nu_{l_2}^I$. But $(a+b,p) \notin \nu_{l_2}^I$ - contradiction.

For the third inequality, choose $t_3 = \frac{1}{2}[\nu^F(a+b,p) + \max\{\nu^F(a,p), \nu^F(b,p)\}]$. Then $\nu^F(a+b,p) > l_3 > \max\{\nu^F(a,p), \nu^F(b,p)\} \Rightarrow (a,p),(b,p) \in \nu_{t_3}^F$ but $(a+b,p) \notin \nu_{p_3}^F$ - contradiction.

Hence the theorem. □

Definition 3.3. For two Q-neutrosophic subsets ν and σ of $S \times Q$, define their intersection by

$$(\nu^T \cap \sigma^T)(a,p) = \min\{\nu^T(a,p), \sigma^T(a,p)\}$$

$$(\nu^I \cap \sigma^I)(a,p) = \min\{\nu^I(a,p), \sigma^I(a,p)\}$$

$$(\nu^F \cap \sigma^F)(a,p) = \max\{\nu^F(a,p), \sigma^F(a,p)\}$$

for all $a \in S$ and $p \in Q$.

Proposition 3.4. *Intersection of any number of Q-neutrosophic left ideals of S is also a Q-neutrosophic left ideal.*

Proof. Assume that $\{\nu_i : c \in C\}$ be a collection of Q-neutrosophic left ideals of S and $a, b \in S$, $p \in Q$. Then

$$\begin{aligned}
(\bigcap_{c\in C}\nu_c^T)(a+b,p) &= \inf_{c\in C}\nu_c^T(a+b,p) \geq \inf_{c\in C}\{\min\{\nu_c^T(a,p),\nu_c^T(b,p)\}\} \\
&= \min\{\inf_{c\in C}\nu_c^T(a,p),\inf_{c\in C}\nu_c^T(b,p)\} \\
&= \min\{(\bigcap_{c\in C}\nu_c^T)(a,p),(\bigcap_{c\in C}\nu_c^T)(b,p)\}
\end{aligned}$$

$$\begin{aligned}
(\bigcap_{c\in C}\nu_c^I)(a+b,p) &= \inf_{c\in C}\nu_c^I(a+b,p) \geq \inf_{c\in C}\tfrac{\nu_c^I(a,p)+\nu_c^I(b,p)}{2} \\
&= \tfrac{\inf_{c\in C}\nu_c^I(a,p)+\inf_{c\in C}\nu_c^I(b,p)}{2} \\
&= \tfrac{\bigcap_{c\in C}\nu_c^I(a,p)+\bigcap_{c\in C}\nu_c^I(b,p)}{2}
\end{aligned}$$

$$\begin{aligned}
(\bigcap_{c\in C}\nu_c^F)(a+b,p) &= \sup_{c\in C}\nu_c^F(a+b,p) \leq \sup_{c\in C}\{\max\{\nu_c^F(a,p),\nu_c^F(b,p)\}\} \\
&= \max\{\sup_{c\in C}\nu_c^F(a,p),\sup_{c\in C}\nu_c^F(b,p)\} \\
&= \max\{(\bigcap_{c\in C}\nu_c^F)(a,p),(\bigcap_{c\in C}\nu_c^F)(b,p)\}
\end{aligned}$$

$$(\bigcap_{c\in C}\nu_c^T)(ab,p) = \inf_{c\in C}\nu_c^T(ab,p) \geq \inf_{c\in C}\nu_c^T(b,p) = (\bigcap_{c\in C}\nu_c^T)(b,p).$$
$$(\bigcap_{c\in C}\nu_c^I)(ab,p) = \inf_{c\in C}\nu_c^I(ab,p) \geq \inf_{c\in C}\nu_c^I(b,p) = (\bigcap_{c\in C}\nu_c^I)(b,p).$$
$$(\bigcap_{c\in C}\nu_c^F)(ab,p) = \sup_{c\in C}\nu_c^F(ab,p) \leq \sup_{c\in C}\nu_c^F(b,p) = (\bigcap_{c\in C}\nu_c^F)(b,p).$$

Therefore $\bigcap_{c\in C}\nu_c$ is a Q-neutrosophic left ideal of S. □

Definition 3.5. For two Q-neutrosophic subsets ν and σ of S, define their cartesian product by

$$(\nu^T \times \sigma^T)((a,b),p) = \min\{\nu^T(a,p),\sigma^T(b,p)\}$$
$$(\nu^I \times \sigma^I)((a,b),p) = \frac{\nu^I(a,p)+\sigma^I(b,p)}{2}$$
$$(\nu^F \times \sigma^F)((a,b),p) = \max\{\nu^F((a,p),\sigma^F(b,p)\}$$

$\forall a, b \in S,\ p \in Q.$

Theorem 3.6. *For two Q-neutrosophic left ideals ν and σ of S, $\nu \times \sigma$ is a Q-neutrosophic left ideal of $S \times S$.*

Proof. Let $(a_1,a_2),(b_1,b_2) \in S \times S$ and $p \in Q$. Now

$$\begin{aligned}
(\nu^T \times \sigma^T)((a_1,a_2)+(b_1,b_2),p) &= (\nu^T \times \sigma^T)((a_1+b_1,a_2+b_2),p) \\
&= \min\{\nu^T(a_1+b_1,p),\sigma^T(a_2+b_2,p)\} \\
&\geq \min\{\min\{\nu^T(a_1,p),\nu^T(b_1,p)\},\min\{\sigma^T(a_2,p),\sigma^T(b_2,p)\}\} \\
&= \min\{\min\{\nu^T(a_1,p),\sigma^T(a_2,p)\},\min\{\nu^T(b_1,p),\sigma^T(b_2,p)\}\} \\
&= \min\{(\nu^T \times \sigma^T)((a_1,a_2),p),(\nu^T \times \sigma^T)((b_1,b_2),p)\}.
\end{aligned}$$

$$\begin{aligned}(\nu^I\times\sigma^I)((a_1,a_2)+(b_1,b_2),p) &= (\nu^I\times\sigma^I)((a_1+b_1,a_2+b_2),p)\\ &= \tfrac{\nu^I(a_1+b_1,p)+\sigma^I(a_2+b_2,p)}{2}\\ &\geq \tfrac{1}{2}\{\tfrac{\nu^I(a_1,p)+\nu^I(b_1,p)}{2}+\tfrac{\sigma^I(a_2,p)+\sigma^I(b_2,p)}{2}\}\\ &= \tfrac{1}{2}\{\tfrac{\nu^I(a_1,p)+\sigma^I(a_2,p)}{2}+\tfrac{\nu^I(b_1,p)+\sigma^I(b_2,p)}{2}\}\\ &= \tfrac{1}{2}\{(\nu^I\times\sigma^I)((a_1,a_2),p)+(\nu^I\times\sigma^I)((b_1,b_2),p)\}.\end{aligned}$$

$$\begin{aligned}(\nu^F\times\sigma^F)((a_1,a_2)+(b_1,b_2),p) &= (\nu^F\times\sigma^F)((a_1+b_1,a_2+b_2),p)\\ &= \max\{\nu^F(a_1+b_1,p),\nu^F(a_2+b_2,p)\}\\ &\leq \max\{\max\{\nu^F(a_1,p),\nu^F(b_1,p)\},\max\{\sigma^F(a_2,p),\sigma^F(b_2,p)\}\}\\ &= \max\{\max\{\nu^F(a_1,p),\sigma^F(a_2,p)\},\max\{\nu^F(b_1,p),\sigma^F(b_2,p)\}\}\\ &= \max\{(\nu^F\times\sigma^F)((a_1,a_2),p),(\nu^F\times\sigma^F)((b_1,b_2),p)\}.\end{aligned}$$

$$\begin{aligned}(\nu^T\times\sigma^T)((a_1,a_2)(b_1,b_2),p) &= (\nu^T\times\sigma^T)((a_1b_1,a_2b_2),p)=\min\{\nu^T(a_1b_1,p),\sigma^T(a_2b_2,p)\}\\ &\geq \min\{\nu^T(b_1,p),\sigma^T(b_2,p)\}=(\nu^T\times\sigma^T)((b_1,b_2),p).\end{aligned}$$

$$\begin{aligned}(\nu^I\times\sigma^I)((a_1,a_2)(b_1,b_2),p) &= (\nu^I\times\sigma^I)((a_1b_1,a_2b_2),p)=\tfrac{\nu^I(a_1b_1,p)+\sigma^I(a_2b_2,p)}{2}\\ &\geq \tfrac{\nu^I(b_1,p)+\sigma^I(b_2,p)}{2}=(\nu^I\times\sigma^I)((b_1,b_2),p).\end{aligned}$$

$$\begin{aligned}(\nu^F\times\sigma^F)((a_1,a_2)(b_1,b_2),p) &= (\nu^F\times\sigma^F)((a_1b_1,a_2b_2),p)=\max\{\nu^F(a_1b_1,p),\sigma^F(a_2b_2,p)\}\\ &\leq \max\{\nu^F(b_1,p),\nu^F(b_2,p)\}=(\nu^F\times\sigma^F)(b_1,b_2,p).\end{aligned}$$

Therefore $\nu\times\sigma$ is a Q-neutrosophic left ideal of $S\times S$. □

Theorem 3.7. *A Q-neutrosophic set ν of S is a Q-neutrosophic left ideal iff $\nu\times\nu$ is a Q-neutrosophic left ideal of $S\times S$.*

Proof. If a Q-neutrosophic subset ν of S is a Q-neutrosophic left ideal then by Theorem 3.6, $\nu\times\nu$ is a Q-neutrosophic left ideal of $S\times S$.

Conversely, suppose $\nu\times\nu$ is a Q-neutrosophic left ideal of $S\times S$ and $a_1,a_2,b_1,b_2\in S$, $p\in Q$. Then

$$\begin{aligned}\min\{\nu^T(a_1+b_1,p),\nu^T(a_2+b_2,p)\} &= (\nu^T\times\nu^T)((a_1+b_1,a_2+b_2),p)\\ &= (\nu^T\times\nu^T)((a_1,a_2)+(b_1,b_2),p)\\ &\geq \min\{(\nu^T\times\nu^T)((a_1,a_2),p),(\nu^T\times\nu^T)((b_1,b_2),p)\}\\ &= \min\{\min\{\nu^T(a_1,p),\nu^T(a_2,p)\},\min\{\nu^T(b_1,p),\nu^T(b_2,p)\}\}.\end{aligned}$$

$$\begin{aligned}\tfrac{\nu^I(a_1+b_1,p)+\nu^I(a_2+b_2,p)}{2} &= (\nu^I\times\nu^I)((a_1+b_1,a_2+b_2),p)\\ &= (\nu^I\times\nu^I)((a_1,a_2)+(b_1,b_2),p)\\ &\geq \tfrac{(\nu^I\times\nu^I)((a_1,a_2),p)+(\nu^I\times\nu^I)((b_1,b_2),p)}{2}\\ &= \tfrac{1}{2}[\tfrac{\nu^I(a_1,p)+\nu^I(a_2,p)}{2}+\tfrac{\nu^I(b_1,p)+\nu^I(b_2,p)}{2}].\end{aligned}$$

$$\begin{aligned}\max\{\nu^F(a_1+b_1,p),\nu^F(a_2+b_2,p)\} &= (\nu^F\times\nu^F)((a_1+b_1,a_2+b_2),p)\\ &= (\nu^F\times\nu^F)((a_1,a_2)+(b_1,b_2),p)\\ &\le \max\{(\nu^F\times\nu^F)((a_1,a_2),p),(\nu^F\times\nu^F)((b_1,b_2),p)\}\\ &= \min\{\max\{\nu^F(a_1,p),\nu^F(a_2,p)\},\max\{\nu^F(b_1,p),\nu^F(b_2,p)\}\}.\end{aligned}$$

Now, putting $a_1 = a, a_2 = 0, b_1 = b$ and $b_2 = 0$, in the above inequalities and noting that $\nu^T(0) \ge \nu^T(x)$, $\nu^I(0) = 0$ and $\nu^F(0) \le \nu^F(x)$ for all $a \in S$ we obtain

$$\begin{aligned}&\nu^T(a+b,p) \ge \min\{\nu^T(a,p),\nu^T(b,p)\}\\ &\nu^I(a+b,p) \ge \tfrac{\nu^I(a,p)+\nu^I(b,p)}{2}\\ &\nu^F(a+b,p) \le \max\{\nu^F(a,p),\nu^F(b,p)\}.\end{aligned}$$

Next, we have

$$\begin{aligned}\min\{\nu^T(a_1b_1),\nu^T(a_2b_2)\} &= (\nu^T\times\nu^T)(a_1b_1,a_2b_2) = (\nu^T\times\nu^T)((a_1,a_2)(b_1,b_2))\\ &\ge (\nu^T\times\nu^T)(b_1,b_2) = \min\{\nu^T(b_1),\nu^T(b_2)\}.\end{aligned}$$

$$\begin{aligned}\tfrac{\nu^I(a_1b_1,q)+\nu^I(a_2b_2,q)}{2} &= (\nu^I\times\nu^I)((a_1,a_2)(b_1,b_2),q)\\ &\ge (\nu^I\times\nu^I)((b_1,b_2),q)\\ &= \tfrac{\nu^I(b_1,q)+\nu^I(b_2,q)}{2}.\end{aligned}$$

$$\begin{aligned}\max\{\nu^F(a_1b_1,q),\nu^F(a_2b_2,q)\} &= (\nu^F\times\nu^F)((a_1b_1,a_2b_2),q) = (\nu^F\times\nu^F)((a_1,a_2)(b_1,b_2),q)\\ &\le (\nu^F\times\nu^F)((b_1,b_2),q) = \max\{\nu^F(b_1,q),\nu^F(b_2,q)\}.\end{aligned}$$

Taking $a_1 = a, b_1 = b$ and $b_2 = 0$, we obtain

$$\begin{aligned}&\nu^T(ab,p) \ge \nu^T(b,p)\\ &\nu^I(ab,p) \ge \nu^I(b,p)\\ &\nu^F(ab,p) \le \nu^F(b,p).\end{aligned}$$

Hence ν becomes a Q-neutrosophic left ideal of S. □

Definition 3.8. For two Q-neutrosophic sets ν and σ of a semiring S, define their composition by

$$\begin{aligned}\nu^T o\sigma^T(a,p) &= \sup_{a=\sum_{c=1}^{m} a_cb_c} \{\min_c\{\nu^T(a_c,p),\sigma^T(b_c,p)\}\}\\ &= 0,\ \text{otherwise}\end{aligned}$$

$$\begin{aligned}\nu^I o\sigma^I(a,p) &= \sup_{a=\sum_{c=1}^{m} a_cb_c} \textstyle\sum_{c=1}^{m} \frac{\nu^I(a_c,p)+\sigma^I(b_c,p)}{2m}\\ &= 0,\ \text{otherwise}\end{aligned}$$

$$\nu^F o\sigma^F(a,p) = \inf_{a=\sum_{c=1}^{m} a_c b_c} \{\max_c\{\nu^F(a_c,p), \sigma^F(b_c,p)\}\}$$
$$= 0, \text{otherwise}$$

where $p \in Q$, $a, a_c, b_c \in S$, $m \in N$-the set of natural number.

Theorem 3.9. *For two Q-neutrosophic left ideals ν and σ of S, $\nu o\sigma$ also forms a Q-neutrosophic left ideal of S.*

Proof. Consider two Q-neutrosophic left ideals ν, σ of S with $a, b \in S$, $p \in Q$. If $(a+b,p)$ has the expression $(\sum_{i=1}^{m} a_c b_c, p)$, where $a_c, b_c \in S$ and $p \in Q$, then the proof is immediate from the definition. So, assume that $a+b$ can be expressed in the said form. Then

$$\begin{aligned}
&(\nu^T o\sigma^T)(a+b,p)\\
&= \sup_{a+b=\sum_{c=1}^{m} a_c b_c} \{\min_c\{\nu^T(a_c,p), \sigma^T(b_c,p)\}\}\\
&\geq \sup_{a=\sum_{c=1}^{m} c_c d_c,\, b=\sum_{c=1}^{m} e_c f_c}\{\min_c\{\nu^T(c_c,p), \sigma^T(d_c,p), \nu^T(e_c,p), \sigma^T(f_c,p)\}\}\\
&= \min\{ \sup_{a=\sum_{c=1}^{m} c_c d_c}\{\min_c\{ \nu^T(c_c,p), \sigma^T(d_c,p)\}\}, \sup_{b=\sum_{c=1}^{m} e_c f_c}\{\min_c\{ \nu^T(e_c,p), \sigma^T(f_c,p)\}\}\}\\
&= \min\{(\nu^T o\sigma^T)(a,p), (\nu^T o\sigma^T)(b,p)\}
\end{aligned}$$

$$\begin{aligned}
&(\nu^I o\sigma^I)(a+b,p)\\
&= \sup_{a+b=\sum_{c=1}^{m} a_c b_c} \sum_{c=1}^{m} \tfrac{\nu^I(a_c,p)+\sigma^I(b_c,p)}{2m}\\
&\geq \sup_{a=\sum_{c=1}^{m} c_c d_c,\, b=\sum_{c=1}^{m} e_c f_c} \sum_{c=1}^{m} \tfrac{\nu^I(c_c,p)+\sigma^I(d_c,p)+\nu^I(e_c,p)+\sigma^I(f_c,p)}{2m}\\
&\geq \tfrac{1}{2}[\sup_{a=\sum_{c=1}^{m} c_c d_c} \sum_{c=1}^{m} \tfrac{\nu^I(c_c,p)+\sigma^I(d_c,p)}{2m}, \sup_{b=\sum_{c=1}^{m} e_c f_c} \sum_{c=1}^{m} \tfrac{\nu^I(e_c,p)+\sigma^I(f_c,p)}{2m}]\\
&= \tfrac{(\nu^I o\sigma^I)(a,p)+(\nu^I o\sigma^I)(b,p)}{2}
\end{aligned}$$

$$
\begin{aligned}
&(\nu^F o\sigma^F)(a+b,p)\\
&= \inf_{a+b=\sum_{c=1}^{m} a_c b_c} \{\max_c\{\nu^F(a_c,p),\sigma^F(b_c,p)\}\}\\
&\leq \inf_{a=\sum_{c=1}^{m} c_c d_c,\, b=\sum_{c=1}^{m} e_c f_c} \{\max_c\{\nu^F(c_c,p),\sigma^F(d_c,p),\nu^F(e_c,p),\sigma^F(f_c,p)\}\}\\
&= \max\{\inf_{a=\sum_{c=1}^{m} c_c d_c}\{\max_c\{\nu^F(c_c,p),\sigma^F(d_c p)\}\},\inf_{b=\sum_{c=1}^{m} e_c f_c}\{\max_c\{\nu^F(e_c,p),\sigma^F(f_c,p)\}\}\}\\
&= \max\{(\nu^F o\sigma^F)(a,p),(\nu^F o\sigma^F)(b,p)\}
\end{aligned}
$$

$$
\begin{aligned}
(\nu^T o\sigma^T)(ab,p) &= \sup_{ab=\sum_{c=1}^{m} a_c b_c} \{\min_c\{\nu^T(a_c,p),\sigma^T(b_c,p)\}\}\\
&\geq \sup_{ab=\sum_{c=1}^{m} ae_c f_c} \{\min_c\{\nu^T(ae_c,p),\sigma^T(f_c,p)\}\}\\
&\geq \sup_{b=\sum_{c=1}^{m} e_c f_c} \{\min_c\{\nu^T(e_c,p),\sigma^T(f_c,p)\}\} = (\nu^T o\sigma^T)(b,p)
\end{aligned}
$$

$$
\begin{aligned}
(\nu^I o\sigma^I)(ab,p) &= \sup_{ab=\sum_{c=1}^{m} a_c b_c} \textstyle\sum_{c=1}^{m} \frac{\nu^I(a_c,p)+\sigma^I(b_c,p)}{2m}\\
&\geq \sup_{ab=\sum_{c=1}^{m} ae_c f_c} \textstyle\sum_{c=1}^{m} \frac{\nu^I(ae_c,p)+\sigma^I(f_c,p)}{2m}\\
&\geq \sup_{b=\sum_{c=1}^{m} e_c f_c} \textstyle\sum_{c=1}^{m} \frac{\nu^I(e_c,p)+\sigma^I(f_c,p)}{2m} = (\nu^I o\sigma^I)(b,p)
\end{aligned}
$$

$$
\begin{aligned}
(\nu^F o\sigma^F)(ab,p) &= \inf_{ab=\sum_{c=1}^{m} a_c b_c} \{\max_c\{\nu^F(a_c,p),\sigma^F(b_c,p)\}\}\\
&\leq \inf_{ab=\sum_{c=1}^{m} ae_c f_c} \{\max_c\{\nu^F(ae_c,p),\sigma^F(f_c,p)\}\}\\
&\leq \inf_{b=\sum_{c=1}^{m} e_c f_c} \{\max_c\{\nu^F(e_c,p),\sigma^F(f_c,p)\}\} = (\nu^F o\nu^F)(b,p)
\end{aligned}
$$

Therefore $\nu o\sigma$ is a Q-neutrosophic left ideal of S. $\square$

Conclusion: In this paper, we have defined Q-neutrosophic ideals of a semiring and studied some its elementary properties. Here also we obtain its characterizations by label subset criteria, cartesian product and composition of two Q-neutrosophic ideals. Our next aim to extend the idea in case of Q-neutrosophic bi-ideals, Q-neutrosophic quasi-ideals and investigate some properties of regular semirings.

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An Introduction to Neutrosophic Minimal Structure Spaces

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Abstract. This paper is an introduction of neutrosophic minimal structure space and addresses properties of neutrosophic minimal structure space. Neutrosophic set has plenty of applications. This motivates us to present the concept of neutrosophic minimal structure space. We defined neutrosophic minimal structure space, closure and interior of a set, subspace. Some properties of neutrosophic minimal structure space are also studied. Finally, Decision making problem solved using score function.

Keywords: Neutrosophic minimal structure; N_m-closure; N_m-interior; N_m-connectedness.

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1. **Introduction**

Zadeh's [23] Fuzzy set laid the foundation of many theories such as intuitionistic fuzzy set and neutrosophic set, rough sets etc. Later, researchers developed K. T. Atanassov's [4] intuitionistic fuzzy set theory in many fields such as differential equations, topology, computer science and so on. F. Smarandache [20, 21] found that some objects have indeterminacy or neutral other than membership and non-membership. So he coined the notion of neutrosophy. Researchers [12, 15–18] applied the concept of neutrosophy when object has inconsistent, incomplete information. The universal set X and $\emptyset$ forms a topology (Munkrer [11]). Popa [14] introduced minimal structures and defined separation axioms using minimal structure. M. Alimohammady, M. Roohi [5] introduced fuzzy minimal structure in lowen sense. S.Bhattacharya (Halder) [6] presented the concept of intuitionistic fuzzy minimal space.

1.1. *Motivation and Objective*

In general topology, the whole set and empty set forms a space with minimal structure. Supra topological space is also a space with neutrosophic minimal structure. These are all the generalization of topological spaces. Our objective is to introduce neutrosophic universal set and neutrosophic null set with neutrosophic minimal structure. It is a generalization of neutrosophic topological space. This paper consisting of basic definitions such as interior, closure, open, closed, subspace with minimal structure and its properties.

1.2. *Limitations*

Neutrosophic topological space, neutrosophic supra topological space are space with neutrosophic minimal structure. The converse is not true that is space with neutrosophic minimal structure is not a neutrosophic supra topological space or neutrosophic topological space.

In section 1, the basic definitions are presented which are useful for our paper and in section 2, the basic definitions of neutrosophic minimal structure space are presented. In further sections some properties of neutrosophic minimal structure space are also investigated. Finally, we introduced an algorithm to solve some applications of neutrosophic minimal structure space. Note that neutrosophic topological space, neutrosophic supra topological space are neutrosophic minimal structure space but converse is not true.

2. **Preliminaries**

In this section, we presented the basic definitions developed by [15, 19–21].

Definition 2.1. [20, 21] A neutrosophic set(in short NS) U on a set $X \neq \emptyset$ is defined by

$U = \{\langle a, T_U(a), I_U(a), F_U(a)\rangle : a \in X\}$ where $T_U : X \to [0,1]$, $I_U : X \to [0,1]$ and $F_U : X \to [0,1]$ denotes the membership of an object, indeterminacy and non-membership of an object, for each $a \in X$ to U, respectively and $0 \leq T_U(a) + I_U(a) + F_U(a) \leq 3$ for each $a \in X$.

Definition 2.2. [19] Let $U = \{\langle a, T_U(a), I_U(a), F_U(a)\rangle : a \in X\}$ be a neutrosophic set.

(i) A neutrosophic set U is an empty set i.e., $U = 0_\sim$ if 0 is membership of an object and 1 is an indeterminacy and non-membership of an object respectively. i.e., $0_\sim = \{x, (0,1,1) : x \in X\}$

(ii) A neutrosophic set U is a universal set i.e., $U = 1_\sim$ if 1 is membership of an object and 0 is an indeterminacy and non-membership of an object respectively. $1_\sim = \{x, (1,0,0) : x \in X\}$

(iii) $U_1 \cup U_2 = \{a, max\{T_{U_1}(a), T_{U_2}(a)\}, min\{I_{U_1}(a), I_{U_2}(a)\}, min\{F_{U_1}(a), F_{U_2}(a)\} : a \in X\}$

(iv) $U_1 \cap U_2 = \{a, min\{T_{U_1}(a), T_{U_2}(a)\}, max\{I_{U_1}(a), I_{U_2}(a)\}, max\{F_{U_1}(a), F_{U_2}(a)\} : a \in X\}$

(v) $U_1^C = \{a, F_U(a), 1 - I_U(a), T_U(a) : a \in X\}$

Definition 2.3. [19] A neutrosophic topology (NT) in Salama's sense on a nonempty set X is a family τ of NSs in X satisfying three axioms:

(1) Empty set ($0_\sim$) and universal set($1_\sim$) are members of τ.

(2) $U_1 \cap U_2 \in \tau$ where $U_1, U_2 \in \tau$.

(3) $\bigcup_{i=1}^{\infty} U_i \in \tau$ where each $U_i \in \tau$.

Each neutrosophic sets in neutrosophic topological spaces are called neutrosophic open sets. Its complements are called neutrosophic closed sets.

Definition 2.4. [19] Let NS U in NTS X. Then a neutrosophic interior of U and a neutrosophic closure of U are defined by

n-int(U) = max $\{F : F\, is\, an\, Neutrosophic\, open\, set\, in\, X\, and\, F \leq U\}$ and

n-cl(U) = min $\{F : F\, is\, an\, Neutrosophic\, closed\, set\, inX\, and\, F\, \geq U\}$ respectively.

Definition 2.5. [15] A neutrosophic supra topology (in short, NST) on a nonempty set X is a family τ of NSs in X satisfying the following axioms:

(1) Empty set ($0_\sim$) and universal set($1_\sim$) are members of τ.

(2) $\bigcup_{i=1}^{\infty} U_i \in \tau$ where each $U_i \in \tau$.

3. Neutrosophic Minimal Structure Spaces

Neutrosophic minimal structure space is defined and studied its properties in this section.

Definition 3.1. Let the neutrosophic minimal structure space over a universal set X be denoted by N_m. N_m is said to be neutrosophic minimal structure space (in short, NMS) over X if it satisfying following the axiom:

(1) $0_\sim, 1_\sim \in N_m$.

A family of neutrosophic minimal structure space is denoted by (X, N_{mX})

Note that neutrosophic empty set and neutrosophic universal set can form a topology and it is known as neutrosophic minimal structure space.

Each neutrosophic set in neutrosophic minimal structure space is neutrosophic minimal open set.

The complement of neutrosophic minimal open set is neutrosophic minimal closed set.

Remark 3.2. Each neutrosophic set in neutrosophic minimal structure space is neutrosophic minimal open set.
The complement of neutrosophic minimal open set is neutrosophic minimal closed set.
In this paper, we refer definition 2.2 for basic operations.

Example 3.3. We know that $0_\sim = \{x, (0,1,1)\}$ & $1_\sim = \{x, (1,0,0)\}$ are neutrosophic minimal open sets. Lets find out their complements.
$0_\sim^C = \{x, (1,0,0)\} = 1_\sim$ and $1_\sim^C = \{x, (0,1,1)\} = 0_\sim$. This clears that $0_\sim$ and $1_\sim$ are both neutrosophic minimal open and closed set.

Remark 3.4. From Definition 3.1. the following are obvious

(1) Neutrosophic supra topological spaces are neutrosophic minimal structure space but converse not true.
(2) Similarly, Neutrosophic topological spaces are neutrosophic minimal structure space but converse is not true.

The following Example 3.5 proves the above Remark 3.4.

Example 3.5. Let $A = \{< 0.6, 0.4, 0.3 >: a\}$, $B = \{< 0.6, 0.5, 0.1 >: a\}$ are neutrosophic sets over the universal set $X = \{a\}$. Then the neutrosophic minimal structure space is $N_m = \{0, 1, A, B\}$. But N_m is not a neutrosophic topological space and not a neutrosophic supra topological space, since arbitrary union and finite intersection doesn't hold in N_m.

Definintion 3.6. A is N_m-closed if and only if $N_m cl(A) = A$.
Similarly, A is a N_m-open if and only if $N_m int(A) = A$.

Definintion 3.7. Let N_m be any neutrosophic minimal structure space and A be any neutrosophic set. Then

(1) Every $A \in N_m$ is open and its complement is closed.
(2) N_m-closure of $A = min\{F : F\ is\ a\ neutrosophic\ minimal\ closed\ set\ and\ F \geq A\}$ and its denoted by $N_m cl(A)$.
(3) N_m-interior of $A = max\{F : F\ is\ a\ neutrosophic\ minimal\ open\ set\ and\ F \leq A\}$ and it is denoted by $N_m int(A)$.

In general $N_m int(A)$ is subset of A and A is a subset of $N_m cl(A)$.

Proposition 3.8. Suppose A and B are any neutrosophic set of neutrosophic minimal structure space N_m over X. Then

i. $N_m^C = \{0, 1, A_i^C\}$ where A_i^C is a complement of neutrosophic set A_i.
ii. $X - N_m int(B) = N_m cl(X - B)$.
iii. $X - N_m cl(B) = N_m int(X - B)$.
iv. $N_m cl(A^C) = (N_m cl(A))^C = N_m int(A)$.
v. N_m closure of an empty set is an empty set and N_m closure of a universal set is a universal set. Similarly, N_m interior of an empty set and universal set respectively an empty and a universal set.
vi. If B is a subset of A then $N_m cl(B) \leq N_m cl(A)$ and $N_m int(B) \leq N_m int(A)$.
vii. $N_m cl(N_m cl(A)) = N_m cl(A)$ and $N_m int(N_m int(A)) = N_m int(A)$.
viii. $N_m cl(A \vee B) = N_m cl(A) \vee N_m cl(B)$
ix. $N_m cl(A \wedge B) = N_m cl(A) \wedge N_m cl(B)$

Proof. (i) We know that $A^C = X - A$. Then $N_m cl(X - A) = N_m cl(A^C) = (N_m cl(A))^C = N_m int(A)$, from (iv).
Similarly for (ii).
(vi) Let $B \leq A$. We know that $B \leq N_m cl(B)$ and $A \leq N_m cl(B)$. So $B \leq N_m cl(B) \leq A \leq N_m cl(A)$. Therefore $N_m cl(B) \leq N_m cl(A)$.
Proof of (vii) is straight forward.
(viii) We know that $A \leq A \vee B$ and $B \leq A \vee B$. $N_m cl(A) \leq N_m cl(A \vee B)$ and $N_m cl(B) \leq N_m cl(A \vee B)$ this implies $N_m cl(A) \vee N_m cl(B) \leq N_m cl(A \vee B) . \longrightarrow (*)$
Also $A \leq N_m cl(A)$ and $B \leq N_m cl(B) \Rightarrow A \vee B \leq N_m cl(A) \vee N_m cl(B)$. $N_m cl(A \vee B) \leq N_m cl(N_m cl(A) \vee N_m cl(B)) = N_m cl(A) \vee N_m cl(B) \longrightarrow (**)$.
From (*) and (**), we have $N_m cl(A \vee B) = N_m cl(A) \vee N_m cl(B)$.

Example 3.9. Consider Example 3.5, the complement of N_m is $\{0, 1, A^C, B^C\}$ where $A^C = \{< 1 - 0.6, 1 - 0.4, 1 - 0.3 > /a : a \in X\} = \{< 0.4, 0.6, 0.7 > /a : a \in X\}$ and $B^C = \{< 1 - 0.6, 1 - 0.5, 1 - 0.1 > /a : a \in X\} = \{< 0.4, 0.5, 0.9 > /a : a \in X\}$.

Defininition 3.10. A function $f : (X, N_{mX}) \to (Y, N_{mY}))$ is called neutrosophic minimal continuous function if and only if $f^{-1}(V) \in N_{mX}$ whenever $V \in N_{mY}$.

Defininition 3.11. Boundary of a neutrosophic set A (in short Bd(A)) of neutrosophic minimal structure (X, N_{mX}) is the intersection of $N_m closure$ of the set A and $N_m closure$ of $X - A$. i.e., $Bd(A) = N_m cl(A) \cap N_m cl(X - A)$

Theorem 3.12. If (X, N_{mX}) and (Y, N_{mY}) are neutrosophic minimal structure space . Then

(1) Identity map from (X, N_{mX}) to (Y, N_{mY}) is a neutrosophic minimal continuous function.

(2) Any constant function which maps from (X, N_{mX}) to (Y, N_{mY}) is a neutrosophic minimal continuous function.

Proof. The proof is obvious.

Theorem 3.13. Let the map f from neutrosophic minimal structure space (X, N_{mX}) to neutrosophic minimal structure space (Y, N_{mY}). Then the following are equivalent,

(1) The map f is a neutrosophic minimal continuous function.

(2) $f^{-1}(V)$ is a neutrosophic minimal closed set for each neutrosophic minimal closed set $V \in N_{mY}$. .

(3) $N_m cl(f^{-1}(V)) \leq f^{-1}(N_m cl(V))$, for each $V \in N_{mY}$.

(4) $N_m cl(f(A)) \geq f(N_m cl(A))$, for each $A \in N_{mX}$.

(5) $N_m int(f^{-1}(V)) \geq f^{-1}(N_m int(V))$, for each $V \in N_{mX}$.

Proof. (1) $\Rightarrow$ (2): Let A be a N_m-closed in Y. Then $f^{-1}(A)^C = f^{-1}(A^C) \in N_{mX}$.

(2) $\Rightarrow$ (3): $N_m cl(f^{-1}(A) = \wedge\{D : f^{-1}(A) \leq D, D^C \in N_{mX}\} \leq \wedge\{f^{-1}(D) : A \leq D, D^C \in N_{mY}\} = f^{-1}(\{D : A \leq D^C \in N_{mY}\}) = f^{-1}(N_m cl(A))$.

(3) $\Rightarrow$ (4): Since $A \leq f^{-1}(f(A))$, then $N_m cl(A) \leq N_m cl f^{-1}(f(A)) \leq f^{-1}(N_m cl(f(A)))$. Therefore $f(N_m cl(A)) \leq N_m cl(f(A))$.

(4) $\Rightarrow$ (5): $f(N_m int(f^{-1}(A)))^C = f(N_m cl(f^{-1}(A))^C) = f(N_m cl(f(A)^C)) \leq N_m cl(f(f^{-1}(A^C))) \leq N_m cl(A^C) = (N_m int(A))^C$. This implies that $N_m int(f^{-1}(B))^C \leq f^{-1}(N_m int(A))^C = (f^{-1}(N_m int(A)))^C$.

Taking complement on both sides, $f^{-1}(N_m int(A)) \leq N_m int(f^{-1}(B))$.

Definition 3.14. Let (X, N_{mX}) be neutrosophic minimal structure space.

i. Arbitrary union of neutrosophic minimal open sets in (X, N_{mX}) is neutrosophic minimal open. (Union Property)

ii. Finite intersection of neutrosophic minimal open sets in (X, N_{mX}) is neutrosophic minimal open. (intersection Property)

4. Neutrosophic Minimal Subspace

In this section, we introduced the neutrosophic minimal subspace and investigate some properties of subspace.

Definition 4.1. Let A be a neutrosophic set in neutrosophic minimal structure space (X, N_{mX}). Then Y is said to be neutrosophic minimal subspace if $(Y, N_{mY}) = \{A \cap U : U \in N_{mY}\}$.

Lemma 4.2. If neutrosophic set b in the basis B for neutrosophic minimal structure space X. Then the collection $B_Y = \{b \cap Y : Y \subset X\}$ is a basis for neutrosophic minimal subspace on Y.

Proof. Given a neutrosophic set A in X and C is a neutrosophic set in both A and subset Y of X. Consider a basis element b of B such that C in b and in Y. Then $C \in B \cap Y \subset U \cap Y$. Hence B_Y is a basis for the neutrosophic minimal subspace on the set Y.

Lemma 4.3. Let (Y, N_{mY}) be a subspace of (X, N_{mX}). If A is a neutrosophic set in Y and $Y \subset X$. Then A is in (X, N_{mX}).

Proof. Given that neutrosophic set A in (Y, N_{mY}). $A = Y \cap B$ for some neutrosophic set $B \in X$. Since Y and B in X. Then A is in X.

Proposition 4.4. Suppose (Y, N_{mY}) is a neutrosophic minimal subspace of (X, τ_X).

(1) If the neutrosophic minimal structure space (X, N_{mX}) has the union property, then the subspace (Y, N_{mY}) also has union property.
(2) If the neutrosophic minimal structure space (X, N_{mX}) has the intersection property, then the subspace (Y, N_{mY}) also has union property.

Proof. Suppose the family of open set $\{V_i : i \in Y\}$ in neutrosophic minimal subspace(Y, N_{mY}) then there exist a family of open sets $\{U_j : j \in X\}$ in neutrosophic minimal structure space (X, N_{mX}) such that $V_i = U_j \cap A, \forall i \in Y$ where $A \in N_{mY}$. $\bigcup_{(i \in Y)} V_i = \bigcup_{(j \in X)} (U_j \cap A) = \bigcup_{(i \in Y)} U_j \cap A$. Since (X, N_{mX}) has union property then (Y, N_{mY}) also has union property. The proof of (ii) is similarly to (i).

Definition 4.5. Suppose (B, N_{mB}) and (C, N_{mC}) are neutrosophic minimal subspaces of neutrosophic minimal structure spaces (Y, N_{mY}) and (Z, N_{mZ}) respectively. Also, suppose that f is a mapping from (Y, N_{mY}) to (Z, N_{mZ}) is a mapping. We say that f is a mapping from (B, N_{mB}) into (C, N_{mC}) if the image of B under f is a subset of C.

Definition 4.6. Suppose (A, N_{mA}) and (B, N_{mB}) are neutrosophic minimal subspaces of neutrosophic minimal structure spaces (Y, N_{mY}) and (Z, N_{mZ}) respectively. The mapping f from (A, N_{mA}) into (B, N_{mB}) is called a

(1) comparative neutrosophic minimal continuous, if $f^1(W) \wedge A \in N_{mA}$ for every neutrosophic minimal structure set W in B,
(2) comparative neutrosophic minimal open, if $f(V) \in N_{mB}$ for every fuzzy set $V \in N_{mA}$.

X/E	c_1	c_2	c_3	....	c_n
a_1	d_{11}	d_{12}	d_{13}	....	d_{1n}
a_2	d_{21}	d_{22}	d_{23}	....	d_{2n}
.	.	.	.	.	.
a_m	d_{m1}	d_{m2}	d_{m3}	....	d_{mn}

TABLE 1. Attributes and Alternative

5. Applications

The application of neutrosophic minimal structure space is based on the minimal element and maximal element. In neutrosophic minimal structure space, $0_\sim$ is the minimal element and $1_\sim$ is the maximal element. The application of neutrosophic minimal structure space used in consumer theory where the customer has only two objective. In consumer theory, the customer has either minimize purchase cost and maximize the quantity or maximize the durability.

The following steps are proposed to take better decision.

Step 1.

Input m Attributes and n alternatives (See TABLE 1).

Step 2. Construct the neutrosophic minimal structure from the data. $\tau_k = \{0_\sim, 1_\sim, U_k\}$ where $U_k = \{d_{1k}, d_{2k}....d_{mk}\}$

Step 3. compute the neutrosophic score function (in short, NF) using the following simple formula, $NF(U_k) = \frac{1}{3m}[\sum_{i=1}^{m}[2 + T_i - I_i - F_i]]$

Step 4. Arrange the score function U_k which we calculated in step 3 in ascending order. Choose the largest score value U_k for better decision.

Lets consider the following example. Let the set of variety of cars be $X = \{C_1, C_2, C_3\}$ and the parameter set $E = \{a = cost\ of\ the\ car, b = safety, c = maintenance\}$. A customer will assign minimum value of $0_\sim$ to bad features,maximum $1_\sim$ to the best feature of the product. Membership, indeterminacy and non-membership values taken from customer's review rating. Membership referred to cost of the car is worth to the model, safe and low maintenance cost. Non-membership referred to cost of the car is not worth to the model, not safe due to break failure or some other reason and high maintenance cost. Indeterminacy referred to neutrality of cost of the car, safe if drive safe and maintenance is neutral. Let us assume TABLE 2. values are taken from customer review rating for the models C_1, C_2 and C_3 with parameters a, b and c.

Step 2. The neutrosophic minimal structure

$\tau_1 = \{0_\sim, 1_\sim, U_1\}$ where $U_1 = \{(0.6, 0.2, 0.4), (0.7, 0.3, 0.4), (0.6, 0.3, 0.4)\}$

Similarly, $\tau_2 = \{0_\sim, 1_\sim, U_2\}$ where $U_2 = \{(0.6, 0.3, 0.4), (0.6, 0.3, 0.4), (0.5, 0.2, 0.4)\}$

X/E	C_1	C_2	C_3
a	(0.6,0.2,0.4)	(0.6,0.3,0.4)	(0.7,0.3,0.4)
b	(0.7,0.3,0.4)	(0.6,0.3,0.4)	(0.8,0.2,0.2)
c	(0.6,0.3,0.4)	(0.5,0.2,0.4)	(0.6,0.2,0.3)

TABLE 2. Input matrix

$\tau_3 = \{0_\sim, 1_\sim, U_3\}$ where $U_3 = \{(0.7, 0.3, 0.4), (0.8, 0.2, 0.2), (0.6, 0.2, 0.3)\}$
Step 3. Neutrosophic score functions are

$$NS(U_1) = 0.6556$$
$$NS(U_2) = 0.6333$$
$$NS(U_3) = 0.7222$$

Step 4. The neutrosophic score functions are arranged in ascending order as follows $U_2 \leq U_1 \leq U_3$. Based on score function, U_3 is the largest score function. U_3 related to the model C_3. Hence Model C_3 is best to buy.

Comparison Analysis: The existing and proposed notion of neutrosophic minimal structure space is compared in the below table.

Spaces	Uncertainty	Truth value of parameter	Uncertainty of parameter	False value of parameter.
Minimal structure space	-	-	-	-
Fuzzy minimal structure space	Present	Present	-	-
Intuitionistic Minimal structure space	Present	Present	Present	-
Neutrosophic minimal structure space	Present	Present	Present	present

TABLE 3. Comparison Table

6. Conclusions

In this paper, Neutrosophic minimal structure space is introduced and some of its properties investigated along with this. Neutrosophic minimal continuous and subspace are also investigated with few properties. Finally, application of neutrosophic minimal structure space is discussed. Future work of this paper is to investigate and study various open sets and separation axioms in neutrosophic minimal structure space. Also the application part discussed in this work leads to analyze in weak structure.

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Interval Valued, m-Polar and m-Polar Interval Valued Neutrosophic Hypersoft Sets

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Abstract: Decision making is a complex issue due to vague, imprecise and indeterminate environment specially, when attributes are more than one, and further bifurcated. To solve such type of problems, concept of neutrosophic hypersoft set (NHSS) was proposed [1]. The purpose of this paper is to provide the extension of NHSS into: Interval Valued, m-Polar and m-Polar interval valued Neutrosophic Hypersoft sets. The definitions of proposed extensions and mathematical operations are discussed in detail with suitable examples. Finally, concluded the present work with the future direction.

Keywords: MCDM, Uncertainty, Soft set (SS), Neutrosophic Soft set (NS's), Hypersoft set (HS's), Neutrosophic Hypersoft set (NHSS)

1. Introduction

The concept of membership was initiated by Zadeh [2] known as fuzzy set (F's). This, concept was extended by Atanassov [3] and known as intuitionistic fuzzy set (IF's) and this concept was extended by Smarandache [4] who proposed the theory of neutrosophic set (N's) with the addition of indeterminacy value along with membership, and non-membership values. The Hybrid within neutrosophic theory was suggested by [5], the hybrids consists of; single-valued neutrosophic set (SVNS), Interval-valued neutrosophic set (IVNS) [6], multi-valued neutrosophic set (MVNS) [7]. After these generalizations many researches related to SVNS have been conducted [8–17]. Broumi *et al.* [18] merged the concept of N's and multi-valued and proposed the new idea; knowns as; multi-valued interval neutrosophic set (MVINS). Many other developments within this structure has been discussed by [19-21]. One of the most important development in the field of fuzzy was made by Molodtsov [22] who provided the idea of soft set (SS), that is very useful to deal with uncertain and vague information. In recent years, the SS theory is extended to many other theories Firstly, Fuzzy soft set theory and its properties was developed by Cagman et al. [23]. The key role in these theories was made by Maji [24] who extended the theory of NS by combining with soft set, named as neutrosophic soft set (NSS). Within this set [25] introduced some basic definitions, operations, and decision-making approaches called as IVNSS. After this, these hybrids were extended to multi-valued neutrosophic soft set (MVNSS) by [26]. Some definitions, operations and applications of MCDM approach-based problems using MVNSS was introduced [27]. Utilizing this idea a few mathematicians have proposed their examination work in various scientific fields [28-37] and this idea is likewise utilized in advancing decision-making calculations [38-42].

Smarandache [43] generalized the concept of soft set (SS) to hypersoft set (HSS) by converting the function into multi-attribute function to deal with uncertainty along with all the hybrids like crisp, fuzzy, intuitionistic and neutrosophic. Saqlain *et al.* [44] proposed the aggregate operators and similarity measure [45] on NHSS. Also, Saqlain et al. [46] developed the generalization of TOPSIS for the NHSS, by using accuracy function they transformed the fuzzy neutrosophic numbers to crisp form.

The purpose of this paper is to overcome the uncertainty problem in more precise way by combing Interval-Valued Neutrosophic set IVNSS, m-Polar Neutrosophic set mpand m-Polar Interval-Valued Neutrosophic set with Hypersoft set.

The paper presentation is as follows. Section 2, provides the basic definitions and major relation with the extension Interval-Valued, multi-polar and multi-polar Interval-valued from NHSS are presented. Section 3, proposes the basic definitions along with: subset, null set and Universal set of each type. Finally, conclusion and future direction is presented in section 4.

1.1 Motivation

From the literature, it is found that Interval Valued, m-Polar and m-Polar Interval Valued Neutrosophic Hypersoft Set respectively has not yet been studied so far. This leads us to the present study.

2.Preliminaries

Definition 2.1: IVNSS [6]

Consider $\mathbb{U}$ and $\mathbb{E}$ be universal and set of attributes respectively and consider $\mathbb{A} \subseteq \mathbb{E}$. The mapping (F, $\mathbb{A}$) is called an IVNSS over $\mathbb{U}$ and is given as;

$\text{₣}: \mathbb{A} \rightarrow \mathbb{P}(\mathbb{U})$ and $(\text{₣}, \mathbb{A}) = \{< u, \mathbb{T}(\text{₣}(\mathbb{A})), \mathbb{I}(\text{₣}(\mathbb{A})), \mathbb{F}(\text{₣}(\mathbb{A})) >, u \in \mathbb{U}\}$

Where $\mathbb{T}(\text{₣}(\mathbb{A})) \subseteq [0,1], \mathbb{I}(\text{₣}(\mathbb{A})) \subseteq [0,1]$ *and* $\mathbb{F}(\text{₣}(\mathbb{A})) \subseteq [0,1]$ are the intervals with side conditions $0 \leq sup\mathbb{T}(\text{₣}(\mathbb{A})) + sup\ \mathbb{I}(\text{₣}(\mathbb{A})) + sup\ \mathbb{F}(\text{₣}(\mathbb{A})) \leq 3$. The terms $\mathbb{T}(\text{₣}(\mathbb{A})), \mathbb{I}(\text{₣}(\mathbb{A})), \mathbb{F}(\text{₣}(\mathbb{A}))$ represent the truthiness, indeterminacy and falsity of u to $\mathbb{A}$ respectively. For our convenience,

we assume that $\mathbb{A} =< \left[\mathbb{T}(\text{₣}(\mathbb{A}))^{-}, \mathbb{T}(\text{₣}(\mathbb{A}))^{+}\right], \left[\mathbb{I}(\text{₣}(\mathbb{A}))^{-}, \mathbb{I}(\text{₣}(\mathbb{A}))^{+}\right], \left[\mathbb{F}(\text{₣}(\mathbb{A}))^{-}, \mathbb{F}(\text{₣}(\mathbb{A}))^{+}\right] >$

where;

$$\mathbb{T}(\text{₣}(\mathbb{A})) = \left[\mathbb{T}(\text{₣}(\mathbb{A}))^{-}, \mathbb{T}(\text{₣}(\mathbb{A}))^{+}\right] \subseteq [0,1], \mathbb{I}(\text{₣}(\mathbb{A})) = \left[\mathbb{I}(\text{₣}(\mathbb{A}))^{-}, \mathbb{I}(\text{₣}(\mathbb{A}))^{+}\right] \subseteq [0,1]\ and$$

$$\mathbb{F}(\text{₣}(\mathbb{A})) = \left[\mathbb{F}(\text{₣}(\mathbb{A}))^{-}, \mathbb{F}(\text{₣}(\mathbb{A}))^{+}\right] \subseteq [0,1].$$

Definition 2.2: m-Polar Neutrosophic Soft Set [26]

Consider $\mathbb{U}$ and $\mathbb{E}$ be universal and set of attributes respectively and consider $\mathbb{A} \subseteq \mathbb{E}$. The mapping (F, $\mathbb{A}$) is called an MVNSS over $\mathbb{U}$ and is given as;

$\text{₣}: \mathbb{A} \rightarrow \mathbb{P}(\mathbb{U})$ and $(\text{₣}, \mathbb{A}) = \left\{\frac{<\mathbb{T}^{x}(\text{₣}(\mathbb{A})), \mathbb{I}^{y}(\text{₣}(\mathbb{A})), \mathbb{F}^{z}(\text{₣}(\mathbb{A}))>}{u}, u \in \mathbb{U}\right\}$

Where $\mathbb{T}(\text{₣}(\mathbb{A})) \subseteq [0,1], \mathbb{I}(\text{₣}(\mathbb{A})) \subseteq [0,1]$ *and* $\mathbb{F}(\text{₣}(\mathbb{A})) \subseteq [0,1]$ are the multi-valued numbers and they are given as;

$$\mathbb{T}^{x}(\text{₣}(\mathbb{A})) = \mathbb{T}^{1}(\text{₣}(\mathbb{A})), \mathbb{T}^{2}(\text{₣}(\mathbb{A})) \dots \mathbb{T}^{x}(\text{₣}(\mathbb{A})),$$

$$\mathbb{I}^{y}(\text{₣}(\mathbb{A})) = \mathbb{I}^{1}(\text{₣}(\mathbb{A})), \mathbb{I}^{2}(\text{₣}(\mathbb{A})) \ldots \mathbb{I}^{y}(\text{₣}(\mathbb{A})),$$
$$\mathbb{F}^{z}(\text{₣}(\mathbb{A})) = \mathbb{F}^{1}(\text{₣}(\mathbb{A})), \mathbb{F}^{2}(\text{₣}(\mathbb{A})) \ldots \mathbb{F}^{z}(\text{₣}(\mathbb{A})).$$

$\mathbb{T}(\text{₣}(\mathbb{A})), \mathbb{I}(\text{₣}(\mathbb{A})), \mathbb{F}(\text{₣}(\mathbb{A}))$ represent the truthiness, indeterminacy and falsity of u to $\mathbb{A}$ respectively.

Definition 2.3: MVINSS [18]

Consider $\mathbb{U}$ and $\mathbb{E}$ be universal and set of attributes respectively and consider $\mathbb{A} \subseteq \mathbb{E}$. The mapping (F, $\mathbb{A}$) is called an MVINSS over $\mathbb{U}$ and is given as;

$$\text{₣}: \mathbb{A} \to \mathbb{P}(\mathbb{U}) \text{ and } (\text{₣}, \mathbb{A}) = \left\{\frac{<\mathbb{T}^{x}(\text{₣}(\mathbb{A})), \mathbb{I}^{y}(\text{₣}(\mathbb{A})), \mathbb{F}^{z}(\text{₣}(\mathbb{A}))>}{u}, u \in \mathbb{U}\right\}$$

Where $\mathbb{T}(\text{₣}(\mathbb{A})) \subseteq [0,1], \mathbb{I}(\text{₣}(\mathbb{A})) \subseteq [0,1]$ *and* $\mathbb{F}(\text{₣}(\mathbb{A})) \subseteq [0,1]$ are the multi-valued numbers and they are given as;

$$\mathbb{T}^{x}(\text{₣}(\mathbb{A})) = \left[\mathbb{T}^{1}(\text{₣}(\mathbb{A}))^{-}, \mathbb{T}^{1}(\text{₣}(\mathbb{A}))^{+}\right], \left[\mathbb{T}^{2}(\text{₣}(\mathbb{A}))^{-}, \mathbb{T}^{2}(\text{₣}(\mathbb{A}))^{+}\right] \ldots \left[\mathbb{T}^{x}(\text{₣}(\mathbb{A}))^{-}, \mathbb{T}^{x}(\text{₣}(\mathbb{A}))^{+}\right]$$

$$\mathbb{I}^{y}(\text{₣}(\mathbb{A})) = \left[\mathbb{I}^{1}(\text{₣}(\mathbb{A}))^{-}, \mathbb{I}^{1}(\text{₣}(\mathbb{A}))^{+}\right], \left[\mathbb{I}^{2}(\text{₣}(\mathbb{A}))^{-}, \mathbb{I}^{2}(\text{₣}(\mathbb{A}))^{+}\right] \ldots \left[\mathbb{I}^{y}(\text{₣}(\mathbb{A}))^{-}, \mathbb{I}^{y}(\text{₣}(\mathbb{A}))^{+}\right],$$

$$\mathbb{F}^{z}(\text{₣}(\mathbb{A})) = \left[\mathbb{F}^{1}(\text{₣}(\mathbb{A}))^{-}, \mathbb{F}^{1}(\text{₣}(\mathbb{A}))^{+}\right], \left[\mathbb{F}^{3}(\text{₣}(\mathbb{A}))^{-}, \mathbb{F}^{3}(\text{₣}(\mathbb{A}))^{+}\right] \ldots \left[\mathbb{F}^{z}(\text{₣}(\mathbb{A}))^{-}, \mathbb{F}^{z}(\text{₣}(\mathbb{A}))^{+}\right].$$

$\mathbb{T}(\text{₣}(\mathbb{A})), \mathbb{I}(\text{₣}(\mathbb{A})), \mathbb{F}(\text{₣}(\mathbb{A}))$ represent the truthiness, indeterminacy and falsity of u to $\mathbb{A}$ respectively.

Definition 2.4: Neutrosophic Hypersoft Set [44]

Let $\mathbb{U} = \{u^{1}, u^{2}, \ldots u^{a}\}$ and $\mathbb{P}(\mathbb{U})$ be the universal set and power set of universal set respectively, also consider $\mathbb{L}_{1}, \mathbb{L}_{2}, \ldots \mathbb{L}_{b}$ for $b \geq 1$, b well defined attributes, and corresponding attributive values are the set $\mathbb{L}_{1}^{a}, \mathbb{L}_{2}^{b}, \ldots \mathbb{L}_{b}^{z}$ and their relation $\mathbb{L}_{1}^{a} \times \mathbb{L}_{2}^{b} \times \ldots \mathbb{L}_{b}^{z}$ where $a, b, c, \ldots z = 1,2, \ldots n$ then the pair $(\mathbb{F}, \mathbb{L}_{1}^{a} \times \mathbb{L}_{2}^{b} \times \ldots \mathbb{L}_{b}^{z})$ is said to be Neutrosophic Hypersoft set over $\mathbb{U}$ where $\mathbb{F}: (\mathbb{L}_{1}^{a} \times \mathbb{L}_{2}^{b} \times \ldots \mathbb{L}_{b}^{z}) \to \mathbb{P}(\mathbb{U})$ and it is define as

$\mathbb{F}: (\mathbb{L}_{1}^{a} \times \mathbb{L}_{2}^{b} \times \ldots \mathbb{L}_{b}^{z}) \to \mathbb{P}(\mathbb{U})$ and

$\mathbb{F}: (\mathbb{L}_{1}^{a} \times \mathbb{L}_{2}^{b} \times \ldots \mathbb{L}_{b}^{z}) = \{< u, \mathbb{T}_{\ell}(u), \mathbb{I}_{\ell}(u), \mathbb{F}_{\ell}(u) > u \in \mathbb{U}, \ell \in (\mathbb{L}_{1}^{a} \times \mathbb{L}_{2}^{b} \times \ldots \mathbb{L}_{b}^{z})\}$ where $\mathbb{T}, \mathbb{I}, \mathbb{F}$ represent the truthiness, indeterminacy and falsity of u to $\mathbb{A}$ respectively such that $\mathbb{T}, \mathbb{I}, \mathbb{F}: \mathbb{U} \to [0,1]$ also $0 \leq \mathbb{T}_{\ell}(u) + \mathbb{I}_{\ell}(u) + \mathbb{F}_{\ell}(u) \leq 3$.

3. Calculations

In this section, NHSS is extended into the following:

Notions: Following abbreviation will be used throughout the article,

- Interval-valued Neutrosophic Hypersoft Set (IVNHSS)
- m-Polar Neutrosophic Hypersoft Set (m-Polar NHSS)
- m-Polar Interval-valued Neutrosophic Hypersoft Set (m-Polar IVNHSS)

Example 1: (*Following formulation and assumptions will be considered throughout as an example*)

Let $\mathbb{U}$ be the set of different schools nominated for best school given as;

$$\mathbb{U} = \{\mathbb{S}^{1}, \mathbb{S}^{2}, \mathbb{S}^{3}, \mathbb{S}^{4}, \mathbb{S}^{5}\}$$

also consider the set of attributes as;

Assumptions:

$$\mathbb{A}_1 = Teaching\ standards, \mathbb{A}_2 = Organization, \mathbb{A}_3 = Ongoing\ Evaluation, \mathbb{A}_4 = Goals$$

And their respective attributes are given as

$\mathbb{A}_1^a = Teaching\ standards = \{high, mediocre, low\}$

$\mathbb{A}_2^b = Organization = \{good, average, poor\}$

$\mathbb{A}_3^c = Ongoing\ Evaluation = \{yes, no\}$

$\mathbb{A}_4^d = Goals = \{effective, commited, upto\ date\}$

Formulation:

$$F: \mathbb{A}_1^a \times \mathbb{A}_2^b \times \mathbb{A}_3^c \times \mathbb{A}_4^d \rightarrow \mathbb{P}(\mathbb{U})$$

Let's assume $F(high\ , average, yes, effective) = \{\mathbb{S}^1, \mathbb{S}^5\}$

Then one of, ***Neutrosophic Hypersoft set NHSS*** of above assumed relation is

$F(high\ , average, yes, effective)$

$= \{\mathbb{S}^1, (high\ \{0.9, 0.3, 0.1\}, average\ \{0.8, 0.2, 0.4\}, yes\ \{0.4, 0.9, 0.6\}, effective\ \{0.6, 0.4, 0.5\}) >,$

$< \mathbb{S}^5(high\ \{0.5, 0.3, 0.8\}, average\ \{0.6, 0.1, 0.2\}, yes\ \{0.6, 0.4, 0.7\}, effective\ \{0.4, 0.5, 0.3\}) >\}$

In this Example,

Case 1: Substituting attributive values as; interval values in the form of neutrosophic, call it, IVNHSS.

Case 2: Substituting attributive values as; m-neutrosophic values, call it, m-polar NHSS.

Case 3: Substituting attributive values as; m-neutrosophic interval values, call it, IVNHSS.

3.1 Interval-valued Neutrosophic Hypersoft Set (IVNHSS)

Definition 3.1.1 IVNHSS

Let $\mathbb{U} = \{u^1, u^2, \dots u^a\}$ and $\mathbb{P}(\mathbb{U})$ be the universal set and power set of universal set respectively, also consider $\mathbb{L}_1, \mathbb{L}_2, \dots \mathbb{L}_{\mathscr{b}}$ for $\mathscr{b} \geq 1$, $\mathscr{b}$ well defined attributes, and corresponding attributive values are the set $\mathbb{L}_1^a, \mathbb{L}_2^b, \dots \mathbb{L}_{\mathscr{b}}^z$ and their relation $\mathbb{L}_1^a \times \mathbb{L}_2^b \times \dots \mathbb{L}_{\mathscr{b}}^z$ where $a, b, c, \dots z = 1,2, \dots n$ then the pair $(\mathbb{F}, \mathbb{L}_1^a \times \mathbb{L}_2^b \times \dots \mathbb{L}_{\mathscr{b}}^z)$ is said to be Interval-Valued Neutrosophic Hypersoft set IVNHSS, over $\mathbb{U}$ where $\mathbb{F}: (\mathbb{L}_1^a \times \mathbb{L}_2^b \times \dots \mathbb{L}_{\mathscr{b}}^z) \rightarrow \mathbb{P}(\mathbb{U})$ and it is define as

$$\mathbb{F}: (\mathbb{L}_1^a \times \mathbb{L}_2^b \times \dots \mathbb{L}_{\mathscr{b}}^z) \rightarrow \mathbb{P}(\mathbb{U})$$

And, $\mathbb{F}: (\mathbb{L}_1^a \times \mathbb{L}_2^b \times \dots \mathbb{L}_{\mathscr{b}}^z) = \{< u, \mathbb{T}_\ell(u), \mathbb{I}_\ell(u), \mathbb{F}_\ell(u) > u \in \mathbb{U}, \ell \in (\mathbb{L}_1^a \times \mathbb{L}_2^b \times \dots \mathbb{L}_{\mathscr{b}}^z)\}$

Where $\mathbb{T}_\ell(u) \subseteq [0,1], \mathbb{I}_\ell(u) \subseteq [0,1]\ and\ \mathbb{F}_\ell(u) \subseteq [0,1]$ are the interval numbers and $0 \leq sup\mathbb{T}_\ell(u) + sup\ \mathbb{I}_\ell(u) + sup\ \mathbb{F}_\ell(u) \leq 3$. The intervals $\mathbb{T}_\ell(u), \mathbb{I}_\ell(u), \mathbb{F}_\ell(u)$ represent the truthiness, indeterminacy and falsity of u to $\mathbb{A}$ respectively. For convenience, we assume that:

$\mathbb{A} =< \left[\left(\mathbb{T}_\ell(u)\right)^-, \left(\mathbb{T}_\ell(u)\right)^+\right], \left[\left(\mathbb{I}_\ell(u)\right)^-, \left(\mathbb{I}_\ell(u)\right)^+\right], \left[\left(\mathbb{F}_\ell(u)\right)^-, \left(\mathbb{F}_\ell(u)\right)^+\right] >$ where

$\mathbb{T}_\ell(u) = \left[\left(\mathbb{T}_\ell(u)\right)^-, \left(\mathbb{T}_\ell(u)\right)^+\right] \subseteq [0,1]$, $\mathbb{I}_\ell(u) = \left[\left(\mathbb{I}_\ell(u)\right)^-, \left(\mathbb{I}_\ell(u)\right)^+\right] \subseteq [0,1]\ and$

$\mathbb{F}_\ell(u) = \left[\left(\mathbb{F}_\ell(u)\right)^-, \left(\mathbb{F}_\ell(u)\right)^+\right] \subseteq [0,1].$

Example:

$$F: \mathbb{A}_1^a \times \mathbb{A}_2^b \times \mathbb{A}_3^c \times \mathbb{A}_4^d \rightarrow \mathbb{P}(\mathbb{U})$$

Let's assume $F(high\ , average, yes, effective) = \{\mathbb{S}^1, \mathbb{S}^5\}$

Then Interval-Valued Neutrosophic Hypersoft set of above assumed relation is

$$Ꞙ(high\ , average, yes, effective) = \{< \mathbb{S}^1, \begin{pmatrix} high < [0.4,0.9], [0.3,0.5], [0.1, 0.7] >, \\ average\ < [0.3,0.7], [0.3,0.4], [0.01, 0.17] >, \\ yes\ < [0.41,0.49], [0.03,0.15], [0.18, 0.28] >, \\ effective\ < [0.12,0.54], [0.23,0.75], [0.51, 0.81] > \end{pmatrix} >,$$

$$(high\ , average, yes, effective) = \{< \mathbb{S}^5, \begin{pmatrix} high < [0.6,0.86], [0.53,0.65], [0.71, 0.89] >, \\ average\ < [0.3,0.4], [0.31,0.55], [0.01, 0.03] >, \\ yes\ < [0.83,0.9], [0.23,0.59], [0.05, 0.09] >, \\ effective\ < [0.23,0.58], [0.03,0.3], [0.01, 0.1] > \end{pmatrix} >\}$$

Definition 3.1.2 Subset of IVNHSS

Let $\boldsymbol{\psi}_\iota, \boldsymbol{\omega}_\iota \in \boldsymbol{IVNHSS}(\mathbb{U})$. Then, $\boldsymbol{\psi}_\iota$ for all $\boldsymbol{x} \in Ꞙ$ is an IVNHSS subset of $\boldsymbol{\omega}_\iota$, denoted by $\boldsymbol{\psi}_\iota \subseteq \boldsymbol{\omega}_\iota$.If $\boldsymbol{\psi}_\iota(\boldsymbol{x}) \subseteq \boldsymbol{\omega}_\iota(\boldsymbol{x})$ for all $\boldsymbol{x} \in Ꞙ$.

Definition 3.1.3 Empty IVNHSS

Let $\boldsymbol{\psi}_\iota \in \boldsymbol{IVNHSS}(\mathbb{U})$. If $\boldsymbol{\psi}_\iota = \boldsymbol{\phi}$ for all $\boldsymbol{x} \in Ꞙ$ then $\mathbb{N}$ is called an empty IVNHSS, denoted by $\widehat{\boldsymbol{\phi}}$.

Definition 3.1.4 Universal Set of IVNHSS

Let $\boldsymbol{\psi}_\iota \in \boldsymbol{IVNHSS}(\mathbb{U})$. If $\boldsymbol{\psi}_\iota = \widehat{Ꞙ}$ for all $\boldsymbol{x} \in Ꞙ$ then $\mathbb{N}$ is called universal set of IVNHSS, denoted by $\widehat{\boldsymbol{\kappa}}$.

3.2 m-Polar Neutrosophic Hypersoft Set (m-Polar NHSS)

Definition 3.2.1 m-Polar NHSS

Let $\mathbb{U} = \{u^1, u^2, \dots u^a\}$ and $\mathbb{P}(\mathbb{U})$ be the universal set and power set of universal set respectively, also consider $\mathbb{L}_1, \mathbb{L}_2, \dots \mathbb{L}_{\mathscr{b}}$ for $\mathscr{b} \geq 1$, $\mathscr{b}$ well defined attributes, and corresponding attributive values are the set $\mathbb{L}_1^a, \mathbb{L}_2^b, \dots \mathbb{L}_{\mathscr{b}}^z$ and their relation $\mathbb{L}_1^a \times \mathbb{L}_2^b \times \dots \mathbb{L}_{\mathscr{b}}^z$ where $a, b, c, \dots z = 1,2, \dots n$ then the pair $(\mathbb{F}, \mathbb{L}_1^a \times \mathbb{L}_2^b \times \dots \mathbb{L}_{\mathscr{b}}^z)$ is said to be m-Polar Neutrosophic Hypersoft set m-Polar NHSS, over $\mathbb{U}$ where $\mathbb{F}: (\mathbb{L}_1^a \times \mathbb{L}_2^b \times \dots \mathbb{L}_{\mathscr{b}}^z) \to \mathbb{P}(\mathbb{U})$ and it is define as

$$\mathbb{F}: (\mathbb{L}_1^a \times \mathbb{L}_2^b \times \dots \mathbb{L}_{\mathscr{b}}^z) \to \mathbb{P}(\mathbb{U})$$

And,

$$\mathbb{F}: \left(\mathbb{L}_1^a \times \mathbb{L}_2^b \times \dots \mathbb{L}_{\mathscr{b}}^z\right) = \left\{ \begin{matrix} < \boldsymbol{u},\ \mathbb{T}_{\boldsymbol{\ell}}{}^{\boldsymbol{i}}(\boldsymbol{u}), \mathbb{I}_{\boldsymbol{\ell}}{}^{\boldsymbol{j}}(\boldsymbol{u}), \mathbb{F}_{\boldsymbol{\ell}}{}^{\boldsymbol{k}}(\boldsymbol{u}) > \boldsymbol{u} \in \mathbb{U}, \boldsymbol{\ell} \in \left(\mathbb{L}_1^a \times \mathbb{L}_2^b \times \dots \mathbb{L}_{\mathscr{b}}^z\right): \\ \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \dots \end{matrix} \right\} \textbf{ Also}$$

$$\mathbf{0} \leq \sum_{i=1}^{p} \mathbb{T}_{\boldsymbol{\ell}}{}^{\boldsymbol{i}}(\boldsymbol{u}) \leq \mathbf{1}, \quad \mathbf{0} \leq \sum_{j=1}^{q} \mathbb{I}_{\boldsymbol{\ell}}{}^{\boldsymbol{j}}(\boldsymbol{u}) \leq \mathbf{1}, \quad \mathbf{0} \leq \sum_{k=1}^{r} \mathbb{F}_{\boldsymbol{\ell}}{}^{\boldsymbol{k}}(\boldsymbol{u}) \leq \mathbf{1}$$

Where $\mathbb{T}_{\boldsymbol{\ell}}{}^{\boldsymbol{i}}(\boldsymbol{u}) \subseteq [\mathbf{0}, \mathbf{1}], \mathbb{I}_{\boldsymbol{\ell}}{}^{\boldsymbol{j}}(\boldsymbol{u}) \subseteq [\mathbf{0}, \mathbf{1}]$ ***and*** $\mathbb{F}_{\boldsymbol{\ell}}{}^{\boldsymbol{k}}(\boldsymbol{u}) \subseteq [\mathbf{0}, \mathbf{1}]$ are the numbers and

$$\mathbf{0} \leq \sum_{i=1}^{p} \mathbb{T}_{\boldsymbol{\ell}}{}^{\boldsymbol{i}}(\boldsymbol{u}) + \sum_{j=1}^{q} \mathbb{I}_{\boldsymbol{\ell}}{}^{\boldsymbol{j}}(\boldsymbol{u}) + \sum_{k=1}^{r} \mathbb{F}_{\boldsymbol{\ell}}{}^{\boldsymbol{k}}(\boldsymbol{u}) \leq \mathbf{3}$$

For our convenience, we assume that

$$\mathbb{T}_\ell{}^i(u) = \mathbb{T}_{\ell 1}{}^1(u), \mathbb{T}_{\ell 2}{}^2(u), \mathbb{T}_{\ell 3}{}^3(u), \dots, \mathbb{T}_{\ell p}{}^p(u)$$
$$\mathbb{I}_\ell{}^{\boldsymbol{j}}(u) = \mathbb{I}_{\ell 1}{}^{\mathbf{1}}(u), \mathbb{I}_{\ell 2}{}^2(u), \mathbb{I}_{\ell 3}{}^3(u), \dots, \mathbb{I}_{\ell} q^q(u)$$
$$\mathbb{F}_\ell{}^{\boldsymbol{k}}(u) = \mathbb{F}_{\ell 1}{}^1(u), \mathbb{F}_{\ell 2}{}^2(u), \mathbb{F}_{\ell 3}{}^{\mathbf{3}}(u), \dots, \mathbb{F}_{\ell r}{}^{\boldsymbol{r}}(u).$$

Example:

$$Ꞙ: \mathbb{A}_1^a \times \mathbb{A}_2^b \times \mathbb{A}_3^c \times \mathbb{A}_4^d \to \mathbb{P}(\mathbb{U})$$

Let's assume $\text{₣}(high\ , average, yes, effective) = \{\mathbb{S}^1, \mathbb{S}^5\}$

Then in Neutrosophic Hypersoft set of above assumed relation is,

$$\text{₣}(high\ , average, yes, effective) = \left\{ < \mathbb{S}^1, \begin{pmatrix} high < (0.01,0.003,0.1,0.023,0.07), (0.092,0.073,0.08,0.2,0.4), (0.2,0.017,0.06,0.13,0.3) >, \\ average\ < (0.2,0.1,0.5,0.019,0.051), (0.21,0.14,0.27,0.009,0.1), (0.113,0.35,0.25,0.12,0.03) >, \\ yes\ < (0.12,0.13,0.14,0.15,0.39), (0.17,0.20,0.24,0.15,0.1), (0.2,0.1,0.5,0.019,0.051) >, \\ effective\ < (0.2,0.017,0.06,0.13,0.3), (0.12,0.025,0.07,0.22,0.074), (0.01,0.003,0.1,0.023,0.07) > \end{pmatrix} > \right\}$$

.

$$\text{₣}(high\ , average, yes, effective) = \left\{ < \mathbb{S}^5, \begin{pmatrix} high < (0.09,0.08,0.7,0.0260.05), (0.04,0.03,0.02,0.1,0.09), (0.32,0.51,0.06,0.03,0.12) >, \\ average\ < (0.12,0.13,0.14,0.15,0.39), (0.17,0.20,0.24,0.15,0.1), (0.2,0.1,0.5,0.019,0.051) >, \\ yes\ < (0.09,0.08,0.7,0.0260.05), (0.04,0.03,0.02,0.1,0.09), (0.32,0.51,0.06,0.03,0.12) >, \\ effective\ < (0.12,0.13,0.14,0.15,0.39), (0.12,0.025,0.07,0.22,0.074), (0.12,0.025,0.07,0.22,0.4) > \end{pmatrix} > \right\}$$

Definition 3.2.2 Subset of m-Polar NHSS

Let $\boldsymbol{\zeta_\iota}, \boldsymbol{\delta_\iota} \in \boldsymbol{m - Polar\ NHSS}(\mathbb{U})$. Then, $\boldsymbol{\zeta_\iota}\ \ \forall\ \boldsymbol{x} \in \text{₣}$ is an m-Polar NHSS subset of $\boldsymbol{\delta_\iota}$, represented by $\boldsymbol{\zeta_\iota} \subseteq \boldsymbol{\delta_\iota}$.If $\boldsymbol{\zeta_\iota}(\boldsymbol{x}) \subseteq \boldsymbol{\delta_\iota}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \text{₣}$.

Definition 3.2.3 Empty m-Polar NHSS

Let $\boldsymbol{\zeta_\iota} \in \boldsymbol{m - Polar\ NHSS}(\mathbb{U})$. If $\boldsymbol{\zeta_\iota} = \boldsymbol{\phi}\ \ \ \forall\ \boldsymbol{x} \in \text{₣}$ then $\mathbb{N}$ is said to be an empty m-polar NHSS, represented by $\widetilde{\boldsymbol{\phi}}$.

Definition 3.2.4 Universal Set of m-Polar NHSS

Let $\boldsymbol{\zeta_\iota} \in \boldsymbol{m - Polar\ NHSS}(\mathbb{U})$. If $\boldsymbol{\zeta_\iota} = \hat{\text{₣}}\ \ \ \forall\ \boldsymbol{x} \in \text{₣}$ then $\mathbb{N}$ is called universal set of m-Polar NHSS, represented by $\widehat{\boldsymbol{\sigma}}$.

3.3 m-Polar Interval-Valued Neutrosophic Hypersoft Set (m-Polar IVNHSS)

Definition 3.3.1: m-Polar IVNHSS)

Let $\mathbb{U} = \{u^1, u^2, \dots u^a\}$ and $\mathbb{P}(\mathbb{U})$ be the universal set and power set of universal set respectively, also consider $\mathbb{L}_1, \mathbb{L}_2, \dots \mathbb{L}_{\mathscr{b}}$ for $\mathscr{b} \geq 1$, $\mathscr{b}$ well defined attributes, and corresponding attributive values are the set $\mathbb{L}_1^a, \mathbb{L}_2^b, \dots \mathbb{L}_{\mathscr{b}}^z$ and their relation $\mathbb{L}_1^a \times \mathbb{L}_2^b \times\ \dots\ \mathbb{L}_{\mathscr{b}}^z$ where $a, b, c, \dots z = 1,2, \dots n$ then the pair $(\mathbb{F}, \mathbb{L}_1^a \times \mathbb{L}_2^b \times\ \dots\ \mathbb{L}_{\mathscr{b}}^z)$ is said to be Interval-Valued Neutrosophic Hypersoft set IVNHSS, over $\mathbb{U}$ where $\mathbb{F}: (\mathbb{L}_1^a \times \mathbb{L}_2^b \times\ \dots\ \mathbb{L}_{\mathscr{b}}^z) \to \mathbb{P}(\mathbb{U})$ and it is define as

$$\mathbb{F}: (\mathbb{L}_1^a \times \mathbb{L}_2^b \times\ \dots\ \mathbb{L}_{\mathscr{b}}^z) \to \mathbb{P}(\mathbb{U})$$

$$\mathbb{F} =: \left\{ \begin{matrix} < u, \mathbb{T}_\ell{}^x(u), \mathbb{I}_\ell{}^y(u), \mathbb{F}_\ell{}^z(u) > u \in\ \mathbb{U}, \ell \in \left(\mathbb{L}_1^a \times \mathbb{L}_2^b \times\ \dots\ \mathbb{L}_{\mathscr{b}}^z\right) : \\ x, y, z = 1,2,3, \dots \end{matrix} \right\}$$

Where,

$$\mathbb{T}_\ell{}^x(u) = \left[\left(\mathbb{T}_\ell{}^x(u)\right)^-, \left(\mathbb{T}_\ell{}^x(u)\right)^+\right] \subseteq [0,1]$$

$$\mathbb{I}_\ell{}^y(u) = \left[\left(\mathbb{I}_\ell{}^y(u)\right)^-, \left(\mathbb{I}_\ell{}^y(u)\right)^+\right] \subseteq [0,1]$$

$$\mathbb{F}_\ell{}^z(u) = \left[\left(\mathbb{F}_\ell{}^z(u)\right)^-, \left(\mathbb{F}_\ell{}^z(u)\right)^+\right] \subseteq [0,1]$$

Also

$$\mathbf{0} \leq \sum_{x=1}^{s} Sup\, \{\mathbb{T}_{\boldsymbol{\ell}}{}^{x}(\boldsymbol{u})\} \leq \mathbf{1}, \qquad \mathbf{0} \leq \sum_{y=1}^{t} Sup\{\mathbb{I}_{\boldsymbol{\ell}}{}^{y}(\boldsymbol{u})\} \leq \mathbf{1}, \quad \mathbf{0} \leq \sum_{z=1}^{v} \{\mathbb{F}_{\boldsymbol{\ell}}{}^{z}(\boldsymbol{u})\} \leq \mathbf{1}$$

And,

$$\mathbf{0} \leq \sum_{x=1}^{s} Sup\, \{\mathbb{T}_{\boldsymbol{\ell}}{}^{x}(\boldsymbol{u})\} + \sum_{y=1}^{t} Sup\{\mathbb{I}_{\boldsymbol{\ell}}{}^{y}(\boldsymbol{u})\} + \sum_{z=1}^{v} \{\mathbb{F}_{\boldsymbol{\ell}}{}^{z}(\boldsymbol{u})\} \leq \mathbf{3}$$

For our convenience, we assume that:

$$\mathbb{T}_{\boldsymbol{\ell}}{}^{x}(\boldsymbol{u}) = < \left[\left(\mathbb{T}_{\ell}{}^{1}(u)\right)^{-}, \left(\mathbb{T}_{\ell}{}^{1}(u)\right)^{+}\right], \left[\left(\mathbb{T}_{\ell}{}^{2}(u)\right)^{-}, \left(\mathbb{T}_{\ell}{}^{2}(u)\right)^{+}\right], \ldots, \left[\left(\mathbb{T}_{\ell}{}^{s}(u)\right)^{-}, \left(\mathbb{T}_{\ell}{}^{s}(u)\right)^{+}\right] >$$

$$\mathbb{I}_{\boldsymbol{\ell}}{}^{y}(\boldsymbol{u}) = < \left[\left(\mathbb{I}_{\boldsymbol{\ell}}{}^{1}(\boldsymbol{u})\right)^{-}, \left(\mathbb{I}_{\boldsymbol{\ell}}{}^{\mathbf{1}}(\boldsymbol{u})\right)^{+}\right], \left[\left(\mathbb{I}_{\boldsymbol{\ell}}{}^{\mathbf{2}}(\boldsymbol{u})\right)^{-}, \left(\mathbb{I}_{\boldsymbol{\ell}}{}^{\mathbf{2}}(\boldsymbol{u})\right)^{+}\right], \ldots, \left[\left(\mathbb{I}_{\boldsymbol{\ell}}{}^{\boldsymbol{t}}(\boldsymbol{u})\right)^{-}, \left(\mathbb{I}_{\boldsymbol{\ell}}{}^{\boldsymbol{t}}(\boldsymbol{u})\right)^{+}\right] >$$

$$\mathbb{F}_{\boldsymbol{\ell}}{}^{z}(\boldsymbol{u}) = < \left[\left(\mathbb{F}_{\boldsymbol{\ell}}{}^{1}(\boldsymbol{u})\right)^{-}, \left(\mathbb{F}_{\boldsymbol{\ell}}{}^{\mathbf{1}}(\boldsymbol{u})\right)^{+}\right], \left[\left(\mathbb{F}_{\boldsymbol{\ell}}{}^{2}(\boldsymbol{u})\right)^{-}, \left(\mathbb{F}_{\boldsymbol{\ell}}{}^{2}(\boldsymbol{u})\right)^{+}\right], \ldots, \left[\left(\mathbb{F}_{\boldsymbol{\ell}}{}^{v}(\boldsymbol{u})\right)^{-}, \left(\mathbb{F}_{\boldsymbol{\ell}}{}^{v}(\boldsymbol{u})\right)^{+}\right] >$$

Example:

$$₣: \mathbb{A}_1^{a} \times \mathbb{A}_2^{b} \times \mathbb{A}_3^{c} \times \mathbb{A}_4^{d} \rightarrow \mathbb{P}(\mathbb{U})$$

Let's assume $₣(high\,, average, yes, effective) = \{\mathbb{S}^1, \mathbb{S}^5\}$

Then m-Polar Interval-Valued Neutrosophic Hypersoft set of above assumed relation is

$$₣(high\,, average, yes, effective)$$

$$= \left\{ < \mathbb{S}^1, \begin{pmatrix} high < ([0.01,0.03],[0.1,0.23],[0.07,0.5]),([0.072,0.073],[0.08,0.2],[0.14,0.32]),([0.12,0.017],[0.06,0.13],[0.3,0.4]) >, \\ average < ([0.2,0.21],[0.15,0.19],[0.02,0.03]),([0.11,0.14],[0.27,0.39],[0.1,0.11]),([0.113,0.35],[0.11,0.12],[0,0.3) >, \\ yes < ([0.12,0.13],[0.14,0.15],[0.39,0.4]),([0.17,0.20],[0.14,0.15],[0.1,0.2]),([0.1,0.11],[0.15,0.19],[0.01,0.09]) >, \\ effective < ([0.15,0.17],[0.06,0.13],[0.3,0.31]),([0.12,0.25],[0.07,0.22],[0.07,0.09]),([0.01,0.03],[0.1,0.23],[0.17,0.19]) > \end{pmatrix} > \right\}$$

.

$$₣(high\,, average, yes, effective)$$

$$= \left\{ < \mathbb{S}^5, \begin{pmatrix} high < ([0.01,0.03],[0.1,0.23],[0.07,0.5]),([0.072,0.073],[0.08,0.2],[0.14,0.32]),([0.12,0.017],[0.06,0.13],[0.3,0.4]) >, \\ average < ([0.2,0.21],[0.15,0.19],[0.02,0.03]),([0.11,0.14],[0.27,0.39],[0.1,0.11]),([0.113,0.35],[0.11,0.12],[0,0.3) >, \\ yes < ([0.12,0.13],[0.14,0.15],[0.39,0.4]),([0.17,0.20],[0.14,0.15],[0.1,0.2]),([0.1,0.11],[0.15,0.19],[0.01,0.09]) >, \\ effective < ([0.15,0.17],[0.06,0.13],[0.3,0.31]),([0.12,0.25],[0.07,0.22],[0.07,0.09]),([0.01,0.03],[0.1,0.23],[0.17,0.19]) > \end{pmatrix} > \right\}$$

Definition 3.3.2 Subset of m-Polar IVNHSS

Let $\boldsymbol{\varpi_\iota}, \boldsymbol{\beta_\iota} \in \boldsymbol{m-Polar\ IVNHSS}(\mathbb{U})$. Then, $\boldsymbol{\varpi_\iota}\ \forall\, \boldsymbol{x} \in ₣$ is an m-Polar IVNHSS subset of $\boldsymbol{\beta_\iota}$, represented by $\boldsymbol{\varpi_\iota} \subseteq \boldsymbol{\beta_\iota}$.If $\boldsymbol{\varpi_\iota}(\boldsymbol{x}) \subseteq \boldsymbol{\beta_\iota}(\boldsymbol{x})$ for all $\boldsymbol{x} \in ₣$.

Definition 3.3.3 Empty IVNHSS

Let $\boldsymbol{\varpi_\iota} \in \boldsymbol{m-Polar\ IVNHSS}(\mathbb{U})$. If $\boldsymbol{\varpi_\iota} = \boldsymbol{\phi}\ \ \forall\ \ \boldsymbol{x} \in ₣$ then $\mathbb{N}$ is said to be an empty m-Polar IVNHSS, represented by $\overline{\overline{\boldsymbol{\phi}}}$.

Definition 3.3.4 Universal Set of IVNHSS

Let $\boldsymbol{\varpi_\iota} \in \boldsymbol{m-Polar\ IVNHSS}(\mathbb{U})$. If $\boldsymbol{\varpi_\iota} = \hat{₣}\ \forall\, \boldsymbol{x} \in ₣$ then $\mathbb{N}$ is called universal set of m-Polar IVNHSS, represented by $\widehat{\boldsymbol{\Omega}}$.

4. Conclusions

In this paper, the concept of Interval Valued NHSS, m-Polar NHSS and m-Polar interval-valued NHSS are proposed. The proposed sets have several significant features. Firstly, they emphasize the hesitant, indeterminate and uncertainty and can be used more practical to solve decision-making problem. Secondly, some basic types of the proposed sets such as; universal set, empty set and subset of each type is defined.

- Since this study has not yet been studied yet, comparative study cannot be done with the existing methods.
- Further, this proposed can be applied immensely in various fields of research. In future, the present work may be extended to other special types of neutrosophic set like neutrosophic rough set etc.
- The sets which are proposed in this paper can be applied in solving supply chain, time series forecasting and decision-making problem such as partner selection, wastewater treatment selection and renewable energy selection, by defining the following:
- the aggregate operators,
- distance measures,
- matrix theory and
- Algorithms.

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