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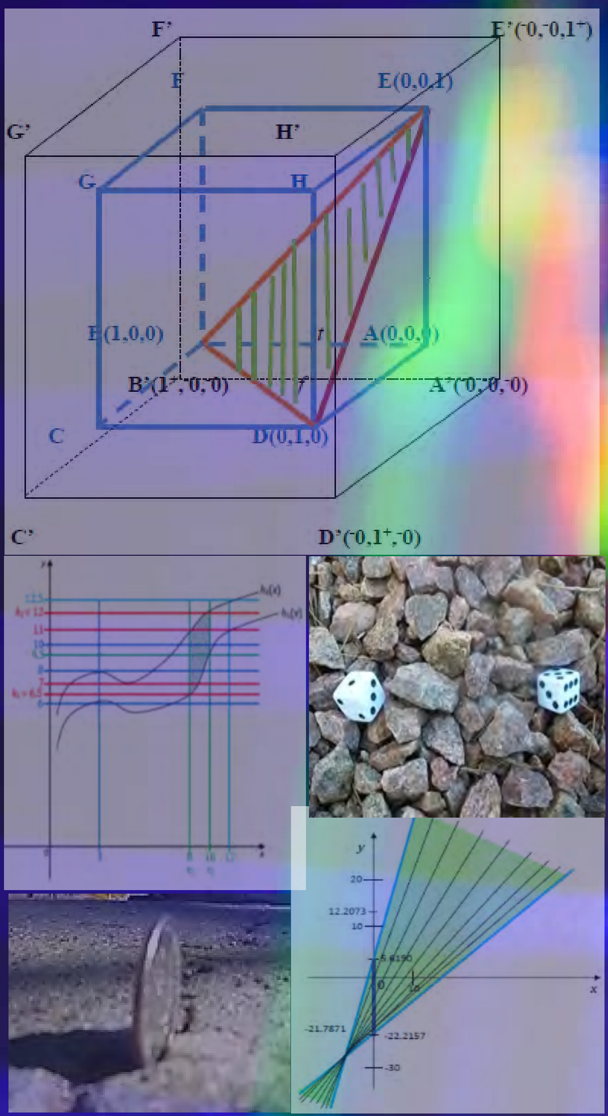

<A> <neutA> <antiA>

Florentin Smarandache . Mohamed Abdel-Basset . Said Broumi
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*"*Neutrosophic Sets and Systems*"* has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Google Dictionaries have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
FOLDOC Dictionary of Computing (**1, 2**), Webster Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

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Contents

Statistical Development of the Neutrosophic Lognormal Model with Application to Environmental Data

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Abstract: The identification of an appropriate probability model is always essential in environmental data analysis. This work presents the notion of the neutrosophic lognormal distribution (NLD) and its application to environmental data. The general structure for the density function of the NLD and its usefulness has been provided. Some more critical distributional properties such as moments, skewness and kurtosis coefficients have been derived. A methodology of estimating the distributional parameters under the neutrosophic environment is developed. The simulation study is conducted to validate the derived results for the proposed model. In the application part, a real dataset on Nitrogen oxides emissions has been analyzed using NLD to highlight the practical significance of the proposed model.

Keywords: Neutrosophic probability model; uncertainty; Lognormal model; Estimation; Simulation

1. Introduction

In pollution studies, observing environmental levels and quantifying the concentrations of different contaminants entering a particular environmental region are of considerable interest because of the potential for adverse side effects. The selection of appropriate statistical models is particularly vital in environmental studies [1]. None of the probability models containing the traditional lognormal has been superior to others [2]. These probability models could estimate the parameters needed to meet up changing information desires of an environmental quality organization [3]. Unfortunately, data on environmental pollution is often biased to the right, with a long slope towards high concentrations [4]. When using the normal distribution to certain types of data, the validity of the appropriateness may be called into question. The modelling of this type of distribution consists of transforming data values to bring changed values closer to the normal distribution [5]. In this context, logarithmic alteration is typically applied to pollution figures. Other distributions may be more appropriate for representing pollution concentration data, despite the lognormal distribution being the most commonly used [6]. Larsen [7] modified the lognormal distribution by including the third parameter, called an increment to deal with air quality data. In the Ghent region of Belgium, Berger et al. [8] analyzed fitness based on the extreme and moderate values in gamma distribution at the daily concentration of sulphur dioxide. Xiang et al. [9] also found that gamma models reflect acid-gas quantities in an industrial zone better than the lognormal model. Neither of the probability models, along with the traditional lognormal, has been preferable to others in a general logic in published literature [10].

Among these generic models, the NLD proposed in this work distribution offers a lot of promise for evaluating environmental contaminant data. Although the NLD is a relatively "unknown" distribution in environmental studies, as indicated above, its skewness and long-tail appear to create it suited for environmental pollutant data. Aside from that, the NLD group is extensive and incorporates many components that are very relatively frequent distribution when it comes to fitting pollutant concentration data. As a result, the NLD appears to be a potential candidate for environmental modeling. Smarandache's work on the idea of neutrosophy provides the inspiration for this generalization [11]. The analysis of false or true statements, but indeterminate, neutral, inconsistent, or something in between, is oriented by Neutrosophy logic [12-14]. In the actual world, there are numerous circumstances where the data that have any kind of indeterminacy [15]. The notion of neutrosophic statistics is used to cope with such data [16]. The term "neutosophic statistics" refers to the extension of conventional statistics [17]. As a result of its advantageous properties, the application of neutrosophic statistics has gained considerable attention in recent years, particularly because conventional statistics cannot be used when our data contains incomplete, vague, unclear or uncertain measurements [18-20]. The use of neutrosophic statistics in the applied research may be seen in literature such as [21–24]. The neutrosophic statistics has given rise to study areas dealing with indeterminacy effects in statistical process control [25-29].

The notion of NLD in the analysis of pollutant concentration data is proposed in the present paper when the moments, kurtosis, and skewness of other distributions do not correspond to log normality. The NLD model, which has a particular form, may offer an excellent alternative to the lognormal distribution and greater flexibility.

The rest of this work is arranged as follows: The neutrosophic extension of the lognormal distribution is established in Section 2. Section 3 explains the mathematical approach used to find unknown distributional parameters. A Monte Carlo simulation is conducted in Section 4 to validate the theoretical results of the neutrosophic model. Section 5 describes an application of the suggested model. Lastly, Section 6 outlines the main research findings.

2. Proposed Model

If $\widetilde{\mathrm{X}} = \ln \mathrm{Z}$ follows a neutrosophic normal distribution, a random variable $\mathrm{Z} > 0$ is said to follow the NLD with the density function:

$$\omega_n(z) = \frac{1}{z\sigma_n\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{ln\, z-\mu_n}{\sigma_n}\right)^2}; z > 0,\ \mu_n\ ,\ \sigma_n > 0 \quad (1)$$

where $\mu_n = [\mu_l, \mu_u]$ is the neutrosophic location, $\sigma_n = [\sigma_l, \sigma_u]$ is neutrosophic shape parameters on the log scale and Z denotes neutrosophic random variable. For the selected values of $\mu_n = [0, 0.2]$ and $\sigma_n = [0.2, 0.8]$, the neutrosophic density $\left(\widetilde{\mathrm{PDF}}\right)$ is graphically portrayed in Figure 1.

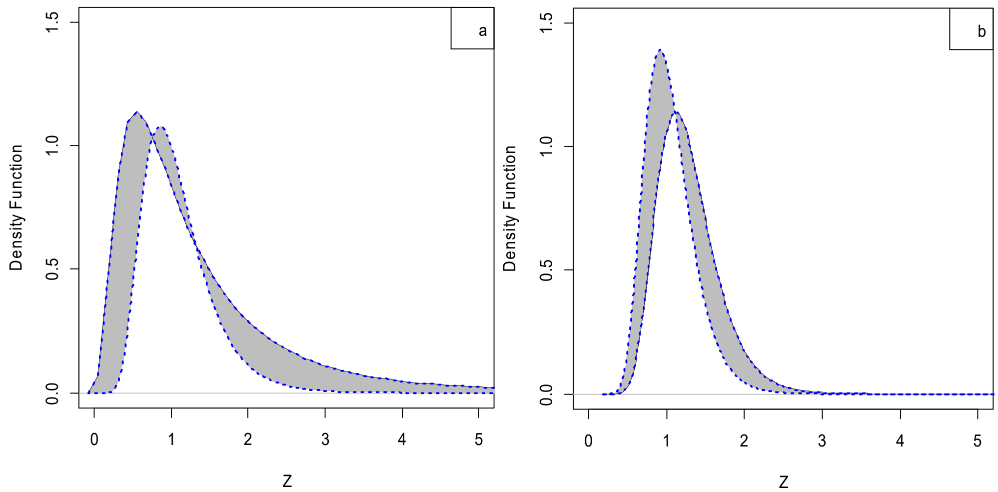

Figure 1. The density of NLD with neutrosophic parameters (a) $\sigma_n = [0.2, 0.8]$ and (b) $\mu_n = [0, 0.2]$

In Figure 1, the area under the curve indicates the interval in which the neutrosophic lognormal variable will fall. The entire area of the graph in this period equals the probability of Z occurring. The grey zone in Figure 1 represents the neutrosophic region due to uncertainties involved in distributional parameters.

Aside from specific pattern of the neutrosophic density, an analyst may be interested in seeking certain additional favourable distributional features of the NLD, which may be established in the form of some theorems below:

Theorem 1 Show that the mode of the NLD is $e^{\mu_n - \sigma_n}$

Proof: The mode of the NLD is the point at which function $\omega_n(z)$ reaches its highest or maximum value.

Therefore differentiating (1) with respect to z implies:

$$\frac{d}{dz}\,\omega_n(z) = \frac{d}{dz}\left[\frac{1}{z\sigma_l\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln z-\mu_l}{\sigma_l}\right)^2}, \qquad \frac{1}{z\sigma_u\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln z-\mu_u}{\sigma_u}\right)^2}\right]$$

$$= \left[\frac{d}{dz}\left\{\frac{1}{z\sigma_l\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln z-\mu_l}{\sigma_l}\right)^2}\right\}, \qquad \frac{d}{dz}\left\{\frac{1}{z\sigma_u\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln z-\mu_u}{\sigma_u}\right)^2}\right\}\right]$$

$$= \left[\left\{-\frac{1}{z^2\sigma_l\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln z-\mu_l}{\sigma_l}\right)^2}\left(\left(\frac{\ln z-\mu_l}{\sigma_l}\right)+1\right)\right\}, \left\{-\frac{1}{z^2\sigma_u\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln z-\mu_u}{\sigma_u}\right)^2}\left(\left(\frac{\ln z-\mu_u}{\sigma_u}\right)+1\right)\right\}\right] \quad (2)$$

To find maxima equating (2) to zero provides:

$$\left[\left\{-\frac{1}{z^2\sigma_l\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln z-\mu_l}{\sigma_l}\right)^2}\left(\left(\frac{\ln z-\ \ _l}{\sigma_l}\right)+1\right)\right\},\ \left\{-\frac{1}{z^2\sigma_u\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln z-\ \ _u}{\sigma_u}\right)^2}\left(\left(\frac{\ln z-\mu_u}{\sigma_u}\right)+1\right)\right\}\right]=[0\ ,0] \qquad (3)$$

Further simplification of (3) implies

$= [e^{\mu_l-\sigma_l},\ e^{\mu_u-\sigma_u}]$,

where $[e^{\mu_l-\sigma_l},\ e^{\mu_u-\sigma_u}] = e^{\mu_n-\sigma_n}$, hence Proved.

Theorem 2: Show that jth moment of the NLD is $e^{j\mu_n+j^2\frac{\sigma_n^2}{2}}$

Proof: By definition, the jth moment of the NLD is given by:

$$\mu'_{jn} = \int_0^\infty \frac{z^j}{z\sigma_n\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln z-\mu_n}{\sigma_n}\right)^2}dz$$

$$= \int_0^\infty z^j\left[\frac{1}{z\sigma_l\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln z-\ \ _l}{\sigma_l}\right)^2},\qquad \frac{1}{z\sigma_u\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln z-\mu_u}{\sigma_u}\right)^2}\right]dz$$

$$= \left[\int_0^\infty \frac{z^j}{z\sigma_l\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln z-\mu_l}{\sigma_l}\right)^2}dz,\ \ \int_0^\infty \frac{z^j}{z\sigma_u\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln z-\mu_u}{\sigma_u}\right)^2}dz\right] \qquad (4)$$

By substituting in (4)

$$y = \frac{\ln z - \mu_n}{\sigma_n}$$

This yielded:

$$\int_0^\infty \frac{z^j}{z\sigma_l\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln z-\mu_l}{\sigma_l}\right)^2} = e^{j\mu_l+j^2\frac{\sigma_l^2}{2}}$$

$$\int_0^\infty \frac{z^j}{z\sigma_u\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln z-\mu_u}{\sigma_u}\right)^2} = e^{j\mu_u+j^2\frac{\sigma_u^2}{2}}$$

Thus (4) becomes

$$\mu_{jn} = \left[e^{j\mu_l+j^2\frac{\sigma_l^2}{2}},\qquad e^{j\mu_u+j^2\frac{\sigma_u^2}{2}}\right]$$

Hence,

$\mu_{jn} = e^{j\mu_n + j^2\frac{\sigma_n^2}{2}}$ where $j = 1,2,3, \ldots$ is a general expression for the jth moment about the origin of the NLD.

By using the following relations, moments about mean for NLD can be derived as:

$$\mu_{1n} = \mu'_{1n} = e^{\mu_n} e^{\frac{\sigma_n^2}{2}}$$

$$\mu_{2n} = \mu'_{2n} - \left(\mu'_{1n}\right)^2 = e^{2\mu_n} e^{\sigma_n^2}\left(e^{\sigma_n^2} - 1\right)$$

$$\mu_{3n} = \mu'_{3n} - 3\mu'_{2n}\mu'_{1n} + 2\left(\mu'_{1n}\right)^3 = e^{3\mu_n} e^{3\sigma_n^2}\left(e^{\sigma_n^2} - 1\right)^2\left(e^{\sigma_n^2} - 2\right)$$

$$\mu_{4n} = \mu'_{4n} - 4\mu'_{3n}\mu'_{1n} + 6\mu'_{2n}\left(\mu'_{1n}\right)^2 - 3\left(\mu'_{1n}\right)^4 = e^{4\mu_n} e^{4\sigma_n^2}\left(e^{\sigma_n^2} - 1\right)^2\left(e^{4\sigma_n^2} + 2e^{3\sigma_n^2} + 3e^{2\sigma_n^2} - 3\right)$$

Theorem 3 Show that the coefficient of skewness for the NLD is $\left(e^{\sigma_n^2} + 2\right)\left(\sqrt{e^{\sigma_n^2} - 1}\right)$

Proof: The coefficient of skewness for NLD is given by:

$$\hat{\gamma}_{1n} = \frac{\mu_{3n}}{\left(\mu_{2n}\right)^{3/2}}, \tag{5}$$

where $\mu_{3n} = e^{3\mu_n} e^{3\sigma_n^2}\left(e^{\sigma_n^2} - 1\right)^2\left(e^{\sigma_n^2} - 2\right)$ and $\mu_{2n} = e^{2\mu_n} e^{\sigma_n^2}\left(e^{\sigma_n^2} - 1\right)$

Substituting in (5) provides:

$$\hat{\gamma}_{1n} = (e^{\sigma_n^2} + 2)\left(\sqrt{e^{\sigma_n^2} - 1}\right),$$

where $\hat{\gamma}_{1n} \in [\gamma_{1l}, \gamma_{1u}]$.

Theorem 4 Show that the coefficient of kurtosis for NLD is $e^{4\sigma_n^2} + 2e^{3\sigma_n^2} + 3e^{2\sigma_n^2} - 3$

Proof: By definition, the coefficient of kurtosis is NLD given by:

$$\hat{\gamma}_{2n} = \frac{\mu_{4n}}{\mu_{2n}^2} \tag{6}$$

where $\mu_{4n} = e^{4\mu_n} e^{4\sigma_n^2}\left(e^{\sigma_n^2} - 1\right)^2\left(e^{4\sigma_n^2} + 2e^{3\sigma_n^2} + 3e^{2\sigma_n^2} - 3\right)$ and $\mu_{2n} = e^{2\mu_n} e^{\sigma_n^2}\left(e^{\sigma_n^2} - 1\right)$

Substituting in (6) and further simplification implies:

$$\hat{\gamma}_{2n} = e^{4\sigma_n^2} + 2e^{3\sigma_n^2} + 3e^{2\sigma_n^2} - 3,$$

where $\hat{\gamma}_{2n} = [\gamma_{2l}, \ \gamma_{2u}]$.

Similarly other characteristics of the defined model may be established in a neutrosophic environment.

3. Estimation Procedure

In this part, a method for estimating the NLD parameters, known as neutrosophic maximum likelihood estimation, is devised. Let us take n samples of $Z_j, j = 1,2,..n$ values from the NLD. The question is, for an observed sample, which values of the neutrosophic parameters should be used? The likelihood function of the neutrosophic model may be used to calculate these values. Because neutrosophy is included in the distributional parameters, joint function of the NLD is given by:

$$\tau_n(\mu_n, \sigma^2{}_n|z) = \prod_{j=1}^{n} \omega_n(z_j) \tag{7}$$

The likelihood function of (7) can be written as:

$$\tau_n(\mu_n, \sigma^2{}_n|z) = \sum_{j=1}^{n} \ln(z_j) - \frac{n\ln(2\pi\sigma^2{}_n)}{2} - \frac{n\mu_n{}^2}{2\sigma^2{}_n} + \frac{\sum_{j=1}^{n}\ln(z_j)}{\sigma^2{}_n} - \frac{\sum_{j=1}^{n}\ln(z_j{}^2)}{2\sigma^2{}_n} \tag{8}$$

The gradient involving unknown values μ_n and $\sigma^2{}_n$, in order to maximize τ_n is given by:

$$\frac{\partial\tau_n(\mu_n,\sigma^2{}_n|z)}{\partial\mu_n} = \frac{\sum_{j=1}^{n}\ln(z_j)}{\sigma^2{}_n} - \frac{2n\mu_n}{2\sigma^2{}_n} \tag{9}$$

$$\frac{\partial\tau_n(\mu_n,\sigma^2{}_n|z)}{\partial\sigma^2{}_n} = -\frac{n}{2\sigma^2{}_n} + \frac{\sum_{j=1}^{n}(\ln(z_j)-\mu_n)^2}{2(\sigma^2{}_n)^2} \tag{10}$$

The simultaneous solution for unknown is obtained by setting the gradians (9) and (10) to zero as:

$$\hat{\mu}_n = \frac{\sum_{j=1}^{n}\ln(z_j)}{n}$$

$$\hat{\sigma}^2{}_n = \frac{\sum_{j=1}^{n}\left(\ln(z_j) - \frac{\sum_{j=1}^{n}\ln(z_j)}{n}\right)^2}{n}$$

where $\hat{\mu}_n = [\hat{\mu}_l, \hat{\mu}_u]$ and $\hat{\sigma}^2{}_n = [\hat{\sigma}^2{}_l, \hat{\sigma}^2{}_u]$ are the required neutrosophic estimators of the parameters μ_n and $\sigma^2{}_n$ respectively.

4. Simulation Study

In this section, analytical results of the NLD for moments, skewness, and kurtosis have been validated using the Monte Carlo simulation. The NLD can be readily simulated in R software to assess the validity of theory-based results. For this, let us set the neutrosophic parameters $\mu_n = [0.5, 1.5]$ and $\sigma_n = [0.25, 0.5]$ in the NLD and 10^5 samples are randomly generated from $U[0, 1]$. Then according to (11), 10^5 pseudo neutrosophic random samples are generated from the NLD.

$$\frac{\ln(Z_i)-\mu_n}{\sigma_n} = F^{-1}(u_i) \tag{11}$$

where $u_i \sim U[0,1]$ for $i = 1,2, \ldots.$

As mentioned in Section 2, these simulated data are used to validate the analytical characteristics. Table 1 shows the exact findings for the mean, variance, mode, skewness, and kurtosis coefficients of the NLD beside the simulated values.

Table 1. Comparisons of the simulated findings with the NLD analytical results

Characteristics	Expected Result	Simulated Result
Mean	[1.700, 5.075]	[1.700, 5.072]
Standard deviation	[0.432, 2.707]	[0.431, 2.709]
Mode	[1.548 , 3.490]	[1.546, 3.488]
Skewness Coefficient	[0.778, 1.750]	[0.771, 1.749]
Kurtosis Coefficient	[4.095, 8.898]	[4.090, 8.891]

Results in Table 1 indicate that the simulated findings match quite well with those obtained from the analytical properties of the NLD.

5. Real Application

To demonstrate the computational method of the proposed NLD model, an actual dataset on yearly Nitrogen oxides emissions for Denmark is provided. The United Nations Statistics Divisions (UNSD) have calculated Nitrogen oxides emissions per capita for the period 1990 to 2018, and the data is accessible on the site [30]. Nitrogen oxides are general names for the two most important air pollutants, nitric oxide and nitrogen dioxide. Smog and acid rain are caused by these substances, which also have an impact on tropospheric ozone. As a result of their presence in the Earth's atmosphere, they constitute one of the most significant pollutants. Naturally, nitric oxide is created during thunderstorms, but it may also be produced during agricultural fertilization. Nitrogen oxides emissions are often calculated using an international methodology based on country information on industrial, energy, waste management and agricultural production. The probability plot and essential CDF plot of the original data are depicted in Figure 2 and Figure 3, respectively.

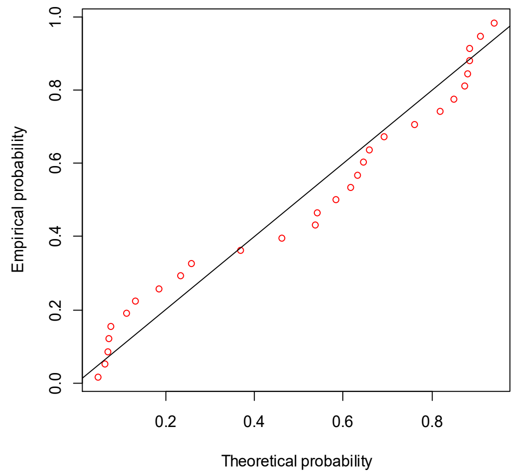

Figure 2. Probability plot for Nitrogen oxides emissions

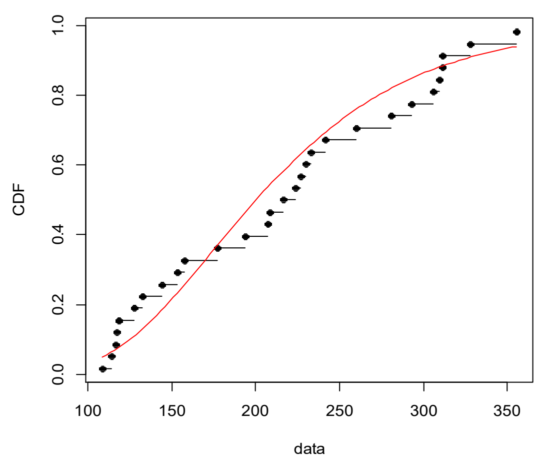

Figure 3. Empirical and theoretical CDF-plots for Nitrogen oxides emissions

Clearly, Figure 3 and Figure 4 indicate that data are skewed to the right and how well the lognormal model fitted the emission measurements. Initially, data are the crisp values however, for the sake of demonstration, we regard data as uncertain sample values for some emission values, as shown in Table 2.

Table 2. Nitrogen oxides emissions for Denmark for the period 1990-2018

Nitrogen oxides emissions
[304.12, 307.82], 355.34, 310.93, 309.47, [**309.12, 312.10**], 292.80, 327.49, 280.33, 259.99, **[238.19, 242.45]**, 229.98, 226.77, 223.57, 233.12, 216.37, 208.16, **[206.30, 209.14]**, 193.44, 177.31, 157.88, 153.18, 143.93, 132.78, 127.87, 118.28, 116.75, 116.96, 114.21, **[106.86, 110.62]**

Table 2 indicates that Nitrogen oxides emissions such as [304.12, 307.82], [309.12, 312.10], [238.19, 242.45], [206.30, 209.14] and [106.86, 110.62] are not accurately recorded to precise values but are given in intervals. Indeed, the existing lognormal model is ineffective due to ambiguity or uncertainties in the sample. On the other hand, the proposed model can easily be employed to analyze neutrosophic set of measurements. The descriptive measures using the proposed NLD are given in Table 3.

Table 3. Numerical characteristics of the for Nitrogen oxides emissions

Descriptive Measures	
Mean ${}_1$)	[214.106, 214.671]
Mode (m)	[174.617, 175.282]
Skewness (${}_{1n}$)	[1.196, 1.200]
Kurtosis (${}_{2n}$)	[5.647, 5.666]

Results in Table 3 show that the essential descriptive statistics of the Nitrogen oxides emissions are in ranges because of vagueness in the observed sample. Thus the proposed model can be applied to analyze the uncertainties involving data, which follows the NLD.

6. Conclusions

In this paper, a new generalization of the lognormal model under the neutrosophic environment, so-called the neutrosophic lognormal distribution has been proposed. This generalization is rooted in the methodology of neutrosophic algebra. The statistical characteristics of the new suggested distribution, such as moments, mode, skewness, and kurtosis, have been studied in detail. A strategy for estimating the neutrosophic distributional parameters has been developed. To investigate the validity of the analytical results produced for the suggested model, a simulation analysis has been performed. Simulated findings matched quite well with those obtained from the analytical properties of the NLD. Due to the variety of statistical properties proposed under the neutrosophic calculus, NLD can effectively be employed in analyzing real-dataset involving uncertainties as described in the application section.

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Conflicts of Interest: The authors declare no conflict of interest.

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Einstein Aggregation Operators of Simplified Neutrosophic Indeterminate Elements and Their Decision-Making Method

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Abstract: Since current decision problems are becoming more and more complex, the decision environment is becoming more and more uncertain. The simplified neutrosophic indeterminate element (SNIE) was defined to adapt to the expression of the indeterminate and inconsistent information in the indeterminate decision-making problems. SNIE consists of the truth, indeterminacy, and falsity neutrosophic numbers and can express a singled value neutrosophic element or an interval value neutrosophic element depending on the value/range of indeterminacy. In this article, we first define some operational rules of SNIEs based on the Einstein T-norm and T-conorm. Next, SNIE Einstein weighted averaging (SNIEEWA) and SNIE Einstein weighted geometric (SNIEEWG) operators are proposed to aggregate SNIEs. In view of the SNIEEWA and SNIEEWG operators, a multi-attribute decision-making (MADM) method is proposed in the case of SNIEs. Finally, the proposed MADM method is applied to solve indeterminate MADM problems in the case of SNIEs. Furthermore, the validity and effectiveness of the proposed method are verified through an illustrative example and comparative analysis.

Keywords: neutrosophic number; simplified neutrosophic indeterminate element; Einstein weighted averaging operator; Einstein weighted geometric operator

1. Introduction

The fuzzy set (FS) [1] can express a degree of truth membership, but does not express a degree of falsity membership. Therefore, an intuitionistic fuzzy set (IFS) was defined by Atanassov [2, 3], it can express the degrees of truth and falsity memberships simultaneously. Then, Atanassov and Gargov [4] introduced interval-valued IFSs (IvIFS) corresponding to the truth and falsity interval membership degrees.

Since FS, IFS, and IvIFS cannot describe inconsistent, incomplete, and indeterminate information, Smarandache [5] proposed a neutrosophic set (NS), where the truth, indeterminacy, and falsity membership degrees were described independently. Then, the three membership degrees belong to the standard interval [0, 1]/nonstandard interval $]^{-}0, 1^{+}[$. Further, Wang et al. presented a single-valued NS (SvNS) [6] and an interval-valued NS (IvNS) [7]. Next, a simplified NS (SNS) implying SvNS and IvNS introduced by Ye [8] can better apply it in real life because the truth, indeterminacy, and falsity membership degrees in SNS are described in the real unit interval [0, 1].

Since then, researches proposed various aggregation operators and decision-making (DM) methods in the cases of SvNSs and IvNSs. Meanwhile, SNSs can be used in multi-attribute decision-making (MADM) problems with SvNSs and IvNSs in the indeterminate and inconsistent situations. By combining SNS with a hesitant fuzzy notion, Liu and Shi [9] proposed single- and interval-valued neutrosophic hesitant fuzzy sets. Then, Ali et al. [10] introduced a neutrosophic cubic set by the combination of both SvNS and IvNS. The neutrosophic cubic information can represent the single- and interval-valued assessment information of decision makers, then the neutrosophic cubic MADM method [10] solved its DM problems with neutrosophic cubic information.

As another branch of neutrosophic theory, NN was first proposed by Smarandache in 1998 [11] and defined as $\delta = t + \varphi\lambda$ for indeterminacy $\lambda \in [\lambda^-, \lambda^+]$ and $t, \varphi \in \Re$, where t and $\varphi\lambda$ indicate the certain and uncertain terms of NN. Then, the NN δ is a changeable interval number $\delta = [t + \varphi\lambda^-, t + \varphi\lambda^+]$ when λ changes in the range of $\lambda \in [\lambda^-, \lambda^+]$. Therefore, NNs have been widely applied to many fields under indeterminate environment, such as optimization programming [12], mechanical fault diagnosis [13] and various DM problems [14].

With the complexity and variability of real DM problems, there may be the indeterminacy of truth, falsity, and indeterminacy degrees in indeterminate DM problems. Since SNS cannot express the indeterminacy of the three membership degrees, Du et al. [15] defined a simplified neutrosophic indeterminate set/element (SNIS/SNIE) by combining the concept of SNS with NNs, which consists of truth, indeterminacy, and falsity NNs to flexibly express the truth, falsity, and indeterminacy degrees. According to different values or ranges of $\lambda \in [\lambda^-, \lambda^+]$, SNIS can express different SvNSs or IvNSs. In [15], Du et al. proposed two weighted aggregation operators of simplified neutrosophic elements (SNEs) and established a MADM method using the SNIE weighted averaging (SNIEWA) and SNIE weighted geometric (SNIEWG) operators. Then, the Einstein T-norm and T-conorm functions [16] have been widely applied to deal with various fuzzy information [17-22], but the Einstein T-norm and T-conorm functions are not applied in the information aggregation of SNIEs. In this study, therefore, we propose the Einstein T-norm and T-conorm aggregation operators and their MADM method.

The main organization of this article is as the following. The concepts of SNS, NN, and SNIS are briefly reviewed, then the score, accuracy, and certainty functions are introduced to rank SNIEs in Section 2. The SNIE Einstein weighted averaging (SNIEEWA) and SNIE Einstein weighted geometric (SNIEEWG) operators are proposed in Section 3. Then, we put forward a MADM approach corresponding to the SNIEEWA and SNIEEWG operators in Section 4. In Section 5, the proposed MADM approach is applied to an investment selection problem of metal mines, and then its validity and flexibility are indicated by the illustrative example and comparative analysis. The last section draws conclusions and indicates future research.

2. Concepts of SNS, NN, and SNIE

Definition 1 [5, 8]. In a universe set $\tau = \{\tau_1, \tau_2, \ldots, \tau_n\}$, $N = \{<\tau_s, T(\tau_s), D(\tau_s), F(\tau_s)> | \tau_s \in \tau\}$ is defined as SNS, where $T(\tau_s), D(\tau_s), F(\tau_s) \in [0, 1]$ or $T(\tau_s), D(\tau_s), F(\tau_s) \subseteq [0, 1]$ ($s = 1, 2, \ldots, n$) are the truth, indeterminacy and falsity membership functions of τ_s to N. The component $<\tau_s, T(\tau_s), D(\tau_s), F(\tau_s)>$ in N is called SNE and is simply denoted as $N_s = <T_s, D_s, F_s>$, which includes the SvNE $N_s = <T_s, D_s, F_s>$ for $T_s, D_s, F_s \in [0, 1]$ and the IvNE $N_s = <[T_s^L, T_s^U], [D_s^L, D_s^U], [F_s^L, F_s^U]>$ for $[T_s^L, T_s^U], [D_s^L, D_s^U], [D_s^L, D_s^U] \subseteq [0, 1]$.

Definition 2 [11]. NN is described as $\delta = t + \varphi\lambda$ for indeterminacy $\lambda \in [\lambda^-, \lambda^+]$ and $t, \varphi \in \Re$, where t and $\varphi\lambda$ describe the certain term and uncertain term of NN, respectively.

Clearly, $\delta = t + \varphi\lambda$ is a changeable interval number $\delta = [t + \varphi\lambda^-, t + \varphi\lambda^+]$ when λ changes in range of $\lambda \in [\lambda^-, \lambda^+]$. It indicates that NN can flexibly express a single value or an indeterminate interval value according to the value/range of $\lambda \in [\lambda^-, \lambda^+]$.

Definition 3 [15]. For a universe set $\tau = \{\tau_1, \tau_2, \ldots, \tau_n\}$, SNIS is defined as $\chi = \{<\tau_s, T(\tau_s, \lambda), D(\tau_s, \lambda), F(\tau_s, \lambda)> | \tau_s \in \tau\}$, where $T(\tau_s, \lambda) = t_s + \alpha_s\lambda \subseteq [0, 1]$, $D(\tau_s, \lambda) = d_s + \beta_s\lambda \subseteq [0, 1]$, and $F(\tau_s, \lambda) = f_s + \gamma_s\lambda \subseteq [0,$

1] (s = 1, 2, …, n) for $\lambda \in [\lambda^-, \lambda^+]$ are the truth, indeterminacy, and falsity membership functions in χ. Then, $\chi_s = \langle T(\tau_s, \lambda), D(\tau_s, \lambda), F(\tau_s, \lambda)\rangle = \langle t_s + \alpha_s\lambda, d_s + \beta_s\lambda, f_s + \gamma_s\lambda\rangle$ is called SNIE and is simply denoted as $\chi_s = \langle T_s(\lambda), D_s(\lambda), F_s(\lambda)\rangle = \langle t_s + \alpha_s\lambda, d_s + \beta_s\lambda, f_s + \gamma_s\lambda\rangle$ for $\lambda \in [\lambda^-, \lambda^+]$.

In order to rank different SNIEs, Du et al. [15] defined three functions to compare two SNIEs.

Let $\chi_s = \langle T_s(\lambda), D_s(\lambda), F_s(\lambda)\rangle = \langle t_s + \alpha_s\lambda, d_s + \beta_s\lambda, f_s + \gamma_s\lambda\rangle$ for $\lambda \in [\lambda^-, \lambda^+]$ be any SNIE and $\lambda^* = \lambda^- + \lambda^+$. Thus, Du et al. [15] defined its score, accuracy, and certainty functions:

$$\begin{aligned} S(\chi_s,\lambda) &= \left\{4+T_s(\lambda^-)+T_s(\lambda^+)-D_s(\lambda^-)-D_s(\lambda^+)-F_s(\lambda^-)-F_s(\lambda^+)\right\}/6 \\ &= \left\{4+2t_s-2d_s-2f_s+(\alpha_s-\beta_s-\gamma_s)\lambda^*\right\}/6 \end{aligned}, \quad S(\chi_s,\lambda)\in[0,1]; \quad (1)$$

$$\begin{aligned} L(\chi_s,\lambda) &= \left\{T_s(\lambda^-)+T_s(\lambda^+)-F_s(\lambda^-)-F_s(\lambda^+)\right\}/2 \\ &= \left\{2t_s-2f_s+(\alpha_s-\gamma_s)\lambda^*\right\}/2 \end{aligned}, \quad L(\chi_s,\lambda)\in[-1,1]; \quad (2)$$

$$C(\chi_s,\lambda)=\left\{T_s(\lambda^-)+T_s(\lambda^+)\right\}/2=(2t_s+\lambda^*)/2,\ C(\chi_s,\lambda)\in[0,1]. \quad (3)$$

Suppose that $\chi_1 = \langle T_1(\lambda), D_1(\lambda), F_1(\lambda)\rangle = \langle t_1 + \alpha_1\lambda, d_1 + \beta_1\lambda, f_1 + \gamma_1\lambda\rangle$ and $\chi_2 = \langle T_2(\lambda), D_2(\lambda), F_2(\lambda)\rangle = \langle t_2 + \alpha_2\lambda, d_2 + \beta_2\lambda, f_2 + \gamma_2\lambda\rangle$ for $\lambda \in [\lambda^-, \lambda^+]$ are two SNIEs. We can rank them by the defined three functions. According to their priority, the ranking method is given as follows:

1) If $S(\chi_1, \lambda) > S(\chi_2, \lambda)$, then $\chi_1 > \chi_2$;
2) If $S(\chi_1, \lambda) = S(\chi_2, \lambda)$ and $L(\chi_1, \lambda) > L(\chi_2, \lambda)$, then $\chi_1 > \chi_2$;
3) If $S(\chi_1, \lambda) = S(\chi_2, \lambda)$, $L(\chi_1, \lambda) = L(\chi_2,\lambda)$ and $C(\chi_1, \lambda) > C(\chi_2, \lambda)$, then $\chi_1 > \chi_2$;
4) If $S(\chi_1, \lambda) = S(\chi_2, \lambda)$, $L(\chi_1, \lambda) = L(\chi_2, \lambda)$ and $C(\chi_1, \lambda) = C(\chi_2, \lambda)$, then $\chi_1 = \chi_2$.

3.The Einstein Aggregation Operations of SNIEs

3.1. Einstein T-norm and T-conorm operations of SNIEs

Definition 4 [16]. If θ and ϕ are real numbers, the Einstein T-norm function $T(\theta, \phi)$ and T-conorm $T^C(\theta, \phi)$ for $(\theta, \phi) \in [0,1]\times[0,1]$ are defined as the following formulae:

$$T(\theta,\phi)=\frac{\theta\phi}{1+(1-\theta)(1-\phi)} \quad (4)$$

$$T^C(\theta,\phi)=\frac{\theta+\phi}{1+\theta\phi} \quad (5)$$

The above functions are increasing strictly and satisfy $T(\theta, \phi), T^C(\theta, \phi) \in [0, 1]$.

According to Eqs. (4) and (5), some operations of SNIEs are defined as follows.

Definition 5. Suppose that $\chi_1 = \langle T_1(\lambda), D_1(\lambda), F_1(\lambda)\rangle = \langle t_1 + \alpha_1\lambda, d_1 + \beta_1\lambda, f_1 + \gamma_1\lambda\rangle$ and $\chi_2 = \langle T_2(\lambda), D_2(\lambda), F_2(\lambda)\rangle = \langle t_2 + \alpha_2\lambda, d_2 + \beta_2\lambda, f_2 + \gamma_2\lambda\rangle$ are two SNIEs for $t_1 + \alpha_1\lambda, d_1 + \beta_1\lambda, f_1 + \gamma_1\lambda, t_2 + \alpha_2\lambda, d_2 + \beta_2\lambda, f_2 + \gamma_2\lambda \subseteq [0, 1]$ and $\lambda \in [\lambda^-, \lambda^+]$.

$$\chi_1\oplus\chi_2=\left\langle \begin{array}{l} \left[\frac{(t_1+\alpha_1\lambda^-)+(t_2+a_2\lambda^-)}{1+(t_1+\alpha_1\lambda^-)(t_2+\alpha_2\lambda^-)},\frac{(t_1+\alpha_1\lambda^+)+(t_2+a_2\lambda^+)}{1+(t_1+\alpha_1\lambda^+)(t_2+\alpha_2\lambda^+)}\right],\left[\frac{(d_1+\beta_1\lambda^-)(d_2+\beta_2\lambda^-)}{1+(1-(d_1+\beta_1\lambda^-))(1-(d_2+\beta_2\lambda^-))},\right. \\ \left.\frac{(d_1+\beta_1\lambda^+)(d_2+\beta_2\lambda^+)}{1+(1-(d_1+\beta_1\lambda^+))(1-(d_2+\beta_2\lambda^+))}\right],\left[\frac{(f_1+\gamma_1\lambda^-)(f_2+\gamma_2\lambda^-)}{1+(1-(f_1+\gamma_1\lambda^-))(1-(f_2+\gamma_2\lambda^-))},\frac{(f_1+\gamma_1\lambda^+)(f_2+\gamma_2\lambda^+)}{1+(1-(f_1+\gamma_1\lambda^+))(1-(f_2+\gamma_2\lambda^+))}\right] \end{array}\right\rangle; \quad (6)$$

$$\chi_1\otimes\chi_2=\left\langle \begin{array}{l} \left[\frac{(t_1+\alpha_1\lambda^-)(t_2+\alpha_2\lambda^-)}{1+(1-(t_1+\alpha_1\lambda^-))(1-(t_2+\alpha_2\lambda^-))},\frac{(t_1+\alpha_1\lambda^+)(t_2+\alpha_2\lambda^+)}{1+(1-(t_1+\alpha_1\lambda^+))(1-(t_2+\alpha_2\lambda^+))}\right],\left[\frac{(d_1+\beta_1\lambda^-)+(d_2+\beta_2\lambda^-)}{1+(d_1+\beta_1\lambda^-)(d_2+\beta_2\lambda^-)},\right. \\ \left.\frac{(d_1+\beta_1\lambda^+)+(d_2+\beta_2\lambda^+)}{1+(d_1+\beta_1\lambda^+)(d_2+\beta_2\lambda^+)}\right],\left[\frac{(f_1+\gamma_1\lambda^-)+(f_2+\gamma_2\lambda^-)}{1+(f_1+\gamma_1\lambda^-)(f_2+\gamma_2\lambda^-)},\frac{(f_1+\gamma_1\lambda^+)+(f_2+\gamma_2\lambda^+)}{1+(f_1+\gamma_1\lambda^+)(f_2+\gamma_2\lambda^+)}\right] \end{array}\right\rangle; \quad (7)$$

$$\rho\chi_1=\left\langle \begin{array}{l} \left[\dfrac{(1+(t_1+\alpha_1\lambda^-))^\rho-(1-(t_1+\alpha_1\lambda^-))^\rho}{(1+(t_1+\alpha_1\lambda^-))^\rho+(1-(t_1+\alpha_1\lambda^-))^\rho},\dfrac{(1+(t_1+\alpha_1\lambda^+))^\rho-(1-(t_1+\alpha_1\lambda^+))^\rho}{(1+(t_1+\alpha_1\lambda^+))^\rho+(1-(t_1+\alpha_1\lambda^+))^\rho}\right],\left[\dfrac{2(d_1+\beta_1\lambda^-)^\rho}{(2-(d_1+\beta_1\lambda^-))^\rho+(d_1+\beta_1\lambda^-)^\rho},\right. \\ \left.\dfrac{2(d_1+\beta_1\lambda^+)^\rho}{(2-(d_1+\beta_1\lambda^+))^\rho+(d_1+\beta_1\lambda^+)^\rho}\right],\left[\dfrac{2(f_1+\gamma_1\lambda^-)^\rho}{(2-(f_1+\gamma_1\lambda^-))^\rho+(f_1+\gamma_1\lambda^-)^\rho},\dfrac{2(f_1+\gamma_1\lambda^+)^\rho}{(2-(f_1+\gamma_1\lambda^+))^\rho+(f_1+\gamma_1\lambda^+)^\rho}\right] \end{array}\right\rangle; \quad (8)$$

$$\chi_1^{\rho}=\left\langle \begin{array}{l} \left[\dfrac{2(t_1+a_1\lambda^-)^\rho}{(2-(t_1+a_1\lambda^-))^\rho+(t_1+a_1\lambda^-)^\rho},\dfrac{2(t_1+a_1\lambda^+)^\rho}{(2-(t_1+a_1\lambda^+))^\rho+(t_1+a_1\lambda^+)^\rho}\right], \\ \left[\dfrac{(1+(d_1+\beta_1\lambda^-))^\rho-(1-(d_1+\beta_1\lambda^-))^\rho}{(1+(d_1+\beta_1\lambda^-))^\rho+(1-(d_1+\beta_1\lambda^-))^\rho},\dfrac{(1+(d_1+\beta_1\lambda^+))^\rho-(1-(d_1+\beta_1\lambda^+))^\rho}{(1+(d_1+\beta_1\lambda^+))^\rho+(1-(d_1+\beta_1\lambda^+))^\rho}\right], \\ \left[\dfrac{(1+(f_1+\gamma_1\lambda^-))^\rho-(1-(f_1+\gamma_1\lambda^-))^\rho}{(1+(f_1+\gamma_1\lambda^-))^\rho+(1-(f_1+\gamma_1\lambda^-))^\rho},\dfrac{(1+(f_1+\gamma_1\lambda^+))^\rho-(1-(f_1+\gamma_1\lambda^+))^\rho}{(1+(f_1+\gamma_1\lambda^+))^\rho+(1-(f_1+\gamma_1\lambda^+))^\rho}\right] \end{array}\right\rangle. \quad (9)$$

3.2. Einstein Weighted Arithmetic Average Operator of SNIEs

Definition 6. Let $\chi = \{\chi_1, \chi_2, \ldots, \chi_n\}$ be SNIS, we can define the SNIEEWA operator:

$$SNIEEWA\left(\chi_1,\chi_2,\ldots,\chi_n\right)=\overset{n}{\underset{k=1}{\oplus}}\rho_k\chi_k, \quad (10)$$

where $\rho_k \in [0, 1]$ is the weight of χ_k for $\sum_{k=1}^{n}\rho_k = 1$.

Theorem 1. Let $\chi_k = \langle T_k(\lambda), D_k(\lambda), F_k(\lambda)\rangle = \langle t_k + \alpha_k\lambda, d_k + \beta_k\lambda, f_k + \gamma_k\lambda\rangle$ ($k = 1, 2, \ldots, n$) for $\lambda \in [\lambda^-, \lambda^+]$ be a group of SNIEs with the related weights $\rho_k \in [0, 1]$ for $\sum_{k=1}^{n}\rho_k = 1$. Thus, the Eq. (10) can be calculated by the following equation:

$$SNIEEWA(\chi_1,\chi_2,\ldots,\chi_n)=\left\langle \begin{array}{l} \left[\dfrac{\prod_{k=1}^{n}(1+t_k+\alpha_k\lambda^-)^{\rho_k}-\prod_{k=1}^{n}(1-t_k-\alpha_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{n}(1+t_k+\alpha_k\lambda^-)^{\rho_k}+\prod_{k=1}^{n}(1-t_k-\alpha_k\lambda^-)^{\rho_k}},\dfrac{\prod_{k=1}^{n}(1+t_k+\alpha_k\lambda^+)^{\rho_k}-\prod_{k=1}^{n}(1-t_k-\alpha_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{n}(1+t_k+\alpha_k\lambda^+)^{\rho_k}+\prod_{k=1}^{n}(1-t_k-\alpha_k\lambda^+)^{\rho_k}}\right], \\ \left[\dfrac{2\prod_{k=1}^{n}(d_k+\beta_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{n}(2-d_k-\beta_k\lambda^-)^{\rho_k}+\prod_{k=1}^{n}(d_k+\beta_k\lambda^-)^{\rho_k}},\dfrac{2\prod_{k=1}^{n}(d_k+\beta_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{n}(2-d_k-\beta_k\lambda^+)^{\rho_k}+\prod_{k=1}^{n}(d_k+\beta_k\lambda^+)^{\rho_k}}\right], \\ \left[\dfrac{2\prod_{k=1}^{n}(f_k+\gamma_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{n}(2-f_k-\gamma_k\lambda^-)^{\rho_k}+\prod_{k=1}^{n}(f_k+\gamma_k\lambda^-)^{\rho_k}},\dfrac{2\prod_{k=1}^{n}(f_k+\gamma_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{n}(2-f_k-\gamma_k\lambda^+)^{\rho_k}+\prod_{k=1}^{n}(f_k+\gamma_k\lambda^+)^{\rho_k}}\right] \end{array}\right\rangle. \quad (11)$$

Proof:

(1) If $n = 2$, by Eqs. (6) and (8), we can get the result:

$SNIEEWA(\chi_1,\chi_2)=\rho_1\chi_1\oplus\rho_2\chi_2$

$$=\left\langle \begin{array}{l} \left[\dfrac{(1+T_1(\lambda^-))^{\rho_1}-(1-T_1(\lambda^-))^{\rho_1}}{(1+T_1(\lambda^-))^{\rho_1}+(1-T_1(\lambda^-))^{\rho_1}},\dfrac{(1+T_1(\lambda^+))^{\rho_1}-(1-T_1(\lambda^+))^{\rho_1}}{(1+T_1(\lambda^+))^{\rho_1}+(1-T_1(\lambda^+))^{\rho_1}}\right],\left[\dfrac{2D_1(\lambda^-)^{\rho_1}}{(2-D_1(\lambda^-))^{\rho_1}+D_1(\lambda^-)^{\rho_1}},\right. \\ \left.\dfrac{2D_1(\lambda^+)^{\rho_1}}{(2-D_1(\lambda^+))^{\rho_1}+D_1(\lambda^+)^{\rho_1}}\right],\left[\dfrac{2F_1(\lambda^-)^{\rho_1}}{(2-F_1(\lambda^-))^{\rho_1}+F_1(\lambda^-)^{\rho_1}},\dfrac{2F_1(\lambda^+)^{\rho_1}}{(2-F_1(\lambda^+))^{\rho_1}+F_1(\lambda^+)^{\rho_1}}\right] \end{array}\right\rangle$$

$$\oplus\left\langle \begin{array}{l} \left[\dfrac{(1+T_2(\lambda^-))^{\rho_2}-(1-T_2(\lambda^-))^{\rho_2}}{(1+T_2(\lambda^-))^{\rho_2}+(1-T_2(\lambda^-))^{\rho_2}},\dfrac{(1+T_2(\lambda^+))^{\rho_2}-(1-T_2(\lambda^+))^{\rho_2}}{(1+T_2(\lambda^+))^{\rho_2}+(1-T_2(\lambda^+))^{\rho_2}}\right],\left[\dfrac{2D_2(\lambda^-)^{\rho_2}}{(2\text{-}D_2(\lambda^-))^{\rho_2}+D_2(\lambda^-)^{\rho_2}},\right. \\ \left.\dfrac{2D_2(\lambda^+)^{\rho_2}}{(2\text{-}D_2(\lambda^+))^{\rho_2}+D_2(\lambda^+)^{\rho_2}}\right],\left[\dfrac{2F_2(\lambda^-)^{\rho_2}}{(2\text{-}F_2(\lambda^-))^{\rho_2}+F_2(\lambda^-)^{\rho_2}},\dfrac{2F_2(\lambda^+)^{\rho_2}}{(2\text{-}F_2(\lambda^+))^{\rho_2}+F_l(\lambda^+)^{\rho_2}}\right] \end{array}\right\rangle$$

$$=\left\langle \begin{array}{l} \left[\dfrac{(1+T_l(\lambda^-))^{\rho_l}(1+T_2(\lambda^-))^{\rho_2}-(1-T_l(\lambda^-))^{\rho_l}(1-T_2(\lambda^-))^{\rho_2}}{(1+T_l(\lambda^-))^{\rho_l}(1+T_2(\lambda^-))^{\rho_2}+(1-T_l(\lambda^-))^{\rho_l}(1-T_2(\lambda^-))^{\rho_2}},\dfrac{(1+T_l(\lambda^+))^{\rho_l}(1+T_2(\lambda^+))^{\rho_2}-(1-T_l(\lambda^+))^{\rho_l}(1-T_2(\lambda^+))^{\rho_2}}{(1+T_l(\lambda^+))^{\rho_l}(1+T_2(\lambda^+))^{\rho_2}+(1-T_l(\lambda^+))^{\rho_l}(1-T_2(\lambda^+))^{\rho_2}}\right], \\ \left[\dfrac{2D_l(\lambda^-)^{\rho_l}D_2(\lambda^-)^{\rho_2}}{(2\text{-}D_l(\lambda^-))^{\rho_l}(2-D_2(\lambda^-))^{\rho_2}+D_l(\lambda^-)^{\rho_l}D_2(\lambda^-)^{\rho_2}},\dfrac{2D_l(\lambda^+)^{\rho_l}D_2(\lambda^+)^{\rho_2}}{(2\text{-}D_l(\lambda^+))^{\rho_l}(2-D_2(\lambda^+))^{\rho_2}+D_l(\lambda^+)^{\rho_l}D_2(\lambda^+)^{\rho_2}}\right], \\ \left[\dfrac{2F_l(\lambda^-)^{\rho_l}F_2(\lambda^-)^{\rho_2}}{(2-F_l(\lambda^-))^{\rho_l}((2-F_2(\lambda^-))^{\rho_2}+F_l(\lambda^-)^{\rho_l}F_2(\lambda^-)^{\rho_2}},\dfrac{2F_l(\lambda^+)^{\rho_l}F_2(\lambda^+)^{\rho_2}}{(2-F_l(\lambda^+))^{\rho_l}((2-F_2(\lambda^+))^{\rho_2}+F_l(\lambda^+)^{\rho_l}F_2(\lambda^+)^{\rho_2}}\right] \end{array}\right\rangle$$

$$=\left\langle \begin{array}{l} \left[\dfrac{\prod_{k=1}^{2}(1+T_k(\lambda^-))^{\rho_k}-\prod_{k=1}^{2}(1-T_k(\lambda^-))^{\rho_k}}{\prod_{k=1}^{2}(1+T_k(\lambda^-))^{\rho_k}+\prod_{k=1}^{2}(1-T_k(\lambda^-))^{\rho_k}},\dfrac{\prod_{k=1}^{2}(1+T_k(\lambda^+))^{\rho_k}-\prod_{k=1}^{2}(1-T_k(\lambda^+))^{\rho_k}}{\prod_{k=1}^{2}(1+T_k(\lambda^+))^{\rho_k}+\prod_{k=1}^{2}(1-T_k(\lambda^+))^{\rho_k}}\right],\left[\dfrac{2\prod_{k=1}^{2}D_k(\lambda^-)^{\rho_k}}{\prod_{k=1}^{2}(2-D_k(\lambda^-))^{\rho_k}+\prod_{k=1}^{2}D_k(\lambda^-)^{\rho_k}},\right. \\ \left.\dfrac{2\prod_{k=1}^{2}D_k(\lambda^+)^{\rho_k}}{\prod_{k=1}^{2}(2-D_k(\lambda^+))^{\rho_k}+\prod_{k=1}^{2}D_k(\lambda^+)^{\rho_k}}\right],\left[\dfrac{2\prod_{k=1}^{2}F_k(\lambda^-)^{\rho_k}}{\prod_{k=1}^{2}(2-F_k(\lambda^-))^{\rho_k}+\prod_{k=1}^{2}F_k(\lambda^-)^{\rho_k}},\dfrac{2\prod_{k=1}^{2}F_k(\lambda^+)^{\rho_k}}{\prod_{k=1}^{2}(2-F_k(\lambda^+))^{\rho_k}+\prod_{k=1}^{2}F_k(\lambda^+)^{\rho_k}}\right] \end{array}\right\rangle$$

$$=\left\langle \begin{array}{l} \left[\dfrac{\prod_{k=1}^{2}(1+t_k+\alpha_k\lambda^-)^{\rho_k}-\prod_{k=1}^{2}(1-t_k-\alpha_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{2}(1+t_k+\alpha_k\lambda^-)^{\rho_k}+\prod_{k=1}^{2}(1-t_k-\alpha_k\lambda^-)^{\rho_k}},\dfrac{\prod_{k=1}^{2}(1+t_k+\alpha_k\lambda^+)^{\rho_k}-\prod_{k=1}^{2}(1-t_k-\alpha_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{2}(1+t_k+\alpha_k\lambda^+)^{\rho_k}+\prod_{k=1}^{2}(1-t_k-\alpha_k\lambda^+)^{\rho_k}}\right], \\ \left[\dfrac{2\prod_{k=1}^{2}(d_k+\beta_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{2}(2-d_k-\beta_k\lambda^-)^{\rho_k}+\prod_{k=1}^{2}(d_k+\beta_k\lambda^-)^{\rho_k}},\dfrac{2\prod_{k=1}^{2}(d_k+\beta_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{2}(2-d_k-\beta_k\lambda^+)^{\rho_k}+\prod_{k=1}^{2}(d_k+\beta_k\lambda^+)^{\rho_k}}\right], \\ \left[\dfrac{2\prod_{k=1}^{2}(f_k+\gamma_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{2}(2-f_k-\gamma_k\lambda^-)^{\rho_k}+\prod_{k=1}^{2}(f_k+\gamma_k\lambda^-)^{\rho_k}},\dfrac{2\prod_{k=1}^{2}(f_k+\gamma_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{2}(2-f_k-\gamma_k\lambda^+)^{\rho_k}+\prod_{k=1}^{2}(f_k+\gamma_k\lambda^+)^{\rho_k}}\right] \end{array}\right\rangle.$$

(2) Set $n = m$. Then, the following formula can hold:

$$SNIEEWA(\chi_1,\chi_2,...,\chi_m)=\left\langle \begin{array}{l} \left[\dfrac{\prod_{k=1}^{m}(1+t_k+\alpha_k\lambda^-)^{\rho_k}-\prod_{k=1}^{m}(1-t_k-\alpha_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{m}(1+t_k+\alpha_k\lambda^-)^{\rho_k}+\prod_{k=1}^{m}(1-t_k-\alpha_k\lambda^-)^{\rho_k}},\dfrac{\prod_{k=1}^{m}(1+t_k+\alpha_k\lambda^+)^{\rho_k}-\prod_{k=1}^{m}(1-t_k-\alpha_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{m}(1+t_k+\alpha_k\lambda^+)^{\rho_k}+\prod_{k=1}^{m}(1-t_k-\alpha_k\lambda^+)^{\rho_k}}\right], \\ \left[\dfrac{2\prod_{k=1}^{m}(d_k+\beta_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{m}(2-d_k-\beta_k\lambda^-)^{\rho_k}+\prod_{k=1}^{m}(d_k+\beta_k\lambda^-)^{\rho_k}},\dfrac{2\prod_{k=1}^{m}(d_k+\beta_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{m}(2-d_k-\beta_k\lambda^+)^{\rho_k}+\prod_{k=1}^{m}(d_k+\beta_k\lambda^+)^{\rho_k}}\right], \\ \left[\dfrac{2\prod_{k=1}^{m}(f_k+\gamma_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{m}(2-f_k-\gamma_k\lambda^-)^{\rho_k}+\prod_{k=1}^{m}(f_k+\gamma_k\lambda^-)^{\rho_k}},\dfrac{2\prod_{k=1}^{m}(f_k+\gamma_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{m}(2-f_k-\gamma_k\lambda^+)^{\rho_k}+\prod_{k=1}^{m}(f_k+\gamma_k\lambda^+)^{\rho_k}}\right] \end{array}\right\rangle. \quad (12)$$

(3) If $n = m + 1$, according to the formulae (6), (8) and (12), we can get

$$SNIEEWA(\chi_1,\chi_2,...,\chi_{m+1}) = SNIEEWA(\chi_1,\chi_2,...,\chi_m)\oplus\rho_{m+1}\chi_{m+1}$$

$$=\left\langle\begin{array}{l}\left[\dfrac{\prod_{k=1}^{m}(1+t_k+\alpha_k\lambda^-)^{\rho_k}-\prod_{k=1}^{m}(1-t_k-\alpha_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{m}(1+t_k+\alpha_k\lambda^-)^{\rho_k}+\prod_{k=1}^{m}(1-t_k-\alpha_k\lambda^-)^{\rho_k}},\dfrac{\prod_{k=1}^{m}(1+t_k+\alpha_k\lambda^+)^{\rho_k}-\prod_{k=1}^{m}(1-t_k-\alpha_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{m}(1+t_k+\alpha_k\lambda^+)^{\rho_k}+\prod_{k=1}^{m}(1-t_k-\alpha_k\lambda^+)^{\rho_k}}\right],\\ \left[\dfrac{2\prod_{k=1}^{m}(d_k+\beta_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{m}(2-d_k-\beta_k\lambda^-)^{\rho_k}+\prod_{k=1}^{m}(d_k+\beta_k\lambda^-)^{\rho_k}},\dfrac{2\prod_{k=1}^{m}(d_k+\beta_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{m}(2-d_k-\beta_k\lambda^+)^{\rho_k}+\prod_{k=1}^{m}(d_k+\beta_k\lambda^+)^{\rho_k}}\right],\\ \left[\dfrac{2\prod_{k=1}^{m}(f_k+\gamma_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{m}(2-f_k-\gamma_k\lambda^-)^{\rho_k}+\prod_{k=1}^{m}(f_k+\gamma_k\lambda^-)^{\rho_k}},\dfrac{2\prod_{k=1}^{m}(f_k+\gamma_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{m}(2-f_k-\gamma_k\lambda^+)^{\rho_k}+\prod_{k=1}^{m}(f_k+\gamma_k\lambda^+)^{\rho_k}}\right]\end{array}\right\rangle$$

$$\oplus\left\langle\begin{array}{l}\left[\dfrac{(1+(t_{m+1}+\alpha_{m+1}\lambda^-))^{\rho_{m+1}}-(1-(t_{m+1}+\alpha_{m+1}\lambda^-))^{\rho_{m+1}}}{(1+(t_{m+1}+\alpha_{m+1}\lambda^-))^{\rho_{m+1}}+(1-(t_{m+1}+\alpha_{m+1}\lambda^-))^{\rho_{m+1}}},\dfrac{(1+(t_{m+1}+\alpha_{m+1}\lambda^+))^{\rho_{m+1}}-(1-(t_{m+1}+\alpha_{m+1}\lambda^+))^{\rho_{m+1}}}{(1+(t_{m+1}+\alpha_{m+1}\lambda^+))^{\rho_{m+1}}+(1-(t_{m+1}+\alpha_{m+1}\lambda^+))^{\rho_{m+1}}}\right],\\ \left[\dfrac{2(d_{m+1}+\beta_{m+1}\lambda^-)^{\rho_{m+1}}}{(2-(d_{m+1}+\beta_{m+1}\lambda^-))^{\rho_{m+1}}+(d_{m+1}+\beta_{m+1}\lambda^-)^{\rho_{m+1}}},\dfrac{2(d_{m+1}+\beta_{m+1}\lambda^+)^{\rho_{m+1}}}{(2-(d_{m+1}+\beta_{m+1}\lambda^+))^{\rho_{m+1}}+(d_{m+1}+\beta_{m+1}\lambda^+)^{\rho_{m+1}}}\right],\\ \left[\dfrac{2(f_{m+1}+\gamma_{m+1}\lambda^-)^{\rho_{m+1}}}{(2-(f_{m+1}+\gamma_{m+1}\lambda^-))^{\rho_{m+1}}+(f_{m+1}+\gamma_{m+1}\lambda^-)^{\rho_{m+1}}},\dfrac{2(f_{m+1}+\gamma_{m+1}\lambda^+)^{\rho_{m+1}}}{(2-(f_{m+1}+\gamma_{m+1}\lambda^+))^{\rho_{m+1}}+(f_{m+1}+\gamma_{m+1}\lambda^+)^{\rho_{m+1}}}\right]\end{array}\right\rangle$$

$$=\left\langle\begin{array}{l}\left[\dfrac{\prod_{k=1}^{m+1}(1+t_k+\alpha_k\lambda^-)^{\rho_k}-\prod_{k=1}^{m+1}(1-t_k-\alpha_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{m+1}(1+t_k+\alpha_k\lambda^-)^{\rho_k}+\prod_{k=1}^{m+1}(1-t_k-\alpha_k\lambda^-)^{\rho_k}},\dfrac{\prod_{k=1}^{m+1}(1+t_k+\alpha_k\lambda^+)^{\rho_k}-\prod_{k=1}^{m+1}(1-t_k-\alpha_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{m+1}(1+t_k+\alpha_k\lambda^+)^{\rho_k}+\prod_{k=1}^{m+1}(1-t_k-\alpha_k\lambda^+)^{\rho_k}}\right],\\ \left[\dfrac{2\prod_{k=1}^{m+1}(d_k+\beta_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{m+1}(2-d_k-\beta_k\lambda^-)^{\rho_k}+\prod_{k=1}^{m+1}(d_k+\beta_k\lambda^-)^{\rho_k}},\dfrac{2\prod_{k=1}^{m+1}(d_k+\beta_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{m+1}(2-d_k-\beta_k\lambda^+)^{\rho_k}+\prod_{k=1}^{m+1}(d_k+\beta_k\lambda^+)^{\rho_k}}\right],\\ \left[\dfrac{2\prod_{k=1}^{m+1}(f_k+\gamma_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{m+1}(2-f_k-\gamma_k\lambda^-)^{\rho_k}+\prod_{k=1}^{m+1}(f_k+\gamma_k\lambda^-)^{\rho_k}},\dfrac{2\prod_{k=1}^{m+1}(f_k+\gamma_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{m+1}(2-f_k-\gamma_k\lambda^+)^{\rho_k}+\prod_{k=1}^{m+1}(f_k+\gamma_k\lambda^+)^{\rho_k}}\right]\end{array}\right\rangle.$$

Thus, we have proved that the Eq. (11) can hold for any k.

The SNIEEWA operator implies the following properties.

(P1) Idempotency: Set $\chi_k = <T_k(\lambda), D_k(\lambda), F_k(\lambda)> = <t_k + \alpha_k\lambda, d_k + \beta_k\lambda, f_k + \gamma_k\lambda>$ as a group of SNIEs for $\lambda\in[\lambda^-, \lambda^+]$ and $k = 1, 2, \ldots, n$. If $\chi_k = \chi$, then $SNIEEWA(\chi_1, \chi_2, \ldots, \chi_n) = \chi$.

(P2) Boundedness: Set $\chi_k = <T_k(\lambda), D_k(\lambda), F_k(\lambda)> = <t_k + \alpha_k\lambda, d_k + \beta_k\lambda, f_k + \gamma_k\lambda>$ as a group of SNIEs for $\lambda\in[\lambda^-, \lambda^+]$ and $k = 1, 2, \ldots, n$. Let the minimum and maximum SNIEs be

$$\chi_{\min}=\left\langle\left[\min_k(t_k+\alpha_k\lambda^-),\min_k(t_k+\alpha_k\lambda^+)\right],\left[\max_k(d_k+\beta_k\lambda^-),\max_k(d_k+\beta_k\lambda^+)\right],\left[\max_k(f_k+\gamma_k\lambda^-),\max_k(f_k+\gamma_k\lambda^+)\right]\right\rangle,$$

$$\chi_{\max}=\left\langle\left[\max_k(t_k+\alpha_k\lambda^-),\max_k(t_k+\alpha_k\lambda^+)\right],\left[\min_k(d_k+\beta_k\lambda^-),\min_k(d_k+\beta_k\lambda^+)\right],\left[\min_k(f_k+\gamma_k\lambda^-),\min_k(f_k+\gamma_k\lambda^+)\right]\right\rangle.$$

Then, there is $\chi_{\min} \le SNIEEWA(\chi_1, \chi_2, \ldots, \chi_n) \le \chi_{\max}$.

(P3) Monotonicity: Set $\chi_k = <T_k(\lambda), D_k(\lambda), F_k(\lambda)>$ and $\chi_k^* = <T_k^*(\lambda), D_k^*(\lambda), F_k^*(\lambda)>$ as two groups of SNIEs for $\lambda\in[\lambda^-, \lambda^+]$ and $k = 1, 2, \ldots, n$. If $\chi_k \subseteq \chi_k^*$, then $SNIEEWA(\chi_1, \chi_2, \ldots, \chi_n) \subseteq SNIEEWA(\chi_1^*, \chi_2^*, \ldots, \chi_n^*)$.

Next, we give proofs of the three properties.

Proof：

(P1) Let $\chi_k = <t_k + \alpha_k\lambda, d_k + \beta_k\lambda, f_k + \gamma_k\lambda> = \chi = <t + \alpha\lambda, d + \beta\lambda, f + \gamma\lambda>$ for $k = 1, 2, \ldots, n$ be SNIE with the related weight $\rho_k \in [0, 1]$ for $\sum_{k=1}^{n}\rho_k = 1$. Then, we can get the result:

$$SNIEEWA(\chi_1,\chi_2,...,\chi_n)$$

$$= \left\langle \begin{array}{l} \left[\frac{\prod_{k=1}^{n}(1+t_k+\alpha_k\lambda^-)^{\rho_k} - \prod_{k=1}^{n}(1-t_k-\alpha_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{n}(1+t_k+\alpha_k\lambda^-)^{\rho_k} + \prod_{k=1}^{n}(1-t_k-\alpha_k\lambda^-)^{\rho_k}}, \frac{\prod_{k=1}^{n}(1+t_k+\alpha_k\lambda^+)^{\rho_k} - \prod_{k=1}^{n}(1-t_k-\alpha_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{n}(1+t_k+\alpha_k\lambda^+)^{\rho_k} + \prod_{k=1}^{n}(1-t_k-\alpha_k\lambda^+)^{\rho_k}} \right], \\ \left[\frac{2\prod_{k=1}^{n}(d_k+\beta_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{n}(2-d_k-\beta_k\lambda^-)^{\rho_k} + \prod_{k=1}^{n}(d_k+\beta_k\lambda^-)^{\rho_k}}, \frac{2\prod_{k=1}^{n}(d_k+\beta_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{n}(2-d_k-\beta_k\lambda^+)^{\rho_k} + \prod_{k=1}^{n}(d_k+\beta_k\lambda^+)^{\rho_k}} \right], \\ \left[\frac{2\prod_{k=1}^{n}(f_k+\gamma_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{n}(2-f_k-\gamma_k\lambda^-)^{\rho_k} + \prod_{k=1}^{n}(f_k+\gamma_k\lambda^-)^{\rho_k}}, \frac{2\prod_{k=1}^{n}(f_k+\gamma_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{n}(2-f_k-\gamma_k\lambda^+)^{\rho_k} + \prod_{k=1}^{n}(f_k+\gamma_k\lambda^+)^{\rho_k}} \right] \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \left[\frac{(1+t+\alpha\lambda^-)^{\sum_{k=1}^{n}\rho_k} - (1-t-\alpha\lambda^-)^{\sum_{k=1}^{n}\rho_k}}{(1+t+\alpha\lambda^-)^{\sum_{k=1}^{n}\rho_k} + (1-t-\alpha\lambda^-)^{\sum_{k=1}^{n}\rho_k}}, \frac{(1+t+\alpha\lambda^+)^{\sum_{k=1}^{n}\rho_k} - (1-t-\alpha\lambda^+)^{\sum_{k=1}^{n}\rho_k}}{(1+t+\alpha\lambda^+)^{\sum_{k=1}^{n}\rho_k} + (1-t-\alpha\lambda^+)^{\sum_{k=1}^{n}\rho_k}} \right], \\ \left[\frac{2(d+\beta\lambda^-)^{\sum_{k=1}^{n}\rho_k}}{(2-d-\beta\lambda^-)^{\sum_{k=1}^{n}\rho_k} + (d+\beta\lambda^-)^{\sum_{k=1}^{n}\rho_k}}, \frac{2(d+\beta\lambda^+)^{\sum_{k=1}^{n}\rho_k}}{(2-d-\beta\lambda^+)^{\sum_{k=1}^{n}\rho_k} + (d+\beta\lambda^+)^{\sum_{k=1}^{n}\rho_k}} \right], \\ \left[\frac{2(f+\gamma\lambda^-)^{\sum_{k=1}^{n}\rho_k}}{(2-f-\gamma\lambda^-)^{\sum_{k=1}^{n}\rho_k} + (f+\gamma\lambda^-)^{\sum_{k=1}^{n}\rho_k}}, \frac{2(f+\gamma\lambda^+)^{\sum_{k=1}^{n}\rho_k}}{(2-f-\gamma\lambda^+)^{\sum_{k=1}^{n}\rho_k} + (f+\gamma\lambda^+)^{\sum_{k=1}^{n}\rho_k}} \right] \end{array} \right\rangle$$

$$= \left\langle \left[t+\alpha\lambda^-, t+\alpha\lambda^+\right], \left[d+\beta\lambda^-, d+\beta\lambda^+\right], \left[f+\gamma\lambda^-, f+\gamma\lambda^+\right] \right\rangle$$

$$= \chi$$

(P2) Assume that there is *SNIEEWA*(χ_1, χ_2, …, χ_n) = <$T(\lambda)$, $D(\lambda)$, $F(\lambda)$>. According to Eq. (11), we know that

$$T(\lambda) = \frac{\prod_{k=1}^{n}(1+t_k+\alpha_k\lambda)^{\rho_k} - \prod_{k=1}^{n}(1-t_k-\alpha_k\lambda)^{\rho_k}}{\prod_{k=1}^{n}(1+t_k+\alpha_k\lambda)^{\rho_k} + \prod_{k=1}^{n}(1-t_k-\alpha_k\lambda)^{\rho_k}}, D(\lambda) = \frac{2\prod_{k=1}^{n}(d_k+\beta_k\lambda)^{\rho_k}}{\prod_{k=1}^{n}(2-d_k-\beta_k\lambda)^{\rho_k} + \prod_{k=1}^{n}(d_k+\beta_k\lambda)^{\rho_k}},$$

$F(\lambda) = \frac{2\prod_{k=1}^{n}(f_k+\gamma_k\lambda)^{\rho_k}}{\prod_{k=1}^{n}(2-f_k-\gamma_k\lambda)^{\rho_k} + \prod_{k=1}^{n}(f_k+\gamma_k\lambda)^{\rho_k}}$ for $\lambda \in [\lambda^-, \lambda^+]$ are increasing functions of λ. So, we can get the following inequations:

$$\min_k(t_k+\alpha_k\lambda^-) \le T(\lambda^-) = \frac{\prod_{k=1}^{n}(1+t_k+\alpha_k\lambda^-)^{\rho_k} - \prod_{k=1}^{n}(1-t_k-\alpha_k\lambda^k)^{\rho_k}}{\prod_{k=1}^{n}(1+t_k+\alpha_k\lambda^-)^{\rho_k} + \prod_{k=1}^{n}(1-t_k-\alpha_k\lambda^k)^{\rho_k}} \le \max_k(t_k+\alpha_k\lambda^-),$$

$$\min_k(t_k+\alpha_k\lambda^+) \le T(\lambda^+) = \frac{\prod_{k=1}^{n}(1+t_k+\alpha_k\lambda^+)^{\rho_k} - \prod_{k=1}^{n}(1-t_k-\alpha_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{n}(1+t_k+\alpha_k\lambda^+)^{\rho_k} + \prod_{k=1}^{n}(1-t_k-\alpha_k\lambda^+)^{\rho_k}} \le \min_k(t_k+\alpha_k\lambda^+),$$

$$\min_k(d_k+\beta_k\lambda^-)\le D(\lambda^-)=\frac{2\prod_{k=1}^{n}(d_{i_k}+\beta_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{n}(2-d_k-\beta_k\lambda^-)^{\rho_i}+\prod_{k=1}^{n}(d_k+\beta_k\lambda^-)^{\rho_k}}\le\max_k(d_k+\beta_k\lambda^-),$$

$$\min_k(d_k+\beta_k\lambda^+)\le D(\lambda^+)=\frac{2\prod_{k=1}^{n}(d_k+\beta_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{n}(2-d_k-\beta_k\lambda^+)^{\rho_k}+\prod_{k=1}^{n}(d_k+\beta_k\lambda^+)^{\rho_k}}\le\max_k(d_k+\beta_k\lambda^+),$$

$$\min_k(f_k+\gamma_k\lambda^-)\le F(\lambda^-)=\frac{2\prod_{k=1}^{n}(f_k+\gamma_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{n}(2-f_k-\gamma_k\lambda^-)^{\rho_k}+\prod_{k=1}^{n}(f_k+\gamma_k\lambda^-)^{\rho_k}}\le\max_k(f_k+\gamma_k\lambda^-),$$

$$\min_k(f_k+\gamma_k\lambda^+)\le F(\lambda^+)=\frac{2\prod_{k=1}^{n}(f_k+\gamma_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{n}(2-f_k-\gamma_k\lambda^+)^{\rho_k}+\prod_{k=1}^{n}(f_k+\gamma_k\lambda^+)^{\rho_k}}\le\max_k(f_k+\gamma_k\lambda^+).$$

According to Eq. (1), we get the score values of $SNIEEWA(\chi_1, \chi_2, \ldots, \chi_n)$, $\chi_{\min}$, and $\chi_{\max}$:

$$S(SNIEEWA(\chi_1,\chi_2,\ldots,\chi_n))=(4+T(\lambda^-)+T(\lambda^+)-D(\lambda^-)-D(\lambda^+)-F(\lambda^-)-F(\lambda^+))/6,$$

$$S(\chi_{\min})=\begin{Bmatrix}(4+\min_k(t_k+\alpha_k\lambda^-)+\min_k(t_k+\alpha_k\lambda^+)-\max_k(d_k+\beta_k\lambda^-)\\-\max_k(d_k+\beta_k\lambda^+)-\max_k(f_k+\gamma_k\lambda^-)-\max_k(f_k+\gamma_k\lambda^+))/6\end{Bmatrix},$$

$$S(\chi_{\max})=\begin{Bmatrix}(4+\max_k(t_k+\alpha_k\lambda^-)+\max_k(t_k+\alpha_k\lambda^+)-\min_k(d_k+\beta_k\lambda^-\\-\min_k(d_k+\beta_k\lambda^+)-\min_k(f_k+\gamma_k\lambda^-)-\min_k(f_k+\gamma_k\lambda^+))/6\end{Bmatrix}.$$

We can get $S(\chi_{\min}) \le S(SNIEEWA(\chi_1, \chi_2,\ldots,\chi_n)) \le S(\chi_{\max})$. Thus $\chi_{\min} \le SNIEEWA(\chi_1, \chi_2, \ldots, \chi_n) \le \chi_{\max}$.

(P3) If $\chi_k = <T_k(\lambda), D_k(\lambda), F_k(\lambda)>$ and $\chi_k^* = <T_k^*(\lambda), D_k^*(\lambda), F_k^*(\lambda)>$ for $\lambda \in [\lambda^-, \lambda^+]$, $k = 1, 2, \ldots, n$, and $\chi_k \subseteq \chi_k^*$, then they satisfy the following constraints: $T_k(\lambda^-) \le T_k^*(\lambda^-)$, $T_k(\lambda^+) \le T_k^*(\lambda^+)$, $D_k(\lambda^-) \ge D_k^*(\lambda^-)$, $D_k(\lambda^+) \ge D_k^*(\lambda^+)$, $F_k(\lambda^-) \ge F_k^*(\lambda^-)$, and $F_k(\lambda^+) \ge F_k^*(\lambda^+)$. We use Eq. (11) to calculate $SNIEEWA(\chi_1, \chi_2, \ldots, \chi_n)$ and $SNIEEWA(\chi_1^*, \chi_2^*, \ldots, \chi_n^*)$: $SNIEEWA(\chi_1, \chi_2, \ldots, \chi_n) = <[T(\lambda^-), T(\lambda^+)], [D(\lambda^-), D(\lambda^+)], [F(\lambda^-)\ F(\lambda^+)]>$ and $SNIEEWA(\chi_1^*, \chi_2^*, \ldots, \chi_n^*) = <[T^*(\lambda^-), T^*(\lambda^+)], [D^*(\lambda^-), D^*(\lambda^+)], [F^*(\lambda^-), F^*(\lambda^+)]>$. Obviously, we can get $T(\lambda^-) \le T^*(\lambda^-)$, $T(\lambda^+) \le T^*(\lambda^+)$, $D(\lambda^-) \ge D^*(\lambda^-)$, $D(\lambda^+) \ge D^*(\lambda^+)$, $F(\lambda^-) \ge F^*(\lambda^-)$, and $F(\lambda^+) \ge F^*(\lambda^+)$. Hence $SNIEEWA(\chi_1, \chi_2, \ldots, \chi_n) \subseteq SNIEEWA(\chi_1^*, \chi_2^*, \ldots, \chi_n^*)$ holds.

3.3. Einstein Weighted Geometric Average Operator of SNIEs

Definition 7. Let $\chi = \{\chi_1, \chi_2, \ldots, \chi_n\}$ be SNIS, we can define the SNIEEWG Operator of SNIEs:

$$SNIEEWG(\chi_1,\chi_2,\ldots,\chi_n)=\bigotimes_{k=1}^{n}\chi_k^{\rho_k}\quad, \tag{13}$$

where $\rho_k \in [0, 1]$ are weights for $\sum_{k=1}^{n}\rho_k = 1$.

Theorem 2. Let $\chi_k = <T_k(\lambda), D_k(\lambda), F_k(\lambda)> = < t_k + \alpha_k\lambda, d_k + \beta_k\lambda, f_k + \gamma_k\lambda >$ for $k = 1, 2, \ldots, n$ and $\lambda \in [\lambda^-, \lambda^+]$ be SNIEs with the related weights $\rho_k \in [0,1]$ for $\sum_{k=1}^{n}\rho_k = 1$. Then according to the operational rules (7) and (9), Eq. (11) can be calculated by

$$SNIEEWG(\chi_1,\chi_2,...,\chi_n)=\left\langle\begin{array}{l}\left[\dfrac{2\prod_{k=1}^{n}(t_k+\alpha_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{n}(2-t_k-\alpha_k\lambda^-)^{\rho_k}+\prod_{k=1}^{n}(t_k+\alpha_k\lambda^-)^{\rho_k}},\dfrac{2\prod_{k=1}^{n}(t_k+\alpha_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{n}(2-t_k-\alpha_k\lambda^+)^{\rho_k}+\prod_{k=1}^{n}(t_k+\alpha_k\lambda^+)^{\rho_k}}\right],\\ \left[\dfrac{\prod_{k=1}^{n}(1+d_k+\beta_k\lambda^-)^{\rho_k}-\prod_{k=1}^{n}(1-d_k-\beta_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{n}(1+d_k+\beta_k\lambda^-)^{\rho_k}+\prod_{k=1}^{n}(1-d_k-\beta_k\lambda^-)^{\rho_k}},\dfrac{\prod_{k=1}^{n}(1+d_k+\beta_k\lambda^+)^{\rho_k}-\prod_{k=1}^{n}(1-d_k-\beta_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{n}(1+d_k+\beta_k\lambda^+)^{\rho_k}+\prod_{k=1}^{n}(1-d_k-\beta_k\lambda^+)^{\rho_k}}\right],\\ \left[\dfrac{\prod_{k=1}^{n}(1+f_k+\gamma_k\lambda^-)^{\rho_k}-\prod_{k=1}^{n}(1-f_k-\gamma_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{n}(1+f_k+\gamma_k\lambda^-)^{\rho_k}+\prod_{k=1}^{n}(1-f_k-\gamma_k\lambda^-)^{\rho_k}},\dfrac{\prod_{k=1}^{n}(1+f_k+\gamma_k\lambda^+)^{\rho_k}-\prod_{k=1}^{n}(1-f_k-\gamma_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{n}(1+f_k+\gamma_k\lambda^+)^{\rho_k}+\prod_{k=1}^{n}(1-f_k-\gamma_k\lambda^+)^{\rho_k}}\right]\end{array}\right\rangle. \quad (14)$$

In view of the same proof of Theorem 1, we can proof Theorem 2, which is omitted here.

4. MADM Method with the SNIEEWA or SNIEEWG Operator

For a MADM problem, there are an alternative set $\eta = \{\eta_1, \eta_2, \ldots, \eta_m\}$ and an attribute set $C = \{C_1, C_2, \ldots, C_n\}$ with the weight vector $\rho = \{\rho_1, \rho_2, \ldots, \rho_n\}$. The assessment values given by the decision makers are in the SNIE form. Thus, the evaluation value of the attribute C_j for the alternative η_i is specified as $\chi_{ij} = \langle T_{ij}(\lambda), D_{ij}(\lambda), F_{ij}(\lambda)\rangle = \langle t_{ij} + \alpha_{ij}\lambda, d_{ij} + \beta_{ij}\lambda, f_{ij} + \gamma_{ij}\lambda\rangle$ for $t_{ij} + \alpha_{ij}\lambda, d_{ij} + \beta_{ij}\lambda, f_{ij} + \gamma_{ij}\lambda \in [0, 1]$ and $\lambda \in [\lambda^-, \lambda^+]$, ($i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$). Hence, all the assessed SNIEs constitute the SNIE decision matrix $\chi = (\chi_{ij})_{m\times n}$. Then, the MADM method is shown as the following steps.

Step1: Calculate the aggregation value of SNIEs χ_{ij} for η_i and some ranges of λ by the following aggregation formula:

$$\chi_i = SNIEEWA(\chi_{i1},\chi_{i2},...,\chi_{in}) = \mathop{\oplus}_{j=1}^{n}\rho_j\chi_{ij}$$
$$=\left\langle\begin{array}{l}\left[\dfrac{\prod_{j=1}^{n}(1+t_{ij}+\alpha_{ij}\lambda^-)^{\rho_j}-\prod_{j=1}^{n}(1-t_{ij}-a_{ij}\lambda^-)^{\rho_j}}{\prod_{j=1}^{n}(1+t_{ij}+\alpha_{ij}\lambda^-)^{\rho_j}+\prod_{j=1}^{n}(1-t_{ij}-a_{ij}\lambda^-)^{\rho_j}},\dfrac{\prod_{j=1}^{n}(1+t_{ij}+\alpha_{ij}\lambda^+)^{\rho_j}-\prod_{j=1}^{n}(1-t_{ij}-a_{ij}\lambda^+)^{\rho_j}}{\prod_{j=1}^{n}(1+t_{ij}+\alpha_{ij}\lambda^+)^{\rho_j}+\prod_{j=1}^{n}(1-t_{ij}-a_{ij}\lambda^+)^{\rho_j}}\right],\\ \left[\dfrac{2\prod_{j=1}^{n}(d_{ij}+\beta_{ij}\lambda^-)^{\rho_j}}{\prod_{j=1}^{n}(2-d_{ij}-\beta_{ij}\lambda^-)^{\rho_j}+\prod_{j=1}^{n}(d_{ij}+\beta_{ij}\lambda^-)^{\rho_j}},\dfrac{2\prod_{j=1}^{n}(d_{ij}+\beta_{ij}\lambda^+)^{\rho_j}}{\prod_{j=1}^{n}(2-d_{ij}-\beta_{ij}\lambda^+)^{\rho_j}+\prod_{j=1}^{n}(d_{ij}+\beta_{ij}\lambda^+)^{\rho_j}}\right],\\ \left[\dfrac{2\prod_{j=1}^{n}(f_{ij}+\gamma_{ij}\lambda^-)^{\rho_j}}{\prod_{j=1}^{n}(2-f_{ij}-\gamma_{ij}\lambda^-)^{\rho_j}+\prod_{j=1}^{n}(f_{ij}+\gamma_{ij}\lambda^-)^{\rho_j}},\dfrac{2\prod_{j=1}^{n}(f_{ij}+\gamma_{ij}\lambda^+)^{\rho_j}}{\prod_{j=1}^{n}(2-f_{ij}-\gamma_{ij}\lambda^+)^{\rho_j}+\prod_{j=1}^{n}(f_{ij}+\gamma_{ij}\lambda^+)^{\rho_j}}\right]\end{array}\right\rangle \quad (15)$$

or

$$\chi_i = SNIEEWG(\chi_{i1},\chi_{i2},...,\chi_{in}) = \mathop{\otimes}_{j=1}^{n}\chi_{ij}^{\rho_j}$$
$$=\left\langle\begin{array}{l}\left[\dfrac{2\prod_{j=1}^{n}(t_{ij}+\alpha_{ij}\lambda^-)^{\rho_j}}{\prod_{j=1}^{n}(2-t_{ij}-\alpha_{ij}\lambda^-)^{\rho_j}+\prod_{j=1}^{n}(t_{ij}+\alpha_{ij}\lambda^-)^{\rho_j}},\dfrac{2\prod_{j=1}^{n}(t_{ij}+\alpha_{ij}\lambda^+)^{\rho_j}}{\prod_{j=1}^{n}(2-t_{ij}-\alpha_{ij}\lambda^+)^{\rho_j}+\prod_{j=1}^{n}(t_{ij}+\alpha_{ij}\lambda^+)^{\rho_j}}\right],\\ \left[\dfrac{\prod_{j=1}^{n}(1+d_{ij}+\beta_{ij}\lambda^-)^{\rho_j}-\prod_{j=1}^{n}(1-d_{ij}-\beta_{ij}\lambda^-)^{\rho_j}}{\prod_{j=1}^{n}(1+d_{ij}+\beta_{ij}\lambda^-)^{\rho_j}+\prod_{j=1}^{n}(1+d_{ij}+\beta_{ij}\lambda^-)^{\rho_j}},\dfrac{\prod_{j=1}^{n}(1+d_{ij}+\beta_{ij}\lambda^+)^{\rho_j}-\prod_{j=1}^{n}(1-d_{ij}-\beta_{ij}\lambda^+)^{\rho_j}}{\prod_{j=1}^{n}(1+d_{ij}+\beta_{ij}\lambda^+)^{\rho_j}+\prod_{j=1}^{n}(1+d_{ij}+\beta_{ij}\lambda^+)^{\rho_j}}\right],\\ \left[\dfrac{\prod_{j=1}^{n}(1+f_{ij}+\gamma_{ij}\lambda^-)^{\rho_j}-\prod_{j=1}^{n}(1-f_{ij}-\gamma_{ij}\lambda^-)^{\rho_j}}{\prod_{j=1}^{n}(1+f_{ij}+\gamma_{ij}\lambda^-)^{\rho_j}+\prod_{j=1}^{n}(1-f_{ij}-\gamma_{ij}\lambda^-)^{\rho_j}},\dfrac{\prod_{j=1}^{n}(1+f_{ij}+\gamma_{ij}\lambda^+)^{\rho_j}-\prod_{j=1}^{n}(1-f_{ij}-\gamma_{ij}\lambda^+)^{\rho_j}}{\prod_{j=1}^{n}(1+f_{ij}+\gamma_{ij}\lambda^+)^{\rho_j}+\prod_{j=1}^{n}(1-f_{ij}-\gamma_{ij}\lambda^+)^{\rho_j}}\right]\end{array}\right\rangle. \quad (16)$$

Step2: Calculate the values of the score function $S(\chi_i, \lambda)$ (accuracy function $L(\chi_i,\lambda)$ and certainty function $C(\chi_i,\lambda)$) by Eq. (1) (Eqs. (2) and (3)).

Step3: Rank the alternatives and define the best one.

5. Illustrative Example, Sensitivity Analysis, and Comparison

5.1. Illustrative Example

In an illustrative example, we apply the proposed MADM method to the risk assessment of the investment selection of metallic mines. The mining projects have great uncertainty and a long cycle. Then, there are investment projects of four candidate mines, denoted as a set of four alternatives $\eta = \{\eta_1, \eta_2, \eta_3, \eta_4\}$. The key evaluation factors/attributes of the four candidate mines contain the economic factor (C_1), the safety factor (C_2), and the environmental risk factor (C_3) in the investment evaluation process. The weight vector ρ = (0.3, 0.36, 0.34) addresses the importance of the three attributes. Because of evaluation information uncertainty in the four candidate mines, the decision makers/experts are required to evaluate each candidate mine on the three attributes in the SNIE form. Their evaluation information is provided by the SNIEs $\chi_{ij} = \langle T_{ij}(\lambda), D_{ij}(\lambda), F_{ij}(\lambda)\rangle = \langle t_{ij} + \alpha_{ij}\lambda, d_{ij} + \beta_{ij}\lambda, f_{ij} + \gamma_{ij}\lambda\rangle$ for j = 1, 2, 3; i = 1, 2, 3, 4. Thus, the decision matrix of all SNIEs is shown below:

$$\chi = \begin{bmatrix} <0.7+0.2\lambda,\ 0.2+0.1\lambda,\ 0.2+0.2\lambda> & <0.7+0.2\lambda,\ 0.1+0.3\lambda,\ 0.1+0.1\lambda> & <0.6+0.2\lambda,\ 0.2+0.2\lambda,\ 0.2+0.2\lambda> \\ <0.7+0.2\lambda,\ 0.2+0.1\lambda,\ 0.3+0.1\lambda> & <0.8+0.1\lambda,\ 0.1+0.2\lambda,\ 0.1+0.3\lambda> & <0.7+0.1\lambda,\ 0.2+0.2\lambda,\ 0.1+0.1\lambda> \\ <0.8+0.1\lambda,\ 0.2+0.1\lambda,\ 0.1+0.2\lambda> & <0.7+0.1\lambda,\ 0.2+0.1\lambda,\ 0.1+0.2\lambda> & <0.7+0.2\lambda,\ 0.3+0.1\lambda,\ 0.2+0.1\lambda> \\ <0.7+0.1\lambda,\ 0.1+0.2\lambda,\ 0.2+0.1\lambda> & <0.8+0.1\lambda,\ 0.1+0.2\lambda,\ 0.2+0.1\lambda> & <0.7+0.1\lambda,\ 0.2+0.1\lambda,\ 0.2+0.2\lambda> \end{bmatrix}.$$

According to the evaluation information and the proposed MADM method, the decision steps are shown below.

Step1: Aggregate SNIEs χ_{ij} for η_i (i = 1, 2, 3, 4; j = 1,2, 3) by Eq. (15) or (16). The indeterminate λ is specified as $\lambda = [\lambda^-, \lambda^+]$ = [0, 0], [0, 0.5], [0, 1], [0, 1.5]. The aggregation values of Eq. (15) or (16) are listed in Tables 1 and 2.

Table 1. The aggregation values corresponding to the SNIEEWA operator

$\lambda = [\lambda^-, \lambda^+]$	Aggregation value
λ= [0, 0]	χ_1 = <[0.6685, 0.6685],[0.1565, 0.1565],[0.1565, 0.1565]>, χ_2 = <[0.7400, 0.7400],[0.1565, 0.1565],[0.1407, 0.1407]>, χ_3 = <[0.7337, 0.7337],[0.2301, 0.2301],[0.1271, 0.1271]>, χ_4 = <[0.7400, 0.7400],[0.1271, 0.1271],[0.2000, 0.2000]>
λ= [0, 0.5]	χ_1 = <[0.6685, 0.7699],[0.1565, 0.2661],[0.1565, 0.2354]>, χ_2 = <[0.7400, 0.8050],[0.1565, 0.2460],[0.1407, 0.2343]>, χ_3 = <[0.7337, 0.8007],[0.2301, 0.2809],[0.1270, 0.2159]>, χ_4 = <[0.7400, 0.7913],[0.1271, 0.2159],[0.2000, 0.2661]>
λ= [0, 1]	χ_1 = <[0.6685, 0.8729],[0.1565, 0.3676],[0.1565, 0.3147]>, χ_2 = <[0.7400, 0.8729],[0.1565, 0.3315],[0.1407, 0.3190]>, χ_3 = <[0.7337, 0.8711],[0.2301, 0.3315],[0.1270, 0.3000]>, χ_4 = <[0.7400, 0.8435],[0.1271, 0.3000],[0.2000, 0.3315]>
λ= [0, 1.5]	χ_1 = <[0.6685, 1.0000],[0.1565, 0.4673],[0.1565, 0.3945]>, χ_2 = <[0.7400, 1.0000],[0.1565, 0.4158],[0.1407, 0.4015]>, χ_3 = <[0.7337, 1.0000],[0.2301, 0.3819],[0.1270, 0.3824]>, χ_4 = <[0.7400, 0.8983],[0.1271, 0.3824],[0.2000, 0.3966]>

Table 2. The aggregation values corresponding to the SNIEEWG operator

$\lambda = [\lambda^-, \lambda^+]$	Aggregation value
λ = [0, 0]	χ_1 = <[0.6651, 0.6651],[0.1645, 0.1644],[0.1644, 0.1644]>, χ_2 = <[0.7354, 0.7354],[0.1645, 0.1644],[0.1617, 0.1617]>, χ_3 = <[0.7294, 0.7294],[0.2346, 0.2346],[0.1343, 0.1343]>, χ_4 = <[0.7354, 0.7354],[0.1343, 0.1343],[0.2000, 0.2000]>
λ = [0, 0.5]	χ_1 = <[0.6651, 0.7654],[0.1645, 0.2672],[0.1644, 0.2473]>, χ_2 = <[0.7354, 0.8006],[0.1645, 0.2495],[0.1617, 0.2477]>, χ_3 = <[0.7294, 0.7967],[0.2346, 0.2847],[0.1343, 0.2171]>, χ_4 = <[0.7354, 0.7855],[0.1343, 0.2171],[0.2000, 0.2672]>
λ = [0, 1]	χ_1 = <[0.6651, 0.8657],[0.1645, 0.3709],[0.1644, 0.3311]>, χ_2 = <[0.7354, 0.8657],[0.1645, 0.3349],[0.1617, 0.3351]>, χ_3 = <[0.7294, 0.8637],[0.2346, 0.3349],[0.1343, 0.3000]>, χ_4 = <[0.7354, 0.8356],[0.1343, 0.3000],[0.2000, 0.3349]>
λ = [0, 1.5]	χ_1 = <[0.6651, 0.9659],[0.1645, 0.4769],[0.1644, 0.4165]>, χ_2 = <[0.7354, 0.9307],[0.1645, 0.4210],[0.1617, 0.4259]>, χ_3 = <[0.7294, 0.9307],[0.2346, 0.3851],[0.1343, 0.3832]>, χ_4 = <[0.7354, 0.8858],[0.1343, 0.3832],[0.2000, 0.4036]>

Step 2: Calculate the scores of $S(\chi_i, \lambda)$ by Eq. (1) and show the results in Tables 3 and 4.

Table 3. Scores and ranking orders corresponding to the SNIEEWA operator

$\lambda = [\lambda^-, \lambda^+]$	Score of $S(\chi_i, \lambda)$	Ranking	The best
λ = [0, 0]	0.7852, 0.8143, 0.7922, 0.8043	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	η_2
λ = [0, 0.5]	0.7706, 0.7946, 0.7801, 0.7870	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	η_2
λ = [0, 1]	0.7577, 0.7775, 0.7694, 0.7708	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	η_2
λ = [0, 1.5]	0.7489, 0.7709, 0.7687, 0.7554	$\eta_2 > \eta_3 > \eta_4 > \eta_1$	η_2

Table 4. Scores and ranking orders corresponding to the SNIEEWG operator

$\lambda = [\lambda^-, \lambda^+]$	Score of $S(\chi_i, \lambda)$	Ranking	The best
λ = [0, 0]	0.7788, 0.8031, 0.7868, 0.8004	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	η_2
λ = [0, 0.5]	0.7645, 0.7854, 0.7759, 0.7837	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	η_2
λ = [0, 1]	0.7500, 0.7675, 0.7649, 0.7670	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	η_2
λ = [0, 1.5]	0.7348, 0.7488, 0.7538, 0.7500	$\eta_3 > \eta_4 > \eta_2 > \eta_1$	η_3

Step 3: The ranking orders are listed in Tables 3 and 4. There are exactly consistent ranking results between the SNIEEWA operator and the SNIEEWG operator when the indeterminate ranges of λ are λ = [0, 0], [0, 0.5], [0, 1], then η_2 is the best alternative. While the indeterminate range is λ= [0, 1.5], the ranking orders are very different between two aggregation operators, then the best alternative is η_2 corresponding to the SNIEEWA operator and η_3 corresponding to the SNIEEWG operator.

5.2. Sensitivity Analysis

An SNIE can represent an SvNE or an IvNE regarding the value or range of the indeterminate λ. In the above example, we have specified several indeterminate ranges of λ to make decisions.

The results show that the ranking orders are the same within certain ranges of λ = [0, 0], [0, 0.5], [0, 1] corresponding to two aggregation operators, while the ranking orders are very different in the range of λ= [0, 1.5]. The above example demonstrates that more ranking orders based on the two aggregation operators are nearly consistent. Due to the variability of the indeterminate λ, the proposed MADM approach is valid and flexible. To further analyze the change of decision results with the indeterminate variety of λ, we show the relational graphs corresponding to the indeterminate value of λ and the score of η_i in Figures 1 and 2**.**

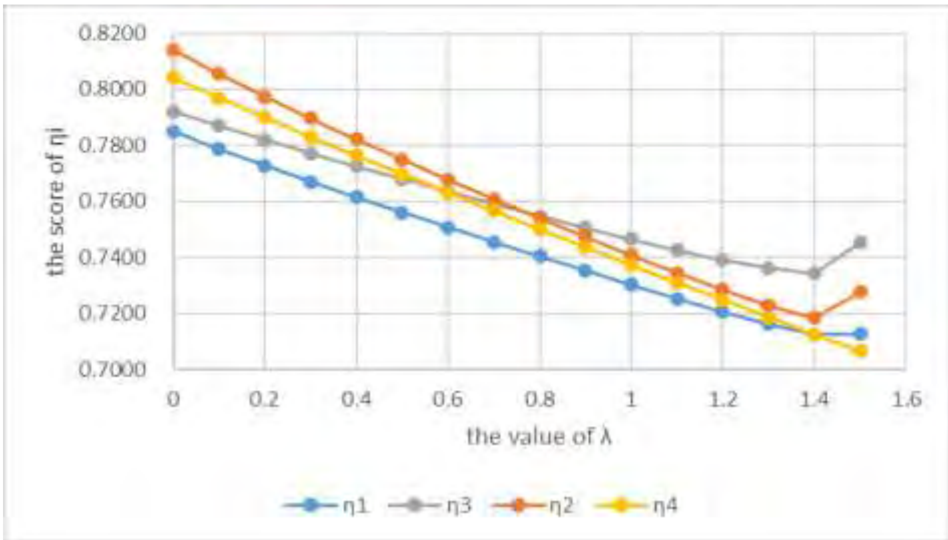

Figure 1. Relationship between the score of η_i and the value of λ corresponding to the SNIEEWA operator

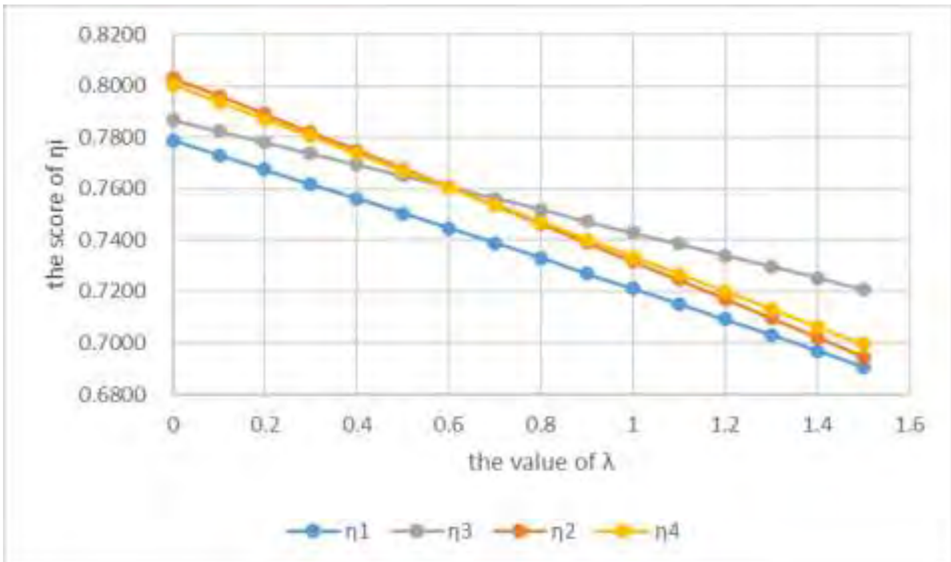

Figure 2. Relationship between the score of η_i and the value of λ corresponding to the SNIEEWG operator

SNIE is reduced to SvNE when the indeterminate λ is a single value. In Figure 1, the ranking order is $\eta_2 > \eta_4 > \eta_3 > \eta_1$ and the best alternative is η_2 when λ is in the range of $\lambda \in [0, 0.8]$ corresponding to the SNIEEWA operator. Then, the best alternative is η_3 in the range of $\lambda \in [0.8, 1.5]$. In Figure 2, the ranking order is $\eta_2 > \eta_4 > \eta_3 > \eta_1$ when the value of λ is less than 0.6 corresponding to the SNIEEWG operator. The ranking order is $\eta_3 > \eta_4 > \eta_2 > \eta_1$ when the value of λ is greater than 0.6. Although the ranking orders are not exactly identical with different aggregation operators, they are some sensitivities to different values/ranges of λ.

5.3. Comparison and Discussion

Du et al. [15] first put forward the concept of SNIE and a MADM approach based on the weighted aggregation operators of SNIEs. To compare the proposed MADM approach with the existing MADM approach [15], the ranking results of the existing MADM approach [15] in specified ranges of λ are indicated corresponding to the SNIEWA and SNIEWG operators and shown in Table 5. The ranking results corresponding to the proposed SNIEEWA operator and the existing SNIEWA operator [15] are identical in all ranges of λ. Corresponding to the proposed SNIEEG operator and the existing SNIEWG operator [15], the ranking results are different only in the range of λ = [0, 1]. In terms of all the results, η_2 is the best investment selection, conversely η_1 is the worst one.

Table 5. Decision results of different methods

	Ranking order in different ranges of λ			
λ	λ = [0, 0]	λ = [0, 0.5]	λ = [0, 1]	λ = [0, 1.5]
SNIEEWA	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	$\eta_2 > \eta_4 > \eta_3 > \eta_1$
SNIEEWG	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	$\eta_3 > \eta_4 > \eta_2 > \eta_1$
SNIEWA [15]	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	$\eta_2 > \eta_4 > \eta_3 > \eta_1$
SNIEWG [15]	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	$\eta_4 > \eta_2 > \eta_3 > \eta_1$	$\eta_3 > \eta_4 > \eta_2 > \eta_1$

6. Conclusions

In this article, we presented the SNIEEWA and SNIEEWG operators of SNIEs with respect to the Einstein t-norm and t-conorm operations. On the basis of the SNIEEWA and SNIEEWG operators, the MADM method was developed and applied to the selection problem of mine investments. In the illustrative example, the decision results were analyzed under the single- and interval-valued neutrosophic indeterminate situations, which indicated some sensitivities to different values/ranges of λ**.** Compared the existing MADM approach [15] in the situation of interval-valued neutrosophic indeterminate information, the ranking results demonstrated that the proposed approach is valid. Since SNIS can flexibly express neutrosophic information according to indetermination ranges of λ, the proposed MADM method reflected its efficiency and flexibility regarding interval indeterminate ranges.

Since SNIS is a flexible form for describing indeterminate and inconsistent assessment information, it can be used in many indeterminate problems. In future research, more aggregation operators, similarity measures, and decision-making methods will be developed and applied to many fields in neutrosophic indeterminate environment.

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Cardiovascular Diseases Risk Analysis using Distance-Based Similarity Measure of Neutrosophic Sets

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Abstract: One of the highest causes of death in many countries in this modern era is cardiovascular diseases. There are a few symptoms that linked significantly to the cardiovascular diseases. The symptoms and diseases relationship can be represented by neutrosophic set values. In general, neutrosophic set gives remarkable contribution in denoising, clustering, segmentation, and classification in handling data of many real applications including in medical field. This study aims to analyse the cardiovascular disease risks by a new distance-based similarity measure motivated from intuitionistic fuzzy set theory. The proof for all the properties is presented clearly. Then, a case study is conducted by using the data on the severity level of the six symptoms found in two different patients. The neutrosophic data are analysed to determine the patients' possibility of having any one or combination of the three types of cardiovascular diseases. A comparative study involving three common distance measures is conducted. The results show that the similarity indexes for all measures of both patients are less than 0.5. This situation can further conclude as both patients are possibly not suffering from cardiovascular diseases.

Keywords: Cardiovascular Disease, Distance Based, Neutrosophic Set, Similarity Measure

1. Introduction

In human body, the cardiovascular system consisting of heart and circulatory system has its crucial role in protection, regulation and transport of nutrients and oxygen to all the tissues of the body [1]. The blood flows through a network of blood vessels in response to the heart pumps and produces a pressure gradient [2]. Unfortunately, diseases related to the cardiovascular system have been proven to be one of the main causes of death in the whole world. Few of the cardiovascular diseases are coronary artery disease, heart failure, congenital heart disease and heart attack [3]. The diseases occur due to factors such as genetic disposition, systolic blood pressure, cholesterol, diabetes, body mass index, depression, and unhealthy diet. Due to the severity cause of these diseases, it is important to make correct diagnosis and to provide appropriate treatment to the patients.

Invasive methods are commonly used in medical diagnosis to identify vascular health conditions. The diagnosis result is used as a guideline by the doctor to provide the appropriate treatment. Alternatively, in the past few decades, many studies used non-invasive method with the purpose of reducing the patients' health risk and clinical utility cost. Mathematical model developed by the physical law of fluid dynamics has the capability in understanding the blood flow behaviour in vascular system [2, 4, 5, 6]. Another aspect in making the correct diagnosis relates to the sufficient patients' information of their medical condition. There is an elegant branch of mathematics which gives us the ability to reduce the possibility in making inaccurate diagnosis despite of the incompleteness or uncertain information. The area uses fuzzy mathematics concept in defining the set theory where it evolves from fuzzy set to many more advance sets e.g intuitionistic fuzzy set and neutrosophic set. The studies on the development of more advance neutrosophic sets and its applications as well in medical or clinical diagnosis are found in [7-10].

Distance and similarity measures are important in various scientific research fields such as decision making, pattern identification, and market forecasting. Lots of studies have been done by adopting fuzzy sets [11], intuitionistic fuzzy sets (IFS) [12,13], and neutrosophic sets [8,14]. The use of similarity measures has significant role in data clustering process and a work of [15] had proven that single-valued neutrosophic set (SVNS) clustering algorithm improved the accuracy in representing the indeterminate or inconsistent information. The most widely used distance measures are Hamming distance and Euclidean distance. [16] introduced a new similarity measure in a real-life decision-making problem and proved its ability to handle multiple existing criteria of incomplete or inconsistent information. Several new similarity measures of the neutrosophic sets with exponential functions in the truth, indeterminacy and falsity memberships were produced by [17]. They concluded that the existing measures failed in some circumstances, while the proposed measures classify them more appropriate and precisely.

To date, the application of neutrosophic set theory is significantly found in decision making studies. The major advantage of the set is its ability to handle uncertainty and incompleteness of the data. Neutrosophic set is an important set for denoising, clustering, segmentation, and classification of real data in many areas which includes medical field. For effective diagnosis systems, neutrosophic set have been integrated with the clustering techniques to reduce ambiguity for competent diagnosis. [3] proposed the neutrosophic clinical decision-making system using explainable artificial intelligence approaches for the proper diagnose of cardiovascular disease risk. Then, [18] extended the same approach to help physicians in early diagnosis, identifying the type of treatment and diagnosis. [7] focused on heart disease diagnosis problem as an application of neutrosophic refined set using distance measure while [8] analysed the medical diagnosis for rough neutrosophic set using Dice and Cosine similarity measures. Meanwhile, [9] created a new model based on Neutrosophic Cognitive Map that integrates diagnosis, treatment, and prognosis processes for supporting clinical decision-making for the treatment of cardiovascular diseases during pregnancy. [10] proposed a novel framework based on Internet of Thing (IoT) and computer supported diagnosis to identify and control heart failure infected patients. They obtained preciseness of diagnosis with vague information and suggested the neutrosophic multi criteria decision making technique in guiding the physician to identify whether a patient is suffering from heart failure. A new distance-based similarity measure has been proposed by [19] for refined neutrosophic sets and they applied the findings in medical diagnosis of few diseases with a common set of symptoms. Further, [20] developed a new hybrid distance-based similarity measure for refined neutrosophic sets. Another approach introduced by [21] is a new parametric divergence measure for neutrosophic sets used not only in medical diagnosis but also in pattern recognition problem, and multi criteria decision making problem.

Based on the related studies stated above, there are less studies focus on the neutrosophic sets with new distance formula modified from IFS. Thus, our present study extended [13] to present the new distance-based similarity measure in analysing the risk of the cardiovascular disease. This new measure is improvised to fulfil the gap within the indeterminate relationship. In this present study, we have compared our new formula with the existing normalized Hamming distance, extended Hausdorff distance and normalised Euclidean distance measures. The study presents significant result on the medical diagnosis of three cardiovascular diseases for two patients. The description of the diseases together with the major factors linked to the diseases are well defined in Section 2. The section also provides the definitions of SVNS that contains three of membership functions (MFs) that are truth (T), indeterminacy (I), and falsity (F) together with several distance measures which being used in the subsequent section. By adopting the distance formula presented in [13] into NS domain, we derive the new formula to obtain the distance measure on SVNSs. Section 3 proves that the new formula satisfies all the four properties of distance measure. By means of the distance and similarity measures, Section 4 uses the clinical data in [3] regarding the symptoms shown on the two patients to relate them with the three cardiovascular diseases.

2. Preliminaries

This section introduced some preliminary notions which will be applied in the final analysis.

2.1. Single Valued Neutrosophic Set

A neutrosophic set which can be used in real scientific and engineering applications is known as Single valued neutrosophic set (SVNS).

Definition 2.1.1 [22]. Let X be a space of points (objects) with a generic element in X denoted by x. A single valued neutrosophic set A in X is characterized by a truth membership function, $T_A(x)$, an indeterminacy membership function, $I_A(x)$, and a falsity membership function $F_A(x)$. Here $T_A(x), I_A(x), F_A(x)$ are real subsets of [0,1].

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\}$$

2.2. Distance-based Similarity Measure of Neutrosophic set

Definition 2.2.1 [23]. Normalized Hamming distance measure $d_{NS}^{NH}(A,B)$ operator between neutrosophic set A and B is defined as follows:

$$d_{NS}^{NH}(A,B) = \frac{1}{3n}\sum_{i=1}^{n}(|T_A(x_i) - T_B(x_i)| + |I_A(x_i) - I_B(x_i)| + |F_A(x_i) - F_B(x_i)|)$$

Definition 2.2.2 [23]. Normalized Euclidean distance measure $d_{NS}^{NE}(A,B)$ operator between neutrosophic set A and B is defined as follows:

$$d_{NS}^{NE}(A,B) = \sqrt{\frac{1}{3n}\sum_{i=1}^{n}((T_A(x_i) - T_B(x_i))^2 + (I_A(x_i) - I_B(x_i))^2 + (F_A(x_i) - F_B(x_i))^2)}$$

Definition 2.2.3 [23]. An extended Hausdorff Distance $d_{NS}^{EH}(A,B)$ operator between neutrosophic set A and B is defined as follows:

$$d_{NS}^{EH}(A,B) = \frac{1}{n}\sum_{i=1}^{n} max\{|T_A(x_i) - T_B(x_i)|, |I_A(x_i) - I_B(x_i)|, |F_A(x_i) - F_B(x_i)|\}$$

Definition 2.2.4 [23]. Let A, B be two neutrosophic sets in X. The similarity measure between the neutrosophic sets A and B can be evaluate from distance measures, as follows:

$$S_N(A,B) = 1 - d_{NS}(A,B)$$

where $d_{NS}(A,B)$ is represent the distance measure between neutrosophic set A and B for all $x_i \in X$.

Proposition 2.2.1: The distance measures for neutrosophic set $d_{NS}(A,B)$and similarity measure for neutrosophic set $S_N(A,B)$ satisfies the following properties:

(C1) $0 \leq d_{NS}(A,B) \leq 1; 0 \leq S_N(A,B) \leq 1;$

(C2) $d_{NS}(A,B) = 0$ if and only if $A = B$; $S_N(A,B) = 1$ if and only if for $A = B$;

(C3) $d_{NS}(A,B) = d_{NS}(B,A)$; $S_N(A,B) = S_N(B,A)$;

(C4) $d_{NS}(A,C) \leq d_{NS}(A,B)$ and $d_{NS}(A,C) \leq d_{NS}(B,C)$ if C is neutrosophic set in X and $A \subseteq B \subseteq C$; $S_N(A,C) \leq S_N(A,B)$ and $S_N(A,C) \leq S_N(B,C)$ if C is neutrosophic set in X and $A \subseteq B \subseteq C$.

All the proof of the proposition are shown in [20-21].

2.3. Distance Measure on Intuitionistic Fuzzy set

Definition 2.3.1 [13]. Let $X = \{x_1, x_2, \dots, x_n\}$ be the universe of discourse. Let $A = \{x_i, T_A(x_i), F_A(x_i)): x_i \in X\}$ and $B = \{x_i, T_B(x_i), F_B(x_i): x_i \in X\}$ be two intuitionistic fuzzy sets. Then, the distance measure between A and B can be defined as:

$$d_{IFS}(A,B) = \frac{2}{n}\sum_{i=1}^{n} \frac{sin\left\{\frac{\pi}{6}|T_A(x_i) - T_B(x_i)|\right\} + sin\left\{\frac{\pi}{6}|F_A(x_i) - F_B(x_i)|\right\}}{1 + sin\left\{\frac{\pi}{6}|T_A(x_i) - T_B(x_i)|\right\} + sin\left\{\frac{\pi}{6}|F_A(x_i) - F_B(x_i)|\right\}}$$

Proposition 2.3.1: The distance measures for intuitionistic fuzzy set $d_{IFS}(A,B)$ satisfies the following properties:

(C1) $0 \leq d_{IFS}(A,B) \leq 1$;

(C2) $d_{IFS}(A,B) = 0$ if and only if $A = B$;

(C3) $d_{IFS}(A,B) = d_{IFS}(B,A)$;

(C4) $d_{IFS}(A,C) \leq d_{IFS}(A,B)$ and $d_{IFS}(A,C) \leq d_{IFS}(B,C)$ if C is intuitionistic fuzzy set in X and $A \subseteq B \subseteq C$.

All the proof of the proposition are shown in [13].

2.4. Major factors of cardiovascular diseases

There is several factors for cardiovascular diseases found in [25]-[27] but this present study only focuses on the six highly significant factors which are:

i. **Cholesterol** - a fatty substance found in all cells in the human body and the bloodstream. The human body needs cholesterol to make hormones, vitamin D, and substances that help you digest food. Cholesterol is usually produced in the liver, but it can be found in a variety of animal -based foods. It can affect your health if you have too high cholesterol levels.

ii. **Depression** - a emotional disorder that can affect a person's daily life. It may be described as prolonged sadness, fatigue, irritability, and loss of interest in daily activities.

iii. **Diabetes** - a disease that occurs when too high blood glucose (blood sugar) in human body caused by the body not being able to produce enough insulin. Insulin is a hormone made by the pancreas that controls the balance of glucose in the body by helping the movement of glucose from the blood into the cells to be used for energy.

iv. **Blood pressure** - the pressure exerted by the blood on the arteries when blood is pumped by the heart throughout the human body. High blood pressure is a silent disease that can lead to complications and even death if left untreated.

v. **Body mass index** - a measure to access person's weight versus height. A high BMI can be an indicator of high body fatness. Being overweight exposes a person to diseases such as heart disease, stroke, diabetes, and high blood pressure.

vi. **Unhealthy diet –** fail to deliver human body with the proper quantities and varieties of nutrients for optimum health, especially when the diet contains high calories and less fruits and vegetables.

2.5. Cardiovascular diseases (CVDs)

Many types of cardiovascular diseases (CVDs) as stated in [24]-[26] that cause the death. This study considers only three types of CVDs, as follows:

i. **Heart attack** - when a blood clot blocks the flow of blood through the blood vessels that feed the heart, perhaps harming or ruining part of the heart muscle. A heart attack can be caused by atherosclerosis

ii. **Heart failure** - heart disease's most frequent complications. When your heart blood pumping ability is not enough to supply blood to comply your body's needs, hence heart failure occurs. Heart failure can be caused by various forms of heart disease, including high blood pressure, heart defects, cardiovascular disease, diabetes, vascular heart disease, heart infections or heart muscle disease.

iii. **Congenital heart disease** - malformations of heart structure existing at birth.

3. A Novel Distance Measure on Neutrosophic Sets

3.1. Distance-based Similarity Measure

Definition 3.1.1: Let $X = \{x_1, x_2, \ldots., x_n\}$ be the universe of discourse. Let $A = \{x_i, T_A(x_i), I_A(x_i), F_A(x_i)): x_i \in X\}$ and $B = \{x_i, T_B(x_i), I_B(x_i), F_B(x_i)): x_i \in X\}$ be two neutrosophic sets. Then, by Definition 2.3.1, a new distance measure can be defined as:

$$d_{New}^{N}(A,B) =$$

$$\frac{2}{n}\sum\nolimits_{i=1}^{n} \frac{\sin\left\{\frac{\pi}{10}|T_A(x_i) - T_B(x_i)|\right\} + \sin\left\{\frac{\pi}{10}|I_A(x_i) - I_B(x_i)|\right\} + \sin\left\{\frac{\pi}{10}|F_A(x_i) - F_B(x_i)|\right\}}{1 + \sin\left\{\frac{\pi}{10}|T_A(x_i) - T_B(x_i)|\right\} + \sin\left\{\frac{\pi}{10}|I_A(x_i) - I_B(x_i)|\right\} + \sin\left\{\frac{\pi}{10}|F_A(x_i) - F_B(x_i)|\right\}} \tag{1}$$

where $\frac{\pi}{6}$ is replaced by the factor $\frac{\pi}{10}$ to fulfil the proof of Proposition 3.1.1.

Proposition 3.1.1: The distance measures $d_{New}^{N}(A,B)$ for neutrosophic sets A and B comply with the following properties:

(C1) $0 \leq d_{New}^{N}(A,B) \leq 1$;
(C2) $d_{New}^{N}(A,B) = 0$ if and only if $A = B$;
(C3) $d_{New}^{N}(A,B) = d_{New}^{N}(B,A)$;
(C4) $d_{New}^{N}(A,C) \leq d_{New}^{N}(A,B)$ and $d_{New}^{N}(A,C) \leq d_{New}^{N}(B,C)$ if C is neutrosophic set in X and $A \subseteq B \subseteq C$.

The new distance measure satisfies all the properties, and the proofs are given below. The degree of truth, indeterminacy, and falsity membership for neutrosophic set maybe in decreasing or increasing order.

Proof:

(C1) $0 \leq d_{New}^{N}(A,B) \leq 1$.

As we know the degree of truth, indeterminacy, and falsity membership for neutrosophic set is $0 \leq T_A(x), I_A(x), F_A(x) \leq 1$. This implies for $A = \{x_i, T_A(x_i), I_A(x_i), F_A(x_i)): x_i \in X\}$ and $B = \{x_i, T_B(x_i), I_B(x_i), F_B(x_i)): x_i \in X\}$.

$0 \le |T_A(x_i) - T_B(x_i)| \le 1, 0 \le |I_A(x_i) - I_B(x_i)| \le 1,$ and $0 \le |F_A(x_i) - F_B(x_i)| \le 1$

$$\Rightarrow 0 \le sin\left\{\frac{\pi}{10}|T_A(x_i) - T_B(x_i)|\right\} \le \frac{1}{3}, 0 \le sin\left\{\frac{\pi}{10}|I_A(x_i) - I_B(x_i)|\right\} \le \frac{1}{3},$$

and

$$0 \le sin\left\{\frac{\pi}{10}|F_A(x_i) - F_B(x_i)|\right\} \le \frac{1}{3}$$

$$\Rightarrow 0 \le sin\left\{\frac{\pi}{10}|T_A(x_i) - T_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|I_A(x_i) - I_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|F_A(x_i) - F_B(x_i)|\right\} \le 1 \quad (2)$$

$$\Rightarrow 0 \le 1 + sin\left\{\frac{\pi}{10}|T_A(x_i) - T_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|I_A(x_i) - I_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|F_A(x_i) - F_B(x_i)|\right\} \le 2 \quad (3)$$

Therefore, from equation (2) and equation (3)

$$\Rightarrow 0 \le 2.\frac{sin\left\{\frac{\pi}{10}|T_A(x_i) - T_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|I_A(x_i) - I_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|F_A(x_i) - F_B(x_i)|\right\}}{1 + sin\left\{\frac{\pi}{10}|T_A(x_i) - T_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|I_A(x_i) - I_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|F_A(x_i) - F_B(x_i)|\right\}} \le 1$$

$$\Rightarrow 0 \le \frac{2}{n}\sum_{i=1}^{n}\frac{sin\left\{\frac{\pi}{10}|T_A(x_i) - T_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|I_A(x_i) - I_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|F_A(x_i) - F_B(x_i)|\right\}}{1 + sin\left\{\frac{\pi}{10}|T_A(x_i) - T_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|I_A(x_i) - I_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|F_A(x_i) - F_B(x_i)|\right\}} \le 1$$

$$\Rightarrow 0 \le d_{New}^N(A,B) \le 1.$$

(C2) $d_{New}^N(A,B) = 0$ if and only if $A = B$.

If $A = B$, then $T_A(x_i) = T_B(x_i), I_A(x_i) = I_B(x_i)$, and $F_A(x_i) = F_B(x_i)$ which states that $|T_A(x_i) - T_B(x_i)| = 0, |I_A(x_i) - I_B(x_i)| = 0,$ and $|F_A(x_i) - F_B(x_i)| = 0$. Hence, $sin\left\{\frac{\pi}{10}|T_A(x_i) - T_B(x_i)|\right\} = 0, sin\left\{\frac{\pi}{10}|I_A(x_i) - I_B(x_i)|\right\} = 0,$ and $sin\left\{\frac{\pi}{10}|F_A(x_i) - F_B(x_i)|\right\} = 0$. Thus, $d_{New}^N(A,B) = 0$.

Conversely,

$$d_{New}^N(A,B) = 0$$

$$\frac{2}{n}\sum_{i=1}^{n}\frac{sin\left\{\frac{\pi}{10}|T_A(x_i) - T_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|I_A(x_i) - I_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|F_A(x_i) - F_B(x_i)|\right\}}{1 + sin\left\{\frac{\pi}{10}|T_A(x_i) - T_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|I_A(x_i) - I_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|F_A(x_i) - F_B(x_i)|\right\}} = 0$$

$$\frac{sin\left\{\frac{\pi}{10}|T_A(x_i) - T_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|I_A(x_i) - I_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|F_A(x_i) - F_B(x_i)|\right\}}{1 + sin\left\{\frac{\pi}{10}|T_A(x_i) - T_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|I_A(x_i) - I_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|F_A(x_i) - F_B(x_i)|\right\}} = 0$$

$$sin\left\{\frac{\pi}{10}|T_A(x_i) - T_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|I_A(x_i) - I_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|F_A(x_i) - F_B(x_i)|\right\} = 0,$$

For this reason

$sin\left\{\frac{\pi}{10}|T_A(x_i) - T_B(x_i)|\right\} = 0, sin\left\{\frac{\pi}{10}|I_A(x_i) - I_B(x_i)|\right\} = 0,$ and $sin\left\{\frac{\pi}{10}|F_A(x_i) - F_B(x_i)|\right\} = 0$

$|T_A(x_i) - T_B(x_i)| = 0, |I_A(x_i) - I_B(x_i)| = 0,$ and $|F_A(x_i) - F_B(x_i)| = 0$

$T_A(x_i) = T_B(x_i), I_A(x_i) = I_B(x_i),$ and $F_A(x_i) = F_B(x_i)$

$\Rightarrow A = B$

Hence $d_{New}^N(A,B) = 0$ if and only if $A = B$.

(C3) $d_{New}^N(A,B) = d_{New}^N(B,A)$.

It is obvious that $T_A(x_i) - T_B(x_i) \ne T_B(x_i) - T_A(x_i)$, $I_A(x_i) - I_B(x_i) \ne I_B(x_i) - I_A(x_i)$, and $F_A(x_i) - F_B(x_i) \ne F_B(x_i) - F_A(x_i)$.

But, $|T_A(x_i) - T_B(x_i)| = |T_B(x_i) - T_A(x_i)|$, $|I_A(x_i) - I_B(x_i)| = |I_B(x_i) - I_A(x_i)|$, and $|F_A(x_i) - F_B(x_i)| = |F_B(x_i) - F_A(x_i)|$.

Hence,

$$d_{New}^N(A,B) = \frac{2}{n}\sum_{i=1}^{n} \frac{sin\left\{\frac{\pi}{10}|T_A(x_i) - T_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|I_A(x_i) - I_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|F_A(x_i) - F_B(x_i)|\right\}}{1 + sin\left\{\frac{\pi}{10}|T_A(x_i) - T_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|I_A(x_i) - I_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|F_A(x_i) - F_B(x_i)|\right\}}$$

$$= \frac{2}{n}\sum_{i=1}^{n} \frac{sin\left\{\frac{\pi}{10}|T_B(x_i) - T_A(x_i)|\right\} + sin\left\{\frac{\pi}{10}|I_B(x_i) - I_A(x_i)|\right\} + sin\left\{\frac{\pi}{10}|F_B(x_i) - F_A(x_i)|\right\}}{1 + sin\left\{\frac{\pi}{10}|T_B(x_i) - T_A(x_i)|\right\} + sin\left\{\frac{\pi}{10}|I_B(x_i) - I_A(x_i)|\right\} + sin\left\{\frac{\pi}{10}|F_B(x_i) - F_A(x_i)|\right\}}$$

$$= d_{New}^N(B,A).$$

(C4) $d_{New}^N(A,C) \le d_{New}^N(A,B)$ and $d_{New}^N(A,C) \le d_{New}^N(B,C)$ if C is a neutrosophic set in X and $A \subseteq B \subseteq C$.

Consider $C = \{x_i, T_C(x_i), I_C(x_i), F_C(x_i)): x_i \in X\}$ is a neutrosophic set in X and let $A \subseteq B \subseteq C$. This implies that $T_A(x) \le T_B(x) \le T_C(x), I_A(x) \le I_B(x) \le I_C(x), F_A(x) \le F_B(x) \le F_C(x)$ for every $x_i \in X$. Then, we will have the following relations:

$a)|T_A(x_i) - T_C(x_i)| \le |T_A(x_i) - T_B(x_i)|$, and $|T_A(x_i) - T_C(x_i)| \le |T_B(x_i) - T_C(x_i)|$

$b)|I_A(x_i) - I_C(x_i)| \le |I_A(x_i) - I_B(x_i)|$, and $|I_A(x_i) - I_C(x_i)| \le |I_B(x_i) - I_C(x_i)|$

$c)|F_A(x_i) - F_C(x_i)| \le |F_A(x_i) - F_B(x_i)|$, and $|F_A(x_i) - F_C(x_i)| \le |F_B(x_i) - F_C(x_i)|$

Then,

$sin\left\{\frac{\pi}{10}|T_A(x_i) - T_C(x_i)|\right\} \le sin\left\{\frac{\pi}{10}|T_A(x_i) - T_B(x_i)|\right\}$ and

$sin\left\{\frac{\pi}{10}|T_A(x_i) - T_C(x_i)|\right\} \le sin\left\{\frac{\pi}{10}|T_B(x_i) - T_C(x_i)|\right\}$

Similarly,

$sin\left\{\frac{\pi}{10}|I_A(x_i) - I_C(x_i)|\right\} \le sin\left\{\frac{\pi}{10}|I_A(x_i) - I_B(x_i)|\right\}$ and

$sin\left\{\frac{\pi}{10}|I_A(x_i) - I_C(x_i)|\right\} \le sin\left\{\frac{\pi}{10}|I_B(x_i) - I_C(x_i)|\right\}$

$sin\left\{\frac{\pi}{10}|F_A(x_i) - F_C(x_i)|\right\} \le sin\left\{\frac{\pi}{10}|F_A(x_i) - F_B(x_i)|\right\}$ and

$sin\left\{\frac{\pi}{10}|F_A(x_i) - F_C(x_i)|\right\} \le sin\left\{\frac{\pi}{10}|F_B(x_i) - F_C(x_i)|\right\}$

Then,

$$sin\left\{\frac{\pi}{10}|T_A(x_i) - T_C(x_i)|\right\} + sin\left\{\frac{\pi}{10}|I_A(x_i) - I_C(x_i)|\right\} + sin\left\{\frac{\pi}{10}|F_A(x_i) - F_C(x_i)|\right\} \le$$
$$sin\left\{\frac{\pi}{10}|T_A(x_i) - T_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|I_A(x_i) - I_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|F_A(x_i) - F_B(x_i)|\right\}.$$

and

$$sin\left\{\frac{\pi}{10}|T_A(x_i) - T_C(x_i)|\right\} + sin\left\{\frac{\pi}{10}|I_A(x_i) - I_C(x_i)|\right\} + sin\left\{\frac{\pi}{10}|F_A(x_i) - F_C(x_i)|\right\} \le$$
$$sin\left\{\frac{\pi}{10}|T_B(x_i) - T_C(x_i)|\right\} + sin\left\{\frac{\pi}{10}|I_B(x_i) - I_C(x_i)|\right\} + sin\left\{\frac{\pi}{10}|F_B(x_i) - F_C(x_i)|\right\}.$$

Hence,

$$\frac{2}{n}\sum_{i=1}^{n} \frac{sin\left\{\frac{\pi}{10}|T_A(x_i) - T_C(x_i)|\right\} + sin\left\{\frac{\pi}{10}|I_A(x_i) - I_C(x_i)|\right\} + sin\left\{\frac{\pi}{10}|F_A(x_i) - F_C(x_i)|\right\}}{1 + sin\left\{\frac{\pi}{10}|T_A(x_i) - T_C(x_i)|\right\} + sin\left\{\frac{\pi}{10}|I_A(x_i) - I_C(x_i)|\right\} + sin\left\{\frac{\pi}{10}|F_A(x_i) - F_C(x_i)|\right\}}$$

$$\le \frac{2}{n}\sum_{i=1}^{n} \frac{sin\left\{\frac{\pi}{10}|T_A(x_i) - T_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|I_A(x_i) - I_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|F_A(x_i) - F_B(x_i)|\right\}}{1 + sin\left\{\frac{\pi}{10}|T_A(x_i) - T_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|I_A(x_i) - I_B(x_i)|\right\} + sin\left\{\frac{\pi}{10}|F_A(x_i) - F_B(x_i)|\right\}}$$

and

$$\frac{2}{n}\sum_{i=1}^{n}\frac{\sin\left\{\frac{\pi}{10}|T_A(x_i)-T_C(x_i)|\right\}+\sin\left\{\frac{\pi}{10}|I_A(x_i)-I_C(x_i)|\right\}+\sin\left\{\frac{\pi}{10}|F_A(x_i)-F_C(x_i)|\right\}}{1+\sin\left\{\frac{\pi}{10}|T_A(x_i)-T_C(x_i)|\right\}+\sin\left\{\frac{\pi}{10}|I_A(x_i)-I_C(x_i)|\right\}+\sin\left\{\frac{\pi}{10}|F_A(x_i)-F_C(x_i)|\right\}}$$
$$\leq\frac{2}{n}\sum_{i=1}^{n}\frac{\sin\left\{\frac{\pi}{10}|T_B(x_i)-T_C(x_i)|\right\}+\sin\left\{\frac{\pi}{10}|I_B(x_i)-I_C(x_i)|\right\}+\sin\left\{\frac{\pi}{10}|F_B(x_i)-F_C(x_i)|\right\}}{1+\sin\left\{\frac{\pi}{10}|T_B(x_i)-T_C(x_i)|\right\}+\sin\left\{\frac{\pi}{10}|I_B(x_i)-I_C(x_i)|\right\}+\sin\left\{\frac{\pi}{10}|F_B(x_i)-F_C(x_i)|\right\}}$$

$\Rightarrow d^N_{New}(A,C)\leq d^N_{New}(A,B)$and $d^N_{New}(A,C)\leq d^N_{New}(B,C)$.
The proof is completed. ■

Example 3.1.1: Let $A=\{x_1,(0.7,0.6,0.2):x_1\in X\}$, $B=\{x_1,(0.8,0.2,0.9):x_1\in X\}$, and $C=\{x_1,(0.4,0.5,0.6):x_1\in X\}$ be a three neutrosophic sets in *X.* Then, by using a new distance-similarity measure as equation (1), the Proposition 3.1.1 is satisfied.

(C1) $0\leq d^N_{New}(A,B)\leq 1$;

$$d^N_{New}(A,B)=2\left(\frac{\sin\{\frac{\pi}{10}|0.7-0.8|\}+\sin\{\frac{\pi}{10}|0.6-0.2|\}+\sin\{\frac{\pi}{10}|0.2-0.9|\}}{1+\sin\{\frac{\pi}{10}|0.7-0.8|\}+\sin\{\frac{\pi}{10}|0.6-0.2|\}+\sin\{\frac{\pi}{10}|0.2-0.9|\}}\right)=0.5452\ \in[0,1].$$

(C2) $d^N_{New}(A,B)=0$ if and only if $A = B$;

$$\text{If } A=B,\ d^N_{New}(A,A)=2\left(\frac{\sin\{\frac{\pi}{10}|0.7-0.7|\}+\sin\{\frac{\pi}{10}|0.6-0.6|\}+\sin\{\frac{\pi}{10}|0.2-0.2|\}}{1+\sin\{\frac{\pi}{10}|0.7-0.7|\}+\sin\{\frac{\pi}{10}|0.6-0.6|\}+\sin\{\frac{\pi}{10}|0.2-0.2|\}}\right)=0.$$

(C3) $d^N_{New}(A,B)=d^N_{New}(B,A)$;

It is obviously that:

$|0.7-0.8| = |0.8-0.7|$, $|0.6-0.2| = |0.2-0.6|$ and $|0.2-0.9| = |0.9-0.2|$.

Then, $d^N_{New}(A,B)=d^N_{New}(B,A)=0.5452$.

(C4) $d^N_{New}(A,C)\leq d^N_{New}(A,B)$ and $d^N_{New}(A,C)\leq d^N_{New}(B,C)$ if C is a neutrosophic set in X and $A\subseteq B\subseteq C$.

$$d^N_{New}(A,C)=2\left(\frac{\sin\left\{\frac{\pi}{10}|0.7-0.4|\right\}+\sin\left\{\frac{\pi}{10}|0.6-0.5|\right\}+\sin\left\{\frac{\pi}{10}|0.2-0.6|\right\}}{1+\sin\left\{\frac{\pi}{10}|0.7-0.4|\right\}+\sin\left\{\frac{\pi}{10}|0.6-0.5|\right\}+\sin\left\{\frac{\pi}{10}|0.2-0.6|\right\}}\right)=0.4005$$
$$d^N_{New}(B,C)=2\left(\frac{\sin\left\{\frac{\pi}{10}|0.8-0.4|\right\}+\sin\left\{\frac{\pi}{10}|0.2-0.5|\right\}+\sin\left\{\frac{\pi}{10}|0.9-0.6|\right\}}{1+\sin\left\{\frac{\pi}{10}|0.8-0.4|\right\}+\sin\left\{\frac{\pi}{10}|0.2-0.5|\right\}+\sin\left\{\frac{\pi}{10}|0.9-0.6|\right\}}\right)=0.4770$$

As a result, $0.4005\leq 0.5452$ and $0.4005\leq 0.4770$.
Therefore, $d^N_{New}(A,C)\leq d^N_{New}(A,B)$ and $d^N_{New}(A,C)\leq d^N_{New}(B,C)$ if C is a neutrosophic set in X and $A\subseteq B\subseteq C$.

4. Methodology

There are three steps to complete the cardiovascular disease risk analysis using distance-based similarity measure of neutrosophic set. This study uses the Normalized Hamming distance, extended Hausdorff distance, Normalized Euclidean distance, and new distance measure of NS as defined in Section 3 for comparative analysis. The steps to complete the analysis are as follows:

Step 1: The extraction of data.

The data on the symptoms experienced by the patients are given in the form of neutrosophic sets values [3]. Further, the relationships between the symptoms and diseases are displayed in binary form and can easily being used as a reference in determining which cardiovascular diseases that put the patients at a high risk.

Step 2: Distance-based similarity measure

Determine the similarity measure of neutrosophic set for each patient by using four different distance measures. Neutrosophic set is used to determine the similarity measure of the relationship between symptoms and diseases, patients, and symptoms by using Definitions 2.2.1 – 2.2.4 and Definition 3.3.1.

Step 3: Discussion of a complete data analysis

Finally, based on the results in step 2, the whole data analysis can be discussed whether the patients' symptoms are close to the diseases. The conclusion can be made depend on the value of the similarity measures. The patient possibly suffers from the disease when the value of similarity measure is bigger than 0.5. Meanwhile, the patient may not possibly suffer from the disease when the value of similarity measure is less than 0.5.

5. Case Study: Implementation in Cardiovascular Disease Risks Analysis

This section discusses on the case study of two patients having five similar symptoms but different in body mass index. The patients' data on the severity degree of the symptoms together with the experts' consensus on the symptoms-cardiovascular diseases relationship are represented in SVNS. The data considers the degree of truth membership, indeterminacy membership and falsity membership for each element set. Let $P = \{p_1, p_2\}$ is a set of patients, $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ is a set of symptoms. Table 1 shows data relationship between patients and symptoms as discussed in [3]. Besides, the relationship between cardiovascular diseases with symptoms is presented in Table 2.

Table 1. The relationship between patients and symptoms [3]

Symptom	Patient 1, p_1	Patient 1, p_2
Cholesterol, s_1	High (0.21, 0.81, 0.8)	High (0.21, 0.81, 0.8)
Depression, s_2	High (0.8, 0.33, 0.25)	High (0.8, 0.33, 0.25)
Diabetes, s_3	High (0.5, 0.58, 0.53)	High (0.5, 0.58, 0.53)
Blood Pressure, s_4	High (0.8, 0.4, 0.29)	High (0.8, 0.4, 0.29)
Body Mass Index, s_5	Med (0, 0.75, 1)	High (0.86, 0.57, 0.2)
Unhealthy Diet, s_6	Frequently (0.5,0.83,0.625)	Frequently (0.5,0.83,0.625)

Table 2. The relationship between cardiovascular disease with symptoms [3]

Symptom	Heart Attack (D1)	Heart Failure(D2)	Congenital Heart Disease (D3)
Cholesterol, s_1	(1,0,0)	(0,0,1)	(0,0,1)
Depression, s_2	(1,0,0)	(0,0,1)	(0,0,1)
Diabetes, s_3	(1,0,0)	(1,0,0)	(1,0,0)
Blood Pressure, s_4	(1,0,0)	(1,0,0)	(0,0,1)
Body Mass Index, s_5	(1,0,0)	(1,0,0)	(0,0,1)
Unhealthy Diet, s_6	(1,0,0)	(0,0,1)	(0,0,1)

According to data collected in Table 1, the truth membership degree for cholesterol for both patients are 0.21, the indeterminacy membership degree for cholesterol for both patients is 0.81 and the falsity membership degree for cholesterol for both patients are 0.8. The same description is indicated for each data. It is obvious that the SVNS value of body mass index (BMI) for the two patients is different. The BMI of patient 2 is categorized as high with the value of the truth membership degree is 0.86. As for patient 1, his BMI falls into the category of medium with 0.75 and 1 indicating the indeterminacy and falsity membership degrees respectively.

Table 3. The distance measure for neutrosophic set

Distance	Normalized Hamming		Extended Hausdorff		Normalized Euclidean	
	P1	**P2**	**P1**	**P2**	**P1**	**P2**
Heart Attack, D1	0.576944	0.474722	0.658333	0.586667	0.631358	0.526889
Heart Failure, D2	0.558611	0.456389	0.736667	0.6650	0.616668	0.509195
Congenital heart disease, D3	0.504167	0.586389	0.761667	0.7800	0.570819	0.623922

Table 4. The new distance measure for neutrosophic set

New distance measure	**P1**	**P2**
Heart Attack, D1	0.6718168	0.5924242
Heart Failure, D2	0.6683397	0.5889471
Congenital heart disease, D3	0.6297732	0.7032119

By using the data in Table 1 and Table 2, the three distance-based similarity measures discussed in Section 2 are calculated and the values are displayed in Table 3. Further, the new distance values obtained by the new formula in Section 3 are presented in Table 4. Then, the associated values of similarity for all the four distance measures are calculated and tabulated in Table 5 and 6 respectively.

Table 5. Similarity measure values of the three-distance measure of neutrosophic set

Similarity	Normalized Hamming		Extended Hausdorff		Normalized Euclidean	
	P1	**P2**	**P1**	**P2**	**P1**	**P2**
Heart Attack, D1	0.423056	**0.525278**	0.341667	**0.413333**	0.368642	**0.473111**
Heart Failure, D2	0.441389	**0.543611**	0.263333	**0.335000**	0.383332	**0.490805**
Congenital heart disease, D3	**0.495833**	0.413611	0.238333	0.220000	**0.429181**	0.376078

Table 6. The new similarity measure for neutrosophic set

New similarity measure	**P1**	**P2**
Heart Attack, D1	0.3281832	**0.4075758**
Heart Failure, D2	0.3316603	**0.4110529**
Congenital heart disease, D3	**0.3702268**	0.2967881

A lower distance or higher similarity measure value implies higher possibility of one patient having a particular disease. The slight difference of symptoms shown in both patients results to distinct conclusion on the type of cardiovascular diseases that they experience. Patient 1 has the highest severity degree of symptoms for congenital heart disease. Meanwhile, Patient 2 is more likely to be diagnosed of having heart attack and heart failure. Concurrently, it is apparent that most of the similarity measure values in Table 5 and 6 are less than 0.5. Hence, it is probable to conclude that both patients are possibly not suffering from any of the three cardiovascular diseases.

6. Conclusions

This research proposes a novel distance measure for single value neutrosophic set which results to the use of similarity measure. The effectiveness of the new developed measure formula is demonstrated by adopted it in the process of medical diagnosis. Its similarity values are found to be consistent with similarity measures of the three existing distance measures i.e Normalized Hamming distance, extended Hausdorff distance and Normalized Euclidean distance. The analyses show that the new distance-based similarity measure is well executed in the case of truth membership, indeterminacy membership and falsity membership functions. In the future study, it is recommended that one might consider additional significant symptoms and other distance or similarity measure to increase the accuracy level in diagnosing a patient with any cardiovascular diseases. Besides, it is also recommended to utilize new entropy-based similarity measures of SVNS [28] to overcome the restriction of the distance similarity measures. The neutrosophic set also can be extend to pythagorean neutrosophic multi set as this can provide many applications to multi attribute group decision making problems in medical diagnosis and many other real-life problems [29].

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Interval-Type Fuzzy Linear Fractional Programming Problem in Neutrosophic Environment: A Fuzzy Mathematical Programming Approach

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Abstract: This article proposes an interval-valued fuzzy linear fractional programming (LFP) problem, where the coefficients in the objective functions are assumed to be single-valued trapezoidal fuzzy neutrosophic numbers. In addition to this, the coefficients in the constraints are represented by interval-valued fuzzy numbers. Auxiliary models according to different criteria are developed. Fuzzy mathematical programming approach is applied for solving each model by defining membership function. In this work, a linear membership function is used to determine the optimal compromise solutions of the auxiliary models. A numerical example is solved for the illustration and clear explanation of the proposed approach.

Keywords: Interval-type; Fractional programming; Fuzzy numbers; Neutrosophic numbers; Interval- valued fuzzy numbers; Trapezoidal fuzzy numbers; Auxiliary models.

1. Introduction:

Linear fractional programming (LFP) model was initially developed to determine the two-objective linear programming problem (LPP). In many applications as cutting stock problem, shipping schedules problem, blending problem, etc., the optimization of ratios provides more insight into the situation than the optimization of numerator and denominator individually. Therefore, maximizing a ratio is seen as the simultaneous maximization of numerator and minimization of denominator, its solution considering one solution among several pareto optimal solutions of two-objective model. In daily life situations related problems, policy maker sometimes might face to examine the ratio between actual cost and standard cost, output and employee, inventory and sales, etc., with both denominator and nominator are linear. A LFP problem is only one ratio under linear constraints. For the benchmark trade-off among the simplicity and accuracy of a real-life model, the fractional programming provides more accuracy, and simultaneously wins to avoid the overload of model under considerations. On the other side, the fractional objective incorporated in standard membership function for fuzzy goal, makes them non-linear.

Charnes and Cooper (1962) solved LFP problem as two LP optimization models. They also suggested several applications to a ship routing problem. The fractional programming problem may also be nonlinear type in nature. Dinkelbach (1967) studied nonlinear fractional programming problems, and their methodology. Bitran and Novaes (1973) presented a LPP including the fractional objective function. Many researchers have studied LFP problems (Schaible, 1976; Charnes et al. 1987, and Craven, 1988). Moore (1979) investigated some methods with applications related to interval programming problem. Later on, Gupta and Chakraborty (1998) have applied fuzzy programming approach for obtaining optimal compromise solution for LFP problem under fuzziness. Ammar and Khalifa (2004) presented a parametric solution methodology to solve the multiple criteria LFP problem. Jain and Saksena (2012) have proposed a method for solving fractional programming in the case of there is no completely functional relationship between the decision variables and the objective function. Guzel (2013) developed a proposal to the solution of multiple objective function LFP problems.

Fuzzy set theory firstly introduced by Zadeh (1965). Fuzzy numerical data can be represented by means of fuzzy subsets of the real line known as fuzzy numbers. Decision making in a fuzzy environment has been an improvement and a great help in the management decision problems (Bellman and Zadeh, 1970). Zimmermann (1974) is one of the pioneer researchers in the fuzzy linear programming (FLP). Once of the difficulties occur in the application of mathematical programming is that the parameters in the formulation are not constants but uncertain. The fuzzy nature in a goal programming problem firstly has been discussed by Zimmermann (1978), and lots of others authors working in that field. The decision maker cannot always articulate the goal precisely in a spite of having his/ her decision making experience. Luhandjula (1984) introduced some fuzzy mathematical approaches for solving the multi objective LFP. Dutta et al. (1993) studied the effect of tolerance in fuzzy LFP problem. Sadjadi et al. (2005) presented a new methodology based on fuzzy concept for solving the multiple objective LFP model developed for inventory control problem. Ammar and Khalifa (2009) described the LFP problem considering the fuzzy parameter. Kumar and Dutta (2015) developed LFP with multiple objective functions as an inventory model of multiple items with price-sensitive demand in fuzzy environment. Veeramani and Sumathi (2017) proposed a solution procedure to solve LFP with triangular fuzzy numbers in the objective function cost, the resources, and the technological coefficients. Stanojevic et al. (2020) have introduced two crisp models for solving fuzzy multiple objective LFP problems.

Several researchers presented their work in stochastic LFP programming in fuzzy environment. Over the years, this area has become popular in fractional programming community by means of fuzzy and probabilistic parameters and variables. Chen (2005) presented the fractional programming approach with its application to two inventory models in stochastic environment. Iskander (2003) presented a case of fuzzy weighted objective function. They used various kinds of dominance criteria to linear multiple objective optimization in stochastic fuzzy environment.

Intuitionistic fuzzy sets were first proposed by Atanassov (1986), which have become a very interested topic of research in the area of fuzzy set. Wu and Liu (2013) presented a methodology to solve the multiple attribute group decision making models involving the interval-valued intuitionistic fuzzy numbers. Singh and Yadav (2016) proposed the fuzzy mathematical programming approach for solving LFP problem in intuitionistic fuzzy environment. Ali et al. (2018) studied the LFP with multiple objective functions in intuitionistic fuzzy environment with an application to inventory management. Dutta and Kumar (2015) presented the fuzzy goal programming approach with an application to solve the multiple objective LFP for inventory problem consisted of deteriorating items. Several authors further extended this problem to uncertain environment. Garai and Garg (2019) further extended introduced multi objective LFP problem to possibility and necessity constraints and generalized intuitionistic fuzzy parameters. Nasseri and Bavandi (2019) studied the stochastic LFP model. They further used the fuzzy based method to determine the single objective LFP problem.

Neutrosophic sets were first introduced by Smarandache (2005) as a new theory dealing with the origin, nature and scope of neutralities, as well as their interactions. Neutrosophic sets are the generalizations of the intuitionistic fuzzy sets. Dubois et al. (2005) presented the terminological type difficulties in the theory of fuzzy sets, which caused the case of intuitionistic fuzzy sets. Tian et al. (2016) presented the simplified neutrosophic linguistic normalized weighted bonferroni mean operator. They also presented some applications of neutrosophic set theory to multi-criteria decision-making problems. Thamaraiselvi and Santhi (2016) introduced a mathematical approach to real transportation model in neutrosophic environment. Rizk-Allah et al. (2018) developed a multiple objective transportation model in neutrosophic set environment. Ahmad et al. (2018) presented an algorithm for the computation of multi objective nonlinear optimization problem with single valued neutrosophic hesitant fuzzy criteria. Chakraborty et al. (2019) studied the various kinds of trapezoidal neutrosophic numbers along with the process of de-neutrosophication techniques. They also presented an application based on time and cost optimization method, in sequencing problem. Khalifa et al. (2020) presented a study on optimizing neutrosophic complex programming with the help of lexicographic order. In the application sometimes, determining the membership functions of fuzzy sets is not easy, but the degree of interval membership is easy to determine.

In this paper, linear programming problem with objective function coefficients represented as neutrosophic numbers and interval- valued fuzzy coefficients in the constraints is presented. In the meaning of different criteria, auxiliary models are obtained. Outlay of this article is as described under:

Section 2 presents the related preliminaries. Section 3 formulates neutrosophic linear programming with interval- valued coefficients. In Section 4, the procedure for the solution of the problem is described. In Section 5, we provide a numerical illustration for the efficiency of the solution approach. In the last, we conclude in Section 6**.**

2. Preliminaries:

In order to discuss the problem under consideration, let us introduce some results related to interval- valued fuzzy set and neutrosophic numbers.

Moore (1979) introduced the concept of closed interval number. Let

$I(\mathbb{R}) = \{[a^-,\ a^+]: a^-,\ a^+ \in \mathbb{R} = (-\infty, \infty), a^- \leq\ a^+\}$

represent the closed intervals on the real line $\mathbb{R}$. Wu and Liu (2013) defined the closed interval of

$[0,1]$ as $I = [0,1]$, $[I] = \{[a,b]: a \leq b, a, b \in I\}$.

Definition 2.1. (Gorzafczang, 1983). Let X be a non-empty crisp set. A mapping $P: X \longrightarrow [I]$ is said to be an interval- valued fuzzy set (IVFS). All IVFSs on X denoted as $IF(X)$.

Definition 2.2. (Wu and Liu, 2013). Assume that $Q \in IF(X)$, and $Q(x) = [Q^-(x), Q^+(x)]$. Ordinary fuzzy set

$Q^-(x): X \longrightarrow I, x \mapsto Q^-(x), Q^+(x): X \longrightarrow I, x \mapsto Q^+(x)$, are called lower and upper fuzzy sets on Q, respectively.

Definition 2.3. (Wu and Liu, 2013). Assume that $Q \in IF(X)$and $[\xi_1,\ \xi_2] \in [I]$. The set

$Q[\xi_1,\ \xi_2] = \{x \in X: \xi_1 \leq Q^-(x),\ \xi_2 \leq\ Q^+(x)\}$, is called a $[\xi_1,\ \xi_2]$ −level set of Q.

Let $\mathrm{IFN}(\mathbb{R})$ denotes all interval- valued fuzzy numbers on the real number fields $\mathbb{R}$.

Lemma 2.1. (Wu and Liu, 2013). $Q \in \mathrm{IFN}(\mathbb{R})$, then for any $[\xi_1, \xi_2] \in [\mathrm{I}]$, the level set of $\mathrm{Q}[\xi_1, \xi_2]$ is an empty set or closed interval.

Lemma 2.2. (Wu and Liu, 2013). $\mathrm{Q} \in \mathrm{IFN}(\mathbb{R})$ if and only if

$$\mathrm{Q}^-(\mathrm{x}) = \begin{cases} \mathrm{R}^-(\mathrm{x}), \mathrm{x} > \psi; \\ \mathrm{L}^-(\mathrm{x}),\ \mathrm{x} \le \psi. \end{cases} \qquad \mathrm{Q}^+(\mathrm{x}) = \begin{cases} \mathrm{R}^+(\mathrm{x}), \mathrm{x} > \lambda; \\ \mathrm{L}^+(\mathrm{x}), \mathrm{x} < \mu; \\ 1,\ \mathrm{x} \in [\mu, \lambda] \ne \emptyset, \end{cases}$$

where, $\mathrm{L}^-(\mathrm{x})\ (0 \le \mathrm{L}^-(\mathrm{x}) \le 1)$ and $\mathrm{L}^+(\mathrm{x})(0 \le \mathrm{L}^+(\mathrm{x}) \le 1)$ are increasing right continuous functions, and $\lim_{\mathrm{x} \to -\infty} \mathrm{L}^-(\mathrm{x}) = \lim_{\mathrm{x} \to -\infty} \mathrm{L}^+(\mathrm{x}) = 0$. In addition, $\mathrm{R}^-(\mathrm{x})\ (0 \le \mathrm{R}^-(\mathrm{x}) \le 1)$ and $\mathrm{R}^+(\mathrm{x})\ (0 \le \mathrm{R}^-(\mathrm{x}) \le 1)$ are decreasing left continuous functions, and $\lim_{\mathrm{x} \to +\infty} \mathrm{R}^-(\mathrm{x}) = \lim_{\mathrm{x} \to +\infty} \mathrm{R}^+(\mathrm{x}) = 0$.

Definition 2.4. (Score and Accuracy functions of single valued trapezoidal neutrosophic number, Thamaraiselvi, 2016). Suppose $\tilde{\mathrm{c}}^{\mathrm{N}} = \langle (\mathrm{a}_1, \mathrm{a}_2,\ \mathrm{a}_2, \mathrm{a}_2);\ \mathrm{w}_c,\ \varpi_c, \mathrm{y}_c \rangle$ is a single- valued trapezoidal fuzzy number. Therefore,

i. Score function $\mathrm{S}(\tilde{\mathrm{c}}^{\mathrm{N}}) = \frac{1}{16}[\mathrm{a}_1 + \mathrm{a}_2 + \mathrm{a}_2 + \mathrm{a}_2] \times \begin{bmatrix} \nu_{\tilde{c}^{\mathrm{N}}} + (1 - \pi_{\tilde{c}^{\mathrm{N}}}) \\ + (1 - \rho_{\tilde{c}^{\mathrm{N}}}) \end{bmatrix}$,

ii. Accuracy function

$$A(\tilde{\mathrm{c}}^{\mathrm{N}}) = \frac{1}{16}[\mathrm{a}_1 + \mathrm{a}_2 + \mathrm{a}_2 + \mathrm{a}_2] \times [\nu_{\tilde{c}^{\mathrm{N}}} + (1 - \pi_{\tilde{c}^{\mathrm{N}}}) + (1 + \rho_{\tilde{c}^{\mathrm{N}}})].$$

3. Problem formulation & solution concepts:

Consider the following interval- valued fuzzy LFP problem in neutrosophic environment as follows:

$$\text{Max}\, \tilde{Z}^N(x) = \frac{\tilde{c}^{N^T} x + \tilde{\alpha}^N}{\tilde{d}^{N^T} x + \tilde{\beta}^N}$$

Subject to (1)

$$x \in \tilde{\Gamma} = \{x : \tilde{A}x \le \tilde{b}, x \ge 0\},$$

where, $\tilde{A} = (\tilde{a}_{ij})_{m \times n}, \tilde{b} = (\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_m), \in \mathrm{IVN}(\mathbb{R})$; $\tilde{c}^N = (\tilde{c}_1^{\ N}, \tilde{c}_2^{\ N} \dots, \tilde{c}_n^{\ N})^T$,

$\tilde{d}^N = \left(\tilde{d}_1{}^N, \dots, \tilde{d}_n{}^N\right)^T, \tilde{\alpha}^N$, and $\tilde{\beta}^N$ are single valued trapezoidal fuzzy neutrosophic numbers.

In accordance with Definition 4, problem (1) converted to problem (2) as follows:

$$\max Z(x) = \frac{c^T x+\alpha}{d^T x+\beta}$$

Subject to (2)

$$x \in \tilde{\Gamma} = \{x: \tilde{A}x \le \tilde{b}, x \ge 0\}.$$

Definition 3.1. If $a_{ij} = [a_{ij}{}^-(x), a_{ij}{}^+(x)]$, and $b_i = [b_i{}^-(x), b_i{}^+(x)]$, then the mappings

$$f_{ij}: \text{IFN}(\mathbb{R}) \longrightarrow \mathbb{R}, a_{ij} \longmapsto f_{ij}(a_{ij}) = a_{ij}^0,$$

$g_i: \text{IFN}(\mathbb{R}) \longrightarrow \mathbb{R}, b_i \mapsto g_i(b_i) = b_i^0$, are called fuzzy- crisp transformations, and hence problem becomes

$$\text{Max}\, \tilde{Z}^N(x) = \frac{c^T x+\alpha}{d^T x+\beta} = \frac{M(x)}{N(x)}$$

Subject to (3)

$$x \in \Gamma = \{x: g(x) = A^0 x - b^0 \le 0, x \ge 0\}.$$

Here, problem (3) is the corresponding auxiliary model of problem (2).

There are numerous criterion for the values of a_{ij}^0, and b_i^0 depending on the decision maker's objective:

- The first criterion "choosing big from small".

Suppose that $\mathrm{Q} \in \text{IFN}(\mathbb{R})$, and $\mathrm{Q(x)} = [\mathrm{Q^-(x)}, \mathrm{Q^+(x)}]$. Then, we have

$$\underset{x\in\mathbb{R}}{\text{Max}}\{\text{Min}(\mathrm{Q^-(x)}, \mathrm{Q^+(x)})\} = \underset{x\in\mathbb{R}}{\text{Max}}\, \mathrm{Q^-(x)} = \mathrm{Q^-(Q^0)}. \tag{4}$$

- The second criterion "choosing big from big". Then

$$\underset{x\in\mathbb{R}}{\text{Max}}\{\text{Max}(\mathrm{Q^-(x)}, \mathrm{Q^+(x)})\} = \underset{x\in\mathbb{R}}{\text{Max}}\, \mathrm{Q^+(x)} = \mathrm{Q^+(Q^0)}. \tag{5}$$

In the case that the first criterion is conservative and the second is risky, the compromise criterion is considered.

- The third criterion "Compromise criterion"

$$\mathrm{Q(x)} = \mathrm{k}\, \mathrm{Q^+(x)} + (1-\mathrm{k})\mathrm{Q^-(x)}. \tag{6}$$

Here, k is called the optimistic coefficient.

Lemma 3. Let $\mathrm{Q} \in \text{IFN}(\mathbb{R})$. Hence

i. If Q is taken according to the first criterion, then $Q^0 \in Q[\xi_1, \xi_2]$, where

$\xi_1 = \underset{x \in \mathbb{R}}{\text{Max}}\, Q^-(x)$, and ξ_2 any value of the interval $[0,1]$.

ii. If Q is taken referring to the second criterion, then $Q^0 \in Q[\xi_1, \xi_2]$, where

$\xi_1 = \underset{x \in \mathbb{R}}{\text{Max}}\, Q^-(x)$, and ξ_2 any value of the interval $[0, \xi_1]$, and

iii. If Q is taken referring to the third criterion, then $Q^0 \in Q[\xi_1, \xi_2]$, where

$\xi_1 = Q^-(x^*)$, and

$$Q^+(x^*) = \underset{x \in \mathbb{R}}{\text{Max}}\, Q^+(x),\ \xi_2 = Q^+(x^{**}),\ Q^-(x^{**}) = \underset{x \in \mathbb{R}}{\text{Max}}\, Q^-(x)$$

Therefore, with the help of the variable transformation (Charnes and Cooper, 1962; Schaible, 1976), we have

$$y = t\,x \ (\,t \text{ is scalar})$$

It is shown that if for $x \in \Gamma, M(x) \geq 0$, problem (3) with $N(x) > 0$ is an equivalent to

$$\begin{aligned} &\text{Max}\, t\, M\left(\tfrac{y}{t}\right) \\ &\text{Subject to} \\ &\quad t\, g\left(\frac{y}{t}\right) \leq 0; t\, N\left(\frac{y}{t}\right) \leq 1; y \geq 0, t \geq 0. \end{aligned} \tag{7}$$

In addition, for some $x \in \Gamma, M(x) < 0$, problem (3) is equivalent to

$$\begin{aligned} &\text{Max}\, t\, N\left(\tfrac{y}{t}\right) \\ &\text{Subject to} \\ &\quad t\, g\left(\frac{y}{t}\right) \leq 0; -t\, M\left(\frac{y}{t}\right) \leq 1; y \geq 0, t \geq 0. \end{aligned} \tag{8}$$

If for $x \in \Gamma, M(x) \geq 0$, then the membership function of the objective function is expressed as follows:

$$\mu(t\,M(x)) = \begin{cases} 1, & \text{if } t\,M(x) \leq \bar{Z}, \\ \frac{t\,M(x) - 0}{\bar{Z} - 0}, & \text{if } 0 < t\,M(x) < \bar{Z}, \\ 0, & \text{if } t\,M(x) \geq \bar{Z} \end{cases} \tag{9}$$

If for all $x \in \Gamma, M(x) < 0$, then the membership function is expressed as follows:

$$\mu(t\,M(x)) = \begin{cases} 1, & \text{if } t\,M(x) \le \underline{Z}, \\ \frac{t\,M(x)-\underline{Z}}{\overline{Z}-\underline{Z}}, & \text{if } \underline{Z} < t\,M(x) < \overline{Z}, \\ 0, & \text{if } t\,M(x) \le \overline{Z} \end{cases} \quad (10)$$

In Equation (10), $\underline{Z}$, and $\overline{Z}$ are aspiration levels for the minimization and maximization of $tM(x)$, respectively. Using the relations (9) and (10), problem (7) reduced into the following linear programming problem using Zadeh's min operator as

$$\text{Max}\,\nu$$

Subject to (11)

$$\mu(t\,M(y/t)) \ge \nu, t\,N\left(\frac{y}{t}\right) \le 1, A\left(\frac{y}{t}\right) - b \le 0, y, t \ge 0.$$

Proposition (Chakraborty and Gupta, 2002). If $d^T > 0$, and $\beta > 0$, then we have

$$Z = \frac{c^T x + \alpha}{d^T x + \beta}, x \ge 0, \text{ has } \overline{Z} = \text{Max}\left\{\frac{c_i}{d_i}, \frac{\alpha}{\beta}, i = 1, 2, \dots, n\right\},$$

$$\text{and } \underline{Z} = \text{Min}\left\{\frac{c_i}{d_i}, \frac{\alpha}{\beta}, i = 1, 2, \dots, n\right\},$$

where, $\bar{Z}$, and $\underline{Z}$ are the maximum and minimum values, respectively.

4. Solution procedure:

The steps of the solution method for solving interval- valued fuzzy LFP problem in neutrosophic environment are as follow:

Step 1: Consider problem (1),

Step 2: Convert problem (1) into problem (2), and hence (3),

Step 3: According to different criteria, obtaining problems (4), (5), and (6),

Step 4: Applying variable transformation method with membership functions as given in Equations (9) and (10) to problem (7) as in problem (8).

Step 5: Applying Zadah's min operator, problem (8) is converted into problem (11) which may be solved using any software (like LINGO 18.0 or MATLAB 2020a) for obtaining the optimal compromise solution.

5. Numerical example:

Consider a fractional programming problem as follows:

$$\operatorname{Max} \tilde{Z}^N = \frac{\tilde{c}_1^N x_1 + \tilde{c}_2^N x_2}{\tilde{d}_1^N x_1 + \tilde{d}_2^N x_2 + \tilde{\beta}^N}$$

Subject to (12)

$$a_{11}x_1 + a_{12}x_2 \leq b_1,$$

$$a_{21}x_1 + a_{22}x_2 \leq b_2,$$

$$x_1,\ x_2 \geq 0\ .$$

In Equation (12), let us consider the coefficients as follows:

$a_{11} = [a_{11}^{-}(x), a_{11}^{+}(x)] \in IFN(\mathbb{R}),\ {a_{11}}^{+} = (3,1,5),\ a_{12} = (3, 2.5, 5.5),$

$b_1 = (15,, 13, 16),\ a_{21} = (1, 0, 2),\ a_{22} = (1, 1, 2),\ b_2 = (1, 0, 3)$ are $L - R$ fuzzy numbers,

$\tilde{c}_1^N = \langle(-14, -10, -8, -5); 0.3, 0.6, 0.6\rangle,\ \tilde{c}_2^N = \langle(1, 3, 4, 6); 0.6, 0.3\ 0.5\rangle,$

$\tilde{d}_1^N = \langle(0, 1, 3, 6); 0.7, 0.5, 0.3\rangle = \tilde{d}_2^N$, and $\tilde{\beta}^N = \langle(5, 9\ 14, 19); 0.3, 0.7, 0.6\rangle$ are single-valued trapezoidal neutrosophic numbers.

$$a_{11}^{-}(x) = \begin{cases} \frac{x-1}{2}, 1 \leq x \leq 2; \\ \frac{-2x+9}{10}, 2 \leq x \leq \frac{5}{2}; \\ \frac{-6x+19}{10}, \frac{5}{2} \leq x \leq 3; \\ \frac{-x+5}{20}, 3 \leq x \leq 5; \\ 0, \text{Otherwise} \end{cases}$$

In accordance with the Definition 4, with the criterion in Equation (4), we formulate the model as presented in Equation (13):

$$\operatorname{Max} Z = \frac{-3x_1 + 2x_2}{x_1 - x_2 + 3}$$

Subject to (13)

$$2x_1 + 3x_2 \leq 15,$$

$$x_1 - x_2 \geq 1,$$

$$x_1 \geq 3,$$

$$x_1, x_2 \geq 0.$$

$$\bar{Z} = \text{Max}\left\{-\frac{3}{1}, -\frac{2}{1}, \frac{0}{3}\right\} = 0, \text{ and } \underline{Z} = \text{Min}\left\{-\frac{3}{1}, -\frac{2}{1}, \frac{0}{3}\right\} = -3.$$

Then, problem (13) is transformed to problem (14) as follows

$$\begin{aligned} &\text{Max}\,\nu \\ &\text{Subject to} \qquad (14) \\ &3y_1 - 2y_2 + 3\nu \leq 3, \\ &y_1 - y_2 + 3t \leq 1, \\ &2y_1 + 3y_2 - 15t \leq 0, \\ &y_1 - y_2 - t \geq 0, \\ &y_1 - 3t \geq 0, \\ &y_1, y_2, t, \nu \geq 0. \end{aligned}$$

The solution of problem (14) is given by

$$y_1 = 5.8887 \times e^{-12}, \; y_2 = 1.3614 \times e^{-12},$$

$$t = 1.8565 \times e^{-12}, \quad \nu = 0.999999,$$

$$x_1 = 3.1719, \quad x_2 = 0.7333,$$

$$\text{and} \quad Z = -0.00299586,$$

$$\tilde{Z}^N = \langle\left(\frac{-24}{12}, \frac{-16}{21}, \frac{-3}{37}, \frac{40}{7}\right); 0.3, 0.7, 0.6\rangle. \qquad (15)$$

Problem (11) according to the third criterion can be presented as follows

$$\begin{aligned} &\text{Max }\nu \\ &\text{Subject to} \qquad (16) \\ &3y_1 - 2y_2 + 3\nu \leq 3, \\ &y_1 - y_2 + 3t \leq 1, \\ &2.5y_1 + 3y_2 - 15t \leq 0, \\ &y_1 - y_2 - t \geq 0, \end{aligned}$$

$$y_1 - 3t \geq 0,$$

and $$y_1, y_2, t, \nu \geq 0.$$

The solution of problem (16) is given by

$y_1 = 3.19387 \times e^{-12}$, $y_2 = 8.54807 \times e^{-13}$,

$t = 9.50564 \times e^{-13}$, $\nu = 0.999999$,

$x_1 = 1.0849$, $x_2 = 0.8993$, and $Z = -0.45709$,

$$\tilde{Z}^N = \left\langle \left(\frac{-24}{12}, \frac{-16}{21}, \frac{-3}{37}, \frac{40}{7} \right); 0.3, 0.7, 0.6 \right\rangle.$$

6. Conclusions:

In this paper, LFP problem with single- valued trapezoidal fuzzy neutrosophic numbers in the objective functions coefficients and interval- valued fuzzy numbers coefficients in the constraints has studied. Auxiliary models according to different criteria have introduced. The solution of the corresponding auxiliary model under its criterion has considered the solution of the original problem according to the subjective and objective factors combination. For further research scope, we can extend the proposed work by introducing the generalized trapezoidal neutrosophic fuzzy number to deal with LFP problems in Neutrosophic fuzzy environment. Second, to extend the proposed work for nonlinear case would be an interesting area of research.

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SVPNS-MADM strategy based on GRA in SVPNS Environment

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Abstract: In the present study, we present a multi-attribute-decision-making (MADM) strategy in Single Valued Pentapartitioned Neutrosophic Set (SVPNS) environment based on Grey Relational Analysis (GRA) which we call SVPNS- MADM strategy. We define Hamming distance between two single valued pentapartitioned neutrosophic sets and prove its basic properties. The notion of pentapartitioned neutrosophic set is a powerful mathematical tool to deal with incomplete, indeterminate, ignorance, and inconsistent information. In this paper, we extend the neutrosophic GRA strategy to pentapartitioned neutrosophic GRA strategy. Then we employ it to an MADM strategy. Further, we demonstrate the developed MADM strategy by solving an illustrative numerical example that reflects the efficiency and applicability of the proposed strategy.

Keywords: Neutrosophic set, Single valued neutrosophic set, Pentapartitioned neutrosophic set; Multi attribute decision making, Grey relational analysis.

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1. Introduction

The idea of neutrosophic set (NS) was presented by Smarandache [38], which was a powerful mathematical tool to deal with incomplete, indeterminate, and inconsistent information. The notion of NS and its various extensions have been successfully applied in the many fields such as decision making [1-13, 16-19, 21, 25-31, 36-38, 43, 45], medical diagnosis [32-33, 44], data mining [20], conflict resolution [35], etc. In the recent past, the NSs [6, 22, 23-24, 39-42] have drawn a great attention during the last two decades. Different models of Multi-Attribute Decision Making (MADM) for crisp set, fuzzy set, intuitionistic fuzzy set and NS environments have been already proposed by so many researchers. Biswas et al. [5] proposed an entropy based grey relational analysis strategy for MADM model under The Single Valued Neutrosophic Set (SVNS) environment. Mondal and Pramanik [16] also proposed a neutrosophic decision making model for clay-brick selection in construction field based on Grey Relational Analysis GRA) in SVNS environment. The notion of Pentapartitioned Neutrosophic Set (PNS) was grounded by Mallick and Pramanik [15] by in the year 2020, which was also very powerful mathematical tool to deal with the data includes incomplete, indeterminate, ignorance, and unknown information. Since PNS has been restricted in the [0,1], we call it Single Valued PNS (SVPNS).

There is no study in the literature relating to MADM in SVPNS environment. To explore the unexplored MADM in SVPNS environment, we present an MADM strategy under in SVPNS environment based on GRA.

The rest of the paper is organized in the following way:
Section 2 recalls some relevant results on SVPNSs. Section 3 presents some new definitions relating to SVPNS that are useful to develop the present paper. Section 4 devotes to develop the GRA based SVPNS-MADM strategy. In section 5, we present a numerical example to demonstrate the proposed SVPNS-MADM strategy. In section 6, we present the concluding remarks and future scope of research.

2. Some Relevant Results:

In this section, we give some existing definitions, properties of PNS. We also provide some illustrative examples on PNS.

Definition 2.1.[15] Suppose that Ψ be a non-empty set. Then a PNS V over Ψ is defined by:
$V=\{(s,T_V(s),C_V(s),G_V(s),U_V(s),F_V(s)): s\in\Psi\}$, where $T_V(s)$, $C_V(s)$, $G_V(s)$, $U_V(s)$, $F_V(s) \in [0,1]$ are the degree of truth, contradiction, ignorance, unknown, and falsity membership of $s\in\Psi$. Therefore $0 \leq T_V(s)+C_V(s)+ G_V(s)+U_V(s)+F_V(s) \leq 5$.

Example 2.1. Let $\Psi = \{r, s\}$. Then $W=\{(r,0.8,0.3,0.4,0.8,0.9), (s,0.9,0.9,0.2,0.4,0.5)\}$ is a PNS over Ψ.

Definition 2.2.[15] Suppose that $V=\{(s, T_V(s), C_V(s), G_V(s), U_V(s), F_V(s)): s\in\Psi\}$ and $Y = \{(s, T_Y(s), C_Y(s), G_Y(s), U_Y(s), F_Y(s)): s\in\Psi\}$ be two PNSs over Ψ. Then

(i)$V\cup Y = \{(s, \max \{T_V(s), T_Y(s)\}, \max \{C_V(s), C_Y(s)\}, \min \{G_V(s), G_Y(s)\}, \min \{U_V(s), U_Y(s)\}, \min \{F_V(s), F_Y(s)\}): s\in\Psi\}$;

(ii)$V\cap Y = \{(s, \min \{T_V(s), T_Y(s)\}, \min \{C_V(s), C_Y(s)\}, \max \{G_V(s), G_Y(s)\}, \max \{U_V(s), U_Y(s)\}, \max \{F_V(s), F_Y(s)\}): s\in\Psi\}$;

(iii) $V^c = \{(s, F_V(s), U_V(s), 1\text{-}G_V(s), C_V(s), T_V(s)): s\in\Psi\}$;

(iv) $V\subseteq Y$ iff $T_V(s)\leq T_Y(s)$, $C_V(s)\leq C_Y(s)$, $G_V(s)\geq G_Y(s)$, $U_V(s)\geq U_Y(s)$, $F_V(s)\geq F_Y(s)$, for all $s\in\Psi$.

Example 2.2. Suppose that $\Psi=\{s, r\}$. Let $V=\{(s,0.3,0.8,0.8,0.5,0.2), (r,0.5,0.9,0.5,0.2,0.3)\}$ and $Y=\{(s,0.9,0.6,0.6,0.7,0.2), (r,0.8,0.6,0.3,0.1,0.9)\}$ be two PNSs over Ψ. Then

(i) $V \cup Y$={(s,0.9,0.8,0.6,0.5,0.2), (r,0.8,0.9,0.3,0.1,0.3)};

(ii) $V \cap Y$={(s,0.3,0.6,0.8,0.7,0.2), (r,0.5,0.6,0.5,0.2,0.9)};

(iii) V^c={(s,0.2,0.5,0.2,0.8,0.3), (r,0.3,0.2,0.5,0.9,0.5)}, Y^c={(s,0.2,0.7,0.4,0.6,0.9), (r,0.9,0.1,0.7,0.6,0.8)}

Example 2.3. Suppose that Ψ={s, r}. Let V={(s,0.3,0.3,0.8,0.7,0.2), (r,0.8,0.9,0.5,0.2,0.3)} and Y={(s,0.9,0.9,0.6,0.7,0.2), (r,0.8,1.0,0.3,0.1,0.1)} be two PNSs over Ψ. Then $V \subseteq Y$.

3. Single Valued Pentapartitioned Neutrosophic Set (SVPNS):

Definition 3.1. An *SVPNS* [15] Y over a fixed set Ψ are characterized by a truth-membership function (T_Y), a contradiction-membership function (C_Y), an ignorance-membership function (G_Y), an unknown-membership function (U_Y), a falsity-membership function (F_Y). Here $T_Y(s)$, $C_Y(s)$, $G_Y(s)$, $U_Y(s)$, $F_Y(s) \in [0,1]$, $\forall s \in \Psi$. The *SVPNS* Y is denoted as follows:

Y= {(s, $T_Y(s)$, $C_Y(s)$, $G_Y(s)$, $U_Y(s)$, $F_Y(s)$): $s \in \Psi$}.

Definition 3.2. Assume that B= {(s, $T_B(s)$, $C_B(s)$, $G_B(s)$, $U_B(s)$, $F_B(s)$): $s \in \Psi$} and D= {(s, $T_D(s)$, $C_D(s)$, $G_D(s)$, $U_D(s)$, $F_D(s)$): $s \in \Psi$} be two *SVPNSs* [15] over Ψ. Then,

(i) $B \subseteq D$ iff $T_B(s) \le T_D(s)$, $C_B(s) \le C_D(s)$, $G_B(s) \ge G_D(s)$, $U_B(s) \ge U_D(s)$, $F_B(s) \ge F_D(s)$, $\forall s \in \Psi$;

(ii) B=D iff $D \subseteq B$ and $B \subseteq D$.

Definition 3.3. Assume that B = {(s, $T_B(s)$, $C_B(s)$, $G_B(s)$, $U_B(s)$, $F_B(s)$): $s \in \Psi$} and D= {(s, $T_D(s)$, $C_D(s)$, $G_D(s)$, $U_D(s)$, $F_D(s)$): $s \in \Psi$} be two SVPNSs over Ψ. Let the cardinality of Ψ be n. The Hamming distance (H_d) between B and D is defined by

$H_d(B, D)$=

$$\sum_{s\in\Psi}(|T_B(s) - T_D(s)| + |C_B(s) - C_D(s)| + |G_B(s) - G_D(s)| + |U_B(s) - U_D(s)| + |F_B(s) - F_D(s)|) \quad (1)$$

where $0 \le H_d(B, D) \le 5n$.

Example 3.1. Suppose that V= {(s, 0.3, 0.3, 0.8, 0.7, 0.2), (r, 0.8, 0.9, 0.5, 0.2, 0.3)} and Y={(s, 0.9, 0.9, 0.6, 0.7, 0.2), (r, 0.8, 1.0, 0.3, 0.1, 0.1)} be two SVPN sets over Ψ={s, r}. Then, the Hamming distance between V and Y is $H_d(V, Y)$ = 2.

Theorem 3.1. The Hamming distance between two SVPNSs is bounded.

Proof. Suppose that B = {(s, $T_B(s)$, $C_B(s)$, $G_B(s)$, $U_B(s)$, $F_B(s)$): $s \in \Psi$} and D= {(s, $T_D(s)$, $C_D(s)$, $G_D(s)$, $U_D(s)$, $F_D(s)$): $s \in \Psi$} be two SVPNSs over Ψ, where cardinality of Ψ is n. Therefore, $0 \le T_B(s) \le 1$, $0 \le C_B(s) \le 1$, $0 \le G_B(s) \le 1$, $0 \le U_B(s) \le 1$, $0 \le F_B(s) \le 1$, $0 \le T_D(s) \le 1$, $0 \le C_D(s) \le 1$, $0 \le G_D(s) \le 1$, $0 \le U_D(s) \le 1$, and $0 \le F_D(s) \le 1$, for each $s \in \Psi$. This implies $0 \le |T_B(s) - T_D(s)| \le 1$, $0 \le |C_B(s) - C_D(s)| \le 1$, $0 \le |G_B(s) - G_D(s)| \le 1$, $0 \le |U_B(s) - U_D(s)| \le 1$, $0 \le |F_B(s) - F_D(s)| \le 1$, for each $s \in \Psi$.

Therefore we have,

$$0 \le |T_B(s) - T_D(s)| + |C_B(s) - C_D(s)| + |G_B(s) - G_D(s)| + |U_B(s) - U_D(s)| + |F_B(s) - F_D(s)| \le 5$$

$$\Rightarrow 0 \le \sum_{s\in\Psi} (|T_B(s) - T_D(s)| + |C_B(s) - C_D(s)| + |G_B(s) - G_D(s)| + |U_B(s) - U_D(s)| + |F_B(s) - F_D(s)|) \le 5n$$

$\Rightarrow 0 \le H_d(B, D) \le 5n$

$\Rightarrow H_d(B, D) \in [0, 5n]$.

Therefore, the Hamming distance between two SVPNSs is bounded.

Theorem 3.2. Suppose that D = {(s, $T_D(s)$, $C_D(s)$, $G_D(s)$, $U_D(s)$, $F_D(s)$): $s\in\Psi$}, B = {(s, $T_B(s)$, $C_B(s)$, $G_B(s)$, $U_B(s)$, $F_B(s)$): $s\in\Psi$} and A= {(s, $T_A(s)$, $C_A(s)$, $G_A(s)$, $U_A(s)$, $F_A(s)$): $s\in\Psi$} be three SVPNSs over Ψ, where cardinality of Ψ is n. If $D \subseteq B \subseteq A$, then

(i) $H_d(D, B) \leq H_d(D, A)$;

(ii) $H_d(B, A) \leq H_d(D, A)$.

Proof. Let D = {(s, $T_D(s)$, $C_D(s)$, $G_D(s)$, $U_D(s)$, $F_D(s)$): $s\in\Psi$}, B = {(s, $T_B(s)$, $C_B(s)$, $G_B(s)$, $U_B(s)$, $F_B(s)$): $s\in\Psi$} and A= {(s, $T_A(s)$, $C_A(s)$, $G_A(s)$, $U_A(s)$, $F_A(s)$): $s\in\Psi$} be three SVPNSs over Ψ, where cardinality of Ψ is n.

(i) Suppose that $D \subseteq B \subseteq A$. So $|T_D(s) - T_B(s)| \leq |T_D(s) - T_A(s)|$, $|C_D(s) - C_B(s)| \leq |C_D(s) - C_A(s)|$, $|G_D(s) - G(s)| \leq |G_D(s) - G_A(s)|$, $|U_D(s) - U_B(s)| \leq |U_D(s) - U_A(s)|$, $|F_D(s) - F_B(s)| \leq |F_D(s) - F_A(s)|$, for each $s\in\Psi$.

Therefore,

$$\sum_{s\in\Psi}(|T_D(s) - T_B(s)| + |C_D(s) - C_B(s)| + |G_D(s) - G_B(s)| + |U_D(s) - U_B(s)| + |F_D(s) - F_B(s)|)$$
$$\leq \sum_{s\in\Psi}(|T_D(s) - T_A(s)| + |C_D(s) - C_A(s)| + |G_D(s) - G_A(s)| + |U_D(s) - U_A(s)| + |F_D(s) - F_A(s)|)$$

Now, we have

$H_d(D, B)$

$$= \sum_{s\in\Psi}(|T_D(s) - T_B(s)| + |C_D(s) - C_B(s)| + |G_D(s) - G_A(s)| + |U_D(s) - U_A(s)| + |F_D(s) - F_A(s)|)$$
$$\leq \sum_{s\in\Psi}(|T_D(s) - T_A(s)| + |C_D(s) - C_A(s)| + |G_D(s) - G_A(s)| + |U_D(s) - U_A(s)| + |F_D(s) - F_A(s)|)$$
$$= H_d(D, A).$$

Hence, $H_d(D, B) \leq H_d(D, A)$.

(ii) Assume that $D \subseteq B \subseteq A$. So $|T_B(s) - T_A(s)| \leq |T_D(s) - T_A(s)|$, $|C_B(s) - C_A(s)| \leq |C_D(s) - C_A(s)|$, $|G_B(s) - G_A(s)| \leq |G_D(s) - G_A(s)|$, $|U_B(s) - U_A(s)| \leq |U_D(s) - U_A(s)|$, $|F_B(s) - F_A(s)| \leq |F_D(s) - F_A(s)|$, for each $s\in\Psi$.

Therefore,

$$\sum_{s\in\Psi}(|T_B(s) - T_A(s)| + |C_B(s) - C_A(s)| + |G_B(s) - G_A(s)| + |U_B(s) - U_A(s)| + |F_B(s) - F_A(s)|)$$
$$\leq \sum_{s\in\Psi}(|T_D(s) - T_A(s)| + |C_D(s) - C_A(s)| + |G_D(s) - G_A(s)| + |U_D(s) - U_A(s)| + |F_D(s) - F_A(s)|)$$

Now, we have

$H_d(B, A)$

$$= \sum_{s\in\Psi}(|T_B(s) - T_A(s)| + |C_B(s) - C_A(s)| + |G_B(s) - G_A(s)| + |U_B(s) - U_A(s)| + |F_B(s) - F_A(s)|)$$
$$\leq \sum_{s\in\Psi}(|T_D(s) - T_A(s)| + |C_D(s) - C_A(s)| + |G_D(s) - G_A(s)| + |U_D(s) - U_A(s)| + |F_D(s) - F_A(s)|)$$
$$= H_d(D, A).$$

Hence, $H_d(B, A) \leq H_d(D, A)$.

Definition 3.4. Assume that B= {(s, $T_B(s)$, $C_B(s)$, $G_B(s)$, $U_B(s)$, $F_B(s)$): $s\in\Psi$} and D= {(s, $T_D(s)$, $C_D(s)$, $G_D(s)$, $U_D(s)$, $F_D(s)$): $s\in\Psi$} be two SVPN sets over Ψ. Let the cardinality of Ψ be n. The normalized Hamming distance (N-H_d) between B and D is defined by

$N\text{-}H_d(B, D)$=

$$\frac{1}{5n}\sum_{s\in\Psi}(|T_B(s) - T_D(s)| + |C_B(s) - C_D(s)| + |G_B(s) - G_D(s)| + |U_B(s) - U_D(s)| + |F_B(s) - F_D(s)|) \quad (2)$$

where $0 \leq N\text{-}H_d(B, D) \leq 1$.

Example 3.2. Suppose that V and Y are two SVPNSs over Ψ={s, r} as shown in Example 3.1. Then $N\text{-}H_d(V, Y) = 0.2$.

Theorem 3.3. The Normalized Hamming distance between two SVPNSs is bounded.

Proof. Suppose that B = {(s, $T_B(s)$, $C_B(s)$, $G_B(s)$, $U_B(s)$, $F_B(s)$): $s\in\Psi$} and D= {(s, $T_D(s)$, $C_D(s)$, $G_D(s)$, $U_D(s)$, $F_D(s)$): $s\in\Psi$} be two SVPNSs over Ψ, where cardinality of Ψ is n. Therefore, $0\le T_B(s)\le 1$, $0\le C_B(s)\le 1$, $0 \le G_B(s) \le 1$, $0 \le U_B(s) \le 1$, $0 \le F_B(s) \le 1$, $0 \le T_D(s) \le 1$, $0 \le C_D(s) \le 1$, $0 \le G_D(s) \le 1$, $0 \le U_D(s) \le 1$, and $0 \le F_D(s) \le 1$. This implies $0 \le |T_B(s) - T_D(s)| \le 1$, $0 \le |C_B(s) - C_D(s)| \le 1$, $0 \le |G_B(s) - G_D(s)| \le 1$, $0 \le |U_B(s) - U_D(s)| \le 1$, $0 \le |F_B(s) - F_D(s)| \le 1$.
Therefore we have,

$0 \le |T_B(s) - T_D(s)| + |C_B(s) - C_D(s)| + |G_B(s) - G_D(s)| + |U_B(s) - U_D(s)| + |F_B(s) - F_D(s)| \le 5$

$\Rightarrow 0 \le \sum_{s\in\Psi}(|T_B(s) - T_D(s)| + |C_B(s) - C_D(s)| + |G_B(s) - G_D(s)| + |U_B(s) - U_D(s)| + |F_B(s) - F_D(s)|) \le 5n$

$$\Rightarrow 0 \le \frac{1}{5n}\sum_{s\in\Psi}\begin{pmatrix}|T_B(s) - T_D(s)| + |C_B(s) - C_D(s)| + |G_B(s) - G_D(s)| + \\ |U_B(s) - U_D(s)| + |F_B(s) - F_D(s)|\end{pmatrix} \le 1$$

$\Rightarrow 0 \le N\text{-}H_d(B, D) \le 1$

$\Rightarrow N\text{-}H_d(B, D) \in [0, 1]$.

Theorem 3.4. Suppose that D = {(s, $T_D(s)$, $C_D(s)$, $G_D(s)$, $U_D(s)$, $F_D(s)$): $s\in\Psi$}, B = {(s, $T_B(s)$, $C_B(s)$, $G_B(s)$, $U_B(s)$, $F_B(s)$): $s\in\Psi$} and A= {(s, $T_A(s)$, $C_A(s)$, $G_A(s)$, $U_A(s)$, $F_A(s)$): $s\in\Psi$} be three SVPNSs over Ψ, where cardinality of Ψ is n. If $B \subseteq D \subseteq A$, then

(i) $N\text{-}H_d(B, D) \le N\text{-}H_d(B, A)$;

(ii) $N\text{-}H_d(D, A) \le N\text{-}H_d(B, A)$.

Proof. Let D = {(s, $T_D(s)$, $C_D(s)$, $G_D(s)$, $U_D(s)$, $F_D(s)$): $s\in\Psi$}, B = {(s, $T_B(s)$, $C_B(s)$, $G_B(s)$, $U_B(s)$, $F_B(s)$): $s\in\Psi$} and A= {(s, $T_A(s)$, $C_A(s)$, $G_A(s)$, $U_A(s)$, $F_A(s)$): $s\in\Psi$} be three SVPNSs over Ψ, where cardinality of Ψ is n.

(i) Suppose that $B \subseteq D \subseteq A$. So $|T_B(s) - T_D(s)| \le |T_B(s) - T_A(s)|$, $|C_B(s) - C_D(s)| \le |C_B(s) - C_A(s)|$, $|G_B(s) - G_D(s)| \le |G_B(s) - G_A(s)|$, $|U_B(s) - U_D(s)| \le |U_B(s) - U_A(s)|$, $|F_B(s) - F_D(s)| \le |F_B(s) - F_A(s)|$, for each $s\in\Psi$.

Therefore,

$$\frac{1}{5n}\sum_{s\in\Psi}(|T_B(s) - T_D(s)| + |C_B(s) - C_D(s)| + |G_B(s) - G_D(s)| + |U_B(s) - U_D(s)| + |F_B(s) - F_D(s)|)$$

$$\le \frac{1}{5n}\sum_{s\in\Psi}(|T_B(s) - T_A(s)| + |C_B(s) - C_A(s)| + |G_B(s) - G_A(s)| + |U_B(s) - U_A(s)| + |F_B(s) - F_A(s)|)$$

Now, we have

$N\text{-}H_d(B, D)$

$$= \frac{1}{5n}\sum_{s\in\Psi}(|T_B(s) - T_D(s)| + |C_B(s) - C_D(s)| + |G_B(s) - G_D(s)| + |U_B(s) - U_D(s)| + |F_B(s) - F_D(s)|)$$

$$\le \frac{1}{5n}\sum_{s\in\Psi}(|T_B(s) - T_A(s)| + |C_B(s) - C_A(s)| + |G_B(s) - G_A(s)| + |U_B(s) - U_A(s)| + |F_B(s) - F_A(s)|)$$

$= N\text{-}H_d(B, A)$.

Hence, $N\text{-}H_d(B, D) \le N\text{-}H_d(B, A)$.

(ii) Assume that $B \subseteq D \subseteq A$. So $|T_D(s) - T_A(s)| \le |T_B(s) - T_A(s)|$, $|C_D(s) - C_A(s)| \le |C_B(s) - C_A(s)|$, $|G_D(s) - G_A(s)| \le |G_B(s) - G_A(s)|$, $|U_D(s) - U_A(s)| \le |U_B(s) - U_A(s)|$, $|F_D(s) - F_A(s)| \le |F_B(s) - F_A(s)|$, for each $s\in\Psi$.

Therefore,

$$\frac{1}{5n}\sum_{s\in\Psi}(|T_D(s) - T_A(s)| + |C_D(s) - C_A(s)| + |G_D(s) - G_A(s)| + |U_D(s) - U_A(s)| + |F_D(s) - F_A(s)|)$$

$\leq \frac{1}{5n}\sum_{s\in\Psi}(|T_B(s) - T_A(s)| + |C_B(s) - C_A(s)| + |G_B(s) - G_A(s)| + |U_B(s) - U_A(s)| + |F_B(s) - F_A(s)|)$

Now, we have

$N\text{-}H_d(D, A)$

$= \frac{1}{5n}\sum_{s\in\Psi}(|T_D(s) - T_A(s)| + |C_D(s) - C_A(s)| + |G_D(s) - G_A(s)| + |U_D(s) - U_A(s)| + |F_D(s) - F_A(s)|)$

$\leq \frac{1}{5n}\sum_{s\in\Psi}(|T_B(s) - T_A(s)| + |C_B(s) - C_A(s)| + |G_B(s) - G_A(s)| + |U_B(s) - U_A(s)| + |F_B(s) - F_A(s)|)$

$= N\text{-}H_d(B, A)$.

Hence, $N\text{-}H_d(D, A) \leq N\text{-}H_d(B, A)$.

4. SVPNS-MADM strategy based on GRA:

Choosing an alternative from a set of possible alternatives based on some attributes is a challenging task for a Decision Maker (DM). For that the DM should have to plan an MADM strategy to take the decision. Assume that $L= \{L_1, L_2, ..., L_p\}$ is the collection of some possible alternatives and $S=\{S_1, S_2, ..., S_q\}$ is the family of attributes. The DM provides their evaluation information for every alternative L_i (i=1, 2,..., p) based on the attribute S_j (j=1, 2, ..., q) in terms of Single Valued Pentapartitioned Neutrosophic Numbers (SVPNNs). So the whole evaluation information of all alternatives can be expressed by a decision matrix.

The steps of the proposed SVPNS-MADM strategy are presented as follows:

Step-1: Construct the decision matrix using SVPNS

The whole evaluation assessment of every alternative L_i (i = 1, 2,..., p) over the attributes S_j (j = 1, 2, ..., q) is presented in terms of SVPNNs E_{L_i}= $\{(S_j,T_{ij}(L_i, S_j),C_{ij}(L_i, S_j),G_{ij}(L_i, S_j),U_{ij}(L_i, S_j),F_{ij}(L_i, S_j)): S_j\in S\}$, where $(T_{ij}(L_i, S_j),C_{ij}(L_i, S_j),G_{ij}(L_i, S_j),U_{ij}(L_i, S_j),F_{ij}(L_i, S_j)) = (T_{ij},C_{ij},G_{ij},U_{ij},F_{ij})$ (in short) is the evaluation assessment of alternative L_i (i = 1, 2,..., n) over the attribute S_j (j = 1, 2, ..., m).

Then the decision matrix (D) is given by:

D	S_1	S_2	…	…	S_m
L_1	$< T_{11}(L_1, S_1),C_{11}(L_1, S_1),$ $G_{11}(L_1, S_1),U_{11}(L_1, S_1),$ $F_{11}(L_1, S_1) >$	$< T_{12}(L_1, S_2),C_{12}(L_1, S_2),$ $G_{12}(L_1, S_2),U_{12}(L_1, S_2),$ $F_{12}(L_1, S_2) >$	....	…	$< T_{1m}(L_1, S_m),C_{1m}(L_1, S_m),$ $G_{1m}(L_1, S_m),U_{1m}(L_1, S_m),$ $F_{1m}(L_1, S_m) >$
L_2	$< T_{21}(L_2, S_1),C_{21}(L_2, S_1),$ $G_{21}(L_2, S_1),U_{21}(L_2, S_1),$ $F_{21}(L_2, S_1) >$	$< T_{22}(L_2, S_2),C_{22}(L_2, S_2),$ $G_{22}(L_2, S_2),U_{22}(L_2, S_2),$ $F_{22}(L_2, S_2) >$	… … …	…	$< T_{2m}(L_2, S_m),C_{2m}(L_2, S_m),$ $G_{2m}(L_2, S_m),U_{2m}(L_2, S_m),$ $F_{2m}(L_2, S_m) >$

. . .	. . .	. . .	. . .	. . .	. . .
L_n	$< T_{n1}(L_n, S_1), C_{n1}(L_n, S_1),$ $G_{n1}(L_n, S_1), U_{n1}(L_n, S_1),$ $F_{n1}(L_n, S_1) >$	$< T_{n2}(L_n, S_2), C_{n2}(L_n, S_2),$ $G_{n2}(L_n, S_2), U_{n2}(L_n, S_2),$ $F_{n2}(L_n, S_2) >$	… … … …	… … … …	$< T_{nm}(L_n, S_m), C_{nm}(L_n, S_m),$ $G_{nm}(L_n, S_m), U_{nm}(L_n, S_m),$ $F_{nm}(L_n, S_m) >$

Step-2: Determine the weights for the attributes.

In every MADM strategy, the weights of the attributes play an important role in making decision. If the weights of the information of all attributes are completely unknown to the decision makers, then by using the following compromise function, the decision maker can find the weights of the attributes.

Compromise Function: The compromise function of L is defined as follows:

$$\xi_j = \sum_{i=1}^{n} (3 + T_{ij}(L_i, S_j) + C_{ij}(L_i, S_j) - G_{ij}(L_i, S_j) - U_{ij}(L_i, S_j) - F_{ij}(L_i, S_j))/5 \quad (3)$$

Then the weights of the jth attribute is defined by $w_j = \frac{\xi_j}{\sum_{j=1}^{m} \xi_j}$ (4)

Here $\sum_{j=1}^{m} w_j = 1$.

Step-3: Construct the Ideal Pentapartitioned Neutrosophic Estimates Reliability Solution (IPNERS) and Ideal Pentapartitioned Neutrosophic Estimates Un-Reliability Solution (IPNEURS) for the decision matrix:

The IPNERS for the decision matrix is presented as:

$$R^{+} = [<T_1^{+}, C_1^{+}, G_1^{+}, U_1^{+}, F_1^{+}>, <T_2^{+}, C_2^{+}, G_2^{+}, U_2^{+}, F_2^{+}>, \ldots\ldots\ldots, <T_m^{+}, C_m^{+}, G_m^{+}, U_m^{+}, F_m^{+}>], \quad (5)$$

where T_j^{+}= max $\{T_{ij}(L_i, S_j)$: i=1, 2, 3, ….., $n\}$, C_j^{+}= max $\{C_{ij}(L_i, S_j)$: i=1, 2, 3, ….., $n\}$, G_j^{+}= min $\{G_{ij}(L_i, S_j)$: i=1, 2, 3, ….., $n\}$, U_j^{+}= min $\{U_{ij}(L_i, S_j)$: i=1, 2, 3, ….., $n\}$, and U_j^{+}= min $\{U_{ij}(L_i, S_j)$: i=1, 2, 3, ….., $n\}$.

The IPNEURS for the decision matrix is presented as:

$$R^{-} = [<T_1^{-}, C_1^{-}, G_1^{-}, U_1^{-}, F_1^{-}>, <T_2^{-}, C_2^{-}, G_2^{-}, U_2^{-}, F_2^{-}>, \ldots\ldots\ldots, <T_m^{-}, C_m^{-}, G_m^{-}, U_m^{-}, F_m^{-}>], \quad (6)$$

where T_j^{-}= min $\{T_{ij}(L_i, S_j)$: i=1, 2, 3, ….., $n\}$, C_j^{+}= min $\{C_{ij}(L_i, S_j)$: i=1, 2, 3, ….., $n\}$, G_j^{+}= max $\{G_{ij}(L_i, S_j)$: i=1, 2, 3, ….., $n\}$, U_j^{+}= max $\{U_{ij}(L_i, S_j)$: i=1, 2, 3, ….., $n\}$, and U_j^{+}= max $\{U_{ij}(L_i, S_j)$: i=1, 2, 3, ….., $n\}$.

Step-4: Determination of Pentapartitioned Neutrosophic Grey Relational Coefficient (PNGRC) of each alternative from IPNERS & IPNEURS.

The PNGRC of each alternative from IPNERS is presented as:

$G_{ij}^{+} = \frac{\min_i \min_j \Delta_{ij}^{+} + k.\max_i \max_j \Delta_{ij}^{+}}{\Delta_{ij}^{+} + k.\max_i \max_j \Delta_{ij}^{+}}$, where $\Delta_{ij}^{+} = H_d(< T_j^{+}, C_j^{+}, G_j^{+}, U_j^{+}, F_j^{+} >, < T_{ij}, C_{ij}, G_{ij}, U_{ij}, F_{ij} >)$, i=1,2,…,n and j=1,2,…,m, and $k \in [0,1]$.

The PNGRC of each alternative from IPNEURS is given below:

$G_{ij}^{-} = \frac{\min_{i}\min_{j}\Delta_{ij}^{-} + k.\max_{i}\max_{j}\Delta_{ij}^{-}}{\Delta_{ij}^{-} + k.\max_{i}\max_{j}\Delta_{ij}^{-}}$, where $\Delta_{ij}^{-}= H_d\,(< T_{ij}, C_{ij}, G_{ij}, U_{ij}, F_{ij} >, < T_j^{-}, C_j^{-}, G_j^{-}, U_j^{-}, F_j^{-} >)$, i=1, 2, …, n, and j=1, 2, …, m, and $k \in[0,1]$.

Here G_{ij}^{+} and G_{ij}^{-} are the identification coefficient used to adjust the range of the comparison environment, and to control level of differences of the relation coefficients. The comparison environment remains unchanged when k =1 and the comparison environment disappears when k = 0. If the identification coefficient is smaller, then the range of grey relational coefficient will become so large. Generally, k = 0.5 is considered for decision making situation.

Step-5: Determine the PNGRC

The PNGRC of each alternative from IPNERS and IPNEURS are defined as follows:

$$G_i^{+}= \sum_{j=1}^{q} w_j G_{ij}^{+} \quad (7)$$

where i = 1, 2, …., n,

and $$G_i^{-}= \sum_{j=1}^{n} w_j G_{ij}^{-} \quad (8)$$

where i =1, 2, …, m.

Step-6: Determine the pentapartitioned neutrosophic relative relational degree.

The pentapartitioned neutrosophic relative relational degree of each alternatives is can be defined as follows:

$$\Re_i = \frac{G_i^{+}}{G_i^{+}+G_i^{-}} \quad (9)$$

where i= 1, 2, …, n.

Step-7: Rank the alternatives.

The ranking order of all alternatives can be determined according to the ascending order of the pentapartitioned relative relational degree. The alternative with highest value of $\Re_i$ indicates the best alternative.

Step-8: End.

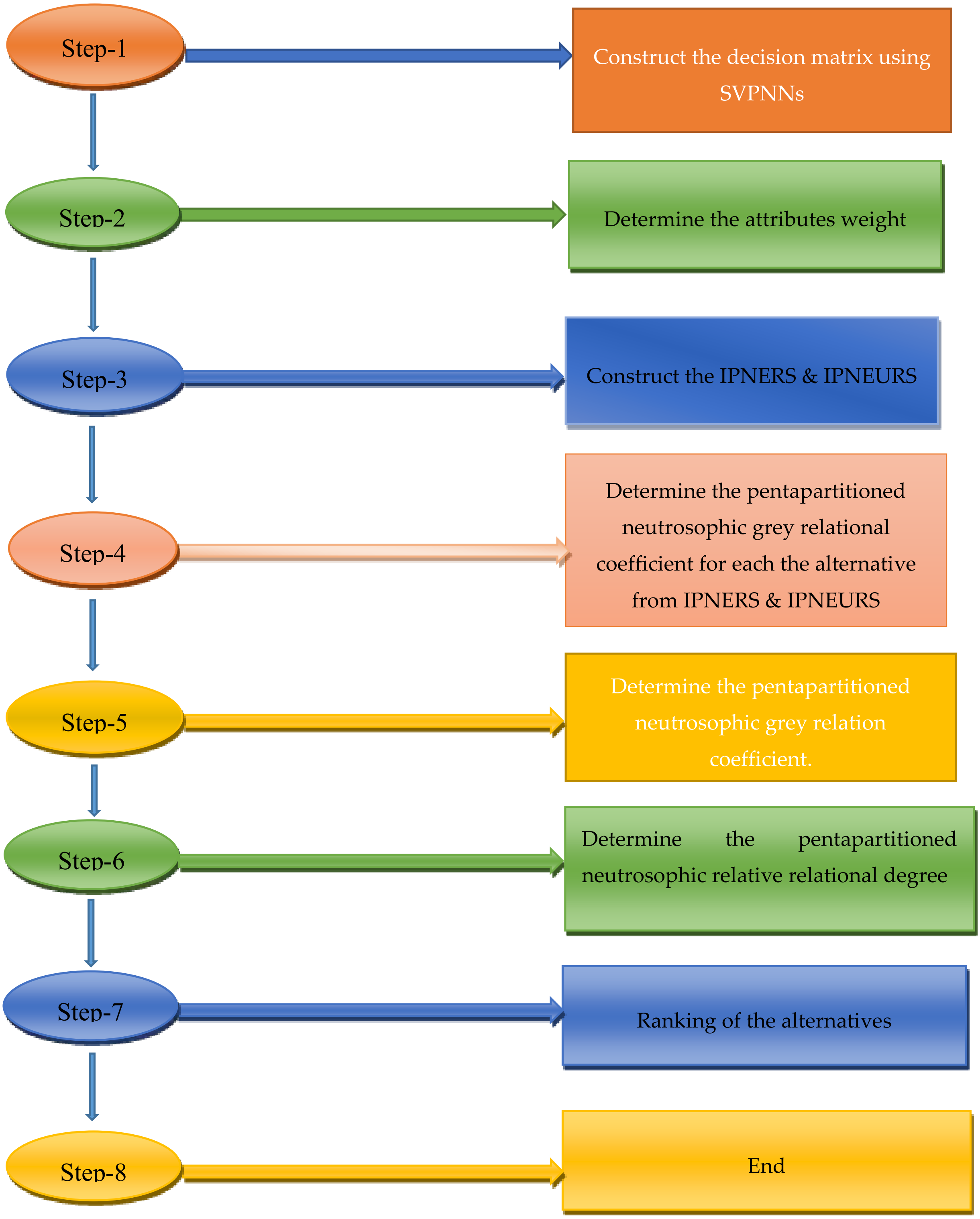

Figure 1: Flow chart of the proposed SVPNS-MADM strategy.

5. Validation of Proposed Model:

In this section, we present a numerical example namely "Selection of supplier to buy electronic goods for an institution" to validate the proposed strategy.

5.1. Selection of supplier to buy electronic goods for an institution:

In every government/private institutions, lots of electronic goods namely Computer, Printer, Scanner, Projector, AC, etc. are required for the purpose of official uses. To buy a particular or all electronic goods, the institutions must select a suitable private company for giving the tender against some attributes. So, the selection of best private company by the institution for purchasing the necessary electronic goods can be considered as an MADM problem. For the selection of suitable private company, the decision maker selects four major attributes namely S_1: Cost of the products; S_2: Quality of the products; S_3: Service of the Company; S_4: Reliability.

Then, the developed MADM strategy is presented using the following steps.

Step-1: Determine the decision matrix in single valued pentapartitioned neutrosophic environment. The decision maker provides the evaluation information for all the alternatives over the attributes as shown in Table-1

Table-1:

	S_1	S_2	S_3	S_4
L_1	(0.9,0.3,0.1,0.5,0.2)	(0.8,0.2,0.2,0.1,0.4)	(0.9,0.1,0.3,0.1,0.3)	(0.9,0.1,0.2,0.3,0.4)
L_2	(0.8,0.1,0.3,0.3,0.2)	(0.9,0.2,0.3,0.4,0.2)	(0.6,0.1,0.2,0.3,0.2)	(0.9,0.2,0.1,0.2,0.2)
L_3	(0.9,0.4,0.2,0.3,0.1)	(0.8,0.3,0.4,0.1,0.1)	(0.5,0.1,0.1,0.2,0.1)	(0.8,0.3,0.1,0.3,0.1)

Step-2: Determine the weights of attributes

By using the eq. (3) and (4), we get the weight vector as follows:

(w_1, w_2, w_3, w_4) = (0.261728, 0.249383, 0.234568, 0.254321).

Step-3: Determine the IPNERS & IPNEURS

The IPNERS (R^+) and IPNEURS (R^-) for the decision matrix are presented in the Table-3.

Table-3:

	S_1	S_2	S_3	S_4
L_1	(0.9,0.3,0.1,0.5,0.2)	(0.8,0.2,0.2,0.1,0.4)	(0.9,0.1,0.3,0.1,0.3)	(0.9,0.1,0.2,0.3,0.4)
L_2	(0.8,0.1,0.3,0.3,0.2)	(0.9,0.2,0.3,0.4,0.2)	(0.6,0.1,0.2,0.3,0.2)	(0.9,0.2,0.1,0.2,0.2)
L_3	(0.9,0.4,0.2,0.3,0.1)	(0.8,0.3,0.4,0.1,0.1)	(0.5,0.1,0.1,0.2,0.1)	(0.8,0.3,0.1,0.3,0.1)
R^+	(0.9,0.4,0.1,0.3,0.1)	(0.9,0.3,0.2,0.1,0.1)	(0.9,0.1,0.1,0.1,0.1)	(0.9,0.3,0.1,0.2,0.1)

R^-	(0.8,0.1,0.3,0.5,0.2)	(0.8,0.2,0.4,0.4,0.4)	(0.5,0.1,0.3,0.3,0.3)	(0.8,0.1,0.2,0.3,0.4)

Step-4: Determine the PNGRCfor each of the alternative from IPNERS & IPNEURS.

The PNGRC of each alternative from IPNERS and IPNEURS is presented in the Table-4 ,Table-5, Table-6, and Table-7, respectively.

Table-4:

	S_1	S_2	S_3	S_4	$min_i\ \Delta_{ij}^+$	$max_i\ \Delta_{ij}^+$
Δ_{1j}^+	0.4	0.5	0.4	0.7	0.4	0.7
Δ_{2j}^+	0.7	0.6	0.7	0.2	0.2	0.7
Δ_{3j}^+	0.1	0.3	0.5	0.2	0.1	0.5
$min_i\ min_j\ \Delta_{ij}^+$					0.1	
$max_i\ max_j\ \Delta_{ij}^+$						0.7

Table-5:

	S_1	S_2	S_3	S_4	$min_i\ \Delta_{ij}^-$	$max_i\ \Delta_{ij}^-$
Δ_{1j}^-	0.5	0.5	0.6	0.1	0.1	0.6
Δ_{2j}^-	0.2	0.4	0.3	0.6	0.2	0.6
Δ_{3j}^-	0.8	0.7	0.5	0.6	0.5	0.8
$min_i\ min_j\ \Delta_{ij}^-$					0.2	
$max_i max_j\ \Delta_{ij}^-$						0.8

Table-6:

G_{ij}^+	S_1	S_2	S_3	S_4
L_1	0.6	0.5294	0.6	0.4286
L_2	0.4286	0.4737	0.4286	0.8182
L_3	1	0.6923	0.5294	0.8182

Table-7:

G_{ij}^-	S_1	S_2	S_3	S_4
L_1	0.5556	0.5556	0.5	1
L_2	0.8333	0.625	0.7143	0.5
L_3	0.4167	0.4546	0.5556	0.5

Step-5: Determine the PNGRC.

The PNGRCs G_i^+ and G_i^- of each alternative (S_i, i = 1, 2, 3, 4) from IPNERS and IPNEURS are presented in Table-8.

Table-8:

	G_i^+	G_i^-

L_1	0.538803	0.655578
L_2	0.538931	0.668675
L_3	0.766642	0.479918

Step-6: Determine the pentapartitioned neutrosophic relative relational degree.

The pentapartitioned neutrosophic relative relational degree ($\Re_i$) of each alternative (A_i, i = 1, 2, 3, 4) is presented in the following Table 9.

Table-9:

	$\Re_i = \frac{G_i^+}{G_i^+ + G_i^-}$
L_1	0.4511148
L_2	0.4462805
L_3	0.6150061

Step-7: Rank the alternatives.

From Table-9, it is clear that $\Re_2 < \Re_1 < \Re_3$. Therefore, L_3 is the best suitable alternative to choose.

5. Conclusions

In the study, we have proposed Hamming distance and proves its basic properties for PNSs. We have further developed a GRA based SVPNS-MADM strategy in PNS environment. We also validate the proposed SVPNS-MADM strategy by solving an illustrative decision-making problem.

The proposed SVPNS-MADM strategy can also be used to deal with the other decision-making problems such as brick selection [19], stock trending analysis [14], logistic center location selection [29], teacher selection [34], etc.

We further hope that the proposed MADM strategy will open up a new avenue of research in pentapartitioned neutrosophic set environments.

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NN-TOPSIS strategy for MADM in neutrosophic number setting

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Abstract: Selecting an appropriate alternative among the feasible selection options is a difficult activity for decision makers. Because of the imprecise information and the complexity of decision problem, it is not easy to evaluate the attributes in terms of crisp numbers. To deal with the problem, neutrosophic numbers can be used during the decision process. Neutrosophic numbers can easily describe cognitive information. In this paper, we use neutrosophic numbers to state evaluation information. We define unit neutrosophic number as an effective tool to express cognitive information. We propose novel NN-TOPSIS strategy in neutrosophic number environment. Moreover, we define Euclidean distance in neutrosophic numbers environment, and propose a tangent function to determine unknown attribute weights. We propose linguistic variables in neutrosophic number environment. We present a numerical example dealing with a decision-making problem based on the proposed NN-TOPSIS strategy We present a sensitivity analysis for reflecting the influence of indeterminacy values. We then conduct a comparison analysis between the proposed NN-TOPSIS and other existing decision- making strategies.

Keywords: Neutrosophic number; Unit neutrosophic number; TOPSIS, NN-TOPSIS; MADM

1. Introduction

Decision-making, in general, refers to a cognitive process that is continuously performed by a human being or a group of human beings. Generally, cognitive process is done consciously or unconsciously [1–3]. Some studies have been appeared in the literature dealing with the utilization of cognitive information for decision making process [4–8]. Zhang et al. [9] opined that specific numerical rating values provided by the decision makers do not always accurately present the behaviors and opinions of the decision makers, especially in the fields of decision-making [10–40] in

general, linear programming problems [41], cloud computing [42, 43], supplier chain management [44–46] image processing [47-48], medical diagnosis [49–52], etc.
Basset et al. [53] proposed an included neutrosophic and SWOT and AHP process for strategic setting up methodology choice. Basset et al. [54] presented a group MADM algorithm based on triangular neutrosophic sets. Chang et al. [55] developed a recycle strategic decision outline framework from theories to practical. Basset et al. [56] proposed a group decision structure based on VIKOR (neutrosophic field) method for e-governance website evaluation. Basset and Mohamed [57] proposed the role of rough sets and SVNS in smart city under defective and incomplete data. Mondal et al. [58] developed similarity measure (with tangent function) based model for interval neutrosophic sets. Pramanik et al. [59] developed NC-VIKOR technique for neutrosophic cubic sets. Mondal et al. [60] proposed hybrid similarity measure (based on logarithm function) under SVNSs assessments. Dalapati et al. [61] developed IN-cross entropy technique for INSs. Mondal et al. [62] introduced sine similarity measures based on hyperbolic function for MADM in SVNS According to Zeleny [63], human decision making involves multi-attributes. So, MADM problems are common in human life. The attribute values in the MADM problems often involve indeterminacy. Therefore, it is difficult to describe the attribute values by the crisp numbers.

Smarandache [64, 65] introduced the neutrosophic number (NN) to deal with partial and indeterminate information. An NN comprises of two components namely, determinate component and indeterminate component. An NN is presented in the form: $N = r + sI$, where r stands for the determinate part and sI stands for the indeterminate part. If $N = sI$, we obtain the worst situation. If $N = r$, we obtain the best situation. Thus, it seems that NNs are capable to deal with the imprecise information in realistic decision-making situation.

Ye [66] initiated to study of linear programming for neutrosophic numbers. Ye et al. [67] developed a nonlinear programming in NN environment and provided general solution. Banerjee and Pramanik [68] proposed a linear goal programming with a single-objective in NN environment. Pramanik and Banerjee [69] developed a goal programming for multiple objective linear programming in neutrosophic number setting. Pramanik and Dey [70] developed a Bi-Level Linear Programming (BLP) model in NN environment. Maiti et al. [[71] extended BLP to Bi-Level Decentralized Programming (BLDP)in NN environment. Pramanik and Dey [72] extended BLP to Multi-Level Programming (MLP) in NN environment. Maiti et al. [73] extended MLP To Multi-Level Multi-Objective Linear Programming (MLMOLP) in NN environment.

Ye [74] studied possibility degree based ranking method and proposed an MADM technique in NN environment. Ye [75] proposed an MAGDM approach based on bidirectional measure in NN environment. Kong et al. [76] defined a cosine similarity for NNs to solve the misfire error finding gasoline engines. Ye [77] defined exponential similarity measure for NNs and used it to the fault analysis of vapor turbine. Ye et al. [78] proposed a joint roughness coefficient using NN functions. Liu and Liu [79] defined weighted power averaging operator and proposed an MAGDM technique in NN environment. Zheng et al. [80] developed an MAGDM policy based on NN fusion weighted averaging operator. Employing aggregation operators of NN-Harmonic mean, Mondal et al. [81] proposed an MAGDM technique in NN environment. Pramanik et al. [82] presented a teacher selection strategy in NN environment. Shi and Ye [83] presented a linguistic NN and presented an

MAGDM technique in NN environment based on cosine measures. Ye [84] defined the hesitant neutrosophic linguistic number and developed an MADM technique based on the probable value and similarity measure in an HNLN environment. The study for MAGDM in NN environment is its infancy. New research is necessary to handle the MAGDM problems in NN environment.

TOPSIS is a well-liked technique to deal with MAGDM. TOPSIS [85] selects the best option, which is the nearest to the solution (ideal) and the farthest from the solution (negative ideal). The TOPSIS technique is based on information of attributes from decision maker/makers. In SVNS setting, Biswas et al. [86] proposed TOPSIS strategy for MAGDM in Single Valued Neutrosophic Set (SVNS) environment. Ye [87] extended TOPSIS approach for MAGDM based on SVNS linguistic numbers. Pramanik et al. [88] presented a TOPSIS technique for neutrosophic cubic set environment. Mondal et al. [89] developed TOPSIS for MAGDM in rough neutrosophic setting. Pramanik et al. [90] presented a TOPSIS technique for MADM in single valued neutrosophic soft expert set. Biswas et al. [91] studied a TOPSIS approach for MADM with trapezoidal neutrosophic numbers. García-Cascales and Lamata [92] proposed an improved version of TOPSIS based on rank reversal technique.

TOPSIS is yet to come into view NN environment. For the research gap, we develop an MAGDM technique based on TOPSIS in NN environment namely, NN-TOPSIS method for solving MAGDM problems.

Contribution of the paper:

- We develop an NN-TOPSIS technique to solve MAGDM problem in NN environment.
- We define an UNN and establish its basic properties.
- We define NN weighted arithmetic aggregation operator (NNWANO) to aggregate NN decision matrices.
- We propose a tangent function to decide unknown weights of attributes in NN environment.
- We propose a linguistic variable to present NN.
- We present sensitivity analysis for different values of I to reflect the influence of I on ranking order of selection options.
- The proposed NN-TOPSIS is comprehensive because when $I = 0$, NN-TOPSIS reduces to classical TOPSIS.

The paper is structured as follows. Section 2 presents several basic ideas of NNs, operations on NNs, unit neutrosophic number (UNN), Euclidean distance between two NNs, tangent function for NNs, NN relative ideal solution (positive) and NN relative ideal solution (negative). Section 3 defines 'NN weighted arithmetic aggregation operator (NNWANO) to aggregate NN decision matrix and develops a novel NN-TOPSIS technique for solving MAGDM problem in NN environment. Section 4 provides an example based on proposed NN-TOPSIS technique. Section 5 conducts sensitivity study to show the impact of ranking for different indeterminacy values. Section 6 presents a comparison analysis with other existing strategies. Section 7 presents conclusion and future scope of research.

2. Preliminaries

In this section, the idea of NN, operations on NNs, unit neutrosophic numbers and Euclidean distance between two NNs, tangent function for NNs, NN relative ideal solution (positive) and NN relative ideal solution (negative) are outlined.

2.1. Neutrosophic numbers (NNs)

An NN [64, 65] is expressed as $z = p + qI$ for $p, q \in R$, where I denotes indeterminacy and R denotes the set of real numbers. An NN z is expressed as a interval number in the form: $z = [p + qI^L, p + qI^U]$ for $z \in Z$ and $I \in [I^L, I^U]$. The interval $I \in [I^L, I^U]$ is regarded as an indeterminate interval.

Here, Z = set of all neutrosophic numbers.

- If $qI = 0$, then $z = p$ *i.e.*, real number or crisp number.
- If $p = 0$, then $z = qI$ *i.e.*, indeterminate number
- If $I^L = I^U$, then z is a crisp number.

Assume that $z_1 = p_1 + q_1I$ and $z_2 = p_2 + q_2I$ for $z_1, z_2 \in Z$ and $I \in [I^L, I^U]$ are two NNs. Some basic operational laws [66] for z_1 and z_2 are presented as follows:

(1) $I^2 = I$

(2) $I.0 = 0$

(3) I/I = Undefined

(4) $z_1 + z_2 = p_1 + p_2 + (q_1 + q_2)I = [p_1 + p_2 + (q_1 + q_2)I^L, p_1 + p_2 + (q_1 + q_2)I^U]$

(5) $z_1 - z_2 = p_1 - p_2 + (q_1 - q_2)I = [p_1 - p_2 + (q_1 - q_2)I^L, p_1 - p_2 + (q_1 - q_2)I^U]$

(6) $z_1 \times z_2 = p_1p_2 + (p_1q_2 + p_2q_1)I + q_1q_2I^2 = p_1p_2 + (p_1q_2 + p_2q_1 + q_1q_2)I$

(7) $\dfrac{z_1}{z_2} = \dfrac{p_1 + q_1I}{p_2 + q_2I} = \dfrac{p_1}{p_2} + \dfrac{p_2q_1 - p_1q_2}{p_2(p_2 + q_2)}I;\ p_2 \neq 0, p_2 \neq -q_2$

(8) $z_1^2 = p_1^2 + (2p_1q_1 + q_1^2)I$

(9) $\lambda z_1 = \lambda p_1 + \lambda q_1 I$

2.2. Unit neutrosophic numbers (UNNs)

In this subsection, we define unit neutrosophic number (UNN).

Definition 1 Let $A=\{\langle a_1+b_1I\rangle,\langle a_2+b_2I\rangle,\cdots,\langle a_n+b_nI\rangle\}$(i = 1, 2, . . . , n) be a set of neutrosophic numbers. Then $A^{\circ}=\left\{\left\langle\frac{a_1+b_1I}{2\sqrt{a_1^2+b_1^2}}\right\rangle,\left\langle\frac{a_2+b_2I}{2\sqrt{a_2^2+b_2^2}}\right\rangle,\cdots,\left\langle\frac{a_n+b_nI}{2\sqrt{a_n^2+b_n^2}}\right\rangle\right\}$, (i = 1, 2, . . . , n) is the set of UNNs.

Example 1 Let A = {3+4I, 4–3I} be a set of NNs. Then, we obtain $A^{\circ}=\left\{\frac{3+4I}{10},\frac{4-3I}{10}\right\}$.

Theorem 1 Each element of the set of UNNs lies in the interval [−1, 1].

Proof Let a_i, $b_i \in R$ (set of real numbers), and $I \in [0, 1]$.

$$\Rightarrow -1\le\frac{a_i}{\sqrt{a_i^2+b_i^2}}\le 1;\quad -1\le\frac{b_iI}{\sqrt{a_i^2+b_i^2}}\le 1.$$

$$\Rightarrow -1\le\frac{a_i+b_iI}{2\sqrt{a_i^2+b_i^2}}\le 1;\ \text{i = 1, 2, …, n.}$$

2.3. Euclidean distance of two Sets of NNs and UNNs

Definition 2 Let $A=\{\langle a_1+b_1I\rangle,\langle a_2+b_2I\rangle,\cdots,\langle a_n+b_nI\rangle\}$ and $X=\{\langle x_1+y_1I\rangle,\langle x_2+y_2I\rangle,\cdots,\langle x_n+y_nI\rangle\}$ (i = 1, 2, . . . , n.) be any two sets of NNs and $I \in [0, 1]$. Then the Euclidean distance of A and X is defined as:

$$D_{Eucl}(A,\ X)=\sqrt{\sum_{i=1}^{n}\left\langle a_i-x_i)+(b_i-y_i)I^*\right\rangle^2} \tag{1}$$

Here, $I^*=\lambda I^L+(1-\lambda)I^U$, and $0\le\lambda\le 1$. Let,

$$A^{\circ}=\left\{\left\langle\frac{a_1+b_1I}{2\sqrt{a_1^2+b_1^2}}\right\rangle,\left\langle\frac{a_2+b_2I}{2\sqrt{a_2^2+b_2^2}}\right\rangle,\cdots,\left\langle\frac{a_n+b_nI}{2\sqrt{a_n^2+b_n^2}}\right\rangle\right\},\ X^{\circ}=\left\{\left\langle\frac{x_1+y_1I}{2\sqrt{x_1^2+y_1^2}}\right\rangle,\left\langle\frac{x_2+y_2I}{2\sqrt{x_2^2+y_2^2}}\right\rangle,\cdots,\left\langle\frac{x_n+y_nI}{2\sqrt{x_n^2+y_n^2}}\right\rangle\right\}$$

be the sets of UNNs. Then the Euclidean distance of A° and X° is defined as:

$$D_{Eucl}(A^{\circ},\ X^{\circ})=\sqrt{\sum_{i=1}^{n}\left\langle\frac{(a_i+b_iI^*)}{2\sqrt{a_i^2+b_i^2}}-\frac{(x_i+y_iI^*)}{2\sqrt{x_i^2+y_i^2}}\right\rangle^2} \tag{2}$$

Definition 3 The normalized Euclidean distance of two sets of UNNs A° and X° is defined as:

$$D_{Eucl}^{N}(A^{\circ}, X^{\circ}) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left\langle \frac{(a_i + b_i I^*)}{2\sqrt{a_i^2 + b_i^2}} - \frac{(x_i + y_i I^*)}{2\sqrt{x_i^2 + y_i^2}} \right\rangle^2} \quad (3)$$

Note 1 In decision making situation we use expectation value (mean value) of λ i.e., $\lambda = 0.5$.

2.4. Tangent function for NNs

Definition 4 The tangent function of a neutrosophic number $P = x_{ij} + y_{ij}I = [x_{ij} + y_{ij}I^L, x_{ij} + y_{ij}I^U]$, where x_{ij} and y_{ij} are not simultaneously zeroes, (i = 1, 2, ..., m; j = 1, 2, ..., n) is defined as:

$$T_j(P) = 1 - \frac{1}{n}\sum_{i=1}^{n} \tan\frac{\pi}{4}\left(\left|\frac{y_{ij}}{\sqrt{x_{ij}^2 + y_{ij}^2}}\right|\right) \quad (4)$$

The weight structure is defined as:

$$\omega_j = \frac{T_j(P)}{\sum_{j=1}^{n} T_j(P)}\text{; j = 1, 2, ..., n} \quad (5)$$

Here, $\sum_{j=1}^{n}\omega_j = 1$.

Theorem 2 The function $T_j(P)$ is bounded.

Proof Since $x_{ij}, y_{ij} \in$ R and x_{ij}, y_{ij} are not both zero, we have.

$$0 \leq \frac{|y_{ij}|}{\left|\sqrt{x_{ij}^2 + y_{ij}^2}\right|} \leq 1 \Rightarrow 0 \leq \tan\frac{\pi}{4}\left(\frac{|y_{ij}|}{\left|\sqrt{x_{ij}^2 + y_{ij}^2}\right|}\right) \leq 1 \Rightarrow 0 \leq 1 - \tan\frac{\pi}{4}\left(\frac{|y_{ij}|}{\left|\sqrt{x_{ij}^2 + y_{ij}^2}\right|}\right) \leq 1 \Rightarrow 0 \leq T_j(P) \leq 1.$$

Hence, the function $T_j(P)$ is bounded.

Theorem 3 The function $T_j(P)$ is monotone decreasing.

Proof Here, $\frac{1}{n}\sum_{i=1}^{n} \tan\frac{\pi}{4}\left(\left|\frac{y_{ij}}{\sqrt{x_{ij}^2 + y_{ij}^2}}\right|\right)$ is monotone increasing in the interval $[0, \pi/4]$ and $0 \leq \frac{|y_{ij}|}{\left|\sqrt{x_{ij}^2 + y_{ij}^2}\right|} \leq 1$.

$\Rightarrow\ 1-\frac{1}{n}\sum_{i=1}^{n}\tan\frac{\pi}{4}\left(\left|\frac{y_{ij}}{\sqrt{x_{ij}^2+y_{ij}^2}}\right|\right)$ is a monotone decreasing function in the interval $[0,\pi/4]$.

$\Rightarrow\ T_j(P)$ is monotone decreasing in the interval $[0,\pi/4]$.

Example 2 Assume that $P_1 = 3 + 2I$, and $P_2 = 3 + 5I$. Then, we obtain $T(P_1) = 0.5345,\ T(P_2) = 0.2020$.

Example 3 Assume that $P_1 = 3 + I$, and $P_2 = 7 + I$. Then, we obtain $T(P_1) = 0.7464,\ T(P_2) = 0.8883$.

Example 4 Assume that $P_1 = 10 + 2I$, and $P_2 = 2 + 10I$. Then, we obtain $T(P_1) = 0.8447,\ T(P_2) = 0.0299$.

2.5. NN relative positive ideal solution and NN relative negative ideal solution

Definition 5 Assume that C^+ and C^- denote respectively two the type modifiers, namely, the benefit attribute and cost attribute. Assume that G_N^+ denotes the NN relative positive ideal solution (NNRPIS) and G_N^- denotes the NN relative negative ideal solution (NNRNIS).

Then G_N^+ is defined as:

$$G_N^+ = \left\langle d_1^{\omega+}, d_2^{\omega+}, \cdots, d_n^{\omega+} \right\rangle \tag{6}$$

Here $d_j^{\omega+} = \left\langle x_j^{\omega+} + y_j^{\omega+} I \right\rangle$ for j = 1, 2, . . . , n.

$$x_j^{\omega+} = \{(\max_i \{x_{ij}^{\omega j}\} / j \in C^+),\ (\min_i (x_{ij}^{\omega j}) / j \in C^-)\}$$

$$y_j^{\omega+} = \{(\min_i \{y_{ij}^{\omega j}\} / j \in C^+),\ (\max_i (y_{ij}^{\omega j}) / j \in C^-)\}$$

Then G_N^- is defined as follows:

$$G_N^- = \left\langle d_1^{\omega-}, d_2^{\omega-}, \cdots, d_n^{\omega-} \right\rangle \tag{7}$$

Here $d_j^{\omega-} = \left\langle x_j^{\omega-} + y_j^{\omega-} I \right\rangle$ for j = 1, 2, . . . , n.

$$x_j^{\omega-} = \{(\min_i \{x_{ij}^{\omega j}\} / j \in C^+),\ (\max_i (x_{ij}^{\omega j}) / j \in C^-)\}$$

$$y_j^{\omega-} = \{(\max_i \{y_{ij}^{\omega j}\} / j \in C^+),\ (\min_i (y_{ij}^{\omega j}) / j \in C^-)\}$$

3. NN-TOPSIS technique for MADM

Assume that an MADM problem is characterized by m selection options and n attributes. Also let $D = (D_1, D_2, \ldots, D_r)$ be the set of decision makers, $K = (K_1, K_2, \ldots, K_m)$ be the set of selection options, and $C = (C_1, C_2, \ldots, C_n)$ be the set of attributes. The ratings offered by the decision maker describe the performance of the selection option K_i against the attribute C_j. Let $\{\omega_1, \omega_2, \cdots, \omega_n\}$ be the weight vector assigned for the attributes $C_1, C_2, \ldots, C_n$. The rating values associated with the selection options with respect to the attributes is presented in the following NN based decision matrix (for r[th] decision maker).

$$D_r[K \mid C_1, C_2, \ldots, C_n] = \begin{array}{c} \\ K_1 \\ K_2 \\ \vdots \\ K_m \end{array} \begin{pmatrix} C_1 & C_2 & \cdots & C_n \\ \langle x_{11}+y_{11}I\rangle_r & \langle x_{12}+y_{12}I\rangle_r & \cdots & \langle x_{1n}+y_{1n}I\rangle_r \\ \langle x_{21}+y_{21}I\rangle_r & \langle x_{22}+y_{22}I\rangle_r & \cdots & \langle x_{2n}+y_{2n}I\rangle_r \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_{m1}+y_{m1}I\rangle_r & \langle x_{m2}+y_{m2}I\rangle_r & \cdots & \langle x_{mn}+y_{mn}I\rangle_r \end{pmatrix} \tag{8}$$

Here, $\langle x_{ij}+y_{ij}I\rangle_r$ is the rating value (in terms of NN) for ij[th] element of the decision matrix of r-th decision maker.

Now the steps (Figure 1) of NN-TOPSIS technique in NN environment are described below:

Step 1: Determine the weights of the decision makers.

Assume that $D = \{D_1, D_2, \ldots, D_r\}$ be a group of decision makers. In decision making situation, the decision maker's weight may be different as their importance is not identical. The importance of each decision maker is expressed in terms of linguistic variables. The linguistic variables are then transformed into NNs (see Table 1).

Table 1 Transformation of linguistic variable into NN

Linguistic terms (LTs)	NNs
Very Important (VI)	5
Important (I)	$4+I$
Medium (M)	$3+2I$
Unimportant (UI)	$2+3I$

Very unimportant (VUI)	1+4I

Assume that $a_r + b_r I$ presents the rating of r[th] decision maker. Then, the weight (ξ_r) of the r[th] decision maker is presented as:

$$\xi_r = \frac{[2a_r + b_r(I^L + I^U)]}{\sum_{t=1}^{r}[2a_t + b_t(I^L + I^U)]} \tag{9}$$

and $\sum_{t=1}^{r}\xi_t = 1$.

Step 2: Calculate the aggregated NN decision matrix based on decision makers' assessments.

Assume that $D^r = \langle a_k + b_k I \rangle_{m\times n}$ denotes the NN decision matrix of the r[th] decision maker and $\xi = (\xi_1, \xi_2, ..., \xi_r)^T$ be the weight format of decision makers such that each $\xi_r \in (0, 1)$. The aggregated matrix is obtained using NN weighted arithmetic mean aggregation operator (NNWAMAO) as:

$$D_{aggr} = (d_{ij})_{m\times n} = \text{NNWAMAO}\left(d_{ij}^1, d_{ij}^2, \cdots, d_{ij}^r\right) = \xi_1\, d_{ij}^1 \oplus \xi_2\, d_{ij}^2 \oplus \cdots \oplus \xi_r\, d_{ij}^r = \left\langle \sum \xi_{ij} a_{ij} + I \sum \xi_{ij} b_{ij} \right\rangle \tag{10}$$

Now the aggregated NN decision matrix is defined as:

$$D_{aggr}[K \mid C_1, C_2, ..., C_n] = \begin{array}{c} \\ K_1 \\ K_2 \\ \vdots \\ K_m \end{array} \begin{pmatrix} C_1 & C_2 & \cdots & C_n \\ \langle x_{11} + y_{11} I \rangle_{aggr} & \langle x_{12} + y_{12} I \rangle_{aggr} & \cdots & \langle x_{1n} + y_{1n} I \rangle_{aggr} \\ \langle x_{21} + y_{21} I \rangle_{aggr} & \langle x_{22} + y_{22} I \rangle_{aggr} & \cdots & \langle x_{2n} + y_{2n} I \rangle_{aggr} \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_{m1} + y_{m1} I \rangle_{aggr} & \langle x_{m2} + y_{m2} I \rangle_{aggr} & \cdots & \langle x_{mn} + y_{mn} I \rangle_{aggr} \end{pmatrix} \tag{11}$$

Here, $\langle x_{ij} + y_{ij} I \rangle_{aggr}$ is the rating value for ij[th] element of aggregated decision

matrix $D_{aggr}[K \mid C_1, C_2, ..., C_n]$

(i = 1, 2, . . . m; j = 1, 2, . .. n).

Step 3: Calculate the attribute weights.

When weights of attributes are completely unknown to decision makers, the entropy measure [93] is used to calculate attribute weights. Entropy method [94] is used to find out completely unknown attribute weights of single valued neutrosophic sets. Method to determine unidentified attribute weights in NN environment is yet to come into view in literature. We define a function for measuring unknown attribute weights (see definition 4).

Step 4: Aggregate the weighted NN decision matrix.

The calculated weights of the attributes and aggregated NN decision matrix are fused to construct the aggregated weighted NN decision matrix. The aggregated weighted NN decision matrix is defined by utilizing the multiplication rules between attribute weights and corresponding rating values of attributes as:

$$D\otimes\omega = \left\langle d^{\omega j}_{ij} \right\rangle_{m\times n} = \left\langle a^{\omega j}_{ij} + b^{\omega j}_{ij} I \right\rangle_{m\times n} = \begin{array}{c} \\ K_1 \\ K_2 \\ \vdots \\ K_m \end{array} \begin{pmatrix} C_1 & C_2 & \cdots & C_n \\ \langle x_{11}+y_{11}I\rangle_{\omega j} & \langle x_{12}+y_{12}I\rangle_{\omega j} & \cdots & \langle x_{1n}+y_{1n}I\rangle_{\omega j} \\ \langle x_{21}+y_{21}I\rangle_{\omega j} & \langle x_{22}+y_{22}I\rangle_{\omega j} & \cdots & \langle x_{2n}+y_{2n}I\rangle_{\omega j} \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_{m1}+y_{m1}I\rangle_{\omega j} & \langle x_{m2}+y_{m2}I\rangle_{\omega j} & \cdots & \langle x_{mn}+y_{mn}I\rangle_{\omega j} \end{pmatrix} \tag{12}$$

Here, $d^{\omega j}_{ij} = \left\langle T^{\omega j}_{ij}, I^{\omega j}_{ij}, F^{\omega j}_{ij} \right\rangle$ denotes the rating value for (ij)th element of the aggregated weighted NN decision matrix $D\otimes\omega$ (i = 1, 2, . . . , m and j = 1, 2, . . . , n).

Step 5: Determine the NNRPIS and the NNRNIS.

Assume that $\left\langle d^{\omega}_{ij} \right\rangle_{m\times n} = \left\langle x_{ij}+y_{ij}I \right\rangle_{m\times n}$ is an NN decision matrix, where, x_{ij} and $y_{ij}I$ are respectively the determinant part and indeterminate part of the evaluation for the attribute C_j with respect to the selection option K_i.

Step 6: Determine the distance measures of each selection option from the NNRPIS and the NNRNIS.

The normalized Euclidean distance measure of all selection option $\left\langle x^{\omega j}_{ij} + y^{\omega j}_{ij} I^{*} \right\rangle$ from the NNRPIS $\left\langle d^{\omega+}_{1}, d^{\omega+}_{2}, \ldots, d^{\omega+}_{n} \right\rangle$ for i = 1, 2, …, m and j = 1, 2, …, n is written as:

$$\Delta^{i+}_{Eucl}(d^{\omega j}_{ij}, d^{\omega+}_{j}) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left\langle \frac{(a^{\omega j}_{ij} + b^{\omega j}_{j} I^{*})}{(a^{\omega j}_{ij} + b^{\omega j}_{j})} - \frac{(a^{\omega+}_{ij} + b^{\omega+}_{j} I^{*})}{(a^{\omega+}_{ij} + b^{\omega+}_{j})} \right\rangle^{2}} \tag{13}$$

The normalized Euclidean distance measure of all selection option $\left\langle x^{\omega j}_{ij} + y^{\omega j}_{ij} I^{*} \right\rangle$ from the NNRNIS $\left\langle d^{\omega -}_{1}, d^{\omega -}_{2}, \ldots, d^{\omega -}_{n} \right\rangle$ for i = 1, 2, …, m and j = 1, 2, …, n is presented as:

$$\Delta^{i-}_{Eucl}(d^{\omega j}_{ij}, d^{\omega -}_{j}) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left\langle \frac{(a^{\omega j}_{ij}+b^{\omega j}_{j} I^{*})}{(a^{\omega j}_{ij}+b^{\omega j}_{j})} - \frac{(a^{\omega -}_{ij}+b^{\omega -}_{j} I^{*})}{(a^{\omega -}_{ij}+b^{\omega -}_{j})} \right\rangle^{2}} \tag{14}$$

Step 7: compute the relative closeness co-efficient (RCC) to the NN ideal solution.

RCC of each selection option K_i with respect to the NN positive ideal solution G^{+}_{N} is defined as:

$$\mathrm{RCC}(K_{\mathrm{i}}) = \frac{\left\langle \Delta^{-}_{Eucl}(d^{\omega j}_{ij}, d^{\omega -}_{j}) \right\rangle}{\left\langle \Delta^{-}_{Eucl}(d^{\omega j}_{ij}, d^{\omega -}_{j}) + \Delta^{+}_{Eucl}(d^{\omega j}_{ij}, d^{\omega +}_{j}) \right\rangle} \tag{15}$$

Here, $0 \le \mathrm{RCC}(K_{\mathrm{i}}) \le 1$.

Step 8: Rank the priority

All the RCC values are arranged in descending order. A set of alternatives is then preference ranked order. We select the alternative corresponding to the highest value of $\mathrm{RCC}(K_{\mathrm{i}})$ as the best choice K_i for i = 1, 2, . . . , m.

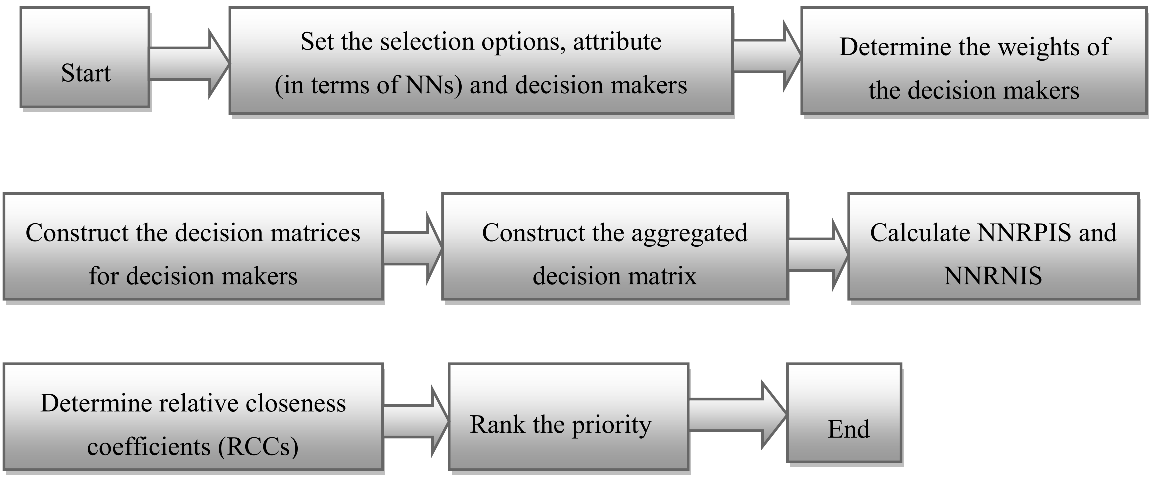

Figure 1: Steps of NN-TOPSIS technique

Step 9: End.

4. Illustrative example

Let a multi-national company wants to recruit managing director for their company. An interview board with three members D_1, D_2, D_3 is formed to select the managing director. The selection options (candidates) are K_1, K_2, K_3, and K_4. Decision makers must take their decisions based on the following attributes (C_1): academic qualification, (C_2): interview performance (C_3): management experience and (C_4): risk factor. Assume that ω_1, ω_2, ω_3 and ω_4 be the weights assigned to the attributes C_1, C_2, C_3, and C_4 respectively. The rating values of the selection options for the MAGDM problem with respect to the attribute are presented in NN based decision matrices (Eqs. (16), (17), and (18)).

Each decision maker uses five-point scale (see Table 1) to express his/her rating values. Each decision maker forms an NN based decision matrix to express rating values. The decision matrices corresponding to decision makers D_1, D_2, and D_3 are shown in (16), (17), and (18) respectively.

$$D_1[K|C_1, C_2, C_3] = \begin{array}{c} \\ K_1 \\ K_2 \\ K_3 \\ K_4 \end{array} \begin{pmatrix} C_1 & C_2 & C_3 & C_4 \\ 1+2I & 5 & 4+I & 2+I \\ 2+I & 1+I & 2+2I & 1+2I \\ 1+3I & 2+2I & 5+I & 2I \\ 3+I & 2+3I & 4+I & I \end{pmatrix} \tag{16}$$

$$D_2[K|C_1, C_2, C_3] = \begin{array}{c} \\ K_1 \\ K_2 \\ K_3 \\ K_4 \end{array} \begin{pmatrix} C_1 & C_2 & C_3 & C_4 \\ 2+3I & 4 & 3+I & I \\ 2+I & 2+2I & 2+I & 2I \\ 3I & 2+2I & 5 & 1+I \\ 1+I & 2+I & 4 & I \end{pmatrix} \tag{17}$$

$$D_3[K|C_1, C_2, C_3] = \begin{array}{c} \\ K_1 \\ K_2 \\ K_3 \\ K_4 \end{array} \begin{pmatrix} C_1 & C_2 & C_3 & C_4 \\ 3+2I & 3 & 4+I & I \\ 1+3I & 2+I & 3 & I \\ 2I & 1+I & 4 & 2I \\ 2+I & 2+I & 3+I & 1+I \end{pmatrix} \tag{18}$$

The problem is solved using the following steps of the proposed NN-TOPSIS technique.

Step 1: Determine the weights of decision makers.

Linguistic variables are employed to represent the weights of decision makers and their corresponding neutrosophic numbers are shown in Table 2.

Table 2 Transformation of linguistic variable into NN

	D_1	D_2	D_3
LTs	Important	Medium	Important
NNs	4+I	3+2I	4+I

Using Eq. (9), we obtain the weights of the decision makers as:

$\xi_1 = 0.346$, $\xi_2 = 0.308$, $\xi_3 = 0.346$.

Step 2: Construct the aggregated NN decision matrix based on the decision makers' assessments.

Using Eq. (10), we calculate aggregated NN decision matrix as:

$$D_{aggr}[K \mid C]= \begin{array}{c} \\ K_1 \\ K_2 \\ K_3 \\ K_4 \end{array} \begin{pmatrix} C_1 & C_2 & C_3 & C_4 \\ 1.997+2.337I & 4.026 & 3.683+I & 0.696+I \\ 1.683+1.673I & 1.667+1.337I & 2.343+1.039I & 0.356+1.683I \\ 0.343+3I & 1.683+1.680I & 4.680+0.356I & 0.337+1.683I \\ 2.026+I & 2+1.712I & 3.683+0.683I & 0.330+I \end{pmatrix} \tag{19}$$

Step 3: Determine the weights of the attributes.

Using Eqs. (4) and (5), we calculate the weights of the attributes as follows:

$\omega_1 = 0.1875$, $\omega_2 = 0.3180$, $\omega_3 = 0.4535$, $\omega_4 = 0.0410$.

Step 4: Aggregate the weighted NN decision matrices

Using Eq. (12), we calculate the aggregated weighted NN decision matrix as follows:

$$\left\langle a^{\omega_j}_{ij} + b^{\omega_j}_{ij} I \right\rangle_{4\times 4} = \begin{array}{c} \\ K_1 \\ K_2 \\ K_3 \\ K_4 \end{array} \begin{pmatrix} C_1 & C_2 & C_3 & C_4 \\ 0.374+0.438I & 1.276 & 1.663+0.4533I & 0.028+0.041I \\ 0.315+0.313I & 0.527+0.422I & 1.054+0.464I & 0.014+0.068I \\ 0.066+0.564I & 0.531+0.531I & 2.115+0.153I & 0.013+0.069I \\ 0.379+0.1875I & 0.636+0.541I & 1.662+0.302I & 0.014+0.041I \end{pmatrix} \tag{20}$$

Step 5: Determine the NNRPIS and the NNRNIS.

Here, C_1, C_2 and C_3 are benefit attributes and C_4 is the cost attribute. Then we obtain NNRPIS and NNRNIS as follows:

$$\text{NNRPIS} = \{(0.379+0.1875I), (1.276+0.422I), (1.054+0.464I), (0.013+0.069I)$$

$\text{NNRNIS} = \{(0.066+0.564I),(0.527+0.541I),(2.115+0.153I),(0.028+0.041I)$

Step 6: Determine the distance measures of each selection option from the NNRPIS and the NNRNIS.

Using Eq. (13), the normalized Euclidean distance measures of all selection options from the NNRPIS are calculated and shown in Table 3. Using Eq. (14), the normalized Euclidean distance measures of all selection options from the NNRNIS are calculated and shown in Table 3.

Step 7: Determine the RCC to the NN ideal solution.

The RCC of each selection option K_i with respect to the NN positive ideal solution is calculated as follows:

$\text{RCC}(K_1)=0.3791$, $\text{RCC}(K_2)=0.3159$, $\text{RCC}(K_3)=0.0723$, $\text{RCC}(K_4)=0.4788$.

Step 8: Ranking the priority.

According to the RCC values, we have,

$\text{RCC}(K_4) \succ \text{RCC}(K_1) \succ \text{RCC}(K_2) \succ \text{RCC}(K_3)$.

Hence, the candidate K_4 is the best selection option.

Table 3 Distance measures and RCC values of selection options

Selection options	Δ_{Eucl}^{i+}	Δ_{Eucl}^{i-}	$\text{RCC}(K_i)$
K_1	0.5562	0.3397	0.3791
K_2	0.5271	0.2434	0.3159
K_3	0.3582	0.0279	0.0723
K_4	0.9395	0.8629	0.4788

Step 9: End.

5. Sensivity study

In this section, we present sensitivity analysis to demonstrate the impact of different values of I on ranking order of selection options (see Figure 2). The ranking order for different intervals of I, is shown in Table 4. Table 4 reflects that the ranking order of selection options are same for selected values of *I*.

Table 4 Ranking order of the selection options for different *I*

I	RCC(K_i)	Ranking order
$I = 0$	RCC(K_1) = 0.3889, RCC(K_2) = 0.3291, RCC(K_3) = 0.1179, RCC(K_4) = 0.4899	$K_4 \succ K_1 \succ K_2 \succ K_3$
$I \in [0, 0.1]$	RCC(K_1) = 0.3876, RCC(K_2) = 0.3274, RCC(K_3) = 0.1077, RCC(K_4) = 0.4881	$K_4 \succ K_1 \succ K_2 \succ K_3$
$I \in [0, 0.2]$	RCC(K_1) = 0.3865, RCC(K_2) = 0.3264, RCC(K_3) = 0.1075, RCC(K_4) = 0.4866	$K_4 \succ K_1 \succ K_2 \succ K_3$
$I \in [0, 0.3]$	RCC(K_1) = 0.3857, RCC(K_2) = 0.3251, RCC(K_3) = 0.1031, RCC(K_4) = 0.4850	$K_4 \succ K_1 \succ K_2 \succ K_3$
$I \in [0, 0.4]$	RCC(K_1) = 0.3842, RCC(K_2) = 0.3241, RCC(K_3) = 0.0992, RCC(K_4) = 0.4838	$K_4 \succ K_1 \succ K_2 \succ K_3$
$I \in [0, 0.5]$	RCC(K_1) = 0.3831, RCC(K_2) = 0.3225, RCC(K_3) = 0.0945, RCC(K_4) = 0.4823	$K_4 \succ K_1 \succ K_2 \succ K_3$
$I \in [0, 0.6]$	RCC(K_1) = 0.3821, RCC(K_2) = 0.3212, RCC(K_3) = 0.0899, RCC(K_4) = 0.4815	$K_4 \succ K_1 \succ K_2 \succ K_3$
$I \in [0, 0.7]$	RCC(K_1) = 0.3812, RCC(K_2) = 0.3196, RCC(K_3) = 0.0952, RCC(K_4) = 0.4807	$K_4 \succ K_1 \succ K_2 \succ K_3$
$I \in [0, 0.8]$	RCC(K_1) = 0.3805, RCC(K_2) = 0.3183, RCC(K_3) = 0.0807, RCC(K_4) = 0.4797	$K_4 \succ K_1 \succ K_2 \succ K_3$
$I \in [0, 0.9]$	RCC(K_1) = 0.3798, RCC(K_2) = 0.3170, RCC(K_3) = 0.0765, RCC(K_4) = 0.4776	$K_4 \succ K_1 \succ K_2 \succ K_3$
$I \in [0, 1]$	RCC(K_1) = 0.3791, RCC(K_2) = 0.3159, RCC(K_3) = 0.0723, RCC(K_4) = 0.4758	$K_4 \succ K_1 \succ K_2 \succ K_3$

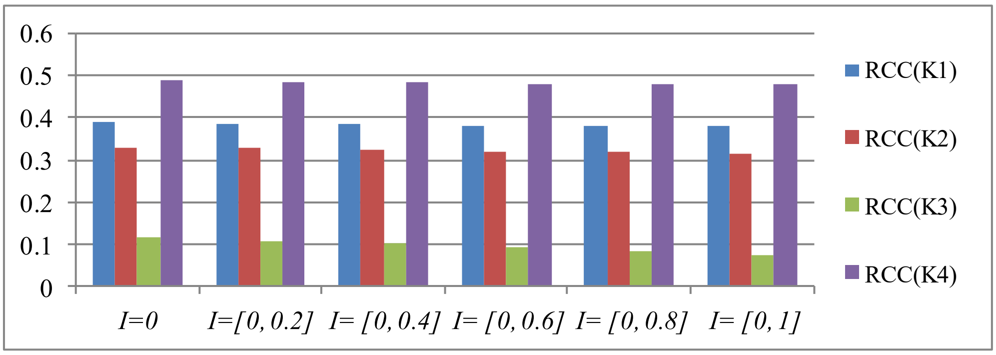

Figure 2: Ranking order of the alternatives with different values of *I*

6. Comparison analysis

In this section, a comparison analysis is presented between the proposed NN-TOPSIS technique and other existing decision-making strategies in NN environment. The ranking results obtained from the existing strategies [71, 76, 77] are furnished in Table 5. From the second column (Ye [71]) of Table 5, we see that K_4 is the best selection option for $I = 0$, $I \in [0, 0.2]$, and $I \in [0, 0.4]$. K_1 is the best selection option for other selected indeterminacy intervals. From the third column (Liu and Liu [76]) of Table 5, we observe that K_4 is the best selection option for $I = 0$, $I \in [0, 0.2]$, $I \in [0, 0.4]$, and $I \in [0, 0.6]$. K_1 is the best selection option for $I \in [0, 0.8]$, and $I \in [0, 1]$. From the fourth column (Zheng et al. [77]) of Table 5, we state that, K_4 is the best selection option for every selected indeterminacy interval. In the proposed technique, ranking order of selection option is unaltered for every selected indeterminacy interval. The comparison of ranking order of selection options between the proposed NN-TOPSIS technique and existing MADM strategies is shown in Table 5.

Table 5 The ranking order of existing strategies with different values of '*I*'

I	Ye [71]	Liu and Liu [76]	Zheng et al. [77]	NN-TOPSIS
$I = 0$	$K_4 \succ K_1 \succ K_2 \succ K_3$	$K_4 \succ K_1 \succ K_2 \succ K_3$	$K_4 \succ K_1 \succ K_2 \succ K_3$	$K_4 \succ K_1 \succ K_2 \succ K_3$
$I \in [0, 0.1]$	$K_4 \succ K_1 \succ K_2 \succ K_3$	$K_4 \succ K_1 \succ K_2 \succ K_3$	$K_4 \succ K_1 \succ K_2 \succ K_3$	$K_4 \succ K_1 \succ K_2 \succ K_3$
$I \in [0, 0.2]$	$K_4 \succ K_1 \succ K_2 \succ K_3$	$K_4 \succ K_1 \succ K_2 \succ K_3$	$K_4 \succ K_1 \succ K_2 \succ K_3$	$K_4 \succ K_1 \succ K_2 \succ K_3$
$I \in [0, 0.3]$	$K_4 \succ K_1 \succ K_2 \succ K_3$	$K_4 \succ K_1 \succ K_2 \succ K_3$	$K_4 \succ K_1 \succ K_2 \succ K_3$	$K_4 \succ K_1 \succ K_2 \succ K_3$
$I \in [0, 0.4]$	$K_4 \succ K_1 \succ K_2 \succ K_3$	$K_4 \succ K_1 \succ K_3 \succ K_2$	$K_4 \succ K_1 \succ K_2 \succ K_3$	$K_4 \succ K_1 \succ K_2 \succ K_3$
$I \in [0, 0.5]$	$K_4 \succ K_1 \succ K_2 \succ K_3$	$K_4 \succ K_1 \succ K_2 \succ K_3$	$K_4 \succ K_1 \succ K_2 \succ K_3$	$K_4 \succ K_1 \succ K_2 \succ K_3$
$I \in [0, 0.6]$	$K_1 \succ K_4 \succ K_2 \succ K_3$	$K_4 \succ K_1 \succ K_3 \succ K_2$	$K_4 \succ K_1 \succ K_2 \succ K_3$	$K_4 \succ K_1 \succ K_2 \succ K_3$
$I \in [0, 0.7]$	$K_1 \succ K_4 \succ K_2 \succ K_3$	$K_1 \succ K_4 \succ K_3 \succ K_2$	$K_4 \succ K_1 \succ K_2 \succ K_3$	$K_4 \succ K_1 \succ K_2 \succ K_3$
$I \in [0, 0.8]$	$K_1 \succ K_4 \succ K_2 \succ K_3$	$K_1 \succ K_4 \succ K_3 \succ K_2$	$K_4 \succ K_1 \succ K_3 \succ K_2$	$K_4 \succ K_1 \succ K_2 \succ K_3$
$I \in [0, 0.9]$	$K_1 \succ K_4 \succ K_2 \succ K_3$	$K_1 \succ K_4 \succ K_3 \succ K_2$	$K_4 \succ K_1 \succ K_3 \succ K_2$	$K_4 \succ K_1 \succ K_2 \succ K_3$
$I \in [0, 1]$	$K_1 \succ K_4 \succ K_2 \succ K_3$	$K_1 \succ K_4 \succ K_3 \succ K_2$	$K_4 \succ K_1 \succ K_3 \succ K_2$	$K_4 \succ K_1 \succ K_2 \succ K_3$

7. Conclusion

In real decision making, indeterminacy plays a very important role. In this article, the selection process is studied based on proposed NN-TOPSIS technique. To develop the NN-TOPSIS technique, we have defined an UNN and proved the basic properties. The defined UNN is an effective mathematical tool to express cognitive information considering the reliability of the information. We have defined Euclidean distance between two sets of NNs. We have defined NN weighted arithmetic aggregation operator (NNWANO) to aggregate NN decision matrices. We have also proposed a tangent function to determine unknown weights of attributes in NN environment. We have proposed a linguistic variable to present NN. We have performed sensitivity analysis for different values of I to show the influence of I on ranking order of selection options. The proposed technique simply and reliably represents human cognition by considering the interactivity of attribute and the cognition towards indeterminacy involved in the problem. The developed NN-TOPSIS technique combines the advantages of NN and TOPSIS. NN-TOPSIS is more comprehensive because when $I = 0$, NN-TOPSIS reduces to classical TOPSIS. Finally, we have addressed a problem of selecting the managing director of a multi-national company based on the proposed NN-TOPSIS technique. Future studies may consider the following problems: (i) the case when I varies for different NNs, (ii) more than 5 point-scale can be employed for rating purpose, (iii) Rank reversal in TOPSIS technique in NN environment, etc.

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Topology on Ultra Neutrosophic Set

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Abstract:

In this paper, we introduce the ultra neutrosophic set, and define some operations and establish a few properties of the ultra neutrosophic sets. Using the notion of topology on ultra neutrosophic sets, we introduce the concept of ultra neutrosophic topology. Further, we define the notion of ultra neutrosophic interior and ultra neutrosophic closure via ultra neutrosophic topological space.

Keywords: *Neutrosophic Set*; *Ultra Neutrosophic Crisp Set*; *Ultra Neutrosophic Set*.

1. Introduction:

The fundamental concept of Neutrosophic Set (NS) was introduced by Smarandache [1], which is the generalization of the Fuzzy Set (FS) [2] and the Intuitionistic Fuzzy Set (IFS) [3]. The concept of Neutrosophic Crisp Set (NCS) was grounded by Alblowi et al. [4] in 2014. Afterwards, Neutrosophic Crisp Topological Space (NCTS) was studied by Salama et al. [5]. In 2015, Salama et al. [6] further studied NCS theory. Later on, the Ultra Neutrosophic Crisp Set (UNCS) was presented by El Ghawably and Salama [7] in 2015.

Research Gap: The Ultra Neutrosophic Set (UNS) and the Ultra Neutrosophic Topology (UNT) on UNSs have not yet been introduced in the literature.

Motivation: To address the research gap, we introduce the Ultra Neutrosophic Set (UNS) and present a few basic properties of UNSs. Also, we present the Ultra Neutrosophic Topology (UNT) on UNSs.

The rest of the paper has been divided into the following sections. Section 2 presents the preliminaries and definitions on NS, NCS, UNCS. In section 3, we introduce the concept of ultra neutrosophic set and ultra neutrosophic topology. Besides, we formulate several results on them. Finally, in section 4, we conclude the paper by stating some future direction of research.

2. Preliminaries & Definitions:

In this section, we present some definitions on neutrosophic crisp set, neutrosophic set and ultra neutrosophic crisp set those are relevant for developing the main results of this article.

Definition 2.1. Suppose that Z be a non-empty fixed set. Then B_N, an NCS [4] is a triplet defined by $B_N=(B_1, B_2, B_3)$, where B_i (i = 1, 2, 3) be any subset of Z.

Definition 2.2. Assume that $B_N=(B_1, B_2, B_3)$ is an NCS. Then, the complement [4] of $B_N=(B_1, B_2, B_3)$ is defined by $B_N^c=(B_1^c, B_2^c, B_3^c)$.

Definition 2.3. Suppose that $B_N=(B_1, B_2, B_3)$ and $A_N=(A_1, A_2, A_3)$ are any two NCSs [4]. Then,

(i) $B_N \subseteq A_N$ iff $B_1 \subseteq A_1$, $B_2 \subseteq A_2$, $B_3 \supseteq A_3$;

(ii) $B_N \cup A_N = (B_1 \cup A_1, B_2 \cup A_2, B_3 \cap A_3)$;

(iii) $B_N \cap A_N = (B_1 \cap A_1, B_2 \cap A_2, B_3 \cup A_3)$.

Definition 2.4. Assume that Z is a fixed set. Then, an UNCS [7] $\widetilde{B_N}$ is defined as follows:

$\widetilde{B_N}=(B_1, B_2, B_3, M_B)$, where $M_B=(\cup_{i=1}^{3} B_i)^c$.

Definition 2.5. Suppose that $\widetilde{B_N}=(B_1, B_2, B_3, M_B)$ be an UNCS. Then, the complement [7] of $\widetilde{B_N}=(B_1, B_2, B_3, M_B)$ is defined by $\widetilde{B_N}^c=(B_1^c, B_2^c, B_3^c, M_B^c)$.

Definition 2.6. Suppose that $\widetilde{B_N}=(B_1, B_2, B_3, M_B)$ and $\widetilde{A_N}=(A_1, A_2, A_3, M_A)$ are two UNCSs. Then [7],

(i) $\widetilde{B_N} \subseteq \widetilde{A_N}$ iff $B_1 \subseteq A_1$, $B_2 \subseteq A_2$, $B_3 \supseteq A_3$, $M_B \supseteq M_A$;

(ii) $\widetilde{B_N} \cup \widetilde{A_N} = (B_1 \cup A_1, B_2 \cup A_2, B_3 \cap A_3, M_B \cap M_A)$;

(iii) $\widetilde{B_N} \cap \widetilde{A_N} = (B_1 \cap A_1, B_2 \cap A_2, B_3 \cup A_3, M_B \cup M_A)$.

Definition 2.7. Assume that Z is a fixed set. Then, an NS [1] U over Z is defined as follows:

$U = \{(\delta, T_U(\delta), I_U(\delta), F_U(\delta)) : \delta \in Z$, and $T_U(\delta), I_U(\delta), F_U(\delta) \in]^-0, 1^+[\}$, where $^-0 \leq T_U(\delta) + I_U(\delta) + F_U(\delta) \leq 3^+$.

Definition 2.8. Suppose that $U = \{(\delta, T_U(\delta), I_U(\delta), F_U(\delta)) : \delta \in Z$, and $T_U(\delta), I_U(\delta), F_U(\delta) \in [0, 1]\}$ be an NS over a fixed set Z. Then, complement [1] of U is $U^c = \{(\delta, 1-T_U(\delta), 1-I_U(\delta), 1-F_U(\delta)) : \delta \in Z\}$.

Definition 2.9. Suppose that $U = \{(\delta, T_U(\delta), I_U(\delta), F_U(\delta)) : \delta \in Z$, and $T_U(\delta), I_U(\delta), F_U(\delta) \in [0, 1]\}$ and $Y = \{(\delta, T_Y(\delta), I_Y(\delta), F_Y(\delta)) : \delta \in Z$, and $T_Y(\delta), I_Y(\delta), F_Y(\delta) \in [0, 1]\}$ be two NSs [1] over Z. Then,

i. $U \subseteq Y$ iff $T_U(\delta) \leq T_Y(\delta)$, $I_U(\delta) \geq I_Y(\delta)$, $F_U(\delta) \geq F_Y(\delta)$, for each $\delta \in Z$.

ii. $U \cup Y = \{(\delta, T_U(\delta) \vee T_Y(\delta), I_U(\delta) \wedge I_Y(\delta), F_U(\delta) \wedge F_Y(\delta)) : \delta \in Z\}$;

iii. $U \cap Y = \{(\delta, T_U(\delta) \wedge T_Y(\delta), I_U(\delta) \vee I_Y(\delta), F_U(\delta) \vee F_Y(\delta)) : \delta \in Z\}$.

Definition 2.11. The null NS (0_N) [1] and the whole NS (1_N) [1] over a fixed set Z are defined as follows:

(i) $0_N = \{(\delta, 0, 0, 1) : \delta \in Z\}$;

(ii) $1_N = \{(\delta, 1, 0, 0) : \delta \in Z\}$.

Clearly, $0_N \subseteq U \subseteq 1_N$, for any NS U over a fixed set Z.

3. Ultra Neutrosophic Set and Ultra Neutrosophic Topology:

In this section, we procure the notion of ultra neutrosophic set and ultra neutrosophic topology. Besides, we establish several results on them.

Definition 3.1. An ultra neutrosophic set R over a non-empty set Z is defined by $R=\{(\delta, T_R(\delta), I_R(\delta), F_R(\delta), M_R(\delta)) : \delta \in Z\}$, where $M_R(\delta) = T_R(\delta) \wedge I_R(\delta) \wedge F_R(\delta)$, $\forall \delta \in Z$ or $M_R(\delta) = T_R(\delta) \vee I_R(\delta) \vee F_R(\delta)$, $\forall \delta \in Z$.

Example 3.1. Consider a fixed set Z = {p, q, r}. Then, R = {(p, 0.3, 0.2, 0.5, 0.2), (q, 0.5, 0.7, 0.6, 0.5), (r, 0.6, 0.4, 0.2, 0.2)} is an UNS over Z.

Remark 3.1. The notion of ultra neutrosophic set is fully different from the notion of ultra neutrosophic crisp set. In an Ultra Neutrosophic Crisp Set $\widetilde{B_N}=(B_1, B_2, B_3, M_B)$, the crisp set M_B is defined by $M_B = (\cup_{i=1}^{3} B_i)^c$. But in the case of Ultra Neutrosophic Set $R=\{(\delta, T_R(\delta), I_R(\delta), F_R(\delta), M_R(\delta)) : \delta \in Z\}$ over Z, where $T_R(\delta)$, $I_R(\delta)$, and $F_R(\delta)$ denote respectively the truth, indeterminacy and false membership values of each $\delta \in Z$, another membership function $M_R : Z \to [0, 1]$ is defined by $M_R(\delta) = T_R(\delta) \wedge I_R(\delta) \wedge F_R(\delta)$, $\forall \delta \in Z$ or $M_R(\delta) = T_R(\delta) \vee I_R(\delta) \vee F_R(\delta)$, $\forall \delta \in Z$, which is totally different from the point of view of ultra neutrosophic crisp set.

Definition 3.2. The complement of an UNS $R = \{(\delta, T_R(\delta), I_R(\delta), F_R(\delta), M_R(\delta)) : \delta \in Z\}$ is defined by $R^c = \{(\delta, 1-T_R(\delta), 1-I_R(\delta), 1-F_R(\delta), 1-M_R(\delta)) : \delta \in Z\}$.

Definition 3.3. Consider any two UNSs $R = \{(\delta, T_R(\delta), I_R(\delta), F_R(\delta), M_R(\delta)) : \delta \in Z\}$ and $S = \{(\delta, T_S(\delta), I_S(\delta), F_S(\delta), M_S(\delta)) : \delta \in Z\}$ over a fixed set Z. Then,

i. $R \subseteq S \Leftrightarrow T_R(\delta) \leq T_S(\delta), I_R(\delta) \geq I_S(\delta), F_R(\delta) \geq F_S(\delta), M_R(\delta) \leq M_S(\delta)$, for each $\delta \in Z$;

ii. R=S if and if $R \subseteq S$ and $S \subseteq R$;

iii. $R \cup S=\{(\delta, T_R(\delta) \vee T_S(\delta), I_R(\delta) \wedge I_S(\delta), F_R(\delta) \wedge F_S(\delta), M_R(\delta) \vee M_S(\delta)) : \delta \in Z\}$;

iv. $R \cap S=\{(\delta, T_R(\delta) \wedge T_S(\delta), I_R(\delta) \vee I_S(\delta), F_R(\delta) \vee F_S(\delta), M_R(\delta) \wedge M_S(\delta)) : \delta \in Z\}$.

Definition 3.4. The null UNS $\widehat{0_N}$ and whole UNS $\widehat{1_N}$ over a fixed set Z is defined as follows:

i. $\widehat{0_N}=\{(\delta, 0, 1, 1, 0) : \delta \in Z\}$;

ii. $\widehat{1_N}=\{(\delta, 1, 0, 0, 1) : \delta \in Z\}$.

Clearly, $\widehat{0_N} \subseteq U \subseteq \widehat{1_N}$, for any UNS U over Z.

Theorem 3.1. Let N, R and C be three UNSs over a fixed set Z. Then, the following results hold:

i. $N \cup [R \cup C] = [N \cup R] \cup C$;

ii. $N \cap [R \cap C] = [N \cap R] \cap C$;

iii. $N \cup R = R \cup N$ and $N \cap R = R \cap N$;

iv. $N \cup N = N$ and $N \cap N = N$;

v. $N \cup [R \cap C] = [N \cup R] \cap [N \cup C]$;

vi. $N \cap [R \cup M] = [N \cap R] \cup [N \cap C]$;

vii. $[N^c]^c = N$.

Proof. Suppose that $N=\{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta\in Z\}$, $R=\{(\delta, T_R(\delta), I_R(\delta), F_R(\delta), M_R(\delta)) : \delta\in Z\}$ and $C=\{(\delta, T_C(\delta), I_C(\delta), F_C(\delta), M_C(\delta)) : \delta\in Z\}$ be three UNSs over a fixed set Z.

i. We have, $N\cup[R\cup C]$

$= \{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta\in Z\} \cup [\{(\delta, T_R(\delta), I_R(\delta), F_R(\delta), M_R(\delta)) : \delta\in Z\} \cup \{(\delta, T_C(\delta), I_C(\delta), F_C(\delta), M_C(\delta)) : \delta\in Z\}]$

$= \{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta\in Z\} \cup \{(\delta, T_R(\delta)\vee T_C(\delta), I_R(\delta)\wedge I_C(\delta), F_R(\delta)\wedge F_C(\delta), M_R(\delta)\vee M_C(\delta)) : \delta\in Z\}$

$= \{(\delta, T_N(\delta)\vee(T_R(\delta)\vee T_C(\delta)), I_N(\delta)\wedge(I_R(\delta)\wedge I_C(\delta)), F_N(\delta)\wedge(F_R(\delta)\wedge F_C(\delta)), M_N(\delta)\vee(M_R(\delta)\vee M_C(\delta))) : \delta\in Z\}$

$= \{(\delta, (T_N(\delta)\vee T_R(\delta))\vee T_C(\delta), (I_N(\delta)\wedge I_R(\delta))\wedge I_C(\delta), (F_N(\delta)\wedge F_R(\delta))\wedge F_C(\delta), (M_N(\delta)\vee M_R(\delta))\vee M_C(\delta)) : \delta\in Z\}$

$= \{(\delta, T_N(\delta)\vee T_R(\delta), I_N(\delta)\wedge I_R(\delta), F_N(\delta)\wedge F_R(\delta), M_N(\delta)\vee M_R(\delta)) : \delta\in Z\} \cup \{(\delta, T_C(\delta), I_C(\delta), F_C(\delta), M_C(\delta)) : \delta\in Z\}$

$=[\{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta\in Z\} \cup \{(\delta, T_R(\delta), I_R(\delta), F_R(\delta), M_R(\delta)) : \delta\in Z\}] \cup \{(\delta, T_C(\delta), I_C(\delta), F_C(\delta), M_C(\delta)) : \delta\in Z\}$

$= [N\cup R]\cup C$

Therefore, $N\cup[R\cup C] = [N\cup R]\cup C$.

ii. We have, $N\cap[R\cap C]$

$= \{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta\in Z\} \cap [\{(\delta, T_R(\delta), I_R(\delta), F_R(\delta), M_R(\delta)) : \delta\in Z\} \cap \{(\delta, T_C(\delta), I_C(\delta), F_C(\delta), M_C(\delta)) : \delta\in Z\}]$

$= \{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta\in Z\} \cap \{(\delta, T_R(\delta)\wedge T_C(\delta), I_R(\delta)\vee I_C(\delta), F_R(\delta)\vee F_C(\delta), M_R(\delta)\wedge M_C(\delta)) : \delta\in Z\}$

$= \{(\delta, T_N(\delta)\wedge(T_R(\delta)\wedge T_C(\delta)), I_N(\delta)\vee(I_R(\delta)\vee I_C(\delta)), F_N(\delta)\vee(F_R(\delta)\vee F_C(\delta)), M_N(\delta)\wedge(M_R(\delta)\wedge M_C(\delta))) : \delta\in Z\}$

$= \{(\delta, (T_N(\delta)\wedge T_R(\delta))\wedge T_C(\delta), (I_N(\delta)\vee I_R(\delta))\vee I_C(\delta), (F_N(\delta)\vee F_R(\delta))\vee F_C(\delta), (M_N(\delta)\wedge M_R(\delta))\wedge M_C(\delta)) : \delta\in Z\}$

$= \{(\delta, T_N(\delta)\wedge T_R(\delta), I_N(\delta)\vee I_R(\delta), F_N(\delta)\vee F_R(\delta), M_N(\delta)\wedge M_R(\delta)) : \delta\in Z\} \cap \{(\delta, T_C(\delta), I_C(\delta), F_C(\delta), M_C(\delta)) : \delta\in Z\}$

$=[\{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta\in Z\} \cap \{(\delta, T_R(\delta), I_R(\delta), F_R(\delta), M_R(\delta)) : \delta\in Z\}] \cap \{(\delta, T_C(\delta), I_C(\delta), F_C(\delta), M_C(\delta)) : \delta\in Z\}$

$= [N\cap R]\cap C$

Therefore, $N\cap[R\cap C] = [N\cap R]\cap C$.

iii. We have, $N\cup R$

$= \{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta\in Z\} \cup \{(\delta, T_R(\delta), I_R(\delta), F_R(\delta), M_R(\delta)) : \delta\in Z\}$

$= \{(\delta, T_N(\delta)\vee T_R(\delta), I_N(\delta)\wedge I_R(\delta), F_N(\delta)\wedge F_R(\delta), M_N(\delta)\vee M_R(\delta)) : \delta\in Z\}$

$= \{(\delta, T_R(\delta) \vee T_N(\delta), I_R(\delta) \wedge I_N(\delta), F_R(\delta) \wedge F_N(\delta), M_R(\delta) \vee M_N(\delta)) : \delta \in Z\}$

$= \{(\delta, T_R(\delta), I_R(\delta), F_R(\delta), M_R(\delta)) : \delta \in Z\} \cup \{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta \in Z\}$

$= R \cup N$

Therefore, $N \cup R = R \cup N$.

Further, we have $N \cap R$

$= \{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta \in Z\} \cap \{(\delta, T_R(\delta), I_R(\delta), F_R(\delta), M_R(\delta)) : \delta \in Z\}$

$= \{(\delta, T_N(\delta) \wedge T_R(\delta), I_N(\delta) \vee I_R(\delta), F_N(\delta) \vee F_R(\delta), M_N(\delta) \wedge M_R(\delta)) : \delta \in Z\}$

$= \{(\delta, T_R(\delta) \wedge T_N(\delta), I_R(\delta) \vee I_N(\delta), F_R(\delta) \vee F_N(\delta), M_R(\delta) \wedge M_N(\delta)) : \delta \in Z\}$

$= \{(\delta, T_R(\delta), I_R(\delta), F_R(\delta), M_R(\delta)) : \delta \in Z\} \cap \{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta \in Z\}$

$= R \cap N$

Therefore, $N \cap R = R \cap N$.

iv. We have, $N \cup N$

$= \{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta \in Z\} \cup \{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta \in Z\}$

$= \{(\delta, T_N(\delta) \vee T_N(\delta), I_N(\delta) \wedge I_N(\delta), F_N(\delta) \wedge F_N(\delta), M_N(\delta) \vee M_N(\delta)) : \delta \in Z\}$

$= \{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta \in Z\}$

$= N$

Therefore, $N \cup N = N$.

Further, we have $N \cap N$

$= \{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta \in Z\} \cap \{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta \in Z\}$

$= \{(\delta, T_N(\delta) \wedge T_N(\delta), I_N(\delta) \vee I_N(\delta), F_N(\delta) \vee F_N(\delta), M_N(\delta) \wedge M_N(\delta)) : \delta \in Z\}$

$= \{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta \in Z\}$

$= N$

Therefore, $N \cap N = N$.

v. We have $N \cup [R \cap C]$

$= \{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta \in Z\} \cup [\{(\delta, T_R(\delta), I_R(\delta), F_R(\delta), M_R(\delta)) : \delta \in Z\} \cap \{(\delta, T_C(\delta), I_C(\delta), F_C(\delta), M_C(\delta)) : \delta \in Z\}]$

$= \{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta \in Z\} \cup [\{(\delta, T_R(\delta) \wedge T_C(\delta), I_R(\delta) \vee I_C(\delta), F_R(\delta) \vee F_C(\delta), M_R(\delta) \wedge M_C(\delta)) : \delta \in Z\}]$

$= [\{(\delta, T_N(\delta) \vee (T_R(\delta) \wedge T_C(\delta)), I_N(\delta) \wedge (I_R(\delta) \vee I_C(\delta)), F_N(\delta) \wedge (F_R(\delta) \vee F_C(\delta)), M_N(\delta) \vee (M_R(\delta) \wedge M_C(\delta))) : \delta \in Z\}]$

Now, we have $[N \cup R] \cap [N \cup C]$

$= [\{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta \in Z\} \cup \{(\delta, T_R(\delta), I_R(\delta), F_R(\delta), M_R(\delta)) : \delta \in Z\}] \cap [\{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta \in Z\} \cup \{(\delta, T_C(\delta), I_C(\delta), F_C(\delta), M_C(\delta)) : \delta \in Z\}]$

$= \{(\delta, T_N(\delta) \vee T_R(\delta), I_N(\delta) \wedge I_R(\delta), F_N(\delta) \wedge F_R(\delta), M_N(\delta) \vee M_R(\delta)) : \delta \in Z\} \cap \{(\delta, T_N(\delta) \vee T_C(\delta), I_N(\delta) \wedge I_C(\delta), F_N(\delta) \wedge F_C(\delta), M_N(\delta) \vee M_C(\delta)) : \delta \in Z\}$

= $\{(\delta, (T_N(\delta)\vee T_R(\delta)) \wedge (T_N(\delta)\vee T_C(\delta)), (I_N(\delta)\wedge I_R(\delta)) \vee (I_N(\delta)\wedge I_C(\delta)), (F_N(\delta)\wedge F_R(\delta)) \vee (F_N(\delta)\wedge F_C(\delta)), (M_N(\delta)\vee M_R(\delta)) \wedge (M_N(\delta)\vee M_C(\delta))) : \delta\in Z\}$

= $\{(\delta, T_N(\delta)\vee(T_R(\delta)\wedge T_C(\delta)), I_N(\delta)\wedge(I_R(\delta)\vee I_C(\delta)), F_N(\delta)\wedge(F_R(\delta)\vee F_C(\delta)), M_N(\delta)\vee(M_R(\delta)\wedge M_C(\delta))) : \delta\in Z\}$

= $N\cup[R\cap C]$

Therefore, $N\cup[R\cap C] = [N\cup R]\cap[N\cup C]$.

vi. We have $N\cap[R\cup C]$

= $\{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta\in Z\} \cap [\{(\delta, T_R(\delta), I_R(\delta), F_R(\delta), M_R(\delta)) : \delta\in Z\} \cup \{(\delta, T_C(\delta), I_C(\delta), F_C(\delta), M_C(\delta)) : \delta\in Z\}]$

= $\{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta\in Z\} \cap [\{(\delta, T_R(\delta)\vee T_C(\delta), I_R(\delta)\wedge I_C(\delta), F_R(\delta)\wedge F_C(\delta), M_R(\delta)\vee M_C(\delta)) : \delta\in Z\}]$

= $[\{(\delta, T_N(\delta)\wedge(T_R(\delta)\vee T_C(\delta)), I_N(\delta)\vee(I_R(\delta)\wedge I_C(\delta)), F_N(\delta)\vee(F_R(\delta)\wedge F_C(\delta)), M_N(\delta)\wedge(M_R(\delta)\vee M_C(\delta))) : \delta\in Z\}]$

Now, we have $[N\cap R]\cup[N\cap C]$

= $[\{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta\in Z\} \cap \{(\delta, T_R(\delta), I_R(\delta), F_R(\delta), M_R(\delta)) : \delta\in Z\}] \cup [\{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta\in Z\} \cap \{(\delta, T_C(\delta), I_C(\delta), F_C(\delta), M_C(\delta)) : \delta\in Z\}]$

= $\{(\delta, T_N(\delta)\wedge T_R(\delta), I_N(\delta)\vee I_R(\delta), F_N(\delta)\vee F_R(\delta), M_N(\delta)\wedge M_R(\delta)) : \delta\in Z\} \cup \{(\delta, T_N(\delta)\wedge T_C(\delta), I_N(\delta)\vee I_C(\delta), F_N(\delta)\vee F_C(\delta), M_N(\delta)\wedge M_C(\delta)) : \delta\in Z\}$

= $\{(\delta, (T_N(\delta)\wedge T_R(\delta))\vee (T_N(\delta)\wedge T_C(\delta)), (I_N(\delta)\vee I_R(\delta))\wedge (I_N(\delta)\vee I_C(\delta)), (F_N(\delta)\vee F_R(\delta))\wedge (F_N(\delta)\vee F_C(\delta)), (M_N(\delta)\wedge M_R(\delta))\vee (M_N(\delta)\wedge M_C(\delta))) : \delta\in Z\}$

= $\{(\delta, T_N(\delta)\wedge(T_R(\delta)\vee T_C(\delta)), I_N(\delta)\vee(I_R(\delta)\wedge I_C(\delta)), F_N(\delta)\vee(F_R(\delta)\wedge F_C(\delta)), M_N(\delta)\wedge(M_R(\delta)\vee M_C(\delta))) : \delta\in Z\}$

= $N\cap[R\cup C]$

Therefore, $N\cap[R\cup C] = [N\cap R]\cup[N\cap C]$.

vii. We have, $N^c = \{(\delta, 1-T_N(\delta), 1-I_N(\delta), 1-F_N(\delta), 1-M_N(\delta)) : \delta\in Z\}$.

Therefore, $(N^c)^c$

= $\{(\delta, 1-(1-T_N(\delta)), 1-(1-I_N(\delta)), 1-(1-F_N(\delta)), 1-(1-M_N(\delta))) : \delta\in Z\}$

= $\{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta\in Z\}$

= N

Hence, $(N^c)^c = N$.

Definition 3.5. Consider a universe of discourse Z and τ be any collection of UNSs over Z. Then, τ is called an Ultra Neutrosophic Topology (Ultra-N-T) on Z if it satisfies the following three axioms:

i. $\widehat{0_N}, \widehat{1_N} \in\tau$;

ii. $C_1, C_2 \in\tau \Rightarrow C_1\cap C_2 \in\tau$;

iii. $\{C_i : i \in \Delta\} \subseteq \tau \Rightarrow \cup C_i \in\tau$.

In that case, (Z, τ) is called an Ultra Neutrosophic Topological Space (Ultra-N-T-S). If K is an element of τ, then K is said to be an Ultra Neutrosophic Open Set (Ultra-N-O-S) and the complement of K is said to be an Ultra Neutrosophic Closed Set (Ultra-N-C-S).

Example 3.2. Assume that Z={c, d} and L={(c, 0.5, 0.3, 0.7, 0.7), (d, 0.4, 0.5, 0.3, 0.5)}, M={(c, 0.4, 0.5, 0.7, 0.4), (d, 0.2, 0.6, 0.5, 0.2)}, N={(c, 0.8, 0.2, 0.5, 0.8), (d, 0.5, 0.4, 0.3, 0.5)} are three UNSs over Z. Then, the collection τ={$\widehat{0_N}$, $\widehat{1_N}$, L, M, N} is an Ultra-N-T on Z.

Definition 3.6. Assume that (Z, τ) is an Ultra-N-T-S and K be an UNS over Z. Then, Ultra Neutrosophic Interior (Ultra-N_{int}) and Ultra Neutrosophic Closure (Ultra-N_{cl}) of U are defined by

Ultra-N_{int} (K) = ∪{L : L is an Ultra-N-O-S in Z and L⊆K},

Ultra-N_{cl} (K) = ∩{H : H is an Ultra-N-C-S in Z and K⊆H}.

It is clear that, Ultra-N_{int}(K) is the largest Ultra-N-O-S over Z which is contained in K and Ultra-N_{cl}(K) is the smallest Ultra-N-C-S over Z which contains K.

Theorem 3.2. Let (Z, τ) be an Ultra-N-T-S. Assume that A and B are any two UNSs over Z. Then, the following properties hold:

i. Ultra-N_{int}(A) ⊆ A ⊆ Ultra-N_{cl}(A);
ii. A ⊆ B ⟹ Ultra-N_{int}(A) ⊆ Ultra-N_{int}(B);
iii. A ⊆ B ⟹ Ultra-N_{cl}(A) ⊆ Ultra-N_{cl}(B);
iv. Ultra-N_{cl}(0_N) = 0_N & Ultra-N_{cl}(1_N) = 1_N;
v. Ultra-N_{int}(0_N) = 0_N & Ultra-N_{int}(1_N) = 1_N;
vi. Ultra-N_{cl}(A ∪ B) = Ultra-N_{cl}(A) ∪ Ultra-N_{cl}(B);
vii. Ultra-N_{int}(A) ∪ Ultra-N_{int}(B) ⊂ Ultra-N_{int}(A∪B);
viii. Ultra-N_{int}(A ∩ B) = Ultra-N_{int}(A) ∩ Ultra-N_{int}(B);
ix. Ultra-N_{cl}(A ∩ B) ⊂ Ultra-N_{cl}(A) ∩ Ultra-N_{cl}(B).

Proof.

i. By Definition 3.6, we have Ultra-N_{int}(A) = ∪{R : R is an Ultra-N-O-S in (Z, τ) and R⊆A}. Since, each R⊆A, so ∪{R : R is an Ultra-N-O-S in (Z, τ) and R⊆A} ⊆ A, i.e., Ultra-N_{int}(A)⊆A.
Again, Ultra-N_{cl}(A) = ∩{W : W is an Ultra-N-C-S in (Z, τ) and A⊆W}. Since, each W⊇A, so ∩{W : W is an Ultra-N-C-S in (Z, τ) and A⊆W} ⊇ A, i.e., Ultra-N_{cl}(A)⊇A. Therefore, Ultra-N_{int}(A)⊆A⊆Ultra-N_{cl}(A).
Hence, Ultra-N_{int}(A) ⊆ A ⊆ Ultra-N_{cl}(A), for any ultra neutrosophic set A over Z.

ii. Suppose that (Z, τ) is an Ultra-N-T-S. Let A and B be any two UNSs over Z such that A⊆B.
Now, we have
Ultra-N_{int}(A) = ∪{W : W is an Ultra-N-O-S in (Z, τ) and W⊆A}
⊆ ∪{W : W is an Ultra-N-O-S in (Z, τ) and W⊆B} [Since A⊆B]
= Ultra-N_{int}(B)
⇒ Ultra-N_{int}(A) ⊆ Ultra-N_{int}(B).

Therefore, $A \subseteq B \Rightarrow$ Ultra-$N_{int}(A) \subseteq$ Ultra-$N_{int}(B)$.

iii. Assume that (Z, τ) is an Ultra-N-T-S. Let A and B be any two UNSs over Z such that $A \subseteq B$. Now, we have

Ultra-$N_{cl}(A) = \cap\{W : W$ is an Ultra-N-C-S in (Z, τ) and $A \subseteq W\}$

$\subseteq \cap\{W : W$ is an Ultra-N-C-S in (Z, τ) and $B \subseteq W\}$ [Since $A \subseteq B$]

= Ultra-$N_{cl}(B)$

$\Rightarrow$ Ultra-$N_{cl}(A) \subseteq$ Ultra-$N_{cl}(B)$.

Therefore, $A \subseteq B \Rightarrow$ Ultra-$N_{cl}(A) \subseteq$ Ultra-$N_{cl}(B)$.

iv. It is known that, Ultra-$N_{cl}(A) = \cap \{W : W$ is an Ultra-N-C-S in (Z, τ) and $A \subseteq W\}$.

We have, Ultra-$N_{cl}(0_N)$

$= \cap \{W : W$ is an Ultra-N-C-S in (Z, τ) and $0_N \subseteq W\}$

$= 0_N \cap \{W : W$ is an Ultra-N-C-S in (Z, τ) and $0_N \subset W\}$

$= 0_N \cap M$, where $M = \cap\{W$: W is an Ultra-N-C-S in (Z, τ) and $0_N \subset W\}$ is a neutrosophic sub-set of (Z, τ).

$= 0_N$

Therefore, Ultra-$N_{cl}(0_N) = 0_N$.

Further, we have

Ultra-$N_{cl}(1_N)$

$= \cap \{W : W$ is an Ultra-N-C-S in (Z, τ) and $1_N \subseteq W\}$

$= 1_N$ [since there exists no Ultra-N-C-S W in (Z, τ) such that $1_N \subset W$]

Therefore, Ultra-$N_{cl}(1_N) = 1_N$.

v. It is known that, Ultra-$N_{int}(A) = \cup \{W : W$ is an Ultra-N-O-S in (Z, τ) and $W \subseteq A\}$.

We have,

Ultra-$N_{int}(0_N)$

$= \cup \{W : W$ is an Ultra-N-O-S in (Z, τ) and $W \subseteq 0_N\}$

$= 0_N$ [since there exists no Ultra-N-O-S W in (Z, τ) such that $W \subset 0_N$]

Therefore, Ultra-$N_{int}(0_N) = 0_N$.

Further, we have Ultra-$N_{int}(1_N)$

$= \cup \{W : W$ is an Ultra-N-O-S in (Z, τ) and $W \subseteq 1_N\}$

$= \cup \{W : W$ is an Ultra-N-O-S in (Z, τ) and $W \subseteq 1_N\} \cup 1_N$

$= M \cup 1_N$, where $M = \cup\{W : W$ is an Ultra-N-O-S in (Z, τ) and $W \subseteq 1_N\}$ is a neutrosophic subset of (Z, τ).

$= 1_N$

Therefore, Ultra-$N_{int}(1_N) = 1_N$.

vi. Let A and B be any two ultra neutrosophic sub-sets of an Ultra-N-T-S (Z, τ). It is known that, $A \subseteq A \cup B$ and $B \subseteq A \cup B$.

Now, $A \subseteq A \cup B$

$\Rightarrow$ Ultra-$N_{cl}(A) \subseteq$ Ultra-$N_{cl}(A \cup B)$;

and $B \subseteq A \cup B$

$\Rightarrow$ Ultra-$N_{cl}(B) \subseteq$ Ultra-$N_{cl}(A \cup B)$.

Therefore, Ultra-$N_{cl}(A) \cup$Ultra-$N_{cl}(B) \subseteq$ Ultra-$N_{cl}(A \cup B)$ (1)

We have, $A \subseteq$Ultra-$N_{cl}(A)$, $B \subseteq$Ultra-$N_{cl}(B)$. Therefore, $A \cup B \subseteq$ Ultra-$N_{cl}(A) \cup$Ultra-$N_{cl}(B)$. Further, it is known that Ultra-$N_{cl}(A) \cup$Ultra-$N_{cl}(B)$ is an Ultra-N-C-S in (Z, τ). It is clear that, Ultra-$N_{cl}(A) \cup$Ultra-$N_{cl}(B)$ is an Ultra-N-C-S in (Z, τ), which contains $A \cup B$. But it is known that Ultra-$N_{cl}(A \cup B)$ is the smallest Ultra-N-C-S in (Z, τ), which contains $A \cup B$. Therefore, Ultra-$N_{cl}(A \cup B) \subseteq$ Ultra-$N_{cl}(A) \cup$Ultra-$N_{cl}(B)$ (2)

From eqs. (1) and (2), we have Ultra-$N_{cl}(A \cup B)$ = Ultra-$N_{cl}(A) \cup$Ultra-$N_{cl}(B)$.

vii. Suppose that A and B are two ultra neutrosophic sub-sets of an Ultra-N-T-S (Z, τ). It is known that $A \subset A \cup B$ and $B \subset A \cup B$.

Thus, we obtain

$A \subset A \cup B$

$\Rightarrow$ Ultra-$N_{int}(A) \subset$ Ultra-$N_{int}(A \cup B)$;

and $B \subset A \cup B$

$\Rightarrow$ Ultra-$N_{int}(B) \subset$ Ultra-$N_{int}(A \cup B)$.

Therefore, Ultra-$N_{int}(A) \cup$Ultra-$N_{int}(B) \subset$ Ultra-$N_{int}(A \cup B)$.

viii. Assume that A and B are any two ultra neutrosophic sub-sets of an Ultra-N-T-S (Z, τ). It is known that $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

Now, we have

$A \cap B \subseteq A$

$\Rightarrow$ Ultra-$N_{int}(A \cap B) \subseteq$ Ultra-$N_{int}(A)$;

and $A \cap B \subseteq B$

$\Rightarrow$ Ultra-$N_{int}(A \cap B) \subseteq$ Ultra-$N_{int}(B)$.

Therefore, Ultra-$N_{int}(A \cap B) \subseteq$ Ultra-$N_{int}(A) \cap$Ultra-$N_{int}(B)$.

For any two ultra neutrosophic sets A and B, we have Ultra-$N_{int}(A) \subseteq A$ & Ultra-$N_{int}(B) \subseteq B$. This implies, Ultra-$N_{int}(A) \cap$ Ultra-$N_{int}(B) \subseteq A \cap B$. It is known that, Ultra-$N_{int}(A) \cap$Ultra-$N_{int}(B)$ is an Ultra-N-O-S in (Z, τ). Therefore, Ultra-$N_{int}(A) \cap$Ultra-$N_{int}(B)$ is an Ultra-N-O-S in (Z, τ), which is contained in $A \cap B$. We know that Ultra-$N_{int}(A \cap B)$ is the largest Ultra-N-O-S in (Z, τ), which is contained in $A \cap B$. Therefore, Ultra-$N_{int}(A) \cap$Ultra-$N_{int}(B) \subseteq A \cap B$.

Hence, Ultra-$N_{int}(A \cap B) \subseteq$ Ultra-$N_{int}(A) \cap$Ultra-$N_{int}(B)$ and Ultra-$N_{int}(A) \cap$Ultra-$N_{int}(B) \subseteq A \cap B$. Therefore, Ultra-$N_{int}(A \cap B)$ = Ultra-$N_{int}(A) \cap$Ultra-$N_{int}(B)$.

ix. Suppose that A and B be any two ultra neutrosophic sub-sets of an Ultra-N-T-S (Z, τ). It is known that $A \cap B \subseteq A$, $A \cap B \subseteq B$.

Now, $A \cap B \subseteq A$

$\Rightarrow$ Ultra-$N_{cl}(A \cap B) \subseteq$ Ultra-$N_{cl}(A)$;

and $A \cap B \subseteq B$

$\Rightarrow$ Ultra-$N_{cl}(A \cap B) \subseteq$ Ultra-$N_{cl}(B)$.

Therefore, Ultra-$N_{cl}(A \cap B) \subseteq$ Ultra-$N_{cl}(A) \cap$Ultra-$N_{cl}(B)$.

Theorem 3.3. Let (Z, τ) be an Ultra-N-T-S.

i. If K is an Ultra-N-C-S in (Z, τ), then Ultra-$N_{cl}(K)$=K;

ii. If K is an Ultra-N-O-S in (Z, τ), then Ultra-$N_{int}(K)$=K.

Proof.

i. Let (Z, τ) be an Ultra-N-T-S , and S be an Ultra-N-C-S in (Z, τ).

Now, Ultra-$N_{cl}(S) = \cap\{W$: W is an Ultra-N-C-S in (Z, τ) and $S \subseteq W\}$. Since, S is an Ultra-N-C-S in (Z, τ), so S is the smallest Ultra-N-C-S, which contains S. This implies, $\cap\{W$: W is an Ultra-N-C-S in (Z, τ) and $S \subseteq W\}$ = S. Therefore, Ultra-$N_{cl}(S)$ = S.

ii. Ley (Z, τ) be an Ultra-N-T-S , and S be an Ultra-N-C-S in (Z, τ).

Now, Ultra-$N_{int}(S) = \cup\{W$: W is an Ultra-N-O-S in (Z, τ) and $W \subseteq S\}$. Since, S is an Ultra-N-O-S in (Z, τ), so S is the largest Ultra-N-O-S, which is contained in S. This implies, $\cup\{W$: W is an Ultra-N-O-S in (Z, τ) and $W \subseteq S\}$ = S. Therefore, Ultra-$N_{int}(S)$ = S.

Theorem 3.4. Assume that N is an Ultra neutrosophic sub-set of an Ultra-N-T-S (Z, τ). Then,

i. (Ultra-$N_{int}(N))^c$ = Ultra-$N_{cl}(N^c)$;

ii. (Ultra-$N_{cl}(N))^c$ = Ultra-$N_{int}(N^c)$.

Proof.

i. Let (Z, τ) be an Ultra-N-T-S, and $N=\{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta \in Z\}$ be an Ultra neutrosophic sub-set of (Z, τ).

Now, we have

Ultra-$N_{int}(N)$

$= \cup\{W_i : i \in \Delta$ and W_i is an Ultra-N-O-S in (Z, τ) such that $W_i \subseteq N\}$

$= \{(\delta, \vee T_{W_i}(\delta), \vee I_{W_i}(\delta), \wedge F_{W_i}(\delta), \vee M_{W_i}(\delta)) : \delta \in Z\}$, where for all $i \in \Delta$ and W_i is an Ultra-N-O-S in (Z, τ) such that $W_i \subseteq N$.

$\Rightarrow$ (Ultra-$N_{int}(N))^c = \{(\delta, \wedge T_{W_i}(\delta), \vee I_{W_i}(\delta), \vee F_{W_i}(\delta), \wedge M_{W_i}(\delta)) : \delta \in Z\}$.

Since, $\wedge T_{W_i}(\delta) \leq T_N(\delta)$, $\vee I_{W_i}(\delta) \leq I_N(\delta)$, $\vee F_{W_i}(\delta) \geq F_N(\delta)$, $\wedge M_{W_i}(\delta) \geq M_N(\delta)$, for each $i \in \Delta$ and $\delta \in Z$, so Ultra-$N_{cl}(N^c) = \{(\delta, \wedge T_{W_i}(\delta), \vee I_{W_i}(\delta), \vee F_{W_i}(\delta), \wedge M_{W_i}(\delta)) : \delta \in Z\}\} = \cap\{W_i : i \in \Delta$ and W_i is an Ultra-N-C-S in (Z, τ) such that $N^c \subseteq W_i\}$.

Therefore, (Ultra-$N_{int}(N))^c$ = Ultra-$N_{cl}(N^c)$.

ii. Assume that (Z, τ) be an Ultra-N-T-S, and $N=\{(\delta, T_N(\delta), I_N(\delta), F_N(\delta), M_N(\delta)) : \delta \in Z\}$ be an Ultra neutrosophic sub-set of (Z, τ).

Now, we have

Ultra-N_{cl}(N)

$= \cap\{W_i : i\in\Delta$ and W_i is an Ultra-N-C-S in (Z, τ) such that $W_i \supseteq N\}$

$= \{(\delta, \wedge T_{W_i}(\delta), \vee I_{W_i}(\delta), \vee F_{W_i}(\delta), \wedge M_{W_i}(\delta)) : \delta\in Z\}$, where for all $i\in\Delta$ and W_i is an Ultra-N-C-S in (Z, τ) such that $W_i \supseteq N$.

$\Rightarrow$ (Ultra-N_{cl}(N))^c $= \{(\delta, \vee T_{W_i}(\delta), \wedge I_{W_i}(\delta), \wedge F_{W_i}(\delta), \vee M_{W_i}(\delta)) : \delta\in\Omega\}$.

Since, $\vee T_{W_i}(\delta) \geq T_N(\delta)$, $\wedge I_{W_i}(\delta) \geq I_N(\delta)$, $\wedge F_{W_i}(\delta) \leq F_N(\delta)$, $\vee M_{W_i}(\delta) \leq M_N(\delta)$, for each $i\in\Delta$ and $\delta\in Z$, so Ultra-N_{int}(N^c) $= \{(\delta, \vee T_{W_i}(\delta), \wedge I_{W_i}(\delta), \wedge F_{W_i}(\delta), \vee M_{W_i}(\delta)) : \delta\in Z\} = \cup\{W_i : i\in\Delta$ and W_i is an Ultra-N-O-S in (Z, τ) such that $W_i \subseteq N^c\}$. Therefore, (Ultra-N_{cl}(N))^c = Ultra-N_{int}(N^c).

Definition 3.7. Assume that (Z,τ) be an Ultra-N-T-S and K be an UNS over Z. Then K is said to be an Ultra Neutrosophic Semi Open (Ultra-N-S-O) Set if and only if
K$\subseteq$Ultra-N_{cl}(Ultra-N_{int}(K)).

Definition 3.8. Assume that (Z,τ) be an Ultra-N-T-S and K be an UNS over Z. Then K is said to be an Ultra Neutrosophic Pre Open (Ultra-N-P-O) Set if and only if K$\subseteq$ Ultra-N_{int}(Ultra-N_{cl}(K)).

Definition 3.9. Assume that (Z,τ) be an Ultra-N-T-S and K be an UNS over Z. Then K is said to be an Ultra Neutrosophic b-Open (Ultra-N-b-O) Set if and only if
K $\subseteq$ Ultra-N_{cl}(Ultra-N_{int}(K)) $\cup$ Ultra-N_{int}(Ultra-N_{cl}(K)).

An UNS H is said to be an Ultra Neutrosophic b-Closed (Ultra-N-b-C) Set iff H^c is an Ultra Neutrosophic b-Open Set.

Theorem 3.5. Assume that (Z, τ) be an Ultra-N-T-S. Then

i. Every Ultra-N-O-S is an Ultra-N-S-O set;
ii. Every Ultra-N-O-S is an Ultra-N-P-O set;
iii. Every Ultra-N-S-O set is an Ultra-N-b-O set;
iv. Every Ultra-N-P-O set is an Ultra-N-b-O set.

Proof.

i. Assume that (Z, τ) is an Ultra-N-T-S, and K is an Ultra-N-O-S in (Z,τ). So Ultra-N_{int}(K)=K. It is known that K$\subseteq$Ultra-N_{cl}(K). This implies, K$\subseteq$Ultra-N_{cl}(Ultra-N_{int}(K)). Therefore, K is an Ultra-N-S-O set in (Z, τ).

ii. Suppose that (Z, τ) is an Ultra-N-T-S. Assume that K is an Ultra-N-O-S in (Z,τ). Therefore, Ultra-N_{int}(K)=K. It is known that, K$\subseteq$ Ultra-N_{cl}(K).

Now, K $\subseteq$ Ultra-N_{cl}(K)

$\Rightarrow$ Ultra-N_{int}(K) $\subseteq$ Ultra-N_{int}(Ultra-N_{cl}(K))

$\Rightarrow$ K = Ultra-N_{int}(K) $\subseteq$ Ultra-N_{int}(Ultra-N_{cl}(K))

$\Rightarrow$ K $\subseteq$ Ultra-N_{int}(Ultra-N_{cl}(K))

Therefore, K is an Ultra-N-P-O set in (Z, τ).

iii. Suppose that (Z, τ) is an Ultra-N-T-S, and K is an Ultra-N-S-O set in (Z, τ). Therefore, K$\subseteq$ Ultra-N_{cl}(Ultra-N_{int}(K)). This implies, K $\subseteq$ Ultra-N_{cl}(Ultra-N_{int}(K)) $\cup$ Ultra-N_{int}(Ultra-N_{cl}(K)). Therefore, K is an Ultra-N-b-O set in (Z, τ).

iv. Assume that (Z, τ) is an Ultra-N-T-S. Suppose that K is an Ultra-N-P-O set in (Z, τ). So K$\subseteq$ Ultra-N_{int}(Ultra-N_{cl}(K)). This implies, K $\subseteq$ Ultra-N_{cl}(Ultra-N_{int}(K)) $\cup$ Ultra-N_{int}(Ultra-N_{cl}(K)). Therefore, K is an Ultra-N-b-O set in (Z, τ).

4. Conclusions:

In this paper, we introduce the ultra neutrosophic sets and investigate some of their basic properties. Also, we introduce the ultra neutrosophic topology, Ultra Neutrosophic interior, Ultra Neutrosophic closure. By defining Ultra Neutrosophic Set, Ultra Neutrosophic Topology, Ultra Neutrosophic interior and Ultra Neutrosophic closure, we formulate and prove some theorems on Ultra-N-T-Ss and few illustrative examples are provided. We hope that based on these notions in Ultra-N-T-Ss, many new investigations can be carried out in future.

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Complex Bipolar- Valued Neutrosophic Soft Set and its Decision Making Method

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Abstract: We establish the hybrid concept of complex bipolar- valued neutrosophic soft set (CBVNSS) as a hybrid model of bipolar neutrosophic soft set (BNSS)and complex fuzzy set (CFS). A CBVNSS is highly suitable for use in real life situations where the decision makers are interested to deal with bi-polarity as well as truth membership, indeterminacy membership and falsity membership grades to the alternatives in an extended range with complex numbers. Certain operations on CBVNSS like complement, subset, union and intersection operations are defined. Some related examples are also given to enhance the understanding of the proposed concept. The basic properties are also verified. We then provide a decision-making method on the CBVNSS. Finally, a numerical example has been presented to verify validity and feasibility of the developed method.

Keywords: neutrosophic set; complex neutrosophic set; neutrosophic soft set; complex neutrosophic soft set; bipolar neutrosophic soft set; decision making

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1. Introduction

Smarandache [1, 2] introduced the notation of neutrosophic set (NS) to examine and process the truth, indeterminate and false information simultaneously to help with making decisions. NS is one of the most effective techniques for presenting uncertainty and vagueness in decision making, which is the more generality of fuzzy set (FS) [3] and intuitionistic fuzzy set (IFS) [4]. Yet, in order to adapt these uncertainty sets with more real complex cases, complex fuzzy sets (CFSs) [5], complex intuitionistic fuzzy sets (CIFSs) [6] and complex neutrosophic sets(CNSs) [7] have been proposed accordingly. In CFS, the degree of membership is expressed by a complex-valued function where the amplitude term and the phase term are both real-valued functions. The CFSs are utilized to represent data with uncertainty and periodicity in the form of amplitude term which handles uncertainty and the phase term to represent periodicity. As a generalization of CFS, CIFS is traded by a complex-valued truth membership function which handles the uncertainty with periodicity and a complex-valued false membership function which handles the falsity with periodicity. In other words, CIFS is used to handle the intuitionistic fuzzy information that happen periodically. Some problems have imprecise, indeterminate, inconsistent, and incomplete information which appear in a periodic manner in our real life. These types of information cannot be handled by CFS and CIFS. To overcome this problem CNS has been introduced as a generalization of CFS and CIFS. A CNS is defined by a complex-valued truth membership function which represents uncertainty with periodicity, complex-valued indeterminacy membership function which represents indeterminacy with periodicity, and a complex-valued falsity membership function which represents falsity with periodicity.

A soft set (SS) is a set-valued map defined by Molodtsov [8], to approximately describe objects using several parameters. Both CNS [7] and neutrosophic soft set (NSS) [9] are improved and generalized models of the neutrosophic set but in different spaces. Complex neutrosophic set handles the neutrosophic data which has the periodic manner, while neutrosophic soft set provides a parameterization tool to hanle the neutrosophic data. Subsequently, these uncertainty sets have been actively applied in various decision making problems to address the uncertainty [10-20].

A wide variety of human decision making is based on double-sided or bipolar judgmental thinking on a positive side and a negative side. A great deal of research have been conducted to integrate the idea of bipolarity in decision making techniques by virtue of the uncertainty sets like fuzzy, intuitionistic fuzzy, complex fuzzy and complex neutrosophic sets [21- 31]. Bipolar complex neutrosophic set (BCNS) [22] and BNSS [28] are the most advanced methods which have the advantages of both bipolarity and neutrosophy which make them superior to all of the aforementioned uncertainty sets. However, the BCNS lacks the adequate parameterization tool to facilitate the representation of parameters. On the other hand, bipolar neutrosophic soft set lacks the phase terms of the complex numbers which have the ability to represent two-dimensional neutrosophic information side by side with the amplitude terms. Motivated by these results and as per our knowledge there is no work available on CBVNSS and its application. Accordingly, we introduce CBVNSS and its operations. CBVNSS is equipped with adequate parameterization. It also has the ability to handle the imprecise, indeterminate, inconsistent and incomplete information that is captured by the amplitude terms and phase terms of the complex numbers, simultaneously. The results of this paper also have been applied to solve a decision-making problem.

2. Preliminaries

We provide a brief overview of some concepts on NSs and CNSs. We begin by defining the concepts of NS [2], NSS [9] and BNSS [28].

Definition 1. [2] Let M be a universe. A NS S in M is defined as $S = \{< m;\ T_S(m),\ I_S(m)\ , F_S(m) >: m \in M\}$, where $T_S(m), I_S(m)$ and $F_S(m)$ are the truth, the indeterminacy and the falsity membership functions, such that $T, I, F: U \to]^{-}0, 1^{+}[$, and $0^{-} \leq T + I + F \leq 3^{+}$.

Definition 2. [9] If M is the universe and A is a set of parameters set. A pair (S, A) is called a NSS over M, where S is a mapping given by $S: A \to \rho(S)$, where $\rho(S\ )$ denotes the power neutrosophic set of M.

Definition 3. [28] Let M be a universe and E be a set of parameters. A BNSS B in M is defined as $B = \{< e, \{T^+(m), I^+(m), F^+(m), T^-(m), I^-(m), F^-(m)\} >: e \in E,\ m \in M\}$, where $T^+, I^+, F^+: M \to [0,1]$ and $T^-, I^-, F^-: M \to [-1,0]$. The membership degrees T^+, I^+, F^+ denote the truth membership, indeterminate membership and false membership of an element corresponding to a bipolar neutrosophic soft set B and the membership degrees T^-, I^-, F^- denote the truth membership,indeterminate membership and false membership of an element $m \in M$ to some implicit counter-property corresponding to a bipolar neutrosophic soft set B.

Now, we define the concepts of CNS and BCNS as follows.

Definition 4. [7] Let R be the universe. A complex neutrosophic set S in R is defined as $S = \{< r;\ T_s(r),\ I_s(r),\ F_s(r) >: r \in R\}$, where $T_s(r),\ I_s(r)$ and $F_s(r)$ are complex-valued truth, indeterminate and false membership functions and are of the form $T_s(r) = P_s(r), e^{j\mu_s(r)},\ I_s(r) = q_s(r), e^{ju_s(r)}$ and $F_s(r) = v_s(r), e^{j\omega_s(r)}$. By definition, $P_s(r), q_s(r), v_s(r)$ and $\mu_s(r),\ u(r),\ \omega_s(r)$ are, respectively, real valued and $P_s(r), q_s(r), v_s(r) \in [0,\ 1]$, such that $0^- \leq P_s(r) + q_s(r) + v_s(r) \leq 3^+$.

Definition 5. [22] A BCNS S in U is defined as:

$S = \{< u;\ p^+e^{i\mu^+},\ q^+e^{iv^+}, r^+e^{i\omega^+},\ p^-e^{i\mu^-},\ q^-e^{iv^-}, r^-e^{i\omega^-} >: u \in U\}$, where $p^+, q^+, r^+: U \to [0,1]$ and $p^-, q^-, r^-: U \to [-1,0]$.

A bipolar complex neutrosophic number can be represented as follows.

$S =< p^+e^{i\mu^+}, q^+e^{iv^+}, r^+e^{i\omega^+}, p^-e^{i\mu^-}, q^-e^{iv^-}, r^-e^{i\omega^-} >$.

3. Complex bipolar- valued neutrosophic soft set

Definition 6. Let X be a universe and A be a set of parameters. A complex bipolar- valued neutrosophic soft set (CBVNSS) (B, A) is defined as:

$(B, A) = \{< a, \{T^+_{B(a)}(x), I^+_{B(a)}(x), F^+_{B(a)}(x), T^-_{B(a)}(x), I^-_{B(a)}(x), F^-_{B(a)}(x)\} >: a \in A,\ x \in X\}$, where $\forall\, a \in A, \forall\, x \in X, T^+_{B(a)}(x) = P^+_{B(a)}(x)e^{2\pi i\mu^+_{B(a)}(x)}, I^+_{B(a)}(x) = q^+_{B(a)}(x)e^{2\pi iv^+_{B(a)}(x)}, F^+_{B(a)}(x) = r^+_{B(a)}(x)e^{2\pi i\omega^+_{B(a)}(x)}, T^-_{B(a)}(x) = P^-_{B(a)}(x)e^{2\pi i\mu^-_{B(a)}(x)}, I^-_{B(a)}(x) = q^-_{B(a)}(x)e^{2\pi iv^-_{B(a)}(x)}$, and $F^-_{B(a)}(x) = r^-_{B(a)}(x)e^{2\pi i\omega^-_{B(a)}(x)}$, such that :

$p^+, q^+, r^+, \mu^+, v^+, \omega^+: X \to [0,1]$ and $p^-, q^-, r^-, \mu^-, v^-, \omega^-: X \to [-1,0]$. The positive membership degrees T^+, I^+, F^+ denote, respectively the complex valued truth, indeterminacy, and falsity membership degrees of an element $x \in X$ to the property corresponding to a CBVNSS (B, A), and the negative membership degrees T^-, I^-, F^- are to denote the complex valued truth, indeterminacy, and falsity membership degrees of an element $x \in X$ to some implicit counter-property corresponding to a CBVNSS (B, A).

To illustrate the above definition, we provide the following example.

Example 1. Suppose $X = \{x_1, x_2\}$ is the universe and $A = \{a_1, a_2\}$ is the parameters set. Then the CBVNSS (B, A) is defined as below:

$(B, A) =$

$$\{< a_1, \left\{\frac{x_1}{<0{,}2\, e^{2\pi i(0{,}5)},\ 0{,}1\, e^{2\pi i(0{,}4)},\ 0{,}3\, e^{2\pi i(0{,}8)}, -0{,}2\, e^{2\pi i(-0{,}5)}, -0{,}8\, e^{2\pi i(-0{,}7)},\ -0{,}1\, e^{2\pi i(-0{,}2)}>}\right\},$$

$$\left\{\frac{x_2}{<0{,}9\, e^{2\pi i(0{,}7)},\ 0{,}2\, e^{2\pi i(0{,}5)},\ 0{,}4\, e^{2\pi i(0{,}1)}, -0{,}3\, e^{2\pi i(-0{,}6)}, -0{,}1\, e^{2\pi i(-0{,}5)},\ -0{,}4\, e^{2\pi i(-0{,}5)}>}\right\}>,$$

$$< a_2, \left\{\frac{x_1}{<0{,}5\, e^{2\pi i(0{,}6)},\ 0{,}4\, e^{2\pi i(0{,}3)},\ 0{,}1\, e^{2\pi i(0{,}5)}, -0{,}2\, e^{2\pi i(-0{,}7)}, -0{,}3\, e^{2\pi i(-0{,}4)},\ -0{,}2\, e^{2\pi i(-0{,}6)}>}\right\},$$

$$\left\{\frac{x_2}{<0{,}8\, e^{2\pi i(0{,}4)},\ 0{,}2\, e^{2\pi i(0{,}4)},\ 0{,}7\, e^{2\pi i(0{,}9)}, -0{,}9\, e^{2\pi i(-0{,}4)}, -0{,}8\, e^{2\pi i(-0{,}2)},\ -0{,}7\, e^{2\pi i(-0{,}5)}>}\right\}>\}.$$

In the following we define the empty CBVNSS and the the absolute CBVNSS.

Definition 7. Let (B, A) be a CBVNSS over M. Then (B, A) is said to be empty CBVNSS denoted by $B_\emptyset$, if $T^+_{B(a)}(m) = 0, I^+_{B(a)}(m) = 1, F^+_{B(a)}(m) = 1$ and $T^-_{B(a)}(m) = 0, I^-_{B(a)}(m) = -1, F^-_{B(a)}(m) = -1, \forall a \in A, \forall m \in M$ and defined as:

$$(B_\emptyset, A) = \{< a, \{0, 1, 1, 0, -1, -1\} >: a \in A, m \in M\}.$$

Definition 8. Let (B, A) be a CBVNSS over M. Then (B, A) is said to be absolute CBVNSS denoted by B_M, if $T^+_{B(a)}(m) = 1, I^+_{B(a)}(m) = 0, F^+_{B(a)}(m) = 0$ and $T^-_{B(a)}(m) = -1, I^-_{B(a)}(m) = 0, F^-_{B(a)}(m) = 0, \forall a \in A, \forall m \in M$ and defined as:

$$(B_M, A) = \{< a, \{1, 0, 0, -1{,}0{,}0\} >: a \in A, m \in M\}.$$

Now, we define the concept of the complement of the CBVNSS.

Definition 9. Let M be a universe of discourse and (B, A) be a CBVNSS on M, which is defined below:

$$(B, A) = \left\{< a, \left\{T^+_{B(a)}(m), I^+_{B(a)}(m), F^+_{B(a)}(m), T^-_{B(a)}(m), I^-_{B(a)}(m), F^-_{B(a)}(m)\right\} >: a \in A, \quad m \in M\right\}.$$

The complement of (B, A) is denoted by $(B, A)^c = (B^c, A)$ and is defined as:

$(B, A)^c = \{< a, \{T^+_{B^c(a)}(m), I^+_{B^c(a)}(m), F^+_{B^c(a)}(m), T^-_{B^c(a)}(m), I^-_{B^c(a)}(m), F^-_{B^c(a)}(m)\} >: a \in A,\ m \in M\}$, where

$T^+_{B^c(a)}(m) = P^+_{B^c(a)}(m)e^{2\pi i \mu^+_{B^c(a)}(m)} = r^+_{B(a)}(m)e^{2\pi i \omega^+_{B(a)}(m)}$, $I^+_{B^c(a)}(m) = q^+_{B^c(a)}(m)e^{2\pi i \nu^+_{B^c(a)}(m)} =$

$\left(1 - q^+_{B(a)}(m)\right) e^{2\pi i(1-\nu^+_{B(a)}(m))}$, $F^+_{B^c(a)}(m) = r^+_{B^c(a)}(m)e^{2\pi i \omega^+_{B^c(a)}(m)} = P^+_{B(a)}(m)e^{2\pi i \mu^+_{B(a)}(m)}$,

$T^-_{B^c(a)}(m) = P^-_{B^c(a)}(m)e^{2\pi i \mu^-_{B^c(a)}(m)} = r^-_{B(a)}(m)e^{2\pi i \omega^-_{B(a)}(m)}$, $I^-_{B^c(a)}(m) = q^-_{B^c(a)}(m)e^{2\pi i \nu^-_{B^c(a)}(m)} =$

$\left(-1 - q^-_{B(a)}(m)\right) e^{2\pi i(-1-\nu^-_{B(a)}(m))}$, $F^-_{B^c(a)}(m) = r^-_{B^c(a)}(m)e^{2\pi i \omega^-_{B^c(a)}(m)} = P^-_{B(a)}(m)e^{2\pi i \mu^-_{B(a)}(m)}$.

Example 2. Consider Example 1. By Definition 9, we get the complement of the CBVNSS (B, A) as:

$(B, A)^{c} =$

$$\{< a_1, \left\{\frac{x_1}{<0{,}3\, e^{2\pi i(0{,}8)},\ 0{,}9\, e^{2\pi i(0{,}6)},\ 0{,}2\, e^{2\pi i(0{,}5)}, -0{,}1\, e^{2\pi i(-0{,}2)}, -0{,}2\, e^{2\pi i(-0{,}3)},\ -0{,}2\, e^{2\pi i(-0{,}5)} >}\right\},$$

$$\left\{\frac{x_2}{<0{,}4\, e^{2\pi i(0{,}1)},\ 0{,}8\, e^{2\pi i(0{,}5)},\ 0{,}9\, e^{2\pi i(0{,}7)}, -0{,}4\, e^{2\pi i(-0{,}5)}, -0{,}9\, e^{2\pi i(-0{,}5)},\ -0{,}3\, e^{2\pi i(-0{,}6)} >}\right\},$$

$$a_2, \left\{\frac{x_1}{<0{,}1\, e^{2\pi i(0{,}5)},\ 0{,}6\, e^{2\pi i(0{,}2)},\ 0{,}5\, e^{2\pi i(0{,}6)}, -0{,}2\, e^{2\pi i(-0{,}6)}, -0{,}7\, e^{2\pi i(-0{,}6)},\ -0{,}2\, e^{2\pi i(-0{,}7)} >}\right\},$$

$$\left\{\frac{x_2}{<0{,}7\, e^{2\pi i(0{,}9)},\ 0{,}8\, e^{2\pi i(0{,}6)},\ 0{,}8\, e^{2\pi i(0{,}4)}, -0{,}7\, e^{2\pi i(-0{,}5)}, -0{,}2\, e^{2\pi i(-0{,}8)},\ -0{,}9\, e^{2\pi i(-0{,}4)} >}\right\} >\}.$$

Proposition 1. If (B, A) is a CBVNSS over the universe X, then $((B,\ A)^c)^c = (B,\ A)$.

Proof. The proof is straightforward from Definition 9. □

Now, we put forward the definition of the subset of two CBVNSSs.

Definition 10. For two CBVNSSs (B, A) and $(B',\ A')$ over a universe U, CBVNSS (B, A) is contained in CBVNSS $(B',\ A')$, denoted as $(B, A) \sqsubseteq (B',\ A')$ if:

(1) $A \subseteq A'$, and (2) $\forall a \in A$, $\forall m \in M$, $P^{+}_{B(a)}(m) \leq P^{+}_{B'(a)}(m)$, $q^{+}_{B(a)}(m) \geq q^{+}_{B'(a)}(m)$, $r^{+}_{B(a)}(m) \geq r^{+}_{B'(a)}(m)$, $\mu^{+}_{B(a)}(m) \leq \mu^{+}_{B'(a)}(m)$, $v^{+}_{B(a)}(m) \geq v^{+}_{B'(a)}(m)$, $\omega^{+}_{B(a)}(m) \geq \omega^{+}_{B'(a)}(m)$ and $P^{-}_{B(a)}(m) \geq P^{-}_{B'(a)}(m)$, $q^{-}_{B(a)}(m) \leq q^{-}_{B'(a)}(m)$, $r^{-}_{B(a)}(m) \leq r^{-}_{B'(a)}(m)$, $\mu^{-}_{B(a)}(m) \geq \mu^{-}_{B'(a)}(m)$, $v^{-}_{B(a)}(m)\, v^{-}_{B'(a)}(m)$, $\omega^{-}_{B(a)}(m) \leq \omega^{-}_{B'(a)}(m)$.

Definition 11. For two CBVNSSs (B, A) and $(B',\ A')$ over a universe M, (B, A) is equal to $(B',\ A')$ and it is denoted as $(B, A)=(B',\ A')$ if and only if $(B, A) \sqsubseteq (B',\ A')$ and $(B',\ A') \sqsubseteq (B, A)$.

We establish the definitions of the union and intersection of two CBVNSSs below.

Definition 12. Lex X be a universe. The union of two CBVNSSs (B, A) and $(B',\ A')$ denoted as

$(B, A) \sqcup (B',\ A')$ is a CBVNSS (C, D), where $D = A \cup A'$ and $\forall \epsilon \in D$, $\forall m \in M$,

$$T^{+}_{C(\epsilon)} = \begin{cases} P^{+}_{B(\epsilon)}(m)e^{2\pi i \mu^{+}_{B(\epsilon)}(m)} & if\ \epsilon \in A - A' \\ P^{+}_{B'(\epsilon)}(m)e^{2\pi i \mu^{+}_{B'(\epsilon)}(m)} & if\ \epsilon \in A' - A \\ \left(P^{+}_{B(\epsilon)}(m) \vee P^{+}_{B'(\epsilon)}(m) \right).e^{2\pi i(\mu^{+}_{B(\epsilon)}(m) \vee \mu^{+}_{B'(\epsilon)}(m))} & if\ \epsilon \in A \cap A' \end{cases}$$

$$I^{+}_{C(\epsilon)} = \begin{cases} q^{+}_{B(\epsilon)}(m)e^{2\pi i v^{+}_{B(\epsilon)}(m)} & if\ \epsilon \in A - A' \\ q^{+}_{B'(\epsilon)}(m)e^{2\pi i v^{+}_{B'(\epsilon)}(m)} & if\ \epsilon \in A' - A \\ \left(q^{+}_{B(\epsilon)}(m) \wedge q^{+}_{B'(\epsilon)}(m) \right).e^{2\pi i(v^{+}_{B(\epsilon)}(m) \wedge v^{+}_{B'(\epsilon)}(m))} & if\ \epsilon \in A \cap A' \end{cases}$$

$$F^{+}_{C(\epsilon)} = \begin{cases} r^{+}_{B(\epsilon)}(m)e^{2\pi i \omega^{+}_{B(\epsilon)}(m)} & if\ \epsilon \in A - A' \\ r^{+}_{B'(\epsilon)}(m)e^{2\pi i \omega^{+}_{B'(\epsilon)}(m)} & if\ \epsilon \in A' - A \\ \left(r^{+}_{B(\epsilon)}(m) \wedge r^{+}_{B'(\epsilon)}(m) \right).e^{2\pi i(\omega^{+}_{B(\epsilon)}(m) \wedge \omega^{+}_{B'(\epsilon)}(m))} & if\ \epsilon \in A \cap A' \end{cases}$$

$$T^{-}_{C(\epsilon)} = \begin{cases} P^{-}_{B(\epsilon)}(m)e^{2\pi i\mu^{-}_{B(\epsilon)}(m)} & if\ \epsilon \in A - A' \\ P^{-}_{B'(\epsilon)}(m)e^{2\pi i\mu^{-}_{B'(\epsilon)}(m)} & if\ \epsilon \in A' - A \\ \left(P^{-}_{B(\epsilon)}(m) \wedge P^{-}_{B'(\epsilon)}(m) \right). e^{2\pi i(\mu^{-}_{B(\epsilon)}(m) \wedge \mu^{-}_{B'(\epsilon)}(m))} & if\ \epsilon \in A \cap A' \end{cases}$$

$$I^{-}_{C(\epsilon)} = \begin{cases} q^{-}_{B(\epsilon)}(m)e^{2\pi i\nu^{-}_{B(\epsilon)}(m)} & if\ \epsilon \in A - A' \\ q^{-}_{B'(\epsilon)}(m)e^{2\pi i\nu^{-}_{B'(\epsilon)}(m)} & if\ \epsilon \in A' - A \\ \left(q^{-}_{B(\epsilon)}(m) \vee q^{-}_{B'(\epsilon)}(m) \right). e^{2\pi i(\nu^{-}_{B(\epsilon)}(m) \vee \nu^{-}_{B'(\epsilon)}(m))} & if\ \epsilon \in A \cap A' \end{cases}$$

$$F^{-}_{C(\epsilon)} = \begin{cases} r^{-}_{B(\epsilon)}(m)e^{2\pi i\omega^{-}_{B(\epsilon)}(m)} & if\ \epsilon \in A - A' \\ r^{-}_{B'(\epsilon)}(m)e^{2\pi i\omega^{-}_{B'(\epsilon)}(m)} & if\ \epsilon \in A' - A \\ \left(r^{-}_{B(\epsilon)}(m) \vee r^{-}_{B'(\epsilon)}(m) \right). e^{2\pi i(\omega^{-}_{B(\epsilon)}(m) \vee \omega^{-}_{B'(\epsilon)}(m))} & if\ \epsilon \in A \cap A' \end{cases}$$

Definition 13. Lex M be a universe. The intersection of two CBVNSSs (B, A) and $(B',\ A')$ denoted as $(B, A) \sqcap (B',\ A')$ is a CBVNSS (C, D), where $D = A\ \cup\ A'$ and $\forall \epsilon \in D,\ \forall m\ \in M,$

$$T^{+}_{C(\epsilon)} = \begin{cases} P^{+}_{B(\epsilon)}(m)e^{2\pi i\mu^{+}_{B(\epsilon)}(m)} & if\ \epsilon \in A - A' \\ P^{+}_{B'(\epsilon)}(m)e^{2\pi i\mu^{+}_{B'(\epsilon)}(m)} & if\ \epsilon \in A' - A \\ \left(P^{+}_{B(\epsilon)}(m) \wedge P^{+}_{B'(\epsilon)}(m) \right). e^{2\pi i(\mu^{+}_{B(\epsilon)}(m) \wedge \mu^{+}_{B'(\epsilon)}(m))} & if\ \epsilon \in A \cap A' \end{cases}$$

$$I^{+}_{C(\epsilon)} = \begin{cases} q^{+}_{B(\epsilon)}(m)e^{2\pi i\nu^{+}_{B(\epsilon)}(m)} & if\ \epsilon \in A - A' \\ q^{+}_{B'(\epsilon)}(m)e^{2\pi i\nu^{+}_{B'(\epsilon)}(m)} & if\ \epsilon \in A' - A \\ \left(q^{+}_{B(\epsilon)}(m) \vee q^{+}_{B'(\epsilon)}(m) \right). e^{2\pi i(\nu^{+}_{B(\epsilon)}(m) \vee \nu^{+}_{B'(\epsilon)}(m))} & if\ \epsilon \in A \cap A' \end{cases}$$

$$F^{+}_{C(\epsilon)} = \begin{cases} r^{+}_{B(\epsilon)}(m)e^{2\pi i\omega^{+}_{B(\epsilon)}(m)} & if\ \epsilon \in A - A' \\ r^{+}_{B'(\epsilon)}(m)e^{2\pi i\omega^{+}_{B'(\epsilon)}(m)} & if\ \epsilon \in A' - A \\ \left(r^{+}_{B(\epsilon)}(m) \vee r^{+}_{B'(\epsilon)}(m) \right). e^{2\pi i(\omega^{+}_{B(\epsilon)}(m) \vee \omega^{+}_{B'(\epsilon)}(m))} & if\ \epsilon \in A \cap A' \end{cases}$$

$$T^{-}_{C(\epsilon)} = \begin{cases} P^{-}_{B(\epsilon)}(m)e^{2\pi i\mu^{-}_{B(\epsilon)}(m)} & if\ \epsilon \in A - A' \\ P^{-}_{B'(\epsilon)}(m)e^{2\pi i\mu^{-}_{B'(\epsilon)}(m)} & if\ \epsilon \in A' - A \\ \left(P^{-}_{B(\epsilon)}(m) \vee P^{-}_{B'(\epsilon)}(m) \right). e^{2\pi i(\mu^{-}_{B(\epsilon)}(m) \vee \mu^{-}_{B'(\epsilon)}(m))} & if\ \epsilon \in A \cap A' \end{cases}$$

$$I^{-}_{C(\epsilon)} = \begin{cases} q^{-}_{B(\epsilon)}(m)e^{2\pi i\nu^{-}_{B(\epsilon)}(m)} & if\ \epsilon \in A - A' \\ q^{-}_{B'(\epsilon)}(m)e^{2\pi i\nu^{-}_{B'(\epsilon)}(m)} & if\ \epsilon \in A' - A \\ \left(q^{-}_{B(\epsilon)}(m) \wedge q^{-}_{B'(\epsilon)}(m) \right). e^{2\pi i(\nu^{-}_{B(\epsilon)}(m) \wedge \nu^{-}_{B'(\epsilon)}(m))} & if\ \epsilon \in A \cap A' \end{cases}$$

$$F^{-}_{C(\epsilon)} = \begin{cases} r^{-}_{B(\epsilon)}(m)e^{2\pi i\omega^{-}_{B(\epsilon)}(m)} & if\ \epsilon \in A - A' \\ r^{-}_{B'(\epsilon)}(m)e^{2\pi i\omega^{-}_{B'(\epsilon)}(m)} & if\ \epsilon \in A' - A \\ \left(r^{-}_{B(\epsilon)}(m) \wedge r^{-}_{B'(\epsilon)}(m) \right). e^{2\pi i(\omega^{-}_{B(\epsilon)}(m) \wedge \omega^{-}_{B'(\epsilon)}(m))} & if\ \epsilon \in A \cap A' \end{cases}$$

Theorem 1. If (B, A) and (B', A') are two CBVNSSs over the universe $\boldsymbol{X}$, then the union $(B, A) \sqcup (B', A')$ is the smallest CBVNSS which contains both these two sets.

Proof. The proof can be easily stated according to Definitions 10 and 12.

Theorem 2. If (B, A) and (B', A') are two CBVNSSs over the universe X, then the intersection $(B, A) \sqcap (B', A')$ is the largest CBVNSS which is contained in both of these two sets.

Proof. The proof can be easily stated according to Definitions 10 and 13.

Proposition 2. The following properties hold for the CBVNSSs (B, A) , (B', A') and (B'', A'').

1. $(B_{\emptyset}, A)^c = (B_X, A)$,
2. $(B_X, A)^c = (B_{\emptyset}, A)$,
3. $(B, A) \sqcup (B_{\emptyset}, A) = (B, A)$,
4. $(B, A) \sqcup (B_X, A) = (B_X, A)$,
5. $(B, A) \sqcap (B_{\emptyset}, A) = (B_{\emptyset}, A)$,
6. $(B, A) \sqcap (B_X, A) = (B, A)$,
7. $(B, A) \sqcup (B', A') = (B', A') \sqcup (B, A)$,
8. $(B, A) \sqcap (B', A') = (B', A') \sqcap (B, A)$,
9. $(B, A) \sqcup ((B', A') \sqcup (B'', A'')) = ((B, A) \sqcup (B', A')) \sqcup (B'', A'')$,
10. $(B, A) \sqcap ((B', A') \sqcap (B'', A'')) = ((B, A) \sqcap (B', A')) \sqcap (B'', A'')$,
11. $(B, A) \sqcup ((B', A') \sqcap (B'', A'')) = ((B, A) \sqcup (B', A')) \sqcap \big((B, A) \sqcup (B'', A'')\big)$,
12. $(B, A) \sqcap ((B', A') \sqcup (B'', A'')) = ((B, A) \sqcap (B', A')) \sqcup \big((B, A) \sqcap (B'', A'')\big)$,
13. $((B, A) \sqcup ((B', A'))^c = (B, A)^c \sqcap (B', A')^c$,
14. $((B, A) \sqcap ((B', A'))^c = (B, A)^c \sqcup (B', A')^c$.

Proof. The proof is straightforward by Definitions 9, 12 and 13.

4. Application of the CBVNSS in the decision making.

In this section, we establish an approach to decision making problem based on the CBVNSS model proposed in this paper.

In fact, all the existing approaches to decision making based on NSS and its extensions theory have solved kinds of decision problem effectively. In 2013, Maji [9] first give the decision method based on NSS theory by using the comparison matrix of the NSS to compute the scores of a set of alternatives. In the same year Broumi and Smarandache [32] also applied the intuitionistic neutrosophic soft set to solve the blouse purchase problem by computing the comparison matrix of the intuitionistic neutrosophic soft set. As an adaptation of the algorithm proposed in [32], Broumi et al. [33] developed an algorithm used the score function and the comparison matrix of CNSS to determine the country with the strongest economic indicators among a set of selected countries. On the other hand, Ali et al. [28] proposed an aggregation bipolar neutrosophic soft operator of a bipolar neutrosophic soft set and developed a decision making algorithm based on bipolar neutrosophic soft sets.

In the following, we establish a new approach to decision making based on the CBVNSS theory. In this approach, a modified algorithm and an accompanying score function is presented as an adaptation of the method proposed in [33], which was then made compatible with the structure of the CBVNSS model. To achieve this, we present the definitions of the comparison matrix of the CBVNSS and the score function as follows.

Definition 14. Assume that $(B, A) = \{< a, \{T^+_{B(a)}(m),\ I^+_{B(a)}(m),\ F^+_{B(a)}(m),\ T^-_{B(a)}(m),\ I^-_{B(a)}(m), F^-_{B(a)}(m)\ \} >: a \in A,\ m\ \in M\}$ is a CBVNSS, where $\forall\ a \in A,\ \forall\ m \in M$,

$$T^+_{B(a)}(m) = P^+_{B(a)}(m)e^{2\pi i\mu^+_{B(a)}(m)},\ I^+_{B(a)}(m) = q^+_{B(a)}(m)e^{2\pi i v^+_{B(a)}(m)},\ F^+_{B(a)}(m) = r^+_{B(a)}(m)e^{2\pi i\omega^+_{B(a)}(m)},$$

$$T^-_{B(a)}(m) = P^-_{B(a)}(m)e^{2\pi i\mu^-_{B(a)}(m)},\ I^-_{B(a)}(m) = q^-_{B(a)}(m)e^{2\pi i v^-_{B(a)}(m)},\ F^-_{B(a)}(m) = r^-_{B(a)}(m)\ e^{2\pi i\omega^-_{B(a)}(m)}.$$

Then the comparison matrix of (B, A) is a matrix whose rows comprise the alternatives variables m_1, m_2, $x_3, \dots m_k$ and the columns represent the parameters variables $a_1, a_2, a_3 \dots, a_n$. The entries μ_{ij} of this matrix are calculated as $\mu_{ij} = (\theta^+_{amp} + \sigma^+_{amp} - \tau^+_{amp}) + (\theta^+_{phase} + \sigma^+_{phase} - \tau^+_{phase}) -$

$(\theta^-_{amp} + \sigma^-_{amp} - \tau^-_{amp}) - (\theta^-_{phase} + \sigma^-_{phase} - \tau^-_{phase})$, where the components of this formula are as defined below for all $m_i, m_k \in M$, such that $i \neq k$.

θ^+_{amp} = The number of times $P^+_{B(a_j)}(m_i)$ exceeds or is equal to $P^+_{B(a_j)}(m_k)$,
σ^+_{amp}= The number of times $q^+_{B(a_j)}(m_i)$ exceeds or is equal to $q^+_{B(a_j)}(m_k)$,
τ^+_{amp} = The number of times $r^+_{B(a_j)}(m_i)$ exceeds or is equal to $r^+_{B(a_j)}(m_k)$,

θ^+_{phase} = The number of times $\mu^+_{B(a_j)}(m_i)$ exceeds or is equal to $\mu^+_{B(a_j)}(m_k)$,
σ^+_{phase}= The number of times $v^+_{B(a_j)}(m_i)$ exceeds or is equal to $v^+_{B(a_j)}(m_k)$,
τ^+_{phase} = The number of times $\omega^+_{B(a_j)}(m_i)$ exceeds or is equal to $\omega^+_{B(a_j)}(m_k)$,

θ^-_{amp} = The number of times $P^-_{B(a_j)}(m_i)$ exceeds or is equal to $P^-_{B(a_j)}(m_k)$,
σ^-_{amp}= The number of times $q^-_{B(a_j)}(m_i)$ exceeds or is equal to $q^-_{B(a_j)}(m_k)$,
τ^-_{amp} = The number of times $r^-_{B(a_j)}(m_i)$ exceeds or is equal to $r^-_{B(a_j)}(m_k)$,
and
θ^-_{phase} = The number of times $\mu^-_{B(a_j)}(m_i)$ exceeds or is equal to $\mu^-_{B(a_j)}(m_k)$,
σ^-_{phase}= The number of times $v^-_{B(a_j)}(m_i)$ exceeds or is equal to $v^-_{B(a_j)}(m_k)$,
τ^-_{phase} = The number of times $\omega^-_{B(a_j)}(m_i)$ exceeds or is equal to $\omega^-_{B(a_j)}(m_k)$.

Definition 15. Score of the complex bipolar-valued neutrosophic element x_i in the universe X is calculated using the score function $\mathcal{R}_i$ as $\mathcal{R}_i = \sum_j \mu_{ij}$.

Example 3. Consider that an automobile manufacturer produces three models of cars, where $X = \{ x_1, x_2, x_3 \}$ represents the set of alternative models. Suppose that the manufacturer wants to examine these three models of cars two times, once before and again after trying a sample of each model of these cars. Then decide to make a decision about the most desirable model of these cars. Suppose that there are three attributes have been approved in this decision making process, where a_1 stands for comfortability, a_2 stands for reliability and a_3 stands for durability.

In this context, the CBVNSS has been applied such that the amplitude terms represent the decision information in first stage (before testing the cars), whereas the phase terms represent the decision information in second stage (after testing the cars. On the other hand, the positive decision information denote the complex valued membership degrees of an element $x \in X$ to the attribute corresponding to a CBVNSS, and the negative decision information denote the complex valued membership degrees of an element $x \in X$ to some implicit counter-attribute corresponding to a CBVNSS. All of these different types of valuable information can be expressed using the CBVNSS (B, A) as follows.

$$(B,A) =$$

$$\{< a_1, \left\{\frac{x_1}{<0{,}5\, e^{2\pi i(0,7)},\ 0{,}1\, e^{2\pi i(0,2)},\ 0{,}1\, e^{2\pi i(0,2)},-0{,}4\, e^{2\pi i(-0,5)},-\ 0{,}8\, e^{2\pi i(-0,7)},\ -0{,}2\, e^{2\pi i(-0,2)}>}\right\},$$

$$\left\{\frac{x_2}{<0{,}8\, e^{2\pi i(0,7)},\ 0{,}4\, e^{2\pi i(0,5)},\ 0{,}1\, e^{2\pi i(0,1)},-0{,}1\, e^{2\pi i(-0,3)},-\ 0{,}1\, e^{2\pi i(-0,5)},\ -0{,}8\, e^{2\pi i(-0,7)}>}\right\},$$

$$\left\{\frac{x_3}{<0{,}1\, e^{2\pi i(0,3)},\ 0{,}5\, e^{2\pi i(0,4)},\ 0{,}8\, e^{2\pi i(0,8)},-0{,}9\, e^{2\pi i(-0,8)},-\ 0{,}3\, e^{2\pi i(-0,4)},\ -0{,}1\, e^{2\pi i(-0,2)}>}\right\}>,$$

$$< a_2, \left\{\frac{x_1}{<0{,}3\, e^{2\pi i(0,4)},\ 0{,}3\, e^{2\pi i(0,4)},\ 0{,}7\, e^{2\pi i(0,8)},-0{,}7\, e^{2\pi i(-0,5)},-\ 0{,}5\, e^{2\pi i(-0,7)},\ -0{,}6\, e^{2\pi i(-0,7)}>}\right\},$$

$$\left\{\frac{x_2}{<0{,}9\, e^{2\pi i(0,8)},\ 0{,}1\, e^{2\pi i(0,3)},\ 0{,}2\, e^{2\pi i(0,4)},-0{,}1\, e^{2\pi i(-0,4)},-\ 0{,}2\, e^{2\pi i(-0,3)},\ -0{,}1\, e^{2\pi i(-0,2)}>}\right\},$$

$$\left\{\frac{x_3}{<0{,}2\, e^{2\pi i(0,3)},\ 0{,}4\, e^{2\pi i(0,3)},\ 0{,}5\, e^{2\pi i(0,6)},-0{,}4\, e^{2\pi i(-0,5)},-\ 0{,}5\, e^{2\pi i(-0,6)},\ -0{,}6\, e^{2\pi i(-0,7)}>}\right\}>,$$

$$< a_3, \left\{\frac{x_1}{<0{,}4\, e^{2\pi i(0,3)},\ 0{,}1\, e^{2\pi i(0,2)},\ 0{,}9\, e^{2\pi i(0,7)},-0{,}5\, e^{2\pi i(-0,5)},-\ 0{,}2\, e^{2\pi i(-0,5)},\ -0{,}7\, e^{2\pi i(-0,6)}>}\right\},$$

$$\left\{\frac{x_2}{<0{,}7\, e^{2\pi i(0,6)},\ 0{,}3\, e^{2\pi i(0,5)},\ 0{,}4\, e^{2\pi i(0,3)},-0{,}3\, e^{2\pi i(-0,4)},-\ 0{,}1\, e^{2\pi i(-0,3)},\ -0{,}5\, e^{2\pi i(-0,6)}>}\right\},$$

$$\left\{\frac{x_3}{<0{,}3\, e^{2\pi i(0,5)},\ 0{,}8\, e^{2\pi i(0,5)},\ 0{,}7\, e^{2\pi i(0,6)},-0{,}7\, e^{2\pi i(-0,5)},-\ 0{,}8\, e^{2\pi i(-0,7)},\ -0{,}4\, e^{2\pi i(-0,2)}>}\right\}>\}.$$

Now, our problem is to determine the most desirable model of cars to the manufacturer.
To solve this decision making problem, we use the above CBVNSS along with a modified algorithm adopted from [33] as follows.

Step 1 : Input the CBVNSS(B, A) .

Step 2: Compute the CBVNSS comparison matrix using the formula given in Definition 14.

Step 3: Compute the score $\mathcal{R}_i$ for each alternative x_i in the universe X using Definition 15.

Step 4: Choose the alternative with the maximum score as the optimal alternative, If there are more than one alternative have the maximum score, anyone can be chosen as the optimal alternative.

Now, we apply the above algorithm to solve our decision making problem.

The comparison matrix of the CBVNSS (B, A) is constructed as in Table 1.

Table 1. Comparison matrix of the CBVNSS (B, A)

Attributes	Alternatives		
	x_1	x_2	x_3
a_1	2	-1	0
a_2	1	1	-1
a_3	-6	1	6

Next, we compute the score values $\mathcal{R}_i$ for each alternative x_i as shown in Table 2.

Table 2. Score values of the alternatives x_i

x_i	x_1	x_2	x_3
$\mathcal{R}_i$	-3	1	5

Clearly the maximum score is 5. Thus, the decision is to choose the alternative x_3 as the optimal Solution. Therefore, we conclude that model x_3 is the most desirable model of cars, followed by model x_2 and model x_1.

Conclusion

We established the notion of CBVNSS as an extension of BNSS by extending its range from the real space to the complex space. The formal definition of the CBVNSS is stated based on the definitions of the bipolar complex neutrosophic set and soft set. Some essential operations such as complement, subset, union and intersection with their properties are defined and verified. A modified algorithm has been presented and its decision steps constructed. It was shown to be workable and successful in producing a desired result as illustrated by the application in decision making. CBVNSS seems to be a promising new concept, open the way toward various future research. We intend to study this model further to conduct some real applications which involve bipolarity, uncertainty and periodicity simultaneously in the field of physics, signal processing, stock marketing, and so on.

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New Generalized Closed Set in ℵeutrosophic Topological Spaces.

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Abstract: The main intention of this paper is to develop the idea of ℵeutrosophic Semi-generalized pre-closed set in neutrosophic topological space. We also study relations and some properties between the existing ℵeutrosophic closed set. The examples are provided wherever necessary. Besides, we discuss some applications of ℵeutrosophic Semi-generalized pre closed set.

Keywords: ℵsgp-closed set,ℵsgp-open set, ℵ-sgp-$T_{1/2}$, ℵ-pc-$T_{1/2}$,ℵ-open set , ℵ-closed set.

1. Introduction

Fuzzy set theory is introduced and studied as a mathematical tool for dealing with uncertainties where each element had a degree of membership, truth(t), by Zadeh[14]. The falsehood (f), the degree of non-membership, was introduced by Atanassov [2] in an intuitionistic fuzzy set. Coker [3] developed intuitionistic fuzzy topology. Neutrality (i), the degree of indeterminacy, as an independent concept, was introduced by Smarandache[8,9,10]. He also defined the neutrosophic set on three components (t, f, i) = (truth, falsehood, indeterminacy). Salama et.al. [6,7] converted Neutrosophic crisp set in to neutrosophic topological spaces. This opened a wide range of investigation in terms of neutrosophic topology and its application in decision making problems. A.A. Salama et al in [7] introduced neutrosophic closed sets and continuous functions. R. Dhavaseelan et al [4] introduced generalized neutrosophic closed sets. Neutrosophic semi-open, pre-open, α-open and semipro-open are presented in [11]. In [13]authors discussed properties of Generalized pre-closed sets in neutrosophic topological space(NTS in short).

This paper is devoted to the study new generalized closed set in Neutrosophic topology called Neutrosophic semi-generalized pre closed set. The basic properties are discussed and compared the new set with existing neutrosophic closed sets. As its applications, we have defined as ℵeutrosophic-sgp-$T_{1/2}$ and as ℵeutrosophic-pc-$T_{1/2}$.

2. Preliminaries

Definition: 2.1[8,9]: Let S_1 be a non-empty fixed set. A neutrosophic set (in short NS) Λ is an object such that $\Lambda = \{\langle x, \mu_\Lambda(x), \sigma_\Lambda(x), \gamma_\Lambda(x)\rangle : x \in S_1\}$ wherein $\mu_\Lambda(x), \sigma_\Lambda(x)$ and $\gamma_\Lambda(x)$ which represents the degree of membership function (viz $\mu_\Lambda(x)$), the degree of indeterminacy (viz $\sigma_\Lambda(x)$) as well as the degree of non-membership (viz $\gamma_\Lambda(x)$) respectively of each element $x \in S_1$ to the set Λ.

Remark: 2.2[8,9]: (i) An N-set $\Lambda = \{\langle x, \mu_\Lambda(x), \sigma_\Lambda(x), \Gamma_\Lambda(x)\rangle : x \in S_1\}$ can be identified to an ordered triple $\langle \mu_\Lambda, \sigma_\Lambda, \Gamma_\Lambda\rangle$ in $]0^-, 1^+[$ on S_1.

(ii) In this paper, we use the symbol $\Lambda = \langle \mu_\Lambda, \sigma_\Lambda, \Gamma_\Lambda\rangle$ for the N-set $\Lambda = \{\langle x, \mu_\Lambda(x), \sigma_\Lambda(x), \Gamma_\Lambda(x)\rangle : x \in S_1\}$.

Definition: 2.3*[8,9]:* Let $S_1 \neq \emptyset$ and the N-sets Λ and Γ be defined as

$\Lambda = \{\langle x, \mu_\Lambda(x), \sigma_\Lambda(x), \Gamma_\Lambda(x)\rangle : x \in S_1\}$, $\Gamma = \{\langle x, \mu_\Gamma(x), \sigma_\Gamma(x), \Gamma_\Gamma(x)\rangle : x \in S_1\}$. Then

I. $\Lambda \subseteq \Gamma$ *iff* $\mu_\Lambda(x) \leq \mu_\Gamma(x)$, $\sigma_\Lambda(x) \leq \sigma_\Gamma(x)$ *and* $\Gamma_\Lambda(x) \geq \Gamma_\Gamma(x)$ *for all* $x \in S_1$;

II. $\Lambda = \Gamma$ *iff* $\Lambda \subseteq \Gamma$ *and* $\Gamma \subseteq \Lambda$;

III. $\bar{\Lambda} = \{\langle x, \Gamma_\Lambda(x), \sigma_\Lambda(x), \mu_\Lambda(x)\rangle : x \in S_1\}$; *[Complement of* Λ*]*

IV. $\Lambda \cap \Gamma = \{\langle x, \mu_\Lambda(x) \wedge \mu_\Gamma(x), \sigma_\Lambda(x) \wedge \sigma_\Gamma(x), \Gamma_\Lambda(x) \vee \Gamma_\Gamma(x)\rangle : x \in S_1\}$;

V. $\Lambda \cup \Gamma = \{\langle x, \mu_\Lambda(x) \vee \mu_\Gamma(x), \sigma_\Lambda(x) \vee \sigma_\Gamma(x), \Gamma_\Lambda(x) \wedge \gamma_\Gamma(x)\rangle : x \in S_1\}$;

VI. $[\ ]\Lambda = \{\langle x, \mu_\Lambda(x), \sigma_\Lambda(x), 1 - \mu_\Lambda(x)\rangle : x \in S_1\}$;

VII. $\langle\rangle\, \Lambda = \{\langle x, 1 - \Gamma_\Lambda(x), \sigma_\Lambda(x), \Gamma_\Lambda(x)\rangle : x \in S_1\}$.

Definition: 2.4[9,10]: Let $\{\Lambda_i : i \in J\}$ be an arbitrary family of N-sets in S_1. Thereupon

I. $\cap \Lambda_i = \{\langle p \wedge \mu_{\Lambda_i}(p), \wedge \sigma_{\Lambda_i}(p), \vee \Gamma_{\Lambda_i}(p)\rangle : p \in S_1\}$;

II. $\cup \Lambda_i = \{\langle p \vee \mu_{\Lambda_i}(p), \vee \sigma_{\Lambda_i}(p), \wedge \Gamma_{\Lambda_i}(p)\rangle : p \in S_1\}$.

The main theme is to construct the tools for developing NTS, so we establish the neutrosophic sets $0_\aleph$ along with $1_\aleph$ in X as follows:

Definition: 2.5[9,10]:$0_\aleph = \{\langle q, 0,0,1\rangle : q \in X\}$ and $1_\aleph = \{\langle q, 1,1,0\rangle : q \in X\}$.

Definition: 2.6[7]: A neutrosophic topology (in short, $\aleph T$)$S_1 \neq \emptyset$ is a family ξ_1 of N-sets in S_1 satisfying the laws given below:

I. $0_\aleph, 1_\aleph \in \xi_1$,

II. $W_1 \cap W_2 \in T$ *being* $W_1, W_2 \in \xi_1$,

III. $\cup W_i \in \xi_1$ *for arbitrary family* $\{W_i | i \in \Lambda\} \subseteq \xi_1$.

In this case the ordered pair (S_1, ξ_1) or simply S_1 is termed as NTS and each NS in ξ_1 is named as neutrosophic open set (in short, ℵOS) . The complement $\overline{\Lambda}$ of an ℵ-open set Λ in S_1 is known as neutrosophic closed set (briefly,ℵCS) in S_1.

Definition: 2.7[7,8]: Let Λ be an NS in an NTS S_1. Thereupon

$\aleph int(\Lambda)$ $=\cup \{G | G$ is an ℵOS in S_1 and $G \subseteq \Lambda\}$ is termed as neutrosophic interior (in brief $\aleph int$) of Λ;

$\aleph cl(\Lambda) = \cap \{G|G$ is an ℵCS in S_1 and $G \supseteq \Lambda\}$ is termed as neutrosophic closure (shortly $\aleph cl$) of Λ.

Definition: 2.8[4]: Let X be a nonempty set. Whenever r, t, s be real standard or non standard subsets of $]0^-, 1^+[$ then the neutrosophic set $x_{r,t,s}$ is termed as neutrosophic point(in short NP)in X given by $x_{r,t,s}(x_p) = \begin{cases} (r,t,s), & if \quad x = x_p \\ (0,0,1), & if \quad x \neq x_p \end{cases}$ for $x_p \in X$ is termed as the support of $x_{r,t,s}$, wherein r indicates the degree of membership value, t indicates the degree of indeterminacy along with s as the degree of non-membership value of $x_{r,t,s}$.

Definition 2.9[11]: For an ℵS D in an NTS (X, T). We have,

(i) ℵeutrosophic semiopen set (ℵSOS) if $D \subseteq \aleph cl(\aleph int(D))$.

(ii) ℵeutrosophic preopen set (ℵPOS) if $D \subseteq \aleph int(\aleph cl(D))$.

(iii) ℵeutrosophic α-open set (ℵαOS) if $D \subseteq \aleph int(\aleph cl(\aleph int(D)))$.

(iv) ℵeutrosophic semi-preopen (ℵSPOS) if $D \subseteq \aleph cl(\aleph int(\aleph cl(D)))$.

The complement of D is an ℵSOS, ℵPOS,ℵαOS, ℵSPOS is called respectively as ℵSCS, ℵPCS, ℵαCS, ℵSPCS

Definition 2.10[13]: Let (X,T) be an NTS then neutrosophic pre-closure of D (in short, pℵCl(D)) is defined as

(i) $p\aleph Cl(D) = \cap\{K: K\ is\ an\ NPC\ in\ T, D \subseteq K\}$.

(ii) $p\aleph Int(D) = \cup\{Q: K\ is\ an\ NPO\ in\ T, D \subseteq Q\}$.

Definition 2.11 [13]: An NS is said to be a neutrosophic generalized pre-closed set (GNPCS in short) in (X, T) if $p\aleph Cl(R) \subseteq Q$ whenever $R \subseteq Q$ and Q is a NOS in (X,T).

Definition 2.12 [13]: AnℵTS (X, S) is named as ℵeutrosophic-gp-$T_{1/2}$ (ℵgp-$T_{1/2}$ in short) space if every GNPCS in X is a ℵPCS.

3. Neutrosophic Semi-Generalized-Pre-Closed Sets.

Definition 3.1: An NS μ of NTS(X, S) is termed as Neutrosophic Semigeneralized pre-closed set (ℵsgp-CS in short) if pℵCl(μ) ⊆ η whenever μ ⊆ η and η is ℵSOS in X.

Definition 3.2: Let (X, S) be a NTS and η be an NS in X. Then the neutrosophic semigeneralized pre-closure and neutrosophic semigeneralized pre-interior of η are deonoted and defined by,

$\aleph sgpCl(\eta) = \cap\{\ \lambda$: λ is a ℵsgp-CS in X and $\eta \subseteq \lambda\}$

$\aleph sgpInt(\eta) = \cup\{\ \lambda$: λ is a ℵsgp-OS in X and $\eta \supseteq \lambda\}$

Proposition 3.3: Consider(X, S) be any NTS and A and B be neutrosophic sets in (X, S). Then the ℵsg-closure and ℵsgp-interior operator satisfy the following properties

i. $\eta \subseteq \aleph sgpCl(\eta)$
ii. $\aleph sgpInt(\eta) \subseteq \eta$
iii. $\eta \subseteq \lambda \Rightarrow \aleph sgpCl(\eta) \subseteq \aleph sgpCl(\lambda)$
iv. $\eta \subseteq \lambda \Rightarrow \aleph sgpInt(\eta) \subseteq \aleph sgpInt(\lambda)$
v. $\aleph sgpCl(\eta \cup \lambda) = \aleph sgpCl(\eta) \cup \aleph sgpCl(\lambda)$
vi. $\aleph sgpInt(\eta \cap \lambda) = \aleph sgpInt(\eta) \cap \aleph sgpInt(\lambda)$
vii. $\aleph \overline{sgpCl(\eta)} = \aleph sgpInt(\bar{\eta})$
viii. $\aleph \overline{shpInt(\eta)} = \aleph sgpCl(\bar{\eta})$

Proposition 3.4: Each ℵCS set is ℵsgp-CS.

Proof: Let μ is ℵCS such that μ ⊆ η and η is ℵSOS in X. As μ is ℵCS, μ = ℵcl(μ). Hence ℵcl(μ) ⊆ μ. But pℵCl(μ) ⊆ ℵcl(μ), therefore pℵCl(μ) ⊆ μ whenever μ ⊆ η and η is ℵSOS in X. Therefore μ is ℵsgp-CS.

Remark 3.5: The example makes clear that converse of the above proposition is not true.

Example 3.6: Let $X = \{a, b, c\}$. Define the neutrosophic sets A,B and C in X as follows A = $\langle x, \left(\frac{a}{0.6}, \frac{b}{0.6}, \frac{c}{0.6}\right), \left(\frac{a}{0.7}, \frac{b}{0.7}, \frac{c}{0.7}\right), \left(\frac{a}{0.3}, \frac{b}{0.3}, \frac{c}{0.3}\right)\rangle$, B = $\langle x, \left(\frac{a}{0.4}, \frac{b}{0.4}, \frac{c}{0.5}\right), \left(\frac{a}{0.4}, \frac{b}{0.4}, \frac{c}{0.5}\right), \left(\frac{a}{0.5}, \frac{b}{0.5}, \frac{c}{0.5}\right)\rangle$ and C = $\langle x, \left(\frac{a}{0.4}, \frac{b}{0.4}, \frac{c}{0.6}\right), \left(\frac{a}{0.4}, \frac{b}{0.4}, \frac{c}{0.6}\right), \left(\frac{a}{0.5}, \frac{b}{0.5}, \frac{c}{0.5}\right)\rangle$. Then the families $\tau = \{0_N, 1_N, A, B\}$ is topology on X and the space (X, τ) is NTS. Then C is a ℵsgp-CS but not a ℵCS.

Proposition 3.7: Every $\aleph\alpha$CS is ℵsgp-CS.

Proof: Let W is ℵCS such that W ⊆ Q and Q is ℵSOS in X. Since W is $\aleph\alpha$CS, W = $\aleph\alpha$cl(μ). Hence $\aleph\alpha$cl(W) ⊆ Q. But pℵCl(W) ⊆ $\aleph\alpha$cl(W), therefore pℵCl(W) ⊆ W whenever W⊆ Q and Q is ℵSOS in X. Therefore μ is ℵsgp-CS.

Remark 3.8: Converse of the above proposition is not true as seen below.

Example 3.9: Consider $X = \{\eta, \beta, \delta\}$. Define the neutrosophic sets P, Q and R in X as follows P = $\langle x, \left(\frac{\eta}{0.6}, \frac{\beta}{0.6}, \frac{\delta}{0.6}\right), \left(\frac{\eta}{0.7}, \frac{\beta}{0.7}, \frac{\delta}{0.7}\right), \left(\frac{\eta}{0.3}, \frac{\beta}{0.3}, \frac{\delta}{0.3}\right)\rangle$, Q = $\langle x, \left(\frac{\eta}{0.4}, \frac{\beta}{0.4}, \frac{\delta}{0.5}\right), \left(\frac{\eta}{0.4}, \frac{\beta}{0.4}, \frac{\delta}{0.5}\right), \left(\frac{\eta}{0.5}, \frac{\beta}{0.5}, \frac{\delta}{0.5}\right)\rangle$ and R = $\langle x, \left(\frac{\eta}{0.4}, \frac{\beta}{0.4}, \frac{\delta}{0.5}\right), \left(\frac{\eta}{0.4}, \frac{\beta}{0.4}, \frac{\delta}{0.5}\right), \left(\frac{\eta}{0.5}, \frac{\beta}{0.5}, \frac{\delta}{0.5}\right)\rangle$. Then the families $\tau = \{0_N, 1_N, P, Q\}$ is topology on X and the space (X, τ) is NTS. Then C is a ℵsgp-CS but not a $\aleph\alpha$CS.

Proposition 3.10: Each ℵPCS is ℵsgp-CS.

Proof: Consider Jas ℵCS with J ⊆ K and K is ℵSOS in X. Since W is ℵPCS, J = pℵCl(J). Hence $p\aleph C$l(J) ⊆ J whenever J ⊆ K and K is ℵSOS in X. Therefore J is ℵsgp-CS.

Remark 3.11: The example shows that the reverse implication of above proposition is not possible.

Example 3.12: For $X = \{p, q, r\}$ the neutrosophic sets K, M and L in X are defined as $A = \langle x, \left(\frac{p}{0.6}, \frac{q}{0.6}, \frac{r}{0.6}\right), \left(\frac{p}{0.7}, \frac{q}{0.7}, \frac{r}{0.7}\right), \left(\frac{p}{0.3}, \frac{q}{0.3}, \frac{r}{0.3}\right)\rangle$, $M = \langle x, \left(\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.5}\right), \left(\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.5}\right), \left(\frac{p}{0.5}, \frac{q}{0.5}, \frac{r}{0.5}\right)\rangle$ and $L = \langle x, \left(\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.6}\right), \left(\frac{p}{0.4}, \frac{q}{0.4}, \frac{r}{0.6}\right), \left(\frac{p}{0.5}, \frac{q}{0.5}, \frac{r}{0.5}\right)\rangle$. Then the families $\tau = \{0_N, 1_N, K, M\}$ is topology on X and the space (X, τ) is NTS. Thereupon C is a ℵsgp-Closed set but not a ℵPCS

Proposition 3.13: Every is ℵsgp-CS is GNPCS.

Proof: Let J is ℵsgp-CS with $J \subseteq$ K and K is ℵOS in X. Since eachℵOS is ℵSOS , K is ℵSOS such that J $\subseteq$ K. By definition 3.1, $p\aleph Cl(J) \subseteq K$ whenever J $\subseteq$ K and K is ℵOS in X. Therefore J is GNPCS.

Proposition 3.14: Let D be a ℵsgp-CS in an ℵTS (X, S) and $D \subseteq E \subseteq p\aleph Cl(D)$. Thereupon E is ℵsgp-CS in X.

Proof: Consider G be a ℵSOS in X thereby $E \subseteq G$. Then $D \subseteq G$ and since D is ℵsgp-CS, $p\aleph Cl(D) \subseteq G$. Now $E \subseteq p\aleph Cl(D)$implies $\aleph cl(E) \subseteq p\aleph Cl(D) \subseteq G$. Consequently E is ℵsgp-CS in X.

Definition 3.15: An NS B of a ℵTS (X,S) is named as neutrosophic semi-generalized open set (ℵsgp-OS in short) if and only if $\bar{B}$ is ℵsgp-CS.

Remark 3.16: For any two Neutrosophic Sets A and B of NTS(X, S). Then
i). A is a $\aleph\ closed$ set iff $\aleph cl(A) = A$.
ii). A is a $\aleph\ open$ set iff $\aleph int(A) = A$.
iii). $\aleph cl(\bar{A}) = \aleph \overline{int}(\bar{A})$
iv). $\aleph int(\bar{A}) = \aleph \overline{cl}(\bar{A})$

Proposition 3.17: An NS F of a ℵTS (X, S) is ℵsgp-OS if $G \subseteq p\aleph Cl(F)$ whenever G is ℵSCS and $G \subseteq F$.

Proof: Follows from definition 3.1 and remark 3.16.

Proposotion 3.18: Let A be a ℵsgp-OS in a ℵTS (X, S) and $p\aleph Int(A) \subseteq B \subseteq A$. Then B is ℵsgp-OS.
Proof: Suppose A is ℵsgp-OS in X and $p\aleph Int(A) \subseteq B \subseteq A$ implies $\bar{A} \subseteq \overline{B} \subseteq (p\aleph \overline{Int}(A))$ implies $\bar{A} \subseteq \overline{B} \subseteq p\aleph Cl(\bar{A})$. Then B is ℵsgp-OS.

4. Applications of Neutrosophic Semi Generalized Pre Closed Set.

Definition 4.1: An ℵTS (X, S) is named as ℵeutrosophic-sgp-$T_{1/2}$ (in short ℵ-sgp-$T_{1/2}$) space if every ℵsgp-CS in X is a ℵCS.

Definition 4.2: An ℵTS (X, S) is named as ℵeutrosophic-pc-$T_{1/2}$ (in short ℵ-pc-$T_{1/2}$) space if every ℵsgp-CS in X is a ℵPCS.

Proposition 4.3: Each ℵ-pc-$T_{1/2}$ space is ℵ-sgp-$T_{1/2}$.

Proof: Consider X to be a ℵ-pc-$T_{1/2}$ space and G be ℵsgp-CS in X. By assumption, G is ℵPCS in X. Since every ℵPCS is ℵsgp-CS , G is ℵsgp-CS in X. Hence, X is ℵ-pc-$T_{1/2}$.

Proposition 4.5: Each ℵ-sgp-$T_{1/2}$ is Ngp$T_{1/2}$.

Proof: Consider X to be a ℵ-sgp-$T_{1/2}$ space and Q be GNPCS in X. By assumption, Q is ℵsgp-CS in X. Since every ℵsgp-CS is GNPCS, Q is GNPCS in X. Hence, X is Ngp-$T_{1/2}$.

Proposition 4.6: Let (X, S) be a ℵTS and ℵ-sgp-$T_{1/2}$. Then the following statements hold.

(i) Any union of ℵsgp-CS is a ℵsgp-CS.

(ii) Any intersection of ℵsgp-CS is a ℵsgp-CS.

Proof:(i) Let $\{B_i\}_{i\epsilon j}$ be a collection of ℵsgp-CS in a ℵ-sgp-$T_{1/2}$ space (X, S). Therefore every ℵsgp-CS is ℵCS. However, the union of ℵCS is a ℵCS. Hence the union of ℵsgp-CS is ℵsgp-CS in X.

(ii) It can be proved by taking complement in (i).

Proposition 4.7: An NTS X is an ℵ-sgp-$T_{1/2}$ iff ℵsgp-OS= ℵPOS.

Proof:(i)Consider K be a ℵsgp-OS in X, thereupon K^c is ℵsgp-CS in X. By presumption, K^c is an ℵPCS in X. Thus, K is ℵsgp-OS in X. Therefore, ℵsgp-OS = ℵPOS.

(ii) Consider ℵsgp-CS in X. Then, K^c is ℵsgp-OS in X. By assumption, K^c is an ℵPOS in X. Then, K is an ℵPCS in X. Thereupon, X is an ℵ-sgp-$T_{1/2}$.

5. Conclusions

The class of neutrosophic semi-generalized pre closed sets in neutrosophic topological space is useful not only increase our understanding of some special features of the already known notions of neutrosophic topology but also useful in developing the neutrosophic multifunction theory in neutrosophic control theory as well as in neutrosophic economy. Some results have been proved to show that how far topological structures are preserved by the new neutrosophic set defined. We have given examples where such properties fail to be preserved. Here we have presented the idea; still some more theoretical research is to be carried out to establish a general frame work for decision making and to define patterns for complex network conceiving and practical application.

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Multi-Objective Inventory Model with Deterioration under Space Constraint: Neutrosophic Hesitant Fuzzy Programming Approach

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Abstract: We have considered a deterministic inventory model with time-dependent demand and holding cost and time varying deterioration where shortages are allowed and partially backlogged. To reduce deterioration we have considered here a preservation condition. In the presence uncertainty we have taken cost parameters as generalized trapezoidal fuzzy number. The proposed model has been solved by neutrosophic hesitant fuzzy programming approach, fuzzy nonlinear programming approach and fuzzy additive goal programming technique. The model is illustrated with numerical example and we presented sensitivity analysis finally.

Keywords: Inventory model, Multi-item, Preservation condition, demand, holding cost, deterioration, generalized trapezoidal fuzzy number, Neutrosophic Hesitant Fuzzy programming approach.

1. Introduction: In inventory models the effect of deterioration plays a very important role but in traditional inventory model the most popular assumptions was that the products kept their characteristic and physical structure while they were stored in the storage of inventory. But this types of assumptions are not be true always for all types of products. Deterioration is defined as erosion, change or spoilage that prevents the item from being used for its original purpose. Food products, drugs, radioactive substances, photograph, electronic components are a few examples of products in which decay may occur during the normal storage period of the units, and that is why this loss we have to consider into account while developing the inventory model. But with the help of some electronic devices like freeze, Micro-oven, etc. or some other type of devices it is possible to reduce the effect of deterioration.

Formulating an inventory model most of the researchers take deteriorating cost, inventory set-up cost ,holding cost as constant but in reality they may not constant always ,so if we take those cost parameters in fuzzy variable then it will be most realistic and interesting.

F. Harris (1915) developed first inventory model. Lotfi A. Zadeh in 1965 introduced the concept of fuzzy set theory in inventory modeling. L. A. Zadeh and R. E. Bellman in 1970 considered an inventory model on decision making in fuzzy environment. In 1934, Wilson gave a formula to obtain EOQ. In 1996, M. Vujosevic, D. Petrovic and R. Petrovic developed an EOQ formula by assuming inventory cost as a fuzzy number. M. Lee developed a fuzzy inventory model by considering backorder as a trapezoidal fuzzy number in 1999. In 1999, Chang and Dye developed an inventory model with time –varying demand and partial backlogging.

Deterioration of a product is the most realistic thing to be considered. Ghare and Schrader developed an inventory model where they took demand rate and deterioration rate as a constant. T.K Roy and M.Maity presented an inventory fuzzy EOQ model with demand-dependent unit cost under limited storage capacity. Vinod Kumar Mishra, Lal Sahab Singh and Rakesh Kumar developed an inventory model for deteriorating items with time-dependent demand and time-varying holding cost under partial backlogging. Zaid T, Balkhi, Lakdere Benkherouf represent an inventory model for deterioration items with stock dependent and time-varying demand rates. In 2001, Horng-Jinh Chang and Chung-Yuan Dye developed an inventory model for deterioration items with partial backlogging and permissible delay in payment. Misra and Singh gave an inventory model for ramp –type demand, time-dependent deterioration items with salvage value and shortages and deteriorating inventory model for time-dependent demand and holding cost with partial backlogging. Sumana Saha and Tripti Chakrabarty in 2012 considered a fuzzy EOQ model with time varying demand and shortages. D. Dutta and Pawan Kumar studied a fuzzy inventory model without shortages using a trapezoidal fuzzy number. In 2013, D. Dutta and Pawan Kumar considered an optimal replenishment for an inventory model without shortages by assuming fuzziness in demand, holding cost and ordering cost.

Smarandache. F introduced the neutrosophic set. Smarandache. F developed a generalization of the intuitionistic fuzzy set and gave geometric interpretation of the Neutrosophic set which is the generalization of the intuitionistic fuzzy set. Ye. J studied on multiple-attribute decision-making method under a single-valued neutrosophic hesitant fuzzy environment. Abdel-Basset et al. proposed a novel approach to solving fully neutrosophic linear programming problem and applied to production planning problem. Ye et.al. formulated neutrosophic number nonlinear programming problem (NN-NPP) and proposed an effective method to solve the problem under neutrosophic number environments. Firoz Ahmad, Ahmad Yusuf Adhami, Florentin Smarandache developed Single valued Neutrosophic Hesitant Fuzzy Computational Algorithm for Multi objective Nonlinear Optimization Problem. C. kar, B. Mondal and T.K Roy developed an Inventory Model under Space Constraint in Neutrosophic Environment by Geometric Programming Approach. T. Garai and T.K. Roy studied on optimization of EOQ Model with Limited Storage Capacity by neutrosophic geometric programming.P. Biswas, S.Pramanik and B.C. Giri presented multi-attribute group decision making based on expected value of neutrosophic trapezoidal numbers. In 2018, I. Ali, S.

Gupta, A. Ahmed represented Multi-objective bi-level supply chain network order allocation problem under fuzziness. V. Charles, G. Gupta and I. Ali developed A Supply Chain Network model in Fuzzy Goal Programming Approach with Pareto-Distributed Random Variables in 2019. In 2021, P. Gautam; S. Maheshwari, A. Kausar and C.K. Jaggi described an inventory Models for Imperfect Quality Items a Two-Decade Review. S. Gupta, A. Haq, I. Ali. And B. Sarkar described the Significance of multi-objective optimization in logistics problem for multi-product supply chain network under the intuitionistic fuzzy environment in 2021.

Research Motivation: In the present scenario in every situation of inventory storage it is very much obvious that there must be some kind of deterioration occur in every product sometimes it could be less or sometimes it could be more. Looking at this types of situation many authors takes many types of deteriorating function while developing their inventory models, sometimes they took constant deterioration, sometimes time depending linear type deterioration or sometimes stock dependent linear or quadratic type deteriorating function ect.But a few authors think about how to control these type of problems that is what should be the necessary step to be taken to reduce the very much effect of deterioration in inventory. In this paper we developed a method to reduce the effect of deterioration, here we introduced a constant "k" in the differential equation which gives some reduction on deterioration. In real life it is very often where many companies use freezer, Micro-oven or some other type of device to reduce the effect of deterioration. After getting the total average cost we have discussed several methodology to find the minimum average cost and what should be the timing to get minimum average cost.

AUTHORS CONTRIBUTION:

Names of authors	Papers title	Contribution
Ghare and Schrader	An EOQ Model for Deteriorating Items with Linear Demand, Variable Deterioration and Partial Backlogging.	Demand rate deterioration rate as a constant rate
T.K Roy and M.Maity	A fuzzy EOQ model with demand-dependent unit cost under limited storage capacity	Demand dependent unit cost under limited storage capacity
Vinod Kumar Mishra, Lal Sahab Singh and Rakesh Kumar	An Inventory Model for Deteriorating Items with Time Dependent Demand and Holding Cost under Partial Backlogging	Inventory model for deteriorating items with time dependent demand and time varying holding cost under partial backlogging

D. Dutta and Pawan Kumar	Fuzzy Inventory Model without Shortage Using Trapezoidal Fuzzy Number with Sensitivity Analysis	Fuzzy inventory model without shortages using trapezoidal fuzzy number
P. Gautam, S.Maheshwari;A.Kausar,C.K.Jaggi	Inventory Models for Imperfect Quality Items: A Two-Decade Review.	an inventory Models for Imperfect Quality Items
F. Smarandache	A Geometric Interpretation of the Neutrosophic Set - A Generalization of the Intuitionistic Fuzzy Set	developed a generalization of the intuitionistic fuzzy set and gave geometric interpretation of the Neutrosophic set
Jun Ye, Surapati Pramanik	Multiple-attribute Decision-Making Method under a Single-Valued Neutrosophic Hesitant Fuzzy Environment	multiple-attribute decision-making method under a single-valued neutrosophic hesitant fuzzy environment
Firoz Ahmad, Ahmad Yusuf Adhami, Florentin Smarandache	Single valued Neutrosophic Hesitant Fuzzy Computational Algorithm for Multi objective Nonlinear Optimization Problem	developed Single valued Neutrosophic Hesitant Fuzzy Computational Algorithm for Multi objective Nonlinear Optimization Problem
I. Ali, S. Gupta, A. Ahmed	Multi-objective linear fractional inventory problem under intuitionistic fuzzy environment.	represent Multi-objective bi-level supply chain network order allocation problem under fuzziness
Sahidul Islam and Kausik Das	Multi-objective inventory model with deterioration under space constrain:Neutrosophic fuzzy programming approach	Inventory model with time dependent demand and holding cost with preservation condition under neutrosophic hesitant

		fuzzy programming approach.

In this present paper we have taken a deterministic inventory model with time-dependent demand and holding cost and time varying deterioration. Shortages are allowed and partially backlogged. In this proposed model we have taken deterioration of item as linearity increasing function of time. We have considered here a preservation condition. In the presence of uncertainty the cost parameters are considered as generalized trapezoidal fuzzy number. This model has been solved by neutrosophic hesitant fuzzy programming approach, fuzzy nonlinear programming approach and fuzzy additive goal programming technique. Numerical examples have been finally.

2. Mathematical Model:

The following notations and assumptions are considered to formulate the model for i'th item.

2.1 Notations For i'th item (i=1, 2, 3,…., n):

A_i : Ordering cost per unit item for i'th item..

H_i: Holding cost per unit time for i'th item..

θ_i : Deterioration rate is time proportional for i'th item.

k_i:The preservation constant for i'th item

D_i : Demand rate is time proportional for i'th item.

C_{1i} : the purchase cost per unit time for i'th item.

C_{2i} : Deterioration cost per unit time for i'th item.

C_{3i} : Shortage cost per unit time for i'th item.

C_{4i} : Opportunity cost(lost sale cost) per unit time for i'th item.

B_i : The backlogging rate , $0 \le B_i \le 1$ for i'th item.

T_i : The length of cycle time for i'th item, $T_i \ge 0$.

t_{0i} : The time when the inventory level starts to reduce due to demand only ,$t_{0i} \ge 0$.

t_{1i} : The time when the inventory level starts to reduce due to both demand and deterioration for i'th item, $t_{1i} \ge 0$.

I_{1i} : the level of positive inventory for i'th item when deterioration is not occurring in time $[0, t_{0i}]$.

I_{2i} : the level of positive inventory for i'th item when deterioration starts to occur in time $[t_{0i}, t_{1i}]$.

I_{3i} : the level of negative inventory due to shortages for i'th item in the time interval $[t_{1i}, T_i]$.

I_{max}: the maximum inventory at time t=0 for i'th item.

S: the maximum back order quantity during stock-out period for i'th item.

$Q_i(I_{max} + S)$: the order quantity for the duration of a cycle of length T_i for i'th item.

$TAC_i(t_{0i}.t_{1i},T_i)$: total average cost per unit for i'th item.

w_i: storage space per unit time for i'th item.

W: total area space.

$\widetilde{A_i}$: Fuzzy ordering cost for i'th item.

$\widetilde{C_{1i}}$ Fuzzy purchase cost for i'th item.

$\widetilde{C_{2i}}$: Fuzzy Deterioration cost per unit time for i'th item.

$\widetilde{C_{3\iota}}$: Fuzzy Shortage cost per unit time for i'th item.

$\widetilde{C_{4\iota}}$:Fuzzy Opportunity cost (lost sale cost) per unit time for i'th item.

$\widetilde{H_\iota}$:Fuzzy Holding cost per unit time for i'th item.

$\widetilde{TAC_\iota}$:Total average cost per unit for i'th item.

$\widetilde{w_\iota}$:Fuzzy storage space per unit time for i'th item.

2.2 Assumptions:

i. The inventory model involves multi-item.

ii. The replenishment occurs instantaneously at infinite rate.

iii. The lead time is negligible.

iv. Demand rate is constant for $t \in [0, t_{0i}]$ and demand rate is time dependent for $t \in [t_{0i}, t_{1i}]$.That is

$$Demand(D) \ = \begin{cases} a_i & if\ t \in [0, t_{0i}] \\ b_i e^{-\alpha_i t} & if\ t \in [t_{0i}, t_{1i}] \end{cases}$$

v. The deterioration rate for i'th item is $\theta_i(t) = \theta_i t.\quad 0 < \theta_i < 1$.

vi. The backlogging rate is $B_i(t)= \frac{1}{1+\delta_i(T_i-t)}$, where $t_{1i} \le t \le T_i.\delta_i$ is the positive back logging parameter.

vii. The constant preservation constant is k_i .

viii. The inventory holding cost $H_i(t)=h_i.t$ is a linear function.

2.3. Mathematical formulation

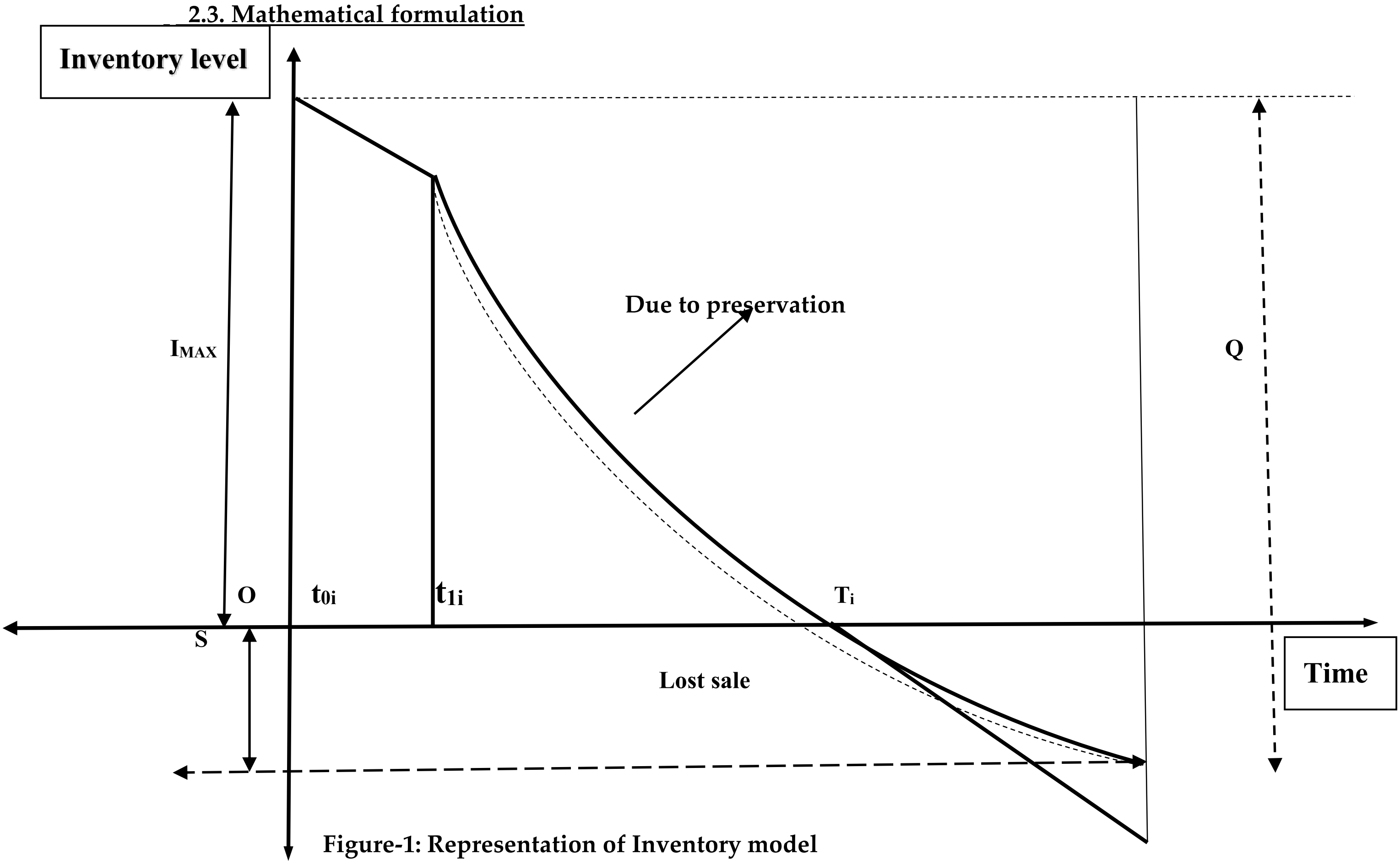

Figure-1: Representation of Inventory model

The initial inventory level is I_{max} unit at time t=0.From t=0 to t=t_{0i}, the inventory reduces just for demand. In time t=t_{0i} to t=t_{1i} the inventory level reduces for both demand as well as deterioration. At this time, shortage is accrued which is partially backlogged at rate $B_i(t)$ for the time [t_{1i},T_i].At the end of the cycle, the inventory reaches a maximum shortage level so as to clear the backlogged and again the inventory level to I_{max} (Figure 1).

The rate of change of inventory during the positive stock period $[0, t_{0i}]$, $[t_{0i}, t_{1i}]$and negative stock period (i.e shortage period) [t_{1i}, T_i] is governed by the following differential equation for i'th item.

$$\frac{dI_{1i}}{dt} = -D_i = -a_i\,, \qquad 0 \le t \le t_{0i} \tag{2.1}$$

$$\frac{dI_{2i}}{dt} = -D_i(t) - (\theta_i(t) - k_i)I_{2i}(t) = -b_i e^{-\alpha_i t} - (\theta_i t - k)I_{2i}(t), \quad t_{0i} \le t \le t_{1i} \tag{2.2}$$

$$\frac{dI_{3i}}{dt} = -\mathrm{a_i B_i(t)} = -\frac{a_i}{1+\delta_i(T_i - t)}\,, \quad t_{1i} \le t \le T_i \tag{2.3}$$

Thus the boundary condition are as follows:

$I_{1i}(0)=I_{max}$; $\quad I_{2i}(t_1)=0$; $\quad I_{3i}(t_1)=0$;

From (2.1) we get, using the initial condition (2.4)

$I_{1i}(t)= I_{max} - a_i t$, $0 \le t \le t_{0i}$

From (2.2) we get using initial condition and neglecting the higher power of θ_i,multiplication of θ_i with k_i and $\alpha_i$also neglecting higher power $of\ k_i$ we get.

$$I_{2i}(t)=b_i[(t_{1i} - t) - \frac{\theta_i}{3}(t_{1i}^3 - t^3) + \frac{k_i - \alpha_i}{2}(t_{1i}^2 - t^2)]\,,\ t_{0i} \le t \le t_{1i} \tag{2.5}$$

Since, $I_{1i}(t_0)$= , $I_{2i}(t_0)$,so we have

$$I_{max} = a_i t_{0i} + b_i[(t_{1i} - t_{0i}) - \frac{\theta_i}{3}(t_{1i}^3 - t_{0i}^3) + \frac{k_i - \alpha_i}{2}(t_{1i}^2 - t_{0i}^2)] \tag{2.6}$$

Hence we have from (2.4)

$$I_{1i}(t)= a_i(t_{0i} - t)+ b_i[(t_{1i} - t_{0i}) - \frac{\theta_i}{3}(t_{1i}^3 - t_{0i}^3) + \frac{k_i - \alpha_i}{2}(t_{1i}^2 - t_{0i}^2)]\,,\ 0 \le t \le t_{0i} \tag{2.7}$$

Now solving (2.3) with the boundary condition we get

$$I_{3i} = -\frac{a_i}{\delta_i}[\log\{1 + \delta_i(T_i - t_{1i})\}]\ ,\ t_{1i} \le t \le T_i \tag{2.8}$$

The maximum backordered unit is

$$S= -\mathrm{I_{3i}}(T_i) = \frac{a_i}{\delta_i}[\log\{1 + \delta_i(T_i - t_{1i})\}] \tag{2.9}$$

Hence, the order size during [0,T_i] is Q_i for i'th item is given by,

$Q_i= I_{max}+S.$

$$= \mathrm{a_i t_{0i}} + \mathrm{b_i}[(t_{1i} - t_{0i}) - \frac{\theta_i}{3}(t_{1i}^3 - t_{oi}^3) + \frac{k_i - \alpha_i}{2}(t_{1i}^2 - t_{0i}^2)] + \frac{a_i}{\delta_i}[\log\{1 + \delta_i(T_i - t_{1i})\}] \tag{2.10}$$

Now,

1. The ordering cost per cycle for i'th item.

$OC_i = A_i$ (2.11)

2. The inventory holding cost per cycle for i'th item.

$IHC_i = \int_0^{t_{0i}} h_i.t.I_{1i}(t)\,dt + \int_{t_{0i}}^{t_{1i}} h_i.t.I_{2i}(t)dt$

$= \frac{a_i h_i t_{0i}^3}{6} + \frac{b_i h_i t_{0i}^2}{2}\left[(t_{1i}-t_{0i}) - \frac{\theta_i}{3}(t_{1i}^3 - t_{0i}^3) + \frac{k_i-\alpha_i}{2}(t_{1i}^2 - t_{0i}^2)\right] +$

$b_i h_i[\frac{t_{1i}}{2}(t_{1i}^2 - t_{0i}^2) - \frac{1}{3}(t_{1i}^3 - t_{0i}^3) - \frac{\theta_i}{3}\{\frac{t_{1i}^3}{2}(t_{1i}^2 - t_{0i}^2) - \frac{1}{5}(t_{1i}^5 - t_{0i}^5)\} + \frac{k_i-\alpha_i}{2}\{\frac{t_{1i}^2}{2}(t_{1i}^2 - t_{0i}^2) - \frac{1}{4}(t_{1i}^4 - t_{0i}^4)\}]$

(2.12)

3. The deterioration cost for the time interval $[t_{0i}, t_{1i}]$.

$DC_i = c_{2i}[\frac{k_i-\alpha_i}{2}(t_{1i}^2 - t_{0i}^2) + \frac{b_i\alpha_i}{2}(t_{1i}^2 - t_{0i}^2) - \frac{\theta_i}{3}(t_{1i}^3 - t_{0i}^3) - k_i(t_{1i} - t_{0i})]$ (2.13)

4. Shortage due to accumulation in the system during the time interval $[t_{1i}, T_i]$

The optimum level of the shortage is present at t = T_i ; Therefore the total shortage cost during this time period is as follows:

$SC_i = \frac{C_{3i}a_i}{\delta_i}(T_i - t_{1i})\ \log\{1 + \delta_i(T_i - t_{1i})\}$ (2.14)

5. Due to stock out during $[t_{1i}, T_i]$ shortage is accumulated, but not all customers are willing to wait for the next lot size arrive. Hence, this results in some loss of sale which accounts to loss in profit.

Opportunity cost (Lost sale cost) calculated for the i'th item as follows

$OC_i = C_{4i}a_i\ [T_i - t_{1i} - \frac{\log\{1+\delta_i(T_i - t_{1i})\}}{\delta_i}]$ (2.15)

6. Purchase cost for i'th item is as follows:

$PC_i = C_{1i}[a_i t_{0i}\ +\ b_i\{(t_{1i} - t_{0i}) - \frac{\theta_i}{3}(t_{1i}^3 - t_{oi}^3) + \frac{k_i-\alpha_i}{2}(t_{1i}^2 - t_{0i}^2)\}$

$+\frac{a_i}{\delta_i}[\log\{1 + \delta_i(T_i - t_{1i})\}]]$ (2.16)

So, the total average cost for i'th item is as follows

$TAC_i(t_{0i}.t_{1i},T_i) = \frac{1}{T_i}[A_i + \frac{a_i h_i t_{0i}^3}{6} + \frac{b_i h_i t_{0i}^2}{2}\left[(t_{1i}-t_{0i}) - \frac{\theta_i}{3}(t_{1i}^3 - t_{0i}^3) + \frac{k_i-\alpha_i}{2}(t_{1i}^2 - t_{0i}^2)\right] + b_i h_i\{\frac{t_{1i}}{2}(t_{1i}^2 - t_{0i}^2) -$

$\frac{1}{3}(t_{1i}^3 - t_{0i}^3) - \frac{\theta_i}{3}\{\frac{t_{1i}^3}{2}(t_{1i}^2 - t_{0i}^2) - \frac{1}{5}(t_{1i}^5 - t_{0i}^5)\} + \frac{k_i-\alpha_i}{2}\{\frac{t_{1i}^2}{2}(t_{1i}^2 - t_{0i}^2) - \frac{1}{4}(t_{1i}^4 - t_{0i}^4)\}\quad\}\quad + c_{2i}\{\frac{k_i-\alpha_i}{2}(t_{1i}^2 -$

$t_{0i}^2) + \frac{b_i\alpha_i}{2}(t_{1i}^2 - t_{0i}^2) - \frac{\theta_i}{3}(t_{1i}^3 - t_{0i}^3) - k_i(t_{1i} - t_{0i})\}\ + \frac{C_{3i}a_i}{\delta_i}(T_i - t_{1i}).\log\{1 + \delta_i(T_i - t_{1i})\}\quad + C_{4i}a_i\ \Big\{T_i -$

$t_{1i} - \frac{\log\{1+\delta_i(T_i - t_{1i})\}}{\delta_i}\Big\} + C_{1i}\ [a_i t_{0i}\ +\ b_i\{(t_{1i} - t_{0i}) - \frac{\theta_i}{3}(t_{1i}^3 - t_{oi}^3) + \frac{k_i-\alpha_i}{2}(t_{1i}^2 - t_{0i}^2)\}\ + \frac{a_i}{\delta_i}\{\log\{1 + \delta_i(T_i -$

$t_{1i})\}\}]]$ (2.17)

From above we have a multi-objective (MOIM) inventory model given below

Minimize { $TAC_1(t_{01}.t_{11},T_1)$, $TAC_2(t_{02}.t_{12},T_2)$,…………………., $TAC_n(t_{0n}.t_{1n},T_n)$}

Subject to: $\sum_{i=i}^{n} w_i Q_i \leq W$ for i=1,2,3,4,……….,n

Where $TAC_i(t_{0i}.t_{1i},T_i)=\frac{1}{T_i}[A_i+\frac{a_i h_i t_{0i}^3}{6}+\frac{b_i h_i t_{0i}^2}{2}\left[(t_{1i}-t_{0i})-\frac{\theta_i}{3}(t_{1i}^3-t_{0i}^3)+\frac{k_i-\alpha_i}{2}(t_{1i}^2-t_{0i}^2)\right]+$

$b_i h_i\{\frac{t_{1i}}{2}(t_{1i}^2-t_{0i}^2)-\frac{1}{3}(t_{1i}^3-t_{0i}^3)-\frac{\theta_i}{3}\{\frac{t_{1i}^3}{2}(t_{1i}^2-t_{0i}^2)-\frac{1}{5}(t_{1i}^5-t_{0i}^5)\}+\frac{k_i-\alpha_i}{2}\{\frac{t_{1i}^2}{2}(t_{1i}^2-t_{0i}^2)-\frac{1}{4}(t_{1i}^4-t_{0i}^4)\}\}$

$+c_{2i}\{\frac{k_i-\alpha_i}{2}(t_{1i}^2-t_{0i}^2)+\frac{b_i\alpha_i}{2}(t_{1i}^2-t_{0i}^2)-\frac{\theta_i}{3}(t_{1i}^3-t_{0i}^3)-k_i(t_{1i}-t_{0i})\}+\frac{C_{3i}\alpha_i}{\delta_i}(T_i-t_{1i}).\log\{1+\delta_i(T_i-t_{1i})\}$

$+C_{4i}a_i\left\{T_i-t_{1i}-\frac{\log\{1+\delta_i(T_i-t_{1i})\}}{\delta_i}\right\}+C_{1i}[a_i t_{0i}+b_i\{(t_{1i}-t_{0i})-\frac{\theta_i}{3}(t_{1i}^3-t_{oi}^3)+\frac{k_i-\alpha_i}{2}(t_{1i}^2-t_{0i}^2)\}$

$+\frac{a_i}{\delta_i}\{\log\{1+\delta_i(T_i-t_{1i})\}\}]]$

And $Q_i=a_i t_{0i}+b_i\{(t_{1i}-t_{0i})-\frac{\theta_i}{3}(t_{1i}^3-t_{oi}^3)+\frac{k_i-\alpha_i}{2}(t_{1i}^2-t_{0i}^2)\}+\frac{a_i}{\delta_i}\{\log\{1+\delta_i(T_i-t_{1i})\}\}$ (2.18)

3. Prerequisite mathematics:

3.1 Fuzzy set:

Let S be the collection of substance called universe of discourse. A fuzzy set is a subset of S denoted by Ã and is defined by a set of ordered pairs Ã= $\{(x,\mu_{\tilde{A}}(x)): x\in S\}$.

Where $\mu_{\tilde{A}}: S\rightarrow[0,1]$ is a membership function of the fuzzy set Ã and $\mu_{\tilde{A}}(x)$ is called the grade of membership of x∈ S in the fuzzy set Ã.

3.2 Equality of fuzzy two sets: Two fuzzy sets Ã and $\tilde{B}$ are called equal iff $\mu_{\tilde{A}}(x)=\mu_{\tilde{B}}(x)$,∀ x∈S.It is usually denoted by Ã=$\tilde{B}$.

3.3 Union of two fuzzy sets: Let Ã and $\tilde{B}$ be two fuzzy sets. The union of Ã and $\tilde{B}$ is a fuzzy set in S is denoted by Ã ∪ $\tilde{B}$ and is defined by

$$\mu_{\tilde{A}\cup\tilde{B}}(x)=\mu_{\tilde{A}}(x)\vee\mu_{\tilde{B}}(x)=\max\{\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)\}\quad\forall\ x\in S.$$

3.4 Intersection of two fuzzy sets: Let Ã and $\tilde{B}$ be two fuzzy sets. The intersection of Ã and $\tilde{B}$ is a fuzzy set in S is denoted by Ã ∩ $\tilde{B}$ and is defined by

$$\mu_{\tilde{A}\cap\tilde{B}}(x)=\mu_{\tilde{A}}(x)\wedge\mu_{\tilde{B}}(x)=\min\{\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)\}\quad\forall\ x\in S.$$

3.5 Trapezoidal fuzzy number (TrFN): Let F ($\mathcal{R}$) be the set of all TrFN in the line $\mathcal{R}$ .A trapezoidal fuzzy number Ã∈ F ($\mathcal{R}$) is parameterized by (a, b, c, d) with the membership function $\mu_{\tilde{A}}:\mathcal{R}\rightarrow[0,1]$ which is defined as

$$\mu_{\tilde{A}}(x)=\begin{cases}\frac{x-a}{b-a} & for\ a\le x\le b\\ 1 & for\ b\le x\le c\\ \frac{d-x}{d-c} & for\ c\le x\le d\\ 0 & otherwise\end{cases}$$

Where a$<b<c<d$.Here a and d represent the lower and upper limits of sustains of a fuzzy set Ã.

3.6 Generalized Trapezoidal fuzzy number (GTrFN):A generalized TrFN Ã can be represented as Ã=(a, b, c, d; w) 0≤w≤ 1 and a,b,c,d∈ $\mathcal{R}$.Here Ã is a fuzzy subset of $\mathcal{R}$,with the membership function $\mu_{\tilde{A}}:\mathcal{R}\rightarrow[0,w]$ is defined as

$$\mu_{\tilde{A}}(x) = \begin{cases} w\frac{x-a}{b-a} & for\ a \le x \le b \\ w & for\ b \le x \le c \\ w\frac{d-x}{d-c} & for\ c \le x \le d \\ 0 & otherwise \end{cases}$$

Where $a < b < c < d$.Here a and d represent the lower and upper limits of sustains of a fuzzy set $\tilde{A}$ and w ∈[0, 1].If w=1 then GTrFN $\tilde{A}$ is called TrFN.

3.7 Neutrosophic set: Let S be a universe of discourse. A neutrosophic set $\widetilde{A^n}$ in S is defined by

$$\widetilde{A^n} = \{(x, T_A(x), I_A(x), F_A(x)) \mid x \in S\}$$

Where $T_A(x)$ is the truth membership function, $I_A(x)$ is the indeterminacy membership function, $F_A(x)$ is falsity membership function.

Where,

$$T_A(x):S\rightarrow]0^-,1^+[$$

$$I_A(x):S\rightarrow]0^-,1^+[$$

$$F_A(x):S\rightarrow]0^-,1^+[$$

So we can say $0^- \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3^+$

3.7.1 Single valued neutrosophic set: Let S be a universe of discourse. A single valued neutrosophic set $\widetilde{A^n}$ in S is defined by

$$\widetilde{A^n} = \{(x, T_A(x), I_A(x), F_A(x)) \mid x \in S\}$$

Where $T_A(x)$ is the truth membership function, $I_A(x)$ is the indeterminacy membership function, $F_A(x)$ is falsity membership function.

Together with,

$$T_A(x): S\rightarrow [0, 1]$$

$$I_A(x): S\rightarrow [0, 1]$$

$$F_A(x): S\rightarrow [0, 1]$$

So we can say $0 \le T_A(x) + I_A(x) + F_A(x) \le 3\ \forall\ x \in S$.

3.8 Hesitant Fuzzy set (HFS): Let S be a non-empty reference set, a hesitant fuzzy set A^h of S is defined by a function $A_h(x)$ which is applied to S returns a finite subset of [0,1]. Whose mathematical representation I given by

$$A^h = \{(x, A_h(x) \mid x \in S\}$$

Here $h_A(x) \in [0, 1]$, representing the possible membership degree of the elements $x \in S$ to A^h.The set $h_A(x)$ is known as hesitant fuzzy element (HFE).

e.g.Let S ={x_1, x_2, x_3} be a reference set . $A_h(x_1)$=(0.1,0.2,0.3), $A_h(x_2)$=(0.2,0.3,0.4), $A_h(x_3)$=(0.1,0.2,0.5) be hesitant elements of x_1, x_2, x_3 to a set A^h.Therefore the hesitant fuzzy set A^h is given by

A^h={(x_1,{ 0.1,0.2,0.3}),(x_2,{0.2,0.3,0.4}).(x_3,{ 0.1,0.2,0.5})}

3.9 Single valued neutrosophic Hesitant Fuzzy Set (NVNHFS):

We take S be non-empty reference set, then a single valued neutrosophic hesitant fuzzy set A on S is defined as

$$A^h = \{(x, \widetilde{T_A}(x), \widetilde{I_A}(x), \widetilde{F_A}(x)) \mid x \in S\}$$

Here $\widetilde{T_A}(x)=\{\mu \mid \mu \in \widetilde{T_A}(x) : x \in S\}$ is known as possible truth membership hesitant degree , $\widetilde{I_A}(x)=\{\rho \mid \{\rho \in \widetilde{I_A}(x) : x \in S\}$ is known as possible indeterminacy membership hesitant degree and $\widetilde{F_A}(x)=\{\sigma \mid \sigma \in \widetilde{F_A}(x) : x \in S\}$ is known as possible falsity membership hesitant degree. These sets takes the different values in [0, 1].Which satisfies the following conditions

$\mu, \rho, \sigma \subseteq [0,1]$ and $0 \le \sup\mu^+ + \sup \rho^+ + \sup\sigma^+ \le 3$

Where $\mu^+ = \cup_{\mu \in \widetilde{T_A}(x)} \max\{\mu\}$, $\rho^+ = \cup_{\rho \in \widetilde{T_A}(x)} \max\{\rho\}$, $\sigma^+ = \cup_{\sigma \in \widetilde{T_A}(x)} \max\{\sigma\}$ for $x \in S$.

3.10 Union of two SVNS sets: Let S be a universe of discourse and $\widetilde{A^n}$ and $\widetilde{B^n}$ are any two subsets of S.

Where $T_A(x)$ is the truth membership function, $I_A(x)$ is the indeterminacy membership function, $F_A(x)$ is falsity membership function of respectively. The union of $\widetilde{A^n}$ and $\widetilde{B^n}$ is denoted by $\widetilde{A^n} \cup \widetilde{B^n}$ and defined by

$\widetilde{A^n} \cup \widetilde{B^n} = \{(x, \max(T_A(x), T_B(x)), \max(I_A(x), I_B(x)), \min(F_A(x), F_B(x))) \mid x \in S\}$

3.11 Intersection of two SVNs sets: Let S be a universe of discourse and $\widetilde{A^n}$ and $\widetilde{B^n}$ are any two subsets of S.

Where $T_A(x)$ is the truth membership function, $I_A(x)$ is the indeterminacy membership function, $F_A(x)$ is falsity membership function of respectively. The union of $\widetilde{A^n}$ and $\widetilde{B^n}$ is denoted by $\widetilde{A^n} \cap \widetilde{B^n}$ and defined by

$\widetilde{A^n} \cap \widetilde{B^n} = \{(x, \min(T_A(x), T_B(x)), \min(I_A(x), I_B(x)), \max(F_A(x), F_B(x))) \mid x \in S\}$

4. Fuzzy model for i'th item

For the presence of uncertainty we take some cost parameters as generalized trapezoidal fuzzy number which are already mentioned above.

Let us take the following fuzzy cost parameters

$\widetilde{a_i} = (a_i^1, a_i^2, a_i^3, a_i^4; w_{a_i})$,$0 \le w_{a_i} \le 1$;

$\widetilde{b_i} = (b_i^1, b_i^2, b_i^3, b_i^4; w_{b_i})$,$0 \le w_{b_i} \le 1$;

$\widetilde{A_i} = (A_i^1, A_i^2, A_i^3, A_i^4; w_{A_i})$,$0 \le w_{A_i} \le 1$;

$\widetilde{C_{1i}} = (C_{1i}^1, C_{1i}^2, C_{1i}^3, C_{1i}^4; w_{C_{1i}})$,$0 \le w_{C_{1i}} \le 1$;

$\widetilde{C_{2i}} = (C_{2i}^1, C_{2i}^2, C_{2i}^3, C_{2i}^4; w_{C_{2i}})$,$0 \le w_{C_{2i}} \le 1$;

$\widetilde{C_{3i}} = (C_{3i}^1, C_{3i}^2, C_{3i}^3, C_{3i}^4; w_{C_{3i}})$,$0 \le w_{C_{3i}} \le 1$;

$\widetilde{C_{4i}} = (C_{4i}^1, C_{4i}^2, C_{4i}^3, C_{4i}^4; w_{C_{4i}})$,$0 \le w_{C_{4i}} \le 1$;

$\widetilde{H_i} = (H_i^1, H_i^2, H_i^3, H_i^4; w_{H_i})$,$0 \le w_{H_i} \le 1$;

$\widetilde{w_i} = (w_i^1, w_i^2, w_i^3, w_i^4; w_{w_i})$,$0 \le w_{w_i} \le 1$;

The fuzzy total average cost is given by:

$$\widetilde{TAC_i}(t_{0i}, t_{1i}, T_i) = \frac{1}{T_i}[\widetilde{A_i} + \frac{\widetilde{a_i}\widetilde{h_i}t_{0i}^3}{6} + \frac{\widetilde{b_i}\widetilde{h_i}t_{0i}^2}{2}\left[(t_{1i} - t_{0i}) - \frac{\theta_i}{3}(t_{1i}^3 - t_{0i}^3) + \frac{k_i - \alpha_i}{2}(t_{1i}^2 - t_{0i}^2)\right] +$$

$$\widetilde{b_i}\widetilde{h_i}\{\frac{t_{1i}}{2}(t_{1i}^2 - t_{0i}^2) - \frac{1}{3}(t_{1i}^3 - t_{0i}^3) - \frac{\theta_i}{3}\{\frac{t_{1i}^3}{2}(t_{1i}^2 - t_{0i}^2) - \frac{1}{5}(t_{1i}^5 - t_{0i}^5)\} + \frac{k_i - \alpha_i}{2}\{\frac{t_{1i}^2}{2}(t_{1i}^2 - t_{0i}^2) - \frac{1}{4}(t_{1i}^4 - t_{0i}^4)\}\}$$

$$+\widetilde{C_{2i}}\{\frac{k_i - \alpha_i}{2}(t_{1i}^2 - t_{0i}^2) + \frac{\widetilde{b_i}\alpha_i}{2}(t_{1i}^2 - t_{0i}^2) - \frac{\theta_i}{3}(t_{1i}^3 - t_{0i}^3) - k_i(t_{1i} - t_{0i})\} + \widetilde{C_{3i}}\frac{\widetilde{a_i}}{\delta_i}(T_i - t_{1i}).\log\{1 +$$

$$\delta_i(T_i - t_{1i})\} + \widetilde{C_{4i}}\widetilde{a_i}\{T_i - t_{1i} - \frac{\log\{1 + \delta_i(T_i - t_{1i})\}}{\delta_i}\} + \widetilde{C_{1i}}\{\widetilde{a_i}t_{0i} + \widetilde{b_i}[(t_{1i} - t_{0i}) - \frac{\theta_i}{3}(t_{1i}^3 - t_{oi}^3) +$$

$$\frac{k_i - \alpha_i}{2}(t_{1i}^2 - t_{0i}^2)] + \frac{\widetilde{a_i}}{\delta_i}\{\log\{1 + \delta_i(T_i - t_{1i})\}\}\}] \quad (4.1)$$

And our Multi-objective inventory model (MOIM) become fuzzy model as

Min { $\widetilde{TAC_1}(t_{01}, t_{11}, T_1), \widetilde{TAC_2}(t_{02}, t_{12}, T_2), \ldots\ldots\ldots\ldots\ldots\ldots, \widetilde{TAC_n}(t_{0n}, t_{1n}, T_n)$}

Subject to: $\sum_{i=i}^{n} w_i Q_i \leq W$ for i=1, 2, 3, 4,……….,n

Where, $\widetilde{TAC_\iota}(t_{0i}, t_{1i}, T_i) = \frac{1}{T_i}[\widetilde{A_\iota} + \frac{\widetilde{a_\iota}\widetilde{h_\iota}t_{0i}^3}{6} + \frac{\widetilde{b_\iota}\widetilde{h_\iota}t_{0i}^2}{2}\left[(t_{1i} - t_{0i}) - \frac{\theta_i}{3}(t_{1i}^3 - t_{0i}^3) + \frac{k_i - \alpha_i}{2}(t_{1i}^2 - t_{0i}^2)\right] +$

$\widetilde{b_\iota}\widetilde{h_\iota}\{\frac{t_{1i}}{2}(t_{1i}^2 - t_{0i}^2) - \frac{1}{3}(t_{1i}^3 - t_{0i}^3) - \frac{\theta_i}{3}\{\frac{t_{1i}^3}{2}(t_{1i}^2 - t_{0i}^2) - \frac{1}{5}(t_{1i}^5 - t_{0i}^5)\} + \frac{k_i - \alpha_i}{2}\{\frac{t_{1i}^2}{2}(t_{1i}^2 - t_{0i}^2) - \frac{1}{4}(t_{1i}^4 - t_{0i}^4)\}\}$

$+\widetilde{C_{2\iota}}\{\frac{k_i - \alpha_i}{2}(t_{1i}^2 - t_{0i}^2) + \frac{\widetilde{b_\iota}\alpha_i}{2}(t_{1i}^2 - t_{0i}^2) - \frac{\theta_i}{3}(t_{1i}^3 - t_{0i}^3) - k_i(t_{1i} - t_{0i})\} + \widetilde{C_{3\iota}}\,\frac{\widetilde{a_\iota}}{\delta_i}(T_i - t_{1i}).\log\{1 +$

$\delta_i(T_i - t_{1i})\}\ + \widetilde{C_{4\iota}}\widetilde{a_\iota}\{T_i - t_{1i} - \frac{\log\{1+\delta_i(T_i - t_{1i})\}}{\delta_i}\}\ + \widetilde{C_{1\iota}}\{\widetilde{a_\iota}t_{0i}\ +\ \widetilde{b_\iota}[(t_{1i} - t_{0i}) - \frac{\theta_i}{3}(t_{1i}^3 - t_{oi}^3) +$

$\frac{k_i - \alpha_i}{2}(t_{1i}^2 - t_{0i}^2)] + \frac{\widetilde{a_\iota}}{\delta_i}\{\log\{1 + \delta_i(T_i - t_{1i})\}\}\}]$

And $Q_i = \widetilde{a_\iota}t_{0i}\ +\ \widetilde{b_\iota}[(t_{1i} - t_{0i}) - \frac{\theta_i}{3}(t_{1i}^3 - t_{oi}^3) + \frac{k_i - \alpha_i}{2}(t_{1i}^2 - t_{0i}^2)] + \frac{\widetilde{a_\iota}}{\delta_i}\{\log\{1 + \delta_i(T_i - t_{1i})\}\}$ (4.2)

Now we have to defuzzify all the fuzzy number we have considered earlier. In this model we have considered GTrFN. Now for defuzzification we have considered here the following technique.

Let $\tilde{A} = (a, b, c, d; w)$ be any GTrFN fuzzy number, then the total λ-integer value of $\tilde{A}$ is

$I_\lambda^w(\tilde{A}) = \lambda\omega\left(\frac{c+d}{2}\right) + (1 - \lambda)\omega\left(\frac{a+b}{2}\right)$

For simplification we have considered $\lambda = \frac{1}{2}$, therefore we get the approximate value of $\tilde{A} = (a, b, c, d, \omega)$ as $\omega\left(\frac{a+b+c+d}{4}\right)$

On the basis of this method all the GTrFN fuzzy number $(\widetilde{a_\iota}, \widetilde{b_\iota}, \widetilde{C_{1\iota}}, \widetilde{C_{2\iota}}, \widetilde{C_{3\iota}}, \widetilde{C_{4\iota}}, \widetilde{H_\iota}, \widetilde{A_\iota}, \widetilde{w_\iota})$ converted to the crisp value as $(\widehat{a_\iota}, \widehat{b_\iota}, \widehat{C_{1\iota}}, \widehat{C_{2\iota}}, \widehat{C_{3\iota}}, \widehat{C_{4\iota}}, \widehat{H_\iota}, \widehat{A_\iota}, \widehat{w_\iota})$.

So the above model (4.2) becomes a crisp model as

Min {$\widehat{TAC_1}(t_{01}, t_{11}, T_1), \widehat{TAC_2}(t_{02}, t_{12}, T_2), \ldots\ldots\ldots\ldots, \widehat{TAC_n}(t_{0n}, t_{1n}, T_n)$}

(4.3)

Subject to, Subject to: $\sum_{i=i}^{n} w_i Q_i \leq W$ for i=1, 2, 3, 4,……….,n

Where,

$$\widehat{TAC_\iota}(t_{0i}, t_{1i}, T_i) = \frac{1}{T_i}[\widehat{A_\iota} + \frac{\widehat{a_\iota}\widehat{h_\iota}t_{0i}^3}{6} + \frac{\widehat{b_\iota}\widehat{h_\iota}t_{0i}^2}{2}\left[(t_{1i} - t_{0i}) - \frac{\theta_i}{3}(t_{1i}^3 - t_{0i}^3) + \frac{k_i - \alpha_i}{2}(t_{1i}^2 - t_{0i}^2)\right] +$$

$\widehat{b_\iota}\widehat{h_\iota}\{\frac{t_{1i}}{2}(t_{1i}^2 - t_{0i}^2) - \frac{1}{3}(t_{1i}^3 - t_{0i}^3) - \frac{\theta_i}{3}\{\frac{t_{1i}^3}{2}(t_{1i}^2 - t_{0i}^2) - \frac{1}{5}(t_{1i}^5 - t_{0i}^5)\} + \frac{k_i - \alpha_i}{2}\{\frac{t_{1i}^2}{2}(t_{1i}^2 - t_{0i}^2) - \frac{1}{4}(t_{1i}^4 - t_{0i}^4)\}\}$

$+\widehat{C_{2\iota}}\{\frac{k_i - \alpha_i}{2}(t_{1i}^2 - t_{0i}^2) + \frac{\widehat{b_\iota}\alpha_i}{2}(t_{1i}^2 - t_{0i}^2) - \frac{\theta_i}{3}(t_{1i}^3 - t_{0i}^3) - k_i(t_{1i} - t_{0i})\} + \widehat{C_{3\iota}}.\frac{\widehat{a_\iota}}{\delta_i}(T_i - t_{1i}).\log\{1 +$

$\delta_i(T_i - t_{1i})\} + \widehat{C_{4\iota}}\widehat{a_\iota}\{T_i - t_{1i} - \frac{\log\{1+\delta_i(T_i - t_{1i})\}}{\delta_i}\} + \widehat{C_{1\iota}}\{\widehat{a_\iota}t_{0i} + \widehat{b_\iota}[(t_{1i} - t_{0i}) - \frac{\theta_i}{3}(t_{1i}^3 - t_{oi}^3) +$

$\frac{k_i - \alpha_i}{2}(t_{1i}^2 - t_{0i}^2)] + \frac{\widehat{a_\iota}}{\delta_i}\{\log\{1 + \delta_i(T_i - t_{1i})\}\}\}]$

And $Q_i = \widehat{a_\iota}t_{0i} + \widehat{b_\iota}[(t_{1i} - t_{0i}) - \frac{\theta_i}{3}(t_{1i}^3 - t_{oi}^3) + \frac{k_i - \alpha_i}{2}(t_{1i}^2 - t_{0i}^2)] ++ \frac{\widehat{a_\iota}}{\delta_i}\{\log\{1 + \delta_i(T_i - t_{1i})\}\}$ for i=1, 2, ..., n.

5. Fuzzy programming technique (FNLP AND FAGP):

To solve the above MOIM (4.3) problem we take one objective at a time and we ignore the others.

Using this we find out the value of every objective function and from this we formulate the pay-of-matrix as follows.

$$\begin{array}{c|cccc} & TAC_1(t_{01},t_{11},T_1) & TAC_2(t_{02},t_{12},T_2) & \dots\dots\dots & TAC_n(t_{0n},t_{1n},T_n) \\ \hline (t_{01}^1,t_{11}^1,T_1^1) & TAC_1^*(t_{01}^1,t_{11}^1,T_1^1) & TAC_2(t_{01}^1,t_{11}^1,T_1^1) & \dots\dots.. & TAC_n(t_{01}^1,t_{11}^1,T_1^1) \\ (t_{02}^2,t_{12}^2,T_2^2) & TAC_1(t_{02}^2,t_{12}^2,T_2^2) & TAC_1^*(t_{02}^2,t_{12}^2,T_2^2) & \dots\dots\dots & TAC_n(t_{02}^2,t_{12}^2,T_2^2) \\ \dots\dots\dots\dots & \dots\dots\dots & \dots\dots\dots\dots & \dots\dots\dots\dots.. & \dots\dots\dots\dots\dots \\ (t_{0n}^n,t_{1n}^n,T_n^n) & TAC_1(t_{0n}^n,t_{1n}^n,T_n^n) & TAC_2(t_{0n}^n,t_{1n}^n,T_n^n) & \dots\dots\dots\dots & TAC_n^*(t_{0n}^n,t_{1n}^n,T_n^n) \end{array}$$

Now we assume that $U^k = \max\{TAC_k(t_{0i}^i, t_{1i}^i, T_i^i), i = 1,2,3,\dots\dots, n\}$ for k= 1, 2, 3, ….., n

And $L^k = \{TAC_k^*(t_{0k}^k, t_{1k}^k, T_k^k), k = 1,2,3,\dots\dots, n\}$

Here $L^k \le TAC_k(t_{0i}^i, t_{2i}^i, T_i^i) \le U^k$ for $i = 1,2,3,\dots\dots, n$; and k= 1, 2, 3, ….., n; (5.1)

Now we take the linear fuzzy membership function $\mu_{TAC_k}(TAC_k(t_{0k}, t_{1k}, T_k))$ for the k'th objective function $TAC_k(t_{0k}, t_{1k}, T_k)$ as follows.

$$\mu_{TAC_k}(TAC_k(t_{0k}, t_{1k}, T_k)) = \begin{cases} 1 & for \quad TAC_k(t_{0k}, t_{1k}, T_k) \le \mathrm{L^k} \\ \frac{U^k - TAC_k(t_{0k}, t_{1k}, T_k)}{U^k - L^k} & for \ L^k \le TAC_k(t_{0k}, t_{1k}, T_k) \le \mathrm{U^k} \\ 0 & for \quad TAC_k(t_{0k}, t_{1k}, T_k) \ge \mathrm{U^k} \end{cases} \quad (5.2)$$

For k=1, 2,3,….., n;

After getting the membership function we formulate the fuzzy non-linear programming problems (FNLP) based on max-min operator as follow:

Max p

Subject to,

$$p(U^k - L^k) + TAC_k(t_{0k}, t_{1k}, T_k) \le U^k \quad \text{For k=1, 2, 3,……, n}$$

$$0 \le p \le 1, \qquad t_{0k} \ge 0, t_{1k} \ge 0, T_k \ge 0; \qquad (5.3)$$

And the same constraints and restriction as the problem (4.3)

Now based on max-additive operator we formulate Fuzzy additive goal programming (FAGP) as follows:

$$\text{Max} \sum_{k=1}^{n} \frac{U^k - TAC_k(t_{0k}, t_{1k}, T_k)}{U^k - L^k} \qquad (5.4)$$

Subject to,$0 \le \mu_{TAC_k}(TAC_k(t_{0k}, t_{1k}, T_k)) \le 1$, for k=1,2,3,…….,n

And the same constraints and restrictions as in the problem (4.3)

Solving the above reduced problem (5.3) and (5.4) by using the above FNLP and FAGP method we shall find the optimal solutions.

6. Weighted Fuzzy programming technique (WFNLP AND WFAGP):

Here we take a positive weight for ω_k for each of the objective $\left(TAC_k(t_{0k}, t_{1k}, T_k)\right)$

(Where k=1, 2, 3,….., n) and $\sum_{k=1}^{n} \omega_k = 1$.

Having the membership function (5.2) WFNLP technique becomes

$$\begin{aligned} &\textbf{Max p} \\ &\text{Subject to,} \\ &\omega_k . \mu_{TAC_k}\left(TAC_k(t_{0k}, t_{1k}, T_k)\right) \geq \text{p} \quad \text{For k=1, 2, 3,……, n} \\ &0 \leq p \leq 1, \quad t_{0k} \geq 0, t_{1k} \geq 0, T_k \geq 0 \text{ and } \sum_{k=1}^{n} \omega_k = 1 \end{aligned} \tag{6.1}$$

And the same constraints and restriction as the problem (4.3)

Having the membership function (5.2) WFAGP technique becomes

$$\text{Max } \sum_{k=1}^{n} \omega_k . \mu_{TAC_k}\left(TAC_k(t_{0k}, t_{1k}, T_k)\right)$$

$$\text{Subject to,} 0 \leq \mu_{TAC_k}\left(TAC_k(t_{0k}, t_{1k}, T_k)\right) \leq 1, \text{ for k=1,2,3,…….,n and } \sum_{k=1}^{n} \omega_k = 1 \tag{6.2}$$

And the same constraints and restrictions as in the problem (4.3)

Solving the problem by using the above WFNLP and WFAGP method we shall find the optimal solution.

7. Neutrosophic Hesitant Fuzzy Non-Linear Programming (NHFNLP) technique:

Using (5.1) we can defined now the different hesitant membership function under neutrosophic hesitant fuzzy environment as follows.

For minimization type objective function

The truth hesitant-membership function:

$$T_{\tilde{h}}^{E_1}(TAC_k(t_{0k}, t_{1k}, T_k)) = \begin{cases} 1 & if \quad TAC_k(t_{0k}, t_{1k}, T_k) < \mathrm{L^k} \\ \zeta_1 \frac{(U^k)^t - (TAC_k(t_{0k}, t_{1k}, T_k))^t}{(U^k)^t - (L^k)^t} & if \ L^k \leq TAC_k(t_{0k}, t_{1k}, T_k) \leq \mathrm{U^k} \\ 0 & if \quad TAC_k(t_{0k}, t_{1k}, T_k) > \mathrm{U^k} \end{cases}$$

$$T_{\tilde{h}}^{E_2}(TAC_k(t_{0k}, t_{1k}, T_k)) = \begin{cases} 1 & if \quad TAC_k(t_{0k}, t_{1k}, T_k) < \mathrm{L^k} \\ \zeta_2 \frac{(U^k)^t - (TAC_k(t_{0k}, t_{1k}, T_k))^t}{(U^k)^t - (L^k)^t} & if \ L^k \leq TAC_k(t_{0k}, t_{1k}, T_k) \leq \mathrm{U^k} \\ 0 & if \quad TAC_k(t_{0k}, t_{1k}, T_k) > \mathrm{U^k} \end{cases}$$

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$$T_{\tilde{h}}^{E_n}(TAC_k(t_{0k}, t_{1k}, T_k)) = \begin{cases} 1 & if \quad TAC_k(t_{0k}, t_{1k}, T_k) < \mathrm{L^k} \\ \zeta_n \frac{(U^k)^t - (TAC_k(t_{0k}, t_{1k}, T_k))^t}{(U^k)^t - (L^k)^t} & if \ L^k \leq TAC_k(t_{0k}, t_{1k}, T_k) \leq \mathrm{U^k} \\ 0 & if \quad TAC_k(t_{0k}, t_{1k}, T_k) > \mathrm{U^k} \end{cases}$$

The indeterminacy hesitant-membership function:

$$I_{\tilde{h}}^{E_1}(TAC_k(t_{0k}, t_{1k}, T_k)) = \begin{cases} 1 & if \quad TAC_k(t_{0k}, t_{1k}, T_k) < \mathrm{L^k} - s^k \\ \eta_1 \frac{(U^k)^t - (TAC_k(t_{0k}, t_{1k}, T_k))^t}{(s^k)^t} & if \ U^k - s^k \leq TAC_k(t_{0k}, t_{1k}, T_k) \leq \mathrm{U^k} \\ 0 & if \quad TAC_k(t_{0k}, t_{1k}, T_k) > \mathrm{U^k} \end{cases}$$

$$I_{\tilde{h}}^{E_2}(TAC_k(t_{0k}, t_{1k}, T_k)) = \begin{cases} 1 & if \quad TAC_k(t_{0k}, t_{1k}, T_k) < \mathrm{L^k} - s^k \\ \eta_2 \frac{(U^k)^t - (TAC_k(t_{0k}, t_{1k}, T_k))^t}{(s^k)^t} & if \ U^k - s^k \leq TAC_k(t_{0k}, t_{1k}, T_k) \leq \mathrm{U^k} \\ 0 & if \quad TAC_k(t_{0k}, t_{1k}, T_k) > \mathrm{U^k} \end{cases}$$

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$$I_{h^-}^{E_n}(TAC_k(t_{0k},t_{1k},T_k)) = \begin{cases} 1 & if \quad TAC_k(t_{0k},t_{1k},T_k) < \mathrm{L^k} - s^k \\ \eta_n \frac{(U^k)^t - (TAC_k(t_{0k},t_{1k},T_k))^t}{(S^k)^t} & if \;\; U^k - s^k \le TAC_k(t_{0k},t_{1k},T_k) \le \mathrm{U^k} \\ 0 & if \quad TAC_k(t_{0k},t_{1k},T_k) > \mathrm{U^k} \end{cases}$$

The falsity hesitant-membership function:

$$F_{h^-}^{E_1}(TAC_k(t_{0k},t_{1k},T_k)) = \begin{cases} 0 & if \;\; TAC_k(t_{0k},t_{1k},T_k) < \mathrm{L^k} + r^k \\ \xi_1 \frac{(TAC_k(t_{0k},t_{1k},T_k))^t - (L^k)^t - (r^k)^t}{(U^k)^t - (L^k)^t - (r^k)^t} & if \;\; L^k + r^k \le TAC_k(t_{0k},t_{1k},T_k) \le \mathrm{U^k} \\ 1 & if \quad TAC_k(t_{0k},t_{1k},T_k) > \mathrm{U^k} \end{cases}$$

$$F_{h^-}^{E_2}(TAC_k(t_{0k},t_{1k},T_k)) = \begin{cases} 0 & if \;\; TAC_k(t_{0k},t_{1k},T_k) < \mathrm{L^k} + r^k \\ \xi_2 \frac{(TAC_k(t_{0k},t_{1k},T_k))^t - (L^k)^t - (r^k)^t}{(U^k)^t - (L^k)^t - (r^k)^t} & if \;\; L^k + r^k \le TAC_k(t_{0k},t_{1k},T_k) \le \mathrm{U^k} \\ 1 & if \quad TAC_k(t_{0k},t_{1k},T_k) > \mathrm{U^k} \end{cases}$$

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$$F_{h^-}^{E_n}(TAC_k(t_{0k},t_{1k},T_k)) = \begin{cases} 0 & if \;\; TAC_k(t_{0k},t_{1k},T_k) < \mathrm{L^k} + r^k \\ \xi_n \frac{(TAC_k(t_{0k},t_{1k},T_k))^t - (L^k)^t - (r^k)^t}{(U^k)^t - (L^k)^t - (r^k)^t} & if \;\; L^k + r^k \le TAC_k(t_{0k},t_{1k},T_k) \le \mathrm{U^k} \\ 1 & if \quad TAC_k(t_{0k},t_{1k},T_k) > \mathrm{U^k} \end{cases}$$

Here t>0 is a parameter and $s^k, r^k \in (0,1) \forall\, k = 1,2,3, \dots\dots, n$ are the indeterminacy and falsity tolerance values. They are assigned by decision making and h^- represents the minimization type hesitant objective function.

$T_{h^-}^{E_1}(TAC_k(t_{0k},t_{1k},T_k))$, $I_{h^-}^{E_1}(TAC_k(t_{0k},t_{1k},T_k))$, $F_{h^-}^{E_1}(TAC_k(t_{0k},t_{1k},T_k))$ are the truth, indeterminacy and falsity hesitant membership degree assigned by 1^{st} expert.

$T_{h^-}^{E_2}(TAC_k(t_{0k},t_{1k},T_k))$, $I_{h^-}^{E_2}(TAC_k(t_{0k},t_{1k},T_k))$, $F_{h^-}^{E_2}(TAC_k(t_{0k},t_{1k},T_k))$ are the truth, indeterminacy and falsity hesitant membership degree assigned by 2^{st} expert.

……………………………

$T_{h^-}^{E_n}(TAC_k(t_{0k},t_{1k},T_k))$, $I_{h^-}^{E_n}(TAC_k(t_{0k},t_{1k},T_k))$, $F_{h^-}^{E_n}(TAC_k(t_{0k},t_{1k},T_k))$ are the truth, indeterminacy and falsity hesitant membership degree assigned by n^{st} expert.

With the help of above membership function our multi-objective inventory model transforms into the following form

$$\text{Max } \frac{\sum_1^n \zeta_i}{n}$$

$$\text{Max } \frac{\sum_1^n \eta_i}{n}$$

$$\text{Max } \frac{\sum_1^n \xi_i}{n}$$

s.t,

$$T_{h^-}^{E_i}\big(TAC_k(t_{0k},t_{1k},T_k)\big) \ge \zeta_i \,;\, I_{h^-}^{E_i}\big(TAC_k(t_{0k},t_{1k},T_k)\big) \ge \eta_i \,;\, F_{h^-}^{E_i}\big(TAC_k(t_{0k},t_{1k},T_k)\big) \le \xi_i$$

$$\sum_{i=i}^n w_i Q_i \le W \qquad \text{For i=1, 2, 3, 4,………., n}$$

$$\text{Where } Q_i = \tilde{a}_i t_{0i} + \tilde{b}_i[(t_{1i} - t_{0i}) - \frac{\theta_i}{3}(t_{1i}^3 - t_{0i}^3) + \frac{k_i - \alpha_i}{2}(t_{1i}^2 - t_{0i}^2)] + \frac{\tilde{a}_i}{\delta_i}\{\log\{1 + \delta_i(T_i - t_{1i})\}\}$$

$$t_{0k}, t_{1k}, T_k \ge 0;\, \zeta_i,\ \eta_i,\ \xi_i \in (0,1);\ \zeta_i + \eta_i + \xi_i \le 3;\, \zeta_i \ge \eta_i;\ \zeta_i \ge \xi_i;\ \ \forall\, i = 1,2,3, \dots\dots, n \tag{7.1}$$

Using the linear membership function our multi-item inventory model can be written as follows:

Max $\frac{\zeta_1+\zeta_2+ \dots\dots\dots +\zeta_n}{n} + \frac{\eta_1+\eta_2+ \dots\dots\dots +\eta_n}{n} + \frac{\xi_1+\xi_2+ \dots\dots\dots +\xi_n}{n}$

Subject to,

$T_{h^-}^{E_1}\left(TAC_k(t_{0k},t_{1k},T_k)\right) \geq \zeta_1, T_{h^-}^{E_2}\left(TAC_k(t_{0k},t_{1k},T_k)\right) \geq \zeta_2, \dots, T_{h^-}^{E_n}\left(TAC_k(t_{0k},t_{1k},T_k)\right) \geq \zeta_n;$

$I_{h^-}^{E_1}\left(TAC_k(t_{0k},t_{1k},T_k)\right) \geq \eta_1, I_{h^-}^{E_2}\left(TAC_k(t_{0k},t_{1k},T_k)\right) \geq \eta_2, \dots\dots, I_{h^-}^{E_n}\left(TAC_k(t_{0k},t_{1k},T_k)\right) \geq \eta_n;$

$F_{h^-}^{E_1}\left(TAC_k(t_{0k},t_{1k},T_k)\right) \leq \xi_1, F_{h^-}^{E_2}\left(TAC_k(t_{0k},t_{1k},T_k)\right) \leq \xi_2, \dots, F_{h^-}^{E_n}\left(TAC_k(t_{0k},t_{1k},T_k)\right) \leq \xi_n;$

$\sum_{i=i}^{n} w_i Q_i \leq W$; For i=1, 2, 3, 4, ………., n

Where $Q_i = \widetilde{\alpha_\iota} t_{0i} + \widetilde{b_\iota}[(t_{1i}-t_{0i}) - \frac{\theta_i}{3}(t_{1i}^3 - t_{oi}^3) + \frac{k_i-\alpha_i}{2}(t_{1i}^2 - t_{0i}^2)] + \frac{\widetilde{\alpha_\iota}}{\delta_i}\{\log\{1+\delta_i(T_i - t_{1i})\}\}$

$$0 \leq \zeta_1,\zeta_2,\dots\dots\dots,\zeta_n \leq 1; 0 \leq \eta_1,\eta_2,\dots\dots\dots,\eta_n \leq 1; 0 \leq \xi_1,\xi_2,\dots\dots\dots,\xi_n \leq 1;$$

$t_{0k},t_{1k},T_k \geq 0;\ \zeta_i,\eta_i,\xi_i \in (0,1);\ \zeta_i + \eta_i + \xi_i \leq 3; \zeta_i \geq \eta_i;\ \zeta_i \geq \xi_i;\ \forall\, i = 1,2,3,\dots\dots,n$ (7.2)

From this we get the optimal solution for our multi-item inventory model.

8. NUMERICAL EXAMPLE:

Let us take our inventory model which consists of two items only. And the several parameter's values are taken at form of generalized fuzzy number and some values are in crisp. Here we take total storage area W=3500m[2].

Minimize { $\widetilde{TAC_1}(t_{01},t_{11},T_1), \widetilde{TAC_2}(t_{02},t_{12},T_2)$}

Subject to:$\sum_{i=i}^{n} w_i Q_i \leq W$ for i=1, 2

Where, $\widetilde{TAC_\iota}(t_{0i}, t_{1i}, T_i) = \frac{1}{T_i}[\widetilde{A_\iota} + \frac{\widetilde{\alpha_\iota}\widetilde{h_\iota}t_{0i}^3}{6} + \frac{\widetilde{b_\iota}\widetilde{h_\iota}t_{0i}^2}{2}\left[(t_{1i}-t_{0i}) - \frac{\theta_i}{3}(t_{1i}^3 - t_{0i}^3) + \frac{k_i-\alpha_i}{2}(t_{1i}^2 - t_{0i}^2)\right] +$

$\widetilde{b_\iota}\widetilde{h_\iota}\{\frac{t_{1i}}{2}(t_{1i}^2 - t_{0i}^2) - \frac{1}{3}(t_{1i}^3 - t_{0i}^3) - \frac{\theta_i}{3}\{\frac{t_{1i}^3}{2}(t_{1i}^2 - t_{0i}^2) - \frac{1}{5}(t_{1i}^5 - t_{0i}^5)\} + \frac{k_i-\alpha_i}{2}\{\frac{t_{1i}^2}{2}(t_{1i}^2 - t_{0i}^2) - \frac{1}{4}(t_{1i}^4 - t_{0i}^4)\}\ \}$

$+\widetilde{C_{2\iota}}\{\frac{k_i-\alpha_i}{2}(t_{1i}^2 - t_{0i}^2) + \frac{\widetilde{b_\iota}\alpha_i}{2}(t_{1i}^2 - t_{0i}^2) - \frac{\theta_i}{3}(t_{1i}^3 - t_{0i}^3) - k_i(t_{1i}-t_{0i})\} + \widetilde{C_{3\iota}}\ \frac{\widetilde{\alpha_\iota}}{\delta_i}(T_i - t_{1i}).\log\{1+\delta_i(T_i -$

$t_{1i})\} + \widetilde{C_{4\iota}}\widetilde{\alpha_\iota}\{T_i - t_{1i} - \frac{\log\{1+\delta_i(T_i-t_{1i})\}}{\delta_i}\} + \widetilde{C_{1\iota}}\ \{\widetilde{\alpha_\iota} t_{0i} + \widetilde{b_\iota}[(t_{1i}-t_{0i}) - \frac{\theta_i}{3}(t_{1i}^3 - t_{oi}^3) + \frac{k_i-\alpha_i}{2}(t_{1i}^2 - t_{0i}^2)] +$

$\frac{\widetilde{\alpha_\iota}}{\delta_i}\{\log\{1+\delta_i(T_i - t_{1i})\}\}\}]$

And $Q_i = \widetilde{\alpha_\iota} t_{0i} + \widetilde{b_\iota}[(t_{1i}-t_{0i}) - \frac{\theta_i}{3}(t_{1i}^3 - t_{oi}^3) + \frac{k_i-\alpha_i}{2}(t_{1i}^2 - t_{0i}^2)] + \frac{\widetilde{\alpha_\iota}}{\delta_i}\{\log\{1+\delta_i(T_i - t_{1i})\}\}$ for i=1, 2

(8.1)

The values of those parameters which are taken as generalized trapezoidal fuzzy number are given in table-1.

Table-1

Input data

Parameters	Item

	Item1	Item2
$\widetilde{a_\iota}$	(45,43,39,33 ;0.9)	(39,41,44,46 ;0.8)
$\widetilde{b_\iota}$	(50,45,41,39 ;0.8)	(49,53,57,61 ;0.6)
$\widetilde{C_{1\iota}}$	(17,15,13,10 ;0.8)	(16,19,22,23 ;0.6)
$\widetilde{C_{2\iota}}$	(15,16,18,19 ;0.5)	(9.3,11.2,13.1,13.9 ;0.8)
$\widetilde{C_{3\iota}}$	(7,6,4,3 ;0.8)	(7,9,11,13 ;0.5)
$\widetilde{C_{4\iota}}$	(2.8,2.6,2.5,2.1 ;0.8)	(2,3,4,6 ;0.8)
$\widetilde{H_\iota}$	(3.5,4.3,5.5,6.7 ;0.5)	(3.2,3.3,4.7,5.8 ;0.6)
$\widetilde{A_\iota}$	(113,101,97,89 ;0.5)	(99,105,101,111 ;0.5)
$\widetilde{w_\iota}$	(2.7,2.5,2.6,2.2 ;0.8)	(2.5,2.6,2.8,2.1 ;0.8)

The values of those parameters which are taken as crisp are given below.

$$\alpha_1 = .017, \delta_1 = .216, k_1 = .001, \theta_1 = .004, t_{01} = 0.12$$

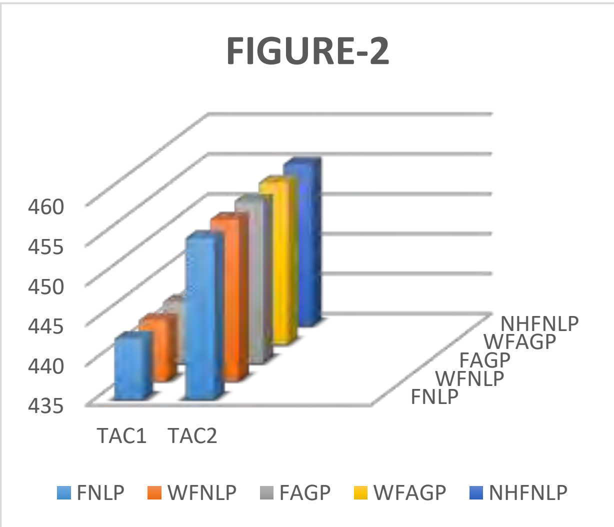
$$\alpha_2 = .018, \delta_2 = .216, k_2 = .001, \theta_2 = .004, t_{02} = 0.12$$

We take the weights w_1=.5, w_2=.5 for WFNLP and weights w_1=.5, w_2=.5 for WFAGP.

Table-2

Optimal solution by FNLP, WFNLP, FAGP, WFAGP, NHFNLP

Methods	t_{11}	T_1	$TAC_1(t^*_{01}, t^*_{11}, T^*_1)$	t_{12}	T_2	$TAC_2(t^*_{02}, t^*_{12}, T^*_2)$
FNLP	1.199857	1.420192	442.6481	1.244292	1.421455	455.2850
WFNLP	1.199205	1.419957	422.6481	1.244292	1.421455	455.2850
FAGP	1.200245	1.420618	442.6481	1.244230	1.421975	455.2850
WFAGP	1.199774	1.419792	442.6481	1.244322	1.421549	455.2850
NHFNLP	1.99023	1.420511	442.6482	1.244454	1.421594	455.2850



We can see from the figure-2 that total average cost of two items is almost same. FNLP, WFNLP, FAGP, WFAGP all are giving the same average cost but NHFNLP gives slide different value of total average cost. So we can say all the methodology applied here are almost identical for this proposed model.

Figure-2: **GRAPH OF TOTAL AVERAGE COST WITH DIFERENT METHODS.**

9. Sensitivity analysis:

Here we have discussed the optimal solution of multi-item (two-item) inventory model with different weights by WFNLP technique.

TABLE-3

Different weight	TAC1($t_{01}^*, t_{11}^*, T_1^*$)	TAC2($t_{02}^*, t_{12}^*, T_2^*$)
w_1=.1, w_2=.9	442.6481	455.4970
w_1=.2, w_2=.8	442.4681	455.4639
w_1=.3, w_2=.7	442.4681	455.4213
w_1=.4, w_2=.6	442.4681	455.3645
w_1=.5, w_2=.5	442.4681	455.2850
w_1=.6, w_2=.4	442.7164	455.2850
w_1=.7, w_2=.3	442.7652	455.2850
w_1=.8, w_2=.2	442.8018	455.2850
w_1=.9, w_2=.1	442.8302	455.2850

Here we can see that if w_1 increases total average cost TAC1 ($t_{01}^*, t_{11}^*, T_1^*$) increases and total average cost TAC2 ($t_{02}^*, t_{12}^*, T_2^*$) decreases. If w_2 increases total average cost TAC1 ($t_{01}^*, t_{11}^*, T_1^*$) decreases and total average cost TAC2 ($t_{02}^*, t_{12}^*, T_2^*$) increases.

Here we have discussed the optimal solution of multi-item (two-item) inventory model with different weight by WFAGP technique.

TABLE-4

Different weight	TAC1($t_{01}^*, t_{11}^*, T_1^*$)	TAC2($t_{02}^*, t_{12}^*, T_2^*$)
w_1=.1, w_2=.9	442.6481	455.2850
w_1=.2, w_2=.8	442.4681	455.2850
w_1=.3, w_2=.7	442.4681	455.2850
w_1=.4, w_2=.6	442.4681	455.2850
w_1=.5, w_2=.5	442.4681	455.2850
w_1=.6, w_2=.4	442.6481	455.2850
w_1=.7, w_2=.3	442.6481	455.2850
w_1=.8, w_2=.2	442.6481	455.2850
w_1=.9, w_2=.1	442.6481	455.2850

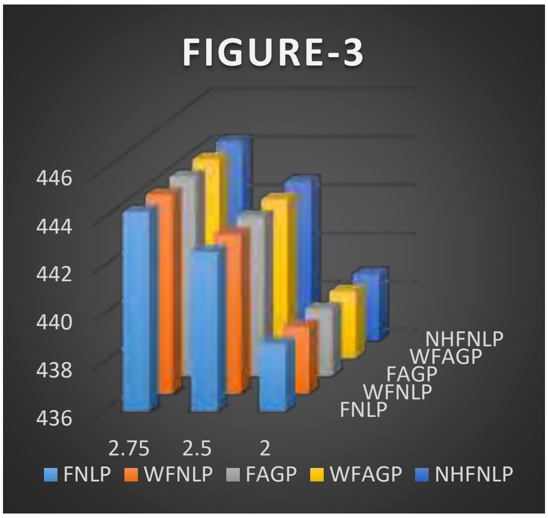
Here we can see that total average cost of TAC1 ($t_{01}^*, t_{11}^*, T_1^*$) and TAC2 ($t_{02}^*, t_{12}^*, T_2^*$) are constant.it does not depend on different values of weight w_1 and w_2.

Now we focus on the change of total average cost depending on various holding cost by FNLP, WFNLP, FAGP, WFAGP and NHFNLP techniques are represented in table -5. Here we take the weights (w_1=w_2=.5) for both WFNLP and WFAGP technique.

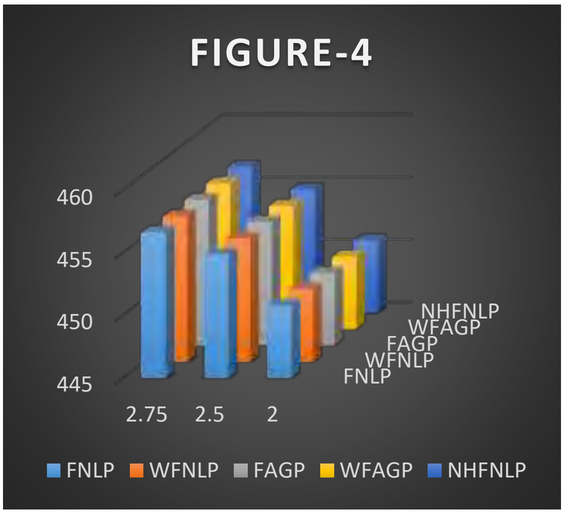
TABLE-5

Methods	HOLDING	TAC1($t_{01}^*, t_{11}^*, T_1^*$)	t_{11}^*	T_1^*	TAC2($t_{02}^*, t_{12}^*, T_2^*$)	t_{12}^*	T_2^*

	COST[h1 (1st item)=h2 (2nd item=h]						
FNLP	2.75	444.3175	1.156815	1.385719	456.7003	1.210349	1.392935
	2.5	442.6481	1.200486	1.421524	454.9165	1.253672	1.429418
	2.0	438.8166	1.307034	1.509127	450.8424	1.365131	1.524196
WFNLP	2.75	444.3175	1.156815	1.385719	456.7003	1.208066	1.390538
	2.5	442.6481	1.199541	1.420801	454.9165	1.253672	1.429418
	2.0	438.8166	1.307034	1.509127	450.8424	1.366006	1.525305
FAGP	2.75	444.3175	1.156815	1.385719	456.7003	1.209225	1.392011
	2.5	442.6481	1.200144	1.420725	454.9125	1.253672	1.429418
	2.0	438.8166	1.307012	1.508977	450.8423	1.364764	1.525177
WFAGP	2.75	444.3175	1.156815	1.385719	456.7003	1.209045	1.391500
	2.5	442.6481	1.199810	1.420557	454.9165	1.253672	1.429418
	2.0	438.8166	1.307034	1.509127	450.8423	1.364750	1.525606
NHFNLP	2.75	444.3177	1.154959	1.384276	456.7004	1.210351	1.393492
	2.5	442.6481	1.199795	1.420212	454.9165	1.253362	1.429104
	2.0	438.8166	1.307033	1.509126	450.8423	1.364590	1.523901



Total average cost of first item in different technique and different values of holding cost.



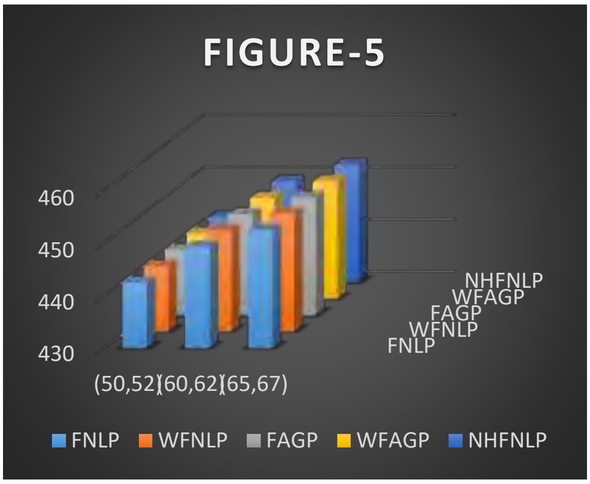
Total average cost of second item in different technique and different values of holding cost.

In above figures (figue-3 and figure-4) we can see that when we increase the value of holding cost (h) corresponding average cost value also increases. And it is true for all methodology we are dealing with.

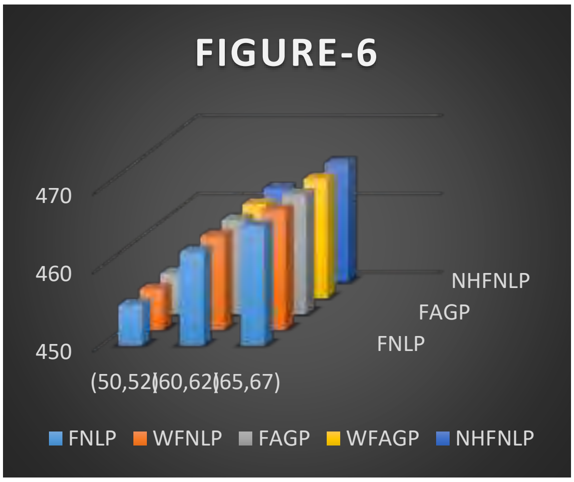
Now we focused on the change of total average cost depending on various ordering cost by FNLP, WFNLP, FAGP, WFAGP and NHFNLP techniques are represented in table-6. Here we take the weights ($w_1=w_2=.5$) for both WFNLP and WFAGP technique.

Table-6

Methods	Ordering cost for first item(A_1)	$TAC_1(t_{01}^*, t_{11}^*, T_1^*)$	t_{11}^*	T_1^*	Ordering cost for second item(A_2)	$TAC_2(t_{02}^*, t_{12}^*, T_2^*)$	t_{12}^*	T_2^*
FNLP	50	442.6481	1.199857	1.420192	52	455.2850	1.244292	1.421455
	60	449.4415	1.268755	1.522320	62	462.0886	1.314198	1.517613
	65	452.6740	1.299211	1.569278	67	465.3352	1.346345	1.562357
WFNLP	50	442.6481	1.199205	1.419957	52	455.2850	1.244292	1.421455
	60	449.4415	1.269526	1.523659	62	462.0886	1.314198	1.517613
	65	452.6740	1.300564	1.569681	67	465.3352	1.346345	1.562357
FAGP	50	442.6481	1.200245	1.420618	52	455.2850	1.244230	1.421975
	60	449.4415	1.268980	1.521919	62	462.0886	1.313272	1.516051
	65	452.6740	1.301278	1.571637	67	465.3352	1.346345	1.562357
WFAGP	50	442.6481	1.199774	1.419792	52	455.2850	1.244322	1.421549
	60	449.4415	1.270147	1.524057	62	462.0886	1.313251	1.516190
	65	452.6740	1.300205	1.569592	67	465.3352	1.346344	1.562291
NHFNLP	50	442.6482	1.99023	1.420511	52	455.2850	1.244454	1.421594
	60	449.4415	1.267611	1.522379	62	462.0886	1.314243	1.5185587
	65	452.6740	1.300222	1.569239	67	465.3353	1.348005	1.563531



Total average cost of first item in different technique and different values of ordering Cost.



Total average cost of second item in different technique and different values of ordering Cost.

In above figures (figue-5 and figure-6) we can see that when we increase the value of ordering cost (A) corresponding average cost value also increases. And it is true for all methodology we are dealing with.

9. CONCLUSION:

This paper presents a multi-objective inventory model in fuzzy environment. Here the model is directly connected with the real life business enterprises which consider the fact that storage space is limited and storage item is deteriorated during storage period and here the demand rate, deterioration and holding cost depend upon time. The model is developed in deterministic manner in which we use a preservation constant to reduce deterioration. After getting the result we have seen that the effect of deterioration can be controlled slightly by using a preservation condition. The model allows for shortages and in the shortage period the demand is partially backlogging. For the presence of uncertainty some cost parameters are considered as generalized trapezoidal fuzzy number. Finally the proposed model has been verified in several techniques as FNLP, WFNLP, FAGP, WFAGP and NHFNLP Techniques. The acquired results assure us the stability and validity of this model.

The proposed model can be implemented by taking more realistic assumptions such as probabilistic demand, power demand, finite replenishment etc. Instead of taking generalized trapezoidal fuzzy number we may also take triangular fuzzy number, pentagonal fuzzy number, parabolic flat fuzzy number etc. for all cost parameters we have considered to develop our multi-item inventory model. It will be also very interesting if we can develop a model where deterioration is depending with preservation condition w.r.t time. Instead of taking preservation condition as a constant function it would be more realistic and interesting if it would be taken as some function of time that means preservation power should be decrease with the increasing of time.

Highlights of the manuscript:

i) An inventory model with time dependent deterioration and time-dependent holding cost.

ii) Storage space is limited for this proposed model

iii) Preservation constant is used for controlling the effect of deterioration.

iv) Different methodologies are used for solving this model that is for finding the minimum total average cost.

v) All the methods gives us more or less same optimum values of total average cost.

vi) Neutrosophic hesitant technique gives us almost same optimum value as FNLP and FAGP methods give.

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Author's Contribution:

Dr Sahidul Islam gave the idea in developing the Inventory model and described how the preservation condition could be used for this model and also discussed various methodologies for solving this model. Kausik Das implemented this concept in this model and describes various methodology for solving the inventory model mainly focused on the technique neutrosophic hesitant fuzzy programming. Both authors participate in developing the manuscript.

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Similarity Measure for m-Polar Interval Valued Neutrosophic Soft Set with Application for Medical Diagnoses

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Abstract:

The similarity measure is used to tackle many issues that include indistinct as well as blurred information excluding is not in a position to deal with the general fuzziness along with obscurity of the problems that have various information. The main purpose of this research is to propose a multi-polar interval-valued neutrosophic soft set (mPIVNSS) with operations and basic properties. We also develop Hamming distance and Euclidean distance by using mPIVNSS and numerical examples and use the developed distances to introduce similarity measures. By using the developed similarity measures a decision-making approach is presented for mPIVNSS. Finally, we used the developed decision-making approach for medical diagnosis.

Keywords: Multipolar interval-valued neutrosophic set; multipolar interval-valued neutrosophic soft set; similarity measures.

1. Introduction

Uncertainty plays a dynamic role in many areas of life (such as modeling, medicine, engineering, etc.). However, researchers raised a general question, that is, how do we express and use the concept of uncertainty in mathematical modeling. Many researchers in the world have proposed and recommended different methods of using uncertainty. First of all, Zadeh proposed the concept of fuzzy sets [1] to solve those problems containing uncertainty and ambiguity. It can be seen that in some cases, fuzzy sets cannot handle situations. To overcome such situations, Turksen [2] proposed the idea of interval-valued fuzzy sets (IVFS). In some cases, we must consider the unbiased value of the appropriate representation of the object under the conditions of uncertainty and vagueness, as the non-membership values of the appropriate representation of the object, these fuzzy sets or IVFS cannot handle. To overcome these difficulties, Atanassov proposed the concept of an Intuitionistic Fuzzy Set (IFS) [3]. Zulqarnain et. [4] introduced the correlation coefficient for interval-valued intuitionistic fuzzy soft sets and established the TOPSIS technique based on their developed correlation measures to solve decision-making complications. The theory proposed by Atanassov only deals with under-considered data and membership and non-membership values. However, the

IFS theory cannot deal with overall incompatibility and imprecise information. To solve this incompatible and imprecise information, Smarandache [5] proposed the idea of NS.

Molodtsov [6] proposed a general mathematical tool to deal with uncertain, ambiguous, and undefined substances, called soft sets (SS). Maji et al. [7] Extended the work of SS and defined some operations and their features. They also used the SS theory to make decisions [8]. Ali et. al. [9] Modified the Maji method of SS and developed some new operations with its properties. Sezgin and Atagun [10] proved De Morgan's SS theory and law by using different operators. Cagman and Enginoglu [11] proposed the concept of soft matrices with operations and discussed their properties. They also introduced a decision-making method to solve problems that contain uncertainty. In [12], they modified the actions proposed by Molottsov's SS. In [13], the author plans to perform some new operations on soft matrices, such as soft differential product, soft limited differential product, soft extended differential product, and weak extended differential product. Zulqarnain et al. [14, 15] discussed the Pythagorean fuzzy soft sets and established the aggregation operator and TOPSIS technique to solve the MCDM problem.

Maji [16] put forward the idea of NSS with necessary operations and characteristics. The idea of NSS possibility was put forward by Karaaslan [17] and introduced a neutrosophic soft decision method to solve those uncertain problems based on And-product. Broumi [18] developed a generalized NSS with certain operations and properties and used the proposed concept for decision-making. To solve the MCDM problem with single-valued neutrosophic numbers (SVNN) proposed by Deli and Subas [19], they constructed the concept of SVNN cut sets. Based on the correlation of IFS, the CC term of SVNS was introduced [20]. In [21], the idea of simplifying NS introduced some operational laws and aggregation operators, such as weighted arithmetic and weighted geometric average operator. They constructed the MCDM method based on the proposed aggregation operator. Mukherjee and Das [22] neutrosophic bipolar vague soft sets and some of its operations using. It is the combination of neutrosophic bipolar vague sets and soft sets neutrosophic bipolar vague soft sets and some of its operations. It is the combination of neutrosophic bipolar vague sets and soft sets. Zulqarnain et al. [23, 24] utilized the neutrosophic TOPSIS model to solve the MCDM problem and for the selection of suppliers in the production industry. Masooma et al. [25] by combining multi-polar fuzzy sets and neutrosophic sets, developed a new concept called multi-polar neutrosophic sets. They also established various representations and instance arithmetic.

In the past few years, many mathematicians have developed various similarity measures, correlation coefficients, aggregation operators, and decision-making applications. These structures are based on different sets and provide better solutions to decision-making problems. It has multiple applications in different fields such as pattern recognition, medical diagnosis, artificial intelligence, social science, business, and multi-attribute decision-making problems. Garg [26] developed the MCDM method based on weighted cosine similarity measures under an intuitionistic fuzzy environment and used the proposed technique for pattern recognition and medical diagnoses. To measure the relative strength of IFS Garg and Kumar [27] presented some new similarity measures, they also formulated a connection number for set pair analysis (SPA) and developed some new similarity measures and weighted similarity measures based on defined SPA. Nguyen et al. [28] defined some similarity measures for PFS by using the exponential function for the membership and non-membership degrees with its several properties and relations. Peng and Garg [29] presented some diverse types of similarity measures for PFS with multiple parameters. In [30] the authors established the concept of mPNSS with its properties and operators, they also developed the distance-based similarity measures and used the proposed similarity measures for decision making and medical diagnoses. Recently, Smarandache [31] extended the concept of the SS to hypersoft set (HSS) by replacing the single-parameter function F with a multi-parameter (sub-attribute) function defined on Cartesian products of n different attributes. The established HSS is more flexible than SS and is more suitable for the decision-making environment. He also introduced the further extension of HSS, such as crisp HSS, fuzzy HSS, intuitionistic fuzzy HSS, neutrosophic HSS, and plithogenic HSS. Nowadays, HSS theory and its extensions are developing rapidly. Many researchers have developed different operators and properties based on HSS and its extensions [32-44].

In this era, professionals believe that real life is moving in the direction of multi-polarization. Therefore, there is no doubt that the multi-polarization of information has played an important role in the prosperity of many fields of science and technology. In neurobiology, multipolar neurons in the brain collect a lot of information from other neurons. In information technology, multi-polar technology can be used to control a wide range of structures. In the full text, the motivation for the expansion and mixed work of this research is gradually given. We proved that under any appropriate circumstances, different hybrid structures containing fuzzy sets will be converted into special privileges of mPIVNSS. The concept of a neutrosophic environment to a multipolar interval-valued neutrosophic soft set is novel. We tend to discuss the effectiveness, flexibility, quality, and advantages of planning work and algorithms. This research will be the most versatile form and will combine data to a considerable extent, as well as appropriate medicine, engineering, artificial intelligence, agriculture, and other daily life complications. In the future, the current work may be competent for other methods and different types of mixed structures.

The following research is organized as follows: In section 2, we recollected some basic definitions which are used in the following sequel such as NS, SS, NSS, and multipolar neutrosophic set. In section 3, we proposed the mPIVNSS with its properties and operations. In section 4, distance-based similarity measures have been developed by using Hamming distance and Euclidean distance between two mPIVNSS. In section 5, we use the developed distance-based similarity measures for medical diagnoses. Finally, the conclusion and future directions are presented in section 6.

2. Preliminaries

In this section, some basic concepts have been recalled such as NS, SS, NSS, and IVNSS, etc. which are used in the following sequel.

Definition 2.1 [7]

Let $\mathcal{U}$ be a universe and $\mathcal{A}$ be an NS on $\mathcal{U}$ is defined as $\mathcal{A} = \{< u, \mathcal{u}_{\mathcal{A}}(u), \mathcal{v}_{\mathcal{A}}(u), \mathcal{w}_{\mathcal{A}}(u) >: u \in \mathcal{U}\}$, where $\mathcal{u}, \mathcal{v}, \mathcal{w}: \mathcal{U} \to]0^-, 1^+[$ and $0^- \leq \mathcal{u}_{\mathcal{A}}(u) + \mathcal{v}_{\mathcal{A}}(u) + \mathcal{w}_{\mathcal{A}}(u) \leq 3^+$.

Definition 2.2 [25]

Let $\mathcal{U}$ be the universal set and $\mathcal{E}$ be the set of attributes concerning $\mathcal{U}$, then $\mathcal{F}_{\mathcal{E}}$ is said to multipolar neutrosophic set if

$\mathcal{F}_{\mathcal{E}} = \{< u, (s_i \bullet \mathcal{u}_e(u), s_i \bullet \mathcal{v}_e(u), s_i \bullet \mathcal{w}_e(u)) >: u \in \mathcal{U}, e \in \mathcal{E}, i = 1, 2, 3, \ldots. m\}$, where $s_i \bullet \mathcal{u}_{\mathcal{E}}$, $s_i \bullet \mathcal{v}_{\mathcal{E}}, s_i \bullet \mathcal{w}_{\mathcal{E}}: \mathcal{U} \to [0, 1]$, and $0 \leq s_i \bullet \mathcal{u}_{\mathcal{E}}(u) + s_i \bullet \mathcal{v}_{\mathcal{E}}(u) + s_i \bullet \mathcal{w}_{\mathcal{E}}(u) \leq 3$; $i = 1, 2, 3, \ldots. m$. $\mathcal{u}_e$, $\mathcal{v}_e$, and $\mathcal{w}_e$ represent the truth, indeterminacy, and falsity of the considered alternative.

Definition 2.3 [3]

Let $\mathcal{U}$ be the universal set and $\mathcal{E}$ be the set of attributes concerning $\mathcal{U}$. Let $\mathcal{P}(\mathcal{U})$ be the power set of $\mathcal{U}$ and $\mathcal{A} \subseteq \mathcal{E}$. A pair $(\mathcal{F}, \mathcal{A})$ is called a soft set over $\mathcal{U}$ and its mapping is given as

$$\mathcal{F}: \mathcal{A} \to \mathcal{P}(\mathcal{U})$$

It is also defined as:

$$(\mathcal{F}, \mathcal{A}) = \{\mathcal{F}(e) \in \mathcal{P}(\mathcal{U}): e \in \mathcal{E}, \mathcal{F}(e) = \emptyset \; if \; e \neq \mathcal{A}\}$$

Definition 2.4 [16]

Let $\mathcal{U}$ be the universal set and $\mathcal{E}$ be the set of attributes concerning $\mathcal{U}$. Let $\mathcal{P}(\mathcal{U})$ be the set of Neutrosophic values of $\mathcal{U}$ and $\mathcal{A} \subseteq \mathcal{E}$. A pair $(\mathcal{F}, \mathcal{A})$ is called a Neutrosophic soft set over $\mathcal{U}$ and its mapping is given as

$$\mathcal{F}: \mathcal{A} \to \mathcal{P}(\mathcal{U})$$

Definition 2.5 [46]

Let $\boldsymbol{\mathcal{U}}$ be a universal set, then interval valued neutrosophic set can be expressed by the set $\boldsymbol{\mathcal{A}} = \{< \boldsymbol{u}, \boldsymbol{u}_{\mathcal{A}}(\boldsymbol{u}), \boldsymbol{v}_{\mathcal{A}}(\boldsymbol{u}), \boldsymbol{w}_{\mathcal{A}}(\boldsymbol{u}) > : \boldsymbol{u} \in \boldsymbol{\mathcal{U}}\}$, where $\boldsymbol{u}_{\mathcal{A}}$, $\boldsymbol{v}_{\mathcal{A}}$, and $\boldsymbol{w}_{\mathcal{A}}$ are truth, indeterminacy, and falsity membership functions for $\boldsymbol{\mathcal{A}}$ respectively, $\boldsymbol{u}_{\mathcal{A}}$, $\boldsymbol{v}_{\mathcal{A}}$, and $\boldsymbol{w}_{\mathcal{A}} \subseteq [0, 1]$ for each $\boldsymbol{u} \in \boldsymbol{\mathcal{U}}$. Where

$\boldsymbol{u}_{\mathcal{A}}(\boldsymbol{u}) = \left[\boldsymbol{u}^{L}_{\mathcal{A}}(\boldsymbol{u}), \boldsymbol{u}^{U}_{\mathcal{A}}(\boldsymbol{u})\right]$

$\boldsymbol{v}_{\mathcal{A}}(\boldsymbol{u}) = \left[\boldsymbol{v}^{L}_{\mathcal{A}}(\boldsymbol{u}), \boldsymbol{v}^{U}_{\mathcal{A}}(\boldsymbol{u})\right]$

$\boldsymbol{w}_{\mathcal{A}}(\boldsymbol{u}) = \left[\boldsymbol{w}^{L}_{\mathcal{A}}(\boldsymbol{u}), \boldsymbol{w}^{U}_{\mathcal{A}}(\boldsymbol{u})\right]$

For each point $\boldsymbol{u} \in \boldsymbol{\mathcal{U}}$, $0 \leq \boldsymbol{u}_{\mathcal{A}}(\boldsymbol{u}) + \boldsymbol{v}_{\mathcal{A}}(\boldsymbol{u}) + \boldsymbol{w}_{\mathcal{A}}(\boldsymbol{u}) \leq 3$ and IVN($\boldsymbol{\mathcal{U}}$) represent the family of all interval valued neutrosophic sets.

Definition 2.6 [45]

Let U be an initial universe set, IVN(U) denotes the set of all interval valued neutrosophic sets of U and $\boldsymbol{\mathcal{E}}$ be a set of parameters that describe the elements of U. An interval-valued neutrosophic soft sets(ivn-soft sets) over U is a set defined by a set-valued function $\boldsymbol{\Upsilon_K}$ representing a mapping $\boldsymbol{v_K}$: $\boldsymbol{\mathcal{E}}$ → IVN(U) It can be written as a set of ordered pairs

$\boldsymbol{\Upsilon_K}$ = {(x, $\boldsymbol{v_K}$(x)): x $\in$ $\boldsymbol{\mathcal{E}}$}

Here, $\boldsymbol{v_K}$, which is interval-valued neutrosophic sets, is called the approximate function of the ivn-soft sets $\boldsymbol{\Upsilon_K}$ and $\boldsymbol{v_K}$(x) is called the x-approximate value of x $\in$ $\boldsymbol{\mathcal{E}}$. The subscript K in the $\boldsymbol{v_K}$ indicates that $\boldsymbol{v_K}$ is the approximate function of $\boldsymbol{\Upsilon_K}$. Generally if $\boldsymbol{v_K}$, $\boldsymbol{v_L}$, $\boldsymbol{v_M}$,… will be used as an approximate function of $\boldsymbol{\Upsilon_K}$, $\boldsymbol{\Upsilon_L}$, $\boldsymbol{\Upsilon_M}$,…, respectively. Note that the sets of all ivn-soft sets over U will be denoted by IVNSS.

3. Multi-Polar Interval Valued Neutrosophic Soft Set with Aggregate Operators and Properties

In this section, we develop the concept of mPIVNSS and introduce some basic operations on mPIVNSS with their properties.

Definition 3.1

Let $\mathcal{U}$ and E are universal and set of attributes respectively, and $\mathcal{A} \subseteq$ E, if there exists a mapping Φ such as

Φ: $\mathcal{A} \to mPIVNSS^{\mathcal{U}}$

Then (Φ, $\mathcal{A}$) is called mPIVNSS over $\mathcal{U}$ defined as follows

Υ_K = (Φ, $\mathcal{A}$) = $\left\{\left(u, \Phi_{\mathcal{A}(e)}(u)\right): e \in E, u \in \mathcal{U}\right\}$, where

$\Phi_{\mathcal{A}}(e) = \{(e, < u, [s_i \bullet inf\ u_{\mathcal{A}(e)}(u), s_i \bullet sup\ u_{\mathcal{A}(e)}(u)], [s_i \bullet inf\ v_{\mathcal{A}(e)}(u), s_i \bullet sup\ v_{\mathcal{A}(e)}(u)], [s_i \bullet inf\ w_{\mathcal{A}(e)}(u), s_i \bullet sup\ w_{\mathcal{A}(e)}(u)] > : u \in \mathcal{U}, e \in E)\}$, and

$0 \leq s_i \bullet sup\ u_{\mathcal{A}(e)}(u) + s_i \bullet sup\ v_{\mathcal{A}(e)}(u) + s_i \bullet sup\ w_{\mathcal{A}(e)}(u) \leq 3$ for all $i \in$ 1, 2, 3,…, m; $e \in E$ and $u \in \mathcal{U}$.

Definition 3.2

Let Υ_A and $\Upsilon_B \in$ mPIVNSS over $\mathcal{U}$, then Υ_A is called a multi-polar interval-valued neutrosophic soft subset of Υ_B. If

$s_i \bullet inf\ u_{A(e)}(u) \leq s_i \bullet inf u_{B(e)}(u),\ s_i \bullet sup\ u_{A(e)}(u) \leq s_i \bullet sup\ u_{B(e)}(u)$

$s_i \bullet inf\ v_{A(e)}(u) \leq s_i \bullet inf\ v_{B(e)}(u),\ s_i \bullet sup\ v_{A(e)}(u) \leq s_i \bullet sup\ v_{B(e)}(u)$

$s_i \bullet inf\ w_{A(e)}(u) \geq s_i \bullet inf\ w_{B(e)}(u),\ s_i \bullet sup\ w_{A(e)}(u) \geq s_i \bullet sup\ w_{B(e)}(u)$

for all $i \in 1, 2, 3, \ldots, m;\ e \in E$ and $u \in \mathcal{U}$.

Example 1 Assume $\mathcal{U} = \{u_1,\ u_2\}$ be a universe of discourse and $E = \{x_1,\ x_2,\ x_3,\ x_4\}$ be a set of attribuites and $A = B = \{x_1,\ x_2\} \subseteq E$. Consider F_A and $G_B \in$ 3-PIVNSS over $\mathcal{U}$ can be represented as follows

$$F_A = \left\{\begin{array}{c} (x_1, \{\langle u_1, ([.5,.8],[.2,.5],[.1,.6]), ([.3,.5],[.1,.3],[.3,.7]), ([.4,.6],[.3,.7],[.8,1]), \\ (u_2, ([.2,.4],[.3,0.4],[.2,.5]), ([.2,.5],[.1,.6],[.3,.8]), ([.3,.8],[.4,.9],[.6,.7]))\rangle, \\ (x_2, \{\langle u_1, ([.3,.6],[.1,.6],[.4,.7]), ([0,.2],[.1,.4],[.5,.9]), ([.3,.6],[.1,.4],[.5,.8])), \\ (u_2, ([.2,.5],[.2,.3],[.5,.6]), ([.3,.5],[.1,.5],[.5,.8]), ([.4,.6],[.3,.5],[.6,.9]))\rangle \end{array}\right\}$$

and

$$G_B = \left\{\begin{array}{c} (x_1, \{\langle u_1, ([.6,.8],[.4,.6],[.1,.4]), ([.4,.7],[.3,.4],[.2,.6]), ([.5,.7],[.4,.7],[.5,1]), \\ (u_2, ([.3,.6],[.5,0.7],[.1,.5]), ([.3,.8],[.2,.6],[.1,.5]), ([.4,1],[.5,.9],[.4,.6]))\rangle, \\ (x_2, \{\langle u_1, ([.4,.7],[.3,.7],[.3,.5]), ([0,.3],[.2,.5],[.3,.7]), ([.4,.9],[.2,.6],[.5,.7])), \\ (u_2, ([.2,.9],[.1,.5],[.3,.6]), ([.6,.9],[.3,.5],[1,1]), ([.5,.7],[.3,.7],[.1,.8]))\rangle \end{array}\right\}.$$

Thus

$F_A \subseteq G_B$.

Definition 3.3

Let Υ_A and $\Upsilon_B \in$ mPIVNSS over $\mathcal{U}$, then $\Upsilon_A = \Upsilon_B$, if

$s_i \bullet inf\ u_{A(e)}(u) \leq s_i \bullet inf\ u_{B(e)}(u),\ s_i \bullet inf\ u_{B(e)}(u) \leq s_i \bullet inf\ u_{A(e)}(u)$

$s_i \bullet sup\ u_{A(e)}(u) \leq s_i \bullet sup\ u_{B(e)}(u),\ s_i \bullet sup\ u_{B(e)}(u) \leq s_i \bullet sup\ u_{A(e)}(u)$

$s_i \bullet inf\ v_{A(e)}(u) \leq s_i \bullet inf\ v_{B(e)}(u),\ s_i \bullet inf\ v_{B(e)}(u) \leq s_i \bullet inf\ v_{A(e)}(u)$

$s_i \bullet sup\ v_{A(e)}(u) \leq s_i \bullet sup\ v_{B(e)}(u),\ s_i \bullet sup\ v_{B(e)}(u) \leq s_i \bullet sup\ v_{A(e)}(u)$

$s_i \bullet inf\ w_{A(e)}(u) \geq s_i \bullet inf\ w_{B(e)}(u),\ s_i \bullet inf\ w_{B(e)}(u) \geq s_i \bullet inf\ w_{A(e)}(u)$

$s_i \bullet sup\ w_{A(e)}(u) \geq s_i \bullet sup\ w_{B(e)}(u),\ s_i \bullet sup\ w_{B(e)}(u) \geq s_i \bullet sup\ w_{A(e)}(u)$

for all $i \in 1, 2, 3, \ldots, m;\ e \in E$ and $u \in \mathcal{U}$.

Definition 3.4

Let $F_A \in$ mPIVNSS over $\mathcal{U}$, then empty mPIVNSS can be represented as $F_{\breve{0}}$, and defined as follows

$F_{\breve{0}} = \{e, < u, ([0,0],[1,1],[1,1]),\ ([0,0],[1,1],[1,1]), \ldots, ([0,0],[1,1],[1,1]) > : e \in E, u \in \mathcal{U}\}$.

Definition 3.5

Let $F_A \in$ mPIVNSS over $\mathcal{U}$, then universal mPIVNSS can be represented as $F_{\breve{E}}$, and defined as follows

$F_{\breve{E}} = \{e, < u, ([1,1],[0,0],[0,0]),\ ([1,1],[0,0],[0,0]), \ldots, ([1,1],[0,0],[0,0]) > : e \in E, u \in \mathcal{U}\}$.

Example 2 Assume $\mathcal{U} = \{u_1,\ u_2\}$ be a universe of discourse and $E = \{x_1,\ x_2,\ x_3,\ x_4\}$ be a set of attributes. The tabular representation of $F_{\breve{0}}$ and $F_{\breve{E}}$ given as follows in table 1 and table 2 respectively.

Table 1: Tablur representation of mPIVNSS $F_{\breve{0}}$

$\mathcal{U}$	u_1	u_2	$\cdots$	u_n
x_1	([0, 0], [1, 1], [1, 1])	([0, 0], [1, 1], [1, 1])	$\cdots$	([0, 0], [1, 1], [1, 1])

x_2	$([0,0],[1,1],[1,1])$	$([0,0],[1,1],[1,1])$	$\cdots$	$([0,0],[1,1],[1,1])$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_n	$([0,0],[1,1],[1,1])$	$([0,0],[1,1],[1,1])$	$\cdots$	$([0,0],[1,1],[1,1])$

Table 2: Tablur representation of mPIVNSS $F_{\breve{E}}$

$\mathcal{U}$	u_1	u_2	$\cdots$	u_n
x_1	$([1,1],[0,0],[0,0])$	$([1,1],[0,0],[0,0])$	$\cdots$	$([1,1],[0,0],[0,0])$
x_2	$([1,1],[0,0],[0,0])$	$([1,1],[0,0],[0,0])$	$\cdots$	$([1,1],[0,0],[0,0])$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
x_n	$([1,1],[0,0],[0,0])$	$([1,1],[0,0],[0,0])$	$\cdots$	$([1,1],[0,0],[0,0])$

Definition 3.6

Let $F_A \in$ mPIVNSS over $\mathcal{U}$, then the complement of mPIVNSS is defined as follows $F_A^c(e) = \{e, < u, [s_i \bullet inf\, w_{A(e)}(u), s_i \bullet sup\, w_{A(e)}(u)], [(1,1,\ldots,1) - s_i \bullet sup\, v_{A(e)}(u), (1,1,\ldots,1) - s_i \bullet inf\, v_{A(e)}(u)], [s_i \bullet inf\, u_{A(e)}(u), s_i \bullet sup\, u_{A(e)}(u)] >: u \in \mathcal{U}\}$, for all $i \in 1, 2, 3, \ldots, m$; $e \in E$ and $u \in \mathcal{U}$.

Example 3 Assume $\mathcal{U} = \{u_1, u_2\}$ be a universe of discourse and $E = \{x_1, x_2, x_3, x_4\}$ be a set of attributes and $A = \{x_1, x_2\} \subseteq E$. Consider $F_A \in$ 3-PIVNSS over $\mathcal{U}$ can be represented as follows

$$F_A = \begin{Bmatrix} (x_1, \{\langle u_1, ([.6,.8],[.4,0.6],[.1,.4]), ([.4,.7],[.3,.4],[.2,.6]), ([.5,.7],[.6,.9],[1,1]), \\ (u_2, ([.3,.6],[.5,0.7],[.1,.5]), ([.3,.8],[.2,.6],[.1,.5]), ([.4,1],[.5,.9],[.4,.6]))\rangle, \\ (x_2, \{\langle u_1, ([.4,.7],[.3,.7],[.3,.5]), ([0,.3],[.2,.5],[.3,.7]), ([.4,.9],[.2,.6],[.5,.7])), \\ (u_2, ([.2,.9],[.1,.5],[.7,.8]), ([.6,.9],[.3,.5],[1,1]), ([.5,.9],[.3,.7],[.1,.8]))\rangle \end{Bmatrix}$$

Then,

$$F_A^c(x) = \begin{Bmatrix} (x_1, \{\langle u_1, ([.1,.4],[.4,0.6],[.6,.8]), [.2,.6](,[.6,.7],[.4,.7]), ([1,1],[.1,.4],[.5,.7]), \\ (u_2, ([.1,.5],[.3,0.5],[.3,.6]), ([.1,.5],[.4,.8],[.3,.8]), ([.4,.6],[.1,.5],[.4,1]))\rangle, \\ (x_2, \{\langle u_1, ([.3,.5],[.3,.7],[.4,.7]), ([.3,.7],[.5,.8],[0,.3]), ([.5,.7],[.4,.8],[.4,.9])), \\ (u_2, ([.7,.8],[.5,.9],[.2,.9]), ([1,1],[.5,.7],[.6,.9]), ([.1,.8],[.3,.7],[.5,.9]))\rangle \end{Bmatrix}$$

Proposition 3.7

If $F_{\mathrm{A}} \in$ mPIVNSS, then

1. $(F_A^c)^c = F_{\mathrm{A}}$
2. $(F_{\breve{0}})^c = F_{\breve{E}}$
3. $(F_{\breve{E}})^c = F_{\breve{0}}$

Proof 1 Let

$$F_{\mathrm{A}}(e) = \begin{Bmatrix} < u, [s_i \bullet inf\, u_{A(e)}(u), s_i \bullet sup\, u_{A(e)}(u)], \\ [s_i \bullet inf\, v_{A(e)}(u), s_i \bullet sup\, v_{A(e)}(u)], \\ [s_i \bullet inf\, w_{A(e)}(u), s_i \bullet sup\, w_{A(e)}(u)] >: u \in \mathcal{U}, e \in E \end{Bmatrix}.$$

Then by using definition 3.6, we get

$$F_A^c(e) = \left\{ \begin{array}{c} < u, [s_i \bullet inf\ w_{A(e)}(u), s_i \bullet sup\ w_{A(e)}(u)], \\ [(1,1,\dots,1) - s_i \bullet sup\ v_{A(e)}(u), (1,1,\dots,1) - s_i \bullet inf\ v_{A(e)}(u)], \\ [s_i \bullet inf\ u_{A(e)}(u), s_i \bullet sup\ u_{A(e)}(u)] > : u \in \mathcal{U}, e \in E \end{array} \right\}$$

Again by using definition 3.6

$(F_A^c(e))^c=$

$$\left\{ \begin{array}{c} < u, [s_i \bullet inf\ u_{A(e)}(u), s_i \bullet sup\ u_{A(e)}(u)], \\ [(1,1,\dots,1) - (1,1,\dots,1) - s_i \bullet inf\ v_{A(e)}(u), (1,1,\dots,1) - (1,1,\dots,1) - s_i \bullet sup\ v_{A(e)}(u)], \\ [s_i \bullet inf\ w_{A(e)}(u), s_i \bullet sup\ w_{A(e)}(u)] > : u \in \mathcal{U}, e \in E \end{array} \right\}$$

$$(F_A^c(e))^c = \left\{ \begin{array}{c} < u, [s_i \bullet inf\ u_{A(e)}(u), s_i \bullet sup\ u_{A(e)}(u)], \\ [s_i \bullet inf\ v_{A(e)}(u), s_i \bullet sup\ v_{A(e)}(u)], \\ [s_i \bullet inf\ w_{A(e)}(u), s_i \bullet sup\ w_{A(e)}(u)] > : u \in \mathcal{U}, e \in E \end{array} \right\}$$

$(F_A^c(e))^c = F_A(e)$.

Similarly, we can prove 2 and 3.

Definition 3.8

Let $F_{A(e)}$ and $G_{B(e)}$ $\in$ mPIVNSS over $\mathcal{U}$, then

$F_{A(e)} \cup G_{B(e)}=$

$$\left\{ \begin{array}{c} (e, < u, [max\{s_i \bullet inf\ u_{A(e)}(u), s_i \bullet inf\ u_{B(e)}(u)\}, max\{s_i \bullet sup\ u_{A(e)}(u), s_i \bullet sup\ u_{B(e)}(u)\}], \\ [min\{s_i \bullet inf\ v_{A(e)}(u), s_i \bullet inf\ v_{B(e)}(u)\}, min\{s_i \bullet sup\ v_{A(e)}(u), s_i \bullet sup\ v_{B(e)}(u)\}], \\ [min\{s_i \bullet inf\ w_{A(e)}(u), s_i \bullet inf\ w_{B(e)}(u)\}, min\{s_i \bullet sup\ w_{A(e)}(u), s_i \bullet sup\ w_{B(e)}(u)\}] > : u \in \mathcal{U}, e \in E) \end{array} \right\}$$

Example 4 Assume $\mathcal{U} = \{u_1, u_2\}$ be a universe of discourse and $E = \{x_1, x_2, x_3, x_4\}$ be a set of attribuites and $A = B = \{x_1, x_2\} \subseteq E$. Consider $F_{A(e)}$ and $G_{B(e)}$ $\in$ 3-PIVNSS over $\mathcal{U}$ can be represented as follows

$$F_{A(x)} = \left\{ \begin{array}{c} (x_1, \{\langle u_1, ([.5,.8],[.2,0.5],[.1,.2]), ([.3,.5],[.1,.3],[.2,.4]), ([.6,.9],[.7,.8],[.8,1]), \\ (u_2, ([.2,.4],[.3,0.4],[.1,.3]), ([.2,.5],[.1,.6],[.1,.3]), ([.8,1],[.6,.9],[.6,.7]))\rangle, \\ (x_2, \{\langle u_1, ([.3,.6],[.1,.6],[.3,.4]), ([0,.2],[.1,.4],[.3,.5]), ([.5,.9],[.3,.8],[.5,.8])), \\ (u_2, ([.2,.5],[.2,.3],[.5,.6]), ([.3,.5],[.1,.5],[.5,.8]), ([.6,.9],[.5,.8],[.6,.9]))\rangle \end{array} \right\}$$

and

$$G_{B(x)} = \left\{ \begin{array}{c} (x_1, \{\langle u_1, ([.4,.8],[.3,0.6],[.2,.5]), ([.2,.7],[.3,.4],[.4,.6]), ([.7,.8],[.4,.9],[.5,1]), \\ (u_2, ([.1,.6],[.5,0.7],[.1,.2]), ([.3,.4],[.2,.5],[.2,.5]), ([.5,.9],[.7,.8],[.4,.6]))\rangle, \\ (x_2, \{\langle u_1, ([.2,.7],[.3,.5],[.2,.6]), ([.1,.3],[.2,.5],[.2,.7]), ([.4,.9],[.4,.7],[.5,.8])), \\ (u_2, ([.1,.6],[.1,.5],[.4,.8]), ([.3,.6],[.3,.4],[1,1]), ([.5,.9],[.3,.7],[.1,.8]))\rangle \end{array} \right\}.$$

Then

$$F_{A(x)} \cup G_{B(x)} = \left\{ \begin{array}{c} (x_1, \{\langle u_1, ([.5,.8],[.2,0.5],[.1,.2]), ([.3,.7],[.1,.3],[.2,.4]), ([.7,.9],[.4,.8],[.5,1]), \\ (u_2, ([.2,.6],[.3,0.4],[.1,.2]), ([.3,.5],[.1,.5],[.1,.3]), ([.8,1],[.6,.8],[.4,.6]))\rangle, \\ (x_2, \{\langle u_1, ([.3,.7],[.1,.5],[.2,.4]), ([.1,.3],[.1,.4],[.2,.5]), ([.5,.9],[.3,.7],[.5,.8])), \\ (u_2, ([.2,.6],[.1,.3],[.4,.6]), ([.3,.6],[.1,.4],[.5,.8]), ([.6,.9],[.3,.7],[.1,.8]))\rangle \end{array} \right\}$$

Proposition 3.9

Let $\mathcal{F}_{\breve{A}}$, $\mathcal{G}_{\breve{B}}$, $\mathcal{H}_{\breve{C}}$ $\in$ mPIVNSS over $\mathcal{U}$. Then

1. $\mathcal{F}_{\breve{A}} \cup \mathcal{F}_{\breve{A}} = \mathcal{F}_{\breve{A}}$
2. $\mathcal{F}_{\breve{A}} \cup \mathcal{F}_{\breve{0}} = \mathcal{F}_{\breve{A}}$
3. $\mathcal{F}_{\breve{A}} \cup \mathcal{F}_{\breve{E}} = \mathcal{F}_{\breve{E}}$
4. $\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}} = \mathcal{G}_{\breve{B}} \cup \mathcal{F}_{\breve{A}}$

5. $(\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}}) \cup \mathcal{H}_{\breve{C}} = \mathcal{F}_{\breve{A}} \cup (\mathcal{G}_{\breve{B}} \cup \mathcal{H}_{\breve{C}})$

Proof 1. As we know that

$F_{\mathrm{A}}(e) = \left\{\begin{matrix} (e, < u, [s_i \bullet inf\ u_{A(e)}(u), s_i \bullet sup\ u_{A(e)}(u)], \\ [s_i \bullet inf\ v_{A(e)}(u), s_i \bullet sup\ v_{A(e)}(u)], \\ [s_i \bullet inf\ w_{A(e)}(u), s_i \bullet sup\ w_{A(e)}(u)] >): u \in \mathcal{U}, e \in E \end{matrix}\right\}$ be an mPIVNSS, then by using definition 3.8, we have

$$\mathcal{F}_{\widetilde{A(e)}} \cup \mathcal{F}_{\widetilde{A(e)}} = \left\{\begin{matrix} (e, < u, [max\{s_i \bullet inf\ u_{A(e)}(u), s_i \bullet inf\ u_{A(e)}(u)\}, max\{s_i \bullet sup\ u_{A(e)}(u), s_i \bullet sup\ u_{A(e)}(u)\}], \\ [min\{s_i \bullet inf\ v_{A(e)}(u), s_i \bullet inf\ v_{A(e)}(u)\}, min\{s_i \bullet sup\ v_{A(e)}(u), s_i \bullet sup\ v_{A(e)}(u)\}], \\ [min\{s_i \bullet inf\ w_{A(e)}(u), s_i \bullet inf\ w_{A(e)}(u)\}, min\{s_i \bullet sup\ w_{A(e)}(u), s_i \bullet sup\ w_{A(e)}(u)\}] > : u \in \mathcal{U}, e \in E) \end{matrix}\right\}$$

$$= \left\{\begin{matrix} (e, < u, [s_i \bullet inf\ u_{A(e)}(u), s_i \bullet sup\ u_{A(e)}(u)], \\ [s_i \bullet inf\ v_{A(e)}(u), s_i \bullet sup\ v_{A(e)}(u)], \\ [s_i \bullet inf\ w_{A(e)}(u), s_i \bullet sup\ w_{A(e)}(u)] >): u \in \mathcal{U}, e \in E \end{matrix}\right\} = \mathcal{F}_{\widetilde{A(e)}}$$

Proof 2. $\mathcal{F}_{\breve{A}} \cup \mathcal{F}_{\breve{0}} = \mathcal{F}_{\breve{A}}$

$$\mathcal{F}_{\breve{A}} \cup \mathcal{F}_{\breve{0}} = \left\{\begin{matrix} (e, < u, [max\{s_i \bullet inf\ u_{A(e)}(u), s_i \bullet inf\ u_{\breve{0}}(u)\}, max\{s_i \bullet sup\ u_{A(e)}(u), s_i \bullet sup\ u_{\breve{0}}(u)\}], \\ [min\{s_i \bullet inf\ v_{A(e)}(u), s_i \bullet inf\ v_{\breve{0}}(u)\}, min\{s_i \bullet sup\ v_{A(e)}(u), s_i \bullet sup\ v_{\breve{0}}(u)\}], \\ [min\{s_i \bullet inf\ w_{A(e)}(u), s_i \bullet inf\ w_{\breve{0}}(u)\}, min\{s_i \bullet sup\ w_{A(e)}(u), s_i \bullet sup\ w_{\breve{0}}(u)\}] > : u \in \mathcal{U}, e \in E) \end{matrix}\right\}$$

$$= \left\{\begin{matrix} (e, < u, [s_i \bullet inf\ u_{A(e)}(u), s_i \bullet sup\ u_{A(e)}(u)], \\ [s_i \bullet inf\ v_{A(e)}(u), s_i \bullet sup\ v_{A(e)}(u)], \\ [s_i \bullet inf\ w_{A(e)}(u), s_i \bullet sup\ w_{A(e)}(u)] >): u \in \mathcal{U}, e \in E \end{matrix}\right\} = \mathcal{F}_{\breve{A}}$$

Proof 3. $\mathcal{F}_{\breve{A}} \cup \mathcal{F}_{\breve{E}} = \mathcal{F}_{\breve{E}}$

$$\mathcal{F}_{\breve{A}} \cup \mathcal{F}_{\breve{E}} = \left\{\begin{matrix} (e, < u, [max\{s_i \bullet inf\ u_{A(e)}(u), s_i \bullet inf\ u_{\breve{0}}(u)\}, max\{s_i \bullet sup\ u_{A(e)}(u), s_i \bullet sup\ u_{\breve{0}}(u)\}], \\ [min\{s_i \bullet inf\ v_{A(e)}(u), s_i \bullet inf\ v_{\breve{0}}(u)\}, min\{s_i \bullet sup\ v_{A(e)}(u), s_i \bullet sup\ v_{\breve{0}}(u)\}], \\ [min\{s_i \bullet inf\ w_{A(e)}(u), s_i \bullet inf\ w_{\breve{0}}(u)\}, min\{s_i \bullet sup\ w_{A(e)}(u), s_i \bullet sup\ w_{\breve{0}}(u)\}] > : u \in \mathcal{U}, e \in E) \end{matrix}\right\}$$

$$= \left\{\begin{matrix} (e, < u, [s_i \bullet inf\ u_{\breve{E}(e)}(u), s_i \bullet sup\ u_{\breve{E}(e)}(u)], \\ [s_i \bullet inf\ v_{\breve{E}(e)}(u), s_i \bullet sup\ v_{\breve{E}(e)}(u)], \\ [s_i \bullet inf\ w_{\breve{E}(e)}(u), s_i \bullet sup\ w_{\breve{E}(e)}(u)] >): u \in \mathcal{U}, e \in E \end{matrix}\right\} = \mathcal{F}_{\breve{E}}.$$

Proof 4. $\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}} = \mathcal{G}_{\breve{B}} \cup \mathcal{F}_{\breve{A}}$

$$\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}} = \left\{\begin{matrix} (e, < u, [max\{s_i \bullet inf\ u_{A(e)}(u), s_i \bullet inf\ u_{B(e)}(u)\}, max\{s_i \bullet sup\ u_{A(e)}(u), s_i \bullet sup\ u_{B(e)}(u)\}], \\ [min\{s_i \bullet inf\ v_{A(e)}(u), s_i \bullet inf\ v_{B(e)}(u)\}, min\{s_i \bullet sup\ v_{A(e)}(u), s_i \bullet sup\ v_{B(e)}(u)\}], \\ [min\{s_i \bullet inf\ w_{A(e)}(u), s_i \bullet inf\ w_{B(e)}(u)\}, min\{s_i \bullet sup\ w_{A(e)}(u), s_i \bullet sup\ w_{B(e)}(u)\}] > : u \in \mathcal{U}, e \in E) \end{matrix}\right\}$$

$$\mathcal{G}_{\breve{B}} \cup \mathcal{F}_{\breve{A}} = \left\{\begin{matrix} (e, < u, [max\{s_i \bullet inf\ u_{B(e)}(u), s_i \bullet inf\ u_{A(e)}(u)\}, max\{s_i \bullet sup\ u_{B(e)}(u), s_i \bullet sup\ u_{A(e)}(u)\}], \\ [min\{s_i \bullet inf\ v_{B(e)}(u), s_i \bullet inf\ v_{A(e)}(u)\}, min\{s_i \bullet sup\ v_{B(e)}(u), s_i \bullet sup\ v_{A(e)}(u)\}], \\ [min\{s_i \bullet inf\ w_{B(e)}(u), s_i \bullet inf\ w_{A(e)}(u)\}, min\{s_i \bullet sup\ w_{B(e)}(u), s_i \bullet sup\ w_{A(e)}(u)\}] > : u \in \mathcal{U}, e \in E) \end{matrix}\right\}$$

$\mathcal{G}_{\breve{B}} \cup \mathcal{F}_{\breve{A}} =$

$$\left\{\begin{matrix} (e, < u, [max\{s_i \bullet inf\ u_{A(e)}(u), s_i \bullet inf\ u_{B(e)}(u)\}, max\{s_i \bullet sup\ u_{A(e)}(u), s_i \bullet sup\ u_{B(e)}(u)\}], \\ [min\{s_i \bullet inf\ v_{A(e)}(u), s_i \bullet inf\ v_{B(e)}(u)\}, min\{s_i \bullet sup\ v_{A(e)}(u), s_i \bullet sup\ v_{B(e)}(u)\}], \\ [min\{s_i \bullet inf\ w_{A(e)}(u), s_i \bullet inf\ w_{B(e)}(u)\}, min\{s_i \bullet sup\ w_{A(e)}(u), s_i \bullet sup\ w_{B(e)}(u)\}] > : u \in \mathcal{U}, e \in E) \end{matrix}\right\}$$

So, $\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}} = \mathcal{G}_{\breve{B}} \cup \mathcal{F}_{\breve{A}}$.

Proof 5. Similar to assertion 4.

Definition 3.10

Let $F_{A(e)}$ and $G_{B(e)} \in$ mPIVNSS over $\mathcal{U}$, then

$F_{A(e)} \cap G_{B(e)} =$

$$\left\{\begin{matrix} (e, < u, [min\{s_i \bullet inf\ u_{A(e)}(u), s_i \bullet inf\ u_{B(e)}(u)\}, min\{s_i \bullet sup\ u_{A(e)}(u), s_i \bullet sup\ u_{B(e)}(u)\}], \\ [max\{s_i \bullet inf\ v_{A(e)}(u), s_i \bullet inf\ v_{B(e)}(u)\}, max\{s_i \bullet sup\ v_{A(e)}(u), s_i \bullet sup\ v_{B(e)}(u)\}], \\ [max\{s_i \bullet inf\ w_{A(e)}(u), s_i \bullet inf\ w_{B(e)}(u)\}, max\{s_i \bullet sup\ w_{A(e)}(u), s_i \bullet sup\ w_{B(e)}(u)\}] > : u \in \mathcal{U}, e \in E) \end{matrix}\right\}$$

Proposition 3.11

Let $\mathcal{F}_{\breve{A}}$, $\mathcal{G}_{\breve{B}}$, $\mathcal{H}_{\breve{C}} \in$ mPIVNSS over $\mathcal{U}$. Then

1. $\mathcal{F}_{\breve{A}} \cap \mathcal{F}_{\breve{A}} = \mathcal{F}_{\breve{A}}$
2. $\mathcal{F}_{\breve{A}} \cap \mathcal{F}_{\breve{0}} = \mathcal{F}_{\breve{A}}$
3. $\mathcal{F}_{\breve{A}} \cap \mathcal{F}_{\breve{E}} = \mathcal{F}_{\breve{A}}$
4. $\mathcal{F}_{\breve{A}} \cap \mathcal{G}_{\breve{B}} = \mathcal{G}_{\breve{B}} \cap \mathcal{F}_{\breve{A}}$
5. $(\mathcal{F}_{\breve{A}} \cap \mathcal{G}_{\breve{B}}) \cap \mathcal{H}_{\breve{C}} = \mathcal{F}_{\breve{A}} \cap (\mathcal{G}_{\breve{B}} \cap \mathcal{H}_{\breve{C}})$

Proof 1. As we know that

$$\mathcal{F}_{\widetilde{A(e)}} = \left\{\begin{matrix} (e, < u, [s_i \bullet inf\ u_{A(e)}(u), s_i \bullet sup\ u_{A(e)}(u)], \\ [s_i \bullet inf\ v_{A(e)}(u), s_i \bullet sup\ v_{A(e)}(u)], \\ [s_i \bullet inf\ w_{A(e)}(u), s_i \bullet sup\ w_{A(e)}(u)] >): u \in \mathcal{U}, e \in E \end{matrix}\right\}$$ be an mPIVNSS, then by using definition 3.8, we have

$\mathcal{F}_{\widetilde{A(e)}} \cap \mathcal{F}_{\widetilde{A(e)}} =$

$$\left\{\begin{matrix} (e, < u, [max\{s_i \bullet inf\ u_{A(e)}(u), s_i \bullet inf\ u_{A(e)}(u)\}, max\{s_i \bullet sup\ u_{A(e)}(u), s_i \bullet sup\ u_{A(e)}(u)\}], \\ [min\{s_i \bullet inf\ v_{A(e)}(u), s_i \bullet inf\ v_{A(e)}(u)\}, min\{s_i \bullet sup\ v_{A(e)}(u), s_i \bullet sup\ v_{A(e)}(u)\}], \\ [min\{s_i \bullet inf\ w_{A(e)}(u), s_i \bullet inf\ w_{A(e)}(u)\}, min\{s_i \bullet sup\ w_{A(e)}(u), s_i \bullet sup\ w_{A(e)}(u)\}] > : u \in \mathcal{U}, e \in E) \end{matrix}\right\}$$

$$= \left\{\begin{matrix} (e, < u, [s_i \bullet inf\ u_{A(e)}(u), s_i \bullet sup\ u_{A(e)}(u)], \\ [s_i \bullet inf\ v_{A(e)}(u), s_i \bullet sup\ v_{A(e)}(u)], \\ [s_i \bullet inf\ w_{A(e)}(u), s_i \bullet sup\ w_{A(e)}(u)] >): u \in \mathcal{U}, e \in E \end{matrix}\right\} = \mathcal{F}_{\widetilde{A(e)}}$$

Proof 2. $\mathcal{F}_{\breve{A}} \cap \mathcal{F}_{\breve{0}} = \mathcal{F}_{\breve{A}}$

$\mathcal{F}_{\breve{A}} \cap \mathcal{F}_{\breve{0}} =$

$$\left\{\begin{matrix} (e, < u, [min\{s_i \bullet inf\ u_{A(e)}(u), s_i \bullet inf\ u_{B(e)}(u)\}, min\{s_i \bullet sup\ u_{A(e)}(u), s_i \bullet sup\ u_{B(e)}(u)\}], \\ [max\{s_i \bullet inf\ v_{A(e)}(u), s_i \bullet inf\ v_{B(e)}(u)\}, max\{s_i \bullet sup\ v_{A(e)}(u), s_i \bullet sup\ v_{B(e)}(u)\}], \\ [max\{s_i \bullet inf\ w_{A(e)}(u), s_i \bullet inf\ w_{B(e)}(u)\}, max\{s_i \bullet sup\ w_{A(e)}(u), s_i \bullet sup\ w_{B(e)}(u)\}] > : u \in \mathcal{U}, e \in E) \end{matrix}\right\}$$

$$\mathcal{F}_{\breve{A}} \cap \mathcal{F}_{\breve{0}} = \left\{\begin{matrix} (e, < u, [s_i \bullet inf\ u_{A(e)}(u), s_i \bullet sup\ u_{A(e)}(u)], \\ [s_i \bullet inf\ v_{A(e)}(u), s_i \bullet sup\ v_{A(e)}(u)], \\ [s_i \bullet inf\ w_{A(e)}(u), s_i \bullet sup\ w_{A(e)}(u)] >): u \in \mathcal{U}, e \in E \end{matrix}\right\} = \mathcal{F}_{\widetilde{A(e)}}$$

Proof 3. $\mathcal{F}_{\breve{A}} \cap \mathcal{F}_{\breve{E}} = \mathcal{F}_{\breve{A}}$

$\mathcal{F}_{\breve{A}} \cap \mathcal{F}_{\breve{E}} =$

$$\begin{Bmatrix} (e, < u, [min\{s_i \bullet inf\ u_{A(e)}(u), s_i \bullet inf\ u_{\breve{E}}(u)\}, min\{s_i \bullet sup\ u_{A(e)}(u), s_i \bullet sup\ u_{\breve{E})}(u)\}], \\ [max\{s_i \bullet inf\ v_{A(e)}(u), s_i \bullet inf\ v_{\breve{E}}(u)\}, max\{s_i \bullet sup\ v_{A(e)}(u), s_i \bullet sup\ v_{\breve{E}}(u)\}], \\ [max\{s_i \bullet inf\ w_{A(e)}(u), s_i \bullet inf\ w_{\breve{E}}(u)\}, max\{s_i \bullet sup\ w_{A(e)}(u), s_i \bullet sup\ w_{B(e)}(u)\}] >: u \in \mathcal{U}, e \in E) \end{Bmatrix}$$

$$\begin{Bmatrix} (e, < u, [min\{s_i \bullet inf\ u_{A(e)}(u), s_i \bullet inf\ u_{B(e)}(u)\}, min\{s_i \bullet sup\ u_{A(e)}(u), s_i \bullet sup\ u_{B(e)}(u)\}], \\ [max\{s_i \bullet inf\ v_{A(e)}(u), s_i \bullet inf\ v_{B(e)}(u)\}, max\{s_i \bullet sup\ v_{A(e)}(u), s_i \bullet sup\ v_{B(e)}(u)\}], \\ [max\{s_i \bullet inf\ w_{A(e)}(u), s_i \bullet inf\ w_{B(e)}(u)\}, max\{s_i \bullet sup\ w_{A(e)}(u), s_i \bullet sup\ w_{B(e)}(u)\}] >: u \in \mathcal{U}, e \in E) \end{Bmatrix}$$

$$= \begin{Bmatrix} (e, < u, [s_i \bullet inf\ u_{A(e)}(u), s_i \bullet sup\ u_{A(e)}(u)], \\ [s_i \bullet inf\ v_{A(e)}(u), s_i \bullet sup\ v_{A(e)}(u)], \\ [s_i \bullet inf\ w_{A(e)}(u), s_i \bullet sup\ w_{A(e)}(u)] >): u \in \mathcal{U}, e \in E \end{Bmatrix} = \mathcal{F}_{\breve{A}}$$

Proof 4. 5. Similar to assertion 3.

Proposition 3.12

Let F_A and $G_B \in$ mPIVNSS over $\mathcal{U}$, then

1. $(F_{A(e)} \cup G_{B(e)})^C = F_{A(e)}{}^C \cap G_{B(e)}{}^C$
2. $(F_{A(e)} \cap G_{B(e)})^C = F_{A(e)}{}^C \cup G_{B(e)}{}^C$

Proof 1 As we know that

$$F_A(e) = \begin{Bmatrix} (e, < u, [s_i \bullet inf\ u_{A(e)}(u), s_i \bullet sup\ u_{A(e)}(u)], \\ [s_i \bullet inf\ v_{A(e)}(u), s_i \bullet sup\ v_{A(e)}(u)], \\ [s_i \bullet inf\ w_{A(e)}(u), s_i \bullet sup\ w_{A(e)}(u)] >): u \in \mathcal{U}, e \in E \end{Bmatrix} \text{ and}$$

$$G_B(e) = \begin{Bmatrix} (e, < u, [s_i \bullet inf\ u_{B(e)}(u), s_i \bullet sup\ u_{B(e)}(u)], \\ [s_i \bullet inf\ v_{B(e)}(u), s_i \bullet sup\ v_{B(e)}(u)], \\ [s_i \bullet inf\ w_{B(e)}(u), s_i \bullet sup\ w_{B(e)}(u)] >): u \in \mathcal{U}, e \in E \end{Bmatrix}$$

By using definition 3.8, we get

$F_{A(e)} \cup G_{B(e)} =$

$$\begin{Bmatrix} (e, < u, [max\{s_i \bullet inf\ u_{A(e)}(u), s_i \bullet inf\ u_{B(e)}(u)\}, max\{s_i \bullet sup\ u_{A(e)}(u), s_i \bullet sup\ u_{B(e)}(u)\}], \\ [min\{s_i \bullet inf\ v_{A(e)}(u), s_i \bullet inf\ v_{B(e)}(u)\}, min\{s_i \bullet sup\ v_{A(e)}(u), s_i \bullet sup\ v_{B(e)}(u)\}], \\ [min\{s_i \bullet inf\ w_{A(e)}(u), s_i \bullet inf\ w_{B(e)}(u)\}, min\{s_i \bullet sup\ w_{A(e)}(u), s_i \bullet sup\ w_{B(e)}(u)\}] >: u \in \mathcal{U}, e \in E) \end{Bmatrix}$$

Now by using definition 3.6, we get

$(F_{A(e)} \cup G_{B(e)})^c =$

$$\begin{Bmatrix} (e, < u, [min\{s_i \bullet inf\ w_{A(e)}(u), s_i \bullet inf\ w_{B(e)}(u)\}, min\{s_i \bullet sup\ w_{A(e)}(u), s_i \bullet sup\ w_{B(e)}(u)\}], \\ [(1,1,\dots,1) - min\{s_i \bullet sup\ v_{A(e)}(u), s_i \bullet sup\ v_{B(e)}(u)\}, (1,1,\dots,1) - min\{s_i \bullet inf\ v_{A(e)}(u), s_i \bullet inf\ v_{B(e)}(u)\}], \\ [max\{s_i \bullet inf\ u_{A(e)}(u),\ s_i \bullet inf\ u_{B(e)}(u)\}, max\{s_i \bullet sup\ u_{A(e)}(u), s_i \bullet sup\ u_{B(e)}(u)\}] >: u \in \mathcal{U}, e \in E) \end{Bmatrix}$$

Now

$$F_{A(e)}{}^C = \begin{Bmatrix} (e, < u, [s_i \bullet inf\ w_{A(e)}(u), s_i \bullet sup\ w_{A(e)}(u)], \\ [(1,1,\dots,1) - s_i \bullet sup\ v_{A(e)}(u), (1,1,\dots,1) - s_i \bullet inf\ v_{A(e)}(u)], \\ [s_i \bullet inf\ u_{A(e)}(u), s_i \bullet sup\ u_{A(e)}(u)] >): u \in \mathcal{U}, e \in E \end{Bmatrix}$$

$$G_{B(e)}{}^C = \begin{Bmatrix} (e, < u, [s_i \bullet inf\ w_{B(e)}(u), s_i \bullet sup\ w_{B(e)}(u)], \\ [(1,1,\dots,1) - s_i \bullet sup\ v_{B(e)}(u), (1,1,\dots,1) - s_i \bullet inf\ v_{B(e)}(u)], \\ [s_i \bullet inf\ u_{B(e)}(u), s_i \bullet sup\ u_{B(e)}(u)] >): u \in \mathcal{U}, e \in E \end{Bmatrix}$$

By using definition 3.10

$F_{A(e)}{}^{C} \cap G_{B(e)}{}^{C} =$

$$\left\{\begin{matrix} (e,< u,\left[min\{s_i \bullet inf\ w_{A(e)}(u), s_i \bullet inf\ w_{B(e)}(u)\}, min\{s_i \bullet sup\ w_{A(e)}(u),\ s_i \bullet up\ w_{B(e)}(u)\}\right], \\ \left[min\left\{(1,1,\ldots,1) - s_i \bullet sup\ v_{A(e)}(u), (1,1,\ldots,1) - s_i \bullet sup\ v_{B(e)}(u)\right\}, min\left\{(1,1,\ldots,1) - s_i \bullet inf\ v_{A(e)}(u), (1,1,\ldots,1) - s_i \bullet inf\ v_{B(e)}(u)\right\}\right] \\ \left[max\{s_i \bullet inf\ u_{A(e)}(u), s_i \bullet inf u_{B(e)}(u)\}, max\{s_i \bullet sup\ u_{A(e)}(u), s_i \bullet sup\ u_{B(e)}(u)\}\right] >: u \in \mathcal{U}, e \in E) \end{matrix}\right\}$$

$F_{A(e)}{}^{C} \cap G_{B(e)}{}^{C} =$

$$\left\{\begin{matrix} (e,< u,\left[min\{s_i \bullet inf\ w_{A(e)}(u), s_i \bullet inf\ w_{B(e)}(u)\}, min\{s_i \bullet sup\ w_{A(e)}(u), s_i \bullet sup\ w_{B(e)}(u)\}\right], \\ \left[(1,1,\ldots,1) - min\{s_i \bullet sup\ v_{A(e)}(u), s_i \bullet sup\ v_{B(e)}(u)\}, (1,1,\ldots,1) - min\{s_i \bullet inf\ v_{A(e)}(u), s_i \bullet inf\ v_{B(e)}(u)\}\right], \\ \left[max\{s_i \bullet inf\ u_{A(e)}(u), s_i \bullet inf\ u_{B(e)}(u)\}, max\{s_i \bullet sup\ u_{A(e)}(u), s_i \bullet sup\ u_{B(e)}(u)\}\right] >: u \in \mathcal{U}, e \in E) \end{matrix}\right\}$$

Hence

$(F_{A(e)} \cup G_{B(e)})^{C} = F_{A(e)}{}^{C} \cap G_{B(e)}{}^{C}$.

Proof 4, 5. Similar to assertion 1.

Proposition 3.13

Let $\mathcal{F}_{\breve{A}(e)}$, $\mathcal{G}_{\overline{B(e)}}$, $\mathcal{H}_{\check{C}(e)} \in$ mPIVNSS over $\mathcal{U}$. Then

1. $\mathcal{F}_{\breve{A}(e)} \cup (\mathcal{G}_{\overline{B(e)}} \cap \mathcal{H}_{\check{C}(e)}) = (\mathcal{F}_{\overline{A(e)}} \cup \mathcal{G}_{\overline{B(e)}}) \cap (\mathcal{F}_{\overline{A(e)}} \cup \mathcal{H}_{\overline{C(e)}})$
2. $\mathcal{F}_{\overline{A(e)}} \cap (\mathcal{G}_{\overline{B(e)}} \cup \mathcal{H}_{\check{C}(e)}) = (\mathcal{F}_{\breve{A}(e)} \cap \mathcal{G}_{\overline{B(e)}}) \cup (\mathcal{F}_{\overline{A(e)}} \cap \mathcal{H}_{\check{C}(e)})$
3. $\mathcal{F}_{\breve{A}(e)} \cup (\mathcal{F}_{\breve{A}(e)} \cap \mathcal{G}_{\overline{B(e)}}) = \mathcal{F}_{\overline{A(e)}}$
4. $\mathcal{F}_{\overline{A(e)}} \cap (\mathcal{F}_{\overline{A(e)}} \cup \mathcal{G}_{\overline{B(e)}}) = \mathcal{F}_{\overline{A(e)}}$

Proof 1 As we know that

$\mathcal{G}_{\overline{B(e)}} \cap \mathcal{H}_{\overline{C(e)}} =$

$$\left\{\begin{matrix} (e,< u,\left[min\{s_i \bullet inf\ u_{\overline{B(e)}}(u), s_i \bullet inf\ u_{\overline{C(e)}}(u)\}, min\{s_i \bullet sup\ u_{\overline{B(e)}}(u), s_i \bullet sup\ u_{\overline{C(e)}}(u)\}\right], \\ \left[max\{s_i \bullet inf\ v_{\overline{B(e)}}(u), s_i \bullet inf\ v_{\overline{C(e)}}(u)\}, max\{s_i \bullet sup\ v_{\overline{B(e)}}(u), s_i \bullet sup\ v_{\overline{C(e)}}(u)\}\right], \\ \left[max\{s_i \bullet inf\ w_{\overline{B(e)}}(u), s_i \bullet inf\ w_{\overline{C(e)}}(u)\}, max\{s_i \bullet sup\ w_{\overline{B(e)}}(u), s_i \bullet sup\ w_{\overline{C(e)}}(u)\}\right] >: u \in \mathcal{U}, e \in E) \end{matrix}\right\}$$

$\mathcal{F}_{\breve{A}(e)} \cup (\mathcal{G}_{\overline{B(e)}} \cap \mathcal{H}_{\overline{C(e)}}) =$

$$\left\{\begin{matrix} (e,< u,\left[max\{s_i \bullet inf\ u_{\overline{A(e)}}(u), min\{s_i \bullet inf\ u_{\overline{B(e)}}(u), s_i \bullet inf\ u_{\overline{C(e)}}(u)\}\}, max\{s_i \bullet sup\ u_{\overline{A(e)}}(u), min\{s_i \bullet sup\ u_{\overline{B(e)}}(u), s_i \bullet sup\ u_{\overline{C(e)}}(u)\}\}\right], \\ \left[min\{s_i \bullet inf\ v_{\overline{A(e)}}(u), max\{s_i \bullet inf\ v_{\overline{B(e)}}(u), s_i \bullet inf\ v_{\overline{C(e)}}(u)\}\}, min\{s_i \bullet sup\ v_{\overline{A(e)}}(u), max\{s_i \bullet sup\ v_{\overline{B(e)}}(u), s_i \bullet sup\ v_{\overline{C(e)}}(u)\}\}\right], \\ \left[min\{s_i \bullet inf\ w_{\overline{A(e)}}(u), max\{s_i \bullet inf\ w_{\overline{B(e)}}(u), s_i \bullet inf\ w_{\overline{C(e)}}(u)\}\}, min\{s_i \bullet sup\ w_{\overline{A(e)}}(u), max\{s_i \bullet sup\ w_{\overline{B(e)}}(u), s_i \bullet sup\ w_{\overline{C(e)}}(u)\}\}\right] >: u \in \mathcal{U}, e \in E) \end{matrix}\right\}$$

$\mathcal{F}_{\overline{A(e)}} \cap \mathcal{G}_{\overline{B(e)}} =$

$$\left\{\begin{matrix} (e,< u,\left[min\{s_i \bullet inf\ u_{\overline{A(e)}}(u), s_i \bullet inf\ u_{\overline{B(e)}}(u)\}, min\{s_i \bullet sup\ u_{\overline{A(e)}}(u), s_i \bullet sup\ u_{\overline{B(e)}}(u)\}\right], \\ \left[max\{s_i \bullet inf\ v_{\overline{A(e)}}(u), s_i \bullet inf\ v_{\overline{B(e)}}(u)\}, max\{s_i \bullet sup\ v_{\overline{A(e)}}(u), s_i \bullet sup\ v_{\overline{B(e)}}(u)\}\right], \\ \left[max\{s_i \bullet inf\ w_{\overline{A(e)}}(u), s_i \bullet inf\ w_{\overline{B(e)}}(u)\}, max\{s_i \bullet sup\ w_{\overline{A(e)}}(u), s_i \bullet sup\ w_{\overline{B(e)}}(u)\}\right] >: u \in \mathcal{U}, e \in E) \end{matrix}\right\}$$

$\mathcal{F}_{\overline{A(e)}} \cap \mathcal{H}_{\overline{C(e)}} =$

$$\left\{\begin{matrix} (e,< u,\left[min\{s_i \bullet inf\ u_{\overline{A(e)}}(u), s_i \bullet inf\ u_{\overline{C(e)}}(u)\}, min\{s_i \bullet sup\ u_{\overline{A(e)}}(u), s_i \bullet sup\ u_{\overline{C(e)}}(u)\}\right], \\ \left[max\{s_i \bullet inf\ v_{\overline{A(e)}}(u), s_i \bullet inf\ v_{\overline{C(e)}}(u)\}, max\{s_i \bullet sup\ v_{\overline{A(e)}}(u), s_i \bullet sup\ v_{\overline{C(e)}}(u)\}\right], \\ \left[max\{s_i \bullet inf\ w_{\overline{A(e)}}(u), s_i \bullet inf\ w_{\overline{C(e)}}(u)\}, max\{s_i \bullet sup\ w_{\overline{A(e)}}(u), s_i \bullet sup\ w_{\overline{C(e)}}(u)\}\right] >: u \in \mathcal{U}, e \in E) \end{matrix}\right\}$$

$(\mathcal{F}_{\widetilde{A(e)}} \cap \mathcal{G}_{\widetilde{B(e)}}) \cup (\mathcal{F}_{\widetilde{A(e)}} \cap \mathcal{H}_{\widetilde{C(e)}}) =$

$$\left\{\begin{matrix} (e, < u, \begin{bmatrix} max\{min\{s_i \bullet inf\ u_{\widetilde{A(e)}}(u), s_i \bullet inf\ u_{\widetilde{B(e)}}(u)\}, min\{s_i \bullet inf\ u_{\widetilde{B(e)}}(u), s_i \bullet inf\ u_{\widetilde{C(e)}}(u)\}\}, \\ max\{min\{s_i \bullet sup\ u_{\widetilde{A(e)}}(u), s_i \bullet sup\ u_{\widetilde{B(e)}}(u)\}, min\{s_i \bullet sup\ u_{\widetilde{B(e)}}(u), s_i \bullet sup\ u_{\widetilde{C(e)}}(u)\}\} \end{bmatrix}, \\ \begin{bmatrix} min\{max\{s_i \bullet inf\ v_{\widetilde{A(e)}}(u), inf\ v_{\widetilde{B(e)}}(u)\}, max\{s_i \bullet inf\ v_{\widetilde{B(e)}}(u), s_i \bullet inf\ v_{\widetilde{C(e)}}(u)\}\}, \\ min\{max\{s_i \bullet sup\ v_{\widetilde{A(e)}}(u), s_i \bullet sup\ v_{\widetilde{B(e)}}(u)\}, max\{s_i \bullet sup\ v_{\widetilde{B(e)}}(u), s_i \bullet sup\ v_{\widetilde{C(e)}}(u)\}\} \end{bmatrix}, \\ \begin{bmatrix} min\{max\{s_i \bullet inf\ w_{\widetilde{A(e)}}(u), s_i \bullet inf\ w_{\widetilde{B(e)}}(u)\}, max\{s_i \bullet inf\ w_{\widetilde{B(e)}}(u), s_i \bullet inf\ w_{\widetilde{C(e)}}(u)\}\}, \\ min\{max\{s_i \bullet sup\ w_{\widetilde{A(e)}}(u), s_i \bullet sup\ w_{\widetilde{B(e)}}(u)\}, max\{s_i \bullet sup\ w_{\widetilde{B(e)}}(u), s_i \bullet sup\ w_{\widetilde{C(e)}}(u)\}\} \end{bmatrix} > : u \in \mathcal{U}, e \in E) \end{matrix}\right\}$$

$(\mathcal{F}_{\widetilde{A(e)}} \cap \mathcal{G}_{\widetilde{B(e)}}) \cup (\mathcal{F}_{\widetilde{A(e)}} \cap \mathcal{H}_{\widetilde{C(e)}}) =$

$$\left\{\begin{matrix} (e, < u, [max\{s_i \bullet inf\ u_{\widetilde{A(e)}}(u), min\{s_i \bullet inf\ u_{\widetilde{B(e)}}(u), s_i \bullet inf\ u_{\widetilde{C(e)}}(u)\}\}, max\{s_i \bullet sup\ u_{\widetilde{A(e)}}(u), min\{s_i \bullet sup\ u_{\widetilde{B(e)}}(u), s_i \bullet sup\ u_{\widetilde{C(e)}}(u)\}\}], \\ [min\{s_i \bullet inf\ v_{\widetilde{A(e)}}(u), max\{s_i \bullet inf\ v_{\widetilde{B(e)}}(u), s_i \bullet inf\ v_{\widetilde{C(e)}}(u)\}\}, min\{s_i \bullet sup\ v_{\widetilde{A(e)}}(u), max\{s_i \bullet sup\ v_{\widetilde{B(e)}}(u), s_i \bullet sup\ v_{\widetilde{C(e)}}(u)\}\}], \\ [min\{s_i \bullet inf\ w_{\widetilde{A(e)}}(u), max\{s_i \bullet inf\ w_{\widetilde{B(e)}}(u), s_i \bullet inf\ w_{\widetilde{C(e)}}(u)\}\}, min\{s_i \bullet sup\ w_{\widetilde{A(e)}}(u), max\{s_i \bullet sup\ w_{\widetilde{B(e)}}(u), s_i \bullet sup\ w_{\widetilde{C(e)}}(u)\}\}] > : u \in \mathcal{U}, e \in E) \end{matrix}\right\}$$

Hence

$\mathcal{F}_{\breve{A}(e)} \cup (\mathcal{G}_{\widetilde{B(e)}} \cap \mathcal{H}_{\breve{C}(e)}) = (\mathcal{F}_{\widetilde{A(e)}} \cup \mathcal{G}_{\widetilde{B(e)}}) \cap (\mathcal{F}_{\widetilde{A(e)}} \cup \mathcal{H}_{\widetilde{C(e)}})$.

Similarly, we can prove other results.

Definition 3.14

Let F_A, G_B $\in$ mPIVNSS, then their difference defined as follows

$F_A \setminus G_B =$

$$\left\{\begin{matrix} (e, < u, [min\{s_i \bullet inf\ u_A(u), s_i \bullet inf\ u_B(u)\}, min\{s_i \bullet sup\ u_A(u), s_i \bullet sup\ u_B(u)\}], \\ [max\{s_i \bullet inf\ v_A(u),\ (1,1,\dots,1) - s_i \bullet sup\ v_B(u)\}, max\{s_i \bullet sup\ v_A(u),\ (1,1,\dots,1) - s_i \bullet inf\ v_B(u)\}], \\ [max\{s_i \bullet inf\ w_A(u), s_i \bullet inf\ w_B(u)\}, max\{s_i \bullet sup\ w_A(u), s_i \bullet sup\ w_B(u)\}] > : u \in \mathcal{U}) \end{matrix}\right\}$$

Definition 3.15

Let F_A, G_B $\in$ mPIVNSS, then their addition defined as follows

$F_A + G_B =$

$$\left\{\begin{matrix} (e, < u, [min\{s_i \bullet inf\ u_A(u) + s_i \bullet inf\ u_B(u), (1,1,\dots,1)\}, min\{s_i \bullet sup\ u_A(u) + s_i \bullet sup\ u_B(u), (1,1,\dots,1)\}], \\ [min\{s_i \bullet inf\ v_A(u) + s_i \bullet inf\ v_B(u), (1,1,\dots,1)\}, min\{s_i \bullet sup\ v_A(u) + s_i \bullet sup\ v_B(u), (1,1,\dots,1)\}], \\ [min\{s_i \bullet inf\ w_A(u) + s_i \bullet inf\ w_B(u), (1,1,\dots,1)\}, min\{s_i \bullet sup\ w_A(u) + s_i \bullet sup\ w_B(u), (1,1,\dots,1)\}] > : u \in \mathcal{U}) \end{matrix}\right\}$$

Definition 3.16

Let F_A $\in$ mPIVNSS, then its scalar multiplication is represented as $F_A.\breve{a}$, where $\breve{a}$ $\in$ [0, 1] and defined as follows

$$F_A.\breve{a} = \left\{\begin{matrix} (e, < u, [min\{s_i \bullet inf\ u_A(u).\breve{a}, (1,1,\dots,1)\}, min\{s_i \bullet sup\ u_A(u).\breve{a}, (1,1,\dots,1)\}], \\ [min\{s_i \bullet inf\ v_A(u).\breve{a}, (1,1,\dots,1)\}, min\{s_i \bullet sup\ v_A(u).\breve{a}, (1,1,\dots,1)\}], \\ [min\{s_i \bullet inf\ w_A(u).\breve{a}, (1,1,\dots,1)\}, min\{s_i \bullet sup\ w_A(u).\breve{a}, (1,1,\dots,1)\}] > : u \in \mathcal{U}) \end{matrix}\right\}.$$

Definition 3.17

Let F_A $\in$ mPIVNSS, then its scalar division is represented as $F_A/\breve{a}$, where $\breve{a}$ $\in$ [0, 1] and defined as follows

$$F_A/\breve{a} = \left\{\begin{matrix} (e, < u, [min\{s_i \bullet inf\ u_A(u)/\breve{a}, (1,1,\dots,1)\}, min\{s_i \bullet sup\ u_A(u)/\breve{a}, (1,1,\dots,1)\}], \\ [min\{s_i \bullet inf\ v_A(u)/\breve{a}, (1,1,\dots,1)\}, min\{s_i \bullet sup\ v_A(u)/\breve{a}, (1,1,\dots,1)\}], \\ [min\{s_i \bullet inf\ w_A(u)/\breve{a}, (1,1,\dots,1)\}, min\{s_i \bullet sup\ w_A(u)/\breve{a}, (1,1,\dots,1)\}] > : u \in \mathcal{U}) \end{matrix}\right\}.$$

4. Distance and Similarity Measure of Multi-Polar Interval Valued Neutrosophic Soft set

In this section, we introduce the Hamming distance and Euclidean distance between two mPIVNSS and develop the similarity measure by using these distances.

Definition 4.1

$\mathcal{U}$ and E are universal set and set of attributes respectively, assume mPIVNSS($\mathcal{U}$) represents the collection of all multi polar interval-valued neutrosophic soft sets. Suppose $(\Phi_{\mathcal{F}}, E)$ and $(\varphi_{\mathcal{G}}, E) \in$ mPIVNSS and there exist a mapping $\Phi_{\mathcal{F}}, \varphi_{\mathcal{G}}: E \rightarrow$ mPIVNSS($\mathcal{U}$), then we define the distances between $(\Phi_{\mathcal{F}}, E)$ and $(\varphi_{\mathcal{G}}, E)$ as follows

Hamming distance

$$d_H(\Phi_{\mathcal{F}}(e), \varphi_{\mathcal{G}}(e)) = \frac{1}{2m}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p} \begin{matrix}\left(\left| s_i \bullet u_{\Phi_{\mathcal{F}}}(u_j) - s_i \bullet u_{\varphi_{\mathcal{G}}}(u_j)\right|\right) + \left(\left| s_i \bullet v_{\Phi_{\mathcal{F}}}(u_j) - s_i \bullet v_{\varphi_{\mathcal{G}}}(u_j)\right|\right) + \\ \left(\left| s_i \bullet w_{\Phi_{\mathcal{F}}}(u_j) - s_i \bullet w_{\varphi_{\mathcal{G}}}(u_j)\right|\right)\end{matrix}\right\} \tag{4.1}$$

Where

$$s_i \bullet u_{\Phi_{\mathcal{F}}}(u_j) = \frac{1}{2}\left(s_i \bullet \inf u_{\Phi_{\mathcal{F}}}(u_j) + s_i \bullet \sup u_{\Phi_{\mathcal{F}}}(u_j)\right)$$

$$s_i \bullet v_{\Phi_{\mathcal{F}}}(u_j) = \frac{1}{2}\left(s_i \bullet \inf v_{\Phi_{\mathcal{F}}}(u_j) + s_i \bullet \sup v_{\Phi_{\mathcal{F}}}(u_j)\right)$$

$$s_i \bullet w_{\Phi_{\mathcal{F}}}(u_j) = \frac{1}{2}\left(s_i \bullet \inf w_{\Phi_{\mathcal{F}}}(u_j) + s_i \bullet \sup w_{\Phi_{\mathcal{F}}}(u_j)\right)$$

$$s_i \bullet u_{\varphi_{\mathcal{G}}}(u_j) = \frac{1}{2}\left(s_i \bullet \inf u_{\varphi_{\mathcal{G}}}(u_j) + s_i \bullet \sup u_{\varphi_{\mathcal{G}}}(u_j)\right)$$

$$s_i \bullet v_{\varphi_{\mathcal{G}}}(u_j) = \frac{1}{2}\left(s_i \bullet \inf v_{\varphi_{\mathcal{G}}}(u_j) + s_i \bullet \sup v_{\varphi_{\mathcal{G}}}(u_j)\right)$$

$$s_i \bullet w_{\varphi_{\mathcal{G}}}(u_j) = \frac{1}{2}\left(s_i \bullet \inf w_{\varphi_{\mathcal{G}}}(u_j) + s_i \bullet \sup w_{\varphi_{\mathcal{G}}}(u_j)\right)$$

Normalized Hamming distance

$$d_{NH}(\Phi_{\mathcal{F}}(e), \varphi_{\mathcal{G}}(e)) = \frac{1}{2mp}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p} \begin{matrix}\left(\left| s_i \bullet u_{\Phi_{\mathcal{F}}}(u_j) - s_i \bullet u_{\varphi_{\mathcal{G}}}(u_j)\right|\right) + \left(\left| s_i \bullet v_{\Phi_{\mathcal{F}}}(u_j) - s_i \bullet v_{\varphi_{\mathcal{G}}}(u_j)\right|\right) + \\ \left(\left| s_i \bullet w_{\Phi_{\mathcal{F}}}(u_j) - s_i \bullet w_{\varphi_{\mathcal{G}}}(u_j)\right|\right)\end{matrix}\right\} \tag{4.2}$$

Euclidean distance

$$d_E(\Phi_{\mathcal{F}}(e), \varphi_{\mathcal{G}}(e)) = \left(\frac{1}{2m}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p} \begin{matrix}\left(\left| s_i \bullet u_{\Phi_{\mathcal{F}}}(u_j) - s_i \bullet u_{\varphi_{\mathcal{G}}}(u_j)\right|\right)^2 + \left(\left| s_i \bullet v_{\Phi_{\mathcal{F}}}(u_j) - s_i \bullet v_{\varphi_{\mathcal{G}}}(u_j)\right|\right)^2 + \\ \left(\left| s_i \bullet w_{\Phi_{\mathcal{F}}}(u_j) - s_i \bullet w_{\varphi_{\mathcal{G}}}(u_j)\right|\right)^2\end{matrix}\right\}\right)^{\frac{1}{2}} \tag{4.3}$$

Normalized Euclidean distance

$$d_{NE}(\Phi_{\mathcal{F}}(e), \varphi_{\mathcal{G}}(e)) = \left(\frac{1}{2mp}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p} \begin{matrix}\left(\left| s_i \bullet u_{\Phi_{\mathcal{F}}}(u_j) - s_i \bullet u_{\varphi_{\mathcal{G}}}(u_j)\right|\right)^2 + \left(\left| s_i \bullet v_{\Phi_{\mathcal{F}}}(u_j) - s_i \bullet v_{\varphi_{\mathcal{G}}}(u_j)\right|\right)^2 + \\ \left(\left| s_i \bullet w_{\Phi_{\mathcal{F}}}(u_j) - s_i \bullet w_{\varphi_{\mathcal{G}}}(u_j)\right|\right)^2\end{matrix}\right\}\right)^{\frac{1}{2}} \tag{4.4}$$

Weighted distance

$d^{w}(\Phi_{\mathcal{F}}(e), \varphi_{\mathcal{G}}(e))$ =

$$\left(\frac{1}{2m}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p} w_i \left\{\begin{matrix}\left(\left| s_i \bullet u_{\Phi_{\mathcal{F}}}(u_j) - s_i \bullet u_{\varphi_{\mathcal{G}}}(u_j)\right|\right)^r + \left(\left| s_i \bullet v_{\Phi_{\mathcal{F}}}(u_j) - s_i \bullet v_{\varphi_{\mathcal{G}}}(u_j)\right|\right)^r + \\ \left(\left| s_i \bullet w_{\Phi_{\mathcal{F}}}(u_j) - s_i \bullet w_{\varphi_{\mathcal{G}}}(u_j)\right|\right)^r\end{matrix}\right\}\right\}\right)^{\frac{1}{r}} \tag{4.5}$$

Where $r > 0$ and $w = (w_1, w_2, w_3, \dots, w_n)^T$ be a weight vector of e_i (i = 1, 2, 3, …, n). If r = 1 and r = 2, then equation 4.5 becomes the weighted hamming and weighted euclidean distances respectively.

Definition 4.2

$\mathcal{U}$ and E are universal set and set of attributes respectively and $(\Phi_{\mathcal{F}}, E)$, $(\varphi_{\mathcal{G}}, E)$ are two mIVNSS($\mathcal{U}$). Then similarity measure based on definition 4.1 between $(\Phi_{\mathcal{F}}, E)$ and $(\varphi_{\mathcal{G}}, E)$ defined as follows

$$S(\Phi_{\mathcal{F}}, \varphi_{\mathcal{G}}) = \frac{1}{1+d(\Phi_{\mathcal{F}}, \varphi_{\mathcal{G}})} \tag{4.6}$$

Another similarity measure between $(\Phi_{\mathcal{F}}, E)$ and $(\varphi_{\mathcal{G}}, E)$ defined as

$$S(\Phi_{\mathcal{F}}, \varphi_{\mathcal{G}}) = e^{-\beta d(\Phi_{\mathcal{F}}, \varphi_{\mathcal{G}})} \tag{4.7}$$

Where β is a steepness measure and a positive real number.

Definition 4.3

$\mathcal{U}$ and E are universal set and set of attributes respectively and $(\Phi_{\mathcal{F}}, E)$, $(\varphi_{\mathcal{G}}, E)$ are two mIVNSS($\mathcal{U}$). Then the following distances between $(\Phi_{\mathcal{F}}, E)$ and $(\varphi_{\mathcal{G}}, E)$ defined as follows

$$d(\Phi_{\mathcal{F}}(e), \varphi_{\mathcal{G}}(e)) = \left(\frac{1}{2m}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p} \begin{matrix}\left(\left| s_i \bullet u_{\Phi_{\mathcal{F}}}(u_j) - s_i \bullet u_{\varphi_{\mathcal{G}}}(u_j)\right|\right)^r + \left(\left| s_i \bullet v_{\Phi_{\mathcal{F}}}(u_j) - s_i \bullet v_{\varphi_{\mathcal{G}}}(u_j)\right|\right)^r + \\ \left(\left| s_i \bullet w_{\Phi_{\mathcal{F}}}(u_j) - s_i \bullet w_{\varphi_{\mathcal{G}}}(u_j)\right|\right)^r\end{matrix}\right\}\right)^{\frac{1}{r}} \tag{4.8}$$

And

$d(\Phi_{\mathcal{F}}(e), \varphi_{\mathcal{G}}(e))$ =

$$\left(\frac{1}{2mp}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p} \begin{matrix}\left(\left| s_i \bullet u_{\Phi_{\mathcal{F}}}(u_j) - s_i \bullet u_{\varphi_{\mathcal{G}}}(u_j)\right|\right)^r + \left(\left| s_i \bullet v_{\Phi_{\mathcal{F}}}(u_j) - s_i \bullet v_{\varphi_{\mathcal{G}}}(u_j)\right|\right)^r + \\ \left(\left| s_i \bullet w_{\Phi_{\mathcal{F}}}(u_j) - s_i \bullet w_{\varphi_{\mathcal{G}}}(u_j)\right|\right)^r\end{matrix}\right\}\right)^{\frac{1}{r}} \tag{4.9}$$

Where $r > 0$, equations 4.8 and 4.9 reduced to 4.1 and 4.2 respectively, if r = 1. Similarly, if r = 2 then equations 4.8 and 4.9 reduced to 4.3 and 4.4 respectively.

Definition 4.4

Similarity measure between two mIVNSS $(\Phi_{\mathcal{F}}, E)$ and $(\varphi_{\mathcal{G}}, E)$ based on the weighted distance of $(\Phi_{\mathcal{F}}, E)$ and $(\varphi_{\mathcal{G}}, E)$ defined as follows

$$S(\Phi_{\mathcal{F}}, \varphi_{\mathcal{G}}) = \frac{1}{1+d^{w}(\Phi_{\mathcal{F}}, \varphi_{\mathcal{G}})} \tag{4.10}$$

Definition 4.5

Let $\Phi_{\mathcal{F}}$ and $\varphi_{\mathcal{G}}$ are mPIVNSS over the universal set, then $\Phi_{\mathcal{F}}$ and $\varphi_{\mathcal{G}}$ are said to be α – similar if and only if $S_{mPIVNSS}(\Phi_{\mathcal{F}}, \varphi_{\mathcal{G}}) \geq \alpha$ for $\alpha \in (0, 1)$. If $S_{mPIVNSS}(\Phi_{\mathcal{F}}, \varphi_{\mathcal{G}}) > \frac{1}{2}$, then we can say that $\Phi_{\mathcal{F}}$ and $\varphi_{\mathcal{G}}$ are significantly similar.

5. Applications of Similarity Measures of mPIVNSS in Medical Diagnoses

In this section, we proposed the algorithm for mPIVNSS by using developed similarity measures. We also used the proposed methods for medical diagnoses.

5.1. Application of Similarity Measure in Medical Diagnoses

We develop the algorithm of mPIVNSS for similarity measure and used the developed similarity measure for medical diagnoses by using the proposed algorithm.

5.1.1. Algorithm for Similarity Measure of mPIVNSS

Step 1. Pick out the set containing parameters.

Step 2. Construct the mPIVNSS according to experts.

Step 3. Construct mPIVNSS $\varphi_{\mathcal{G}}^{t}$ for the evaluation of different decision-makers, where t = 1, 2,…,m.

Step 4. Find the distance between two mPIVNSS by using the distance formula.

$d_{mPIVNSS}^{H}(\Phi_{\mathcal{F}}(e), \varphi_{\mathcal{G}}(e))=$

$$\frac{1}{2m}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p}\begin{matrix}\left(\left| s_i \bullet u_{\Phi_{\mathcal{F}}}(u_j) - s_i \bullet u_{\varphi_{\mathcal{G}}}(u_j)\right| \right) + \left(\left| s_i \bullet v_{\Phi_{\mathcal{F}}}(u_j) - s_i \bullet v_{\varphi_{\mathcal{G}}}(u_j)\right| \right) + \\ \left(\left| s_i \bullet w_{\Phi_{\mathcal{F}}}(u_j) - s_i \bullet w_{\varphi_{\mathcal{G}}}(u_j)\right| \right)\end{matrix}\right\}$$

Step 5. Compute the similarity measure between two mPIVNSS by utilizing the following formula

$$S_{mPIVNSS}(\Phi_{\mathcal{F}},\ \varphi_{\mathcal{G}}) = \frac{1}{1+d(\Phi_{\mathcal{F}},\varphi_{\mathcal{G}})}$$

Step 6. Analyze the result.

5.2. Problem Formulation and Application of Similarity Measure of mPIVNSS For Disease Diagnoses

The general proposed algorithm can be used in diagnosis complications, then we are giving one numerical example containing way out those diagnosis problems in the general lighted of scientific discipline. This planned algorithm may be obtained from immoderate medical disease diagnosis complications. We consider typhoid disease as a diagnosis problem, so whether a well-advised patient has typhoid or not, as many containing the overall signs and symptoms of typhoid are going to be compatible as well as other diseases such as malaria. For a verbal description of the disease, we tend dispensed similarity measures along the mPIVNSS structure to attain an insured person as well as high-fidelity consequences. The general m-polar anatomical structure offers us a record of medical experts rating for the extraordinary disease.

5.2.1. Application of Similarity Measure

Now we assume the universal set as follows $\mathcal{U}$ = {u_1 = typhoid, u_2 = not typhoid} and E be a set of parameters which consist of symptoms of typhoid disease such as E = {x_1 = flu, x_2 = body pain, x_3 = headache}. Assume $\mathcal{F}$ and $\mathcal{G} \subseteq E$, then we construct the 3-PIVNSS of $\mathcal{F}$ and $\mathcal{G}$ such as $\Phi_{\mathcal{F}}(x)$ and $\varphi_{\mathcal{G}}(x)$ according to experts given as follows.

Table 3: 3-PIVNSS of $\mathcal{F}_{\ddot{A}}$ according to experts

$\boldsymbol{\Phi_{\mathcal{F}}(x)}$	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$
$\boldsymbol{u_1}$	$\begin{pmatrix}([.5,.8],[.2,.5],[.1,.2]),([.3,.5],[.1,.3],[.2,.4]),\\([.6,.9],[.7,.8],[.8,1])\end{pmatrix}$	([.2,.4],[.3,0.4],[.1,.3]),([.2,.5],[.1,.6],[.1,.3]), ([.8,1],[.6,.9],[.6,.7])

u_2	([.3,.6],[.1,.6],[.3,.4]),([0,.2],[.1,.4],[.3,.5]),([.5,.9],[.3,.8],[.5,.8])	([.2,.5],[.2,.3],[.5,.6]),([.3,.5],[.1,.5],[.5,.8]),([.6,.9],[.5,.8],[.6,.9])

Table 4: 3-PIVNSS of $\mathcal{G}_{\breve{B}}$ according to experts

$\varphi_{\mathcal{G}}(x)$	x_1	x_2
u_1	([.4,.8],[.3,0.6],[.2,.5]),([.2,.7],[.3,.4],[.4,.6]),([.7,.8],[.4,.9],[.5,1])	([.1,.6],[.5,0.7],[.1,.2]),([.3,.4],[.2,.5],[.2,.5]),([.5,.9],[.7,.8],[.4,.6])
u_2	([.2,.7],[.3,.5],[.2,.6]),([.1,.3],[.2,.5],[.2,.7]),([.4,.9],[.4,.7],[.5,.8])	([.1,.6],[.1,.5],[.4,.8]),([.3,.6],[.3,.4],[1,1]),([.5,.9],[.3,.7],[.1,.8])

Now we compute distances between $\Phi_{\mathcal{F}}(x)$ and $\varphi_{\mathcal{G}}(x)$ by using definition 4.1 given as follows.

$d^{H}_{3-PIVNSS}(\Phi_{\mathcal{F}}(e), \varphi_{\mathcal{G}}(e)) = 0.55$

$d^{NH}_{3-PIVNSS}(\Phi_{\mathcal{F}}(e), \varphi_{\mathcal{G}}(e)) = 0.275$

$d^{E}_{3-PIVNSS}(\Phi_{\mathcal{F}}(e), \varphi_{\mathcal{G}}(e)) = 0.31111$

$d^{NE}_{3-PIVNSS}(\Phi_{\mathcal{F}}(e), \varphi_{\mathcal{G}}(e)) = 0.22$

Now by using the above-calculated distances we will find the similarity measure between $\Phi_{\mathcal{F}}(e)$ as well as $\varphi_{\mathcal{G}}(e)$ given as follows

$S^{H}_{3-PIVNSS}(\Phi_{\mathcal{F}}, \varphi_{\mathcal{G}}) = 0.6452 > 0.5$

$S^{NH}_{3-PIVNSS}(\Phi_{\mathcal{F}}, \varphi_{\mathcal{G}}) = 0.7843 > 0.5$

$S^{E}_{3-PIVNSS}(\Phi_{\mathcal{F}}, \varphi_{\mathcal{G}}) = 0.7627 > 0.5$

$S^{NE}_{3-PIVNSS}(\Phi_{\mathcal{F}}, \varphi_{\mathcal{G}}) = 0.8197 > 0.5$

According to the above calculation analyze that $S_{3-PIVNSS}(\Phi_{\mathcal{F}}, \varphi_{\mathcal{G}}) \geq 0.5$, so 3-PIVNSS of $\Phi_{\mathcal{F}}$ and $\varphi_{\mathcal{G}}$ are significantly similar which shows that the patient suffering from typhoid.

6. Conclusion

In this article, we studied IVNSS and proposed the idea of mPIVNSS with some basic operations and properties. We use attributes and numerical examples to develop some basic operators. By using Hamming distance and Euclidean distance and their characteristics, a distance-based mPIVNSS similarity measure was also developed in this research. By using the presented distance-based similarity measure, a decision-making method has been developed for mPIVNSS. Finally, the developed technique has been used in medical diagnosis. In the future, the concept of mIVPNSS will be extended to neutrosophic fuzzy soft sets, interval-valued neutrosophic fuzzy soft sets, m-polar neutrosophic fuzzy soft sets, m-polar interval neutrosophic fuzzy soft sets, etc., and will be used to solve different real-life Problems, such as medical diagnosis, decision making, etc.

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Complex Neutrosophic Matrix with Some Algebraic Operations and Matrix Norm Convergence

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Abstract: In this article, a new concept of the Complex Neutrosophic Matrix is introduced to solve different problems related to uncertainties. Based on the proposed matrix, we have provided various algebraic operations like addition, subtraction, union many others which will be of great help in estabalishing the fundamental concepts. The results obtained through the above operations will be consequently utilized in solving the higher-dimensional problems due to their matrix properties. This novel concept lays the foundation of various research solutions in the field of the complex neutrosophic matrix, which are yet to be considered by researchers.The matrix norm convergence of the proposed matrix has also beenstudied for the necessary foundation of the complex neutrosophic matrix. Also, the future studies of the proposed concept have been proposed.

Keywords: Complex neutrosophic matrix, Complex neutrosophic set, Matrix Norm, Convergence, Matrix operations

1. Introduction

Zadeh [1] in 1965 introduced the concept of the fuzzy set to deal with the problems created by the uncertainty components. Atanassov [2] in 1993 further extended the theory of the fuzzy set to the intuitionistic fuzzy set (IFS) in which the non-membership function was added to the membership function. IFS played a significant role in solving the unsolved problems, but the indeterminacy/neutrality function was the dependent concept in this case which started creating the problem andwas later resolved by Samarandache in 1995 by introducing the theory of neutrosophic set. *Neutrosophic set is the branch of philosophy that deals with neutrality and its interaction with the different philosophical spectra.* Samarandache [3] defined the concept of neutrosophic set in the form of degree of truth, indeterminacy and falsity membership functions. Further, the notion of the neutrosophic set has been extended to a single-valued neutrosophic set [4], complex neutrosophic set [5], neutrosophic hypergraph [6] etc. Wang et al. [4] extended the theory of single valued neutrosophic sets by introducing the various properties and set-theoretic operators. Many difficulties related to the various fields of medicine [7-9], decision-making[10-12] and pattern recognition in the range [0,1] have been solved using these theories. Later, Ramot et al. [13] presented the novel theory of a complex fuzzy set (CFS) in 2010, which extended the range of fuzzy set to the unit circle in the complex plane. "*A complex fuzzy set preserves the* *concept of uncertainty through amplitude in [0,1] with the addition of the phase term in* $[0,2\pi]$. *Further, the properties of union, intersection and complement with its application in the case of solar activity and signals have been defined. Alkouri and Salleh [14] introduced the concept of a complex intuitionistic fuzzy set (CIFS), which added the complex non-membership function to the complex membership function. Along with the various properties related to its complement, union, T-norm and S-norm have been described. Later, Ali and Samarandache [5] proposed the concept of a complex neutrosophic set (CNS) which added the degree of complex neutrality function to membership and non-membership functions followed by various properties like union, intersection, a product with their application.*"

Further, the concept was extended in direction of matrices. Thomson [15] initiated by introducing the concept of the fuzzy matrix in 1977 and defined the convergence of fuzzy matrix. Kim and Roush [16] contributed by defining the fuzzy matrices concept as an extension of Boolean algebra. Later, the determinant of the intuitionistic fuzzy matrix was proposed by Pal [17] and interval-valued intuitionistic fuzzy matrices were introduced by Khan and Pal [18]. In 2014, Dhar et al. [19] gave the concept of neutrosophic fuzzy matrices, which was extended by Kandasamy et al. [20] who proposed the concept of neutrosophic interval matrices with its application.

Various researchers have extended their study in the direction of extension of fuzzy theories, which later turned to complex fuzzy matrices by Zhao and Ma [21] in 2016. They defined the complex fuzzy matrices in the form of $C = \left(A_{ij}(x) + \mathrm{i}B_{ij}(x)\right)$ and also explained the norm convergence. Khan et al. [22] extended the concept of complex fuzzy matrices to complex fuzzy soft matrices in 2020 and also proposed some theorems, which have been explained with the help of its application in DMP (Decision-Making Problems).

Various extensions have been done in the theory of the neutrosophic sets and information in literature. This has been explained with the help of the following figure:

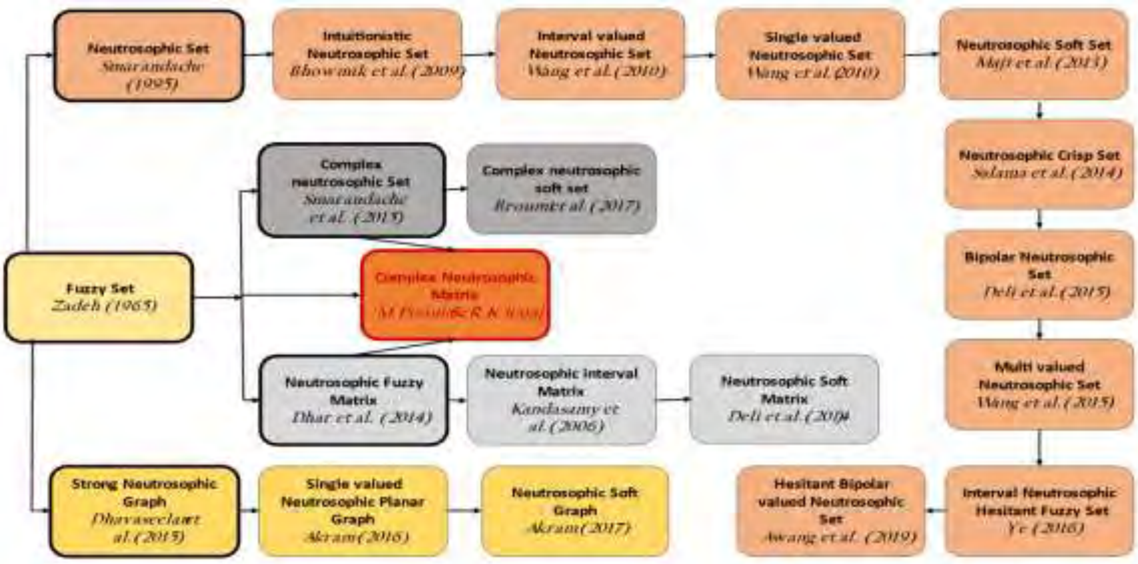

Fig 1: Literature Survey of Neutrosophic Theory.

Now, after considering the importance of matrix form in solving a large number as well as higher dimension problems in a single interval of time motivated us to extend these advantages of the matrix form from the real plane to the complex pane of unit range. Thus, we extended the theory of neutrosophic matrices to the complex plane and introduced the novel concept of a complex neutrosophic matrix. In this article, we have defined the concept of complex neutrosophic matrices and explained it with the help of algebraic operations. In continuation, the concepts like union, intersection have been also explained for a better understanding of the concept. Also, the concept of matrix norm and power convergence have been provided along with various results.

The manuscript has been organized as follows. Section 2 provides the basic and fundamental preliminaries related to the proposed concept. The formal notion of the complex neutrosophic matrix has been detailed and studied in Section 3. Different algebraic operations and properties related to the proposed matrix along with norm convergence have been included in Sections 4 and 5. Finally, the conclusion and scope for future work have been presented in Section 6.

2. Preliminaries

In this section, we briefly discuss the basic preliminaries and definitions related to the complex neutrosophic matrix present in the literature.

Definition 2.1. ***(Neutrosophic set) [4]*** "Let U be a space of points and $x\epsilon U$. A neutrosophic set S in U is characterized by a truth membership function $\Gamma_S(x)$, an indeterminacy membership function $\mathrm{I}_S(x)$ and falsity membership function $\Pi_S(x)$. $\Gamma_S(x)$, $\mathrm{I}_S(x)$ and$\Pi_S(x)$ are a real-valued subset of $]0^-, 1^+[$. The neutrosophic set can be represented as

$$S = \{(x, \Gamma_S(x) = a_\Gamma, \mathrm{I}_S(x) = a_I, \Pi_S(x) = a_\Pi) : x\epsilon U\}$$

where $^-0 \le \Gamma_S(x) + \mathrm{I}_S(x) + \Pi_S(x) \le 3^+$."

Definition 2.2. ***(Complex fuzzy set) [13]*** "A complex fuzzy set S, defined on a universe of discourse U, is characterized by a membership function $\mu_S(x)$ that assigns any element $x\epsilon U$ a complex-valued grade of membership in S. By definition, the values $\mu_S(x)$ may receive all values lie within in the unit circle in the complex plane, and are thus of form $\mathrm{R}_S(x)e^{\mathrm{i}\gamma_S(x)}$, where $\mathrm{i} = \sqrt{-1}$, $\mathrm{R}_S(x)$ and $\gamma_S(x)$ are both real-valued, and $\mathrm{R}_S(x)\epsilon[0,1]$. The complex fuzzy set S may be represented as the set of the ordered pairs

$$S = \{(x, \mu_S(x)) : x\epsilon U\}."$$

Definition 2.3. ***(Complex Neutrosophic set) [5]*** "A complex neutrosophic set S defined on the universe of discourse U, which is characterized by a truth membership function $\Gamma_S(x)$, an indeterminacy membership function $\mathrm{I}_S(x)$ and falsity membership function $\Pi_S(x)$ that assigns a complex-valued grade of $\Gamma_S(x)$, $\mathrm{I}_S(x)$ and$\Pi_S(x)$ in S for any $x\epsilon U$. The values $\Gamma_S(x), \mathrm{I}_S(x), \Pi_S(x)$ and their sum may all lie within the unit circle in the complex plane and so is of the following form,

$\Gamma_S(x) = P_S(x)e^{\mathrm{i}\alpha_S(x)}, \mathrm{I}_S(x) = \mathrm{Q}_S(x)e^{\mathrm{i}\beta_S(x)}$and $\Pi_S(x) = \mathrm{R}_S(x)e^{\mathrm{i}\gamma_S(x)}$
where $P_S(x)$, $Q_S(x)$, $R_S(x)$ and $\alpha_S(x)$, $\beta_S(x)$, $\gamma_S(x)$ are respectively, real-valued and $P_S(x), Q_S(x), R_S(x)\epsilon[0,1]$ such that $^-0 \le P_S + Q_S + R_S \le 3^+$.

$$S = \{(x, \Gamma_S(x) = a_\Gamma, \mathrm{I}_S(x) = a_I, \Pi_S(x) = a_\Pi): x\epsilon U\},$$

where $|a_\Gamma| \le 1, |a_\mathrm{I}| \le 1, |a_\Pi| \le 1 \,\&|a_\Gamma + a_\mathrm{I} + a_\Pi| \le 3$."

Example 1: Let $U = \{u_1, u_2, u_3\}$be the universe of discourse. Then, consider a complex neutrosophic set S in U which is denoted by:

$$S = \frac{\left(\frac{3}{5}e^{\mathrm{i}0.8}, \frac{2}{5}e^{i\frac{\pi}{4}}, \frac{1}{2}e^{i\frac{3\pi}{4}}\right)}{u_1} + \frac{\left(\frac{3}{10}e^{\mathrm{i}0.1}, \frac{2}{5}e^{\mathrm{i}\frac{3\pi}{4}}, \frac{1}{10}e^{\mathrm{i}\frac{\pi}{4}}\right)}{u_2} + \frac{\left(\frac{1}{5}e^{\mathrm{i}0.7}, \frac{1}{10}e^{\mathrm{i}\frac{5\pi}{4}}, \frac{2}{5}e^{\mathrm{i}\frac{\pi}{4}}\right)}{u_3}$$

Definition 2.4. ***(Complex fuzzy matrix) [21]*** "Suppose$S = \left(\Gamma_{ij}(x) + \mathrm{i}\,\Pi_{ij}(x)\right)_{m\times n}$ is the matrix, all of the $\Gamma_{ij}(x) + \mathrm{i}\Pi_{ij}(x)$ is a complex fuzzy number for $i, j(1 \le i \le n, 1 \le j \le n)$, then called S is a complex fuzzy matrix."

3. Notion of Complex Neutrosophic Matrix

In this segment of the current article, we proposed a new kind of a complex neutrosophic matrix. The formal definition and an example of the complex neutrosophic matrix have been provided along with the operation of union and intersection for a better understanding of the concept.

Definition 3.1. "A complex neutrosophic fuzzy matrix$S_{m\times n}$ defined on a universe of discourse U, which can be characterized by a truth membership function $\Gamma_S(x_{ij})$, an indeterminacy membership function $\mathrm{I}_S(x_{ij})$ and a falsity membership function $\Pi_S(x_{ij})$ that assign complex value functions of the form,

$$\Gamma_S(x_{ij}) = P_S(x_{ij})e^{\mathrm{i}\alpha_S(x_{ij})},$$

$$\mathrm{I}_S(x_{ij}) = \mathrm{Q}_S(x_{ij})e^{\mathrm{i}\beta_S(x_{ij})}$$

and

$$\Pi_S(x_{ij}) = \mathrm{R}_S(x_{ij})e^{\mathrm{i}\gamma_S(x_{ij})}$$

in $S_{m\times n}$ for any $x_{ij}\epsilon U$, where $P_S, Q_S, R_S \in [0,1]$ s.t. $^-0 \le P_S + Q_S + R_S \le 3^+$. The values and the sum of Γ_S, I_S and Π_S may always lie within the unit circle in the complex plane."
Then, the complex neutrosophic fuzzy matrix $S_{m\times n}$is represented as

$$S_{m\times n} = \left\{\left[\Gamma_S(x_{ij}), \mathrm{I}_S(x_{ij}), \Pi_S(x_{ij})\right]_{m\times n} | x_{ij} \in U\right\}$$

where $|\Gamma_S| \le 1, |\mathrm{I}_S| \le 1, |\Pi_S| \le 1 \,\&|\Gamma_S + \mathrm{I}_S + \Pi_S| \le 3$.

Example 2. We could represent example 1 in matrix form or order 3×1 i.e.

$$S_{3\times 1} = \begin{bmatrix} \left(\frac{3}{5}e^{\mathrm{i}0.8}, \frac{2}{5}e^{\mathrm{i}\frac{\pi}{4}}, \frac{1}{2}e^{\mathrm{i}\frac{3\pi}{4}}\right) \\ \left(\frac{3}{10}e^{\mathrm{i}0.1}, \frac{2}{5}e^{\mathrm{i}\frac{3\pi}{4}}, \frac{1}{10}e^{\mathrm{i}\frac{\pi}{4}}\right) \\ \left(\frac{1}{5}e^{\mathrm{i}0.7}, \frac{1}{10}e^{\mathrm{i}\frac{5\pi}{4}}, \frac{2}{5}e^{\mathrm{i}\frac{\pi}{4}}\right) \end{bmatrix}$$

Definition 3.2. ***(Complement)***

The complement of the complex neutrosophic matrix can be written in the form of

$$(S_{m\times n})^c = \left(\left[\Gamma_S(x_{ij}), \mathrm{I}_S(x_{ij}), \Pi_S(x_{ij})\right]_{m\times n}\right)^c = \left[{\Gamma_S}^c(x_{ij}), {\mathrm{I}_S}^c(x_{ij}), {\Pi_S}^c(x_{ij})\right]_{m\times n}$$

$$= \left[\left(P_S(x_{ij})e^{\mathrm{i}\alpha_S(x_{ij})}\right)^c, \left(\mathrm{Q}_S(x_{ij})e^{i\beta_S(x_{ij})}\right)^c, \left(\mathrm{R}_S(x_{ij})e^{\mathrm{i}\gamma_S(x_{ij})}\right)^c\right]_{m\times n}$$

where $\left(P_S(x_{ij})\right)^c = \mathrm{R}_S(x_{ij})$and $\left(e^{\mathrm{i}\alpha_S(x_{ij})}\right)^c = e^{\mathrm{i}(2\pi-\alpha_S(x_{ij}))}$.

Similarly, $\left(R_S(x_{ij})\right)^c = \mathrm{P}_S(x_{ij})$ and $\left(e^{\mathrm{i}\gamma_S(x_{ij})}\right)^c = e^{\mathrm{i}(2\pi-\gamma_S(x_{ij}))}$ and

Finally, $\left(Q_S(x_{ij})\right)^c = 1 - Q_S(x_{ij})$ and $\left(e^{\mathrm{i}\beta_S(x_{ij})}\right)^c = e^{\mathrm{i}(2\pi-\beta_S(x_{ij}))}$.

Example3. Suppose $S_{3\times 1}$be a complex neutrosophic matrix. Then, the complement of $(S_{3\times 1})^c$ will be given by

$$S_{3\times 1} = \begin{bmatrix} \left(\frac{3}{5}e^{\mathrm{i}0.8}, \frac{2}{5}e^{\mathrm{i}\frac{\pi}{4}}, \frac{1}{2}e^{\mathrm{i}\frac{3\pi}{4}}\right) \\ \left(\frac{3}{10}e^{\mathrm{i}0.1}, \frac{2}{5}e^{\mathrm{i}\frac{3\pi}{4}}, \frac{1}{10}e^{\mathrm{i}\frac{\pi}{4}}\right) \\ \left(\frac{1}{5}e^{\mathrm{i}0.7}, \frac{1}{10}e^{\mathrm{i}\frac{5\pi}{4}}, \frac{2}{5}e^{\mathrm{i}\frac{\pi}{4}}\right) \end{bmatrix}, \quad (S_{3\times 1})^c = \begin{bmatrix} \left(\frac{2}{5}e^{\mathrm{i}\left(2\pi-\frac{\pi}{225}\right)}, \frac{3}{5}e^{\mathrm{i}\frac{7\pi}{4}}, \frac{1}{2}e^{\mathrm{i}\frac{5\pi}{4}}\right) \\ \left(\frac{7}{10}e^{\mathrm{i}\left(2\pi-\frac{\pi}{1800}\right)}, \frac{3}{5}e^{\mathrm{i}\frac{5\pi}{4}}, \frac{9}{10}e^{\mathrm{i}\frac{7\pi}{4}}\right) \\ \left(\frac{4}{5}e^{\mathrm{i}\left(2\pi-\frac{7\pi}{1800}\right)}, \frac{9}{10}e^{\mathrm{i}\frac{3\pi}{4}}, \frac{3}{5}e^{\mathrm{i}\frac{7\pi}{4}}\right) \end{bmatrix}$$

Definition 3.3. ***(Union of the complex neutrosophic matrix)*** Consider two complex neutrosophic matrices $S^1_{m\times n} = \left[\Gamma^1_S(x_{ij}), \mathrm{I}^1_S(x_{ij}), \Pi^1_S(x_{ij})\right]_{m\times n}$ and

$S^2_{m\times n} = \left[\Gamma^2_S(x_{ij}), \mathrm{I}^2_S(x_{ij}), \Pi^2_S(x_{ij})\right]_{m\times n}$ respectively. Then the union of these two matrices will be given by

$$S^1_{m\times n} \cup S^2_{m\times n} = \left\{\max\left\{\Gamma^1_S(x_{ij}), \Gamma^2_S(x_{ij})\right\}, \min\left\{\mathrm{I}^1_S(x_{ij}), \mathrm{I}^2_S(x_{ij})\right\}, \min\left\{\Pi^1_S(x_{ij}), \Pi^2_S(x_{ij})\right\}\right\}_{m\times n}$$

where

$$\max\left\{\Gamma^1_S(x_{ij}), \Gamma^2_S(x_{ij})\right\} = \max\left\{\mathrm{P}^1_S(x_{ij}), \mathrm{P}^2_S(x_{ij})\right\} e^{\mathrm{i}\max\left\{\alpha^1_S(x_{ij}), \alpha^2_S(x_{ij})\right\}},$$

$\min\left\{\mathrm{I}^1_S(x_{ij}), \mathrm{I}^2_S(x_{ij})\right\} = \min\left\{\mathrm{Q}^1_S(x_{ij}), \mathrm{Q}^2_S(x_{ij})\right\}e^{\mathrm{i}\min\left\{\beta^1_S(x_{ij}), \beta^2_S(x_{ij})\right\}}$ and

$$\min\{\Pi_S^1(x_{ij}),\Pi_S^2(x_{ij})\}=\min\{\mathrm{R}_S^1(x_{ij}),\mathrm{R}_S^2(x_{ij})\}e^{\mathrm{i}\min\{\gamma_S^1(x_{ij}),\gamma_S^2(x_{ij})\}}$$

Example 4. Consider two complex neutrosophic matrices

$$S_{3\times1}^1=\begin{bmatrix}\left(\frac{3}{5}e^{\mathrm{i}0.8},\frac{2}{5}e^{\mathrm{i}\frac{\pi}{4}},\frac{1}{2}e^{\mathrm{i}\frac{3\pi}{4}}\right)\\\left(\frac{1}{10}e^{\mathrm{i}0.7},\frac{1}{5}e^{\mathrm{i}\frac{\pi}{4}},\frac{9}{10}e^{\mathrm{i}\frac{5\pi}{4}}\right)\\\left(\frac{1}{5}e^{\mathrm{i}0.7},\frac{1}{10}e^{\mathrm{i}\frac{5\pi}{4}},\frac{2}{5}e^{\mathrm{i}\frac{\pi}{4}}\right)\end{bmatrix},\qquad S_{3\times1}^2=\begin{bmatrix}\left(\frac{1}{10}e^{\mathrm{i}0.2},\frac{3}{10}e^{\mathrm{i}\frac{3\pi}{4}},\frac{7}{10}e^{\mathrm{i}\frac{\pi}{4}}\right)\\\left(\frac{1}{5}e^{\mathrm{i}0.5},\frac{1}{2}e^{\mathrm{i}\frac{\pi}{4}},\frac{3}{10}e^{\mathrm{i}\frac{\pi}{4}}\right)\\\left(\frac{3}{5}e^{\mathrm{i}0.7},\frac{1}{5}e^{\mathrm{i}\frac{\pi}{4}},\frac{1}{2}e^{\mathrm{i}\frac{\pi}{4}}\right)\end{bmatrix}$$

$$S_{3\times1}^1\cup S_{3\times1}^2=\begin{bmatrix}\left(\frac{3}{5}e^{\mathrm{i}0.8},\frac{2}{5}e^{\mathrm{i}\frac{3\pi}{4}},\frac{7}{10}e^{\mathrm{i}\frac{3\pi}{4}}\right)\\\left(\frac{1}{10}e^{\mathrm{i}0.5},\frac{1}{5}e^{\mathrm{i}\frac{\pi}{4}},\frac{3}{10}e^{\mathrm{i}\frac{\pi}{4}}\right)\\\left(\frac{1}{5}e^{\mathrm{i}0.7},\frac{1}{10}e^{\mathrm{i}\frac{\pi}{4}},\frac{2}{5}e^{\mathrm{i}\frac{\pi}{4}}\right)\end{bmatrix}$$

Definition 3.4. ***(Intersection of the complex neutrosophic matrix)*** Consider two complex neutrosophic matrices $S_{m\times n}^1=\left[\Gamma_S^1(x_{ij}),\mathrm{I}_S^1(x_{ij}),\Pi_S^1(x_{ij})\right]_{m\times n}$ and $S_{m\times n}^2=\left[\Gamma_S^2(x_{ij}),\mathrm{I}_S^2(x_{ij}),\Pi_S^2(x_{ij})\right]_{m\times n}$ respectively. Then, the intersection of these two matrices will be given by

$$S_{m\times n}^1\cap S_{m\times n}^2=\left\{\min\{\Gamma_S^1(x_{ij}),\Gamma_S^2(x_{ij})\},\max\{\mathrm{I}_S^1(x_{ij}),\mathrm{I}_S^2(x_{ij})\},\max\{\Pi_S^1(x_{ij}),\Pi_S^2(x_{ij})\}\right\}_{m\times n}$$

where,

$$\min\{\Gamma_S^1(x_{ij}),\Gamma_S^2(x_{ij})\}=\min\{\mathrm{P}_S^1(x_{ij}),\mathrm{P}_S^2(x_{ij})\}\,e^{\mathrm{i}\min\{\alpha_S^1(x_{ij}),\alpha_S^2(x_{ij})\}},$$

$\max\{\mathrm{I}_S^1(x_{ij}),\mathrm{I}_S^2(x_{ij})\}=\max\{\mathrm{Q}_S^1(x_{ij}),\mathrm{Q}_S^2(x_{ij})\}e^{\mathrm{i}\max\{\beta_S^1(x_{ij}),\beta_S^2(x_{ij})\}}$ and

$$\max\{\Pi_S^1(x_{ij}),\Pi_S^2(x_{ij})\}=\max\{\mathrm{R}_S^1(x_{ij}),\mathrm{R}_S^2(x_{ij})\}e^{\mathrm{i}\max\{\gamma_S^1(x_{ij}),\gamma_S^2(x_{ij})\}}$$

Example 5. Consider two complex neutrosophic matrices

$$S_{3\times1}^1=\begin{bmatrix}\left(\frac{3}{5}e^{\mathrm{i}0.8},\frac{2}{5}e^{\mathrm{i}\frac{\pi}{4}},\frac{1}{2}e^{\mathrm{i}\frac{3\pi}{4}}\right)\\\left(\frac{1}{10}e^{\mathrm{i}0.7},\frac{1}{5}e^{\mathrm{i}\frac{\pi}{4}},\frac{9}{10}e^{\mathrm{i}\frac{5\pi}{4}}\right)\\\left(\frac{1}{5}e^{\mathrm{i}0.7},\frac{1}{10}e^{\mathrm{i}\frac{5\pi}{4}},\frac{2}{5}e^{\mathrm{i}\frac{\pi}{4}}\right)\end{bmatrix},\qquad S_{3\times1}^2=\begin{bmatrix}\left(\frac{1}{10}e^{\mathrm{i}0.2},\frac{3}{10}e^{\mathrm{i}\frac{3\pi}{4}},\frac{7}{10}e^{\mathrm{i}\frac{\pi}{4}}\right)\\\left(\frac{1}{5}e^{\mathrm{i}0.5},\frac{1}{2}e^{\mathrm{i}\frac{\pi}{4}},\frac{3}{10}e^{\mathrm{i}\frac{\pi}{4}}\right)\\\left(\frac{3}{5}e^{\mathrm{i}0.7},\frac{1}{5}e^{\mathrm{i}\frac{\pi}{4}},\frac{1}{2}e^{\mathrm{i}\frac{\pi}{4}}\right)\end{bmatrix}$$

$$S_{3\times1}^1\cap S_{3\times1}^2=\begin{bmatrix}\left(\frac{1}{10}e^{\mathrm{i}0.2},\frac{3}{10}e^{\mathrm{i}\frac{\pi}{4}},\frac{1}{2}e^{\mathrm{i}\frac{\pi}{4}}\right)\\\left(\frac{1}{5}e^{\mathrm{i}0.7},\frac{1}{2}e^{\mathrm{i}\frac{\pi}{4}},\frac{9}{10}e^{\mathrm{i}\frac{5\pi}{4}}\right)\\\left(\frac{3}{5}e^{\mathrm{i}0.7},\frac{1}{5}e^{\mathrm{i}\frac{5\pi}{4}},\frac{1}{2}e^{\mathrm{i}\frac{\pi}{4}}\right)\end{bmatrix}$$

4. Algebraic Operations On Complex Neutrosophic Matrices

In this segment of the current article, we have discussed the theoretical operations on the complex neutrosophic matrices. This section begins with the basic definition related to the concept and is followed by the theorem, multiplication and additive identity.

Definition 4.1. A be a $m \times n$neutrosophic matrix. If all of its entries are $\langle 0,0,1e^{i0}\rangle$, then $A$is called zero complex neutrosophic matrices and denoted by 0.
If all of its entries are $\langle 1e^{i0}, 1e^{i0}, 0\rangle$, then A is called universal complex neutrosophic matrix and denoted by J.

Theorem 1. The matrix $S_{m\times n}$ are a complex neutrosophic fuzzy algebra under the component-wise addition and multiplication operations $(+,\odot)$ represented as:

For $S_1 = [\Gamma_{S_1}(x_{ij}), \mathrm{I}_{S_1}(x_{ij}), \Pi_{S_1}(x_{ij})]_{m\times n}$and $S_2 = [\Gamma_{S_2}(x_{ij}), \mathrm{I}_{S_2}(x_{ij}), \Pi_{S_2}(x_{ij})]_{m\times n}$ in $S_{m\times n}$,

$$S_1 + S_2 = (\sup\{S_1, S_2\})$$
$$= (\sup\{\Gamma_{S_1}(x_{ij}), \Gamma_{S_2}(x_{ij})\}, \sup\{\mathrm{I}_{S_1}(x_{ij}), \mathrm{I}_{S_2}(x_{ij})\}, \inf\{\Pi_{S_1}(x_{ij}), \Pi_{S_2}(x_{ij})\})$$
$$S_1 \odot S_2 = (\inf\{S_1, S_2\}) = (\inf\{\Gamma_{S_1}(x_{ij}), \Gamma_{S_2}(x_{ij})\}, \inf\{\mathrm{I}_{S_1}(x_{ij}), \mathrm{I}_{S_2}(x_{ij})\}, \sup\{\Pi_{S_1}(x_{ij}), \Pi_{S_2}(x_{ij})\})$$

where $S_1 = [\Gamma_{S_1}(x_{ij}), \mathrm{I}_{S_1}(x_{ij}), \Pi_{S_1}(x_{ij})]_{m\times n}$or

$$S_1 = \left[P_{S_1}(x_{ij})e^{\mathrm{i}\alpha_{S_1}(x_{ij})}, \mathrm{Q}_{S_1}(x_{ij})e^{\mathrm{i}\beta_{S_1}(x_{ij})}, \mathrm{R}_{S_1}(x_{ij})e^{\mathrm{i}\gamma_{S_1}(x_{ij})}\right]_{m\times n} \text{and}$$

$$S_2 = \left[P_{S_2}(x_{ij})e^{\mathrm{i}\alpha_{S_2}(x_{ij})}, \mathrm{Q}_{S_2}(x_{ij})e^{\mathrm{i}\beta_{S_2}(x_{ij})}, \mathrm{R}_{S_2}(x_{ij})e^{\mathrm{i}\gamma_{S_2}(x_{ij})}\right]_{m\times n}$$

Proof. Every matrix in complex neutrosophic algebra should also satisfy the properties of fuzzy algebra. Therefore, $S_1 + O = S_1$ and $S_1 \odot J = S_1 \forall\, S_1 \epsilon S_{m\times n}$, hence in this case the addition and multiplication identities are denoted by the zero matrix O and the universal matrix J respectively. Thus, the identity element relative to the operations $(+,\odot)$ exist. Further, $S_1 + J = J$ and $S_1 \odot O = O$. This proves that Universal bound holds.
Similarly, we can prove for Idempotence, Commutativity, Associative and Absorption properties.
Now, in the case of Distributivity property, we have to prove
$S_1 \odot (S_2 + S_3) = (S_1 \odot S_2) + (S_1 \odot S_3)$
where $S_1 = \left[P_{S_1}(x_{ij})e^{\mathrm{i}\alpha_{S_1}(x_{ij})}, \mathrm{Q}_{S_1}(x_{ij})e^{\mathrm{i}\beta_{S_1}(x_{ij})}, \mathrm{R}_{S_1}(x_{ij})e^{\mathrm{i}\gamma_{S_1}(x_{ij})}\right]_{m\times n}$,

$$S_2 = \left[P_{S_2}(x_{ij})e^{\mathrm{i}\alpha_{S_2}(x_{ij})}, \mathrm{Q}_{S_2}(x_{ij})e^{\mathrm{i}\beta_{S_2}(x_{ij})}, \mathrm{R}_{S_2}(x_{ij})e^{\mathrm{i}\gamma_{S_2}(x_{ij})}\right]_{m\times n} \text{ and}$$

$$S_3 = \left[P_{S_3}(x_{ij})e^{\mathrm{i}\alpha_{S_3}(x_{ij})}, \mathrm{Q}_{S_3}(x_{ij})e^{\mathrm{i}\beta_{S_3}(x_{ij})}, \mathrm{R}_{S_3}(x_{ij})e^{\mathrm{i}\gamma_{S_3}(x_{ij})}\right]_{m\times n}.$$

Next, if $S_1 \le S_2 (or) S_3$ i.e. $\Gamma_{S_1}(x_{ij}) \le \Gamma_{S_2}(x_{ij})\ or\ \Gamma_{S_3}(x_{ij})$, $\mathrm{I}_{S_1}(x_{ij}) \le \mathrm{I}_{S_2}(x_{ij})\ or\ \mathrm{I}_{S_3}(x_{ij})$, $\Pi_{S_1}(x_{ij}) \ge \Pi_{S_2}(x_{ij})\ or\ \Pi_{S_3}(x_{ij})$ then in both cases

$\inf\{S_1, \sup\{S_2, S_3\}\} = S_1$ and $sup\ \{\inf\{S_1, S_2\}, \inf\{S_1, S_3\}\} = S_1$.

In a similar manner,

$S_1 + (S_2 \odot S_3) = (S_1 + S_2) \odot (S_1 + S_3)$

Next, if $S_1 \geq S_2 (or) S_3$ then in both cases

$\sup\{S_1, \inf\{S_2, S_3\}\} = S_1$ and $inf\{\sup\{S_1, S_2\}, \sup\{S_1, S_3\}\} = S_1$.

Hence, all the properties are proved.

Definition 4.2. Multiplication of two complex neutrosophic matrices

Consider two complex neutrosophic matrices given by $S^1_{3\times2}$ and $S^2_{2\times1}$ on the unit circle in complex plane i.e.

$$S^1_{3\times2} = \begin{bmatrix} (a_1e^{i\theta_1}, a_2e^{i\theta_2}, a_3e^{i\theta_3}) (a_4e^{i\theta_4}, a_5e^{i\theta_5}, a_6e^{i\theta_6}) \\ (b_1e^{i\sigma_1}, b_2e^{i\sigma_2}, b_3e^{i\sigma_3}) (b_4e^{i\sigma_4}, b_5e^{i\sigma_5}, b_6e^{i\sigma_6}) \\ (c_1e^{i\rho_1}, c_2e^{i\rho_2}, c_3e^{i\rho_3}) (c_4e^{i\rho_4}, c_5e^{i\rho_5}, c_6e^{i\rho_6}) \end{bmatrix},$$

$$S^2_{2\times1} = \begin{bmatrix} (p_1e^{i\alpha_1}, p_2e^{i\alpha_2}, p_3e^{i\alpha_3}) \\ (q_1e^{i\beta_1}, q_2e^{i\beta_2}, q_3e^{i\beta_3}) \end{bmatrix}$$

Now the product of two matrices is given by

$$S^1_{3\times2}.S^2_{2\times1} = \begin{bmatrix} d_{11} \\ d_{21} \\ d_{31} \end{bmatrix}$$

where,

$$"d_{11} = \big(\sup\{\inf(a_1e^{i\theta_1}, p_1e^{i\alpha_1}), \inf(a_4e^{i\theta_4}, q_1e^{i\beta_1})\}, \sup\{\inf(a_2e^{i\theta_2}, p_2e^{i\alpha_2}), \inf(a_5e^{i\theta_5}, q_2e^{i\beta_2})\}, \inf\{\sup(a_3e^{i\theta_3}, p_3e^{i\alpha_3}), \sup(a_6e^{i\theta_6}, q_3e^{i\beta_3})\}\big)$$

$$d_{21} = \big(\sup\{\inf(b_1e^{i\sigma_1}, p_1e^{i\alpha_1}), \inf(b_4e^{i\sigma_4}, q_1e^{i\beta_1})\}, \sup\{\inf(b_2e^{i\sigma_2}, p_2e^{i\alpha_2}), \inf(b_5e^{i\sigma_5}, q_2e^{i\beta_2})\}, \inf\{\sup(b_3e^{i\sigma_3}, p_3e^{i\alpha_3}), \sup(b_6e^{i\sigma_6}, q_3e^{i\beta_3})\}\big)$$

$$d_{31} = \big(\sup\{\inf(c_1e^{i\rho_1}, p_1e^{i\alpha_1}), \inf(c_4e^{i\rho_4}, q_1e^{i\beta_1})\}, \sup\{\inf(c_2e^{i\rho_2}, p_2e^{i\alpha_2}), \inf(c_5e^{i\rho_5}, q_2e^{i\beta_2})\}, \inf\{\sup(c_3e^{i\rho_3}, p_3e^{i\alpha_3}), \sup(c_6e^{i\rho_6}, q_3e^{i\beta_3})\}\big)."$$

Example 6. Let us consider two matrices given below

$$S^1_{3\times2} = \begin{bmatrix} \left(\frac{3}{5}e^{i0.8}, \frac{2}{5}e^{i\frac{\pi}{4}}, \frac{1}{2}e^{i\frac{3\pi}{4}}\right) & \left(\frac{1}{2}e^{i0.4}, \frac{1}{5}e^{i\frac{3\pi}{4}}, \frac{1}{10}e^{i\frac{5\pi}{4}}\right) \\ \left(\frac{1}{10}e^{i0.7}, \frac{1}{5}e^{i\frac{\pi}{4}}, \frac{9}{10}e^{i\frac{5\pi}{4}}\right) & \left(\frac{7}{10}e^{i0.3}, \frac{1}{10}e^{i\frac{3\pi}{4}}, \frac{1}{2}e^{i\frac{\pi}{4}}\right) \\ \left(\frac{1}{5}e^{i0.7}, \frac{1}{10}e^{i\frac{5\pi}{4}}, \frac{2}{5}e^{i\frac{\pi}{4}}\right) & \left(\frac{7}{10}e^{i0.1}, \frac{9}{10}e^{i\frac{\pi}{4}}, \frac{1}{5}e^{i\frac{3\pi}{4}}\right) \end{bmatrix}, \quad S^2_{2\times1} = \begin{bmatrix} \left(\frac{1}{10}e^{i0.2}, \frac{3}{10}e^{i\frac{3\pi}{4}}, \frac{7}{10}e^{i\frac{\pi}{4}}\right) \\ \left(\frac{1}{5}e^{i0.5}, \frac{1}{2}e^{i\frac{\pi}{4}}, \frac{3}{10}e^{i\frac{\pi}{4}}\right) \end{bmatrix}$$

$$S^1_{3\times2}.S^2_{2\times1} = \begin{bmatrix} \left(\sup\left\{\frac{1}{10}e^{i0.2}, \frac{1}{5}e^{i0.4}\right\}, \sup\left\{\frac{3}{10}e^{i\frac{\pi}{4}}, \frac{1}{5}e^{i\frac{\pi}{4}}\right\}, \inf\left\{\frac{7}{10}e^{i\frac{3\pi}{4}}, \frac{3}{10}e^{i\frac{5\pi}{4}}\right\}\right) \\ \left(\sup\left\{\frac{1}{10}e^{i0.2}, \frac{1}{5}e^{i0.3}\right\}, \sup\left\{\frac{1}{5}e^{i\frac{\pi}{4}}, \frac{1}{10}e^{i\frac{\pi}{4}}\right\}, \inf\left\{\frac{9}{10}e^{i\frac{5\pi}{4}}, \frac{1}{2}e^{i\frac{\pi}{4}}\right\}\right) \\ \left(\sup\left\{\frac{1}{10}e^{i0.2}, \frac{1}{5}e^{i0.1}\right\}, \sup\left\{\frac{1}{10}e^{i\frac{3\pi}{4}}, \frac{1}{2}e^{i\frac{\pi}{4}}\right\}, \inf\left\{\frac{7}{10}e^{i\frac{\pi}{4}}, \frac{3}{10}e^{i\frac{3\pi}{4}}\right\}\right) \end{bmatrix}$$

$$S^1_{3\times2}.S^2_{2\times1}=\begin{bmatrix}\left(\frac{1}{5}e^{\mathrm{i}0.4},\frac{3}{10}e^{\mathrm{i}\frac{\pi}{4}},\frac{3}{10}e^{\mathrm{i}\frac{3\pi}{4}}\right)\\ \left(\frac{1}{5}e^{\mathrm{i}0.3},\frac{1}{5}e^{\mathrm{i}\frac{\pi}{4}},\frac{1}{2}e^{\mathrm{i}\frac{\pi}{4}}\right)\\ \left(\frac{1}{5}e^{\mathrm{i}0.2},\frac{1}{2}e^{\mathrm{i}\frac{3\pi}{4}},\frac{3}{10}e^{\mathrm{i}\frac{\pi}{4}}\right)\end{bmatrix}$$

Definition 4.3. The identity element for addition

Consider two neutrosophic matrices $S_{2\times2}$ and $I_{2\times2}$ respectively, where $I_{2\times2}$ is an identity matrix. Then,

$$S_{2\times2}=\begin{bmatrix}\left(\frac{1}{10}e^{\mathrm{i}0.3},\frac{7}{10}e^{\mathrm{i}\frac{\pi}{4}},\frac{1}{5}e^{\mathrm{i}\frac{5\pi}{4}}\right) & \left(\frac{7}{10}e^{\mathrm{i}0.4},\frac{3}{5}e^{\mathrm{i}\frac{5\pi}{4}},\frac{1}{10}e^{\mathrm{i}\frac{\pi}{4}}\right)\\ \left(\frac{1}{5}e^{\mathrm{i}0.2},\frac{4}{5}e^{\mathrm{i}\frac{5\pi}{4}},\frac{1}{2}e^{\mathrm{i}\frac{3\pi}{4}}\right) & \left(\frac{3}{5}e^{\mathrm{i}0.7},\frac{1}{2}e^{\mathrm{i}\frac{\pi}{4}},\frac{2}{5}e^{\mathrm{i}\frac{3\pi}{4}}\right)\end{bmatrix}$$

$$I_{2\times2}=\begin{bmatrix}(0,0,1e^{\mathrm{i}0}) & (0,0,1e^{\mathrm{i}0})\\ (0,0,1e^{\mathrm{i}0}) & (0,0,1e^{\mathrm{i}0})\end{bmatrix}$$

$$S_{2\times2}+I_{2\times2}=\begin{bmatrix}d_{11} & d_{12}\\ d_{21} & d_{22}\end{bmatrix}=S_{2\times2}$$

where,

$$d_{11}=\left(\sup\left(\frac{1}{10}e^{\mathrm{i}0.3},0\right),\sup\left(\frac{7}{10}e^{\mathrm{i}\frac{\pi}{4}},0\right),\inf\left(\frac{1}{5}e^{\mathrm{i}\frac{5\pi}{4}},1\right)\right)=\left(\frac{1}{10}e^{\mathrm{i}0.3},\frac{7}{10}e^{\mathrm{i}\frac{\pi}{4}},\frac{1}{5}e^{\mathrm{i}\frac{5\pi}{4}}\right)$$

$$d_{12}=\left(\sup\left(\frac{7}{10}e^{\mathrm{i}0.4},0\right),\sup\left(\frac{3}{5}e^{\mathrm{i}\frac{5\pi}{4}},0\right),\inf\left(\frac{1}{10}e^{\mathrm{i}\frac{\pi}{4}},1\right)\right)=\left(\frac{7}{10}e^{\mathrm{i}0.4},\frac{3}{5}e^{\mathrm{i}\frac{5\pi}{4}},\frac{1}{10}e^{\mathrm{i}\frac{\pi}{4}}\right)$$

$$d_{21}=\left(\sup\left(\frac{1}{5}e^{\mathrm{i}0.2},0\right),\sup\left(\frac{4}{5}e^{\mathrm{i}\frac{5\pi}{4}},0\right),\inf\left(\frac{1}{2}e^{\mathrm{i}\frac{3\pi}{4}},1\right)\right)=\left(\frac{1}{5}e^{\mathrm{i}0.2},\frac{4}{5}e^{\mathrm{i}\frac{5\pi}{4}},\frac{1}{2}e^{\mathrm{i}\frac{3\pi}{4}}\right)$$

$$d_{22}=\left(\sup\left(\frac{3}{5}e^{\mathrm{i}0.7},0\right),\sup\left(\frac{1}{2}e^{\mathrm{i}\frac{\pi}{4}},0\right),\inf\left(\frac{2}{5}e^{\mathrm{i}\frac{3\pi}{4}},1\right)\right)=\left(\frac{3}{5}e^{\mathrm{i}0.7},\frac{1}{2}e^{\mathrm{i}\frac{\pi}{4}},\frac{2}{5}e^{\mathrm{i}\frac{3\pi}{4}}\right)$$

5. Norm Convergence for Complex Neutrosophic Matrix

This section includes the norm convergence of the complex neutrosophic matrix, followed by some basic properties, definitions and theorem.

Definition 5.1. [35] "Suppose $F = R$ or C, V in linear space over F. If the real vector function $\|*\|$ on V verify the properties given below:

a) For arbitrary $u \in V$, $\|u\| \geq 0$, and $\|u\| = 0 \Leftrightarrow u = 0$.
b) For arbitrary $a \in F, u \in V$ get $\|au\| = \|a\|\|u\|$,
c) For arbitrary $u, v \in V$, get $\|u+v\| \leq \|u\| + \|v\|$,

Then $\|u\|$ is called the vector norm of X in V."

Definition 5.2. Consider $\|*\|$ is a non-negative real function on $F^{n\times n}$, if

$$\left\| C_1 C_2 R\left(\Gamma_S(x_{ij})\right) \right\| \leq \left\| C_1 R\left(\Gamma_S(x_{ij})\right) \right\| \left\| C_2 R\left(\Gamma_S(x_{ij})\right) \right\|$$

$$\left\| C_1 C_2 \tau\left(\Gamma_S(x_{ij})\right) \right\| \leq \left\| C_1 \tau\left(\Gamma_S(x_{ij})\right) \right\| \left\| C_2 \tau\left(\Gamma_S(x_{ij})\right) \right\|$$

where $R\left(\Gamma_S(x_{ij})\right) \& \tau\left(\Gamma_S(x_{ij})\right)$ is the real and imaginary part of the complex neutrosophic matrix.

Similarly, for $\mathrm{I}_S(x_{ij}) \& \Pi_S(x_{ij})$.Then, this known as $\|*\|$ is $CNFM(n,m)$.

Definition 5.3. "Suppose $(V, \|*\|)$ is a n-dimensional normed linear space, $p_1, p_2, \dots, p_k, \dots$ is a vector sequence of V, δ is a fixed vector V, if

$$\lim_{k\to\infty} \|p_k - \delta\| = 0$$

Then called vector sequence convergence in the norm, $\mathcal{M}$ is a limit of a sequence, note as: $\lim_{k\to\infty} p_k = \delta$ or $p_k \to \delta$

The vector sequence does not converge called divergence."

Definition 5.4. "Suppose $(V, \|*\|)$ is a n-dimensional normed linear space, $p_1, p_2, \dots, p_k, \dots$ where $p_1 = \left(\Gamma_p^1(x_{ij}), \mathrm{I}_p^1(x_{ij}), \Pi_p^1(x_{ij})\right), p_2 = \left(\Gamma_p^2(x_{ij}), \mathrm{I}_p^2(x_{ij}), \Pi_p^2(x_{ij})\right), \dots$ is a complex neurotrophic matrix sequence of V, $p(k) = \left(\Gamma_p^k(x_{ij}), \mathrm{I}_p^k(x_{ij}), \Pi_p^k(x_{ij})\right)$ constitutes a complex neutrosophic function $\delta = \left(\Gamma_\delta(x_{ij}), \mathrm{I}_\delta(x_{ij}), \Pi_\delta(x_{ij})\right)$ is a fixed complex neutrosophic matrix of V, if

$$\lim_{k\to\infty} \left\| pR\left(\Gamma_p^k(x_{ij})\right) - \delta R\left(\Gamma_\delta(x_{ij})\right) \right\| = 0, \lim_{k\to\infty} \left\| p\tau\left(\Gamma_p^k(x_{ij})\right) - \delta\tau\left(\Gamma_\delta(x_{ij})\right) \right\| = 0$$

where $R\left(\Gamma_p^k(x_{ij})\right), R\left(\Gamma_\delta(x_{ij})\right) \& \tau\left(\Gamma_p^k(x_{ij})\right), \tau\left(\Gamma_\delta(x_{ij})\right)$ is the real and imaginary part of the complex neutrosophic matrix and function respectively."

Similarly, for the case of indeterminacy and falsity components of the matrix i.e.

$$\lim_{k\to\infty} \left\| pR\left(\mathrm{I}_p^k(x_{ij})\right) - \delta R\left(\mathrm{I}_\delta(x_{ij})\right) \right\| = 0, \lim_{k\to\infty} \left\| p\tau\left(\mathrm{I}_p^k(x_{ij})\right) - \delta\tau\left(\mathrm{I}_\delta(x_{ij})\right) \right\| = 0$$

$$\lim_{k\to\infty} \left\| pR\left(\Pi_p^k(x_{ij})\right) - \delta R\left(\Pi_\delta(x_{ij})\right) \right\| = 0, \lim_{k\to\infty} \left\| p\tau\left(\Pi_p^k(x_{ij})\right) - \delta\tau\left(\Pi_\delta(x_{ij})\right) \right\| = 0$$

Then, it is known as complex neutrosophic matrix sequence, $p_1, p_2, \dots, p_k, \dots$ converges in the norm, $\delta = \left(\Gamma_\delta(x_{ij}), \mathrm{I}_\delta(x_{ij}), \Pi_\delta(x_{ij})\right)$ is the limit of the sequence.

5.1. The Convergence of Power of Complex Neutrosophic Matrix

Definition 5.1.1. Consider $\mathcal{M}\left(\Gamma_{\mathcal{M}}(x_{ij}), \mathrm{I}_{\mathcal{M}}(x_{ij}), \Pi_{\mathcal{M}}(x_{ij})\right) \in CNFM(n,n)$ power K of $\mathcal{M}$ is defined as $\mathcal{M}^k$, among them $\mathcal{M}^1 = \mathcal{M},\ \mathcal{M}^k = \mathcal{M}^{k-1}\mathcal{M}$.

Theorem. Consider $\mathcal{M}\left(\Gamma_{\mathcal{M}}(x_{ij}), \mathrm{I}_{\mathcal{M}}(x_{ij}), \Pi_{\mathcal{M}}(x_{ij})\right) \in CNFM(n,n)$, there exists a positive integer a and K, such that $\forall k \geq K$ has $\mathcal{M}^{k+a} = \mathcal{M}^k$.

Proof. Suppose $\forall k \geq 1$,

$$\mathcal{M}\left(\Gamma_{\mathcal{M}}(x_{ij}), \mathrm{I}_{\mathcal{M}}(x_{ij}), \Pi_{\mathcal{M}}(x_{ij})\right)$$

$$= \left(R\left(\Gamma_{\mathcal{M}}(x_{ij}), \mathrm{I}_{\mathcal{M}}(x_{ij}), \Pi_{\mathcal{M}}(x_{ij})\right) + \mathrm{i}\left(\tau\left(\Gamma_{\mathcal{M}}(x_{ij}), \mathrm{I}_{\mathcal{M}}(x_{ij}), \Pi_{\mathcal{M}}(x_{ij})\right)\right)\right)_{n\times n}$$

$$\mathcal{M}^k = \left(R\left(\Gamma_{\mathcal{M}}(x_{ij}), \mathrm{I}_{\mathcal{M}}(x_{ij}), \Pi_{\mathcal{M}}(x_{ij})\right) + \mathrm{i}\left(\tau\left(\Gamma_{\mathcal{M}}(x_{ij}), \mathrm{I}_{\mathcal{M}}(x_{ij}), \Pi_{\mathcal{M}}(x_{ij})\right)\right)\right)^k_{n\times n}$$

$$= \left(R^k\left(\Gamma_{\mathcal{M}}(x_{ij}), \mathrm{I}_{\mathcal{M}}(x_{ij}), \Pi_{\mathcal{M}}(x_{ij})\right) + \mathrm{i}\left(\tau^k\left(\Gamma_{\mathcal{M}}(x_{ij}), \mathrm{I}_{\mathcal{M}}(x_{ij}), \Pi_{\mathcal{M}}(x_{ij})\right)\right)\right)_{n\times n}$$

$$R\left(\Gamma_{\mathcal{M}}(x_{ij}), \mathrm{I}_{\mathcal{M}}(x_{ij}), \Pi_{\mathcal{M}}(x_{ij})\right)$$

$$= \bigvee_{1\leq p_1,\dots,p_{k-1}\leq n}\left(R\left(\Gamma_{\mathcal{M}}(x_{ip_1}), \mathrm{I}_{\mathcal{M}}(x_{ip_1}), \Pi_{\mathcal{M}}(x_{ip_1})\right) \wedge \dots \wedge R\left(\Gamma_{\mathcal{M}}(x_{i,p_{k-1}}), \mathrm{I}_{\mathcal{M}}(x_{i,p_{k-1}}), \Pi_{\mathcal{M}}(x_{i,p_{k-1}})\right)\right)$$

$$\tau\left(\Gamma_{\mathcal{M}}(x_{ij}), \mathrm{I}_{\mathcal{M}}(x_{ij}), \Pi_{\mathcal{M}}(x_{ij})\right)$$

$$= \bigwedge_{1\leq p_1,\dots,p_{k-1}\leq n}\left(\tau\left(\Gamma_{\mathcal{M}}(x_{ip_1}), \mathrm{I}_{\mathcal{M}}(x_{ip_1}), \Pi_{\mathcal{M}}(x_{ip_1})\right) \vee \dots \vee \tau\left(\Gamma_{\mathcal{M}}(x_{i,p_{k-1}}), \mathrm{I}_{\mathcal{M}}(x_{i,p_{k-1}}), \Pi_{\mathcal{M}}(x_{i,p_{k-1}})\right)\right).$$

It is known that $\vee$ & $\wedge$ are closed, therefore, the number of the elements of $\{\mathcal{M}^k, k \geq 1\}$ will not be greater than $(n^{4n})^n$.

Then there exists a positive integer a and K, s.t. $\mathcal{M}^{k+a} = \mathcal{M}^k$, thus $k \geq K$ has

$$\mathcal{M}^{k+a} = \mathcal{M}^{(k-k)+k+a} = \mathcal{M}^{k-k}\mathcal{M}^{k+a} = \mathcal{M}^{k-k}\mathcal{M}^k = \mathcal{M}^k.$$

6. Conclusions & Future Work

In the current study, a novel concept of complex neutrosophic matrices is presented and explained with the help of a few algebraic operations and properties, which will be of great help for the researchers to understand the basics of the concept. The outcomes of these operations will lay the foundation of the fundamental rules to solve or design methodology to solve a complex problem. The numerical examples presented in the current manuscript will be of great help to understand the process more clearly. This could be the initial research in the direction of the novel concept. Further, the matrix norm convergence and power convergence of the complex neutrosophic matrix is discussed thoroughly. These results can be applied in further study of the complex neutrosophic theory. In future, the complex neutrosophic matrices concept may be applied to various applications related to pattern recognition, decision-making, medical diagnosis etc.

Declarations & Compliance with ethical standards

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Neutrosophic Fuzzy Boundary Value Problem under Generalized Hukuhara Differentiability

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Abstract: In this article, the main definitions and differentiation concepts of neutrosophic fuzzy environment will be reviewed. This article will introduce an analytical methodology for solving the second-order linear ordinary differential problem with neutrosophic fuzzy boundary values, this analysis will be under generalized Hukuhara differentiability to show the analytical solutions from a different point of view for the uncertain system, some of these solutions may be decreasing in uncertainty or maybe reflecting the behavior of some real-world systems better. Some applications and numeral examples will be solved to show the behavior of the solution.

Keywords: Fuzzy; neutrosophic fuzzy number; Hukuhara differentiability; neutrosophic fuzzy differential equation.

1. Introduction

The fuzzy differential equation as a topic has been developed in last years so rapidly and it can be used as a suitable way to model dynamic systems under possible uncertainty. The concept of fuzzy and fuzzy derivative was initiated by Zadeh and Chang [1] then it was followed up by a wide group of researchers to develop many different methods as Dubois, Prade, Puri, Goetschel, and Voxman [2]. After Kaleva [3] formulated the first concept of differential equations in a fuzzy environment, Hukuhara changed the concept of difference and differentiability, and to overcome the difficulty of no solution for BVPs, the generalized Hukuhara differentiability was developed by Stefanini and Bede [4:6].

As time goes, the definition of the possible uncertainty was developed and generalized, which was firstly introduced by Atanassov [7,8] and it is called intuitionistic fuzzy which generalized the definition of the fuzzy environment from depending on the definition of the membership only into definition depends on membership degree and non-membership degree which summed to less than one to highlight the non-belongingness and add too many questions on the degree of hesitation. Wide applications are solved by Mondal and Roy [9,10].

It is fact that the world is always searching for a more generalized definition for the fuzzy environment to model more reliable systems. So it was important to find a more generalized fuzzy set and functions that depend not only on the membership and non-membership but also describe the degree of hesitation, and it was firstly initiated by Smarandache [11:15] who calls it a neutrosophic fuzzy set which is considered as a generalization for fuzzy set and intuitionistic fuzzy set because it adds the concept of indeterminacy. As an extension of the neutrosophic Logic, A. A. Salama introduced the Neutrosophic Crisp Sets Theory as a generalization of crisp sets theory and developed, inserted, and formulated new concepts in the fields of mathematics, statistics, computer science, and information systems through neutrosophic [16:21].

In this paper, the second-order homogenous ordinary differential equation via generalized neutrosophic fuzzy numbers as boundaries will be solved under strongly generalized Hukuhara differentiability. And the solution will be applied to numeral problems.

2. Definitions and theories

We begin this section by defining the notation and theories we will use in this paper.

2.1. Definition [11] neutrosophic memberships

Let X be universe set and a neutrosophic set A on X is defined as $A = \{(T_A(x), I_A(x), F_A(x)): x\epsilon X\}$ represents the degree of membership $T_A(x)$, the degree of indeterministic $I_A(x)$ and the degree of non-membership $F_A(x)$. Such that $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

2.2. Definition [11](α, β, γ)-cuts

The (α, β, γ)-cuts are fixed values on set A where $A_{\alpha,\beta,\gamma} = \{(T_A(x), I_A(x), F_A(x)): x\epsilon X, T_A(x) \geq \alpha, I_A(x) \leq \beta, F_A(x) \leq \gamma\}$ which define each of $T_A(x), I_A(x), F_A(x)$ in terms of lower and upper functions of (α, β, γ)-cuts.

2.3. Definition [11,12] neutrosophic number

A neutrosophic set A defined on a universal set of real numbers R is said to be a neutrosophic number

i) A is normal if $x_a \epsilon R\ T_A(x_a) = 1, I_A(x_a) = F_A(x_a) = 0$.

ii) A is a convex set on truth function $T_A(x)$where $T_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min(T_A(x_1), T_A(x_2))$

iii) A is a concave set on indeterministic and falsity functions $I_A(x), F_A(x)$where $I_A(\lambda x_1 + (1 - \lambda)x_2) \geq \max(I_A(x_1), I_A(x_2))$

$$F_A(\lambda x_1 + (1 - \lambda)x_2) \geq \max(F_A(x_1), F_A(x_2))$$

2.4. Definition Triangular neutrosophic number

Let A be a Generalized triangle neutrosophic number $A_{GTN} = (a, b, c: \omega, \eta, \xi)$

$$T_A(x) = \begin{cases} \frac{x-a}{b-a}\omega & a \leq x < b \\ \omega & x = b \\ \frac{c-x}{c-b}\omega & b < x \leq c \end{cases} \text{ and zero otherwise}$$

$$I_A(x) = \begin{cases} \frac{b-x}{b-a}\eta & a \leq x < b \\ \eta & x = b \\ \frac{b-x}{b-c}\eta & b < x \leq c \end{cases} \text{ and 1 otherwise}$$

$$F_A(x) = \begin{cases} \frac{b-x}{b-a}\xi & a \leq x < b \\ \xi & x = b \\ \frac{b-x}{b-c}\xi & b < x \leq c \end{cases} \text{ and 1 otherwise}$$

And we can represent (α, β, γ)-cuts on the generalized triangle neutrosophic number as

$$A_{\alpha,\beta,\gamma} = \left[\underline{A(\alpha)}, \overline{A(\alpha)}\right], \left[\underline{A(\beta)}, \overline{A(\beta)}\right], \left[\underline{A(\gamma)}, \overline{A(\gamma)}\right]$$

$$\left[\underline{A(\alpha)}, \overline{A(\alpha)}\right] = \left[\left(a + \frac{\alpha}{\omega}(b-a)\right), \left(c - \frac{\alpha}{\omega}(c-b)\right)\right]$$

$$\left[\underline{A(\beta)}, \overline{A(\beta)}\right] = \left[\left(a + \frac{1-\beta}{1-\eta}(b-a)\right), \left(c - \frac{1-\beta}{1-\eta}(c-b)\right)\right]$$

$$\left[\underline{A(\gamma)}, \overline{A(\gamma)}\right] = \left[\left(a + \frac{1-\gamma}{1-\xi}(b-a)\right), \left(c - \frac{1-\gamma}{1-\xi}(c-b)\right)\right]$$

2.5. Definition Trapezoidal neutrosophic number

Let A be a generalized Trapezoidal neutrosophic number $A_{GTRN} = (a, b, c, d\colon \omega, \eta, \xi)$

$$T_A(x) = \begin{cases} \frac{x-a}{b-a}\omega & a \le x \le b \\ \omega & b \le x \le c \\ \frac{d-x}{d-c}\omega & c \le x \le d \end{cases} \text{ and zero otherwise}$$

$$I_A(x) = \begin{cases} \frac{b-x}{b-a}\eta & a \le x \le b \\ \eta & b \le x \le c \\ \frac{c-x}{c-d}\eta & c \le x \le d \end{cases} \text{ and 1 otherwise}$$

$$F_A(x) = \begin{cases} \frac{b-x}{b-a}\xi & a \le x \le b \\ \xi & b \le x \le c \\ \frac{c-x}{c-d}\xi & c \le x \le d \end{cases} \text{ and 1 otherwise}$$

And we can represent (α, β, γ)-cuts on the generalized trapezoidal neutrosophic number as

$$A_{\alpha,\beta,\gamma} = \left[\underline{A(\alpha)}, \overline{A(\alpha)}\right], \left[\underline{A(\beta)}, \overline{A(\beta)}\right], \left[\underline{A(\gamma)}, \overline{A(\gamma)}\right]$$

$$\left[\underline{A(\alpha)}, \overline{A(\alpha)}\right] = \left[\left(a + \frac{\alpha}{\omega}(b-a)\right), \left(d - \frac{\alpha}{\omega}(d-c)\right)\right]$$

$$\left[\underline{A(\beta)}, \overline{A(\beta)}\right] = \left[\left(a + \frac{1-\beta}{1-\eta}(b-a)\right), \left(d - \frac{1-\beta}{1-\eta}(d-c)\right)\right]$$

$$\left[\underline{A(\gamma)}, \overline{A(\gamma)}\right] = \left[\left(a + \frac{1-\gamma}{1-\xi}(b-a)\right), \left(d - \frac{1-\gamma}{1-\xi}(d-c)\right)\right]$$

2.6. Definition [4,6]Let $F\colon (a, b) \to \mathcal{F}(R)$, if the next limits

$$\lim_{h \to 0+} \frac{F(x_0 + h) \ominus_H F(x_0)}{h}, \qquad \lim_{h \to 0+} \frac{F(x_0) \ominus_H F(x_0 - h)}{h}$$

exist and equal some elements $F'_H(x_0) \in \mathcal{F}(R)$, then F is Hukuhara differentiable at x_0, and $F'_H(x_0)$ is its derivative at x_0.

Theorem 1 [7,8] Let $F\colon (a, b) \to \mathcal{F}(R)$ be a generalized Hukuhara differentiable Function if and only if (a) or (b) are satisfied

(a) $\underline{f'}_\alpha(x)$ is increasing and $\overline{f'}_\alpha(x)$ is decreasing

(b) $\underline{f'}_\alpha(x)$ is decreasing and $\overline{f'}_\alpha(x)$ is increasing

Then,

$$\left[F'_{gH}(x)\right]_{\boldsymbol{\alpha}} = \left[min\left(\underline{f'}_\alpha(x), \overline{f'}_\alpha(x)\right), max\left(\underline{f'}_\alpha(x), \overline{f'}_\alpha(x)\right)\right]$$

This concept is so near to the generalized differentiability, but may be it focuses on the cases of the function and both of these differentiability change the concept of derivatives, so let us show these changes in the next definition of generalized Hukuhara derivatives.

2.7. Definition [8,9]The generalized Hukuhara first derivative of a fuzzy parametric function is defined as; $f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)\ominus_{gh} f(x_0)}{h}$, From the definition, we have two classes:

(i)-differentiable at x_0 : $[f'(x_0)]_\alpha = [\underline{f'}_\alpha(x_0), \overline{f'}_\alpha(x_0)]$

(ii)-differentiable at x_0 : $[f'(x_0)]_\alpha = [\overline{f'}_\alpha(x_0), \underline{f'}_\alpha(x_0)]$

2.8. Definition [8,9]The generalized Hukuhara second derivative of the fuzzy function is defined as

$$f''(x_0) = \lim_{h\to 0} \frac{f'(x_0+h) \ominus_{gh} f'(x_0)}{h}$$

According to the last definitions, we have the following classes:

$$f'(x_0) \text{ is } (i)\text{-differentiable if: } f''(x_0) = \begin{Bmatrix} \left[\underline{f''}_\alpha(x_0), \overline{f''}_\alpha(x_0)\right] \ if\ f\ is\ (i) - differentiable \\ class(1,1) \\ \left[\overline{f''}_\alpha(x_0), \underline{f''}_\alpha(x_0)\right] \ if\ f\ is\ (ii) - differentiable \\ class(2,2) \end{Bmatrix}$$

$$f'(x_0) \text{ is } (ii)\text{-differentiable if: } f''(x_0) = \begin{Bmatrix} \left[\overline{f''}_\alpha(x_0), \underline{f''}_\alpha(x_0)\right] \ if\ f\ is\ (i) - differentiable \\ class(1,2) \\ \left[\underline{f''}_\alpha(x_0), \overline{f''}_\alpha(x_0)\right] \ if\ f\ is\ (ii) - differentiable \\ class(2,1) \end{Bmatrix}$$

2.9. Definition The solution$\tilde{y}(t, \alpha, \beta, \gamma)$ of the neutrosophic fuzzy differential equation is strong only if, $\frac{\partial \underline{y}}{\partial \alpha} > 0, \frac{d\bar{y}}{d\alpha} < 0$ but $\frac{\partial \underline{y}}{\partial \beta} < 0, \frac{d\bar{y}}{d\beta} > 0$ and $\frac{\partial \underline{y}}{\partial \gamma} < 0, \frac{d\bar{y}}{d\gamma} > 0$.

3 . Methodology

As it is defined before both Hukuhara theory and neutrosophic fuzzy definition, so we can deal with differential problems by applying the generalized Hukuhara differentiability on Neutrosophic Fuzzy differential equations

3.1 Neutrosophic second order differential equation under generalized Hukuhara:

Choosing second order differential equation in linear homogenous form with full terms of differentiability in different cases of coefficients signs.

$$\tilde{y}''(t) = \pm p\tilde{y}'(t) \pm q\tilde{y}(t)$$

Neutrosophic boundary conditions

$$\tilde{y}(t_0) = \tilde{a} \quad \tilde{y}(T) = \tilde{b}$$

Using $(\alpha, \beta, \gamma) - cut$ to apply generalized Hukuhara differentiability classes. Where $\tilde{a}\ and\ \tilde{b}$are two generalized trapezoidal neutrosophic numbers.

$$\tilde{a} = (a_1, a_2, a_3, a_4; \omega, \eta, \xi)$$
$$\tilde{b} = (b_1, b_2, b_3, b_4; \omega, \eta, \xi)$$

3.1.1. Case (1): Positive sign for p and q (+ +)

The analysis and solution will be introduced in case of positive signs of p, q.
According to generalized Hukuhara, the problem will be studied in 4 classes

Class (1,1)

$\bar{y}''(t,\alpha,\beta,\gamma) = p.\bar{y}'(t,\alpha,\beta,\gamma) + q.\bar{y}(t,\alpha,\beta,\gamma),$

$\underline{y}''(t,\alpha,\beta,\gamma) = p.\underline{y}'(t,\alpha,\beta,\gamma) + q.\underline{y}(t,\alpha,\beta,\gamma),$

Class (1,2)

$\underline{y}''(t,\alpha,\beta,\gamma) = p.\bar{y}'(t,\alpha,\beta,\gamma) + q.\bar{y}(t,\alpha,\beta,\gamma),$

$\overline{y''}(t,\alpha,\beta,\gamma) = p.\underline{y}'(t,\alpha,\beta,\gamma) + q.\underline{y}(t,\alpha,\beta,\gamma),$

Class (2,1)

$\bar{y}''(t,\alpha,\beta,\gamma) = p.\bar{y}'(t,\alpha,\beta,\gamma) + q.\underline{y}(t,\alpha,\beta,\gamma),$

$\underline{y}''(t,\alpha,\beta,\gamma) = p.\underline{y}'(t,\alpha,\beta,\gamma) + q.\bar{y}(t,\alpha,\beta,\gamma),$

Class (2, 2)

$\bar{y}''(t,\alpha,\beta,\gamma) = p.\underline{y}'(t,\alpha,\beta,\gamma) + q.\bar{y}(t,\alpha,\beta,\gamma)$

$\underline{y}''(t,\alpha,\beta,\gamma) = p.\bar{y}'(t,\alpha,\beta,\gamma) + q.\underline{y}(t,\alpha,\beta,\gamma)$

to find the solution of each class as a function in the general values of coefficients p and q, the methodology of getting the analytical solution of this system will be described in class(1,1) in detail as an example.

Class (1,1)

$$\bar{y}''(t,\boldsymbol{\alpha},\beta,\gamma) = p.\bar{y}'(t,\boldsymbol{\alpha},\beta,\gamma) + q.\bar{y}(t,\boldsymbol{\alpha},\beta,\gamma), \quad \bar{y}(t_0,\boldsymbol{\alpha},\beta,\gamma) = \bar{a}, \bar{y}(T,\boldsymbol{\alpha},\beta,\gamma) = \bar{b}$$

$$\underline{y}''(t,\boldsymbol{\alpha},\beta,\gamma) = p.\underline{y}'(t,\boldsymbol{\alpha},\beta,\gamma) + q.\underline{y}(t,\boldsymbol{\alpha},\beta,\gamma), \quad \underline{y}(t_0,\boldsymbol{\alpha},\beta,\gamma) = \underline{a}, \underline{y}(T,\boldsymbol{\alpha},\beta,\gamma) = \underline{b}$$

Let

$$\underline{y} = x, \quad \bar{y} = z$$

$$z'' = pz' + qz \qquad z' = v$$

$$x'' = px' + qx \qquad x' = u$$

$$\begin{bmatrix} x \\ u \\ z \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ q & p & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & q & p \end{bmatrix} \begin{bmatrix} x \\ u \\ z \\ v \end{bmatrix}$$

From this system, the eign values of the matrix will appear in the solution, so these definitions will be used

$d_1 = p + \sqrt{p^2 + 4q} \qquad d_2 = p + \sqrt{p^2 - 4q} \quad d_3 = p - \sqrt{p^2 + 4q} \qquad d_4 = p - \sqrt{p^2 - 4q}$

General solution:

$$x = \underline{y}(t,\boldsymbol{\alpha},\beta,\gamma) = c_2 e^{\frac{(d_1)t}{2}} + c_4 e^{\frac{(d_3)t}{2}}$$

$$z = \bar{y}(t,\boldsymbol{\alpha},\beta,\gamma) = c_1 e^{\frac{(d_1)t}{2}} + c_3 e^{\frac{(d_3)t}{2}}$$

By using the boundary condition to get $c_1, c_2, c_3 and\ c_4$:

At $t = t_0$

$$\underline{y}(t_0,\boldsymbol{\alpha},\beta,\gamma) = \underline{a} = c_2 e^{\frac{(d_1)t_0}{2}} + c_4 e^{\frac{(d_3)t_0}{2}}$$

$$\bar{y}(t_0,\boldsymbol{\alpha},\beta,\gamma) = \bar{a} = c_1 e^{\frac{(d_1)t_0}{2}} + c_3 e^{\frac{(d_3)t_0}{2}}$$

At $t = T$

$$\underline{y}(T,\boldsymbol{\alpha},\beta,\gamma) = \underline{b} = c_2 e^{\frac{(d_1)T}{2}} + c_4 e^{\frac{(d_3)T}{2}}$$

$$\bar{y}(T,\boldsymbol{\alpha},\beta,\gamma) = \bar{b} = c_1 e^{\frac{(d_1)T}{2}} + c_3 e^{\frac{(d_3)T}{2}}$$

$$\begin{bmatrix} \underline{a} \\ \overline{a} \\ \underline{b} \\ \overline{b} \end{bmatrix} = \begin{bmatrix} 0 & e^{\frac{(d_1)t_0}{2}} & 0 & e^{\frac{(d_3)t_0}{2}} \\ e^{\frac{(d_1)t_0}{2}} & 0 & e^{\frac{(d_3)t_0}{2}} & 0 \\ 0 & e^{\frac{(d_1)T}{2}} & 0 & e^{\frac{(d_3)T}{2}} \\ e^{\frac{(d_1)T}{2}} & 0 & e^{\frac{(d_3)T}{2}} & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

The solution of C's:

$$c_1 = \frac{-(\overline{a}e^{\frac{(d_3)T}{2}} - \overline{b}e^{\frac{(d_3)t_0}{2}})}{(e^{\frac{(d_1)T}{2}}e^{\frac{(d_3)t_0}{2}} - e^{\frac{(d_3)T}{2}}e^{\frac{(d_1)t_0}{2}})}, \quad c_2 = \frac{-(\underline{a}e^{\frac{(d_3)T}{2}} - \underline{b}e^{\frac{(d_3)t_0}{2}})}{(e^{\frac{(d_1)T}{2}}e^{\frac{(d_3)t_0}{2}} - e^{\frac{(d_3)T}{2}}e^{\frac{(d_1)t_0}{2}})},$$

$$c_3 = \frac{(\overline{a}e^{\frac{(d_1)T}{2}} - \overline{b}e^{\frac{(d_1)t_0}{2}})}{(e^{\frac{(d_1)T}{2}}e^{\frac{(d_3)t_0}{2}} - e^{\frac{(d_3)T}{2}}e^{\frac{(d_1)t_0}{2}})}, \quad c_4 = \frac{(\underline{a}e^{\frac{(d_1)T}{2}} - \underline{b}e^{\frac{(d_1)t_0}{2}})}{(e^{\frac{(d_1)T}{2}}e^{\frac{(d_3)t_0}{2}} - e^{\frac{(d_3)T}{2}}e^{\frac{(d_1)t_0}{2}})}$$

By using the value of $c, \underline{a}, \overline{a}, \underline{b}$ $and \overline{b}$ then get $\underline{y}(t, \boldsymbol{\alpha}, \beta, \gamma)$ and $\overline{y}(t, \boldsymbol{\alpha}, \beta, \gamma)$

Then, the other classes can be obtained by the same method.

3.1.2. Case (2): Positive sign for (+p) but the negative sign for (-q)

Also, any case of negative sign of the coefficient is needed to be shown to realize the effect of the negative sign on the generalized Hukuhara differentiability definition for the system.

$$\boldsymbol{\widetilde{y''}(t) = p\widetilde{y}'(t) - q\widetilde{y}(t)}$$

According to the Hukuhara definition, it is known that the negative sign turns the lower term into the upper term and vice versa.

Class (1, 1)

$$\overline{y}''(t, \boldsymbol{\alpha}, \beta, \gamma) = p.\overline{y}'(t, \boldsymbol{\alpha}, \beta, \gamma) - q.\underline{y}(t, \boldsymbol{\alpha}, \beta, \gamma)$$
$$\underline{y''}(t, \boldsymbol{\alpha}, \beta, \gamma) = p.\underline{y}'(t, \boldsymbol{\alpha}, \beta, \gamma) - q.\overline{y}(t, \boldsymbol{\alpha}, \beta, \gamma)$$

Class (1, 2)

$$\overline{y}''(t, \boldsymbol{\alpha}, \beta, \gamma) = p.\underline{y}'(t, \boldsymbol{\alpha}, \beta, \gamma) - q.\overline{y}(t, \boldsymbol{\alpha}, \beta, \gamma)$$
$$\underline{y''}(t, \boldsymbol{\alpha}, \beta, \gamma) = p.\overline{y}'(t, \boldsymbol{\alpha}, \beta, \gamma) - q.\underline{y}(t, \boldsymbol{\alpha}, \beta, \gamma)$$

Class (2, 1)

$$\overline{y}''(t, \boldsymbol{\alpha}, \beta, \gamma) = p.\overline{y}'(t, \boldsymbol{\alpha}, \beta, \gamma) - q.\overline{y}(t, \boldsymbol{\alpha}, \beta, \gamma)$$
$$\boldsymbol{\underline{y''}(t, \alpha, \beta, \gamma) = p.\underline{y}'(t, \alpha, \beta, \gamma) - q.\underline{y}(t, \alpha, \beta, \gamma)}$$

Class (2, 2)

$$\underline{y''}(t, \boldsymbol{\alpha}, \beta, \gamma) = p.\overline{y}'(t, \boldsymbol{\alpha}, \beta, \gamma) - q.\overline{y}(t, \boldsymbol{\alpha}, \beta, \gamma)$$
$$\overline{y''}(t, \boldsymbol{\alpha}, \beta, \gamma) = p.\underline{y}'(t, \boldsymbol{\alpha}, \beta, \gamma) - q.\underline{y}(t, \boldsymbol{\alpha}, \beta, \gamma)$$

Also to find the analytical solution of each class, the same methodology will be used and it will be described in solving first class as an example

Class (1, 1)

$$\overline{y}''(t, \boldsymbol{\alpha}, \beta, \gamma) = p.\overline{y}'(t, \boldsymbol{\alpha}, \beta, \gamma) - q.\underline{y}(t, \boldsymbol{\alpha}, \beta, \gamma)$$

$$\underline{y''}(t, \boldsymbol{\alpha}, \beta, \gamma) = p.\underline{y}'(t, \boldsymbol{\alpha}, \beta, \gamma) - q.\overline{y}(t, \boldsymbol{\alpha}, \beta, \gamma)$$

Let $\underline{y} = x, \ \overline{y} = z$

$$x'' = px' - qz$$

$$x' = u$$

$$z'' = pz' - qx$$

$$z' = v$$

$$\begin{bmatrix} x \\ u \\ z \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & p & -q & 0 \\ 0 & 0 & 0 & 1 \\ -q & 0 & 0 & p \end{bmatrix} \begin{bmatrix} x \\ u \\ z \\ v \end{bmatrix}$$

The solution

$$\boldsymbol{x = \underline{y}(t,\alpha) = c_1 e^{\frac{t}{2}d_2} + c_2 e^{\frac{t}{2}d_1} + c_3 e^{\frac{t}{2}d_4} + c_4 e^{\frac{t}{2}d_3}}$$

$$\boldsymbol{z = \overline{y}(t,\alpha) = c_1 e^{\frac{t}{2}d_2} - c_2 e^{\frac{t}{2}d_1} + c_3 e^{\frac{t}{2}d_4} - c_4 e^{\frac{t}{2}d_3}}$$

After studying all classes of all cases of different signs of coefficients, a table of collected general solutions of all cases will be introduced to help in solving applications by substituting

Table 1. General solutions of N.F Differential equations

	Class(1,1)	Class(1,2)	Class(2,1)	Class(2,2)
+p, +q	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ q & p & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & q & p \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & q & p \\ 0 & 0 & 0 & 1 \\ q & p & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & p & q & 0 \\ 0 & 0 & 0 & 1 \\ q & 0 & 0 & p \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ q & 0 & 0 & p \\ 0 & 0 & 0 & 1 \\ 0 & p & q & 0 \end{bmatrix}$
G. S	$\underline{y}(t,\alpha,\beta,\gamma)$ $= c_2 e^{\frac{(d_1)t}{2}} + c_4 e^{\frac{(d_3)t}{2}}$ $\overline{y}(t,\alpha,\beta,\gamma)$ $= c_1 e^{\frac{(d_1)t}{2}} + c_3 e^{\frac{(d_3)t}{2}}$	$\underline{y}(t,\alpha,\beta,\gamma)$ $= c_1 e^{\frac{-t}{2}d_2} + c_2 e^{\frac{-t}{2}d_4}$ $+ c_3 e^{\frac{t}{2}d_3} + c_4 e^{\frac{t}{2}d_1}$ $\overline{y}(t,\alpha,\beta,\gamma)$ $= c_1 e^{\frac{-t}{2}d_2}\left(1 - d_4\left({}^{p}/_{q}\right)\right)$ $+ c_2 e^{\frac{-t}{2}d_4}\left(1 - d_2\left({}^{p}/_{q}\right)\right)$ $+ c_3 e^{\frac{t}{2}d_3} + c_4 e^{\frac{t}{2}d_1}$	$\underline{y}(t,\alpha,\beta,\gamma)$ $= c_1 e^{\frac{t}{2}d_2} + c_2 e^{\frac{t}{2}d_1} + c_3 e^{\frac{t}{2}d_4}$ $+ c_4 e^{\frac{t}{2}d_3}$ $\overline{y}(t,\alpha,\beta,\gamma)$ $= -c_1 e^{\frac{t}{2}d_2} + c_2 e^{\frac{t}{2}d_1}$ $- c_3 e^{\frac{t}{2}d_4} + c_4 e^{\frac{t}{2}d_3}$	$\underline{y}(t,\alpha)$ $= c_1 e^{\frac{-t}{2}d_3} + c_2 e^{\frac{t}{2}d_3}$ $+ c_3 e^{\frac{-t}{2}d_1} + c_4 e^{\frac{t}{2}d_1}$ $\overline{y}(t,\alpha,\beta,\gamma)$ $= -c_1 e^{\frac{-t}{2}d_3} + c_2 e^{\frac{t}{2}d_3}$ $- c_3 e^{\frac{-t}{2}d_1} + c_4 e^{\frac{t}{2}d_1}$
-p -q	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -q & -p \\ 0 & 0 & 0 & 1 \\ -q & -p & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -q & -p & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -q & -p \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -q & 0 & 0 & -p \\ 0 & 0 & 0 & 1 \\ 0 & -p & -q & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -p & -q & 0 \\ 0 & 0 & 0 & 1 \\ -q & 0 & 0 & -p \end{bmatrix}$
G S	$\underline{y}(t,\alpha,\beta,\gamma)$ $= c_1 e^{\frac{-t}{2}d_2} + c_3 e^{\frac{-t}{2}d_4}$ $+ c_4 e^{\frac{t}{2}d_3} + c_2 e^{\frac{t}{2}d_1}$ $\overline{y}(t,\alpha,\beta,\gamma) =$ $-c_1 e^{\frac{-t}{2}d_2}\left(1 - d_4\left({}^{p}/_{q}\right)\right) -$ $c_3 e^{\frac{-t}{2}d_4}\left(1 - d_2\left({}^{p}/_{q}\right)\right)$ $- c_4 e^{\frac{t}{2}d_3} - c_2 e^{\frac{t}{2}d_1}$	$\underline{y}(t,\alpha,\beta,\gamma)$ $= c_2 e^{\frac{(-d_4)t}{2}} + c_4 e^{\frac{(-d_2)t}{2}}$ $\overline{y}(t,\alpha,\beta,\gamma)$ $= c_1 e^{\frac{(-d_4)t}{2}} + c_3 e^{\frac{(-d_2)t}{2}}$	$\underline{y}(t,\alpha,\beta,\gamma)$ $= c_1 e^{\frac{t}{2}d_2} + c_2 e^{\frac{-t}{2}d_4}$ $+ c_3 e^{\frac{-t}{2}d_2} + c_4 e^{\frac{t}{2}d_4}$ $\overline{y}(t,\alpha,\beta,\gamma)$ $= -c_1 e^{\frac{t}{2}d_2} + c_2 e^{\frac{-t}{2}d_4}$ $+ c_3 e^{\frac{-t}{2}d_2} - c_4 e^{\frac{t}{2}d_4}$	$\underline{y}(t,\alpha,\beta,\gamma)$ $= c_1 e^{\frac{-t}{2}d_2} + c_2 e^{\frac{-t}{2}d_1}$ $+ c_3 e^{\frac{-t}{2}d_4} + c_4 e^{\frac{-t}{2}d_3}$ $\overline{y}(t,\alpha,\beta,\gamma)$ $= c_1 e^{\frac{-t}{2}d_2} - c_2 e^{\frac{-t}{2}d_1}$ $+ c_3 e^{\frac{-t}{2}d_4} - c_4 e^{\frac{-t}{2}d_3}$
+p -q	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & p & -q & 0 \\ 0 & 0 & 0 & 1 \\ -q & 0 & 0 & p \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -q & 0 & 0 & p \\ 0 & 0 & 0 & 1 \\ 0 & p & -q & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -q & p & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -q & p \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -q & p \\ 0 & 0 & 0 & 1 \\ -q & p & 0 & 0 \end{bmatrix}$
G S	$\underline{y}(t,\alpha,\beta,\gamma)$ $= c_1 e^{\frac{t}{2}d_2} + c_2 e^{\frac{t}{2}d_1} + c_3 e^{\frac{t}{2}d_4}$ $+ c_4 e^{\frac{t}{2}d_3}$	$\underline{y}(t,\alpha,\beta,\gamma)$ $= c_1 e^{\frac{-t}{2}d_2} + c_2 e^{\frac{t}{2}d_2}$ $+ c_3 e^{\frac{-t}{2}d_4} + c_4 e^{\frac{t}{2}d_4}$	$\underline{y}(t,\alpha,\beta,\gamma)$ $= c_2 e^{\frac{(d_2)t}{2}} + c_4 e^{\frac{(d_4)t}{2}}$ $\overline{y}(t,\alpha,\beta,\gamma)$ $= c_1 e^{\frac{(d_2)t}{2}} + c_3 e^{\frac{(d_4)t}{2}}$	$\underline{y}(t,\alpha,\beta,\gamma)$ $= c_1 e^{\frac{t}{2}d_2} + c_2 e^{\frac{-t}{2}d_1}$ $+ c_3 e^{\frac{t}{2}d_4} + c_4 e^{\frac{-t}{2}d_3}$

	$\overline{y}(t,\alpha,\beta,\gamma)$ $= c_1 e^{\frac{t}{2}d_2} - c_2 e^{\frac{t}{2}d_1} + c_3 e^{\frac{t}{2}d_4}$ $- c_4 e^{\frac{t}{2}d_3}$	$\overline{y}(t,\alpha,\beta,\gamma)$ $= -c_1 e^{\frac{-t}{2}d_2} + c_2 e^{\frac{t}{2}d_2}$ $- c_3 e^{\frac{-t}{2}d_4} + c_4 e^{\frac{t}{2}d_4}$		$\overline{y}(t,\alpha,\beta,\gamma)$ $= -c_1 e^{\frac{t}{2}d_2}\left(1 - d_4\left(p/q\right)\right)$ $- c_3 e^{\frac{t}{2}d_4}\left(1 - d_2\left(p/q\right)\right)$ $-c_2 e^{\frac{-t}{2}d_1} - c_4 e^{\frac{-t}{2}d_3}$
-p +q	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ q & 0 & 0 & -p \\ 0 & 0 & 0 & 1 \\ 0 & -p & q & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -p & q & 0 \\ 0 & 0 & 0 & 1 \\ q & 0 & 0 & -p \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & q & -p \\ 0 & 0 & 0 & 1 \\ q & -p & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ q & -p & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & q & -p \end{bmatrix}$
G S	$\underline{y}(t,\alpha,\beta,\gamma)$ $= c_1 e^{\frac{-t}{2}d_1} + c_2 e^{\frac{t}{2}d_1}$ $+ c_3 e^{\frac{-t}{2}d_3} + c_4 e^{\frac{t}{2}d_3}$ $\overline{y}(t,\alpha,\beta,\gamma)$ $= c_1 e^{\frac{-t}{2}d_1} - c_2 e^{\frac{t}{2}d_1}$ $+ c_3 e^{\frac{-t}{2}d_3} - c_4 e^{\frac{t}{2}d_3}$	$\underline{y}(t,\alpha,\beta,\gamma)$ $= c_1 e^{\frac{-t}{2}d_4} + c_2 e^{\frac{-t}{2}d_3}$ $+ c_3 e^{\frac{-t}{2}d_2} + c_4 e^{\frac{-t}{2}d_1}$ $\overline{y}(t,\alpha,\beta,\gamma)$ $= -c_1 e^{\frac{-t}{2}d_4} + c_2 e^{\frac{-t}{2}d_3}$ $- c_3 e^{\frac{-t}{2}d_2} + c_4 e^{\frac{-t}{2}d_1}$	$\underline{y}(t,\alpha,\beta,\gamma)$ $= c_1 e^{\frac{t}{2}d_2} + c_2 e^{\frac{t}{2}d_4}$ $+ c_3 e^{\frac{-t}{2}d_3} + c_4 e^{\frac{-t}{2}d_1}$ $\overline{y}(t,\alpha,\beta,\gamma)$ $= c_1 e^{\frac{t}{2}d_2}\left(1 - d_4\left(p/q\right)\right)$ $+ c_2 e^{\frac{t}{2}d_4}\left(1 - d_2\left(p/q\right)\right)$ $+c_3 e^{\frac{-t}{2}d_3} + c_4 e^{\frac{-t}{2}d_1}$	$\underline{y}(t,\alpha,\beta,\gamma)$ $= c_2 e^{\frac{(-d_1)t}{2}} + c_4 e^{\frac{(-d_3)t}{2}}$ $\overline{y}(t,\alpha,\beta,\gamma)$ $= c_1 e^{\frac{(-d_1)t}{2}} + c_3 e^{\frac{(-d_3)t}{2}}$

4. Applications

4.1. Example 1

$$\widetilde{y''}(t) = 5\widetilde{y}'(t) + 4\tilde{y}(t),$$
$$\tilde{y}(t_0 = 0) = \tilde{a} = (0.8,1,1.4; 0.8,0.2,0.3)$$
$$\tilde{y}(T = 1) = \tilde{b} = (2.6,3,3.1, ; 0.8,0.2,0.3)$$

Solution:

- **First step: Analyzing the neutrosophic boundary points**

  According to Definition 4 the generalized NF triangle point, then

$$\left[\underline{a(\alpha)}, \overline{a(\alpha)}\right] = \left[\left(0.8 + \frac{\alpha}{4}\right), \left(1.4 - \frac{\alpha}{2}\right)\right]$$
$$\left[\underline{a(\beta)}, \overline{a(\beta)}\right] = \left[0.8 + \frac{(1-\beta)}{4}, 1.4 - \frac{(1-\beta)}{2}\right]$$
$$\left[\underline{a(\gamma)}, \overline{a(\gamma)}\right] = \left[\left(0.8 + \frac{2(1-\gamma)}{7}\right), \left(1.4 - \frac{4(1-\gamma)}{7}\right)\right]$$
$$\left[\underline{b(\alpha)}, \overline{b(\alpha)}\right] = \left[\left(2.6 + \frac{\alpha}{2}\right), \left(3.1 - \frac{\alpha}{8}\right)\right]$$
$$\left[\underline{b(\beta)}, \overline{b(\beta)}\right] = \left[\left(2.6 + \frac{(1-\beta)}{2}\right), \left(3.1 - \frac{(1-\beta)}{8}\right)\right]$$
$$\left[\underline{b(\gamma)}, \overline{b(\gamma)}\right] = \left[\left(2.6 + \frac{4(1-\gamma)}{7}\right), \left(3.1 - \frac{1-\gamma}{7}\right)\right]$$

- **Second step: Solving the problem according to methodology or use Table 1**

  For positives signs of coefficients and substituting the values of p and q

Class (1, 1):

$$\underline{y}(t,\alpha,\beta,\gamma) = c_2 e^{\frac{(5+\sqrt{41})t}{2}} + c_4 e^{\frac{(5-\sqrt{41})t}{2}}$$

$$\overline{y}(t,\alpha,\beta,\gamma) = c_1 e^{\frac{(5+\sqrt{41})t}{2}} + c_3 e^{\frac{(5-\sqrt{41})t}{2}}$$

To find the values of constants for each $\alpha, \beta, \gamma,$ then use the boundary points

$$\underline{y}(t_0,\alpha,\beta,\gamma) = c_2 e^{\frac{(5+\sqrt{41})t_0}{2}} + c_4 e^{\frac{(5-\sqrt{41})t_0}{2}} = [0.8 + \frac{\alpha}{4}, 0.8 + \frac{(1-\beta)}{4}, 0.8 + \frac{2(1-\gamma)}{7}]$$

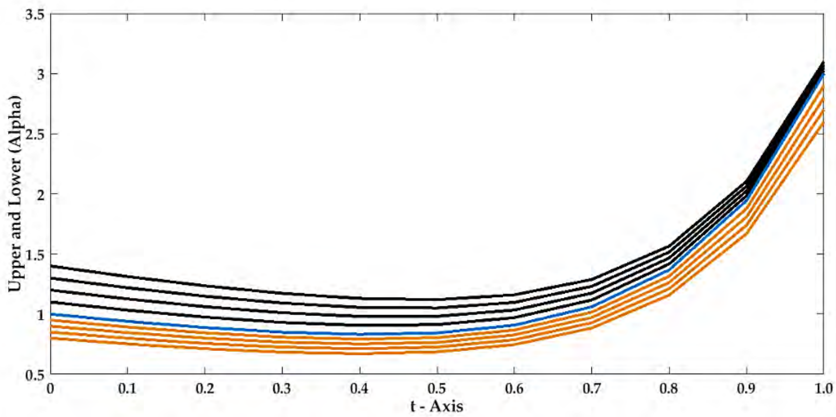
$$\overline{y}(t_0,\alpha,\beta,\gamma) = c_1 e^{\frac{(5+\sqrt{41})t_0}{2}} + c_3 e^{\frac{(5-\sqrt{41})t_0}{2}} = [1.4 - \frac{\alpha}{2}, 1.4 - \frac{(1-\beta)}{2}, 1.4 - \frac{4(1-\gamma)}{7}]$$

$$\underline{y}(T,\alpha,\beta,\gamma) = c_2 e^{\frac{(5+\sqrt{41})T}{2}} + c_4 e^{\frac{(5-\sqrt{41})T}{2}} = [2.6 + \frac{\alpha}{2}, 2.6 + \frac{(1-\beta)}{2}, 2.6 + \frac{4(1-\gamma)}{7}]$$

$$\overline{y}(T,\alpha,\beta,\gamma) = c_1 e^{\frac{(5+\sqrt{41})T}{2}} + c_3 e^{\frac{(5-\sqrt{41})T}{2}} = [3.1 - \frac{\alpha}{8} 3.1 - \frac{(1-\beta)}{8}, 3.1 - \frac{1-\gamma}{7}]$$

Table (2): The Results of Upper and Lower for (α,β,γ) at $t = 0.5$

α	$\underline{y}(0.5,\alpha)$	$\overline{y}(0.5,\alpha)$	β	$\underline{y}(0.5,\beta)$	$\overline{y}(0.5,\beta)$	γ	$\underline{y}(0.5,\gamma)$	$\overline{y}(0.5,\gamma)$
0.0	0.685682	0.713451	0.2	0.843218	0.843218	0.3	0.843218	0.843218
0.2	0.725066	0.745893	0.4	0.803834	0.912266	0.5	0.798207	0.922130
0.4	0.764450	0.778334	0.6	0.764450	0.981315	0.7	0.753197	1.001043
0.6	0.803834	0.810776	0.8	0.725066	1.050363	0.9	0.708187	1.079956
0.8	0.907334	0.907334	1.0	0.685682	1.119412	1.0	0.685682	1.119412



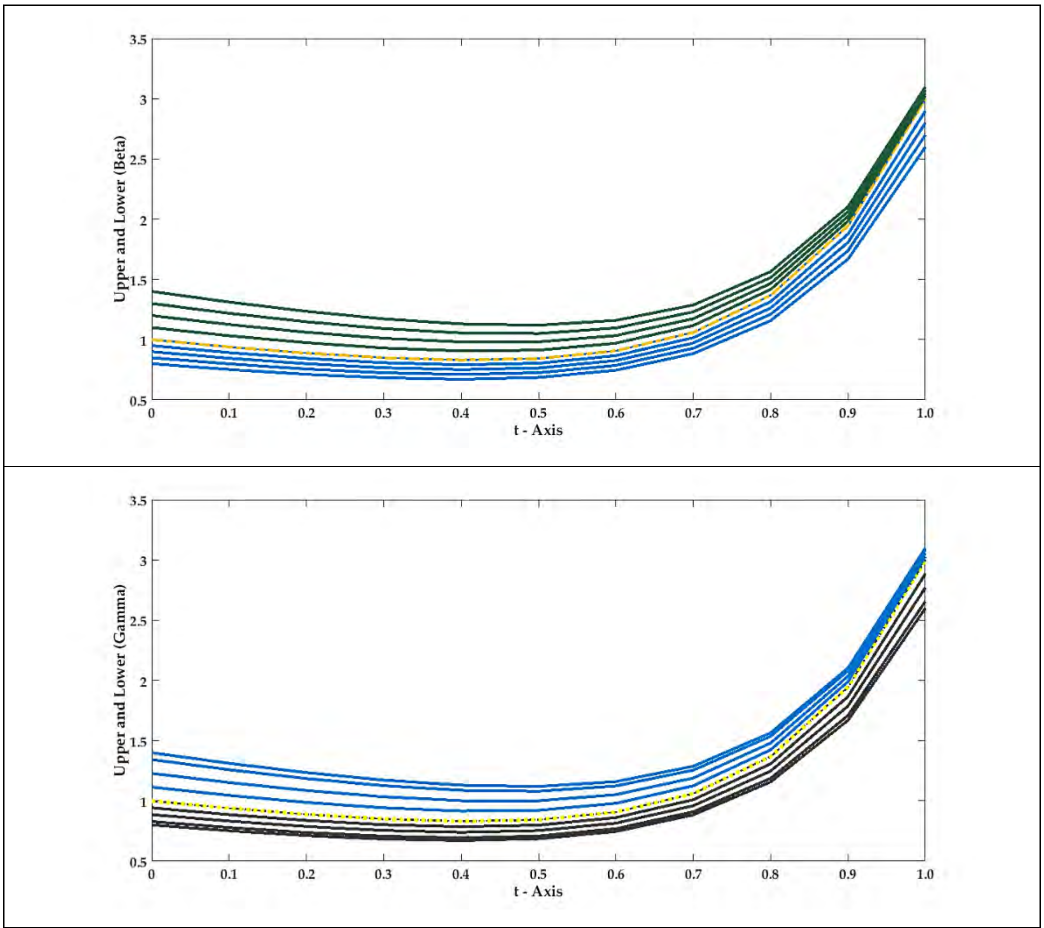


Fig. 1: The Results of Upper and Lower for (α, β, γ) **of Class(1,1).**

Class (1, 2)

$$\underline{y}(\boldsymbol{t}, \boldsymbol{\alpha}, \beta, \gamma) = \boldsymbol{c_1} e^{-4t} + (\boldsymbol{c_2}) e^{-t} + \boldsymbol{c_3} e^{\frac{(5-\sqrt{41})t}{2}} + \boldsymbol{c_4} e^{\frac{(5+\sqrt{41})t}{2}},$$

$$\overline{y}(\boldsymbol{t}, \boldsymbol{\alpha}, \beta, \gamma) = \boldsymbol{c_1} e^{-4t}\left(-\left(3/2\right)\right) + \boldsymbol{c_2} e^{-t}(-9) + \boldsymbol{c_3} e^{\frac{(5-\sqrt{41})t}{2}} + \boldsymbol{c_4} e^{\frac{(5+\sqrt{41})t}{2}}.$$

To find the values of constants for each α, β, γ, then use the boundary points

$$\underline{y}(t_0, \boldsymbol{\alpha}, \beta, \gamma) = \boldsymbol{c_1} e^{-4t_0} + (\boldsymbol{c_2}) e^{-t_0} + \boldsymbol{c_3} e^{\frac{(5-\sqrt{41})t_0}{2}} + \boldsymbol{c_4} e^{\frac{(5+\sqrt{41})t_0}{2}} = 0.8 + \frac{\alpha}{4}, 0.8 + \frac{(1-\beta)}{4}, 0.8 + \frac{2(1-\gamma)}{7}$$

$$\overline{y}(t_0, \boldsymbol{\alpha}, \beta, \gamma) = -\mathbf{1.5}\boldsymbol{c_1} e^{-4t_0} - 9\boldsymbol{c_2} e^{-t_0} + \boldsymbol{c_3} e^{\frac{(5-\sqrt{41})t}{2}} + \boldsymbol{c_4} e^{\frac{(5+\sqrt{41})t}{2}} = 1.4 - \frac{\alpha}{2}, 1.4 - \frac{(1-\beta)}{2}, 1.4 - \frac{4(1-\gamma)}{7}$$

$$\underline{y}(T, \boldsymbol{\alpha}, \beta, \gamma) = \boldsymbol{c_1} e^{-4T} + (\boldsymbol{c_2}) e^{-T} + \boldsymbol{c_3} e^{\frac{(5-\sqrt{41})T}{2}} + \boldsymbol{c_4} e^{\frac{(5+\sqrt{41})T}{2}} = 2.6 + \frac{\alpha}{2}, 2.6 + \frac{(1-\beta)}{2}, 2.6 + \frac{4(1-\gamma)}{7}$$

$$\overline{y}(T, \boldsymbol{\alpha}, \beta, \gamma) = -\mathbf{1.5}\boldsymbol{c_1} e^{-4T} - 9\boldsymbol{c_2} e^{-T} + \boldsymbol{c_3} e^{\frac{(5-\sqrt{41})T}{2}} + \boldsymbol{c_4} e^{\frac{(5+\sqrt{41})T}{2}} = 3.1 - \frac{\alpha}{8}, 3.1 - \frac{(1-\beta)}{8}, 3.1 - \frac{1-\gamma}{7}$$

Table (3): The Results of Upper and Lower for (α,β,γ) at t = 0.5

α	$\underline{y}(0.5,\alpha)$	$\overline{y}(0.5,\alpha)$	β	$\underline{y}(0.5,\beta)$	$\overline{y}(0.5,\beta)$	γ	$\underline{y}(0.5,\gamma)$	$\overline{y}(0.5,\gamma)$
0.0	0.525047	1.265411	0.2	0.843218	0.843218	0.3	0.843218	0.843218
0.2	0.604589	1.159863	0.4	0.763675	0.948766	0.5	0.752312	0.963844
0.4	0.684132	1.054314	0.6	0.684132	1.054314	0.7	0.661406	1.084471
0.6	0.763675	0.948766	0.8	0.604589	1.159863	0.9	0.570500	1.205098
0.8	0.907334	0.843218	1.0	0.525047	1.265411	1.0	0.525047	1.265411

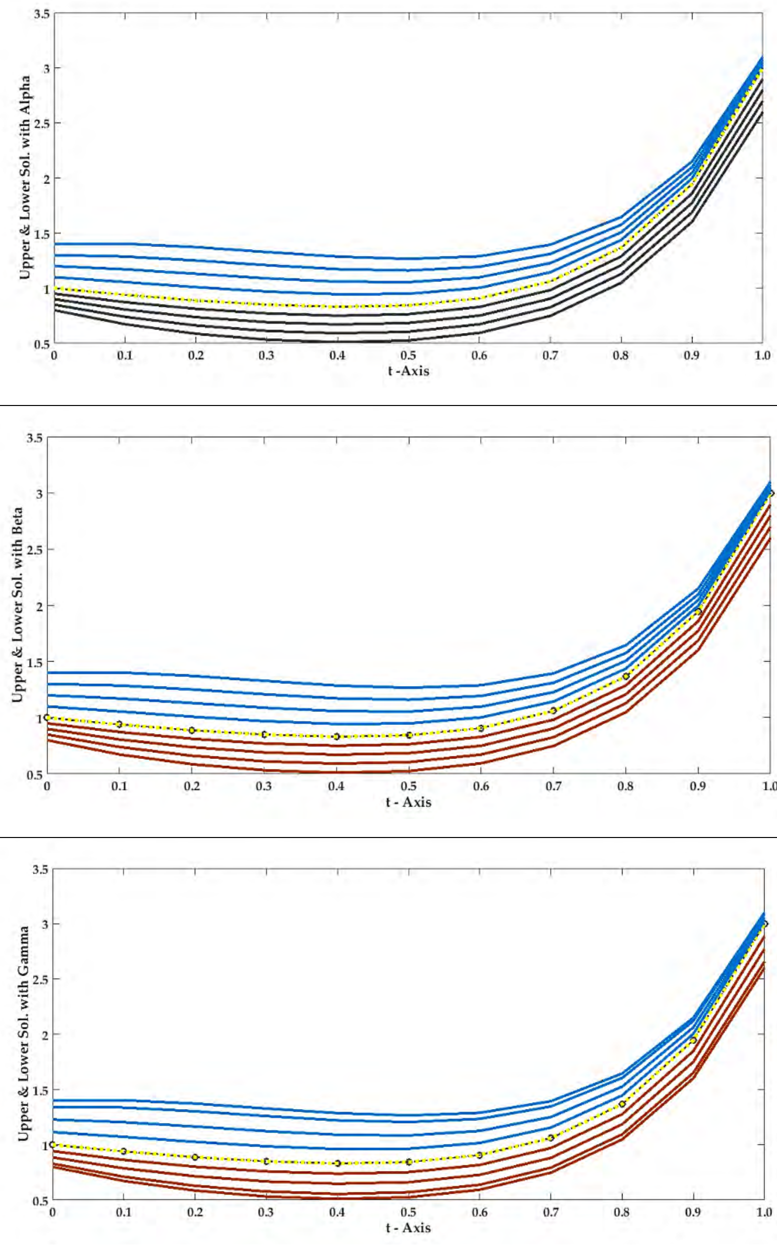


Fig. 2: The Results of Upper and Lower for (α,β,γ) **of Class(1,2).**

Class (2, 1)

$$\underline{y}(t,\boldsymbol{\alpha},\beta,\gamma) = \boldsymbol{c_1}e^{4t} + \boldsymbol{c_2}e^{\frac{(5+\sqrt{41})t}{2}} + \boldsymbol{c_3}e^{t} + \boldsymbol{c_4}e^{\frac{(5-\sqrt{41})t}{2}},$$

$$\bar{y}(t,\boldsymbol{\alpha},\beta,\gamma) = -\boldsymbol{c_1}e^{4t} + \boldsymbol{c_2}e^{\frac{(5+\sqrt{41})t}{2}} - \boldsymbol{c_3}e^{t} + \boldsymbol{c_4}e^{\frac{(5-\sqrt{41})t}{2}}$$

To find the values of constants for each α, β, γ, then use the boundary points

$$\underline{\boldsymbol{y}}(t_0,\boldsymbol{\alpha},\beta,\gamma) = \boldsymbol{c_1}e^{4t_0} + \boldsymbol{c_2}e^{\frac{(5+\sqrt{41})t_0}{2}} + \boldsymbol{c_3}e^{t_0} + \boldsymbol{c_4}e^{\frac{(5-\sqrt{41})t_0}{2}} = 0.8 + \frac{\alpha}{4}, 0.8 + \frac{(1-\beta)}{4}, 0.8 + \frac{2(1-\gamma)}{7}$$

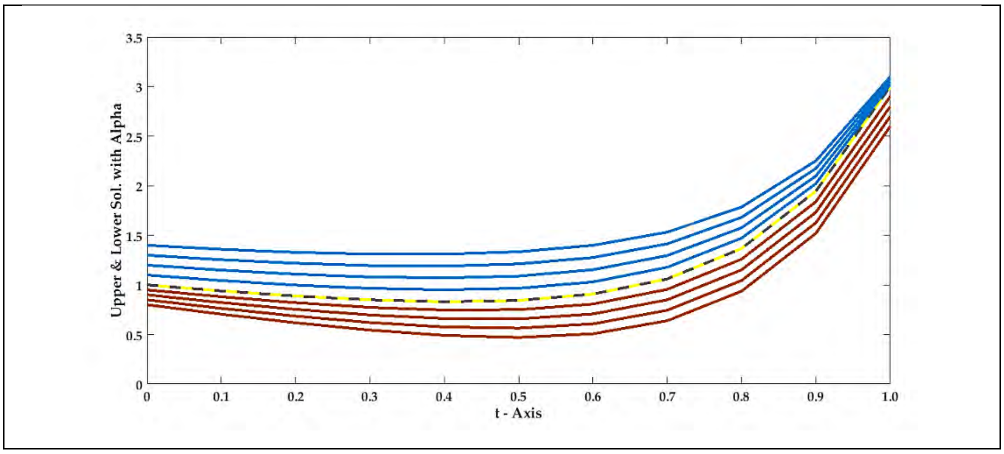
$$\bar{\boldsymbol{y}}(t_0,\boldsymbol{\alpha},\beta,\gamma) = -\boldsymbol{c_1}e^{4t_0} + \boldsymbol{c_2}e^{\frac{(5+\sqrt{41})t_0}{2}} - \boldsymbol{c_3}e^{t_0} + \boldsymbol{c_4}e^{\frac{(5-\sqrt{41})t_0}{2}} = 1.4 - \frac{\alpha}{2}, 1.4 - \frac{(1-\beta)}{2}, 1.4 - \frac{4(1-\gamma)}{7}$$

$$\underline{\boldsymbol{y}}(T,\boldsymbol{\alpha},\beta,\gamma) = \boldsymbol{c_1}e^{4T} + \boldsymbol{c_2}e^{\frac{(5+\sqrt{41})T}{2}} + \boldsymbol{c_3}e^{T} + \boldsymbol{c_4}e^{\frac{(5-\sqrt{41})T}{2}} = 2.6 + \frac{\alpha}{2}, 2.6 + \frac{(1-\beta)}{2}, 2.6 + \frac{4(1-\gamma)}{7}$$

$$\bar{\boldsymbol{y}}(T,\boldsymbol{\alpha},\beta,\gamma) = -\boldsymbol{c_1}e^{4T} + \boldsymbol{c_2}e^{\frac{(5+\sqrt{41})T}{2}} - \boldsymbol{c_3}e^{T} + \boldsymbol{c_4}e^{\frac{(5-\sqrt{41})T}{2}} = 3.1 - \frac{\alpha}{8}, 3.1 - \frac{(1-\beta)}{8}, 3.1 - \frac{1-\gamma}{7}$$

Table (4): The Results of Upper and Lower for (α,β,γ) at t = 0.5

α	$\underline{y}(0.5,\alpha)$	$\bar{y}(0.5,\alpha)$	β	$\underline{y}(0.5,\beta)$	$\bar{y}(0.5,\beta)$	γ	$\underline{y}(0.5,\gamma)$	$\bar{y}(0.5,\gamma)$
0.0	0.470499	1.334594	0.2	0.843218	0.843218	0.3	0.843218	0.843218
0.2	0.563679	1.211750	0.4	0.750038	0.966062	0.5	0.736727	0.983611
0.4	0.656859	1.088906	0.6	0.656859	1.088906	0.7	0.630236	1.124004
0.6	0.750038	0.966062	0.8	0.563679	1.211750	0.9	0.523745	1.264398
0.8	0.907334	0.843218	1.0	0.470500	1.334594	1.0	0.470499	1.334594



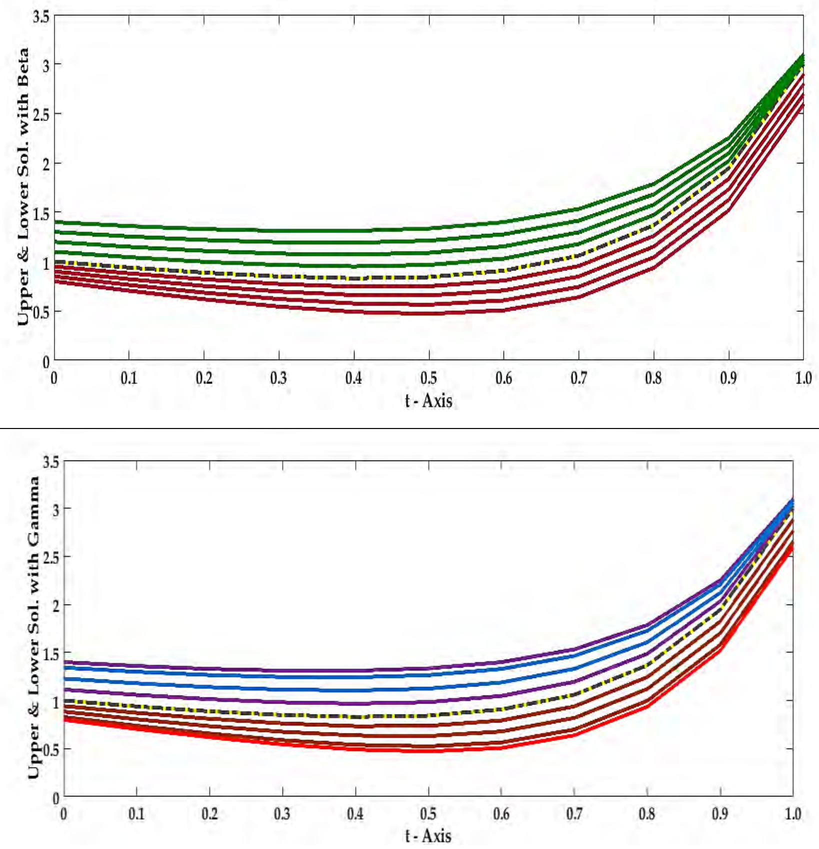


Fig. 3: The Results of Upper and Lower for (α,β,γ) **of Class(2,1).**

Class (2, 2)

$$\underline{y}(t,\boldsymbol{\alpha},\beta,\gamma)=\boldsymbol{c_1}e^{\frac{-(5-\sqrt{41})t}{2}}+\boldsymbol{c_2}e^{\frac{(5-\sqrt{41})t}{2}}+\boldsymbol{c_3}e^{\frac{-(5+\sqrt{41})t}{2}}+\boldsymbol{c_4}e^{\frac{(5+\sqrt{41})t}{2}},$$

$$\bar{y}(t,\boldsymbol{\alpha},\beta,\gamma)=-\boldsymbol{c_1}e^{\frac{-(5-\sqrt{41})t}{2}}+\boldsymbol{c_2}e^{\frac{(5-\sqrt{41})t}{2}}-\boldsymbol{c_3}e^{\frac{-(5+\sqrt{41})t}{2}}+\boldsymbol{c_4}e^{\frac{(5+\sqrt{41})t}{2}}$$

To find the values of constants for each $\alpha,\beta,\gamma,$ then use the boundary points

$$\underline{\boldsymbol{y}}(t_0,\boldsymbol{\alpha},\beta,\gamma)=\ \boldsymbol{c_1}e^{\frac{-(5-\sqrt{41})t_0}{2}}+\boldsymbol{c_2}e^{\frac{(5-\sqrt{41})t_0}{2}}+\boldsymbol{c_3}e^{\frac{-(5+\sqrt{41})t_0}{2}}+\boldsymbol{c_4}e^{\frac{(5+\sqrt{41})t_0}{2}}=0.8+\frac{\alpha}{4}\ 0.8+\frac{(1-\beta)}{4},0.8+\frac{2(1-\gamma)}{7}$$

$$\bar{\boldsymbol{y}}(t_0,\boldsymbol{\alpha},\beta,\gamma)=-\boldsymbol{c_1}e^{\frac{-(5-\sqrt{41})t_0}{2}}+\boldsymbol{c_2}e^{\frac{(5-\sqrt{41})t_0}{2}}-\boldsymbol{c_3}e^{\frac{-(5+\sqrt{41})t_0}{2}}+\boldsymbol{c_4}e^{\frac{(5+\sqrt{41})t_0}{2}}=1.4-\frac{\alpha}{2},1.4-\frac{(1-\beta)}{2},1.4-\frac{4(1-\gamma)}{7},$$

$$\underline{\boldsymbol{y}}(T,\boldsymbol{\alpha},\beta,\gamma)=\ \boldsymbol{c_1}e^{\frac{-(5-\sqrt{41})T}{2}}+\boldsymbol{c_2}e^{\frac{(5-\sqrt{41})T}{2}}+\boldsymbol{c_3}e^{\frac{-(5+\sqrt{41})T}{2}}+\boldsymbol{c_4}e^{\frac{(5+\sqrt{41})T}{2}}=2.6+\frac{\alpha}{2},2.6+\frac{(1-\beta)}{2},2.6+\frac{4(1-\gamma)}{7}$$

$$\bar{\boldsymbol{y}}(T,\boldsymbol{\alpha},\beta,\gamma)=\ -\boldsymbol{c_1}e^{\frac{-(5-\sqrt{41})T}{2}}+\boldsymbol{c_2}e^{\frac{(5-\sqrt{41})T}{2}}-\boldsymbol{c_3}e^{\frac{-(5+\sqrt{41})T}{2}}+\boldsymbol{c_4}e^{\frac{(5+\sqrt{41})T}{2}}=3.1-\frac{\alpha}{8},3.1-\frac{(1-\beta)}{8},3.1-\frac{1-\gamma}{7}$$

Table (5): The Results of Upper and Lower for (α,β,γ) at t = 0.5

α	$\underline{y}(0.5,\alpha)$	$\overline{y}(0.5,\alpha)$	β	$\underline{y}(0.5,\beta)$	$\overline{y}(0.5,\beta)$	γ	$\underline{y}(0.5,\gamma)$	$\overline{y}(0.5,\gamma)$
0.0	0.716735	1.088359	0.2	0.843218	0.843218	0.3	0.843218	0.843218
0.2	0.748356	1.027074	0.4	0.811597	0.904503	0.5	0.807080	0.913258
0.4	0.779976	0.965788	0.6	0.779976	0.965788	0.7	0.770942	0.983298
0.6	0.811597	0.904503	0.8	0.748356	1.027074	0.9	0.734804	1.053339
0.8	0.843218	0.843218	1.0	0.716735	1.088359	1.0	0.716735	1.088359

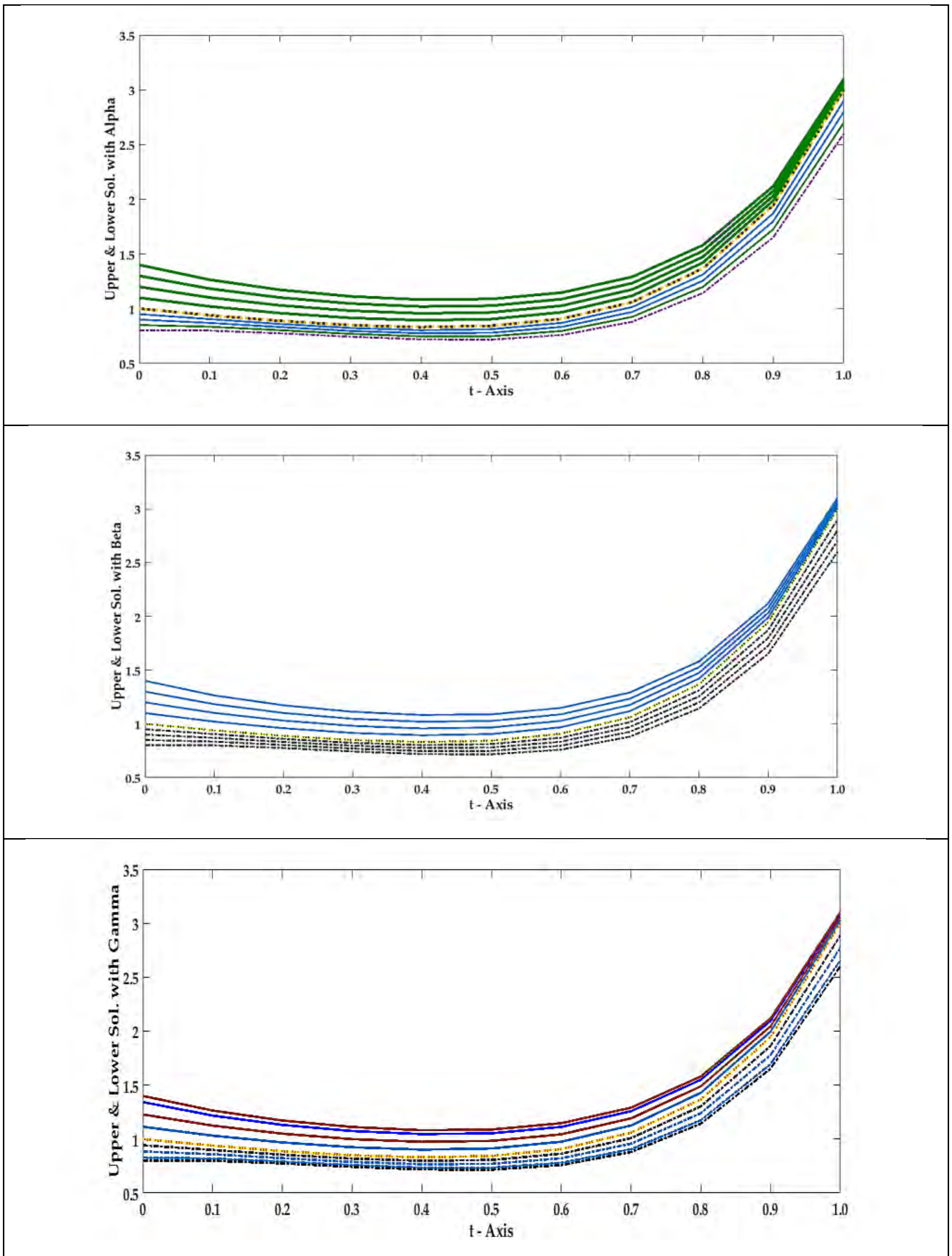


Fig. 4: The Results of Upper and Lower for (α,β,γ) **of Class(2,2).**

In this example, All the last figures which describe the upper and lower solution of each class for each $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ gave a notification that the solutions bands are the strong solutions because there was no overlapping between the bands, Also by applying the meaning of strong solution from definition 11, the tables show that the solution values at $\boldsymbol{t} = \mathbf{0.5}$ are strong because it is observed that the lower solution is increasing ascending with $\boldsymbol{\alpha}$ but the upper is descending with it, and the lower solution is decreasing with $\boldsymbol{\beta}$ ***and*** $\boldsymbol{\gamma}$ but the upper is increasing with them.

4.2. Example 2

$$\widetilde{\boldsymbol{y}''}(\boldsymbol{t}) = \mathbf{3}\widetilde{\boldsymbol{y}'}(\boldsymbol{t}) - \mathbf{2}\tilde{\boldsymbol{y}}(\boldsymbol{t})$$

$$\tilde{y}(t_0 = 0) = \tilde{a} = (0.8,1,1.4; 0.8,0.2,0.3)$$

$$\tilde{y}(T = 1) = \tilde{b} = (2.6,3,3.1,; 0.8,0.2,0.3)$$

Class (1, 1)

$$\underline{\boldsymbol{y}}(\boldsymbol{t}, \alpha, \beta, \gamma) = \boldsymbol{c_1} \boldsymbol{e}^{2t} + \boldsymbol{c_2} \boldsymbol{e}^{\frac{(3+\sqrt{17})t}{2}} + \boldsymbol{c_3} \boldsymbol{e}^{t} + \boldsymbol{c_4} \boldsymbol{e}^{\frac{(3-\sqrt{17})t}{2}},$$

$$\overline{\boldsymbol{y}}(\boldsymbol{t}, \alpha, \beta, \gamma) = \boldsymbol{c_1} \boldsymbol{e}^{2t} - \boldsymbol{c_2} \boldsymbol{e}^{\frac{(3+\sqrt{17})t}{2}} + \boldsymbol{c_3} \boldsymbol{e}^{t} - \boldsymbol{c_4} \boldsymbol{e}^{\frac{(3-\sqrt{17})t}{2}}.$$

To find the values of constants for each α, β, γ, then use the boundary points

$$\underline{y}(t_0, \alpha, \beta, \gamma) = c_1 + c_2 + c_3 + c_4 = 0.8 + \frac{\alpha}{4}, 0.8 + \frac{(1-\beta)}{4}, 0.8 + \frac{2(1-\gamma)}{7}$$

$$\overline{y}(t_0, \alpha, \beta, \gamma) = c_1 - c_2 + c_3 - c_4 = 1.4 - \frac{\alpha}{2}, 1.4 - \frac{(1-\beta)}{2}, 1.4 - \frac{4(1-\gamma)}{7}$$

$$\underline{y}(T, \alpha, \beta, \gamma) = c_1 e^2 + c_2 e^{\frac{(3+\sqrt{17})}{2}} + c_3 e^1 + c_4 e^{\frac{(3-\sqrt{17})}{2}} = 2.6 + \frac{\alpha}{2}, 2.6 + \frac{(1-\beta)}{2}, 2.6 + \frac{4(1-\gamma)}{7}$$

$$\overline{y}(T, \alpha, \beta, \gamma) = c_1 e^2 - c_2 e^{\frac{(3+\sqrt{17})}{2}} + c_3 e^1 - c_4 e^{\frac{(3-\sqrt{17})}{2}} = 3.1 - \frac{\alpha}{8}, 3.1 - \frac{(1-\beta)}{8}, 3.1 - \frac{1-\gamma}{7}$$

Table (6): The Results of Upper and Lower for (α,β,γ) at t = 0.5

α	$\underline{y}(0.5, \alpha)$	$\overline{y}(0.5, \alpha)$	β	$\underline{y}(0.5, \beta)$	$\overline{y}(0.5, \beta)$	γ	$\underline{y}(0.5, \gamma)$	$\overline{y}(0.5, \gamma)$
0.0	1.543156	2.019864	0.2	1.713232	1.713232	0.3	1.713232	1.713232
0.2	1.585675	1.943206	0.4	1.670713	1.789890	0.5	1.664639	1.800841
0.4	1.628194	1.866548	0.6	1.628194	1.866548	0.7	1.616045	1.888450
0.6	1.670713	1.789890	0.8	1.585675	1.943206	0.9	1.567452	1.976059
0.8	1.713232	1.713232	1.0	1.543156	2.019864	1.0	1.543156	2.019864

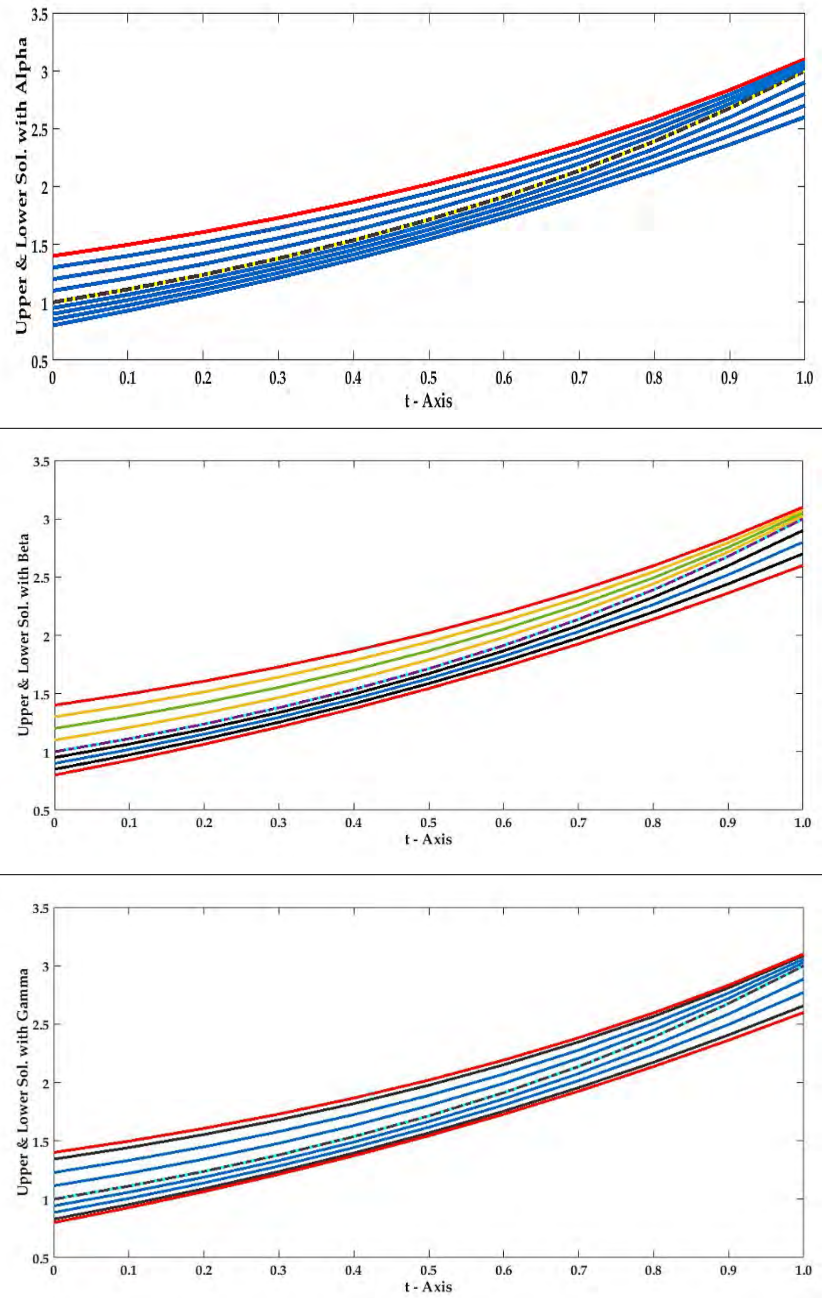


Fig. 5: The Results of Upper and Lower for (α, β, γ) **of Class(1,1).**

Class (1, 2)

$$\underline{y}(t, \alpha) = c_1 e^{-2t} + c_2 e^{2t} + c_3 e^{-t} + c_4 e^{t}$$

$$\bar{y}(t, \alpha) = -c_1 e^{-2t} + c_2 e^{2t} - c_3 e^{-t} + c_4 e^{t}$$

To find the values of constants for each α, β, γ, then use the boundary points

$$\underline{y}(t_0, \alpha, \beta, \gamma) = c_1 + c_2 + c_3 + c_4 = 0.8 + \frac{\alpha}{4}, 0.8 + \frac{(1-\beta)}{4}, 0.8 + \frac{2(1-\gamma)}{7}$$

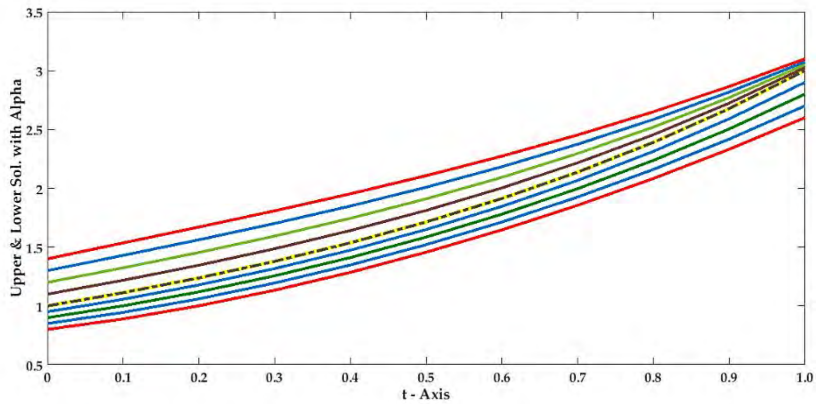
$$\bar{y}(t_0, \alpha, \beta, \gamma) = -c_1 + c_2 - c_3 + c_4 = 1.4 - \frac{\alpha}{2}, 1.4 - \frac{(1-\beta)}{2}, 1.4 - \frac{4(1-\gamma)}{7}$$

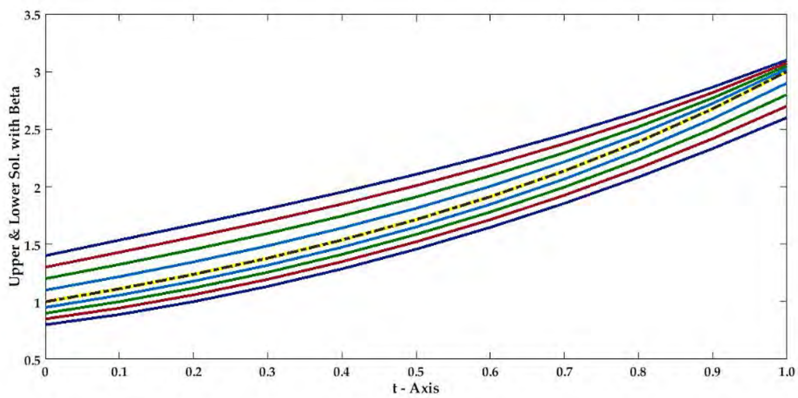
$$\underline{y}(T,\alpha,\beta,\gamma) = c_1e^{-2} + c_2e^2 + c_3e^{-1} + c_4e^1 == 2.6 + \frac{\alpha}{2}, 2.6 + \frac{(1-\beta)}{2}, 2.6 + \frac{4(1-\gamma)}{7}$$

$$\bar{y}(T,\alpha,\beta,\gamma) = -c_1e^{-2} + c_2e^2 - c_3e^{-1} + c_4e^1 = 3.1 - \frac{\alpha}{8}, 3.1 - \frac{(1-\beta)}{8}, 3.1 - \frac{1-\gamma}{7}$$

Table (7): The Results of Upper and Lower for (α,β,γ) at t = 0.5

α	$\underline{y}(0.5,\alpha)$	$\bar{y}(0.5,\alpha)$	β	$\underline{y}(0.5,\beta)$	$\bar{y}(0.5,\beta)$	γ	$\underline{y}(0.5,\gamma)$	$\bar{y}(0.5,\gamma)$
0.0	1.456247	2.106772	0.2	1.713232	1.713232	0.3	1.713232	1.713232
0.2	1.520493	2.008387	0.4	1.648986	1.811617	0.5	1.639808	1.825672
0.4	1.584740	1.910002	0.6	1.584740	1.910002	0.7	1.566383	1.938112
0.6	1.648986	1.811617	0.8	1.520493	2.008387	0.9	1.492959	2.050552
0.8	1.713232	1.713232	1.0	1.456247	2.106772	1.0	1.456247	2.106772





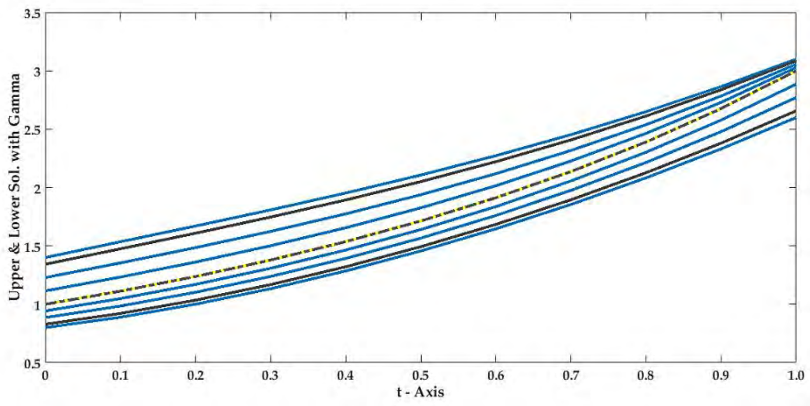


Fig. 6: The Results of Upper and Lower for (α, β, γ) **of Class(1,2).**

Class (2, 1)

$$\underline{y}(t, \alpha) = c_2 e^{2t} + c_4 e^t$$

$$\overline{y}(t, \alpha) = c_1 e^{2t} + c_3 e^t$$

To find the values of constants for each α, β, γ, then use the boundary points

$$\underline{\boldsymbol{y}}(t_0, \alpha, \beta, \gamma) = \boldsymbol{c_2} + \boldsymbol{c_4} = 0.8 + \frac{\alpha}{4}, 0.8 + \frac{(1-\beta)}{4}, 0.8 + \frac{2(1-\gamma)}{7}$$

$$\overline{\boldsymbol{y}}(t_0, \alpha, \beta, \gamma) = \boldsymbol{c_1} + \boldsymbol{c_3} = 1.4 - \frac{\alpha}{2}, 1.4 - \frac{(1-\beta)}{2}, 1.4 - \frac{4(1-\gamma)}{7}$$

$$\underline{\boldsymbol{y}}(T, \alpha, \beta, \gamma) = \boldsymbol{c_2} e^2 + \boldsymbol{c_4} e^1 = 2.6 + \frac{\alpha}{2}, 2.6 + \frac{(1-\beta)}{2}, 2.6 + \frac{4(1-\gamma)}{7}$$

$$\overline{\boldsymbol{y}}(T, \alpha, \beta, \gamma) = \boldsymbol{c_1} e^2 + \boldsymbol{c_3} e^1 = 3.1 - \frac{\alpha}{8}, 3.1 - \frac{(1-\beta)}{8}, 3.1 - \frac{1-\gamma}{7}$$

Table (8): The Results of Upper and Lower for (α,β,γ) at t=0.5

α	$\underline{\boldsymbol{y}}(0.5, \alpha)$	$\overline{\boldsymbol{y}}(0.5, \alpha)$	β	$\underline{\boldsymbol{y}}(0.5, \beta)$	$\overline{\boldsymbol{y}}(0.5, \beta)$	γ	$\underline{\boldsymbol{y}}(0.5, \gamma)$	$\overline{\boldsymbol{y}}(0.5, \gamma)$
0.0	1.416384	2.146636	0.2	1.713232	1.713232	0.3	1.713232	1.713232
0.2	1.490596	2.038285	0.4	1.639020	1.821583	0.5	1.628418	1.837062
0.4	1.564808	1.929934	0.6	1.564808	1.929934	0.7	1.543604	1.960891
0.6	1.639020	1.821583	0.8	1.490596	2.038285	0.9	1.458790	2.084721
0.8	1.713232	1.713232	1.0	1.416384	2.146636	1.0	1.416384	2.146636

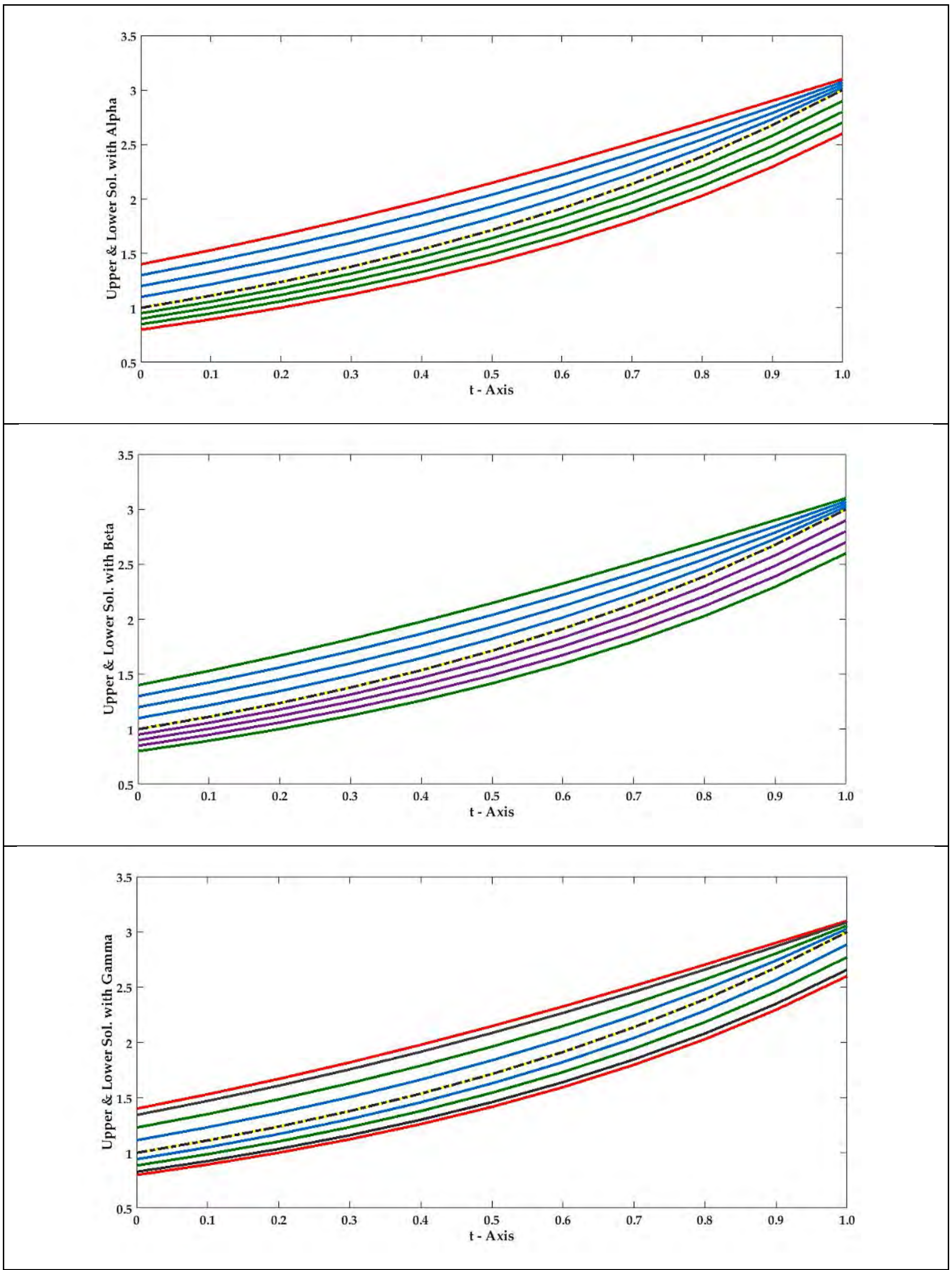


Fig. 7: The Results of Upper and Lower for (α,β,γ) **of Class(2,1).**

Class (2, 2)

$$\underline{\boldsymbol{y}}(t,\boldsymbol{\alpha},\beta,\gamma) = \boldsymbol{c_1}e^{2t} + \boldsymbol{c_2}e^{\frac{-(3+\sqrt{17})t}{2}} + \boldsymbol{c_3}e^{t} + \boldsymbol{c_4}e^{\frac{-(3-\sqrt{17})t}{2}}$$

$$\overline{\boldsymbol{y}}(t,\boldsymbol{\alpha},\beta,\gamma) = -\boldsymbol{c_1}e^{2t}(-\boldsymbol{2}) - \boldsymbol{c_3}e^{t}(-\boldsymbol{5}) - \boldsymbol{c_2}e^{\frac{-(3+\sqrt{17})t}{2}} - \boldsymbol{c_4}e^{\frac{-(3-\sqrt{17})t}{2}}$$

To find the values of constants for each $\alpha,\beta,\gamma,$ then use the boundary points

$$\underline{\boldsymbol{y}}(t_0,\boldsymbol{\alpha},\beta,\gamma) = \boldsymbol{c_1} + \boldsymbol{c_2} + \boldsymbol{c_3} + \boldsymbol{c_4} = 0.8 + \frac{\alpha}{4},\ 0.8 + \frac{(1-\beta)}{4},\ 0.8 + \frac{2(1-\gamma)}{7}$$

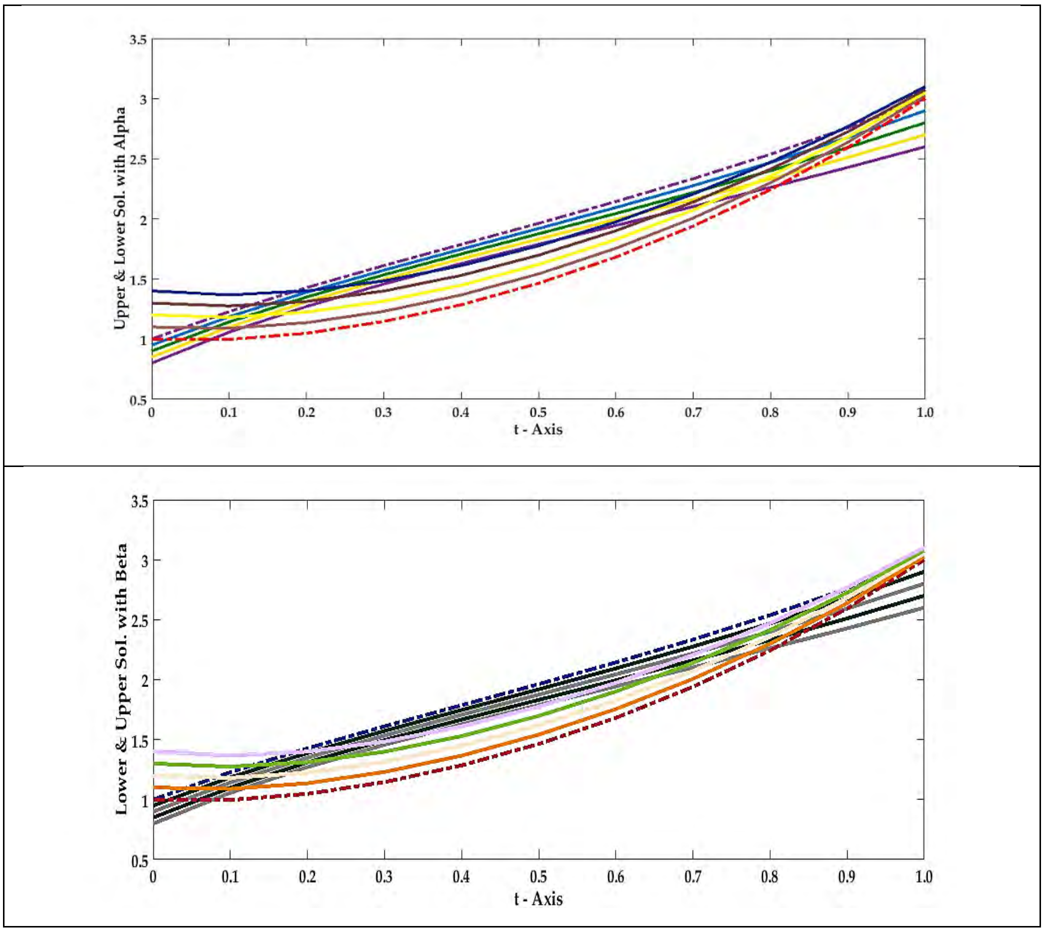
$$\overline{\boldsymbol{y}}(t_0,\boldsymbol{\alpha},\beta,\gamma) = \boldsymbol{2c_1} + \boldsymbol{5c_3} - \boldsymbol{c_2} - \boldsymbol{c_4} = 1.4 - \frac{\alpha}{2},\ 1.4 - \frac{(1-\beta)}{2},\ 1.4 - \frac{4(1-\gamma)}{7}$$

$$\underline{y}(T,\alpha,\beta,\gamma) = c_1 e^2 + c_2 e^{\frac{-(3+\sqrt{17})}{2}} + c_3 e^1 + c_4 e^{\frac{-(3-\sqrt{17})}{2}} = 2.6+\frac{\alpha}{2}, 2.6+\frac{(1-\beta)}{2}, 2.6+\frac{4(1-\gamma)}{7}$$

$$\overline{y}(T,\alpha,\beta,\gamma) = -c_1 e^2(-2) - c_3 e^{11}(-5) - c_2 e^{\frac{-(3+\sqrt{17})}{2}} - c_4 e^{\frac{-(3-\sqrt{17})}{2}} = 3.1-\frac{\alpha}{8}, 3.1-\frac{(1-\beta)}{8}, 3.1-\frac{1-\gamma}{7}$$

Table (9): The Results of Upper and Lower for (α,β,γ) at t = 0.5

α	$\underline{y}(0.5,\alpha)$	$\overline{y}(0.5,\alpha)$	β	$\underline{y}(0.5,\beta)$	$\overline{y}(0.5,\beta)$	γ	$\underline{y}(0.5,\gamma)$	$\overline{y}(0.5,\gamma)$
0.0	1.787831	1.775188	0.2	1.962740	1.463724	0.3	1.962740	1.463724
0.2	1.831558	1.027074	0.4	1.919013	0.904503	0.5	1.912766	1.552714
0.4	1.875285	0.965788	0.6	1.875285	0.965788	0.7	1.862792	1.641703
0.6	1.919013	0.904503	0.8	1.831558	1.027074	0.9	1.812818	1.730693
0.8	1.962740	1.463724	1.0	1.787831	1.775188	1.0	1.787831	1.775188



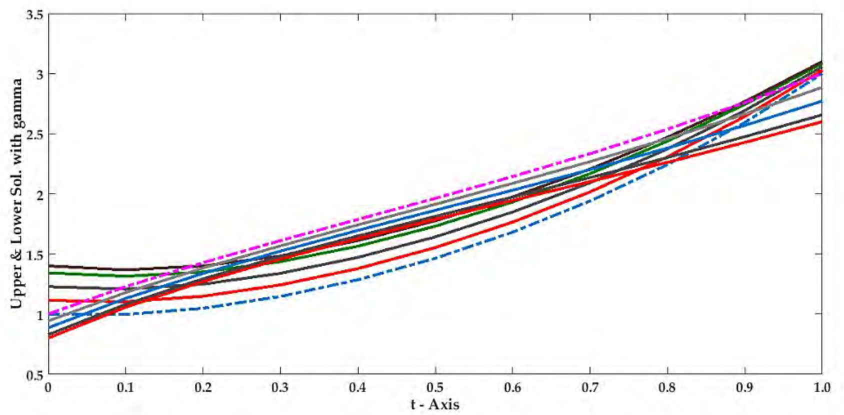


Fig. 8: The Results of Upper and Lower for (α, β, γ) **of Class(2,2).**

In this example, we noticed that classes (1,1), (1,2), and (2,1) have no overlapping between uppers and lowers cuts also it is approved from the tables of these classes have the strong solution, But Class (2,2) in Fig. 8 for α, β *and* γ have very clear overlapping between uppers and lowers and it can be observed easily from class(2,2) table of solution values at t = 0.5 that $\frac{\partial \underline{y}}{\partial \alpha} < 0, \frac{d\bar{y}}{d\alpha} > 0$ but $\frac{\partial \underline{y}}{\partial \beta} > 0, \frac{d\bar{y}}{d\beta} < 0, \frac{\partial \underline{y}}{\partial \gamma} > 0, \frac{d\bar{y}}{d\gamma} < 0$, $\bar{\boldsymbol{y}}(0.5, \alpha, \beta, \gamma) < \underline{\boldsymbol{y}}(0.5, \alpha, \beta, \gamma)$ which approve that class(2,2) solution is weak.

5. Conclusions

In this paper, after solving the neutrosophic fuzzy boundary value problem by analytical solution we conclude that it is a generalization of the fuzzy boundary value problem solution by determining the degree of membership $\boldsymbol{\alpha}$, the degree of indeterministic $\boldsymbol{\beta}$ and the degree of non-membership $\boldsymbol{\gamma}$ and it is discussed clearly in applications where the solution of each class founded by constructing an algorithm using Matlab and graphical representation is also interpreted.

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Generalized closed sets and pre-closed sets via Bipolar single-valued neutrosophic Topological Spaces

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Abstract: The purpose of the paper is to introduce a new class of sets namely bipolar single-valued neutrosophic generalized closed sets and bipolar single-valued neutrosophic generalized pre- closed sets in bipolar single-valued neutrosophic topological spaces. Also we analysis the properties and its applications.

Keywords: Bipolar single-valued neutrosophic generalized closed sets, bipolar single-valued neutrosophic generalized pre- closed sets, and bipolar single-valued neutrosophic generalized pre-open sets, BSVN $T_{1/2}$ space, BSVN $_{p}T_{1/2}$ space, BSVN $_{gp}T_{1/2}$ space, BSVN $_{gp}T_{p}$ space.

1. Introduction

Zadeh [37], the Father of the Fuzzy Logic who imported the fuzzy sets in 1965 where the Fuzzy logic feature the human decision making technique and it is a tool in research logical subject. The concept of fuzzy sets is to deal with contrasting types of uncertainties. Fuzzy topology was introduced by Chang [5] in 1967 after the introduction of fuzzy sets. In 1970, Levine [21] studied the generalized closed sets in general topology. In 1991, Binshahan [4] introduced and investigate the notion of fuzzy pre-open and fuzzy pre-closed sets. The concept of generalized fuzzy closed set was introduced by Balasubramanian and Sundaram [3]. Fukutake et al. [19] gave the generalized pre-closed fuzzy sets in fuzzy topological spaces.

In 1994, Zhang [38] introduced the notion of a bipolar fuzzy set. Azhagappan and Kamaraj [2] investigated bipolar valued fuzzy topological spaces. Bipolar fuzzy topological spaces were proposed by Kim J, Samanta S. K, Lim P. K, Lee J. G and Hur K[20]. An intuitionistic fuzzy set was introduced by Atanassov [1] in 1986 as the extension of Zadeh's Fuzzy Sets besides the degree of membership and degree of non-membership. Dogan Coker [18] who gave introduction to intuitionistic fuzzy topological spaces. Rajarajeswari and Senthil Kumar [29] introduced Generalized pre-closed sets in Intuitionistic fuzzy Topological spaces.

Smarandache [31] introduced the neutrosophic set which is the base for the new mathematical theories. Neutrosophic set has the capability to induce classical sets, fuzzy set, Intuitionistic fuzzy sets. Introducing the components of the neutrosophic set are True (T), Indeterminacy (I), False (F) which represent the membership, indeterminacy, and non-membership values respectively. The notion of classical set, fuzzy set, interval-valued fuzzy set, Intuitionistic fuzzy, etc were generalized by the neutrosophic set. Neutrosophic topological spaces were presented by Salama et al. [30]. The concept of generalized closed sets and generalized pre-closed sets in neutrosophic Topological spaces were introduced by Wadei Al-Omeri et al.[33]. The neutrosophic pre-open and pre-closed sets in neutrosophic topology were extended by Venkateswara Rao et al.[32] who introduce neutrosophic topological space and open sets, closed sets, semi-open and semi closed sets. Generalized neutrosophic closed sets was introduced and some of their characterizations were also discussed by Dhavaseelan and Jafari [17]. Many Researchers [6-15, 26] have studied Neutrosophic in different areas with applications and the results.

Deli et al.[16] developed bipolar neutrosophic sets and study their application in decision making problem. The notation of bipolar neutrosophic soft set was proposed by Mumtaz Ali et al.[27]. Single-valued neutrosophic sets (in sort, SVN) were proposed by Wang et al.[35] by simplifying the Neutrosophic set. Single-valued neutrosophic topological space was given by YL Liu and HL Yang [22] and discussed the relationships between single valued neutrosophic approximation spaces and single valued neutrosophic topological spaces. Many researchers have studied the applications of SVNSs as well as theory. Ye [36] proposed decision making based on correlation coefficients and weighted correlation coefficient of SVNSs, and gave the application of proposed methods. Majumdar and Samant [23] studied distance, similarity and entropy of SVNSs from a theoretical aspect. Bipolar single-valued neutrosophic set was introduced by Mohana et al. [25] and also they give bipolar single-valued neutrosophic topological spaces.

In the paper, we introduce a new class of sets namely bipolar single-valued neutrosophic generalized closed sets and bipolar single-valued neutrosophic generalized pre-closed sets in bipolar single-valued neutrosophic topological spaces. Further we examine the interesting properties and some applications with counter examples.

2. Preliminaries

2.1 Definition [31]: Let a universe U of discourse. Then $K=\{<x, T_K(x), I_K(x), F_K(x)>x\in X\}$ defined as a neutrosophic set where truth-membership function T_K, an indeterminacy-membership function I_K and a falsity-membership function F_K. T_K, I_K, F_K are real or non-standard elements of $]\,0^-, 1^+\,[$.
No restriction on the sum of $T_K(x)$, $I_K(x)$ and $F_K(x)$, so $0^- \leq \sup T_K(x) \leq \sup I_K(x) \leq \sup F_K(x) \leq 3^+$.

2.2 Definition [30]: A Neutrosophic topology [NT for short] is a non-empty set X is a family of Neutrosophic subsets in X satisfying the following axioms:
(NT_1) $0_N, 1_N \in \tau$,
(NT_2) $G_1 \cap G_2 \in \tau$, for any $G_1, G_2 \in \tau$,
(NT3) $\cup G_i \in \tau$, for every $\{G_i : i \in J\} \subseteq \tau$.
The pair (X, τ) is called a Neutrosophic topological space (NTS for short). The elements of τ are called Neutrosophic open sets [NOS for short]. A complement C(A) of a NOS A in NTS (X, τ) is called a Neutrosophic closed set [NCS for short] in X.

2.3 Definition: [30]: Let (X, τ) be NTS and $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle : x \in X\}$ be a NS in X. Then the Neutrosophic closure and Neutrosophic interior of A are defined by NCl(A) = {K : K is a NCS in X and A K} NInt(A) = {G : G is a NOS in X and G A} It can be also shown that NCl(A) is NCS and NInt(A) is a NOS in X. a) A is NOS if and only if A = NInt(A), b) A is NCS if and only if A = NCl(A).

2.4 Definition: [34]: A Neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle : x \in X\}$ in a NTS (X, τ) is said to be
(i) Neutrosophic regular closed set (NRCS for short) if A = NCl(NInt(A)),
(ii) Neutrosophic regular open set (NROS for short) if A = NInt(NCl(A)),
(iii) Neutrosophic semi closed set (NSCS for short) if NInt(NCl(A)) ⊆ A,
(iv) Neutrosophic semi open set (NSOS for short) if A ⊆ NCl(NInt(A)),
(v) Neutrosophic pre closed set (NPCS for short) if NCl(NInt(A)) ⊆ A,
(vi) Neutrosophic pre-open set (NPOS for short) if A ⊆ NInt(NCl(A)),
(vii) Neutrosophic α- closed set (NSCS for short) if NCl(NInt(NCl(A))) ⊆ A,
(viii) Neutrosophic α- open set (NSOS for short) if A ⊆ NInt(NCl(NInt(A))).

2.5 Definition: [33]: Let (X, τ) be NTS and A = {⟨x, $\mu_A(x)$, $\sigma_A(x)$, $\nu_A(x)$⟩: x ∈ X} be a NS in X. Then the Neutrosophic pre closure and Neutrosophic pre interior of A are defined by NPCl(A) = {K : K is a NPCS in X and A⊆ K}, NPInt(A) = {G : G is a NPOS in X and G⊆ A}.

2.6 Definition: [28] :A Neutrosophic set A = {⟨x, $\mu_A(x)$, $\sigma_A(x)$, $\nu_A(x)$⟩: x ∈ X} in a NTS (X, τ) is said to be a Neutrosophic generalized closed set (NGCS for short) if NCl(A) U whenever A U and U is a NOS in (X, τ). A Neutrosophic set A of a NTS (X, τ) is called a Neutrosophic generalized open set (NGOS for short) if C (A) is a NGCS in (X, τ).

2.7 Definition: [33]: A Neutrosophic set A = {⟨x, μA(x), σA(x), νA(x)⟩: x ∈ X} in a NTS (X, τ) is said to be a Neutrosophic α- generalized closed set (NαGCS for short) if NαCl(A) U whenever A U and U is a NOS in (X, τ). A Neutrosophic set A of a NTS (X, τ) is called a Neutrosophic α- generalized open set (NαGOS for short) if C (A) is an NαGCS in (X, τ).

2.8 Definition: [24]: A Neutrosophic set A = {⟨x, μA(x), σA(x), νA(x)⟩: x ∈ X} in a NTS (X, τ) is said to be a Neutrosophic regular generalized closed set (NRGCS for short) if NCl(A) U whenever A U and U is a NROS in (X, τ). A Neutrosophic set A of a NTS (X, τ) is called a Neutrosophic regular generalized open set (NRGOS for short) if C(A) is a NRGCS in (X, τ).

2.9 Definition: [33]: A Neutrosophic set A = {⟨x, μA(x), σA(x), νA(x)⟩: x ∈ X} in a NTS (X, τ) is said to be a Neutrosophic generalized pre closed set (NGPCS for short) if NPCl(A) U whenever A U and U is a NOS in (X, τ). A Neutrosophic set A of a NTS (X, τ) is called a Neutrosophic generalized pre-open set (NGPOS for short) if C(A) is a NGPCS in (X, τ).

2.10 Definition [35]: Let a universe X of discourse. Then A_{NS}={<x,$T_A(x)$,$I_A(x)F_A(x)$>x∈X} defined as a single-valued neutrosophic set(SVNS in short) where truth-membership function T_A:X→[0,1],an indeterminacy-membership function I_A: X→[0,1] and a falsity-membership function F_A: X → [0,1].No restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so 0≤sup $T_A(x)$ ≤sup $I_A(x)$ ≤sup$F_A(x)$ ≤3. $\widetilde{A}$ =<T, I, F> is denoted as a single-valued neutrosophic number.

2.11 Definition [22]: A Single-valued neutrosophic topology on a non-empty set U is a family τ of SVNSs in U that satisfies the following conditions:

(T_1) $\widetilde{\phi}$, $\widetilde{U}$ ∈ τ,

(T_2) $\widetilde{A} \cap \widetilde{B}$ ∈ τ for any $\widetilde{A}$, $\widetilde{B}$ ∈ τ,

(T_3) $\cup_{i\in I} \widetilde{A}_i$ ∈ τ for any $\widetilde{A}_i$ ∈ τ, i∈ I, where I is an index set

The pair (U,τ) is called Single valued neutrosophic topological space and each SVNS $\widetilde{A}$ in τ is referred to as a single valued neutrosophic open set in (U, τ). The complement of a single valued neutrosophic open set in (U, τ) is called a single valued neutrosophic closed set in (U, τ).

2.12 Definition [16]: In X, a bipolar neutrosophic set B is defined in the form
B=<x, ($T^+(x)$, $I^+(x)$, $F^+(x)$, $T^-(x)$, $I^-(x)$, $F^-(x)$):x∈X>
Where T^+, I^+, F^+: X→ [1, 0] and T^- , I^- , F^- : X [-1, 0].The positive membership degree denotes the truth membership $T^+(x)$, indeterminate membership I^+ (x) and false membership F^+ (x) of an element x∈X corresponding to the set A and the negative membership degree denotes the truth membership $T^-(x)$,

indeterminate membership $I^-(x)$ and false membership $F^-(x)$ of an element $x\in X$ to some implicit counter-property corresponding to a bipolar neutrosophic set .

2.13 Definition [25]: A Bipolar Single-Valued Neutrosophic set (BSVNs) S in X is defined in the form of BSVN (S) $=<v,(T_{BSVN}^+,T_{BSVN}^-),(I_{BSVN}^+,I_{BSVN}^-),(F_{BSVN}^+,F_{BSVN}^-):v\in X>$ → (I)
where $(T_{BSVN}^+,I_{BSVN}^+,F_{BSVN}^+):X\to[0,1]$ and $(T_{BSVN}^-,I_{BSVN}^-,F_{BSVN}^-):X\to[-1,0]$ .In this definition, there T_{BSVN}^+ and T_{BSVN}^- are acceptable and unacceptable in past. Similarly I_{BSVN}^+ and I_{BSVN}^- are acceptable and unacceptable in future. F_{BSVN}^+ and F_{BSVN}^- are acceptable and unacceptable in present respectively.

2.14 Definition [25]: Let two bipolar single-valued neutrosophic sets $BSVN_1(S)$ and $BSVN_2(S)$ in X defined as
$BSVN_1(S)=<v,(T_{BSVN}^+(1),T_{BSVN}^-(1)),(I_{BSVN}^+(1),I_{BSVN}^-(1)),(F_{BSVN}^+(1),F_{BSVN}^-(1)):v\in X>$ and
$BSVN_2(S)=<v,(T_{BSVN}^+(2),T_{BSVN}^-(2)),(I_{BSVN}^+(2),I_{BSVN}^-(2)),(F_{BSVN}^+(2),F_{BSVN}^-(2)):v\in X>$. Then the operators are defined as follows:

(i) Complement

$BSVN^c(S) = \{< v,(1-T_{BSVN}^+), (-1-T_{BSVN}^-), (1-I_{BSVN}^+), (-1-I_{BSVN}^-), (1-F_{BSVN}^+), (-1-F_{BSVN}^-): v\in X >\}$

(ii) Union of two BSVN

$BSVN_1(S) \cup BSVN_2(S)=$

$$\left\langle \begin{matrix} \max(T_{BSVN}^+(1),T_{BSVN}^+(2)),\min(I_{BSVN}^+(1),I_{BSVN}^+(2)),\min(F_{BSVN}^+(1),F_{BSVN}^+(2)) \\ \max(T_{BSVN}^-(1),T_{BSVN}^-(2)),\min(I_{BSVN}^-(1),I_{BSVN}^-(2)),\min(F_{BSVN}^-(1),F_{BSVN}^-(2)) \end{matrix} \right\rangle$$

(iii) Intersection of two BSVN

$BSVN_1(S)\cap BSVN_2(S)=$

$$\left\langle \begin{matrix} \min(T_{BSVN}^+(1),T_{BSVN}^+(2)),\max(I_{BSVN}^+(1),I_{BSVN}^+(2)),\max(F_{BSVN}^+(1),F_{BSVN}^+(2)) \\ \min(T_{BSVN}^-(1),T_{BSVN}^-(2)),\max(I_{BSVN}^-(1),I_{BSVN}^-(2)),\max(F_{BSVN}^-(1),F_{BSVN}^-(2)) \end{matrix} \right\rangle$$

2.15 Definition [25]: Let two bipolar single-valued neutrosophic sets be $BSVN_1$ and $BSVN_2$ in X defined as
$BSVN_1(S)=<v,(T_{BSVN}^+(1),T_{BSVN}^-(1)),(I_{BSVN}^+(1),I_{BSVN}^-(1)),(F_{BSVN}^+(1),F_{BSVN}^-(1)):v\in X>$ and
$BSVN_2(S)=<v, (T_{BSVN}^+(2),T_{BSVN}^-(2)),(I_{BSVN}^+(2),I_{BSVN}^-(2)), (F_{BSVN}^+(2),F_{BSVN}^-(2)):v\in X>$.

(i) Then $S_1\subseteq S_2$ if and only if

$T_{BSVN}^+(1) \le T_{BSVN}^+(2)$, $I_{BSVN}^+(1) \ge I_{BSVN}^+(2)$, $F_{BSVN}^+(1) \ge F_{BSVN}^+(2)$,
$T_{BSVN}^-(1) \le T_{BSVN}^-(2)$, $I_{BSVN}^-(1) \ge I_{BSVN}^-(2)$, $F_{BSVN}^-(1) \ge F_{BSVN}^-(2)$ for all $v\in X$.

(ii) Then $S_1=S_2$ if and only if

$T_{BSVN}^+(1) = T_{BSVN}^+(2)$, $I_{BSVN}^+(1) = I_{BSVN}^+(2)$, $F_{BSVN}^+(1) = F_{BSVN}^+(2)$,
$T_{BSVN}^-(1) = T_{BSVN}^-(2)$, $I_{BSVN}^-(1) = I_{BSVN}^-(2)$, $F_{BSVN}^-(1) = F_{BSVN}^-(2)$ for all $v\in X$.

2.16 Definition [25]: A bipolar single-valued neutrosophic topology (BSVNT) on a non-empty set X is a τ of BSVN sets satisfying the axioms

(i) 0_{BSVN} , $1_{BSVN} \in \tau$
(ii) $S_1 \cap S_2 \in \tau$ for any $S_1,S_2 \in \tau$
(iii) $\cup S_i \in \tau$ for any arbitrary family $\{S_i : i\in j\} \in \tau$

The pair (X,τ) is called BSVN topological space(BSVNTS). Any BSVN set in τ is called as BSVN open set(BSVNOs) in X. The complement S^c of BSVN set in BSVN topological space (X, τ) is called a BSVN closed set(BSVNCs).

2.17 Definition [25]: Let (X,τ) be a BSVN topological space (BSVNTS) and
BSVN (S) =<v, (T_{BSVN^+},T_{BSVN^-}), (I_{BSVN^+},I_{BSVN^-}),(F_{BSVN^+},F_{BSVN^-}):v∈X> be a BSVN set in X. Then the closure and interior of A is defined as
Int (S) = U {F: F is a BSVN open set (BSVNOs) in X and F⊆S}
Cl (S) = ∩ {F: F is a BSVN closed set (BSVNCs) in X and S⊆F}.
Here cl(S) is a BSVNCs and int (S) is a BSVNOs in X.

(a) S is a BSVNCs in X iff cl (S) =S.
(b) S is a BSVNOs in X iff int (S) =S.

2.18 Proposition [25]: Let BSVNTS of (X,τ) and S,T be BSVNs's in X. Then the properties hold:

i. int (S) ⊆ S and S ⊆ cl(S)
ii. S⊆ T ⇒int(S) ⊆ int(T)
S⊆ T ⇒cl(S) ⊆ cl(T)
iii. int(int(S))=int(S)
cl(cl(S))=cl(S)
iv. int(S∩T)=int(S)∩int(T)
cl(SUT)=cl(S)Ucl(T)
v. int(1_{BSVN})=1_{BSVN}
cl(0_{BSVN})=0_{BSVN}

3. Bipolar Single-Valued Neutrosophic Generalized Closed Sets

For our convenience, we take (I) as S = {<x, (T_S^+ (x), I_S^+(x), F_S^+(x), T_S^-(x), I_S^-(x), F_S^-(x))>: x ∈X}.

3.1 Definition: A BSVNs S of a BSVNTS (X, τ) is said to be bipolar single-valued neutrosophic generalized closed set (BSVNGCs) if BSVN cl (S) $\subseteq$ U whenever S $\subseteq$ U and U is BSVNOs in X.

3.2 Definition: Let 0_{BSVN} and 1_{BSVN} be BSVNS in X defined as
0_{BSVN} = {<x, 0, 1, 1, -1, 0, 0: x ∈ X >} is said to be Null or Empty bipolar single-valued neutrosophic set.
1_{BSVN} = {<x, 1, 0, 0, 0, -1, -1: x ∈ X >} is said to be Absolute or Unit bipolar single-valued neutrosophic set.

3.3 Example: Let X ={p, q} and

$$S=\begin{Bmatrix} < p,(0.3,0.5,0.1,-0.2,-0.4,-0.3) > \\ < q,(0.2,0.8,0.2,-0.4,-0.6,-0.9) > \end{Bmatrix} \quad T=\begin{Bmatrix} < p,(0.4,0.4,0.1,-0.1,-0.5,-0.4) > \\ < q,(0.3,0.7,0.1,-0.3,-0.6,-0.9) > \end{Bmatrix}$$

Then τ= {0_{BSVN}, 1_{BSVN}, S, T} is a BSVNT on X. The BSVNs

$$R=\begin{Bmatrix} < p,(0.2,0.3,0.7,-0.8,-0.1,-0.3) > \\ < q,(0.3,0.8,0.3,-0.4,-0.1,-0.4) > \end{Bmatrix} \text{ is BSVNGCs in X.}$$

3.4 Definition: A BSVNs S = {<x, (T_S^+(x), I_S^+(x), F_S^+(x), T_S^-(x), I_S^- (x), F_S^-(x))>: x ∈ X} in BSVNTS

(X, τ) is said to be

(1) Bipolar single-valued neutrosophic semi closed set (BSVNSCs) if BSVN int (BSVN cl (S)) $\subseteq$ S,
(2) Bipolar single-valued neutrosophic semi open set (BSVNSOs) if S $\subseteq$ BSVN cl (BSVN int (S)),

(3) Bipolar single-valued neutrosophic pre-closed set (BSVNPCs) if BSVN cl (BSVN int (S)) $\subseteq$ S,

(4) Bipolar single-valued neutrosophic pre-open set (BSVNPOs) if S $\subseteq$ BSVN int (BSVN cl (S)),

(5) Bipolar single-valued neutrosophic α-closed set (BSVN αCs) if
BSVN cl (BSVN int (BSVN cl (S))) $\subseteq$ S,

(6) Bipolar single-valued neutrosophic α-open set (BSVN αOs) if
S $\subseteq$ BSVN int (BSVN cl (BSVN int (S))),

(7) Bipolar single-valued neutrosophic semi pre-closed set (BSVNSPCs) if
BSVN int (BSVN cl (BSVN int (S))) $\subseteq$ S,

(8) Bipolar single-valued neutrosophic semi pre-open set (BSVNSPOs) if
S$\subseteq$ BSVN cl (BSVN int (BSVN cl (S))),

(9) Bipolar single-valued neutrosophic regular open set (BSVNROs) if S = BSVN int (BSVN cl (S)),

(10) Bipolar single-valued neutrosophic regular closed set (BSVNRCs) if S = BSVN cl (BSVN int (S)).

3.5 Definition: Let (X, τ) be BSVNTS and S be BSVNs in X. Then the bipolar single-valued neutrosophic generalized interior and bipolar single-valued neutrosophic generalized closure are denoted by

(1) BSVNG int (S) = $\cup$ {G / G is a BSVNGOs in X and G$\subseteq$S}

(2) BSVNG cl (S) = $\cap${K / K is a BSVNGCs in X and S$\subseteq$K}

3.6 Definition: Let (X, τ) be any BSVNTS and let S and T be BSVNs in X. Then the bipolar single-valued neutrosophic generalized closure operator satisfies the properties:

1. S $\subseteq$ BSVN cl(S)
2. BSVN int(S) $\subseteq$S
3. S$\subseteq$T $\Rightarrow$BSVN cl(S) $\subseteq$ BSVN cl(T)
4. S$\subseteq$T $\Rightarrow$BSVN int(S) $\subseteq$ BSVN int(T)
5. BSVN cl(S$\cup$T)= BSVN cl(S) $\cup$ BSVN cl(T)
6. BSVN int(S$\cap$T)= BSVN int(S) $\cap$BSVN int(S)
7. (BSVN cl(S))^c = BSVN int(S^c)
8. (BSVN cl(S))^c = BSVN int(S^c)

Proof:

1. BSVN cl(S) = $\cap${K / K is a BSVNGCs in X and S$\subseteq$K}. Thus S $\subseteq$ BSVN cl(S).
2. BSVNG int (S) = $\cup$ {G / G is a BSVNGOs in X and G$\subseteq$S}. Thus BSVN int(S) $\subseteq$S.
3. BSVN cl(T) = $\cap${K / K is a BSVNGCs in X and T$\subseteq$K},
   $\supseteq$ $\cap${K / K is a BSVNGCs in X and S$\subseteq$K},
   $\supseteq$ BSVN cl(S). Thus BSVN cl(S) $\subseteq$ BSVN cl(T).
4. BSVN int (T) = $\cup$ {G / G is a BSVNGOs in X and G$\subseteq$T},
   $\supseteq$$\cup$ {G / G is a BSVNGOs in X and G$\subseteq$S},
   $\supseteq$ BSVN int (S). Thus BSVN int (S) $\subseteq$ BSVN int (T).
5. BSVN cl(S$\cup$T)= $\cap${K / K is a BSVNGCs in X and S$\cup$T$\subseteq$K},
   ($\cap${K / K is a BSVNGCs in X and S$\subseteq$K})$\cup$($\cap${K / K is a BSVNGCs in X and T$\subseteq$K}),
   = BSVN cl(S) $\cup$ BSVN cl(T). Thus BSVN cl(S$\cup$T)= BSVN cl(S) $\cup$ BSVN cl(T).

6. BSVN int($S \cap T$)= $\cup$ {G / G is a BSVNGOs in X and G $\subseteq$ $S \cap T$ },
   ($\cup$ {G / G is a BSVNGOs in X and G $\subseteq$ S}) $\cap$ ($\cup$ {G / G is a BSVNGOs in X and G $\subseteq$ T}),
   = BSVN int(S) $\cap$ BSVN int(S). Thus BSVN int($S \cap T$)= BSVN int(S) $\cap$ BSVN int(S).
7. (BSVN cl(S)) = $\cap$ {K / K is a BSVNGCs in X and S $\subseteq$ K},
   $(\text{BSVN cl}(S))^c$ = $\cap$ {K^c / K^c is a BSVNGCs in X and $S^c \subseteq K^c$},
   = BSVN int(S^c). Thus $(\text{BSVN cl}(S))^c$ = BSVN int(S^c).
8. BSVNG int (S) = $\cup$ {G / G is a BSVNGOs in X and G $\subseteq$ S},
9. $(\text{BSVNG int }(S))^c$ = $\cup$ {G / G is a BSVNGOs in X and $G^c \subseteq S^c$}= BSVN int(S^c)
   Thus $(\text{BSVN cl}(S))^c$ = BSVN int(S^c).

3.6 Definition: Let (X, τ) be a BSVNTS and S be a BSVNs in X. The bipolar single-valued neutrosophic pre interior of S and denoted by BSVN pint (S) and bipolar single-valued neutrosophic pre-closure of S is denoted by BSVN pcl (S).

(1) BSVN pint (S) = $\cup$ {G / G is a BSVNPOs in X and G $\subseteq$ S}
(2) BSVN pcl (S) = $\cap$ {K / K is a BSVNPCs in X and S $\subseteq$ K}

3.7 Result 3.21: Let A be BSVNs of a BSVNTS (X, τ), then
(1). BSVN pcl (S) = S $\bigcup$ BSVN cl (BSVN int (S)),
(2). BSVN pint (S) = S $\bigcap$ BSVN int (BSVN cl (S)).

3.8 Definition: Let S be BSVNs of a BSVNTS (X, τ). Then the bipolar single-valued neutrosophic semi interior of S (BSVN sint (S)) and bipolar single-valued neutrosophic semi closure of S (BSVN scl (S)) are defined by

(1) BSVN sint (S) = $\cup$ {G / G is a BSVNSOs in X and G $\subseteq$ S}
(2) BSVN scl (S) = $\cap$ {K / K is a BSVNSCs in X and S $\subseteq$ K}

3.9 Result: Let S be BSVNs of a BSVNTS (X, τ), then
(1) BSVN scl (S) = S $\bigcup$ BSVN int (BSVN cl (S)),
(2). BSVN sint (S) = S $\bigcap$ BSVN cl (BSVN int (S)).

3.10 Definition: Let S be BSVNs of a BSVNTS (X, τ). Then the bipolar single-valued neutrosophic alpha interior of S (BSVN αint (S) and bipolar single-valued neutrosophic alpha closure of S (BSVN αcl (S)) is defined by

(1) BSVN αint (S) = $\cup$ {G / G is a BSVNαOs in X and G $\subseteq$ S}
(2) BSVN αcl (S) = $\cap$ {K / K is a BSVNαCs in X and S $\subseteq$ K}

3.11 Result: Let S be BSVNs of a BSVNTS (X, τ), then
(1) BSVN αcl (S) = S $\bigcup$ BSVN cl (BSVN int (BSVN cl (S))),
(2) BSVN αint (S) = S $\bigcap$ BSVN int (BSVN cl (BSVN int (S))).

3.12 Definition: Let A be BSVNs of a BSVNTS (X, τ). Then the bipolar single-valued neutrosophic semi-pre interior of S (BSVN spint (S)) and bipolar single-valued neutrosophic semi-pre closure of S (BSVN spcl (S)) are defined by

(1) BSVN spint (S) = $\cup$ {G / G is a BSVNSPOs in X and G $\subseteq$ S}

(2) BSVN spcl (S) = $\cap$ {K / K is a BSVNSPCs in X and S $\subseteq$ K}

3.13 Definition: A BSVNs S of a BSVNTS (X, τ) is said to be bipolar single-valued neutrosophic generalized semi closed set (BSVNGSCS) if BSVN scl (S) $\subseteq$ U whenever S $\subseteq$ U and U is BSVNOs in X.

3.14 Definition: A BSVNs S of a BSVNTS (X, τ) is said to be bipolar single-valued neutrosophic alpha generalized closed set (BSVN αGCS) if BSVN αcl (S) $\subseteq$ U whenever S $\subseteq$ U and U is BSVNOs in X.

3.15 Definition: A BSVNs S of a BSVNTS (X, τ) is said to be bipolar single-valued neutrosophic generalized semi-pre closed set (BSVNGSPCs) if BSVN spcl (S) $\subseteq$ U whenever S $\subseteq$ U and U is BSVNOs in X.

3.16 Definition: Let {A_i : i $\in$ J} be an arbitrary family of BSVNs in X. Then

(1). $\bigcap S_i$= {<x, min ($T^+_{S_i}$ (x)), max ($I^+_{S_i}$ (x)), max ($F^+_{S_i}$ (x)),

min ($T^-_{S_i}$ (x)), max ($I^-_{S_i}$ (x)), max ($F^-_{S_i}$ (x))>}

(2). $\bigcup A_i$= {<x, max ($T^+_{S_i}$ (x)), min ($I^+_{S_i}$ (x)), min ($F^+_{S_i}$ (x),

max ($T^-_{S_i}$ (x)),min ($I^-_{S_i}$ (x)), min ($F^-_{S_i}$ (x))>}

3.18 Corollary: Let S, T, M and N be bipolar single-valued neutrosophic set in X. Then

(1) S $\subseteq$ T and M $\subseteq$ N $\Rightarrow$ S $\bigcup$ M $\subseteq$ T $\bigcup$ N and S $\bigcap$ M $\subseteq$ T $\bigcap$ N

(2) S $\subseteq$ T and S $\subseteq$ M $\Rightarrow$ S $\subseteq$ T $\bigcap$ M

(3) S $\subseteq$ M and T $\subseteq$ M $\Rightarrow$ S $\bigcup$ T $\subseteq$ M

(4) S $\subseteq$ T and T $\subseteq$ M $\Rightarrow$ S $\subseteq$ M

(5) $(S \bigcup T)^c = S^c \bigcap T^c$

(6) $(S \bigcap T)^c = S^c \bigcup T^c$

(7) S $\subseteq$ T $\Rightarrow$ $T^c \subseteq S^c$

(8) $(S^c)^c = S$

(9) $0^C_{BSVN} = 1_{BSVN}$

(10) $1^C_{BSVN} = 0_{BSVN}$

Proof: The proof is obvious.

3.19 Theorem: Every bipolar single-valued neutrosophic closed set is bipolar single-valued neutrosophic generalized closed set.

Proof. Let S be BSVNCs in X. Suppose U is BSVNOs in X, such that S $\subseteq$ U. Then BSVN cl (S) = S $\subseteq$ U. Hence S is BSVNGCs in X.

3.20 Remark: The converse of the above theorem is not true which is shown in the example.

3.21 Example: Let X ={p, q} and

S= $\begin{Bmatrix} < p,(0.3,0.5,0.1,-0.2,-0.4,-0.3) > \\ < q,(0.2,0.8,0.2,-0.4,-0.6,-0.9) > \end{Bmatrix}$ T= $\begin{Bmatrix} < p,(0.4,0.4,0.1,-0.1,-0.5,-0.4) > \\ < q,(0.3,0.7,0.1,-0.3,-0.6,-0.9) > \end{Bmatrix}$

Then τ= {0_{BSVN}, 1_{BSVN}, S, T} is a BSVNT on X. The BSVNs

R= $\begin{Bmatrix} < p,(0.2,0.3,0.7,-0.8,-0.1,-0.3) > \\ < q,(0.3,0.8,0.3,-0.4,-0.1,-0.4) > \end{Bmatrix}$ is BSVNGCs in X but not BSVNCs in X.

3.22 Remark: The Intersection of two BSVNGCs is need not be true. Shown in the following example.

3.23 Example: Let X ={p, q} and

S= $\begin{Bmatrix} < p,(0.3,0.5,0.1,-0.2,-0.4,-0.3) > \\ < q,(0.2,0.8,0.2,-0.4,-0.6,-0.9) > \end{Bmatrix}$ T= $\begin{Bmatrix} < p,(0.4,0.4,0.1,-0.1,-0.5,-0.4) > \\ < q,(0.3,0.7,0.1,-0.3,-0.6,-0.9) > \end{Bmatrix}$

Then τ= {0_{BSVN}, 1_{BSVN}, S, T} is a BSVNT on X. The BSVNs

R= $\begin{Bmatrix} < p,(0.2,0.3,0.7,-0.8,-0.1,-0.3) > \\ < q,(0.3,0.8,0.3,-0.4,-0.1,-0.4) > \end{Bmatrix}$ V= $\begin{Bmatrix} < p,(0.6,0.6,0.9,-0.9,-0.5,-0.6) > \\ < q,(0.7,0.3,0.9,-0.7,-0.1,-0.1) > \end{Bmatrix}$ are BSVNGCs in X but R∩V is not BSVNGCs in X**.**

3.24 Proposition: Let (X, τ) be BSVNTS. If S is a bipolar single-valued neutrosophic generalized closed set and S $\subseteq$ T $\subseteq$ BSVN cl (S) then T is bipolar single-valued neutrosophic generalized closed set.

Proof: Let G be a bipolar single-valued neutrosophic open set in (X, τ), such that T $\subseteq$ G. Since S $\subseteq$ T, S $\subseteq$ G. Now S is a bipolar single-valued neutrosophic generalized closed set and BSVN cl (S) $\subseteq$ G. But BSVN cl (T) $\subseteq$ BSVN cl (S). Since BSVN cl (T) $\subseteq$ BSVN cl (S) $\subseteq$ G. BSVN cl (T) $\subseteq$ G.

Hence T is a bipolar single-valued neutrosophic generalized closed set.

3.25 Proposition: Let (X, τ) be BSVNTS and a BSVNs S is a bipolar single-valued neutrosophic generalized open if and only if T $\subseteq$BSVN int (S) whenever T is bipolar single-valued neutrosophic closed set and T$\subseteq$S.

Proof: Let S is a bipolar single-valued neutrosophic generalized open set and T be a bipolar single-valued neutrosophic closed set, such that T $\subseteq$S. Now T $\subseteq$ S$\Rightarrow$$S^c$ $\subseteq$$T^c$ and S^c is a bipolar single-valued neutrosophic generalized closed set implies that BSVN cl (S^c) $\subseteq$$T^c$. (i.e) T=$(T^c)^c$$\subseteq$(BSVN cl($S^c$))^c. But (BSVN cl ($S^c$))^c = BSVN int (S). Thus T$\subseteq$BSVN int (S).

Conversely, suppose that S be a bipolar single-valued neutrosophic set, such that T$\subseteq$BSVN int(S) whenever T is bipolar single-valued neutrosophic closed and T$\subseteq$S. Let $S^c$$\subseteq$T whenever T is bipolar single-valued neutrosophic open. Now $S^c$$\subseteq$T $\Rightarrow$$T^c$ $\subseteq$S. Hence by the assumption, T^c $\subseteq$ BSVN int (S). (i.e) (BSVN int (S))^c $\subseteq$ T. But (BSVN int (S))^c = BSVN cl (S^c). Hence (BSVN int (S))^c $\subseteq$BSVN cl (S^c) . (i.e) S^c is bipolar single-valued neutrosophic generalized closed set. Therefore, S is bipolar single-valued neutrosophic generalized open set. Hence proved.

3.26 Proposition: If BSVN int (S) $\subseteq$T $\subseteq$S and if S is bipolar single-valued neutrosophic generalized open set then T is also bipolar single-valued neutrosophic generalized open set.

Proof: Now S^c $\subseteq$ $T^c$$\subseteq$ (BSVN int (S))^c= BSVN cl (S^c). As S is a bipolar single-valued neutrosophic generalized open, S^c is bipolar single-valued neutrosophic generalized closed set. Then by the proposition 3.24, T is bipolar single-valued neutrosophic generalized open set. Hence Proved.

4. Bipolar Single-Valued Neutrosophic Generalized Pre-Closed Set

4.1 Definition: A BSVNs S is said to be bipolar single-valued neutrosophic generalized pre-closed set (BSVNGPCs) in (X, τ) if BSVN pcl (S) $\subseteq$ U whenever S $\subseteq$ U and U is BSVNOs in X. The family of all BSVNGPCs's of a BSVNTS (X, τ) is denoted by BSVNGPC (X).

4.2 Example: Let X ={p, q} and

S=$\begin{Bmatrix} < p,(0.1,-0.7),(0.3,-0.8),(0.5,-0.1) > \\ < q,(0.2,0.4,0.6,-0.8,-0.2,-0.4) > \end{Bmatrix}$ T=$\begin{Bmatrix} < p,(0.1,0.2,0.3,-0.7,-0.9,-0.9) > \\ < q,(0.4,0.3,0.6,-0.1,-0.3,-0.5) > \end{Bmatrix}$

Then τ= {0_{BSVN}, 1_{BSVN}, S, T} is a BSVNT on X. The BSVNs

R=$\begin{Bmatrix} < p,(0.8,0.7,0.8,-0.7,-0.2,-0.3) > \\ < q,(0.1,0.7,0.9,-0.2,-0.7,-0.2) > \end{Bmatrix}$ is BSVNGPCs in X.

4.3 Theorem:

(1) Every bipolar single-valued neutrosophic closed set is bipolar single-valued neutrosophic generalized pre-closed set.

(2) Every bipolar single-valued neutrosophic generalized closed set is bipolar single-valued neutrosophic generalized pre-closed set.

(3) Every bipolar single-valued neutrosophicα closed set is bipolar single-valued neutrosophic generalized pre-closed set.
(4) Every bipolar single-valued neutrosophic regular closed set is bipolar single-valued neutrosophic generalized pre-closed set.
(5) Every bipolar single-valued neutrosophic pre-closed set is bipolar single-valued neutrosophic generalized pre-closed set.
(6) Every bipolar single-valued neutrosophicα generalized closed set is bipolar single-valued neutrosophic generalized pre-closed set.
(7) Every bipolar single-valued neutrosophic generalized pre-closed set is bipolar single-valued neutrosophic semi-pre closed set.
(8) Every bipolar single-valued neutrosophic generalized pre-closed set is bipolar single-valued neutrosophic generalized semi-pre closed set.

Proof. (1) Let S be BSVNCs in X and let S $\subseteq$U and U be BSVNOs in X. Since BSVN pcl (S) $\subseteq$ BSVN cl (S) and S is BSVNCs in X, BSVN pcl (S) $\subseteq$ BSVN cl (S) = S $\subseteq$ U. Therefore S is BSVNGPCs in X.

(2) Let S be BSVNGCs in X and let S $\subseteq$ U and U is BSVNOs in (X, τ). Since
BSVN pcl (S) $\subseteq$ BSVN cl (S) and by hypothesis, BSVN pcl (S) $\subseteq$ U. Therefore S is BSVNGPCs in X.

(3) Let S be BSVN αCS in X and let S $\subseteq$ U and U be BSVNOs in X. By hypothesis,
BSVN cl (BSVN int (BSVN cl (S))) $\subseteq$ S. Since S$\subseteq$ BSVN cl (S);
BSVN cl (BSVN int (S)) $\subseteq$ BSVN cl (BSVN int (BSVN cl (S))) $\subseteq$ S. Hence BSVN pcl (S) $\subseteq$ S $\subseteq$ U. Therefore S is BSVNGPCs in X.

(4) Let S be a BSVNRCs in X. By Definition S = BSVN cl (BSVN int (S)). This implies
BSVN cl (S) = BSVN cl (BSVN int (S)). Therefore BSVN cl (S) = S. (i.e) S is BSVNCs in X. S is BSVNGPCs in X.

(5) Let S be BSVNPCs in X and let S $\subseteq$ U and U is BSVNOs in X. By Definition, BSVN cl (BSVN int (S)) $\subseteq$ S. This implies BSVN pcl (S) = S $\bigcup$ BSVN cl (BSVN int (S)) $\subseteq$ S. Therefore
BSVN pcl (S) $\subseteq$ U. Hence S is BSVNGPCs in X.

(6) Let S be BSVN αGCs in X and let S $\subseteq$ U and U is BSVNOs in (X, τ). By Result 3.11,
S$\bigcup$ BSVN cl (BSVN int (BSVN cl (S))) $\subseteq$U. This implies BSVN cl (BSVN int (BSVN cl (S))) $\subseteq$ U and BSVN cl (BSVN int (S)) $\subseteq$ U. Thus BSVN pcl (S) = S$\bigcup$ BSVN cl (BSVN int (S)) $\subseteq$U. Hence S is BSVNGPCs in X.
(7) Let S be BSVNGPCs in X , this implies BSVN pcl (S) $\subseteq$ U whenever S $\subseteq$ U and U is BSVNOs in X. By hypothesis BSVN cl (BSVN int (S)) $\subseteq$ S. Therefore
BSVN int (BSVN cl (BSVN int (S))) $\subseteq$ BSVN int (S) $\subseteq$ S. Therefore
BSVN int (BSVN cl (BSVN int (S))) $\subseteq$S. Hence S is BSVNSPCs in X.

(8) Let s be BSVNGPCsin X and let S $\subseteq$ U and U is BSVNOs in X. By hypothesis
BSVN cl (BSVN int (S)) $\subseteq$ S $\subseteq$ U. Therefore
BSVN int (BSVN cl (BSVN int (S))) $\subseteq$ BSVN int (S) $\subseteq$ U. This implies BSVN spcl (S) $\subseteq$ U whenever S $\subseteq$ U and U is BSVNOs in X. Therefore S is BSVNGSPCs in X.

4.4 Remark: The converse of the above theorem 4.3 (1-8) is not true which is shown in the example.

4.5 Example:

(1) Let X ={p, q} and

$$S=\begin{Bmatrix} < p,(0.1,0.3,0.5,-0.7,-0.8,-0.1) > \\ < q,(0.2,0.4,0.6,-0.8,-0.2,-0.4) > \end{Bmatrix} \quad T=\begin{Bmatrix} < p,(0.1,0.2,0.3,-0.7,-0.9,-0.9) > \\ < q,(0.4,0.3,0.6,-0.1,-0.3,-0.5) > \end{Bmatrix}$$

Then τ= {0_{BSVN}, 1_{BSVN}, S, T} is a BSVNT on X. The BSVNs

$$R=\begin{Bmatrix} < p,(0.8,0.7,0.8,-0.7,-0.2,-0.3) > \\ < q,(0.1,0.7,0.9,-0.2,-0.7,-0.2) > \end{Bmatrix}$$ is BSVNGPCs in X but not BSVNCs in X.

(2) Let X ={p, q} and

$$S=\begin{Bmatrix} < p,(0.1,0.3,0.2,-0.3,-0.4,-0.6) > \\ < q,(0.2,0.4,0.5,-0.1,-0.1,-0.3) > \end{Bmatrix} \quad T=\begin{Bmatrix} < p,(0.1,0.3,0.4,-0.4,-0.1,-0.4) > \\ < q,(0.2,0.5,0.6,-0.3,-0.1,-0.1) > \end{Bmatrix}$$

Then τ= {0_{BSVN}, 1_{BSVN}, S, T} is a BSVNT on X. The BSVNs

$$R=\begin{Bmatrix} < p,(0.1,0.5,0.4,-0.5,-0.3,-0.1) > \\ < q,(0.1,0.6,0.5,-0.2,-0.1,-0.3) > \end{Bmatrix}$$ is BSVNGPCs in X but not BSVNGCs in X.

(3) Let X ={p, q} and

$$S=\begin{Bmatrix} < p,(0.5,0.4,0.1,-0.6,-0.5,-0.4) > \\ < q,(0.5,0.1,0.1,-0.3,-0.1,-0.2) > \end{Bmatrix} \quad T=\begin{Bmatrix} < p,(0.4,0.5,0.3,-0.6,-0.3,-0.1) > \\ < q,(0.2,0.3,0.6,-0.4,-0.2,-0.1) > \end{Bmatrix}$$

Then τ= {0_{BSVN}, 1_{BSVN}, S, T} is a BSVNT on X. The BSVNs

$$R=\begin{Bmatrix} < p,(0.1,0.5,0.5,-0.7,-0.3,-0.4) > \\ < q,(0.5,0.9,0.2,-0.3,-0.2,-0.1) > \end{Bmatrix}$$ is BSVNGPCs in X but not BSVNαCs in X.

(4) Let X ={p, q} and

$$S=\begin{Bmatrix} < p,(0.1,0.3,0.2,-0.3,-0.4,-0.6) > \\ < q,(0.2,0.4,0.5,-0.1,-0.1,-0.3) > \end{Bmatrix} \quad T=\begin{Bmatrix} < p,(0.1,0.3,0.4,-0.4,-0.1,-0.4) > \\ < q,(0.2,0.5,0.6,-0.3,-0.1,-0.1) > \end{Bmatrix}$$

Then τ= {0_{BSVN}, 1_{BSVN}, S, T} is a BSVNT on X. The BSVNs

R=$\begin{Bmatrix} < p,(0.1,0.5,0.4,-0.5,-0.3,-0.1) > \\ < q,(0.1,0.6,0.5,-0.2,-0.1,-0.3) > \end{Bmatrix}$ is BSVNGPCs in X but not BSVNRCs in X.

(5) Let X ={p, q} and

S=$\begin{Bmatrix} < p,(0.5,0.4,0.3,-0.6,-0.4,-0.2) > \\ < q,(0.2,0.5,0.1,-0.5,-0.3,-0.1) > \end{Bmatrix}$ T=$\begin{Bmatrix} < p,(0.6,0.2,0.1,-0.5,-0.6,-0.8) > \\ < q,(0.3,0.1,0.1,-0.4,-0.4,-0.3) > \end{Bmatrix}$

Then τ= {0_{BSVN}, 1_{BSVN}, S, T} is a BSVNT on X. The BSVNs

R=$\begin{Bmatrix} < p,(0.5,0.3,0.2,-0.1,-0.7,-0.3) > \\ < q,(0.2,0.4,0.1,-0.3,-0.4,-0.1) > \end{Bmatrix}$ is BSVNGPCs in X but not BSVNPCs in X.

(6) Let X ={p, q} and

S=$\begin{Bmatrix} < p,(0.1,0.3,0.6,-0.2,-0.4,-0.5) > \\ < q,(0.2,0.4,0.5,-0.1,-0.9,-0.5) > \end{Bmatrix}$ T=$\begin{Bmatrix} < p,(0.1,0.3,0.6,-0.7,-0.3,-0.2) > \\ < q,(0.2,0.6,0.7,-0.8,-0.4,-0.1) > \end{Bmatrix}$

Then τ= {0_{BSVN}, 1_{BSVN}, S, T} is a BSVNT on X. The BSVNs

R=$\begin{Bmatrix} < p,(0.1,0.4,0.7,-0.8,-0.2,-0.1) > \\ < q,(0.2,0.7,0.7,-0.9,-0.1,-0.1) > \end{Bmatrix}$ is BSVNGPCs in X but not BSVN αGCs in X.

(7) Let X ={p, q} and

S=$\begin{Bmatrix} < p,(0.1,0.5,0.5,-0.5,-0.1,-0.4) > \\ < q,(0.3,0.5,0.6,-0.7,-0.1,-0.2) > \end{Bmatrix}$ T=$\begin{Bmatrix} < p,(0.3,0.2,0.3,-0.2,-0.3,-0.5) > \\ < q,(0.4,0.3,0.1,-0.1,-0.4,-0.5) > \end{Bmatrix}$

Then τ= {0_{BSVN}, 1_{BSVN}, S, T} is a BSVNT on X. The BSVNs

R=$\begin{Bmatrix} < p,(0.1,0.3,0.5,-0.4,-0.1,-0.5) > \\ < q,(0.4,0.3,0.1,-0.1,-0.2,-0.3) > \end{Bmatrix}$ is BSVNSPCs in X but not BSVNGPCs in X.

(8) Let X ={p, q} and

S=$\begin{Bmatrix} < p,(0.4,0.7,0.4,-0.5,-0.4,-0.2) > \\ < q,(0.3,0.2,0.4,-0.3,-0.1,-0.1) > \end{Bmatrix}$ T=$\begin{Bmatrix} < p,(0.3,0.8,0.8,-0.7,-0.3,-0.1) > \\ < q,(0.2,0.3,0.7,-0.4,-0.1,-0.1) > \end{Bmatrix}$

Then τ= {0_{BSVN}, 1_{BSVN}, S, T} is a BSVNT on X. The BSVNs

R= $\left\{\begin{matrix} < p,(0.3,0.8,0.5,-0.6,-0.3,-0.2) > \\ < q,(0.2,0.3,0.7,-0.3,-0.1,-0.1) > \end{matrix}\right\}$ is BSVNGSPCs in X but not BSVNGPCs in X.

4.6 Proposition: BSVNSCs and BSVNGPCs are independent to each other which are shown in the example.

4.7 Example: Let X ={p, q} and

S= $\left\{\begin{matrix} < p,(0.5,0.4,0.2,-0.1,-0.2,-0.7) > \\ < q,(0.7,0.6,0.3,-0.6,-0.1,-0.5) > \end{matrix}\right\}$ T= $\left\{\begin{matrix} < p,(0.8,0.3,0.1,-0.1,-0.3,-0.8) > \\ < q,(0.8,0.2,0.3,-0.4,-0.5,-0.6) > \end{matrix}\right\}$

Then τ= {0_{BSVN}, 1_{BSVN}, S, T} is a BSVNT on X. The BSVNs

R= $\left\{\begin{matrix} < p,(0.2,0.5,0.3,-0.1,-0.1,-0.7) > \\ < q,(0.6,0.7,0.4,-0.7,-0.1,-0.2) > \end{matrix}\right\}$ is BSVNGPCs in X but not BSVNSCs in X.

4.8 Example: Let X ={p, q} and

S= $\left\{\begin{matrix} < p,(0.1,0.7,0.6,-0.8,-0.2,-0.5) > \\ < q,(0.3,0.7,0.7,-0.8,-0.2,-0.2) > \end{matrix}\right\}$ T= $\left\{\begin{matrix} < p,(0.9,0.5,0.5,-0.3,-0.4,-0.7) > \\ < q,(0.5,0.5,0.3,-0.2,-0.7,-0.8) > \end{matrix}\right\}$

Then τ= {0_{BSVN}, 1_{BSVN}, S, T} is a BSVNT on X. The BSVNs

R= $\left\{\begin{matrix} < p,(0.2,0.6,0.5,-0.7,-0.3,-0.5) > \\ < q,(0.3,0.7,0.7,-0.8,-0.2,-0.2) > \end{matrix}\right\}$ is BSVNSCs in X but not BSVNGPCs in X.

4.9 Proposition: BSVNGSCs and BSVNGPCs are independent to each other which are shown in the example.

4.10 Example: Let X ={p, q} and

S= $\left\{\begin{matrix} < p,(0.9,0.5,0.6,-0.3,-0.8,-0.5) > \\ < q,(0.9,0.1,0.3,-0.2,-0.6,-0.6) > \end{matrix}\right\}$ T= $\left\{\begin{matrix} < p,(0.8,0.7,0.7,-0.4,-0.8,-0.5) > \\ < q,(0.8,0.8,0.7,-0.3,-0.5,-0.4) > \end{matrix}\right\}$

Then τ= {0_{BSVN}, 1_{BSVN}, S, T} is a BSVNT on X. The BSVNs

R= $\left\{\begin{matrix} < p,(0.9,0.8,0.7,-0.6,-0.5,-0.4) > \\ < q,(0.3,0.2,0.3,-0.2,-0.3,-0.4) > \end{matrix}\right\}$ is BSVNGPCs in X but not BSVNGSCs in X.

4.11 Example: Let X ={p, q} and

$$S=\left\{\begin{matrix}< p,(0.1,0.6,0.9,-0.9,-0.1,-0.1) > \\ < q,(0.2,0.8,0.9,-0.8,-0.3,-0.2) >\end{matrix}\right\} \quad T=\left\{\begin{matrix}< p,(0.2,0.5,0.8,-0.8,-0.1,-0.2) > \\ < q,(0.4,0.7,0.8,-0.8,-0.3,-0.4) >\end{matrix}\right\}$$

Then τ= {0_{BSVN}, 1_{BSVN}, S, T} is a BSVNT on X. The BSVNs

$$R=\left\{\begin{matrix}< p,(0.2,0.5,0.8,-0.8,-0.1,-0.2) > \\ < q,(0.4,0.7,0.8,-0.8,-0.3,-0.4) >\end{matrix}\right\}$$ is BSVNGSCs in X but not BSVNGPCs in X.

Figure 1: The Diagram represents the implication of the above theorem 4.3.

1. BSVNGPCs 2.BSVNCs 3.BSVNGCs 4.BSVNαCs 5.BSVNαGCs 6.BSVNRCs 7.BSVNPCs 8.BSVNSPCs 9.BSVNGSPCs 10.BSVNSCs 11.BSVNGSCs.

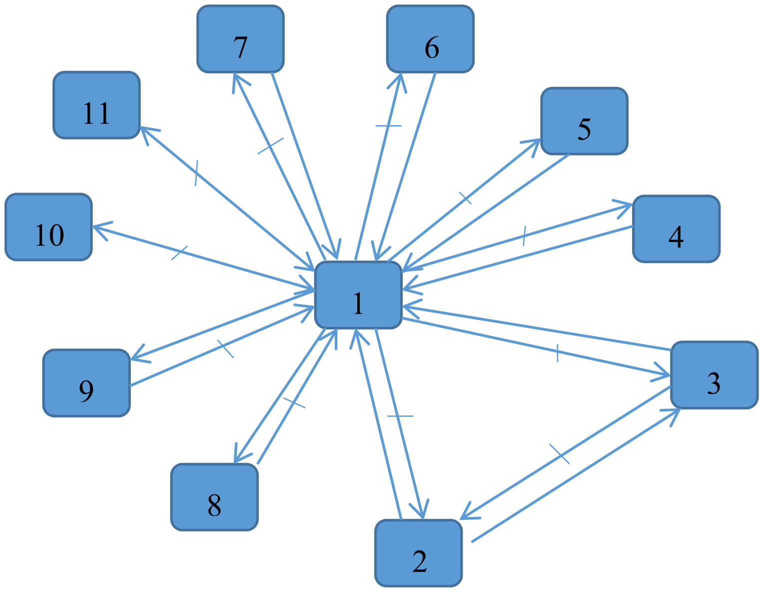

Figure 1

4.12 Remark: The union of any two BSVNGPCs's is not BSVNGPCs in general as seen in the following example.

4.13 Example: Let X ={p, q} and

$$S=\left\{\begin{matrix}< p,(0.1,0.5,0.5,-0.7,-0.3,-0.4) > \\ < q,(0.5,0.9,0.8,-0.3,-0.2,-0.1) >\end{matrix}\right\} \quad T=\left\{\begin{matrix}< p,(0.4,0.5,0.4,-0.5,-0.3,-0.6) > \\ < q,(0.7,0.6,0.5,-0.2,-0.4,-0.3) >\end{matrix}\right\}$$

Then τ= {0_{BSVN}, 1_{BSVN}, S, T} is a BSVNT on X. The BSVNs

R= $\left\{\begin{matrix} < p,(0.2,0.5,0.4,-0.6,-0.2,-0.6) > \\ < q,(0.7,0.9,0.8,-0.9,-0.2,-0.2) > \end{matrix}\right\}$, V= $\left\{\begin{matrix} < p,(0.1,0.6,0.7,-0.8,-0.3,-0.3) > \\ < q,(0.4,0.9,0.9,-0.3,-0.2,-0.1) > \end{matrix}\right\}$ are BSVNGPCs in X but R $\cup$ V is not BSVNGPCs in X.

5. Bipolar Single-Valued Neutrosophic generalized Pre-Open Set

5.1 Definition: A BSVNs S is said to be bipolar single-valued neutrosophic generalized pre-open set (BSVNGPOs) in (X, τ) if the complement S^c is BSVNGPCs in (X, τ). The family of all BSVNGPOs's of BSVNTS (X, τ) is denoted by BSVNGPO (X).

5.2 Example: Let X ={p, q} and

S= $\left\{\begin{matrix} < p,(0.1,0.5,0.8,-0.9,-0.4,-0.2) > \\ < q,(0.2,0.6,0.7,-0.9,-0.3,-0.4) > \end{matrix}\right\}$ T= $\left\{\begin{matrix} < p,(0.3,0.3,0.8,-0.2,-0.5,-0.3) > \\ < q,(0.4,0.5,0.5,-0.1,-0.4,-0.4) > \end{matrix}\right\}$

Then τ= {0_{BSVN}, 1_{BSVN}, S, T} is a BSVNT on X. The BSVNs

R= $\left\{\begin{matrix} < p,(0.7,0.5,0.3,-0.2,-0.5,-0.7) > \\ < q,(0.7,0.4,0.3,-0.1,-0.7,-0.6) > \end{matrix}\right\}$ is BSVNGPOs in X.

5.3 Theorem: For any BSVNTS (X, τ), we have the following results.
(1). Every BSVNOs is BSVNGPOs.
(2). Every BSVNROs is BSVNGPOs.
(3). Every BSVN αOs is BSVNGPOs.
(4). Every BSVNPOs is BSVNGPOs.

5.4 Remark: The converse of the above theorem need not be true which can be seen from the following examples.

5.5 Example: Let X ={p, q} and

S= $\left\{\begin{matrix} < p,(0.1,0.5,0.8,-0.9,-0.4,-0.2) > \\ < q,(0.2,0.6,0.7,-0.9,-0.3,-0.4) > \end{matrix}\right\}$ T= $\left\{\begin{matrix} < p,(0.3,0.3,0.8,-0.2,-0.5,-0.3) > \\ < q,(0.4,0.5,0.5,-0.1,-0.4,-0.4) > \end{matrix}\right\}$

Then τ= {0_{BSVN}, 1_{BSVN}, S, T} is a BSVNT on X. The BSVNs

R= $\left\{\begin{matrix} < p,(0.7,0.5,0.3,-0.2,-0.5,-0.7) > \\ < q,(0.7,0.4,0.3,-0.1,-0.7,-0.6) > \end{matrix}\right\}$ is BSVNGPOs in X but not BSVNOs in X.

5.6 Example: Let X ={p, q} and

S= $\left\{\begin{matrix} < p,(0.5,0.4,0.5,-0.7,-0.2,-0.3) > \\ < q,(0.2,0.4,0.5,-0.3,-0.1,-0.4) > \end{matrix}\right\}$ T= $\left\{\begin{matrix} < p,(0.5,0.3,0.4,-0.5,-0.3,-0.5) > \\ < q,(0.5,0.3,0.4,-0.2,-0.1,-0.5) > \end{matrix}\right\}$

Then τ= {0_{BSVN}, 1_{BSVN}, S, T} is a BSVNT on X. The BSVNs

R= $\left\{\begin{matrix} < p,(0.5,0.5,0.5,-0.3,-0.9,-0.4) > \\ < q,(0.8,0.5,0.4,-0.7,-0.9,-0.5) > \end{matrix}\right\}$ is BSVNGPOs in X but not BSVNROs in X.

5.7 Example: Let X ={p, q} and

S= $\left\{\begin{matrix} < p,(0.5,0.4,0.5,-0.7,-0.2,-0.3) > \\ < q,(0.2,0.4,0.5,-0.3,-0.1,-0.4) > \end{matrix}\right\}$ T= $\left\{\begin{matrix} < p,(0.5,0.3,0.4,-0.5,-0.3,-0.5) > \\ < q,(0.5,0.3,0.4,-0.2,-0.1,-0.5) > \end{matrix}\right\}$

Then τ= {0_{BSVN}, 1_{BSVN}, S, T} is a BSVNT on X. The BSVNs

R= $\left\{\begin{matrix} < p,(0.5,0.5,0.5,-0.3,-0.9,-0.4) > \\ < q,(0.8,0.5,0.4,-0.7,-0.9,-0.5) > \end{matrix}\right\}$ is BSVNGPOs in X but not BSVNαOs in X.

5.8 Example: Let X ={p, q} and

S= $\left\{\begin{matrix} < p,(0.7,0.6,0.5,-0.8,-0.9,-0.7) > \\ < q,(0.4,0.6,0.7,-0.8,-0.9,-0.8) > \end{matrix}\right\}$ T= $\left\{\begin{matrix} < p,(0.7,0.8,0.6,-0.9,-0.9,-0.6) > \\ < q,(0.3,0.7,0.8,-0.9,-0.9,-0.7) > \end{matrix}\right\}$

Then τ= {0_{BSVN}, 1_{BSVN}, S, T} is a BSVNT on X. The BSVNs

R= $\left\{\begin{matrix} < p,(0.3,0.3,0.5,-0.3,-0.1,-0.3) > \\ < q,(0.5,0.5,0.3,-0.2,-0.1,-0.1) > \end{matrix}\right\}$ is BSVNGPOs in X but not BSVNPOs in X.

5.9 Remark: The intersection of any two BSVNGPOs's is not BSVNGPOs in general and it is shown in the following example.

5.10 Example: Let X ={p, q} and

S= $\left\{\begin{matrix} < p,(0.8,0.4,0.3,-0.1,-0.3,-0.5) > \\ < q,(0.5,0.4,0.3,-0.8,-0.5,-0.6) > \end{matrix}\right\}$ T= $\left\{\begin{matrix} < p,(0.4,0.6,0.7,-0.9,-0.2,-0.4) > \\ < q,(0.4,0.5,0.4,-0.9,-0.4,-0.5) > \end{matrix}\right\}$

Then τ= {0_{BSVN}, 1_{BSVN}, S, T} is a BSVNT on X. The BSVNs

R= $\left\{\begin{matrix} < p,(0.7,0.3,0.3,-0.1,-0.2,-0.8) > \\ < q,(0.7,0.4,0.3,-0.1,-0.7,-0.7) > \end{matrix}\right\}$, V= $\left\{\begin{matrix} < p,(0.6,0.4,0.1,-0.1,-0.2,-0.9) > \\ < q,(0.9,0.2,0.6,-0.1,-0.6,-0.7) > \end{matrix}\right\}$ are BSVNGPOs in X but R∩V is not BSVNGPOs in X.

5.11 Theorem: Let (X, τ) be BSVNTS. If S ∈ BSVNGPO(X) then V ⊆ BSVN int (BSVN cl (S)) whenever V ⊆ S and V is BSVNCs in X.

Proof. Let S ∈ BSVNGPO(X). Then S^c is BSVNGPCs in X. Therefore BSVN pcl (S^c) ⊆ U whenever S^c ⊆ U and U is BSVNOS in X. That is BSVN cl (BSVN int (S^c)) ⊆ U. This implies U^c ⊆ BSVN int (BSVN cl (S)) whenever U^c ⊆ S and U^c is BSVNCs in X. Replacing U^c by V , we get V ⊆ BSVN int (BSVN cl (S)) whenever V ⊆ S and V is BSVNCs in X.

5.12 Theorem: Let (X, τ) be BSVNTS. Then for every S ∈ BSVNGPO(X) and for every T ∈ BSVNs(X), BSVN pint (S) ⊆ T ⊆ S implies T ⊆ BSVNGPO(X).

Proof. By hypothesis S^c ⊆ T^c ⊆ (BSVN pint (S))^c. Let T^c ⊆ U and U be BSVNOs. Since S^c ⊆ T^c, S^c ⊆ U. But S^c is BSVNGPCs, BSVN pcl (S^c) ⊆ U. Also T^c ⊆ (BSVN pint (S))^c = BSVN pcl (S^c). Therefore BSVN pcl (T^c) ⊆ BSVN pcl (S^c) ⊆ U. Hence T^c is BSVNGPCs. Which implies T is BSVNGPOs of X.

5.13 Theorem: A BSVNs S of BSVNTS (X, τ) is BSVNGPOs if and only if F ⊆ BSVN pint (S) whenever F is BSVNCs and F ⊆ S.

Proof. <u>Necessity:</u> Suppose S is BSVNGPOs in X. Let F be BSVNCs and F ⊆ S. Then F^c is BSVNOs in X such that S^c ⊆ F^c. Since S^c is BSVNGPCs, we have BSVN pcl (S^c) ⊆ F^c. Hence (BSVN pint (S))^c ⊆ F^c. Therefore F ⊆ BSVN pint (S).

<u>Suffciency:</u> Let S be BSVNs of X and let F ⊆ BSVN pint (S) whenever F is BSVNCs and F ⊆ S. Then S^c ⊆ F^c and F^c is BSVNOs. By hypothesis, (BSVN pint (S))^c ⊆ F^c. This implies BSVN pcl (S^c) ⊆ F^c. Therefore S^c is BSVNGPCs of X. Hence S is BSVNGPOs of X.

5.14 Corollary: A BSVNs S of a BSVNTS (X, τ) is BSVNGPOs if and only if F ⊆ BSVN int (BSVN cl (S)) whenever F is BSVNCS and F ⊆ S.

Proof. <u>Necessity:</u> Suppose S is BSVNGPOs in X. Let F be BSVNCs and F ⊆ S. Then F^c is BSVNOs in X such that S^c ⊆ F^c. Since S^c is BSVNGPCs, we have BSVN pcl (S^c) ⊆ F^c. Therefore BSVN cl (BSVN int (S^c)) ⊆ F^c. Hence (BSVN int (BSVN cl (S)))^c ⊆ F^c. This implies F ⊆ BSVN int (BSVN cl (S)).

<u>Suffciency:</u> Let S be BSVNs of X and let F ⊆ BSVN int (BSVN cl (S)) whenever F is BSVNCs and F ⊆ S. Then S^c ⊆ F^c and F^c is BSVNOs. By hypothesis, (BSVN int (BSVN cl (S)))^c ⊆ F^c. Hence BSVN cl (BSVN int (S^c)) ⊆ F^c, which implies BSVN pcl (S^c) ⊆ F^c. Hence S is BSVNGPOs of X.

5.15 Theorem: For a BSVNs S, S is BSVNOs and BSVNGPCs in X if and only if S is BSVNROs in X.

Proof. <u>Necessity:</u> Let S be BSVNOs and BSVNGPCs in X. Then BSVN pcl (S) ⊆ S. This implies BSVN cl (BSVN int (S)) ⊆ S. Since S is BSVNOs, it is BSVNPOs. Hence S ⊆ BSVN int (BSVN cl (S)). Therefore S = BSVN int (BSVN cl (S)). Hence S is BSVNROs in X.

Suffciency: Let S be BSVNROs in X. Therefore S = BSVN int (BSVN cl (S)). Let S $\subseteq$ U and U is BSVNOs in X. This implies BSVN pcl (S) $\subseteq$ S. Hence S is BSVNGPCs in X.

6. Applications Of Bipolar Single-Valued Neutrosophic generalized Pre-Closed Sets

6.1 Definition: A BSVNTS (X, τ) is said to be bipolar single-valued neutrosophic $T_{1/2}$ space (BSVN $T_{1/2}$ space) if every BSVNGCs in X is BSVNCs in X.

6.2 Definition: A BSVNTS (X, τ) is said to be bipolar single-valued neutrosophic ${}_pT_{1/2}$ space (BSVN ${}_pT_{1/2}$ space) if every BSVNPCs in X is BSVNCs in X.

6.3 Definition: A BSVNTS (X, τ) is said to be bipolar single-valued neutrosophic ${}_{gp}T_{1/2}$ space (BSVN ${}_{gp}T_{1/2}$ space) if every BSVNGPCs in X is BSVNCs in X.

6.4 Definition: A BSVNTS (X, τ) is said to be a bipolar single-valued neutrosophic ${}_{gp}T_p$ space (BSVN ${}_{gp}T_p$ space) if every BSVNGPCs in X is BSVNPCs in X.

6.5 Theorem: Every BSVN $T_{1/2}$ space is BSVN ${}_{gp}T_p$ space.

Proof. Let X is BSVN $T_{1/2}$ space and let S be BSVNGCs in X, we know that every BSVNGCs is BSVNGPCs; hence S is BSVNGPCs in X. By hypothesis S is BSVNCs in X. Since every BSVNCs is BSVNPCs, S is BSVNPCs in X. Hence X is BSVN ${}_{gp}T_p$ space.

6.6 Remark: The converse of the above theorem is not true which is shown in the example.

6.7 Example: Let X ={p, q} and

$$S=\begin{Bmatrix} < p,(0.1,0.3,0.5,-0.3,-0.5,-0.1) > \\ < q,(0.2,0.4,0.6,-0.4,-0.6,-0.3) > \end{Bmatrix} \quad T=\begin{Bmatrix} < p,(0.3,0.2,0.4,-0.2,-0.6,-0.3) > \\ < q,(0.3,0.3,0.5,-0.2,-0.7,-0.3) > \end{Bmatrix}$$

Then τ= {0_{BSVN}, 1_{BSVN}, S, T} is a BSVNT on X. The BSVNs

R= $\begin{Bmatrix} < p,(0.3,0.4,0.5,-0.6,-0.6,-0.3) > \\ < q,(0.2,0.4,0.3,-0.3,-0.1,-0.2) > \end{Bmatrix}$. Then (X, τ) is BSVN $_{gp}$ T $_{p}$ space. But not BSVN T $_{1/2}$ space. Since R is BSVNGCs but not BSVNCs in X.

6.8 Theorem: Every BSVN $_{gp}$ T $_{1/2}$ space is BSVN $_{gp}$ T $_{p}$ space.

Proof. Let X is BSVN $_{gp}$ T $_{p}$ space and let S be BSVNGPCs in X. By hypothesis S is BSVNCs in X. Since every BSVNCs is BSVNPCs, S is BSVNPCs in X. Hence X is BSVN $_{gp}$ T $_{p}$ space.

6.9 Remark: The converse of the above theorem is not true which is shown in the example.

6.10 Example: Let X ={p, q} and

S= $\begin{Bmatrix} < p,(0.2,0.4,0.7,-0.5,-0.3,-0.4) > \\ < q,(0.6,0.9,0.8,-0.7,-0.1,-0.2) > \end{Bmatrix}$ T= $\begin{Bmatrix} < p,(0.3,0.3,0.6,-0.3,-0.4,-0.5) > \\ < q,(0.7,0.8,0.7,-0.6,-0.1,-0.3) > \end{Bmatrix}$

Then τ= {0_{BSVN}, 1_{BSVN}, S, T} is a BSVNT on X. The BSVNs

R= $\begin{Bmatrix} < p,(0.1,0.5,0.2,-0.7,-0.3,-0.6) > \\ < q,(0.4,0.8,0.9,-0.2,-0.4,-0.5) > \end{Bmatrix}$. Then (X, τ) is BSVN $_{gp}$ T $_{p}$ space. But not $_{gp}$ T $_{1/2}$ space.

Since R is BSVNGPCs but not BSVNCs in X.

6.11 Theorem: Let (X, τ) be BSVNTS and X is BSVN $_{gp}$ T $_{1/2}$ space then,

(1). Any union of BSVNGPCs's is BSVNGPCs.
(2). Any intersection of BSVNGPOs's is BSVNGPOS.
Proof.

(1). Let $\{A_i\}_{i \in J}$ is a collection of BSVNGPCs's in BSVN $_{gp}$ T $_{1/2}$ space (X, τ). Therefore every BSVNGPCs is BSVNCs. But the union of BSVNCs is BSVNCs. Hence the union of BSVNGPCs is BSVNGPCs in X.
(2). Take complement of (1) to prove.

6.12 Theorem: A BSVNTS X is BSVN $_{gp}$ T $_{1/2}$ space if and only if BSVNGPO(X) = BSVNPO(X).

Proof. Necessity: Let S be BSVNGPOs in X, then S^c is BSVNGPCs in X. By hypothesis S^c is BSVNGPCs in X. Therefore S is BSVNPOs in X. Hence BSVNGPO(X) = BSVNPO(X).

Sufficiency: Let S be BSVNGPCs in X. Then S^c is BSVNGPOs in X. By hypothesis S^c is BSVNGPOs in X. Therefore S is BSVNPCs in X. Hence X is BSVN $_{gp}T_{1/2}$ space.

7. Conclusions

We introduced a new class of sets namely bipolar single-valued neutrosophic generalized closed sets and bipolar single-valued neutrosophic generalized pre- closed sets in bipolar single-valued neutrosophic topological spaces. We also analyzed the properties and its applications with some examples.

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A New Approach for Solving Bi-Level Multi-Objective Non-Linear Programming Model under Neutrosophic Environment

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Abstract

Multi-level programming problems (MLPPs) are considered very large decentralized decision problems, occur in hierarchical decision-making organizations where a decision maker (DM) is present at each decision-making level and is assigned the task of optimizing one or more objective functions. In this paper, a new computational algorithm using neutrosophic technique to solve bi-level multi-objective non-linear programming (BL-MONLP) problem is presented. Neutrosophic set theory is played an important role for dealing the inaccuracy and complexity of data found in solving real life problems. We compared also the performance of the optimal solution between fuzzy and neutrosophic optimization techniques through numerical example which has demonstrated the evolved algorithm.

Keywords: Multi-level programming with multiple objectives; Non-Linear programming problems; Fuzzy Programming; Neutrosophtic set; Satisfactory solution.

1. Introduction

In a hierarchical organization with several interacting decision-makers, multi-level programming problems (MLPPS), which are the primary mathematical optimization problems for representing large decentralized decision problems, commonly used in industry [1], agriculture [2], transport [3], public policy [4], finance [5], planning [6], municipal waste system [7] and supply chain management [8]. Decision makers make decisions in order from the top to the bottom level. The upper- level is a priority over the lower- level, but it still depends on reactions at the lower- level. Furthermore, the objective function is optimized by each decision-maker to the extent possible.

In particular, many authors have researched bi-level programming problems (BLPPs) [9-11] and tri-level programming problems (TLPPS) [12-13], noted that MLPPS was NP-hard and when the number of levels was greater than two, the decent method [14], the approach based on the conditions of Karush-Kuhn-Tucker [15], the cutting plane algorithm [16], the penalty function approach [17], the heuristics technique [18] and the vertex enumeration [19] are represented six main approaches for solving MLPPS.

There is a possibility that their methods lead to undesirable solutions due to the differences between fuzzy goals of objective functions and decision variable. An interactive fuzzy programming (FP) for MLPPS was presented to solve this situation with the elimination of the fuzzy goals of decision variables [20-21].

Although fuzzy set theory (FST) is very helpful for dealing the uncertainties, it does not resolve certain instances of uncertainty where it is difficult to use a special value to define the degree of membership. The intuitionistic fuzzy set (IFS) is considered an extension of FST to solve the non-member degree knowledge lake [22-23]. The membership grade and the non-membership grade in IFS [24] are attached to each variable in a collection, where the sum of these two grades is limited to less than or equal to one. For a specific element, the degree of non-belonging is equal to 1 minus the degree of belonging [25].

Moreover, some researchers have used IFS for different types of decision-making problems. IFS has been applied to the multi-attribute decision making model and methodology in recent years [26-27]. The problem of multi-objective optimization of reliability [28-30], the problems of transport [31-32], the problem of multi-level programming [33]. Although the development of FST and IFS still lacks a general framework in which indeterminate information cannot be handled to deal with all kinds of uncertainty in different areas. This problem is beyond the scope of FST and IFS, so dealing with a kind of infinite situation of unknown data is certainly a real problem.

The neutrosophic set studied by FST and IFS has recently been a generalized form of [34]. It offers a more general framework and a more suitable shape to solve the existing problem. Neutrosophic means neutral information, and the primary difference between fuzzy and intuitionist fuzzy logic is this neutral. The neutrosophic Set (NS) is created on a logical basis in which elements of the universe are presented in three degrees. That is, the degree of truth, the degree of indeterminacy and the degree of falsity, they are somewhere between [0,1]. It differs from the intuitionist fuzzy sets, where the uncertainty involved depends on the degree of belonging and non-belonging, here the uncertainty present, i.e., the component of indeterminacy is independent of the values of truth and falsity. Some emphasis has been built on optimization aspects since its inception by [34-36].

The main purpose of the solution proposed is to include a general framework to help deal with the impressions and uncertainties of the knowledge available. In addition, managing model memberships will produce the best compromise outcome that not only meets the desires of the decision-maker, but it also makes an undominated contribution to deal with experiences by considering the membership of truth, indeterminacy membership and falsity membership associated with satisfaction to some degree and dissatisfaction with objectives in finding the best compromise solution (BCS) respectively. It can also be done to cover a broad range of BCSs by interactively managing the membership functions. This is first time that the best of our experience is to broaden the principles of Zimmermann to solve the problem of BL-MONLP. This paper focuses on exploring the best compromise solution to the problem of bi-level multi-objective non-linear programming (BL-MONLP) under neutrosophic compromise programming approach (NS-CPA). The proposed NS-CPA is built by expanding the principles of Zimmermann [37] to the neutrosophic environment and a new neutrosophic BL-MONLP is provided by using three memberships to obtain the best compromise solution: membership of truth, membership of indeterminacy, and membership of falsity. NS-CPA is a modern way of dealing with unreliable, ambiguous, incomplete and contradictory data that is very common in science and engineering situations.

The paper is structured as follows: Section 2 outlines some fundamental principles applicable to the neutrosophic set; Section 3 presents the technique of neutrosophic optimization to solve the problem of bi-level multi-objective non-linear programming; Section 4 explains in a numerical example the new approach and compares this new strategy with the problem of fuzzy programming, we present the conclusion and future direction of research in section 5.

2. Prerequisite Mathematics

Definition-1 (Fuzzy set) [37]

Le X be fixed set. The fuzzy set A of X is defined the set of an object that has the form $\widetilde{A}=\{(x,\mu_A(x)),\ x\in X\}$ where the function $\mu_A(x):X\to[0,1]$ determines the element's $x\in X$ true membership to the set A.

Definition -2 (Intuitionistic fuzzy set) [22]

Let X be fixed set. The set of the form

$\widetilde{A}^i = \left\{ < X, \mu_A(x),\ \upsilon_A^{\upsilon}(x) > \middle| \ x \in X \right\}$ is defined an intuitionistic fuzzy set (IFS) in X where $\mu_A(x): X \to [0,1]$ and $\upsilon_A(x): X \to [0,1]$ are called the Truth-membership and Falsity-membership respectively, for every element of $x \in X, 0 \le \mu_A(x) + \upsilon_A(x) \le 1$.

Definition -3 (Neutrosophic set) [34]

Let X be a set of objects and defined as $x \in X$. A neutrosophic set $\widetilde{A}^n$ in X is defined by a truth-membership function $\mu_A(x)$, an indeterminacy membership function $\sigma_A(x)$ and a falsity-membership function $\upsilon_A(x)$, having the form:

$$\widetilde{A}^n = \left\{ < X,\ \mu_A(x), \sigma_A(x),\ \upsilon_A(x) > \middle| x \in X \right\}.$$

$\mu_A(x), \sigma_A(x)$ and $\upsilon_A(x)$ are real standard or non-standard elements of $\left]0^-, 0^+\right[$, that is: $\mu_A(x): X \to \left]0^-, 1^+\right[, \sigma_A(x): X \to \left]0^-, 1^+\right[, \upsilon_A(x): X \to \left]0^-, 1^+\right[$

There is no limit to the sum of $\mu_A(x), \sigma_A(x)$ *and* $\upsilon_A(x)$, so

$$^-0 \le \sup\ \mu_A(x) + \sup \sigma_A(x) + \sup\ \upsilon_A(x) \le 3^+$$

Definition -4 (Complement) [38]

A single valued neutrosophic set A is called complement, denoted by C(A) if is defined by

$$\mu_{C(A)}(x) = \upsilon_A(x)\ ,\ \sigma_{C(A)}(x) = 1 - \sigma_A(x)\ ,\ \upsilon_{C(A)}(x) = \mu_A(x)\ ,$$

for all *x's* in X.

Definition -5 (Union) [38]

A single valued neutrosophic set C is union of two single valued neutrosophic sets A and B, written as C = AUB, having the functions of truth-membership, indeterminacy- membership and falsity-membership defined by:

$$\mu_{C(A)}(x) = \max(\mu_A(x), \mu_B(x))$$
$$\sigma_{C(A)}(x) = \max(\sigma_A(x), \sigma_B(x))$$
$$\upsilon_{C(A)}(x) = \min(\upsilon_A(x), \upsilon_B(x)) \quad \text{forall } x \text{ in } X$$

Definition-6 (Intersection) [38]

The intersection of two single valued neutrosophic sets A and B is a single valued neutrosophic set C written as $C = A \cap B$, having the functions of truth-membership, indeterminacy-membership and falsity-membership defined by:

$$\mu_{C(A)}(x) = \min\left(\mu_A(x), \mu_B(x)\right) = \min\left(\sigma_A(x), \sigma_B(x)\right)$$
$$\upsilon_{C(A)}(x) = \max\left(\upsilon_{(A)}(x),\ \upsilon_B(x)\right)$$

Here, we note that single valued neutrosophic sets satisfy the most characteristics as the classic set, fuzzy set and intuitionistic fuzzy set by the concept of complement, union and intersection of single valued neutrosophic sets. The Fuzzy collection does not satisfy the middle exclusion principle.

3. Neutrosophic technique to solve bi-level multi-objective non-linear programming (NS-BLMONLP) Problem.

3.1. Problem Formulation

In the problem of bi-level multi-objective non-linear programming (BL-MONLP), multiple decision-makers (DMs) exist at each decision-making level and are known to optimize one or more objective functions as bi-level multi-leader and / or multi-follower decision problem. The BL-MONLP problem's mathematical formulation can be written as:

$$\min_{x_1, x_2}\ F_k(x) = \left(F_1(x),\ F_2(x), ..., F_m(x)\right) \qquad \text{(The upper-level (UL))} \qquad (1)$$

where x_2 solves:

$$\min_{x_2} F_k(x) = \left(F_{m+1}(x), F_{m+2}(x), ..., F_N(x)\right) \qquad \text{(The lower-level (LL))} \tag{2}$$

s.t .

$$G_j = \left\{ \mathrm{x} \;\middle|\; g_j\ (x)\ \le b_j \qquad , \qquad j = 1,2,...., J \right\} \tag{3}$$

where $x_i = \{x_{ir}\}$, $i = 1,2$ and $r = 1, 2, ..., n_i$, $x_i \in R^{n_i}$, $(x_1, x_2) \subset R^{n=n_1+n_2}$ and $F(x)$ are the decision variables and the objective functions of the UL and LL respectively. G is the feasible set of problem form [(1) – (3)].

3.2. Neutrosophic Optimization Technique for solving MONLP problem.

Consider the following MONLP problem represented by problem from (1) or (2) and (3). For all objectives as $F_k(x)$, $k = 1,2,..., m$, find firstly the best values ℓ_k (minimum values) and the worst values u_k (maximum values). So, ℓ_k can be calculated by:

$$\ell_k = \min_{x \in G} F_k(x) \tag{4}$$

If the feasible set G is bounded, then u_k can be derived by:

$$u_k = \max_{x \in G} F_k(x) \tag{5}$$

Otherwise, let the solutions of (4) are $\bar{x}_k$ then u_k are calculated by:

$$u_k = \max_{k} F_k(\bar{x}_k), \tag{6}$$

Now, a mapping $\mu_k : x \to [0,1]$ is known as the membership function and the acceptable degrees of decision makers for a solution can be expressed. It is possible to state a membership function as:

$$\mu_k(F_k) = \begin{cases} 1 & , \quad F_k(x) \le \ell_k \\ \dfrac{u_k - F_k}{u_k - \ell_k} & , \quad \ell_k \le F_k(x) \le u_k \\ 0 & , \quad F_k(x) \ge u_k \end{cases} \tag{7}$$

Here, we present a new approach under the set of constraints centered on a neutrosophic set to solve the MONLP problem. A creating information into indeterminacy treatment is introduced by the neutrosophic approach (NSA), which is described in the main optimization problem as the aim of maximizing the degree of truth (T) at the same time and minimizing the degrees of falsity (F) and indeterminacy (I)of a neutrosophic decision set (D_s).

Generally, we describe what is called a combination of neutrosophic objectives and constraints as:

$$D_s = \left(\bigcap_{k=1}^{m} Z_k \right) \bigcap \left(\bigcap_{j=1}^{J} C_j \right) = \left\{ \left(x, T_{D_S}(x), I_{D_S}(x), F_{D_S}(x) \right) \right\} \tag{8}$$

where, $T_{D_S}(x)$: the function of truth membership, $I_{D_S}(x)$: the function of indeterminacy membership and $F_{D_S}(x)$: the function of falsity membership of neutrosophic decision set D_s which is defined as:

$$T_{D_s}(x) = \min \begin{Bmatrix} T_{z_1}(x) & , & T_{z_2}(x) & , & \ldots & , & T_{z_k}(x); \\ T_{c_1}(x) & , & T_{c_2}(x) & , & \ldots & , & T_{c_j}(x) \end{Bmatrix} \text{ for all } x \in X \tag{9}$$

$$I_{D_s}(x) = \max \begin{Bmatrix} I_{z_1}(x) & , & I_{z_2}(x) & , & \ldots & , & I_{z_k}(x); \\ I_{c_1}(x) & , & I_{c_2}(x) & , & \ldots & , & I_{c_j}(x) \end{Bmatrix} \text{ for all } x \in X \tag{10}$$

$$F_{D_s}(x) = \max \begin{Bmatrix} F_{z_1}(x) & , & F_{z_2}(x) & , & \ldots & , & F_{z_k}(x); \\ F_{c_1}(x) & , & F_{c_2}(x) & , & \ldots & , & F_{c_j}(x) \end{Bmatrix} \text{ for all } x \in X \tag{11}$$

To formulate the membership functions for NS-MONLP problem, we evaluate firstly the lower (ℓ_k) and upper (u_k) bounds for each objective function by using (4) and (6), then the bounds for NS can be determined as:

$$u_k^T = u_k \text{ and } l_k^T = \ell_k \text{ , for truth membership} \left(T_k(F_k)\right) \tag{12}$$

$$u_k^I = l_k^T + s_k\left(u_k^T - \ell_k^T\right) \text{and } l_k^I = l_k^T \text{ ,for indeterminacy membership} \left(I_k(F_k)\right) \tag{13}$$

$$u_k^F = u_k^T \text{ and } l_k^F = l_k^T + t_k\left(u_k^T - l_k^T\right), \text{ for falsity membership} \left(F_k(F_k)\right) \tag{14}$$

where t_k and s_k real numbers are predetermined in (0,1).

We can define now the membership functions (7) according to the above bounds as:

$$T_k(F_k(x)) = \begin{cases} 1 & \text{,if } \quad F_k(x) < l_k^T \\ \dfrac{u_k^T - F_k(x)}{u_k^T - \ell_k^T} & \text{,if } \quad l_k^T \le F_k(x) \le u_k^T \\ 0 & \text{,if } \quad F_k(x) > u_k^T \end{cases} \tag{15}$$

$$I_k(F_k(x)) = \begin{cases} 0 & \text{,if } \quad F_k(x) < l_k^I \\ \dfrac{F_k(x) - l_k^I}{u_k^I - l_k^I} & \text{,if } \quad l_k^I \le F_k(x) \le u_k^I \\ 1 & \text{,if } \quad F_k(x) > u_k^I \end{cases} \tag{16}$$

$$F_k(F_k(x)) = \begin{cases} 0 & \text{,if } \quad F_k(x) < l_k^F \\ \dfrac{F_k - l_k^F}{u_k^F - l_k^F} & \text{,if } \quad l_k^F \le F_k(x) \le u_k^F \\ 1 & \text{,if } \quad F_k(x) > u_k^F \end{cases} \tag{17}$$

The problem with NS-MONLP can be stated as:

$$\max \min_{k=1,2,\ldots,m} T_k(F_k(x)) \tag{18}$$

$$\min \max_{k=1,2,\ldots,m} I_k(F_k(x)) \tag{19}$$

$$\min \max_{k=1,2,\ldots,m} F_k(F_k(x)) \tag{20}$$

s.t .

$$x \in G \tag{21}$$

Problem from [(18) – (21)] can be taken the following form as:

$$\max \alpha \tag{22}$$
$$\min \gamma \tag{23}$$
$$\min \beta \tag{24}$$
s.t .
$$T_k(F_k(x)) \geq \alpha\ ,\ I_k(F_k(x)) \leq \gamma\ ,\ F_k(F_k(x)) \leq \beta, \tag{25}$$
$$*\, x \in G(x), x \geq 0, \tag{26}$$
$$\alpha + \gamma + \beta \leq 3\ ,\ \alpha \geq \beta\ ,\ \alpha \geq \gamma\ ,\ \alpha, \gamma, \beta \in [0,1]. \tag{27}$$

Problem from [(22) – (27)] can be simplified to non-linear programming (NLP) problem by using neutrosophic model as:

$$\max\ \alpha - \gamma - \beta \tag{28}$$
s.t .
$$F_k(x) + \left(u_k^T - l_k^T\right)\alpha \leq u_k^T\ , \tag{29}$$
$$F_k(x) - \left(u_k^I - l_k^I\right)\gamma \leq l_k^I\ , \tag{30}$$
$$F_k(x) - \left(u_k^F - l_k^F\right)\beta \leq l_k^F\ , \tag{31}$$
$$x \in G(x)\ \ ,\ x \geq 0, \tag{32}$$
$$\alpha + \gamma + \beta \leq 3\ ,\ \alpha \geq \beta\ ,\ \alpha \geq \gamma\ ,\ \alpha, \gamma, \beta \in [0,1] \tag{33}$$

3.3. Algorithm (ALG (1)) for solving NS-ULMONLP problem

In this section, a new approach which discussed above is simplified to find the neutrosophic optimal solution of UL-MONLP problem.

Step1: Solve each UL-MONLP problem form (1) and (3) individually as a single objective NLP problem. Let $x_k, k = 1,2,...,m,$ be the respective optimal solution for the k[th] different objective $F_k(x), k = 1,2,...,m$ and calculate each objective value for each these k[th] optimal solutions. If two optimal solutions are different and far from the set of optimal solution and has the different bound values, then go to step2. Otherwise, if all $F_k(x)$ have the same solution, $x_1 = x_2 = ... = x_k$, then choose one of them as the optimal compromise solution and go to step 6.

Step 2: Find lower and upper bounds for all objectives by using (4) and (6).

Step 3: Calculate the NS bounds for each objective $F_k(x)$ to find the lower bound l_k and the upper bound $u_k, k = 1,2,...,m.$ for truth membership $(T_k(F_k))$, indeterminacy membership $(I_k(F_k))$, and falsity membership $(F_k(F_k))$, of objectives as (12), (13) and (14).

Step 4: Use step (3) to construct the membership functions as (15), (16) and (17).

Step 5: Define and solve NS for UL-MONLP problem as in [(28)-(33)] to get the best compromise solution. Also, find the values of the $F_k(x), k = 1,...,m$ .at the best solution.

Step 6: stop.

3.4. Algorithm (ALG (2)) for solving NS-LLMONLP problem

As in the previous section considers LL-MONLP problem is (2) and (3). The procedures for using the neutrosophic technique for solving LL-MONLP problem to get the NS optimal solution can be summarized as:

Step1: Solve the problem LL-MONLP (2) and (3) as a single non-linear (NL) objective problem k times, $k = m+1,..., N,$ for each problem by taking one of the goals at a time and ignoring the others subject to the set of constraints G in order to obtain the set of solutions $x_k, k = m+1,..., N,$ and calculate the values of each objectives at x_k.

Step2: If all $F_k(x), k = m+1,..., N,$ have the same solutions, then select and stop one of them as the optimal compromise solution. Otherwise, proceed to step 3.

Step3: Calculate the NS boundaries for each objective $F_k(x), k = m+1,..., N,$ in order to find the lower l_k and upper $u_k, k = m+1,..., N,$ boundaries for $T_k(F_k), I_k(F_k)$ and $F_k(F_k)$ as in (12), (13) and (14).

Step 4: Use step (3) to construct the membership functions as (15), (16) and (17).

Step5: Solve NS for LL-MONLP problem as [(28) – (33)] to find the best compromise solution. Also, find the values of $F_k(x), k = m+1,..., N,$ at the best solution.

3.5. Algorithm (ALG (3)) for solving NS-BLMONLP problem

To solve a bi-level multi-objective non-linear programming problem with a linear membership function by neutrosophic methodology, the following algorithm steps are used to find an optimal compromise solution.

Step 1: Substitute by optimal compromise solution x_k^U of NS-ULMONLP problem in $F_k, k = m+1,..., N,$ and it is denoted by $\hat{u}_k, k = m+1,..., N.$ Also, substituting by optimal compromise solution x_k^L of NS-LLMONLP problem in $F_k, k = 1,..., m,$ to obtain $u'_k, k = 1,..., m.$

Step 2: Use the optimal decision x^U of NS-ULMONLP problem as a control factor for NS-LLMONLP problem. It is not practical, so we find some tolerance that gives the NS-LLMONLP problem a feasible region to search for his/ her optimal solution. The range of the decision variable *x* should be found around x^U with its maximum tolerances *k*.

Step 3: Formulate the following membership function which specify x^U as:

$$T_X(x) = \begin{cases} \dfrac{X - (x^U - k)}{k} & , \quad \text{if} \quad x^U - k \le X \le x^U, \\ \dfrac{(x^U + k) - X}{k} & , \quad \text{if} \quad x^U \le X \le x^U + k \\ 0 & , \quad \text{otherwise} \end{cases} \tag{34}$$

$$I_X(x)=\begin{cases}\dfrac{X-(x^U-k)}{k} & , \text{ if } \quad x^U-k\le X\le x^U\\ \dfrac{(x^U+k)-X}{k} & , \text{ if } \quad x^U\le X\le x^U+k\\ 0 & , \qquad \text{otherwise}\end{cases} \tag{35}$$

$$F_X(x)=\begin{cases}\dfrac{X-(x^U-k)}{k} & , \text{ if } \quad x^U-k\le X\le x^U\\ \dfrac{(x^U+k)-X}{k} & , \text{ if } \quad x^U\le X\le x^U+k\\ 0 & , \qquad \text{otherwise}\end{cases} \tag{36}$$

where x^U is the most preferred solution [39].

Step 4: Construct the membership functions of NS-ULMONLP problem at all F_k where

$l_k\le F_k\le u'_k \ , \ u'_k=F_k(x^L), k=1,2,...m,$ as:

$$T_k(F_k)=\begin{cases}1 & , \text{ if } \quad F_k(x)<l_k^T\\ \dfrac{u_k'^T-F_k(x)}{u_k'^T-l_k^T} & , \text{ if } \quad l_k^T\le F_k(x)\le u_k'^T\\ 0 & , \text{ if } \quad F_k(x)>u_k'^T\end{cases} \tag{37}$$

$$I_k(F_k)=\begin{cases}0 & , \text{ if } \quad F_k(x)<l_k^I\\ \dfrac{F_k(x)-l_k^I}{u_k'^I-l_k^I} & , \text{ if } \quad l_k^I\le F_k(x)\le u_k'^I\\ 1 & , \text{ if } \quad F_k(x)>u_k'^I\end{cases} \tag{38}$$

$$F_k(F_k)=\begin{cases}0 & , \text{ if } \quad F_k(x)<l_k^F\\ \dfrac{F_k(x)-l_k^F}{u_k'^F-l_k^F} & , \text{ if } \quad l_k^F\le F_k(x)\le u_k'^F\\ 1 & , \text{ if } \quad F_k(x)>u_k'^F\end{cases} \tag{39}$$

Step 5: Define the membership functions of NS-LLMONLP problem for all goals F_k

where $l_k\le F_k\le \hat{u}_k \ , \ \hat{u}_k=F_k(x^U), k=m+1,...,N,$ as:

$$T_k(F_k)=\begin{cases}1 & , \text{ if } \quad F_k(x)<l_k^T\\ \dfrac{\hat{u}_k^T-F_k(x)}{\hat{u}_k^T-l_k^T} & , \text{ if } \quad l_k^T\le F_k(x)\le \hat{u}_k^T\\ 0 & , \text{ if } \quad F_k(x)>\hat{u}_k^T\end{cases} \tag{40}$$

$$I_k(F_k)=\begin{cases}0 & , \text{ if } \quad F_k(x)<l_k^I \\ \dfrac{F_k(x)-l_k^I}{\hat{u}_k^I-l_k^I} & , \text{ if } \quad l_k^I \le F_k(x) \le \hat{u}_k^I \\ 1 & , \text{ if } \quad F_k(x)>\hat{u}_k^I\end{cases} \tag{41}$$

$$F_k(F_k)=\begin{cases}0 & , \text{ if } \quad F_k(x)<l_k^F \\ \dfrac{F_k(x)-l_k^F}{\hat{u}_k^F-l_k^F} & , \text{ if } \quad l_k^F \le F_k(x) \le \hat{u}_k^F \\ 1 & , \text{ if } \quad F_k(x)>\hat{u}_k^F\end{cases} \tag{42}$$

Step 6: Generate a suitable solution that is also a pareto optimal solution for both decision makers (DMs) with an overall satisfaction by solving the following Tchebycheff problem [40] which is considered NS-BLMONLP problem as:

$$Max\ \alpha-\gamma-\beta \tag{43}$$

s.t .

$$\left[X-(x^u-k)\right]/K \ge \alpha \text{ I}, \qquad \left[(x^U+k)-X\right]/K \ge \alpha \text{ I},$$

$$\left[X-(x^U-k)\right]/K \le \gamma \text{ I}, \qquad \left[(x^U+k)-X\right]/K \ge \gamma \text{ I}, \tag{44}$$

$$\left[X-(x^U-k)\right]/K \le \beta \text{ I}, \qquad \left[(x^U+k)-k\right]/K \le \beta \text{ I},$$

$$\begin{aligned}
&F_k(x)+\left(u_k'^T-l_k^T\right)\alpha \le u_k'^T, \quad k=1,2,\dots,m,\\
&F_k(x)-\left(u_k'^I-l_k^I\right)\gamma \le l_k^I, \quad k=1,2,\dots,m,\\
&F_k(x)-\left(u_k'^F-l_k^F\right)\beta \le l_k^F, \quad k=1,2,\dots,m,\\
&F_k(x)+\left(\hat{u}_k^T-l_k^T\right)\alpha \le \hat{u}_k^T, \quad k=m+1,\dots,N,\\
&F_k(x)-\left(\hat{u}_k^I-l_k^I\right)\gamma \le l_k^I, \quad k=m+1,\dots,N,\\
&F_k(x)-\left(\hat{u}_k^F-l_k^F\right)\beta \le l_k^F, \quad k=m+1,\dots,N,
\end{aligned} \tag{45}$$

$$x \in G \quad , \quad x \ge 0, \tag{46}$$

$$\alpha+\beta+\gamma \le 3,\ \alpha \ge \beta,\ \alpha \ge \gamma,\ \alpha,\beta,\gamma \in [0,1] \tag{47}$$

where α, β and γ are the overall satisfaction and I, with all elements = 1 and the same direction as x, is the column vector.

4. Numerical Example

An example in [41] is used to explain and comparison of optimal solutions by fuzzy programming (FP) for BL-MOGPP and NS for BL-MONLP problem.

Let us consider BL-MONLP problem as:

(UL-MONLP): $\min\limits_{x_1} F_1(x)=20\,x_1^{-1}x_2^{-3}x_3^{-5}+60\,x_1^{-1}\,x_2^{-1}$

$$\min_{x_1} F_2(x) = 50\ x_1^{-1} x_2^{-2} x_3^{-2} + 60\ x_1^{3}\ x_2^{-2} x_3^{-3}$$
$$\text{s.t.}$$
$$G_1(x) = x_1 x_2 x_3^2 + x_2 x_3 \ \le 3,$$
$$x = (x_1, x_2, x_3) > 0$$

where x_2 and x_3 solve LL-MONLP problem as:

(LL-MONLP): $$\min_{x_2, x_3} F_3(x) = x_1^{-2} + 0.25\ x_2^2\ x_3^{-1},$$
$$\min_{x_2, x_3} F_4(x) = 2x_1^{-1} x_2^{-1} x_3^{-1} + 2x_1 x_2$$
$$\text{s.t.}$$
$$G_2(x) = \frac{3}{4} x_1^2 x_2^{-2} + \frac{3}{8} x_2 x_3^2 \le 1,$$
$$x = (x_1, x_2, x_3) > 0.$$

First: For UL- MONLP problem

$$\ell_1 = \min_{x_1 \in G_1} F_1(x_1) = 20.666 \text{ for } x_1 = (1.952,\quad 4.462,\quad 0.384)$$
$$\ell_2 = \min_{x_2 \in G_1} F_2(x_2) = 18.207 \text{ for } x_2 = (0.471,\quad 11.442,\quad 0.236)$$
$$u_1 = \max F_1(x_2) = F_1(0.471,\ 11.442,\ 0.236) = 49.891$$
$$u_2 = \max F_2(x_1) = F_2(1.952,\ 4.462,\ 0.384) = 404.016$$
$$\therefore\ 20.666 \le F_1 \le 49.891\ \textit{and}\ 18.207 \le F_2 \le 404.016$$

Here the upper and lower bounds for NS can be calculates from [(12)-(14)] as:

For F1: $l_1^T = 20.666,\ u_1^T = 49.891,\ l_1^I = 20.666,$
$$u_1^I = 20.666 + 29.25 s_1, l_1^F = 20.666 + 29.225 t_1,\ u_1^F = 49.891.$$

For F2: $l_2^T = 18.207,\ u_2^T = 404.016,\ l_2^I = 18.207,$
$$u_2^I = 18.207 + 385.809 s_1, l_2^F = 18.207 + 385.809 t_1,\ u_2^F = 404.016.$$

Neutrosophic optimization method for UL-MONLP problem can be constituted from [(28) – (33)] as:

$$\max \alpha_1 - \gamma_1 - \beta_1$$
$$\text{s.t.}$$
$$\left(20x_1^{-1} x_2^{-3} x_3^{-5} + 60\ x_1^{-1} x_2^{-1}\right) + 29.225\ \alpha_1 \le 49.891,$$
$$\left(20x_1^{-1} x_2^{-3} x_3^{-5} + 60\ x_1^{-1} x_2^{-1}\right) - 29.225\ s_1 \gamma_1 \le 20.666,$$
$$\left(20x_1^{-1} x_2^{-3} x_3^{-5} + 60\ x_1^{-1} x_2^{-1}\right) - (29.225 - 29.22\ 5\ t_1)\beta_1 \le 20.666 + 29.225 t_1$$

$$\left(50x_1^{-1}x_2^{-2}x_3^{-2}+60x_1^{3}x_2^{-2}x_3^{-3}\right)+385.809\alpha_1 \le 404.016\,,$$
$$\left(50x_1^{-1}x_2^{-2}x_3^{-2}+60x_1^{3}x_2^{-2}x_3^{-3}\right)-385.809\, s_1\gamma_1 \le 18.207\,,$$
$$\left(50x_1^{-1}x_2^{-2}x_3^{-2}+60x_1^{3}x_2^{-2}x_3^{-3}\right)-(385.809-385.809t_1)\ \beta_1 \le$$
$$18.207+385.809t_1,$$
$$x_1\, x_2\, x_3^2+x_2x_3 \le 3\ \ ,\ (x_1,x_2,x_3)>0,$$
$$\alpha_1+\gamma_1+\beta_1 \le 3\ \ ,\ \ \alpha_1 \ge \beta_1\ \ ,\ \ \alpha_1 \ge \gamma_1 \text{ and } \alpha_1,\gamma_1,\beta_1 \in [0,1].$$

The solution of the above problem can be given as the comparison of optimal solutions between the solution by using fuzzy programming problem (FPP) and neutrosophic technique (NS) as:

Optimization technique	$x^U=\left(x_1^U,x_2^U,x_3^U\right)$	$F^U=\left(F_1^U,F_2^U\right)$	$\alpha_1^*,\gamma_1^*,\beta_1^*$	Sum of optimal objective values
FPP	$x_1^U=1.072595$ $x_2^U=5.0157517$ $x_3^U=0.4053974$	$F_1^U=23.25882$ $F_2^U=52.44017$	$\alpha^*=0.91127$	0.91127
NS	$x_1^U=1.07258$ $x_2^U=5.157534$ $x_3^U=0.4053977$	$F_1^U=23.259$ $F_2^U=5244$	$\alpha_1^*=0.911275$ $\gamma_1^*=0.088724$ $\beta_1^*=0.0000$	0.82255

and $s_1 = 1.000$, $t_1 = 0.4622$

Second: for LL-MONLP problem

$$\ell_3=\min_{x_3\in G_2} F_3(x_3)=1.172\ \ ,\ \ x_3=(1.24,\ 1.27,\ 0.775)$$
$$\ell_4=\min_{x_4\in G_2} F_4(x_4)=3.504\ \ ,\ x_4=(0.623,\ 1.206,\ 1.33)$$
$$u_3=\max\ F_3(x_4)=2.852\ \ ,\ u_4=\max\ F_4\ (x_3)=4.789$$
$$\therefore 1.172\le F_3 \le 2.852\ \ ,3.504\le\ F_4 \le 4.789$$

For F3: $l_3^T=1.172\ \ ,\ u_3^T=2.852\ ,\ l_3^I=1.172,$

$u_3^I=1.172{+}1.68s_2\,,\ l_3^F=3.504+1.285\,t_2\ \ ,\ u_3^F=2.852$

For F4: $l_4^T=3.504,\ u_4^T=4.789,\ l_4^I=3.504, u_4^I=3.504+1.285\,s_2,$

$l_4^F=3.504+1.285\,t_2\ \ ,\ u_4^F=4.789$

Neturosophic optimization method for LL-MONLP problem from [(28)–(33)] can be simplified as:

$$
\begin{aligned}
&\max\ \alpha_2-\gamma_2-\beta_2\\
&\text{s.t}\\
&x_1^{-2}+0.25\,x_2^{2}\,x_3^{-1}+1.68\,\alpha_2\le 2.852\\
&x_1^{-2}+0.25\,x_2^{2}\,x_3^{-1}-1.68\,s_2\gamma_2\le 1.172\\
&x_1^{-2}+0.25\,x_2^{2}\,x_3^{-1}-(1.68-1.68t_2)\,\beta_2\le 1.172+1.68\,t_2,\\
&2x_1^{-1}x_2^{-1}\,x_3^{-1}+2x_1x_2+1.285\,\alpha_2\le 4.789\\
&2x_1^{-1}x_2^{-1}\,x_3^{-1}+2x_1x_2-1.285s_2\,\gamma_2\le 3.504\\
&2x_1^{-1}x_2^{-1}\,x_3^{-1}+2x_1x_2-(1.285-1.285\,t_2)\,\beta_2\le 3.504+1.285t_2,\\
&\frac{3}{4}\,x_1^{2}x_2^{-2}+\frac{3}{8}x_2x_3^{2}\le 1\quad,\quad (x_1,x_2,x_3)>0,\\
&\alpha_2+\gamma_2+\beta_2\le 3\ ,\ \alpha_2\ge\beta_2\ ,\ \alpha_2\ge\gamma_2\ \text{and}\ \alpha_2,\gamma_2,\beta_2\in[0,1].
\end{aligned}
$$

The comparison of optimal solution between FPP and NS technique can be summarized as:

Optimal technique	$x^L=\left(x_1^L,x_2^L,x_3^L\right)$	$F^L=\left(F_3^L,F_4^L\right)$	$\alpha_2^*,\gamma_2^*,\beta_2^*$	Sum of optimal objective values
FPP	$x_1^L=0.8932686$ $x_2^L=1.135477$ $x_3^L=1.121792$	$F_3^L=1.5405769$ $F_4^L=3.786322$	$\beta=0.780441$	0.91127
NS	$x_1^L=0.8932$ $x_2^L=1.1355$ $x_3^L=1.1219$	$F_3^L=1.54$ $F_4^L=3.786$	$\alpha_2=0.7805$ $\gamma_2=0.2195$ $\beta_2=0.000$	0.82255

and $s_2=1.000$, $t_2=0.242$

Third: Substituting by optimal compromise solution $\left(x_1^L, x_2^L, x_3^L\right)$ in F_1 and F_2 respectively, we get:

$32.259 < F_1 < 67.764$ and $52.44 < F_2 < 54.25$

We are also substituted by optimum compromise solution $\left(x_1^U, x_2^U, x_3^U\right)$ in F_3 and F_4, respectively we get:

$1.54 < F_3 < 17.237$ and $3.786 < F_4 < 11.956$

Application of steps algorithm (3) and assuming the control decision is x_1^U with tolerance k = 1 of NS-UL-MONLP problem. The NS-BLMONLP problem from [(43) – (47)] can be generated as:

$$
\begin{aligned}
&\max\ \alpha_3 - \gamma_3, -\beta_3\\
&\text{s.t.}\\
&x_1 - 0.07 \ge \alpha_3 \quad , \quad 2.07 - x_1 \ge \alpha_3,\\
&x_1 - 0.07 \le \gamma_3 \quad , \quad 2.07 - x_1 \le \gamma_3,\\
&x_1 - 0.07 \le \beta_3 \quad , \quad 2.07 - x_1 \le \beta_3,\\
&20x_1^{-1}x_2^{-3}x_3^{-5} + 60\, x_1^{-1}x_2^{-1} + 44.505\alpha_3 \le 67.764,\\
&50x_1^{-1}x_2^{-2}x_3^{-2} + 60\, x_1^{3}x_2^{-2}x_3^{-3} + 1.81\alpha_3 \le 54.25 \quad ,\\
&x_1^{-2} + 0.25\, x_2^{2}x_3^{-1} + 15.733\,\alpha_3 \le 17.273 \quad ,\\
&2x_1^{-1}x_2^{-1}x_3^{-1} + 2\, x_1x_2 + 8.17\,\alpha_3 \le 11.956 \quad ,\\
&20x_1^{-1}x_2^{-3}x_3^{-5} + 60\, x_1^{-1}x_2^{-1} - 44.505 s_3\gamma_3 \le 23.259,\\
&50x_1^{-1}x_2^{-2}x_3^{-2} + 60\, x_1^{3}x_2^{-2}x_3^{-3} - 1.81 s_3\,\gamma_3 \le 52.44,\\
&x_1^{-2} + 0.25\, x_2^{2}x_3^{-1} - 15.733 s_3\,\gamma_3 \le 1.54,\\
&2x_1^{-1}x_2^{-1}x_3^{-1} + 2\, x_1x_2 - 8.17 s_3\,\gamma_3 \le 3.786,\\
&20x_1^{-1}x_2^{-3}x_3^{-5} + 60\, x_1^{-1}x_2^{-1} - (44.505 - 44.505t_3)\,\beta_3 \le 23.259 + 44.505t_3,\\
&50x_1^{-1}x_2^{-2}x_3^{-2} + 60\, x_1^{3}x_2^{-2}x_3^{-3} - (1.81 - 1.81t_3)\,\beta_3 \le 52.44 + 1.81t_3,\\
&x_1^{-2} + 0.25\, x_2^{2}x_3^{-1} - (15.733 - 15.733\,t_3)\beta_3 \le 1.54 + 15.733t_3,\\
&2x_1^{-1}x_2^{-1}x_3^{-1} + 2x_1x_2 - (8.17 - 8.17t_3)\,\beta_3 \le 3.786 + 8.17t_3,\\
&x_1x_2x_3^2 + x_1x_2 \le 3,\\
&\frac{3}{4}x_1^2x_2^{-2} + \frac{3}{8}x_2x_3^2 \le 1, \quad (x_1, x_2, x_3) > 0,\\
&\alpha_3 + \gamma_3 + \beta_3 \le 3, \quad \alpha_3 \ge \beta_3\ , \alpha_3 \ge \gamma_3 \ \text{and}\ \alpha_3, \gamma_3, \beta_3 \in [0,1].
\end{aligned}
$$

The comparison of optimal solutions between FPP and NS-technique can be summarized as:

Optimization technique	$x^* = (x_1^*, x_2^*, x_3^*)$	$F^* = (F_1^*, F_2^*, F_3^*, F_4^*)$	$\alpha_3^*, \gamma_3^*, \beta_3^*$	Sum of optimal objective values
FPP	$x_1^* = 1.071753$ $x_2^* = 2.114856$ $x_3^* = 0.7749304$	$F_1^* = 33.9266984$ $F_2^* = 53.86580161$ $F_3^* = 2.318923741$ $F_4^* = 5.665864$	$\delta = 0.77$	0.77
NS	$x_1^* = 1.071751$ $x_2^* = 2.114812$	$F_1^* = 33.9267$ $F_2^* = 538658$ $F_3^* = 2.3189$ $F_4^* = 5.6659$	$\alpha_3^* = 0.7691842$ $\gamma_3^* = 0.230815$ $\beta_3^* = 0.000$	0.538368

and $s_3 = 1.000$, $t_3 = 0.287599$

Here, we demonstrate that the technique of neutrousophic optimization results better than the problem of fuzzy programming.

5. Conclusions and research directions

Neutrosophic set theory considers an important role in resolving the inaccuracy and uncertainty of data in solving real-life problems. The known methods as fuzzy theory, not sufficient in many bi-level programming situations for dealing with these situations in which indeterminacy is necessarily involved. This paper introduces a new neutrosophic compromise programming strategy (NS-CPA) to resolve the problem of bi-level multi-objective non-linear programming (NS-BLMONLP) under fuzziness.

At the same time, this method is characterized by maximizing the degree of truth (satisfaction), minimizing the degrees of both falsity (dissatisfaction) and indeterminacy (satisfaction to some extent) of neutrosophic decision-making. The analysis of the results obtained for the problem undertaken clearly indicates that neutrosophic optimization is superior to fuzzy optimization. To apply the steps of the NS-CPA, a numerical problem is solved.

We hope that in the neutrosophic setting, the bi-level non-linear programming technique can open a new avenue of study for future neutrsophic researchers. In addition, we agree that the propose neutrosophic compromise programming solution can be useful in addressing multi-objective geometric programming, Multi-objective decentralized bi-level non-linear programming, multi-objective decentralized multi-level non-linear programming, multi-objective non-linear programming problems based on priority, real-world decision-making problems such as agriculture, production of bio-fuels, selection of portfolios, transport.

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Split Domination in Neutrosophic Graphs

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Abstract: This paper demonstrates a concept of split domination in neutrosophic graphs.Minimal split domination, lower and upper split dominations in neutrosophic graphs are discussed. Theorems are derived for minimal split domination in neutrosophic graphs with suitable examples.

Keywords: Neutrosophic graph, Domination in neutrosophic graph, Split domination, Cardinality, Split domination number, Isolated vertex.

1 Introduction

A mathematical frame work to describe the phenomena of uncertainly in real life situation is first suggested by L.A.Zadeh in 1965[26]. Rosenfeld[16] introduced the notion of fuzzy graphs and several fuzzy analogs of graph theoretic concepts such as path, cycle and connectedness. The study of dominating sets in graphs was begun by Orge and Berge. Many authors discussed the concept of various dominations in graph, fuzzy and intuitionistic fuzzy graphs in [2,7,6,9,10,11,14,15, 18,19,24]. Q.M.Mahyoub and N.D.Soner[8] initiate the split dominating set and split domination number in fuzzy graphs. Also, the split domination number and its properties in Intuitionistic Fuzzy Graphs (IFGS) were studied by [11]. Neutrosophic set proposed by Smarandache[1] is powerful tool for dealing incomplete and indeterminate problems in the real world. It is the generalization of fuzzy sets [3] and Intuitionistic fuzzy sets [4,5]. Fuzzy graph and Intuitionistic approaches are failed in some applications when indeterminacy occurs. So Smarandache defined four main categories of Neutrosophic graphs in [20,21,22,23]. M.Mullai [27] introduced the concept of domination in neutrosophic graphs. By considering the existing split dominating sets, in this proposed work, the split domination in neutrosophic graph is developed with suitable examples to know the advantages of neutrosophic split domination in real world applications than other existing split dominations.

2 Preliminaries

Definition 2.1. [12].

An intuitionisic fuzzy graph is of the form G=(V, E) where
(1) V = $\{v_1,v_2,..........,v_n\}$ such that μ_1: $V \rightarrow [0,1]$ and γ_1: $V \rightarrow [0,1]$ denote the degree of membership and non-membership of the element $v_1 \in$ V, respectively, and $0 \leq \mu_1(v_i)$+$\gamma_1(v_i) \leq$1, for every $v_i \in$ V,(i=1,2,3,.......,n),
(2) E $\subseteq$ V$\times$ V, where μ_2 : V $\times V \rightarrow [0,1]$ and γ_2: V$\times V \rightarrow [0,1]$ are such that
μ_2 $(v_i, v_j) \leq$ min[$\mu_1(v_i)$,$\mu_1(v_j)$] and $\gamma_2(v_i, v_j) \geq$ max[$\gamma_1(v_i)$,$\gamma_1(v_j)$] and
0$\leq$ $\mu_2(v_i$,$v_j)$+$\gamma_2(v_i$,$v_j)$$\leq$ 1, for every $(v_i$,$v_j)$ $\in$E, (i,j=1,2,.........,n).

Definition 2.2. [4].

Let G =(V,E) be an intuitionistic fuzzy graph (IFG). Then the cardinality of G is defined to be
$|G| =| \sum\limits_{v_i \in V} \frac{1+\mu_1(v_i)-\gamma_1(v_i)}{2} + \sum\limits_{v_i,v_j \in V} \frac{1+\mu_2(v_i,v_j)-\gamma_2(v_i,v_j)}{2} |$.

Definition 2.3. [6].

A vertex u $\in V$ of an IFG G = (V,E) is said to be an isolated vertex if $\mu_2(u,v) = 0$ and $\gamma_2(u,v) = 0$ for all v $\in$ V. That is N(u)= $\emptyset$. Thus, an isolated vertex does not dominate any other vertex in G.

Definition 2.4. [6].

Let G= (V, E) be an IFG and let u,v $\in$ V, we say that u dominates v in G if there exists a strong arc between them. A subset D $\subseteq$ V is said to be dominating set in G if for every v $\in$ V-D, there exists u $\in$ D dominates v.

Definition 2.5. [4].

A dominating set D of IFG is said to be minimal dominating set if no proper subset S of D is a dominating set. Minimum cardinality among all minimal dominating sets is called the inituitionistic fuzzy dominating number, and is denoted by γ_{if}(G).

Definition 2.6. [12].

A dominating set D of a intuitionistic fuzzy graph G = (V,E) is a split dominating set if the induced fuzzy subgraph H = ($< V - D >$,$V\prime, E\prime$) is disconnected. The minimum fuzzy cardinality of a split dominating set is called a split domination number and id denotedby $\gamma_s(G)$.

Definition 2.7. [12].

A split dominating set D of a intuitionistic fuzzy graph G is said to be a minimal split dominating set if no proper subset of D is a split dominating set of G with $|D'| < 1$.

Definition 2.8. [12].

Minimum cardinality among all minimal split dominating set is called lower split domination number of IFG of G and is denoted by $D_S(G)$.

Definition 2.9. [12].

Maximum cardinality among all minimal split dominating set is called upper split domination number of IFG of G and is denoted by $D_S(G)$.

Definition 2.10. [16].

Let X be a space of points(objects) with generic elements in X denoted by X, then the neutrosophic sets A (NS A) is an object having the form
A = $\{< x : T_A(x), I_A(x), F_A(x) >, x \in X\}$,
where the functions T, I, F: X→ $[0^-, 1^+]$ define respectively the truth membership function, indeterminacy membership function , and a falsity membership function of the element x ∈ X to the set A with the condition $0^- \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$,
the functions $T_A(x), I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $[0^-, 1^+]$.

Definition 2.11. [16].

Let X be a space of points (objects) with generic elements in X denoted by X. A single valued neutrosophic set A (SVNS A) is characterized by truth membership function $T_A(X)$, an indeterminacy membership function $I_A(X)$ and a falsity membership function $F_A(X)$. For each point x in X, $T_A(X), I_A(X)$ and $F_A(X) \in [0, 1]$. A SVNS A can be written as
A =$\{< x : T_A(x), I_A(x), F_A(x) >, x \in X\}$.

Definition 2.12. [16].

Let A = (T_A, I_A, F_A) and B = (T_B, I_B, F_B) be single valued neutrosophic sets on a set X. If A = (T_A, I_A, F_A) is a single valued neutrosophic relation on a set X, then A = (T_A, I_A, F_A) is called a single valued neutrosophic relation on B = (T_B, I_B, F_B), if $T_B(x, y) \leq min(T_A(x), T_A(y))$, $I_B(x, y) \geq max(I_A(x), I_A(y))$, $F_B(x, y) \geq max(F_A(x), F_A(y))$ for all x,y in X.
A Single valued neutrosophic relation A on X is called symmetric if,
$T_A(x, y) = T_A(y, x)$, $I_A(x, y) = I_A(y, x)$, $F_A(x, y) = F_A(y, x)$, and
$T_B(x, y) = T_B(y, x)$, $I_B(x, y) = I_B(y, x)$,$F_B(x, y) = F_B(y, x)$, for all x,y in X.

Definition 2.13. [16].

A single valued neutrosophic graph (SVN- graph) with underlying set V is defined to be a pair G = (A,B) where,
(1) The functions $T_A : V \to [0, 1]$, $I_A : V \to [0, 1]$, $F_A : V \to [0, 1]$ denote the degree of truth membership, degree of indeterminacy membership and degree of falsity membership of the element $v_i \in V$, respectively and
$0 \leq T_A(v_i) + I_A(v_i) + F_A(v_i) \leq 3$, for all $v_i \in V$, (i = 1,2,3,......,n)
(2) The functions $T_B : E \subseteq V \times V \to [0, 1]$, $I_B : E \subseteq V \times V \to [0, 1]$ and $F_B : E \subseteq V \times V \to [0, 1]$ are defined by $T_B(\{v_i, v_j\}) \leq min(T_A(v_i), T_A(v_j))$, $I_B(\{v_i, v_j\}) \geq max(I_A(v_i), I_A(v_j))$ and $F_B(\{v_i, v_j\}) \geq max(F_A(v_i), F_A(v_j))$.
denote the degree of truth membership, degree of indeterminacy membership and degree of falsity membership of the edge $(v_i, v_j) \in$ E respectively, where
$0 \leq T_B(\{v_i, v_j\}) + I_B(\{v_i, v_j\}) + F_B(\{v_i, v_j\}) \leq 3$ for all $\{v_i, v_j\} \in E$,(i,j = 1,,2,....,n).

Definition 2.14. [16].

Let G = (A,B) be a single valued neutrosophic graph on the vertex set V and x,y ∈ V. x dominates y in G if $T_A(x, y) = min\{T_B(x), T_B(y)\}$,
$I_A(x, y) = min\{I_B(x), I_B(y)\}$ and $F_A(x, y) = min\{F_B(x), F_B(y)\}$. A subset D^N of V is called a dominating set in G if for every vertex v ∈ V-D^N there exists u ∈ D^N such that u dominates v.

Definition 2.15. [16].

The minimum cardinality of a dominating set in a neutrosophic graph G is called the domination number of G and is denoted by γ^N(G) (or) γ^N.

Definition 2.16. [16].

Let G be a neutrosophic graph. A dominating set D^N of G is said to be a minimal dominating set if no proper subset of D^N is a dominating set of G

Definition 2.17. [16].

A vertex x of a neutrosophic graph G is said to be an isolated vertex if,
$T_B(x,y) < min\{T_B(x), T_B(y)\}$
$I_B(x,y) < max\{I_B(x), I_B(y)\}$ and
$F_B(x,y) < max\{F_B(x), F_B(y)\}$, for all y $\in$ V-$\{x\}$,
(ie)N(x) = $\emptyset$.

Definition 2.18. [16].

A set of vertices D^N of a neutrosophic graph G is said to be independent
$T_A(xy) < min\{T_A(x), T_A(y)\}$,
$I_A(xy) < max\{I_A(x), I_A(y)\}$ and
$F_A(xy) < max\{F_A(x), F_A(y)\}$, for all x,y $\in D^N$.

3 Split domination in neutrosophic graphs

Definition 3.1. A dominating set D^N of a neutrosophic graph G = (A,B) is a split dominating set if the induced neutrosophic subgraph H = ($< V - D^N >$,V', E') is disconnected. The minimum neutrosophic cardinality of a split dominating set is called a split domination number and id denoted by $\gamma_s^N(G)$.

Example 3.2. Here strong arcs e_1, e_2, e_4 and e_6.
$[ie, T(v_1, v_3) > T^\infty(v_1, v_3), T(v_1, v_2) > T^\infty(v_1, v_2), T(v_2, v_4) > T^\infty(v_2, v_4)$ and $T(v_3, v_4) > T^\infty(v_3, v_4)]$.
Dominating set in neutrosophic is D^N = [v_2, v_3, v_5], V-D^N = $\{v_1, v_4\}$.

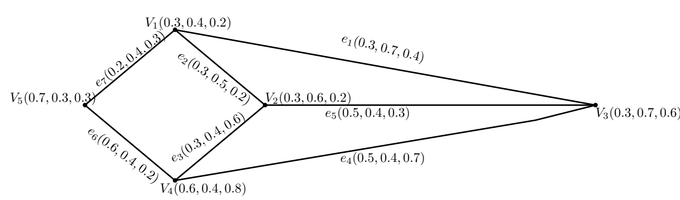

Figure 1: Neutrosophic Graphs

For every, v $\in$ V-D^N, there exists u $\in D^N$ and V-D^N is a induced neutrosophic subgraph and it is disconnected. (i.e)There exists two isolated vertices v_1 and v_4. The minimum cardinality of a split dominating set is called split domination number $\gamma_s^N(G)$ = 1.35

Theorem 3.3. *For any neutrosophic graph G = (A,B),*
(i)W(D^N) $\geq \delta(G)$
(ii) W(D^N) $\leq \Delta(G)$, where W(D^N) is a weight of a split dominating set.
Proof:

Consider Fig.1,
The strong arcs are e_1, e_2, e_4 and e_6. The minimum degree δ is
$\delta_T(G) = min\{d_T(v_i)/v_i \in V\}$ *= 0.6*

$\delta_I(G) = min\{d_I(v_i)/v_i \in V\}$ *= 0.8*
$\delta_F(G) = min\{d_F(v_i)/v_i \in V\}$ *= 0.8*
The minimum degree of G is $\delta(G) = min\{d_T(v_i), d_I(v_i), d_F(v_i)/v_i \in V\}$ *= (0.6,0.8,0.8)*
The maximum T degree is
$\Delta_T(G) = max\{d_T(v_i)/v_i \in V\}$ *= 0.9*
The maximum I degree is
$\Delta_I(G) = max\{d_I(v_i)/v_i \in V\}$ *= 1.6*
The maximum F degree is
$\Delta_F(G) = max\{d_F(v_i)/v_i \in V\}$ *= 1.6*
The maximum degree of G is
$\Delta(G) = max\{d_T(v_i), d_I(v_i), d_F(v_i)/v_i \in V\}$ *= (0.9,1.6,1.6)*
$W_T(D) = W_T\{v_2, v_3, v_5\} = W_T(v_2) + W_T(v_3) + W_T(v_5)$ *= 0.6*
$W_I(D) = W_I\{v_2, v_3, v_5\} = W_I(v_2) + W_I(v_3) + W_I(v_5)$ *= 1.2*
$W_F(D) = W_F\{v_2, v_3, v_5\} = W_F(v_2) + W_F(v_3) + W_F(v_5)$ *= 1.2*
W(D^N) = (0.6,1.2,1.2)
Also, $\delta(G)$ *= (0.6,0.8,0.8) and* $\Delta(G)$ *= (0.9,1.6,1.6)*
Therefore, W(D^N)$\geq \delta(G)$ and W(D^N)$\leq \Delta(G)$.
Hence the proof.

Theorem 3.4. *A dominating set D^N of a neutrosophic graph G is a split dominating set if and only if there exists two neutrosophic vertices u,v $\in$ V-D^N such that every u-v path contains an neutrosophic vertex of D^N.*
Proof:
Let D^N be a split dominating set of a neutrosophic graph G. Then induced neutrosophic subgraph $< V - D^N >$ is disconnected. Hence, there exist two vertices u,v $\in$ V-D^N such that every u-v path contains a neutrosophic vertex of D^N.
Let D^N be a dominating set. Then induced subgraph V-D^N of a neutrosophic graph G is connected (or) disconnected.
If it is connected, then there exist two vertices u,v in V-D^N such that some u-v path does not contain a neutrosophic vertex of D^N, which is a contradiction. Hence V-D^N is disconnected, which implies D^N is a neutrosophic split dominating set of G.

Theorem 3.5. *For any neutrosophic graph G = (A,B), $\gamma_S^N(G) \geq \beta(G)$, where $\beta(G)$ is a fuzzy vertex covering number of G.*
Proof:
Let S be a subset of A which is an independent set satisfying the condition $T_B(u,v) < T_B^\infty(u,v)$, $I_B(u,v) < I_B^\infty(u,v)$ and $F_B(u,v) < F_B^\infty(u,v)$ for all u,v $\in$ S. If S is maximal independent set then for every vertex v $\subset$ V , S the set S $\cup\{v\}$ is not independent.
That is, there exists strong neighbor adjacent to every vertex in S.
Hence, the minimum cardinality of a split dominating set is greater than maximum cardinality of independent set $\beta(G)$, a vertex covering of neutrosophic G. Hence, $\gamma_S^N(G) \geq \beta(G)$.

Definition 3.6. A split dominating set D^N of a neutrosophic graph G is said to be a minimal split dominating set if no proper subset of D^N is a split dominating set of G with $|D^{N'}| < 1$.

Theorem 3.7. *A split dominating set D^N of neutrosophic graph G is minimal if and only if for each vertex v $\in$ D^N one of the following conditions holds,*

(i) there exists a vertex u $\in$ *V-*D^N *such that N(u)*$\cup D^N$ *=* $\{v\}$*.*
(ii) v is an isolate in $< D^N >$*.*
(iii) $< V - (D^N)' >$*is connected.*

Proof:

Suppose that D^N *is minimal and there exists a vertex v* $\in D^N$ *such that v does not satisfy any of the above conditions.*
(ie) by condition (i), there exists a vertex u $\in$ *V-*D^N *such that N(u)*$\cup D^N \neq \{v\}$ *and by condition(ii), v is not an isolate vertex of the induced subgraph* $< D^N >$*.*
Let $(D^N)'$ *=* D^N*-*$\{V\}$*, then* D^N *is a split dominating set, which satisfies above two conditions.*
Hence, the induced subgraph $< V - (D^N) >$ *is disconnected. which contradicts the third condition.*
This implies a vertex v is in D^N*.*
Therefore, D^N *is minimal split dominating set, which satisfies one of the above conditions.*

Theorem 3.8. *For any neutrosophic graph G = (A,B) with neutrosophic end vertex* $\gamma_2^N(G) \geq \gamma^N(G)$*, Furthermore, there exists a split dominating set of G containing some vertices adjacent to neutrosophic end vertices.*

Proof:

Let v be a neutrosophic end vertex of neutrosophic graph G, then there exists neutrosophic cut vertex $'w'$ *such that* $T(u, v) > 0$ *,* $I(u, v) > 0$ *and* $F(u, v) > 0$*.*
Let D^N *be a dominating set of neutrosophic G. Suppose that w* $\in D^N$*, then* D^N *is a split dominating set of G. Repeating this process for all such cut vertices adjacent to neutrosophic end vertices, we obtain a split dominating set of G containing some vertices adjacent to the end- vertices.*

Definition 3.9. Minimum cardinality among all minimal split dominating set is called lower split domination number of neutrosophic graph G and is denoted by $d_S^N(G)$.

Definition 3.10. Maximum cardinality among all minimal split dominating set is called upper split domination number of neutrosophic graph G and is denoted by $D_S^N(G)$.

Example 3.11. Here, strong arcs are $(e_2, e_6, e_8, e_9, e_{10})$
Let the neutrosophic dominating sets are $D_1^N = \{v_1, v_2, v_4, v_5\}$; $D_2^N = \{v_1, v_3, v_4, v_6\}$;

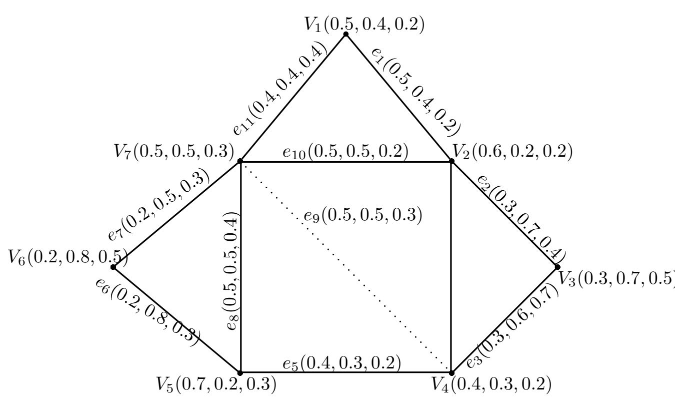

Figure 2: Neutrosophic minimal split dominating sets

$D_3^N = \{v_1, v_2, v_4, v_6\}$; $D_4^N = \{v_1, v_2, v_5, v_7\}$ and $D_5^N = \{v_1, v_3, v_5, v_7\}$.... etc.
Out of these neutrosophic dominating sets, minimal split dominating sets are
$D_1^N = \{v_1, v_2, v_4, v_5\}$, V-$D_1^N = \{v_3, v_6, v_7\}$

Therefore, $\gamma_S(D_1^N)$ = 2.55
$D_3^N = \{v_1, v_2, v_4, v_6\}$, V-$D_3^N = \{v_3, v_5, v_7\}$
Therefore, $\gamma_S(D_3^N)$ = 2
$D_5^N = \{v_1, v_3, v_5, v_7\}$, V-$D_5^N = \{v_2, v_4, v_6\}$
Therefore, $\gamma_S(D_5^N)$ = 2.1
Here upper split domination number of neutrosophic graph G is $D_S^N(G)$ = 2.55 and lower split domination number of G is $d_S^N(G)$ = 2.

Theorem 3.12. *For any neutrosophic graph G = (A,B)*
(i) $\gamma(G) \geq P/(\Delta_T(G) + 1)$
(ii) $\gamma(G) \geq P/(\Delta_I(G) + 1)$ *(iiI)* $\gamma(G) \geq P/(\Delta_F(G) + 1)$ *where* $\Delta_T(G)$ *is the maximum T -degree of G and* $\Delta_I(G)$ *is the maximum I - degree of G and* $\Delta_F(G)$ *is the maximum F- degree of G.*
Proof:
(i) Let D^N *be a neutrosophic dominating set of G with* $|D^N|$ = γ *and since every vertex in V-*D^N *is adjacent to some vertices in* D^N*, we have*
$$|V - D^N| \leq \sum_{i=1}^{r} d(v_i) \leq \gamma.\Delta_\mu(G)$$
$$\Rrightarrow p - \gamma(G) \leq \gamma.\Delta = \gamma(G).\Delta_\mu(G).$$
$$\Rrightarrow p \leq \gamma(G) + \gamma(G).\Delta_\mu(G) = \gamma(G).(1 + \Delta_\mu).$$
$$\Rrightarrow P/(\Delta_\mu(G) + 1) \leq \gamma(G).$$
Similarly, $\gamma(G) \geq P/(\Delta_\gamma(G) + 1)$.

Theorem 3.13. *For any neutrosophic graph G = (A,B)*
(i) $\gamma_S(G) \geq P.\Delta - T(G)/(\Delta_T(G) + 1)$
(ii) $\gamma_S(G) \geq P.\Delta_I(G)/(\Delta_I(G) + 1)$
(ii) $\gamma_S(G) \geq P.\Delta_F(G)/(\Delta_F(G) + 1)$
Proof:
The proof is similar to the above theorem.

Theorem 3.14. *If neutrosophic graph G = (A,B) has one neutrosophic cut vertex v and at least two neutrosophic blocks* H_1 *and* H_2 *with v adjacent to all vertices of* H_1 *and* H_2*, then v is in every split dominating set of G.*
Proof:
Let D^N *be a split dominating set of neutrosophic of G.*
Suppose v is a cut vertex does not belong to D^N*.*
This implies v $\in$ *V-*D^N*, then each of* H_1 *and* H_2 *contributes at least one vertex to* D^N *say u and w, which is adjacent to v. If V-*D^N *includes v then every V-*D^N *is connected.*
This implies, D^N *is not a split dominating set, which is a contradiction.*
Hence, v is in every split dominating set of neutrosophic of G.

Example 3.15. In fig.3, strong arcs are e_1, e_3, e_5, e_6 and e_9.
Let $H_1 = v_2, v_3, v_5, v_6$ and $H_2 = v_3, v_4, v_6$ be two blocks with one cut vertex v_3, which is adjacent to all vertices of H_1 and H_2.
Then v_3 is in every split dominating set of neutrosophic of G.

Theorem 3.16. *Let v be a neutrosophic cut vertex of neutrosophic graph G, if there is a block H in G such that v is the only cut vertex of H and v is adjacent to all vertices of H. Then there is a split dominating set of G*

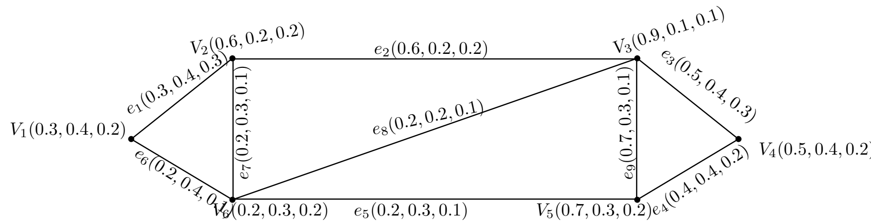

Figure 3: Neutrosophic split dominating sets

containing v.

Proof:

If there exists two blocks in neutrosophic graph G, satisfying the given condition, then by above theorem, v is in every split dominating set of neutrosophic graph G.

Theorem 3.17. *If $\gamma_S^N(G) \leq \gamma_D^N(G)$, then for any split dominating set D^N of neutrosophic graph G, V-D^N is a split dominating set of G.*

Proof:

Since D^N is minimal split dominating set in neutrosophic graph G, we know that $V - D^N$ is dominating set of G and D^N is a split dominating set, since $< D^N >$ is disconnected.

Theorem 3.18. *Let G be a neutrosophic graph such that both G and $\overline{G}$ are connected, then $\gamma_S(G) + \gamma_S(\overline{G}) \geq 2|V|$.*

Proof:

We know that $\gamma_S(G) \geq \beta(G)$. Since both G and $\overline{G}$ are connected, $\Delta(G), \Delta(\overline{G}) > P$ this implies $\alpha_0(G), \alpha_0(\overline{G}) \leq 0$. Hence $\gamma_S(G) \geq |v|$.
Similarly, $\gamma_S(\overline{G}) \geq p$, which implies $\gamma_S(G) + \gamma_S(\overline{G}) \geq |v| + |v| \geq 2|v|$.
Hence the theorem

Conclusion

Neutrosophic set is the generalization of fuzzy set and intuitionistic fuzzy set. Neutrosophic models in real world applications are flexible and compatiblethan fuzzy and intuitionistic fuzzy models. In this proposed work, the definition of split domination number in a neutrosophic set is defined with suitable examples and some theorems in split domination in neutrosophic graph are developed. Also, the bound on split domination number related to the above concepts are studied.Neutrosophic split dominating set gives more efficient results than other existing split dominating sets. In future, the concept of split domination in neutrosophic graphs will be extented and applied to many real life situation problems.

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Single Valued Neutrosophic VIKOR and Its Application to Wastewater Treatment Selection

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Abstract: The fuzzy VIsekriterijumska optimizacija i KOmpromisno Resenje (VIKOR) has been used in solving various multi-criteria decision-making problems where triangular fuzzy numbers are utilized in defining decision-makers' linguistic judgements. Most of the fuzzy VIKOR are built from linguistic variables based on fuzzy sets and its generalization such as intuitionistic fuzzy sets. Recent literature suggests that single-valued neutrosophic sets (SVNSs) can offer a better alternative particularly when fuzzy sets have some extent of limitations in handling indeterminacy and uncertainty. This paper proposes the SVNS-VIKOR where single-valued neutrosophic numbers are utilized in defining linguistic variables of VIKOR. Differently from the typical fuzzy VIKOR which directly utilizes fuzzy numbers with a single membership, the proposed method introduces three independent memberships of truth, indeterminacy and false to enhance judgments in the group decision-making environment. The obtained solutions would help policy makers in identifying the best solution that could enhance the efficiency of treating wastewater.

Keywords: multi-criteria decision-making; single-valued neutrosophic; VIKOR; linguistic variables; wastewater treatment; compromise solution

1. Introduction

The multi-criteria decision-making (MCDM) methods are got significant consideration in decision sciences discipline. In recent years, the necessity of concurrent consideration to the criteria and alternatives in decision problems is more vital especially in presence of uncertain data sets. So, decision makers use subjective evaluation methods to deal with this obstacle. Zadeh [1] introduced Fuzzy Sets (FSs) theory to overcome on uncertain and imprecise data sets. Besides, generalized FSs such as interval-valued fuzzy set [2], fuzzy soft set [3], interval type-2 fuzzy set [4], hesitant fuzzy set [5], intuitionistic fuzzy set [6], [7], intuitionistic fuzzy soft set [8] and interval-valued intuitionistic fuzzy set [9] were developed for the same purpose of FS. Although FS theory has been developed and extended, it still cannot deal with all possible uncertainties. For instance, when a decision-maker is asked on the possibility of the true answer, he/she may think the possibility is equal to 0.5, the possibility of the false answer is 0.8 and the degree of uncertainty is 0.3. This problem is beyond the scopes of FS and intuitionistic fuzzy set (IFS). Therefore, Smarandache [10] proposed the neutrosophic logic and neutrosophic set to generalized the concepts of the classic set, FS, IFS, etc. In neutrosophic set (NS), the truth-membership, indeterminacy membership, and false-membership are completely independent and lie in the nonstandard unit interval $]0^{-}, 1^{+}[$. From scientific point of view, neutrosophic set and set-theoretic view, it is difficult to apply in the real situation since operators need to be specified. Hence, Wang et al [11] defined a single-valued neutrosophic set (SVNS) and proposed the set theoretic operations and some properties of SVNSs.

There are several researches about using the SVNSs with MCDM methods in the literature. For example, Biswas et al [12] extended the grey relational analysis method to the neutrosophic environment and applied it to the selection of the investment sector. In this study, neutrosophic grey relational coefficient is calculated by using Hamming distance between each alternative to ideal neutrosophic estimates reliability solution and the ideal neutrosophic estimates un-reliability solution. Then, Then, for ranking/ordering all alternatives, the neutrosophic relational degree is determined [12]. Gomes and Lima [13] developed TODIM (An acronym in Portuguese of interactive and decision-making method named Tomada de decisao interativa e multicrite´vio) method to consider the risk preferences of decision-makers. Xu et al. [14] continued the TODIM method to the MCDM with the single-valued neutrosophic numbers (SVNNs). The authors proposed the extended classical TODIM method to solve the selection of emerging technology enterprise with the SVNNs. The complete evaluation of the decision-makers' bounded rationality, which is genuine action in the decision-making process, is a significant element of this method [14]. Biswas, et al. [15] introduced a new approach for multi-attribute group decision- making problems by extending the technique for order preference by similarity to ideal solution (TOPSIS) method to single-valued neutrosophic environment. They used SVNS to rank alternatives based on the characteristic, which expresses the opinion of the decision-makers based on the information provided [15]. The applicability and effectiveness of the proposed approach are shown with the tablet selection case study. Stanujkij [16] applied SVNs with Multi-Objective Optimization by a Ratio Analysis (MULTIMOORA) method for selection of communication circuit designs case study. The proposed method has the potential to be more efficient in handling a large number of complicated decision issues involving imprecise and insufficient data sets [16]. Sahin and Yigider [17] used single valued neutrosophic information with the TOPSIS method for supplier selection decision problems. The author agreed about the usage of SVNS beside MCDM methods in vast knowledge domains of the real-life as business, management, environmental sustainability, financial, scientific, and engineering. They addressed wastewater treatment (WWT) decision problems from sustainability engineering processes which get considerable attention in the usage of SVNS with MCDM methods [17].

Various methods of MCDM in selecting WWT technologies have been extensively discussed in many literatures [18]–[22]. Kalbar [23] developed the multiple attribute decision-making methodology TOPSIS and applied it to the WWT selection. The four most commonly used WWT technologies for treatment of municipal wastewater in India are ranked in many scenarios. A commonly used compensatory method, TOPSIS has been most preferred as the best to rank the WWT alternatives [23]. Abdullah [24] selected the most suitable WWT technology with the participation of three decision makers for providing WWT information and evaluating criteria using Fuzzy simple additive weighting (SAW) method . Ilangkumaran [25] recommended the best WWT technology using Fuzzy analytic hierarchy process (AHP) to assign the weights to the criteria and applying grey relation analysis technique to rank alternatives. Molinos-Senante [26] determined weights to the attributes and selected the most sustainable WWT technology using the AHP method. Abdullah and Rahman [27] implemented the analytic network process (ANP) to rank the alternatives and select the best WWT for the related case study. Zhou et al. [28] introduced a group decision-making model for WWT selection utilizing the intuitionistic fuzzy set to deal with uncertainties associated with the decision problem.

Apart from TOPSIS, AHP, ANP and SAW methods, VIKOR is another MCDM method which helps decision makers to solve MCDM problems in presence of conflicting and incommensurable criteria. VIKOR stands for VlseKriterijumska Optimizaciji I Kompromisno Resenje was translated in English as Multi-criteria Optimization and Compromise Solution. It was first developed in 1998 by Opricovic [29] that solves discrete decision problems with conflicting criteria. VIKOR rates alternatives by compromising solution from a set of conflicting criteria and comparing the proximity to the ideal solution [30]. The main advantage of the VIKOR method is its practicality in real case problems and the final results can be accomplished owing to the initial characteristics and capabilities. Moreover, regarding VIKOR ability is solving MCDM problems with discrete data sets and, the obtained compromise solution gives a maximum group utility for the ''majority'' and a minimum individual regret for the ''opponent''. The multi-criteria ranking index is ranked based on the particular measure of ''closeness'' to the ideal solution [30].

Recently, some approaches are introduced to generalize the crisp VIKOR method into fuzzy environment to cover uncertain information [31]–[33], however, there is a lack of a proper approach for solving the multi-criteria group decision-making with IFS. So, an extension of fuzzy VIKOR method, namely intuitionistic fuzzy VIKOR is proposed for handling fuzzy MCDM problems based on the IFSs [34]–[36], where the characteristics of the alternatives and attributes are represented by the IFS. The generalizing into interval-valued intuitionistic fuzzy VIKOR models was done [37]–[39] because the interval-valued intuitionistic fuzzy set (IVIFS) is more suitable for dealing with imprecise and uncertain information than FS or IFS. Although there is success in solving decision-making problems, still fuzzy-based VIKOR method cannot handle the problems with neutrosophic information. Therefore, it is essential to describe the generalization of the MCDM approaches under the NSs environment [40]–[44].

Since its introduction, the VIKOR method has received much attention from a number of scholars to solve MCDM problems. For instance, Ghorabaee [45] introduced VIKOR with interval type-2 fuzzy numbers to handle the robot selection decision problem. Liu et al [46] proposed an interval 2-tuple linguistic VIKOR method for solving the material selection under uncertain and incomplete information considering the subjective and objective weights of criteria simultaneously. The proposed method has exact characteristics and could avoid information distortion and loss in linguistic information processing. Devi [34] introduced an intuitionistic fuzzy VIKOR to solve robot selection problems and expressed the performance rating values as well as the weights of criteria with linguistic terms using triangular IFSs. Peng et al. [35] presented an efficient VIKOR method which optimizes multi-response problems in intuitionistic fuzzy environments. They evaluated the importance weights of various responses in terms of IFSs and applied the proposed approach in two case studies which are plasma-enhanced chemical vapor deposition and a double-sided surface mount technology electronic assembly operation. Other than that, VIKOR method has been widely applied in many other decision-making problems such as supplier selection [47]–[49], e-government website evaluation [50], doctor selection [51], robot selection [34], [45], renewable energy selection [52], [53], WWT technology [19], [54], insurance company selection [55], [56] and material selection [57]–[59].

To deal with imprecise and inconsistent information, we extend the crisp VIKOR method applying the SVNS environment, namely single-valued neutrosophic based VIKOR (SVN-VIKOR). The main advantage of the proposed method is dealing with imprecise and inconsistent information. Nowadays, the way of rating alternatives and selecting the best one based on uncertain conditions and given requirement based on SVNSs is an interesting and significant research topic that is motivated us to do this research. This paper has twofold purposes. Firstly, we would like to present the SVNSs into the VIKOR method as a new approach that is the extension of traditional FSs. Comparing results of crisp, fuzzy, IFS, and IVIFS with the results of SVNS provides significantly greater flexibility, which can be helpful to solve decision-making problems associated with ambiguity and uncertain. Secondly, the proposed approach contributes to ease the interpretation of a complex decision-making problems. For testing the proposed method, a case study of WWT selection problem is used to rate and select the best WWT technology. The paper is organized as follows. In second section, we review some basic concepts of NS and SVNS. The third section explains the proposed SVN-VIKOR. Subsequently, we illustrate a case study to show the decision-making steps and the applicability of the proposed method to WWT decision-making case study. The last section refers to the conclusion and future works.

2. Preliminaries

In this section, some definitions with regard to neutrosophic set, single-valued neutrosophic set and entropy are reviewed. These definitions can be retrieved from the references [51, 10].

2.1. Neutrosophic set [10]

Let X be a nonempty set. A neutrosophic set A of X defined as $\boldsymbol{A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X, T(x), I_A(x), F_A(x) \in\ ]^{-}0, 1^{+}[\ \}}$, where $\boldsymbol{T_A(x)}$, $\boldsymbol{I_A(x)}$ and $\boldsymbol{F_A(x)}$ are truth membership function, indeterminacy-membership function, and falsity-membership function respectively.

2.2. Single-valued neutrosophic set [11]

Let X be a nonempty set. Single-valued neutrosophic set A of X is defined as $\boldsymbol{A = \{\langle x, T(x), I_A(x), F_A(x)\rangle | x \in X\}}$ where $\boldsymbol{T_A(x)}$, $\boldsymbol{I_A(x)}$ and $\boldsymbol{F_A(x) \in [0, 1]}$ for each $\boldsymbol{x \in X}$ and $\boldsymbol{0 \le T_A(x) + I_A(x) + F_A(x) \le 3}$.

2.2.1. Set theoretic operations and relations [11]

The notion of union, intersection, inclusion and equality have been defined on single-valued neutrosophic sets as follows.
Given that two SVNS sets $\boldsymbol{H_1}$ and $\boldsymbol{H_2}$ in U:

1. Union of two sets formed $\boldsymbol{H_3}$ written as:
$$\boldsymbol{H_3 = H_1 \cup H_2}$$
$$\boldsymbol{T_{H_3}(u) = \max\{T_{H_1}(u), T_{H_2}(u)\}},$$
$$\boldsymbol{I_{H_3}(u) = \min\ \{I_{H_1}(u), I_{H_2}(u)\}},$$
$$\boldsymbol{F_{H_3}(u) = \min\ \{F_{H_1}(u), F_{H_2}(u)\}}$$

2. Intersection of two sets denoted by $\boldsymbol{H_4}$ defined as:
$$\boldsymbol{H_4 = H_1 \cap H_2}$$
$$\boldsymbol{T_{H_4}(u) = \min\ \{T_{H_1}(u), T_{H_2}(u)\}},$$
$$\boldsymbol{I_{H_4}(u) = \max\ \{I_{H_1}(u), I_{H_2}(u)\}},$$
$$\boldsymbol{F_{H_4}(u) = \max\ \{F_{H_1}(u), F_{H_2}(u)\}}$$

3. Inclusion of two sets denoted by $\boldsymbol{H_1 \subseteq H_2}$ defined as:
$\boldsymbol{T_{H_1}(u) \le T_{H_2}(u), I_{H_1}(u) \ge I_{H_2}(u), F_{H_1}(u) \ge F_{H_2}(u)}$ for all $\boldsymbol{u \in U}$.

4. Equality of two sets denoted by $\boldsymbol{H_1 = H_2}$ defined as:
$\boldsymbol{T_{H_1}(u) = T_{H_2}(u), I_{H_1}(u) = I_{H_2}(u), F_{H_1}(u) = F_{H_2}(u)}$ for all $\boldsymbol{u \in U}$.

2.2.2. Axiomatic of Entropy

Let $N(X)$ be all SVNSs on x and $A \in N(x)$. An entropy on SVNSs is a function $E_N: N(X) \to [0,1]$ which satisfies the following axioms:
The entropy of SVNS set A is:

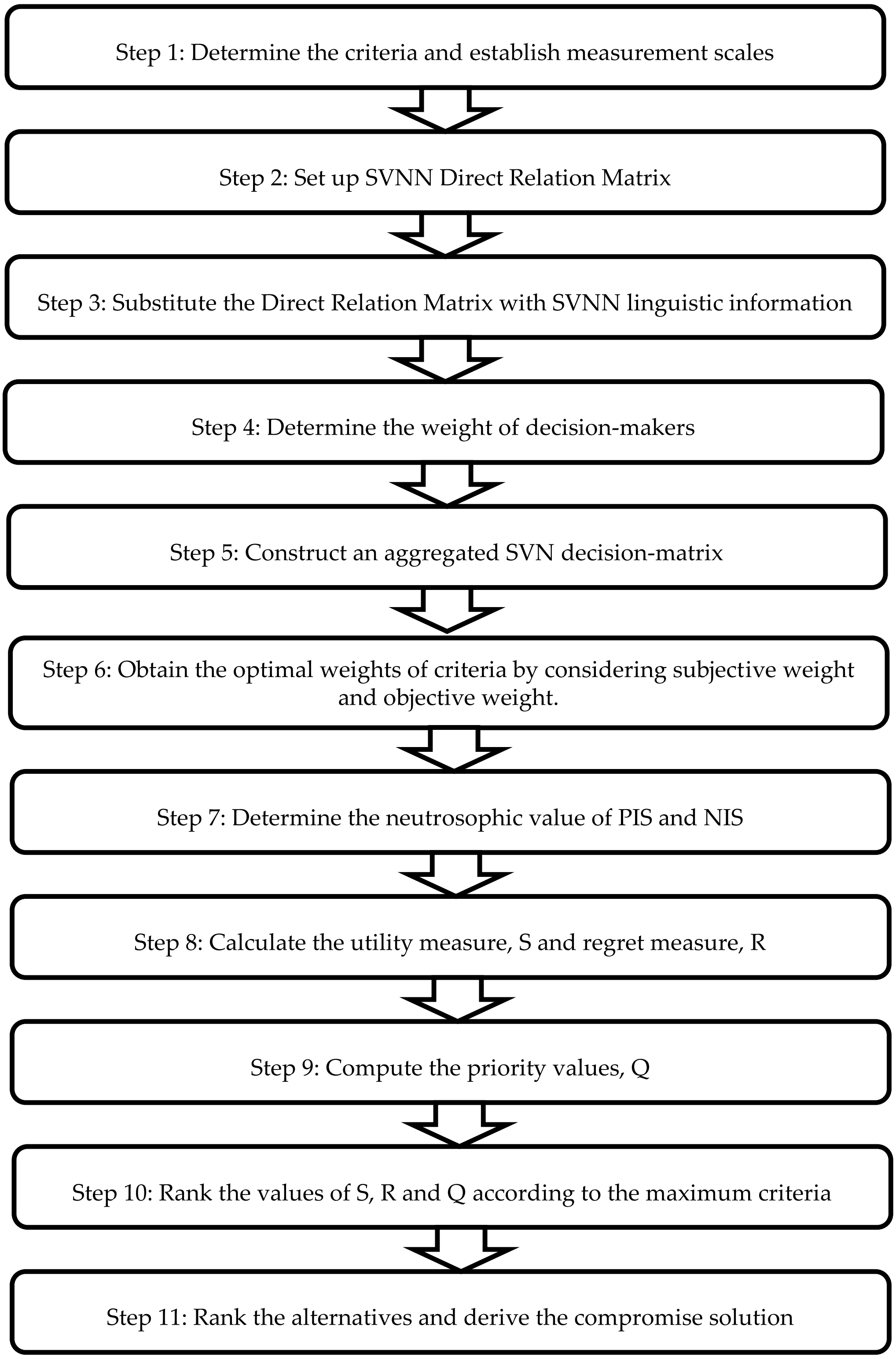
$$E_N(A) = \frac{1}{n}\sum_{t=1}^{n}\left(1 - \frac{1}{b-a}\int_a^b |T_A(x_i) - F_A(x_i)|\left|I_A(x_i) - I_{A}c(x_i)\right|dx\right) \text{ for all } x \in X.$$

1. $E_N(A) = 0$ if A is crisp set.
2. $E_N(A) = 1$ if $\left(T_A(x), I_A(x), F_A(x)\right) = (0.5, 0.5, 0.5)$ for all $x \in X$.
3. $E_N(A) \geq E_N(B)$ if $A \subset B$.
4. $E_N(A) = E_N(A^C)$ for all $A \in N(X)$.

3. Proposed SVN-VIKOR method

MCDM evaluation is a complex, imprecise and time-consuming process. Moreover, it is a very significant process for choosing the best alternative. In this section, we extend the VIKOR method to solve MCDM problems in which all preference information provided by decision-makers are expressed as single-valued neutrosophic values, and the interaction phenomena among the preference of individual decision-makers and conflicting criteria are taken into account. To deal with vague and inconsistent information, we apply the SVN-VIKOR approach since it can compromise the multiple, conflict criteria and dealing efficiently with vague and inconsistent information. The SVN-VIKOR method is developed to provide a rational, systematic decision-making process by which one discovers the best solution and a compromise solution that can be used to resolve a MCDM problem in neutrosophic environment. The extended VIKOR decision procedure of MCDM based on SVNS is summarized as in Figure 1.

Figure 1 A general procedure of the proposed SVN-VIKOR method



Based on the Figure 1, the proposed method consists of eleven steps. Differently with other VIKOR based methods, the proposed SVN-VIKOR method has two types of optimal weights of criteria which are subjective weight and objective weight.

Let $D^{(k)}$ be a committee of decision-makers where $k = 1, 2, \dots, p$. A_i represents alternatives, where $i = 1, 2, \dots, m$ and C_j represents criteria, where $j = 1, 2, \dots, n$. The criteria can be classified as cost criteria and benefit criteria.

Step 1: Define criteria and establish measurement scales

Step 2: Set up single valued neutrosophic number (SVNN) Direct-Relation Matrix [42]

We will get a single-valued neutrosophic decision-matrix $X_k, (k = 1, 2, \dots, p)$ for k th decision-maker as shown as follows:

$$X_k = \begin{array}{c} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{ccccc} C_1 & \cdots & C_j & \cdots & C_n \end{array} \\ \begin{bmatrix} \langle t_{11}^k, i_{11}^k, f_{11}^k \rangle & \cdots & \langle t_{1j}^k, i_{1j}^k, f_{1j}^k \rangle & \cdots & \langle t_{1n}^k, i_{1n}^k, f_{1n}^k \rangle \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \langle t_{i1}^k, i_{i1}^k, f_{i1}^k \rangle & \cdots & \langle t_{ij}^k, i_{ij}^k, f_{ij}^k \rangle & \cdots & \langle t_{in}^k, i_{in}^k, f_{in}^k \rangle \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \langle t_{m1}^k, i_{m1}^k, f_{m1}^k \rangle & \cdots & \langle t_{mj}^k, i_{mj}^k, f_{mj}^k \rangle & \cdots & \langle t_{mn}^k, i_{mn}^k, f_{mn}^k \rangle \end{bmatrix} \end{array}$$

Step 3: Substitute the Direct Relation Matrix with SVNN linguistic information [12]

Step 4: Determine the weight of decision-makers

The weight of the k th decision-maker can be obtained through the following formula [17]:

$$\psi_k = \frac{T_k + I_k\left(\frac{T_k}{T_k+F_k}\right)}{\sum_{k=1}^{p}\left(T_k + I_k\left(\frac{T_k}{T_k+F_k}\right)\right)}, where\ \psi_k \geq 0, \sum_{k=1}^{p} \psi_k = 1. \tag{1}$$

and

T_k represents the truth-membership function of k th decision-maker;
I_k represents the indeterminacy-membership function of k th decision-maker;
F_k represents the falsity-membership function of k th decision-maker.

Step 5: Construct an aggregated single-valued neutrosophic decision-matrix

Let $D^{(k)} = (D_{ij}^k)_{mxn}$ be a single-valued decision-matrix of the k th decision-maker. The simplified neutrosophic weighted averaging (SNWA) operator [60] is used to aggregate all individual decision-matrices of $D^{(k)} = (D_{ij}^k)_{mxn}$ where k = 1, 2, …, p, i = 1, 2, …, m and j = 1, 2, …, n into a collective decision-matrix $D = (d_{ij})_{mxn}$.

$$d_{ij} = SNWA\left(d_{ij}^{(1)}, d_{ij}^{(2)}, \dots, d_{ij}^{(p)}\right)$$

$$= \left(1 - \prod_{k=1}^{p}\left(1 - T_{ij}^{(k)}\right)^{\frac{1}{p}}, \prod_{k=1}^{p}\left(1 - I_{ij}^{(k)}\right)^{\frac{1}{p}}, \prod_{k=1}^{p}\left(1 - F_{ij}^{(k)}\right)^{\frac{1}{p}}\right) \tag{2}$$

The aggregated decision-matrix D is defined as follows:

$$D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{m1} & d_{m2} & \cdots & d_{mn} \end{bmatrix}, \text{ where } d_{ij} = \left(T_{ij}, I_{ij}, F_{ij}\right).$$

Step 6: Obtain the optimal weights of criterion:

In this section, there are two types of weight of criteria that need to be considered which are subjective weight and objective weight.

1. Subjective weight

The rating of alternatives with regards to each criterion is collecting through decision-makers' opinion. The weight of importance of the criteria correspond to alternatives are identified through linguistic rating scale as follows:

Table 1 Five-point linguistic rating scale and its linguistic terms.

Linguistic Terms	Influence Score	SVNNs
Very Low	1	⟨0.1, 0.8, 0.9⟩
Low	2	⟨0.35, 0.6 ,0.7⟩
Medium	3	⟨0.5, 0.4, 0.45⟩
High	4	⟨0.8, 0.2, 0.15⟩
Very High	5	⟨0.9, 0.1, 0.1⟩

Assume that the weight of the criterion is obtained using eq (3):

$$w_j = SVNSWA\left(w_j^1, w_j^2, \ldots, w_j^l\right)$$
$$= \left(1 - \prod_{k=1}^{l}\left(1 - T_{ij}^{(k)}\right)^{\psi_k}, \prod_{k=1}^{l}\left(I_{ij}^{(k)}\right)^{\psi_k}, \prod_{k=1}^{l}\left(F_{ij}^{(k)}\right)^{\psi_k}\right) \tag{3}$$

where $j = 1,2,\ldots,n$ and $w_j = \left(T_j, I_j, F_j\right)$ is the importance weight of the j th criterion. Normalized subjective weight of each criterion can be obtained using eq (4). Assume that our decision group has k decision-makers and $A_j = \left(T_j, I_j, F_j\right)$ is an SVNN expresses j th decision-maker.

$$w_j^s = T_j + I_j\left(\frac{T_j}{T_j+F_j}\right)\left[-\left(\frac{1}{\ln m}\right)\sum_{i=1}^{m} T_j + I_j\left(\frac{T_j}{T_j+F_j}\right)\right]^{-1} \tag{4}$$

where $j = 1,2,\ldots,n,$ and $\sum_{j=1}^{n} w_j^s = 1.$

2. Objective weight [46]

The evaluation criterion should be normalized by using eq (5):

$$P_{ij} = \frac{x_{ij}}{\sum_{i=1}^{m} x_{ij}} \tag{5}$$

where P_{ij} is the projected outcome of criterion *j*.

Next, entropy E_j of the set of projected outcomes of criterion *j* should be calculated using eq (6):

$$E_j = -\left(\frac{1}{\ln m}\right)\sum_{i=1}^{m} P_{ij}\ln P_{ij} \tag{6}$$

where m is the number of criteria and $0 \le E_j \le 1$.

After that, the divergence, div_j which is the divergence degree of the intrinsic information of criterion j, should be defined in order to obtain the objective weights of the criteria.

$$div_j = 1 - E_j \tag{7}$$

The greater the divergence degree of the criterion, the more important the criterion in the decision-making process. Finally, objective weights can be obtained using eq (8).

$$w_j^o = \frac{div_j}{\sum_{j=1}^{n} div_j} \tag{8}$$

Step 7: Determine the neutrosophic value of positive ideal solution (PIS) $f_j^+ = \langle T_j^+, I_j^+, F_j^+ \rangle$ and the neutrosophic value of negative ideal solution (NIS) $f_j^- = \langle T_j^-, I_j^-, F_j^- \rangle$ for all criteria rating, $j = 1,2,\ldots,n$.

$$f_j^- = \begin{Bmatrix} \min\limits_i r_{ij}, \text{for Benefit Criteria} \\ \max\limits_i r_{ij}, \text{for Cost Criteria} \end{Bmatrix} \tag{9}$$

$$f_j^+ = \begin{Bmatrix} \max\limits_i r_{ij}, \text{for Benefit Criteria} \\ \min\limits_i r_{ij}, \text{for Cost Criteria} \end{Bmatrix} \tag{10}$$

where $j = 1,2, \ldots, n$.

Step 8: Calculate the utility measure, (S_i), and the regret measure, (R_i) for the alternative as follow:

$$S_i = \sum_{j=1}^{n} \frac{w_j \|\tilde{f}_j^+ - \tilde{f}_{1j}\|}{\|\tilde{f}_j^+ - \tilde{f}_j^-\|}, i = 1,2,\ldots,m \tag{11}$$

$$R_i = \max_j \left\{ \frac{w_j \|\tilde{f}_j^+ - \tilde{f}_{1j}\|}{\|\tilde{f}_j^+ - \tilde{f}_j^-\|} \right\}, i = 1,2,\ldots,m \tag{12}$$

where w_j indicates the combination weight for each criterion.

$$w_j = v w_j^s + (1 - v) w_j^o \tag{13}$$

where v denotes relative importance between subjective weights and objective weights. It can be taken in any value from 0 to 1 but usually it is set as 0.5.

Step 9: Compute the priority values, $Q_i, i = 1,2,\ldots,m$ by using the formula as follow:

$$Q_i = v \frac{(S_i - S^+)}{(S^- - S^+)} + (1 - v) \frac{(R_i - R^+)}{(R^- - R^+)} \tag{14}$$

where $S^+ = \min\limits_i S_i$, $S^- = \max\limits_i S_i$, $R^+ = \min\limits_i R_i$, $R^- = \max\limits_i R_i$,

v indicates the weight of the strategy of the majority of criteria, usually it is assumed as 0.5.

Step 10: Rank the value of S_i, R_i, and Q_i according to maximum criteria. Rank the alternatives in decreasing order results.

Step 11: Rank the alternatives and derive the compromise solution.

The alternative $A^{(1)}$ (top alternative) that ranks the best in minimum value of Q fulfills the following two conditions:

Condition 1: Acceptable advantages

$$Q(A^{(2)}) - Q(A^{(1)}) \geq \frac{1}{m-1} \quad (15)$$

where $A^{(1)}$ and $A^{(2)}$ are the top two alternatives in Q_i.

Condition 2: Acceptable Stability

The top alternatives should be the best ranked by S_i and R_i.

If one of the above conditions cannot be satisfied, a set of compromise solutions has been proposed:

1. Alternatives $A^{(1)}$ and $A^{(2)}$ are accepted if only stability condition is not satisfied;
2. Alternatives $A^{(1)}$, $A^{(2)}$, …, $A^{(u)}$ are accepted if advantage condition is not satisfied. $A^{(u)}$ is determined by the relation $Q(A^{(u)}) - Q(A^{(1)}) \geq \frac{1}{m-1}$ for maximum u (the positions of these alternatives are in closeness).

4. Implementation

In order to test the efficiency and effectiveness of proposed method for the WWT selection, we present a case study which are includes scenario, implementation methods and data analysis. The WWT selection is a complex process, where alternatives and criteria are inherited some extent of imprecise information. In the midst of this complexity, it is a very vital process for finding the best alternative to solve the problem. Our case study includes the evaluation process of various WWT which contains five alternatives as follows: Activated Sludge (A1), Aerated Lagoons (A2), Rotating Biological Contactors (A3), Oxidation Ditch (A4) and Trickling Filter (A5). Three interview sessions of three decision-makers were conducted to evaluate the importance of alternatives with respect to criteria of WWT.

The decision-making for the best WWT alternatives is done by considering the basis of nine core factors or criteria. The amount of pollutants removed (C1), lifetime (C2), operation and maintenance cost (C3), reliability (C4), capital cost (C5), environmental impacts (C6), sustainability (C7), land area requirement (C8) and safety risk to worker (C9) are nine criteria of WWT. The procedure of decision-making is presented as follows:

Step 1: Define criteria and establish measurement scales

Structured interviews were used to choose three distinct decision-makers (DM1, DM2, DM3) as expert groups in order to gather their perspectives. The DMs were carefully chosen to ensure that they are well recognised and specialists in WWT technology. Tables 2 and 3 describe decision-makers' perspectives on the weight of importance of criteria and the evaluation of WWT alternatives against the criteria.

Table 2 Influence score of three decision-makers on importance of criteria.

	DM1	**DM2**	**DM3**
C1	5	4	5
C2	3	5	5
C3	5	5	4
C4	3	4	5

C5	5	3	5
C6	5	3	3
C7	4	3	5
C8	4	4	3
C9	5	2	3

Table 3 Rating on evaluation of WWT alternatives with respected to criteria.

		C1	C2	C3	C4	C5	C6	C7	C8	C9
A1	DM1	7	7	7	7	7	5	7	5	6
	DM2	7	7	7	7	5	7	7	7	7
	DM3	7	7	8	8	6	5	4	4	6
A2	DM1	6	6	5	6	4	4	5	1	6
	DM2	7	7	7	7	5	7	7	3	7
	DM3	6	6	4	8	3	4	6	7	6
A3	DM1	7	7	5	7	6	8	7	5	6
	DM2	7	5	7	7	7	7	7	7	7
	DM3	7	7	4	7	6	8	7	5	6
A4	DM1	6	6	5	6	4	4	5	1	6
	DM2	7	7	7	5	5	7	7	5	7
	DM3	7	7	5	6	4	4	5	3	6
A5	DM1	7	6	4	7	4	8	7	6	5
	DM2	7	5	3	7	5	7	7	7	7
	DM3	7	6	5	5	4	6	7	3	5

Step 2 & Step 3: Construct SVNN Direct Relation Matrix

SVNNs Linguistic Phrases are substituted into table 3 in order to construct SVNN Direct Relation Matrix (see Table 4).

Table 4 SVNN Direct Relation Matrix.

		C1	C2	C3	C4	C5	C6	C7	C8	C9
A1	DM1	0.8	0.8	0.8	0.8	0.8	0.5	0.8	0.5	0.65
		0.2	0.2	0.2	0.2	0.2	0.5	0.2	0.5	0.35
		0.15	0.15	0.15	0.15	0.15	0.45	0.15	0.45	0.3
	DM2	0.8	0.8	0.8	0.8	0.5	0.8	0.8	0.8	0.8
		0.2	0.2	0.2	0.2	0.5	0.2	0.2	0.2	0.2
		0.15	0.15	0.15	0.15	0.45	0.15	0.15	0.15	0.15
	DM3	0.8	0.8	0.9	0.9	0.65	0.5	0.35	0.35	0.65
		0.2	0.2	0.1	0.1	0.35	0.5	0.65	0.65	0.35
		0.15	0.15	0.05	0.05	0.3	0.45	0.6	0.6	0.3
A2	DM1	0.65	0.65	0.5	0.65	0.35	0.35	0.5	0.05	0.65
		0.35	0.35	0.5	0.35	0.65	0.65	0.5	0.9	0.35

		0.3	0.3	0.45	0.3	0.6	0.6	0.45	0.95	0.3
		0.8	0.8	0.8	0.8	0.5	0.8	0.8	0.2	0.8
	DM2	0.2	0.2	0.2	0.2	0.5	0.2	0.2	0.75	0.2
		0.15	0.15	0.15	0.15	0.45	0.15	0.15	0.8	0.15
		0.65	0.65	0.35	0.9	0.2	0.35	0.65	0.8	0.65
	DM3	0.35	0.35	0.65	0.1	0.75	0.65	0.35	0.2	0.35
		0.3	0.3	0.6	0.05	0.8	0.6	0.3	0.15	0.3
		0.8	0.8	0.5	0.8	0.65	0.9	0.8	0.5	0.65
	DM1	0.2	0.2	0.5	0.2	0.35	0.1	0.2	0.5	0.35
		0.15	0.15	0.45	0.15	0.3	0.05	0.15	0.45	0.3
		0.8	0.5	0.8	0.8	0.8	0.8	0.8	0.8	0.8
A3	DM2	0.2	0.5	0.2	0.2	0.2	0.2	0.2	0.2	0.2
		0.15	0.45	0.15	0.15	0.15	0.15	0.15	0.15	0.15
		0.8	0.8	0.35	0.8	0.65	0.9	0.8	0.5	0.65
	DM3	0.2	0.2	0.65	0.2	0.35	0.1	0.2	0.5	0.35
		0.15	0.15	0.6	0.15	0.3	0.05	0.15	0.45	0.3
		0.65	0.65	0.5	0.65	0.35	0.35	0.5	0.05	0.65
	DM1	0.35	0.35	0.5	0.35	0.65	0.65	0.5	0.9	0.35
		0.3	0.3	0.45	0.3	0.6	0.6	0.45	0.95	0.3
		0.8	0.8	0.8	0.5	0.5	0.8	0.8	0.5	0.8
A4	DM2	0.2	0.2	0.2	0.5	0.5	0.2	0.2	0.5	0.2
		0.15	0.15	0.15	0.45	0.45	0.15	0.15	0.45	0.15
		0.8	0.8	0.5	0.65	0.35	0.35	0.5	0.2	0.65
	DM3	0.2	0.2	0.5	0.35	0.65	0.65	0.5	0.75	0.35
		0.15	0.15	0.45	0.3	0.6	0.6	0.45	0.8	0.3
		0.8	0.65	0.35	0.8	0.35	0.9	0.8	0.65	0.5
	DM1	0.2	0.35	0.65	0.2	0.65	0.1	0.2	0.35	0.5
		0.15	0.3	0.6	0.15	0.6	0.05	0.15	0.3	0.45
		0.8	0.5	0.2	0.8	0.5	0.8	0.8	0.8	0.8
A5	DM2	0.2	0.5	0.75	0.2	0.5	0.2	0.2	0.2	0.2
		0.15	0.45	0.8	0.15	0.45	0.15	0.15	0.15	0.15
		0.8	0.65	0.5	0.5	0.35	0.65	0.8	0.2	0.5
	DM3	0.2	0.35	0.5	0.5	0.65	0.35	0.2	0.75	0.5
		0.15	0.3	0.45	0.45	0.6	0.3	0.15	0.8	0.45

Step 4: Determine the weights of decision-makers

The weights of decision-makers are obtained as in eq (1).

$$T_k + I_k\left(\frac{T_k}{T_k + F_k}\right)$$

I: $0.8 + 0.2\left(\frac{0.8}{0.8+0.15}\right) = 0.9684$

I: $0.8+0.2\left(\frac{0.8}{0.8+0.15}\right)=0.9684$

M: $0.5+0.4\left(\frac{0.5}{0.5+0.45}\right)=0.7105$

$$\sum_{k=1}^{p}\mu_k+\pi_k\left(\frac{\mu_k}{\mu_k+v_k}\right)=0.9684+0.9684+0.7105=2.6474$$

ψ_k, where $k=1,2,3$

DM1 = $\frac{0.9684}{2.6474}=0.3658$, DM2 = $\frac{0.9684}{2.6474}=0.3658$, DM3 = $\frac{0.7105}{0.6474}=0.2684$

Step 5: Construct an aggregated SVN decision-matrix

The importance weight of decision-makers is shown in Table 5.

Table 5 Importance weight of decision-makers.

DM	Linguistic Variable	IFNs	Weights
1	Important	$\langle 0.8, 0.2, 0.15\rangle$	0.3658
2	Important	$\langle 0.8, 0.2, 0.15\rangle$	0.3658
3	Medium	$\langle 0.5, 0.4, 0.45\rangle$	0.2684

To fuse the weight of all decision-makers into one, SNWA operator is applied by using eq (2).
$T_{1,1}=1-\left((1-0.8)^{0.3658}\times(1-0.8)^{0.3658}\times(1-0.8)^{0.2684}\right)=0.8$
$I_{1,1}=0.2^{0.3658}\times 0.2^{0.3658}\times 0.2^{0.2684}=0.2$
$F_{1,1}=0.15^{0.3658}\times 0.15^{0.3658}\times 0.15^{0.2684}=0.15$
The rest of calculations are calculated in similarly. The detailed calculation of aggregated SVNS matrix is shown in the Table 6.

Table 6 Aggregated SVNS matrix.

	C1	C2	C3	C4	C5	C6	C7	C8	C9
	0.8	0.8	0.8339	0.8340	0.6750	0.6424	0.7256	0.6163	0.7148
A1	0.2	0.2	0.1660	0.1660	0.3250	0.3576	0.2744	0.3837	0.2852
	0.15	0.15	0.1116	0.1117	0.2700	0.3011	0.2176	0.3252	0.2328
	0.7148	0.7148	0.6163	0.79623	0.3756	0.5777	0.6750	0.4128	0.7148
A2	0.2852	0.2852	0.3837	0.20377	0.6136	0.4223	0.3250	0.5623	0.2852
	0.2328	0.2328	0.3252	0.14393	0.5834	0.3613	0.2700	0.5436	0.2328
	0.8	0.7204	0.6163	0.8	0.7148	0.8711	0.8	0.6424	0.7148
A3	0.2	0.2796	0.3837	0.2	0.2852	0.1289	0.2	0.3576	0.2852
	0.15	0.2242	0.3252	0.15	0.2328	0.0747	0.15	0.3011	0.2328
A4	0.7546	0.7546	0.6424	0.6012	0.4095	0.5777	0.6424	0.2827	0.7148

	0.2454	0.2454	0.3576	0.3988	0.5905	0.4223	0.3576	0.6912	0.2852
	0.1932	0.1932	0.3011	0.3479	0.5401	0.3613	0.3011	0.6902	0.2328
	0.8	0.6012	0.3464	0.7442	0.4095	0.8196	0.8	0.6439	0.6424
A5	0.2	0.3988	0.6384	0.2558	0.5905	0.1804	0.2	0.3499	0.3577
	0.15	0.3480	0.6171	0.2014	0.5401	0.1209	0.15	0.3029	0.3011

Step 6: Obtain the optimal weight of criterion

The linguistic variables are substituted into Table 2. Aggregated subjective weight of criterion is calculated by using eq (3).

$T_1 = 1 - \left((1-0.9)^{0.3658} \times (1-0.8)^{0.3658} \times (1-0.9)^{0.2684}\right) = 0.8711$

$I_1 = 0.1^{0.3658} \times 0.2^{0.3658} \times 0.1^{0.2684} = 0.1289$

$F_1 = 0.1^{0.3658} \times 0.15^{0.3658} \times 0.1^{0.2684} = 0.116$

The result of calculation of aggregated subjective weight is shown in Table 7.

Table 7 Aggregated subjective weight.

	T	***I***	***F***
C1	0.8711399	0.12886	0.115989
C2	0.8198282	0.166049	0.17336
C3	0.8795537	0.120446	0.111496
C4	0.7678305	0.213971	0.201078
C5	0.8198282	0.166049	0.17336
C6	0.7224871	0.240893	0.259576
C7	0.7678305	0.213971	0.201078
C8	0.7442398	0.240893	0.20144
C9	0.694533	0.279408	0.30511

The subjective weight of criterion is calculated using eq (4) and presented as follows:

$$T_j + I_j\left(\frac{T_j}{T_j + F_j}\right)$$

$$0.8711 + 0.129\left(\frac{0.8711}{0.8711 + 0.116}\right) = 0.985$$
$$0.8198 + 0.166\left(\frac{0.8198}{0.8198 + 0.173}\right) = 0.957$$
$$0.8796 + 0.12\left(\frac{0.8796}{0.8796 + 0.111}\right) = 0.986$$
$$0.7678 + 0.214\left(\frac{0.7678}{0.7678 + 0.201}\right) = 0.937$$
$$0.8198 + 0.166\left(\frac{0.8198}{0.8198 + 0.173}\right) = 0.957$$
$$0.7225 + 0.241\left(\frac{0.7225}{0.7225 + 0.26}\right) = 0.9$$
$$0.7678 + 0.214\left(\frac{0.7678}{0.7678 + 0.201}\right) = 0.937$$
$$0.7442 + 0.241\left(\frac{0.7442}{0.7442 + 0.201}\right) = 0.934$$
$$0.6945 + 0.279\left(\frac{0.6945}{0.6945 + 0.305}\right) = 0.889$$

$$\sum_{j=1}^{n} T_j + I_j\left(\frac{T_j}{T_j + F_j}\right) = 0.985 + 0.957 + 0.986 + 0.937 + 0.957 + 0.9 + 0.937 + 0.934 + 0.889$$
$$= 8.482$$

$$w_1^s = \frac{0.985}{8.482} = 0.116$$

$$w_2^s = \frac{0.957}{8.482} = 0.113$$

$$w_3^s = \frac{0.986}{8.482} = 0.116$$

The rest answers are given as follow:

$w_4^s = 0.111,\quad w_5^s = 0.113,\quad w_6^s = 0.106,\quad w_7^s = 0.111,\quad w_8^s = 0.11,\quad w_9^s = 0.105$

Crisp Value are calculated using the following equation:

$$s(x_{ij}) = \frac{2 + T_{ij} - I_{ij} - F_{ij}}{3}$$

$$s(x_{11}) = \frac{2 + 0.8 - 0.2 - 0.15}{3} = 0.8167$$

$$s(x_{21}) = \frac{2 + 0.7148 - 0.2852 - 0.2328}{3} = 0.7323$$

The aggregated crisp matrix is given in Table 8.

Table 8 Aggregated crisp matrix.

	C1	C2	C3	C4	C5	C6	C7	C8	C9
A1	0.8167	0.8167	0.8521	0.8521	0.6934	0.6612	0.7445	0.6358	0.7323
A2	0.7323	0.7323	0.6358	0.8162	0.3929	0.598	0.6934	0.4356	0.7323
A3	0.8167	0.7388	0.6358	0.8167	0.7323	0.8892	0.8167	0.6612	0.7323

A4	0.7719	0.7719	0.6612	0.6182	0.4263	0.598	0.6612	0.3004	0.7323
A5	0.8167	0.6182	0.3637	0.7623	0.4263	0.8395	0.8167	0.6637	0.6612

After that, the evaluation of criterion is normalized by using eq (5).

$$P_{11} = \frac{0.8167}{0.8167 + 0.7323 + 0.8167 + 0.7719 + 0.8167} = 0.2065$$
$$P_{21} = \frac{0.7323}{0.8167 + 0.7323 + 0.8167 + 0.7719 + 0.8167} = 0.1852$$
$$P_{31} = \frac{0.8167}{0.8167 + 0.7323 + 0.8167 + 0.7719 + 0.8167} = 0.2065$$
$$P_{41} = \frac{0.7719}{0.8167 + 0.7323 + 0.8167 + 0.7719 + 0.8167} = 0.1952$$
$$P_{51} = \frac{0.8167}{0.8167 + 0.7323 + 0.8167 + 0.7719 + 0.8167} = 0.2065$$

Next, entropy E_j is calculated by using eq (6).

$\ln P_{ij}$:

$\ln P_{11} = \ln 0.2065 = -1.5773$
$\ln P_{21} = \ln 0.1852 = -1.6864$
$\ln P_{31} = \ln 0.2065 = -1.5773$
$\ln P_{41} = \ln 0.1952 = -1.6336$
$\ln P_{51} = \ln 0.2065 = -1.5773$

$$\sum_{i=1}^{m} P_{ij} \ln P_{ij} = (0.2065 \times -1.5773) + (0.1852 \times -1.6864) + (0.20644 \times -1.5773)$$
$$+(0.1952 \times -1.6336) + (0.2065 \times -1.5773)$$
$$= -1.6085$$

$$\therefore E_1 = -\left(\frac{1}{\ln 5}\right)(-1.6085) = 0.9994$$

After that, the divergence is calculated as formula in eq (7).

$$div_1 = 1 - 0.9994 = 0.0006$$

$$\sum_{j=1}^{n} div_j = 0.0005 + 0.0026 + 0.0202 + 0.0037 + 0.0229 + 0.0091 + 0.0022 + 0.0249 + 0.0005$$
$$= 0.0867$$

Objective weights are calculated through the formula given in eq (8).

$$w_1^o = \frac{0.0005821}{0.08671} = 0.0067$$

The rest of objective weights are calculated in similar manner. The result of calculated objective weight and subjective weight are shown in the Table 9.

Table 9 Subjective weight and objective weight of criteria.

	Subjective Weight	**Objective Weight**
C1	0.1161	0.0067

C2	0.1128	0.0295
C3	0.1163	0.2334
C4	0.1105	0.0427
C5	0.1128	0.2644
C6	0.1061	0.1045
C7	0.1105	0.0257
C8	0.1101	0.2874
C9	0.1048	0.0057

The result of subjective weight for each criterion implies that operational and maintenance cost (C3) is the most important criteria and safety risk to worker (C9) is the least important based on decision-makers.

For analysis of data for objective weight of criteria, we found that capital cost of WWT (C5) is the most important criteria while safety risk to worker (C9) is the least important.

Step 7: Determine the neutrosophic value of Positive Ideal Solution (PIS) and Negative Ideal Solution (NIS)

Benefit Criteria and Cost Criteria are categorized. Benefit Criteria includes the amount of pollutant removed (C1), Lifetime (C2), Reliability (C4), Sustainability (C7), and Safety Risk to Worker (C9). On the other hand, the cost criteria are Operational and Maintenance Cost (C3), Capital Cost (C5), Environmental Impacts (C6), and Safety Risk to Worker (C8).

The PIS and NIS from Table 8 is obtained using eq (9) and eq (10) respectively. The PIS and NIS of all criteria are given in Table 10.

Table 10 The PIS and NIS of all criteria.

	PIS	**NIS**
C1	0.8167	0.7279
C2	0.8167	0.6182
C3	0.1114	0.2741
C4	0.8521	0.6182
C5	0.1114	0.2741
C6	0.1114	0.2741
C7	0.8167	0.6612
C8	0.1114	0.2741
C9	0.7326	0.6565

Step 8: The utility measure (S_i), and the regret measure, (R_i) for the alternative are calculated using eq (11) and eq (12) respectively. The combination weight for each criterion is given in eq (13).

Let $v = 0.5$, $w_{C1} = (0.5)0.1161 + (1 - 0.5)0.0067 = 0.0614$,

Then,

$$\frac{w_{C1}||\tilde{f}_1^+ - \tilde{f}_{11}||}{||\tilde{f}_1^+ - \tilde{f}_1^-||} = \frac{0.0614121(0.8167 - 0.8167)}{0.8167 - 0.7323} = 0$$

So,

$S_{A1} = 0 + 0 + 0.1748 + 0 + 0.167 + 0.0229 + 0.0316 + 0.1835 + 0 = 0.5798$
$S_{A2} = 0.0614 + 0.0303 + 0.0974 + 0.0118 + 0 + 0 + 0.0541 + 0.074 + 0 = 0.3289$
$S_{A3} = 0 + 0.0279 + 0.0974 + 0.0116 + 0.1886 + 0.1053 + 0 + 0.1974 + 0 = 0.62818$
$S_{A4} = 0.0325 + 0.016 + 0.1065 + 0.0166 + 0.0185 + 0 + 0.068126 + 0 + 0 = 0.3184$
$S_{A5} = 0 + 0.0711 + 0 + 0.0294 + 0.0186 + 0.0873 + 0 + 0.1987 + 0.0552 = 0.4604$

and

$R_1 = 0.1098$
$R_2 = R_3 = R_4 = R_5 = 0.1123.$

The rest of S_i and R_i are calculated in similar manner.

Step 9: Compute priority value, Q_i as in eq (14).

From above,
$S^+ = \min_i S_i = 0.3184$
$S^- = \max_i S_i = 0.6282$
$R^+ = \min_i R_i = 0.09741$
$R^- = \max_i R_i = 0.19874$

$$Q_{A1} = (0.5)\frac{(0.4003 - 0.4003)}{(0.4866 - 0.4003)} + (1 - 0.5)\frac{(0.1098 - 0.1098)}{(0.1123 - 0.1098)} = 0$$

Other calculations are obtained in the similar manner.
Step 10: S_i, R_i, Q_i are ranked according to the maximum criteria.

Table 11 The result of S_i, R_i, Q_i and their ranking according to the maximum criteria

	S	Ranking	R	Ranking	Q	Ranking
A1	0.57978575	4	0.18347188	3	0.84656246	4
A2	0.31841157	1	0.09741361	1	0.0168618	1
A3	0.62817945	5	0.19739361	4	0.99337543	5
A4	0.32885806	2	0.10652321	2	0.04495352	2
A5	0.46039547	3	0.19873605	5	0.72917789	3

Step 11: Ranking the alternatives
According to the results of the analysis, the ranking of alternatives based on S_i and R_i values are obtained as $A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$. The ranking order according to Q value is obtained as $A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$. Since $0.045 - 0.0169 \leq \frac{1}{4}$ (condition 1), the acceptable advantage does not fulfill.

To fulfill the condition 2 which is acceptable stability, the top alternatives should be the best ranked by S_i and R_i. From the Table 11, alternative A_2 presents the best ranked S_i and R_i. Because of one of the conditions does not fulfill, thus alternative A_2 is not the best ranking. Therefore, a set of compromise solutions is proposed. From the result, only the acceptable advantage condition is not satisfied. Which mean, alternative A_u and the top two alternatives represent a group of compromise solutions. The closeness of these options is determined.

Based on the closeness formula, $0.72917789 - 0.0168618 \geq \frac{1}{5-1}$, it shows that the third rank of the alternatives is alternative A_u. As a result, the compromise solutions found using the SVN-VIKOR approach are Aerated Lagoons (A2), Oxidation Ditch (A4), and Trickling Filter (A5), indicating that these three alternatives are in close competition for the best position. Dursun [19], Wongburi and Park [61] and Maurya et al. [62]'s results are comparable. Dursun [19] revealed that aerated lagoons is the best WWT alternative. According to the findings of Wongburi and Park [61]'s research, aerated lagoon is also the best option, followed by oxidation ditch. However, Maurya et al. [62] discovered that Trickling Filter is the best WWT option. They came to the conclusion that the best WWT alternatives are 'Aerated Lagoon,' 'Oxidation Ditch,' and 'Trickling Filter'. This comparable finding demonstrates the efficacy of the SVN-VIKOR method as well.

5. Conclusions

In this study, we presented the definitions related to SVNS, entropy weight, set-theoretic operations and relations of SVNs, and described the steps of the VIKOR method for MCDM problems. Next, the ratings of each alternative and the weights of each criterion were interpreted in linguistic terms expressed by single-valued neutrosophic numbers regarding the importance of different elements in the decision-making procedure, namely the weight of decision-makers, the weight of criteria and the impact of alternatives on criteria with respect to decision-makers. Other than that, SNWA operator is used to aggregating all individual decision-makers' opinions in SVN assessments. Then, we determine the weight of criteria by considering the subjective weight and objective weight. The PIS and NIS of all criteria were also determined. Next, the utility measure, regret measure and priority values were calculated ranked according to the maximum criteria. Finally, the alternatives are ranked according to the previous value and the compromise solution was derived if one of the conditions cannot be fulfilled. To show the applicability of the proposed method, a case study of WWT alternatives selection was used. The obtained results show that Aerated Lagoons (A2), Oxidation Ditch (A4) and Trickling Filter (A5) are the three alternatives that have a close position as the best alternatives for the WWT. Our findings appear to be correlated when compared to those of other studies on WWT selection [19], [61], [62], demonstrating the feasibility and efficacy of the proposed approach.

The benefit of the SVN-VIKOR method is that it is more beneficial for addressing MCDM problems since it takes into account the significance of decision-makers and may be used to identify the best solution in a conflicting criteria environment. Moreover, considering that IFSs are sometimes unable to deal with ambiguity and uncertainty, the SVN-VIKOR method was proposed in this work to handle MCDM issues. WWT selection was investigated using the suggested SNV VIKOR technique. The findings would be extremely useful to policymakers in determining the best technology for WWT. Our proposed method deals efficiently with imprecise, inconsistent and inadequate information by considering all aspects of the decision-making process. Therefore, SVN-VIKOR can be preferable for dealing with incomplete and unpredictable information in MCDM problems such as supplier selection, landfill sites selection and many other decision-making problems. There are some recommendations suggested for future research. In the future, a sensitivity analysis with alteration of some parameters will be presented to analyze changes in the results. Furthermore, we will consider more decision-makers with specific years in the related field. The linguistic variable used may consider decision-makers' opinions to increase the accuracy of the calculation in decision-making.

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P and R Order of Plithogenic Neutrosophic Cubic sets

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Abstract: The paper presents a new concept called *P*-Order (Union and Intersection) and *R*- Order (Union and Intersection) of the Plithogenic Neutrosophic Cubic Sets (PNCS). We derived some of the primary properties of the internal and external PNCS of *P* and *R*- Order. We also proved that *P*-Union and *P*- intersection of Truth (T) (resp. falsity (F), indeterminacy(I)) external PNCS may not be T (resp. F, I) external PNCS and R-Union and R-intersection of T (resp. F, I) internal PNCS may not be T (resp. F, I) internal PNCS with the numerical examples. This principle is extremely appropriate for analyzing problems that involve multi-attribute decision making since this PNCS is defined by many values of attribute and the reliability of the data is also so accurate.

Keywords: Cubic set, Neutrosophic set, Plithogenic set, Plithogenic neutrosophic cubic set, internal and external plithogenic neutrosophic cubic set

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1. Introduction

The definition of fuzzy sets, proposed by Zadeh [19] in 1965, paved the way for fuzzy mathematics development. In many areas of mathematics, this concept has a wide range of applications, such as reasoning, set theory, number theory, fuzzy set theory, real analysis, metric theory, and topology. In 1975, Zadeh [19] developed the concepts of interval-valued fuzzy sets as an extension of fuzzy sets, that is, fuzzy sets with interval-valued membership functions.

A notable concept has been developed by Jun et al. [3], namely the cubic sets theory. An interval-valued fuzzy set and a fuzzy set constitute this structure. In addition, the definition of an internal cubic set and an external cubic set was also implemented by Jun et al. [3].

Smarandache [9, 10] proposed Neutrosophic sets (NSs), a generalisation of FS and IFS, which is highly helpful for dealing with inadequate, uncertain, and varying data that exists in the real life. NSs are characterised by functions of truth (T), indeterminacy (I) and falsity (F) belonging functions. This concept is very essential in several areas of application since indeterminacy is clearly enumerated and the truth, indeterminacy and falsity membership functions are independent.

Wang, Smarandache, Zhang and Sunderraman[18] anticipated the definition of an interval valued neutrosophic set (IVNS) as an extension of NS. The IVNS could reflect indeterminate, inaccurate, inadequate and unreliable data that occurs in the reality.

Plithogeny is the foundation, establishment, construction and development of new articles from the combination of consistent or inconsistent multiple old articles. Smarandache[13] introduced the plithogenic set as a generalisation of neutrosophy in 2017.

The elements of Plithogenic sets are denoted by one or many number of attributes and each of it have several values. Each values of attribute has its respective (fuzzy, intuitionistic fuzzy or neutrosophic) appurtenance degree for the component x (say) to the plithogenic set P (say) with respect to certain constraints. For the first time, Smarandache[12] introduced the inconsistency degree between each value of attribute and the dominant value of attribute which results in getting the better accuracy for the plithogenic aggregation operators(fuzzy, intuitionistic fuzzy or neutrosophic).

Priyadharshini et al [8] introduced the new concept called plithogenic cubic sets which has a wide range of application in multi criteria decision making problems. In particular, the ideology of Plithogenic neutrosophic cubic set helped in this paper to learn the P, R-union and the P, R-intersection of PNCS and derived some of its core properties.

We proved that the P-union and the P-intersection of T (resp. F, I) internal PNCS are also T (resp. F, I) internal PNCS. We provide examples to show that the P-union and the P-intersection of T (resp. F, I) external PNCS may not be T (resp. F, I) external PNCS, and the R-union and the R intersection of T (resp. F, I) internal PNCS may not be T (resp. F, I) internal PNCS. The conditions for the R-union and R-intersection of two T (resp. F, I) internal PNCS to be a T (resp. F, I) internal PNCS.

We believe that the suggested theorems and examples will be effective in resolving multi-attribute group decision-making concerns in a Plithogenic neutrosophic cubic set environment.

The rest of the article is as follows. Section 2 is concerned with the preliminary concepts and definitions that are absolutely vital for the proposed work. Section 3 provides a brief description of the P and R order of Plithogenic neutrosophic Cubic sets along with numerical examples. Section 4 summarizes the conclusion and the scope of future work.

2. Preliminaries

Definition 2.1 [9, 10] Let N be a non-void set. The set $A=\{\langle n,\lambda_A,\phi_A,\gamma_A\rangle n\in N\}$ is called a neutrosophic set (in short, NS) of N where the function $\lambda_A : N\to[0,1]$, $\phi_A : N\to[0,1]$ and $\gamma_A : N\to[0,1]$ denotes the membership degree (say $\lambda_A(n)$), indeterminacy degree (say $\phi_A(n)$), and non-membership degree (say $\gamma_A(n)$)of each element $n\in N$ to the set A and satisfies the constraint that $0\le\lambda_A(n)+\phi_A(n)+\gamma_A(n)\le 3$.

Definition 2.2 [11] Let R be a non-void set. An interval valued neutrosophic set (INS) A in R is described by the functions of the truth-value (A_T) , the indeterminacy (A_I) and the falsity-value (A_F) for each point $r\in R$, $A_T(r),A_I(r),A_F(r)\subseteq[0,1]$.

Definition 2.3 [3] Let E be a non-void set. By a cubic set in E, we construct a set which has the form $\Psi = \{< e, B(e), \mu(e) > | e \in E\}$ in which B is an interval valued fuzzy set (IVFS) in E and μ is a fuzzy set in E.

Definition 2.4 [3] Let E be a non-void set. If $B^{-}(e) \leq \mu(e) \leq B^{+}(e)$ for all $e \in E$ then the cubic set $\Psi = < B, \mu >$ in E is called an internal cubic set (briefly IPS).

Definition 2.5 [3] Let E be a non-void set. If $\mu(e) \notin (B^{-}(e), B^{+}(e))$ for all $e \in E$ then the cubic set $\Psi = < B, \mu >$ in E is called an external cubic set (briefly ECS).

Definition 2.6 [8] Let Ω be an universal set and Y be a non-void set. The structure $\Lambda = \{< y, B(y), \lambda(y) > | y \in Y\}$ is said to be Plithogenic Neutrosophic cubic set (PNCS) in Y, where $B = \left\{\left(B_{d_i}{}^{T}(y), B_{d_i}{}^{I}(y), B_{d_i}{}^{F}(y)\right)\right\}$ is an interval valued Plithogenic Neutrosophic set in Y and $\lambda = \{(\lambda_i{}^{T}(y), \lambda_i{}^{I}(y), \lambda_i{}^{F}(y)\}$ is a neutrosophic set in Y.

The pair $\Lambda = < B, \lambda >$ is called plithogenic neutrosophic cubic set over Ω where Λ is a mapping given by $\Lambda : B \to NC(\Omega)$ .The set of all plithogenic neutrosophic cubic sets(PNCS) over Ω will be denoted by $P_N{}^{\Omega}$

Definition 2.7 [8] For a non-void set Y, the plithogenic neutrosophic cubic set $\Lambda = < B, \lambda >$ in Y is called truth internal, indeterminacy internal, falsity internal respectively if the following equations hold

(i) $B_{d_i}{}^{-T}(y) \leq \lambda_i{}^{T}(y) \leq B_{d_i}{}^{+T}(y)$ (3.1)

(ii) $B_{d_i}{}^{-I}(y) \leq \lambda_i{}^{I}(y) \leq B_{d_i}{}^{+I}(y)$ (3.2)

(iii) $B_{d_i}{}^{-F}(y) \leq \lambda_i{}^{F}(y) \leq B_{d_i}{}^{+F}(y)$ (3.3)

Where for all $y \in Y$ and d_i represents the dissimilarity measure and their respective value of attributes.

If a PNCS in Y satisfies the above equations we conclude that Λ is an internal plithogenic neutrosophic cubic set (IPNCS) in Y.

Definition 2.8 [8] For a non-void set Y, the PNCS $\Lambda = < B, \lambda >$ in Y is called truth external, indeterminacy external, falsity external respectively if the following equations hold

(i) $\lambda_i{}^{T}(y) \notin (B_{d_i}{}^{-T}(y), B_{d_i}{}^{+T}(y))$ (3.4)

(ii) $\lambda_i^{I}(y) \notin (B_{d_i}^{-I}(y), B_{d_i}^{+I}(y))$ (3.5)

(iii) $\lambda_i^{F}(y) \notin (B_{d_i}^{-F}(y), B_{d_i}^{+F}(y))$ (3.6)

Where for all $y \in Y$ and d_i represents the dissimilarity measure and their respective value of attributes.

If a PNCS in Y satisfies the above equations, we conclude that Λ is an external plithogenic neutrosophic cubic set (EPNCS) in Y.

3. P and R Order of PNCS

Definition 3.1 Let $C = <\mathrm{K}, \Phi>$ and $D = <\mathrm{M}, \Omega>$ be a PNCS in a non-void set H where

$$\mathrm{K} = \{\left\langle h : K_{d_j}{}^{T}(h), K_{d_j}{}^{I}(h), K_{d_j}{}^{F}(h)\right\rangle \mid h \in H\},$$
$$\Phi = \{\left\langle h : \phi_{d_j}{}^{T}(h), \phi_{d_j}{}^{I}(h), \phi_{d_j}{}^{F}(h)\right\rangle \mid h \in H\},$$
$$\mathrm{M} = \{\left\langle h : \mathrm{M}_{d_j}{}^{T}(h), \mathrm{M}_{d_j}{}^{I}(h), \mathrm{M}_{d_j}{}^{F}(h)\right\rangle \mid h \in H\},$$
$$\Omega = \{\left\langle h : \delta_{d_j}{}^{T}(h), \delta_{d_j}{}^{I}(h), \delta_{d_j}{}^{F}(h)\right\rangle \mid h \in H\}.$$

Then we define the equality, P-order and R-order as follows:

(i) $C = D \Leftrightarrow K = M \ and \ \Phi = \Omega.$ (Equality)

(ii) $C \subseteq_P D \Leftrightarrow K \subseteq M \ and \ \Phi \leq \Omega.$ (*P*-Order)

(iii) $C \subseteq_R D \Leftrightarrow K \subseteq M \ and \ \Phi \geq \Omega.$ (*R*-Order)

The *P*-Union, *P*-Intersection, *R*-Union, *R*-Intersection of PNCS are described as follows.

Definition 3.2 For any PNCS $C_{d_j} = <\mathrm{K}_{d_j}, \Phi_{d_j}>$ in a non-void set H where

$$K = \{\left\langle h : K_{d_j}{}^{T}(h), K_{d_j}{}^{I}(h), K_{d_j}{}^{F}(h)\right\rangle \mid h \in H\},$$
$$\Phi = \{\left\langle h : \phi_{d_j}{}^{T}(h), \phi_{d_j}{}^{I}(h), \phi_{d_j}{}^{F}(h)\right\rangle \mid h \in H\}.$$

For $j \in B$ and B is any index set, we define

(i) $\bigcup_{j \in B} {}_P K_{d_j} = (\bigcup_{j \in B} K_{d_j}, \bigvee_{j \in B} \Phi_{d_j})$ (*P*-Union)

(ii) $\bigcap_{j \in B} {}_P K_{d_j} = (\bigcap_{j \in B} K_{d_j}, \bigwedge_{j \in B} \Phi_{d_j})$ (*P*-Intersection)

(iii) $\bigcup_{j \in B} {}_R K_{d_j} = (\bigcup_{j \in B} K_{d_j}, \bigwedge_{j \in B} \Phi_{d_j})$ (*R*-Union)

(iv) $\bigcap_{j \in B} {}_R K_{d_j} = (\bigcap_{j \in B} K_{d_j}, \bigvee_{j \in B} \Phi_{d_j})$ (*R*-Intersection)

Where

$$\bigcup_{j\in B} K_{d_j} = \left\{ \left\langle h : \left(\bigcup_{j\in B} K_{d_j}{}^{T} \right)(h), \left(\bigcup_{j\in B} K_{d_j}{}^{I} \right)(h), \left(\bigcup_{j\in B} K_{d_j}{}^{F} \right)(h) \right\rangle \mid h \in H \right\},$$

$$\bigvee_{j\in B} \Phi_{d_j} = \left\{ \left\langle h : \left(\bigvee_{j\in B} \phi_{d_j}{}^{T} \right)(h), \left(\bigvee_{j\in B} \phi_{d_j}{}^{I} \right)(h), \left(\bigvee_{j\in B} \phi_{d_j}{}^{F} \right)(h) \right\rangle \mid h \in H \right\},$$

$$\bigcap_{j\in B} K_{d_j} = \left\{ \left\langle h : \left(\bigcap_{j\in B} K_{d_j}{}^{T} \right)(h), \left(\bigcap_{j\in B} K_{d_j}{}^{I} \right)(h), \left(\bigcap_{j\in B} K_{d_j}{}^{F} \right)(h) \right\rangle \mid h \in H \right\},$$

$$\bigwedge_{j\in B} \Phi_{d_j} = \left\{ \left\langle h : \left(\bigwedge_{j\in B} \phi_{d_j}{}^{T} \right)(h), \left(\bigwedge_{j\in B} \phi_{d_j}{}^{I} \right)(h), \left(\bigwedge_{j\in B} \phi_{d_j}{}^{F} \right)(h) \right\rangle \mid h \in H \right\}.$$

Remarks

(i) $\left(\bigcup_{P\, j\in B} C_{d_j} \right)' = \bigcap_{P\, j\in B} C'_{d_j}$

(ii) $\left(\bigcap_{P\, j\in B} C_{d_j} \right)' = \bigcup_{P\, j\in B} C'_{d_j}$

(iii) $\left(\bigcup_{R\, j\in B} C_{d_j} \right)' = \bigcap_{R\, j\in B} C'_{d_j}$

(iv) $\left(\bigcap_{R\, j\in B} C_{d_j} \right)' = \bigcup_{R\, j\in B} C'_{d_j}$

Proposition 3.3

For any PNCS $C = <\mathrm{K}, \Phi>$, $D = <\mathrm{M}, \Omega>$, $X = <L, \psi>$ and $Y = <\mathrm{N}, \Lambda>$ in a non-void set H, we have

(i) If $C \subseteq_P D$ and $D \subseteq_P X$ then $C \subseteq_P X$

(ii) If $C \subseteq_P D$ then $D' \subseteq_P C'$

(iii) If $C \subseteq_P D$ and $C \subseteq_P X$ then $C \subseteq_P D \bigcap_P X$

(iv) If $C \subseteq_P D$ and $X \subseteq_P D$ then $C \bigcup_P X \subseteq_P D$

(v) If $C \subseteq_P D$ and $X \subseteq_P Y$ then $C \bigcup_P X \subseteq_P D \bigcup_P Y$ and $C \bigcap_P X \subseteq_P D \bigcap_P Y$

(vi) If $C \subseteq_R D$ and $D \subseteq_R X$ then $C \subseteq_R X$

(vii) If $C \subseteq_R D$ then $D' \subseteq_R C'$

(viii) If $C \subseteq_R D$ and $C \subseteq_R X$ then $C \subseteq_R D \bigcap_R X$

(ix) If $C \subseteq_R D$ and $X \subseteq_R D$ then $C \bigcup_R X \subseteq_R D$

(x) If $C \subseteq_R D$ and $X \subseteq_R Y$ then $C \bigcup_R X \subseteq_R D \bigcup_R Y$ and $C \bigcap_R X \subseteq_R D \bigcap_R Y$

Proposition 3.4

Let $\{C_{d_j} =< \mathrm{K}_{d_j}, \Phi_{d_j} > | j \in B\}$ be a family of F- IPNCS in a non-void set H, then the *P*-Order of $\{C_{d_j} =< \mathrm{K}_{d_j}, \Phi_{d_j} > | j \in B\}$ are F-IPNCS in H.

Proof

Since $C_{d_j} =< \mathrm{K}_{d_j}, \Phi_{d_j} >$ is an F-IPNCS in a non-void set H, We have

$$(K_{d_j}{}^F)^-(h) \le \phi_{d_j}{}^F(h) \le (K_{d_j}{}^F)^+(h) \, for \, j \in B$$

It follows that

$$(\bigcup_{j\in B} K_{d_j}{}^F)^-(h) \le (\bigvee_{j\in B} \phi_{d_j}{}^F(h) \le (\bigcup_{j\in B} K_{d_j}{}^F)^+(h)$$

and

$$(\bigcap_{j\in B} K_{d_j}{}^F)^-(h) \le (\bigwedge_{j\in B} \phi_{d_j}{}^F(h) \le (\bigcap_{j\in B} K_{d_j}{}^F)^+(h)$$

Therefore

$\bigcup_{P\, j\in B} C_{d_j} = (\bigcup_{j\in B} K_{d_j}, \bigvee_{j\in B} \phi_{d_j})$ and $\bigcap_{P\, j\in B} C_{d_j} = (\bigcap_{j\in B} K_{d_j}, \bigwedge_{j\in B} \phi_{d_j})$ are F-INPCS in H.

Correspondingly the following propositions holds.

Proposition 3.5

Let $\{C_{d_j} =< \mathrm{K}_{d_j}, \Phi_{d_j} > | j \in B\}$ be a family of T-IPNCS in a non-void set H, then the *P*-Order of $\{C_{d_j} =< \mathrm{K}_{d_j}, \Phi_{d_j} > | j \in B\}$ are T-IPNCS in H.

Proposition 3.6

Let $\{C_{d_j} =< \mathrm{K}_{d_j}, \Phi_{d_j} > | j \in B\}$ be a family of I-IPNCS in a non-void set H, then the *P*-Order of $\{C_{d_j} =< \mathrm{K}_{d_j}, \Phi_{d_j} > | j \in B\}$ are I-IPNCS in H.

Corollary 3.7

Let $\{C_{d_j} =< \mathrm{K}_{d_j}, \Phi_{d_j} > | j \in B\}$ be a family of IPNCS in a non-void set H, then the *P*-Order of $\{C_{d_j} =< \mathrm{K}_{d_j}, \Phi_{d_j} > | j \in B\}$ are IPNCS in H.

The subsequent illustration indicates that the P -Union and P- intersection of T, I -IPNCS may not be T, I-IPNCS.

Example 3.8

Let us consider the attribute values National Electronic Fund Transfer (NEFT), Real Time Gross Settlement (RTGS), Immediate Payment Service (IMPS), Unified Payment Interface (UPI) for transferring money and $C = <\mathrm{K}, \Phi>$ and $D = <\mathrm{M}, \Omega>$ be the PNCS in H with the Table values 1 & 2 correspondingly.

Table 1. $C = <\mathrm{K}, \Phi>$

Dissimilarity Measure	Value of attributes	Appurtenance Measure $K(h)$	$\Phi(h)$
0	NEFT	([0.35, 0.4], [0.7, 0.9], [0.2, 0.5])	(0.2, 0.65, 0.4)
0.5	RTGS	([0.6, 0.8], [0.5, 0.7], [0.3, 0.7])	(0.2, 0.85, 0.6)
0.75	IMPS	([0.45, 0.6], [0.2, 0.4], [0.1, 0.3])	(0.7, 0.6, 0.2)
1	UPI	([0.2, 0.3], [0.1, 0.3], [0.6, 0.9])	(0.1, 0.4, 0.95)

Table 2. $D = <\mathrm{M}, \Omega>$

Dissimilarity Measure	Value of attributes	Appurtenance Measure $M(h)$	$\Omega(h)$
0	NEFT	([0.1, 0.3], [0.7, 0.8], [0.6, 0.9])	(0.7, 0.92, 0.7)
0.5	RTGS	([0.6, 0.7], [0.5, 0.6], [0.2, 0.4])	(0.9, 0.2, 0.3)
0.75	IMPS	([0.45, 0.8], [0.2, 0.5], [0.5, 0.7])	(0.35, 0.1, 0.6)
1	UPI	([0.5, 0.6], [0.4, 0.5], [0.4, 0.7])	(0.45, 0.3, 0.45)

Table 3. $C \cup_P D = (K \cup M, \Phi \vee \Omega)$

Dissimilarity Measure	Value of attributes	Appurtenance Measure $(K \cup M)(h)$	$(\Phi \vee \Omega)(h)$
0	NEFT	([0.35, 0.4], [0.7, 0.9], [0.6, 0.9])	(0.7, 0.92, 0.7)
0.5	RTGS	([0.6, 0.8], [0.5, 0.7], [0.3, 0.7])	(0.9, 0.85, 0.6)
0.75	IMPS	([0.45, 0.8], [0.2, 0.5], [0.5, 0.7])	(0.7, 0.6, 0.6)
1	UPI	([0.5, 0.6], [0.4, 0.5], [0.6, 0.9])	(0.45, 0.4, 0.95)

Table 4. $C \cap_P D = (K \cap M, \Phi \wedge \Omega)$

Dissimilarity	Value of	Appurtenance Measure	$(\Phi \wedge \Omega)(h)$

Measure	attributes	$(K \cap M)(h)$	
0	NEFT	([0.1, 0.3], [0.7, 0.8], [0.2, 0.5])	(0.2, 0.65, 0.4)
0.5	RTGS	([0.6, 0.7], [0.5, 0.6], [0.2, 0.4])	(0.2, 0.2, 0.3)
0.75	IMPS	([0.45, 0.6], [0.2, 0.4], [0.1, 0.3])	(0.35, 0.1, 0.2)
1	UPI	([0.2, 0.3], [0.1, 0.3], [0.4, 0.7])	(0.1, 0.3, 0.45)

Then $C = <\mathrm{K}, \Phi>$ and $D = <\mathrm{M}, \Omega>$ are both T-external and I-external PNCS in H. $C \cup_P D = (K \cup M, \Phi \vee \Omega)$ and $C \cap_P D = (K \cap M, \Phi \wedge \Omega)$ are given by Tables 3 & 4 correspondingly.

Then $C \cup_P D = (K \cup M, \Phi \vee \Omega)$ is neither an I-EPNCS nor T-EPNCS in H since

$$((\Phi^T \vee \Omega^T)(IMPS) = 0.7 \in (0.45, 0.8) = ((C^T \cup D^T)^-(IMPS), (C^T \cup D^T)^+(IMPS)))$$

And

$$(\Phi^I \vee \Omega^I)(UPI) = 0.4 \in (0.4, 0.5) = ((C^I \cup D^I)^-(UPI), (C^I \cup D^I)^+(UPI))$$

Also $C \cap_P D = (K \cap M, \Phi \wedge \Omega)$ is neither an I-EPNCS nor T-EPNCS in H since

$$(\Phi^T \wedge \Omega^T)(NEFT) = 0.2 \in (0.1, 0.3) = ((C^T \cap D^T)^-(NEFT), (C^T \cap D^T)^+(NEFT))$$

$$(\Phi^I \wedge \Omega^I)(UPI) = 0.3 \in (0.1, 0.3) = ((C^I \cap D^I)^-(UPI), (C^I \cap D^I)^+(UPI))$$

We provide conditions for the *R*-Union of two T-internal (resp. I- internal and F-Internal) PNCS to be a T- internal (resp. I- internal and F-Internal) PNCS.

Proposition 3.9

If $C = <\mathrm{K}, \Phi>$ and $D = <\mathrm{M}, \Omega>$ be T-IPNCS in a non-void set H such that

$$(\forall z \in Z)(\max\{(K^T)^-(h), (M^T)^-(h)\} \leq (\phi^T \wedge \delta^T)(h)). \qquad (3.1)$$

Then the *R*-Union of $C = <\mathrm{K}, \Phi>$ and $D = <\mathrm{M}, \Omega>$ is a T-INPCS in H.

Proof

Let $C = <\mathrm{K}, \Phi>$ and $D = <\mathrm{M}, \Omega>$ be T-IPNCS in a non-void set H which satisfies the constraints (3.1). Then

$$(K^T)^-(h) \leq \phi^T(h) \leq (K^T)^+(h) \text{ and } (M^T)^-(h) \leq \delta^T(h) \leq (M^T)^+(h),$$

and so $(\phi^T \wedge \delta^T)(h) \leq (K^T \cup M^T)^+(h)$.

It follows from (3.1) that

$$(K^T \cup M^T)^-(h) = \max\{(K^T)^-(h), (M^T)^-(h)\} \leq (\phi^T \wedge \delta^T)(h) \leq (K^T \cup M^T)^+(h)$$

Hence $C \cup_R D = (K \cup M, \phi \cup \delta)$ is a T-INPCS in Z.

Correspondingly the following propositions hold.

Proposition 3.10

If $C = <\mathrm{K}, \Phi>$ and $D = <\mathrm{M}, \Omega>$ be I-IPNCS in a non-void set H such that

$$(\forall h \in H)(\max\{(K^I)^-(h), (M^I)^-(h)\} \le (\phi^I \wedge \delta^I)(h)). \qquad (3.2)$$

Then the *R*-Union of $C = <\mathrm{K}, \Phi>$ and $D = <\mathrm{M}, \Omega>$ is a I-INPCS in H.

Proposition 3.11

If $C = <\mathrm{K}, \Phi>$ and $D = <\mathrm{M}, \Omega>$ be F-IPNCS in a non-void set H such that

$$(\forall h \in H)(\max\{(K^F)^-(h), (M^F)^-(h)\} \le (\phi^F \wedge \delta^F)(h)). \qquad (3.3)$$

Then the *R*-Union of $C = <\mathrm{K}, \Phi>$ and $D = <\mathrm{M}, \Omega>$ is a F-INPCS in H.

Corollary 3.12

If two INPCS $C = <\mathrm{K}, \Phi>$ and $D = <\mathrm{M}, \Omega>$ satisfies the constraints (3.1), (3.2), (3.3), then the *R*-Union of $C = <\mathrm{K}, \Phi>$ and $D = <\mathrm{M}, \Omega>$ is an INPCS in H.

We provide conditions for the *R*-Intersection of two T-internal (resp. I- internal and F-Internal) PNCS to be a T- internal (resp. I- internal and F-Internal) PNCS.

Proposition 3.13

If $C = <\mathrm{K}, \Phi>$ and $D = <\mathrm{M}, \Omega>$ be T-IPNCS in a non-void set H such that

$$(\forall h \in H)((\phi^T \vee \delta^T)(h) \le \min\{(K^T)^+(h), (M^T)^+(h)\}). \qquad (3.4)$$

Then the *R*-Intersection of $C = <\mathrm{K}, \Phi>$ and $D = <\mathrm{M}, \Omega>$ is an T-INPCS in H.

Proof

Assume that the condition (4.4) is valid. Then

$(K^T)^-(h) \le \phi^T(h) \le (K^T)^+(h)$ and $(M^T)^-(h) \le \delta^T(h) \le (M^T)^+(h)$ for all $h \in H$.

It follows from (4.4) that

$(K^T \cap M^T)(h) \le (\phi^T \vee \delta^T)(h) \le \min\{(K^T)^+, (M^T)^+(h)\} = (A^T \cap B^T)^+(h)$ for all $h \in H$.

Therefore $C \cap_R D = (K \cap M, \phi \vee \delta)$ is a T-INPCS.

Correspondingly the subsequent propositions hold.

Proposition 3.14

If $C = <\mathrm{K}, \Phi>$ and $D = <\mathrm{M}, \Omega>$ be I-IPNCS in a non-void set H such that

$$(\forall h \in H)((\phi^I \vee \delta^I)(h) \leq \min\{(K^I)^+(h),(M^I)^+(h)\}). \qquad (3.5)$$

Then the *R*-Intersection of $C=<K,\Phi>$ and $D=<M,\Omega>$ is an I-INPCS in H.

Proposition 3.15

If $C=<K,\Phi>$ and $D=<M,\Omega>$ be F-IPNCS in a non-void set H such that

$$(\forall h \in H)((\phi^F \vee \delta^F)(h) \leq \min\{(K^F)^+(h),(M^F)^+(h)\}). \qquad (3.6)$$

Then the *R*-Intersection of $C=<K,\Phi>$ and $D=<M,\Omega>$ is an F-INPCS in H.

Corollary 3.16

If two INPCS $C=<K,\Phi>$ and $D=<M,\Omega>$ satisfies the constraints (3.4), (3.5), (3.6), then the *R*-Intersection of $C=<K,\Phi>$ and $D=<M,\Omega>$ is an INPCS in H.

The subsequent illustration indicates that the R -Union and R- intersection of T, F -EPNCS may not be T, F-EPNCS

Example 3.17

Let us consider the attribute values of life insurance policies 'Whole life Insurance (WLI), Term Life Insurance(TLI),Universal Life Insurance(ULI) and Variable Life Insurance (VLI) and $C=<K,\Phi>$ and $D=<M,\Omega>$ be the PNCS in H with the Table values 5 & 6 respectively.

Table 5. $C=<K,\Phi>$

Dissimilarity Measure	Value of attributes	Appurtenance Measure $K(h)$	$\Phi(h)$
0	WLI	([0.8, 0.9], [0.2, 0.4], [0.3, 0.5])	(0.8, 0.25, 0.5)
0.5	TLI	([0.7, 0.9], [0.3, 0.6], [0.7, 0.8])	(0.8, 0.5, 0.75)
0.75	ULI	([0.5, 0.8], [0.3, 0.5], [0.1, 0.6])	(0.7, 0.4, 0.1)
1	VLI	([0.1, 0.2], [0.1, 0.6], [0.4, 0.8])	(0.2, 0.7, 0.6)

Table 6. $D=<M,\Omega>$

Dissimilarity Measure	Value of attributes	Appurtenance Measure $M(h)$	$\Omega(h)$
0	WLI	([0.4, 0.7], [0.3, 0.6], [0.4, 0.6])	(0.6, 0.45, 0.6)
0.5	TLI	([0.1, 0.5], [0.3, 0.4], [0.6, 0.8])	(0.3, 0.3, 0.7)
0.75	ULI	([0.6, 0.8], [0.6, 0.7], [0.2, 0.9])	(0.6, 0.7, 0.9)
1	VLI	([0.5, 0.7], [0.1, 0.3], [0.2, 0.5])	(0.65, 0.2, 0.4)

Table 7. $C \cup_R D = (K \cup M, \Phi \wedge \Omega)$

Dissimilarity Measure	Value of attributes	Appurtenance Measure $(K \cup M)(h)$	$(\Phi \wedge \Omega)(h)$
0	WLI	([0.8, 0.9], [0.3, 0.6], [0.4, 0.6])	(0.6, 0.25, 0.5)
0.5	TLI	([0.7, 0.9], [0.3, 0.6], [0.6, 0.8])	(0.3, 0.3, 0.7)
0.75	ULI	([0.6, 0.8], [0.6, 0.7], [0.2, 0.9])	(0.6, 0.4, 0.1)
1	VLI	([0.5, 0.7], [0.1, 0.6], [0.4, 0.8])	(0.2, 0.2, 0.4)

Table 8. $C \cap_R D = (K \cap M, \Phi \vee \Omega)$

Dissimilarity Measure	Value of attributes	Appurtenance Measure $(K \cap M)(h)$	$(\Phi \vee \Omega)(h)$
0	WLI	([0.4, 0.7], [0.2, 0.4], [0.3, 0.5])	(0.8, 0.45, 0.6)
0.5	TLI	([0.1, 0.5], [0.3, 0.4], [0.7, 0.8])	(0.8, 0.5, 0.75)
0.75	ULI	([0.5, 0.8], [0.3, 0.5], [0.1, 0.6])	(0.7, 0.7, 0.9)
1	VLI	([0.1, 0.2], [0.1, 0.3], [0.2, 0.5])	(0.65, 0.7, 0.6)

Then $C = <\mathrm{K}, \Phi>$ and $D = <\mathrm{M}, \Omega>$ are both T-internal and F-internal PNCS in H. $C \cup_R D = (K \cup M, \Phi \wedge \Omega)$ and $C \cap_R D = (K \cap M, \Phi \vee \Omega)$ are given by Tables 7 & 8 respectively.

Then $C \cup_R D = (K \cup M, \Phi \wedge \Omega)$ is neither an I-IPNCS nor T-IPNCS in H since

$$(\Phi^T \wedge \Omega^T)(WLI) = 0.6 \notin (0.8,0.9) = ((C^T \cup D^T)^-(WLI), (C^T \cup D^T)^+(WLI));$$

$$(\Phi^T \wedge \Omega^T)(TLI) = 0.3 \notin (0.7,0.9) = ((C^T \cup D^T)^-(TLI), (C^T \cup D^T)^+(TLI));$$

$$(\Phi^T \wedge \Omega^T)(VLI) = 0.2 \notin (0.5,0.7) = ((C^T \cup D^T)^-(VLI), (C^T \cup D^T)^+(VLI)).$$

And

$$(\Phi^F \wedge \Omega^F)(ULI) = 0.1 \notin (0.2,0.9) = ((C^F \cup D^F)^-(ULI), (C^F \cup D^F)^+(ULI)).$$

Also $C \cap_R D = (K \cap M, \Phi \vee \Omega)$ is neither an I-IPNCS nor T-IPNCS in H since

$$(\Phi^T \vee \Omega^T)(WLI) = 0.8 \notin (0.4,0.7) = ((C^T \cap D^T)^-(WLI), (C^T \cap D^T)^+(WLI));$$

$$(\Phi^T \vee \Omega^T)(TLI) = 0.8 \notin (0.1,0.5) = ((C^T \cap D^T)^-(TLI), (C^T \cap D^T)^+(TLI));$$

$$(\Phi^T \vee \Omega^T)(VLI) = 0.65 \notin (0.1,0.2) = ((C^T \cap D^T)^-(VLI), (C^T \cap D^T)^+(VLI)).$$

And

$$(\Phi^{F} \vee \Omega^{F})(WLI) = 0.6 \notin (0.3,0.5) = ((C^{F} \cap D^{F})^{-}(WLI),(C^{F} \cap D^{F})^{+}(WLI));$$

$$(\Phi^{F} \vee \Omega^{F})(ULI) = 0.9 \notin (0.1,0.6) = ((C^{F} \cap D^{F})^{-}(ULI),(C^{F} \cap D^{F})^{+}(ULI));$$

$$(\Phi^{F} \vee \Omega^{F})(VLI) = 0.6 \notin (0.2,0.5) = ((C^{F} \cap D^{F})^{-}(VLI),(C^{F} \cap D^{F})^{+}(VLI)).$$

4. Conclusion and Future Work

In this paper we studied the *P*-Order and *R*- Order (Union and Intersection) of the PNCS. The theorems and results we derived will be useful in the multi criteria decision making problems. Applications in various engineering, technical, medical, and so on areas of learning P and R-Order should be assessed. In future we can extend the concept to plithogenic neutrosophic soft sets which may help abundantly in the areas related with decision making.

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New Notions On Neutrosophic Random Variables

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Abstract. In this paper, new general definitions of neutrosophic random variables are introduced and their properties are studied. Notions of neutrosophic random vector, joint probability function, joint distribution function, neutrosophic random vector expected value, neutrosophic random vector variance, neutrosophic random vector covariance and some examples supported our results are presented which show the power of the study.

Keywords: Neutrosophic random vector; Neutrosophic random vector expected Value; Neutrosophic random vector Variance; joint probability function; joint distribution function; Neutrosophic Logic.

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1. **Introduction and Preliminaries**

Neutrosophic logic is an extension of intuitionistic fuzzy logic by adding indeterminacy component (I) where $I^2 = I, ..., I^n = I, 0.I = 0; n \in \mathbb{N}$ and I^{-1} is undefined [20], [32]. Neutrosophic logic has a huge brand of applications in many fields including decision making [27], [19], [25], machine learning [7], [28], intelligent disease diagnosis [30], [12], communication services [9], pattern recognition [29], social network analysis and e-learning systems [21], physics [34], sequences spaces [14] and so on.

In probability theory, F. Smarandache defined the neutrosophic probability measure as a mapping $NP : X \to [0, 1]^3$ where X is a neutrosophic sample space, and defined the probability function to take the form $NP(A) = (ch(A), ch(neutA), ch(antiA)) = (\alpha, \beta, \gamma)$ where $0 \leq \alpha, \beta, \gamma \leq 1$ and $0 \leq \alpha + \beta + \gamma \leq 3$ [33]. Besides, many researchers have introduced many neutrosophic probability distributions like Poisson, exponential, binomial, normal, uniform, Weibull and so on. [32], [4], [18], [26]. Additionally, researchers have presented the concept of neutrosophic queueing theory in [35], [36] that is one branch of neutrosophic stochastic modelling. Furthermore, researchers have studied neutrosophic time series prediction and modelling in many cases like neutrosophic moving averages, neutrosophic logarithmic models, neutrosophic linear models and so on. [2], [3], [13].

On the other hand, neutrosophic logic has solved many decision-making problems efficiently like evaluating green credit rating, personnel selection, etc. [22], [23], [24], [1].

In this paper we will introduce a generalization to classical random vector to deal with imprecise, uncertainty, ambiguity, vagueness, enigmatic adding the indeterminacy part to its form, then we will show and prove several characteristics of this neutrosophic random vector including expected value, variance, covariance, joint function probability, joint distribution function and study its properties. This extension lets us build and study many stochastic models in the future that help us in modelling, simulation, decision making, prediction and classification specially in the cases of incomplete data and indeterminacy. For more notions associated to neutrosophic theory, we refer the reader to [10, 11, 14–17].

Now, we show some well-known definitions and properties of neutrosophic logic and neutrosophic probability which are useful for the developing of this paper.

Definition 1.1. (see [31]) Let X be a non-empty fixed set. A neutrosophic set A is an object having the form $\{x, (\mu A(x), \delta A(x), \gamma A(x)) : x \in X\}$, where $\mu A(x), \delta A(x)$ and $\gamma A(x)$ represent the degree of membership, the degree of indeterminacy , and the degree of non-membership respectively of each element $x \in X$to the set A.

Definition 1.2. (see [6]) Let K be a field, the neutrosophic filed generated by K and I is denoted by $\langle K \cup I \rangle$ under the operations of K, where I is the neutrosophic element with the property $I^2 = I$.

Definition 1.3. (see [32]) Classical neutrosophic number has the form $a + bI$ where a, b are real or complex numbers and I is the indeterminacy such that $0.I = 0$ and $I^2 = I$ which results that $I^n = I$ for all positive integers n.

Definition 1.4. (see [33]) The neutrosophic probability of event A occurrence is $NP(A) = (ch(A), ch(neutA), ch(antiA)) = (T, I, F)$ where T, I, F are standard or non-standard subsets of the non-standard unitary interval $]^-0, 1^+[$.

Recently, Bisher and Hatip [8] introduced and studied the notions of neutrosophic random variables by using the concepts presented by [33], these notions were defined as follows:

Definition 1.5. Consider the real valued crisp random variable X which is defined as follows:

$$X : \Omega \to \mathbb{R}$$

where Ω is the events space. Now, they defined a neutrosophic random variable X_N as follows:

$$X_N : \Omega \to \mathbb{R}(I)$$

and

$$X_N = X + I$$

where I is indeterminacy.

Theorem 1.6. *Consider the neutrosophic random variable $X_N = X + I$ where cumulative distribution function of X is $F_X(x) = P(X \leq x)$. Then, the following statements hold:*

(1) $F_{X_N}(x) = F_X(x - I)$,
(2) $f_{X_N}(x) = f_X(x - I)$.

Where F_{X_N} and f_{X_N} are cumulative distribution function and probability density function of X_N, respectively.

Theorem 1.7. *Consider the neutrosophic random variable $X_N = X + I$, expected value can be found as follows:*

$$E(X_N) = E(X) + I.$$

Proposition 1.8 (Properties of expected value of a neutrosophic random variable)**.** *Let X_N and Y_N be neutrosophic random variables, then the following properties holds:*

(1) $E(aX_N + b + cI) = aE(X_N) + b + cI; a, b, c \in \mathbb{R}$,
(2) *If X_N and Y_N are neutrosophic random variables, then $E(X_N \pm E_(Y_N) = E(X_N) \pm E(Y_N)$,*
(3) $E[(a + bI)X_N] = aE(X_N) + bIE(X_N); a, b \in \mathbb{R}$,
(4) $|E(X_N)| \leq E|X_N|$.

Theorem 1.9. *Consider the neutrosophic random variable $X_N = X + I$, variance of X_N is equal to variance of X, i.e. $V(X_N) = V(X)$.*

For supporting above definitions and their implications, we present some examples by using exponential distribution on a neuttrosophic random variable which show how importance neutrosophic random variable is in neutrosophic probability theory. For more examples on neutrosophic random variable and its difference between classical random variable see [5].

Example 1.10. Let X_N be a neutrosophic continuous random variable which has an exponential distribution with parameter $\lambda > 0$, and we will denote this as $X_N \sim exp(\lambda)$, its density function is defined as follows:

$$f_{X_N}(x) = f_X(x - I) = \begin{cases} \lambda e^{-\lambda(x-I)} & if\ x > I, \\ \\ 0 & if\ x \leq I. \end{cases}$$

Applying Theorems 1.7 and 1.9, we can check that $E(X_N) = 1/\lambda + I$ and $V(X_N) = 1/\lambda^2$.

Example 1.11. Suppose that the time in minutes that a user spends checking his email follows an exponential parameter distribution $\lambda = 2$. Calculate the probability that the user stay connected to the mail server for less than a minute with an indeterminacy I.

Solution: Let X_N the connection time to the mail server, by example 1.10 we have

$$P(X_N < 1) = P(X + I < 1) = P(X < 1 - I) = \int_0^{1-I} 2e^{-2(x-I)}dx = e^{2I} - e^{-2(1-2I)},$$

if we take $I = 0.5$, we have $\int_0^{0.5} 2e^{-2(x-0.5)}dx = e - 1 \backsimeq 0.368$

Example 1.12. Let X_N be a neutrosophic random variable with exponential distribution $exp(\lambda)$. We will show that the function generating moments of X_N is the function M_{X_N} that appears below.

$$M_{X_N}(t) = \frac{\lambda}{\lambda - t}e^{tI}, \text{ for all } t < \lambda.$$

Besides, we will find its expected.

Solution:

$M_{X_N} = E(e^{tX_N}) = E(e^{t(X+I)}) = E(e^{tx+tI}) = e^{tI}E(e^{tX}) = e^{tI}\int_0^{\infty} e^{tx}\lambda e^{-\lambda x}dx = e^{tI}\frac{\lambda}{\lambda - t}$, if $t < \lambda$.

Now, we will find its expected, Bisher and Hatip [8] proved $\frac{d_{M_{X_N}}(0)}{dt} = E(X_N)$. Therefore we have

$$\frac{d_{M_{X_N}}(t)}{dt} = e^{tI}\frac{\lambda}{(\lambda - t)^2} + \frac{\lambda}{\lambda - t}Ie^{tI},$$

if we take $t = 0$, we obtain

$$\frac{d_{M_{X_N}}(0)}{dt} = \frac{\lambda}{\lambda^2} + \frac{\lambda}{\lambda}I = \frac{1}{\lambda} + I = E(X_N)$$

as was shown in example 1.10.

2. Main Results

In [33] Smarandache presented the neutrosophic random variable that it is a variable that may have and indeterminate outcome, and later [8] Bisher and Hatip represented that indeterminacy by mathematical formula on neutrosophic random variables. Now, we are going to find the properties of joint neutrosophic random variable by using notions mentioned above, with this we proved that covariance of neutrosophic random variables X_N and Y_N is equal to covariance of X and Y as can be seen in Theorem 2.15. But first, we have to introduce and study the following definitions:

Definition 2.1. A neutrosophic random vector of two dimension is a vector (X_N, Y_N) in which each coordinate is a neutrosophic random variable. Analogously, we can define a neutrosophic random vector multidimensional as follows $(X_{N_1}, X_{N_2}, ..., X_{N_n})$ in which $X_{N_1}, X_{N_2}, ..., X_{N_n}$ are neutrosophic random variables for each $n = 1, 2, ...$

Definition 2.2. Let (X_N, Y_N) be a neutrosophic random vector in which X_N takes the value $x_1, x_2, ...$ and Y_N takes the value $y_1, y_2, ...$ Then, joint probability function of a neutrosophic discrete random vector (X_N, Y_N) $f_N(x, y) : \mathbb{R}^2 \to [0, 1]$ and any $(x, y) \in \mathbb{R}^2$, it is defined as follows

$$f_{(X_N,Y_N)}(x, y) = f_{(X,Y)}(x - I, y - I) =$$
$$\begin{cases} P(X = x - I, Y = y - I) = & \displaystyle\sum_{u \leq x-I} \sum_{v \leq y-I} f_{(X_N,Y_N)}(u, v) \\ & if \ (x - I, y - I) \in \{x_1 - I, x_2 - I, ...\} \times \{y_1 - I, y_2 - I, ...\} \\ 0 & otherwise. \end{cases}$$

Definition 2.3. Let (X_N, Y_N) be a neutrosophic random vector, we define probability function of a neutrosophic continuous random vector (X_N, Y_N). Then, joint probability neutrosophic function of a discrete random vector (X_N, Y_N) $f_N(x, y) : \mathbb{R}^2 \to [0, \infty)$ in which is non-negative and integrable, and for any $(x, y) \in \mathbb{R}^2$, it is defined as follows

$$P(X_N \leq x, Y_N \leq y) = P(X \leq x - I, Y \leq y - I) = \int_{-\infty}^{y-I} \int_{-\infty}^{x-I} f_{(X_N,Y_N)}(u, v) dvdu$$

Example 2.4. We will show that $g : \mathbb{R}^2 \to [0, 1]$ is a joint probability neutrosophic function where.

$$g(x, y) = \begin{cases} 4xy & if \quad 0 < x < 1, 0 < y < 1 \\ 0 & otherwise. \end{cases}$$

Taking into account that $g_N(x, y) = g(x - I, y - I)$,

$$g_{(X_N,Y_N)}(x, y) = g(x - I, y - I) =$$
$$\begin{cases} 4(x - I)(y - I) & if \quad I < x < 1 + I, I < y < 1 + I \\ 0 & otherwise. \end{cases}$$

We can see that $g(x, y) \geq 0$ for any $(x, y) \in \mathbb{R}^2$. Now, $\int_I^{1+I} \int_I^{1+I} 4(x - I)(y - I)dxdy = \int_I^{1+I} 2(y - I)[\int_I^{1+I} 2(x - I)dx]dy = \int_I^{1+I} 2(y - I)dy = 1.$

Definition 2.5. Let (X_N, Y_N) be a neutrosophic random vetor, we define neutrosophic joint distribution function which will be denoted by $F_{(X_N,Y_N)}(x, y) = P(X_N \leq x, Y_N \leq y) = P(X \leq x - I, Y \leq y - I)$.

Remark 2.6. The little comma which appears in the right means intersection of $(X_N \leq x)$ and $(Y_N \leq y)$, i.e. $F_{(X_N,Y_N)}(x,y)$ is the probability of $(X_N \leq x) \cap (Y_N \leq y) = (X \leq x - I) \cap (Y \leq y - I)$.

Remark 2.7. If we know joint probability neutrosophic function of a random vector, we can find neutroosphic joint distribution function as

(1) $F_{(X_N,Y_N)}(x,y) = \int_{-\infty}^{y-I} \int_{-\infty}^{x-I} f_{(X_N,Y_N)}(u,v)dvdu$ **(Continuous case)**.

(2) $F_{(X_N,Y_N)}(x,y) = \sum_{u \leq x-I} \sum_{v \leq y-I} f_{(X_N,Y_N)}(u,v)$ **(Discrete case)**.

Besides, if we know neutrosophic joint distribution function, we can find joint probability neutrosophic function as

(3) $f_{(X_N,Y_N)}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{(X_N,Y_N)}(x,y)$ **(continuous case)**.

(4) $f_{(X_N,Y_N)}(x,y) = F_N(x,y) - F_N(x^-,y) - F_N(x,y^-) + F_N(x^-,y^-)$ **(Discrete case)**, where $F_N(x^-,y)$ means the limit of $F_{(X_N,Y_N)}(x,y)$ in the point (x,y) taking into account that y is a constant and we approximate x by left.

Here, we will show a little proof of part four: $(X_N \leq x) = (X \leq x - I)$ can be written as $(X < x-I) \cup (X = x-I)$ where $(X < x-I) \cap (X = x-I) = \emptyset$. Analogously, $(Y_N \leq y) = (Y \leq y - I) = (Y < y - I) \cup (Y = y - I)$. if we write $(X_N \leq x, Y_N \leq y) = (X \leq x - I, Y \leq y - I)$ takes into account mentioned above, the proof follows.

Proposition 2.8. *Let $F_{(X_N,Y_N)}(x,y)$ and $G_{(X_N,Y_N)}(x,y)$ be two neutrosophic joint distribution functions . Then, for any $\lambda \in [0,1]$, $(x,y) \to \lambda F_{(X_N,Y_N)}(x,y) + (1-\lambda)G_{(X_N,Y_N)}(x,y)$ is a neutrosophic joint distribution function.*

Example 2.9. Let (X_N, Y_N) be a continuous neutrosophic random vector with joint probability neutrosophic function

$$h_{(X_N,Y_N)}(x,y) = \begin{cases} 1 & if \quad 0 < x,y < 1 \\ \\ 0 & otherwise. \end{cases}$$

Taking into account that $h_{(X_N,Y_N)}(x,y) = h(x - I, y - I)$, we have

$$h_{(X_N,Y_N)}(x,y) = h(x - I, y - I) = \begin{cases} 1 & if \quad I < x,y < 1 + I \\ \\ 0 & otherwise. \end{cases}$$

we can see that neutrosophic joint distribution function is determined by

$$F_{(X_N,Y_N)}(x,y) = F_{(X,Y)}(x-I,y-I) = \begin{cases} 0 & if \quad x \leq I \ or \ y \leq I \\ \\ (x-I)(y-I) & if \quad I < x, y < 1+I \\ \\ (x-I) & if \quad I < x < 1+I, y \geq 1+I \\ \\ (y-I) & if \quad I < y < 1+I, x \geq 1+I \\ \\ 1 & if \quad x, y \geq 1+I. \end{cases}$$

Definition 2.10. Let $f_{(X_N,Y_N)}(x,y)$ be a joint probability neutrosophic function of a continuous random variable (X_N, Y_N). We define neutrosophic marginal function of X_N as follows:

$$f_{X_N}(x) = \int_{-\infty}^{+\infty} f_{(X_N,Y_N)}(x,y)dy$$

and we define neutrosophic marginal function of Y_N as follows:

$$f_{Y_N}(y) = \int_{-\infty}^{+\infty} f_{(X_N,Y_N)}(x,y)dx$$

Example 2.11. Let (X_N, Y_N) be a continuous neutrosophic random vector with joint probability neutrosophic function

$$g_{(X_N,Y_N)}(x,y) = \begin{cases} 4xy & if \quad 0 < x < 1, 0 < y < 1 \\ \\ 0 & otherwise. \end{cases}$$

Then, neutrosophic marginal function of X_N is $f_{X_N}(x) = \int_I^{1+I} 4(x-I)(y-I)dy = 4(x-I)\int_I^{1+I}(y-I)dy = 2(x-I)$. Therefore,

$$f_{X_N}(x,y) = \begin{cases} 2(x-I) \ if \quad I < x < 1+I \\ \\ 0 \quad otherwise. \end{cases}$$

Analogously, we can show that

$$f_{Y_N}(x,y) = \begin{cases} 2(y-I) \ if \quad I < y < 1+I \\ \\ 0 \quad otherwise. \end{cases}$$

Definition 2.12. Let $f_{(X_N,Y_N)}(x,y)$ be a joint probability neutrosophic function of a discrete random variable (X_N, Y_N). We define neutrosophic marginal function of X_N as follows:

$$f_{X_N}(x) = \sum_y f_{(X_N,Y_N)}(x,y)$$

and we define neutrosophic marginal function of Y_N as follows:

$$f_{Y_N}(y) = \sum_x f_{(X_N,Y_N)}(x,y)$$

Example 2.13. Let (X_N, Y_N) be a discrete random variable with joint probability neutrosophic function

$$f_{(X_N,Y_N)}(x,y) = \begin{cases} \dfrac{x+2y}{30}\ if \quad (x,y) \in \{1,2,3\} \times \{1,2\} \\ \\ 0 \quad otherwise. \end{cases}$$

We can see that $f_{(X_N,Y_N)}(x,y)$ is a joint probability neutrosophic function, we show a little proof,

Since $f_{(X_N,Y_N)}(x,y) = f_{(X,Y)}(x-I, y-I)$, we have

$$f_{X_N}(x,y) = \sum_{y=1+I}^{2+I} f_{(X_N,Y_N)}(x,y) = \sum_{y=1+I}^{2+I} \frac{x+2y-3I}{30} = \frac{x+3-I}{15}$$

for $x \in \{1+I, 2+I, 3+I\}$ and 0 otherwise. In the same way

$$f_{Y_N}(x,y) = \sum_{x=1+I}^{3+I} f_{(X_N,Y_N)}(x,y) = \sum_{x=1+I}^{3+I} \frac{x+2y-3I}{30} = \frac{1+y-I}{5}$$

for $y \in \{1+I, 2+I\}$ and 0 otherwise.

Now, the neutrosophic marginal functions $f_{X_N}(x)$ and $f_{Y_N}(y)$ are

$$f_{X_N}(x,y) = \sum_{y=1+I}^{2+I} = \begin{cases} 8/30\ if \quad x = 1+I \\ \\ 10/30 \quad if\ x = 2+I \\ \\ 12/30 \quad if\ x = 3+I \\ \\ 0 \qquad otherwise. \end{cases}$$

and

$$f_{Y_N}(x,y) = \sum_{x=1+I}^{3+I} = \begin{cases} 12/30\ if \quad y = 1+I \\ \\ 18/30 \quad if\ y = 2+I \\ \\ 0 \qquad otherwise. \end{cases}$$

Definition 2.14. Expected of a neutrosophic random vector (X_N, Y_N) in which expected of X_N and Y_N exist, we define $E(X_N, Y_N) = (E(X_N), E(Y_N))$.

Theorem 2.15. *Consider the neutrosophic random variables $X_N = X + I$ and $Y_N = Y + I$, covariance of (X_N, Y_N) is equal to covariance of (X, Y), i.e. $Cov(X_N, Y_N) = Cov(X, Y)$.*

Proof: Let $Cov(X_N, Y_N) = E[(X_N - E(X_N)(Y_N - E(Y_N)]$,

$Cov(X_N, Y_N) = E[(X_N - E(X_N)(Y_N - E(Y_N)]$
$= E[(X + I - E(X) - I)(Y + I - E(Y) - I)]$

$E[(X-E(X))(Y-E(Y))]$
$=Cov(X,Y).$

Example 2.16. Let (X_N, Y_N) be a continuous neutrosophic random vector with joint probability neutrosophic function

$$g_{(X_N,Y_N)}(x,y)=\begin{cases} 4xy & if \quad 0<x<1, 0<y<1 \\ \\ 0 & otherwise. \end{cases}$$

Then, covariance of (X_N, Y_N) is calculated as

$$Cov(X_N,Y_N)=Cov(X,Y)=\int_0^1\int_0^1(x-\frac{2}{3})(y-\frac{2}{3})4xydxdy=$$
$$\int_0^1 4y(y-\frac{2}{3})[\int_0^1(x^2-\frac{2}{3}x)dx]dy=0.$$

Remark 2.17. It is clear that $Cov(X_N,Y_N)=Cov(Y_N,X_N)$.

Theorem 2.18. *Variance of a neutrosophic random vector (X_N, Y_N) is equal to variance of a random vector (X,Y), i.e. $Var(X_N,Y_N)=Var(X,Y)$. In others words,*

$$Var(X_N,Y_N)=\begin{pmatrix} Var(X_N) & Cov(X_N,Y_N) \\ Cov(Y_N,X_N) & Var(Y_N)\end{pmatrix}=\begin{pmatrix} Var(X) & Cov(X,Y) \\ Cov(Y,X) & Var(Y)\end{pmatrix}=Var(X,Y). \tag{1}$$

Proof: The proof is followed by Theorems 1.9 and 2.15.

Example 2.19. Let (X_N, Y_N) be a continuous neutrosophic random vector with normal distribution and parameters $(\mu_1,\sigma_1^2,\mu_2,\sigma_2^2,\rho)$. This distribution is defined as follows:
$f_{(X_N,Y_N)}(x,y) = f_{(X,Y)}(x-I,y-I) = \dfrac{1}{2\pi\sigma_1^2\sigma_2^2\sqrt{1-\rho^2}}\exp(-\dfrac{1}{2(1-\rho^2)}[\dfrac{(x-I-\mu_1)^2}{\sigma_1^2} - \dfrac{2\rho}{\sigma_1^2\sigma_2^2}(x-I-\mu_1)(y-I-\mu_2)+\dfrac{(y-I-\mu_2)^2}{\sigma_2^2}])$ where $\mu_1,\mu_2\in\mathbb{R}$; $\sigma_1^2,\sigma_2^2>0$ and $-1<\rho<1$.

When, $\mu_1=\mu_2=0$ and $\sigma_1^2=\sigma_2^2$, we have neutrosophic random vector with standard normal distribution, and $f_{(X_N,Y_N)}(x,y)$ is reduced as:
$f_{(X_N,Y_N)}(x,y)=f_{(X,Y)}(x-I,y-I)=\dfrac{1}{2\pi\sqrt{1-\rho^2}}\exp(-\dfrac{1}{2(1-\rho^2)}((x-I)^2-2\rho(x-I)(y-I)+(y-I)^2))$.

Then, we will show that:

(1) $E(X_N,Y_N)=(\mu_1+I,\mu_2+I)$.
(2) $Cov(X_N,Y_N)=\rho\sigma_1^2\sigma_2^2$.
(3) $Var(X_N,Y_N)=\begin{pmatrix}\sigma_1^2 & \rho\sigma_1^2\sigma_2^2\\ \rho\sigma_1^2\sigma_2^2 & \sigma_2^2\end{pmatrix}$.

Solution:

(1) Since $E(X_N,Y_N)=(E(X_N),E(Y_N))=(E(X)+I,E(Y)+I)=(\mu_1+I,\mu_2+I)$.

(2) $Cov(X_N, Y_N) = Cov(X, Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{(X_N,Y_N)}(x, y)dxdy - \mu_1\mu_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{(X,Y)}(x, y)dxdy - \mu_1\mu_2 = \mu_1\mu_2 + \rho_{12}^{22} - \mu_1\mu_2 = \rho\sigma_1^2\sigma_2^2$.

(3) By part (2) of this example and takes into account that $Var(X_N) = Var(X)$. We have, $Var(X_N, Y_N) = Var(X, Y) = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1^2\sigma_2^2 \\ \rho\sigma_1^2\sigma_2^2 & \sigma_2^2 \end{pmatrix}$.

3. Conclusion

The results that are presented in this paper can be applied to define several notions in neutrosophic probability theory that are not defined and not studied yet including independence random variables, convergence in random variables, stochastic processes, reliability theory models, quality control techniques. where all depend on the concept of neutrosophic random variables and its properties. Besides, these results can be applied in stochastic modelling and random numbers generating which is very important in simulation of probabilistic models.

We are looking forward to studying the properties of joint neutrosophic probability distributions when the distribution of random vector changes $(X_N, Y_N) = (X + I, Y + I)$ i.e., when the random vector contains an indeterminant part so we can model and simulate many stochastic problems.

In this research, we firstly obtained a new general definitions of neutrosophic random vector, concepts of joint probability distribution function and joint distribution function. We focused on the neutrosophic representation and proved some properties. Additionally, we showed various examples in which can help to applied them in several applications problems.

Conflicts of Interest

The author declares no conflict of interest

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Generalized Pythagorean Neutrosophic Sets In the Study of Group Theory

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Abstract: In 2019, Jansi et al. present the notion of the Pythagorean neutrosophic set (PNS) as an extension of a neutrosophic set with dependent neutrosophic components whenever $0 \le \mu_A(x)^2 + \varsigma_A(x)^2 + \gamma_A(x)^2 \le 2$. But due to the more complexity involved in a decision-making problem, there is a serious need to generalize the PNS for dealing with indeterminate, incomplete, and inconsistent data present in the belief system. The main objective of this paper is to elicit the notion of (α,β,δ)-Pythagorean neutrosophic set as a generalization of PNS. The (α,β,δ)-PNS provides a more powerful tool to model the various types of uncertainty with high precision and accuracy. Concerning to the idea of (α,β,δ)-PNS, we propose a new (α,β,δ)-Pythagorean neutrosophic subgroup (PNSG) and thus investigate some properties based on the proposed subgroup. Moreover, we discuss the impact of (α,β,δ)-Pythagorean neutrosophic subgroups in solving real-world problems with an aid of a suitable example.

Keywords: Pythagorean neutrosophic set; (α,β,δ) -Pythagorean neutrosophic set; Pythagorean neutrosophic subgroup, (α,β,δ) -Pythagorean neutrosophic subgroup, (α,β,δ) -Pythagorean neutrosophic normal subgroup.

1. Introduction

Before the invention of the fuzzy set (FS), suggested by Zadeh[1], we deal with uncertainty in an unorganized manner. The FS provides a general framework to handle uncertainty systematically. It makes a huge impact on real decision-making problems. Later, Zimmermann [2] elaborately studied the FS theory and discussed its practical applications. Every FS is a set of couples in which for every member of the set of discourse there exists a membership value, belongs to [0, 1]. So, we need to assign a membership function to design the uncertainty. Thus, the FS is useful in modeling vagueness with a proper methodology. It grows rapidly over the decades and it has huge applications in different disciplines such as medicine, economics, social science, computer science, engineering, etc. Relying on the need and the importance of FS theory in the advancement of modern technologies and researches, by embedding this idea, many new theories such as fuzzy logic [3], fuzzy graph

[4], fuzzy topology [5], fuzzy optimization [6], fuzzy image processing [7], fuzzy neural network [8], fuzzy sets in artificial intelligence [9], fuzzy Boolean algebra [10], linguistic fuzzy logic game theory [11], etc. have been developed and they have huge applications in decision-making. The fuzzy set theory has been extended to the intuitionistic fuzzy set(IFS) proposed by Atanassov [12], which incorporates the concept of non-membership as well as membership. Some other extensions of fuzzy sets are given in [13-16].

Nowadays the concept of two-valued logic does not justify the concept of imprecision in various situations. Smarandache[17] established the neutrosophic set(NS) as a result of this. There are three independent components in the neutrosophic set. It is a recent topic to handle uncertainty, incompleteness, and indeterminacy. Wang et al. [18] defined the single-valued neutrosophic set for research purposes. Also, [19-24] discusses some more aspects of the neutrosophic set.

Group theory is the study of groups where several sets are equipped with an operation. It is the building block of abstract algebra and it has been used in nearly all branches of mathematics such as cryptography, number theory, harmonic analysis, algebraic geometry, crystallography, etc. In the study of combinatory, we use a symmetric group. We also use the concept of group theory in physics, Chemistry, Molecular biology, Material science, etc. Rosenfeld proposed fuzzy groups [25] in 1971, introducing the notion of degree in the fuzzy set. Anthony et al. [26] redefined fuzzy groups. Das [27], in 1981, gave the idea of fuzzy level subgroup, Ajmal et al. [28] introduced fuzzy coset and fuzzy normal subgroup, Zhan and Tan [29] studied intuitionistic fuzzy subgroup, complex intuitionistic fuzzy groups were proposed by Husban et al. [30]. In a different way, Agboola et al. [31] defined neutrosophic groups and subgroups.

As an extension of fuzzy set, Atanassov developed the notion of intuitionistic fuzzy set in [12], where $\mu(x)+\gamma(x)\leq 1$ and $\mu(x)$, $\gamma(x)\in[0, 1]$. But, there are some situations under which $\mu(x)+\gamma(x)>1$. To put it another way, neither the fuzzy set nor the intuitionistic fuzzy set can resolve such uncertainty. Yager[32] proposed the Pythagorean fuzzy set(PFS) as a result of this. It is the latest tool developed to deal with imprecision with a wider scope of applications. It is a generalization of the fuzzy set and intuitionistic fuzzy set. The Pythagorean fuzzy set has a close connection with the intuitionistic fuzzy set. Ejegwa [33] gave an application of PFS in career placement, Liang et al. [34] investigated the use of PFS to extend TOPSIS to multi-criteria decision making. Lin et al.[35] studied the Pythagorean TOPSIS method and its application, which is based on unique correlation measurements. Garg[36] presented the generalized Pythagorean fuzzy geometric aggregation operator using Einstein t-norm and t-co-norm for multi-criteria decision-making. In 2018, Garg [37] defined the linguistic Pythagorean fuzzy sets and their applications, Naz et al.[38] used a novel approach to decision-making with Pythagorean fuzzy information. The complex Pythagorean fuzzy environment for decision-making was proposed by Akram and Naz[39]. The generalized interval-valued Pythagorean fuzzy aggregation operators were discovered by Rahman et al. [40]. The Pythagorean neutrosophic fuzzy graph was proposed by Ajay et al.[41]. To deal with indeterminacy along with incompleteness, in 2019, Jansi et al.[42] introduced the PNS withT and F as dependent components where the truth-membership, falsity -membership, and indeterminate-membership satisfy the criteria $0\leq(\mu(x))^2+(\gamma(x))^2+(\varsigma(x))^2\leq 2$, with $\mu(x)$, $\gamma(x)$ and $\varsigma(x)\in[0, 1]$.

The primary objective for writing this study is to present a novel strategy for generalizing the PNS concept described in [42]. When the PNS condition fails, i.e., $(\mu(x))^2+(\gamma(x))^2+(\varsigma(x))^2>2$, we have to figure out what to do. For example, information about an object in a neutrosophic environment is delivered to the decision-maker in the form $(0.9, 0.8, 0.8)$, which PNS cannot address because $0.9^2 + 0.8^2 + 0.8^2 > 2$. This led to the foundation of introducing $(\alpha, \beta, \delta)-$ PNSs and formation of their associated groups and subgroups. Then, we study some properties of $(\alpha, \beta, \delta)-$ Pythagorean neutrosophic groups and justify them with examples. We also discuss the concept of $(\alpha, \beta, \delta)-$ Pythagorean neutrosophic coset, $(\alpha, \beta, \delta)-$ Pythagorean neutrosophic normal subgroup. Finally, we give a real-world example that justifies the newly proposed theory and its contribution to the field of group theory appropriately.

The remainder of the paper is organized as follows: The proposed study requires certain basic definitions, which are included in Section 2. Section 3 examines some of the features and assertions of different sorts of (α, β, δ) -Pythagorean neutrosophic subgroups. A real-life implementation of the suggested study has been carried out in section 4. Section 5 presents the proposed study's conclusion and future scope.

2. Preliminaries

In this section, we provide the foundation of knowledge that helps us to unveil the newly proposed theory.

2.1 Definition [1, 2]

Let X be a crisp set and A be a subset of X. Then the fuzzy set defined on X is defined as a set of ordered pairs of the form $\{(x, \mu_A(x)) : x \in X\}$, where $\mu_A : X \to [0,1]$.

2.2 Definition [12]

Let X be a crisp set and A be a subset of X. Then the intuitionistic fuzzy set defined on X is defined as a set of ordered triples of the type $\{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$, where $\mu_A : X \to [0,1], \gamma_A : X \to [0,1]$ and satisfies the condition $0 \le \mu_A(x) + \gamma_A(x) \le 1$.

2.3 Definition [17]

Let x be a generic element in X, a universe of discourse. A neutrosophic set A in X is characterized by a truth-membership function T_A, an indeterminacy-membership function I_A, and a falsity-membership function F_A. Here, T_A, I_A and F_A are real standard and non-standard subsets of [0,1] and its set-theoretic representation is given by

$$A = \{\langle x, (T_A(x), I_A(x), F_A(x))\rangle : x \in X, T_A(x), I_A(x), F_A(x) \in [0,1]\}$$

As there are no restrictions on the sum of $T_A(x), I_A(x)$ and $F_A(x)$, as a result, we take $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

2.4 Definition [42]

Let X be a universal set. A neutrosophic set A on X is of the type $A = \left\{ \left\langle x, \left(T_A(x), I_A(x), F_A(x) \right) \right\rangle : x \in X, T_A(x), I_A(x), F_A(x) \in [0,1] \right\}$.

If $T_A(x)$ and $F_A(x)$ are both dependent components, and $I_A(x)$ is an independent component, then

$0 \le T_A(x) + I_A(x) + F_A(x) \le 2, \forall x \in X$.

2.5 Definition [32]

Let X be a crisp set. A PFS χ on X is an object of the type $\chi = \left\{ \left(x, \mu(x), \gamma(x) \right) : x \in X \right\}$,

where $\mu(x), \gamma(x) \in [0,1]$ which satisfies the condition $0 \le \mu^2(x) + \gamma^2(x) \le 1$.

2.6 Definition [42]

Let X be a universal set. A PNS A, with $\mu_A(x)$ and $\gamma_A(x)$ are dependent neutrosophic components and $\varsigma_A(x)$ is an independent component, is an object on X is of the form $A = \left\{ \left\langle x, \left(\mu_A(x), \varsigma_A(x), \gamma_A(x) \right) \right\rangle : x \in X \right\}$, where $\mu_A(x), \varsigma_A(x)$ and $\gamma_A(x) \in [0,1]$ and

$0 \le \left(\mu_A(x) \right)^2 + \left(\varsigma_A(x) \right)^2 + \left(\gamma_A(x) \right)^2 \le 2$

2.7 Definition [25]

Let $(G, \circ)$ be a group and μ be a fuzzy subset of G. Then μ is regarded as a fuzzy subgroup of $(G, \circ)$ if

$\mu(x \circ y) \ge \mu(x) \wedge \mu(y)$ and $\mu\left(x^{-1}\right) \ge \mu(x)$, $\forall x, y \in G$.

2.8 Definition [29]

Let $(G, \circ)$ be a group and $A = \left\{ \left(x, \mu(x), \gamma(x) \right) : x \in G \right\}$ be an intuitionistic fuzzy set of G. Then A is described as an intuitionistic fuzzy subgroup of $(G, \circ)$ if

$\mu(x \circ y) \ge \mu(x) \wedge \mu(y)$ and $\gamma(x \circ y) \le \mu(x) \vee \mu(y)$

$\mu\left(x^{-1}\right) \ge \mu(x)$ and $\gamma\left(x^{-1}\right) \le \gamma(x)$, $\forall x, y \in G$.

2.9 Definition [32]

Let $A=\left\{\left(x,\mu(x),\gamma(x)\right):x\in \breve{G}\right\}$ be a PFS of a group $(\breve{G},\circ)$, then A is said to be a Pythagorean fuzzy subgroup (PFSG) of $\breve{G}$ if it fulfills the conditions

$\mu^2(x\circ y)\geq \mu^2(x)\wedge \mu^2(y)$ and $\gamma^2(x\circ y)\leq \gamma^2(x)\vee \gamma^2(y)$

$\mu^2(x^{-1})\geq \mu^2(x)$ and $\gamma^2(x^{-1})\leq \gamma^2(x)$, $\forall x,y\in \breve{G}$

2.10 Definition

Let $(\breve{G},\circ)$ be any group and $A=\left\{\left(x,\mu(x),\varsigma(x),\gamma(x)\right):x\in \breve{G}\right\}$ be a NS of $\breve{G}$. Then A is called a neutrosophic subgroup of $(\breve{G},\circ)$ if

$\mu(x\circ y)\leq \mu(x)\vee \mu(y)$, $\varsigma(x\circ y)\leq \varsigma(x)\vee \varsigma(y)$ and $\gamma(x\circ y)\geq \gamma(x)\wedge \gamma(y)$

$\mu(x^{-1})\leq \mu(x)$, $\varsigma(x^{-1})\leq \varsigma(x)$ and $\gamma(x^{-1})\geq \gamma(x)$, $\forall x,y\in \breve{G}$

2.11 Definition

Let $(\breve{G},\circ)$ be a group and $A=\left\{\left\langle x,\left(\mu_A(x),\varsigma_A(x),\gamma_A(x)\right)\right\rangle:x\in \breve{G}\right\}$ be a PNS of $\breve{G}$. Then A is called a Pythagorean neutrosophic subgroup (PNSG) of $\breve{G}$ if it satisfies the following

$\mu^2(x\circ y)\geq \mu^2(x)\wedge \mu^2(y)$, $\varsigma^2(x\circ y)\geq \varsigma^2(x)\wedge \varsigma^2(y)$ and $\gamma^2(x\circ y)\leq \gamma^2(x)\vee \gamma^2(y)$

$\mu^2(x^{-1})\geq \mu^2(x)$, $\varsigma^2(x^{-1})\geq \varsigma^2(x)$ and $\gamma^2(x^{-1})\leq \gamma^2(x)$, $\forall x,y\in \breve{G}$.

3. Different Types of (α,β,δ)-Pythagorean neutrosophic Subgroups (PNSGs) and their properties

This section contains the (α,β,δ)-PNSs, (α,β,δ)-PNSGs and investigate some of their properties.

3.1 Definition

Let X be a crisp set and α,β and $\delta\in[0,1]$ such that $0\leq \alpha^2+\beta^2+\delta^2\leq 2$. A (α,β,δ)-PNS χ in X is an item of the form

$\chi=\left\{\left(x,\mu^\alpha(x),\varsigma^\beta(x),\gamma^\delta(x)\right):x\in X\right\}$, where $\mu^\alpha(x)=\mu(x)\vee\alpha$, $\varsigma^\beta(x)=\varsigma(x)\vee\beta$ and $\gamma^\delta(x)=\gamma(x)\wedge\delta$, where $0\leq\left(\mu^\alpha(x)\right)^2+\left(\varsigma^\beta(x)\right)^2+\left(\gamma^\delta(x)\right)^2\leq 2$

We are now going to describe some operations on (α,β,δ)-PNSs.

Let χ_1 and χ_2 be two $(\alpha,\beta,\delta)-$PNSs such that $\chi_1=\left\{\left(x,\mu_1^{\alpha}(x),\varsigma_1^{\beta}(x),\gamma_1^{\delta}(x)\right):x\in X\right\}$ and $\chi_2=\left\{\left(x,\mu_2^{\alpha}(x),\varsigma_2^{\beta}(x),\gamma_2^{\delta}(x)\right):x\in X\right\}$ then we have the following operations defined on two $(\alpha,\beta,\delta)-$PNSs

(a) $\chi_1\cup\chi_2=\left\{\left(x,\left\langle\mu_1^{\alpha}(x)\vee\mu_2^{\alpha}(x),\varsigma_1^{\beta}(x)\vee\varsigma_2^{\beta}(x),\gamma_1^{\delta}(x)\wedge\gamma_2^{\delta}(x)\right\rangle\right):x\in X\right\}$

(b) $\chi_1\cap\chi_2=\left\{\left(x,\left\langle\mu_1^{\alpha}(x)\wedge\mu_2^{\alpha}(x),\varsigma_1^{\beta}(x)\wedge\varsigma_2^{\beta}(x),\gamma_1^{\delta}(x)\vee\gamma_2^{\delta}(x)\right\rangle\right):x\in X\right\}$

(c) $\left(\chi_1\cup\chi_2\right)^c=\chi_1^c\cap\chi_2^c$ and $\left(\chi_1\cap\chi_2\right)^c=\chi_1^c\cup\chi_2^c$ (Demorgan's laws)

(d) $\chi_1\subseteq\chi_2$ if $\mu_1^{\alpha}(x)\le\mu_2^{\alpha}(x),\varsigma_1^{\beta}(x)\le\varsigma_2^{\beta}(x)$ and $\gamma_1^{\delta}(x)\ge\gamma_2^{\delta}(x)$

(e) $\chi_1=\chi_2$ if $\mu_1^{\alpha}(x)=\mu_2^{\alpha}(x),\varsigma_1^{\beta}(x)=\varsigma_2^{\beta}(x)$ and $\gamma_1^{\delta}(x)=\gamma_2^{\delta}(x)$

Proof: Proofs of (a), (b), (d), and (e) are obvious. We only show the proof of (c) given in the following:

We define the complement of χ_1 and χ_2 as follows

$\chi_1^c=\left\{\left(x,\gamma_1^{\delta}(x),\varsigma_1^{\beta}(x),\mu_1^{\alpha}(x)\right):x\in X\right\}$ and $\chi_2^c=\left\{\left(x,\gamma_2^{\delta}(x),\varsigma_2^{\beta}(x),\mu_2^{\alpha}(x)\right):x\in X\right\}$

Then,
$$\begin{aligned}\left(\chi_1\cup\chi_2\right)^c&=\left\{\left(x,\left\langle\mu_1^{\alpha}(x)\vee\mu_2^{\alpha}(x),\varsigma_1^{\beta}(x)\vee\varsigma_2^{\beta}(x),\gamma_1^{\delta}(x)\wedge\gamma_2^{\delta}(x)\right\rangle\right):x\in X\right\}^c\\&=\left\{\left(x,\left\langle\gamma_1^{\delta}(x)\vee\gamma_2^{\delta}(x),\varsigma_1^{\beta}(x)\wedge\varsigma_2^{\beta}(x),\mu_1^{\alpha}(x)\wedge\mu_2^{\alpha}(x)\right\rangle\right):x\in X\right\}\end{aligned}$$

$$\chi_1^c\cap\chi_2^c=\left\{\left(x,\left\langle\gamma_1^{\delta}(x)\vee\gamma_2^{\delta}(x),\varsigma_1^{\beta}(x)\wedge\varsigma_2^{\beta}(x),\mu_1^{\alpha}(x)\wedge\mu_2^{\alpha}(x)\right\rangle\right):x\in X\right\}$$

Thus, $\left(\chi_1\cup\chi_2\right)^c=\chi_1^c\cap\chi_2^c$

We can also demonstrate that $\left(\chi_1\cap\chi_2\right)^c=\chi_1^c\cup\chi_2^c$

3.1.1 Example

Let $X=\{a,b,c\}$ be a crisp set. The membership-function (μ), indeterminacy-function (ς) and the non-membership function (γ) on X are all defined as follows:

$$\mu(a)=0.8, \varsigma(a)=0.5, \gamma(a)=0.8$$
$$\mu(b)=0.7, \varsigma(a)=0.6, \gamma(a)=0.8$$
$$\mu(c)=0.8, \varsigma(c)=0.9, \gamma(c)=0.8$$

Clearly, $\mu(x)+\varsigma(x)+\gamma(x)>2, \forall x \in X$.

By definition **2.4**, $N=\{(x,(\mu(x),\varsigma(x),\gamma(x))): x \in X\}$ is not a NS with the dependent neutrosophic component.

Again, $\mu^2(c)+\varsigma^2(c)+\gamma^2(c)=2.09>2$. So, by definition **2.6**, N is not a PNS.

Taking, $\alpha=0.7, \beta=0.6$ and $\delta=0.7$ such that $\alpha^2+\beta^2+\delta^2=1.34<2$.

Now, $\mu^{\alpha}(a)=\mu(a)\vee\alpha=0.8$, $\varsigma^{\beta}(a)=\varsigma(a)\vee\beta=0.6$ and $\gamma^{\delta}(a)=\gamma(a)\wedge\delta=0.7$

We can easily verify that $0 \le \mu^{\alpha}(x)+\varsigma^{\beta}(x)+\gamma^{\delta}(x) \le 2, \forall x \in X$.

Thus, the set $N=\{(x,(\mu(x),\varsigma(x),\gamma(x))): x \in X\}$ is $(\alpha,\beta,\delta)-$PNS.

3.2 Definition

Let $(G,\circ)$ be a group and $N=\{\langle x,(\mu^{\alpha},\varsigma^{\beta},\gamma^{\delta})\rangle : x \in G\}$ be a $(\alpha,\beta,\delta)-$PNS.

Then, N is said to be the (α,β,δ) – PNSG of G if it satisfies the following conditions:

$$\mu^{\alpha}(x\circ y) \le \mu^{\alpha}(x)\vee\mu^{\alpha}(y); \varsigma^{\beta}(x\circ y) \le \varsigma^{\beta}(x)\vee\varsigma^{\beta}(y) \text{ and } \gamma^{\delta}(x\circ y) \ge \gamma^{\delta}(x)\wedge\gamma^{\delta}(y)$$

$$\mu^{\alpha}(x^{-1}) \le \mu^{\alpha}(x); \varsigma^{\beta}(x^{-1}) \le \varsigma^{\beta}(x) \text{ and } \gamma^{\delta}(x^{-1}) \ge \gamma^{\delta}(x)\ , \ \forall x,y \in G.$$

3.2.1 Example

Let $(G,*)$ be a group where $G=\{1,\omega,\omega^2\}$ and $'*'$ denotes the usual multiplication. In this group 1 is the identity element and ω represents the cube root of unity such that $\omega^3=1$. Let us define the membership functions as follows

$$\mu(1)=0.8, \varsigma(1)=0.7, \gamma(1)=0.6$$
$$\mu(\omega)=0.7, \varsigma(\omega)=0.6, \gamma(\omega)=0.5$$
$$\mu(\omega^2)=0.6, \varsigma(\omega^2)=0.8, \gamma(\omega^2)=0.7$$

We take, $\alpha=0.65, \beta=0.55$ and $\delta=0.67$

Now,

$$\mu^{\alpha}(1*\omega)=\mu^{\alpha}(\omega)=\mu(\omega)\vee\alpha=0.7 \qquad ;$$

$$\varsigma^{\beta}(1*\omega)=\varsigma^{\beta}(\omega)=\varsigma(\omega)\vee\beta=0.6; \gamma^{\delta}(1*\omega)=\gamma^{\delta}(\omega)=\gamma(\omega)\wedge\delta=0.5$$

Again, $\mu^{\alpha}(1)\vee\mu^{\alpha}(\omega)$=0.8; $\varsigma^{\beta}(1)\vee\varsigma^{\beta}(\omega)$=0.7 and $\gamma^{\delta}(1)\wedge\gamma^{\delta}(\omega)$=0.5

$\mu^{\alpha}(\omega^{-1})=\mu^{\alpha}(\omega^{2})=0.65 \le \mu^{\alpha}(\omega)=0.7$; $\varsigma^{\beta}(\omega^{-1})=\varsigma^{\beta}(\omega^{2})=0.8\le\varsigma^{\beta}(\omega)=0.55$ and

$\gamma^{\delta}(\omega^{-1})=\gamma^{\delta}(\omega^{2})=0.67\ge\gamma^{\delta}(\omega)=0.5$

So, for all $x,y\in\breve{G}$, we can easily verify that the set $N^{\breve{}}=\left\{\left\langle x,\left(\mu^{\alpha},\varsigma^{\beta},\gamma^{\delta}\right)\right\rangle : x\in G\right\}$ is a (α,β,δ) –PNSG of $\breve{G}$.

It can easily verify that every PNSG is a (α,β,δ) –PNSG, but the converse is not true.

3.2.2 Example

Consider the Klein's four-group $\breve{G}=\{e,a,b,\mathrm{c}\}$ with four elements, in which each element is self-inverse and the composition $'*'$ on any two elements results in the third element. Let us check whether it satisfies the criteria of being (α,β,δ)–PNSG or not.

Let us consider the membership values of each element of $\breve{G}$ as follows:

$$\mu(e)=0.6, \varsigma(e)=0.7, \gamma(e)=0.5$$
$$\mu(a)=0.65, \varsigma(a)=0.55, \gamma(a)=0.45$$
$$\mu(b)=0.35, \varsigma(b)=0.4, \gamma(b)=0.75$$
$$\mu(c)=0.6, \varsigma(c)=0.7, \gamma(c)=0.55$$

We consider $\alpha=0.5, \beta=0.6$ and $\delta=0.65$

$$\mu^{\alpha}(\mathrm{a*b})=\mu^{\alpha}(c)=\mu(c)\vee\alpha=0.6;\ \varsigma^{\beta}(\mathrm{a*b})=\varsigma^{\beta}(c)=\varsigma(c)\vee\beta=0.7$$

$$\gamma^{\delta}(\mathrm{a*b})=\gamma^{\delta}(c)=\gamma(c)\wedge\delta=0.55$$

$\mu^{\alpha}(a)\vee\mu^{\alpha}(b)$=0.65; $\varsigma^{\beta}(a)\vee\varsigma^{\beta}(b)$=0.6 and $\gamma^{\delta}(a)\wedge\gamma^{\delta}(b)$=0.45

Since, the condition $\varsigma^{\beta}(\mathrm{a*b})\le\varsigma^{\beta}(a)\vee\varsigma^{\beta}(b)$ does not hold, therefore the group $\breve{G}=\{e,a,b,\mathrm{c}\}$ is not a (α,β,δ) –PNSG.

3.2.3Example

Let us consider the group = U_{10} $\{[a]\in Z_{10}:0<a<10\ \&\ \gcd(a,10)=1\}$. Then,

$U_{10}=\{[1],[3],[7],[9]\}$.

We can easily show that it is (α,β,δ) –PNSG.

The calculation part is left as an exercise for the readers.

3.3 Theorem

If $A=\left\{\left(x,\mu(x),\varsigma(x),\gamma(x)\right):x\in \overset{\square}{G}\right\}$ is a neutrosophic subgroup of the group $(\overset{\square}{G},\circ)$ then

$N^{\square}=\left\{\left\langle x,\left(\mu^{\alpha},\varsigma^{\beta},\gamma^{\delta}\right)\right\rangle : x\in G\right\}$ is a (α,β,δ) –PNSG of $(\overset{\square}{G},\circ)$.

Proof. Since $\left\{\left(x,\mu(x),\varsigma(x),\gamma(x)\right):x\in \overset{\square}{G}\right\}$ is a neutrosophic subgroup of the group $(\overset{\square}{G},\circ)$,

Then,

$$\mu^2(x\circ y)\le \mu^2(x)\vee\mu^2(y),\varsigma^2(x\circ y)\le\varsigma^2(x)\vee\varsigma^2(y) \text{ and } \gamma^2(x\circ y)\ge\gamma^2(x)\wedge\gamma^2(y)$$

$$\mu^2\left(x^{-1}\right)\le\mu^2(x),\varsigma^2\left(x^{-1}\right)\le\varsigma^2(x) \text{ and } \gamma^2\left(x^{-1}\right)\ge\gamma^2(x),\quad \forall x,y\in \overset{\square}{G}$$

Now,

$$\mu^{\alpha}(x\circ y)=\mu(x\circ y)\vee\alpha\le\left[\mu(x)\vee\alpha\right]\vee\left[\mu(y)\vee\alpha\right]=\mu^{\alpha}(x)\vee\mu^{\alpha}(y)$$

Similarly,

$$\varsigma^{\beta}(x\circ y)\le\varsigma^{\beta}(x)\vee\varsigma^{\beta}(y) \text{ and } \gamma^{\delta}(x\circ y)\ge\gamma^{\delta}(x)\wedge\gamma^{\delta}(y)$$

Again,

$$\mu^{\alpha}\left(x^{-1}\right)=\mu\left(x^{-1}\right)\vee\alpha\le\mu(x)\vee\alpha=\mu^{\alpha}(x)$$

Similarly, $\varsigma^{\beta}\left(x^{-1}\right)\le\varsigma^{\beta}(x)$ and $\gamma^{\delta}\left(x^{-1}\right)\ge\gamma^{\delta}(x)$, $\forall x,y\in \overset{\square}{G}$.

This proves the theorem.

3.4 Proposition A (α,β,δ) -PNS $N^{\square}=\left\{\left\langle x,\left(\mu^{\alpha},\varsigma^{\beta},\gamma^{\delta}\right)\right\rangle : x\in G\right\}$ of $(\overset{\square}{G},\circ)$ is a (α,β,δ) – PNSG of $(\overset{\square}{G},\circ)$ iff

$$\mu^{\alpha}\left(x\circ y^{-1}\right)\le\mu^{\alpha}(x)\vee\mu^{\alpha}\left(y^{-1}\right);\varsigma^{\beta}\left(x\circ y^{-1}\right)\le\varsigma^{\beta}(x)\vee\varsigma^{\beta}\left(y^{-1}\right) \text{ and } \gamma^{\delta}\left(x\circ y^{-1}\right)\ge\gamma^{\delta}(x)\wedge\gamma^{\delta}\left(y^{-1}\right)$$

Proof. It is obvious.

3.5 Theorem The intersection of two (α,β,δ) –PNSGs of $(\overset{\square}{G},\circ)$ is a (α,β,δ) –PNSG of $(\overset{\square}{G},\circ)$.

Proof.

Let $N_1^{\square}=\left\{\mu_1^{\alpha},\varsigma_1^{\beta},\gamma_1^{\delta}\right\}$ and $N_2^{\square}=\left\{\mu_2^{\alpha},\varsigma_2^{\beta},\gamma_2^{\delta}\right\}$ be two (α,β,δ) –PNSGs of $(\overset{\square}{G},\circ)$.

Then, their intersection defined as

$N_1^{\square}\overset{\square}{\cap}N_2^{\square}=\left\{\mu^{\alpha},\varsigma^{\beta},\gamma^{\delta}\right\}$, where $\mu^{\alpha}=\mu_1^{\alpha}\vee\mu_2^{\alpha}$, $\varsigma^{\beta}=\varsigma_1^{\beta}\vee\varsigma_2^{\beta}$ and $\gamma^{\delta}=\gamma_1^{\delta}\wedge\gamma_2^{\delta}$.

Now, for all $x,y\in \overset{\square}{G}$,

$$\begin{aligned}\mu^{\alpha}\left(x\circ y^{-1}\right)&=\mu_1^{\alpha}\left(x\circ y^{-1}\right)\vee\mu_2^{\alpha}\left(x\circ y^{-1}\right)\le\left(\mu_1^{\alpha}(x)\vee\mu_1^{\alpha}(y)\right)\vee\left(\mu_2^{\alpha}(x)\vee\mu_2^{\alpha}(y)\right)\\&=\left(\mu_1^{\alpha}(x)\vee\mu_2^{\alpha}(x)\right)\vee\left(\mu_1^{\alpha}(y)\vee\mu_2^{\alpha}(y)\right)\\&=\mu^{\alpha}(x)\vee\mu^{\alpha}(y)\end{aligned}$$

Similarly, $\varsigma^{\beta}\left(x\circ y^{-1}\right)\le\varsigma^{\beta}(x)\vee\varsigma^{\beta}(y)$ and $\gamma^{\delta}\left(x\circ y^{-1}\right)\ge\gamma^{\delta}(x)\wedge\gamma^{\delta}(y)$

This proves the theorem.

Note: The union of two (α,β,δ) –PNSGs of $(\overset{\square}{G},\circ)$ may not be a (α,β,δ) –PNSG of $(\overset{\square}{G},\circ)$.

3.6 Definition

Let $N^{\square}=\left(\mu^{\alpha},\varsigma^{\beta},\gamma^{\delta}\right)$ be a (α,β,δ)-PNSG of a group $\left(\overset{\square}{G},\circ\right)$. Then for all $x,y\in\overset{\square}{G}$, $N^{\square}$ is said to be a normalized (α,β,δ)-PNSG of a group $\left(\overset{\square}{G},\circ\right)$, if the following conditions hold:

(a) $\mu^{\alpha}(x\circ y)\le\mu^{\alpha}(x)\vee\mu^{\alpha}(y),\varsigma^{\beta}(x\circ y)\le\varsigma^{\beta}(x)\vee\varsigma^{\beta}(y)$ and $\gamma^{\delta}(x\circ y)\ge\gamma^{\delta}(x)\wedge\gamma^{\delta}(y)$

(b) $\mu^{\alpha}\left(x^{-1}\right)=\mu^{\alpha}(x),\varsigma^{\beta}\left(x^{-1}\right)=\varsigma^{\beta}(x)$ and $\gamma^{\delta}\left(x^{-1}\right)=\gamma^{\delta}(x)$

(c) $\mu^{\alpha}\left(e^{\square}\right)=1,\varsigma^{\beta}\left(e^{\square}\right)=1$ and $\gamma^{\delta}\left(e^{\square}\right)=0$, where $e^{\square}$ is the identity element.

3.7 Theorem

Let $N^{\square}=\left(\mu^{\alpha},\varsigma^{\beta},\gamma^{\delta}\right)$ be a (α,β,δ) -PNSG of a group $\left(\overset{\square}{G},\circ\right)$. Then the set $M^{\square}=\left\{x\in\overset{\square}{G}:\mu^{\alpha}(x)=\mu^{\alpha}\left(e^{\square}\right),\varsigma^{\beta}(x)=\varsigma^{\beta}\left(e^{\square}\right),\gamma^{\delta}(x)=\gamma^{\delta}\left(e^{\square}\right)\right\}$ forms a (α,β,δ)-PNSG of a group $\left(\overset{\square}{G},\circ\right)$.

Proof: Clearly, the set $M^{\square}$ is the non-empty set, as $e^{\square}\in M^{\square}$.

Since, $M^{\square}$ is the (α,β,δ)-PNSG of a group $\left(\overset{\square}{G},\circ\right)$, then for all $x,y\in\overset{\square}{G}$

$$\begin{aligned}\mu^{\alpha}\left(x\circ y^{-1}\right)&\le\mu^{\alpha}(x)\vee\mu^{\alpha}\left(y^{-1}\right)\\&=\mu^{\alpha}(x)\vee\mu^{\alpha}(y)\\&=\mu^{\alpha}\left(e^{\square}\right)\end{aligned}$$

Similarly, $\varsigma^{\beta}\left(x\circ y^{-1}\right)\leq\varsigma^{\beta}\left(e^{\square}\right)$ and $\gamma^{\delta}\left(x\circ y^{-1}\right)\geq\gamma^{\delta}\left(e^{\square}\right)$

Thus, $x\circ y^{-1}\in M^{\square}$.

Therefore, $\left(M^{\square},\circ\right)$ is the subgroup of $\left(G^{\square},\circ\right)$

3.8 Definition

Let $N^{\square}=\left(\mu^{\alpha},\varsigma^{\beta},\gamma^{\delta}\right)$ be a (α,β,δ) -PNSG of a group $\left(\overset{\square}{G},\circ\right)$. Then for all $x\in\overset{\square}{G}$, (α,β,δ)-Pythagorean neutrosophic left coset(PNLC) of $N^{\square}$ is the (α,β,δ)-PNS such that $xN^{\square}$ = $\left(x\mu^{\alpha},x\varsigma^{\beta},x\gamma^{\delta}\right)$ and it is defined by $\left(x\mu^{\alpha}\right)(k)=\mu^{\alpha}\left(x^{-1}\circ k\right),\left(x\varsigma^{\beta}\right)(k)=\varsigma^{\beta}\left(x^{-1}\circ k\right)$ and $\left(x\gamma^{\delta}\right)(k)=\gamma^{\delta}\left(x^{-1}\circ k\right)$ for all $k\in\overset{\square}{G}$.

Similarly, the (α,β,δ)-Pythagorean neutrosophic right coset(PNRC) of $N^{\square}$ is defined by

$(k)\left(x\mu^{\alpha}\right)=\mu^{\alpha}\left(k\circ x^{-1}\right),(k)\left(x\varsigma^{\beta}\right)=\varsigma^{\beta}\left(k\circ x^{-1}\right)$ and $(k)\left(x\gamma^{\delta}\right)=\gamma^{\delta}\left(k\circ x^{-1}\right)$ for all $k\in\overset{\square}{G}$.

3.9 Definition

Let $N^{\square}=\left(\mu^{\alpha},\varsigma^{\beta},\gamma^{\delta}\right)$ be a (α,β,δ)-PNSG of a group $\left(\overset{\square}{G},\circ\right)$. Then $N^{\square}$ is a (α,β,δ)-Pythagorean neutrosophic normal subgroup(PNNSG) of the group $\left(\overset{\square}{G},\circ\right)$, if every (α,β,δ)- PNLC of $N^{\square}$ is also a (α,β,δ)- PNRC of $N^{\square}$.

3.9.1 Example

Let us consider the group $\overset{\square}{G}=\left(Z_4,+_4\right)$, denotes the integers modulo 4 under addition. Firstly we construct the composition table as follows:

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

For $3 \in \overset{\square}{G}$ and for all $k \in \overset{\square}{G}$, we define the (α,β,δ)-PNLC and the (α,β,δ)-PNRC as follows:

$$\left(3\mu^{\alpha}\right)(k) = \mu^{\alpha}\left(3^{-1} +_4 k\right), \left(3\varsigma^{\beta}\right)(k) = \varsigma^{\beta}\left(3^{-1} +_4 k\right) \text{ and } \left(3\gamma^{\delta}\right)(k) = \gamma^{\delta}\left(3^{-1} +_4 k\right)$$

Again, $(k)\left(3\mu^{\alpha}\right) = \mu^{\alpha}\left(k +_4 3^{-1}\right), (k)\left(3\varsigma^{\beta}\right) = \varsigma^{\beta}\left(k +_4 3^{-1}\right)$ and $(k)\left(3\gamma^{\delta}\right) = \gamma^{\delta}\left(k +_4 3^{-1}\right)$

For k=0,

$$\left(3\mu^{\alpha}\right)(0) = \mu^{\alpha}\left(3^{-1} +_4 0\right) = \mu^{\alpha}\left(1 +_4 0\right) = \mu^{\alpha}(1)$$

$$(0)\left(3\mu^{\alpha}\right) = \mu^{\alpha}\left(0 +_4 3^{-1}\right) = \mu^{\alpha}\left(0 +_4 1\right) = \mu^{\alpha}(1)$$

Therefore, $\left(3\mu^{\alpha}\right)(0) = (0)\left(3\mu^{\alpha}\right)$

Similarly, we can show that $\left(3\varsigma^{\beta}\right)(0) = (0)\left(3\varsigma^{\beta}\right)$ and $\left(3\gamma^{\delta}\right)(0) = (0)\left(3\gamma^{\delta}\right)$

We also verify the above result for, k=1, 2, 3.

Therefore, the group $\overset{\square}{G} = \left(Z_4, +_4\right)$ is a (α,β,δ)-PNNSG.

3.10 Proposition

Let $N^{\square} = \left(\mu^{\alpha}, \varsigma^{\beta}, \gamma^{\delta}\right)$ be a (α,β,δ)-PNSG of $\left(\overset{\square}{G}, \circ\right)$. Then for every $x, y \in \overset{\square}{G}$, $N^{\square}$ is a (α,β,δ)-PNNSG of a group $\left(\overset{\square}{G}, \circ\right)$ iff

$$\mu^{\alpha}(x \circ y) = \mu^{\alpha}(y \circ x), \varsigma^{\beta}(x \circ y) = \varsigma^{\beta}(y \circ x) \text{ and } \gamma^{\delta}(x \circ y) = \gamma^{\delta}(y \circ x)$$

Proof:

Let $N^{\square} = \left(\mu^{\alpha}, \varsigma^{\beta}, \gamma^{\delta}\right)$ be a (α,β,δ)-PNSG of a group $\left(\overset{\square}{G}, \circ\right)$. Then for every $x \in \overset{\square}{G}$, $xN^{\square} = N^{\square}x$. Now, for every $x, k \in \overset{\square}{G}$, $\left(x\mu^{\alpha}\right)(k) = \left(\mu^{\alpha}x\right)(k)$, $\left(x\varsigma^{\beta}\right)(k) = \left(\varsigma^{\beta}x\right)(k)$, and $\left(x\gamma^{\delta}\right)(k) = \left(\gamma^{\delta}x\right)(k)$.

Also, for every $x, k \in \overset{\square}{G}$, $\mu^{\alpha}\left(x^{-1} \circ k\right) = \mu^{\alpha}\left(k \circ x^{-1}\right)$, $\varsigma^{\beta}\left(x^{-1} \circ k\right) = \varsigma^{\beta}\left(k \circ x^{-1}\right)$, and $\gamma^{\delta}\left(x^{-1} \circ k\right) = \gamma^{\delta}\left(k \circ x^{-1}\right)$.

Therefore, $\mu^{\alpha}(x \circ y) = \mu^{\alpha}\left(x \circ \left(y^{-1}\right)^{-1}\right) = \mu^{\alpha}\left(\left(y^{-1}\right)^{-1} \circ x\right) = \mu^{\alpha}(y \circ x)$

Similarly, $\varsigma^{\beta}\left(x \circ y\right) = \varsigma^{\beta}\left(y \circ x\right)$, and $\gamma^{\delta}\left(x \circ y\right) = \gamma^{\delta}\left(y \circ x\right)$.

Conversely, for every $x, \mathrm{y} \in \overset{\square}{G}$, $\mu^{\alpha}\left(x \circ y\right) = \mu^{\alpha}\left(y \circ x\right)$, $\varsigma^{\beta}\left(x \circ y\right) = \varsigma^{\beta}\left(y \circ x\right)$, and $\gamma^{\delta}\left(x \circ y\right) = \gamma^{\delta}\left(y \circ x\right)$.

This gives, $\mu^{\alpha}\left(x \circ \left(y^{-1}\right)^{-1}\right) = \mu^{\alpha}\left(\left(y^{-1}\right)^{-1} \circ x\right)$, $\varsigma^{\beta}\left(x \circ \left(y^{-1}\right)^{-1}\right) = \varsigma^{\beta}\left(\left(y^{-1}\right)^{-1} \circ x\right)$, and $\gamma^{\delta}\left(x \circ \left(y^{-1}\right)^{-1}\right) = \gamma^{\delta}\left(\left(y^{-1}\right)^{-1} \circ x\right)$.

Put $y^{-1} = z$. Then for every $x, \mathrm{z} \in \overset{\square}{G}$, $\mu^{\alpha}\left(x \circ z^{-1}\right) = \mu^{\alpha}\left(z^{-1} \circ x\right)$, $\varsigma^{\beta}\left(x \circ z^{-1}\right) = \varsigma^{\beta}\left(z^{-1} \circ x\right)$, and $\gamma^{\delta}\left(x \circ z^{-1}\right) = \gamma^{\delta}\left(z^{-1} \circ x\right)$.

$\Rightarrow \left(\mu^{\alpha} z\right)\left(x\right) = \left(z \mu^{\alpha}\right)\left(x\right), \left(\varsigma^{\beta} z\right)\left(x\right) = \left(z \varsigma^{\beta}\right)\left(x\right)$, $\left(\gamma^{\delta} z\right)\left(x\right) = \left(z \gamma^{\delta}\right)\left(x\right)$

$\Rightarrow \mu^{\alpha} z = z \mu^{\alpha}, \varsigma^{\beta} z = z \varsigma^{\beta}$, and $\gamma^{\delta} z = z \gamma^{\delta}$

$\Rightarrow N^{\square} z = z N^{\square}$ for every $\mathrm{z} \in \overset{\square}{G}$.

As a result, $N^{\square}$ is a $\left(\alpha, \beta, \delta\right)$-PNNSG of a group $\left(\overset{\square}{G}, \circ\right)$.

3.11 Proposition

Let $N^{\square} = \left(\mu^{\alpha}, \varsigma^{\beta}, \gamma^{\delta}\right)$ be a $\left(\alpha, \beta, \delta\right)$-PNSG of a group $\left(\overset{\square}{G}, \circ\right)$. Then for every $x, y \in \overset{\square}{G}$, $N^{\square}$ is a $\left(\alpha, \beta, \delta\right)$-PNNSG of a group $\left(\overset{\square}{G}, \circ\right)$ iff

$\mu^{\alpha}\left(y \circ x \circ y^{-1}\right) = \mu^{\alpha}\left(x\right), \varsigma^{\beta}\left(y \circ x \circ y^{-1}\right) = \varsigma^{\beta}\left(x\right)$ and $\gamma^{\delta}\left(y \circ x \circ y^{-1}\right) = \gamma^{\delta}\left(x\right)$

Proof:

Let $N^{\square} = \left(\mu^{\alpha}, \varsigma^{\beta}, \gamma^{\delta}\right)$ be a $\left(\alpha, \beta, \delta\right)$-PNSG of a group $\left(\overset{\square}{G}, \circ\right)$.

Then for every $x, y \in \overset{\square}{G}$, $\mu^{\alpha}\left(x \circ y\right) = \mu^{\alpha}\left(y \circ x\right)$, $\varsigma^{\beta}\left(x \circ y\right) = \varsigma^{\beta}\left(y \circ x\right)$, and $\gamma^{\delta}\left(x \circ y\right) = \gamma^{\delta}\left(y \circ x\right)$.

Therefore,

$$\begin{aligned}\mu^{\alpha}\left(y\circ x\circ y^{-1}\right)&=\mu^{\alpha}\left(\left(y\circ x\right)\circ y^{-1}\right)\\&=\mu^{\alpha}\left(y^{-1}\circ\left(y\circ x\right)\right)\\&=\mu^{\alpha}\left(y^{-1}\circ y\circ x\right)\\&=\mu^{\alpha}\left(e\circ x\right)\\&=\mu^{\alpha}\left(x\right)\end{aligned}$$

Similarly, we can write $\varsigma^{\beta}\left(y\circ x\circ y^{-1}\right)=\varsigma^{\beta}\left(x\right)$, and $\gamma^{\delta}\left(y\circ x\circ y^{-1}\right)=\gamma^{\delta}\left(x\right)$.

Conversely,

for all $x,y\in\overset{\square}{G}$, $\mu^{\alpha}\left(y\circ x\circ y^{-1}\right)=\mu^{\alpha}\left(x\right),\varsigma^{\beta}\left(y\circ x\circ y^{-1}\right)=\varsigma^{\beta}\left(x\right)$ and $\gamma^{\delta}\left(y\circ x\circ y^{-1}\right)=\gamma^{\delta}\left(x\right)$

Therefore,

$$\begin{aligned}\mu^{\alpha}\left(x\circ y\right)&=\mu^{\alpha}\left(y^{-1}\circ y\circ x\circ y\right)\\&=\mu^{\alpha}\left(\left(y^{-1}\right)\circ\left(y\circ x\right)\circ\left(y^{-1}\right)^{-1}\right)\\&=\mu^{\alpha}\left(y\circ x\right)\end{aligned}$$

Similarly, $\varsigma^{\beta}\left(x\circ y\right)=\varsigma^{\beta}\left(y\circ x\right)$, and $\gamma^{\delta}\left(x\circ y\right)=\gamma^{\delta}\left(y\circ x\right)$.

Hence by using proposition **3.10,** $N^{\square}$ is a $\left(\alpha,\beta,\delta\right)$-PNNSG of a group $\left(\overset{\square}{G},\circ\right)$.

3.12 Theorem

Let $N^{\square}=\left(\mu^{\alpha},\varsigma^{\beta},\gamma^{\delta}\right)$ be a $\left(\alpha,\beta,\delta\right)$ -PNSG of a group $\left(\overset{\square}{G},\circ\right)$. Then

$S=\left\{x\in\overset{\square}{G}:\mu^{\alpha}\left(x\right)=\mu^{\alpha}\left(e^{\square}\right),\varsigma^{\beta}\left(x\right)=\varsigma^{\beta}\left(e^{\square}\right)\text{ and }\gamma^{\delta}\left(x\right)=\gamma^{\delta}\left(e^{\square}\right)\right\}$ is a normal subgroup of the group $\left(\overset{\square}{G},\circ\right)$.

Proof.

As $e^{\square}\in S$, S is a non-empty set. Clearly, S is a subgroup of $\left(\overset{\square}{G},\circ\right)$.

Let $x\in\overset{\square}{G}$ and $s\in S$, then $\mu^{\alpha}\left(s\right)=\mu^{\alpha}\left(e^{\square}\right),\varsigma^{\beta}\left(s\right)=\varsigma^{\beta}\left(e^{\square}\right)$ and $\gamma^{\delta}\left(s\right)=\gamma^{\delta}\left(e^{\square}\right)$

By proposition **3.11**,

$$\mu^{\alpha}\left(x\circ s\circ x^{-1}\right)=\mu^{\alpha}\left(s\right)=\mu^{\alpha}\left(e^{\square}\right),\varsigma^{\beta}\left(x\circ s\circ x^{-1}\right)=\varsigma^{\beta}\left(s\right)=\varsigma^{\beta}\left(e^{\square}\right)\text{and }\gamma^{\delta}\left(x\circ s\circ x^{-1}\right)=\gamma^{\delta}\left(s\right)=\gamma^{\delta}\left(e^{\square}\right)$$

Therefore, $x\circ s\circ x^{-1}\in S$.

Thus, $(S,\circ)$ is a normal subgroup of $\left(\overset{\square}{G},\circ\right)$.

4. An Application

When we are dealing with an object that appears symmetric, group theory can help us for its study and analysis. This concept applies to geometric figures, which remain invariant under some transformations (reflection or rotation or both). In mathematics, a dihedral group is the group of symmetry of a regular polygon which includes rotation and reflection. Decorative motifs on floor tiles, buildings, and artwork are frequently based on dihedral groups. Chemists and mineralogists utilize dihedral groups to categorize the structure of molecules and crystals, respectively. Many advertising agencies are employed symmetric groups to design the logo for the companies.

In this section, we study the dihedral group for an asymmetric molecule having a tetrahedron structure. We express the dihedral group in terms of (α,β,δ) -Pythagorean neutrosophic subgroup. For this, we take the dihedral group D_4 (symmetries of squares).

D_4 is a symmetric group of order 8 and it is defined by $D_4=\left\{R_0,R_{90},R_{180},R_{270},H,V,D,D^{/}\right\}$, where R_0 is the rotation of 0°, R_{90} is the rotation of 90°, R_{180} is the rotation of 180°, R_{270} is the rotation of 270°, H is a reflection about x -axis, V is a reflection about y -axis, D is a reflection about main diagonal, and $D^{/}$ is the reflection about other diagonal. Now we assign the membership, indeterminacy, and non-membership values to each element of D_4, which is displayed in the form of a table given by,

	R_0	R_{90}	R_{180}	R_{270}	H	V	D	$D^{/}$
μ	*0.8*	*0.65*	*0.75*	*0.9*	*0.4*	*0.7*	*0.85*	*0.5*
ς	*0.7*	*0.55*	*0.67*	*0.5*	*0.8*	*0.6*	*0.55*	*0.9*
γ	*0.6*	*0.9*	*0.7*	*0.8*	*0.9*	*0.8*	*0.75*	*0.7*

Table1. Membership, indeterminacy, and non-membership values to each element of D_4

We take α =0.8, β =0.6 and δ =0.7. For the elements H and V,

$$\mu^{\alpha}\left(H\circ V^{-1}\right)=\mu^{\alpha}\left(R_{180}\right)=\mu\left(R_{180}\right)\vee\alpha=0.8\leq\mu^{\alpha}\left(H\right)\vee\mu^{\alpha}\left(V^{-1}\right)=0.8$$

$$\varsigma^{\beta}\left(H\circ V^{-1}\right)=\varsigma^{\beta}\left(R_{180}\right)=\varsigma\left(R_{180}\right)\vee\beta=0.6\leq\varsigma^{\beta}\left(H\right)\vee\varsigma^{\beta}\left(V^{-1}\right)=0.8$$

$$\gamma^{\delta}\left(H\circ V^{-1}\right)=\gamma^{\delta}\left(R_{180}\right)=\gamma\left(R_{180}\right)\wedge\delta=0.7\geq\gamma^{\delta}\left(H\right)\wedge\gamma^{\delta}\left(V^{-1}\right)=0.7$$

For every pair of elements in D_4, the above conditions hold.

Therefore, $N^{\square}=\left(\mu^{\alpha},\varsigma^{\beta},\gamma^{\delta}\right)$ is a-PNSG.

5. Conclusions

PFS, PIFS, PNS, etc. are the latest mathematical tools that are developed to design the uncertainty with high precision. But, there exist some events where all these tools are unable to make a decision. This led to the motivation of the introduction of (α,β,δ)-PNS-as an extension of a PNS. Concerning to this notion we derive (α,β,δ) -PNSGs and study their various results and properties. We also discuss (α,β,δ)-PNNSGs, (α,β,δ)-PNCs. Finally, we give an application where we use the dihedral group and by using it we model the (α,β,δ)-PNSG. In the future, there are a lot of scopes to use this concept in the study of cryptography, crystallography, graph theory, molecular chemistry, natural science, artificial intelligence, data mining, etc.

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On Neutrosophic Vague Binary $\mathbf{BZMV^{dM}}$ Sub - algebra of $\mathbf{BZMV^{dM}}$- algebra In Neutrosophic Vague Binary Sets

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Abstract: In Model theory, common algebraic structures found are Lattices and Boolean Algebras. In the broad field of research, various algebraic structures can be introduced for a set. BCK, BCI, BCH, BH etc. are some of them. In this paper, a comparatively novel mixed structure namely, de Morgan BZMV – algebra, is presented for neutrosophic vague binary sets. Obviously, this is a mixed output pattern and more effective than the already existing single output approaches. Instead of our usual Boolean approach, this model is a kind of de Morgan lattice extension. Additionally, it takes the effects of Lukasiewicz many- valued logic, combined with BZ –lattices. The logical connective operators, MV conjunction operator $\odot$ and MV disjunction operator $\oplus$ have shown the behavior of idempotency, as same as, their underlying logical patterns, framed of 'usual conjunction $\wedge$ and disjunction $\vee$ '. Both kind of orthocomplementations or negations, one as a fuzzy type and other one as an intuitionistic type are implemented by BZ lattices. That is, Kleene (fuzzy or Zadeh) orthocomplementation $\neg$ and Brouwer orthocomplementation $\sim$ are got implemented in BZ- structure. Here $\neg$ is a fuzzy type negation and $\sim$ is an intuitionistic type negation. In our new type, de Morgan effects are given to these orthocomplementations and hence in this paper, instead of usual negations, de Morgan negations or orthocomplementations are used. Some ideals for this new concept have also got constructed. Behaviour of its direct sum, properties and some of its related theorems are also mentioned in this paper.

Keywords: vague $\mathrm{BZMV^{dM}}$ – subalgebra, neutrosophic $\mathrm{BZMV^{dM}}$ – subalgebra , neutrosophic vague $\mathrm{BZMV^{dM}}$ – subalgebra, neutrosophic vague binary $\mathrm{BZMV^{dM}}$ – subalgebra, prime ideal of $\mathrm{BZMV^{dM}}$ – subalgebra, normal ideal of $\mathrm{BZMV^{dM}}$ – subalgebra, $\sim$ ideal of $\mathrm{BZMV^{dM}}$ – subalgebra, direct sum of neutrosophic vague binary $\mathrm{BZMV^{dM}}$ – subalgebra

Notations: NVBS - neutrosophic vague binary set, NVBSS - neutrosophic vague binary subset, NVBI - neutrosophic vague binary ideal. Throughout this paper NVB indicates neutrosophic vague binary, NV denotes neutrosophic vague, N denotes neutrosophic and V denote vague

1. Introduction

For a set, different kinds of structures can be defined. Poset or partially ordered set is one such! In a poset, elements may or may not be comparable. $(\mathrm{P}, \leq)$ is a poset indicated by a relation $\leq$ defined on P and does not give any sense to the actual meaning of 'less than or equal to'. Chang [4] invented MV

(Many − Valued) – algebra in 1958. As algebraic semantics of Lukasiewicz many- valued (or multi-valued) logic, MV-algebra is considered to be an algebraic system with one binary operation, 2 unary operations and with one constant which satisfy certain axioms. In 1994, Vijay K. Khanna [24] explored lattices and Boolean algebra in his book. In 2010, N.O. Alshehri [1] introduced a new concept called an additive derivation of MV – algebra and investigated its several properties. It is possible to introduce a lattice structure from any MV – algebra. In 2017, Jean B. Nganou [13] introduced Stone MV – algebras and strongly complete MV – algebras. In 2008, Shokoofeh Ghorbani, Hasankhani. A and Esfandiar Eslami [23] introduced hyper MV – algebras. In 2012, Musa Hasankhani. M and Borumand Saeid. A [21] studied, hyper MV – algebras defined by bipolar-valued fuzzy sets. In 2014, Yongwei Yang, Xiaolong Xin and Pengfei He [28] gave some characterizations of MV – Algebras based on the theory of falling shadows. In 2003, Gianpiero Cattaneo and David Ciucci [11] gave an algebraic approach to Shadowed sets. In 2003, Gianpiero Cattaneo and David Ciucci [12] also discussed on shadowed sets and related algebraic structures. In 2004, Cattaneo. G, Ciucci.D, Giuntini. R and Konig. M [3] discussed on algebraic structures related to many valued logical systems. In 1984, Komori. Y [15] introduced the concept of a BCC algebra. In 2018, Mozahir Anwar and Jay Nandan Prasad Singh [20] discussed on the direct sum of BCC algebra. In 1991, Ye. R. F [27] discussed on BZ-algebras. Several authors like Zhang. X. H in Korea and China termed weak BCC algebra as BZ algebra. In 2003, Xiaohong Zhang, Yongquan Wang and Wieslaw. A. Dudek [26] studied T-ideals in BZ (Brower − Zadeh) -algebras and T-type BZ-algebras. In 2009, Wieslaw A. Dudek, Xiaohong Zhang, Yongquan Wang [25] gave a study on ideals and atoms of BZ Algebras. In 2017, Yousefi. A and Borumand.Saeid. A [29] discussed on various types of ideals in $\mathrm{BZMV^{dM}}$-algebra. In 2020, Mohamed Abdel-Basset, Abduallah Gamal, Le Hoang Son and Florentin Smarandache [16] provided a new approach for professional selection using bipolar neutrosophic multi criteria decision making method. In 2020, Mohamed Abdel – Basset, Weiping Ding, Rehab Mohamed and Noura Metawa [17] explained a new plithogenic Multicriteria approach, which is beneficial to the manufacturing industries, for evaluation of financial performance. For smart product service systems, in 2020, Mohamed Abdel-Basst, Rehab Mohamed, Mohamed Elhoseny [18] discussed on a novel framework to evaluate innovative value proposition. In 2020, Mohamed Abdel-Basset, Rehab Mohamed, Abd El-Nasser, H. Zaied, Abduallah Gamal, Florentin Smarandache [19] developed a plithogenic model based on best – worst method in supply chain problem.

Heyting Algebra is introduced by a dutch mathematician and logician, Arend Heyting [29] in 1930, to formulize intuitionistic logic. Wajesberg algebra is introduced in 1984 by Joseph. Maria Font, Antonio. J. Rodriguez, Antonio Torrens [14] and it is an extension of BCK algebras. Ortho-algebra is introduced by Foulis, D. P, Greechie. R. J and Rüttimann. G. T [5] in the year 1992. An effect algebra is considered as a generalization of ortho-algebra. MV – algebra too, is a generalization of ortho- algebra. Effect algebra, which is a tool for the studies of unsharp elements in quantum mechanics, is introduced by Foulis. D. J and Bennett. M.K [6] in 1994. A special example for MV – algebra is effect algebra. An orthogonal algebra is an effect algebra. Heyting Wajesberg algebra is introduced by Gianpiero Cattaneo and Davide Ciucci [11] in the year 2002. Sometimes, HW algebra is called as pseudo Boolean algebra or even Brower Lattice. Boolean algebra, which is a generalization of power set algebra, is considered to be a complemented distributed lattice in an abstract algebra.

In 1999, G. Cattaneo, R. Giuntini, R. Pilla [2] introduced $\mathrm{BZMV^{dM}}$-algebra. It is an algebraic structure with a binary operation $\oplus$ (which is both commutative and associative), with 2 unary operations $\neg$ (Kleene orthocomplementation) and $\sim$ (Brouwer orthocomplementation) and with two constants 0 and 1, satisfying certain axioms. $\sim$ is also known as intutionistic complement, since neither excluded middle law, $(a \vee \neg a = 0)$ nor double negation law, $(\neg \neg a = a)$ got satisfied by it, but it behaves well with non – contradiction law, $(a \wedge \neg a = 0)$. Any brouwer distributive lattice (resp. de Morgan Brouwer distributive lattice) becomes pre –brouwer distributive lattice (resp. de Morgan pre - Brouwer distributive lattice), if it satisfies additionally the above non – contradiction law. So any Brouwer lattice is pre –Brouwer but the converse does not hold, generally. An open problem is given in paper [2] which rightly points towards the behavioral difference between MV – algebras and $\mathrm{BZMV^{dM}}$-algebras. [0, 1] is an example for a $\mathrm{BZMV^{dM}}$-algebra but it is not a $\mathrm{MVBZ^3}$ – algebra. Every equation which holds in every MV – algebra based on [0, 1] holds in every MV – algebra. But the open problem is that it is not known still, whether it is true for $BZMV^{dM}$-algebras. But it is true for every $MVBZ^3$- algebra and the concrete $MVBZ^3$- algebra based on $\left\{0, \left(\frac{1}{2}\right), 1\right\}$, that is, shadowed sets. Any stonian MV-algebra will act as a $BZMV^{dM}$-algebra and vice versa.

In 1993, Gau. W. L and Buehrer. D. J [9] introduced vague set theory. In 2005, Florentin Smarandache [8] introduced neutrosophic sets. In 2019, Emimanancy. M and Francina Shalini. A [7] introduced Bipolar-valued fuzzy $BZMV^{dM}$ sub- algebra. In 2019, Remya. P.B and Francina Shalini. A [22] developed neutrosophic vague binary sets. These sets are more useful than fuzzy, since neutrosophic criteria is implemented and it will provide three values for each data set values collected. Fuzzy sets limitation is got reduced in neutrosophic zone. In this paper, $BZMV^{dM}$-algebraic concepts are developed to neutrosophic vague binary sets. Ideals, direct sum, theorems are also got discussed to this novel concept.

In this paper the following points are newly presented

- V $BZMV^{dM}$ – subalgebra for Different sets [Section 3]
  - ✓ V $BZMV^{dM}$ subalgebra [Definition 3.1]
  - ✓ N $BZMV^{dM}$ subalgebra [Definition 3.2]
  - ✓ NV $BZMV^{dM}$ subalgebra [Definition 3.3]
- NVB $BZMV^{dM}$ – subalgebra [Section 4]
  - ✓ NVB $BZMV^{dM}$ – subalgebra [Definition 4.1]
- Ideal in Neutrosophic Vague Binary $BZMV^{dM}$ Sub - algebra [Section 5]
  - ✓ NVB $BZMV^{dM}$- Ideal [Definition 5.1]
- Various Ideals in Neutrosophic Vague Binary $BZMV^{dM}$- subalgebra [Section 5]
  - ✓ Prime ideal, normal ideal,∼ ideal of NVB $BZMV^{dM}$–subalgebra [Definition 5.2]
- Direct sum of two NVB $BZMV^{dM}$ – subalgebra [Section 6]

2. Preliminaries

In this section some preliminaries are given.

Definition 2.1 [22] (Neutrosophic vague binary set)
A neutrosophic vague binary set M_{NVB} (NVBS in short) over a common universe $\left\{U_1 = \{x_j / 1 \leq j \leq n\}; U_2 = \{y_k/1 \leq k \leq p\}\right\}$ is an object of the form

$$M_{NVB} = \left\{\langle\frac{\hat{T}_{M_{NVB}}(x_j),\ \hat{I}_{M_{NVB}}(x_j),\ \hat{F}_{M_{NVB}}(x_j)}{x_j};\ \forall\, x_j \in U_1\rangle\, \langle\frac{\hat{T}_{M_{NVB}}(y_k),\ \hat{I}_{M_{NVB}}(y_k),\ \hat{F}_{M_{NVB}}(y_k)}{y_k};\ \forall\, y_k \in U_2\rangle\right\}$$

is defined as
$\hat{T}_{M_{NVB}}(x_j) = [T^-(x_j), T^+(x_j)]$, $\hat{I}_{M_{NVB}}(x_j) = [I^-(x_j), I^+(x_j)]$ and $\hat{F}_{M_{NVB}}(x_j) = [F^-(x_j), F^+(x_j)]$; $x_j \in U_1$ and
$\hat{T}_{M_{NVB}}(y_k) = [T^-(y_k), T^+(y_k)]$, $\hat{I}_{M_{NVB}}(y_k) = [I^-(y_k), I^+(y_k)]$ and $\hat{F}_{M_{NVB}}(y_k) = [F^-(y_k), F^+(y_k)]$; $y_k \in U_2$
where (1) $T^+(x_j) = 1 - F^-(x_j)$; $F^+(x_j) = 1 - T^-(x_j)$; $\forall\, x_j \in U_1$ and
$T^+(y_k) = 1 - F^-(y_k)$; $F^+(y_k) = 1 - T^-(y_k)$; $\forall\, y_k \in U_2$

(2) $^-0 \leq T^-(x_j) + I^-(x_j) + F^-(x_j) \leq 2^+$; $^-0 \leq T^-(y_k) + I^-(y_k) + F^-(y_k) \leq 2^+$

or

$^-0 \leq T^-(x_j) + I^-(x_j) + F^-(x_j) + T^-(y_k) + I^-(y_k) + F^-(y_k) \leq 4^+$

and

$$^{-}0 \le T^{+}(x_j) + I^{+}(x_j) + F^{+}(x_j) \le 2^{+} ; \quad ^{-}0 \le T^{+}(y_k) + I^{+}(y_k) + F^{+}(y_k) \le 2^{+}$$

or

$$^{-}0 \le T^{+}(x_j) + I^{+}(x_j) + F^{+}(x_j) + T^{+}(y_k) + I^{+}(y_k) + F^{+}(y_k) \le 4^{+}$$

(3) $T^{-}(x_j),\ I^{-}(x_j),\ F^{-}(x_j) : V(U_1) \longrightarrow [0,1]$ and $T^{-}(y_k),\ I^{-}(y_k),\ F^{-}(y_k) : V(U_2) \longrightarrow [0,1]$
$T^{+}(x_j),\ I^{+}(x_j),\ F^{+}(x_j) : V(U_1) \longrightarrow [0,1]$ and $T^{+}(y_k),\ I^{+}(y_k),\ F^{+}(y_k) : V(U_2) \longrightarrow [0,1]$

Here $V(U_1),\ V(U_2)$ denotes power set of vague sets on $U_1,\ U_2$ respectively.

Definition 2.2 [24]

A poset $(L, \le)$ is said to form a lattice if for every a, b $\in$ L, Sup $\{a, b\}$ and Inf $\{a, b\}$ exist in L

In that case, we write

$\text{Sup } \{a, b\} = (a \vee b)$ [read a join b]

$\text{Inf } \{a, b\} = (a \wedge b)$ [read a meet b]

Other notations like $(a + b)$ and $(a \,.\, b)$ or $(a \cup b)$ and $(a \cap b)$ are also used for Sup$\{a, b\}$& Inf $\{a, b\}$

Definition 2.3 [1] (MV- algebra)

An MV-algebra is a structure (M, $\oplus$, $\star$, 0) where $\oplus$ is a binary operation, $\star$ is a unary operation, and 0 is a constant such that the following axioms are satisfied for any a, b $\in$ M:

(MV1) (M, $\oplus$, 0) is a commutative monoid

(MV2) $(a^{\star})^{\star} = a$

(MV3) $(0^{\star} \oplus a) = 0^{\star}$

(MV4) $(a^{\star} \oplus b)^{\star} \oplus b = (b^{\star} \oplus a)^{\star} \oplus a$

If we define the constant $1 = 0^{\star}$ and the auxiliary operations $\odot$, $\vee$ and $\wedge$ by

$a \odot b = (a^{\star} \oplus b^{\star})^{\star}$, $a \vee b = a \oplus (b \odot a^{\star})$, $a \wedge b = a \odot (b \oplus a^{\star})$,

then (M, $\odot$, 1) is a commutative monoid and the structure (M, $\vee$, $\wedge$, 0, 1) is a bounded distributive lattice. Also, we define the binary operation $\ominus$ by $x \ominus y = x \odot y^{\star}$. A subset X of an MV- algebra M is called sub algebra of M if and only if X is closed under the MV-operations defined in M. In any MV-algebras, one can define a partial order $\le$ by putting $x \le y$ if and only if $x \wedge y = x$ for each x, y $\in$ M. If the order relation $\le$, defined over M, is total, then we say that M is linearly ordered. For an MV-algebra M, if we define $B(M) = \{x \in M: x \oplus x = x\} = \{x \in M: x \odot x = x\}$. Then, (B(M), $\oplus$, $\star$, 0) is both a largest subalgebra of M and a Boolean algebra

An MV-algebra M has the following properties for all x, y, z $\in$ M,

(1) $(x \oplus 1) = 1$

(2) $(x \oplus x*) = 1$

(3) $(x \odot x^{*}) = 0$

(4) If $(x \odot y) = 0$, then $x = y = 0$

(5) If $(x \odot y) = 1$, then $x = y = 1$

(6) If $x \le y$, then $(x \vee z) \le (y \vee z)$ and $(x \wedge z) \le (y \wedge z)$

(7) If $x \le y$, then $(x \oplus z) \le (y \oplus z)$ and $(x \odot z) \le (y \odot z)$

(8) $x \le y$ if and only $y^{*} \le x^{*}$

(9) $(x \oplus y) = y$ if and only if $(x \odot y) = x$

Definition 2.4 [20] (BZ-algebra)

An algebra $\langle X, *, 0 \rangle$ of type $(2, 0)$ is called a BZ- algebra if it satisfies the following axioms :

For any $x, y, z \in X$,

(1) $[(x * z) * (y * z)] * (x * y) = 0$

(2) $(x * 0) = x$

(3) $(x * y) = (y * x) = 0$ implies $x = y$

A partial ordering $\leq$ can be defined by $x \leq y$ if and only if $(x * y) = 0$

Remark 2.5 [20]

A weak BCC algebra was also termed as BZ algebra in Korea and China by several authors e.g., X. H. Zhang.

Definition 2.6 [25, 26] $\left(\textbf{distributive Brouwer} - \textbf{Zadeh} - \textbf{lattice}\ (\mathrm{BZ} - \mathrm{lattice})\right)$

A distributive Brouwer - Zadeh (BZ)-lattice is a structure $\langle \sum, \vee, \wedge, \neg, \sim, 0 \rangle$, where

(a) $\langle \sum, \vee, \wedge, 0 \rangle$ is a (nonempty) distributive lattice with minimum element 0

(b) The mapping $\neg : \sum \rightarrow \sum$ is a Kleene orthocomplementation, that is

(doc-1) $\neg(\neg a) = a$

(doc-2) $\neg (a \vee b) = (\neg a \wedge \neg b)$

(re) $(a \wedge \neg a \leq b \vee \neg b)$

(c) The mapping $\sim : \sum \rightarrow \sum$ is a Brouwer orthocomplementation, that is

(woc-1) $(a \wedge \sim \sim a) = a$

(woc-2) $\sim (a \vee b) = (\sim a \wedge \sim b)$

(woc-3) $(a \wedge \sim a) = 0$

(d) The two orthocomplementations are linked by the following interconnection rule:

(in) $\neg \sim a = \sim \sim a$

The mapping $\neg$ is also called the Lukasiewicz [or fuzzy (Zadeh)] orthocomplementation while the mapping $\sim$ is an intuitionistic – like orthocomplementation. The element $1 := \sim 0 = \neg 0$ is the greatest element of $\sum$

Definition 2.7 [2] $\left(\textbf{distributive de Morgan BZ} - \textbf{lattice}(\mathbf{BZ^{dM}} - \textbf{lattice})\right)$

A distributive de Morgan BZ -lattice $\left(\mathrm{BZ^{dM}} - \mathrm{lattice}\right)$ is a distributive BZ- lattice for which the following hold: $\sim (a \wedge b) = (\sim a \vee \sim b)$

Definition 2.8 [11, 12] (Brouwer – Zadeh many – valued algebra (BZMV - algebra))

By pasting of BZ -lattices and MV – algebras one obtains so – called BZMV – algebra

A Brouwer Zadeh Many Valued (BZMV) algebra is a system $\mathcal{A} = \langle A, \oplus, \neg, \sim, 0 \rangle$ where A is a non-empty set, 0 is a constant, $\neg$ and $\sim$ are unary operations, $\oplus$ a binary operator, obeying the following axioms:

(BZMV1) $(a \oplus b) \oplus c = (b \oplus c) \oplus a$ (BZMV2) $(a \oplus 0) = a$

(BZMV3) $\neg(\neg a) = a$ (BZMV4) $\neg(\neg a \oplus b) \oplus b = \neg(a \oplus \neg b) \oplus a$

(BZMV5) $\sim a \oplus \sim \sim a = \neg 0$ (BZMV6) $a \oplus \sim \sim a = \sim \sim a$

(BZMV7) $\sim \neg [(\neg(a \oplus \neg b) \oplus b] = \neg(\sim \sim a \oplus \neg \sim \sim b) \oplus \neg \sim \sim b$

Definition 2.9 [2] (de Morgan BZMV (BZMV$^{\mathbf{dM}}$) algebra

A de Morgan BZMV (BZMV$^{\mathrm{dM}}$) algebra, is a BZMV algebra $\mathcal{A} = \langle A, \oplus, \neg, \sim, 0 \rangle$ where axiom (BZMV7) is replaced by the following:

(BZMV7′) $\sim \neg [(\neg (a \oplus \neg b) \oplus \neg b)] = \neg(\sim \sim a \oplus \neg \sim \sim b) \oplus \neg \sim \sim b$

Connectives $\vee$ and $\wedge$ are the algebraic realization of logical disjunction and conjunction of a distributive lattice. In particular, they are idempotent operators. Connectives $\oplus$ and $\odot$ are the well known MV disjunction and MV conjunction operators, which are idempotent.

$$(a \odot b) := \neg\ (\neg a \oplus \neg b)$$
$$(a \vee b) := \neg\ (\neg a \oplus b) \oplus b \quad ; \quad (a \wedge b) := \neg\ (a \oplus \neg b) \oplus \neg b$$

A partial order can be naturally induced by the lattice operators as:
$a \leq b$ iff $(a \wedge b) = a$ (equivalently, $a \vee b = b$)
Let us notice that, since it is possible to prove that $\sim 0 = \neg 0$, in the sequel we set $1 := \sim 0 = \neg 0$
With respect to the just defined partial order we have that the lattice is bounded: $\forall\ a \in A$, $0 \leq a \leq 1$
The unary operation $\neg : A \to A$ is a Kleene (or Zadeh) orthocomplementation (negation). In other words, it satisfies the properties:
(K1) $\neg\ (\neg a) = a$ (K2) $\neg\ (a \vee b) = \neg a \wedge \neg b$ (K3) $a \wedge \neg a \leq b \vee \neg b$
Let us recall that under (K1), condition (K1), condition (K2) is equivalent to the dual de Morgan law. In general, neither the non-contradiction law,
$\forall\ a : a \wedge \neg a = 0$, nor the excluded middle law, $\forall\ a : a \vee \neg a = 1$, are satisfied by this negation
The unary operation $\sim : A \to A$ is a Brouwer orthocomplementation (negation). In other words, it satisfies the properties:
(B1) $a \wedge \sim\sim a = a$ (equivalently, $a \leq \sim\sim a$) (B2) $\sim (a \vee b) = \sim a \wedge \sim b$ (B3) $a \wedge \sim a = 0$

Remark 2. 10 [29]

BZW and BZMV – algebras are equivalent structures.

Definition 2.11 [29] (Linear $BZMV^{dM}$-algebra)

In linear $BZMV^{dM}$-algebra A, Brouwer orthocomplementation $\sim$ is uniquely defined in the following way for all $a \in A$.

$$\sim a = \begin{cases} 1 & ;\ \text{if } a = 0 \\ 0 & ;\ \text{otherwise} \end{cases}$$

Theorem 2.12 [9]

If A is a $BZMV^{dM}$--algebra then the following results are true:

(1) $(x \oplus y) = (y \oplus x)$ (2) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ (3) $(x \oplus 1) = 1$

(4) $(x \oplus \neg x) = 1$ (5) $\neg (x \oplus \sim\sim x) \oplus \sim\sim x = 1$ (6) $\neg x \oplus \sim\sim x = 1$

(7) $x \wedge \sim\sim x = x$ (8) $\neg \sim x = \sim\sim x$ (9) $\sim (x \wedge y) = \sim x \vee \sim y$

(10) $\sim (x \vee y) = \sim x \wedge \sim y$ (Equivalently, $x \leq y$ implies $\sim y \leq \sim x$)

(11) $x \wedge \sim x = 0$ (12) $\sim x = \sim\sim\sim x$ (13) $\sim x \oplus \sim x = \sim x$ (14) $\neg 0 = \sim 0$

Definition 2.13 [29] (Ideal in $BZMV^{dM}$- algebra)

Let A be an $BZMV^{dM}$--algebra. An ideal of A is any subset I of A such that the following conditions are satisfied:

(i) $0 \in I$ (ii) $x, y \in I$ imply $(x \oplus y) \in I$ (iii) $x \in I$ and $y \in A$ then $(x \odot y) \in I$

Definition 2.14 [29] (prime ideal of $BZMV^{dM}$ algebra)

An ideal I is called a prime ideal if and only if $\forall\ x, y \in A$, $(x \odot \neg y) \in I$ or $(\neg x \odot y) \in I$

Definition 2.15 [29] (normal ideal of $BZMV^{dM}$ algebra)

Let I be an ideal of A. I is called normal ideal whenever, $\forall\ x, y \in A$, $(\neg x \odot y) \in I$ if and only if $(\sim x \odot y \in I)$

Definition 2.16 [29] ($\sim$ ideal of $BZMV^{dM}$ algebra)

Let A be a $BZMV^{dM}$ – algebra. Let I be an ideal of A. Then it is called $\sim$ ideal of A if it satisfies the following condition: $\forall\ x, y \in A$, if $(x \odot y) \in I$, then $(\sim\sim x \odot \sim\neg y \in I)$

Lemma 2.17 [21] (Fuzzy sub – hyper MV algebra)

A fuzzy subset μ of M is called a fuzzy sub-hyper-MV -algebra of M if it satisfies:

(1) $(\forall\ x \in M)$, $\mu(x) \leq \mu(x^*)$ (2) $(\forall\ x, y \in M)$, $\inf_{v \in x \oplus y} \mu(v) \geq \min \{\mu(x), \mu(y)\}$

Definition 2.18 [7] (Bipolar valued fuzzy BZMV algebra)

Let $M = \langle M, \oplus, \neg, \sim, 0\rangle$ be a BZMV – algebra with a non – empty set M, a binary operation $\oplus$, two unary operations $\neg$ and $\sim$ and with a constant 0. A bipolar valued fuzzy set $B = (\mu^{-1}, \mu^{+})$ is called bipolar valued fuzzy BZMV algebra of M, if for every x, y in M it satisfies:

(1) $\mu^{+}(x) \leq \mu^{+}(\neg x)$ (2) $\mu^{-}(x) \geq \mu^{+}(\neg x)$ (3) $\mu^{+}(x) \leq \mu^{+}(\sim x)$ (4) $\mu^{-}(x) \geq \mu^{+}(\sim x)$

(5) $\inf_{v \in x \oplus y} \mu^{+}(v) \geq \min\{\mu^{+}(x), \mu^{+}(y)\}$ (6) $\sup_{v \in x \oplus y} \mu^{-}(v) \leq \max\{\mu^{-}(x), \mu^{-}(y)\}$

3. $\mathbf{BZMV^{dM}}$ – algebra for different uncertain sets

In this section $\mathbf{BZMV^{dM}}$ algebraic structure is developed for vague and neutrosophic sets. Then it is extended to its hybrid set 'neutrosophic vague' with single universe.

Definition 3. 1 (Vague $\mathbf{BZMV^{dM}}$ subalgebra)

A vague $BZMV^{dM}$ – subalgebra (V $BZMV^{dM}$ - subalgebra, in short) is a structure $\mathfrak{M}_A = \left(U^{\mathfrak{M}_A}, \oplus, \neg, \sim, 0\right)$ which satisfies, the following two $\mathfrak{M}_A$ inequalities:

$\mathfrak{M}_A$ ***inequality* (1) :** $\left(\inf_{\mathbf{u_v} \in \mathbf{u_x} \oplus \mathbf{u_y}} \mathbf{V_A(u_v)}\right) \succcurlyeq r\min\{\mathbf{V_A(u_x)}, \mathbf{V_A(u_y)}\}$; $\forall$ $\mathbf{u_v}, \mathbf{u_x}, \mathbf{u_y} \in \mathbf{U}$

$$\text{i.e., } \left(\inf_{u_v \in (u_x \oplus u_y)} t_A(u_v)\right) \geq \min\{t_A(u_x), t_A(u_y)\} ;$$

$$\left(\inf_{u_v \in (u_x \oplus u_y)} f_A(u_v)\right) \leq \max\{f_A(u_x), f_A(u_y)\}$$

$\mathfrak{M}_A$ ***inequality* (2)** :

(i) $\mathbf{V_A(u_x) \succcurlyeq V_A(\neg u_x)}$

i.e., $t_A(u_x) \geq t_A(\neg u_x)$ & $f_A(u_x) \leq f_A(\neg u_x)$

(ii) $\mathbf{V_A(u_x) \succcurlyeq V_A(\sim u_x)}$

i.e., $t_A(u_x) \geq t_A(\sim u_x)$ & $f_A(u_x) \leq f_A(\sim u_x)$

Here,

- A is a non- empty vague set with universe **U**
- $U^{\mathfrak{M}_A} = (U, \oplus, \neg, \sim, 0)$ is a $BZMV^{dM}$– algebraic structure of the underlying universal set U of a vague set A, with a binary operation $\oplus$, two unary operations $\neg, \sim$ and a constant 0 which satisfies the following $BZMV^{dM}$ axioms : $\forall$ $u_x, u_y, u_z \in U$

  (1) $(u_x \oplus u_y) \oplus u_z = (u_y \oplus u_z) \oplus u_x$ (2) $(u_x \oplus 0) = u_x$ (3) $\neg(\neg u_x) = u_x$

  (4) $\neg(\neg u_x \oplus u_y) \oplus u_y = \neg(u_x \oplus \neg u_y) \oplus u_x$ (5) $\sim u_x \oplus \sim\sim u_x = \neg 0$ (6) $u_x \oplus \sim\sim u_x = \sim\sim u_x$

  (7) $\sim\neg[(\neg(u_x \oplus \neg u_y) \oplus \neg u_y)] = \neg(\sim\sim u_x \oplus \neg\sim\sim u_y) \oplus \neg\sim\sim u_y$

Definition 3.2 (Neutrosophic $\mathbf{BZMV^{dM}}$ subalgebra)

A neutrosophic $BZMV^{dM}$– subalgebra (N $BZMV^{dM}$- sub algebra, in short) is a structure $\mathfrak{M}_{M_N} = \left(U^{\mathfrak{M}_{M_N}}, \oplus, \neg, \sim, 0\right)$ which satisfies, the following two $\mathfrak{M}_{M_N}$ inequalities:

$\mathfrak{M}_{M_N}$ ***inequality* (1):**

$$\left(\begin{matrix}\inf\\ u_v \in (u_x \oplus u_y)\end{matrix} N_{M_N}(v)\right) \succcurlyeq r \min\{N_{M_N}(u_x), N_{M_N}(u_y)\} \ ; \ \forall\ u_v,\ u_x,\ u_y\ \in U$$

$$\left(\begin{matrix}\inf\\ u_v \in (u_x \oplus u_y)\end{matrix} T_{M_N}(u_v)\right) \geq \min\{T_{M_N}(u_x),\ T_{M_N}(u_y)\};$$
$$\left(\begin{matrix}\inf\\ u_v \in (u_x \oplus u_y)\end{matrix} I_{M_N}(u_v)\right) \leq \max\{I_{M_N}(u_x),\ I_{M_N}(u_y)\} \ ;$$
$$\left(\begin{matrix}\inf\\ u_v \in (u_x \oplus u_y)\end{matrix} F_{M_N}(u_v)\right) \leq \max\{F_{M_N}(u_x),\ F_{M_N}(u_y)\}$$

$\mathfrak{M}_{M_N}$***inequality* (2) :**

(i) $N_{M_N}(u_x) \succcurlyeq N_{M_N}(\neg\, u_x)$; i.e., $T_{M_N}(u_x) \geq T_{M_N}(\neg\, u_x)$; $I_{M_N}(u_x) \leq I_{M_N}(\neg\, u_x)$; $F_{M_N}(u_x) \leq F_{M_N}(\neg\, u_x)$

(ii) $N_{M_N}(u_x) \succcurlyeq N_{M_N}(\sim u_x)$; i.e., $T_{M_N}(u_x) \geq T_{M_N}(\sim u_x)$; $I_{M_N}(u_x) \leq I_{M_N}(\sim u_x)$; $F_{M_N}(u_x) \leq F_{M_N}(\sim u_x)$

Here,

- M_N is a non- empty neutrosophic set with universe U
- $U^{\mathfrak{M}_{M_N}} = (U,\ \oplus,\ \neg,\ \sim, 0)$ is a $BZMV^{dM}$ – algebraic structure of the underlying universal set U of a neutrosophic set M_N with a binary operation $\oplus$, two unary operations $\neg$, $\sim$ and a constant 0 which satisfies the following $BZMV^{dM}$ - axioms : $\forall\ u_x,\ u_y,\ u_z\ \in U$

  (1) $(u_x \oplus u_y) \oplus u_z = (u_y \oplus u_z) \oplus u_x$ (2) $(u_x \oplus 0) = u_x$ (3) $\neg\,(\neg\, u_x) = u_x$

  (4) $\neg\,(\neg\, u_x \oplus u_y) \oplus u_y = \neg\,(u_x \oplus \neg\, u_y) \oplus u_x$ (5) $\sim u_x \oplus \sim\sim u_x = \neg\, 0$ (6) $u_x \oplus \sim\sim u_x = \sim\sim u_x$

  (7) $\sim\neg\,[(\neg\,(u_x \oplus \neg\, u_y) \oplus \neg\, u_y)] = \neg\,(\sim\sim u_x \oplus \neg\sim\sim u_y) \oplus \neg\sim\sim u_y$

Definition 3.3 (Neutrosophic Vague $BZMV^{dM}$ – subalgebra)

A neutrosophic vague $BZMV^{dM}$ – subalgebra (NV $BZMV^{dM}$ sub algebra, in short) is a structure $\mathfrak{M}_{M_{NV}} = (U^{\mathfrak{M}_{M_{NV}}},\ \oplus,\ \neg,\ \sim, 0)$ with, $\hat{T} = [T^{-1}, T^{+}]$; $\hat{I} = [I^{-1}, I^{+}]$; $\hat{F} = [F^{-1}, F^{+}]$, which satisfies, the following two $\mathfrak{M}_{M_{NV}}$ inequalities:

$\mathfrak{M}_{M_{NV}}$ ***inequality* (1)** :

$$\left(\begin{matrix}\inf\\ u_v \in (u_x \oplus u_y)\end{matrix} NV_{M_{NV}}(u_v)\right) \succcurlyeq r \min\{NV_{M_{NV}}(u_x), NV_{M_{NV}}(u_y)\};\ \forall\ u_v,\ u_x,\ u_y\ \in U$$

i.e.,

$$\left(\begin{matrix}\inf\\ u_v \in (u_x \oplus u_y)\end{matrix} \hat{T}_{M_{NV}}(u_v)\right) \geq \min\{\hat{T}_{M_{NV}}(u_x),\ \hat{T}_{M_{NV}}(u_y)\};$$
$$\left(\begin{matrix}\inf\\ u_v \in (u_x \oplus u_y)\end{matrix} \hat{I}_{M_{NV}}(u_v)\right) \leq \max\{\hat{I}_{M_{NV}}(u_x),\ \hat{I}_{M_{NV}}(u_y)\} \ ;$$
$$\left(\begin{matrix}\inf\\ u_v \in (u_x \oplus u_y)\end{matrix} \hat{F}_{M_{NV}}(u_v)\right) \leq \max\{\hat{F}_{M_{NV}}(u_x),\ \hat{F}_{M_{NV}}(u_y)\}$$

$\mathfrak{M}_{M_{NV}}$ ***inequality* (2)** :

(i) $NV_{M_{NV}}(u_x) \succcurlyeq NV_{M_{NV}}(\neg\, u_x)$ i.e., $\hat{T}_{M_{NV}}(u_x) \geq \hat{T}_{M_{NV}}(\neg\, u_x)$; $\hat{I}_{M_{NV}}(u_x) \leq \hat{I}_{M_{NV}}(\neg\, u_x)$; $\hat{F}_{M_{NV}}(u_x) \leq \hat{F}_{M_{NV}}(\neg\, u_x)$

(ii) $NV_{M_{NV}}(u_x) \succcurlyeq NV_{M_{NV}}(\sim u_x)$ i.e., $\hat{T}_{M_{NV}}(u_x) \geq \hat{T}_{M_{NV}}(\sim u_x)$; $\hat{I}_{M_{NV}}(u_x) \leq \hat{I}_{M_{NV}}(\sim u_x)$; $\hat{F}_{M_{NV}}(u_x) \leq \hat{F}_{M_{NV}}(\sim u_x)$

Here,

- M_{NV} is a non- empty neutrosophic vague set with a single universe U

- $U^{\mathfrak{M}_{M_{NV}}} = (U, \oplus, \neg, \sim, 0)$ is a $BZMV^{dM}$ – algebraic structure of the underlying single universal set U of a neutrosophic vague set M_{NV}, with a binary operation $\oplus$, two unary operations $\neg$, $\sim$ and a constant 0 which satisfies the following $BZMV^{dM}$ - axioms: $\forall\ u_x, u_y, u_z \in U$

(1) $(u_x \oplus u_y) \oplus u_z = (u_y \oplus u_z) \oplus u_x$ (2) $(u_x \oplus 0) = u_x$ (3) $\neg(\neg u_x) = u_x$

(4) $\neg(\neg u_x \oplus u_y) \oplus u_y = \neg(u_x \oplus \neg u_y) \oplus u_x$ (5) $\sim u_x \oplus \sim\sim u_x = \neg 0$ (6) $u_x \oplus \sim\sim u_x = \sim\sim u_x$

(7) $\sim\neg[(\neg(u_x \oplus \neg u_y) \oplus \neg u_y)] = \neg(\sim\sim u_x \oplus \neg\sim\sim u_y) \oplus \neg\sim\sim u_y$

4. Neutrosophic Vague Binary $BZMV^{dM}$-subalgebra

In this section neutrosophic vague $\mathbf{BZMV^{dM}}$-subalgebra is extended to its binary concept.

Definition 4.1 (Neutrosophic Vague Binary $BZMV^{dM}$ subalgebra)

A neutrosophic vague binary $BZMV^{dM}$- subalgebra (NVB $BZMV^{dM}$ Sub-algebra, in short) is a structure $\mathfrak{M}_{M_{NVB}} = \left(U^{\mathfrak{M}_{M_{NVB}}}, \oplus, \neg, \sim, 0\right)$ with, $\widehat{T} = [T^{-1}, T^{+}]$; $\widehat{I} = [I^{-1}, I^{+}]$; $\widehat{F} = [F^{-1}, F^{+}]$, which satisfies, the following two $\mathfrak{M}_{M_{NVB}}$ inequalities:

$\boldsymbol{\mathfrak{M}_{M_{NVB}}}$ ***inequality* (1)** :

$$\left(\inf_{u_v \in (u_x \oplus u_y)} NVB_{M_{NVB}}(u_v)\right) \succcurlyeq r\min\left\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\right\};\ \forall\ u_v, u_x, u_y \in U$$

i.e.,

$$\left(\inf_{u_v \in (u_x \oplus u_y)} \widehat{T}_{M_{NVB}}(u_v)\right) \geq \min\left\{\widehat{T}_{M_{NVB}}(u_x), \widehat{T}_{M_{NVB}}(u_y)\right\}$$
$$\left(\inf_{u_v \in (u_x \oplus u_y)} \widehat{I}_{M_{NVB}}(u_v)\right) \leq \max\left\{\widehat{I}_{M_{NVB}}(u_x), \widehat{I}_{M_{NVB}}(u_y)\right\}$$
$$\left(\inf_{u_v \in (u_x \oplus u_y)} \widehat{F}_{M_{NVB}}(u_v)\right) \leq \max\left\{\widehat{F}_{M_{NVB}}(u_x), \widehat{F}_{M_{NVB}}(u_y)\right\}$$

$\mathfrak{M}_{M_{NVB}}$ *inequality* (2) :

(i) $NVB_{M_{NVB}}(u_x) \succcurlyeq NVB_{M_{NVB}}(\neg u_x)$

i.e., $\widehat{T}_{M_{NVB}}(u_x) \geq \widehat{T}_{M_{NVB}}(\neg u_x)$; $\widehat{I}_{M_{NVB}}(u_x) \leq \widehat{I}_{M_{NVB}}(\neg u_x)$; $\widehat{F}_{M_{NVB}}(u_x) \leq \widehat{F}_{M_{NVB}}(\neg u_x)$

(ii) $NVB_{M_{NVB}}(u_x) \succcurlyeq NVB_{M_{NVB}}(\sim u_x)$

i.e., $\widehat{T}_{M_{NVB}}(u_x) \geq \widehat{T}_{M_{NVB}}(\sim u_x)$; $\widehat{I}_{M_{NVB}}(u_x) \leq \widehat{I}_{M_{NVB}}(\sim u_x)$; $\widehat{F}_{M_{NVB}}(u_x) \leq \widehat{F}_{M_{NVB}}(\sim u_x)$

Here,

- M_{NVB} is a non- empty neutrosophic vague binary set with two universes U_1, U_2

- $U^{\mathfrak{M}_{M_{NVB}}} = (U = \{U_1 \cup U_2\}, \oplus, \neg, \sim, 0)$ is a $BZMV^{dM}$ – algebraic structure of the underlying universal set U which is got by combining the two universes of the given neutrosophic vague binary set M_{NVB}, with a binary operation $\oplus$, two unary operations $\neg$, $\sim$ and a constant 0 which satisfies the following $BZMV^{dM}$ - axioms: $\forall\ u_x, u_y, u_z \in U$

(1) $(u_x \oplus u_y) \oplus u_z = (u_y \oplus u_z) \oplus u_x$ (2) $(u_x \oplus 0) = u_x$ (3) $\neg(\neg u_x) = u_x$

(4) $\neg(\neg u_x \oplus u_y) \oplus u_y = \neg(u_x \oplus \neg u_y) \oplus u_x$ (5) $\sim u_x \oplus \sim\sim u_x = \neg 0$ (6) $u_x \oplus \sim\sim u_x = \sim\sim u_x$

(7) $\sim\neg[(\neg(u_x \oplus \neg u_y) \oplus \neg u_y)] = \neg(\sim\sim u_x \oplus \neg\sim\sim u_y) \oplus \neg\sim\sim u_y$

Remark 4.2

(i) In a NVB $BZMV^{dM}$ – subalgebra, possible operations can also be further derived

(1) $(u_x \odot u_y) = \neg(\neg u_x \odot \neg u_y)$

(2) $(u_x \vee u_y) = \neg(\neg u_x \oplus u_y) \oplus u_y$

(3) $\left(u_x \wedge u_y\right) = \neg\ \left(\neg\left(u_x \oplus \neg\ u_y\right) \oplus \neg\ u_y\right)$

(ii) In neutrosophic vague binary concept $U=\{U_1 \cup U_2\}$. But in neutrosophic vague it is single universe U

Example 4.3

Let $U_1 = \{0,\ u_p,\ u_q, 1\}$ and $U_2 = \{0,\ u_r,\ u_s,\ 1\}$ be two universes with neutrosophic vague binary membership grades as given below:

$$\forall\ u_p \in U_1,\ NVB_{M_{NVB}}(u_p) = \begin{cases} [0.8, 0.9][0.1, 0.6][0.1, 0.2] & ;\ u_p = 0 \text{ and } 1 \\ [0.9, 0.9][0.1, 0.2][0.1, 0.1] & ;\ 0 < u_p < 1 \end{cases}$$

$$\forall\ u_q \in U_2,\ NVB_{M_{NVB}}(u_q) = \begin{cases} [0.7, 0.9][0.2, 0.5][0.1, 0.3] & ;\ u_q = 0 \text{ and } 1 \\ [0.8, 0.9][0.1, 0.4][0.1, 0.2] & ;\ 0 < u_q < 1 \end{cases}$$

Corresponding neutrosophic vague binary set (in short, NVBS) is given as below:

$$\therefore\ M_{NVB} = \left\{ \begin{matrix} \left\langle \frac{[0.8,0.9][0.1,0.6][0.1,0.2]}{0}, \frac{[0.9,0.9][0.1,0.2][0.1,0.1]}{u_p}, \frac{[0.9,0.9][0.1,0.2][0.1,0.1]}{u_q}, \frac{[0.8,0.9][0.1,0.6][0.1,0.2]}{1} \right\rangle \\ \left\langle \frac{[0.7,0.9][0.2,0.5][0.1,0.3]}{0}, \frac{[0.8,0.9][0.1,0.4][0.1,0.2]}{u_r}, \frac{[0.8,0.9][0.1,0.4][0.1,0.2]}{u_s}, \frac{[0.7,0.9][0.2,0.5][0.1,0.3]}{1} \right\rangle \end{matrix} \right\}$$

Combined universal set is $U = \{U_1 \cup U_2\} = \{0, u_p, u_q, u_r, u_s, 1\}$ & $\{U_1 \cap U_2\} = \{0, 1\}$

Neutrosophic vague binary union of common elements are given by,

$NVB_{M_{NVB}}(0) = [0.8, 0.9][0.1, 0.6][0.1, 0.2] \cup [0.7, 0.9][0.2, 0.5][0.1, 0.3]$

$= [0.8, 0.9][0.1, 0.5][0.1, 0.2] = NVB_{M_{NVB}}(1)$

Combined neutrosophic vague binary membership grades are given as follows:

$$NVB_{M_{NVB}}(u_t) = \begin{cases} [0.8, 0.9][0.1, 0.5][0.1, 0.2]\ ;\ u_t = 0 \\ [0.9, 0.9][0.1, 0.2][0.1, 0.1]\ ;\ u_t = u_p \\ [0.9, 0.9][0.1, 0.2][0.1, 0.1]\ ;\ u_t = u_q \\ [0.8, 0.9][0.1, 0.4][0.1, 0.2]\ ;\ u_t = u_r \\ [0.8, 0.9][0.1, 0.4][0.1, 0.2]\ ;\ u_t = u_s \\ [0.8, 0.9][0.1, 0.5][0.1, 0.2]\ ;\ u_t = 1 \end{cases}$$

Algebraic structure $U^{\mathfrak{M}_{M_{NVB}}} = (U = \{U_1 \cap U_2\},\ *,\ \neg,\ \sim, 0)$ with binary and unary operations defined as in Cayley table given below clearly indicates a $BZMV^{dM}$ – subalgebra.

Cayley table for unary operations $\neg$ and $\sim$ are given below:

	0	$\mathbf{u_p}$	$\mathbf{u_q}$	$\mathbf{u_r}$	$\mathbf{u_s}$	1
$\neg$	1	$\mathbf{u_r}$	$\mathbf{u_s}$	$\mathbf{u_s}$	$\mathbf{u_r}$	0
$\sim$	1	**0**	**0**	0	**0**	**0**

Cayley table for binary operation ‘$*$’ is given below

$*$	0	$\mathbf{u_p}$	$\mathbf{u_q}$	$\mathbf{u_r}$	$\mathbf{u_s}$	1
0	**0**	$\mathbf{u_p}$	$\mathbf{u_q}$	$\mathbf{u_r}$	$\mathbf{u_s}$	1
$\mathbf{u_p}$	$\mathbf{u_p}$	$\mathbf{u_p}$	$\mathbf{u_r}$	$\mathbf{u_r}$	1	1

$\mathbf{u_q}$	$\mathbf{u_q}$	$\mathbf{u_r}$	$\mathbf{u_s}$	1	$\mathbf{u_s}$	1
$\mathbf{u_r}$	$\mathbf{u_r}$	$\mathbf{u_r}$	1	1	1	1
$\mathbf{u_s}$	$\mathbf{u_s}$	**1**	$\mathbf{u_s}$	1	$\mathbf{u_s}$	1
1	1	1	1	1	1	1

Now have to verify, neutrosophic vague binary concept!
For that check the inequalities given in definition 4.1.

$\mathfrak{M}_{M_{NVB}}$ inequality (1): (Binary Operation)

$$\forall\, u_x,\ u_y \in U,\ \left(\underset{u_v \in (u_x * u_y)}{\inf} NVB_{M_{NVB}}(u_v)\right) \geqslant r\min\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\}$$

$$\Rightarrow \left(\underset{u_v \in \{0,\ u_p,\ u_q,\ u_r,\ u_s, 1\}}{\inf} NVB_{M_{NVB}}(u_v)\right) \geqslant r\min\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\}$$

$$\Rightarrow \left(\underset{u_v \in \{0,\ u_p,\ u_q,\ u_r,\ u_s, 1\}}{\mathrm{glb}} NVB_{M_{NVB}}(u_v)\right) \geqslant r\min\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\}$$

$$\Rightarrow \left(\underset{u_v \in \{0,\ u_p,\ u_q,\ u_r,\ u_s, 1\}}{\min} NVB_{M_{NVB}}(u_v)\right) \geqslant r\min\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\}$$

$$= \min\left\{\begin{array}{l}[0.8, 0.9][0.1, 0.5][0.1, 0.2]\ ;\ u_v = 0\\ [0.9, 0.9][0.1, 0.2][0.1, 0.1]\ ;\ u_v = u_p\\ [0.9, 0.9][0.1, 0.2][0.1, 0.1]\ ;\ u_v = u_q\\ [0.8, 0.9][0.1, 0.4][0.1, 0.2]\ ;\ u_v = u_r\\ [0.8, 0.9][0.1, 0.4][0.1, 0.2]\ ;\ u_v = u_s\\ [0.8, 0.9][0.1, 0.5][0.1, 0.2]\ ;\ u_v = 1\end{array}\right\} = \cap\left\{\begin{array}{l}[0.8, 0.9][0.1, 0.5][0.1, 0.2]\ ;\ u_v = 0\\ [0.9, 0.9][0.1, 0.2][0.1, 0.1]\ ;\ u_v = u_p\\ [0.9, 0.9][0.1, 0.2][0.1, 0.1]\ ;\ u_v = u_q\\ [0.8, 0.9][0.1, 0.4][0.1, 0.2]\ ;\ u_v = u_r\\ [0.8, 0.9][0.1, 0.4][0.1, 0.2]\ ;\ u_v = u_s\\ [0.8, 0.9][0.1, 0.5][0.1, 0.2]\ ;\ u_v = 1\end{array}\right\} = [0.8, 0.9][0.1, 0.5][0.1, 0.2]$$

[In neutrosophic concept, minimum concept has been taken as intersection. ie., in this case, (Min, Max, Max)]

$$\therefore \left(\underset{u_v \in (u_x * u_y)}{\inf} NVB_{M_{NVB}}(u_v)\right) = [0.8, 0.9][0.1, 0.5][0.1, 0.2]$$

In this case, for any pair of elements from U,

$$\left(\underset{u_v \in (u_x \oplus u_y)}{\inf} NVB_{M_{NVB}}(u_v)\right) \geqslant r\min\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\},$$ got satisfied.

$\mathfrak{M}_{M_{NVB}}$ inequality (2): (Unary operations)

Next to check, the 2 inequalities of $\mathfrak{M}_{M_{NVB}}(2)$ for all elements of U.
(i) From Cayley table for unary operation $\neg$ (Kleene or Zadeh or fuzzy orthocomplementation)

$NVB_{M_{NVB}}(u_x) \geqslant NVB_{M_{NVB}}(\neg\, u_x)\ ;\ \forall\ \ u_x \in U$ as showed in table.

u_x	$NVB_{M_{NVB}}(u_x)$	$NVB_{M_{NVB}}(\neg\, u_x)$
0	[0.8, 0.9][0.1, 0.5][0.1, 0.2]	[0.8, 0.9][0.1, 0.5][0.1, 0.2]
u_p	[0.9, 0.9][0.1, 0.2][0.1, 0.1]	[0.8, 0.9][0.1, 0.4][0.1, 0.2]
u_q	[0.9, 0.9][0.1, 0.2][0.1, 0.1]	[0.8, 0.9][0.1, 0.4][0.1, 0.2]
u_r	[0.8, 0.9][0.1, 0.4][0.1, 0.2]	[0.8, 0.9][0.1, 0.4][0.1, 0.2]

u_s	[0.8, 0.9][0.1, 0.4][0.1, 0.2]	[0.8, 0.9][0.1, 0.4][0.1, 0.2]
1	[0.8, 0.9][0.1, 0.5][0.1, 0.2]	[0.8, 0.9][0.1, 0.5][0.1, 0.2]

(ii) From Cayley table for unary operation $\sim$ (Brower orthocomplementation)

$NVB_{M_{NVB}}(u_x) \succcurlyeq NVB_{M_{NVB}}(\sim u_x)$; $\forall$ $u_x \in U$ as showed in table

u_x	$NVB_{M_{NVB}}(u_x)$	$NVB_{M_{NVB}}(\sim u_x)$
0	[0.8, 0.9][0.1, 0.5][0.1, 0.2]	[0.8, 0.9][0.1, 0.5][0.1, 0.2]
u_p	[0.9, 0.9][0.1, 0.2][0.1, 0.1]	[0.8, 0.9][0.1, 0.5][0.1, 0.2]
u_q	[0.9, 0.9][0.1, 0.2][0.1, 0.1]	[0.8, 0.9][0.1, 0.5][0.1, 0.2]
u_r	[0.8, 0.9][0.1, 0.4][0.1, 0.2]	[0.8, 0.9][0.1, 0.5][0.1, 0.2]
u_s	[0.8, 0.9][0.1, 0.4][0.1, 0.2]	[0.8, 0.9][0.1, 0.5][0.1, 0.2]
1	[0.8, 0.9][0.1, 0.5][0.1, 0.2]	[0.8, 0.9][0.1, 0.5][0.1, 0.2]

So given example is a $\mathfrak{M}_{M_{NVB}}$ with structure $\left(U^{\mathfrak{M}_{M_{NVB}}}, *, \neg, \sim, 0\right)$

Remark 4.4

It is to be noted that,

(i) first column of the Cayley table for binary operation will be a copy of column of operands, using definition 2.8 (BZMV2)

(ii) last row and column of the Cayley table for binary operation will be always 1 for a $BZMV^{dM} -$ algebra, by using (3) and (4) of theorem 2.12

Theorem 4.5

If M_{NVB} is a $\mathfrak{M}_{M_{NVB}}$ then the following results are true:

(1) $NVB_{M_{NVB}}(u_x \oplus u_y) = NVB_{M_{NVB}}(u_y \oplus u_x)$ $\quad$ [i. e., commutative law holds for Binary Operation]

(2) $NVB_{M_{NVB}}\left((u_x \oplus u_y) \oplus u_z\right) = NVB_{M_{NVB}}\left(u_x \oplus (u_y \oplus u_z)\right)$

[i. e., associative law holds for Binary Operation]

(3) $NVB_{M_{NVB}}(u_x \oplus 1) = NVB_{M_{NVB}}(1)$

(4) $NVB_{M_{NVB}}(u_x \oplus \neg u_x) = NVB_{M_{NVB}}(1)$

$\left[\begin{array}{c}\text{neutrosophic vague binary membership grade of an element binary operated with it's kleene complement} \\ \text{always produce the neutrosophic vague binary membership grade of the maximum element 1}\end{array}\right]$

(5) $NVB_{M_{NVB}}(\neg (u_x \oplus \sim\sim u_x) \oplus \sim\sim u_x) = NVB_{M_{NVB}}(1)$

(6) $NVB_{M_{NVB}}(\neg u_x \oplus \sim\sim u_x) = NVB_{M_{NVB}}(1)$

(7) $NVB_{M_{NVB}}(u_x \wedge \sim\sim u_x) = NVB_{M_{NVB}}(u_x)$

(8) $\mathrm{NVB}_{\mathrm{M_{NVB}}}(\neg \sim \mathrm{u_x}) = \mathrm{NVB}_{\mathrm{M_{NVB}}}(\sim \sim \mathrm{u_x})$

(9) $\mathrm{NVB}_{\mathrm{M_{NVB}}}\left(\sim (\mathrm{u_x} \wedge \mathrm{u_y})\right) = \mathrm{NVB}_{\mathrm{M_{NVB}}}(\sim \mathrm{u_x} \vee \sim \mathrm{u_y})$

(10) $\mathrm{NVB}_{\mathrm{M_{NVB}}}\left(\sim (\mathrm{u_x} \vee \mathrm{u_y})\right) = \mathrm{NVB}_{\mathrm{M_{NVB}}}(\sim \mathrm{u_x} \wedge \sim \mathrm{u_y})$

(Equivalently, $\mathrm{u_x} \leq \mathrm{u_y}$ implies $\sim \mathrm{u_y} \leq \sim \mathrm{u_x}$)

(11) $\mathrm{NVB}_{\mathrm{M_{NVB}}}(\mathrm{u_x} \wedge \sim \mathrm{u_x}) = \mathrm{NVB}_{\mathrm{M_{NVB}}}(0)$

(12) $\mathrm{NVB}_{\mathrm{M_{NVB}}}(\sim \mathrm{u_x}) = \mathrm{NVB}_{\mathrm{M_{NVB}}}(\sim \sim \sim \mathrm{u_x})$

(13) $\mathrm{NVB}_{\mathrm{M_{NVB}}}(\sim \mathrm{u_x} \oplus \sim \mathrm{u_x}) = \mathrm{NVB}_{\mathrm{M_{NVB}}}(\sim \mathrm{u_x})$

(14) $\mathrm{NVB}_{\mathrm{M_{NVB}}}(\neg\, 0) = \mathrm{NVB}_{\mathrm{M_{NVB}}}(\sim 0)$

Proof

(1) $\mathrm{NVB}_{\mathrm{M_{NVB}}}(\mathrm{u_x} \oplus \mathrm{u}_y) = \mathrm{NVB}_{\mathrm{M_{NVB}}}\left((\mathrm{u_x} \oplus \mathrm{u}_y) \oplus 0\right)$,

by putting $\mathrm{u_x} = (\mathrm{u_x} \oplus \mathrm{u_y})$ in definition 4.1 (2), $(\mathrm{u_x} \oplus 0) = \mathrm{u_x}$

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}\left((\mathrm{u_y} \oplus 0) \oplus \mathrm{u_x}\right)$, by using definition 4.1 (1)

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}(\mathrm{u_y} \oplus \mathrm{u_x})$, by using definition 4.1 (2)

(2) $\mathrm{NVB}_{\mathrm{M_{NVB}}}\left((\mathrm{u_x} \oplus \mathrm{u_y}) \oplus \mathrm{u_z}\right) = \mathrm{NVB}_{\mathrm{M_{NVB}}}\left((\mathrm{u_y} \oplus \mathrm{u_z}) \oplus \mathrm{u_x}\right)$, by using definition 4.1 (1)

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}\left(\mathrm{u_x} \oplus (\mathrm{u_y} \oplus \mathrm{u_z})\right)$, by using theorem 4.5 (1)

(3) $\mathrm{NVB}_{\mathrm{M_{NVB}}}(\mathrm{u_x} \oplus 1) = \mathrm{NVB}_{\mathrm{M_{NVB}}}(\mathrm{u_x} \oplus \neg\, 0)$, since $1 = \neg\, 0$

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}\left(\mathrm{u_x} \oplus (\sim \mathrm{u_x} \oplus \sim \sim \mathrm{u_x})\right)$, by using definition 4.1 (5)

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}\left((\sim \mathrm{u_x} \oplus \sim \sim \mathrm{u_x}) \oplus \mathrm{u_x}\right)$, by using theorem 4.5 (1)

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}\left((\sim \sim \mathrm{u_x} \oplus \mathrm{u_x}) \oplus \sim \mathrm{u_x}\right)$, by using definition 4.1 (1)

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}\left((\mathrm{u_x} \oplus \sim \sim \mathrm{u_x}) \oplus \sim \mathrm{u_x}\right)$, by using theorem 4.5 (1)

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}(\sim \sim \mathrm{u_x} \oplus \sim \mathrm{u_x})$, by using theorem 4.1 (6)

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}(\sim \mathrm{u_x} \oplus \sim \sim \mathrm{u_x})$, by theorem 4.5 (1)

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}(\neg\, 0)$, by using definition 4.1 (5)

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}(1)$, since $\neg\, 0 = 1$

(4) $\mathrm{NVB}_{\mathrm{M_{NVB}}}(\mathrm{u_x} \oplus \neg\, \mathrm{u_x})$

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}(\neg \neg\, \mathrm{u_x} \oplus \neg\, \mathrm{u_x})$, by using definition 4.1 (3)

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}(\neg (\neg\, \mathrm{u_x} \oplus 0) \oplus \neg\, \mathrm{u_x})$, by using definition 4.1 (2)

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}(\neg (0 \oplus \neg\, \mathrm{u_x}) \oplus \neg\, \mathrm{u_x})$, by theorem 4.5 (1)

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}(\neg (\mathrm{u_x} \oplus \neg\, 0) \oplus \neg\, 0)$, by theorem 4.1 (4)

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}(\neg (\mathrm{u_x} \oplus 1) \oplus 1)$, since $\neg\, 0 = 1 = \mathrm{NVB}_{\mathrm{M_{NVB}}}(\neg\, 0)$

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}(1)$, since $\neg\, 0 = 1$

(5) $\mathrm{NVB}_{\mathrm{M_{NVB}}}(\neg (\mathrm{u_x} \oplus \sim \sim \mathrm{u_x}) \oplus \sim \sim \mathrm{u_x})$

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}(\neg (\sim \sim \mathrm{u_x}) \oplus \sim \sim \mathrm{u_x})$, by using definition 4.1 (6)

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}\left(\sim \sim \mathrm{u_x} \oplus \neg (\sim \sim \mathrm{u_x})\right)$, by theorem 4.5 (1)

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}(1)$, by theorem 4.5 (4)

(6) $\mathrm{NVB}_{\mathrm{M_{NVB}}}(\neg\, \mathrm{u_x} \oplus \sim \sim \mathrm{u_x})$

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}\left(\neg\, \mathrm{u_x} \oplus (\mathrm{u_x} \oplus \sim \sim \mathrm{u_x})\right)$, by using definition 4.1(6)

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}\left((\neg\, \mathrm{u_x} \oplus \mathrm{u_x}) \oplus \sim \sim \mathrm{u_x}\right)$, by theorem 4.5 (2)

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}\left((\mathrm{u_x} \oplus \neg\, \mathrm{u_x}) \oplus \sim \sim \mathrm{u_x}\right)$, by theorem 4.5 (1)

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}(1 \oplus \sim\sim u_x)$, by theorem 4.5 (4)
$= \mathrm{NVB}_{\mathrm{M_{NVB}}}(\sim\sim u_x \oplus 1)$, by theorem 4.5 (1)
$= \mathrm{NVB}_{\mathrm{M_{NVB}}}(1)$, by putting $u_x = \sim\sim u_x$ in theorem 4.5 (3)
(7) $\mathrm{NVB}_{\mathrm{M_{NVB}}}(u_x \wedge \sim\sim u_x)$
$= \mathrm{NVB}_{\mathrm{M_{NVB}}}\big(\neg\,(\neg\,(u_x \oplus \neg \sim\sim u_x) \oplus \neg \sim\sim u_x)\big)$

$\big[$From remark 4.2 (i) 3, we have, $(u_x \wedge u_y) = \neg\,(\neg\,(u_x \oplus \neg\, u_y) \oplus \neg\, u_y)$, by putting $u_y = \sim\sim u_x\big]$

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}\big(\neg\,(\neg\,(\neg \sim\sim u_x \oplus u_x) \oplus \neg \sim\sim u_x)\big)$

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}\big(\neg\,(\neg\,(u_x \oplus \neg \sim\sim u_x) \oplus \neg \sim\sim u_x)\big) = \mathrm{NVB}_{\mathrm{M_{NVB}}}\big(\neg\,(\neg\,(u_x \oplus \neg \sim\sim u_x) \oplus \sim\sim\sim u_x)\big)$

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}\big(\neg\,(\neg\,(u_x \oplus \neg \sim\sim u_x) \oplus u_x)\big) = \mathrm{NVB}_{\mathrm{M_{NVB}}}\big(\neg\,(\neg\,(u_x \oplus \neg \sim\sim u_x) \oplus u_x)\big)$

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}\big(\neg(\neg\,(\neg\, u_x \oplus \sim\sim u_x) \oplus \sim\sim u_x)\big)$

$\big[$From definition 4.1(4), $\neg\,(\neg\, u_x \oplus u_y) \oplus u_y = \neg\,(u_x \oplus \neg\, u_y) \oplus u_x\big]$

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}\big(\neg\,(\neg\, 1 \oplus \sim\sim u_x)\big)$, by definition 4.5 (6) $= \mathrm{NVB}_{\mathrm{M_{NVB}}}\big(\neg\,(0 \oplus \sim\sim u_x)\big)$, since $\neg\, 1 = 0$

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}\big(\neg\,(\sim\sim u_x \oplus 0)\big)$, from theorem 4.5 (1) $= \mathrm{NVB}_{\mathrm{M_{NVB}}}\big(\neg\,(\sim\sim u_x)\big)$, from theorem 4.1 (1)

$= \mathrm{NVB}_{\mathrm{M_{NVB}}}(\sim\sim\sim u_x)$, since $\neg\,(\sim\sim u_x) = \sim\sim\sim u_x$ $= \mathrm{NVB}_{\mathrm{M_{NVB}}}(u_x)$, since $\sim\sim\sim u_x = u_x$

(8) Consider definition 4.1 (7),
$\mathrm{NVB}_{\mathrm{M_{NVB}}}\big(\sim \neg\,[\neg\,(u_x \oplus \neg\, u_y) \oplus \neg\, u_y]\big) = \mathrm{NVB}_{\mathrm{M_{NVB}}}\,\big(\neg\,(\sim\sim u_x \oplus \neg \sim\sim u_y) \oplus \neg \sim\sim u_y\big)$
By putting $u_y = u_x$ in the above,
$\Rightarrow \mathrm{NVB}_{\mathrm{M_{NVB}}}(\sim \neg\,[\neg\,(u_x \oplus \neg\, u_x) \oplus \neg\, u_x]) = \mathrm{NVB}_{\mathrm{M_{NVB}}}(\neg\,(\sim\sim u_x \oplus \neg \sim\sim u_x) \oplus \neg \sim\sim u_x)$,
$\Rightarrow \mathrm{NVB}_{\mathrm{M_{NVB}}}(\sim \neg\,[\neg\, 1 \oplus \neg\, u_x]) = \mathrm{NVB}_{\mathrm{M_{NVB}}}(\neg\, 1 \oplus \neg \sim\sim u_x)$, by using definition 4.5 (4)
$\Rightarrow \mathrm{NVB}_{\mathrm{M_{NVB}}}(\sim \neg\,[0 \oplus \neg\, u_x]) = \mathrm{NVB}_{\mathrm{M_{NVB}}}(0 \oplus \neg \sim\sim u_x)$
$\Rightarrow \mathrm{NVB}_{\mathrm{M_{NVB}}}(\sim \neg\,[\neg\, u_x \oplus 0]) = \mathrm{NVB}_{\mathrm{M_{NVB}}}(\neg \sim\sim u_x \oplus 0)$, by theorem 4.5 (1)
$\Rightarrow \mathrm{NVB}_{\mathrm{M_{NVB}}}(\sim \neg\,[\neg\, u_x]) = \mathrm{NVB}_{\mathrm{M_{NVB}}}(\neg \sim\sim u_x)$, by definition 4.1 (2)
$\Rightarrow \mathrm{NVB}_{\mathrm{M_{NVB}}}(\sim u_x) = \mathrm{NVB}_{\mathrm{M_{NVB}}}(\neg \sim\sim u_x)$, by definition 4.1 (3)
$\Rightarrow \mathrm{NVB}_{\mathrm{M_{NVB}}}(\neg \sim u_x) = \mathrm{NVB}_{\mathrm{M_{NVB}}}(\neg\neg \sim\sim u_x)$, by applying $\neg$ on both sides.
$\Rightarrow \mathrm{NVB}_{\mathrm{M_{NVB}}}(\neg \sim u_x) = \mathrm{NVB}_{\mathrm{M_{NVB}}}(\sim\sim u_x)$, by using definition 4.1(3)
(9) Consider definition 4.1 (7)
$\Rightarrow \mathrm{NVB}_{\mathrm{M_{NVB}}}\,\big(\sim \neg\,[\neg\,(u_x \oplus \neg\, u_y) \oplus \neg\, u_y]\big) = \mathrm{NVB}_{\mathrm{M_{NVB}}}\,\big(\neg\,(\sim\sim u_x \oplus \neg \sim\sim u_y) \oplus \neg \sim\sim u_y\big)$

$\Rightarrow \mathrm{NVB}_{\mathrm{M_{NVB}}}\,\big(\sim (u_x \wedge u_y)\big) = \mathrm{NVB}_{\mathrm{M_{NVB}}}\big(\neg\,(\neg \sim u_x \oplus \neg\neg \sim u_y) \oplus \neg\neg \sim u_y\big)$

$\big[$By using auxiliary operation, $(u_x \wedge u_y) = \neg\,[\neg\,(u_x \oplus \neg\, u_y) \oplus \neg\, u_y]\big]$ &

[By using theorem 4.1 (8), $\neg \sim u_x = \sim\sim u_x$]

$\Rightarrow \mathrm{NVB}_{\mathrm{M_{NVB}}}\,\big(\sim (u_x \wedge u_y)\big) = \mathrm{NVB}_{\mathrm{M_{NVB}}}\big(\neg\,(\sim\sim u_x \oplus \sim u_y) \oplus \sim u_y\big)$, by using definition 4.1 (3)

$\Rightarrow NVB_{M_{NVB}}\left(\sim(u_x \wedge u_y)\right) = NVB_{M_{NVB}}(\neg(\neg\sim u_x \oplus \sim u_y) \oplus \sim u_y)$, by using $\sim\sim u_x = \neg\sim u_x$

$\Rightarrow NVB_{M_{NVB}}\left(\sim(u_x \wedge u_y)\right) = NVB_{M_{NVB}}(\sim u_x \vee \sim u_y)$

[By using auxiliary operation, $NVB_{M_{NVB}}(u_x \vee u_y) = NVB_{M_{NVB}}(\neg[\neg(u_x \oplus u_y) \oplus u_y])$]

(10) Let $NVB_{M_{NVB}}(u_x) \leq NVB_{M_{NVB}}(u_y) \Rightarrow NVB_{M_{NVB}}(u_x) = NVB_{M_{NVB}}(u_x \wedge u_y)$

$\Rightarrow NVB_{M_{NVB}}(\sim u_x) = NVB_{M_{NVB}}\left(\sim(u_x \wedge u_y)\right)$, by applying $\sim$ on both sides and by using (9)

$\Rightarrow NVB_{M_{NVB}}(\sim u_x) = NVB_{M_{NVB}}(\sim u_x \vee \sim u_y)$, by theorem 4.5 (9)

$\Rightarrow NVB_{M_{NVB}}(\sim u_y) \leq NVB_{M_{NVB}}(\sim u_x)$.

Using theorem 4.5 (7), $\forall\ u_x$, $NVB_{M_{NVB}}(u_x) \leq NVB_{M_{NVB}}(\sim\sim u_x)$.

So now the contraposition law is equivalent to the de Morgan law:

$NVB_{M_{NVB}}\left(\sim(u_x \vee u_y)\right) = NVB_{M_{NVB}}(\sim u_x \wedge \sim u_y)$

(11) $NVB_{M_{NVB}}(u_x \wedge \sim u_x) = NVB_{M_{NVB}}\left(\neg(\neg(u_x \oplus \neg\sim u_x) \oplus \neg\sim u_x)\right)$

$= NVB_{M_{NVB}}\left(\neg(\neg(u_x \oplus \sim\sim u_x) \oplus \sim\sim u_x)\right)$, by theorem 2.12 (8) & by definition 4.5 (8)

$= NVB_{M_{NVB}}(\neg 1)$, by definition 4.5 (5) $= NVB_{M_{NVB}}(0)$

(12) From definition 4.5 (8), $NVB_{M_{NVB}}(\neg\sim u_x) = NVB_{M_{NVB}}(\sim\sim u_x)$

Put $u_x = \sim u_x$, in the above then, $NVB_{M_{NVB}}(\neg\sim\sim u_x) = NVB_{M_{NVB}}(\sim\sim\sim u_x)$

$\Rightarrow NVB_{M_{NVB}}(\neg\neg\sim u_x) = NVB_{M_{NVB}}(\sim\sim\sim u_x)$, since $\sim\sim u_x = \neg\sim u_x$

$\Rightarrow NVB_{M_{NVB}}(\sim u_x) = NVB_{M_{NVB}}(\sim\sim\sim u_x)$, since $\neg\neg\sim u_x = \sim u_x$

(13) $NVB_{M_{NVB}}(u_x \wedge \sim\sim u_x) = NVB_{M_{NVB}}(\sim\sim u_x)$, by definition 4.1 (6)

Taking Brouwerian orthocomplementation to both sides,

$NVB_{M_{NVB}}(\sim u_x \wedge \sim\sim\sim u_x) = NVB_{M_{NVB}}(\sim\sim\sim u_x)$

$\Rightarrow NVB_{M_{NVB}}(\sim u_x \wedge \sim u_x) = NVB_{M_{NVB}}(\sim u_x)$ [By theorem 4.5(12), $\sim\sim\sim u_x = \sim u_x$]

(14) To prove that, $NVB_{M_{NVB}}(\neg 0) = NVB_{M_{NVB}}(\sim 0)$.

It is enough to prove that,

$NVB_{M_{NVB}}(\neg 0) \preccurlyeq NVB_{M_{NVB}}(\sim 0)$ and $NVB_{M_{NVB}}(\sim 0) \preccurlyeq NVB_{M_{NVB}}(\neg 0)$

We know that, $NVB_{M_{NVB}}(1) = NVB_{M_{NVB}}(\neg 0)$

$\forall\ u_x \in M_{NVB}$, where M_{NVB} is a $\mathfrak{M}_{M_{NVB}}$,

$NVB_{M_{NVB}}(u_x) \preccurlyeq NVB_{M_{NVB}}(1)$, since 1 is the maximum element;

In particular, $NVB_{M_{NVB}}(\sim 0) \preccurlyeq NVB_{M_{NVB}}(1) \Rightarrow NVB_{M_{NVB}}(\sim 0) \preccurlyeq NVB_{M_{NVB}}(\neg 0)$

Similarly, being the least element, $\forall\ u_x \in M_{NVB}$, where M_{NVB} is a $\mathfrak{M}_{M_{NVB}}$,

$NVB_{M_{NVB}}(u_x) \preccurlyeq NVB_{M_{NVB}}(\sim\sim u_x) \preccurlyeq NVB_{M_{NVB}}(\sim 0)$.

In particular, $NVB_{M_{NVB}}(\neg 0) \preccurlyeq NVB_{M_{NVB}}(\sim 0)$

Theorem 4.6

Let M_{NVB} is a $\mathfrak{M}_{M_{NVB}}$. Then

(i) $\forall\ u_x, u_y \in M_{NVB}$, $NVB_{M_{NVB}}(u_x \wedge u_y) = NVB_{M_{NVB}}(0) \Leftrightarrow NVB_{M_{NVB}}(y) \preccurlyeq NVB_{M_{NVB}}(\sim u_x)$

Equivalently, $NVB_{M_{NVB}}(u_x \wedge u_y) = NVB_{M_{NVB}}(0) \Leftrightarrow NVB_{M_{NVB}}(u_x) \preccurlyeq NVB_{M_{NVB}}(\sim u_y)$

(ii) Let $u_x \in M_{NVB}$ be such that $NVB_{M_{NVB}}(u_x \oplus u_x) = NVB_{M_{NVB}}(u_x)$,

Then $\forall\ u_y \in M_{NVB}$, $NVB_{M_{NVB}}(u_x \wedge u_y) = NVB_{M_{NVB}}(0) \Leftrightarrow NVB_{M_{NVB}}(u_x) \preccurlyeq NVB_{M_{NVB}}(\neg u_y)$

Proof

(i) Assume $(u_x \wedge u_y) = 0$.

Now, $NVB_{M_{NVB}}(u_y \wedge \sim u_x) = NVB_{M_{NVB}}((u_y \wedge \sim u_x) \vee 0)$, since in any lattice $(u_x \vee 0) = u_x$

$\Rightarrow NVB_{M_{NVB}}(u_y \wedge \sim u_x) = NVB_{M_{NVB}}((u_y \wedge \sim u_x) \vee (u_y \wedge \sim u_y))$,

by a result $(u_y \wedge \sim u_y) = 0$, of $BZMV^{dM}$ algebra

$= NVB_{M_{NVB}}(u_y \wedge (\sim u_x \vee \sim u_y))$, by theorem 4.5 (9) $= NVB_{M_{NVB}}(u_y \wedge \sim (u_x \wedge u_y))$, by theorem 4.5 (9)

$= NVB_{M_{NVB}}(u_y \wedge \sim 0)$ [by assumption]

$= NVB_{M_{NVB}}(u_y) = NVB_{M_{NVB}}(u_y \wedge 1)$, since $\sim 0 = \neg\ 0 = 1$

$= NVB_{M_{NVB}}(u_y)$, [since in any lattice $(y \wedge 1) = y \Rightarrow NVB_{M_{NVB}}(y \wedge 1) = NVB_{M_{NVB}}(y)$]

$\Rightarrow NVB_{M_{NVB}}(u_y) \preccurlyeq NVB_{M_{NVB}}(\sim u_x)$

Conversely, suppose $NVB_{M_{NVB}}(y) \preccurlyeq NVB_{M_{NVB}}(\sim u_x)$ then $NVB_{M_{NVB}}(u_x \wedge y)$

$= NVB_{M_{NVB}}\ (u_x \wedge (u_y \wedge \sim u_x)) = NVB_{M_{NVB}}(u_y \wedge\ (u_x \wedge \sim u_x))$, by associativity

$= NVB_{M_{NVB}}(u_y \wedge 0)$ [by theorem 4.5 (11)]

$= NVB_{M_{NVB}}(0)$, since in any lattice $(u_y \wedge 0) = u_y$

Equivalently, $NVB_{M_{NVB}}(u_x \wedge u_y) = NVB_{M_{NVB}}(0) \Leftrightarrow NVB_{M_{NVB}}(u_x) \preccurlyeq NVB_{M_{NVB}}(\sim u_y)$, can be proved

(ii) Suppose $NVB_{M_{NVB}}(u_x) = NVB_{M_{NVB}}(u_x \oplus u_x)$.

Then, $NVB_{M_{NVB}}(u_x \odot u_x) = NVB_{M_{NVB}}(u_x \wedge u_y)$.

Thus we got, $NVB_{M_{NVB}}(u_x) \preccurlyeq NVB_{M_{NVB}}(\neg\ u_y)$ iff $NVB_{M_{NVB}}(u_x \wedge u_y) = NVB_{M_{NVB}}(0)$

Theorem 4. 7

In a $\mathfrak{M}_{M_{NVB}}$ the following holds :

$NVB_{M_{NVB}}(\sim\sim u_x) = NVB_{M_{NVB}}(u_x) \Leftrightarrow NVB_{M_{NVB}}(\sim u_x \oplus u_x) = NVB_{M_{NVB}}(1)$

$\Leftrightarrow NVB_{M_{NVB}}(u_x \oplus u_x) = NVB_{M_{NVB}}(u_x)$

Proof

Assume $\sim\sim u_x = u_x$. Then, $NVB_{M_{NVB}}(\sim\sim u_x) = NVB_{M_{NVB}}(u_x)$,

Definition 4.1 (6) $\Rightarrow NVB_{M_{NVB}}(u_x \oplus \sim\sim u_x) = NVB_{M_{NVB}}(\sim\sim u_x)$

$\Rightarrow NVB_{M_{NVB}}(u_x \oplus u_x) = NVB_{M_{NVB}}(u_x)$

Again, from definition 4.1 (5) $\Rightarrow NVB_{M_{NVB}}(\sim u_x \oplus \sim\sim u_x) = NVB_{M_{NVB}}(\neg\ 0)$

$\Rightarrow NVB_{M_{NVB}}(\sim u_x \oplus u_x) = NVB_{M_{NVB}}(1)$

Assume, $NVB_{M_{NVB}}(u_x \oplus u_x) = NVB_{M_{NVB}}(u_x)$. Then by theorem 4.6 (ii), since $\neg\ u_x \in \mathfrak{M}_{M_{NVB}}$ and $(u_x \wedge \neg\ u_x) = 0 \Rightarrow NVB_{M_{NVB}}(u_x \wedge \neg\ u_x) = NVB_{M_{NVB}}(0) \Rightarrow NVB_{M_{NVB}}(\neg\ u_x) \preccurlyeq NVB_{M_{NVB}}(\sim u_x)$

$\Rightarrow NVB_{M_{NVB}}(\neg \sim u_x) \preccurlyeq NVB_{M_{NVB}}(\sim\sim u_x)$, by putting $u_x = \sim u_x$

$\Rightarrow NVB_{M_{NVB}}(\sim\sim u_x) = NVB_{M_{NVB}}(u_x)$

Under condition $NVB_{M_{NVB}}(\sim u_x \oplus u_x) = NVB_{M_{NVB}}(1)$, it is clear that,

$NVB_{M_{NVB}}(u_x\ \wedge \sim\sim u_x) = NVB_{M_{NVB}}(\sim\sim u_x) \Rightarrow NVB_{M_{NVB}}(\sim\sim u_x) \preccurlyeq NVB_{M_{NVB}}(u_x)$.

In fact, $NVB_{M_{NVB}}(u_x \wedge \sim\sim u_x) = NVB_{M_{NVB}}(\neg\ [\neg\ (u_x \oplus \neg \sim\sim u_x) \oplus \neg \sim\sim u_x])$

[Using definition of $\wedge$]

$= NVB_{M_{NVB}}(\neg\ [\neg\ (u_x \oplus \neg\neg \sim u_x) \oplus \neg\neg \sim u_x])$, [since $\neg \sim u_x = \sim\sim u_x$]

$= NVB_{M_{NVB}}(\neg\ [\neg\ (u_x \oplus \sim u_x) \oplus \sim u_x])$, [since $\neg\neg\ u_x = u_x$]

$= NVB_{M_{NVB}}(\neg\ (0 \oplus \sim u_x))$

$= \text{NVB}_{M_{NVB}}(\neg (\sim u_x \oplus 0))$, [by theorem 4.5 (1)]

$= \text{NVB}_{M_{NVB}}(\neg \sim u_x)$, [by definition 4.1 (2)]

$= \text{NVB}_{M_{NVB}}(\sim \sim u_x)$. [since $\neg \sim u_x = \sim \sim u_x$]

Similarly, we get, $\text{NVB}_{M_{NVB}}(u_x) = \text{NVB}_{M_{NVB}}(\sim \sim u_x)$. Hence proved.

Theorem 4.8

Let M_{NVB} be a $\mathfrak{M}_{M_{NVB}}$; then for any $u_x, u_y, u_z \in M_{NVB}$,

$\text{NVB}_{M_{NVB}}\left((u_x \oplus u_y) \wedge u_z\right) = \text{NVB}_{M_{NVB}}(0)$ iff $\text{NVB}_{M_{NVB}}(u_x \wedge u_z) = \text{NVB}_{M_{NVB}}(0)$ and $\text{NVB}_{M_{NVB}}(u_y \wedge u_z) = \text{NVB}_{M_{NVB}}(0)$

Proof

Suppose, $\text{NVB}_{M_{NVB}}\left((u_x \oplus u_y) \wedge u_z\right) = \text{NVB}_{M_{NVB}}(0)$

$\Rightarrow \text{NVB}_{M_{NVB}}(u_x \wedge u_z) \preccurlyeq \text{NVB}_{M_{NVB}}\left((u_x \oplus u_y) \wedge u_z\right) = \text{NVB}_{M_{NVB}}(0)$ and

$\text{NVB}_{M_{NVB}}(u_y \wedge u_z) \preccurlyeq \text{NVB}_{M_{NVB}}\left((u_x \oplus u_y) \wedge u_z\right) = \text{NVB}_{M_{NVB}}(0)$, by theorem 4.6

$\Rightarrow \text{NVB}_{M_{NVB}}(u_x \wedge u_z) = \text{NVB}_{M_{NVB}}(0) = \text{NVB}_{M_{NVB}}(u_y \wedge u_z)$, trivial

Conversely, let $\text{NVB}_{M_{NVB}}(u_x \wedge u_z) = \text{NVB}_{M_{NVB}}(0)$ and $\text{NVB}_{M_{NVB}}(u_y \wedge u_z) = \text{NVB}_{M_{NVB}}(0)$.

$\text{NVB}_{M_{NVB}}(u_x \wedge \neg u_z) = \text{NVB}_{M_{NVB}}(\neg u_z) = \text{NVB}_{M_{NVB}}(u_y \wedge \neg u_z)$.

So, $\text{NVB}_{M_{NVB}}\left((u_x \oplus u_y) \oplus \neg u_z\right) = \text{NVB}_{M_{NVB}}\left(u_x \oplus (u_y \oplus \neg u_z)\right)$, by definition 4.5 (2)

$= \text{NVB}_{M_{NVB}}(u_x \oplus \neg u_z) = \text{NVB}_{M_{NVB}}(\neg u_z)$, from assumption

$\therefore \text{NVB}_{M_{NVB}}\left((u_x \oplus u_y) \wedge u_z\right) = \text{NVB}_{M_{NVB}}(\neg [\neg (u_x \oplus \neg \sim \sim u_x) \oplus \neg \neg \sim u_x])$

$= \text{NVB}_{M_{NVB}}\left(\left((u_x \oplus u_y) \oplus \neg u_z\right) \odot \neg u_z\right) = \text{NVB}_{M_{NVB}}(0 \odot \neg u_z) = \text{NVB}_{M_{NVB}}(u_x \odot 0) = \text{NVB}_{M_{NVB}}(0)$

5. Ideals in Neutrosophic Vague Binary BZMV^{dM} subalgebra

Concept of ideal with three different kinds are developed in this section

Definition 5.1 (NVB BZMV^{dM}- ideal)

Let M_{NVB} be a $\mathfrak{M}_{M_{NVB}}$ and I_{NVB} be a nonempty subset of M_{NVB}. Then I_{NVB} is a neutrosophic vague binary BZMV^{dM}- ideal $(\text{NVB} - \text{BZMV}^{\text{dM}}\text{ ideal})$ if the following inequalities got satisfied:

(i) $\text{NVB}_{I_{NVB}}(0) \succcurlyeq \text{NVB}_{I_{NVB}}(u_x)$; $\forall\ u_x \in I_{NVB}$

(ii) $\text{NVB}_{I_{NVB}}(u_x \oplus u_y) \succcurlyeq \text{r min}\left\{\text{NVB}_{I_{NVB}}(u_x), \text{NVB}_{I_{NVB}}(u_y)\right\}$; $\forall\ u_x, u_y \in I_{NVB}$

(iii) $\text{NVB}_{I_{NVB}}(u_y) \succcurlyeq \text{r min}\left\{\text{NVB}_{I_{NVB}}(u_x), \text{NVB}_{I_{NVB}}(u_y \leq u_x)\right\}$; $\forall\ u_x, u_y \in I_{NVB}$

Definition 5.2 $(\textbf{prime ideal}, \sim \textbf{ideal}, \textbf{normal ideal of a NVB BZMV}^{\textbf{dM}} - \textbf{subalgebra})$

Let M_{NVB} be a $\mathfrak{M}_{M_{NVB}}$ and I_{NVB} be a NVB BZMV^{dM}- ideal of M_{NVB}. I_{NVB} is called,

(i) a neutrosophic vague binary BZMV^{dM}- prime ideal $(\text{NVB BZMV}^{\text{dM}} - \text{prime ideal})$ of M_{NVB}

$\Leftrightarrow \{\text{NVB}_{I_{NVB}}(u_x \odot \neg u_y) \in M_{NVB}$ or $\text{NVB}_{I_{NVB}}(\neg u_x \odot u_y) \in M_{NVB}$; $\forall\ u_x, u_y \in M_{NVB}$

(ii) a neutrosophic vague binary $\text{BZMV}^{\text{dM}} \sim$ ideal $(\text{NVB BZMV}^{\text{dM}} \sim \text{ideal})$ of M_{NVB} if it satisfies:

$\text{NVB}_{I_{NVB}}(\sim \sim u_x \odot \sim \neg u_y \in I_{NVB}) \succcurlyeq \text{NVB}_{I_{NVB}}(u_x \odot u_y)$; $\forall\ u_x, u_y \in M_{NVB}$

(iii) a neutrosophic vague binary BZMV^{dM} normal ideal $(\text{NVB BZMV}^{\text{dM}}\text{ normal ideal})$ of M_{NVB} whenever

$\text{NVB}_{I_{NVB}}(\neg u_x \odot u_y) \succcurlyeq \text{NVB}_{I_{NVB}}(\sim u_x \odot u_y)$ and $\text{NVB}_{I_{NVB}}(\neg u_x \odot u_y) \preccurlyeq \text{NVB}_{I_{NVB}}(\sim u_x \odot u_y)$

i.e., $\text{NVB}_{I_{NVB}}(\neg u_x \odot u_y) \Leftrightarrow \text{NVB}_{I_{NVB}}(\sim u_x \odot u_y)$, $\forall\ u_x, u_y \in M_{NVB}$

Theorem 5.3

I_{NVB} is a NVB $BZMV^{dM}$– p ideal of $\mathfrak{M}_{M_{NVB}}$ $\Leftrightarrow \begin{Bmatrix} NVB_{I_{NVB}}(u_x) \succcurlyeq NVB_{I_{NVB}}(u_x \wedge u_y) \\ \text{or} \\ NVB_{I_{NVB}}(u_y) \succcurlyeq NVB_{I_{NVB}}(u_x \wedge u_y) \end{Bmatrix}$

Proof

Assume I_{NVB} is a NVB $BZMV^{dM}$ – p ideal of $\mathfrak{M}_{M_{NVB}}$

$$\Rightarrow \begin{cases} NVB_{I_{NVB}}(u_x \odot \neg u_y) \succcurlyeq r\min\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\};\ \forall u_x,\ u_y \in M_{NVB} \\ \qquad\qquad or \\ NVB_{I_{NVB}}(\neg u_x \odot u_y) \succcurlyeq r\min\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\};\ \forall u_x,\ u_y \in M_{NVB} \end{cases}$$

Without loss of generality, consider $NVB_{I_{NVB}}(\neg u_x \odot u_y) \succcurlyeq r\min\{NVB_{M_{NVB}}(u_x),\ NVB_{M_{NVB}}(u_y)\}$

$NVB_{I_{NVB}}(u_x \wedge u_y) = NVB_{I_{NVB}}(u_y \wedge u_x)$ [using commutativity of $\wedge$]

$= NVB_{I_{NVB}}(u_y \odot (u_x \oplus \neg u_y))$ [by using $(u_a \wedge u_b) = u_a \odot (u_b \oplus \neg u_a)$]

$= NVB_{I_{NVB}}(\neg(\neg u_y \oplus \neg(u_x \oplus \neg u_y)))$ [by using $(u_a \odot u_b) = \neg(\neg u_a \oplus \neg u_b)$]

$= NVB_{I_{NVB}}(\neg(\neg u_y \oplus \neg(\neg u_y \oplus u_x)))$ [by commutativity of $\oplus$]

$= NVB_{I_{NVB}}(\neg(\neg u_y \oplus \neg(\neg u_y \oplus \neg\neg u_x)))$ [since $\neg\neg u_x = u_x$]

$= NVB_{I_{NVB}}(\neg(\neg u_y \oplus (u_y \odot \neg u_x)))$ [since $\neg(\neg u_y \oplus \neg\neg u_x) = (u_y \odot \neg u_x)$] $\in I_{NVB}$

$NVB_{I_{NVB}}(\neg(\neg u_y \oplus (u_y \odot \neg u_x))) \succcurlyeq NVB_{I_{NVB}}(u_x \wedge u_y)$ and

$NVB_{I_{NVB}}(\neg u_x \odot u_y) \succcurlyeq r\min\{NVB_{M_{NVB}}(u_x), NVB_{M_{NVB}}(u_y)\}$

$\Rightarrow NVB_{I_{NVB}}(\neg(\neg u_y \oplus (u_y \odot \neg u_x)) \oplus (\neg u_x \odot u_y)) \in I_{NVB}$

$\Rightarrow NVB_{I_{NVB}}(\neg(\neg u_y \oplus (u_y \odot \neg u_x)) \oplus (u_y \odot \neg u_x)) \in I_{NVB}$

Hence, $NVB_{I_{NVB}}(u_y \vee (u_y \odot \neg u_x)) \in I_{NVB} \Rightarrow NVB_{I_{NVB}}(u_y) \in I_{NVB}$, since $(u_y \odot \neg u_x) \leq u_y$

$\therefore NVB_{I_{NVB}}(u_y) \succcurlyeq NVB_{I_{NVB}}(\neg u_x \odot u_y)$

Similarly, if $NVB_{I_{NVB}}(\neg u_y \odot u_x) \in I_{NVB} \Rightarrow NVB_{I_{NVB}}(u_x) \in I_{NVB}$

$\Rightarrow NVB_{I_{NVB}}(u_x) \succcurlyeq NVB_{I_{NVB}}(\neg u_y \odot u_x)$

Conversely, $NVB_{I_{NVB}}((\neg u_x \odot u_y) \wedge (u_x \odot \neg u_y)) = NVB_{I_{NVB}}(0) \in I_{NVB}$

[from definition of NVB $BZMV^{dM}$ ideal]

$\Rightarrow NVB_{I_{NVB}}(\neg u_x \odot u_y) \in I_{NVB}$ or $NVB_{I_{NVB}}(u_x \odot \neg u_y) \in I_{NVB}$

$\Rightarrow I_{NVB}$ is a NVB $BZMV^{dM}$ prime ideal of $\mathfrak{M}_{M_{NVB}}$

Theorem 5.4

Let I_{NVB} be an NVB ideal of a neutrosophic vague binary $BZMV^{dM}$ – subalgebra $\mathfrak{M}_{M_{NVB}}$ and $\sim\sim u_x = u_x$ for all $u_x \in M_{NVB}$. Then the following conditions are equivalent :

(1) I_{NVB} is a NVB $BZMV^{dM}$ normal ideal (2) I_{NVB} is a NVB $BZMV^{dM}$ $\sim$ ideal

(3) $NVB_{I_{NVB}}(\sim u_x) \in I_{NVB} \Leftrightarrow NVB_{I_{NVB}}(\neg u_x) \in I_{NVB}$

Proof

(1) $\Rightarrow$ (2)

Let I_{NVB} is a NVB $BZMV^{dM}$- normal ideal of $\mathfrak{M}_{M_{NVB}}$

Then, $\forall\ u_x,\ u_y \in M_{NVB},\ NVB_{M_{NVB}}(u_x \odot u_y) \in I_{NVB}$

$\Rightarrow NVB_{M_{NVB}}(\neg\neg u_x \odot u_y) \in I_{NVB}$ [by using $u_x = \neg\neg u_x$]

$\Rightarrow NVB_{M_{NVB}}(\sim\neg u_x \odot u_y) \in I_{NVB}$ [by property of $BZMV^{dM}$ – subalgebra $\neg\neg u_x = \sim\neg u_x$]

$\Rightarrow NVB_{M_{NVB}}(\sim\neg u_x \odot \sim\sim u_y) \in I_{NVB}$ [since given $\sim\sim u_x = u_x$]

$\Rightarrow NVB_{M_{NVB}}(\sim\sim u_y \odot \sim\neg u_x) \in I_{NVB}$ [by commutativity]

$\therefore NVB_{M_{NVB}}(\sim\sim u_y \odot \sim\neg u_x) \geqslant NVB_{M_{NVB}}(u_x \odot u_y),\ \forall\ u_x,\ u_y \in M_{NVB}$

$\Rightarrow I_{NVB}$ is a neutrosophic vague binary $BZMV^{dM} \sim$ ideal of M_{NVB}[by definition 5.2 (ii)]

(2) $\Rightarrow$ (1)

Let I_{NVB} be a neutrosophic vague binary $BZMV^{dM} \sim$ ideal.

Then, $\forall\ u_x,\ u_y \in M_{NVB},\ NVB_{M_{NVB}}(\neg u_x \odot u_y) \in I_{NVB}$

$\Rightarrow NVB_{M_{NVB}}(\sim\neg(\neg u_x) \odot \sim\sim u_y) \in I_{NVB} \Rightarrow NVB_{M_{NVB}}(\sim(\neg\neg u_x) \odot \sim\sim u_y) \in I_{NVB}$

$\Rightarrow NVB_{M_{NVB}}(\sim u_x \odot u_y) \in I_{NVB} \Rightarrow NVB_{M_{NVB}}(\sim\neg(\sim u_x) \odot \sim\sim u_y) \in I_{NVB}$

$\Rightarrow NVB_{M_{NVB}}(\sim(\neg\sim u_x) \odot \sim\sim u_y) \in I_{NVB} \Rightarrow NVB_{M_{NVB}}(\sim\sim\sim u_x \odot \sim\sim u_y) \in I_{NVB}$, since $[\neg\sim u_x = \sim\sim u_x]$

$\Rightarrow NVB_{M_{NVB}}(\neg(\sim\sim u_x) \odot \sim\sim u_y) \in I_{NVB}$

$\Rightarrow NVB_{M_{NVB}}(\neg u_x \odot u_y) \in I_{NVB}$, so I_{NVB} is a NVB $BZMV^{dM}$ normal ideal of M_{NVB}.

(1) $\Rightarrow$ (3)

Let I_{NVB} be a NVB $BZMV^{dM}$ normal ideal of M_{NVB}

$\Rightarrow NVB_{I_{NVB}}(\neg u_x \odot u_y) \Leftrightarrow NVB_{I_{NVB}}(\sim u_x \odot u_y),\ \ \forall u_x,\ u_y \in M_{NVB}$

$\Rightarrow NVB_{I_{NVB}}(\neg u_x \odot 1) \Leftrightarrow NVB_{I_{NVB}}(\sim u_x \odot 1),\ \ \forall u_x,\ u_y \in M_{NVB}$ [by putting $u_y = 1$]

$\Rightarrow NVB_{I_{NVB}}(\neg(\neg\neg u_x \oplus \neg 1)) \Leftrightarrow NVB_{I_{NVB}}(\neg(\neg\sim u_x \oplus \neg 1)),\ \ \forall u_x,\ u_y \in M_{NVB}$ [by definition of $\odot$]

$\Rightarrow NVB_{I_{NVB}}(\neg(\neg\neg u_x \oplus 0)) \Leftrightarrow NVB_{I_{NVB}}(\neg(\neg\sim u_x \oplus 0)),\ \ \forall u_x,\ u_y \in M_{NVB}$

$\Rightarrow NVB_{I_{NVB}}(\neg(\neg\neg u_x)) \Leftrightarrow NVB_{I_{NVB}}(\neg(\neg\sim u_x)),\ \ \forall u_x,\ u_y \in M_{NVB}$ [since $(u_x \oplus 0) = u_x$]

$\Rightarrow NVB_{I_{NVB}}(\neg\neg(\neg u_x)) \Leftrightarrow NVB_{I_{NVB}}(\neg\neg(\sim u_x)),\ \ \forall u_x,\ u_y \in M_{NVB}$

$\Rightarrow NVB_{I_{NVB}}(\neg u_x) \Leftrightarrow NVB_{I_{NVB}}(\sim u_x),\ \ \forall u_x,\ u_y \in M_{NVB}$ [since $\neg\neg(u_x) = u_x$]

(3) $\Rightarrow$ (1)

Suppose, $NVB_{I_{NVB}}(\sim u_x) \in I_{NVB} \Leftrightarrow NVB_{I_{NVB}}(\neg u_x) \in I_{NVB}$

$NVB_{I_{NVB}}(\sim u_x \odot u_y) \in I_{NVB} \Leftrightarrow NVB_{I_{NVB}}(\sim u_x \odot \sim\sim u_y) \in I_{NVB}$ [since $u_y = \sim\sim u_y$, by definition]

$\Rightarrow NVB_{I_{NVB}}(\neg(\neg\sim u_x \oplus \neg u_y)) \in I_{NVB}$ [by definition of $\odot$]

$\Rightarrow NVB_{I_{NVB}}(\neg(\sim\sim u_x \oplus \neg u_y)) \in I_{NVB}$ [$\neg\sim u_x = \sim\sim u_x$]

$\Rightarrow NVB_{I_{NVB}}(\neg(u_x \oplus \sim u_y)) \in I_{NVB}$ [given $\sim\sim u_x = u_x$] $\Rightarrow NVB_{I_{NVB}}(\neg(u_x \odot \neg\sim u_y)) \in I_{NVB}$

$\Rightarrow NVB_{I_{NVB}}(\neg u_x \odot \neg\sim u_y) \in I_{NVB}$ [by property of $\odot$]$\Rightarrow NVB_{I_{NVB}}(\neg u_x \odot \sim\sim u_y) \in I_{NVB}$ [$\neg\sim u_y = \sim\sim u_y$]

$\Rightarrow NVB_{I_{NVB}}(\neg u_x \odot u_y) \in I_{NVB}$ [by using the given property $\sim\sim u_x = u_x$]. $\therefore I_{NVB}$ is a NVB $BZMV^{dM}$- ideal

6. Direct sum of neutrosophic vague binary $BZMV^{dM}$ – subalgebra

In this section, a method is provided as a theorem to obtain a NVB $BZMV^{dM}$ - subalgebra by joining two NVB $BZMV^{dM}$ - subalgebras having $\{0, 1\}$ as common elements.

Theorem 6.1

Let $\mathfrak{M}_{M_{NVB}} = \langle U^{\mathfrak{M}_{M_{NVB}}}, \oplus_1, \neg_1, \sim_1, 0, 1\rangle$ and $\mathfrak{M}_{P_{NVB}} = \langle U^{\mathfrak{M}_{P_{NVB}}}, \oplus_2, \neg_2, \sim_2, 0, 1\rangle$ be two NVB $BZMV^{dM}$ – subalgebras such that $\left(U^{\mathfrak{M}_{M_{NVB}}} \cap U^{\mathfrak{M}_{P_{NVB}}}\right) = \{0, 1\}$. Let $U^{\mathfrak{M}_{W_{NVB}}} = \left(U^{\mathfrak{M}_{M_{NVB}}} \cup U^{\mathfrak{M}_{P_{NVB}}}\right)$ and let a binary operation $\ominus$ be defined on Z as follows:

$$(u_a \ominus u_b) = \begin{cases} u_a \oplus_1 u_b & \text{if } u_a,\ u_b \in U^{\mathfrak{M}_{M_{NVB}}} \\ u_a \oplus_2 u_b & \text{if } u_a,\ u_b \in U^{\mathfrak{M}_{P_{NVB}}} \\ u_a & \text{otherwise} \end{cases}$$

$$\neg^{\circledast} u_a = \begin{cases} \neg_1 u_a & \text{if } u_a \in U^{\mathfrak{M}_{M_{NVB}}} \\ \neg_2 u_a & \text{if } u_a \in U^{\mathfrak{M}_{P_{NVB}}} \end{cases}$$

$$\sim^{\circledast} u_a = \begin{cases} \sim_1 u_a & \text{if } u_a \in U^{\mathfrak{M}_{M_{NVB}}} \\ \sim_2 u_a & \text{if } u_a \in U^{\mathfrak{M}_{P_{NVB}}} \end{cases}$$

Then, $\langle U^{\mathfrak{M}_{W_{NVB}}}, \ominus, \neg^{\circledast}, \sim^{\circledast}, 0, 1\rangle$ is a neutrosophic vague binary $BZMV^{dM}$ – subalgebra $\mathfrak{M}_{Z_{NVB}}$.
Here, $\ominus$ denotes direct sum.

Proof

(1) Let $u_a,\ u_b \in U^{\mathfrak{M}_{M_{NVB}}}$ and $u_c \in U^{\mathfrak{M}_{P_{NVB}}}$

$NVB_{M_{NVB}}\left((u_a \oplus u_b) \oplus u_c\right) = NVB_{M_{NVB}}\left((u_a \oplus_1 u_b) \oplus u_c\right) = NVB_{M_{NVB}}(u_a \oplus_1 u_b)$

$NVB_{M_{NVB}}\left((u_b \oplus u_c) \oplus u_a\right) = NVB_{M_{NVB}}(u_b \oplus u_a) = NVB_{M_{NVB}}(u_a \oplus u_b) = NVB_{M_{NVB}}(u_a \oplus_1 u_b)$

(2) Let $u_a \in U^{\mathfrak{M}_{M_{NVB}}}$ and $0 \in U^{\mathfrak{M}_{P_{NVB}}}$. $NVB_{M_{NVB}}(u_a \oplus 0) = NVB_{M_{NVB}}(u_a)$

Similarly, all the axioms for a $BZMV^{dM}$– subalgebra can be verified.

Case (i) : $u_a,\ u_b \in U^{\mathfrak{M}_{M_{NVB}}}$, $\left(\forall\, u_a,\ u_b \in U^{\mathfrak{M}_{M_{NVB}}}\right)$

(1) $\left(\inf_{u_v \in (u_a \ominus u_b)} NVB_{M_{NVB}}(u_v)\right)$

$= \left(\inf_{u_v \in (u_a \oplus_1 u_b)} NVB_{M_{NVB}}(u_v)\right) \succcurlyeq r\min\left\{NVB_{M_{NVB}}(u_a), NVB_{M_{NVB}}(u_b)\right\}$

(2) $\left(\forall\, u_a \in U^{\mathfrak{M}_{M_{NVB}}}\right)$

(i) $NVB_{M_{NVB}}(u_a) \succcurlyeq NVB_{M_{NVB}}\left(\neg^{\circledast} u_a\right) \Rightarrow NVB_{M_{NVB}}(u_a) \succcurlyeq NVB_{M_{NVB}}(\neg_1 u_a)$

(ii) $NVB_{M_{NVB}}(u_a) \succcurlyeq NVB_{M_{NVB}}\left(\sim^{\circledast} u_a\right) \Rightarrow NVB_{M_{NVB}}(u_a) \succcurlyeq NVB_{M_{NVB}}(\sim_1 u_a)$

$\left[\text{Since, } \langle U^{\mathfrak{M}_{M_{NVB}}}, \oplus_1, \neg_1, \sim_1, 0, 1\rangle \text{ is a NVB } BZMV^{dM} \text{ – subalgebra}\right]$

Case (ii) : $u_a,\ u_b \in U^{\mathfrak{M}_{P_{NVB}}}$, $\left(\forall\, u_a,\ u_b \in U^{\mathfrak{M}_{P_{NVB}}}\right)$

(1) $\left(\inf_{u_v \in (u_a \ominus u_b)} NVB_{M_{NVB}}(u_v)\right)$

$= \left(\inf_{u_v \in (u_a \oplus_2 u_b)} NVB_{M_{NVB}}(u_v)\right) \succcurlyeq r\min\left\{NVB_{M_{NVB}}(u_a), NVB_{M_{NVB}}(u_b)\right\}$

(2) $\left(\forall\, u_a \in U^{\mathfrak{M}_{P_{NVB}}}\right)$

(i) $NVB_{M_{NVB}}(u_a) \succcurlyeq NVB_{M_{NVB}}\left(\neg^{\circledast} u_a\right) \Rightarrow NVB_{M_{NVB}}(u_a) \succcurlyeq NVB_{M_{NVB}}(\neg_2 u_a)$

(ii) $NVB_{M_{NVB}}(u_a) \succcurlyeq NVB_{M_{NVB}}\left(\sim^{\circledast} u_a\right) \Rightarrow NVB_{M_{NVB}}(u_a) \succcurlyeq NVB_{M_{NVB}}(\sim_2 u_a)$

$\left[\text{Since, } \langle U^{\mathfrak{M}_{P_{NVB}}}, \oplus_2, \neg_2, \sim_2, 0, 1\rangle \text{ is a NVB } BZMV^{dM}\text{– subalgebra}\right]$

Case (iii) : $\forall\ u_a \in U^{\mathfrak{M}_{M_{NVB}}},\ u_b \in U^{\mathfrak{M}_{P_{NVB}}}$ or $u_a \in U^{\mathfrak{M}_{P_{NVB}}},\ u_b \in U^{\mathfrak{M}_{M_{NVB}}}$

(1) $\left(\forall\ u_a \in U^{\mathfrak{M}_{M_{NVB}}},\ u_b \in U^{\ddagger} \text{ or } u_a \in U^{\mathfrak{M}_{P_{NVB}}},\ u_b \in U^{\mathfrak{M}_{M_{NVB}}}\right)$

$\left(\inf_{u_v \in (u_a \ominus u_b)} NVB_{M_{NVB}}(u_v)\right) = \left(\inf_{u_v \in (u_a)} NVB_{M_{NVB}}(v)\right) \succcurlyeq r\min\left\{NVB_{M_{NVB}}(u_a),\ NVB_{M_{NVB}}(b)\right\}$

Being a unary operation, 2nd axiom does not exists.

Clearly all the conditions for a NVB $\mathrm{BZMV^{dM}}$ – subalgebra is verified. It is clear that combining of two NVB $\mathrm{BZMV^{dM}}$– subalgebras, will produce the same.

7. Conclusions

Binary concept leads us to handle the situations with two universal sets which are found to be common in real-life. In this paper neutrosophic vague binary $\mathbf{BZMV^{dM}}$ Sub-algebra of $\mathbf{BZMV^{dM}}$- algebra is developed. This idea will provide a combined effect of the distributive Brouwer Zadeh lattice with Many –Valued or Multi - Valued algebra when stipulated into the de-Morgan's zone. Its basic-ideal with various sub kinds are also developed. Some theorems, properties, direct sum for this new concept are explored. This paper is an attempt to discuss with mixed patterns and an investigation towards its wide scope. It could be further extended to higher dimensions. Pre -$\mathbf{BZMV^{dM}}$ algebraic structure is a weaker structure than $\mathbf{BZMV^{dM}}$ – algebra. In some times, violation of theorems in stronger structure may come into a control mode in their relaxed forms. That also could be verified by comparing both these structures. $\mathbf{BZMV^{dM}}$ –algebra can be considered as a strong $\mathbf{MV}$- algebraic structure in de- Morgan's environment. Same way, one more strengthened $\mathbf{MV}$ – algebraic structure namely neutrosophic vague binary - $\mathbf{BZMV_{\Delta}}$ ($\mathbf{NVB - BZMV_{\Delta} - subalgebra}$) can also be developed. Here Δ is an additional unary operator in the context. It can be also extended towards the de – Morgan's atmosphere and can get a more tightened structure neutrosophic vague binary $\mathbf{BZMV_{\Delta}{}^{dM}}$ Sub-algebra of $\mathbf{BZMV_{\Delta}{}^{dM}}$ - algebra ($\mathbf{NVB\ BZMV_{\Delta}{}^{dM}\ Sub - algebra\ of\ BZMV_{\Delta}{}^{dM} - algebra}$). In future, its applications can be extended to a number of areas like geology, unmanned aerial vehicle, business analysis, chemistry, mechatronics, aerospace, biomedical etc and have to be discussed in detail. Numerous applications can be tried out in the field of working algorithms of vacuum cleaners, washing machines etc., and in stock trading, medical diagnosis and treatment plans, weather forecasting systems, 3D animations etc. Neutrosophic Vague Binary ideas with its logical $\mathbf{BZMV^{dM}}$ Sub - algebraic pattern can be hopefully developed towards this area and to its working algorithms to produce more accuracy in this digital world.

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On Neutrosophic Γ-Semirings

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Abstract. In this paper, we introduce and study the concept of Neutrosophic Γ-semiring and study various properties. Also, we prove that there is a one-to-one correspondence between Neutrosophic Γ-semirings and sub Γ-semirings of a Γ-semiring. Further, we prove that the set of all neutrosophic Γ-semirings is a De-Morgan algebra. Moreover, we establish that the homomorphic image and inverse image of a Neutrosophic Γ-semiring is also a Neutrosophic Γ-semiring.

Keywords: Γ-semiring, fuzzy set, neutrosophic set, neutrosophic Γ-semiring.

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1. **Introduction**

In 1965, Zadeh, L.A. [14] introduced the concept of fuzzy sets. In 1986, Atanassov, K. [4] proposed intuitionistic fuzzy set theory as an extension of the fuzzy set theory. Next, in 1998, Smarandache F. [13] introduced the notion of neutrosophic sets, which are a common generalization of fuzzy sets and intuitionistic sets.

Recently, Smarandache F. [11,12] defined the NeutroAlgebraic structures and AntiAlgebraic structures. Al-Tahan, M. et al. defined the neutrosophic quadruple H_v-rings, neutrosophic quadruple H_v-subrings, and neutrosophic quadruple homomorphism and studied their various properties [3]. Muzaffar, A. et al. summarized the previous work carried out in the field of neutrosophic logic, set, measure, and also classification techniques in neutrosophy and the relevant research work has been discussed and they investigated some various of applications in the field of neutrosophy [7]. Neutrosophic quadruple algebraic structures and hyperstructures are discussed in [1,2,6]. Further, Rezaei, A. et al. introduced the notions of neutrosemihypergroup and antisemihypergroup and investigated some of their properties [10].

In 1996, Rao, M.K. [8] introduced the concept of Γ-semiring as a generalization of semiring as well as Γ-ring (also see [9]). It is known that, the notion of Γ-semirings is an extension of the ternary semirings. Then Bhargavi, Y. et al. studied on fuzzy Γ-semirings and investigated some of their properties [5].

In this paper, we introduce and study the concept of neutrosophic Γ-semiring and study various properties. Further, we prove that the set of all neutrosophic Γ-semirings is a De-Morgan algebra. Also, we establish that the homomorphic image and inverse image of a neutrosophic Γ-semiring is also a neutrosophic Γ-semiring.

2. Preliminaries

We recall the basic notions and definitions regrading Γ-semirings used in the paper.

Definition 2.1. ([9]) Let E and Γ be two additive commutative semigroups. Then E is called Γ-semiring if there exists a mapping $E \times \Gamma \times E \to E$ image to be denoted by $e\alpha f$ if it satisfies the following conditions: for all $e, f, g \in E; \alpha, \beta \in \Gamma$.
(ΓSR1) $e\alpha(f+g) = e\alpha f + e\alpha g$,
(ΓSR2) $(e+f)\alpha g = e\alpha g + f\alpha g$,
(ΓSR3) $e(\alpha+\beta)f = e\alpha f + e\beta f$,
(ΓSR4) $e\alpha(f\beta g) = (e\alpha f)\beta g$.

Definition 2.2. ([9]) A nonempty subset φ of a Γ-semiring E is said to be a sub Γ-semiring of E if $(F,+)$ is a sub semigroup of $(E,+)$ and $e\alpha f \in F$, for all $e, f \in F; \alpha \in \Gamma$.

Definition 2.3. ([9]) Let E and φ be two Γ-semirings. Then $\varphi : E \to F$ is called a homomorphism if
1. $\varphi(e+f) = \varphi(e) + \varphi(f)$,
2. $\varphi(e\gamma f) = \varphi(e)\gamma\varphi(f)$, for all $e, f \in E; \gamma \in \Gamma$.

Definition 2.4. ([13]) Let E be a space of points (objects), with a generic element in E denoted by E. A neutrosophic set ψ in E is characterized by a truth-membership function $\psi_T(e)$, an indeterminacy-membership function $\psi_I(e)$ and a falsity-membership function $\psi_F(e)$. Then, a simple valued neutrosophic set A can be denoted by

$$\psi = \{< e, \psi_T(e), \psi_I(e), \psi_F(e) >: e \in E\},$$

where $\psi_T(e), \psi_I(e), \psi_F(e) \in [0,1]$ for each point E in E. Therefore, the sum of $\psi_T(e), \psi_I(e), \psi_F(e)$ satisfies the condition $0 \leq \psi_T(e) + \psi_I(e) + \psi_F(e) \leq 3$.

For convenience, simple valued neutrosophic set is abbreviated to neutrosophic set later.

Definition 2.5. ([13]) Let $\psi = (\psi_T, \psi_I, \psi_F)$ and $\phi = (\phi_T, \phi_I, \phi_F)$ be two neutrosophic sets of a universe of discourse E.

The complement of ψ is denoted by ψ^c or ψ' and is defined as

$$\psi^c_T(e) = \psi_F(e),\ \psi^c_I(e) = 1 - \psi_I(e),\ \psi^c_F(e) = \psi_T(e).$$

The intersection of ψ and ϕ is defined as $\psi \cap \phi = ((\psi \cap \phi)_T, (\psi \cap \phi)_I, (\psi \cap \phi)_F)$, where $(\psi \cap \phi)_T(e) = \min\{\psi_T(e), \phi_T(e)\}$, $(\psi \cap \phi)_I(e) = \max\{\psi_I(e), \phi_I(e)\}$ and $(\psi \cap \phi)_F(e) = \max\{\psi_F(e), \phi_F(e)\}$.

The union of ψ and ϕ is defined as $\psi \cup \phi = ((\psi \cup \phi)_T, (\psi \cup \phi)_I, (\psi \cup \phi)_F)$, where $(\psi \cup \phi)_T(e) = \max\{\psi_T(e), \phi_T(e)\}$, $(\psi \cup \phi)_I(e) = \min\{\psi_I(e), \phi_I(e)\}$ and $(\psi \cup \phi)_F(e) = \min\{\psi_F(e), \phi_F(e)\}$.

A neutrosophic set ψ is contained in another neutrosophic set ϕ, defined as follows:

$$\psi \subseteq \phi \text{ if and only if } \psi_T(e) \leq \phi_T(e),\ \psi_I(e) \geq \phi_I(e) \,\text{and}\, \psi_F(e) \geq \phi_F(e), \text{ for all}\, e \in E.$$

Definition 2.6. ([13]) Let $\psi = (\psi_T, \psi_I, \psi_F)$ be a neutrosophic set of a universe of discourse E. For $\alpha, \beta, \gamma \in [0, 1]$ with $0 \leq \alpha + \beta + \gamma \leq 3$, the (α, β, γ)- cut or neutrosophic cut of ψ is the crisp subset of E is given by

$$\psi_{(\alpha,\beta,\gamma)} = \{e \in E : \psi_T(e) \geq \alpha, \psi_I(e) \leq \beta, \psi_F(e) \leq \gamma\}.$$

Definition 2.7. ([13]) Let φ be a mapping from a set E into a set F. Let ψ be a neutrosophic set in E. Then the image $\varphi(\psi)$ of ψ is the neutrosophic set in F defined by:

$$(\varphi(\psi_T))(f) = \begin{cases} \sup\limits_{z \in \varphi^{-1}(f)} \psi_T(z) & \text{if } \varphi^{-1}(f) \neq \phi \\ 0 & otherwise \end{cases},$$

$$(\varphi(\psi_I))(f) = \begin{cases} \inf\limits_{z \in \varphi^{-1}(f)} \psi_I(z) & \text{if } \varphi^{-1}(f) \neq \phi \\ 1 & otherwise \end{cases}$$

and

$$(\varphi(\psi_F))(f) = \begin{cases} \inf\limits_{z \in \varphi^{-1}(f)} \psi_F(z) & \text{if } \varphi^{-1}(f) \neq \phi \\ 1 & otherwise \end{cases},$$

for all $f \in F$, where $\varphi^{-1}(f) = \{e : \varphi(e) = f\}$.

Let ϕ be a neutrosophic set in F. Then the inverse image of $\varphi^{-1}(\phi)$ of ϕ is the neutrosophic set in E by $\varphi^{-1}(\phi)(e) = \phi(\varphi(e))$, for all $e \in E$.

Definition 2.8. ([5]) A fuzzy set μ in a Γ-semiring E is called fuzzy Γ-semiring if it satisfies the following properties: for all $e, f \in E; \gamma \in \Gamma$

$(FI1)$ $\mu(e + f) \geq \min\{\mu(e), \mu(f)\}$,

$(FI2)$ $\mu(e\gamma f) \geq \min\{\mu(e), \mu(f)\}$.

3. On Neutrosophic Γ-semirings

This section presents some important properties of neutrosophic Γ-semirings and characterize neutrosophic Γ-semirings to the crisp Γ-semirings, and we prove that the set of all neutrosophic Γ-semirings is a De-Morgan algebra.

Throughout this section E stands for a Γ-semiring unless otherwise mentioned.
Now, we introduce the following.

Definition 3.1. A neutrosophic set $A = (\psi_T, \psi_I, \psi_F)$ in a Γ-semiring E is called a Neutrosophic Γ-semiring if it satisfies the following properties: for all $e, f \in E; \gamma \in \Gamma$
$(N\Gamma SR1)$ $\psi_T(e+f) \geq \min\{\psi_T(e), \psi_T(f)\}$,
$(N\Gamma SR2)$ $\psi_I(e+f) \leq \max\{\psi_I(e), \psi_I(f)\}$,
$(N\Gamma SR3)$ $\psi_F(e+f) \leq \max\{\psi_F(e), \psi_F(f)\}$,
$(N\Gamma SR4)$ $\psi_T(e\gamma f) \geq \min\{\psi_T(e), \psi_T(f)\}$,
$(N\Gamma SR5)$ $\psi_I(e\gamma f) \leq \max\{\psi_I(e), \psi_I(f)\}$,
$(N\Gamma SR6)$ $\psi_F(e\gamma f) \leq \max\{\psi_F(e), \psi_F(f)\}$.

Example 3.2: Let E be the set of negative integers and Γ be the set of negative even integers. Then E, Γ are additive commutative semigroups. Define the mapping $E \times \Gamma \times E \to E$ by $e\alpha f$ usual product of e, α, f, for all $e, f \in E$; $\alpha \in \Gamma$. Then E is a Γ-semiring. Let $\psi = (\psi_T, \psi_I, \psi_F)$, where $\psi_T : E \to [0, 1]$, $\psi_I : E \to [0, 1]$ and $\psi_F : E \to [0, 1]$ defined by:

$$\psi_T(e) = \begin{cases} 0.6 & \text{if } \ e = -1 \\ 0.7 & \text{if } \ e = -2 \\ 0.9 & \text{if } \ e < -2 \end{cases},$$

$$\psi_I(e) = \begin{cases} 0.5 & \text{if } \ e = -1 \\ 0.3 & \text{if } \ e = -2 \\ 0.2 & \text{if } \ e < -2 \end{cases}$$

and

$$\psi_F(e) = \begin{cases} 0.4 & \text{if } \ e = -1 \\ 0.2 & \text{if } \ e = -2 \\ 0.1 & \text{if } \ e < -2 \end{cases}.$$

Thus ψ is a Neutrosophic Γ-semiring of E.

Example 3.2. Let E be the set of real numbers and Γ be the set of positive numbers. Then E, Γ are additive commutative semigroups. Define the mapping $E \times \Gamma \times E \to E$ by $e\alpha f$ usual product of e, α, f, for all $e, f \in E; \alpha \in \Gamma$. Then E is a Γ-semiring. Let $\psi = (\psi_T, \psi_I, \psi_F)$, where

$\psi_T : E \to [0,1]$, $\psi_I : E \to [0,1]$ and $\psi_F : E \to [0,1]$ defined by:

$$\psi_T(e) = \begin{cases} 0.9 & \text{if } \ e = 0 \\ 0.7 & \text{if } \ e \ is \ positive \\ 0.6 & \text{if } \ e \ is \ negative \end{cases},$$

$$\psi_I(e) = \begin{cases} 0.2 & \text{if } \ e = 0 \\ 0.3 & \text{if } \ e \ is \ positive \\ 0.5 & \text{if } \ e \ is \ negative \end{cases}$$

and

$$\psi_F(e) = \begin{cases} 0.1 & \text{if } \ e = 0 \\ 0.2 & \text{if } \ e \ is \ positive \\ 0.4 & \text{if } \ e \ is \ negative \end{cases}.$$

Thus ψ is a Neutrosophic Γ-semiring of E.

Theorem 3.3. *A neutrosophic set $\psi = (\psi_T, \psi_I, \psi_F)$ is a neutrosophic Γ-semiring of E if and only if ψ_T, $1 - \psi_I$ and $1 - \psi_F$ are fuzzy Γ-semirings of E.*

Proof. Suppose $\psi = (\psi_T, \psi_I, \psi_F)$ is a neutrosophic Γ-semiring of E. Let $e, f \in E$; $\gamma \in \Gamma$.Then
(i) $\psi_T(e + f) \geq \min\{\psi_T(e), \psi_T(f)\}$,
(ii) $\psi_I(e + f) \leq \max\{\psi_I(e), \psi_I(f)\}$, i.e., $1 - \psi_I(e + f) \geq \min\{1 - \psi_I(e), 1 - \psi_I(f)\}$,
(iii) $\psi_F(e + f) \leq \max\{\psi_F(e), \psi_F(f)\}$, i.e., $1 - \psi_F(e + f) \geq \min\{1 - \psi_F(e), 1 - \psi_F(f)\}$,
(iv) $\psi_T(e\gamma f) \geq \min\{\psi_T(e), \psi_T(f)\}$,
(v) $\psi_I(e\gamma f) \leq \max\{\psi_I(e), \psi_I(f)\}$, i.e., $1 - \psi_I(e\gamma f) \geq \min\{1 - \psi_I(e), 1 - \psi_I(f)\}$,
(vi) $\psi_F(e\gamma f) \leq \max\{\psi_F(e), \psi_F(f)\}$, i.e., $1 - \psi_F(e\gamma f) \geq \min\{1 - \psi_F(e), 1 - \psi_F(f)\}$.
Thus, ψ_T, $1 - \psi_I$ and $1 - \psi_F$ are fuzzy Γ-semiring of E. The converse part is obvious from the definition. $\square$

Theorem 3.4. *A neutrosophic set psi $= (\psi_T, \psi_I, \psi_F)$ of E is neutrosophic Γ- semiring of E if and only if for all $\alpha, \beta, \gamma \in [0,1]$, the (α, β, γ)-cut $\psi_{(\alpha,\beta,\gamma)}$ is a sub Γ-semiring of E.*

Proof. Suppose $\psi = (\psi_T, \psi_I, \psi_F)$ of E is a neutrosophic Γ-semiring. Let $e, f \in \psi_{(\alpha,\beta,\gamma)}$; $\eta \in \Gamma$. Then $\psi_T(e), \psi_T(f) \geq \alpha$, $\psi_I(e), \psi_I(f) \leq \beta$, $\psi_F(e), \psi_F(f) \leq \gamma$. Since A is neutrosophic Γ-semiring, we have:
(i) $\psi_T(e + f) \geq \min\{\psi_T(e), \psi_T(f)\} \geq \alpha$,
(ii) $\psi_I(e + f) \leq \max\{\psi_I(e), \psi_I(f)\} \leq \beta$,
(iii) $\psi_F(e + f) \leq \max\{\psi_F(e), \psi_F(f)\} \leq \gamma$,
which implies $e + f \in \psi_{(\alpha,\beta,\gamma)}$.
Also, since
(iv) $\psi_T(e\eta f) \geq \min\{\psi_T(e), \psi_T(f)\} \geq \alpha$,

(v) $\psi_I(e\eta f) \leq \max\{\psi_I(e), \psi_I(f)\} \leq \beta$,
(vi) $\psi_F(e\eta f) \leq \max\{\psi_F(e), \psi_F(f)\} \leq \gamma$,
which implies $e\eta f \in \psi_{(\alpha,\beta,\gamma)}$.
Thus, $\psi_{(\alpha,\beta,\gamma)}$ is a sub Γ-semiring of E.

Conversely, suppose $\psi_{(\alpha,\beta,\gamma)}$ is a sub Γ-semiring of E. Let $e, f \in E$; $\eta \in \Gamma$. Let $\psi_T(e) > \alpha 1,\ \psi_I(e) < \beta\psi_1,\ \psi_F(e) < \gamma\psi_1$ and $\psi_T(f) > \alpha 2,\ \psi_I(f) < \beta\psi_2,\ \psi_F(f) < \gamma_2$.
Put $\alpha = \min\{\alpha 1, \alpha 2\},\ \beta = \max\{\beta\psi_1, \beta\psi_2\}$ *and* $\gamma = \max\{\gamma\psi_1, \gamma\psi_2\}$. Then $e, f \in \psi_{(\alpha,\beta,\gamma)}$, and so $e + f \in \psi_{(\alpha,\beta,\gamma)}$ and $e\eta f \in \psi_{(\alpha,\beta,\gamma)}$. Hence $\psi_T(e + f) \geq \alpha = \min\{\psi_T(e), \psi_T(f)\}$, $\psi_I(e+f) \leq \beta = \max\{\psi_I(e), \psi_I(f)\}$, $\psi_F(e+f) \leq \gamma = \max\{\psi_F(e), \psi_F(f)\}$ and $\psi_T(e\eta f) \geq \alpha = \min\{\psi_T(e), \psi_T(f)\}$, $\psi_I(e\eta f) \leq \beta = \max\{\psi_I(e), \psi_I(f)\}$, $\psi_F(e\eta f) \leq \gamma = \max\{\psi_F(e), \psi_F(f)\}$.
Thus, ψ is a neutrosophic Γ-semiring of E. □

Theorem 3.5. *Let $\psi = (\psi_T, \psi_I, \psi_F)$ be a neutrosophic set of E. The two neutrosophic cuts $\psi_{(\alpha_1,\beta_1,\gamma_1)}$ and $\psi_{(\alpha_2,\beta_2,\gamma_2)}$ of E are equal, where $\alpha_1,\ \alpha_2,\ \alpha_3,\ \beta_1,\ \beta_2,\ \beta_3,\ \gamma_1,\ \gamma_2,\ \gamma_3 \in [0, 1]$ with $\alpha_1 < \alpha_2$, $\beta_1 > \beta_2$, $\gamma_1 > \gamma_2$ if and only if there is no $e \in E$ such that $\alpha_1 \leq \psi_T(e) < \alpha_2,\ \beta_1 \geq \psi_I(e) > \beta_2,\ \gamma_1 \geq \psi_F(e) > \gamma_2$.*

Proof. Suppose $\psi_{(\alpha 1,\beta_1,\gamma_1)}$ and $\psi_{(\alpha_2,\beta_2,\gamma_2)}$ of E are equal. Suppose if possible there exists $e \in E$ such that $\alpha_1 \leq \psi_T(e) < \alpha_2,\ \beta_1 \geq \psi_I(e) > \beta_2,\ \gamma_1 \geq \psi_F(e) > \gamma_2$. Then $e \in \psi_{(\alpha_1,\beta_1,\gamma_1)} = \psi_{(\alpha_2,\beta_2,\gamma_2)}$, and so $\psi_T(e) \geq \alpha_2,\ \psi_I(e) \leq \beta_2,\ \geq \psi_F(e) \leq \gamma_2$. Which is a contradiction. Hence there exists no $e \in E$ such that $\alpha_1 \leq \psi_T(e) < \alpha_2,\ \beta_1 \geq \psi_I(e) > \beta_2,\ \gamma_1 \geq \psi_F(e) > \gamma_2$.

Conversely, suppose that there exists no $e \in E$ such that $\alpha_1 \leq \psi_T(e) < \alpha_2,\ \beta_1 \geq \psi_I(e) > \beta_2,\ \gamma_1 \geq \psi_F(e) > \gamma_2$. Suppose if possible $\psi_{(\alpha_1,\beta_1,\gamma_1)} \neq \psi_{(\alpha_2,\beta_2,\gamma_2)}$. Then there exists $e \in \psi_{(\alpha_1,\beta_1,\gamma_1)}$ and $e \notin \psi_{(\alpha_2,\beta_2,\gamma_2)}$, i.e., $\psi_T(e) \geq \alpha_1,\ \psi_I(e) \leq \beta_1,\ \psi_F(e) \leq \gamma_1$ and $\psi_T(e) < \alpha_2,\ \psi_I(e) > \beta_2,\ \psi_F(e) > \gamma_2$. So, there exists $e \in E$ such that $\alpha_1 \leq \psi_T(e) < \alpha_2,\ \beta_1 \geq \psi_I(e) > \beta_2,\ \gamma_1 \geq \psi_F(e) > \gamma_2$. Which is a contradiction. Thus, $\psi_{(\alpha_1,\beta_1,\gamma_1)} = \psi_{(\alpha_2,\beta_2,\gamma_2)}$. □

Theorem 3.6. *If $\psi = (\psi_T, \psi_I, \psi_F)$ and $\phi = (\phi_T, \phi_I, \phi_F)$ are two neutrosophic Γ-semirings of E, then $\psi \cap \phi$ is a neutrosophic Γ-semiring of E.*

Proof. Let $e, f \in E$; $\eta \in \Gamma$. Then

$$\begin{aligned}
(\psi \cap \phi)_T(e + f) &= \min\{\psi_T(e + f), \phi_T(e + f)\} \\
&\geq \min\{\min\{\psi_T(e), \psi_T(f)\}, \min\{\phi_T(e), \phi_T(f)\}\} \\
&\geq \min\{\min\{\psi_T(e), \phi_T(e)\}, \min\{\psi_T(f), \phi_T(f)\}\} \\
&= \min\{(\psi \cap \phi)_T(e), (\psi \cap \phi)_T(f)\},
\end{aligned}$$

$$\begin{aligned}(\psi\cap\phi)_I(e+f)&=\max\{\psi_I(e+f),\phi_I(e+f)\}\\&\leq\max\{\max\{\psi_I(e),\psi_I(f)\},\max\{\phi_I(e),\phi_I(f)\}\}\\&\leq\max\{\max\{\psi_I(e),\phi_I(e)\},\max\{\psi_I(f),\phi_I(f)\}\}\\&=\max\{(\psi\cap\phi)_I(e),(\psi\cap\phi)_I(f)\}\end{aligned}$$

and

$$\begin{aligned}(\psi\cap\phi)_F(e+f)&=\max\{\psi_F(e+f),\phi_F(e+f)\}\\&\leq\max\{\max\{\psi_F(e),\psi_F(f)\},\max\{\phi_F(e),\phi_F(f)\}\}\\&\leq\max\{\max\{\psi_F(e),\phi_F(e)\},\max\{\psi_F(f),\phi_F(f)\}\}\\&=\max\{(\psi\cap\phi)_F(e),(\psi\cap\phi)_F(f)\}.\end{aligned}$$

Also, we get

$$\begin{aligned}(\psi\cap\phi)_T(e\eta f)&=\min\{\psi_T(e\eta f),\phi_T(e\eta f)\}\\&\geq\min\{\min\{\psi_T(e),\psi_T(f)\},\min\{\phi_T(e),\phi_T(f)\}\}\\&\geq\min\{\min\{\psi_T(e),\phi_T(e)\},\min\{\psi_T(f),\phi_T(f)\}\}\\&=\min\{(\psi\cap\phi)_T(e),(\psi\cap\phi)_T(f)\},\end{aligned}$$

$$\begin{aligned}(\psi\cap\phi)_I(e\eta f)&=\max\{\psi_I(e\eta f),\phi_I(e\eta f)\}\\&\leq\max\{\max\{\psi_I(e),\psi_I(f)\},\max\{\phi_I(e),\phi_I(f)\}\}\\&\leq\max\{\max\{\psi_I(e),\phi_I(e)\},\max\{\psi_I(f),\phi_I(f)\}\}\\&=\max\{(\psi\cap\phi)_I(e),(\psi\cap\phi)_I(f)\}\end{aligned}$$

and

$$\begin{aligned}(\psi\cap\phi)_F(e\eta f)&=\max\{\psi_F(e\eta f),\phi_F(e\eta f)\}\\&\leq\max\{\max\{\psi_F(e),\psi_F(f)\},\max\{\phi_F(e),\phi_F(f)\}\}\\&\leq\max\{\max\{\psi_F(e),\phi_F(e)\},\max\{\psi_F(f),\phi_F(f)\}\}\\&=\max\{(\psi\cap\phi)_F(e),(\psi\cap\phi)_F(f)\}.\end{aligned}$$

Thus, $\psi\cap\phi$ is a neutrosophic Γ-semiring of E. □

Corollary 3.7. *The intersection of arbitrary family of neutrosophic Γ-semirings is a neutrosophic Γ-semiring.*

The following example shows that the union of two neutrosophic Γ-semirings may not be a neutrosophic Γ-semiring, in general.

Example 3.8. consider the additive abelian group $Z_4 = \{0, 1, 2, 3\}$ and the subgroup $\Gamma = \{0, 2\}$. Define $Z_4 \times \Gamma \times Z_4 \to Z_4$ by $e\alpha f$ usual product of $e, \alpha, f, \ \forall\ e, f \in\ Z_4;\ \alpha \in \Gamma$.
Then Z_4 is a Γ-semiring.
Let $\psi = (\psi_T, \psi_I, \psi_F)$, where $\psi_T : Z_4 \to [0, 1]$, $\psi_I : Z_4 \to [0, 1]$ and $\psi_F : Z_4 \to [0, 1]$ defined by:

$$\psi_T(e) = \begin{cases} 0.8 & \text{if } \ e = 0; \\ 0.6 & \text{if } \ e = 1; \\ 0.4 & \text{otherwise} \end{cases}$$

$$\psi_I(e) = \begin{cases} 0.2 & \text{if } \ e = 0; \\ 0.4 & \text{if } \ e = 1; \\ 0.5 & \text{otherwise} \end{cases}$$

$$\psi_F(e) = \begin{cases} 0.2 & \text{if } \ e = 0; \\ 0.3 & \text{if } \ e = 1; \\ 0.5 & \text{otherwise} \end{cases}$$

Let $\phi = (\phi_T, \phi_I, \phi_F)$, where $\phi_T : Z_4 \to [0, 1]$, $\phi_I : Z_4 \to [0, 1]$ and $\phi_F : Z_4 \to [0, 1]$ defined by:

$$\phi_T(e) = \begin{cases} 0.6 & \text{if } \ e = 0; \\ 0.5 & \text{if } \ e = 2; \\ 0.2 & \text{otherwise} \end{cases}$$

$$\phi_F(e) = \begin{cases} 0.2 & \text{if } \ e = 0; \\ 0.3 & \text{if } \ e = 2; \\ 0.4 & \text{otherwise} \end{cases}$$

$$\phi_F(e) = \begin{cases} 0.3 & \text{if } \ e = 0; \\ 0.4 & \text{if } \ e = 2; \\ 0.5 & \text{otherwise} \end{cases}$$

Thus, ψ and ϕ are neutrosophic Γ-semirings of Z_4, but $\psi \cup \phi$ is not a neutrosophic Γ-semiring of Z_4.

In particular we have the following:

Theorem 3.9. *If $\psi = (\psi_T, \psi_I, \psi_F)$ and $\phi = (\phi_T, \phi_I, \phi_F)$ are two neutrosophic Γ-semirings of E, then $\psi \cup \phi$ is a neutrosophic Γ-semiring of E only if $\psi \subseteq \phi$ or $\phi \subseteq \psi$.*

Proof. Assume that $e, f \in E$; $\eta \in \Gamma$. Suppose $A \subseteq B$. Then

$$\begin{aligned}(\psi \cup \phi)_T(e+f) &= \max\{\psi_T(e+f), \phi_T(e+f)\}\\ &= \phi_T(e+f)\\ &\geq \min\{\phi_T(e), \phi_T(f)\}\\ &= \min\{\max\{\psi_T(e), \phi_T(e)\}, \max\{\psi_T(f), \phi_T(f)\}\}\\ &= \min\{(\psi \cup \phi)_T(e), (\psi \cup \phi)_T(f)\},\end{aligned}$$

$$\begin{aligned}(\psi \cup \phi)_I(e+f) &= \min\{\psi_I(e+f), \phi_I(e+f)\}\\ &= \phi_I(e+f)\\ &\leq \max\{\phi_I(e), \phi_I(f)\}\\ &= \max\{\min\{\psi_I(e), \phi_I(e)\}, \min\{\psi_I(f), \phi_I(f)\}\}\\ &= \max\{(\psi \cup \phi)_I(e), (\psi \cup \phi)_I(f)\}\end{aligned}$$

and

$$\begin{aligned}(\psi \cup \phi)_F(e+f) &= \min\{\psi_F(e+f), \phi_F(e+f)\}\\ &= \phi_F(e+f)\\ &\leq \max\{\phi_F(e), \phi_F(f)\}\\ &= \max\{\min\{\psi_F(e), \phi_F(e)\}, \min\{\psi_F(f), \phi_F(f)\}\}\\ &= \max\{(\psi \cup \phi)_F(e), (\psi \cup \phi)_F(f)\}.\end{aligned}$$

Also, we have

$$\begin{aligned}(\psi \cup \phi)_T(e\eta f) &= \max\{\psi_T(e\eta f), \phi_T(e\eta f)\}\\ &= \phi_T(e\eta f)\\ &\geq \min\{\phi_T(e), \phi_T(f)\}\\ &= \min\{\max\{\psi_T(e), \phi_T(e)\}, \max\{\psi_T(f), \phi_T(f)\}\}\\ &= \min\{(\psi \cup \phi)_T(e), (\psi \cup \phi)_T(f)\},\end{aligned}$$

$$\begin{aligned}(\psi \cup \phi)_I(x\eta y) &= \min\{\psi_I(x\eta y), \phi_I(x\eta y)\}\\ &= \phi_I(e+f)\\ &\leq \max\{\phi_I(e), \phi_I(f)\}\\ &= \max\{\min\{\psi_I(e), \phi_I(e)\}, \min\{\psi_I(f), \phi_I(f)\}\}\\ &= \max\{(\psi \cup \phi)_I(e), (\psi \cup \phi)_I(f)\}\end{aligned}$$

and

$$
\begin{aligned}
(\psi \cup \phi)_F(e\eta f) &= \min\{\psi_F(e\eta f), \phi_F(e\eta f)\} \\
&= \phi_F(e\eta f) \\
&\leq \max\{\phi_F(e), \phi_F(f)\} \\
&\leq \max\{\min\{\psi_F(e), \phi_F(e)\}, \min\{\psi_F(f), \phi_F(f)\}\} \\
&= \max\{(\psi \cup \phi)_F(e), (\psi \cup \phi)_F(f)\}.
\end{aligned}
$$

Similarly, we can prove if $\phi \subseteq \psi$. Thus, $\psi \cup \phi$ is a neutrosophic Γ-semiring of E. □

Lemma 3.10. *Let $A(E)$ be the set of all neutrosophic Γ-semirings of E. Then $(A(E), \subseteq)$ is a poset.*

Proof. Let $A, B, C \in A(E)$.
1. Always $A \subseteq A$, for all $A \in A(E)$. So, $\subseteq$ is reflexive.
2. Let $A \subseteq B$ and $B \subseteq A$
$\Rightarrow A = B$.
So, $\subseteq$ is anti symmetric.
3.Let $A \subseteq B$ and $B \subseteq C$
$\Rightarrow A \subseteq C$.
So, $\subseteq$ is transitive.
Thus $\subseteq$ is partial ordering and hence $(A(E), \subseteq)$ is a poset. □

Theorem 3.11. $(A(E), \cup, \cap, ', 0, 1)$ *is a De-Morgan Algebra.*

Proof. We will show that
1. $(A(E), \cup, \cap, ', 0, 1)$ is a bounded distributive lattice
2. $(\psi')' = \psi$, $(\psi \cup \phi)' = \psi' \cap \phi'$ and $(\psi \cap \phi)' = \psi' \cup \psi'$, for all $\psi, \phi \in A(E)$.
Let $\psi = (\psi_T, \psi_I, \psi_F)$, $\phi = (\phi_T, \phi_I, \phi_F)$, $\sigma = (\sigma T, \sigma I, \sigma F) \in A(E)$.

1. Since $0 \leq \psi_T(e) \leq 1$, $0 \leq \psi_I(e) \leq 1$ and $0 \leq \psi_F(e) \leq 1$, for all $x \in R$. So, $A(E)$ is bounded.

<u>Idempotency</u>:

$$
\psi \cap \psi = (\psi_T, \psi_I, \psi_F) \cap (\psi_T, \psi_I, \psi_F) = (\psi_T, \psi_I, \psi_F) = \psi,
$$
$$
\psi \cup \psi = (\psi_T, \psi_I, \psi_F) \cup (\psi_T, \psi_I, \psi_F) = (\psi_T, \psi_I, \psi_F) = \psi.
$$

Commutativity:

$$\begin{aligned}
\psi \cap \phi &= (\psi_T, \psi_I, \psi_F) \cap (\phi_T, \phi_I, \phi_F)\\
&= (\min\{\psi_T, \phi_T\}, \max\{\psi_I, \phi_I\}, \max\{\psi_F, \phi_F\})\\
&= (\min\{\phi_T, \psi_T\}, \max\{\phi_I, \psi_I\}, \max\{\phi_F, \psi_F\})\\
&= (\phi_T, \phi_I, \phi_F) \cap (\psi_T, \psi_I, \psi_F)\\
&= \phi \cap \psi,\\
\psi \cup \phi &= (\psi_T, \psi_I, \psi_F) \cup (\phi_T, \phi_I, \phi_F)\\
&= (\max\{\psi_T, \phi_T\}, \min\{\psi_I, \phi_I\}, \min\{\psi_F, \phi_F\})\\
&= (\max\{\phi_T, \psi_T\}, \min\{\phi_I, \psi_I\}, \min\{\phi_F, \psi_F\})\\
&= (\phi_T, \phi_I, \phi_F) \cup (\psi_T, \psi_I, \psi_F)\\
&= \phi \cup \psi.
\end{aligned}$$

Associativity:

$$\begin{aligned}
\psi \cap (\phi \cap \sigma) &= (\psi_T, \psi_I, \psi_F) \cap ((\phi_T, \phi_I, \phi_F) \cap (\sigma_T, \sigma_I, \sigma_F))\\
&= (\min\{\psi_T, \min\{\phi_T, \sigma_T\}\}, \max\{\psi_I, \max\{\phi_I, \sigma_I\}\}, \max\{\psi_F, \max\{\phi_F, \sigma_F\}\})\\
&= (\min\{\min\{\psi_T, \phi_T\}, \sigma_T\}, \max\{\max\{\psi_I, \phi_I\}, \sigma_I\}, \max\{\max\{\psi_F, \phi_F\}, \sigma_F\})\\
&= (\psi \cap \phi) \cap \sigma,\\
\psi \cup (\phi \cup \sigma) &= (\psi_T, \psi_I, \psi_F) \cup ((\phi_T, \phi_I, \phi_F) \cup (\sigma_T, \sigma_I, \sigma_F))\\
&= (\max\{\psi_T, \max\{\phi_T, \sigma_T\}\}, \min\{\psi_I, \min\{\phi_I, \sigma_I\}\}, \min\{\psi_F, \min\{\phi_F, \sigma_F\}\})\\
&= (\max\{\max\{\psi_T, \phi_T\}, \sigma_T\}, \min\{\min\{\psi_I, \phi_I\}, \sigma_I\}, \min\{\min\{\psi_F, \phi_F\}, \sigma_F\})\\
&= (\psi \cup \phi) \cup \sigma.
\end{aligned}$$

Absorption:

$$\begin{aligned}
\psi \cap (\psi \cup \phi) &= (\min\{\psi_T, \max\{\psi_T, \phi_T\}\}, \max\{\psi_I, \min\{\psi_I, \phi_I\}\}, \max\{\psi_F, \min\{\psi_F, \phi_F\}\})\\
&= (\psi_T, \psi_I, \psi_F),\\
&= \psi,\\
\psi \cup (\psi \cap \phi) &= (\max\{\psi_T, \min\{\psi_T, \phi_T\}\}, \min\{\psi_I, \max\{\psi_I, \phi_I\}\}, \min\{\psi_F, \max\{\psi_F, \phi_F\}\})\\
&= (\psi_T, \psi_I, \psi_F)\\
&= \psi.
\end{aligned}$$

<u>**Distributivity**</u>:

$$\begin{aligned}
\psi \cap (\phi \cup \sigma) &= (\min\{\psi_T, \max\{\phi_T, \sigma_T\}\}, \max\{\psi_I, \min\{\phi_I, \sigma_I\}\}, \max\{\psi_F, \min\{\phi_F, \sigma_F\}\}) \\
&= (\max\{\min\{\psi_T, \phi_T\}, \min\{\psi_T, \sigma_T\}\}, \min\{\max\{\psi_I, \phi_I\}, \max\{\psi_I, \sigma_I\}\}, \\
&\quad \min\{\max\{\psi_F, \phi_F\}, \max\{\psi_F, \sigma_F\}\}) \\
&= (\psi \cap \phi) \cup (\psi \cap \sigma), \\
\psi \cup (\phi \cap \sigma) &= (\max\{\psi_T, \min\{\phi_T, \sigma_T\}\}, \min\{\psi_I, \max\{\phi_I, \sigma_I\}\}, \min\{\psi_F, \max\{\phi_F, \sigma_F\}\}) \\
&= (\min\{\max\{\psi_T, \phi_T\}, \max\{\psi_T, \sigma_T\}\}, \max\{\min\{\psi_I, \phi_I\}, \min\{\psi_I, \sigma_I\}\}, \\
&\quad \max\{\min\{\psi_F, \phi_F\}, \min\{\psi_F, \sigma_F\}\}) \\
&= (\psi \cup \phi) \cap (\psi \cup \sigma).
\end{aligned}$$

Thus, $(A(E), \cup, \cap, ', 0, 1)$ is a bounded distributive lattice.

2. Now, we show that $(\psi')' = \psi$, $(\psi \cup \phi)' = \psi' \cap \phi'$ and $(\psi \cap \phi)' = \psi' \cup \phi'$.

$$\begin{aligned}
(\psi \cap \phi)' &= (\min\{\psi_T, \phi_T\}, \max\{\psi_I, \phi_I\}, \max\{\psi_F, \phi_F\})' \\
&= (\max\{\psi_F, \phi_F\}, \min\{1 - \psi_I, 1 - \phi_I\}, \min\{\psi_T, \phi_T\}) \\
&= \psi' \cup \phi'.
\end{aligned}$$

Therefore $(\psi \cap \phi)' = \psi' \cup \phi'$. Similarly, we can show that $(\psi \cup \phi)' = \psi' \cap \phi'$. Also, we have $\psi' = (\psi_F, 1 - \psi_I, \psi_T)$, and so $(\psi')' = \psi$. Thus, $(A(E), \cup, \cap, ', 0, 1)$ is a De-Morgan algebra. □

4. Hommorphic image and Pre-image of Neutrosophic Γ-semirings

Theorem 4.1. *Let φ be a homomorphism from a Γ-semiring E onto a Γ-semiring F and let ϕ be a neutrosophic Γ-semiring of F. Then the pre-image $\varphi^{-1}(\phi)$ of ϕ is a neutrosophic Γ-semiring of E.*

Proof. Assume that $e, f \in E$; $\eta \in \Gamma$. Then

$$\begin{aligned}
(\varphi^{-1}(\phi_T))(e + f) &= \phi_T(\varphi(e + f)) \\
&= \phi_T(\varphi(e) + \varphi(f)) \\
&\geq \min\{\phi_T(\varphi(e)), \phi_T(\varphi(f))\} \\
&= \min\{\varphi^{-1}(\phi_T)(e), \varphi^{-1}(\phi_T)(f)\},
\end{aligned}$$

$$\begin{aligned}(\varphi^{-1}(\phi_I))(e+f) &= \phi_I(\varphi(e+f))\\ &= \phi_I(\varphi(e)+\varphi(f))\\ &\leq \max\{\phi_I(\varphi(e)),\phi_I(\varphi(f))\}\\ &= \max\{\varphi^{-1}(\phi_I)(e),\varphi^{-1}(\phi_I)(f)\},\end{aligned}$$

$$\begin{aligned}(\varphi^{-1}(\phi_F))(e+f) &= \phi_F(\varphi(e+f))\\ &= \phi_F(\varphi(e)+\varphi(f))\\ &\leq \max\{\phi_F(\varphi(e)),\phi_F(\varphi(f))\}\\ &= \max\{\varphi^{-1}(\phi_F)(e),\varphi^{-1}(\phi_F)(f)\}\end{aligned}$$

and

$$\begin{aligned}(\varphi^{-1}(\phi_T))(x\eta y) &= \phi_T(\varphi(e\eta f))\\ &= \phi_T(\varphi(e)\eta\varphi(f))\\ &\geq \min\{\phi_T(f(e)),\phi_T(f(f))\}\\ &= \min\{(\varphi^{-1}\phi_T(e),(\varphi^{-1}(\phi_T)(f)\},\end{aligned}$$

$$\begin{aligned}(\varphi^{-1}(\phi_I))(e\eta f) &= \phi_I(\varphi(e\eta f))\\ &= \phi_I(\varphi(e)\eta\varphi(f))\\ &\leq \max\{\phi_I(\varphi(e)),\phi_I(\varphi(f))\}\\ &= \max\{\varphi^{-1}(\phi_I)(e),\varphi^{-1}(\phi_I)(f)\},\end{aligned}$$

$$\begin{aligned}(\varphi^{-1}(\phi_F))(e\eta f) &= \phi_F(\varphi(e\eta f))\\ &= \phi_F(\varphi(e)\eta\varphi(f))\\ &\leq \max\{\phi_F(\varphi(e)),\phi_F(\varphi(f))\}\\ &= \max\{\varphi^{-1}(\phi_F)(e),\varphi^{-1}(\phi_F)(f)\}.\end{aligned}$$

Thus, $\varphi^{-1}(\phi)$ is a neutrosophic Γ-semiring of E. □

Theorem 4.2. *Let φ be a homomorphism from a Γ-semiring E onto a Γ-semiring F. Let ψ be a neutrosophic Γ-semiring of E. Then the homomorphic image $\varphi(\psi)$ of ψ is a neutrosophic Γ-semiring of F.*

Proof. Let $p, q \in F$; $\gamma \in \Gamma$. If either $\varphi^{-1}(p)$ or $\varphi^{-1}(q)$ is empty, then the result is trivially satisfied.

Suppose $\varphi^{-1}(p)$ and $\varphi^{-1}(q)$ are non-empty. Since $p, q \in F$, then there exist $e, f \in E$ such that $e = f(p), f = f(q)$. Then

$$\begin{aligned}(\varphi(\psi_T))(p+q) &= \sup_{z\in\varphi^{-1}(p+q)} \psi_T(z)\\ &= \sup\{\psi_T(e+f) : e, f \in E, e = \varphi(p), f = \varphi(q)\}\\ &\geq \sup\{\min\{\psi_T(e), \psi_T(f)\} :\ e, f \in E, e = \varphi(p), f = \varphi(q)\}\\ &= \min\{\sup\{\psi_T(e) : e \in E, x = \varphi(p)\}, \sup\{\psi_T(f) : f \in E, f = \varphi(q)\}\\ &= \min\{(\varphi(\psi_T))(p), (\varphi(\psi_T))(q)\},\end{aligned}$$

$$\begin{aligned}(\varphi(\psi_I))(p+q) &= \inf_{z\in\varphi^{-1}(p+q)} \psi_I(z)\\ &= \inf\{\psi_I(e+f) :\ e, f \in E, x = \varphi(p), f = \varphi(q)\}\\ &\leq \inf\{\max\{\psi_I(e), \psi_I(f)\} :\ e, f \in E, x = \varphi(p), f = \varphi(q)\}\\ &= \max\{\inf\{\psi_I(e) : e \in E, e = \varphi(p)\}, \inf\{\psi_I(f) : f \in E, f = \varphi(q)\}\\ &= \max\{(\varphi(\psi_I))(p), (\varphi(\psi_I))(q)\},\end{aligned}$$

$$\begin{aligned}(\varphi(\psi_F))(p+q) &= \inf_{z\in\varphi^{-1}(p+q)} \psi_F(z)\\ &= \inf\{\psi_F(e+f) : e, f \in E, e = \varphi(p), f = \varphi(q)\}\\ &\leq \inf\{\max\{\psi_F(e), \psi_F(f)\} :\ e, f \in E, e = \varphi(p), f = \varphi(q)\}\\ &= \max\{\inf\{\psi_F(e) : e \in E, e = \varphi(p)\}, \inf\{\psi_F(f) : f \in E, f = \varphi(q)\}\\ &= \max\{(\varphi(\psi_F))(p), (\varphi(\psi_F))(q)\}\end{aligned}$$

and

$$\begin{aligned}(\varphi(\psi_T))(p\gamma q) &= \sup_{z\in\varphi^{-1}(p\gamma q)} \psi_T(z)\\ &= \sup\{\psi_T(e\gamma f) : e, f \in E, e = \varphi(p), f = \varphi(q)\}\\ &\geq \sup\{\min\{\psi_T(e), \psi_T(f)\} :\ e, f \in E, e = \varphi(p), f = \varphi(q)\}\\ &= \min\{\sup\{\psi_T(e) : e \in E, e = \varphi(p)\}, \sup\{\psi_T(f) : f \in E, f = \varphi(q)\}\\ &= \min\{(\varphi(\psi_T))(p), (\varphi(\psi_T))(q)\},\end{aligned}$$

$$
\begin{aligned}
(\varphi(\psi_I))(p\gamma q) &= \inf_{z\in\varphi^{-1}(p\gamma q)} \psi_I(z) \\
&= \inf\{\psi_I(e\gamma f) : e, f \in E, e = \varphi(p), f = \varphi(q)\} \\
&\le \inf\{\max\{\psi_I(e), \psi_I(f)\} :\ e, f \in E, e = \varphi(p), f = \varphi(q)\} \\
&= \max\{\inf\{\psi_I(e) : e \in E, e = \varphi(p)\}, \inf\{\psi_I(f) : f \in E, f = \varphi(q)\} \\
&= \max\{(\varphi(\psi_I))(p), (\varphi(\psi_I))(q)\},
\end{aligned}
$$

and

$$
\begin{aligned}
(\varphi(\psi_F))(p\gamma q) &= \inf_{z\in\varphi^{-1}(p\gamma q)} \psi_F(z) \\
&= \inf\{\psi_F(e\gamma f) : e, f \in E, e = \varphi(p), f = \varphi(q)\} \\
&\le \inf\{\max\{\psi_F(e), \psi_F(f)\} :\ e, f \in E, e = \varphi(p), f = \varphi(q)\} \\
&= \max\{\inf\{\psi_F(e) : e \in E, e = \varphi(p)\}, \inf\{\psi_F(f) : f \in E, f = \varphi(q)\} \\
&= \max\{(\varphi(\psi_F))(p), (\varphi(\psi_F))(q)\}.
\end{aligned}
$$

Thus, $\varphi(\psi)$ is neutrosophic Γ-semiring of F. □

5. **Conclusions and future works**

In this paper, we introduce the notion of a neutrosophic Γ-semiring and characterized the neutrosophic Γ-semiring in terms of crisp Γ-semirings and obtained some properties. In continuity of this paper, we study neutrosophic ideals, neutrosophic bi-ideals, neutrosophic quasi ideals, neutrosophic interior ideals of Γ-semiring.

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An Efficient Enumeration Technique for a Transportation Problem in Neutrosophic Environment

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Abstract. In this paper, we have given a technique to solve transportation problems in a neutrosophic environment. The proposed technique gives solutions in the best possible and the worst possible manner. The concept is based on (α, β, γ) cut-sets of neutrosophic sets. It converts the problem into an interval-valued problem, which is further solved to give the best possible and the worst possible solution to the considered problem. Moreover, the technique proposed in this paper produces a direct optimal solution. The gained results ensure that the proposed method is better than the traditional ones as it is computationally much more efficient. The proposed technique has been suitably modified by varying (α, β, γ) according to the decision maker's knowledge of supply and demand requirements. Numerical illustrations have been provided to demonstrate the methodology and to prove our claim on efficiency.

Keywords: Neutrosophic Transportation Problem, Single Valued Trapezoidal Neutrosophic Number, (α, β, γ) Cut-Sets, Decision Making Problem

1. Introduction

The transportation problem (TP) is a unique magnificence network-based linear programming problem with utmost significance in literature. TP was first introduced by Hitchcock [1]. These days transportation problem is used in many fields like management [2], job scheduling [3], investment [4], inventory [5], production [6], etc. To model these real-life problems we need to know some parameters values such as transportation cost, demand and supply. However, in real-life situations the parameters depend on various factors such as travel time, traffic jam, prices of diesel/petrol, weather condition, and so on. Similarly, the demands of several wearing products depends on the season, discount offers, fashion trends, etc. To deal with these obstacles amicably, the parameters of the problems can be represented as imprecise numbers having some uncertainty and vagueness.

The fuzzy set theory (FS) symbolizes the uncertainty introduced by Zadeh [7] in the given data, which is characterized by the grade of membership. A transportation problem discussed in fuzzy environment is called a fuzzy transportation problem (FTP), which has been solved by many researchers ([8]- [14]). However, sometimes the grade of accuracy or membership is not enough to describe ambiguity of the problem. Thus, Atanassove [15] developed the theory of intuitionistic fuzzy set (IFS). An IFS distinguish between the grade of membership and non-membership of each element in the set. The IFS approach is much applicable in real-life decision-making problems. The solution approach of transportation in IFS has been applied by many researchers ([16]- [20]).

Apart from the uncertainty or vagueness of the parameters of the transportation problem in real life, there is some indeterminacy due to various reasons such as imperfection of the data, ignorance of the problem, poor status forecasting, etc. Inconsistency and indeterminacy in information can not be well handled by an IFS. To overcome such uncertainties, Smarandache [21]developed the concept of neutrosophic set (NS) theory, a generalization of the IFS. In the neutrosophic set, the grade of indeterminacy membership is independent of the grade of truth-membership and the grade of falsity-membership. When the grade of the uncertainty of NS equals the grade of hesitation, NS becomes IFS. The single value neutrosophic set (SVN) was developed by Wang et al. [22] for the use of the neutrosophic set in practical decision-making problems and supply management problems in real life. The transportation problem discussed in the neutrosophic environment is known as the neutrosophic transportation problem. Thamaraiselvi and Santhi [23] presented a technique to solved transportation problems in a neutrosophic environment. Singh et al. [24] developed modified method of [23] by correcting mathematical assumptions and introduced a new method to solve the neutrosophic transportation problems. Later, many researchers have explored neutrosophic set in decision-making problems [25], [26], [27].

In spite of the above-mentioned developments, this article aims at providing a simple yet efficient technique for solving neutrosophic transportation problems with easy application in day-to-day situations. The major advantages of the proposed technique are as follows:

- The proposed technique produces the optimal solution for the considered problem in the best possible and the worst possible solution mode.
- The proposed technique is based on (α, β, γ) cut-set values and decision makes can vary these parameters according to their requirements.

- The method gives direct optimal solution for the considered problem.
- The proposed method is better than the traditional or oriented previous methods as it is computationally much more efficient.

The rest of the paper has been configuration as follows: The basic definitions of SVN-numbers are presented in the next Section. In Sect. 3, the arithmetic operations on Single valued trapezoidal neutrosophic (SVTN) numbers are discussed. The concept of (α, β, γ) cut-sets on SVTN-number is presented in Sect. 4. In Sect. 5, the mathematical formulation of transportation problem in neutrosophic environment is discussed. The developed methodology is presented in Sect. 6. The numerical example is illustrated in Sect. 7. The result and discussion are given in Sect. 8. The article is then concluded in Section 9.

2. **Mathematical preliminaries**

In this section, a brief overview of neutrosophic sets followed by some elementry definitions. Throughout this article, S and R represent the set of all neutrosophic sets and the set of real numbers respectively.

Definition 2.1. [21] The neutrosophic set N is characterized by three membership functions, which are the truth-membership function T_S, indeterminacy-membership function I_S, and falsity-membership function F_S, where P is the universe of discourse and $\forall\ u \in P$, $T_S(u), I_S(u), F_S(u) \subseteq\ ]^-0, 1^+[$, and $^-0 \leq inf\ T_S(u) + inf\ I_S(u) + inf\ F_S(u) \leq sup\ T_S(u) + sup\ I_S(u) + sup\ F_S(u) \leq 3^+$.

See that according to Definition 2.1, $T_S(u), I_S(u), F_S(u)$ are real standard or non-standard subsets of $]^-0, 1^+[$and hence, $T_S(u), I_S(u), F_S(u)$ can be subintervals of $[0, 1]$.

Remark 2.2. [28] If $T_S(u) + I_S(u) + F_S(u) = 1$, where $F_S(u) \leq 1 - T_S(u)$ (i.e. $I_S(u) \geq 0$ may exist) then neutrosophic set is known as IFS.

Remark 2.3. [28] If $T_S(u) + I_S(u) + F_S(u) = 1$, where $F_S(u) = 1 - T_S(u)$ (i.e. $I_S(u) = 0$ does not exist) the neutrosophic set is known as a fuzzy set.

Definition 2.4. [22] The single-valued neutrosophic set N over P is $S = \langle T_S(u), I_S(u), F_S(u); u \in P\rangle$ where $T_S : P \to [0, 1], I_S : P \to [0, 1]$, and $F_S : P \to [0, 1]$, $0 \leq T_S(u) + I_S(u) + F_S(u) \leq 3$.

The single-valued neutrosophic number is symbolized by $N = (t, i, f)$, such that $0 \leq t, i, f \leq$ and $0 \leq t + i + f \leq 3$.

Definition 2.5. [26] A single valued trapezoidal neutrosophic number is defined by $\tilde{p} = \langle (p_1, p_2, p_3, p_4); m_{\tilde{p}}, n_{\tilde{p}}, o_{\tilde{p}}\rangle$, where $m_{\tilde{p}}, n_{\tilde{p}}, o_{\tilde{p}} \in [0, 1]$ and p_1, p_2, p_3, p_4 in $\mathbb{R}$ with condition that $p_1 \leq p_2 \leq p_3 \leq p_4$. The truth-membership, indeterminacy-membership, and falsity-membership functions of $\tilde{p}$ are given as follows:

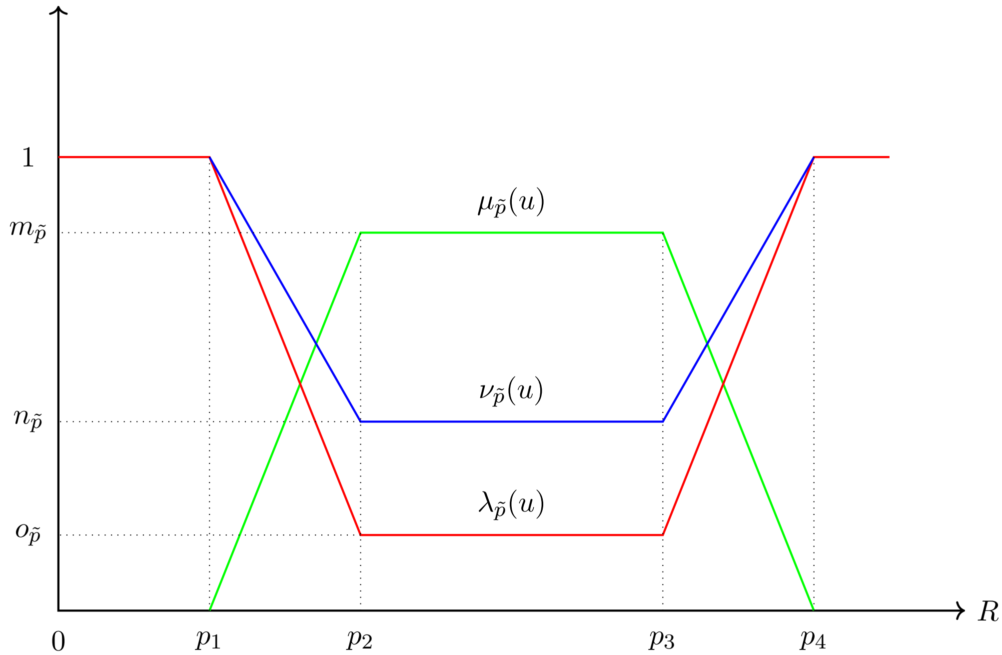

Figure 1. SVTN-number

$$\mu_{\tilde{p}}(u) = \begin{cases} m_{\tilde{p}}(\frac{u-p_1}{p_2-p_1}); & p_1 \leq u \leq p_2 \\ m_{\tilde{p}}; & p_2 \leq u \leq p_3 \\ m_{\tilde{p}}(\frac{p_4-u}{p_4-p_3}); & p_3 \leq u \leq p_4 \\ 0; & \text{otherwise,} \end{cases}$$

$$\nu_{\tilde{p}}(u) = \begin{cases} \frac{p_2-u+n_{\tilde{p}}(u-p_1)}{p_2-p_1}; & p_1 \leq u \leq p_2 \\ n_{\tilde{p}}; & p_2 \leq u \leq p_3 \\ \frac{u-p_3+n_{\tilde{p}}(p_4-u)}{p_4-p_3}; & p_3 \leq u \leq p_4 \\ 1; & \text{otherwise} \end{cases}$$

$$\lambda_{\tilde{p}} = \begin{cases} \frac{p_2-u+o_{\tilde{p}}(u-p_1)}{p_2-p_1}; & p_1 \leq u \leq p_2 \\ o_{\tilde{p}}; & p_2 \leq u \leq p_3 \\ \frac{u-p_3+o_{\tilde{p}}(p_4-u)}{p_4-p_3}; & p_3 \leq u \leq p_4 \\ 1; & \text{otherwise,} \end{cases}$$

where $m_{\tilde{p}}$, $n_{\tilde{p}}$ and $o_{\tilde{p}}$ are represents the maximum truth-membership grade, minimum-indeterminacy membership grade, minimum falsity-membership grade respectively. The geometrical representation of SVTN-number is shown by Fig. 1.

3. Arithmetic operations on SVTN-number

In this section, the arithmetic operations on single-valued trapezoidal neutrosophic numbers are defined. Let $\tilde{p} = \langle (p_1, p_2, p_3, p_4); m_{\tilde{p}}, n_{\tilde{p}}, o_{\tilde{p}} \rangle$ and $\tilde{q} = \langle (q_1, q_2, q_3, q_4); m_{\tilde{q}}, n_{\tilde{q}}, o_{\tilde{q}} \rangle$ be two single valued trapezoidal neutrosophic numbers and $k \neq 0$ be any number and the operators $(\wedge,\vee)$ are the max, min respectively then the operations on them are defined as follows [25]:

(1) $\tilde{p} \oplus \tilde{q} = \langle (p_1 + q_1, p_2 + q_2, p_3 + q_3, p_4 + q_4); m_{\tilde{p}} \wedge m_{\tilde{q}}, n_{\tilde{p}} \vee n_{\tilde{q}}, o_{\tilde{p}} \vee o_{\tilde{q}} \rangle$,

(2) $\tilde{p} \ominus \tilde{q} = \langle (p_1 - q_4, p_2 - q_3, p_3 - q_2, p_4 - q_1); m_{\tilde{p}} \wedge m_{\tilde{q}}, n_{\tilde{p}} \vee n_{\tilde{q}}, o_{\tilde{p}} \vee o_{\tilde{q}} \rangle$,

(3)

$$\tilde{p} \otimes \tilde{q} = \begin{cases} \langle (\frac{p_1}{q_4}, \frac{p_2}{q_3}, \frac{p_3}{q_2}, \frac{p_4}{q_1}); m_{\tilde{p}} \wedge m_{\tilde{q}}, n_{\tilde{p}} \vee n_{\tilde{q}}, o_{\tilde{p}} \vee o_{\tilde{q}} \rangle & \text{if } p_4 > 0, q_4 > 0 \\ \langle (\frac{p_4}{q_4}, \frac{p_3}{q_3}, \frac{p_2}{q_2}, \frac{p_1}{q_1}); m_{\tilde{p}} \wedge m_{\tilde{q}}, n_{\tilde{p}} \vee n_{\tilde{q}}, o_{\tilde{p}} \vee o_{\tilde{q}} \rangle & \text{if } p_4 < 0, q_4 > 0 \\ \langle (\frac{p_4}{q_1}, \frac{p_3}{q_2}, \frac{p_2}{q_3}, \frac{p_1}{q_4}); m_{\tilde{p}} \wedge m_{\tilde{q}}, n_{\tilde{p}} \vee n_{\tilde{q}}, o_{\tilde{p}} \vee o_{\tilde{q}} \rangle & \text{if } p_4 < 0, q_4 < 0 \end{cases}$$

(4)

$$c\tilde{p} = \begin{cases} \langle (cp_1, cp_2, cp_3, cp_4); m_{\tilde{p}}, n_{\tilde{p}}, o_{\tilde{p}} \rangle & \text{if } c > 0 \\ \langle (cp_4, cp_3, cp_2, cp_1); m_{\tilde{p}}, n_{\tilde{p}}, o_{\tilde{p}} \rangle & \text{if } c < 0 \end{cases}$$

(5) $\tilde{p}^{-1} = \langle (\frac{1}{p_4}, \frac{1}{p_3}, \frac{1}{p_2}, \frac{1}{p_1}); m_{\tilde{p}}, n_{\tilde{p}}, o_{\tilde{p}} \rangle$, where $\tilde{p} \neq 0$.

4. Concepts of Cut-Sets for SVTN-number

In this section, the (α, β, γ) cut-sets for single valued trapezoidal neutrosophic numbers are discussed [25].

The cut-sets for SVTN-number $\tilde{p} = \langle (p_1, p_2, p_3, p_4); m_{\tilde{p}}, n_{\tilde{p}}, o_{\tilde{p}} \rangle$ are defined as follows:

An (α, β, γ)-cut set of $\tilde{p}$ is a crisp subset of R defined as :

$$\tilde{p}_{\langle \alpha, \beta, \gamma \rangle} = \{u : \mu_{\tilde{p}}(u) \geq \alpha, \nu_{\tilde{p}}(u) \leq \beta, \lambda_{\tilde{p}}(u) \leq \gamma\}$$

which satisfies the conditions as follows:

$0 \leq \alpha \leq m_{\tilde{p}},\ n_{\tilde{p}} \leq \beta \leq 1,\ o_{\tilde{p}} \leq \gamma \leq 1$ and $0 \leq \alpha + \beta + \gamma \leq 3$.

An α-cut of $\tilde{p}$ is a crisp subset of R defined as:

$$\tilde{p}_\alpha = \{u : \mu_{\tilde{p}}(u) \geq \alpha, u \in R\}$$

where $\alpha \in [0, m_{\tilde{a}}]$.

Clearly, any α-cut set of $\tilde{p}$ for truth-membership function is a closed interval, denoted by

$$\tilde{p}_\alpha = [L_{\tilde{p}}(\alpha), R_{\tilde{p}}(\alpha)] = [\tfrac{(m_{\tilde{p}} - \alpha)p_1 + \alpha p_2}{m_{\tilde{p}}}, \tfrac{(m_{\tilde{p}} - \alpha)p_4 + \alpha p_3}{m_{\tilde{p}}}]$$

An β-cut set of $\tilde{p}$ is a crisp subset of R defined as:

$$\tilde{p}_\beta = \{u : \nu_{\tilde{p}}(u) \leq \beta, u \in R\}$$

where $\beta \in [n_{\tilde{p}}, 1]$.

Clearly, β-cut set of $\tilde{p}$ for indeterminacy-membership is a closed interval, denoted by

$$\tilde{p}_\beta = [L^*_{\tilde{p}}(\beta), R^*_{\tilde{p}(\beta)}] = [\frac{(1-\beta)p_2+(\beta-n_{\tilde{p}})p_1}{1-n_{\tilde{p}}}, \frac{(1-\beta)p_3+(\beta-n_{\tilde{p}})p_4}{1-n_{\tilde{p}}}]$$

An γ-cut set of $\tilde{p}$ is a crisp subset of R defined as:

$$\tilde{p}_\gamma = \{\lambda_{\tilde{p}}(u) \leq \gamma, u \in R\}$$

where $\gamma \in [o_{\tilde{p}}, 1]$.

Clearly, γ-cut set of $\tilde{p}$ for falsity-membership is a closed interval, denoted by

$$\tilde{p}_\gamma = [L^{**}_{\tilde{p}}(\gamma), R^{**}_{\tilde{p}}] = [\frac{(1-\gamma)p_2+(\gamma-o_{\tilde{p}})p_1}{1-o_{\tilde{p}}}, \frac{(1-\gamma)p_3+(\gamma-o_{\tilde{p}})p_4}{1-o_{\tilde{p}}}]$$

Thus, it can be easily concluded from the definitions of $\tilde{p}_\alpha$, $\tilde{p}_\beta$ and $\tilde{p}_\gamma$ cut sets as follows:

$$\tilde{p}_{\langle\alpha,\beta,\gamma\rangle} = \tilde{p}_\alpha \wedge \tilde{p}_\beta \wedge \tilde{p}_\gamma.$$

5. Mathematical Formulation of Transportation Problem in Neutrosophic Environment

In this section, the transportation problem in a neutrosophic environment is considered. The cost parameter of the problem is taken as SVTN-number. Other parameters supply and demand of problem are assume to be precisely known. Thus, it is assumed that decision maker is indeterminate in considering the value of transportation cost, but there is no hesitation about the demand and supply of the commodity. The mathematical formulation of SVTN- number transportation problem under consideration is as follows:

$$\min \tilde{Z}^N = \sum_{i=1}^{m}\sum_{j=1}^{n} \mathcal{X}_{ij} \otimes \tilde{\mathcal{C}}^{\mathcal{N}}_{ij}$$

Subject to

$$\sum_{j=1}^{n} \mathcal{X}_{ij} = \mathcal{S}_i \ ; \ i = 1, 2, ..., m,$$

$$\sum_{i=1}^{m} \mathcal{X}_{ij} = \mathcal{D}_j \ ; \ j = 1, 2, ..., n,$$

and $\mathcal{X}_{ij} \geq 0$.

Where,

- m and n denote total number of supply sources and total number of demand points, respectively.
- $\mathcal{S}_i$ denotes available commodity at ith source.
- $\mathcal{D}_j$ denotes demand of the commodity at jth destination.
- $\tilde{\mathcal{C}}^{\mathcal{N}}_{ij} = (c_{ij,1}, c_{ij,2}, c_{ij,3}, c_{ij,4} \ ; w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}})$ denotes the neutrosophic transportation cost of a unit commodity from ith source to jth destination.
- $\mathcal{X}_{ij}$ denotes number of units of the commodity to be transported from ith source to jth destination.

6. **Proposed technique**

In this section, the developed methodology to solve the transportation problem is described in which cost parameter is SVTN-number, supply and demand are precisely known. The step-by-step procedures of proposed algorithm is as follows:

Step 1 Consider the transportation problem in a neutrosophic environment where the cost parameter is taken as SVTN-number.

Step 2 Apply the cut set ranking function as defined in section 4 to convert the transportation cost into interval.

Step 3 Take the most optimistic (least value) members of the interval from each cell of the table to convert the interval-valued transportation problem to crisp transportation problem.

Step 4 Select the minimum element in each row and subtract it from each cell of the corresponding row.

Step 5 Select the minimum element from each column and subtract it from each cell of the corresponding column.

Step 6 In this process, at least one zero value cell in each row and each column is obtained. Calculate S_{ij} by using the following formula corresponding to each zero cell:

$$S_{ij} = \frac{\text{Sum of cost of cell adjacent to the (i,j)-cell}}{\text{Number of non-zero value ranked cells adjacent to (i,j)-cell}}$$

Step 7 Choose the cell which has a maximum value of S_{ij} and assign the maximum possible demand to that particular cell. Delete either row/column for which the demand is exhausted.

Step 8 If there occurs a situation in which two or more cell have the same rank then choose a cell which assigns the maximum possible demand.

Step 9 Repeat the process by applying Step 6 to Step 8 until the total demand fulfilled.

Step 10 The required optimal solution is denoted by $\mathcal{X}_{ij}$ and corresponding optimal value can be obtained by $\sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{\mathcal{C}}_{ij}^{\mathcal{N}} \otimes \mathcal{X}_{ij}$. This is the best possible solution of the given transportation problem in the neutrosophic environment for given α, β, γ.

The worst possible solution of the given transportation problem in the neutrosophic environment for the values of the given α, β, γ can be obtained by considering the most pessimistic (greatest value) member of the interval from each cell in step 3 and hence following steps 4 to 10.

7. **Numerical Example**

In this section, a neutrosophic transportation problem with three sources A, B, C and four destination W, X, Y, Z is considered. The parameters of the problem are taken as single valued trapezoidal neutrosophic numbers. The input data of the problem SVTN-TP is denoted in Table 1 as follows [24]:

Table 1. SVTN transportation problem

Destination → Sources ↓	W	X	Y	Z	Supply
A	$(3,5,6,8\,;\,0.6,0.5,0.6)$	$(5,8,10,14\,;\,0.3,0.6,0.6)$	$(12,15,19,22\,;\,0.6,0.4,0.5)$	$(14,17,21,28;0.8,0.2,0.6)$	26
B	$(0,1,3,6\,;\,0.7,0.5,0.3)$	$(5,7,9,11\,;\,0.9,0.7,0.5)$	$(15,17,19,22\,;\,0.4,0.8,0.4)$	$(9,11,14,16\,;\,0.5,0.4,0.7)$	24
C	$(4,8,11,15\,;\,0.6,0.3,0.2)$	$(1,3,4,6\,;\,0.6,0.3,0.5)$	$(5,7,8,10\,;\,0.5,0.4,0.7)$	$(5,9,14,19\,;\,0.3,0.7,0.6)$	30
Demand	17	23	28	12	

For $(\alpha, \beta, \gamma) = (0.3, 0.8, 0.7)$, applying the cut set ranking function as discussed in section 4. The problem is converted to an inter-valued problm as shown in table 2.

Table 2. Obtained interval valued transportation problem

Destination → Sources ↓	W	X	Y	Z	Supply
A	$[4, 7]$	$[8, 10]$	$[13.8, 20.2]$	$[16.2, 22.7]$	26
B	$[.4, 4.7]$	$[6.3, 9.6]$	$[17, 19]$	$[11, 14]$	24
C	$[6, 13]$	$[2.7, 4.8]$	$[7, 8]$	$[9, 14]$	30
Demand	17	23	28	12	

Now, taking the most optimistic and most pessimistic values of intervals to get two separate transportation problems as shown in Table 3 and Table 4 respectively.

By solving problem in Table 3 by our proposed technique we will get best possible solution for the problem and by solving problem in Table 4 by our methodology we will get the worst possible solution for the given problem.

After solving the problem in Table 3, we have obtained optimal solution $\{(370, 543, 694, 938); 0.3, 0.7, 0.7\}$, which represent the best possible solution of the problem. Similarly,

Table 3. Optimistic values problem

Destination $\rightarrow$ Sources $\downarrow$	W	X	Y	Z	Supply
A	4	8	13.8	16.2	26
B	.4	6.3	17	11	24
C	6	2.7	7	9	30
Demand	17	23	28	12	

Table 4. Pessimistic values problem

Destination $\rightarrow$ Sources $\downarrow$	W	X	Y	Z	Supply
A	7	10	20.2	22.7	26
B	4.7	9.6	19	14	24
C	13	4.8	8	14	30
Demand	17	23	28	12	

solving problem in Table 4 by our technique, we have obtained optimal solution $\{(380, 549, 696, 940); 0.7, 0.6, 0.7\}$.

8. Result and discussion

The result of the above numerical example gives two solutions the best and the worst possible solutions. For $(\alpha, \beta, \gamma) = (0.3, 0.8, 0.7)$, the best possible solution for the problem is $\{(370, 543, 694, 938); 0.3, 0.7, 0.7\}$. The obtained solution represents 30 percent level of truthfulness, 70 level of percent indeterminacy and 70 percent level of falsity. The other values for level (grade) of truthfulness or acceptance $\mu(x)$, indeterminacy $\nu(x)$ and falsity $\lambda(x)$ are

$$\mu(x) = \begin{cases} 0.3(\frac{x-370}{543-370}), & 370 \leq x \leq 543 \\ 0.3, & 543 \leq x \leq 694 \\ 0.3(\frac{938-x}{938-694}), & 694 \leq x \leq 938 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \nu(x) = \lambda(x) = \begin{cases} \frac{543-x+0.7(x-370)}{543-370}, & 370 \leq x \leq 694 \\ 0.7, & 543 \leq x \leq 694 \\ \frac{x-694+0.7(938-x)}{938-694}, & 694 \leq x \leq 938 \\ 1, & \text{otherwise.} \end{cases}$$

respectively.

For $(\alpha, \beta, \gamma) = (0.3, 0.8, 0.7)$, the worst possible solution of the problem is $\{(380, 549, 696, 940); 0.7, 0.6, 0.7\}$. The solution represent 70 percent level of truthfulness, 60 percent level of indeterminacy and 70 percent level of falsity. The other values for $\mu(x)$, $\nu(x)$ and $\lambda(x)$ are

$$\mu(x) = \begin{cases} 0.7(\frac{x-380}{549-380}), & 380 \leq x \leq 549 \\ 0.7, & 549 \leq x \leq 696 \\ 0.7(\frac{940-x}{940-696}), & 696 \leq x \leq 940 \\ 0, & \text{otherwise} \end{cases}$$

$$\nu(x) = \begin{cases} \frac{549-x+0.6(x-380)}{549-380}, & 380 \leq x \leq 696 \\ 0.6, & 549 \leq x \leq 696 \\ \frac{x-696+0.6(940-x)}{940-696}, & 696 \leq x \leq 940 \\ 1, & \text{otherwise.} \end{cases}$$

and

$$\lambda(x) = \begin{cases} \frac{549-x+0.7(x-380)}{549-380}, & 380 \leq x \leq 696 \\ 0.7, & 549 \leq x \leq 696 \\ \frac{x-696+0.7(940-x)}{940-696}, & 696 \leq x \leq 940 \\ 1, & \text{otherwise.} \end{cases}$$

respectively. Therefore, with the help of $\mu(x)$, $\nu(x)$ and $\lambda(x)$, the decision maker can decide the total neutrosophic transportation cost to schedule the transportation and budget allocation.

The proposed approach in comparison to the methods of Thamaraiselvi and Sathi [23] and Singh et.al [24] is computationally much more efficient as it is producing direct optimal solution without finding an initial basic feasible solution. Also, our method gives the best and the worst possible solutions under neutrosophic of transportation problem, which enables the decision maker to choose the compromise solution. The proposed model gives direct optimal solution, which makes it computationally less time consuming than other existing methods. Moreover, the proposed technique can be modified by the decision maker by choosing the different values of $(\alpha,\ \beta,\ \gamma)$ to get satisfactory result. With different values of $(\alpha,\ \beta,\ \gamma)$, the neutrosophic optimal solutions of the considered problem are shown in Table 5.

9. **Conclusion**

In this article, we have discussed a transportation problem under a neutrosophic environment and proposed a technique to obtaining an optimal solution of the considered problem. The proposed technique has been useful to solve transportation problems in which the cost parameters are taken as single-valued trapezoidal neutrosophic numbers. In this article, we have used cut sets to convert the given problem to an interval-valued problem and is then solved by the proposed technique. The proposed technique is easy to apply in real-life transportation problems. The proposed approach in comparison to the methods of Thamaraiselvi

Table 5. Solutions for different values of (α, β, γ)

α	β	γ	**Best possible solution**	**Worst possible solution**
0.3	0.8	0.7	$(370, 543, 694, 938); 0.3, 0.7, 0.7$	$(380, 549, 696, 940); 0.7, 0.6, 0.7$
0	0.8	0.7	$(370, 543, 694, 938); 0.3, 0.7, 0.7$	$(370, 543, 694, 938); 0.3, 0.7, 0.7$
0	1	1	$(370, 543, 694, 938); 0.3, 0.7, 0.7$	$(370, 539, 676, 890); 0.3, 0.6, 0.7$
0.1	0.9	1	$(370, 543, 694, 938); 0.3, 0.7, 0.7$	$(438, 620, 770, 1044); 0.3, 0.7, 0.7$
0.3	0.8	0.9	$(370, 539, 676, 890); 0.3, 0.6, 0.7$	$(380, 549, 696, 940); 0.3, 0.6, 0.7$
0.1	0.9	1	$(370, 543, 694, 938); 0.3, 0.7, 0.7$	$(370, 543, 694, 930); 0.3, 0.7, 0.7$
0.2	0.9	0.7	$(370, 543, 694, 938); 0.3, 0.7, 0.7$	$(380, 549, 696, 940); 0.7, 0.6, 0.7$

and Sathi [23] and Singh et.al [24] is computationally much more efficient as it is producing direct optimal solution for the problem. Also, our method gives the best and the worst possible solutions under neutrosophic of transportation problem, which enables the decision maker to choose the compromise solution. Moreover, the proposed technique can be modified by the decision maker by choosing the different values of (α, β, γ) to get satisfactory result.

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A Nonlinear Programming Model to Solve Matrix Games with Pay-offs of Single-valued Neutrosophic Numbers

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Abstract. Single-valued neutrosophic number (SVNN) is an appropriate extension of the ordinary fuzzy number. The key feature of the SVNN is that it can capture indeterminacy in the imprecise information. In real-life problems, there are many situations where players of a matrix game can not assess their payoffs in terms of ordinary fuzzy or intuitionistic fuzzy numbers. The SVNN is used as an excellent tool to handle such situations. This paper explores matrix games with SVNN payoffs and investigates a non-linear programming approach to solve such a game. First, two auxiliary non-linear multi-objective programming problems have been formulated. Then, each of the two multi-objective programming problems is converted into two non-linear bi-objective programming problems. Finally, the lexicographic method is used to solve the reduced bi-objective programming problems. It is worth mentioning that the values of the game for both the players are obtained in SVNN forms, which is desirable. The applicability of the proposed approach is illustrated with a market share problem and results are compared and analyzed with an existing method.

Keywords: Matrix game; single-valued neutrosophic number; multi-objective optimization; lexicographic method.

1. Introduction

Matrix game theory [17] gives a mathematical framework to conceive strategies that help to overcome real-life conflicting situations. There are several types of mathematical games [22, 24, 34], which have been broadly studied and successfully utilized in many areas. Many of the real-life situations are uncertain due to the imprecision of data, asymmetric information, and conflict of interest between opponents in the same field of business. So, it is difficult to evaluate payoffs precisely. The players only approximate the payoffs with some imprecise degrees. The most vital and argued issue among the researchers is to settle how to handle the uncertainty. Crisp data can not express most of these complicated structures correctly.

Fuzzy set (FS) was the first to successfully encounter the uncertainty which is not due to the randomness of an event. FS represents each element with a degree of membership (DOM) which lies between 0 and 1. In the literature, matrix games with fuzzy pay-offs have been broadly studied and analyzed by numerous researchers. Campos [6] used a linear programming approach to solve fuzzy matrix games. Bector et al. [3] used fuzzy linear programming duality to solve matrix games with fuzzy goals and fuzzy pay-offs. Li [15] evolved several methods to solve matrix games with payoffs as triangular fuzzy numbers. Verma and Kumar [30] proposed the Mehar method for fuzzy matrix games. Seikh et al. [19] implemented an α-cut based approach to solve fuzzy matrix games. Very recently, Seikh et al. [21] developed a methodology to solve matrix games with hesitant fuzzy payoffs. Some recent references on fuzzy matrix games are [12, 23, 25, 36].

FS considers only the DOM of the elements in a universe, which is a single value, and cannot provide any additional information regarding the incomplete concept of the elements. Atanassov [2] generalized the idea of FS to intuitionistic fuzzy set (IFS) where the degree of non-membership (DONM) $\nu_X(x) \in [0,1]$ is also attached with DOM $\mu_X(x) \in [0,1]$ for each element $x \in X$ in a universe where $0 \leq \mu_X(x) + \nu_X(x) \leq 1$. IFS outlines uncertainty more correctly and descriptively than FS, as IFS considers both complete and incomplete imprecise data. Nan et al. [16] proposed solution methodologies to study matrix games with payoffs of triangular intuitionistic fuzzy numbers (TIFNs) [18]. Seikh et al. [20] accomplished an approach to solve matrix games with intuitionistic fuzzy payoffs. Xing and Qiu [35] used the accuracy function method to solve a matrix game where the payoffs are considered as TIFN.

However, in reality, the available information always contains some imprecise data which consists of conflicting, unpredictable, and indeterminate information. The FS can not express the DONM and the IFS does not control the indeterminacy of information. Neutrosophic sets (NSs) [27] considers the degree of indeterminacy (DOI) $\omega_X(x) \in [0,1]$ together with the DOM and DONM. Therefore, NS can capture more realistic data than that of FS and IFS. NSs are represented by DOM (μ), DOI (ω), and DONM (ν) which are independent and $0 \leq \mu,\ \omega,\ \nu \leq 1$ provided $0 \leq \mu + \omega + \nu \leq 3$. NS generalizes the classical set when $\omega = 0,\ \mu,\ \nu$ either 0 or 1 and $\mu + \omega + \nu = 1$; the FS when $\omega = 0,\ 0 \leq \mu,\ \omega,\ \nu \leq 1$ and $\mu + \omega + \nu = 1$; the IFS when $0 \leq \mu,\ \omega,\ \nu \leq 1$ and $0 < \mu + \omega + \nu < 1$. This concludes the fact that NS is a generalization of a classical set, FS and IFS.

Wang et al. [31] conceptualized a single-valued neutrosophic set (SVNS), a special form of NS for realistic applications. SVNNs are a distinctive case of SVNSs. SVNN is an extension of a fuzzy number (FN) and an intuitionistic fuzzy number (IFN). SVNS outlined the variables which are completely relevant for human prediction because of the imperfection of information that human observes from the outer world. For example, the proposition "The newly launched car Ψ would be the best seller", has not any precise answer for the human brain as far as yes or no, since indeterminacy is the segment of ignorance of the value of a proposition between truth and lie. For that reason, the three components of the neutrosophic set are very much suitable to exhibit the indeterminacy and inconsistency in the information.

To express the uncertainty of a conflicting circumstance, SVNNs are better to utilize instead of FS/IFS. The following example shows the implication and the relevance of the use of SVNN in real-life problems.

Suppose a smartphone manufacturing company '*Alpha*' is going to launch a new item 'Ω' in the new year 2021. The company *Alpha* wants to estimate the number of selling a unit of new products before starting its production. The selling of the new item depends on various uncertain parameters such as new attracting features, the capacity of supply, the selling price, advertisement of the product, etc. But, the company wants to know whether the guaranteed selling unit would be 1 billion in the year or less'. The existence of this uncertain guaranteed selling unit always contains some knowledge of 'neutral' (indeterminate/unknown) thought besides 'truth/membership' and 'falsehood/non-membership components that lie in FS/IFS. This situation can not be revealed by FNs or IFNs and SVNN comes into consideration. Some experts (say α, β, γ) are consulted for their opinions and they express their views in SVNNs. Let the expert α give its opinion about the guaranteed selling unit as $\langle 0.8, 0.3, 0.4\rangle$. This implies that the company has 80% chance to meet the goal positively and is unable to reach the guaranteed selling unit by 40%. In this case, the expert α has a neutral thought that *Alpha* has an indeterminacy to meet the goal by 30%.

Hence, the application of SVNS theory has been growing quickly in many research areas [4, 29, 32, 37, 38]. Selvachandran et al. [26] designed a modified TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) with maximizing deviation method based on the SVNS model. Garg and Nancy [9] discussed some new distance measures under the SVNS environment and developed an algorithm for single-valued neutrosophic decision making based on TOPSIS method. Garai et al. [8] investigated the weighted possibility mean for SVNNs and developed a rigorous ranking methodology to solve multi-attribute decision making. Haque et al. [11] presented a new exponential operational law for trapezoidal neutrosophic numbers and studied pollution-related MCGDM problems in megacities. Chakraborty et al. [7] explored the classification of trapezoidal bipolar neutrosophic numbers and implemented the de-bipolarization in cloud service-based MCGDM problem. Ahmed et al. [1] developed a new approach to solve linear programming problems in bipolar single-valued neutrosophic environments.

2. Motivation

With the use of SVNNs, the decision-maker can set greater flexibility and better dependability in the strategy-making process. Many uncertain situations are better to present by SVNNs than FS/IFS. Moreover, in matrix games due to lack of information in the available data, the DOI plays a vital role while assessing the payoff values. So, there are many uncertain situations where players can assess payoffs of the matrix game problems in SVNN forms. This useful requirement motivates us to investigate the matrix game with SVNN payoffs.

Here we develop a new methodology to solve matrix game problems with SVNN payoffs. First, two multi-objective non-linear programming problems are constructed to get the optimal value and strategies for the players. Then multi-objective programming problems are transferred into bi-objective non-linear programming problems by considering the same importance of the objective functions. Then the Lexicographic method [14] is used to solve the reduced bi-objective programming problems. A market share problem is illustrated to show the validity of the proposed methodology. The obtained results are discussed and compared and the physical significance of the obtained results is explored. The key contributions of this paper are augmented as under.

(1) This is the first attempt to solve matrix game with SVNN payoffs.
(2) To get the optimal strategies and optimal values of the players, two different non-linear multi-objective programming problems are constructed. These multi-objective programming problems are transferred to bi-objective problems by considering the same importance of the two objective functions. Then Lexicographic method is used to solve the bi-objective problems.
(3) Owen [17] proposed the concept of optimal strategies of the players and value of the game for the crisp matrix game problem. Here, we extend the definition for SVNN matrix game.
(4) The optimal values of the game for both the players are obtained in the SVNN forms, which is desirable.
(5) A real-life market share problem is illustrated to check the applicability and validity of the proposed approach.

The remainder of this paper is sorted out as follows. Some preliminaries on NS are recalled in Section 3. Section 4 conceptualized the idea of SVNN matrix games. Section 5 is dedicated to the solution process. In Section 6, a market share problem is illustrated to validate our proposed methodology. Section 7 concludes the paper.

3. Preliminaries

Neutrosophic sets and their basic operations are recalled in this section.

Definition 3.1. [28] $\tilde{M} = \{\langle \xi, (\mu_{\tilde{M}}(\xi), \omega_{\tilde{M}}(\xi), \nu_{\tilde{M}}(\xi))\rangle | \xi \in \Omega, \mu_{\tilde{M}}(\xi), \omega_{\tilde{M}}(\xi), \nu_{\tilde{M}}(\xi) \in]^-0, 1^+[\}$ is said to be a neutrosophic set (NS) over a universe Ω, where $\mu_{\tilde{M}} : \Omega \to]^-0, 1^+[$; $\omega_{\tilde{M}} : \Omega \to]^-0, 1^+[$ and $\nu_{\tilde{M}} : \Omega \to]^-0, 1^+[$ are called respectively the membership function, indeterminacy membership function and non-membership function and $^-0 \leq \mu_{\tilde{M}}(\xi) + \omega_{\tilde{M}}(\xi) + \nu_{\tilde{M}(\xi)} \leq 3^+$.

Definition 3.2. [31] $\tilde{M} = \{\langle \xi, (\mu_{\tilde{M}}(\xi), \omega_{\tilde{M}}(\xi), \nu_{\tilde{M}}(\xi))\rangle | \xi \in \Omega, \mu_{\tilde{M}}(\xi), \omega_{\tilde{M}}(\xi), \nu_{\tilde{M}}(\xi) \in [0, 1]\}$ is said to be a single-valued neutrosophic set (SVNS) over a universe Ω, where $\mu_{\tilde{M}} : \Omega \to [0, 1]$, $\omega_{\tilde{M}} : \Omega \to [0, 1]$ and $\nu_{\tilde{M}} : \Omega \to [0, 1]$ are called respectively the membership function, indeterminacy membership function and non-membership function and $0 \leq \mu_{\tilde{M}}(\xi) + \omega_{\tilde{M}}(\xi) + \nu_{\tilde{M}}(\xi) \leq 3$.

For convenience, $\langle \mu_{\tilde{M}}(\xi), \omega_{\tilde{M}}(\xi), \nu_{\tilde{M}}(\xi)\rangle$ is called a single-valued neutrosophic number (SVNN) which is usually represented by $\tilde{\alpha} = \langle \mu, \omega, \nu\rangle$.

Definition 3.3. [31] Let $\tilde{M}$ and $\tilde{M}'$ be two SVNSs in the set Ω. $\tilde{M} \subseteq \tilde{M}'$ if and only if $\mu_{\tilde{M}}(\xi) \leq \mu_{\tilde{M}'}(\xi)$, $\omega_{\tilde{M}}(\xi) \geq \omega_{\tilde{M}'}(\xi)$ and $\nu_{\tilde{M}}(\xi) \geq \nu_{\tilde{M}'}(\xi)$ for any $\xi \in \Omega$. Again $\tilde{M} \cong \tilde{M}'$ if and only if $\mu_{\tilde{M}}(\xi) = \mu_{\tilde{M}'}(\xi)$, $\omega_{\tilde{M}}(\xi) = \omega_{\tilde{M}'}(\xi)$ and $\nu_{\tilde{M}}(\xi) = \nu_{\tilde{M}'}(\xi)$ for any $\xi \in \Omega$.

Definition 3.4. [31] Let $\tilde{M}$ and $\tilde{M}'$ be two SVNSs in the set Ω. The intersection of $\tilde{M}$ and $\tilde{M}'$ is a SVNS $\tilde{C}$, written as $\tilde{C} = \tilde{M} \cap \tilde{M}'$, whose membership function, indeterminacy membership function and non-membership functions are related to those of $\tilde{M}$ and $\tilde{M}'$ by $\mu_{\tilde{C}}(\xi) = \min\{\mu_{\tilde{M}}(\xi), \mu_{\tilde{M}'}(\xi)\}$, $\omega_{\tilde{C}}(\xi) = \max\{\omega_{\tilde{M}}(\xi), \omega_{\tilde{M}'}(\xi)\}$ and $\nu_{\tilde{C}}(\xi) = \max\{\nu_{\tilde{M}}(\xi), \nu_{\tilde{M}'}(\xi)\}$ for any $\xi \in \Omega$.

Definition 3.5. [5] Let $\tilde{\alpha} = \langle \mu, \omega, \nu\rangle$, $\tilde{\alpha}_1 = \langle \mu_1, \omega_1, \nu_1\rangle$ and $\tilde{\alpha}_2 = \langle \mu_2, \omega_2, \nu_2\rangle$ be three SVNNs, and $\lambda > 0$, then their algebraic operations are defined as follows:

1. $\tilde{\alpha}_1 + \tilde{\alpha}_2 = \langle \mu_1 + \mu_2 - \mu_1\mu_2, \omega_1\omega_2, \nu_1\nu_2\rangle$;
2. $\tilde{\alpha}_1 - \tilde{\alpha}_2 = \langle 1 - (1 - \mu_1)(1 - \mu_2)^{-1}, \omega_1{\omega_2}^{-1}, \nu_1{\nu_2}^{-1}\rangle$;
3. $\tilde{\alpha}_1 \times \tilde{\alpha}_2 = \langle \mu_1\mu_2, \omega_1 + \omega_2 - \omega_1\omega_2, \nu_1 + \nu_2 - \nu_1\nu_2\rangle$;
4. $\lambda\tilde{\alpha} = \langle 1 - (1 - \mu)^\lambda, \omega^\lambda, \nu^\lambda\rangle$;
5. $\tilde{\alpha}^\lambda = \langle \mu^\lambda, 1 - (1 - \omega)^\lambda, 1 - (1 - \nu)^\lambda\rangle$.

4. **Matrix Games with Pay-offs represented by SVNN**

Consider $\Re^n_+$ as the non-negative orthant of n-dimensional Euclidean space. Let the pure strategies ϵ_h and ζ_k are chosen by Player-I and II with probabilities u_h and v_k respectively, for $h \in \triangle_1$ and $k \in \triangle_2$ where $\triangle_1 = \{1, 2, \ldots, p\}$ and $\triangle_2 = \{1, 2, \ldots, q\}$. If $\sum\limits_{h=1}^{p} u_h = 1$ and $\sum\limits_{k=1}^{q} v_k = 1$ for $(\mathbf{u}, \mathbf{v}) \in \Re^p_+ \times \Re^q_+$ where $\mathbf{u} = (u_1, u_2, \ldots, u_p)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_q)$, then $\mathbf{u}$

and $\mathbf{v}$ are called the mixed strategies for Player-I and II, respectively. Let the sets of all mixed strategies for Player-I and Player-II are denoted by U and V respectively, where

$$U = \Big\{\mathbf{u} = (u_1, u_2, \ldots, u_p) \in \Re_+^p : \sum_{h=1}^{p} u_h = 1\Big\},$$
$$V = \Big\{\mathbf{v} = (v_1, v_2, \ldots, v_q) \in \Re_+^q : \sum_{k=1}^{q} v_k = 1\Big\}.$$

Let us consider the matrix $\tilde{N} = (\tilde{n}_{hk})_{p\times q}$, where $\tilde{n}_{hk} = \langle \mu_{hk}, \omega_{hk}, \nu_{hk}\rangle$ is a SVNN, which represents the payoff for the Player-I. Then, the matrix game with SVNN payoffs is represented by $\{U, V, \tilde{N}\}$. From this, the two person matrix game $\{U, V, \tilde{N}\}$ with payoffs of SVNNs is supposed to call as a SVNN matrix game $\tilde{N}$.

For the choice of mixed strategy $(\mathbf{u}, \mathbf{v}) \in U \times V$ by Player-I and II, the expected payoff $\tilde{E}(\mathbf{u}, \mathbf{v})$ for Player-I will be calculated as

$$\begin{aligned}
\tilde{E}(\mathbf{u}, \mathbf{v}) &= \mathbf{u}^T \tilde{A} \mathbf{v} = \sum_{h=1}^{p}\sum_{k=1}^{q} \tilde{n}_{hk} u_h v_k \\
&= \sum_{h=1}^{p}\sum_{k=1}^{q} \langle \mu_{hk}, \omega_{hk}, \nu_{hk}\rangle \ u_h v_k \\
&= \Big\langle 1 - \prod_{k=1}^{q}\prod_{h=1}^{p}(1-\mu_{hk})^{u_h v_k}, \ \prod_{k=1}^{q}\prod_{h=1}^{p} {\omega_{hk}}^{u_h v_k}, \prod_{k=1}^{q}\prod_{h=1}^{p} \nu_{hk}^{u_h v_k} \Big\rangle,
\end{aligned}$$

which is still a SVNN.

Irrespective of the use of best strategies of the players, the maximum guaranteed gain (or the minimum possible loss) is the value of the game for Player-I (or Player-II). According to the maximin and minimax principles for Player-I and Player-II respectively, if for some $(\mathbf{u^0}, \mathbf{v^0}) \in U \times V$, such that

$$\mathbf{u}^{0^T} \tilde{N} \mathbf{v}^0 = \max_{\mathbf{u}\in U} \min_{\mathbf{v}\in V} \{\mathbf{u^T} \tilde{N} \mathbf{v}\} = \min_{\mathbf{v}\in V} \max_{\mathbf{u}\in U} \{\mathbf{u^T} \tilde{N} \mathbf{v}\},$$

then in the sense of Definition 3.3, $\mathbf{u}^0$ and $\mathbf{v}^0$ are called optimal strategies for Player-I and Player-II, respectively and $\mathbf{u^0}^T \tilde{N} \mathbf{v^0}$ is the value of SVNN matrix game $\tilde{N}$.

Bector et al. [3] introduced the concept of a reasonable solution of the fuzzy matrix game. Here we extend the definition of reasonable solution and solution for the SVNN matrix game in the following definitions.

Definition 4.1. Let $\tilde{\theta}$ and $\tilde{\phi}$ be two SVNNs. Assume that there exist $\mathbf{u}^* \in U$ and $\mathbf{v}^* \in V$ such that $\mathbf{u}^* \tilde{N} \mathbf{v} \subset \tilde{\theta}$ and $\mathbf{u} \tilde{N} \bar{\mathbf{v}}^* \supset \tilde{\phi}$ hold for any $\mathbf{u} \in U$ and $\mathbf{v} \in V$, then $(\mathbf{u}^*, \mathbf{v}^*, \tilde{\theta}, \tilde{\phi})$ is called a reasonable solution of the SVNN matrix game $\tilde{N}$.

In this case, $\tilde{\theta}$ and $\tilde{\phi}$ are called reasonable values and $\mathbf{u}^* \in U$ and $\mathbf{v}^* \in V$ are called reasonable strategies for Player-I and Player-II, respectively. The reasonable solution, which is defined in the above definition, does not represent the solution of the SVNN matrix game $\tilde{N}$. In the following definition, the concept of solution of SVNN matrix game $\tilde{N}$ is explored.

Definition 4.2. Assume that Θ and Φ are the sets of reasonable values for Player-I and II respectively and $\tilde{\theta}^* \in \Theta$ and $\tilde{\phi}^* \in \Phi$. If there do not exist $\tilde{\theta}' \in \Theta$ ($\tilde{\theta}' \neq \tilde{\theta}^*$) and $\tilde{\phi}' \in \Phi$ ($\tilde{\phi}' \neq \tilde{\phi}^*$) such that $\tilde{\theta}^* \subset \tilde{\theta}'$ and $\tilde{\phi}^* \supset \tilde{\phi}'$, then $(\bar{\mathbf{u}}^*, \bar{\mathbf{v}}^*, \tilde{\theta}^*, \tilde{\phi}^*)$ is said to be a solution of the SVNN matrix game $\tilde{N}$.

In this case, $\mathbf{u}^*$ and $\mathbf{v}^*$ are respectively called the maximin strategy for Player-I and minimax strategy for Player-II. $\tilde{\theta}^*$ is the *gain floor* for Player-I and $\tilde{\phi}^*$ is the *loss ceiling* for Player-II.

5. Mathematical Model and Solution Approach for SVNN Matrix game

Suppose $E(\mathbf{u})_k$ is the expected payoff for Player-I when Player-I uses the mixed strategy $\mathbf{u} \in U$ and Player-II chooses the pure strategy $\zeta_k,\ k \in \triangle_2$. Then

$$E(\mathbf{u})_k = \langle 1 - \prod_{h=1}^{p}(1-\mu_{hk})^{u_h},\ \prod_{h=1}^{p}{\omega_{hk}}^{u_h}, \prod_{h=1}^{p}\nu_{hk}^{u_h}\rangle.$$

Let $\rho = \langle \mu, \omega, \nu\rangle$ is the minimum of $E(\mathbf{u})_k$. Then, following Definition 3.3, we have

$$\rho = \langle \mu, \omega, \nu\rangle = \langle \min_{k=1}^{q}\{1 - \prod_{h=1}^{p}(1-\mu_{hk})^{u_h}\}, \max_{k=1}^{q}\{\prod_{h=1}^{p}{\omega_{hk}}^{u_h}\}, \max_{k=1}^{q}\{\prod_{h=1}^{p}{\nu_{hk}}^{u_h}\}\rangle. \qquad (1)$$

Obviously ρ is a function of $\mathbf{u}$. Now to get the maximin strategy $\mathbf{u}^* \in U$ and gain-floor ρ^*, Player-I should choose the mixed strategy $\mathbf{u}^* \in U$ in such a way that ρ is maximized. Therefore following Definition 3.3 and Definition 4.2, we have

$$\rho^* = \langle \mu^*, \omega^*, \nu^*\rangle = \langle \max_{\mathbf{u}\in U}\min_{k=1}^{q}\{1 - \prod_{h=1}^{p}(1-\mu_{hk})^{u_h}\}, \min_{\mathbf{u}\in U}\max_{k=1}^{q}\{\prod_{h=1}^{p}{\omega_{hk}}^{u_h}\}, \min_{\mathbf{u}\in U}\max_{k=1}^{q}\{\prod_{h=1}^{p}{\nu_{hk}}^{u_h}\}\rangle. \quad (2)$$

Similarly, let $E(\mathbf{v})_h$ is the expected payoff for Player-II, when Player-II chooses the mixed strategy $\mathbf{v} \in V$ against Player-I's pure strategy $\epsilon_h,\ h \in \triangle_1$. Then

$$E(\mathbf{v})_h = \langle 1 - \prod_{k=1}^{q}(1-\mu_{hk})^{v_k},\ \prod_{k=1}^{q}{\omega_{hk}}^{v_k}, \prod_{k=1}^{q}\nu_{hk}^{v_k}\rangle.$$

Let $\eta = \langle \alpha, \beta, \gamma\rangle$ is the maximum of $E(\mathbf{v})_h$. Then, following Definition 3.3 and Definition 4.2, we have

$$\eta = \langle \alpha, \beta, \gamma\rangle = \langle \max_{h=1}^{p}\{1 - \prod_{k=1}^{q}(1-\mu_{hk})^{v_k}\}, \min_{h=1}^{p}\{\prod_{k=1}^{q}{\omega_{hk}}^{v_k}\}, \min_{h=1}^{p}\{\prod_{k=1}^{q}{\nu_{hk}}^{v_k}\}\rangle. \qquad (3)$$

Clearly, η is a function of $\mathbf{v}$. Now, to obtain minimax strategy $\mathbf{v}^* \in V$ and loss-ceiling η^*, Player-II should choose the mixed strategy $\mathbf{v}^* \in V$ in such a way that η is minimized. Then following Definition 3.3, we have

$$\eta^* = \langle \alpha^*, \beta^*, \gamma^*\rangle = \langle \min_{\mathbf{v}\in V}\max_{h=1}^{p}\{1 - \prod_{k=1}^{q}(1-\mu_{hk})^{v_k}\}, \max_{\mathbf{v}\in V}\min_{h=1}^{p}\{\prod_{k=1}^{q}{\omega_{hk}}^{v_k}\}, \max_{\mathbf{v}\in V}\min_{h=1}^{p}\{\prod_{k=1}^{q}{\nu_{hk}}^{v_k}\}\rangle. \quad (4)$$

Theorem 5.1. *For SVNN matrix game $\tilde{N}$, Player-I's gain floor does not exceed Player-II's loss ceiling, i.e., $\rho^* \subseteq \eta^*$.*

Proof. For any $\mathbf{u} \in U$, it implies that

$$\min_{\mathbf{v}\in V}\{\tilde{E}(\mathbf{u},\mathbf{v})\} \subseteq \tilde{E}(\mathbf{u},\mathbf{v}).$$

Similarly, for any $\mathbf{v} \in V$, we have

$$\tilde{E}(\mathbf{u},\mathbf{v}) \subseteq \max_{\mathbf{u}\in U}\{\tilde{E}(\mathbf{u},\mathbf{v})\}.$$

Thus, for any $\mathbf{u} \in U$ and $\mathbf{v} \in V$, we obtain

$$\min_{\mathbf{v}\in V}\{\tilde{E}(\mathbf{u},\mathbf{v})\} \subseteq \max_{\mathbf{u}\in U}\{\tilde{E}(\mathbf{u},\mathbf{v})\}.$$

Therefore, $\min_{\mathbf{v}\in V}\{\tilde{E}(\mathbf{u},\mathbf{v})\} \subseteq \min_{\mathbf{v}\in V}\max_{\mathbf{u}\in U}\{\tilde{E}(\mathbf{u},\mathbf{v})\}$.
Hence,

$$\max_{\mathbf{u}\in U}\min_{\mathbf{v}\in V}\{\tilde{E}(\mathbf{u},\mathbf{v})\} \subseteq \min_{\mathbf{v}\in V}\max_{\mathbf{u}\in U}\{\tilde{E}(\mathbf{u},\mathbf{v})\},$$

i.e., $\rho^* \subseteq \eta^*$. □

Following Equation (1) and Equation (2), maximin strategy $\mathbf{u}^*$ and the gain floor $\rho^* = \langle \mu^*, \omega^*, \nu^* \rangle$ for Player-I can be obtained by solving the following multi-objective programming problem (5)

$$\begin{aligned}
& \max\{\mu\}, \min\{\omega\}, \min\{\nu\} \\
\text{s.t.} \quad & 1 - \prod_{h=1}^{p}(1-\mu_{hk})^{u_h} \geq \mu,\ \prod_{h=1}^{p}{\omega_{hk}}^{u_h} \leq \omega,\ \prod_{h=1}^{p}{\nu_{hk}}^{u_h} \leq \nu,\ (k \in \triangle_2) \\
& \sum_{h=1}^{p} u_h = 1,\ u_h \geq 0, \\
& \mu \geq 0,\ \omega \geq 0,\ \nu \geq 0,\ 0 \leq \mu+\omega+\nu \leq 3
\end{aligned} \tag{5}$$

where $\mu = \min_{k=1}^{q}\{1 - \prod_{h=1}^{p}(1-\mu_{hk})^{u_h}\}$, $\omega = \max_{k=1}^{q}\{\prod_{h=1}^{p}{\omega_{hk}}^{u_h}$ and $\nu = \max_{k=1}^{q}\{\prod_{h=1}^{p}{\nu_{hk}}^{u_h}\}$.

Similarly, the following multi-objective programming problem (6) is constructed by following Equation (3) and Equation (4) to obtain the minimax strategy $\mathbf{v}^*$ and the loss-ceiling $\eta^* = \langle \alpha^*, \beta^*, \gamma^* \rangle$ for Player-II.

$$\begin{aligned}
& \min\{\alpha\}, \max\{\beta\}, \max\{\gamma\} \\
\text{s.t.} \quad & 1 - \prod_{k=1}^{q}(1-\mu_{hk})^{v_k} \leq \alpha,\ \prod_{k=1}^{q}{\omega_{hk}}^{v_k} \geq \beta,\ \prod_{k=1}^{q}{\nu_{hk}}^{v_k} \geq \gamma,\ (h \in \triangle_1) \\
& \sum_{k=1}^{q} v_k = 1,\ v_k \geq 0, \\
& \alpha \geq 0,\ \beta \geq 0,\ \gamma \geq 0,\ 0 \leq \alpha+\beta+\gamma \leq 3,
\end{aligned} \tag{6}$$

where $\alpha=\max_{h=1}^{p}\{1-\prod_{k=1}^{q}(1-\mu_{hk})^{v_k}\}$, $\beta=\min_{h=1}^{p}\{\prod_{k=1}^{q}{\omega_{hk}}^{v_k}\}$ and $\gamma=\min_{h=1}^{p}\{\prod_{k=1}^{q}{\nu_{hk}}^{v_k}\}$.

In Problem (5), the objective functions ω and ν have the same importance, so we take the average of these two functions. Again maximization of μ is equivalent to minimization of $1-\mu$ and therefore Problem (5) reduces to the following Problem (7) as follows.

$$\begin{aligned}
&\min\{1-\mu\}, \min\left\{\frac{\omega+\nu}{2}\right\},\\
\text{s.t.}\quad &1-\prod_{h=1}^{p}(1-\mu_{hk})^{u_h}\geq\mu,\ \prod_{h=1}^{p}{\omega_{hk}}^{u_h}\leq\omega,\ \prod_{h=1}^{p}{\nu_{hk}}^{u_h}\leq\nu,\ (k\in\triangle_2)\\
&\sum_{h=1}^{p}u_h=1,\ u_h\geq 0,\\
&\mu\geq 0,\ \omega\geq 0,\ \nu\geq 0,\ 0\leq\mu+\omega+\nu\leq 3
\end{aligned}\tag{7}$$

Again, in Problem (6), the objective functions β and γ have the same importance, so we take the average of these two functions. Again minimization of α is equivalent to the maximization of $1-\alpha$ and therefore Problem (6) changes to Problem (8) as follows.

$$\begin{aligned}
&\max\{1-\alpha\}, \max\left\{\frac{\beta+\gamma}{2}\right\}\\
\text{s.t.}\quad &1-\prod_{k=1}^{q}(1-\mu_{hk})^{v_k}\leq\alpha,\ \prod_{k=1}^{q}{\omega_{hk}}^{v_k}\geq\beta,\ \prod_{k=1}^{q}{\nu_{hk}}^{v_k}\geq\gamma,\ (h\in\triangle_1)\\
&\sum_{k=1}^{q}v_k=1,\ v_k\geq 0,\\
&\alpha\geq 0,\ \beta\geq 0,\ \gamma\geq 0,\ 0\leq\alpha+\beta+\gamma\leq 3
\end{aligned}\tag{8}$$

Clearly, Problem (7) and Problem (8) are bi-objective programming problems. There are several methods to solve such problems. The notion of Pareto optimal is commonly-used to solve such problems. Here, we use the Lexicographic method [14] to solve Problem (7) and Problem (8).

For Problem (7), the following Problem (9) is constructed first.

$$\begin{aligned}
&\min\{1-\mu\}\\
\text{s.t.}\quad &1-\prod_{h=1}^{p}(1-\mu_{hk})^{u_h}\geq\mu,\ \prod_{h=1}^{p}{\omega_{hk}}^{u_h}\leq\omega,\ \prod_{h=1}^{p}{\nu_{hk}}^{u_h}\leq\nu,\ (k\in\triangle_2)\\
&\sum_{h=1}^{p}u_h=1,\ u_h\geq 0,\\
&\mu\geq 0,\ \omega\geq 0,\ \nu\geq 0,\ 0\leq\mu+\omega+\nu\leq 3
\end{aligned}\tag{9}$$

Let the solution of the non-linear programming Problem (9) is denoted by $(\mathbf{u}',\mu',\omega',\nu')$.

Following the Lexicographic approach, Problem (10) is constructed by combining the Problem (7) and the solution of Problem (9) as follows.

$$
\begin{aligned}
& \min\left\{\frac{\omega+\nu}{2}\right\} \\
\text{s.t.} \quad & 1-\prod_{i=1}^{p}(1-\mu_{hk})^{u_h} \geq \mu,\ \prod_{h=1}^{p}{\omega_{hk}}^{u_h} \leq \omega,\ \prod_{h=1}^{p}{\nu_{hk}}^{u_h} \leq \nu,\ (k \in \triangle_2) \\
& \sum_{h=1}^{p} u_h = 1,\ u_h \geq 0, \\
& \mu \geq 0,\ \omega \geq 0,\ \nu \geq 0,\ 0 \leq \mu+\omega+\nu \leq 3 \\
& 1-\mu \leq 1-\mu',\ \omega \leq \omega',\ \nu \leq \nu'
\end{aligned}
\tag{10}
$$

The optimal solution $(\mathbf{u}^*, \mu^*, \omega^*, \nu^*)$ will be obtained by solving Problem (10). Then it is obvious to prove that $(\mathbf{u}^*, \mu^*, \omega^*, \nu^*)$ is a Pareto optimal solution of Problem (7).

Similarly, Problem (8) turns into solving Problem (11) at first according to the Lexicographic method.

$$
\begin{aligned}
& \max\{1-\alpha\} \\
\text{s.t.} \quad & 1-\prod_{k=1}^{q}(1-\mu_{hk})^{v_j} \leq \alpha,\ \prod_{k=1}^{q}{\omega_{hk}}^{v_k} \geq \beta,\ \prod_{k=1}^{q}{\nu_{hk}}^{v_k} \geq \gamma,\ (h \in \triangle_1) \\
& \sum_{k=1}^{q} v_k = 1,\ v_k \geq 0 \\
& \alpha \geq 0,\ \beta \geq 0,\ \gamma \geq 0,\ 0 \leq \alpha+\beta+\gamma \leq 3
\end{aligned}
\tag{11}
$$

Let $(\mathbf{v}', \alpha', \beta', \gamma')$ is the optimal solution of Problem (11). Then Problem (12) is formulated by following the lexicographic approach as follows.

$$
\begin{aligned}
& \max\left\{\frac{\beta+\gamma}{2}\right\} \\
\text{s.t.} \quad & 1-\prod_{k=1}^{q}(1-\mu_{hk})^{v_k} \leq \alpha,\ \prod_{k=1}^{q}{\omega_{hk}}^{v_k} \geq \beta,\ \prod_{k=1}^{q}{\nu_{hk}}^{v_k} \geq \gamma,\ (h \in \triangle_1) \\
& \sum_{k=1}^{q} v_k = 1,\ v_k \geq 0 \\
& \alpha \geq 0,\ \beta \geq 0,\ \gamma \geq 0,\ 0 \leq \alpha+\beta+\gamma \leq 3 \\
& 1-\alpha \geq 1-\alpha',\ \beta \geq \beta',\ \gamma \geq \gamma'
\end{aligned}
\tag{12}
$$

Let $(\mathbf{v}^*, \alpha^*, \beta^*, \gamma^*)$ is the optimal solution of Problem (12). Then obviously the solution $(\mathbf{v}^*, \alpha^*, \beta^*, \gamma^*)$ will be the Pareto optimal solution of Problem (8).

5.1. **Algorithm**

The algorithm for solving SVNN matrix game by the proposed approach is abstracted as follows.

Step-1: Consider a matrix game with payoffs of SVNNs.

Step-2: To solve the game, two nonlinear multi-objective programming problems are formulated.

Step-3: Considering the same preference to the DOI and DONM, the multi-objective programming problems are converted to corresponding bi-objective programming problems.

Step-4: Lexicographic method is used to counter the bi-objective programming problems and the optimal strategies for the players are obtained.

The corresponding algorithmic flowchart of the proposed approach is presented in Figure 1.

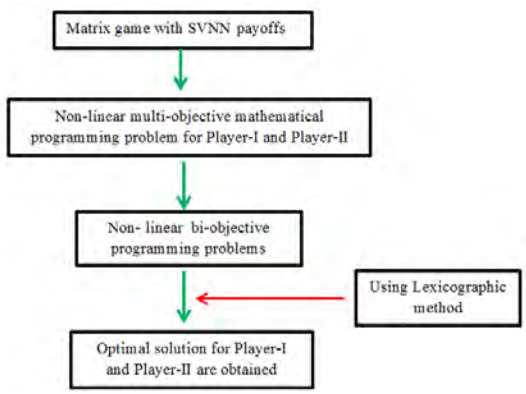

FIGURE 1. Algorithmic flowchart of the proposed approach

6. **Numerical Example**

This subsection provides a market share problem to illustrate the solution procedure of a SVNN matrix game.

6.1. *New factory set up-management problem*

Assume that a renowned foreign car manufacturing company 'X' is going to launch a new factory in India. For smooth functioning of the factory, the industrial manufacturing organization of the company 'X' suggests setting up the new factory either in the state Gujrat or in Hariyana, after considering various government policies and environmental conditions. The respective Ministry of Labour and Welfare department looks into the matter on behalf of their state governments and they jump to take the opportunity for a new industrial agreement, though their resources are limited. In that case, the respective Ministry of both states acts wisely and anticipates opponents' moves. The relationship of the two departments of the states can be regarded as the two players of the game. Here, we assume that the Ministry for Gujrat and Hariyana as Player-I and Player-II, respectively, and the opportunity to get the agreement as payoff values. Player-I and II has limited resources but they frame the game problem as to develop respective strategies and maximize the opportunity to build the new factory. In a zero-sum game, the opportunity gain by Player-I is positive and negative for Player-II. It is unrealistic for the decision-maker (DM) to get accurate and complete information on the payoff values. In this case, the SVNNs are used to express uncertainty.

Suppose that the marketing research team (MRT) of the company 'X' receives the information from the respective Ministry of the two states by following mainly three aspects like- Availability of Raw materials (strategy-I), Labour supply- including workers with the right skills (strategy-II) and Grants and financial incentives- usually from the government (strategy-III). These aspects may be considered as strategies of the two different states. Then according to the view of the MRT of the company 'X', the opportunity for Player-I is estimated and evaluated by using linguistic terms as follows.

$$\tilde{N} = \begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \end{array} \begin{array}{c} \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \begin{pmatrix} \text{Very High} & \text{High} & \text{Medium} \\ \text{Low} & \text{Very High} & \text{High} \\ \text{Medium} & \text{Very Low} & \text{Very High} \end{pmatrix} \end{array}.$$

Table 1 shows the corresponding relations between linguistic terms and SVNNs. Then the

TABLE 1. Assigned SVNN corresponding to linguistic term

Linguistic term	Very High	High	Medium	Low	Very Low
SVNN	$\langle 0.95, 0.07, 0.05\rangle$	$\langle 0.7, 0.5, 0.25\rangle$	$\langle 0.5, 0.4, 0.4\rangle$	$\langle 0.25, 0.3, 0.7\rangle$	$\langle 0.05, 0.1, 0.95\rangle$

pay-off matrix $\tilde{A}$ with payoffs of SVNNs may be transformed according to Table 1 as follows.

$$\tilde{A} = \begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \end{array} \begin{array}{c} \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \begin{pmatrix} \langle 0.95, 0.07, 0.05\rangle & \langle 0.70, 0.50, 0.25\rangle & \langle 0.50, 0.40, 0.40\rangle \\ \langle 0.25, 0.30, 0.70\rangle & \langle 0.95, 0.07, 0.05\rangle & \langle 0.70, 0.50, 0.25\rangle \\ \langle 0.50, 0.40, 0.40\rangle & \langle 0.05, 0.10, 0.95\rangle & \langle 0.95, 0.07, 0.05\rangle \end{pmatrix} \end{array},$$

where $\langle 0.95, 0.07, 0.05\rangle$ in the matrix $\tilde{A}$ is a SVNN which represents that Player-I will get the opportunity to set up the new project positively by 95%, unable to get the opportunity by 5%. Player-I has an indeterminacy about the establishment of the new project by 7%. These situation occur only when both of the players use the strategy I simultaneously. Other SVNNs in $\tilde{A}$ have also a similar explanation.

6.2. *The solution procedure and result discussion*

Problem (13) is formulated by following Problem (9) as follows.

$$\begin{aligned}
&\min\{1-\mu\}\\
\text{s.t. }& 1-0.05^{u_1}0.75^{u_2}0.5^{u_3}\geq\mu,\ 1-0.3^{u_1}0.05^{u_2}0.95^{u_3}\geq\mu,\ 1-0.5^{u_1}0.3^{u_2}0.05^{u_3}\geq\mu,\\
&0.07^{u_1}0.3^{u_2}0.4^{u_3}\leq\omega,\ 0.5^{u_1}0.07^{u_2}0.1^{u_3}\leq\omega,\ 0.4^{u_1}0.5^{u_2}0.07^{u_3}\leq\omega,\\
&0.05^{u_1}0.7^{u_2}0.4^{u_3}\leq\nu, 0.25^{u_1}0.05^{u_2}0.95^{u_3}\leq\nu,\ 0.4^{u_1}0.25^{u_2}0.05^{u_3}\leq\nu,\\
&u_1+u_2+u_3=1,\ u_1,u_2,u_3\geq 0,\\
&0\leq\mu+\omega+\nu\leq 3,\ \mu,\omega,\nu\geq 0.
\end{aligned}\tag{13}$$

Solving Equation (13), the obtained optimal solution is $\mu'=0.772989$, $\omega'=1$ and $\nu'=1$.

According to Equation (10), Problem (14) is constructed as follows.

$$\begin{aligned}
&\min\left\{\frac{\omega+\nu}{2}\right\}\\
\text{s.t.}\quad & 1-0.05^{u_1}0.75^{u_2}0.5^{u_3}\geq\mu,\ 1-0.3^{u_1}0.05^{u_2}0.95^{u_3}\geq\mu,\ 1-0.5^{u_1}0.3^{u_2}0.05^{u_3}\geq\mu,\\
&0.07^{u_1}0.3^{u_2}0.4^{u_3}\leq\omega,\ 0.5^{u_1}0.07^{u_2}0.1^{u_3}\leq\omega,\ 0.4^{u_1}0.5^{u_2}0.07^{u_3}\leq\omega\\
&0.05^{u_1}0.7^{u_2}0.4^{u_3}\leq\nu,\ 0.25^{u_1}0.05^{u_2}0.95^{u_3}\leq\nu, 0.4^{u_1}0.25^{u_2}0.05^{u_3}\leq\nu\\
&1-\mu\leq 0.227011,\ \omega\leq 1,\ \nu\leq 1\\
&0\leq\mu+\omega+\nu\leq 3,\ \mu,\omega,\nu\geq 0\\
&u_1+u_2+u_3=1,\ u_1,u_2,u_3\geq 0.
\end{aligned}\tag{14}$$

Solving Problem (14), the obtained optimal solution is $\mathbf{u}^*=(0.3151,0.3138,0.3711)$, and $\rho^*=\langle\mu^*,\omega^*,\nu^*\rangle=\langle 0.7228,0.2476,0.2476\rangle$.

Now for Player-II, Problem (15) is constructed by following Equation (11), as follows.

$$\begin{aligned}
&\max\{1-\alpha\}\\
\text{s.t.}\quad & 1-0.05^{v_1}0.3^{v_2}0.5^{v_3}\leq\alpha, 1-0.75^{v_1}0.05^{v_2}0.3^{v_3}\leq\alpha, 1-0.5^{v_1}0.95^{v_2}0.05^{v_3}\leq\alpha,\\
&0.07^{v_1}0.5^{v_2}0.4^{v_3}\geq\beta, 0.3^{v_1}0.07^{v_2}0.5^{v_3}\geq\beta,\ 0.4^{v_1}0.1^{v_2}0.07^{v_3}\geq\beta,\\
&0.05^{v_1}0.25^{v_2}0.4^{v_3}\geq\gamma,\ 0.7^{v_1}0.05^{v_2}0.25^{v_3}\geq\gamma,\ 0.4^{v_1}0.95^{v_2}0.05^{v_3}\geq\gamma,\\
&v_1+v_2+v_3=1,\ v_1,v_2,v_3\geq 0,\\
&0\leq\alpha+\beta+\gamma\leq 3,\ \alpha,\beta,\gamma\geq 0.
\end{aligned}\tag{15}$$

Solving Equation (15), the obtained optimal solution is $\alpha'=0.7346452$, $\beta'=0$ and $\gamma'=0$.

Then following Equation (12), Problem (16) is constructed as follows.

$$\begin{aligned}
&\max\left\{\frac{\beta+\gamma}{2}\right\}\\
\text{s.t.}\quad & 1-0.05^{v_1}0.3^{v_2}0.5^{v_3}\le\alpha,\ 1-0.75^{v_1}0.05^{v_2}0.3^{v_3}\le\alpha,\ 1-0.5^{v_1}0.95^{v_2}0.05^{v_3}\le\alpha,\\
& 0.07^{v_1}0.5^{v_2}0.4^{v_3}\ge\beta,\ 0.3^{v_1}0.07^{v_2}0.5^{v_3}\ge\beta,\ 0.4^{v_1}0.1^{v_2}0.07^{v_3}\ge\beta,\\
& 0.05^{v_1}0.25^{v_2}0.4^{v_3}\ge\gamma,\ 0.7^{v_1}0.05^{v_2}0.25^{v_3}\ge\gamma,\ 0.4^{v_1}0.95^{v_2}0.05^{v_3}\ge\gamma,\\
& 1-\alpha\ge 0.2653548,\ 0\le\alpha+\beta+\gamma\le 3,\ \alpha,\beta,\gamma\ge 0, \qquad (16)\\
& v_1+v_2+v_3=1,\ v_1,v_2,v_3\ge 0.
\end{aligned}$$

Solving Problem (16), the optimal obtained solution is $\mathbf{v}^*=(0.1807,0.4255,0.3938)$, and $\eta^*=\langle\alpha^*,\beta^*,\gamma^*\rangle=\langle 0.7346,0.1116,0.1518\rangle$.

The following observations can be made for the results obtained.

(1) $\rho^*=\langle\mu^*,\omega^*,\nu^*\rangle=\langle 0.7228,0.2476,0.2476\rangle$ represents that the state Gujrat has the opportunity to set up the new factory positively by 72.28% and unable to get the opportunity by 24.76%. Also, it is indeterminate to assume that the state Gujrat will get the opportunity by 24.76%. In this case, strategies I, II and III are chosen with probability 0.3151, 0.3138 and 0.3711, respectively.

(2) $\eta^*=\langle\alpha^*,\beta^*,\gamma^*\rangle=\langle 0.7346,0.1116,0.1518\rangle$ represents that the state Hariyana has the opportunity to set up the new factory positively by 73.46% and unable to get the opportunity by 11.16%. Whereas, it is indeterminate to say that the state Hariyana will get the opportunity by 15.18%. In this case, strategies I, II and III are chosen with probability 0.1807,0.4255, and 0.3938, respectively.

(3) ρ^* and η^* are obtained as SVNN, which is desirable.

(4) It is clear that $\mu^*\le\alpha^*$, $\omega^*\ge\beta^*$, $\nu^*\ge\gamma^*$, then $\rho^*\subseteq\eta^*$, which follows Theorem 5.1.

(5) For the maximin strategy $\mathbf{u}^*=(0.3151,0.3138,0.3711)$ and minimax strategy $\mathbf{v}^*=(0.1807,0.4255,0.3938)$, the expected payoff for Player-I is $E(\mathbf{u}^*,\mathbf{v}^*)=\langle 0.7713,0.1862,0.2083\rangle$, which is the value of SVNN matrix game $\tilde{N}$. From Table 1, it is to conclude that the expected payoff for Player-I is between "Very High" and "High" in terms of linguistic terms.

6.3. *Analysis and comparison of results with Li and Nan [13] approach*

NS takes into consideration the indeterminacy together with the membership and non-membership whereas intuitionistic fuzzy set (IFS) consider membership and non-membership and ignore indeterminacy of the elements. So NS is the generalization of IFS. Therefore Atanassov's intuitionistic fuzzy number [2] is a particular case of SVNN as we can simply consider the sum of three independent functions as equal to 1.

To verify the efficiency of the proposed approach, at first, we transfer the payoff matrix $\tilde{N}$ with payoffs as SVNN to a payoff matrix $\tilde{N}'$ with Atanassov's intuitionistic fuzzy sets by considering the DOI as zero, where

$$\tilde{N}' = \begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \end{array} \begin{array}{c} \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \begin{pmatrix} \langle 0.95, 0.05\rangle & \langle 0.70, 0.25\rangle & \langle 0.50, 0.4\rangle \\ \langle 0.25, 0.70\rangle & \langle 0.95, 0.05\rangle & \langle 0.70, 0.25\rangle \\ \langle 0.50, 0.40\rangle & \langle 0.05, 0.95\rangle & \langle 0.95, 0.05\rangle \end{pmatrix} \end{array}.$$

Here, the payoff matrix $\tilde{N}'$ is the same as the payoff matrix considered in the paper of Li and Nan [13]. Li and Nan [13] proposed a nonlinear programming approach to solve matrix game with payoffs of Atanassov's intuitionistic fuzzy sets. Li and Nan [13] use the weighted average method to solve a pair of nonlinear programming models which are derived from two auxiliary nonlinear bi-objective programming models. We use the proposed technique for the payoff matrix $\tilde{N}'$ and obtained results are shown in Table 2. Table 2 also shows the results obtained by Li and Nan [13] approach for the mid-value 0.5 of the interval of the weight λ.

TABLE 2. Results for the matrix game with pay-off matrix $\tilde{A}'$

Articles	$\mathbf{u}^*$	$\mathbf{v}^*$	$E(\mathbf{u}^*, \mathbf{v}^*)$
Li and Nan [13] (for $\lambda = 0.5$)	(0.4076,0.3325,0.2599)	(0.2681,0.2958,0.4361)	$\langle 0.7729, 0.2035\rangle$
Our proposed technique	(0.4009,0.3292,0.2699)	(0.2770,0.2972,0.4258)	$\langle 0.7730, 0.2037\rangle$

Table 2 shows that the optimal expected value obtained by the proposed technique discussed in this paper is $\langle 0.7729, 0.2035\rangle$ and obtained by Li and Nan [13] approach is $\langle 0.7730, 0.2037\rangle$, which implies the fact that the optimal value for both of the players obtained by the proposed method and Li and Nan [13] approach are approximately the same. Table 2 shows that Player-I chooses three strategies (I, II, and III) with probability 0.4076,0.3325, 0.2599 respectively which is obtained through the proposed approach, whereas Player-I choose the same strategies with probability 0.4009, 0.3292, and 0.2699 respectively which is obtained through Li and Nan [13] approach. This implies that the optimal strategies for both players are very similar. The approach discussed in this paper considers DOI together with the DOM and the DONM. Therefore, the proposed method is an extension of Li and Nan [13] method and counter uncertainty in a larger sense. This manifests the effectiveness and validity of the proposed technique.

7. Conclusion

SVNN is a vital tool to tackle uncertainty in decision-making problems. This paper uses SVNNs to represent the imprecise payoffs so that players can consider the neutrality of the elements better. In this paper, a solution procedures is established to solve a new matrix game where the payoffs are represented by SVNNs. Two non-linear multi-objective programming problems are formulated to obtain the optimal values and optimal strategies for the players. These multi-objective programming problems are transformed to bi-objective programming problems by considering the same importance of the objective functions. Then Lexicographic method is applied to counter the bi-objective programming problems.

The proposed approach is illustrated by solving a market share problem, which implies the validity and effectiveness of the proposed approach. The optimal solutions are obtained in SVNN form, which is desirable. We consider the matrix game problem from Li and Nan [13] with payoffs as IFSs and the proposed approach is illustrated by considering the DOI as zero. The obtained results are compared with the results obtained by Li and Nan's [13] approach and observed that the optimal strategies for both players are very similar. This demonstrates the reliability of our approach.

The limitation of the proposed approach is that it does not find the solution to the game problem directly, as it considers the construction of multi-objective programming problems. Also, we do not make any conclusion about the existence of the solution of the SVNN matrix game in the sense discussed in this paper. Therefore these problems need a further investigation in the future.

The proposed methods are indeed able to solve SVNN matrix games. Moreover, the concept of this work is readily applicable to other games such as two-person non-zero-sum games, multi-objective matrix game problems. Although the discussed approach is applied to solve a market share problem, the proposed approach may be applied in management science, war science, economics, advertising, cyber security related problems. In addition, a pair of nonlinear bi-objective programming problems is derived from the two auxiliary nonlinear multi-objective programming problems and countered by using the Lexicographic method. Therefore, other new methods for solving matrix games with payoffs of the SVNNs may be investigated in the near future.

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Fixed Point Theorem on Neutrosophic Triplet b-Metric Space

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Abstract. The notion of neutrosophic triplet, in the form of (p, n_p, a_p) is a recent subject of neutrosophy, where n_p is the neutral of the element p and a_p is the opposite of p. In this paper, neutrosophic triplet b-metric spaces are investigated. Then some new definitions and examples are given for neutrosophic triplet b-metric space. Based on these definitions, new theorems are given and proven. A neutrosophic triplet topology induced by neutrosophic triplet b-metric is obtained. Furthermore, a contraction map is defined for neutrosophic triplet b-metric space, and finally, a fixed point theorem is given for it.

Keywords: Neutrosophic triplet set; neutrosophic triplet topology; neutrosophic triplet metric; neutrosophic triplet b-metric.

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1. Introduction

The classical mathematical methods may not be sufficient to answer some of the complex problems encountered in such disciplines as usual set theory, fuzzy set theory, and probability theory. Many fields, such as economics, engineering, and environmental science, need to model linguistic values and uncertainties mathematically to continue their studies.

Zadeh defined the fuzzy set concept for handling uncertainty problems in 1965 [1]. A fuzzy set is substantially a function that appoints a membership degree (truth value) from the range [0,1] to each element in the universal set. In 1986 [2], Atanasov, as an expansion of the fuzzy set, described the notion of intuitionistic fuzzy set, via defining a non-membership degree in addition to the membership degree.

These theories can deal with real-world problems, but not with the indeterminate data. Based on this, Smarandache published his work in 2005 [3], containing the concept of neutrosophic sets and some of its applications. In this theory, truth, falsity, and indeterminacy are defined as independent of each other. A neutrosophic set is a triplet of the functions

(T, I, F), each of them is from the universal set to [0,1]; T assigns truth-value, I assigns indeterminate-value and F assigns false-value.

Lately, some researchers have been preoccupied with neutrosophic set theory [4–9].

By using the basic rule of neutrosophy, Smarandache established neutrosophic triplet set theory as a generalization of the classical group in 2017 [10]. An neutrosophic triplet space is dissimilar from the usual group because for all element p in a neutrosophic triplet space A with an operation $*$, there is a neutral of p, indicated by n_p and opposite of p, indicated by a_p such that $p * n_p = n_p * p = p$ and $p * a_p = a_p * p = n_p$. The neutral of any element p is distinct from the standard algebraic unit element, and it is not unique. A neutrosophic triplet is of the form $< p, n_p, a_p >$.

For a neutrosophic triplet space $(A, *)$ and a subset $B \subset A$, if B also forms a neutrosophic triplet space under the operation $*$, then $(B, *)$ is called a neutrosophic triplet subset of A.

Smarandache and Ali established a group structure on neutrosophic triplet space in 2018 [11]. Therefore, a new group form is assigned to some non-classical group structures.

Şahin, Kargın and Smarandache introduced neutrosophic triplet topology in [12]. Neutrosophic triplet base was defined for a neutrosophic triplet topology in [13]. Şahin and Kargin defined neutrosophic triplet metric space in 2017 [14] and neutrosophic triplet normed ring space in 2018 [15]. Also, in 2019 [16], they described neutrosophic triplet metric topology and investigated some features of metric topology in neutrosophic triplet metric space. In [17–20], researchers studied neutrosophic triplet metric spaces.

In recent years, many generalizations of classical metric spaces appeared. b-metric space is one of these generalizations, which was introduced by Czerwik in 1993 [21]. Then some fixed point theorems were extended to b-metric spaces. Şahin and Kargin extended b-metric to neutrosophic triplet metric space and defined convergence sequence and Cauchy sequence in neutrosophic triplet b-metric spaces [22]. In 2019 [23], Şahin and Kargın defined neutrosophic triplet b-metric topology by using the open balls.

The main idea of this paper is to describe a contraction for neutrosophic triplet b-metric space and prove a fixed point theorem for neutrosophic triplet b-metric space. Section 2 gives some basic definitions and examples for b-metric and neutrosophic triplet space. Section 3 presents the neutrosophic triplet b-metric space. Many examples are given in this section to illustrate the difference between neutrosophic triplet metric and neutrosophic triplet b-metric. Also, a sufficient condition is given for a convergence sequence in a neutrosophic triplet b-metric space to be a Cauchy sequence. Then a neutrosophic triplet topological structure is established by using neutrosophic triplet b-metric. Section 4 defines a contraction for neutrosophic triplet b-metric space. It is proven that the contraction is continuous respect to the induced topology by neutrosophic triplet b-metric. Also, a fixed point theorem is given and proven for neutrosophic triplet b-metric space. Finally, Section 5 and 6 present a brief of the paper and some future works.

2. Preliminaries

A review of essential concepts of neutrosophic triplet set are presented in this section.

Definition 2.1. [21] Let A be set and $d : A \times A \to \mathbb{R}^+ \cup \{0\}$. If the following statements are satisfied, then d is called a b-metric;

bm$_1$) $u = v$ if and only if $d(u, v) = 0$

bm$_2$) $d(u, v) = d(v, u)$

bm$_3$) for all $u, v, w \in A$, there is a real number $b \geq 1$ such that $d(u, v) \leq b\,[d(u, w) + d(w, v)]$.

The pair (A, d) is called a b-metric space with coefficient b.

It is clear that every metric space is a b-metric space with coefficient 1, but the converse is not always true. For example, $d : A \times A \to \mathbb{R}, d(u, v) = (u - v)^2$ is a b-metric with a coefficient $b{\geq}2$, but it is not a metric. Also, the reader can realize that if d is a b-metric with the coefficient b, then d is a b-metric with any coefficient $k{\geq}b$.

Definition 2.2. [11] Let A be a set and $*$ be a binary operation on A. If for each $p \in A$ there exists n_p and a_p in A such that

$$p * n_p = n_p * p = p$$

and

$$p * a_p = a_p * p = n_p$$

then $(A, *)$ is called neutrosophic triplet set. Also, n_p and a_p are called neutral and opposite of p, respectively. A neutrosophic triplet p is indicated by $p =< p, n_p, a_p >$. Besides, $p_1 = p_2$ if and only if $n_{p_1} = n_{p_2}$ and $a_{p_1} = a_{p_2}$.

Definition 2.3. [13] Let $(A, *)$ be a neutrosophic triplet set, $P(A)$ be set of each subset of A and $\tau \subset P(A)$. If

1) $\varnothing, A \in \tau$
2) The intersection of a finite number of sets in τ is also in τ
3) The union of an arbitrary number of sets in τ is also in τ

then τ is called as a neutrosophic triplet topology on A and, $((A, *), \tau)$ is called as a neutrosophic triplet topological space.

Definition 2.4. [15] A neutrosophic triplet metric on a neutrosophic triplet space $(A, *)$ is a function $d_T : A \times A \to \mathbb{R}$ such that, for all $u, v \in A$,

m$_1$) $u * v \in A$

m$_2$) $d_T(u, v) \geq 0$

m$_3$) If $u = v$ then $d_T(u, v) = 0$

m$_4$) $d_T(u, v) = d_T(v, u)$

m$_5$) If there exists at least an element $w \in A$ for each $u, v \in A$ such that $d_T(u, v) \leq d_T(u, v * n_w)$, then

$$d_T(u, v * n_w) \leq d_T(u, w) + d_T(w, v).$$

$((A, *), d_T)$ is called a neutrosophic triplet metric space.

Example 2.5. [14] Let A be a set. Then $(P(A), \cup)$ is a neutrosophic triplet space with $n_U = a_U = U$ for all $U \in P(A)$, where $P(A)$ is the power set of A.

Define a map $d : P(A) \times P(A) \to \mathbb{R}$ such that $d(U, V) = |m(U) - m(V)|$, where $m(U)$ and $m(V)$ denote the numbers of elements of U and V, respectively. Then $((P(A), \cup), d)$ is a neutrosophic triplet metric space.

3. Neutrosophic Triplet b-metric

This section gives some new examples and definitions for neutrosophic triplet b-metric space. Also it defines neutrosophic triplet b-metric topology.

Definition 3.1. [22] A neutrosophic triplet b-metric is a function $d_b : A \times A \to \mathbb{R}$ such that, for all $u, v \in A$,

b$_1$) $u * v \in A$

b$_2$) $d_b(u, v) \geq 0$

b$_3$) if $u = v$ then $d_b(u, v) = 0$

b$_4$) $d_b(u, v) = d_b(v, u)$

b$_5$) If there exists at least an element $w \in A$ for each $u, v \in A$ such that $d_b(u, v) \leq d_b(u, v * n_w)$, then for a real number $b \geq 1$,

$$d_b(u, v * n_w) \leq b[d_b(u, w) + d_b(w, v)].$$

$((A, *), d_b)$ is called a neutrosophic triplet b-metric space.

neutrosophic triplet b-metric is distinct from the usual b-metric because of the binary operation and neutral element.

It is clear that all neutrosophic triplet metric space is a neutrosophic triplet b-metric space with coefficient 1, but the opposite is not true always like the following example indicates.

Example 3.2. Let $A = \{0, 2, 3, 4\}$. Then A is a neutrosophic triplet space with the multiplication module 6 in $\mathbb{Z}$. Neutrosophic triplets are $< 0,0,0 >, < 2,4,2 >, < 3,3,3 >$ and $< 4,4,4 >$. Let $d : A \times A \to \mathbb{R}$ be defined such that $d(u,v) = (u-v)^2$, for every $u, v \in A$, . Then it is clear that d_b satisfies conditions b_1, b_2, b_3 and b_4.

b_5) For $u = 0, v = 2$,

$d'(0,2) \leq d'(0,2.n_2)$. Also $d'(0,2.n_2) \leq d'(0,2) + d'(2,2)$.

$d'(0,2) \leq d'(0,2.n_4)$. Also $d'(0,2.n_4) \leq d'(0,4) + d'(4,2)$.

For $u = 0, v = 3$,

$d'(0,3) \leq d'(0,3.n_3)$. Also $d'(0,3.n_3) \leq d'(0,3) + d'(3,3)$.

For $u = 0, v = 4$

$d'(0,4) \leq d'(0,4.n_2)$. Also $d'(0,4.n_2) \leq 2\,[d'(0,2) + d'(2,4)]$.

$d'(0,4) \leq d'(0,4.n_4)$. Also $d'(0,4.n_4) \leq d'(0,4) + d'(4,4)$.

For $u = 2, v = 3$,

$d'(2,3) \leq d'(2,3.n_0)$. Also $d'(2,3.n_0) \leq d'(2,0) + d'(0,3)$.

$d'(2,3) \leq d'(2,3.n_2)$. Also $d'(2,3.n_2) \leq 4\,[d'(2,2) + d'(2,3)]$.

$d'(2,3) \leq d'(2,3.n_4)$. Also $d'(2,3.n_4) \leq 2\,[d'(2,4) + d'(4,3)]$.

For $u = 2, v = 4$,

$d'(2,4) \leq d'(2,4.n_0)$. Also $d'(2,4.n_0) \leq d'(2,0) + d'(0,4)$.

$d'(2,4) \leq d'(2,4.n_2)$. Also $d'(2,4.n_2) \leq d'(2,2) + d'(2,4)$.

$d'(2,4) \leq d'(2,4.n_3)$. Also $d'(2,4.n_3) \leq 2\,[d'(2,3) + d'(3,4)]$.

$d'(2,4) \leq d'(2,4.n_4)$. Also $d'(2,4.n_4) \leq d'(2,4) + d'(4,4)$.

For $u = 3, v = 4$,

$d'(3,4) \leq d'(3,4.n_0)$. Also $d'(3,4.n_0) \leq d'(3,0) + d'(0,4)$.

$d'(3,4) \leq d'(3,4.n_3)$. Also $d'(3,4.n_3) \leq d'(3,3) + d'(3,4)$.

Thus d is not a neutrosophic triplet metric, but a neutrosophic triplet b-metric with a coefficient $b \geq 4$.

Example 3.3. Let $A \neq \varnothing$ be a finite subset of the set of natural numbers $\mathbb{N}$ and $P(A)$ be the power set of A. Then $(P(A), \cup)$ is a neutrosophic triplet space. Let $d : P(A) \times P(A) \to \mathbb{R}$ be defined as, if $U = V, d(U,V) = 0$ and if $U \neq V$,

$$d(U,V) = \begin{cases} \left|2^{m(U)} - 2^{m(V)}\right|, & \text{if } m(U) \text{ and } m(V) \text{ are even} \\ 3, & \text{if } m(U) \text{ and } m(V) \text{ are odd} \\ 1, & \text{otherwise} \end{cases},$$

where $m(U)$ denotes the number of elements of U. Then d is not a neutrosophic triplet metric but a neutrosophic triplet b-metric. For example let $m(A) = 10$ and let $m(U) = 8, m(V) = 6, m(W) = 3$ and $W \subset V$. Then

$$d(U,V) \leq d(U, V \cup n_W) = d(U, V \cup W) = d(U,V).$$

But $d(U,V) \geq d(U,W) + d(W,V)$ since $d(U,V) = 192, d(U,W) = 1$ and $d(W,V) = 1$. So d is not a neutrosophic triplet metric. But for $b \geq 1023$, d is a neutrosophic triplet b-metric. (If we take $U = A$ and $V = W = \varnothing$, then b must be at least $\left|2^{10} - 2^0\right|$).

For generally d is a neutrosophic triplet b-metric with a coefficient $b \geq |2^{m(A)} - 2|$.

Example 3.4. Let $A = \{1,2,3,4,5,6,7,8,9,10\}$. Then $(P(A), \cap)$ is a neutrosophic triplet space with neutrosophic triplets in the form $< U, U, U >$.

Let define a map $d : P(A) \times P(A) \to \mathbb{R}$ such that

$$d(U,V) = \begin{cases} 1, & \text{if } U \cap V = \varnothing, \\ \frac{1}{m(U \cap V)}, & \text{if } U \cap V \neq \varnothing, U \neq V, \\ 0, & \text{if } U = V. \end{cases}$$

If we take $U = \{1,2,3\}$, $V = \{4,5,6,7\}$, and $W = \{1,2,3,4,5,6,7\}$. Then

$$d(U,V) = 1 \leq d(U, V \cap n_W) = d(U,V).$$

But

$$d(U, V \cap n_W) \geq d(U,W) + d(W,V) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$

Then d is not a neutrosophic triplet metric. But $((P(A), \cap), d)$ is a neutrosophic triplet b-metric space for any coefficient $b \geq \frac{9}{2}$. For generally, d_b is a neutrosophic triplet b-metric with a coefficient $b \geq \frac{m(A)-1}{2}$.

Now, let give the concept of convergent sequence in neutrosophic triplet b-metric space and some properties.

Definition 3.5. [22] Let $((A,*), d_b)$ be a neutrosophic triplet b-metric space, $\{p_n\}$ be a sequence in A. If for all $\varepsilon > 0$, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$, $d_b(p, p_n) < \varepsilon$, then $\{p_n\}$ converges to p in $((A,*), d_b)$, denoted by $\lim_{n\to\infty} p_n = p$ or $p_n \to p$.

Definition 3.6. [22] Let $((A,*),d_b)$ be a neutrosophic triplet b-metric space, $\{p_n\}$ be a sequence in A. If for all $\varepsilon > 0$, there exists an $M \in \mathbb{N}$ such that, for all $n,m \geq M$, $d_b(p_n,p_m) \leq \varepsilon$, then $\{p_n\}$ is a Cauchy sequence.

Theorem 3.7. *Let $((A,*),d_b)$ be a neutrosophic triplet b-metric space, $\{p_n\}$ be a sequence in A. If for all $\varepsilon > 0$, there exists an $M \in \mathbb{N}$ such that, for all $n,m \geq M$, $d_b(p_n,p) < \varepsilon$ and $d_b(p_n,p_m) \leq d_b(p_n,p_m * n_p)$, then $\{p_n\}$ is a Cauchy sequence.*

Proof. By the hypothesis, for all $\varepsilon > 0$, and for all $n,m \geq M$, $d_b(p,p_n) < \frac{\varepsilon}{2b}$ and $d_b(p,p_m) < \frac{\varepsilon}{2b}$. Since $d_b(p_n,p_m) \leq d_b(p_n,p_m * n_p)$, it is clear that

$$\begin{aligned} d_b(p_n,p_m * n_p) &\leq b\left[d_b(p_n,p) + d_b(p,p_m)\right] \\ &< b\left[\frac{\varepsilon}{2b} + \frac{\varepsilon}{2b}\right] = \varepsilon. \end{aligned}$$

Thus $\{p_n\}$ is a Cauchy sequence. □

Definition 3.8. Let $((A,*),d_b)$ be a neutrosophic triplet b-metric space. If every Cauchy sequent $\{x_n\}$ in A is convergent in A, then $((A,*),d_b)$ is called a complete neutrosophic triplet b-metric space.

Now, let define neutrosophic triplet b-metric topology and give some properties. First, we recall the definition of neutrosophic triplet topology.

Definition 3.9. Let $(A,*)$ be a neutrosophic triplet space, $P(A)$ be the power set of A and $\tau \subseteq P(A)$. If the following conditions are satisfied, then τ is called a neutrosophic triplet topology on A,

T_1) $U * V \in A$, for all $U,V \in A$

T_2) $\varnothing, A \in \tau$

T_3) If $U_i \in \tau$, for all $i \in I$, then $\bigcup\limits_{i\in I} U_i \in A$

T_4) If $U_i \in \tau$, for all $i \in J$ ($J \subset I$, J is finite), then $\bigcap\limits_{i\in J} U_i \in A$.

$((A,*),\tau)$ is called a neutrosophic triplet topological space.

Now let recall the definition of neutrosophic triplet b-metric topology in [23].

Definition 3.10. [23] Let $((A,*),\tau)$ be a neutrosophic triplet topological space and $\mathcal{B} \subset P(A)$. Then the family

$$\mathcal{B}^{\#} = \{Y \subset A \mid Y = \cup Z, Z \subset Y\}$$

is called as the base of the neutrosophic topology τ.

Definition 3.11. [23] Let $((A,*),d_b)$ be a neutrosophic triplet b-metric space, $a \in A$ and $r > 0$. Then

i) $B(a,r) = \{x \in A \mid d_b(a,x) < r\}$ is called as an open ball with center a and radius r.

ii) $B[a,r] = \{x \in A \mid d_b(a,x) \leq r\}$ is called as a closed ball with center a and radius r.

iii) $S(a,r) = \{x \in A \mid d_b(a,x) = r\}$ is called as a disk with center a and radius r.

Definition 3.12. [23] Let $((A,*),d_b)$ be a neutrosophic triplet b-metric space and $\mathcal{B} = \{B(a,\epsilon) \mid a \in A, \epsilon > 0\}$. Then the family $\mathcal{B}^{\#} = \{Y \subset A \mid Y = \cup Z, Z \subset Y\}$ is a neutrosophic triplet topology on A, called as neutrosophic triplet b-metric topology.

Now, let define a neutrosophic triplet topology that arises from neutrosophic triplet b-metric.

For a neutrosophic triplet b-metric $d_b : A \times A \to \mathbb{R}$, let $B^n_\varepsilon(p) = \{q \in A \mid nd_b(p,q) < \varepsilon, \text{ for any } n \in \mathbb{N}\}$.

Theorem 3.13. *Let $((A,*),d_b)$ be a neutrosophic triplet b-metric space. Then*

$$\tau_{d_b} = \{U \subset A \mid \exists\varepsilon > 0, \exists n \in \mathbb{N},\ B^n_\varepsilon(p) \subset U \text{ for all } p \in U\}$$

is a neutrosophic triplet topology.

Proof. T_1 and T_2 are clear. So let prove the other conditions.

T_3) Let $U_i \in \tau_{d_b}$ and $p \in \bigcup_{i\in I} U_i$. Then there is $j \in I$ such that $p \in U_j$. Since $U_j \in \tau_{d_b}$, $\exists\varepsilon > 0$ such that

$$B^n_\varepsilon(p) \subseteq U_j \Rightarrow B^n_\varepsilon(p) \subseteq \bigcup_{i\in I} U_i.$$

Therefore, $\bigcup_{i\in I} U_i \in \tau_{d_b}$.

T_4) Let $U_i \in \tau_{d_b}$, for all $i \in J (J \subset I, J$ is finite) and $p \in \bigcap_{i\in J} U_i$. Then $p \in U_i$ for all $i \in J$. Also $B^{n_i}_{\varepsilon_i}(p) \subseteq U_i$ for $\exists\varepsilon_i > 0, \exists n_i \in \mathbb{N}$, since $U_i \in \tau_{d_b}$.

Let $\inf\{\varepsilon_i \mid i \in J\} = \varepsilon$ and $q \in B^{\sum_{i\in J} n_i}_\varepsilon(p)$. Then for all $i \in J$,

$$n_i d_b(p,q) < \left(\sum_{i\in J} n_i\right).d_b(p,q) < \varepsilon.$$

So $n_i d_b(p,q) < \varepsilon_i$ for all $i \in J$. Thus $q \in B^{n_i}_{\varepsilon_i}(p) \Rightarrow q \in U_i$, for all $i \in J$, since $B^{n_i}_{\varepsilon_i}(p) \subseteq U_i$. That means $q \in \bigcap_{i\in J} U_i$ and thus $B^{n_i}_{\varepsilon_i}(p) \subseteq \bigcap_{i\in J} U_i$, for all $i \in J$. So $\bigcap_{i\in J} U_i \in \tau_{d_b}$.

Consequently, τ_{d_b} is an neutrosophic triplet topology.

This topology called as neutrosophic triplet topology induced by neutrosophic triplet b-metric. □

4. **Fixed Point Theorem For Neutrosophic Triplet b-Metric Space**

In this section, first, a contraction is defined for neutrosophic triplet b-metric space and a fixed point theorem for neutrosophic triplet b-metric space is given and proven.

Definition 4.1. Let $((A, *), d_b)$ be a neutrosophic triplet b-metric space and $S : A \to A$ be a map. If for all $p, q \in A$,

i) there exists any element $z \in A$ such that $d_b(p, q) \leq d_b(p, q * n_z)$
ii) there is $t \in [0, 1)$ such that $d_b(S(p), S(q)) \leq t.d_b(p, q)$, where $t < \frac{1}{b^2}$

then S is called a contraction with the bound t for $((A, *), d_b)$ if for all $p, q \in A$.

Theorem 4.2. *Let $((A, *), d_b)$ be a neutrosophic triplet b-metric space and S be a contraction with the bound t. Then S is continuous respect to τ_{d_b}.*

Proof. Let for each $r > 0$ and $p \in A$, there be $k > 0$ such that $d_b(p, q) \leq k$. Let $k = \frac{r}{t}$, then $d_b(S(p), S(q)) < t.d_b(p, q) < t.k = t.\frac{r}{t} = r$. Therefore, S is continuous. □

Theorem 4.3. *Let $((A, *), d_b)$ be a neutrosophic triplet b-metric space with the coefficient $b > 1$ and $S : A \to A$ be a contraction with a bound t for $((A, *), d_b)$.*

If $d_b(p, S^m(p)) \leq d_b(p, S^m(p) * n_{S(p)})$ for all $p \in A$ and for all $m \in \mathbb{N} \cup \{0\}$, then,

$$d_b(p, S^m(p)) \leq \sum_{i=1}^{m} b^i t^{i-1} d_b(p, S(p)).$$

Proof. Let prove the theorem by induction on m.

For $m = 0$ and for all $p \in A$, $d_b(p, S^0(p)) = d_b(p, p) = 0 \leq \sum_{i \in \varnothing} d(p, S(p))$.

By the hypothesis and the condition b_5 of neutrosophic triplet b-metric,

$$\begin{aligned} d_b\left(p, S^{m+1}(p)\right) &\leq d_b\left(p, S^{m+1}(p) * n_{S(p)}\right) \leq b\left[d_b\left(p, S(p)\right) + d_b\left(S(p), S^{m+1}(p)\right)\right] \\ &= b\left[d_b\left(p, S(p)\right) + d_b\left(S(p), S^m\left(S(p)\right)\right)\right] \\ &= b.d_b\left(p, S(p)\right) + b.d_b\left(S(p), S^m(S(p))\right). \end{aligned}$$

By the inductive hypothesis, $d_b\left(S(p), S^m(S(p))\right) \leq \left(\sum_{i=1}^{m} b^i t^{i-1}\right) d_b\left(S(p), S(S(p))\right)$.

Since S is a contraction with the bound t, $d_b\left(S(p), S(S\left(p\right))\right) \leq t.d_b\left(p, S\left(p\right)\right)$. Therefore

$$\begin{aligned} d_b\left(p, S^{m+1}\left(p\right)\right) &\leq b.d_b\left(p, S\left(p\right)\right) + b\left(\sum_{i=1}^{m} b^i t^{i-1}\right) t.d_b\left(p, S\left(p\right)\right) \\ &= b.d_b\left(p, S\left(p\right)\right) + \left(\sum_{i=1}^{m} b^{i+1} t^i\right) d_b\left(p, S\left(p\right)\right) \\ &= b.d_b\left(p, S\left(p\right)\right) + \sum_{i=2}^{m+1} b^i t^{i-1}\left(d_b\left(p, S\left(p\right)\right)\right) \\ &= \left(b + \sum_{i=2}^{m+1} b^i t^{i-1}\right) d_b\left(p, S\left(p\right)\right) \\ &= \sum_{i=1}^{m+1} b^i t^{i-1} d_b\left(p, S\left(p\right)\right). \end{aligned}$$

This completes the proof. □

Next, we give a fixed point theorem for neutrosophic triplet b-metric space.

Theorem 4.4. *Let $((A, *), d_b)$ be a complete neutrosophic triplet b-metric space. If $S : A \to A$ is a contraction with a bound t, then S has a fixed point.*

Proof. Let $p_0 \in A$ be an unique element and define a sequence $\{p_n\}$ such that $p_n = S(p_{n-1})$, for all $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, $p_n = S^n(p_0)$.

Now, let show p_n is a Cauchy sequence. For $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that

$$\frac{1}{b^{2M-2}}.\frac{1}{b-1}.d_b(p_0, p_1) < \varepsilon.$$

Let $m \geq n \geq M$. Since S is a contraction,

$$\begin{aligned} d_b\left(p_n, p_m\right) &= d_b\left(S^n\left(p_0\right), S^m\left(p_0\right)\right) = d_b\left(S\left(S^{n-1}\left(p_0\right)\right), S\left(S^{m-1}\left(p_0\right)\right)\right) \\ &\leq t.d_b\left(S^{n-1}\left(p_0\right), S^{m-1}\left(p_0\right)\right) \\ &\leq t^2.d_b\left(S^{n-2}\left(p_0\right), S^{m-2}\left(p_0\right)\right) \\ &. \\ &. \\ &\leq t^n.d_b\left(p_0, S^{m-n}\left(p_0\right)\right). \end{aligned}$$

By Theorem 4.3,

$$d_b\left(p_0, S^{m-n}\left(p_0\right)\right) \leq \left(\sum_{i=1}^{m-n} b^i t^{i-1} d_b\left(p_0, S\left(p_0\right)\right)\right).$$

Therefore, since $t < \frac{1}{b^2}$,

$$
\begin{aligned}
d_b\left(p_n, p_m\right) &\leq t^n.\left(\sum_{i=1}^{m-n} b^i t^{i-1} d_b\left(p_0, S\left(p_0\right)\right)\right) \leq \frac{1}{b^{2n}}\left(\sum_{i=1}^{m-n} \frac{1}{b^{i-2}}\right) d_b\left(p_0, p_1\right) \\
&\leq \frac{1}{b^{2n}}\left(\sum_{i=1}^{\infty} \frac{1}{b^{i-2}}\right) d_b\left(p_0, p_1\right) = \frac{1}{b^{2n-2}}\left(\sum_{i=1}^{\infty} \frac{1}{b^{i}}\right) d_b\left(p_0, p_1\right) \\
&= \frac{1}{b^{2n-2}}.\frac{1}{b-1} d_b\left(p_0, p_1\right) \\
&\leq \frac{1}{b^{2M-2}}.\frac{1}{b-1}.d_b\left(p_0, p_1\right) \\
&< \varepsilon.
\end{aligned}
$$

So $\{p_n\}$ is a Cauchy sequence. Then $\{p_n\}$ is converges a point $p \in A$, i.e $p_n \to p$ because $((A, *), d_b)$ is a complete neutrosophic triplet b-metric space. Since S is continuous, $S(p_n) \to S(p)$. □

5. Conclusion

In this paper, some new concepts, properties, and examples were given for neutrosophic triplet b-metric spaces as follows:

1) The neutrosophic triplet space $A = \{0, 2, 3, 4\}$, with the multiplication module 6 in $\mathbb{Z}$, is a neutrosophic triplet b-metric space with $d : A \times A \to \mathbb{R}$, $d(u, v) = (u - v)^2$ where $b \geq 4$.
2) The neutrosophic triplet space $(P(A), \cup)$ is a neutrosophic triplet b-metric space with $d : P(A) \times P(A) \to \mathbb{R}$ defined as, if $U = V$, $d(U, V) = 0$ and if $U \neq V$,

$$
d(U,V) = \begin{cases} \left|2^{m(U)} - 2^{m(V)}\right|, & \text{if } m(U) \text{ and } m(V) \text{ are even} \\ 3, & \text{if } m(U) \text{ and } m(V) \text{ are odd} \\ 1, & \text{otherwise} \end{cases},
$$

where $b \geq (2^{|(A)|} - 2)$.
3) The neutrosophic triplet space $(P(A), \cap)$ is a neutrosophic triplet b-metric space where $A \subset \mathbb{N}$ is finite, with

$$
d(U,V) = \begin{cases} 1, & \text{if } U \cap V = \varnothing, \\ \frac{1}{m(U \cap V)}, & \text{if } U \cap V \neq \varnothing, U \neq V, \\ 0, & \text{if } U = V. \end{cases}
$$

where $b \geq \frac{m(A)-1}{2}$.

4) A sequence $\{p_n\}$ in a neutrosophic triplet b-metric space $((A,*),d_b)$ is a Cauchy sequence if for all $\varepsilon > 0$, there exists an $M \in \mathbb{N}$ such that, for all $n, m \geq M$, $d_b(p_n, p) < \varepsilon$ and

$$d_b(p_n, p_m) \leq d_b(p_n, p_m * n_p).$$

Since S is continuous. Since $S(p_{n-1}) = p_n$ then $S(p_n) = p_{n+1} \to p$. Thus $S(p) = p$.

5) The family $\tau_{d_b} = \{U \subset A \mid \exists \varepsilon > 0, \exists n \in \mathbb{N}, B^n_\varepsilon(p) \subset U$ for all $p \in U\}$ is a neutrosophic triplet topology on a neutrosophic triplet b-metric space $((A,*),d_b)$, called as neutrosophic triplet topology induced by neutrosophic triplet b-metric d_b.

6) A map $S : A \to A$ is called a contraction with the bound $t > 0$ if,
   i) there exists any element $z \in A$ such that $d_b(p,q) \leq d_b(p, q * n_z)$
   ii) there is $t \in [0,1)$ such that $d_b(S(p), S(q)) \leq t.d_b(p,q)$, where $t < \frac{1}{b^2}$

7) A contraction S on a neutrosophic triplet b-metric space $((A,*),d_b)$ is continuous respect to τ_{d_b}.

8) For contraction S on neutrosophic triplet b-metric space $((A,*),d_b)$, if $d_b(p, S^m(p)) \leq d_b(p, S^m(p) * n_{S(p)})$ for all $p \in A$ and for all $m \in \mathbb{N} \cup \{0\}$, then, $d_b(p, S^m(p)) \leq \sum_{i=1}^{m} b^i t^{i-1} d_b(p, S(p))$.

$$d_b(p, S^m(p)) \leq \sum_{i=1}^{m} b^i t^{i-1} d_b(p, S(p)).$$

9) A contraction S on a neutrosophic triplet b-metric space $((A,*),d_b)$ has a fixed point.

6. Future Work

By utilizing these results, the researcher can define neutrosophic triplet partial b-metric, neutrosophic triplet quasi-partial b-metric, neutrosophic triplet rectangular b-metric and can investigate fixed point theorems in these spaces.

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Some Topological Character of Neutrosophic normed spaces

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Abstract. The concept of neutrosophic normed spaces was introduced by Murat Kirisci* and Necip Simsek [3]. In this paper, by using the compact operator $\psi(z_j)$ and the concept of statistical convergence. we introduce some neutrosophic statistical convergence of sequence spaces defined by compact operator and study of neutrosophic norm and derive the relation between statistical convergence in NNS with the help of compact operator. We also focus on some topological and algebraic properties of these convergent sequence spaces.

Keywords: NNS; t-norm; t-conorm; Statistical convergence; compact linear operator; Bounded linear operator; neutrosophic set.

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1. Introduction

The notion of fuzzy set was presented by Zadeh [4], since then several kinds of research have appeared and fuzzication of many classical theories has also been made. Fuzzy sets hypothesis is an amazing handset for demonstrating vulnerability and dubiousness in different problems and issues emerging in field of science and designing. fuzzy topology is perhaps the absolute significant and generally utilized outfits and it ends up being valuable for managing such circumstances where utilization of old style speculations breakdown. [13] he used the concept of fuzzy sets and examine the fuzzy real valued p-absolutely summable multiple sequences in probabilistic normed spaces. Atanassov [29] generalized the fuzzy sets theory and studied the concepts of intuitionistic fuzzy sets (IFS). In 2004, [6] Park researched the idea of (IFS), further Saadati and Park analyzed this concept in the norm. The idea of $IFNS$ and the thought of statistical convergence is a valuable utilitarian apparatus for contemplating the convergence issues of mathematical problems (double sequence) herewith the idea of thickness. [14]. The idea of neutrosophic sets (NS) was introduced by Smarandache [23]. This set is an expansion of IFS regardless, if the summation of neutrosophic segments is < 1, or > 1, or $= 1$. For the situation when the aggregate of the components is 1 ($as\ in\ IFS$), in the wake of satisfying the condition by applying the neutrosophic set operators, different outcomes can be acquired by applying the intuitionistic fuzzy (IF) operators, since the IF operators disregard the $(indeterminacy)$, while the NS operators taken into cognizance of the indeterminacy at a similar level as $(truth - membership)$ and $(falsehood - nonmembersip)$ are taken. [26] using the idea of neutrosophic sets defined the notion of Neutrosophic Bipolar Vague Soft Set and Its Application to Decision Making Problems. Further, Smarandache [9, 10, 12, 35] investigated neutroalgebra which is generalization of partial algebra , neutroalgebraic structures and antialgebraic structures. NS is likewise more adaptable and effective in light of the fact that it handles, aside from autonomous (free) components, additionally partially independent and dependent components,while IFS can't manage these cases. Moreover [38] define neutrosophic fuzzy matrices and Some Algebraic Operations. Smarandache [1, 34] analyzed the conflict between neutrosophic rationale, intuitionistic fuzzy rationale, and the comparing NS and IFS. [11] defined Neutrosophic simply soft open set in neutrosophic soft topological space. Moreover, Bera and Mahapatra [24] introduced the neutrosophic soft linear space. Bera and Mahapatra [25] studied convexity, metric, Cauchy sequence, and neutrosophic soft norm linear space (NSNLS). [15] A lot of developments have been made in this areas after the work of Das, S and Pramanik, S defined the Generalized neutrosophic b-open sets in neutrosophic topological space. Further, [21] examine Neutrosophic Multiset Topological Space. [17] another term find Statistically pre-Cauchy fuzzy real-valued sequences defined by Orlicz function. Later on, the concepts of statistical convergence of double sequences have been analyzed in IFNS by Mursaleen and Mohiuddin [14]. Quite recently, Kirisci and Simsek [3] introduced the notion of NNS and statistical convergence. Further [8] examine On almost statistical convergence of new type of generalized difference sequence of fuzzy numbers. Since NNS is a natural generalization of IFNS and statistical convergence. [19] defined a new concept On pointwise statistical convergence of order alpha of sequences of fuzzy mappings. In this paper we aim to define novel statistical convergence of sequence spaces. sequence spaces using neutrosophic norm and using compact operator as a tool and discussed their topological and algebraic properties. We mention the following notions that will be put to use in the paper further.

2. Preliminaries

:

Definition 2.1. A sequence $z = (z_j)$ is called a δ -convergent to the number ξ for each $\varepsilon > 0$, the set $Y(\epsilon)$ has δ-density zero, where

$$Y_\epsilon(F) = \{j \in \mathbb{N} : |z_j - \xi| \geq \epsilon\} \tag{1}$$

we write $S_\delta - \lim z = \xi$ or $z_j \to \xi(S_\delta)$.

Definition 2.2. [14] A sequence $z = (z_j)$ is called statistically Cauchy sequences if $\exists$, a number $\mathcal{K} = \mathcal{K}(\epsilon)$ such that for each $\epsilon > 0$

$$\lim_{j\to\infty} \frac{1}{j}|\{i \leq j : |z_i - (z_{\mathcal{N}})| \geq \epsilon\}| = 0. \tag{2}$$

Definition 2.3. [7] Let $X \neq \varnothing$, the $\mathcal{W}$ intuitionistic fuzzy set, $\mathcal{W} \subset X$ is defined by

$$\mathcal{W} = \{< z, \mathbf{T}(z),\ \mathbf{F}(z) >: z \in X\}, \tag{3}$$

where $\mathbf{T}(z), \mathbf{F}(z) : X \to [0,1]$, $\mathbf{T}(z) =$ (Truth) and $\mathbf{F}(z) =$ (Falsehood) respectively.

$$0 \leq \mathbf{T}(z) + \mathbf{F}(z) \leq 1$$

Definition 2.4. [31] Let $X \neq \phi$ and $\mathcal{W} \subset X$ Then,

$$\mathcal{W}_{NS} = \{< z, \mathbf{T}(z), \mathbf{I}(z), \mathbf{F}(z) >: z \in X\},$$

where $\mathbf{T}(z), \mathbf{I}(z), \mathbf{F}(z) : X \to [0,1]$, $\mathbf{T}(z) =$ Truth, $\mathbf{I}(z) =$ Indeterminacy, and $\mathbf{F}(z) =$ Falsehood respectively.

$$0 \leq \mathbf{T}(z) + \mathbf{I}(z) + \mathbf{F}(z) \leq 3.$$

The components of neutrosophic are $\mathbf{T}(z), \mathbf{I}(z)$ and $\mathbf{F}(z)$ independent of each other.

Definition 2.5. [1] Suppose $X \neq \phi$, Q and R are neutrosophic sets in X. Then,

(a) $Q \subset R \iff \mathbf{T}_Q(z) \leq \mathbf{T}_R(z), \mathbf{I}_Q(z) \leq \mathbf{I}_R(z), \mathbf{F}_Q(z) \geq \mathbf{F}_R(z)\ \forall\ z \in X$

(b) $Q = R \iff \mathbf{T}_Q(z) = \mathbf{T}_R(z), \mathbf{I}_Q(z) = \mathbf{I}_R(z), \mathbf{F}_Q(z) = \mathbf{F}_R(z)\ \forall\ z \in X$

(c) $Q \cap R = \{\langle z, \min(\mathbf{T}_Q(z), \mathbf{T}_R(z)), \min(\mathbf{I}_Q(z), \mathbf{I}_R(z)), \min(\mathbf{F}_Q(z), \mathbf{F}_R(z))\rangle \mid z \in X\}$

(d) $Q \cup R = \{\langle z, \max(\mathbf{T}_Q(z), \mathbf{T}_R(z, \max(\mathbf{I}_Q(z), \mathbf{I}_R(z)), \max(\mathbf{F}_Q(z), \mathbf{F}_R(z))\rangle, z \in X\}$

(e) $Q^c = \{\langle z, \mathbf{F}_Q(z), 1 - \mathbf{I}_R(z)), \mathbf{T}_Q(z)\rangle \mid z \in X\}$

(f) $Q \backslash R = \{\langle z, \mathbf{T}_Q(z) \min \mathbf{F}_R(z), \mathbf{I}_Q(z) \min 1 - \mathbf{I}_R(z), \mathbf{F}_Q(z) \max \mathbf{T}_R(z)\rangle \mid z \in X\}$.

Definition 2.6. [40] Consider a binary operation $\diamond$ on the interval.if satisfying following axioms,

$$\diamond : [0,1]^2 \longrightarrow [0,1]$$

(1) $\hat{\alpha} \diamond \hat{\beta} = \hat{\beta} \diamond \hat{\alpha}$

(2) $(\hat{\alpha} \diamond \hat{\beta}) \diamond \hat{\lambda} = \hat{\alpha} \diamond (\hat{\beta} \diamond \hat{\lambda})$

(3) $\hat{\alpha} \diamond 1 = \hat{\alpha}\ \forall\ \hat{\alpha} \in [0,1]$,

(4) $\hat{\alpha} \diamond \hat{\beta} \leq \hat{\lambda} \diamond \hat{\upsilon}$ whenever $\hat{\alpha} \leq \hat{\lambda}$ and $\hat{\beta} \leq \hat{\upsilon}$ for each $\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\upsilon} \in [0,1]$.

is called a continuous t-norm.
where $\wedge = \min$ and $\vee = \max$.

Example 2.7. For $\hat{\alpha}, \hat{\beta} \in [0,1]$, define $\hat{\alpha} \diamond \hat{\beta} = \hat{\alpha}\hat{\beta}$ or $\hat{\alpha} \diamond \hat{\beta} = \wedge\{\hat{\alpha}, \hat{\beta}\}$. then $\diamond$ is continuous t-norm.

Definition 2.8. [40] Consider a binary operation $\star$ on the interval.if satisfying following axioms,

$$\star : [0,1]^2 \longrightarrow [0,1]$$

(1) $\hat{\alpha} \star \hat{\beta} = \hat{\beta} \star \hat{\alpha}$
(2) $(\hat{\alpha} \star \hat{\beta}) \star \hat{\lambda} = \hat{\alpha} \star (\hat{\beta} \star \hat{\lambda})$
(3) $\hat{\alpha} \star 0 = \hat{\alpha} \ \forall \ \hat{\alpha} \in [0,1]$,
(4) $\hat{\alpha} \star \hat{\beta} \leq \hat{\lambda} \star \hat{\upsilon}$ whenever $\hat{\alpha} \leq \hat{\lambda}$ and $\hat{\beta} \leq \hat{\upsilon}$ for each $\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\upsilon} \in [0,1]$.
is called a continuous t-conorm.
where $\wedge = \min$ and $\vee = \max$.

Example 2.9. Consider $\hat{\alpha}, \hat{\beta} \in [0,1]$. define $\hat{\alpha} \star \hat{\beta} = \wedge\{\hat{\alpha} + \hat{\beta}, 1\}$ or $\hat{\alpha} \star \hat{\beta} = \vee\{\hat{\alpha}, \hat{\beta}\}$. Then $\star$ is continuous t-conorm

Definition 2.10. [3] Take X as a vector space and $\mathcal{M} = \{< x, \mathbf{T}(z), \mathbf{I}(z), \mathbf{F}(z) >: z \in X\}$ be a normed space such that $\mathbf{T}(z), \mathbf{I}(z), \mathbf{F}(z) : X \times R^+ \to [0,1]$. Assume $\diamond$ show the continuous t-norm and $\star$ be a continuous t-conorm respectively, then $\mathcal{V} = (X, \mathcal{M}, \diamond, \star)$ is called (NNS). if the subsequent terms holds; $\forall \ z, a \in X$ and $s, k \ > 0$

(i) $0 \leq \mathbf{T}(z,s) \leq 1$, $0 \leq \mathbf{I}(z,s) \leq 1$, $0 \leq \mathbf{F}(z,s) \leq 1$, $s \in R^+$,

(ii) $\mathbf{T}(z,s) + \mathbf{I}(z,s) + \mathbf{F}(z,s) \leq 3$, for $s \in R^+$,

(iii) $\mathbf{T}(z,s) = 1$ for $s > 0$ iff $z = 0$

(iv) $\mathbf{T}(\alpha x, s) = \mathbf{T}(z, \frac{s}{|\alpha|})$,

(v) $\mathbf{T}(z,s) \diamond \mathbf{T}(a,s) \leq \mathbf{T}(z+a, s+k)$,

(vi) $\mathbf{T}(z, \diamond)$ is continuous non-decreasing function

(vii) $\lim_{s\to\infty} \mathbf{T}(z,s) = 1$

(viii) $\mathbf{I}(z,s) = 0$ for $s > 0$ iff $z = 0$

(ix) $\mathbf{I}(\alpha z,s) = \mathbf{I}(y,\frac{s}{|\alpha|})$, for each $\alpha \neq 0$,

(x) $\mathbf{I}(z,s) \star \mathbf{I}(a,k) \geq \mathbf{I}(z+a,s+k)$,

(xi) $\mathbf{I}(z,\star)$ is continuous non-increasing function,

(xii) $\lim_{s\to\infty} \mathbf{I}(z,s) = 0$,

(xiii) $\mathbf{F}(z,s) = 0$ for $s > 0$ iff $z = 0$

(xiv) $\mathbf{F}(\alpha z,s) = \mathbf{F}(z,\frac{s}{|\alpha|})$, for each $\alpha \neq 0$,

(xv) $\mathbf{F}(z,s) \star \mathbf{F}(z,k) \geq \mathbf{F}(z+a,s+k)$,

(xvi) $\mathbf{F}(z,.)$ is continuous non-increasing function,

(xvii) $\lim_{s\to\infty} \mathbf{F}(z,s) = 0$,

(xviii) If $s \leq 0$, then $\mathbf{T}(z,s) = 0$, $\mathbf{I}(z,s) = 1$, $\mathbf{F}(z,s) = 1$.

In this case $\mathcal{M} = (\mathbf{T},\mathbf{I},\mathbf{F})$ is said to be neutrosophic normed (NNS).

Example 2.11. [3] Suppose $(X,\|\,.\,\|)$ be a NNS. Give the operations as TC $z\diamond a = z+a-za$ and TN $z\star a = \min(z,a)$. For $s > ||z||$,

$$\mathbf{T}_0(z,s) = \frac{s}{s+||z||},\ \mathbf{I}_0(z,s) = \frac{||z||}{s+||z||},\ \mathbf{F}_0(z,s) = \frac{||z||}{s}. \tag{4}$$

for all $z,a \in [0,1]$ and $s > 0$. If we take $s \leq ||z||$, and let $\mathbf{T}_0,\ \mathbf{I}_0$ *and* $\mathbf{F}_0$ be neutrosophic sets on $X \times (0,\infty)$ then

$$\mathbf{T}_0(z,s) = 0, \mathbf{I}_0(z,s) = 1 \text{ and } \mathbf{F}_0(z,s) = 1.$$

Then, $(X,\mathcal{M},\diamond,\star,)$ is neutrosophic normed space such that $\mathcal{M} : X \times R^+ \to [0,1]$.

3. Main Results

In this article, we examine the algebriac and topological properties on the space of neutrosophic statistical convegence given by :

Definition 3.1. [3] Suppose $(X, \mathcal{M}, \star, \diamond)$ be a NNS and $z = (z_j)$ is called a Cauchy sequence with respect to $\mathcal{M}$, if for each $\epsilon > 0$ and $s > 0$ $\exists$, $u \in \mathbb{N}$ such that $\mathbf{T}(z_j - a_k, s) > 1 - \epsilon$, $\mathbf{I}(z_j - a_k, s) < \epsilon$ and $\mathbf{F}(z_j - a_k, s) < \epsilon$ for all $j, k \geq u$.

Definition 3.2. Let X be a NNS, the sequence $z = (z_j)$ in X is called convergent at $\xi \in X$ iff $\exists$, $\mathring{N} \in \mathbb{N}$, with respect NN $(\mathbf{T}, \mathbf{I}, \mathbf{F})$ if for every $\epsilon > 0$, $s > 0$

$$\mathbf{T}(\psi(z_j) - \xi, s) > 1 - \epsilon, \mathbf{I}(\psi(z_j) - \xi, s) < \epsilon \text{ and } \mathbf{F}(\psi(z_j) - \xi, s) < \epsilon \tag{5}$$

for all $j \geq \mathring{N}$, i.e.,

$$\lim_{j\to\infty} \mathbf{T}(\psi(z_j) - \xi, s) = 1,\ \lim_{j\to\infty} \mathbf{I}(\psi(z_j) - \xi, s) = 0 \text{ and } \lim_{j\to\infty} \mathbf{F}(\psi(z_j) - \xi, s) = 0.$$

In such case, we denote $\mathcal{M} - \lim z_j = \xi$.

Definition 3.3. (See [2]).Suppose A and B be two normed linear spaces (NLS) and $\psi : \mathcal{D} \to\ B$ be a linear operator, where $\mathcal{D}(\psi) \subset A$ Then, the operator ψ is called a bounded, if $\exists$, a +ve real no c' such that

$$||\psi z|| \leq c'||z||,\ \forall z \in \mathcal{D}(\psi).$$

The set of all bounded linear operators $\mathbf{B}(A, B)$ [2] is a (NLS) normed by

$$||\psi|| = \sup_{z\in A,\ ||z||=1} ||\psi z||$$

and $\mathbf{B}(A, B)$ is a Banach space if B is a Banach space.

Definition 3.4. Let us Consider two NLS A and B

$$\psi : A \to B$$

is called a compact linear operator, if

(i) ψ is linear

(ii) ψ maps every bounded sequence (z_j) in A on to a sequence $\psi(z_j)$ in B which has a convergent subsequence.

The set of all compact linear operators $\mathcal{C}(A,B)$ is a closed subspace of $\mathbf{B}(A,B)$ and $\mathcal{C}(A,B)$ is Banach space, if B is a Banach space.

Inspire by this [5], we proposed the previous sequence spaces with the assist of compact operator in NNS:

$$\mathcal{S}^{N}_{(\mathbf{T},\mathbf{I},\mathbf{F})}(\psi) = \big\{(z_j) \in \ell_\infty : \ \exists\, k \in \mathbb{N}, \forall\, j \geq k, \mathbf{T}(\psi(z_j) - \xi, s) > 1-\epsilon \ \text{ or } \ \mathbf{I}(\psi(z_j) - \xi, s) \geq \epsilon \\ \mathbf{F}(\psi(z_j) - \xi, s) \geq \epsilon\big\}.$$

$$S^{N}_{0(\mathbf{T},\mathbf{I},\mathbf{F})}(\psi) = \big\{(z_j) \in \ell_\infty : \ \exists\, k \in \mathbb{N}, \forall\, j \geq k, \mathbf{T}(\psi(z_j) - \xi, s) > 1-\epsilon \text{ or } \mathbf{I}(\psi(z_j) - \xi, s) \geq \epsilon \\ \mathbf{F}(\psi(z_j) - \xi, s) \geq \epsilon\big\}.$$

Definition 3.5. [5] Let $\mathcal{V} = (X, M, \diamond, \star)$ is a NNS. For $s > 0$. we define a open ball (OB) $\mathcal{B}(r,z,s)(\psi)$ with centre $z \in X$ and raduis $0 < r < 1$, there exist $k \in \mathbb{N}$ such that for all $j \geq k$

$$\mathcal{B}(z,r,s)(\psi) = \{(a_j) \in \ell_\infty : \mathbf{T}(\psi(z_j) - \psi(a_k), s) < 1-\epsilon \text{ or } \mathbf{I}(\psi(z_j) - \psi(a_k), s) < \epsilon,\ \mathbf{F}(\psi(z_j) - \psi(a_k), s) < \epsilon\}.$$

Presently, we are prepared to state and demonstrate our primary outcomes. This hypothesis depends on the linearity of new characterize sequence spaces which is expressed as follows.

Theorem 3.6. *$\mathcal{S}^{N}_{(\mathbf{T},\mathbf{I},\mathbf{F})}(\psi)$ and $S^{N}_{0(\mathbf{T},\mathbf{I},\mathbf{F})}(\psi)$ are linear spaces.*

Proof. We shall prove the result for $\mathcal{S}^{N}_{(\mathbf{T},\mathbf{I},\mathbf{F})}(\psi)$ the proof for the different space will take accordingly, let $z = (z_j), a = (a_j) \in \mathcal{S}^{N}_{(\mathbf{T},\mathbf{I},\mathbf{F})}(\psi)$ and $\hbar, \eta$ be scalars then for a given $\epsilon > 0$, we have

$$\mathcal{W}_1 = \Big\{j \in \mathbb{N} : \mathbf{T}\Big(\psi(z_j) - \xi_1, \frac{s}{2|\hbar|}\Big) \leq 1-\epsilon \text{ or } \mathbf{I}\Big(\psi(z_j) - \xi_1, \frac{s}{2|\hbar|}\Big) \geq \epsilon,\ \mathbf{F}\Big(z_j - \xi_1, \frac{s}{2|\hbar|}\Big) \geq \epsilon\Big\}$$

$$\mathcal{W}_2 = \Big\{j \in \mathbb{N} : \mathbf{T}\Big(\psi(a_j) - \xi_2, \frac{s}{2|\eta|}\Big) \leq 1-\epsilon \text{ or } \mathbf{I}\Big(\psi(a_j) - \xi_2, \frac{s}{2|\eta|}\Big) \geq \epsilon, \mathbf{F}\Big(\psi(a_j) - \xi_2, \frac{s}{2|\eta|}\Big) \geqslant \epsilon\Big\}$$

$$\mathcal{W}_1^c = \Big\{j \in \mathbb{N} : \mathbf{T}\Big(\psi(z_j) - \xi_1, \frac{s}{2|\hbar|}\Big) > 1-\epsilon \text{ or } \mathbf{I}\Big(\psi(z_j) - \xi_1, \frac{s}{2|\hbar|}\Big) < \epsilon, \mathbf{F}(\psi(z_j) - \xi_1, \frac{s}{2|\hbar|}) < \epsilon\Big\}$$

$$\mathcal{W}_2^c = \Big\{j \in \mathbb{N} : \mathbf{T}\Big(\psi(a_j) - \xi_2, \frac{s}{2|\eta|}\Big) > 1-\epsilon \text{ or } \mathbf{I}\Big(\psi(a_j) - \xi_2, \frac{s}{2|\eta|}\Big) < \epsilon, \mathbf{F}\Big(\psi(a_j) - \xi_2, \frac{s}{2|\eta|}) < \epsilon\Big\}$$

Define the set $\mathcal{W}_3 = \mathcal{W}_1 \cup \mathcal{W}_2$ *so that* $\mathcal{W}_3 \in \mathcal{M}$. It follow that $\mathcal{W}_3^c \neq \phi$. we shall show that for each $(z_j), (a_j) \in \mathcal{S}^N_{(\mathbf{T},\mathbf{I},\mathbf{F})}(\psi)$.

$$\mathcal{W}_3^c \subset \Big\{ j \in \mathbb{N} : \ \mathbf{T}\Big((\hbar\psi(z_j) + \eta\psi(a_j)) - (\hbar\xi_1 + \eta\xi_2), s\Big) > 1 - \epsilon \ or \ \mathbf{I}\Big(\hbar\psi(z_j) + \eta\psi(a_j)) - (\hbar\xi_1 + \eta\xi_2), s\Big) < \epsilon, \\ \mathbf{F}\Big((\hbar\psi(z_j) + \eta\psi(a_j)) - (\hbar\xi_1 + \eta\xi_2), s\Big) < \epsilon \Big\}$$

let $m \in \mathcal{W}_3^c$. In this case,

$$\mathbf{T}\Big(\psi(z_m) - \xi_1, \tfrac{s}{2|\hbar|}\Big) > 1 - \epsilon \ or \ \mathbf{I}\Big(\psi(z_m) - \xi_1, \tfrac{s}{2|\hbar|}\Big) < \epsilon, \mathbf{F}\Big(\psi(z_m) - \xi_1, \tfrac{s}{2|\hbar|}\Big) < \epsilon$$

and

$$\mathbf{T}\Big(\psi(a_m) - \xi_2, \tfrac{s}{2|\eta|}\Big) > 1 - \epsilon \ or \ \mathbf{I}\Big(\psi(a_m) - \xi_2, \tfrac{s}{2|\eta|}\Big) < \epsilon, \mathbf{F}\Big(\psi(a_m) - \xi_2, \tfrac{s}{2|\eta|}\Big) < \epsilon.$$

We have

$$\mathbf{T}\Big((\hbar\psi(z_m) + \eta\psi(a_m)) - (\hbar\xi_1 + \eta\xi_2), s\Big)$$

$$\geq \mathbf{T}\Big(\hbar\psi(z_m) - \hbar\xi_1, \frac{t}{2}\Big) \diamond \mathbf{T}\Big(\eta\psi(z_m) - \eta\xi_2, \frac{t}{2}\Big)$$

$$= \mathbf{T}\Big(\psi(z_m) - \xi_1, \frac{s}{2|\hbar|}\Big) \star \mathbf{T}\Big(\psi(z_m) - \xi_2, \frac{s}{2|\eta|}\Big)$$

$$> (1 - \epsilon) \diamond (1 - \epsilon) = 1 - \epsilon.$$

In similar way,

$$\mathbf{I}\Big((\hbar\psi(z_m) + \eta\psi(a_m)) - (\hbar\xi_1 + \eta\xi_2), s\Big)$$

$$\leq \mathbf{I}\Big(\hbar\psi(z_m) - \hbar\xi_1, \frac{s}{2}\Big) \star \mathbf{I}\Big(\eta\psi(z_m) - \eta\xi_2, \frac{s}{2}\Big)$$

$$= \mathbf{I}\Big(\psi(z_m) - \xi_1, \frac{s}{2|\hbar|}\Big) \star \mathbf{I}\Big(\psi(z_m) - \xi_2, \frac{s}{2|\eta|}\Big)$$

$$< \epsilon \star \epsilon < \epsilon.$$

and,

$$\mathbf{F}\Big((\hbar\psi(z_m) + \eta\psi(a_m)) - (\hbar\xi_1 + \eta\xi_2), s\Big)$$

$$\leq \mathbf{F}\Big(\hbar\psi(z_m) - \hbar\xi_1, \frac{s}{2}\Big) \star \mathbf{F}\Big(\eta\psi(z_m) - \eta\xi_2, \frac{s}{2}\Big)$$

$$= \mathbf{F}\Big(\psi(z_m) - \xi_1, \frac{s}{2|\hbar|}\Big) \star \mathbf{F}\Big(\psi(z_m) - \xi_2, \frac{s}{2|\eta|}\Big)$$

$$< \epsilon \star \epsilon < \epsilon.$$

This implies that,

$$\mathcal{W}_3^c \subset \Big\{ j \in \mathbb{N} : \begin{array}{l} \mathbf{T}\Big((\hbar\psi(z_j) + \eta\psi(a_j)) - (\hbar\xi_1 + \eta\xi_2), s\Big) > 1 - \epsilon \ or \ \mathbf{I}\Big(\hbar\psi(z_j) + \eta\psi(a_j)) - (\hbar\xi_1 + \eta\xi_2), s\Big) < \epsilon, \\ \mathbf{F}\Big((\hbar\psi(z_j) + \eta\psi(a_j)) - (\hbar\xi_1 + \eta\xi_2), s\Big) < \epsilon \Big\}. \end{array}$$

Hence $\mathcal{S}^N_{(\mathbf{T},\mathbf{I},\mathbf{F})}(\psi)$ is a linear space. □

Remark 3.7. $\mathcal{S}^N_{(\mathbf{T},\mathbf{I},\mathbf{F})}(\psi)$ is an NNS.

Define $\mathcal{T}^N_{(\mathbf{T},\mathbf{I},\mathbf{F})}(\psi) = \{W \subset \mathcal{S}^N_{(\mathbf{T},\mathbf{I},\mathbf{F})}(\psi)$: for every $z \in W$, $\exists$, $s > 0$ and $r \in (0,1)$ such that $\mathcal{B}(z, r, s)(\psi) \subset W\}$.

Theorem 3.8. *The topology $\mathcal{T}^N_{(\mathbf{T},\mathbf{I},\mathbf{F})}(\psi)$ on $\mathcal{S}^N_{(\mathbf{T},\mathbf{I},\mathbf{F})}(\psi)$ is first countable.*

Proof. $\{B_x(\frac{1}{n}, \frac{1}{n}) : n = 1, 2, 3, ...\}$ is a local base at x, $\mathcal{T}^N_{(\mathbf{T},\mathbf{I},\mathbf{F})}(\psi)$ on $\mathcal{S}^N_{(\mathbf{T},\mathbf{I},\mathbf{F})}(\psi)$ is first countable. $\mathcal{S}^N_{(\mathbf{T},\mathbf{I},\mathbf{F})}(\psi)$ is Hausdorff. □

Theorem 3.9. *$\mathcal{S}^N_{(\mathbf{T},\mathbf{I},\mathbf{F})}(\psi)$ is an NNS and $\mathcal{T}^N_{(\mathbf{T},\mathbf{I},\mathbf{F})}(\psi)$ is a topology on $\mathcal{S}^N_{(\mathbf{T},\mathbf{I},\mathbf{F})}(\psi)$ Then a sequence $(z_j) \in \mathcal{S}^N_{(\mathbf{T},\mathbf{I},\mathbf{F})}(\psi), (z_j) \to z$,if and only if $\mathbf{T}((\psi(z_j) - z), t) \to 1, \mathbf{I}((\psi(z_j) - z), s) \to 1$ and $\mathbf{F}((\psi(z_j) - z), s) \to 1$ as $j \to \infty$.*

Proof. Fix $s_0 > 0$. suppose $\psi(z_j) \to z$. Then for $r \in (0,1)$ $\exists$, $n_0 \in \mathbb{N}$ such that $(\psi(z_j)) \in B_z(r,s)$ for all $j \geq n_0$,

$$B_z(r,s) = \Big\{ j \in \mathbb{N} : \mathbf{T}(\psi(z_j) - z, s) \ or \ \mathbf{I}(\psi(z_j) - z, s) < r, \mathbf{F}(\psi(z_j) - z, s) < r \Big\}$$

Such that, $B_z^c(r,s)$ Then,

$$1 - \mathbf{T}((\psi(z_j) - z), s) < r \ and \ \mathbf{I}((\psi(z_j) - z), s) < r, \mathbf{F}((\psi(z_j) - z), s) < r.$$

Hence,

$$\mathbf{T}((\psi(z_j) - z), s) \to 1 \ and \ \mathbf{I}((\psi(z_j - z), s) \to 0, \mathbf{F}((\psi(z_j - z), s) \to 0 \ as \ j \to \infty.$$

$\Longleftarrow$

if for every $s > 0$, $\mathbf{T}((\psi(z_j) - z), s) \to 1$ and $\mathbf{I}((\psi(z_j) - z), s) \to 0$, $\mathbf{F}((\psi(x_k) - x), s) \to 0$ as $j \to \infty$ then for $r \in (0,1)$ $\exists, n_0 \in \mathbb{N}$ such that $1 - \mathbf{T}((\psi(z_j) - z), s) < r$ and $\mathbf{I}((\psi(z_j) - z), s) < r$, $\mathbf{F}((\psi(z_j) - z), s) < r$ for all $j \geq n_0$.

It follows that $\mathbf{T}((\psi(z_j) - z), s) > 1 - r$ and $\mathbf{I}((\psi(z_j) - z), s) < r$, $\mathbf{F}((\psi(z_j) - z), s) < r$ for all $j \geq n_0$.Thus $(\psi(z_j)) \in B_z^c(r,s)$ for all $j \geq n_0$ and hence $\to \psi(z_j)$ □

Theorem 3.10. *A sequence $z = (z_k) \in \mathcal{S}^N_{(\mathbf{T},\mathbf{I},\mathbf{F})}(\psi)$ is $\mathcal{M}$-convergent iff for each $\epsilon > 0$ and $s > 0$ $\exists$, a number $\mathcal{K} = \mathcal{K}(z, \epsilon, t)$ such that*

$$\Big\{ j \in \mathbb{N} : \mathbf{T}\Big((\psi(z_j) - \ell), \frac{s}{2} \Big) > 1 - \epsilon \ or \ \mathbf{I}\Big(\psi((z_j) - \ell), \frac{s}{2} \Big) < \epsilon \ or \ \mathbf{F}\Big((\psi(z_j) - \ell), \frac{s}{2} \Big) < \epsilon \Big\}$$

Proof. Suppose $\mathcal{M} - \lim x = \ell$. For a given $\epsilon > 0$, Let $s > 0$ such that $(1 - \epsilon) \diamond (1 - \epsilon) > 1 - s$ and $\epsilon \star \epsilon < s, \epsilon \star \epsilon < s$. Then
for every $z \in S^N_{(\mathbf{T},\mathbf{I},\mathbf{F})}(\psi)$

$$\mathcal{W} = \Big\{ j \in \mathbb{N} : \mathbf{T}\Big((\psi(z_j) - \ell), \frac{s}{2} \Big) \leq 1 - \epsilon \ or \ \mathbf{I}\Big((\psi(z_j) - \ell), \frac{s}{2} \Big) \geq \epsilon, \mathbf{F}((\psi(z_j) - \ell), \frac{s}{2}) \geq \epsilon \Big\}$$

which implies that

$$\mathcal{W}^c = \Big\{ j \in \mathbb{N} : \mathbf{T}\Big((\psi(z_j) - \ell), \frac{s}{2} \Big) > 1 - \epsilon \ or \ \mathbf{I}\Big((\psi(z_j) - \ell), \frac{s}{2} \Big) < \epsilon, \mathbf{F}\Big((\psi(z_j) - \ell), \frac{s}{2} \Big) < \epsilon \Big\}$$

Conversely, take $N \in \mathcal{W}$. Then

$$\mathbf{T}\Big((\psi(z_j) - \ell), \frac{s}{2} \Big) > 1 - \epsilon \ or \ \mathbf{I}\Big((\psi(z_j) - \ell), \frac{s}{2} \Big) < \epsilon, \mathbf{F}\Big((\psi(z_j) - \ell), \frac{s}{2} \Big) < \epsilon.$$

Now we have to prove that $\exists$, a number $\mathcal{K} = \mathcal{K}(z, \epsilon, t)$ such that,

$$\Big\{ j \in \mathbb{N} : \mathbf{T}(\psi(z_j) - \psi(z_N), s) \leq 1 - \gamma \ or \ \mathbf{I}(\psi(z_k) - \psi(z_N)), s) \geq \gamma, \mathbf{F}(\psi(z_j) - \psi(z_N), s) \geq \gamma \Big\}$$

for each $z \in S^N_{(\mathbf{T},\mathbf{I},\mathbf{F})}(\psi)$

$$B = \Big\{ j \in \mathbb{N} : \mathbf{T}(\psi(z_j) - \psi(z_N), s) \leq 1 - \gamma \ or \ \mathbf{I}(\psi(z_j) - \psi(z_N), s) \geq \gamma, \mathbf{F}(\psi(z_j) - \psi(z_N), s) \geq \gamma \Big\}.$$

Now we will prove that $B \subset \mathcal{W}$. consider that $B \subseteq \mathcal{W}$ Then $\exists$, $n \in B$ and $n \notin \mathcal{W}$ so we get

$$\mathbf{T}(\psi(z_n) - \psi(x_N)), s) \leq 1 - s \ or \ \mathbf{I}(\psi(z_n - \ell), \frac{s}{2}) > 1 - \epsilon, \mathbf{F}\Big(\psi(z_n - \ell), \frac{s}{2}) > 1 - \epsilon$$

In a diffrent way $\mathbf{T}\Big((\psi(z_N) - \ell), \frac{s}{2}\Big) > 1 - \epsilon$ Therefore we have

$$1 - \gamma \geq \mathbf{T}(\psi(z_j) - \psi(z_N)), s) \geq \mathbf{T}\Big((\psi(z_n) - \ell), \frac{s}{2}\Big) \diamond \mathbf{T}\Big((\psi(z_N) - \ell), \frac{s}{2}\Big) \geq (1 - \epsilon) \diamond (1 - \epsilon) > 1 - \gamma,$$

which is impossible. At the same time,

$$\mathbf{I}(\psi(z_n) - \psi(z_N)), s) \geq \gamma \ or \ \mathbf{I}\Big((\psi(z_n) - \ell), \frac{s}{2}\Big) < \epsilon,$$

In a similar way,

$$\mathbf{I}\Big((\psi(z_N) - \ell), \frac{s}{2}\Big) < \epsilon,$$

$$\gamma \leq \mathbf{I}(\psi(z_n) - \psi(z_N), s) \leq \mathbf{I}\Big((\psi(z_n) - \ell), \frac{s}{2}\Big) \star \mathbf{I}\Big((\psi(z_N) - \ell), \frac{s}{2}\Big) \leq \epsilon \star \epsilon < \gamma,$$

which is impossible.At the same time,

$$\mathbf{F}(\psi(z_n) - (z_N)), s) \geq \gamma \ or \ \mathbf{F}\Big((\psi(x_n) - \ell), \frac{s}{2}\Big) < \epsilon,$$

In particular,

$$\mathbf{F}\Big(\psi(z_N) - \ell), \frac{s}{2}\Big) < \epsilon$$

$$\gamma \leq \mathbf{F}(\psi(z_n) - \psi(z_N), s) \leq \mathbf{F}\Big(\psi(z_n) - \ell), \frac{s}{2}\Big) \star \mathbf{F}\Big((\psi(z_N) - \ell), \frac{s}{2}\Big) \leq \epsilon \star \epsilon < \gamma,$$

which is impossible. Therefore $B \subset \mathcal{W}$. Hence $\mathcal{W} \in \mathcal{M}$ implies $B \in \mathcal{M}$. □

Definition 3.11. Let $\mathcal{V} = (X, M, \diamond, \star)$ is a NNS. For $0 < s < 1$ we fix a closed ball $\mathcal{B}[z, r, s]$ with centre $z \in X$ and raduis $r > 0$, there exist $k \in \mathbb{N}$ such that for all $j \geq k$

$$\mathcal{B}[z, r, s] = \{(a_k) \in \ell_\infty : \mathbf{T}(\psi(z_j) - \psi(a_k), s) \geq 1 - r \text{ or } \mathbf{I}(\psi(z_j) - \psi(a_k), s) \leq r, \ \mathbf{F}(\psi(z_j) - \psi(a_k), s) \leq r\}.$$

Lemma 3.12. *Every closed ball $\mathcal{B}[z, r, s](\psi)$ is a closed set (CS).*

Proof. Consider $a = (a_j) \in \ell_\infty$ such that $a \in \overline{\mathcal{B}[z,r,s](\psi)}$. Since X is first countable, $\exists$, $a = (a_j) \in \overline{\mathcal{B}[z,r,s](\psi)}$ such that $a_j \longrightarrow a$ as $j \longrightarrow \infty$. implies the set

$$X = \Big\{(a_j) \in \ell_\infty : \mathbf{T}(\psi(z_j) - \psi(a_j), s) \geq 1 - r \text{ or } \mathbf{I}(\psi(z_j) - \psi(a_j), s) \leq r,\ \mathbf{F}(\psi(z_j) - \psi(a_j), s) \leq r\Big\}.$$

Since $a_j \longrightarrow a$, $\mathbf{T}\big(\psi(a_j) - \psi(a), s\big) \longrightarrow 1$ and $\mathbf{I}\big(\psi(a_j) - \psi(a), s\big) \longrightarrow 0$, $\mathbf{F}\big(\psi(a_i) - \psi(a), s\big) \longrightarrow 0$, as $j \longrightarrow \infty$. for all s. For a given $\epsilon > 0$,

Hence for $j \in X$

$$\mathbf{T}\big(\psi(z) - \psi(a), s + \epsilon\big) \geq \lim_{j \longrightarrow \infty} \mathbf{T}\big(\psi(z) - \psi(a_j), s\big) \diamond \mathbf{T}\big(\psi(a_j) - \psi(a), \epsilon\big) \geq 1 \diamond (1 - r) = 1 - r$$

and

$$\mathbf{I}\big(\psi(z) - \psi(a), s + \epsilon\big) \leq \lim_{j \longrightarrow \infty} \mathbf{I}\big(\psi(z) - \psi(a_j), s\big) \star \mathbf{I}\big(\psi(a_j) - \psi(a), \epsilon\big) \leq 0 \star r = r.$$

$$\mathbf{F}\big(\psi(z) - \psi(a), s + \epsilon\big) \leq \lim_{j \longrightarrow \infty} \mathbf{F}\big(\psi(z) - \psi(a_j), s\big) \star \mathbf{F}\big(\psi(a_j) - \psi(a), \epsilon\big) \leq 0 \star r = r.$$

In particuler for $b \in \mathbb{N}$, take $\epsilon = \frac{1}{b}$. Then,

$$\mathbf{T}\big(\psi(z) - \psi(a), s\big) = \lim_{b \longrightarrow \infty} \mathbf{T}\Big(\psi(z) - \psi(a), s + \frac{1}{b}\Big) \geq 1 - r.$$

and

$$\mathbf{I}\big(\psi(z) - \psi(a), s\big) = \lim_{b \longrightarrow \infty} \mathbf{I}\Big(\psi(z) - \psi(a), r + \frac{1}{b}\Big) \leq r.$$

$$\mathbf{F}\big(\psi(z) - \psi(a), s\big) = \lim_{b \longrightarrow \infty} \mathbf{F}\Big(\psi(z) - \psi(a), s + \frac{1}{b}\Big) \leq r.$$

$\Longrightarrow$ The set

$$\Big\{(a_j) \in \ell_\infty : \mathbf{T}(\psi(z_j) - \psi(a_j), s) \geq 1 - r \text{ or } \mathbf{I}(\psi(z_j) - \psi(a_j), s) \leq r,\ \mathbf{F}(\psi(z_j) - \psi(a_j), s) \leq r\Big\}.$$

$\Longrightarrow$ $a \in B[z,r,s](\psi)$. Therefore $B[z,r,s](\psi)$ is a closed set. $\square$

4. Conclusions

In this review, we have studied the concept of statistical convergence using neutrosophic norm space with the help of compact operator, which has an important place in the literature. We have defined the compact operator form statistical convergence of sequence spaces $\mathcal{S}^N_{(\mathbf{T},\mathbf{I},\mathbf{F})}(\psi)$ and $S^N_{0(\mathbf{T},\mathbf{I},\mathbf{F})}(\psi)$ and investigated basic properties. These are illustrated by proper examples.

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Pythagorean Neutrosophic Dombi Fuzzy Graphs with an Application to MCDM

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Abstract. Pythagorean Neutrosophic fuzzy graph, an extension of Pythagorean and Neutrosophic graph, is more efficient in representing relationship between various objects where the relation between the objects is uncertain, while the Dombi operators with adaptable operational parameter is very useful by taking distinct values. The Pythagorean Neutrosophic Dombi fuzzy graphs (PNDFG) is a novel concept proposed in this research paper by integrating the concepts Pythagorean Neutrosophic fuzzy graph and Dombi operator. Various basic graphical ideas using Dombi operator have been introduced for Pythagorean Neutrosophic fuzzy graphs. The main important part is the MCDM model which is proposed for the developed PNDFG and demonstrated with an illustrative example for choosing the best alternative.

Keywords: Pythagorean Neutrosophic sets; Pythagorean Neutrosophic Dombi fuzzy graphs; Dombi; Pythagorean fuzzy graphs.

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1. **Introduction**

In real-life situations, fuzzy set theory [1] plays an important role in resolving incomplete and ambiguous information. A fuzzy set is a variant of a regular set in which elements have a membership degree between 0 and 1. Fuzzy set and its conceptual development have wide-ranging applications in fields like engineering, computer science, mathematics, artificial intelligence,decision making and image analysis. The passage presented below gives some of the recent advancements in fuzzy set theory.

Atanassov [2] extended the fuzzy set to intuitionistic fuzzy set, which gives each element a membership and non-membership degree. An intuitionistic fuzzy set is one that meets the requirement that the sum of both membership and non-membership values is between 0 and 1. Smarandache's Neutrosophic set [3] is a generalization of the theory of fuzzy and intuitionistic fuzzy sets [1, 4] which deals with imprecise information. Elements with truth, indeterminacy, and false membership degrees that lie within the interval [0,1] characterize the single valued neutrosophic set which was introduced by Wang et al [5].

Yager [6-8] introduced Pythagorean fuzzy sets as an extension of intuitionistic fuzzy sets to deal with complex imprecision and uncertainty when the sum of squares of membership and non-membership degrees is between 0 and 1. Hence, Pythagorean fuzzy set accounts for larger amount of uncertainty than intuitionistic fuzzy set. The degree of dependence among components of fuzzy sets and neutrosophic sets was introduced by Smarandache and was developed further. Out of three membership functions of neutrosophic sets, one special case with independent indeterminacy and dependent truth and falsity are chosen with the constraint that the total of squares of membership, indeterminacy and non-membership lies between 0 and 2 and it is termed as Pythagorean Neutrosophic set [23].

Graphs are pictorial representations of objects and their relationships. More uncertainties occur in relations among objects which results in the need for framing fuzzy graph model rather than ordinary graph, which has the same structure. Using Zadeh's fuzzy relation, Kaufmann [9] introduced the concept of fuzzy graphs. Various fundamental and theoretical ideas like bridges, cycles and connectedness were defined and developed by Rosenfeld [10]. Karunambigai and Parvathi [11] instituted the intuitionistic fuzzy graphs which was further extended to intuitionistic fuzzy hypergraph and its applications have been explored [12]. Broumi et al. presented single-valued neutrosophic graphs [13] with examples and properties, and the properties of degree and regular single valued neutrosophic graphs were also examined [14]. The concept of fuzzy graph was advanced to pythagorean fuzzy graphs in [15]. The new emerging concept of Pythagorean neutrosophic fuzzy graph [16] were advanced by blending the concept of Pythagorean Neutrosophic sets and fuzzy graphs. Ashraf et al. [17] proposed the idea of Dombi fuzzy graphs. Subsequently, many researchers worked on this Dombi fuzzy graphs and made advancements like interval valued Dombi fuzzy neutrosophic graph [18], decision making using Dombi fuzzy graphs [19], Pythagorean Dombi fuzzy graphs [20], picture Dombi fuzzy graph [21] and Dombi bipolar fuzzy graph [22]. Application of the fuzzy theory in decision making is an effective way for solving real-life problems, and it is advanced using all the new concept developments most recently in [24-30]. This paper presents Pythagorean neutrosophic Dombi graphs as a generalization of Pythagorean and Neutrosophic Dombi fuzzy graphs (PNDFG).

The following is the layout of the research paper: In section 2, the paper's basic terminologies are explained. In section 3, we define the complement, homomorphism, isomorphism, strength, and completeness of PNDFG. Section 4 proposes an algorithm for Multi-criteria decision making based on the PNDFG, which is demonstrated with an example and concluded in section 5.

2. Preliminaries

Definition 2.1.[1] On a universe $\mathfrak{U}$, $\mathfrak{A} = \{\langle \mathfrak{s}, \mu_{\mathfrak{A}}(\mathfrak{s})\rangle | \mathfrak{s} \in \mathfrak{U}\}$ is a fuzzy set (FS) where $\mu_{\mathfrak{A}} : \mathfrak{U} \to [0,1]$ symbolizes the membership grade of $\mathfrak{A}$.

Definition 2.2.[1] A fuzzy relation on a fuzzy set $\mathfrak{X}$ is $\mathfrak{X} \times \mathfrak{X}$, represented by $\mathfrak{B} = \{\langle \mathfrak{st}, \mu_{\mathfrak{B}}(\mathfrak{st})\rangle | \mathfrak{st} \in \mathfrak{X} \times \mathfrak{X}\}$, where $\mu_{\mathfrak{B}} : \mathfrak{X} \times \mathfrak{X} \to [0,1]$ is the membership grades of $\mathfrak{A}$.

Definition 2.3.[9] A fuzzy graph is a duo $G = (\mathfrak{A}, \mathfrak{B})$ on $\mathfrak{X}$ with A a FS on $\mathfrak{X}$ and $\mathfrak{B}$ a FR on $\mathfrak{X}$ such that $\mu_{\mathfrak{B}}(\mathfrak{st}) \leq \mu_{\mathfrak{A}}(\mathfrak{s}) \wedge \mu_{\mathfrak{A}}(\mathfrak{t})\ \forall\ s, t \in \mathfrak{X}$, where $\mathfrak{A} : \mathfrak{X} \to [0,1]$ and $\mathfrak{B}$ from $\mathfrak{X} \times \mathfrak{X}$ to $[0,1]$.

Definition 2.4.[6] A Pythagorean fuzzy set (PFS) on a universe $\mathfrak{X}$ is $\mathfrak{A} = \{\langle s, \mu_{\mathfrak{A}}(\mathfrak{s}), \vartheta_{\mathfrak{A}}(\mathfrak{s})\rangle | \mathfrak{s} \in \mathfrak{X}\}$, where $\mu_{\mathfrak{A}} : \mathfrak{X} \to [0,1]$ and $\vartheta_{\mathfrak{A}} : \mathfrak{X} \to [0,1]$ signify the membership and non-membership grades of $\mathfrak{A}$, and $\mu_{\mathfrak{A}}, \vartheta_{\mathfrak{A}}$ satisfying $0 \leq \mu^2_{\mathfrak{A}}(s) + \vartheta^2_{\mathfrak{A}}(\mathfrak{s}) \leq 1\ \forall\ \mathfrak{s} \in \mathfrak{X}$.

Definition 2.5.[6] A Pythagorean fuzzy set on $\mathfrak{X} \times \mathfrak{X}$ is called a Pythagorean fuzzy relation (PFR) on $\mathfrak{X}$, represented by $\mathfrak{B} = \{\langle \mathfrak{st}, \mu_{\mathfrak{B}}(\mathfrak{st}), \vartheta_{\mathfrak{B}}(\mathfrak{st})\rangle | \mathfrak{s} \in \mathfrak{X}\}$, where $\mu_{\mathfrak{B}} : \mathfrak{X} \times \mathfrak{X} \to [0,1]$ and $\vartheta_{\mathfrak{A}} : \mathfrak{X} \times \mathfrak{X} \to [0,1]$ signify the membership and non-membership grades of $\mathfrak{B}$, correspondingly, such that $0 \leq \mu^2_{\mathfrak{B}}(\mathfrak{st}) + \vartheta^2_{\mathfrak{B}}(\mathfrak{st}) \leq 1\ \forall\ \mathfrak{s}, \mathfrak{t} \in \mathfrak{X}$.

Definition 2.6.[15] A Pythagorean fuzzy graph (PFG) on a non-empty set $\mathfrak{X}$ is a pair $G = (\mathfrak{A}, \mathfrak{B})$ with $\mathfrak{A}$ a PFS on $\mathfrak{X}$ and $\mathfrak{B}$ a PFR on $\mathfrak{X}$ such that $\mu_{\mathfrak{B}}(\mathfrak{st}) \leq \mu_{\mathfrak{A}}(\mathfrak{s}) \wedge \mu_{\mathfrak{A}}(\mathfrak{t})$, $\vartheta_{\mathfrak{B}}(\mathfrak{st}) \geq \vartheta_{\mathfrak{A}}(\mathfrak{s}) \vee \vartheta_{\mathfrak{A}}(\mathfrak{t})$ and $0 \leq \mu^2_{\mathfrak{B}}(\mathfrak{st}) + \vartheta^2_{\mathfrak{B}}(\mathfrak{st}) \leq 1\ \forall\ \mathfrak{s}, \mathfrak{t} \in \mathfrak{X}$. where $\mu_{\mathfrak{B}} : \mathfrak{X} \times \mathfrak{X} \to [0,1]$ and $\vartheta_{\mathfrak{B}} : \mathfrak{X} \times \mathfrak{X} \to [0,1]$ symbolize the membership and non-membership grades of $\mathfrak{B}$, correspondingly.

Definition 2.7.[19] A binary function $\mathfrak{T} : [0,1] \times [0,1] \to [0,1]$ is named as t-norm if $\forall$ $\mathfrak{a}, \mathfrak{b}, \mathfrak{u} \in [0,1]$, it fulfills the following criteria:

1. $\mathfrak{T}(\mathfrak{a}, 1) = \mathfrak{a}$,
2. $\mathfrak{T}(\mathfrak{a}, \mathfrak{b}) = \mathfrak{T}(\mathfrak{b}, \mathfrak{a})$,

3. $\mathfrak{T}(\mathfrak{a}, \mathfrak{T}(\mathfrak{b}, u)) = \mathfrak{T}(\mathfrak{T}(\mathfrak{a}, \mathfrak{b}), \mathfrak{u})$,
4. $\mathfrak{T}(\mathfrak{a}, \mathfrak{b}) \leq \mathfrak{T}(\mathfrak{u}, \mathfrak{v})$ if $\mathfrak{a} \leq \mathfrak{u}$ and $\mathfrak{b} \leq \mathfrak{v}$.

The Dombi's t-norm and t-conorm are given by $\frac{1}{1+[(\frac{1-\mathfrak{a}}{\mathfrak{a}})^{\gamma}+(\frac{1-\mathfrak{b}}{\mathfrak{b}})^{\gamma}]^{\frac{1}{\gamma}}}, \gamma > 0$. and $\frac{1}{1+[(\frac{1-\mathfrak{a}}{\mathfrak{a}})^{-\gamma}+(\frac{1-\mathfrak{b}}{\mathfrak{b}})^{-\gamma}]^{\frac{1}{-\gamma}}}, \gamma > 0$ respectively.
By putting $\gamma = 1$ in Dombi's t-norm and t-conorm one gets the other set of T-operators $\mathfrak{T}(\mathfrak{a}, \mathfrak{b}) = \frac{\mathfrak{ab}}{\mathfrak{a}+\mathfrak{b}-\mathfrak{ab}}$ and $P(\mathfrak{a}, t) = \frac{\mathfrak{a}+\mathfrak{b}-2\mathfrak{ab}}{1-\mathfrak{ab}}$,.

Definition 2.8.[16] Pythagorean Neutrosophic Fuzzy Graph (PNFG) is $G = (V, E)$, where $V = \{v_1, v_2, ..., v_n\}$ such that μ_1, β_1, σ_1 are from V to $[0, 1]$ with $0 \leq \mu_1(v_i)^2+\beta_1(v_i)^2+\sigma_1(v_i)^2 \leq 2$ $\forall v_i \in V$ indicates the membership, indeterminacy and non-membership functions and μ_2, β_2, σ_2 are from $V \times V$ to $[0, 1]$ such that $\mu_2(v_iv_j) \leq \mu_1(v_i) \wedge \mu_1(v_j)$,$\beta_2(v_iv_j) \leq \beta_1(v_i) \wedge \beta_1(v_j)$ and $\sigma_2(v_iv_j) \leq \sigma_1(v_i) \vee \sigma_1(v_j)$ with $0 \leq \mu_2(v_iv_j)^2 + \beta_2(v_iv_j)^2 + \sigma_2(v_iv_j)^2 \leq 2$ $\forall v_iv_j \in V \times V$.

Definition 2.9.[17] A Dombi fuzzy graph on V is a pair that has been ordered as $G = (\mathfrak{A}, \mathfrak{B})$, where $\mathfrak{A} : V \to [0, 1]$ is contained in V and $\mathfrak{B} : V \times V \to [0, 1]$ is a symmetric fuzzy relation on $\mathfrak{A}$ such that $\mu_{\mathfrak{B}}(st) \leq \frac{\mu_{\mathfrak{A}}(s)\mu_{\mathfrak{A}}(t)}{\mu_{\mathfrak{A}}(s)+\mu_{\mathfrak{A}}(t)-\mu_{\mathfrak{A}}(s)\mu_A(t)}$ $\forall$ $s, t \in V$, where $\mu_{\mathfrak{A}}$ and $\mu_{\mathfrak{B}}$ symbolize the membership grades of $\mathfrak{A}$ and $\mathfrak{B}$, correspondingly.

3. Pythagorean Neutrosophic Dombi Fuzzy Graphs

Definition 3.1. A Pythagorean Neutrosophic Dombi Fuzzy Graph (PNDFG) with finite underlying set $\mathfrak{V}$ is a pair $G = (\eta, \zeta)$, where $\eta = (\mu_1, \beta_1, \sigma_1)$ from $\mathfrak{V}$ to $[0, 1]$ is a Pythagorean neutrosophic subset in $\mathfrak{V}$ and $\zeta = (\mu_2, \beta_2, \sigma_2)$ from $\mathfrak{V} \times \mathfrak{V}$ to $[0, 1]$ is a symmetric Pythagorean Neutrosophic fuzzy relation on η such that

$$\mu_2(\mathfrak{gh}) \leq \frac{\mu_1(\mathfrak{g})\mu_1(\mathfrak{h})}{\mu_1(\mathfrak{g}) + \mu_1(\mathfrak{h}) - \mu_1(\mathfrak{g}) \cdot \mu_1(\mathfrak{gh})}$$

$$\beta_2(\mathfrak{gh}) \leq \frac{\beta_1(\mathfrak{g})\beta_1(\mathfrak{h})}{\beta_1(\mathfrak{g}) + \beta_1(\mathfrak{h}) - \beta_1(\mathfrak{g}) \cdot \beta_1(\mathfrak{gh})}$$

$$\sigma_2(\mathfrak{gh}) \leq \frac{\sigma_1(\mathfrak{g}) + \sigma_1(\mathfrak{h}) - 2\ \sigma_1(\mathfrak{g})\ \sigma_1(\mathfrak{h})}{1 - \sigma_1(\mathfrak{g})\ \sigma_1(\mathfrak{h})}$$

and $0 \leq \mu_2^2(\mathfrak{gh}) + \beta_2^2(\mathfrak{gh}) + \sigma_2^2(\mathfrak{gh}) \leq 2$ for all $\mathfrak{g}, \mathfrak{h} \in \mathfrak{V}$.
η and ζ denote the Pythagorean Neutrosophic Dombi fuzzy vertex and edge sets of G.

Example 1. Consider a PNDFG over $\mathfrak{V} = \{a, b, c, d, e, f\}$ defined by
$\langle(a, .5, .6, .5), (b, .8, .4, .2), (c, .4, .6, .2), (d, .6, .5, .7), (e, .7, .4, .3), (f, .3, .6, .7)\rangle$
$\langle(ab, .44, .316, .556), (af, .231, .429, .769), (bc, .432, .316, .33), (cd, .316, .376, .721), (de, .477, .286, .734), (ef, .266, .316, .734), (ad, .375, .375, .769), (be, .596, .25, .404), (cf, .207, .429, .721)\rangle$

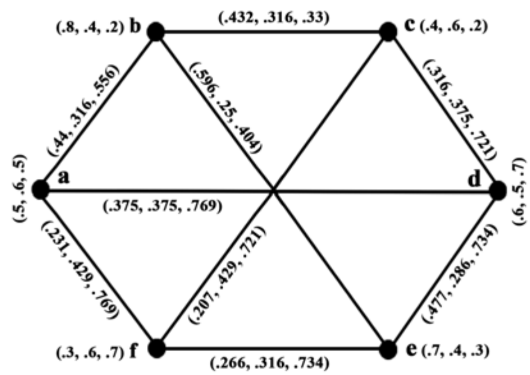

FIGURE 1. Pythagorean Neutrosophic Dombi Fuzzy graph

Definition 3.2. Let $\zeta = \{(\mathfrak{gh}, \mu_2(\mathfrak{gh}), \beta_2(\mathfrak{gh}), \sigma_2(\mathfrak{gh})) / \mathfrak{gh} \in \mathfrak{E}\}$ be a PN Dombi fuzzy edge set in PNDFG G; then

1. The order of G is represented by
$$O(G) = \left(\sum_{\mathfrak{g}\in\mathfrak{V}} \mu_1(\mathfrak{g}), \sum_{\mathfrak{g}\in\mathfrak{V}} \beta_1(\mathfrak{g}), \sum_{\mathfrak{g}\in\mathfrak{V}} \sigma_1(\mathfrak{g})\right)$$
2. The size of G is symbolized as $S(G)$ and is defined by
$$S(G) = \left(\sum_{\mathfrak{gh}\in\mathfrak{E}} \mu_2(\mathfrak{gh}), \sum_{\mathfrak{gh}\in\mathfrak{E}} \beta_2(\mathfrak{gh}), \sum_{\mathfrak{gh}\in\mathfrak{E}} \sigma_2(\mathfrak{gh})\right)$$

Example 2. For the PNDFG in Example 1, the order of G as (3.3, 3.1, 2.6) and size of the PNDFG as (3.34, 3.092, 5.738).

Definition 3.3. Let $\zeta = \{(\mathfrak{gh}, \mu_2(\mathfrak{gh}), \beta_2(\mathfrak{gh}), \sigma_2(\mathfrak{gh})) / \mathfrak{gh} \in \mathfrak{E}\}$ be a PN Dombi fuzzy edge set in PNDFG G; then the degree of vertex $\mathfrak{g} \in \mathfrak{V}$ is symbolized by $(D)_G(\mathfrak{g})$ and defined as $(D)_G(\mathfrak{g}) = ((D)_\mu(\mathfrak{g}), (D)_\beta(\mathfrak{g}), (D)_\sigma(\mathfrak{g}))$, where

$$(D)_\mu(\mathfrak{g}) = \sum_{\mathfrak{g},h\neq\mathfrak{g}\in\mathfrak{V}} \mu_2(gh) = \sum_{\mathfrak{g},\mathfrak{h}\neq\mathfrak{g}\in\mathfrak{V}} \frac{\mu_1(\mathfrak{g})\ \mu_1(\mathfrak{h})}{\mu_1(\mathfrak{g}) + \mu_1(\mathfrak{h}) - \mu_1(\mathfrak{g})\ \mu_1(\mathfrak{h})}$$
$$(D)_\beta(\mathfrak{g}) = \sum_{\mathfrak{g},h\neq\mathfrak{g}\in\mathfrak{V}} \beta_2(\mathfrak{gh}) = \sum_{\mathfrak{g},\mathfrak{h}\neq\mathfrak{g}\in\mathfrak{V}} \frac{\beta_1(\mathfrak{g})\ \beta_1(\mathfrak{h})}{\beta_1(\mathfrak{g}) + \beta_1(\mathfrak{h}) - \beta_1(\mathfrak{g})\ \beta_1(\mathfrak{h})}$$

$$(D)_\sigma(\mathfrak{g}) = \sum_{\mathfrak{g},h\neq\mathfrak{g}\in\mathfrak{V}} \sigma_2(\mathfrak{g}\mathfrak{h}) = \sum_{\mathfrak{g},\mathfrak{h}\neq\mathfrak{g}\in\mathfrak{V}} \frac{\sigma_1(\mathfrak{g}) + \sigma_1(\mathfrak{h}) - 2\ \sigma_1(\mathfrak{g})\ \sigma_1(\mathfrak{h})}{1 - \sigma_1(\mathfrak{g})\ \sigma_1(\mathfrak{h})}$$

The total degree of vertex $\mathfrak{g} \in \mathfrak{V}$ is symbolized by $(TD)_G(\mathfrak{g})$ and defined as $(TD)_G(\mathfrak{g}) = ((TD)_\mu(\mathfrak{g}), (TD)_\beta(\mathfrak{g}), (TD)_\sigma(\mathfrak{g}))$, where

$$(TD)_\mu(\mathfrak{g}) = \sum_{\mathfrak{g},\mathfrak{h}\neq\mathfrak{g}\in\mathfrak{V}} \mu_2(\mathfrak{g}\mathfrak{h}) + \mu_1(\mathfrak{g}) = \sum_{\mathfrak{g},\mathfrak{h}\neq\mathfrak{g}\in\mathfrak{V}} \frac{\mu_1(\mathfrak{g})\ \mu_1(\mathfrak{h})}{\mu_1(\mathfrak{g}) + \mu_1(\mathfrak{h}) - \mu_1(\mathfrak{g})\ \mu_1(\mathfrak{h})} + \mu_1(\mathfrak{g}),$$
$$(TD)_\beta(\mathfrak{g}) = \sum_{\mathfrak{g},\mathfrak{h}\neq\mathfrak{g}\in\mathfrak{V}} \beta_2(\mathfrak{g}\mathfrak{h}) + \beta_1(\mathfrak{g}) = \sum_{\mathfrak{g},\mathfrak{h}\neq\mathfrak{g}\in\mathfrak{V}} \frac{\beta_1(\mathfrak{g})\ \beta_1(\mathfrak{h})}{\beta_1(\mathfrak{g}) + \beta_1(\mathfrak{h}) - \beta_1(\mathfrak{g})\ \beta_1(\mathfrak{h})} + \beta_1(\mathfrak{g}),$$
$$(TD)_\sigma(\mathfrak{g}) = \sum_{\mathfrak{g},\mathfrak{h}\neq\mathfrak{g}\in\mathfrak{V}} \sigma_2(\mathfrak{g}\mathfrak{h}) + \sigma_1(\mathfrak{g}) = \sum_{\mathfrak{g},\mathfrak{h}\neq\mathfrak{g}\in\mathfrak{V}} \frac{\sigma_1(\mathfrak{g}) + \sigma_1(\mathfrak{h}) - 2\ \sigma_1(\mathfrak{g})\ \sigma_1(\mathfrak{h})}{1 - \sigma_1(\mathfrak{g})\ \sigma_1(\mathfrak{h})} + \sigma_1(\mathfrak{g}).$$

Example 3. For the PNDFG in Figure 1, the degree and the total degree of the vertices are
$D_G(a) = (1.046, 1.12, 2.094)$, $TD_G(a) = (1.546, 1.72, 2.594)$
$D_G(b) = (1.468, 0.882, 1.29)$, $TD_G(b) = (2.268, 1.282, 1.49)$
$D_G(c) = (0.995, 1.12, 1.772)$, $TD_G(c) = (1.355, 1.72, 1.972)$
$D_G(d) = (1.168, 1.036, 2.224)$, $TD_G(d) = (1.768, 1.536, 2.924)$
$D_G(e) = (1.339, 0.852, 1.872)$, $TD_G(e) = (2.039, 1.252, 2.172)$
$D_G(f) = (0.704, 1.174, 2.224)$, $TD_G(f) = (1.004, 1.774, 2.924)$

Definition 3.4. The complement of a PNDFG $G = (\eta, \zeta)$ is a PNDFG $\bar{G} = (\bar{\eta}, \bar{\zeta})$ which is defined by

1. $\overline{\mu_1(\mathfrak{g})} = \mu_1(\mathfrak{g})$, $\overline{\beta_1(\mathfrak{g})} = \beta_1(\mathfrak{g})$ and $\overline{\sigma_1(\mathfrak{g})} = \sigma_1(\mathfrak{g})$.
2. $\overline{\mu_2(\mathfrak{g}\mathfrak{h})} = \begin{cases} \frac{\mu_1(\mathfrak{g})\mu_1(\mathfrak{h})}{\mu_1(\mathfrak{g})+\mu_1(\mathfrak{h})-\mu_1(\mathfrak{g})\mu_1(\mathfrak{h})} & if\ \mu_2(\mathfrak{g}\mathfrak{h}) = 0, \\ \frac{\mu_1(\mathfrak{g})\mu_1(\mathfrak{h})}{\mu_1(\mathfrak{g})+\mu_1(\mathfrak{h})-\mu_1(\mathfrak{g})\mu_1(\mathfrak{h})} - \mu_2(\mathfrak{g}\mathfrak{h}) & if\ 0 < \mu_2(\mathfrak{g}\mathfrak{h}) \leq 1 \end{cases}$
3. $\overline{\beta_2(\mathfrak{g}\mathfrak{h})} = \begin{cases} \frac{\beta_1(\mathfrak{g})\beta_1(\mathfrak{h})}{\beta_1(\mathfrak{g})+\beta_1(\mathfrak{h})-\beta_1(\mathfrak{g})\beta_1(\mathfrak{h})} & if\ \beta_2(\mathfrak{g}\mathfrak{h}) = 0, \\ \frac{\beta_1(\mathfrak{g})\beta_1(h)}{\beta_1(\mathfrak{g})+\beta_1(\mathfrak{h})-\beta_1(\mathfrak{g})\beta_1(\mathfrak{h})} - \beta_2(\mathfrak{g}\mathfrak{h}) & if\ 0 < \beta_2(\mathfrak{g}\mathfrak{h}) \leq 1 \end{cases}$
4. $\overline{\sigma_2(\mathfrak{g}\mathfrak{h})} = \begin{cases} \frac{\sigma_1(\mathfrak{g})+\sigma_1(\mathfrak{h})-2\sigma_1(\mathfrak{g})\sigma_1(\mathfrak{h})}{1-\sigma_1(\mathfrak{g})\sigma_1(\mathfrak{h})} & if\ \sigma_2(\mathfrak{g}\mathfrak{h}) = 0, \\ \frac{\sigma_1(\mathfrak{g})+\sigma_1(\mathfrak{h})-2\sigma_1(\mathfrak{g})\sigma_1(\mathfrak{h})}{1-\sigma_1(\mathfrak{g})\sigma_1(\mathfrak{h})} - \sigma_2(\mathfrak{g}\mathfrak{h}) & if\ 0 < \sigma_2(\mathfrak{g}\mathfrak{h}) \leq 1 \end{cases}$

Theorem 1. If $G = (\eta, \zeta)$ is a PNDFG, then $\overline{\overline{G}} = G$.

Proof: Consider G as a PNDFG. By definition of complement of PNDFG, we have
$\overline{\overline{\mu_1(\mathfrak{g})}} = \overline{\mu_1(\mathfrak{g})} = \mu_1(\mathfrak{g})$, $\overline{\overline{\beta_1(\mathfrak{g})}} = \overline{\beta_1(\mathfrak{g})} = \beta_1(\mathfrak{g})$, $\overline{\overline{\sigma_1(\mathfrak{g})}} = \overline{\sigma_1(\mathfrak{g})} = \sigma_1(\mathfrak{g})$, for all $\mathfrak{g} \in \mathfrak{V}$.
If $\mu_2(\mathfrak{g}\mathfrak{h}) = 0$, $\beta_2(\mathfrak{g}\mathfrak{h}) = 0$, $\sigma_2(\mathfrak{g}\mathfrak{h}) = 0$, then

$$\overline{\overline{\mu_2(\mathfrak{g}\mathfrak{h})}} = \frac{\overline{\mu_1(\mathfrak{g})\ \mu_1(\mathfrak{h})}}{\overline{\overline{\mu_1(\mathfrak{g})} + \overline{\mu_1(\mathfrak{h})} - \overline{\mu_1(\mathfrak{g})\ \mu_1(\mathfrak{h})}}} = \frac{\mu_1(\mathfrak{g})\ \mu_1(\mathfrak{h})}{\mu_1(\mathfrak{g}) + \mu_1(\mathfrak{h}) - \mu_1(\mathfrak{g})\ \mu_1(\mathfrak{h})} = \mu_2(\mathfrak{g}\mathfrak{h}),$$

$$\overline{\overline{\beta_2(\mathfrak{g}\mathfrak{h})}} = \frac{\overline{\beta_1(\mathfrak{g})\ \beta_1(\mathfrak{h})}}{\overline{\beta_1(\mathfrak{g})} + \overline{\beta_1(\mathfrak{h})} - \overline{\beta_1(\mathfrak{g})\ \beta_1(\mathfrak{h})}} = \frac{\beta_1(\mathfrak{g})\ \beta_1(\mathfrak{h})}{\beta_1(\mathfrak{g}) + \beta_1(\mathfrak{h}) - \beta_1(\mathfrak{g})\ \beta_1(\mathfrak{h})} = \beta_2(\mathfrak{g}\mathfrak{h}),$$

$$\overline{\overline{\sigma_2(\mathfrak{g}\mathfrak{h})}} = \frac{\overline{\sigma_1(\mathfrak{g})} + \overline{\sigma_1(\mathfrak{h})} - 2\ \overline{\sigma_1(\mathfrak{g})\ \sigma_1(\mathfrak{h})}}{1 - \overline{\sigma_1(\mathfrak{g})\ \sigma_1(\mathfrak{h})}} = \frac{\sigma_1(\mathfrak{g}) + \sigma_1(\mathfrak{h}) - 2\ \sigma_1(\mathfrak{g})\ \sigma_1(\mathfrak{h})}{1 - \sigma_1(\mathfrak{g})\ \sigma_1(\mathfrak{h})} = \sigma_2(\mathfrak{g}\mathfrak{h}).$$

If $0 < \mu_2(\mathfrak{g}\mathfrak{h}), \beta_2(\mathfrak{g}\mathfrak{h}), \sigma_2(\mathfrak{g}\mathfrak{h}) \leq 1$, then

$$\overline{\overline{\mu_2(\mathfrak{g}\mathfrak{h})}} = \frac{\overline{\mu_1(\mathfrak{g})\ \mu_1(\mathfrak{h})}}{\overline{\mu_1(\mathfrak{g})} + \overline{\mu_1(\mathfrak{h})} - \overline{\mu_1(\mathfrak{g})\ \mu_1(\mathfrak{h})}} - \overline{\mu_2(\mathfrak{g}\mathfrak{h})}$$

$$= \frac{\mu_1(\mathfrak{g})\ \mu_1(\mathfrak{h})}{\mu_1(\mathfrak{g}) + \mu_1(\mathfrak{h}) - \mu_1(\mathfrak{g})\ \mu_1(\mathfrak{h})} - [\frac{\mu_1(\mathfrak{g})\ \mu_1(\mathfrak{h})}{\mu_1(\mathfrak{g}) + \mu_1(\mathfrak{h}) - \mu_1(\mathfrak{g})\ \mu_1(\mathfrak{h})} - \mu_2(\mathfrak{g}\mathfrak{h})] = \mu_2(\mathfrak{g}\mathfrak{h}),$$

$$\overline{\overline{\beta_2(\mathfrak{g}\mathfrak{h})}} = \frac{\overline{\beta_1(\mathfrak{g})\ \beta_1(\mathfrak{h})}}{\overline{\beta_1(\mathfrak{g})} + \overline{\beta_1(\mathfrak{h})} - \overline{\beta_1(\mathfrak{g})\ \beta_1(\mathfrak{h})}} - \overline{\beta_2(\mathfrak{g}\mathfrak{h})}$$

$$= \frac{\beta_1(\mathfrak{g})\ \beta_1(\mathfrak{h})}{\beta_1(\mathfrak{g}) + \beta_1(\mathfrak{h}) - \beta_1(\mathfrak{g})\ \beta_1(\mathfrak{h})} - [\frac{\beta_1(\mathfrak{g})\ \beta_1(\mathfrak{h})}{\beta_1(\mathfrak{g}) + \beta_1(\mathfrak{h}) - \beta_1(\mathfrak{g})\ \beta_1(\mathfrak{h})} - \beta_2(\mathfrak{g}\mathfrak{h})] = \beta_2(\mathfrak{g}\mathfrak{h}),$$

$$\overline{\overline{\sigma_2(\mathfrak{g}\mathfrak{h})}} = \frac{\overline{\sigma_1(\mathfrak{g})} + \overline{\sigma_1(\mathfrak{h})} - 2\ \overline{\sigma_1(\mathfrak{g})\ \sigma_1(\mathfrak{h})}}{1 - \overline{\sigma_1(\mathfrak{g})\ \sigma_1(\mathfrak{h})}} - \sigma_2(\mathfrak{g}\mathfrak{h})$$

$$= \frac{\sigma_1(\mathfrak{g}) + \sigma_1(\mathfrak{h}) - 2\ \sigma_1(\mathfrak{g})\ \sigma_1(\mathfrak{h})}{1 - \sigma_1(\mathfrak{g})\ \sigma_1(\mathfrak{h})} - [\frac{\sigma_1(\mathfrak{g}) + \sigma_1(\mathfrak{h}) - 2\ \sigma_1(\mathfrak{g})\ \sigma_1(\mathfrak{h})}{1 - \sigma_1(\mathfrak{g})\ \sigma_1(\mathfrak{h})} - \sigma_2(\mathfrak{g}\mathfrak{h})] = \sigma_2(\mathfrak{g}\mathfrak{h})\ \forall \mathfrak{g}, \mathfrak{h} \in \mathfrak{V}.$$

Hence, the complement of a complement PNDFG is a PNDFG itself.

Definition 3.5. A homomorphism $H : G_1 \to G_2$ of two PNDFGs $G_1 = (\eta_1, \zeta_1)$ and $G_2 = (\eta_2, \zeta_2)$ is a mapping $H : \mathfrak{V}_1 \to \mathfrak{V}_2$ satisfying

(1) $\mu_{\eta_1}(\mathfrak{g}) \leq \mu_{\eta_2}(H(\mathfrak{g}))$,

$\beta_{\eta_1}(\mathfrak{g}) \leq \beta_{\eta_2}(H(\mathfrak{g}))$,

$\sigma_{\eta_1}(\mathfrak{g}) \leq \sigma_{\eta_2}(H(\mathfrak{g}))$.

(2) $\mu_{\zeta_1}(\mathfrak{g}\mathfrak{t}) \leq \mu_{\zeta_2}(H(\mathfrak{g})H(\mathfrak{t}))$,

$\beta_{\zeta_1}(\mathfrak{g}\mathfrak{t}) \leq \beta_{\zeta_2}(H(\mathfrak{g})H(\mathfrak{t}))$,

$\sigma_{\zeta_1}(\mathfrak{g}\mathfrak{t}) \leq \sigma_{\zeta_2}(H(\mathfrak{g})H(\mathfrak{t})) \ \forall \ \mathfrak{g} \in \mathfrak{V}_1, \mathfrak{g}\mathfrak{t} \in \mathfrak{E}_1$.

Definition 3.6. An isomorphism $H : G_1 \to G_2$ of two PNDFGs $G_1 = (\eta_1, \zeta_1)$ and $G_2 = (\eta_2, \zeta_2)$ is a bijective mapping $H : \mathfrak{V}_1 \to \mathfrak{V}_2$ satisfying

(1) $\mu_{\eta_1}(\mathfrak{g}) = \mu_{\eta_2}(H(\mathfrak{g}))$,

$\beta_{\eta_1}(\mathfrak{g}) = \beta_{\eta_2}(H(\mathfrak{g}))$,

$\sigma_{\eta_1}(\mathfrak{g}) = \sigma_{\eta_2}(H(\mathfrak{g}))$.

(2) $\mu_{\zeta_1}(\mathfrak{g}\mathfrak{t}) = \mu_{\zeta_2}(H(\mathfrak{g})H(\mathfrak{t}))$,

$\beta_{\zeta_1}(\mathfrak{g}\mathfrak{t}) = \beta_{\zeta_2}(H(\mathfrak{g})H(\mathfrak{t}))$,

$\sigma_{\zeta_1}(\mathfrak{g}\mathfrak{t}) = \sigma_{\zeta_2}(H(\mathfrak{g})H(\mathfrak{t})) \ \forall \ \mathfrak{g} \in \mathfrak{V}_1, \mathfrak{g}\mathfrak{t} \in \mathfrak{E}_1$.

Definition 3.7. A weak isomorphism $H : G_1 \to G_2$ of two PNDFGs $G_1 = (\eta_1, \zeta_1)$ and $G_2 = (\eta_2, \zeta_2)$ is a bijective mapping $H : \mathfrak{V}_1 \to \mathfrak{V}_2$ satisfying

(1) H is a homomorphism.

(2) $\mu_{\eta_1}(\mathfrak{g}) = \mu_{\eta_2}(H(\mathfrak{g}))$,
$\beta_{\eta_1}(\mathfrak{g}) = \beta_{\eta_2}(H(\mathfrak{g}))$,
$\sigma_{\eta_1}(\mathfrak{g}) = \sigma_{\eta_2}(H(\mathfrak{g})) \ \mathfrak{g} \in \mathfrak{V}_1$.

Definition 3.8. A co-weak isomorphism $H : G_1 \to G_2$ of two PNDFGs $G_1 = (\eta_1, \zeta_1)$ and $G_2 = (\eta_2, \zeta_2)$ is a bijective mapping $H : \mathfrak{V}_1 \to \mathfrak{V}_2$ satisfying

(1) H is a homomorphism.

(2) $\mu_{\zeta_1}(\mathfrak{gt}) = \mu_{\zeta_2}(H(\mathfrak{g})H(\mathfrak{t}))$,

$\beta_{\zeta_1}(\mathfrak{gt}) = \beta_{\zeta_2}(H(\mathfrak{g})H(\mathfrak{t}))$,

$\sigma_{\zeta_1}(\mathfrak{gt}) = \sigma_{\zeta_2}(H(\mathfrak{g})H(t))\ \forall\ \mathfrak{g} \in \mathfrak{V}_1, \mathfrak{gt} \in \mathfrak{E}_1$.

Definition 3.9. A PNDFG $G = (\eta, \zeta)$ is called self-complement if $\overline{G} \cong G$.

Proposition 1. If $G = (\eta, \zeta)$ is a self-complementary PNDFG, then

$$\sum_{\mathfrak{g}\neq t} \mu_\zeta(\mathfrak{gt}) = \frac{1}{2} \sum_{\mathfrak{g}\neq\mathfrak{t}} \frac{\mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})}{\mu_\eta(\mathfrak{g}) + \mu_\eta(\mathfrak{t}) - \mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})},$$

$$\sum_{\mathfrak{g}\neq\mathfrak{t}} \beta_\zeta(\mathfrak{gt}) = \frac{1}{2} \sum_{\mathfrak{g}\neq\mathfrak{t}} \frac{\beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})}{\beta_\eta(\mathfrak{g}) + \beta_\eta(\mathfrak{t}) - \beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})},$$

$$\sum_{\mathfrak{g}\neq\mathfrak{t}} \sigma_\zeta(\mathfrak{gt}) = \frac{1}{2} \sum_{\mathfrak{g}\neq\mathfrak{t}} \frac{\sigma_\eta(\mathfrak{g}) + \sigma_\eta(\mathfrak{t}) - 2\ \sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})}{1 - \sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})}.$$

Proof: Assume that G is a self-complementary PNDFG; then there exists an isomorphism $H : \mathfrak{V} \to \mathfrak{V}$ such that

$\overline{\mu_\eta(H(\mathfrak{g}))} = \mu_\eta(\mathfrak{g})$, $\overline{\beta_\eta(H(\mathfrak{g}))} = \beta_\eta(\mathfrak{g})$, $\overline{\sigma_\eta(H(\mathfrak{g}))} = \sigma_\eta(\mathfrak{g})$. $\forall\ \mathfrak{g} \in \mathfrak{V}$

$\overline{\mu_\zeta(H(\mathfrak{g})H(\mathfrak{t}))} = \mu_\zeta(\mathfrak{gt})$, $\overline{\beta_\zeta(H(\mathfrak{g})H(\mathfrak{t}))} = \beta_\zeta(\mathfrak{gt})$, $\overline{\sigma_\zeta(H(\mathfrak{g})H(\mathfrak{t}))} = \sigma_\zeta(\mathfrak{gt})$. $\forall\ \mathfrak{gt} \in \mathfrak{E}$

By definition of complement of G, we have

$$\overline{\mu_\zeta(H(\mathfrak{g})H(\mathfrak{t}))} = \frac{\overline{\mu_\eta(H(\mathfrak{g}))\ \mu_\eta(H(\mathfrak{t}))}}{\overline{\mu_\eta(H(\mathfrak{g}))} + \overline{\mu_\eta(H(\mathfrak{t}))} - \overline{\mu_\eta(H(\mathfrak{g}))\ \mu_\eta(H(\mathfrak{t}))}} - \mu_\zeta(H(\mathfrak{g})H(\mathfrak{t}))$$

$$\mu_\zeta(\mathfrak{gt}) = \frac{\mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})}{\mu_\eta(\mathfrak{g}) + \mu_\eta(\mathfrak{t}) - \mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})} - \mu_\zeta(H(\mathfrak{g})H(\mathfrak{t}))$$

$$\sum_{\mathfrak{g}\neq\mathfrak{t}} \mu_\zeta(\mathfrak{gt}) = \sum_{\mathfrak{g}\neq\mathfrak{t}} \frac{\mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})}{\mu_\eta(\mathfrak{g}) + \mu_\eta(\mathfrak{t}) - \mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})} - \sum_{\mathfrak{g}\neq\mathfrak{t}} \mu_\zeta(H(\mathfrak{g})H(\mathfrak{t}))$$

$$\sum_{\mathfrak{g}\neq\mathfrak{t}} \mu_\zeta(\mathfrak{gt}) + \sum_{\mathfrak{g}\neq\mathfrak{t}} \mu_\zeta(H(\mathfrak{g})H(\mathfrak{t})) = \sum_{\mathfrak{g}\neq\mathfrak{t}} \frac{\mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})}{\mu_\eta(\mathfrak{g}) + \mu_\eta(\mathfrak{t}) - \mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})}$$

$$2\sum_{\mathfrak{g}\neq\mathfrak{t}} \mu_\zeta(\mathfrak{gt}) = \sum_{\mathfrak{g}\neq\mathfrak{t}} \frac{\mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})}{\mu_\eta(\mathfrak{g}) + \mu_\eta(\mathfrak{t}) - \mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})}$$

$$\sum_{\mathfrak{g}\neq\mathfrak{t}} \mu_\zeta(\mathfrak{g}\mathfrak{t}) = \frac{1}{2}\sum_{\mathfrak{g}\neq\mathfrak{t}} \frac{\mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})}{\mu_\eta(\mathfrak{g}) + \mu_\eta(\mathfrak{t}) - \mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})}$$

Similarly for indeterminacy membership grade, we have

$$\overline{\beta_\zeta(H(\mathfrak{g})H(\mathfrak{t}))} = \frac{\overline{\beta_\eta(H(\mathfrak{g}))\ \beta_\eta(H(\mathfrak{t}))}}{\overline{\beta_\eta(H(\mathfrak{g}))} + \overline{\beta_\eta(H(\mathfrak{t}))} - \overline{\beta_\eta(H(\mathfrak{g}))\ \beta_\eta(H(\mathfrak{t}))}} - \beta_\zeta(H(\mathfrak{g})H(\mathfrak{t}))$$

$$\beta_\zeta(\mathfrak{g}\mathfrak{t}) = \frac{\beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})}{\beta_\eta(\mathfrak{g}) + \beta_\eta(\mathfrak{t}) - \beta_\eta(\mathfrak{g})\ \beta_\eta(t)} - \beta_\zeta(H(\mathfrak{g})H(\mathfrak{t}))$$

$$\sum_{\mathfrak{g}\neq\mathfrak{t}} \beta_\zeta(\mathfrak{g}\mathfrak{t}) = \sum_{\mathfrak{g}\neq\mathfrak{t}} \frac{\beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})}{\beta_\eta(\mathfrak{g}) + \beta_\eta(\mathfrak{t}) - \beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})} - \sum_{\mathfrak{g}\neq\mathfrak{t}} \beta_\zeta(H(\mathfrak{g})H(t))$$

$$\sum_{\mathfrak{g}\neq\mathfrak{t}} \beta_\zeta(\mathfrak{g}\mathfrak{t}) + \sum_{\mathfrak{g}\neq\mathfrak{t}} \beta_\zeta(H(\mathfrak{g})H(\mathfrak{t})) = \sum_{\mathfrak{g}\neq\mathfrak{t}} \frac{\beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})}{\beta_\eta(\mathfrak{g}) + \beta_\eta(\mathfrak{t}) - \beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})}$$

$$2\sum_{\mathfrak{g}\neq\mathfrak{t}} \beta_\zeta(\mathfrak{g}\mathfrak{t}) = \sum_{\mathfrak{g}\neq\mathfrak{t}} \frac{\beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})}{\beta_\eta(\mathfrak{g}) + \beta_\eta(\mathfrak{t}) - \beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})}$$

$$\sum_{\mathfrak{g}\neq\mathfrak{t}} \beta_\zeta(\mathfrak{g}\mathfrak{t}) = \frac{1}{2}\sum_{\mathfrak{g}\neq\mathfrak{t}} \frac{\beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})}{\beta_\eta(\mathfrak{g}) + \beta_\eta(\mathfrak{t}) - \beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})}$$

Likewise, for non-membership, we have

$$\overline{\sigma_\zeta(H(\mathfrak{g})H(\mathfrak{t}))} = \frac{\overline{\sigma_\eta(H(\mathfrak{g}))} + \overline{\sigma_\eta(H(\mathfrak{t}))} - \overline{2\sigma_\eta(H(\mathfrak{g}))\ \sigma_\eta(H(\mathfrak{t}))}}{\overline{1 - \sigma_\eta(H(\mathfrak{g}))\ \sigma_\eta(H(\mathfrak{t}))}} - \sigma_\zeta(H(\mathfrak{g})H(\mathfrak{t}))$$

$$\sigma_\zeta(\mathfrak{g}\mathfrak{t}) = \frac{\sigma_\eta(\mathfrak{g}) + \sigma_\eta(\mathfrak{t}) - 2\sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})}{1 - \sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})} - \sigma_\zeta(H(\mathfrak{g})H(\mathfrak{t}))$$

$$\sum_{\mathfrak{g}\neq\mathfrak{t}} \sigma_\zeta(\mathfrak{g}\mathfrak{t}) = \sum_{\mathfrak{g}\neq t} \frac{\sigma_\eta(\mathfrak{g}) + \sigma_\eta(\mathfrak{t}) - 2\sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})}{1 - \sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})} - \sum_{\mathfrak{g}\neq\mathfrak{t}} \sigma_\zeta(H(\mathfrak{g})H(\mathfrak{t}))$$

$$\sum_{\mathfrak{g}\neq\mathfrak{t}} \sigma_\zeta(\mathfrak{g}\mathfrak{t}) + \sum_{\mathfrak{g}\neq\mathfrak{t}} \sigma_\zeta(H(\mathfrak{g})H(\mathfrak{t})) = \sum_{\mathfrak{g}\neq\mathfrak{t}} \frac{\sigma_\eta(\mathfrak{g}) + \sigma_\eta(\mathfrak{t}) - 2\sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})}{1 - \sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})}$$

$$2\sum_{\mathfrak{g}\neq\mathfrak{t}} \sigma_\zeta(\mathfrak{g}\mathfrak{t}) = \sum_{\mathfrak{g}\neq\mathfrak{t}} \frac{\sigma_\eta(\mathfrak{g}) + \sigma_\eta(\mathfrak{t}) - 2\sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})}{1 - \sigma_\eta(\mathfrak{g})\ \sigma_\eta(t)}$$

$$\sum_{\mathfrak{g}\neq\mathfrak{t}} \sigma_\zeta(\mathfrak{g}\mathfrak{t}) = \frac{1}{2}\sum_{\mathfrak{g}\neq\mathfrak{t}} \frac{\sigma_\eta(\mathfrak{g}) + \sigma_\eta(t) - 2\sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})}{1 - \sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})}$$

This completes the proof.

Proposition 2. Let $G = (\eta, \zeta)$ be a PNDFG. If

$\mu_\zeta(\mathfrak{gt}) = \frac{1}{2}\left(\frac{\mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})}{\mu_\eta(\mathfrak{g})+\mu_\eta(\mathfrak{t})-\mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})}\right)$,

$\beta_\zeta(\mathfrak{gt}) = \frac{1}{2}\left(\frac{\beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})}{\beta_\eta(\mathfrak{g})+\beta_\eta(\mathfrak{t})-\beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})}\right)$,

$\sigma_\zeta(\mathfrak{gt}) = \frac{1}{2}\left(\frac{\sigma_\eta(\mathfrak{g})+\sigma_\eta(\mathfrak{t})-2\sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})}{1-\sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})}\right)\ \forall\ \mathfrak{g},\mathfrak{t}\in\mathfrak{V}$ then G is self-complementary.

Proof: Assume that G is a PNDFG that satisfies

$\mu_\zeta(\mathfrak{gt}) = \frac{1}{2}\left(\frac{\mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})}{\mu_\eta(\mathfrak{g})+\mu_\eta(\mathfrak{t})-\mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})}\right)$,

$\beta_\zeta(\mathfrak{gt}) = \frac{1}{2}\left(\frac{\beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})}{\beta_\eta(\mathfrak{g})+\beta_\eta(\mathfrak{t})-\beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})}\right)$,

$\sigma_\zeta(\mathfrak{gt}) = \frac{1}{2}\left(\frac{\sigma_\eta(\mathfrak{g})+\sigma_\eta(\mathfrak{t})-2\sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})}{1-\sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})}\right)\ \forall\ \mathfrak{g},t\in\mathfrak{V}$.

Then the identify mapping $I:\mathfrak{V}\to\mathfrak{V}$ is an isomorphism from G to $\overline{G}$ that satisfies the following condition:

$\mu_\eta(\mathfrak{g}) = \overline{\mu_\eta(I(\mathfrak{g}))}$, $\beta_\eta(\mathfrak{g}) = \overline{\beta_\eta(I(\mathfrak{g}))}$, and $\sigma_\eta(\mathfrak{g}) = \overline{\sigma_\eta(I(\mathfrak{g}))}\ \forall\ \mathfrak{g}\in\mathfrak{V}$.

The membership grade of an edge $\mathfrak{gt}$ is given by

$\mu_\zeta(\mathfrak{gt}) = \frac{1}{2}\left(\frac{\mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})}{\mu_\eta(\mathfrak{g})+\mu_\eta(\mathfrak{t})-\mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})}\right)\ \forall\ \mathfrak{g},\mathfrak{t}\in\mathfrak{V}$.

we have $\overline{\mu_\zeta(I(\mathfrak{g})I(\mathfrak{t}))} = \overline{\mu_\zeta(\mathfrak{gt})}$

$= \frac{\overline{\mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})}}{\overline{\mu_\eta(\mathfrak{g})+\mu_\eta(\mathfrak{t})-\mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})}} - \mu_\zeta(\mathfrak{gt})$

$= \frac{\mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})}{\mu_\eta(\mathfrak{g})+\mu_\eta(\mathfrak{t})-\mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})} - \frac{1}{2}\left(\frac{\mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})}{\mu_\eta(\mathfrak{g})+\mu_\eta(\mathfrak{t})-\mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})}\right)$

$= \frac{1}{2}\left(\frac{\mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})}{\mu_\eta(\mathfrak{g})+\mu_\eta(\mathfrak{t})-\mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})}\right) = \mu_\zeta(\mathfrak{gt})$

Likewise, the indeterminacy grade of an edge $\mathfrak{gt}$ is

$\beta_\zeta(\mathfrak{gt}) = \frac{1}{2}\left(\frac{\beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})}{\beta_\eta(\mathfrak{g})+\beta_\eta(\mathfrak{t})-\beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})}\right)\ \forall\ \mathfrak{g},\mathfrak{t}\in\mathfrak{V}$.

we have $\overline{\beta_\zeta(I(\mathfrak{g})I(\mathfrak{t}))} = \overline{\beta_\zeta(\mathfrak{gt})}$

$= \frac{\overline{\beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})}}{\overline{\beta_\eta(\mathfrak{g})+\beta_\eta(\mathfrak{t})-\beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})}} - \beta_\zeta(\mathfrak{gt})$

$= \frac{\beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})}{\beta_\eta(\mathfrak{g})+\beta_\eta(\mathfrak{t})-\beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})} - \frac{1}{2}\left(\frac{\beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})}{\beta_\eta(\mathfrak{g})+\beta_\eta(\mathfrak{t})-\beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})}\right)$

$= \frac{1}{2}\left(\frac{\beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})}{\beta_\eta(\mathfrak{g})+\beta_\eta(\mathfrak{t})-\beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})}\right) = \beta_\zeta(\mathfrak{gt})$

Similarly for the non-membership grade of an edge $\mathfrak{gt}$ is,

$\sigma_\zeta(\mathfrak{g}\mathfrak{t}) = \frac{1}{2}\left(\frac{\sigma_\eta(\mathfrak{g})+\sigma_\eta(\mathfrak{t})-2\sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})}{1-\sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})}\right)\ \forall\ \mathfrak{g},\mathfrak{t} \in \mathfrak{V}$.

So, we have

$\overline{\sigma_\zeta(I(\mathfrak{g})I(\mathfrak{t}))} = \overline{\sigma_\zeta(\mathfrak{g}\mathfrak{t})} = \frac{\overline{\sigma_\eta(\mathfrak{g})}+\overline{\sigma_\eta(\mathfrak{t})}-2\ \overline{\sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})}}{1-\overline{\sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})}} - \sigma_\zeta(\mathfrak{g}\mathfrak{t})$

$= \frac{\sigma_\eta(\mathfrak{g})+\sigma_\eta(\mathfrak{t})-2\ \sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})}{1-\sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})} - \frac{1}{2}\left(\frac{\sigma_\eta(\mathfrak{g})+\sigma_\eta(\mathfrak{t})-2\ \sigma_\eta(\mathfrak{g})\ \sigma_\eta(t)}{1-\sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})}\right)$

$= \frac{1}{2}\left(\frac{\sigma_\eta(\mathfrak{g})+\sigma_\eta(\mathfrak{t})-2\ \sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})}{1-\sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})}\right) = \sigma_\zeta(\mathfrak{g}\mathfrak{t})$

Since the conditions of isomorphism $\overline{\mu_\zeta(I(\mathfrak{g})I(\mathfrak{t}))} = \mu_\zeta(\mathfrak{g}\mathfrak{t})$, $\overline{\beta_\zeta(I(\mathfrak{g})I(\mathfrak{t}))} = \beta_\zeta(\mathfrak{g}\mathfrak{t})$ and $\overline{\sigma_\zeta(I(\mathfrak{g})I(\mathfrak{t}))} = \sigma_\zeta(\mathfrak{g}\mathfrak{t})$ are satisfied by I, $G = (\eta, \zeta)$ is self-complementary.

Proposition 3. If $G_1 = (\eta_1, \zeta_1)$ and $G_2 = (\eta_2, \zeta_2)$ are two isomorphic PNDFGs, then the complement of G_1 and G_2 are also isomorphic to each other and the converse also holds.

Proof: Assume that G_1 and G_2 are two isomorphic PNDFGs. Then by definition of isomorphism, there exists a bijective mapping $H : \mathfrak{V}_1 \to \mathfrak{V}_2$ that satisfies

$\mu_{\eta_1}(\mathfrak{g}) = \mu_{\eta_2}(H(\mathfrak{g}))$, $\beta_{\eta_1}(\mathfrak{g}) = \beta_{\eta_2}(H(\mathfrak{g}))$ and $\sigma_{\eta_1}(\mathfrak{g}) = \sigma_{\eta_2}(H(\mathfrak{g}))$. $\forall\ \mathfrak{g} \in \mathfrak{V}_1$,

$\mu_{\zeta_1}(\mathfrak{g}\mathfrak{t}) = \mu_{\zeta_2}(H(\mathfrak{g})H(\mathfrak{t}))$, $\beta_{\zeta_1}(\mathfrak{g}\mathfrak{t}) = \beta_{\zeta_2}(H(\mathfrak{g})H(\mathfrak{t}))$ and $\sigma_{\zeta_1}(\mathfrak{g}\mathfrak{t}) = \sigma_{\zeta_2}(H(\mathfrak{g})H(\mathfrak{t}))$. $\forall\ \mathfrak{g}\mathfrak{t} \in \mathfrak{E}_1$.

By using the definition of complement of PNDFG, the membership grade of an edge $\mathfrak{g}\mathfrak{t}$ is

$\overline{\mu_{\zeta_1}(\mathfrak{g}\mathfrak{t})} = \frac{\mu_{\eta_1}(\mathfrak{g})\ \mu_{\eta_1}(\mathfrak{t})}{\mu_{\eta_1}(\mathfrak{g})+\mu_{\eta_1}(t)-\mu_{\eta_1}(\mathfrak{g})\ \mu_{\eta_1}(\mathfrak{t})} - \mu_{\zeta_1}(\mathfrak{g}\mathfrak{t})$

$= \frac{\mu_{\eta_2}(H(\mathfrak{g}))\ \mu_{\eta_2}(H(\mathfrak{t}))}{\mu_{\eta_2}(H(\mathfrak{g}))+\mu_{\eta_2}(H(\mathfrak{t}))-\mu_{\eta_2}(H(\mathfrak{g}))\ \mu_{\eta_2}(H(t))} - \mu_{\zeta_2}(H(\mathfrak{g})H(\mathfrak{t}))$

$= \overline{\mu_{\zeta_2}(H(\mathfrak{g})H(\mathfrak{t}))}$.

Similarly,

$\overline{\beta_{\zeta_1}(\mathfrak{g}\mathfrak{t})} = \frac{\beta_{\eta_1}(\mathfrak{g})\ \beta_{\eta_1}(\mathfrak{t})}{\beta_{\eta_1}(\mathfrak{g})+\beta_{\eta_1}(\mathfrak{t})-\beta_{\eta_1}(\mathfrak{g})\ \beta_{\eta_1}(\mathfrak{t})} - \beta_{\zeta_1}(\mathfrak{g}\mathfrak{t})$

$= \frac{\beta_{\eta_2}(H(\mathfrak{g}))\ \beta_{\eta_2}(H(t))}{\beta_{\eta_2}(H(\mathfrak{g}))+\beta_{\eta_2}(H(\mathfrak{t}))-\beta_{\eta_2}(H(\mathfrak{g}))\ \beta_{\eta_2}(H(\mathfrak{t}))} - \beta_{\zeta_2}(H(\mathfrak{g})H(\mathfrak{t}))$

$= \overline{\beta_{\zeta_2}(H(\mathfrak{g})H(t))}$.

Also, the non-membership grade of an edge $\mathfrak{g}\mathfrak{t}$ is,

$\overline{\sigma_{\zeta_1}(\mathfrak{g}\mathfrak{t})} = \frac{\sigma_{\eta_1}(\mathfrak{g})+\sigma_{\eta_1}(\mathfrak{t})-2\sigma_{\eta_1}(\mathfrak{g})\ \sigma_{\eta_1}(\mathfrak{t})}{1-\sigma_{\eta_1}(\mathfrak{g})\ \sigma_{\eta_1}(\mathfrak{t})} - \sigma_{\zeta_1}(\mathfrak{g}\mathfrak{t})$

$= \frac{\sigma_{\eta_2}(H(\mathfrak{g}))+\sigma_{\eta_2}(H(\mathfrak{t}))-2\ \sigma_{\eta_2}(H(\mathfrak{g}))\ \sigma_{\eta_2}(H(\mathfrak{t}))}{1-\sigma_{\eta_2}(H(\mathfrak{g}))\ \sigma_{\eta_2}(H(\mathfrak{t}))} - \sigma_{\zeta_2}(H(\mathfrak{g})H(\mathfrak{t}))$

$= \overline{\sigma_{\zeta_2}(H(\mathfrak{g})H(\mathfrak{t}))}$.

Hence, the complement of G_1 is isomorphic to the complement of G_2. Similarly, the converse can be proved.

Definition 3.10. A PNDFG is said to be complete if

$$\mu_\zeta(\mathfrak{g}\mathfrak{t}) = \frac{\mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})}{\mu_\eta(\mathfrak{g})+\mu_\eta(\mathfrak{t})-\mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})},$$

$$\beta_\zeta(\mathfrak{g}\mathfrak{t}) = \frac{\beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})}{\beta_\eta(\mathfrak{g})+\beta_\eta(\mathfrak{t})-\beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})},$$

$$\sigma_\zeta(\mathfrak{g}\mathfrak{t}) = \frac{\sigma_\eta(\mathfrak{g})+\sigma_\eta(\mathfrak{t})-2\sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})}{1-\sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})}\ \forall\ \mathfrak{g},\mathfrak{t}\in\mathfrak{V}.$$

The above mentioned properties are satisfied for the PNDFG in Example 1, thus the PNDFG is a complete PNDFG.

Definition 3.11. A PNDFG is said to be strong if

$$\mu_\zeta(\mathfrak{g}\mathfrak{t}) = \frac{\mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})}{\mu_\eta(\mathfrak{g})+\mu_\eta(\mathfrak{t})-\mu_\eta(\mathfrak{g})\ \mu_\eta(\mathfrak{t})},$$

$$\beta_\zeta(\mathfrak{g}\mathfrak{t}) = \frac{\beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})}{\beta_\eta(\mathfrak{g})+\beta_\eta(\mathfrak{t})-\beta_\eta(\mathfrak{g})\ \beta_\eta(\mathfrak{t})},$$

$$\sigma_\zeta(\mathfrak{g}\mathfrak{t}) = \frac{\sigma_\eta(\mathfrak{g})+\sigma_\eta(\mathfrak{t})-2\sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})}{1-\sigma_\eta(\mathfrak{g})\ \sigma_\eta(\mathfrak{t})}\ \forall\ \mathfrak{g}\mathfrak{t}\in\mathfrak{E}.$$

Example 4. The PNDFG over $\mathfrak{V} = \{m_1, m_2, m_3, m_4, m_5, m_6\}$
$\langle(m_1, .5, .6, .5), (m_2, .8, .4, .2), (m_3, .4, .6, .2), (m_4, .6, .5, .7), (m_5, .7, .4, .3), (m_6, .3, .6, .7)\rangle$
and the edge set
$\langle(m_1m_2, .44, .316, .556), (m_1m_6, .231, .429, .769), (m_2m_3, .432, .316, .33), (m_2m_5, .596, .25, .404),$
$(m_3m_4, .316, .375, .721), (m_4m_5, .477, .286, .734), (m_5m_6, .266, .316, .734)\rangle$ is strong.

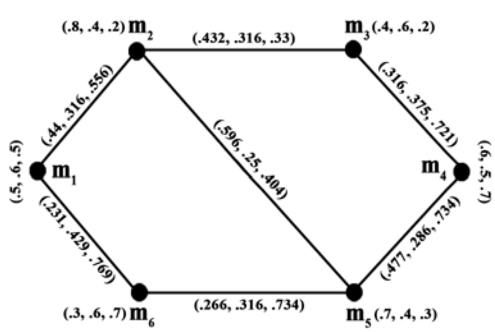

FIGURE 2. Strong PNDFG

4. **Numerical Approach**

We solve a decision-making problem involving the selection of the best money-transfer applications in this section to demonstrate the suitability of the proposed Pythagorean Neutrosophic Dombi fuzzy graphs concept in a real-world scenario.

4.1 Algorithm:

The following algorithm depicts our proposed technique for multi-criteria decision making. The algorithm for the selection of the best money transferring application.

Step 1: Input the attributes $M = \{m_1, m_2, ...m_k\}$ and set of criteria $C = \{c_1, c_2, ...c_n\}$ with weight vector $W = \{w_1, w_2, ...w_n\}$ and construct Pythagorean fuzzy relation $Q^{(g)} = (q_{lp}^{(g)})_{k\times k}$ corresponding to each criterion.

Step 2: Aggregate all $q_{lp}^{(g)} = (\mu_{lp}^{(g)}, \beta_{lp}^{(g)}, \sigma_{lp}^{(g)})$ $(l, p = 1, 2, .., k)$ regarding criteria $c_l(l = 1, 2, 3, 4, 5)$ and get $Q = (q_{lp})_{k\times k}$, where $q_{lp} = (\mu_{lp}, \beta_{lp}, \sigma_{lp})$ is the value assigned for the alternative m_l over m_p with respect to all the considered criteria C_l by using Pythagorean Neutrosophic Dombi fuzzy weighted arithmetic averaging (PNDFWAA) operator given by

$$q_{lp} = PNDFWAA(q_{lp}^{(1)}, q_{lp}^{(2)}, ..., q_{lp}^{(n)},) =$$

$$\left(\sqrt{1-\frac{1}{1+\left[\sum_{j=1}^{n} w_j\left(\frac{(\beta_{lp}^{j})^2}{1-(\beta_{lp}^{g})^2}\right)^{\rho}\right]^{\frac{1}{\rho}}}}, \sqrt{1-\frac{1}{1+\left[\sum_{j=1}^{n} w_j\left(\frac{(\mu_{lp}^{j})^2}{1-(\beta_{lp}^{g})^2}\right)^{\rho}\right]^{\frac{1}{\rho}}}}, \frac{1}{1+\left[\sum_{j=1}^{n} w_j\left(\frac{1-(\sigma_{lp}^{j})}{(\sigma_{lp}^{g})}\right)^{\rho}\right]^{\frac{1}{\rho}}}\right)$$

Step 3: According to Q, draw the Pythagorean Neutrosophic fuzzy directed graph.

Step 4: By considering the condition $\mu_{lp} \geq 0.5$ $(l, p = 1, 2, ..., k)$, draw the Pythagorean Neutrosophic fuzzy partial directed graph.

Step 5: Calculate the out degrees $out - d(M_i)$ $(i = 1, 2, ..k)$ of all the alternatives M_i in the Pythagorean Neutrosophic fuzzy partial directed graph.

Step 6: Arrange the alternatives according to the diminishing value of the membership degrees of $out - d(M_i)$.

Step 7: The optimal alternative is the alternative with the maximum membership degree of $out - d(M_i)$.

4.2 Selection of the best money transferring application:

In this modern technology-filled world, everything is turning to online and digital cards. There is no need of searching for money exchanges as money can be easily transferred in online mode or through online transferring. According to existing trends, the fastest and easiest way of transferring is pivotal. Let us consider the following case. A person who has created a new

bank account wants to sync with this existing trends and techniques and want to have a money transferring app. Let us consider five money transferring applications $M_i (i = 1, 2, 3, 4, 5)$ that are doing really well on the market. The decision-making expert makes a comparison between five money transferring apps with respect to five criteria $C_l (l = 1, 2, 3, 4, 5)$ which are given as
C_1 = Safety & security
C_2 = Fast transfer without delay
C_3 = Money remittance
C_4 = User friendly
C_5 = Offers & gifts
with the respective weight $W = (0.3, 0.2, 0.3, 0.1, 0.1)$ and presents preferable information
$Q^{(g)} = (q^{(g)}_{lp})_{5\times5}$ $(g = 1, 2, 3, 4, 5)$, where
$q^{(g)}_{lp} = (\mu^{(g)}_{lp}, \beta^{(g)}_{lp}, \sigma^{(g)}_{lp})$ is the Pythagorean Neutrosophic number assigned by decision-making expert $\mu^{(g)}_{lp}, \beta^{(g)}_{lp}$ and $\sigma^{(g)}_{lp}$ are the degree to which money transferring application M_l is preferred and not preferred over the application M_p regarding the given criteria, respectively. The relations $Q^{(g)} = (q^{(g)}_{lp})_{5\times5}$ are given in the following tables (1-5).

Table 1. Comparision for Criteria 1

$Q^{(1)}$	M_1	M_2	M_3	M_4	M_5
M_1	(.5,.5,.5)	(.8,.4,.2)	(.7,.3,.1)	(.6,.2,.4)	(.4,.4,.6)
M_2	(.2,.4,.8)	(.5,.5,.5)	(.6,.5,.4)	(.8,.2,.6)	(.6,.4,.7)
M_3	(.1,.3,.7)	(.4,.5,.6)	(.5,.5,.5)	(.8,.4,.3)	(.7,.5,.2)
M_4	(.4,.2,.6)	(.6,.2,.8)	(.3,.4,.8)	(.5,.5,.5)	(.8,.4,.7)
M_5	(.6,.4,.4)	(.7,.4,.6)	(.2,.5,.7)	(.7,.4,.8)	(.5,.5,.5)

Table 2. Comparision for Criteria 2

$Q^{(2)}$	M_1	M_2	M_3	M_4	M_5
M_1	(.5,.5,.5)	(.9,.8,.5)	(.7,.4,.4)	(.9,.4,.7)	(.8,.6,.6)
M_2	(.5,.8,.9)	(.5,.5,.5)	(.8,.4,.5)	(.5,.4,.6)	(.7,.6,.8)
M_3	(.4,.4,.7)	(.5,.4,.8)	(.5,.5,.5)	(.7,.8,.6)	(.7,.7,.8)
M_4	(.7,.4,.9)	(.6,.4,.5)	(.6,.8,.7)	(.5,.5,.5)	(.7,.8,.6)
M_5	(.6,.6,.8)	(.8,.6,.7)	(.8,.7,.7)	(.6,.8,.7)	(.5,.5,.5)

Table 3. Comparision for Criteria 3

$Q^{(3)}$	M_1	M_2	M_3	M_4	M_5
M_1	(.5,.5,.5)	(.9,.6,.5)	(.8,.4,.6)	(.9,.8,.4)	(.7,.6,.5)
M_2	(.5,.6,.9)	(.5,.5,.5)	(.8,.7,.6)	(.9,.8,.7)	(.6,.6,.5)
M_3	(.6,.4,.8)	(.6,.7,.8)	(.5,.5,.5)	(.8,.2,.7)	(.7,.8,.7)
M_4	(.4,.8,.9)	(.7,.8,.9)	(.7,.2,.8)	(.5,.5,.5)	(.9,.6,.7)
M_5	(.5,.6,.7)	(.5,.6,.6)	(.7,.8,.7)	(.7,.6,.9)	(.5,.5,.5)

Table 4. Comparision for Criteria 4

$Q^{(4)}$	M_1	M_2	M_3	M_4	M_5
M_1	(.5,.5,.5)	(.9,.7,.5)	(.6,.3,.4)	(.7,.5,.6)	(.8,.3,.6)
M_2	(.5,.7,.9)	(.5,.5,.5)	(.9,.6,.5)	(.6,.4,.7)	(.7,.6,.6)
M_3	(.4,.3,.6)	(.5,.6,.9)	(.5,.5,.5)	(.9,.6,.4)	(.6,.7,.8)
M_4	(.6,.5,.7)	(.7,.4,.6)	(.4,.6,.9)	(.5,.5,.5)	(.9,.7,.6)
M_5	(.6,.3,.8)	(.6,.6,.7)	(.8,.7,.6)	(.6,.7,.9)	(.5,.5,.5)

Table 5. Comparision for Criteria 5

$Q^{(5)}$	M_1	M_2	M_3	M_4	M_5
M_1	(.5,.5,.5)	(.8,.6,.4)	(.7,.6,.8)	(.6,.6,.8)	(.9,.3,.6)
M_2	(.4,.6,.8)	(.5,.5,.5)	(.8,.7,.6)	(.7,.5,.4)	(.8,.6,.5)
M_3	(.8,.6,.7)	(.6,.7,.8)	(.5,.5,.5)	(.9,.5,.4)	(.8,.5,.7)
M_4	(.8,.6,.6)	(.4,.5,.7)	(.4,.5,.9)	(.5,.5,.5)	(.9,.8,.7)
M_5	(.6,.3,.9)	(.5,.6,.8)	(.7,.5,.8)	(.7,.8,.9)	(.5,.5,.5)

The Pythagorean Neutrosophic directed graphs for $Q^{(g)}(g = 1, 2, 3, 4, 5)$ in Tables 1-5 are displayed in Figure 3.

With the purpose to complete the grouped $q_{lp} = (\mu_{lp}, \beta_{lp}, \sigma_{lp})$ $(e, p = 1, 2, 3, 4, 5)$ of the money transferring application M_l over M_p regarding all considered criteria $e^{(g)}(g = 1, 2, 3, 4, 5)$, the PNDFWAA operator is defined as

$$q_{lp} = PNDFWAA(q_{lp}^{(1)}, q_{lp}^{(2)}, ..., q_{lp}^{(n)},) =$$
$$\left(\sqrt{1 - \frac{1}{1+\left[\sum_{j=1}^{n} w_j \left(\frac{(\beta_{lp}^{j})^2}{1-(\beta_{lp}^{g})^2}\right)^{\rho}\right]^{\frac{1}{\rho}}}}, \sqrt{1 - \frac{1}{1+\left[\sum_{j=1}^{n} w_j \left(\frac{(\mu_{lp}^{j})^2}{1-(\beta_{lp}^{g})^2}\right)^{\rho}\right]^{\frac{1}{\rho}}}}, \frac{1}{1+\left[\sum_{j=1}^{n} w_j \left(\frac{1-(\sigma_{lp}^{j})}{(\sigma_{lp}^{g})}\right)^{\rho}\right]^{\frac{1}{\rho}}} \right)$$

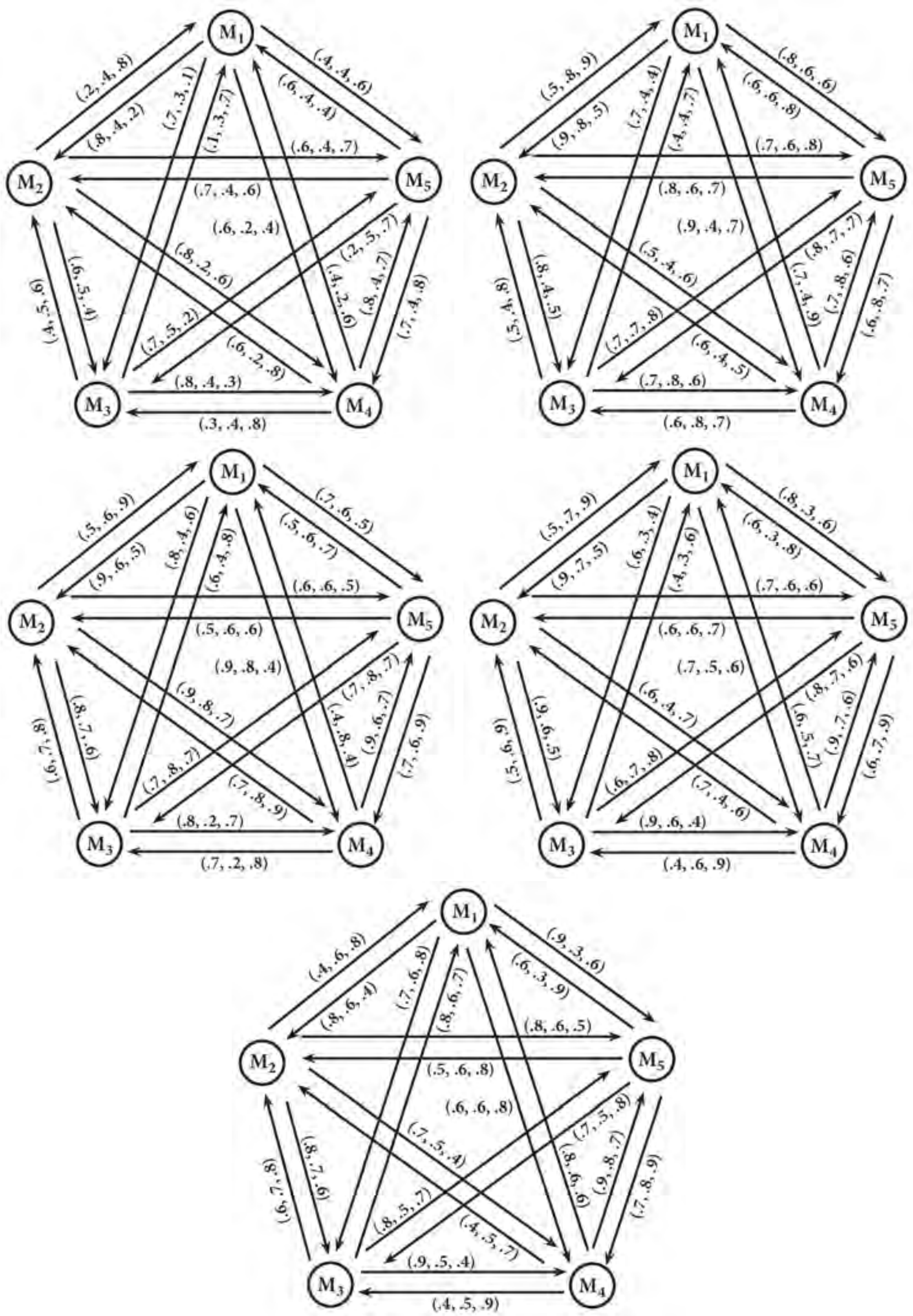

FIGURE 3. Pythagorean Neutrosophic directed graph for $Q^{(g)}(g = 1, 2, 3, 4, 5)$

In the above equation, we consider $\rho = 1$ as in Dombi's t-norm and t-conorm to obtain the corresponding $Q^g = (q_{lp}^{(g)})_{5\times 5}$, which is shown in Table 6.

Table 6. Combined Pythagorean Neutrosophic fuzzy relation

Q	M_1	M_2	M_3	M_4	M_5
M_1	(.5,.5,.5)	(.875,.651,.357)	(.734,.606,.239)	(.843,.635,.481)	(.752,.514,.565)
M_2	(.432,.651,.858)	(.5,.5,.5)	(.79,.605,.496)	(.819,.624,.602)	(.671,.557,.545)
M_3	(.538,.401,.715)	(.525,.605,.734)	(.5,.5,.5)	(.824,.583,.444)	(.708,.644,.409)
M_4	(.593,.635,.732)	(.637,.624,.705)	(.565,.583,.795)	(.5,.5,.5)	(.861,.679,.666)
M_5	(.575,.514,.601)	(.674,.558,.64)	(.694,.699,.697)	(.676,.679,.822)	(.5,.5,.5)

The Pythagorean Neutrosophic directed graphs according to Q, is in Figure 4.

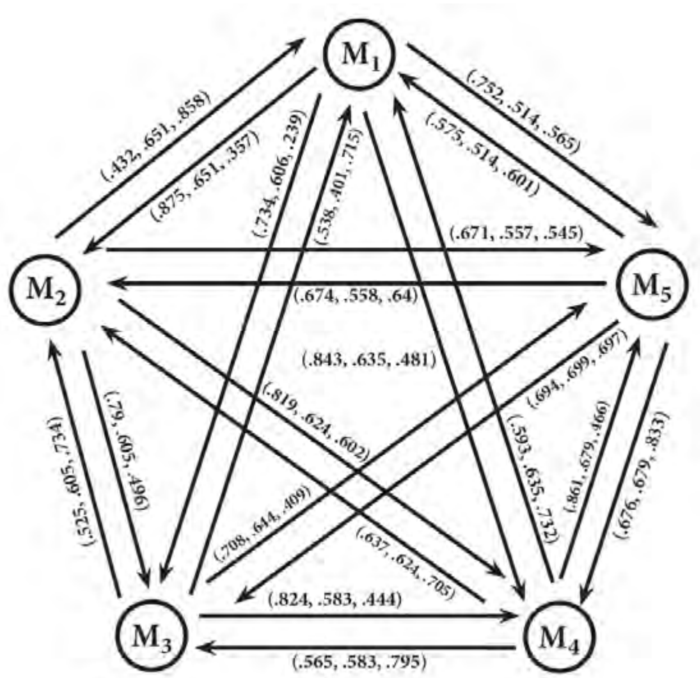

FIGURE 4. Pythagorean Neutrosophic directed graph according to Q

We consider the condition of $\mu_{lp} \geq 0.5$ $(l, p = 1, 2, 3, 4, 5)$ a partial directed graph is drawn in Figure 5.

The out-degrees $out - d(M_l)$ $(l = 1, 2, 3, 4, 5)$ of the money transferring app in the partial graph are calculated as

$out - d(M_1) = (3.204, 2.4061, 1.642)$

$out - d(M_2) = (2.28, 1.786, 1.643)$

$out - d(M_3) = (1.532, 1.227, 0.853)$

$out - d(M_4) = (0.861, 0.679, 0.666)$

$out - d(M_5) = (0, 0, 0)$

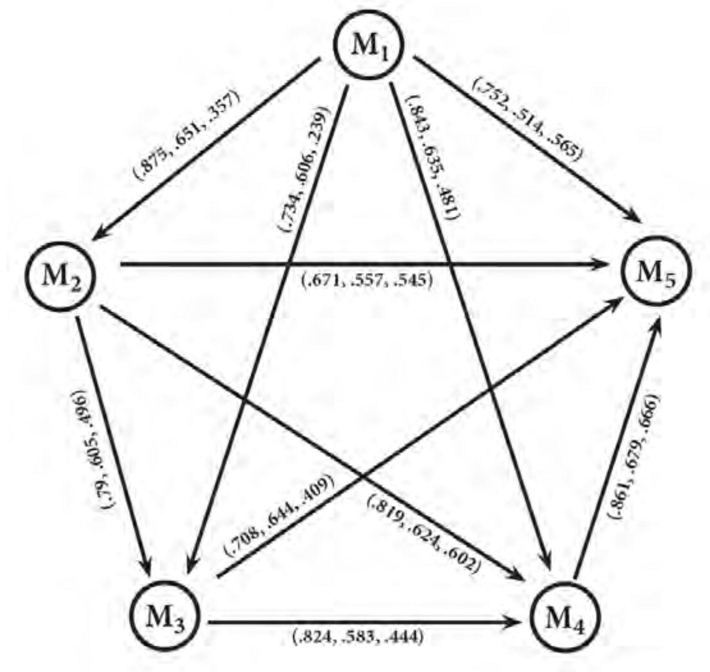

FIGURE 5. Partial directed Pythagorean Neutrosophic directed graph

According to the membership value of out-degrees of $M_l(l = 1, 2, 3, 4, 5)$ we get the optimal ranking order as:

$$M_1 \succ M_2 \succ M_3 \succ M_4 \succ M_5$$

On the basis of ranking, we conclude that M_1 is the best money transferring application.

5. **Conclusion**

In this paper, the concept of Pythagorean Neutrosophic Dombi Fuzzy graph has been introduced. Along with the introduction of this new model of Pythagorean neutrosophic Dombi fuzzy graph some of its definitions and few properties have been discussed. An application in decision making has been done by using the graph model of Pythagorean Neutrosophic Dombi Fuzzy graphs using the newly defined PNDFWAA operator. In the proposed MCDM, the limitation of μ_{lp} is restrained to be greater than 0.5, because the value less than 0.5 will have very low chances to be in the obtained alternative and it is defined to be more precise for making decision. This work on the concept of Pythgorean Neutrosophic Dombi Fuzzy graph can be extended further to investigate the operations of PNDFG and bipolar PNDFG along with some real life applications.

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Selection a Suitable Supplier for Enhancing Supply Chain Management under Neutrosophic Environment

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Abstract: In supply chain management (SCM), selecting a sustainable supplier has improved as one of the most urgent issues. Several previous research has attempted to determine how to choose a sustainable supplier using various strategies and approaches. A suitable supplier needs to be chosen to improve the quality of products, services, reduce prices of purchasing, and regulate time. This paper aims to Enhance Supply Chain Management for the Suitable Supplier selection by integrating the neutrosophic Analytical Hierarchy Process (AHP) with Multiplicative Multi-Objective Optimization based on Ratio Analysis (MULTIMOORA) approach. Furthermore, this is a Multi-Criteria Decision-Making (MCDM) issue, selecting sustainable providers is complex. because includes unclear information, decision-makers' loss of accuracy, indeterminate, uncertainty, and inconsistent information. For this, we use a single value neutrosophic set (SVNS) to can handle unclear information, the knowledge that is inaccurate, and indeterminate, uncertain information. The proposed study applies integrating Single Valued Neutrosophic AHP for calculating the weights of different criteria taking into account their interdependencies. Then the MULTIOORA technique ranks the different alternatives, then chooses the best provider. A case study is utilized on the pharmaceutical manufacturing company. The case study uses the integrated approach to analyze and choose the ideal supplier.

Keywords: Supply Chain Management; Neutrosophic; AHP; MULTIMOORA; MCDM

1. Introduction

Nowadays, Organizations have understood that to compete in local and global marketplaces, they must implement effective strategies to strengthen the supply chain and achieve a competitive advantage over their competitors. Supply chain management (SCM) is a way of referring to a collection of three or more enterprises (organizations or persons) that participate in the input and output flows of products, resources, finances, and information from a resource to a consumer[1]. SCM allows organizations to efficiently integrate products and services to build long-term relationships[2]. SCM is a concept whose main goal is to "integrate and manage the source, flow, and control of materials across various functions using a whole systems approach." Supply chains include both suppliers and consumers so, the SCM method may be used in the greatest organizations to control the flow of information, goods, and materials and to be more reactive in an organizational environment including Lowering prices, ensuring quick delivery, and reducing shipping times.

The Supplier Selection Problem (SSP) is a great issue for most industrial companies and supply chain management. The fundamental goal of a Supplier Selection Problem is really to identify one supplier with the best chance of meeting the company's demands while lowering costs. A well-known and fundamental component of supply chain planning challenges is the supplier selection problem. The best suppliers are chosen based on a variety of factors, including overall performance ratings, item rejection rates, timely delivery, and meeting aggregate demand. SSP is the main issue we want to solve[3]. Furthermore, we proposed neutrosophic logic to solve it. Neutrosophic logic (NL) is one of the most current fuzzy system suggestions, Smarandache [4] proposed it work with inconsistent, fuzzy, imprecise, and incomplete information simultaneously. Various ways for approximation and uncertain reasoning have been created to deal with an ambiguous choice process involving imprecise, partial, and incomplete information. One is fuzzy logic, intuitionistic, and interval logic. The neutrosophic set is an expansion of the fuzzy sets. It contains three terms (trusty, indeterminacy, falsity) to characterize the uncertain membership at the same time[5]. The neutrosophic set assists specialists and decision-makers in comprehending information in an ambiguous context and expressing their opinions more clearly. Wang et al. [6]proposed a single-valued neutrosophic set as an example of a neutrosophic set (SVNS). The classic set, fuzzy set, interval-valued fuzzy set, and intuitionistic fuzzy set are all extensions of the single-valued neutrosophic set. The terms inclusion, complement, union, and intersection have all been defined on SVNS.

Supplier selection is a multi-criteria decision-making (MCDM) problem that incorporates a variety of criteria, options, and decision-makers in determining the best candidate for the company. MCDM problems using neutrosophic or single-valued neutrosophic information, like the neutrosophic Analytic Hierarchy Process (AHP), have been researched by many researchers[7]. Multi-Criteria Decision-Making (MCDM) techniques use a group of experts and decision-makers to address goal evaluation to numerous criteria utilizing computational and mathematical models to tackle real-life decision-making problems.

Engineering, management, technology, and science are all sectors where MCDM approaches are commonly employed. The use of MCDM methodologies to construct specific MCDM models is a viable solution to such situations. These models differ from one another because they use various MCDM methodologies or have distinct requirements. In this paper, we use integrated AHP with a MULTIMOORA method. The AHP uses a hierarchical structure to express and analyze the link between the criteria, sub-criteria and to assess them.

The MULTIMOORA method is an enhanced version of multi-objective optimization (MOORA), a simple and effective multi-attribute decision making (MADM) tool [8]. MULTIMOORA is now the most reliable multi-objective optimization system, as evidenced by the fact that it is the only multi-criteria methodology capable of meeting numerous criteria employing three or more techniques [9]. Many authors proposed a MULTIMOORA to aid decision-makers in selecting a suitable supplier by assessing multiple risks and benefits and knowing how they impact the supply chain[10]. The main benefit of adopting the combination fuzzy AHP- MULTIMOORA method is the ability to categorize and evaluate the criteria, sub-criteria, and alternatives for each risk. So we proposed a hybrid model AHP and MULTIMOORA by using SVNS for selecting a supplier to enhance the supply chain management SCM.

The paper is organized as follows. Section 2, reviews the literature for sustainable supplier selection. Section 3, the preliminaries of the neutrosophic set and SVNS are provided. Section 4, presents the proposed methodology of AHP and MULTIMOORA. Section 5, the results of the case study are presented and analyzed. Section 6, the conclusion and future work are presented.

2. Literature review

In SCM, one of the really difficult multi-criteria decision-making tasks is supplier selection. especially when it comes to sustainability. Several approaches for determining supplier criteria, evaluations, and selection have been published in major scientific journals. Fallahpour et al (2017). [11]produced a fuzzy AHP-TOPSIS approach to aid the supplier evaluation and selection criteria. The fuzzy preference programming technique was used to generate the relative fuzzy weights of the ranking criterion, and the ranking of possible suppliers was determined using fuzzy TOPSIS. Luthra et al (2017). [12] AHP and VIKOR approaches were utilized to develop a sustainable supplier selection strategy. The suggested framework includes 22 criteria for each of the three sustainability pillars. The sustainable supplier selection (SSS) criteria were weighted using the AHP method, and the VIKOR approach was used to choose the most efficient sustainable supplier. Ghorabaee et al (2017).

[13] A study of multi-attribute decision-making procedures for analyzing and selecting the best providers in a fuzzy environment. Tavana et al (2017). [14] Introduced an integrated ANP-QFD method to calculate the decision criteria and sub-criteria to achieve customer needs and sustainable supplier selection. Qin et al (2017). [15] developed a TODIM method for supplier assessment using fuzzy sets of interval type 2. Liu et al (2018). [16] proposed ANP and VIKOR methods to assess supplier choice using interval type-2 fuzzy sets. The criteria are weighted using an ANP technique, and the VIKOR method was used to select and rank sustainable suppliers.

Abdel-Basset et al (2018). [17] combined ANP with TOPSIS for solving the sustainable supplier selection issue using interval-valued neutrosophic numbers. Kumar et al (2018). [18] a Combined method of fuzzy theory and AHP-DEMATEL to aid the automobile industry in optimizing their supplier selection process for capital procurement. Jain et al (2018). [19] fuzzy multi-criteria decision-making techniques that are combined In an Indian car firm, AHP and the approach for order of preference by similarity to ideal solution (TOPSIS) were applied to the problem of supplier selection. Van et al (2018) [20] Proposed Quality Function Deployment (QFD) for supplier selection and assessment using a neutrosophic interval set. Chen et al (2018). [21] For sustainable supplier selection, OWA distance was used in conjunction with a single-valued neutrosophic linguistic (SVNL) based TOPSIS method. Abdel-Basset et al (2018). [22] For sustainable supplier selection challenges, integrate AHP- TOPSIS with interval-valued neutrosophic sets and a multi-criteria decision-making technique. They calculated criterion weights using the AHP approach and SSS using TOPSIS. Abdel-Basset et al (2018). [23] used a neutrosophic set and the DEMATEL approach for decision making and analysis. to examine and determine the factors impacting the selection of supply chain management vendors. Abdel-Basset et al (2018). [24] combined neutrosophic (AHP) with quality function deployment (QFD) to choose the best provider. Neutrosophic set determines three-way judgments based on the categorization into three parts (acceptance, rejection, and not sure). For selected best supplier should satisfy company requirements, so use (QFD) to determine efficient business requirements. Wang et al (2018). [25] Introduced a 2-tuple linguistic neutrosophic numbers (2TLNNs) operator to handle the MADM difficulty and identify sustainable suppliers.

Sinha et al (2018). [26] developed a decision framework for sustainable supplier selection based on a combination of MCDM approaches and graph theory. Song et al (2019). [27] proposed a large-scale decision-making model that involves many stakeholders in the decision-making process. SSS is accomplished using TOPSIS, and their approach comprises partial language phrases depending on risk attitudes. Matic et al (2019). [28] On a sustainable supply chain, created a novel hybrid MCDM model for evaluating and choosing suppliers. Abdel-Basset et al (2019). [29] Developed the ANP and VIKOR approaches for sustainable supplier selection based on triangular neutrosophic numbers (TriNs). Islam et al (2019). [30] To handle supplier selection difficulties, they developed a neutrosophic goal programming technique based on triangular neutrosophic numbers. to identify the optimum compromise for the company, the authors employ neutrosophic goal programming. A multi-objective linear programming problem (MOLP) is an example of this.

Jain et al (2020). [31] suggested Multi-Criteria Decision Making (MCDM) with Fuzzy Inference System (FIS) techniques for weighting and ordering to Suitable Supplier Selection, and fuzzy AHP and fuzzy TOPSIS algorithms. Wang et al (2020) [32]. created a fuzzy ANP-PROMETHEE II to help the textile sector evaluate and choose suppliers. Selection criteria for the proposed model are based on the widely used Supply Chain Operation Reference (SCOR) model. Zeng and al (2020). [33] Proposed a single value neutrosophic for sustainable supplier selection based on ambiguous information given by decision-makers and established a single value neutrosophic hybrid weighted similarity (SVNHWS) measure. Yalcin et al (2020). [34] combined ANP technique and TODIM approach, under interval-valued neutrosophic sets (IVNSs) to Sustainable Supplier's choice. Amiri et al (2020). [35] presented a novel approach multi-criteria decision method based on Best-Worst technique and α-cut for suitable supplier selection. Pamucar et al (2020). [36] the suggested fuzzy neutrosophic technique for robust supplier selection based on trapezoidal linguistic factors. Tavana et al (2021). [37] For a suitable provider, combine AHP with fuzzy multiplicative multi-objective based on (MULTIMOORA). The MULTIMOORA is also used to select the providers and the AHP is used to evaluate the importance of company risks and benefits. Yazdani et al (2021). [38] proposed criteria Correlation and compromised solution under neutrosophic environment and uses multiple alternatives for evaluation and selection of suppliers. Uluas et al (2021). [39] introduced a novel MULTIMOOSRAL method to the supplier selection issue. From previous works, it is the first study to hybrid the AHP and MULTIMMORA method with large dimension data to enhance the SCM by evaluating the best supplier.

3. Preliminaries

This section covers the most important definitions of neutrosophic sets, as well as single-valued neutrosophic sets and their processes.

3.1. Neutrosophic sets

Definition 1. [40]Let L be a collection of elements (objects). A neutrosophic set B in L is characterized by a truth-membership function $T_A(l)$, an indeterminacy membership function $I_A(l)$, and a falsity-membership function $F_A(l)$. The functions $T_A(l)$, $I_A(l)$, $F_A(l)$ are there any actual normal or non-standard subsets of]$^-$0, 1$^+$[. That is $T_A(l)$: L →]$^-$0, 1$^+$[, $I_A(l)$: L →]$^-$0, 1$^+$[and $F_A(l)$: L →]$^-$0, 1$^+$[.

The total amount is unrestricted of $T_A(l)$, $I_A(l)$, and $F_A(l)$. So, $0 \le T_A(l) + I_A(l) + F_A(l) \le 3$.

From a philosophical viewpoint, a neutrosophic set derives its worth from a real norm or non-standard subsets of]$^-$0, 1$^+$[. Because in real-life scientific circumstances, using a neutrosophic set with values from genuine standard or non-standard subsets of]–0, 1+[is impossible. the neutrosophic set (single-valued neutrosophic set) whose value is determined by the subset [0, 1]. Now, we'll make use of the notions. m_A, s_A and z_A instead of notions T_A, I_A and F_A, respectively.

Neutrosophic set is a recent proposal for a strong generic set theory. Therefore, from a technical standpoint, the neutrosophic set should be specified. in this manner, In 2010, Wang et al. [41]presented the single-valued neutrosophic set (SVNS) that is an example of the neutrosophic set.

3.2. Single valued neutrosophic sets

Wang [41] proposed a single-valued neutrosophic as follows:

Definition 2: [41] that L be a universe of discourse with general components indicated by l An SVNS A in L is described by truth-membership function $m_A(l)$, indeterminacy-membership function $s_A(l)$, and falsity-membership function $z_A(l)$. A single valued neutrosophic set An over L is an object with the form:

$$A = \{ \langle l, m_A(l), s_A(l), z_A(l) \rangle \mid l \in L\}, \quad \textbf{(1)}$$

Where m_A : L → [0, 1] , s_A :L → [0, 1] and z_A : L → [0, 1] with the condition

$$0 \le m_A(l) + s_A(l) + z_A(l) \le 3, \ \forall l \in L \quad \textbf{(2)}$$

Definition 3: Let X and B be two single-valued neutrosophic sets,

$$X = \{ \langle l, m_X(l), s_X(l), z_X(l) \rangle : l \in L\} \text{ and} \quad \textbf{(3)}$$

$$B = \{ \langle l, m_B(l), s_B(l), z_B(l) \rangle : l \in L\}. \quad \textbf{(4)}$$

Then some operations can be defined as follows:

1. $X \cup B = \{ \langle l : \max\{m_X(l), m_B(l)\},$
   $\min\{s_X(l), s_B(l)\}\ , \min\{\ z_X(l),\ \ z_B(l)\} \rangle \}$;
2. $X \cap B = \{ \langle l : \min\{m_X(l), m_B(l)\}, \max\{s_X(l), s_B(l)\}\ , \max\{\ z_X(l),\ \ z_B(l)\} \rangle \}$;
3. A ⊆ B if and only if $m_X(l) \leq m_B(l)$, $s_X(l) \geq s_B(l)$, and $z_X(l) \geq z_B(l)$, $\forall\ l \in L$
4. X = B if and only if X ⊆ B and B ⊆ X;
5. $X^c = \{ \langle l, z_X(l), 1 - s_X(l), m_X(l) \rangle \}$.

Because each membership value is independent of the others, there are definitions of distinct neutrosophic empty sets and, as a result, absolute neutrosophic sets in single-valued neutrosophic information.

Definition 4: Let X be a single-valued neutrosophic set on L.

1. A single valued neutrosophic set X is empty, denoted by 0 = {0, 1, 1} if $m_X(l) = 0$, $s_X(l) = 1$ and $z_X(l) = 1$ for each $l \in L$.

2. A single valued neutrosophic set X is absolute, denoted by 1 = {1, 0, 0} if $m_X(l) = 1$, $s_X(l) = 0$ and $z_X(l) = 0$ for each $l \in L$.

Definition 5: Alternative ratings are in the form of INS $\bar{x}_{ab} = [x_{ab}^M, x_{ab}^U]$ in Interval Target value Based on MULTIMOORA. The maximum, minimum, and ordering of INs are determined using the preference matrix. As well, the IN interval distance is used. In this approach, the normalization ratio $\bar{x}_{ab}^*$ is defined as follows:

$$\bar{x}_{ab}^* = \left[x_{ab}^{*,M}, x_{ab}^{*,U}\right] = \exp\left(-\frac{\bar{l}^*(\bar{x}_{ab}, \bar{t}_b)}{\max\limits_a l^*(\bar{x}_{ab}, \bar{t}_b)}\right) \quad \textbf{(5)}$$

$$= \exp\left\{-\frac{\begin{Bmatrix}\left[\min\{|x_{ab}^M - t_b^U|, |x_{ab}^U - t_b^M|\}, \left|\left((x_{ab}^M + x_{ab}^U)/2\right) - \left((t_b^M + t_b^U)/2\right)\right|\right], & \text{if } \bar{x}_{ab} \cap \bar{t}_b = \emptyset \\ \left[0, \left|\left((x_{ij}^L + x_{ij}^U)/2\right) - \left((t_j^L + t_j^U)/2\right)\right|\right], & \text{if } \bar{x}_{ab} \cap \bar{t}_b \neq \emptyset\end{Bmatrix}}{\max\limits_b \begin{Bmatrix}\left(\min\{|x_{ab}^M - t_b^U|, |x_{ab}^U - t_b^M|\} + \left|\left((x_{ab}^M + x_{ab}^U)/2\right) - \left((t_b^M + t_b^U)/2\right)\right|\right)/2, & \text{if } \bar{x}_{ab} \cap \bar{t}_b = \emptyset \\ \left|\left((x_{ab}^M + x_{ab}^U)/2\right) - \left((t_b^M + t_b^U)/2\right)\right|/2, & \text{if } \bar{x}_{ab} \cap \bar{t}_b \neq \emptyset\end{Bmatrix}}\right\} \quad \textbf{(6)}$$

where $\bar{t}_b$ is the interval target value of each criterion and is computed as $\bar{t}_b = [t_b^M, t_b^U]$ = $\left\{\max\limits_a\ \bar{x}_{ab},\ if\ b \in A;\ \min\limits_a\ \bar{x}_{ab},\ if\ b \in B;\ \bar{s}_b,\ if\ b \in K\right\}$ where A, B, and K are the sets of beneficial, non-beneficial, and target-based criteria, together. And, $\bar{s}_b$ is the interval goal number of each target-based criterion. The utility values of the Ratio system, Good Reference

Method, and Complete Multipliers Form interval target-based methods, i.e., $\bar{y}_a^F$, $\bar{T}_a^F$, and $\bar{Z}_a^F$, respectively, are obtained as follows:

are the sets of benefit, non-beneficial, and target-based criteria, together. And, $\bar{s}_b$ is the target-based criterion's interval goal number. Interval target-based approaches are formed by the utility values of the Comparison system, Great Reference Method, i.e., $\bar{y}_a^F$, $\bar{T}_a^F$, and $\bar{Z}_a^F$, are obtained in the following way:

$$\bar{y}_b^F = \left[y_a^{F,M}, y_a^{F,U}\right] = \textstyle\sum_{b=1}^{s} V_b \bar{x}_{ab}^* = \left[\sum_{j=b}^{s} V_b x_{ab}^{*,M}, \sum_{j=b}^{s} V_b x_{ab}^{*,U}\right] \tag{7}$$

$$\begin{aligned}\bar{T}_a^F &= \left[T_a^{F,M}, T_a^{F,U}\right] = \max_b \bar{l}^*(V_b[1,1], V_b \bar{x}_{ab}^*) \\ &= \max_b \left(V_b \cdot \left\{\left[min\{|1 - x_{ab}^U|, |1 - x_{ab}^M|\}, \left|1 - \left((x_{ab}^M + x_{ab}^U)/2\right)\right|\right],\right.\right. \\ &\quad \left.\left.\text{if } [1,1] \cap \bar{x}_{ab}^* = \emptyset; \left[0, \left|1 - \left((x_{ab}^M + x_{ab}^U)/2\right)\right|\right], \text{if } [1,1] \cap \bar{x}_{ab}^* \neq \emptyset\right\}\right)\end{aligned} \tag{8}$$

$$\bar{Z}_a^F = \left[Z_a^{F,M}, Z_a^{F,U}\right] = \textstyle\prod_{b=1}^{s} (\bar{x}_{ab}^*)^{V_b} = \left[\prod_{b=1}^{s} \left(x_{ab}^{*,M}\right)^{V_b}, \prod_{b=1}^{s} \left(x_{ab}^{*,U}\right)^{V_b}\right] \tag{9}$$

4. Research Methodology

A hybrid MULTIMOORA method with neutrosophic analytical hierarchy process (AHP) is used to select the best supplier. In this section, we present a summary of the two methods utilized in our proposed research. Figure. 1 summarized the proposed method and its steps.

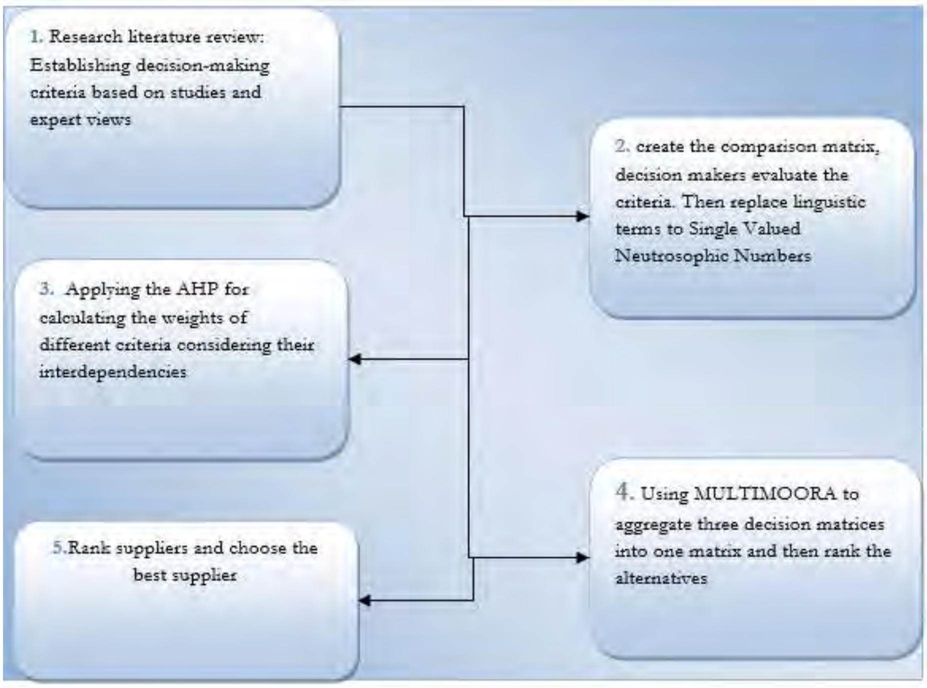

Figure 1. Proposed method framework

4.1 The AHP Approach

Saaty 1980 was the first to introduce the analytical hierarchy approach. The method has been used to solve a wide range of decision-making issues. It also provides a way for calculating the weights of criteria in a structured manner.

Phase 1: In a neutrosophic environment, get expert information.

- Decide on the study's objective, criteria, and alternative.
- Create a pairwise matrix of decision making judgments using the following form:

$$A^U = \begin{bmatrix} M_{11}^U & \cdots & M_{1b}^U \\ \vdots & \ddots & \vdots \\ M_{x1}^U & \cdots & M_{xb}^U \end{bmatrix} \quad \textbf{(10)}$$

Where **u** presents the decision makers, u=1, 2,….d, **x** presents the criteria and **b** presents the alternatives; x=1,2,….a; b =1,2,….c

- Using the score function of M_{ac}, convert neutrosophic scales to crisp values.

$$h(M_{ac}) = \frac{2+T-I-F}{3} \quad \textbf{(11)}$$

where T,I, F presents the truth, indeterminacy, and falsity membership degrees.

- Aggregate a pairwise matrix by:

$$M_{ac} = \frac{\sum_{u=1}^{d} M_{xb}^U}{u} \quad \textbf{(12)}$$

- Create the first pairwise comparison matrix as follows:

$$M = \begin{bmatrix} M_{11} & \cdots & M_{1b} \\ \vdots & \ddots & \vdots \\ M_{X1} & \cdots & M_{xb} \end{bmatrix} \quad \textbf{(13)}$$

Phase 2: Then calculate the weights of criteria.

- Compute the average of row

$$P_a = \frac{\sum_{x=1}^{a} (M_{\ xb})}{a}; x = 1,2,3, \dots \dots . a; b = 1,2,3, \dots \dots c \quad \textbf{(14)}$$

- The following equation is used to compute crisp value normalization**.**

$$P_a^X = \frac{P_a}{\sum_{a=1}^{x} P_a}; x = 1,2,3, \dots \dots . a \quad \textbf{(15)}$$

4.2 The MULTIMOORA Approach

Phase 3: Forming the decision matrix M is the first stage in the MULTIMOORA technique.

- In which M_{xb} presents the performance index of bth alternative respecting $\boldsymbol{x}$ th attribute x = 1, 2, …a and b = 1, 2, …c , and

$$M = [M_{xb}]_{ac} \tag{17}$$

- In the MULTIMOORA approach, to make performance indices comparable, these parameters should be dimensionless. As a result, the choice matrix is a normalized ratio of comparison between each alternative's response to criteria as a numerator and a denominator that represents all alternative performances on that attribute as a denominator.

$$N_{xb}^{*} = \frac{M_{xb}}{\sqrt[2]{\sum_{x=1}^{a} M_{xb}^{2}}} \tag{18}$$

where, N_{xb}^{*} denotes the normalized performance index of $\boldsymbol{b}$th alternative respecting $\boldsymbol{x}$ th attribute x = 1, 2, …a and b = 1, 2, …c and M_{xb} The performance index is displayed..

- Determine the total assessment

$$y_x^{*} = \sum_{b=1}^{g} P_b^c N_{xb}^{*} - \sum_{b=g+1}^{c} P_b^c N_{xb}^{*} \tag{19}$$

y_x^{*} denotes the total assessment of alternative $\boldsymbol{b}$th for subjective importance coefficients of all attributes **x**th, where **g** indicates the objectives to be maximized and (n-g) indicates the objectives to be minimized

- An ordinal ordering of the y_x^{*} with the highest assessment value is the best option based on the ratio system :

$$Z_{ac}^{*} = \left\{ Z_b \mid \max_b M_b^{*} \right\} \tag{20}$$

- The MULTIMOORA approach's second stage is built on the foundation of the ratio scheme displayed in eq. (20). In the procedure, a maximum objective reference point is also established in this form :

$$u_a = \begin{cases} \max_m M_{ac}^{*} & \text{in case of maximization} \\ \max_m M_{ac}^{*} & \text{in case of minimization} \end{cases} \tag{21}$$

Where u_a is the maximal objective reference point vector's $\boldsymbol{a}$th co-ordinate.

- A performance index's deviation from the reference point u_l can be represented as $(u_a - M_{ac}^{*})$. The greatest value of the deviation for each alternative t_a may then be computed using subjective significance coefficients for all criterion V_l^{j} and as follows:

$$t_a^* = \max_l \left| \left(P_b^g u_b - P_b^g M_{ac}^* \right) \right| \tag{22}$$

The best option is determined using the reference point approach, which involves determining the smallest number in prior calculations. Equation **(22)**. Demonstrated shown as:

$$Z_{ac}^* = \left\{ Z_a \mid \min_a t_a^* \right\} \tag{23}$$

The MULTIMOORA approach's third stage is proposed by Brauers and Zavadskas in 2010 is based on an idea from economic mathematics. As seen in this Equation, the formula for the full multiplicative form may well be calculated as demonstrated.

$$W_a' = \frac{\prod_{c=1}^{g} (M_{ac})^{P_c^g}}{\prod_{c=g+1}^{b} (M_{ac})^{P_c^g}} \tag{24}$$

where g refers to the maximized objectives and (b-g) refers to the minimized objectives. The product of performance indices of **a**th alternative related to advantageous attributes is the numerator of Eq. (23). The product of performance indices of **m**th alternative responding to non-beneficial features respects subjective importance coefficients of each attribute $\boldsymbol{P_c^g}$ is the denominator of Eq. (23). In the MULTIMOORA method, to keep all elements of the computations in harmony. Equation (23). Presents the whole multiplicative form in its normalized form. It is based on the search for the maximum among all assessment values of W_a^* similar to the ratio system calculation of the best alternative.

$$Z_{ac}^* = \left\{ Z_a \mid \max_a W_a^* \right\} \tag{25}$$

Figure 2. The hierarchical structure of SSS

5. Case Study

The proposed framework in this paper was applied for sustainable supplier selection (SSS) in an Egyptian pharmaceutical manufacturing company. The company makes and develops a variety of pharmaceuticals and tries to deal with various regulations relating to drug production, and drug marketing. This company's logistics section is in charge of supplying raw materials and chemicals. One of the company objectives is to create a framework for analyzing and finding top sustainable suppliers to improve efficiency and stay competitive. It is critical to choose the proper specialists while making decisions. We employ the integrated neutrosophic AHP-MULTIMOORA technique to choose the best one.

5.1 Results and Discussion

In this subsection, we discuss the results of the case study by using three experts to evaluate twenty criteria and three alternatives. The collected twenty criteria from the previous studies. Figure. 2 shows the hierarchical structure of sustainable supplier selection**.** We use the Single Valued Neutrosophic Scale for this study[42]. First steps, let decision-makers assess the criteria to build the comparison matrix. Then replace the linguistic terms with the Single Valued Neutrosophic Numbers. Then apply the score function to obtain the crisp value. Then aggregate the opinions of three experts into one matrix. Then compute the weights of main and sub-criteria. The weights of criteria show as CT1 =0.017554, CT2= 0.015736, CT3 =0.037136, CT4= 0. 035797, CT5=0.056704, Eg1 =0.039392, Eg2=0.074838, Eg3=0.105902, Eg4=0.09291, SA1 = 0.053527, SA2 =0.095759, SA3=0.088025, SA4=0.097024, SA5=0.189697. Figure 3. Show the weights of criteria.

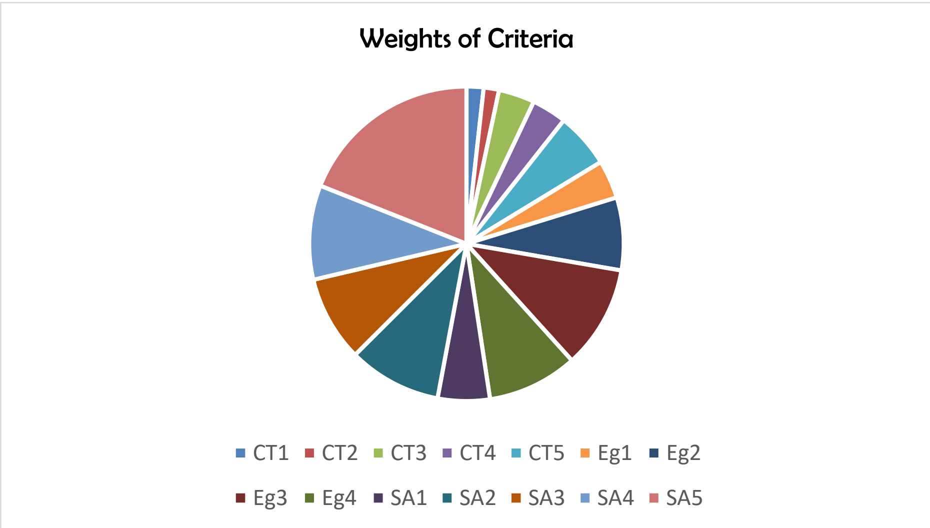

Figure 3. The Weights of Criteria

In the MULTIMOORA method, we let decision-makers evaluate the criteria and alternatives to obtain three decision matrices. Then convert these linguistic terms to neutrosophic numbers. Then convert these neutrosophic numbers into a crisp value. Then aggregate three decision matrices into one matrix in Table 1. Then compute the normalized decision matrix. Then compute the weighted normalized decision matrix by multiplying the normalized decision matrix by the weights of criteria in Table 2. Then compute the deviation references of the decision matrix in Table 3. Then compute the full multiplicative decision matrix in Table 4. Then rank the ratio system, reference point, and full multiplicative form in Table 5. From Table 7 we found that S2 is the best supplier and S1 is the worst Supplier in the ratio system. In the Reference point, we found that S2 is the best supplier and S3 is the worst supplier. But in the full multiplicative form, we found that S1 is the best supplier and S3 is the worst supplier. In the final rank, we conclude that S2 is the best supplier and s1 is the worst supplier. Figure 4 shows the rank of the supplier.

Table 1. The Combined Opinions of Three Experts.

Criteria/ Suppliers	CT_1	CT_2	CT_3	CT_4	CT_5	Eg_1	Eg_2	Eg_3	Eg_4	SA_1	SA_2	SA_3	SA_4	SA_5
S_1(Alternative)	0.62776	0.69443	0.8833	0.47223	0.6611	0.47223	0.4389	0.84996	0.37223	0.62776	0.37223	0.57223	0.59443	0.81663
S_2(Alternative)	0.72776	0.4389	0.4389	0.84996	0.59443	0.5611	0.4167	0.3389	0.3167	0.84996	0.3167	0.56043	0.8833	0.69443
S_3(Alternative)	0.7833	0.47223	0.81663	0.5611	0.2167	0.4167	0.33892	0.84996	0.72776	0.84996	0.4167	0.5389	0.5711	0.59443

Table 2. The Weighted Normalized of Decision matrix

Criteria/ Suppliers	CT_1	CT_2	CT_3	CT_4	CT_5	Eg_1	Eg_2	Eg_3	Eg_4	SA_1	SA_2	SA_3	SA_4	SA_5
S_1	0.3746	0.1323	0.1276	0.2276	0.0740	0.1659	0.1673	0.1099	0.0927	0.1224	0.0617	0.0626	0.0319	0.0263
S_2	0.3231	0.2094	0.2568	0.1264	0.0823	0.1396	0.1762	0.2757	0.1090	0.0904	0.0725	0.0639	0.0215	0.0309
S_3	0.3002	0.1946	0.1380	0.1915	0.2260	0.1880	0.2167	0.1099	0.0474	0.0904	0.0551	0.0665	0.0332	0.0361

Table 3. The Deviation References of Decision matrix.

Criteria/ Suppliers	CT_1	CT_2	CT_3	CT_4	CT_5	Eg_1	Eg_2	Eg_3	Eg_4	SA_1	SA_2	SA_3	SA_4	SA_5
S_1	0	0.0770	0.1292	0	0.1519	0.0221	0.0493	0.1658	0.0162	0	0.0108	0.0038	0.0013	0.0098
S_2	0.0514	0	0	0.1011	0.1436	0.0484	0.0404	0	0	0.0320	0	0.0025	0.0117	0.0052
S_3	0.0743	0.0147	0.1187	0.0360	0	0	0	0.1658	0.0615	0.0320	0.0174		0	0

Table 4. The Full Multiplicative Form

Criteria/ Suppliers	CT_1	CT_2	CT_3	CT_4	CT_5	Eg_1	Eg_2	Eg_3	Eg_4	SA_1	SA_2	SA_3	SA_4	SA_5
S_1	0.6277	0.6944	0.8833	0.4722	0.6611	0.4722	0.4389	0.8499	0.3722	0.6277	0.3722	0.5722	0.5944	0.8166
S_2	0.7277	0.4389	0.4389	0.8499	0.5944	0.5611	0.4167	0.3389	0.3167	0.8499	0.3167	0.5604	0.8833	0.6944
S_3	0.7833	0.4722	0.8166	0.5611	0.2167	0.4167	0.3389	0.8499	0.7277	0.8499	0.4167	0.5389	0.5711	0.5944

Table 5. Rank of MULTIMOORA Approach

Suppliers	Ratio System	Reference Point	Multiplicative Form	Final Rank
S_1	3	2	1	3
S_2	1	1	3	1
S_3	2	3	2	2

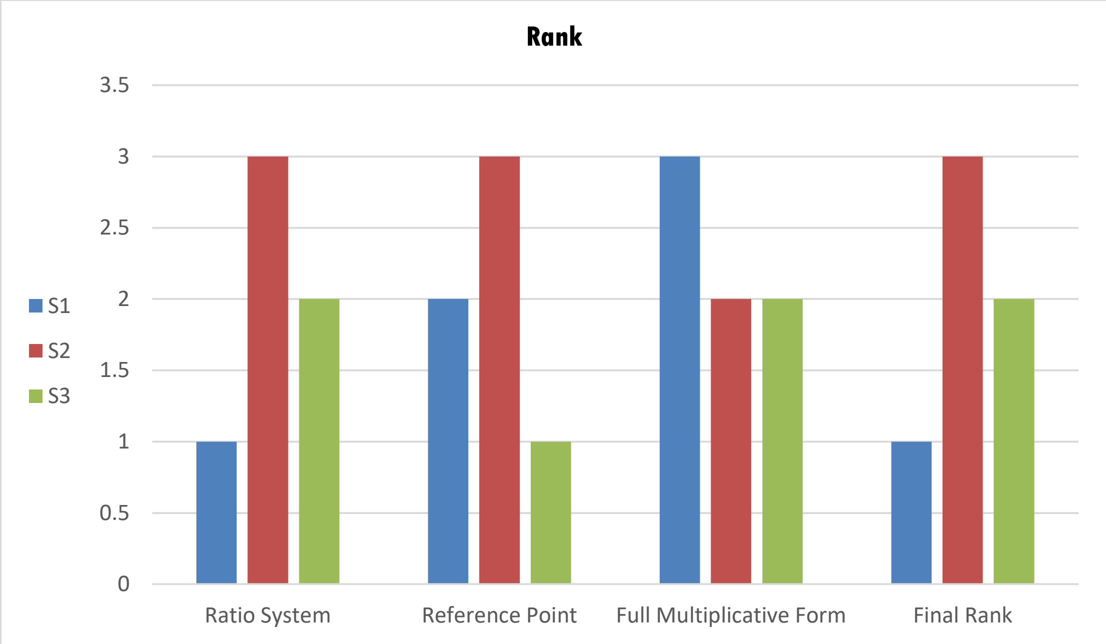

Figure 4. Rank of alternatives by the MULTIMOORA approach

6. Conclusions

Recently decision-making issues contain imprecision, vagueness, ambiguity, inconsistency, incompleteness, and indeterminacy, Neutrosophic set and logic are gaining traction as solutions and has been used to solve the various problem as a critical path problem.

Neutrosophic set helped to deal with ambiguous details, imprecise understanding, missing information, and linguistic imprecision through the neutrosophic environment. three membership degrees include the truth, indeterminacy, and falsity degrees, which are the main parts of a neutrosophic set. This feature is critical in a variety of applications including helping experts and decision-makers understand the information in an uncertain environment and make more precisely expressing their judgments to select the best supplier in supply chain management. NS is employed to evaluate and enhance supply chain management. The proposed study integrates neutrosophic analytical hierarchy process (AHP) and MULTIMOORA technique to suitable supplier selection. The AHP is used to calculate the weight of criteria and sub-criteria. The MULTIMOORA rank in different criteria. A case study is applied to a pharmaceutical manufacturing company demonstrates the efficacy of the suggested method and offers the final judgment to choose the best-qualified applicant for company success. The future work aims to use multiple multi-criteria decision-making strategies and display them in a neutrosophic environment utilizing the DEMETAL with MULTIMOORA method to solve the sustainable supplier selection problem.

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Single Valued Neutrosophic Hypersoft Expert Set with Application in Decision Making

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Abstract. Soft set deals with single set of attributes whereas its generalization i.e. hypersoft set deals with multiple disjoint attribute-valued sets corresponding to distinct attributes. In this paper, we first introduced the concept of single valued neutrosophic hypersoft expert sets (SVNHESs) which combines single valued neutrosophic sets and hypersoft expert sets. Some fundamental properties (i.e. subset, not set and equal set), results (i.e. commutative, associative, distributive and D' Morgan Laws) and set-theoretic operations (i.e. complement, union intersection AND, and OR) are discussed. An algorithm is proposed to solve decision-making problems and applied to select the best product.

Keywords: Soft Set; Soft Expert Set; Neutrosophic set; Single Valued Neutrosophic set; Hypersoft Set; Single Valued Neutrosophic Hypersoft Expert Set.

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1. **Introduction**

Neutrosophy has been introduced by Smarandache [1–3] as a new branch of philosophy and generalization of fuzzy logic, intuitionistic fuzzy logic, para-consistent logic. Fuzzy sets [4] and intuitionistic fuzzy sets [5] are defined by membership functions while intuitionistic fuzzy sets are characterized by membership and nonmembership functions, respectively. In some real life problems for proper description of an object in uncertain and ambiguous environment, we need to handle the indeterminate and incomplete information. But fuzzy sets and intuitionistic fuzzy sets do not handle the indeterminate and inconsistent information. Thus neutrosophic set (NS) is defined by Smarandache, as a new mathematical tool for dealing with problems involving incomplete, indeterminacy, inconsistent knowledge. In NS, the indeterminacy is quantified explicitly and truth-membership, indeterminacy membership, and false-membership are completely independent. From scientific or engineering point of view, the neutrosophic set and set- theoretic view, operators need to be specified. Otherwise, it will be difficult to apply in the real applications. Therefore, Wang et al [6] defined a single valued neutrosophic set (SVNS) and then provided the set theoretic operations and various properties of single valued neutrosophic sets. Broumi et al. [7] defined single valued neutrosophic soft expert sets and applied it in decision making.

Molodtsov [8] conceptualized soft set theory as a new parameterized family of subsets of the universe of discourse. Maji et al. [9] developed fuzzy soft set as a parametrization tool to deal with uncertainty. The fundamentals of soft set like subset, union, intersection, relations, functions etc., have been investigated by researchers [10–15]. Alkhazaleh et al. [16, 17] conceptualized soft expert set and fuzzy soft expert set. They discussed their applications in decision making. Broumi et al. [18] conceptualized intuitionistic fuzzy soft expert sets and presented its application in decision making.

In 2018, Smarandache [19] generalized soft set to hypersoft set by replacing single attribute-valued function to multi-attribute valued function. Saeed et al. [20] and Mujahid et al. [21] discussed the rudiments of hypersoft sets along with illustrative examples. Rahman et al. [22–30] discussed the notions of complex set, convex set, parameterization, bijection, neutrosophic graph and rough set under hypersoft set environment. Saeed et al. [31–36] explored the concepts of complex multi-fuzzy set, mappings and neutrosophic graph with hypersoft settings. They discussed application of these models in decision-making problems. Ihsan et al. [37, 38] introduced the expert system with multi-decisive opinions embedded with hypersoft set scenario. Some decision-making techniques i.e. TOPSIS etc. have been discussed for hypersoft set and its hybrids by researchers [39–43].

Having motivation from above literature, new notions of single valued neutrosophic hypersoft expert set are developed and an application is discussed in decision making through a proposed method. The pattern of rest of the paper is: section 2 reviews the notions of soft sets, fuzzy soft set, fuzzy soft expert set, hypersoft set and relevant definitions used in the proposed work. Section 3, presents notions of single valued neutrosophic hypersoft expert set with properties. Section 4, demonstrates an application of this concept in a decision-making problem. Section 5, concludes the paper.

2. Preliminaries

In this section, some basic definitions and terms regarding the main study, are presented from the literature.

Definition 2.1. [8]

Let $P(\Omega)$ denote power set of Ω(universe of discourse) and F be a collection of parameters defining Ω. A *soft set* Ψ_M is defined by mapping

$$\Psi_M : F \to P(\Omega).$$

Definition 2.2. [9] Suppose Ω be a set of universe, while F is a set of parameters. Here I^{Ω} represents the power set of all fuzzy subsets of Ω. Let $C \subseteq F$. A pair (R, C) is called a fuzzy soft set with R is a mapping given by

$$R : C \to I^{\Omega}.$$

Definition 2.3. [16]
Assume that Y be a set of specialists (operators) and $\ddot{O}$ be a set of conclusions, $T = F \times Y \times \ddot{O}$ with $S \subseteq T$ where Ω denotes the universe , F a set of parameters.
A pair(Φ, S) is known as a *soft expert set* over Ω, where H is a mapping given by

$$\Phi : S \to P(\Omega).$$

Definition 2.4. [17] A pair (H, C) is called a fuzzy soft expert set over Ω where F is a mapping given by

$$H : C \to I^{\Omega}$$

where I^{Ω} the set of all fuzzy subsets of Ω.

Definition 2.5. [2] Suppose Ω denotes the universe of discourse then the neutrosophic set N is an object with the form
$\mathrm{N} = \{< \beta : \mu_N(\beta), \nu_N(\beta), \omega_N(\beta) >, \beta \in \Omega\}$
While the functions $\mu_N(\beta), \nu_N(\beta), \omega_N(\beta) : \Omega \to]^{-}0, 1^{+}[$ denote the degree of membership, indeterminacy and non membership respectively for all $\beta \in \Omega$ with the condition
$^{-}0 \leq \mu_N(\beta) + \nu_N(\beta) + \omega_N(\beta) \leq 3^{+}$.

Definition 2.6. [6] Let Ω be a set of points (objects), with a generic element in Ω denoted by β. A single valued neutrosophic set (SVNS) N in Ω is defined by truth-membership function T_N, indeterminacy-membership function I_N and falsity-membership function F_N.
T_N, I_N, $F_N \in [0, 1]$ for all β in Ω with the condition
$0 \leq T_N(\beta) + I_N(\beta) + F_N(\beta) \leq 3$.

Definition 2.7. [19]
Let $h_1, h_2, h_3, ..., h_m$, for $m \geq 1$, be m distinct attributes, whose corresponding attribute values are respectively the sets $H_1, H_2, H_3, ..., H_m$, with $H_i \cap H_j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, 3, ..., m\}$. Then the pair (Ψ, G), where $G = H_1 \times H_2 \times H_3 \times ..... \times H_m$ and $\Psi : G \to P(\Omega)$ is called a *hypersoft Set* over Ω.

3. Single Valued Neutrosophic Hypersoft Expert set (SVNHSE-Set)

In this section, a new structure of single valued neutrosophic hypersoft expert set is developed and some properties are discussed.

Definition 3.1. Fuzzy Hypersoft Expert set (FHSE-Set)
A pair$(\xi, \mathbb{S})$ is known as a *fuzzy hypersoft expert set* over $\coprod$, where

$$\xi : \mathbb{S} \to I^{\coprod}$$

where

- $I^{\coprod}$ is collection of all fuzzy subsets of $\coprod$
- $\mathbb{S} \subseteq \mathcal{H} = \mathcal{G} \times \mathcal{D} \times \mathbb{C}$
- $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{G}_3 \times .... \times \mathcal{G}_p$ where $\mathcal{G}_i$ are disjoint attributive-valued sets corresponding to distinct attributes $g_i, i = 1, 2, 3, ..., p$
- $\mathcal{D}$ be a set of specialists (operators)
- $\mathbb{C}$ be a set of conclusions

For simplicity, $\mathbb{C} = \{0 = disagree, 1 = agree\}$.

Definition 3.2. Single Valued Neutrosophic Hypersoft Expert set (SVNHSE-Set)
A pair$(\xi, \mathbb{S})$ in definition 3.2, is known as a *single valued neutrosophic hypersoft expert set* over $\coprod$ if

$$\xi : \mathbb{S} \to SVNF^{\coprod}$$

with $SVNF^{\coprod}$ is collection of all single valued neutrosophic subsets of $\coprod$

Example 3.3. Suppose that a multi-national company aims to proceed the evaluation of certain specialists about its certain products. Let $\coprod = \{m_1, m_2, m_3, m_4\}$ be a set of products and
$\mathcal{G}_1 = \{q_{11}, q_{12}\}$
$\mathcal{G}_2 = \{q_{21}, q_{22}\}$
$\mathcal{G}_3 = \{q_{31}, q_{32}\}$
be disjoint attributive sets for distinct attributes q_1= simple to utilize, q_2= nature, q_3= modest.
Now $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{G}_3$

$$\mathcal{G} = \left\{ \begin{array}{l} \mu_1 = (q_{11}, q_{21}, q_{31}), \mu_2 = (q_{11}, q_{21}, q_{32}), \mu_3 = (q_{11}, q_{22}, q_{31}), \mu_4 = (q_{11}, q_{22}, q_{32}), \\ \mu_5 = (q_{12}, q_{21}, q_{31}), \mu_6 = (q_{12}, q_{21}, q_{32}), \mu_7 = (q_{12}, q_{22}, q_{31}), \mu_8 = (q_{12}, q_{22}, q_{32}) \end{array} \right\}$$

Now $\mathcal{H} = \mathcal{G} \times \mathcal{D} \times \mathbb{C}$

$$\mathcal{H} = \begin{Bmatrix} (\mu_1,s,0),(\mu_1,s,1),(\mu_1,t,0),(\mu_1,t,1),(\mu_1,u,0),(\mu_1,u,1), \\ (\mu_2,s,0),(\mu_2,s,1),(\mu_2,t,0),(\mu_2,t,1),(\mu_2,u,0),(\mu_2,u,1), \\ (\mu_3,s,0),(\mu_3,s,1),(\mu_3,t,0),(\mu_3,t,1),(\mu_3,u,0),(\mu_3,u,1), \\ (\mu_4,s,0),(\mu_4,s,1),(\mu_4,t,0),(\mu_4,t,1),(\mu_4,u,0),(\mu_4,u,1), \\ (\mu_5,s,0),(\mu_5,s,1),(\mu_5,t,0),(\mu_5,t,1),(\mu_5,u,0),(\mu_5,u,1), \\ (\mu_6,s,0),(\mu_6,s,1),(\mu_6,t,0),(\mu_6,t,1),(\mu_6,u,0),(\mu_6,u,1), \\ (\mu_7,s,0),(\mu_7,s,1),(\mu_7,t,0),(\mu_7,t,1),(\mu_7,u,0),(\mu_7,u,1), \\ (\mu_8,s,0),(\mu_8,s,1),(\mu_8,t,0),(\mu_8,t,1),(\mu_8,u,0),(\mu_8,u,1) \end{Bmatrix}$$

let

$$\mathbb{S} = \begin{Bmatrix} (\mu_1,s,0),(\mu_1,s,1),(\mu_1,t,0),(\mu_1,t,1),(\mu_1,u,0),(\mu_1,u,1), \\ (\mu_2,s,0),(\mu_2,s,1),(\mu_2,t,0),(\mu_2,t,1),(\mu_2,u,0),(\mu_2,u,1) \\ (\mu_3,s,0),(\mu_3,s,1),(\mu_3,t,0),(\mu_3,t,1),(\mu_3,u,0),(\mu_3,u,1), \end{Bmatrix}$$

be a subset of $\mathcal{H}$ and $\mathcal{D} = \{s,t,u,\}$ be a set of specialists.

Following survey depicts choices of three specialists:

$$\xi_1 = \xi(\mu_1,s,1) = \left\{ \frac{m_1}{<0.2,0.5,0.4>}, \frac{m_2}{<0.7,0.2,0.5>}, \frac{m_3}{<0.5,0.4,0.6>}, \frac{m_4}{<0.1,0.3,0.6>} \right\},$$

$$\xi_2 = \xi(\mu_1,t,1) = \left\{ \frac{m_1}{<0.4,0.2,0.3>}, \frac{m_2}{<0.8,0.1,0.5>}, \frac{m_3}{<0.4,0.5,0.6>}, \frac{m_4}{<0.2,0.5,0.3>} \right\},$$

$$\xi_3 = \xi(\mu_1,u,1) = \left\{ \frac{m_1}{<0.7,0.2,0.3>}, \frac{m_2}{<0.5,0.3,0.6>}, \frac{m_3}{<0.6,0.3,0.7>}, \frac{m_4}{<0.3,0.5,0.6>} \right\},$$

$$\xi_4 = \xi(\mu_2,s,1) = \left\{ \frac{m_1}{<0.9,0.1,0.3>}, \frac{m_2}{<0.4,0.5,0.4>}, \frac{m_3}{<0.7,0.2,0.6>}, \frac{m_4}{<0.3,0.4,0.8>} \right\},$$

$$\xi_5 = \xi(\mu_2,t,1) = \left\{ \frac{m_1}{<0.4,0.5,0.6>}, \frac{m_2}{<0.8,0.1,0.7>}, \frac{m_3}{<0.3,0.6,0.5>}, \frac{m_4}{<0.2,0.6,0.7>} \right\},$$

$$\xi_6 = \xi(\mu_2,u,1) = \left\{ \frac{m_1}{<0.5,0.4,0.7>}, \frac{m_2}{<0.3,0.6,0.4>}, \frac{m_3}{<0.6,0.2,0.5>}, \frac{m_4}{<0.8,0.1,0.6>} \right\},$$

$$\xi_7 = \xi(\mu_3,s,1) = \left\{ \frac{m_1}{<0.2,0.7,0.5>}, \frac{m_2}{<0.9,0.1,0.4>}, \frac{m_3}{<0.4,0.5,0.7>}, \frac{m_4}{<0.5,0.4,0.8>} \right\},$$

$$\xi_8 = \xi(\mu_3,t,1) = \left\{ \frac{m_1}{<0.4,0.3,0.2>}, \frac{m_2}{<0.6,0.3,0.1>}, \frac{m_3}{<0.7,0.2,0.3>}, \frac{m_4}{<0.9,0.1,0.4>} \right\},$$

$$\xi_9 = \xi(\mu_3,u,1) = \left\{ \frac{m_1}{<0.7,0.2,0.6>}, \frac{m_2}{<0.3,0.5,0.7>}, \frac{m_3}{<0.5,0.4,0.5>}, \frac{m_4}{<0.2,0.7,0.8>} \right\},$$

$$\xi_{10} = \xi(\mu_1,s,0) = \left\{ \frac{m_1}{<0.3,0.2,0.1>}, \frac{m_2}{<0.2,0.4,0.5>}, \frac{m_3}{<0.4,0.5,0.8>}, \frac{m_4}{<0.1,0.8,0.3>} \right\},$$

$$\xi_{11} = \xi(\mu_1,t,0) = \left\{ \frac{m_1}{<0.1,0.8,0.4>}, \frac{m_2}{<0.9,0.1,0.2>}, \frac{m_3}{<0.6,0.3,0.4>}, \frac{m_4}{<0.2,0.7,0.5>} \right\},$$

$$\xi_{12} = \xi(\mu_1, u, 0) = \left\{ \frac{m_1}{<0.2, 0.7, 0.5>}, \frac{m_2}{<0.1, 0.8, 0.6>}, \frac{m_3}{<0.3, 0.5, 0.7>}, \frac{m_4}{<0.5, 0.4, 0.6>} \right\},$$
$$\xi_{13} = \xi(\mu_2, s, 0) = \left\{ \frac{m_1}{<0.8, 0.1, 0.6>}, \frac{m_2}{<0.3, 0.6, 0.7>}, \frac{m_3}{<0.5, 0.4, 0.8>}, \frac{m_4}{<0.7, 0.2, 0.9>} \right\},$$
$$\xi_{14} = \xi(\mu_2, t, 0) = \left\{ \frac{m_1}{<0.7, 0.2, 0.5>}, \frac{m_2}{<0.2, 0.6, 0.4>}, \frac{m_3}{<0.9, 0.1, 0.6>}, \frac{m_4}{<0.4, 0.5, 0.7>} \right\},$$
$$\xi_{15} = \xi(\mu_2, u, 0) = \left\{ \frac{m_1}{<0.6, 0.2, 0.5>}, \frac{m_2}{<0.7, 0.2, 0.4>}, \frac{m_3}{<0.3, 0.5, 0.4>}, \frac{m_4}{<0.2, 0.7, 0.6>} \right\},$$
$$\xi_{16} = \xi(\mu_3, s, 0) = \left\{ \frac{m_1}{<0.1, 0.7, 0.5>}, \frac{m_2}{<0.4, 0.5, 0.7>}, \frac{m_3}{<0.7, 0.2, 0.9>}, \frac{m_4}{<0.8, 0.2, 0.4>} \right\},$$
$$\xi_{17} = \xi(\mu_3, t, 0) = \left\{ \frac{m_1}{<0.2, 0.7, 0.4>}, \frac{m_2}{<0.9, 0.1, 0.6>}, \frac{m_3}{<0.8, 0.2, 0.4>}, \frac{m_4}{<0.3, 0.5, 0.7>} \right\},$$
$$\xi_{18} = \xi(\mu_3, u, 0) = \left\{ \frac{m_1}{<0.5, 0.4, 0.2>}, \frac{m_2}{<0.3, 0.6, 0.1>}, \frac{m_3}{<0.6, 0.3, 0.2>}, \frac{m_4}{<0.1, 0.8, 0.3>} \right\}$$

The single valued neutrosophic hypersoft expert set can be described as

$$(\xi, \mathbb{S}) = \left\{ \begin{array}{l}
\left((\mu_1, s, 1), \left\{ \frac{m_1}{<0.2,0.5,0.4>}, \frac{m_2}{<0.7,0.2,0.5>}, \frac{m_3}{<0.5,0.4,0.6>}, \frac{m_4}{<0.1,0.3,0.6>} \right\} \right), \\
\left((\mu_1, t, 1), \left\{ \frac{m_1}{<0.4,0.2,0.3>}, \frac{m_2}{<0.8,0.1,0.5>}, \frac{m_3}{<0.4,0.5,0.6>}, \frac{m_4}{<0.2,0.5,0.3>} \right\} \right), \\
\left((\mu_1, u, 1), \left\{ \frac{m_1}{<0.7,0.2,0.3>}, \frac{m_2}{<0.5,0.3,0.6>}, \frac{m_3}{<0.6,0.3,0.7>}, \frac{m_4}{<0.3,0.5,0.6>} \right\} \right), \\
\left((\mu_2, s, 1), \left\{ \frac{m_1}{<0.9,0.1,0.3>}, \frac{m_2}{<0.4,0.5,0.4>}, \frac{m_3}{<0.7,0.2,0.6>}, \frac{m_4}{<0.3,0.4,0.8>} \right\} \right), \\
\left((\mu_2, t, 1), \left\{ \frac{m_1}{<0.4,0.5,0.6>}, \frac{m_2}{<0.8,0.1,0.7>}, \frac{m_3}{<0.3,0.6,0.5>}, \frac{m_4}{<0.2,0.6,0.7>} \right\} \right), \\
\left((\mu_2, u, 1), \left\{ \frac{m_1}{<0.5,0.4,0.7>}, \frac{m_2}{<0.3,0.6,0.4>}, \frac{m_3}{<0.6,0.2,0.5>}, \frac{m_4}{<0.8,0.1,0.6>} \right\} \right), \\
\left((\mu_3, s, 1), \left\{ \frac{m_1}{<0.2,0.7,0.5>}, \frac{m_2}{<0.9,0.1,0.4>}, \frac{m_3}{<0.4,0.5,0.7>}, \frac{m_4}{<0.5,0.4,0.8>} \right\} \right), \\
\left((\mu_3, t, 1), \left\{ \frac{m_1}{<0.4,0.3,0.2>}, \frac{m_2}{<0.6,0.3,0.1>}, \frac{m_3}{<0.7,0.2,0.3>}, \frac{m_4}{<0.9,0.1,0.4>} \right\} \right), \\
\left((\mu_3, u, 1), \left\{ \frac{m_1}{<0.7,0.2,0.6>}, \frac{m_2}{<0.3,0.5,0.7>}, \frac{m_3}{<0.5,0.4,0.5>}, \frac{m_4}{<0.2,0.7,0.8>} \right\} \right), \\
\left((\mu_1, s, 0), \left\{ \frac{m_1}{<0.3,0.2,0.1>}, \frac{m_2}{<0.2,0.4,0.5>}, \frac{m_3}{<0.4,0.5,0.8>}, \frac{m_4}{<0.1,0.8,0.3>} \right\} \right), \\
\left((\mu_1, t, 0), \left\{ \frac{m_1}{<0.1,0.8,0.4>}, \frac{m_2}{<0.9,0.1,0.2>}, \frac{m_3}{<0.6,0.3,0.4>}, \frac{m_4}{<0.2,0.7,0.5>} \right\} \right), \\
\left((\mu_1, u, 0), \left\{ \frac{m_1}{<0.2,0.7,0.5>}, \frac{m_2}{<0.1,0.8,0.6>}, \frac{m_3}{<0.3,0.5,0.7>}, \frac{m_4}{<0.5,0.4,0.6>} \right\} \right), \\
\left((\mu_2, s, 0), \left\{ \frac{m_1}{<0.8,0.1,0.6>}, \frac{m_2}{<0.3,0.6,0.7>}, \frac{m_3}{<0.5,0.4,0.8>}, \frac{m_4}{<0.7,0.2,0.9>} \right\} \right), \\
\left((\mu_2, t, 0), \left\{ \frac{m_1}{<0.7,0.2,0.5>}, \frac{m_2}{<0.2,0.6,0.4>}, \frac{m_3}{<0.9,0.1,0.6>}, \frac{m_4}{<0.4,0.5,0.7>} \right\} \right), \\
\left((\mu_2, u, 0), \left\{ \frac{m_1}{<0.6,0.2,0.5>}, \frac{m_2}{<0.7,0.2,0.4>}, \frac{m_3}{<0.3,0.5,0.4>}, \frac{m_4}{<0.2,0.7,0.6>} \right\} \right), \\
\left((\mu_3, s, 0), \left\{ \frac{m_1}{<0.1,0.7,0.5>}, \frac{m_2}{<0.4,0.5,0.7>}, \frac{m_3}{<0.7,0.2,0.9>}, \frac{m_4}{<0.8,0.2,0.4>} \right\} \right), \\
\left((\mu_3, t, 0), \left\{ \frac{m_1}{<0.2,0.7,0.4>}, \frac{m_2}{<0.9,0.1,0.6>}, \frac{m_3}{<0.8,0.2,0.4>}, \frac{m_4}{<0.3,0.5,0.7>} \right\} \right), \\
\left((\mu_3, u, 0), \left\{ \frac{m_1}{<0.5,0.4,0.2>}, \frac{m_2}{<0.3,0.6,0.1>}, \frac{m_3}{<0.6,0.3,0.2>}, \frac{m_4}{<0.1,0.8,0.3>} \right\} \right)
\end{array} \right\}$$

Definition 3.4. Single Valued Neutrosophic Hypersoft Expert subset

A single valued neutrosophic hypersoft expert set $(\xi_1, \mathbb{S})$ is said to be single valued neutrosophic hypersoft expert subset of (ξ_2, R) over $\coprod$, if

(i) $\mathbb{S} \subseteq R$,

(ii) $\forall\ \alpha \in \mathbb{S}, \xi_1(\alpha) \subseteq \xi_2(\alpha)$.

and denoted by $(\xi_1, \mathbb{S}) \subseteq (\xi_2, R)$. Similarly (ξ_2, R) is said to be *single valued neutrosophic hypersoft expert superset* of $(\xi_1, \mathbb{S})$.

Example 3.5. Considering Example 3.3, Suppose

$$A_1 = \Big\{ (\mu_1, s, 1), (\mu_3, s, 0), (\mu_1, t, 1), (\mu_3, t, 1), (\mu_3, t, 0), (\mu_1, u, 0), (\mu_3, u, 1) \Big\}$$

$$A_2 = \Big\{ (\mu_1, s, 1), (\mu_3, s, 0), (\mu_3, s, 1), (\mu_1, t, 1), (\mu_3, t, 1), (\mu_1, t, 0), (\mu_3, t, 0), (\mu_1, u, 0), (\mu_3, u, 1), (\mu_1, u, 1) \Big\}$$

It is clear that$A_1 \subset A_2$.Suppose (ξ_1, A_1) and (ξ_2, A_2) be defined as following

$$(\xi_1, \mathrm{A}_1) = \left\{ \begin{array}{l}
\left((\mu_1, s, 1), \left\{ \frac{m_1}{<0.1,0.6,0.7>}, \frac{m_2}{<0.6,0.5,0.8>}, \frac{m_3}{<0.4,0.6,0.9>}, \frac{m_4}{<0.1,0.8,0.6>} \right\} \right), \\
\left((\mu_1, t, 1), \left\{ \frac{m_1}{<0.3,0.4,0.5>}, \frac{m_2}{<0.6,0.4,0.6>}, \frac{m_3}{<0.2,0.5,0.7>}, \frac{m_4}{<0.1,0.5,0.6>} \right\} \right), \\
\left((\mu_3, t, 1), \left\{ \frac{m_1}{<0.2,0.6,0.4>}, \frac{m_2}{<0.5,0.4,0.7>}, \frac{m_3}{<0.6,0.5,0.8>}, \frac{m_4}{<0.8,0.6,0.4>} \right\} \right), \\
\left((\mu_3, u, 1), \left\{ \frac{m_1}{<0.6,0.4,0.3>}, \frac{m_2}{<0.2,0.7,0.6>}, \frac{m_3}{<0.4,0.5,0.3>}, \frac{m_4}{<0.1,0.7,0.4>} \right\} \right), \\
\left((\mu_1, u, 0), \left\{ \frac{m_1}{<0.1,0.6,0.3>}, \frac{m_2}{<0.1,0.7,0.4>}, \frac{m_3}{<0.2,0.7,0.6>}, \frac{m_4}{<0.1,0.6,0.7>} \right\} \right), \\
\left((\mu_3, s, 0), \left\{ \frac{m_1}{<0.1,0.8,0.6>}, \frac{m_2}{<0.3,0.6,0.5>}, \frac{m_3}{<0.6,0.3,0.4>}, \frac{m_4}{<0.7,0.2,0.6>} \right\} \right), \\
\left((\mu_3, t, 0), \left\{ \frac{m_1}{<0.1,0.7,0.4>}, \frac{m_2}{<0.6,0.3,0.6>}, \frac{m_3}{<0.7,0.2,0.5>}, \frac{m_4}{<0.2,0.7,0.4>} \right\} \right)
\end{array} \right\}$$

$$(\xi_2, \mathrm{A}_2) = \left\{ \begin{array}{l}
\left((\mu_1, s, 1), \left\{ \frac{m_1}{<0.2,0.3,0.6>}, \frac{m_2}{<0.7,0.4,0.7>}, \frac{m_3}{<0.5,0.4,0.8>}, \frac{m_4}{<0.2,0.4,0.5>} \right\} \right), \\
\left((\mu_1, t, 1), \left\{ \frac{m_1}{<0.4,0.3,0.4>}, \frac{m_2}{<0.8,0.3,0.5>}, \frac{m_3}{<0.4,0.3,0.6>}, \frac{m_4}{<0.2,0.6,0.5>} \right\} \right), \\
\left((\mu_3, s, 1), \left\{ \frac{m_1}{<0.1,0.3,0.4>}, \frac{m_2}{<0.9,0.1,0.3>}, \frac{m_3}{<0.4,0.5,0.4>}, \frac{m_4}{<0.5,0.3,0.4>} \right\} \right), \\
\left((\mu_3, t, 1), \left\{ \frac{m_1}{<0.4,0.2,0.3>}, \frac{m_2}{<0.6,0.3,0.6>}, \frac{m_3}{<0.7,0.4,0.5>}, \frac{m_4}{<0.9,0.5,0.2} \right\} \right), \\
\left((\mu_1, u, 1), \left\{ \frac{m_1}{<0.7,0.2,0.4>}, \frac{m_2}{<0.5,0.2,0.6>}, \frac{m_3}{<0.6,0.2,0.7>}, \frac{m_4}{<0.3,0.5,0.8} \right\} \right), \\
\left((\mu_3, u, 1), \left\{ \frac{m_1}{<0.7,0.3,0.1>}, \frac{m_2}{<0.3,0.5,0.4>}, \frac{m_3}{<0.5,0.4,0.2>}, \frac{m_4}{<0.2,0.6,0.3>} \right\} \right), \\
\left((\mu_1, u, 0), \left\{ \frac{m_1}{<0.2,0.5,0.1>}, \frac{m_2}{<0.2,0.6,0.3>}, \frac{m_3}{<0.3,0.5,0.4>}, \frac{m_4}{<0.5,0.3,0.5>} \right\} \right), \\
\left((\mu_1, t, 0), \left\{ \frac{m_1}{<0.1,0.6,0.4>}, \frac{m_2}{<0.9,0.1,0.6>}, \frac{m_3}{<0.6,0.3,0.8>}, \frac{m_4}{<0.2,0.6,0.8>} \right\} \right), \\
\left((\mu_3, s, 0), \left\{ \frac{m_1}{<0.2,0.7,0.4>}, \frac{m_2}{<0.4,0.5,0.3>}, \frac{m_3}{<0.7,0.2,0.1>}, \frac{m_4}{<0.8,0.1,0.5>} \right\} \right), \\
\left((\mu_3, t, 0), \left\{ \frac{m_1}{<0.2,0.5,0.1>}, \frac{m_2}{<0.7,0.2,0.3>}, \frac{m_3}{<0.8,0.2,0.4>}, \frac{m_4}{<0.3,0.5,0.2>} \right\} \right)
\end{array} \right\}$$

which implies that $(\xi_1, A_1) \subseteq (\xi_2, A_2)$.

Definition 3.6. Two single valued neutrosophic hypersoft expert sets (ξ_1, A_1) and (ξ_2, A_2) over $\coprod$ are said to be equal if (ξ_1, A_1) is a single valued neutrosophic hypersoft expert subset of (ξ_2, A_2) and (ξ_2, A_2) is a single valued neutrosophic hypersoft expert subset of (ξ_1, A_1).

Definition 3.7. The complement of a single valued neutrosophic hypersoft expert set $(\xi, \mathbb{S})$, denoted by $(\xi, \mathbb{S})^c$, is defined by

$(\xi, \mathbb{S})^c = \tilde{c}(\xi(\beta))\ \forall\ \beta \in \coprod$ while $\tilde{c}$ is a NF complement.

Example 3.8. Taking complement of single valued neutrosophic hypersoft expert set determined in 3.3, we have

$$(\xi, \mathbb{S})^{c} = \left\{ \begin{array}{l}
\left((\mu_1, s, 1), \left\{ \frac{m_1}{<0.4,0.5,0.2>}, \frac{m_2}{<0.5,0.8,0.7>}, \frac{m_3}{<0.6,0.6,0.5>}, \frac{m_4}{<0.6,0.7,0.1>} \right\} \right), \\
\left((\mu_1, t, 1), \left\{ \frac{m_1}{<0.3,0.8,0.6>}, \frac{m_2}{<0.5,0.9,0.8>}, \frac{m_3}{<0.6,0.5,0.4>}, \frac{m_4}{<0.3,0.5,0.2>} \right\} \right), \\
\left((\mu_1, u, 1), \left\{ \frac{m_1}{<0.3,0.8,0.7>}, \frac{m_2}{<0.6,0.7,0.5>}, \frac{m_3}{<0.7,0.7,0.6>}, \frac{m_4}{<0.6,0.5,0.3>} \right\} \right), \\
\left((\mu_2, s, 1), \left\{ \frac{m_1}{<0.3,0.9,0.9>}, \frac{m_2}{<0.4,0.5,0.4>}, \frac{m_3}{<0.6,0.8,0.7>}, \frac{m_4}{<0.8,0.6,0.3>} \right\} \right), \\
\left((\mu_2, t, 1), \left\{ \frac{m_1}{<0.6,0.5,0.4>}, \frac{m_2}{<0.7,0.9,0.8>}, \frac{m_3}{<0.5,0.4,0.3>}, \frac{m_4}{<0.7,0.4,0.2>} \right\} \right), \\
\left((\mu_2, u, 1), \left\{ \frac{m_1}{<0.7,0.6,0.5>}, \frac{m_2}{<0.4,0.4,0.3>}, \frac{m_3}{<0.5,0.8,0.6>}, \frac{m_4}{<0.6,0.9,0.8>} \right\} \right), \\
\left((\mu_3, s, 1), \left\{ \frac{m_1}{<0.5,0.3,0.2>}, \frac{m_2}{<0.4,0.9,0.9>}, \frac{m_3}{<0.7,0.5,0.4>}, \frac{m_4}{<0.8,0.6,0.5>} \right\} \right), \\
\left((\mu_3, t, 1), \left\{ \frac{m_1}{<0.2,0.7,0.4>}, \frac{m_2}{<0.1,0.7,0.6>}, \frac{m_3}{<0.3,0.8,0.7>}, \frac{m_4}{<0.4,0.9,0.9>} \right\} \right), \\
\left((\mu_3, u, 1), \left\{ \frac{m_1}{<0.6,0.8,0.7>}, \frac{m_2}{<0.7,0.5,0.3>}, \frac{m_3}{<0.5,0.6,0.5>}, \frac{m_4}{<0.8,0.3,0.2>} \right\} \right), \\
\left((\mu_1, s, 0), \left\{ \frac{m_1}{<0.1,0.8,0.3>}, \frac{m_2}{<0.5,0.6,0.2>}, \frac{m_3}{<0.8,0.5,0.4>}, \frac{m_4}{<0.3,0.2,0.1>} \right\} \right), \\
\left((\mu_1, t, 0), \left\{ \frac{m_1}{<0.4,0.2,0.1>}, \frac{m_2}{<0.2,0.9,0.9>}, \frac{m_3}{<0.4,0.7,0.6>}, \frac{m_4}{<0.5,0.3,0.2>} \right\} \right), \\
\left((\mu_1, u, 0), \left\{ \frac{m_1}{<0.5,0.3,0.2>}, \frac{m_2}{<0.6,0.2,0.1>}, \frac{m_3}{<0.7,0.5,0.3>}, \frac{m_4}{<0.6,0.6,0.5>} \right\} \right), \\
\left((\mu_2, s, 0), \left\{ \frac{m_1}{<0.6,0.9,0.8>}, \frac{m_2}{<0.7,0.4,0.3>}, \frac{m_3}{<0.8,0.6,0.5>}, \frac{m_4}{<0.9,0.8,0.7>} \right\} \right), \\
\left((\mu_2, t, 0), \left\{ \frac{m_1}{<0.5,0.8,0.7>}, \frac{m_2}{<0.4,0.4,0.2>}, \frac{m_3}{<0.6,0.9,0.9>}, \frac{m_4}{<0.7,0.5,0.4>} \right\} \right), \\
\left((\mu_2, u, 0), \left\{ \frac{m_1}{<0.5,0.8,0.6>}, \frac{m_2}{<0.4,0.8,0.7>}, \frac{m_3}{<0.4,0.5,0.3>}, \frac{m_4}{<0.6,0.3,0.2>} \right\} \right), \\
\left((\mu_3, s, 0), \left\{ \frac{m_1}{<0.5,0.3,0.1>}, \frac{m_2}{<0.7,0.5,0.3>}, \frac{m_3}{<0.9,0.8,0.8>}, \frac{m_4}{<0.4,0.8,0.8>} \right\} \right), \\
\left((\mu_3, t, 0), \left\{ \frac{m_1}{<0.4,0.3,0.2>}, \frac{m_2}{<0.6,0.9,0.9>}, \frac{m_3}{<0.4,0.8,0.8>}, \frac{m_4}{<0.7,0.5,0.3>} \right\} \right), \\
\left((\mu_3, u, 0), \left\{ \frac{m_1}{<0.2,0.6,0.5>}, \frac{m_2}{<0.1,0.4,0.3>}, \frac{m_3}{<0.2,0.7,0.6>}, \frac{m_4}{<0.3,0.2,0.1>} \right\} \right)
\end{array} \right\}$$

Definition 3.9. An agree-single valued neutrosophic hypersoft expert set $(\xi, \mathbb{S})_{ag}$ over $\coprod$, is a single valued neutrosophic hypersoft expert subset of $(\xi, \mathbb{S})$ and is characterized as $(\xi, \mathbb{S})_{ag} = \{\xi_{ag}(\beta) : \beta \in \mathcal{G} \times \mathcal{D} \times \{1\}\}$.

Example 3.10. Finding agree-single valued neutrosophic hypersoft expert set determined in 3.3, we get

$$(\xi,\mathbb{S}) = \left\{ \begin{array}{l}
\left((\mu_1,s,1),\left\{\frac{m_1}{<0.2,0.5,0.4>},\frac{m_2}{<0.7,0.2,0.5>},\frac{m_3}{<0.5,0.4,0.6>},\frac{m_4}{<0.1,0.3,0.6>}\right\}\right),\\
\left((\mu_1,t,1),\left\{\frac{m_1}{<0.4,0.2,0.3>},\frac{m_2}{<0.8,0.1,0.5>},\frac{m_3}{<0.4,0.5,0.6>},\frac{m_4}{<0.2,0.5,0.3>}\right\}\right),\\
\left((\mu_1,u,1),\left\{\frac{m_1}{<0.7,0.2,0.3>},\frac{m_2}{<0.5,0.3,0.6>},\frac{m_3}{<0.6,0.3,0.7>},\frac{m_4}{<0.3,0.5,0.6>}\right\}\right),\\
\left((\mu_2,s,1),\left\{\frac{m_1}{<0.9,0.1,0.3>},\frac{m_2}{<0.4,0.5,0.4>},\frac{m_3}{<0.7,0.2,0.6>},\frac{m_4}{<0.3,0.4,0.8>}\right\}\right),\\
\left((\mu_2,t,1),\left\{\frac{m_1}{<0.4,0.5,0.6>},\frac{m_2}{<0.8,0.1,0.7>},\frac{m_3}{<0.3,0.6,0.5>},\frac{m_4}{<0.2,0.6,0.7>}\right\}\right),\\
\left((\mu_2,u,1),\left\{\frac{m_1}{<0.5,0.4,0.7>},\frac{m_2}{<0.3,0.6,0.4>},\frac{m_3}{<0.6,0.2,0.5>},\frac{m_4}{<0.8,0.1,0.6>}\right\}\right),\\
\left((\mu_3,s,1),\left\{\frac{m_1}{<0.2,0.7,0.5>},\frac{m_2}{<0.9,0.1,0.4>},\frac{m_3}{<0.4,0.5,0.7>},\frac{m_4}{<0.5,0.4,0.8>}\right\}\right),\\
\left((\mu_3,t,1),\left\{\frac{m_1}{<0.4,0.3,0.2>},\frac{m_2}{<0.6,0.3,0.1>},\frac{m_3}{<0.7,0.2,0.3>},\frac{m_4}{<0.9,0.1,0.4>}\right\}\right),\\
\left((\mu_3,u,1),\left\{\frac{m_1}{<0.7,0.2,0.6>},\frac{m_2}{<0.3,0.5,0.7>},\frac{m_3}{<0.5,0.4,0.5>},\frac{m_4}{<0.2,0.7,0.8>}\right\}\right),
\end{array} \right\}$$

Definition 3.11. A disagree-single valued neutrosophic hypersoft expert set $(\xi,\mathbb{S})_{dag}$ over $\coprod$, is a single valued neutrosophic hypersoft expert subset of $(\xi,\mathbb{S})$ and is characterized as $(\xi,\mathbb{S})_{dag} = \{\xi_{dag}(\beta) : \beta \in \mathbb{G} \times \mathcal{D} \times \{0\}\}$.

Example 3.12. Getting disagree-single valued neutrosophic hypersoft expert set determined in 3.3,

$$(\xi,\mathbb{S}) = \left\{ \begin{array}{l}
\left((\mu_1,s,0),\left\{\frac{m_1}{<0.3,0.2,0.1>},\frac{m_2}{<0.2,0.4,0.5>},\frac{m_3}{<0.4,0.5,0.8>},\frac{m_4}{<0.1,0.8,0.3>}\right\}\right),\\
\left((\mu_1,t,0),\left\{\frac{m_1}{<0.1,0.8,0.4>},\frac{m_2}{<0.9,0.1,0.2>},\frac{m_3}{<0.6,0.3,0.4>},\frac{m_4}{<0.2,0.7,0.5>}\right\}\right),\\
\left((\mu_1,u,0),\left\{\frac{m_1}{<0.2,0.7,0.5>},\frac{m_2}{<0.1,0.8,0.6>},\frac{m_3}{<0.3,0.5,0.7>},\frac{m_4}{<0.5,0.4,0.6>}\right\}\right),\\
\left((\mu_2,s,0),\left\{\frac{m_1}{<0.8,0.1,0.6>},\frac{m_2}{<0.3,0.6,0.7>},\frac{m_3}{<0.5,0.4,0.8>},\frac{m_4}{<0.7,0.2,0.9>}\right\}\right),\\
\left((\mu_2,t,0),\left\{\frac{m_1}{<0.7,0.2,0.5>},\frac{m_2}{<0.2,0.6,0.4>},\frac{m_3}{<0.9,0.1,0.6>},\frac{m_4}{<0.4,0.5,0.7>}\right\}\right),\\
\left((\mu_2,u,0),\left\{\frac{m_1}{<0.6,0.2,0.5>},\frac{m_2}{<0.7,0.2,0.4>},\frac{m_3}{<0.3,0.5,0.4>},\frac{m_4}{<0.2,0.7,0.6>}\right\}\right),\\
\left((\mu_3,s,0),\left\{\frac{m_1}{<0.1,0.7,0.5>},\frac{m_2}{<0.4,0.5,0.7>},\frac{m_3}{<0.7,0.2,0.9>},\frac{m_4}{<0.8,0.2,0.4>}\right\}\right),\\
\left((\mu_3,t,0),\left\{\frac{m_1}{<0.2,0.7,0.4>},\frac{m_2}{<0.9,0.1,0.6>},\frac{m_3}{<0.8,0.2,0.4>},\frac{m_4}{<0.3,0.5,0.7>}\right\}\right),\\
\left((\mu_3,u,0),\left\{\frac{m_1}{<0.5,0.4,0.2>},\frac{m_2}{<0.3,0.6,0.1>},\frac{m_3}{<0.6,0.3,0.2>},\frac{m_4}{<0.1,0.8,0.3>}\right\}\right),
\end{array} \right\}$$

Proposition 3.13. *If $(\xi,\mathbb{S})$ is a single valued neutrosophic hypersoft expert set over $\coprod$, then*

(1). $((\xi,\mathbb{S})^c)^c = (\xi,\mathbb{S})$

(2). $(\xi,\mathbb{S})^c_{ag} = (\xi,\mathbb{S})_{dag}$

(3). $(\xi,\mathbb{S})^c_{dag} = (\xi,\mathbb{S})_{ag}$

Definition 3.14. The union of $(\xi_1,\mathbb{S})$ and $(\xi_2,\mathbb{R})$ over $\coprod$ is (ξ_3, L) with $L = \mathbb{S}\cup\mathbb{R}$, defined as

$$\xi_3(\beta) = \begin{cases} \xi_1(\beta) & ; \beta \in \mathbb{S} - \mathbb{R} \\ \xi_2(\beta) & ; \beta \in \mathbb{R} - \mathbb{S} \\ \cup(\xi_1(\beta), \xi_2(\beta)) & ; \beta \in \mathbb{S} \cap \mathbb{R} \end{cases}$$

where $\cup(\xi_1(\beta), \xi_2(\beta)) = \{< u, \max\{\mu_1(\beta), \mu_2(\beta)\}, \min\{\nu_1(\beta), \nu_2(\beta)\}, \min\{\omega_1(\beta), \omega_2(\beta)\} >: u \in U\}$.

Example 3.15. Taking into consideration the concept of example 3.3, consider the following two sets

$$A_1 = \Big\{ (\mu_1, s, 1), (\mu_3, s, 0), (\mu_1, t, 1), (\mu_3, t, 1), (\mu_3, t, 0), (\mu_1, u, 0), (\mu_3, u, 1) \Big\}$$

$$A_2 = \Big\{ (\mu_1, s, 1), (\mu_3, s, 0), (\mu_3, s, 1), (\mu_1, t, 1), (\mu_3, t, 1), (\mu_1, u, 1), (\mu_3, t, 0), (\mu_1, u, 0), (\mu_3, u, 1), (\mu_1, t, 0) \Big\}$$

Suppose (ξ_1, A_1) and (ξ_2, A_2) over $\coprod$ are two single valued neutrosophic hypersoft expert sets such that

$$(\xi_1, \mathrm{A}_1) = \left\{ \begin{array}{l}
\left((\mu_1, s, 1), \left\{\frac{m_1}{<0.1,0.6,0.4>}, \frac{m_2}{<0.6,0.3,0.2>}, \frac{m_3}{<0.4,0.5,0.1>}, \frac{m_4}{<0.1,0.8,0.5>}\right\}\right), \\
\left((\mu_1, t, 1), \left\{\frac{m_1}{<0.3,0.4,0.5>}, \frac{m_2}{<0.6,0.2,0.3>}, \frac{m_3}{<0.2,0.5,0.6>}, \frac{m_4}{<0.1,0.5,0.3>}\right\}\right), \\
\left((\mu_3, t, 1), \left\{\frac{m_1}{<0.2,0.6,0.7>}, \frac{m_2}{<0.5,0.2,0.3>}, \frac{m_3}{<0.6,0.3,0.5>}, \frac{m_4}{<0.8,0.1,0.9>}\right\}\right), \\
\left((\mu_3, u, 1), \left\{\frac{m_1}{<0.6,0.2,0.4>}, \frac{m_2}{<0.2,0.7,0.6>}, \frac{m_3}{<0.4,0.3,0.5>}, \frac{m_4}{<0.1,0.5,0.4>}\right\}\right), \\
\left((\mu_1, u, 0), \left\{\frac{m_1}{<0.1,0.3,0.5>}, \frac{m_2}{<0.1,0.7,0.6>}, \frac{m_3}{<0.2,0.7,0.4>}, \frac{m_4}{<0.4,0.6,0.8>}\right\}\right), \\
\left((\mu_3, s, 0), \left\{\frac{m_1}{<0.1,0.6,0.9>}, \frac{m_2}{<0.3,0.6,0.7>}, \frac{m_3}{<0.6,0.1,0.2>}, \frac{m_4}{<0.7,0.2,0.3>}\right\}\right), \\
\left((\mu_3, t, 0), \left\{\frac{m_1}{<0.1,0.7,0.3>}, \frac{m_2}{<0.8,0.1,0.2>}, \frac{m_3}{<0.7,0.2,0.4>}, \frac{m_4}{<0.2,0.7,0.6>}\right\}\right)
\end{array} \right\}$$

$$(\xi_2, \mathrm{A}_2) = \left\{ \begin{array}{l}
\left((\mu_1, s, 1), \left\{\frac{m_1}{<0.2,0.3,0.4>}, \frac{m_2}{<0.7,0.4,0.5>}, \frac{m_3}{<0.5,0.4,0.6>}, \frac{m_4}{<0.2,0.4,0.7>}\right\}\right), \\
\left((\mu_1, t, 1), \left\{\frac{m_1}{<0.4,0.3,0.8>}, \frac{m_2}{<0.8,0.3,0.5>}, \frac{m_3}{<0.4,0.3,0.5>}, \frac{m_4}{<0.2,0.6,0.7>}\right\}\right), \\
\left((\mu_3, s, 1), \left\{\frac{m_1}{<0.1,0.3,0.6>}, \frac{m_2}{<0.9,0.1,0.7>}, \frac{m_3}{<0.4,0.5,0.8>}, \frac{m_4}{<0.5,0.3,0.5>}\right\}\right), \\
\left((\mu_3, t, 1), \left\{\frac{m_1}{<0.4,0.2,0.3>}, \frac{m_2}{<0.6,0.3,0.5>}, \frac{m_3}{<0.7,0.4,0.5>}, \frac{m_4}{<0.9,0.5,0.7}\right\}\right), \\
\left((\mu_1, u, 1), \left\{\frac{m_1}{<0.7,0.2,0.3>}, \frac{m_2}{<0.5,0.2,0.4>}, \frac{m_3}{<0.6,0.2,0.4>}, \frac{m_4}{<0.3,0.5,0.6}\right\}\right), \\
\left((\mu_3, u, 1), \left\{\frac{m_1}{<0.7,0.3,0.7>}, \frac{m_2}{<0.3,0.5,0.6>}, \frac{m_3}{<0.5,0.4,0.3>}, \frac{m_4}{<0.2,0.6,0.4>}\right\}\right), \\
\left((\mu_1, u, 0), \left\{\frac{m_1}{<0.2,0.5,0.4>}, \frac{m_2}{<0.2,0.6,0.3>}, \frac{m_3}{<0.3,0.5,0.6>}, \frac{m_4}{<0.5,0.3,0.7>}\right\}\right), \\
\left((\mu_1, t, 0), \left\{\frac{m_1}{<0.1,0.6,0.3>}, \frac{m_2}{<0.9,0.1,0.2>}, \frac{m_3}{<0.6,0.3,0.4>}, \frac{m_4}{<0.2,0.6,0.3>}\right\}\right), \\
\left((\mu_3, s, 0), \left\{\frac{m_1}{<0.2,0.7,0.5>}, \frac{m_2}{<0.4,0.5,0.6>}, \frac{m_3}{<0.7,0.2,0.3>}, \frac{m_4}{<0.8,0.1,0.4>}\right\}\right), \\
\left((\mu_3, t, 0), \left\{\frac{m_1}{<0.2,0.5,0.4>}, \frac{m_2}{<0.7,0.2,0.3>}, \frac{m_3}{<0.8,0.2,0.6>}, \frac{m_4}{<0.3,0.5,0.7>}\right\}\right)
\end{array} \right\}$$

Then $(\xi_1, A_1) \cup (\xi_2, A_2) = (\xi_3, A_3)$

$$(\xi_3, \mathrm{A}_3) = \left\{ \begin{array}{l}
\left((\mu_1, s, 1), \left\{ \frac{m_1}{<0.2,0.3,0.4>}, \frac{m_2}{<0.7,0.3,0.2>}, \frac{m_3}{<0.5,0.4,0.1>}, \frac{m_4}{<0.2,0.4,0.5>} \right\} \right), \\
\left((\mu_1, t, 1), \left\{ \frac{m_1}{<0.4,0.3,0.5>}, \frac{m_2}{<0.8,0.2,0.3>}, \frac{m_3}{<0.4,0.3,0.5>}, \frac{m_4}{<0.2,0.5,0.3>} \right\} \right), \\
\left((\mu_3, s, 1), \left\{ \frac{m_1}{<0.1,0.3,0.6>}, \frac{m_2}{<0.9,0.1,0.7>}, \frac{m_3}{<0.4,0.5,0.8>}, \frac{m_4}{<0.5,0.3,0.5>} \right\} \right), \\
\left((\mu_3, t, 1), \left\{ \frac{m_1}{<0.4,0.2,0.3>}, \frac{m_2}{<0.6,0.2,0.3>}, \frac{m_3}{<0.7,0.3,0.5>}, \frac{m_4}{<0.9,0.1,0.7} \right\} \right), \\
\left((\mu_1, u, 1), \left\{ \frac{m_1}{<0.7,0.2,0.3>}, \frac{m_2}{<0.5,0.2,0.4>}, \frac{m_3}{<0.6,0.2,0.4>}, \frac{m_4}{<0.3,0.5,0.6} \right\} \right), \\
\left((\mu_3, u, 1), \left\{ \frac{m_1}{<0.7,0.2,0.4>}, \frac{m_2}{<0.3,0.5,0.6>}, \frac{m_3}{<0.5,0.3,0.3>}, \frac{m_4}{<0.2,0.5,0.4>} \right\} \right), \\
\left((\mu_1, u, 0), \left\{ \frac{m_1}{<0.2,0.3,0.4>}, \frac{m_2}{<0.2,0.6,0.3>}, \frac{m_3}{<0.3,0.5,0.4>}, \frac{m_4}{<0.5,0.3,0.7>} \right\} \right), \\
\left((\mu_1, t, 0), \left\{ \frac{m_1}{<0.1,0.6,0.3>}, \frac{m_2}{<0.9,0.1,0.2>}, \frac{m_3}{<0.6,0.3,0.4>}, \frac{m_4}{<0.2,0.6,0.3>} \right\} \right), \\
\left((\mu_3, s, 0), \left\{ \frac{m_1}{<0.2,0.6,0.5>}, \frac{m_2}{<0.4,0.5,0.6>}, \frac{m_3}{<0.7,0.1,0.2>}, \frac{m_4}{<0.8,0.1,0.3>} \right\} \right), \\
\left((\mu_3, t, 0), \left\{ \frac{m_1}{<0.2,0.5,0.3>}, \frac{m_2}{<0.8,0.1,0.2>}, \frac{m_3}{<0.8,0.2,0.4>}, \frac{m_4}{<0.3,0.5,0.6>} \right\} \right)
\end{array} \right\}$$

Proposition 3.16. *If (ξ_1, A_1),(ξ_2, A_2) and (ξ_3, A_3) are three single valued neutrosophic hypersoft expert sets over $\coprod$, then*

(1). $(\xi_1, A_1) \cup (\xi_2, A_2) = (\xi_2, A_2) \cup (\xi_1, A_1)$

(2). $((\xi_1, A_1) \cup (\xi_2, A_2)) \cup (\xi_3, A_3) = (\xi_1, A_1) \cup ((\xi_2, A_2) \cup (\xi_3, A_3))$

Definition 3.17. The intersection of $(\xi_1, \mathbb{S})$ and $(\xi_2, \mathbb{R})$ over $\coprod$ is (ξ_3, L) with $L = \mathbb{S} \cap \mathbb{R}$, defined as

$$\xi_3(\beta) = \left\{ \begin{array}{cl} \xi_1(\beta) & ; \beta \in \mathbb{S} - \mathbb{R} \\ \xi_2(\beta) & ; \beta \in \mathbb{R} - \mathbb{S} \\ \cap(\xi_1(\beta), \xi_2(\beta)) & ; \beta \in \mathbb{S} \cap \mathbb{R} \end{array} \right.$$

where $\cap(\xi_1(\beta), \xi_2(\beta)) = \{< u, \min\{\mu_1(\beta), \mu_2(\beta)\}, \max\{\nu_1(\beta), \nu_2(\beta)\}, \max\{\omega_1(\beta), \omega_2(\beta)\} >: u \in U\}$.

Example 3.18. Taking into consideration the concept of example 3.3, consider the following two sets

$$A_1 = \Big\{ (\mu_1, s, 1), (\mu_3, s, 0), (\mu_1, t, 1), (\mu_3, t, 1), (\mu_3, t, 0), (\mu_1, u, 0), (\mu_3, u, 1) \Big\}$$

$$A_2 = \Big\{ (\mu_1, s, 1), (\mu_3, s, 0), (\mu_3, s, 1), (\mu_1, t, 1), (\mu_3, t, 1), (\mu_1, t, 0), (\mu_3, t, 0), (\mu_1, u, 0), (\mu_3, u, 1), , (\mu_1, u, 1) \Big\}$$

Suppose (ξ_1, A_1) and (ξ_2, A_2) over $\coprod$ are two single valued neutrosophic hypersoft expert sets such that

$$(\xi_1, \mathrm{A}_1) = \left\{ \begin{array}{l} \left((\mu_1, s, 1), \left\{ \frac{m_1}{<0.1,0.6,0.4>}, \frac{m_2}{<0.6,0.3,0.2>}, \frac{m_3}{<0.4,0.5,0.1>}, \frac{m_4}{<0.1,0.8,0.5>} \right\}\right), \\ \left((\mu_1, t, 1), \left\{ \frac{m_1}{<0.3,0.4,0.5>}, \frac{m_2}{<0.6,0.2,0.3>}, \frac{m_3}{<0.2,0.5,0.6>}, \frac{m_4}{<0.1,0.5,0.3>} \right\}\right), \\ \left((\mu_3, t, 1), \left\{ \frac{m_1}{<0.2,0.6,0.7>}, \frac{m_2}{<0.5,0.2,0.3>}, \frac{m_3}{<0.6,0.3,0.5>}, \frac{m_4}{<0.8,0.1,0.9>} \right\}\right), \\ \left((\mu_3, u, 1), \left\{ \frac{m_1}{<0.6,0.2,0.4>}, \frac{m_2}{<0.2,0.7,0.6>}, \frac{m_3}{<0.4,0.3,0.5>}, \frac{m_4}{<0.1,0.5,0.4>} \right\}\right), \\ \left((\mu_1, u, 0), \left\{ \frac{m_1}{<0.1,0.3,0.5>}, \frac{m_2}{<0.1,0.7,0.6>}, \frac{m_3}{<0.2,0.7,0.4>}, \frac{m_4}{<0.4,0.6,0.8>} \right\}\right), \\ \left((\mu_3, s, 0), \left\{ \frac{m_1}{<0.1,0.6,0.9>}, \frac{m_2}{<0.3,0.6,0.7>}, \frac{m_3}{<0.6,0.1,0.2>}, \frac{m_4}{<0.7,0.2,0.3>} \right\}\right), \\ \left((\mu_3, t, 0), \left\{ \frac{m_1}{<0.1,0.7,0.3>}, \frac{m_2}{<0.8,0.1,0.2>}, \frac{m_3}{<0.7,0.2,0.4>}, \frac{m_4}{<0.2,0.7,0.6>} \right\}\right) \end{array} \right\}$$

$$(\xi_2, \mathrm{A}_2) = \left\{ \begin{array}{l} \left((\mu_1, s, 1), \left\{ \frac{m_1}{<0.2,0.3,0.4>}, \frac{m_2}{<0.7,0.4,0.5>}, \frac{m_3}{<0.5,0.4,0.6>}, \frac{m_4}{<0.2,0.4,0.7>} \right\}\right), \\ \left((\mu_1, t, 1), \left\{ \frac{m_1}{<0.4,0.3,0.8>}, \frac{m_2}{<0.8,0.3,0.5>}, \frac{m_3}{<0.4,0.3,0.5>}, \frac{m_4}{<0.2,0.6,0.7>} \right\}\right), \\ \left((\mu_3, s, 1), \left\{ \frac{m_1}{<0.1,0.3,0.6>}, \frac{m_2}{<0.9,0.1,0.7>}, \frac{m_3}{<0.4,0.5,0.8>}, \frac{m_4}{<0.5,0.3,0.5>} \right\}\right), \\ \left((\mu_3, t, 1), \left\{ \frac{m_1}{<0.4,0.2,0.3>}, \frac{m_2}{<0.6,0.3,0.5>}, \frac{m_3}{<0.7,0.4,0.5>}, \frac{m_4}{<0.9,0.5,0.7} \right\}\right), \\ \left((\mu_1, u, 1), \left\{ \frac{m_1}{<0.7,0.2,0.3>}, \frac{m_2}{<0.5,0.2,0.4>}, \frac{m_3}{<0.6,0.2,0.4>}, \frac{m_4}{<0.3,0.5,0.6} \right\}\right), \\ \left((\mu_3, u, 1), \left\{ \frac{m_1}{<0.7,0.3,0.7>}, \frac{m_2}{<0.3,0.5,0.6>}, \frac{m_3}{<0.5,0.4,0.3>}, \frac{m_4}{<0.2,0.6,0.4>} \right\}\right), \\ \left((\mu_1, u, 0), \left\{ \frac{m_1}{<0.2,0.5,0.4>}, \frac{m_2}{<0.2,0.6,0.3>}, \frac{m_3}{<0.3,0.5,0.6>}, \frac{m_4}{<0.5,0.3,0.7>} \right\}\right), \\ \left((\mu_1, t, 0), \left\{ \frac{m_1}{<0.1,0.6,0.3>}, \frac{m_2}{<0.9,0.1,0.2>}, \frac{m_3}{<0.6,0.3,0.4>}, \frac{m_4}{<0.2,0.6,0.3>} \right\}\right), \\ \left((\mu_3, s, 0), \left\{ \frac{m_1}{<0.2,0.7,0.5>}, \frac{m_2}{<0.4,0.5,0.6>}, \frac{m_3}{<0.7,0.2,0.3>}, \frac{m_4}{<0.8,0.1,0.4>} \right\}\right), \\ \left((\mu_3, t, 0), \left\{ \frac{m_1}{<0.2,0.5,0.4>}, \frac{m_2}{<0.7,0.2,0.3>}, \frac{m_3}{<0.8,0.2,0.6>}, \frac{m_4}{<0.3,0.5,0.7>} \right\}\right) \end{array} \right\}$$

Then $(\xi_1, A_1) \cap (\xi_2, A_2) = (\xi_3, A_3)$

$$(\xi_3, \mathrm{A}_3) = \left\{ \begin{array}{l} \left((\mu_1, s, 1), \left\{ \frac{m_1}{<0.1,0.6,0.4>}, \frac{m_2}{<0.6,0.4,0.5>}, \frac{m_3}{<0.4,0.5,0.6>}, \frac{m_4}{<0.1,0.8,0.7>} \right\}\right), \\ \left((\mu_1, t, 1), \left\{ \frac{m_1}{<0.3,0.4,0.8>}, \frac{m_2}{<0.6,0.3,0.5>}, \frac{m_3}{<0.2,0.5,0.6>}, \frac{m_4}{<0.1,0.6,0.7>} \right\}\right), \\ \left((\mu_3, t, 1), \left\{ \frac{m_1}{<0.2,0.6,0.7>}, \frac{m_2}{<0.5,0.4,0.5>}, \frac{m_3}{<0.6,0.4,0.5>}, \frac{m_4}{<0.8,0.1,0.7>} \right\}\right), \\ \left((\mu_3, u, 1), \left\{ \frac{m_1}{<0.6,0.3,0.7>}, \frac{m_2}{<0.2,0.7,0.6>}, \frac{m_3}{<0.4,0.4,0.5>}, \frac{m_4}{<0.1,0.6,0.4>} \right\}\right), \\ \left((\mu_1, u, 0), \left\{ \frac{m_1}{<0.1,0.5,0.5>}, \frac{m_2}{<0.1,0.6,0.6>}, \frac{m_3}{<0.2,0.7,0.6>}, \frac{m_4}{<0.4,0.6,0.8>} \right\}\right), \\ \left((\mu_3, s, 0), \left\{ \frac{m_1}{<0.1,0.7,0.9>}, \frac{m_2}{<0.3,0.6,0.7>}, \frac{m_3}{<0.6,0.2,0.3>}, \frac{m_4}{<0.7,0.2,0.4>} \right\}\right), \\ \left((\mu_3, t, 0), \left\{ \frac{m_1}{<0.1,0.7,0.4>}, \frac{m_2}{<0.8,0.2,0.3>}, \frac{m_3}{<0.7,0.2,0.6>}, \frac{m_4}{<0.2,0.7,0.7>} \right\}\right) \end{array} \right\}$$

Proposition 3.19. *If (ξ_1, A_1),(ξ_2, A_2) and (ξ_3, A_3) are three single valued neutrosophic hypersoft expert sets over $\coprod$, then*

(1). $(\xi_1, A_1) \cap (\xi_2, A_2) = (\xi_2, A_2) \cap (\xi_1, A_1)$

(2). $((\xi_1, A_1) \cap (\xi_2, A_2)) \cap (\xi_3, A_3) = (\xi_1, A_1) \cap ((\xi_2, A_2) \cap (\xi_3, A_3))$

Proposition 3.20. *If (ξ_1, A_1),(ξ_2, A_2) and (ξ_3, A_3) are three single valued neutrosophic hypersoft expert sets over $\coprod$, then*

$$(1).\ (\xi_1, A_1) \cup ((\xi_2, A_2) \cap (\xi_3, A_3)) = ((\xi_1, A_1) \cup ((\xi_2, A_2)) \cap ((\xi_1, A_1) \cup (\xi_3, A_3))$$
$$(2).\ (\xi_1, A_1) \cap ((\xi_2, A_2) \cup (\xi_3, A_3)) = ((\xi_1, A_1) \cap ((\xi_2, A_2)) \cup ((\xi_1, A_1) \cap (\xi_3, A_3))$$

Definition 3.21. If (ξ_1, A_1) and (ξ_2, A_2) are two single valued neutrosophic hypersoft expert sets over $\coprod$ then (ξ_1, A_1) AND (ξ_2, A_2) denoted by $(\xi_1, A_1) \wedge (\xi_2, A_2)$ is defined by
$(\xi_1, A_1) \wedge (\xi_2, A_2) = (\xi_3, A_1 \times A_2)$,
while $\xi_3(\beta, \gamma) = \xi_1(\beta) \cap \xi_2(\gamma), \forall (\beta, \gamma) \in A_1 \times A_2$.

Example 3.22. Taking into consideration the concept of example 3.3, let two sets

$$A_1 = \left\{ (\mu_1, s, 1), (\mu_1, t, 1), (\mu_3, s, 0) \right\}$$

$$A_2 = \left\{ (\mu_1, s, 0), (\mu_3, s, 1) \right\}$$

Suppose (ξ_1, A_1) and (ξ_2, A_2) over $\coprod$ are two single valued neutrosophic hypersoft expert sets such that

$$(\xi_1, \mathrm{A}_1) = \left\{ \begin{array}{l} \left((\mu_1, s, 1), \left\{ \frac{m_1}{<0.1,0.6,0.4>}, \frac{m_2}{<0.6,0.4,0.5>}, \frac{m_3}{<0.4,0.5,0.6>}, \frac{m_4}{<0.1,0.8,0.7>} \right\}\right), \\ \left((\mu_1, t, 1), \left\{ \frac{m_1}{<0.3,0.4,0.8>}, \frac{m_2}{<0.6,0.3,0.5>}, \frac{m_3}{<0.2,0.5,0.6>}, \frac{m_4}{<0.1,0.6,0.7>} \right\}\right), \\ \left((\mu_3, s, 0), \left\{ \frac{m_1}{<0.1,0.6,0.9>}, \frac{m_2}{<0.3,0.6,0.7>}, \frac{m_3}{<0.6,0.1,0.2>}, \frac{m_4}{<0.7,0.2,0.3>} \right\}\right) \end{array} \right\}$$

$$(\xi_2, \mathrm{A}_2) = \left\{ \begin{array}{l} \left((\mu_1, s, 0), \left\{ \frac{m_1}{<0.2,0.1,0.3>}, \frac{m_2}{<0.7,0.2,0.4>}, \frac{m_3}{<0.5,0.2,0.5>}, \frac{m_4}{<0.2,0.3,0.6>} \right\}\right), \\ \left((\mu_3, s, 1), \left\{ \frac{m_1}{<0.1,0.5,0.6>}, \frac{m_2}{<0.4,0.2,0.5>}, \frac{m_3}{<0.7,0.1,0.2>}, \frac{m_4}{<0.8,0.1,0.4>} \right\}\right), \end{array} \right\}$$

Then $(\xi_3, A_3) \wedge (\xi_2, A_2) = (\xi_3, A_1 \times A_2)$,

$$(\xi_3, \mathrm{A}_1 \times \mathrm{A}_2) = \left\{ \begin{array}{l} \left(((\mu_1, s, 1), (\mu_1, s, 0)), \left\{ \frac{m_1}{<0.1,0.6,0.4>}, \frac{m_2}{<0.6,0.4,0.5>}, \frac{m_3}{<0.4,0.5,0.6>}, \frac{m_4}{<0.1,0.8,0.7>} \right\}\right), \\ \left(((\mu_1, t, 1), (\mu_1, s, 0)), \left\{ \frac{m_1}{<0.2,0.4,0.8>}, \frac{m_2}{<0.6,0.3,0.5>}, \frac{m_3}{<0.2,0.5,0.6>}, \frac{m_4}{<0.1,0.6,0.7>} \right\}\right), \\ \left(((\mu_1, t, 1), (\mu_3, s, 1)), \left\{ \frac{m_1}{<0.1,0.5,0.8>}, \frac{m_2}{<0.4,0.3,0.5>}, \frac{m_3}{<0.2,0.5,0.6>}, \frac{m_4}{<0.1,0.6,0.7>} \right\}\right), \\ \left(((\mu_1, s, 1), (\mu_3, s, 1)), \left\{ \frac{m_1}{<0.1,0.6,0.6>}, \frac{m_2}{<0.4,0.4,0.5>}, \frac{m_3}{<0.4,0.5,0.6>}, \frac{m_4}{<0.1,0.8,0.7>} \right\}\right), \\ \left(((\mu_3, s, 0), (\mu_1, s, 0)), \left\{ \frac{m_1}{<0.1,0.6,0.9>}, \frac{m_2}{<0.3,0.6,0.7>}, \frac{m_3}{<0.5,0.2,0.5>}, \frac{m_4}{<0.2,0.3,0.6>} \right\}\right), \\ \left(((\mu_3, s, 0), (\mu_3, s, 1)), \left\{ \frac{m_1}{<0.1,0.6,0.9>}, \frac{m_2}{<0.3,0.6,0.7>}, \frac{m_3}{<0.6,0.1,0.2>}, \frac{m_4}{<0.7,0.2,0.4>} \right\}\right) \end{array} \right\}$$

Definition 3.23. If (ξ_1, A_1) and (ξ_2, A_2) are two single valued neutrosophic hypersoft expert sets over $\coprod$ then (ξ_1, A_1) OR (ξ_2, A_2) denoted by $(\xi_1, A_1) \vee (\xi_2, A_2)$ is defined by
$(\xi_1, A_1) \vee (\xi_2, A_2) = (\xi_3, A_1 \times A_2)$,
while $\xi_3(\beta, \gamma) = \xi_1(\beta) \cup \xi_2(\gamma), \forall (\beta, \gamma) \in A_1 \times A_2$.

Example 3.24. Taking into consideration the concept of example 3.3, suppose the following sets

$$A_1 = \left\{ (\mu_1, s, 1), (\mu_1, t, 1), (\mu_3, s, 0) \right\}$$

$$A_2 = \left\{ (\mu_1, s, 0), (\mu_3, s, 1) \right\}$$

Suppose (ξ_1, A_1) and (ξ_2, A_2) over $\coprod$ are two single valued neutrosophic hypersoft expert sets such that

$$(\xi_1, \mathrm{A}_1) = \left\{ \begin{array}{l} \left((\mu_1, s, 1), \left\{ \frac{m_1}{<0.1,0.6,0.4>}, \frac{m_2}{<0.6,0.4,0.5>}, \frac{m_3}{<0.4,0.5,0.6>}, \frac{m_4}{<0.1,0.8,0.7>} \right\} \right), \\ \left((\mu_1, t, 1), \left\{ \frac{m_1}{<0.3,0.4,0.8>}, \frac{m_2}{<0.6,0.3,0.5>}, \frac{m_3}{<0.2,0.5,0.6>}, \frac{m_4}{<0.1,0.6,0.7>} \right\} \right), \\ \left((\mu_3, s, 0), \left\{ \frac{m_1}{<0.1,0.6,0.9>}, \frac{m_2}{<0.3,0.6,0.7>}, \frac{m_3}{<0.6,0.1,0.2>}, \frac{m_4}{<0.7,0.2,0.3>} \right\} \right) \end{array} \right\}$$

$$(\xi_2, \mathrm{A}_2) = \left\{ \begin{array}{l} \left((\mu_1, s, 0), \left\{ \frac{m_1}{<0.2,0.1,0.3>}, \frac{m_2}{<0.7,0.2,0.4>}, \frac{m_3}{<0.5,0.2,0.5>}, \frac{m_4}{<0.2,0.3,0.6>} \right\} \right), \\ \left((\mu_3, s, 1), \left\{ \frac{m_1}{<0.1,0.5,0.6>}, \frac{m_2}{<0.4,0.2,0.5>}, \frac{m_3}{<0.7,0.1,0.2>}, \frac{m_4}{<0.8,0.1,0.4>} \right\} \right), \end{array} \right\}$$

Then $(\xi_3, A_3) \vee (\xi_2, A_2) = (\xi_3, A_1 \times A_2)$,

$$(\xi_3, \mathrm{A}_1 \times \mathrm{A}_2) = \left\{ \begin{array}{l} \left(((\mu_1, s, 1), (\mu_1, s, 0)), \left\{ \frac{m_1}{<0.2,0.1,0.3>}, \frac{m_2}{0<0.7,0.2,0.4>}, \frac{m_3}{<0.5,0.2,0.5>}, \frac{m_4}{<0.2,0.3,0.6>} \right\} \right), \\ \left(((\mu_1, t, 1), (\mu_1, s, 0)), \left\{ \frac{m_1}{<0.3,0.1,0.3>}, \frac{m_2}{<0.7,0.2,0.4>}, \frac{m_3}{<0.5,0.2,0.5>}, \frac{m_4}{<0.2,0.3,0.6>} \right\} \right), \\ \left(((\mu_1, t, 1), (\mu_3, s, 1)), \left\{ \frac{m_1}{<0.3,0.4,0.6>}, \frac{m_2}{<0.6,0.2,0.5>}, \frac{m_3}{<0.7,0.1,0.2>}, \frac{m_4}{<0.8,0.1,0.4>} \right\} \right), \\ \left(((\mu_1, s, 1), (\mu_3, s, 1)), \left\{ \frac{m_1}{<0.1,0.5,0.4>}, \frac{m_2}{<0.6,0.2,0.5>}, \frac{m_3}{<0.7,0.1,0.2>}, \frac{m_4}{<0.8,0.1,0.4>} \right\} \right), \\ \left(((\mu_3, s, 0), (\mu_1, s, 0)), \left\{ \frac{m_1}{<0.2,0.1,0.3>}, \frac{m_2}{<0.7,0.2,0.4>}, \frac{m_3}{<0.6,0.1,0.2>}, \frac{m_4}{<0.7,0.2,0.3>} \right\} \right), \\ \left(((\mu_3, s, 0), (\mu_3, s, 1)), \left\{ \frac{m_1}{<0.1,0.5,0.6>}, \frac{m_2}{<0.4,0.2,0.5>}, \frac{m_3}{<0.7,0.1,0.2>}, \frac{m_4}{<0.8,0.1,0.4>} \right\} \right) \end{array} \right\}$$

Proposition 3.25. *If (ξ_1, A_1),(ξ_2, A_2) and (ξ_3, A_3) are three single valued neutrosophic hypersoft expert sets over $\coprod$, then*

(1). $((\xi_1, A_1) \wedge (\xi_2, A_2))^c = ((\xi_1, A_1))^c \vee ((\xi_2, A_2))^c$

(2). $((\xi_1, A_1) \vee (\xi_2, A_2))^c = ((\xi_1, A_1))^c \wedge ((\xi_2, A_2))^c$

Proposition 3.26. *If (ξ_1, A_1),(ξ_2, A_2) and (ξ_3, A_3) are three single valued neutrosophic hypersoft expert sets over $\coprod$, then*

(1). $((\xi_1, A_1) \wedge (\xi_2, A_2)) \wedge (\xi_3, A_3) = (\xi_1, A_1) \wedge ((\xi_2, A_2) \wedge (\xi_3, A_3))$

(2). $((\xi_1, A_1) \vee (\xi_2, A_2)) \vee (\xi_3, A_3) = (\xi_1, A_1) \vee ((\xi_2, A_2) \vee (\xi_3, A_3))$

(3). $(\xi_1, A_1) \vee ((\xi_2, A_2) \wedge (\xi_3, A_3) = ((\xi_1, A_1) \vee ((\xi_2, A_2)) \wedge ((\xi_1, A_1) \vee (\xi_3, A_3))$

(4). $(\xi_1, A_1) \wedge ((\xi_2, A_2) \vee (\xi_3, A_3)) = ((\xi_1, A_1) \wedge ((\xi_2, A_2)) \vee ((\xi_1, A_1) \wedge (\xi_3, A_3))$

4. An Application to Single valued Neutrosophic Hypersoft expert set

In this section, an application of single valued neutrosophic hypersoft expert set theory in a decision making problem, is presented.

Statement of the problem

Mr. John wants to purchase a mobile from a mobile market for his personal use. He takes help from his some friends (Stephen, Thomas and Umar) who have expertise in mobile purchase.

Proposed Algorithm

The following algorithm is adopted for this selection (purchase).

(1). Construct SVNHSES (ξ, K),

(2). Determine the values of $\mu(c_i)-\nu(c_i)-\omega(c_i)$ for each $c_i \in \coprod$ where $\mu(c_i)$ is a membership function, $\nu(c_i)$indeterminacy function and $\omega(c_i)$ is a non membership function for each element of $\coprod$.

(3). Calculate the the highest numerical grade for the agree-SVNHSES and disagree-SVNHSES,

(4). Determine the score of each element $c_i \in \coprod$ by taking the sum of the products of the numerical grade of each element for the agree- SVNHSES and disagree SVNHSES,denoted by G_i and H_i respectively

(5). Determine $j_i = G_i - H_i$ for each element $c_i \in \coprod$,

(6). Compute n, for which $M=$ max j_i. Then the decision is to choose element as the optimal or best solution to the problem.

Step-1

Let four categories of mobile are there which form the universe of discourse $coprod = \{c_1, c_2, c_3, c_4\}$ and X $= \{E_1 = Stephen, E_2 = Thomas, E_3 = Umar\}$ be a set of experts for this purchase. The following are the attribute-valued sets for prescribed attributes:

$L_1 = Brand = \{X = l_1, Y = l_2\}$

$L_2 = Price = \{20,000 = l_3, 15,000 = l_4\}$

$L_3 = Colour = \{White = l_5, Blue = l_6\}$

$L_4 = Memory = \{6GB = l_7, 4GB = l_8\}$

$L_5 = Resolution(size) = \{5inch = l_9, 6inch = l_{10}\}$

and then

$L = L_1 \times L_2 \times L_3 \times L_4 \times L_5$

$$L = \left\{ \begin{array}{l} (l_1, l_3, l_5, l_7, l_9), (l_1, l_3, l_5, l_7, l_{10}), (l_1, l_3, l_5, l_8, l_9), (l_1, l_3, l_5, l_8, l_{10}), (l_1, l_3, l_6, l_7, l_9), \\ (l_1, l_3, l_6, l_7, l_{10}), (l_1, l_3, l_6, l_8, l_9), (l_1, l_3, l_6, l_8, l_{10}), (l_1, l_4, l_5, l_7, l_9), (l_1, l_4, l_5, l_7, l_{10}), \\ (l_1, l_4, l_5, l_8, l_9), (l_1, l_4, l_5, l_8, l_{10}), (l_1, l_4, l_6, l_7, l_9), (l_1, l_4, l_6, l_7, l_{10}), (l_1, l_4, l_6, l_8, l_9), \\ (l_1, l_4, l_6, l_8, l_{10}), (l_2, l_3, l_5, l_7, l_9), (l_2, l_3, l_5, l_7, l_{10}), (l_2, l_3, l_5, l_8, l_9), (l_2, l_3, l_5, l_8, l_{10}), \\ (l_2, l_3, l_6, l_7, l_9), (l_2, l_3, l_6, l_7, l_{10}), (l_2, l_3, l_6, l_8, l_9), (l_2, l_3, l_6, l_8, l_{10}), (l_2, l_4, l_5, l_7, l_9), \\ (l_2, l_4, l_5, l_7, l_{10}), (l_2, l_4, l_5, l_8, l_9), (l_2, l_4, l_5, l_8, l_{10}), (l_2, l_4, l_6, l_7, l_9), (l_2, l_4, l_6, l_7, l_{10}), \\ (l_2, l_4, l_6, l_8, l_9), (l_2, l_4, l_6, l_8, l_{10}) \end{array} \right\}$$

Now take $K \subseteq L$ as

$K = \{k_1 = (l_1, l_3, l_5, l_7, l_9), k_2 = (l_1, l_3, l_6, l_7, l_{10}), k_3 = (l_1, l_4, l_6, l_8, l_9), k_4 = (l_2, l_3, l_6, l_8, l_9), k_5 = (l_2, l_4, l_6, l_7, l_{10})\}$

$$(\xi, \mathrm{A})_1 = \left\{ \begin{array}{l} \left((k_1, E_1, 1), \left\{\frac{c_1}{<0.9,0.1,0.7>}, \frac{c_2}{<0.3,0.4,0.5>}, \frac{c_3}{<0.6,0.2,0.4>}, \frac{c_4}{<0.3,0.7,0.2>}\right\}\right), \\ \left((k_1, E_2, 1), \left\{\frac{c_1}{<0.8,0.2,0.7>}, \frac{c_2}{<0.1,0.2,0.3>}, \frac{c_3}{<0.6,0.2,0.8>}, \frac{c_4}{<0.3,0.6,0.5>}\right\}\right), \\ \left((k_1, E_3, 1), \left\{\frac{c_1}{<0.7,0.3,0.1>}, \frac{c_2}{<0.3,0.7,0.2>}, \frac{c_3}{<0.3,0.1,0.4>}, \frac{c_4}{<0.3,0.6,0.7>}\right\}\right), \\ \left((k_2, E_1, 1), \left\{\frac{c_1}{<0.6,0.4,0.8>}, \frac{c_2}{<0.4,0.2,0.1>}, \frac{c_3}{<0.7,0.1,0.6>}, \frac{c_4}{<0.5,0.2,0.6>}\right\}\right), \\ \left((k_2, E_2, 1), \left\{\frac{c_1}{<0.5,0.2,0.3>}, \frac{c_2}{<0.6,0.4,0.5>}, \frac{c_3}{<0.3,0.6,0.7>}, \frac{c_4}{<0.4,0.2,0.9>}\right\}\right), \\ \left((k_2, E_3, 1), \left\{\frac{c_1}{<0.4,0.3,0.1>}, \frac{c_2}{<0.3,0.2,0.4>}, \frac{c_3}{<0.3,0.2,0.5>}, \frac{c_4}{<0.8,0.2,0.7>}\right\}\right), \\ \left((k_3, E_1, 1), \left\{\frac{c_1}{<0.2,0.4,0.9>}, \frac{c_2}{<0.5,0.2,0.6>}, \frac{c_3}{<0.6,0.2,0.7>}, \frac{c_4}{<0.7,0.2,0.6>}\right\}\right), \\ \left((k_3, E_2, 1), \left\{\frac{c_1}{<0.2,0.3,0.4>}, \frac{c_2}{<0.4,0.2,0.5>}, \frac{c_3}{<0.6,0.2,0.4>}, \frac{c_4}{<0.3,0.2,0.3>}\right\}\right), \\ \left((k_3, E_3, 1), \left\{\frac{c_1}{<0.3,0.4,0.1>}, \frac{c_2}{<0.6,0.3,0.2>}, \frac{c_3}{<0.4,0.2,0.3>}, \frac{c_4}{<0.3,0.6,0.4>}\right\}\right), \\ \left((k_4, E_1, 1), \left\{\frac{c_1}{<0.9,0.1,0.7>}, \frac{c_2}{<0.1,0.3,0.8>}, \frac{c_3}{<0.5,0.1,0.7>}, \frac{c_4}{<0.4,0.3,0.2>}\right\}\right), \\ \left((k_4, E_2, 1), \left\{\frac{c_1}{<0.8,0.1,0.4>}, \frac{c_2}{<0.4,0.3,0.5>}, \frac{c_3}{<0.2,0.7,0.6>}, \frac{c_4}{<0.7,0.2,0.7>}\right\}\right), \\ \left((k_4, E_3, 1), \left\{\frac{c_1}{<0.6,0.2,0.4>}, \frac{c_2}{<0.1,0.3,0.2>}, \frac{c_3}{<0.3,0.5,0.9>}, \frac{c_4}{<0.6,0.1,0.3>}\right\}\right), \\ \left((k_5, E_1, 1), \left\{\frac{c_1}{<0.6,0.3,0.2>}, \frac{c_2}{<0.2,0.8,0.4>}, \frac{c_3}{<0.1,0.2,.03>}, \frac{c_4}{<0.1,0.7,0.4>}\right\}\right), \\ \left((k_5, E_2, 1), \left\{\frac{c_1}{<0.5,0.3,0.4>}, \frac{c_2}{<0.6,0.2,0.4>}, \frac{c_3}{<0.4,0.3,0.5>}, \frac{c_4}{<0.6,0.1,0.2>}\right\}\right), \\ \left((k_5, E_3, 1), \left\{\frac{c_1}{<0.4,0.3,0.1>}, \frac{c_2}{<0.6,0.1,0.2>}, \frac{c_3}{<0.3,0.5,0.6>}, \frac{c_4}{<0.5,0.4,0.2>}\right\}\right), \end{array} \right\}$$

and

$$(\xi, \mathrm{K})_0 = \left\{ \begin{array}{l} \left((k_1, E_1, 0), \left\{\frac{c_1}{<0.4,0.3,0.1>}, \frac{c_2}{<0.9,0.1,0.2>}, \frac{c_3}{<0.8,0.2,0.3>}, \frac{c_4}{<0.6,0.1,0.8>}\right\}\right), \\ \left((k_1, E_2, 0) \left\{\frac{c_1}{<0.7,0.2,0.3>}, \frac{c_2}{<0.6,0.4,0.2>}, \frac{c_3}{<0.2,0.7,0.1>}, \frac{c_4}{<0.8,0.2,0.7>}\right\}\right), \\ \left((k_1, E_3, 0), \left\{\frac{c_1}{<0.2,0.3,0.4>}, \frac{c_2}{<0.6,0.2,0.5>}, \frac{c_3}{<0.1,0.7,0.2>}, \frac{c_4}{<0.1,0.4,0.5>}\right\}\right), \\ \left((k_2, E_1, 0), \left\{\frac{c_1}{<0.1,0.4,0.5>}, \frac{c_2}{<0.4,0.3,0.7>}, \frac{c_3}{<0.2,0.7,0.4>}, \frac{c_4}{<0.3,0.2,0.6>}\right\}\right), \\ \left((k_2, E_2, 0), \left\{\frac{c_1}{<0.2,0.5,0.6>}, \frac{c_2}{<0.7,0.2,0.3>}, \frac{c_3}{<0.6,0.3,0.2>}, \frac{c_4}{<0.5,0.3,0.4>}\right\}\right), \\ \left((k_2, E_3, 0), \left\{\frac{c_1}{<0.7,0.2,0.1>}, \frac{c_2}{<0.3,0.5,0.2>}, \frac{c_3}{<0.1,0.7,0.8>}, \frac{c_4}{<0.3,0.4,0.3>}\right\}\right), \\ \left((k_3, E_1, 0), \left\{\frac{c_1}{<0.9,0.1,0.6>}, \frac{c_2}{<0.4,0.2,0.7>}, \frac{c_3}{<0.5,0.1,0.9>}, \frac{c_4}{<0.3,0.2,0.1>}\right\}\right), \\ \left((k_3, E_2, 0), \left\{\frac{c_1}{<0.8,0.2,0.1>}, \frac{c_2}{<0.2,0.7,0.4>}, \frac{c_3}{<0.3,0.6,0.7>}, \frac{c_4}{<0.5,0.2,0.6>}\right\}\right), \\ \left((k_3, E_3, 0), \left\{\frac{c_1}{<0.6,0.2,0.4>}, \frac{c_2}{<0.6,0.1,0.3>}, \frac{c_3}{<0.5,0.4,0.9>}, \frac{c_4}{<0.4,0.5,0.2>}\right\}\right), \\ \left((k_4, E_1, 0), \left\{\frac{c_1}{<0.6,0.3,0.2>}, \frac{c_2}{<0.3,0.5,0.6>}, \frac{c_3}{<0.2,0.6,0.5>}, \frac{c_4}{<0.3,0.4,0.6>}\right\}\right), \\ \left((k_4, E_2, 0), \left\{\frac{c_1}{<0.5,0.4,0.3>}, \frac{c_2}{<0.5,0.2,0.1>}, \frac{c_3}{<0.3,0.1,0.2>}, \frac{c_4}{<0.6,0.2,0.1>}\right\}\right), \\ \left((k_4, E_3, 0), \left\{\frac{c_1}{<0.4,0.5,0.3>}, \frac{c_2}{<0.7,0.2,0.1>}, \frac{c_3}{<0.6,0.1,0.4>}, \frac{c_4}{<0.9,0.1,0.6>}\right\}\right), \\ \left((k_5, E_1, 0), \left\{\frac{c_1}{<0.2,0.5,0.7>}, \frac{c_2}{<0.6,0.2,0.8>}, \frac{c_3}{<0.9,0.1,0.3>}, \frac{c_4}{<0.9,0.2,0.5>}\right\}\right), \\ \left((k_5, E_2, 0), \left\{\frac{c_1}{<0.3,0.6,0.1>}, \frac{c_2}{<0.2,0.8,0.3>}, \frac{c_3}{<0.3,0.6,0.2>}, \frac{c_4}{<0.4,0.3,0.2>}\right\}\right), \\ \left((k_5, E_3, 0), \left\{\frac{c_1}{<0.1,0.7,0.3>}, \frac{c_2}{<0.5,0.2,0.4>}, \frac{c_3}{<0.5,0.1,0.7>}, \frac{c_4}{<0.4,0.6,0.3>}\right\}\right) \end{array} \right\}$$

are single valued neutrosophic hypersoft expert sets.

Step-2

Table 1 represents the values of $\mu(c_i)$-$\nu(c_i)$-$\omega(c_i)$

Step-(2-5)

Table 2 and table 3 represent the grade values of agree and disagree single valued neutrosophic hypersoft expert set respectively. Table 4 depicts the difference of scores of agree and disagree SVNHSES. The scores for agree SVNHSES are :

$S(c_1) = 0.6$, $S(c_2) = 0.5$, $S(c_3) = 0.4$ and $S(c_4) = 1.5$

whereas scores for disagree SVNHSES are:

$S(c_1) = 1.4$, $S(c_2) = 0.7$, $S(c_3) = 0.5$ and $S(c_4) = -0.2$.

Step-6; Decision

As j_4 is maximum, so category c_4 is preferred to be best.

Table 1. Agree-single valued neutrosophic hypersoft expert set

C	c_1	c_2	c_3	c_4	C	c_1	c_2	c_3	c_4
$(k_1, E_1, 1)$	0.1	-0.6	0.0	-0.6	$(k_1, E_1, 0)$	0.0	0.6	0.3	-0.3
$(k_1, E_2, 1)$	-0.1	-0.4	-0.4	0.2	$(k_1, E_2, 0)$	0.2	0.0	-0.6	-0.1
$(k_1, E_3, 1)$	0.3	-0.2	-0.2	0.4	$(k_1, E_3, 0)$	-0.5	-0.1	-0.8	-0.8
$(k_2, E_1, 1)$	-0.6	0.1	0.0	-0.3	$(k_2, E_1, 0)$	-0.8	-0.6	-0.9	-0.5
$(k_2, E_2, 1)$	0.0	-0.3	0.4	-0.7	$(k_2, E_2, 0)$	-0.9	0.2	0.1	-0.2
$(k_2, E_3, 1)$	0.0	-0.3	-0.4	-0.1	$(k_2, E_3, 0)$	0.4	-0.4	-1.4	-0.4
$(k_3, E_1, 1)$	-1.1	-0.3	-0.3	-0.1	$(k_3, E_1, 0)$	0.2	-0.5	-0.5	0.0
$(k_3, E_2, 1)$	-0.5	-0.3	0.0	-0.2	$(k_3, E_2, 0)$	0.5	-0.9	-1.0	-0.3
$(k_3, E_3, 1)$	-0.2	0.1	-0.1	-0.7	$(k_3, E_3, 0)$	0.0	0.2	-0.8	-0.3
$(k_4, E_1, 1)$	0.1	-1.0	-0.3	-0.1	$(k_4, E_1, 0)$	0.1	-0.8	-0.9	-0.7
$(k_4, E_2, 1)$	0.3	-0.6	-1.1	-0.2	$(k_4, E_2, 0)$	-0.1	0.2	0.0	0.3
$(k_4, E_3, 1)$	0.0	-0.4	-1.1	0.2	$(k_4, E_3, 0)$	-1.4	0.4	0.1	0.2
$(k_5, E_1, 1)$	0.1	-0.7	-0.4	-1.0	$(k_5, E_1, 0)$	-1.0	-0.4	0.5	0.2
$(k_5, E_2, 1)$	-0.2	0.0	-0.4	0.3	$(k_5, E_2, 0)$	-0.4	-0.9	-0.5	0.1
$(k_5, E_3, 1)$	0.0	0.3	-0.8	-0.1	$(k_5, E_3, 0)$	-0.9	-0.1	-0.3	-0.5

Table 2. Numerical Grades of agree SVNHSES

	c_i	Highest Numerical Grade
$(k_1, E_1, 1)$	c_1	0.1
$(k_1, E_2, 1)$	c_4	0.2
$(k_1, E_3, 1)$	c_4	0.4
$(k_2, E_1, 1)$	c_2	0.1
$(k_2, E_2, 1)$	c_3	0.4
$(k_2, E_3, 1)$	c_4	0.6
$(k_3, E_1, 1)$	c_1	0.0
$(k_3, E_2, 1)$	c_3	0.0
$(k_3, E_3, 1)$	c_2	0.1
$(k_4, E_1, 1)$	c_1	0.1
$(k_4, E_2, 1)$	c_1	0.3
$(k_4, E_3, 1)$	c_4	0.2
$(k_5, E_1, 1)$	c_1	0.1
$(k_5, E_2, 1)$	c_4	0.3
$(k_5, E_3, 1)$	c_2	0.3

5. **Conclusions**

In this paper, the fundamentals of single valued neutrosophic hypersoft expert set are established and some basic properties, laws and operations are generalized. A decision-making

TABLE 3. Numerical Grades of disagree SVNHSES

	c_i	Highest Numerical Grade
$(k_1, E_1, 0)$	c_2	0.6
$(k_1, E_2, 0)$	c_1	0.2
$(k_1, E_3, 0)$	c_2	-0.1
$(k_2, E_1, 0)$	c_4	-0.5
$(k_2, E_2, 0)$	c_2	0.2
$(k_2, E_3, 0)$	c_1	0.4
$(k_3, E_1, 0)$	c_1	0.2
$(k_3, E_2, 0)$	c_1	0.5
$(k_3, E_3, 0)$	c_2	0.2
$(k_4, E_1, 0)$	c_1	0.1
$(k_4, E_2, 0)$	c_4	0.3
$(k_4, E_3, 0)$	c_2	0.4
$(k_5, E_1, 0)$	c_3	0.5
$(k_5, E_2, 0)$	c_2	0.1
$(k_5, E_3, 0)$	c_2	-0.1

TABLE 4. Numerical values of $j_i = G_i - H_i$

G_i	H_i	$j_i = G_i - H_i$
$S(c_1) = 0.6$	$S(c_1) = 1.4$	-1.8
$S(c_2) = 0.5$	$S(c_2) = 0.7$	-0.2
$S(c_3) = 0.4$	$S(c_3) = 0.5$	-0.1
$S(c_4) = 1.5$	$S(c_4) = -0.2$	1.7

application regarding the selection of the best product is presented with the help of proposed algorithm. Future work may include the extension of the presented work for other single valued neutrosophic hypersoft-like hybrids.

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Optimal Supplier Selection Via Decision-Making Algorithmic Technique Based on Single-Valued Neutrosophic Fuzzy Hypersoft Set

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Abstract. Hypersoft set, an extension of soft set, is more flexible and useful as it tackles the limitation of soft set for dealing with scenarios where distinct attributes are further classified into disjoint attribute-valued sets. It replaces single-argument approximate function of soft set with multi-argument approximate function. The main goal of this research is to align existing literature on single-valued neutrosophic fuzzy soft sets with the need for such a multi-argument function. Firstly, the novel notions of single-valued neutrosophic fuzzy hypersoft sets are characterized. Some of its essential basic properties and set theoretic operations are discussed with illustrated numerical examples. Secondly, fuzzy decision-making algorithm based on single-valued neutrosophic fuzzy hypersoft set matrix is proposed. Explicatory application is presented which depicts the structural validity of proposed structure for successful application to the problems involving vagueness and uncertainties. Lastly, a comparison of the proposed structure with existing structures, is made under appropriate indicators.

Keywords: Fuzzy set; Neutrosophic set; Single-valued neutrosophic set; Single-valued neutrosophic soft set; Single-valued neutrosophic fuzzy soft set; Hypersoft set.

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1. Introduction

In different mathematical disciplines, fuzzy sets theory (FS-Theory) [1] and intuitionistic fuzzy set theory (IFS-Theory) [2] are considered apt mathematical modes to tackle several intricate problems involving various uncertainties. The former emphasizes on a certain object's degree of true belongingness from the initial sample space, while the latter emphasizes degree of true membership and degree of non-membership with the state of their interdependence. These theories portray some kind of inadequacy in terms of providing due status to a degree of indeterminacy. The implementation of neutrosophic set theory (NS-Theory) [3, 4] overcomes this impediment by taking into account not only the proper status of degree of indeterminacy but also the state of dependence. This theory is more adaptable and suitable for dealing with inconsistent data. Wang et el [5] conceptualized single-valued neutrosophic set in which truth membership degree, indeterminacy degree and falsity degree are restricted within unit closed interval. Many researchers [6]- [14] have been drawn to NS-Theory for further application in statistics, topological spaces, and the construction of some neutrosophic-like blended structures with other existing models for useful applications in decision making. Edalatpanah [15] studied a system of neutrosophic linear equations (SNLE) based on the embedding approach. He used (α, β, γ)-cut for transformation of SNLE into a crisp linear system. Kumar et al. [16] exhibited a novel linear programming approach for finding the neutrosophic shortest path problem (NSSPP) considering Gaussian valued neutrosophic number.

FS-Theory, IFS-Theory and NS-Theory have some kind of complexities which restrain them to solve problem involving uncertainty professionally. The reason for these hurdles is, possibly, the inadequacy of the parametrization tool. It demands a mathematical tool free of all such impediments to tackle such issues. This scantiness is resolved with the development of soft set theory (SS-Theory) [17] which is a new parameterized family of subsets of the universe of discourse. The researchers [18]- [27] studied and investigated some elementary properties, operations, laws and hybrids of SS-Theory with applications in decision making. The gluing concept of NS-Theory and SS-Theory, is studied in [28] to make the NS-Theory adequate with parameterized tool. In many real life situations, distinct attributes are further partitioned in disjoint attribute-valued sets but existing SS-Theory is insufficient for dealing with such kind of attribute-valued sets. Hypersoft set theory (HS-Theory) [29] is developed to make the SST in line with attribute-valued sets to tackle real life scenarios. HS-Theory is an extension of SS-Theory as it transforms the single argument function into a multi-argument function. Certain elementary properties, aggregation operations, laws, relations and functions of HS-Theory, are investigated by [30]- [32] for proper understanding and further utilization in different fields. The applications of HS-Theory in decision making is studied by [33]- [36] and the intermingling study of HS-Theory with complex sets, convex and concave sets is studied by [37, 38]. Deli [39] characterized hybrid set structures under uncertainly parameterized hypersoft sets with theory and applications. Gayen et al. [40] analyzed some essential aspects of plithogenic hypersoft algebraic structures. They also investigated the notions and basic properties of plithogenic hypersoft subgroups ie plithogenic fuzzy hypersoft subgroup, plithogenic intuitionistic fuzzy hypersoft subgroup, plithogenic neutrosophic hypersoft subgroup. Saeed et al. [41, 42] discussed decision making techniques for neutrosophic hypersoft mapping and complex multi-fuzzy hypersoft set. Rahman et al. [43–45] studied decision making applications based on neutrosophic parameterized hypersoft Set, fuzzy parameterized hypersoft set and rough hypersoft set. Ihsan et al. [46] investigated hypersoft expert set with application in decision making for the best selection of product.

1.1. *Motivation*

The attributes must be further partitioned into attribute values in a variety of real-world applications for a more vivid understanding. As a generalization of soft set, hypersoft set overcomes this restriction and emphasizes the disjoint attribute-valued sets for distinct attributes. This generalization shows that using the hypersoft set with neutrosophic, intuitionistic, and fuzzy set theory to build a relation between alternatives and attributes would be extremely useful. In addition, the hypersoft set will reduce the case study's complexity. It's important to note that the hypersoft theory can be applied to every decision-making challenge, regardless of the decision-makers' ability to choose values. Multi-criteria decision making (MCDM), Multi-criteria group decision making (MCGDM), shortest path selection, employee selection, e-learning, graph theory, medical diagnosis, and other applications may all benefit from this theory. The current literature on soft sets should be adequate with regard to the presence and consideration of attribute-valued sets, so the main goal of this research is to extend the concept presented in [48, 49] and to develop novel theory of single-valued neutrosophic fuzzy hypersoft set (an embedded structure of fuzzy set, single-valued neutrosophic set and hypersoft set). After characterizing its basic essential properties and operations, decision-making based algorithm is proposed to solve a real life problem relating to the purchase of most suitable and appropriate supplier with the help of single-valued neutrosophic fuzzy hypersoft set matrix.

1.2. *Paper Organization*

The remaining paper is systemized as:

Section 2 : Some essential definitions and terminologies are recalled.

Section 3 : Theory of single-valued neutrosophic fuzzy hypersoft set is developed with suitable examples.

Section 4 : Decision-making based algorithm and application of single-valued neutrosophic fuzzy hypersoft set are presented.

Section 5 : Comparison Analysis of proposed structures is discussed.

Section 6 : Paper is summarized with future directions.

2. Preliminaries

In the following, we present a short survey of definitions which are necessary to this paper. Here some basic terms are recalled from existing literature to support the proposed work. Throughout the paper, $\hat{\mathcal{V}}$ and $\mathbb{I}$ will denote the universe of discourse and closed unit interval respectively. In this work, algorithmic approach is followed from decision making method stated in [49].

Definition 2.1. [1]
A *fuzzy set* $\mathcal{F}$ defined as $\mathcal{F} = \{(\hat{u}, A_{\mathcal{F}}(\hat{u}))|\hat{u} \in \hat{\mathcal{V}}\}$ such that $A_{\mathcal{F}} : \hat{\mathcal{V}} \to \mathbb{I}$ where $A_{\mathcal{F}}(\hat{u})$ denotes the belonging value of $\hat{u} \in \mathcal{F}$.

Definition 2.2. [2]
An *intuitionistic fuzzy set* $\mathcal{Y}$ defined as $\mathcal{Y} = \{(\hat{u}, < A_{\mathcal{Y}}(\hat{u}), B_{\mathcal{Y}}(\hat{u}) >)|\hat{u} \in \hat{\mathcal{V}}\}$ such that $A_{\mathcal{Y}} : \ddot{\mathcal{U}} \to \mathbb{I}$ and $B_{\mathcal{Y}} : \ddot{\mathcal{U}} \to \mathbb{I}$, where $A_{\mathcal{Y}}(\hat{u})$ and $B_{\mathcal{Y}}(\hat{u})$ denote the belonging value and not-belonging value of $\hat{u} \in \mathcal{Y}$ with condition of $0 \leq A_{\mathcal{Y}}(\hat{u}) + B_{\mathcal{Y}}(\hat{u}) \leq 1$.

Definition 2.3. [3]
A *neutrosophic set* $\mathcal{Z}$ defined as
$\mathcal{Z} = \{(\hat{u}, < \mathcal{A}_{\mathcal{Z}}(\hat{u}), \mathcal{B}_{\mathcal{Z}}(\hat{u}), \mathcal{C}_{\mathcal{Z}}(\hat{u}) >)|\hat{u} \in \ddot{\mathcal{U}}\}$
such that $\mathcal{A}_{\mathcal{Z}}(\hat{u}), \mathcal{B}_{\mathcal{Z}}(\hat{u}), \mathcal{C}_{\mathcal{Z}}(\hat{u}) : \hat{\mathcal{V}} \to (^{-}0, 1^{+})$,
where $\mathcal{A}_{\mathcal{Z}}(\hat{u}), \mathcal{B}_{\mathcal{Z}}(\hat{u})$ and $\mathcal{C}_{\mathcal{Z}}(\hat{u})$ denote the degrees of membership, indeterminacy and non-membership of $\hat{u} \in \mathcal{Z}$ with condition of $^{-}0 \leq \mathcal{A}_{\mathcal{Z}}(\hat{u}) + \mathcal{B}_{\mathcal{Z}}(\hat{u}) + \mathcal{C}_{\mathcal{Z}}(\hat{u}) \leq 3^{+}$.

Note: If $\mathcal{A}_{\mathcal{Z}}(\hat{u}) \in \mathbb{I}, \mathcal{B}_{\mathcal{Z}}(\hat{u}) \in \mathbb{I}$ and $\mathcal{C}_{\mathcal{Z}}(\hat{u}) \in \mathbb{I}$, with

$$0 \leq \mathcal{A}_{\mathcal{Z}}(\hat{u}) + \mathcal{B}_{\mathcal{Z}}(\hat{u}) + \mathcal{C}_{\mathcal{Z}}(\hat{u}) \leq 3,$$

then neutrosophic set $\mathcal{Z}$ is called single-valued neutrosophic set [5].

Definition 2.4. [17]
A pair $(\mathfrak{F}_S, \Lambda)$ is called a *soft set* over $\hat{\mathcal{V}}$, where $F_S : \Lambda \to \mathbb{P}(\hat{\mathcal{V}})$ and Λ be a subset of a set of attributes $\mathfrak{E}$.

For more detail on soft set, see [18, 19].

Definition 2.5. [29]
The pair $(\mathcal{W}, \mathcal{H})$ is called a hypersoft set over $\hat{\mathcal{V}}$, where $\mathcal{H}$ is the cartesian product of n disjoint sets $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, ..., \mathcal{H}_n$ having attribute values of n distinct attributes $\hat{h}_1, \hat{h}_2, \hat{h}_3, ..., \hat{h}_n$ respectively and $\mathcal{W} : \mathcal{H} \to \mathbb{P}(\hat{\mathcal{V}})$.

Note: If $\mathbb{P}(\hat{\mathcal{V}})$ is replaced with $\mathbb{N}(\hat{\mathcal{V}})$(A collection of neutrosophic subsets) in Definition 2.5, then it becomes neutrosophic hypersoft set [28]. For more definitions and operations of hypersoft set, see [30–32].

3. Single-Valued Neutrosophic Fuzzy Hypersoft Set(SV-NFHS-Set)

Definition 3.1. The pair $(\hat{\Psi}_{HS}, \hat{\mathcal{W}})$ is said to be *single-valued neutrosophic fuzzy hypersoft set* (SV-NFHS-Set) over $\hat{\mathcal{V}}$, denoted by $(SVNFHS)^{\hat{\mathcal{V}}\hat{\mathcal{W}}}$, in $(HS)^{\hat{\mathcal{V}}\hat{\mathcal{W}}}$ if

$$\hat{\Psi}_{(\hat{w})} : \hat{\mathcal{W}} \rightarrow SVNF(\hat{\mathcal{V}})$$

defined by

$$\hat{\Psi}_{(\hat{w})} = \left\{ \begin{array}{l} \dfrac{\left(\mathcal{P}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), \mathcal{Q}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), \mathcal{R}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), \xi(\hat{v})\right)}{\hat{v}} \Bigg| \\ \hat{w} \in \hat{\mathcal{W}}, \hat{v} \in \hat{\mathcal{V}}, \\ 0 \leq \mathcal{P}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) + \mathcal{Q}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) + \mathcal{R}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \leq 3 \end{array} \right\}$$

where

(i) $\hat{\mathcal{W}} = \hat{\mathcal{E}}_1 \times \hat{\mathcal{E}}_2 \times \hat{\mathcal{E}}_3 \times .... \times \hat{\mathcal{E}}_n$ for n disjoint attribute-valued sets $\hat{\mathcal{E}}_i, i = 1, 2, ..., n$ corresponding to n distinct attributes $\hat{\varepsilon}_j, j = 1, 2, ..., n$ from set of attributes $\hat{\mathcal{E}}$,

(ii) $SVNF(\hat{\mathcal{V}})$ is the collection of all single-valued neutrosophic fuzzy subsets over $\hat{\mathcal{V}}$,

(iii) All components i.e. $\mathcal{P}_{\hat{\Psi}_{(\hat{w})}}(\hat{v})$, $\mathcal{Q}_{\hat{\Psi}_{(\hat{w})}}(\hat{v})$, $\mathcal{R}_{\hat{\Psi}_{(\hat{w})}}(\hat{v})$ and $\xi(\hat{v})$ belong to $\mathbb{I}_{\bullet}^{\hat{\mathcal{V}}}$,

(iv) $(HS)^{\hat{\mathcal{V}}\hat{\mathcal{W}}}$ is a hypersoft universe.

Note: The collection of all $(SVNFHS)^{\hat{\mathcal{V}}\hat{\mathcal{W}}}$ over $(HS)^{\hat{\mathcal{V}}\hat{\mathcal{W}}}$ is denoted by $\uplus_{svnfhs}$.

Example 3.2. Consider $\hat{\mathcal{V}} = \{\hat{v}_1, \hat{v}_2, \hat{v}_3\}$ consists of three kinds of mobile and $\hat{\mathcal{E}} = \{\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{\varepsilon}_3\}$ be the set of attributes, where $\hat{\varepsilon}_1$ is storage(GB), $\hat{\varepsilon}_2$ is camera resolution (pixels) and $\hat{\varepsilon}_3$ is battery power(mAh). Their corresponding attribute-valued sets are $\hat{\mathcal{E}}_1 = \{j_{11} = 32, j_{12} = 64\}$, $\hat{\mathcal{E}}_2 = \{j_{21} = 8, j_{22} = 16\}$ and $\hat{\mathcal{E}}_3 = \{j_{31} = 4000\}$. Now $\hat{\mathcal{W}} = \hat{\mathcal{E}}_1 \times \hat{\mathcal{E}}_2 \times \hat{\mathcal{E}}_3$,
$\hat{\mathcal{W}} = \{(\hat{w}_{11}, \hat{w}_{21}, \hat{w}_{31}), (\hat{w}_{11}, \hat{w}_{22}, \hat{w}_{31}), (\hat{w}_{12}, \hat{w}_{21}, \hat{w}_{31}), (\hat{w}_{12}, \hat{w}_{22}, \hat{w}_{31})\}$.
or
$\hat{\mathcal{W}} = \{\hat{w}_1, \hat{w}_2, \hat{w}_3, \hat{w}_4\}$
Then for $\xi \in \mathbb{I}_{\bullet}^{\hat{\mathcal{V}}}$, $\hat{\Psi}_{(\hat{w}_i)} \in \uplus_{svnfhs}$ are defined as follows $(i = 1, 2, 3, 4)$:

$$\hat{\Psi}_{(\hat{w}_1)} = \left\{ \frac{(0.2, 0.6, 0.4, 0.4)}{\hat{v}_1}, \frac{(0.2, 0.8, 0.6, 0.4)}{\hat{v}_2}, \frac{(0.3, 0.6, 0.7, 0.4)}{\hat{v}_3} \right\},$$

$$\hat{\Psi}_{(\hat{w}_2)} = \left\{ \frac{(0.6, 0.8, 0.7, 0.5)}{\hat{v}_1}, \frac{(0.3, 0.6, 0.6, 0.4)}{\hat{v}_2}, \frac{(0.7, 0.9, 0.8, 0.3)}{\hat{v}_3} \right\},$$

$$\hat{\Psi}_{(\hat{w}_3)} = \left\{ \frac{(0.1, 0.5, 0.5, 0.4)}{\hat{v}_1}, \frac{(0.4, 0.6, 0.7, 0.5)}{\hat{v}_2}, \frac{(0.3, 0.6, 0.5, 0.2)}{\hat{v}_3} \right\},$$

$$\hat{\Psi}_{(\hat{w}_4)} = \left\{ \frac{(0.3, 0.6, 0.4, 0.2)}{\hat{v}_1}, \frac{(0.2, 0.5, 0.7, 0.4)}{\hat{v}_2}, \frac{(0.6, 0.4, 0.7, 0.4)}{\hat{v}_3} \right\}.$$

It can be represented in matrix form as

$$\hat{\Psi} = \begin{pmatrix} \hat{\mathcal{W}} & \hat{v}_1 & \hat{v}_2 & \hat{v}_3 \\ \hat{w}_1 & (0.2, 0.6, 0.4, 0.4) & (0.2, 0.8, 0.6, 0.4) & (0.3, 0.6, 0.7, 0.4) \\ \hat{w}_2 & (0.6, 0.8, 0.7, 0.5) & (0.3, 0.6, 0.6, 0.4) & (0.7, 0.9, 0.8, 0.3) \\ \hat{w}_3 & (0.1, 0.5, 0.5, 0.4) & (0.4, 0.6, 0.7, 0.5) & (0.3, 0.6, 0.5, 0.2) \\ \hat{w}_4 & (0.3, 0.6, 0.4, 0.2) & (0.2, 0.5, 0.7, 0.4) & (0.6, 0.4, 0.7, 0.4) \end{pmatrix}$$

Definition 3.3. Let $\hat{\Psi}^1_{(\hat{w})}$, $\hat{\Psi}^2_{(\hat{w})} \in \biguplus_{svnfhs}$ and $\xi^1, \xi^2 \in \mathbb{I}_{\bullet}^{\hat{\mathcal{V}}}$, where

$$\hat{\Psi}^1_{(\hat{w})} = \left\{ \begin{array}{l} \frac{\mathcal{P}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), \mathcal{Q}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), \mathcal{R}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), \xi^1(\hat{v})}{\hat{v}} \Bigg| \\ \hat{w} \in \hat{\mathcal{W}}, \hat{v} \in \hat{\mathcal{V}}, \\ 0 \leq \mathcal{P}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) + \mathcal{Q}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) + \mathcal{R}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \leq 3 \end{array} \right\}$$

and

$$\hat{\Psi}^2_{(\hat{w})} = \left\{ \begin{array}{l} \frac{\mathcal{P}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), \mathcal{Q}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), \mathcal{R}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), \xi^2(\hat{v})}{\hat{v}} \Bigg| \\ \hat{w} \in \hat{\mathcal{W}}, \hat{v} \in \hat{\mathcal{V}}, \\ 0 \leq \mathcal{P}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) + \mathcal{Q}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) + \mathcal{R}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \leq 3 \end{array} \right\}.$$

Then $\hat{\Psi}^1_{(\hat{w})}$ is said to be a SV-NFHS- subset of $\hat{\Psi}^2_{(\hat{w})}$, expressed by $\hat{\Psi}^1_{(\hat{w})} \subseteq \hat{\Psi}^2_{(\hat{w})}$, if

(i) $\xi^1 \leq \xi^2$,

(ii) $\mathcal{P}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \leq \mathcal{P}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), \mathcal{Q}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \geq \mathcal{Q}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), \mathcal{R}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \geq \mathcal{R}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v})$.

Example 3.4. Considering data from Example 3.2, let another $\hat{\Psi}'_{(\hat{w})} \in \biguplus_{svnfhs}$ be defined as below:

$$\hat{\Psi}' = \begin{pmatrix} \hat{\mathcal{W}} & \hat{v}_1 & \hat{v}_2 & \hat{v}_3 \\ \hat{w}_1 & (0.4, 0.5, 0.3, 0.4) & (0.3, 0.6, 0.5, 0.5) & (0.4, 0.5, 0.4, 0.6) \\ \hat{w}_2 & (0.7, 0.7, 0.6, 0.6) & (0.5, 0.4, 0.5, 0.6) & (0.8, 0.8, 0.7, 0.5) \\ \hat{w}_3 & (0.2, 0.5, 0.3, 0.7) & (0.7, 0.5, 0.4, 0.7) & (0.4, 0.5, 0.4, 0.3) \\ \hat{w}_4 & (0.5, 0.4, 0.3, 0.6) & (0.4, 0.4, 0.6, 0.6) & (0.7, 0.3, 0.3, 0.5) \end{pmatrix}$$

Thus, $\hat{\Psi}_{(\hat{w})} \subseteq \hat{\Psi}'_{(\hat{w})}$ for all $\hat{w}_i \in \hat{\mathcal{W}}, i = 1, 2, 3, 4$

Definition 3.5. Assuming $\hat{\Psi}^1_{(\hat{w})}$, $\hat{\Psi}^2_{(\hat{w})} \in \biguplus_{svnfhs}$ and $\xi^1, \xi^2 \in \mathbb{I}_{\bullet}^{\hat{\mathcal{V}}}$, as stated in Definition 3.3, then $\hat{\Psi}^1_{(\hat{w})}$ and $\hat{\Psi}^2_{(\hat{w})}$ are said to be SV-NFHS-equal sets, denoted by $\hat{\Psi}^1_{(\hat{w})} = \hat{\Psi}^2_{(\hat{w})}$, if $\hat{\Psi}^1_{(\hat{w})} \subset \hat{\Psi}^2_{(\hat{w})}$ and $\hat{\Psi}^2_{(\hat{w})} \subset \hat{\Psi}^1_{(\hat{w})}$.

Definition 3.6. Let $\hat{\Psi}_{(\hat{w})} \in \biguplus_{svnfhs}$ and $\xi \in \mathbb{I}_{\bullet}^{\hat{\mathcal{V}}}$, then

(i) $\hat{\Psi}_{(\hat{w})}$ is known to be a SV-NFHS-null set, represented by $\hat{\emptyset}_{(\hat{w})}$, if

$$\mathcal{P}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) = 0, \mathcal{Q}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) = 1, \mathcal{R}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) = 1, \xi(\hat{v}) = 0$$

.

(ii) $\hat{\Psi}_{(\hat{w})}$ is known to be a SV-NFHS-universal set, denoted by $\hat{\mathcal{V}}_{(\hat{w})}$, if

$$\mathcal{P}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) = 1, \mathcal{Q}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) = 0, \mathcal{R}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) = 0, \xi(\hat{v}) = 1.$$

Definition 3.7. Let us suppose $\hat{\Psi}^1_{(\hat{w})}$, $\hat{\Psi}^2_{(\hat{w})} \in \biguplus_{svnfhs}$ and $\xi^1, \xi^2 \in \mathbb{I}_\bullet^{\hat{\mathcal{V}}}$, as described in Definition 3.3, then,

(i) the union of $\hat{\Psi}^1_{(\hat{w})}$ and $\hat{\Psi}^2_{(\hat{w})}$, denoted by $\hat{\Psi}^1_{(\hat{w})} \hat{\sqcup} \hat{\Psi}^2_{(\hat{w})}$, is a SV-NFHS-set $\hat{\Omega}^1_{(\hat{w})}$ such that

$$\hat{\Omega}^1_{(\hat{w})} = \left\{ \begin{array}{c|} \frac{\left(\mathcal{P}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \circledcirc \mathcal{P}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}),\, \mathcal{Q}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \circledast \mathcal{Q}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}),\, \mathcal{R}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \circledast \mathcal{R}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}),\, \xi^1(\hat{v}) \circledcirc \xi^2(\hat{v}) \right)}{\hat{v}} \\ \hat{w} \in \hat{\mathcal{W}}, \hat{v} \in \hat{\mathcal{V}} \end{array} \right\}.$$

(ii) the intersection of $\hat{\Psi}^1_{(\hat{w})}$ and $\hat{\Psi}^2_{(\hat{w})}$, denoted by $\hat{\Psi}^1_{(\hat{w})} \hat{\sqcap} \hat{\Psi}^2_{(\hat{w})}$, is a SV-NFHS-set $\hat{\Omega}^2_{(\hat{w})}$ such that

$$\hat{\Omega}^2_{(\hat{w})} = \left\{ \begin{array}{c|} \frac{\left(\mathcal{P}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \circledast \mathcal{P}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}),\, \mathcal{Q}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \circledcirc \mathcal{Q}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}),\, \mathcal{R}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \circledcirc \mathcal{R}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}),\, \xi^1(\hat{v}) \circledast \xi^2(\hat{v}) \right)}{\hat{v}} \\ \hat{w} \in \hat{\mathcal{W}}, \hat{v} \in \hat{\mathcal{V}} \end{array} \right\}.$$

In above definitions, $\circledast$ and $\circledcirc$ denote the t-norm ($\wedge$) and t-conorm ($\vee$) respectively. If $a, b \in \mathbb{I}_\bullet^{\hat{\mathcal{V}}}$, then $a \circledast b = a \wedge b$ and $a \circledcirc b = a \vee b$.

Example 3.8. Supposing the matrix representations of SV-NFHS-sets as determined in Example 3.2 and Example 3.4, then we have

$$\hat{\Omega}^1_{(\hat{w})} = \begin{pmatrix} \hat{\mathcal{W}} & \hat{v}_1 & \hat{v}_2 & \hat{v}_3 \\ \hat{w}_1 & (0.4, 0.5, 0.3, 0.4) & (0.3, 0.6, 0.5, 0.5) & (0.4, 0.5, 0.4, 0.6) \\ \hat{w}_2 & (0.7, 0.7, 0.6, 0.6) & (0.5, 0.4, 0.5, 0.6) & (0.8, 0.8, 0.7, 0.5) \\ \hat{w}_3 & (0.2, 0.5, 0.3, 0.7) & (0.7, 0.5, 0.4, 0.7) & (0.4, 0.5, 0.4, 0.3) \\ \hat{w}_4 & (0.5, 0.4, 0.3, 0.6) & (0.4, 0.4, 0.6, 0.6) & (0.7, 0.3, 0.3, 0.5) \end{pmatrix}$$

and

$$\hat{\Omega}^2_{(\hat{w})} = \begin{pmatrix} \hat{\mathcal{W}} & \hat{v}_1 & \hat{v}_2 & \hat{v}_3 \\ \hat{w}_1 & (0.2, 0.6, 0.4, 0.4) & (0.2, 0.8, 0.6, 0.4) & (0.3, 0.6, 0.7, 0.4) \\ \hat{w}_2 & (0.6, 0.8, 0.7, 0.5) & (0.3, 0.6, 0.6, 0.4) & (0.7, 0.9, 0.8, 0.3) \\ \hat{w}_3 & (0.1, 0.5, 0.5, 0.4) & (0.4, 0.6, 0.7, 0.5) & (0.3, 0.6, 0.5, 0.2) \\ \hat{w}_4 & (0.3, 0.6, 0.4, 0.2) & (0.2, 0.5, 0.7, 0.4) & (0.6, 0.4, 0.7, 0.4) \end{pmatrix}$$

Proposition 3.9. *Let $\hat{\emptyset}_{(\hat{w})}, \hat{\mathcal{V}}_{(\hat{w})}, \hat{\Psi}_{(\hat{w})} \in \biguplus_{svnfhs}$ and $\xi \in \mathbb{I}_\bullet^{\hat{\mathcal{V}}}$. Then the following properties hold:*

(i) $\hat{\Psi}_{(\hat{w})} \hat{\sqcup} \hat{\Psi}_{(\hat{w})} = \hat{\Psi}_{(\hat{w})}$,

(ii) $\hat{\Psi}_{(\hat{w})} \hat{\sqcap} \hat{\Psi}_{(\hat{w})} = \hat{\Psi}_{(\hat{w})}$,

(iii) $\hat{\Psi}_{(\hat{w})} \hat{\sqcup} \hat{\emptyset}_{(\hat{w})} = \hat{\Psi}_{(\hat{w})}$,

(iv) $\hat{\Psi}_{(\hat{w})} \hat{\sqcap} \hat{\emptyset}_{(\hat{w})} = \hat{\emptyset}_{(\hat{w})}$,
(v) $\hat{\Psi}_{(\hat{w})} \hat{\sqcup} \hat{\mathcal{V}}_{(\hat{w})} = \hat{\mathcal{V}}_{(\hat{w})}$,
(vi) $\hat{\Psi}_{(\hat{w})} \hat{\sqcap} \hat{\mathcal{V}}_{(\hat{w})} = \hat{\Psi}_{(\hat{w})}$.

Proof. The above properties (i)-(vi) can easily be proved with the help of Definition 3.6 and Definition 3.7. □

Proposition 3.10. *Let $\hat{\Psi}^1_{(\hat{w})}$, $\hat{\Psi}^2_{(\hat{w})}$, $\hat{\Psi}^3_{(\hat{w})} \in \biguplus_{svnfhs}$ and $\xi^1, \xi^2, \xi^3 \in \mathbb{I}_{\bullet}^{\hat{\mathcal{V}}}$. Then the results (given below) hold:*

(i) $\hat{\Psi}^1_{(\hat{w})} \hat{\sqcup} \hat{\Psi}^2_{(\hat{w})} = \hat{\Psi}^2_{(\hat{w})} \hat{\sqcup} \hat{\Psi}^1_{(\hat{w})}$,
(ii) $\hat{\Psi}^1_{(\hat{w})} \hat{\sqcap} \hat{\Psi}^2_{(\hat{w})} = \hat{\Psi}^2_{(\hat{w})} \hat{\sqcap} \hat{\Psi}^1_{(\hat{w})}$,
(iii) $\hat{\Psi}^1_{(\hat{w})} \hat{\sqcup} \left(\hat{\Psi}^2_{(\hat{w})} \hat{\sqcup} \hat{\Psi}^3_{(\hat{w})}\right) = \left(\hat{\Psi}^1_{(\hat{w})} \hat{\sqcup} \hat{\Psi}^2_{(\hat{w})}\right) \hat{\sqcup} \hat{\Psi}^3_{(\hat{w})}$,
(iv) $\hat{\Psi}^1_{(\hat{w})} \hat{\sqcap} \left(\hat{\Psi}^2_{(\hat{w})} \hat{\sqcap} \hat{\Psi}^3_{(\hat{w})}\right) = \left(\hat{\Psi}^1_{(\hat{w})} \hat{\sqcap} \hat{\Psi}^2_{(\hat{w})}\right) \hat{\sqcap} \hat{\Psi}^3_{(\hat{w})}$,
(v) $\hat{\Psi}^1_{(\hat{w})} \hat{\sqcap} \left(\hat{\Psi}^2_{(\hat{w})} \hat{\sqcup} \hat{\Psi}^3_{(\hat{w})}\right) = \left(\hat{\Psi}^1_{(\hat{w})} \hat{\sqcap} \hat{\Psi}^2_{(\hat{w})}\right) \hat{\sqcup} \left(\hat{\Psi}^1_{(\hat{w})} \hat{\sqcap} \hat{\Psi}^3_{(\hat{w})}\right)$,
(vi) $\hat{\Psi}^1_{(\hat{w})} \hat{\sqcup} \left(\hat{\Psi}^2_{(\hat{w})} \hat{\sqcap} \hat{\Psi}^3_{(\hat{w})}\right) = \left(\hat{\Psi}^1_{(\hat{w})} \hat{\sqcup} \hat{\Psi}^2_{(\hat{w})}\right) \hat{\sqcap} \left(\hat{\Psi}^1_{(\hat{w})} \hat{\sqcup} \hat{\Psi}^3_{(\hat{w})}\right)$.

Proof. Proofs of (i)-(vi) are straightforward by following the concept of Definition 3.7. □

Proposition 3.11. *Let $\hat{\Psi}^1_{(\hat{w})}$, $\hat{\Psi}^2_{(\hat{w})} \in \biguplus_{svnfhs}$ and $\xi^1, \xi^2 \in \mathbb{I}_{\bullet}^{\hat{\mathcal{V}}}$ and $\hat{\Psi}^1_{(\hat{w})} \subseteq \hat{\Psi}^2_{(\hat{w})}$, then the below given results hold:*

(i) $\hat{\Psi}^1_{(\hat{w})} \hat{\sqcup} \hat{\Psi}^2_{(\hat{w})} = \hat{\Psi}^2_{(\hat{w})}$,
(ii) $\hat{\Psi}^1_{(\hat{w})} \hat{\sqcap} \hat{\Psi}^2_{(\hat{w})} = \hat{\Psi}^1_{(\hat{w})}$.

Proof. By following the concept of Definition 3.3 and Definition 3.7, proofs are straightforward.
□

Definition 3.12. Let $\hat{\Psi}_{(\hat{w})} \in \biguplus_{svnfhs}, \xi \in \mathbb{I}_{\bullet}^{\hat{\mathcal{V}}}$, (as described in Definition 3.1) then, its complement $\hat{\Psi}^c_{(\hat{w})}$ is stated as

$$\hat{\Psi}^c_{(\hat{w})} = \left\{ \begin{array}{c|} \frac{\left(\mathcal{R}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), 1-\mathcal{Q}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), \mathcal{P}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), 1-\xi(\hat{v})\right)}{\hat{v}} \\ \\ \hat{w} \in \hat{\mathcal{W}}, \hat{v} \in \check{\mathcal{V}} \end{array} \right\}.$$

Example 3.13. The complement of SV-NFHS-set $\hat{\Psi}_{(\hat{w})}$ (as determined in Example 3.2) is calculated as

$$\hat{\Psi}^c_{(\hat{w})} = \begin{pmatrix} \hat{\mathcal{W}} & \hat{v}_1 & \hat{v}_2 & \hat{v}_3 \\ \hat{w}_1 & (0.3, 0.5, 0.4, 0.6) & (0.6, 0.2, 0.2, 0.6) & (0.4, 0.5, 0.4, 0.6) \\ \hat{w}_2 & (0.7, 0.2, 0.6, 0.5) & (0.6, 0.4, 0.3, 0.6) & (0.8, 0.1, 0.7, 0.7) \\ \hat{w}_3 & (0.5, 0.5, 0.1, 0.6) & (0.7, 0.4, 0.4, 0.5) & (0.5, 0.4, 0.3, 0.8) \\ \hat{w}_4 & (0.4, 0.4, 0.3, 0.8) & (0.7, 0.5, 0.2, 0.6) & (0.7, 0.6, 0.6, 0.6) \end{pmatrix}.$$

Proposition 3.14. *Let $\hat{\emptyset}_{(\hat{w})}, \hat{\mathcal{V}}_{(\hat{w})}, \hat{\Psi}_{(\hat{w})} \in \biguplus_{svnfhs}$ and $\xi \in \mathbb{I}_{\bullet}^{\hat{\mathcal{V}}}$. Then, we have the following results:*

(i) $\hat{\emptyset}^c_{(\hat{w})} = \hat{\mathcal{V}}_{(\hat{w})}$,

(ii) $\hat{\mathcal{V}}^c_{(\hat{w})} = \hat{\emptyset}_{(\hat{w})}$,

(iii) $\left(\hat{\Psi}^c_{(\hat{w})}\right)^c = \hat{\Psi}_{(\hat{w})}$.

Proof. These results can easily be proved with the help of concepts stated in Definition 3.6 and Definition 3.12. □

Remark 3.15. In general, the following results are not hold.

(i) $\hat{\Psi}_{(\hat{w})} \mathbin{\hat{\sqcup}} \hat{\Psi}^c_{(\hat{w})} = \hat{\mathcal{V}}_{(\hat{w})}$,

(ii) $\hat{\Psi}_{(\hat{w})} \mathbin{\hat{\sqcap}} \hat{\Psi}^c_{(\hat{w})} = \hat{\emptyset}_{(\hat{w})}$.

Example 3.16. The Remark 3.15 can easily be verified by considering SV-NFHS-set $\hat{\Psi}_{(\hat{w})}$ from Example 3.2 and its complement $\hat{\Psi}^c_{(\hat{w})}$ from Example 3.13. So we have

$$\hat{\Psi} \mathbin{\hat{\sqcup}} \hat{\Psi}^c = \begin{pmatrix} \hat{\mathcal{W}} & \hat{v}_1 & \hat{v}_2 & \hat{v}_3 \\ \hat{w}_1 & (0.3, 0.5, 0.4, 0.6) & (0.6, 0.2, 0.2, 0.6) & (0.4, 0.5, 0.4, 0.6) \\ \hat{w}_2 & (0.7, 0.2, 0.6, 0.5) & (0.6, 0.4, 0.3, 0.6) & (0.8, 0.1, 0.7, 0.7) \\ \hat{w}_3 & (0.5, 0.5, 0.1, 0.6) & (0.7, 0.4, 0.4, 0.5) & (0.5, 0.4, 0.3, 0.8) \\ \hat{w}_4 & (0.4, 0.4, 0.3, 0.8) & (0.7, 0.5, 0.2, 0.6) & (0.7, 0.6, 0.6, 0.6) \end{pmatrix}.$$

And

$$\hat{\Psi} \mathbin{\hat{\sqcap}} \hat{\Psi}^c = \begin{pmatrix} \hat{\mathcal{W}} & \hat{v}_1 & \hat{v}_2 & \hat{v}_3 \\ \hat{w}_1 & (0.2, 0.6, 0.4, 0.4) & (0.2, 0.8, 0.6, 0.4) & (0.3, 0.6, 0.7, 0.4) \\ \hat{w}_2 & (0.6, 0.8, 0.7, 0.5) & (0.3, 0.6, 0.6, 0.4) & (0.7, 0.9, 0.8, 0.3) \\ \hat{w}_3 & (0.1, 0.5, 0.5, 0.4) & (0.4, 0.6, 0.7, 0.5) & (0.3, 0.6, 0.5, 0.2) \\ \hat{w}_4 & (0.3, 0.6, 0.4, 0.2) & (0.2, 0.5, 0.7, 0.4) & (0.6, 0.4, 0.7, 0.4) \end{pmatrix}.$$

Hence the Remark 3.15 is verified.

Proposition 3.17. *Let $\hat{\Psi}^1_{(\hat{w})}, \hat{\Psi}^2_{(\hat{w})} \in \biguplus_{svnfhs}$ and $\xi^1, \xi^2 \in \mathbb{I}_{\bullet}^{\hat{\mathcal{V}}}$. Then, the following results hold:*

(i) $\left(\hat{\Psi}^1_{(\hat{w})} \mathbin{\hat{\sqcup}} \hat{\Psi}^2_{(\hat{w})}\right)^c = \left(\hat{\Psi}^1_{(\hat{w})}\right)^c \mathbin{\hat{\sqcap}} \left(\hat{\Psi}^2_{(\hat{w})}\right)^c$,

(ii) $\left(\hat{\Psi}^1_{(\hat{w})} \hat{\cap} \hat{\Psi}^2_{(\hat{w})}\right)^c = \left(\hat{\Psi}^1_{(\hat{w})}\right)^c \hat{\cup} \left(\hat{\Psi}^2_{(\hat{w})}\right)^c$.

Proof. Consider the concept stated in Definition 3.7, for $\hat{w} \in \hat{\mathcal{W}}, \hat{v} \in \hat{\mathcal{V}}$, we have

(i). Since $\left(\hat{\Psi}^1_{(\hat{w})} \hat{\cup} \hat{\Psi}^2_{(\hat{w})}\right)^c (\hat{v})$

$$= \left(\left\{ \frac{\left(\mathcal{P}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \circledast \mathcal{P}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), \mathcal{Q}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \circledcirc \mathcal{Q}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), \mathcal{R}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \circledcirc \mathcal{R}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), \xi^1(\hat{v}) \circledast \xi^2(\hat{v})\right)}{\hat{v}} \right\}\right)^c$$

$$= \left\{ \frac{\left(\mathcal{R}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \circledcirc \mathcal{R}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), 1-\left(\mathcal{Q}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \circledcirc \mathcal{Q}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v})\right), \mathcal{P}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \circledast \mathcal{P}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), 1-\left(\xi^1(\hat{v}) \circledast \xi^2(\hat{v})\right)\right)}{\hat{v}} \right\}$$

$$= \left\{ \frac{\left(\mathcal{R}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \wedge \mathcal{R}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), 1-\left(\mathcal{Q}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \wedge \mathcal{Q}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v})\right), \mathcal{P}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \vee \mathcal{P}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), 1-\left(\xi^1(\hat{v}) \vee \xi^2(\hat{v})\right)\right)}{\hat{v}} \right\}$$

$$= \left\{ \frac{\left(\mathcal{R}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \wedge \mathcal{R}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), 1-\mathcal{Q}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \vee 1-\mathcal{Q}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), \mathcal{P}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \vee \mathcal{P}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), 1-\xi^1(\hat{v}) \wedge 1-\xi^2(\hat{v})\right)}{\hat{v}} \right\}$$

$$= \left\{ \frac{\left(\mathcal{R}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \circledcirc \mathcal{R}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), 1-\mathcal{Q}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \circledast 1-\mathcal{Q}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), \mathcal{P}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \circledast \mathcal{P}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), 1-\xi^1(\hat{v}) \circledcirc 1-\xi^2(\hat{v})\right)}{\hat{v}} \right\}$$

$$= \left\{ \frac{\mathcal{R}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), 1-\mathcal{Q}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), \mathcal{P}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), 1-\xi^1(\hat{v})}{\hat{v}} \right\} \hat{\cap} \left\{ \frac{\mathcal{R}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), 1-\mathcal{Q}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), \mathcal{P}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), 1-\xi^2(\hat{v})}{\hat{v}} \right\}$$

$$= \left(\hat{\Psi}^1_{(\hat{w})}\right)^c \hat{\cap} \left(\hat{\Psi}^2_{(\hat{w})}\right)^c.$$

(ii). Since $\left(\hat{\Psi}^1_{(\hat{w})} \hat{\cap} \hat{\Psi}^2_{(\hat{w})}\right)^c$

$$= \left(\left\{ \frac{\left(\mathcal{P}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \circledcirc \mathcal{P}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), \mathcal{Q}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \circledast \mathcal{Q}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), \mathcal{R}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \circledast \mathcal{R}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), \xi^1(\hat{v}) \circledcirc \xi^2(\hat{v})\right)}{\hat{v}} \right\}\right)^c$$

$$= \left\{ \frac{\left(\mathcal{R}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \circledast \mathcal{R}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), 1-\left(\mathcal{Q}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \circledast \mathcal{Q}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v})\right), \mathcal{P}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \circledcirc \mathcal{P}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), 1-\left(\xi^1(\hat{v}) \circledcirc \xi^2(\hat{v})\right)\right)}{\hat{v}} \right\}$$

$$= \left\{ \frac{\left(\mathcal{R}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \vee \mathcal{R}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), 1-\left(\mathcal{Q}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \vee \mathcal{Q}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v})\right), \mathcal{P}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \wedge \mathcal{P}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), 1-\left(\xi^1(\hat{v}) \wedge \xi^2(\hat{v})\right)\right)}{\hat{v}} \right\}$$

$$= \left\{ \frac{\left(\mathcal{R}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \vee \mathcal{R}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), 1-\mathcal{Q}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \wedge 1-\mathcal{Q}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), \mathcal{P}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \wedge \mathcal{P}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), 1-\xi^1(\hat{v}) \vee 1-\xi^2(\hat{v})\right)}{\hat{v}} \right\}$$

$$= \left\{ \frac{\left(\mathcal{R}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \circledast \mathcal{R}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), 1-\mathcal{Q}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \circledcirc 1-\mathcal{Q}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), \mathcal{P}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \circledcirc \mathcal{P}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), 1-\xi^1(\hat{v}) \circledast 1-\xi^2(\hat{v})\right)}{\hat{v}} \right\}$$

$$= \left\{ \frac{\mathcal{R}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), 1-\mathcal{Q}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), \mathcal{P}^1{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), 1-\xi^1(\hat{v})}{\hat{v}} \right\} \hat{\cup} \left\{ \frac{\mathcal{R}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), 1-\mathcal{Q}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), \mathcal{P}^2{}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}), 1-\xi^2(\hat{v})}{\hat{v}} \right\}$$

$$= \left(\hat{\Psi}^1_{(\hat{w})}\right)^c \hat{\cup} \left(\hat{\Psi}^2_{(\hat{w})}\right)^c. \square$$

4. **Application of SV-NFHS-Set in Decision-Making**

In this section, a decision-making algorithm is proposed and is elaborated by daily life decision-making problem.

Algorithm 4.1. Step 1: Input the SV-NFHS- set $\hat{\Phi}_{(\hat{w})} \in \biguplus_{svnfhs}$ as follows:

$$\hat{\Phi}_{(\hat{w})} = \left\{ \begin{array}{l} \left. \frac{\left(\mathcal{A}_{\hat{\Phi}_{(\hat{w})}}(\hat{u}), \mathcal{B}_{\hat{\Phi}_{(\hat{w})}}(\hat{u}), \mathcal{C}_{\hat{\Phi}_{(\hat{w})}}(\hat{u}), \xi(\hat{u})\right)}{\hat{u}} \right| \\ \hat{w} \in \hat{\mathcal{W}}, \hat{u} \in \hat{\mathcal{V}}, \\ 0 \le \mathcal{A}_{\hat{\Phi}_{(\hat{w})}}(\hat{u}) + \mathcal{B}_{\hat{\Phi}_{(\hat{w})}}(\hat{u}) + \mathcal{C}_{\hat{\Phi}_{(\hat{w})}}(\hat{u}) \le 3 \end{array} \right\},$$

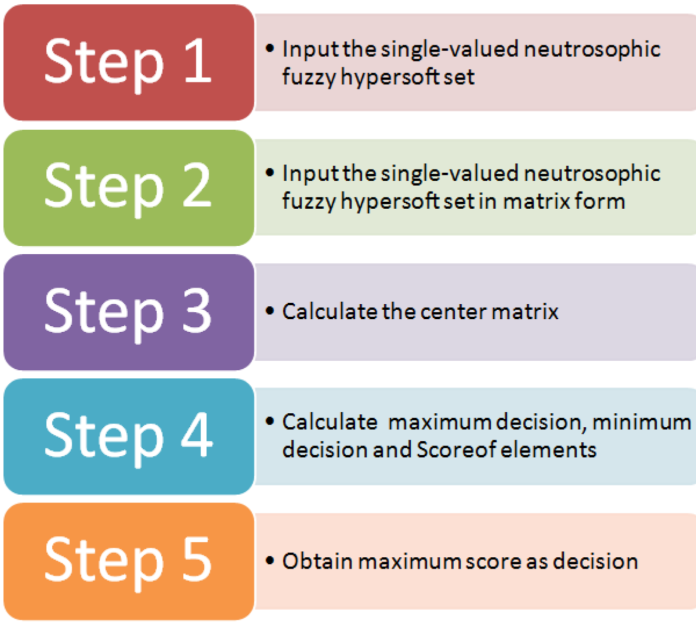

FIGURE 1. Optimal Decision Based on a SV-NFHS-Set matrix

to be evaluated by a group of experts n to element x on parameter i.

Step 2: Input the SV-NFHS- set in matrix form (written as $\mathcal{M}_{l\times k}, k, l \in N$).

$$\mathcal{M}_{l\times k} = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1k} \\ m_{21} & m_{22} & \cdots & m_{2k} \\ m_{31} & m_{32} & \cdots & m_{3k} \\ \vdots & \vdots & \ddots & \vdots \\ m_{l1} & m_{l2} & \cdots & m_{lk} \end{bmatrix}$$

where

$m_{11} = \left(\mathcal{A}_{\hat{\Phi}_{(\hat{w}_1)}}(\hat{u}_1), \mathcal{B}_{\hat{\Phi}_{(\hat{w}_1)}}(\hat{u}_1), \mathcal{C}_{\hat{\Phi}_{(\hat{w}_1)}}(\hat{u}_1), \xi(\hat{u}_1)\right)$

$m_{12} = \left(\mathcal{A}_{\hat{\Phi}_{(\hat{w}_1)}}(\hat{u}_2), \mathcal{B}_{\hat{\Phi}_{(\hat{w}_1)}}(\hat{u}_2), \mathcal{C}_{\hat{\Phi}_{(\hat{w}_1)}}(\hat{u}_2), \xi(\hat{u}_2)\right)$

....

....

....

$m_{1k} = \left(\mathcal{A}_{\hat{\Phi}_{(\hat{w}_1)}}(\hat{u}_k), \mathcal{B}_{\hat{\Phi}_{(\hat{w}_1)}}(\hat{u}_k), \mathcal{C}_{\hat{\Phi}_{(\hat{w}_1)}}(\hat{u}_k), \xi(\hat{u}_k)\right)$

$m_{21} = \left(\mathcal{A}_{\hat{\Phi}_{(\hat{w}_2)}}(\hat{u}_1), \mathcal{B}_{\hat{\Phi}_{(\hat{w}_2)}}(\hat{u}_1), \mathcal{C}_{\hat{\Phi}_{(\hat{w}_2)}}(\hat{u}_1), \xi(\hat{u}_1)\right)$

$m_{22} = \left(\mathcal{A}_{\hat{\Phi}_{(\hat{w}_2)}}(\hat{u}_2), \mathcal{B}_{\hat{\Phi}_{(\hat{w}_2)}}(\hat{u}_2), \mathcal{C}_{\hat{\Phi}_{(\hat{w}_2)}}(\hat{u}_2), \xi(\hat{u}_2)\right)$

....

....

....

$m_{2k} = \left(\mathcal{A}_{\hat{\Phi}_{(\hat{w}_2)}}(\hat{u}_k), \mathcal{B}_{\hat{\Phi}_{(\hat{w}_2)}}(\hat{u}_k), \mathcal{C}_{\hat{\Phi}_{(\hat{w}_2)}}(\hat{u}_k), \xi(\hat{u}_k)\right)$

$m_{31} = \left(\mathcal{A}_{\hat{\Phi}_{(\hat{w}_3)}}(\hat{u}_1), \mathcal{B}_{\hat{\Phi}_{(\hat{w}_3)}}(\hat{u}_1), \mathcal{C}_{\hat{\Phi}_{(\hat{w}_3)}}(\hat{u}_1), \xi(\hat{u}_1)\right)$
$m_{32} = \left(\mathcal{A}_{\hat{\Phi}_{(\hat{w}_3)}}(\hat{u}_2), \mathcal{B}_{\hat{\Phi}_{(\hat{w}_3)}}(\hat{u}_2), \mathcal{C}_{\hat{\Phi}_{(\hat{w}_3)}}(\hat{u}_2), \xi(\hat{u}_2)\right)$
....
....
....
$m_{3k} = \left(\mathcal{A}_{\hat{\Phi}_{(\hat{w}_3)}}(\hat{u}_k), \mathcal{B}_{\hat{\Phi}_{(\hat{w}_3)}}(\hat{u}_k), \mathcal{C}_{\hat{\Phi}_{(\hat{w}_3)}}(\hat{u}_k), \xi(\hat{u}_k)\right)$
....
....
....
$m_{l1} = \left(\mathcal{A}_{\hat{\Phi}_{(\hat{w}_l)}}(\hat{u}_1), \mathcal{B}_{\hat{\Phi}_{(\hat{w}_l)}}(\hat{u}_1), \mathcal{C}_{\hat{\Phi}_{(\hat{w}_l)}}(\hat{u}_1), \xi(\hat{u}_1)\right)$
$m_{l2} = \left(\mathcal{A}_{\hat{\Phi}_{(\hat{w}_l)}}(\hat{u}_2), \mathcal{B}_{\hat{\Phi}_{(\hat{w}_l)}}(\hat{u}_2), \mathcal{C}_{\hat{\Phi}_{(\hat{w}_l)}}(\hat{u}_2), \xi(\hat{u}_2)\right)$
....
....
....
$m_{lk} = \left(\mathcal{A}_{\hat{\Phi}_{(\hat{w}_l)}}(\hat{u}_k), \mathcal{B}_{\hat{\Phi}_{(\hat{w}_l)}}(\hat{u}_k), \mathcal{C}_{\hat{\Phi}_{(\hat{w}_l)}}(\hat{u}_k), \xi(\hat{u}_k)\right)$

Step 3: Determine the core matrix (i.e.,
$\delta_{\hat{\Phi}_{(\hat{w})}}(\hat{u}_k) = \left(\mathcal{A}_{\hat{\Phi}_{(\hat{w})}}(\hat{u}_k) + \mathcal{B}_{\hat{\Phi}_{(\hat{w})}}(\hat{u}_k) + \mathcal{C}_{\hat{\Phi}_{(\hat{w})}}(\hat{u}_k)\right) - \xi(\hat{u}_k))$:

$$G_{l\times k} = \begin{pmatrix} \delta_{\hat{\Phi}_{(\hat{w}_1)}}(\hat{u}_1) & \delta_{\hat{\Phi}_{(\hat{w}_1)}}(\hat{u}_2) & \dots & \delta_{\hat{\Phi}_{(\hat{w}_1)}}(\hat{u}_k) \\ \delta_{\hat{\Phi}_{(\hat{w}_2)}}(\hat{u}_1) & \delta_{\hat{\Phi}_{(\hat{w}_2)}}(\hat{u}_2) & \dots & \delta_{\hat{\Phi}_{(\hat{w}_2)}}(\hat{u}_k) \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{\hat{\Phi}_{(\hat{w})}}(\hat{u}_1) & \delta_{\hat{\Phi}_{(\hat{w})}}(\hat{u}_2) & \dots & \delta_{\hat{\Phi}_{(\hat{w})}}(\hat{u}_k) \end{pmatrix}.$$

Step 4: Calculate the $\mathfrak{D}^{\max}(\hat{u}_j)$ (Max. decision), $\mathfrak{D}^{\min}(\hat{u}_j)$ (Min.decision), and $\mathfrak{S}(\hat{u}_j)$ (score) of all $\hat{u}_j$ (j = 1,2,..., k):

$$\mathfrak{D}^{\max}(\hat{u}_j) = \sum_{i=1}^{l}\left(1 - \delta_{\hat{\Phi}_{(\hat{w})}}(\hat{u}_j)\right)^2, \mathfrak{D}^{\min}(\hat{u}_j) = \sum_{i=1}^{l}\left(\delta_{\hat{\Phi}_{(\hat{w})}}(\hat{u}_j)\right)^2$$

$$\mathfrak{S}(\hat{u}_j) = \mathfrak{D}^{\max}(\hat{u}_j) + \mathfrak{D}^{\min}(\hat{u}_j).$$

(to understand the motivation behind this method, let ρ be the Euclidean metric on R^l, $0 = (0, ..., 0)^T \in R^l, 1 = (1, ..., 1)^T \in R^l$, and $\theta_j = \left(\theta_{1,\hat{u}_j}, \theta_{2,\hat{u}_j}, ..., \theta_{l,\hat{u}_j}\right)^T \in R^l$. Thus $\mathfrak{S}(\hat{u}_j) = [\rho(\theta_j, 1)]^2 + [\rho(\theta_j, 0)]^2 \, (j = 1, 2, ..., k)$.

Step 5: Find optimal decision ρ with the help of following criterion

$$\hat{u}_k = Max\{\mathfrak{S}(\hat{u}_1), \mathfrak{S}(\hat{u}_2), ..., \mathfrak{S}(\hat{u}_k)\}.$$

Now Algorithm 4.1 is elaborated with the help of following example.

Example 4.2. A manufacturing company is wants to select a supplier for one of its manufactures products. Consider a SV-NFHS-set which illustrates the suitability of suppliers and enables the company to select the most suitable supplier for its product. There are five alternatives suppliers and these form the discourse of universe $\mathcal{Z} = \{z_1, z_2, z_3, z_4, z_5\}$. Company constitutes a committee (i.e. Decision-Makers)consisting of its procurement supervisor (chairman of committee) and buyers for the evaluation of these suppliers. With mutual consultation, the chairman develops a mutual consensus and designs a set of parameters $\mathcal{H} = \{h_1, h_2, h_3, h_4\}$, for this evaluation, where h_1, h_2, h_3 and h_4 represent quality and reliability, cost, service and process and design capabilities respectively. These attributes are further classified into attribute-valued sets $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ and $\mathcal{J}_4$ respectively on the basis of their respective categorical criteria, where
$\mathcal{J}_1 = \{j^{11}, j^{12}\}$
$\mathcal{J}_2 = \{j^{21}, j^{22}\}$
$\mathcal{J}_3 = \{j^{31}, j^{32}\}$
and
$\mathcal{J}_4 = \{j^{41}, j^{42}\}$.
Therefore,
$\mathcal{K} = \mathcal{J}_1 \times \mathcal{J}_2 \times \mathcal{J}_3 \times \mathcal{J}_4 = \{k^1, k^2, ..., k^{16}\}$, where each $k^i, i = 1, 2, ..., 16$ is a $4 - tuple$ element of $\mathcal{K}$. For the sake of convenience, k^1, k^3, k^5 and k^{14} are given preference during the evaluation process by the committee. The evaluation process is completed with the help of proposed Algorithm 4.1.

Step 1 The whole scenario is interpreted in the form of SV-NFHS-set $\hat{\Phi}_{(k^i)} \in \biguplus_{svnfhs}$ and is given below

$\mathcal{K}$	z_1	z_2	z_3	z_4	z_5
k^1	$(0.8, 0.4, 0.3, 0.2)$	$(0.7, 0.5, 0.4, 0.3)$	$(0.6, 0.6, 0.5, 0.4)$	$(0.5, 0.7, 0.6, 03)$	$(0.4, 0.8, 0.7, 0.2)$
k^3	$(0.7, 0.5, 0.4, 0.3)$	$(0.6, 0.6, 0.5, 0.4)$	$(0.5, 0.7, 0.6, 0.5)$	$(0.4, 0.8, 0.7, 0.4)$	$(0.3, 0.9, 0.9, 0.4)$
k^5	$(0.6, 0.4, 0.5, 0.4)$	$(0.5, 0.7, 0.6, 0.5)$	$(0.4, 0.8, 0.7, 0.6)$	$(0.3, 0.9, 0.8, 0.5)$	$(0.2, 0.1, 0.1, 0.2)$
k^{14}	$(0.5, 0.5, 0.5, 0.5)$	$(0.4, 0.8, 0.7, 0.6)$	$(0.3, 0.9, 0.8, 0.7)$	$(0.2, 0.3, 0.9, 0.6)$	$(0.1, 0.2, 0.2, 0.3)$

Step 2

$$\mathcal{M}_{5\times4} = \begin{pmatrix} (0.8, 0.4, 0.3, 0.2) & (0.7, 0.5, 0.4, 0.3) & (0.6, 0.4, 0.5, 0.4) & (0.5, 0.5, 0.5, 0.5) \\ (0.7, 0.5, 0.4, 0.3) & (0.6, 0.6, 0.5, 0.4) & (0.5, 0.7, 0.6, 0.5) & (0.4, 0.8, 0.7, 0.6) \\ (0.6, 0.6, 0.5, 0.4) & (0.5, 0.7, 0.6, 0.5) & (0.4, 0.8, 0.7, 0.6) & (0.3, 0.9, 0.8, 0.7) \\ (0.5, 0.7, 0.6, 03) & (0.4, 0.8, 0.7, 0.4) & (0.3, 0.9, 0.8, 0.5) & (0.2, 0.3, 0.9, 0.6) \\ (0.4, 0.8, 0.7, 0.2) & (0.3, 0.9, 0.9, 0.4) & (0.2, 0.1, 0.1, 0.2) & (0.1, 0.2, 0.2, 0.3) \end{pmatrix}.$$

Step 3

$$C_{3\times 5} = \begin{pmatrix} 1.3 & 1.3 & 1.1 & 1.0 \\ 1.3 & 1.3 & 1.3 & 1.3 \\ 1.3 & 1.3 & 1.3 & 1.3 \\ 1.5 & 1.5 & 1.5 & 0.8 \\ 1.7 & 1.7 & 0.2 & 0.2 \end{pmatrix}.$$

Step 4 The values of $\mathfrak{D}^{\max}(z_j)$, $\mathfrak{D}^{\min}(z_j)$ and $S(z_j)$ for $j = 1, 2, 3, 4, 5$ are given in the below table

	z_1	z_2	z_3	z_4	z_5
$\mathfrak{D}^{\max}(z_j)$	0.19	0.36	0.36	0.79	2.26
$\mathfrak{D}^{\min}(z_j)$	5.59	6.76	6.76	7.39	5.86
$S(z_j)$	5.78	7.12	7.12	8.18	8.12

Step 5 It is vivid that z_5 is selected as the best decision due to its highest score.

5. Comparison Analysis

There are many cases where consideration of only attributes is not sufficient, all available distinct attributes are further partitioned into their respective disjoint attribute-valued sets. Decision making techniques based on most of existing models are inadequate for such cases. Therefore, our proposed model not only emphasizes on the due status of such partitioning of attributes but also facilitates the decision makers to deal daily life problems with great ease. Table 1 and Table 2 present a vivid comparison of our proposed model with some existing relevant models under the evaluating indicators MD (Membership Degree), NMD (Non-membership Degree), ID (Indeterminacy Degree), SAAF (Single Argument Approximate Function) and MAAF (Multi Argument Approximate Function).

5.1. *Discussion*

In this subsection, we present the generalization of our proposed structure. Our proposed structure SV-NFHS-set is very useful for tackling many decisive systems and it is the most generalized structure (see Figure 2) as it transforms to:

(i) Single-Valued Neutrosophic Hypersoft Set (SV-NHS-set) if fuzzy valued degree is omitted.

(ii) Neutrosophic Hypersoft Fuzzy Set (SV-NFHS-set) if

$$0 \leq \mathcal{P}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) + \mathcal{Q}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) + \mathcal{R}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \leq 3$$

is replaced with

$$^{-}0 \leq \mathcal{P}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) + \mathcal{Q}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) + \mathcal{R}_{\hat{\Psi}_{(\hat{w})}}(\hat{v}) \leq 3^{+}$$

.

TABLE 1. Comparison with existing models

Authors	Structure	Remarks
Broumi et al. [10]	Intuitionistic NS-Set	(i) Single set of parameters is used with intuitionistic fuzzy values, (ii) Approximate function is the subset of universal set
Das et al. [48]	Neutrosophic fuzzy set	(i) Single set of parameters is used with intuitionistic fuzzy values, (ii) Approximate function is the subset of intuitionistic fuzzy set
Khalil et al. [49]	Single-valued NFS-Set	(i) Single set of parameters is used with neutrosophic values, (ii) Approximate function is the subset of universal set
Proposed Model	Single-valued NFHS-Set	(i) Single set of parameters is further classified into disjoint attribute-valued sets used with intuitionistic fuzzy values, (ii) Approximate function is the subset of neutrosophic set

TABLE 2. Comparison with existing models under appropriate features

Authors	Structure	MD	NMD	ID	SAAF	MAAF
Broumi et al. [10]	Intuitionistic NS-Set	✓	✓	✓	✓	×
Das et al. [48]	Neutrosophic fuzzy set	✓	✓	✓	×	×
Khalil et al. [49]	Single-valued NFS-Set	✓	✓	✓	✓	×
Proposed Model	Single-valued NFHS-Set	✓	✓	✓	✓	✓

(iii) Intuitionistic Fuzzy Hypersoft Set (IFHS-set) if indeterminacy degree and fuzzy valued degree are ignored and remaining uncertain components be restricted within closed unit interval in approximate function of SV-NFHS-set.

(iv) Intuitionistic Fuzzy Soft Set (IFS-set) if
- attribute-valued sets are replaced with only attributes,
- indeterminacy degree and fuzzy valued degree are ignored,
- remaining two uncertain components are made interdependent within closed unit interval in approximate function of SV-NFHS-set.

(v) Fuzzy Hypersoft Set (FHS-set) if
- only fuzzy membership is focussed,

- falsity, indeterminacy degree and fuzzy valued degree are ignored.

(vi) Hypersoft Set (HS-set) if all uncertain components membership, non-membership, indeterminacy and fuzzy valued degrees are ignored.

(vii) Soft Set (S-set) if

- attribute-valued sets are replaced with only attributes,
- all uncertain components membership, non-membership, indeterminacy and fuzzy valued degrees are ignored.

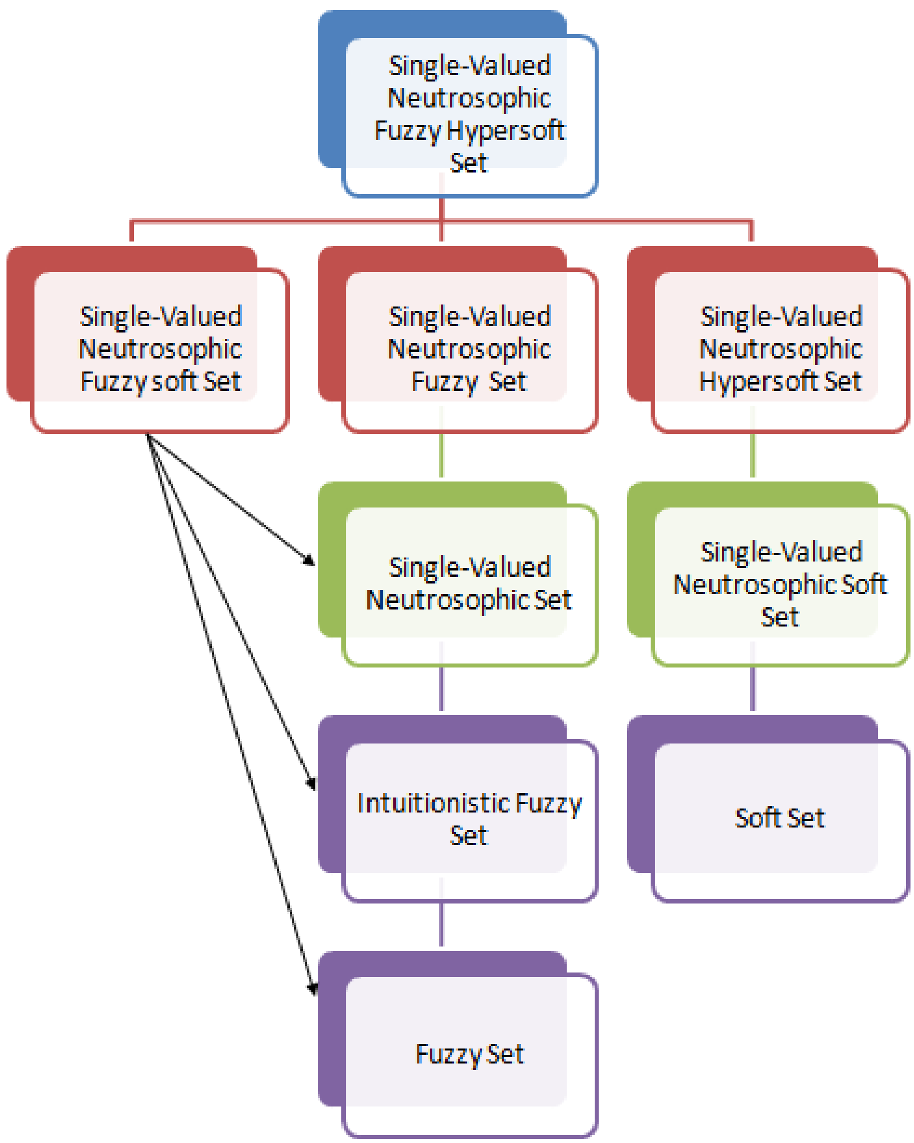

FIGURE 2. Generalization of IV-NFHS-Set

6. Conclusions

In this study, single-valued neutrosophic fuzzy hypersoft set is conceptualized with some of its elementary properties and theoretic operations. Novel algorithm is proposed for decision making and is validated with the help of illustrative examples for appropriate choosing of suitable supplier. Future work may include the extension of this work for:

- The development of algebraic structures i.e. topological spaces, vector spaces etc.,
- Dealing with decision making problems with multi-criteria decision making techniques,
- Applying in medical diagnosis and optimization for agricultural yield,
- Investigating and determining similarity, distance, dissimilarity measures and entropies between the proposed structures.

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Games Based on Simplified Neutrosophic Multiplicative Soft Sets and Their Applications

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Abstract. In this paper, we firstly define simplified neutrosophic multiplicative soft sets by combining simplified neutrosophic multiplicative sets and soft sets. Meanwhile, we introduce some basic operations of simplified neutrosophic multiplicative soft sets and discuss their related properties. Later, we describe two person simplified neutrosophic multiplicative soft games, and give different types of solution models of these games which are simplified neutrosophic multiplicative soft saddle points, simplified neutrosophic multiplicative soft upper and lower values, simplified neutrosophic multiplicative soft dominated strategies and simplified neutrosophic multiplicative soft Nash equilibrium. Moreover, the solution models of two person simplified neutrosophic multiplicative soft games are applied to a real-world problem and supported by comparison analysis. Finally, the framework of n-person simplified neutrosophic multiplicative soft games is presented.

Keywords: Simplified neutrosophic multiplicative sets; Soft sets; Simplified neutrosophic multiplicative soft sets; Simplified neutrosophic multiplicative soft games

1. Introduction

Almost all of the mathematical models proposed until the middle of the 19th century were not suitable for dealing with uncertainty and vagueness. In 1965, Zadeh [47] described the concept of fuzzy sets (FSs) that allow the representation of uncertainty in a mathematical way. While fuzzy sets were based on the truth-membership value of uncertainty, Atanassov [7] generalized the FSs by including the falsity-membership value, and thus proposed the idea of intuitionistic fuzzy sets (IFSs). In 1998, Smarandache [39] introduced the neutrosophic set (NS) to reflect the values of truth-membership, indeterminacy-membership and falsity-membership simultaneously. However, due to the difficulty in applying to real-world problems when the values of truth-membership, indeterminacy-membership and falsity-membership are real non-standard subsets of $]0^-, 1^+[$, Wang et al. [43] and Ye [46] derived single-valued neutrosophic set (SVNS) and simplified neutrosophic set (SNS) as specific descriptions of NSs, respectively. Recently, works on (single-valued/simplified) NS theory has been progressing rapidly and is presenting applications in a wide variety of fields, for instance; aggregation operators [16, 17, 19, 21] and information measures [3, 32] and various solution models for real-life problems [1–5, 42] of (single-valued/simplified) NS.

Although the FSs, IFSs and NSs are powerful mathematical models for dealing with uncertainties, these sets use the 0-1 scale, which is distributed symmetrically and uniformly. However, there are real-life problems that need to be scaled as unsymmetrically and non-uniformly. The grading system of universities can be given as the most obvious example of this situation [18]. In 1990, Saaty [36] proposed the 1-9 scale (or $\frac{1}{9}-9$ scale) as an useful tool to deal with such problems that need to be scaled unsymmetrically and non-uniformly whilst assigning the variable grades. These different scales lead to the construction of multiplicative preference relation [37]. By inspired this idea, Xia et al. [45] demonstrated that the interval-valued fuzzy preference relations can be equivalent to the intuitionistic fuzzy preference relations, and then introduced the intuitionistic multiplicative sets (IMSs) and the intuitionistic multiplicative preference relations (IMPRs). Moreover, they present a comparison between 0.1-0.9 and $\frac{1}{9}-9$ scales as in Table 1.

Table 1. The comparison between 0.1-0.9 and $\frac{1}{9}-9$ scales [45]

$\frac{1}{9}-9$ scale	0.1-0.9 scale	Meaning
$\frac{1}{9}$	0.1	Extremely not preferrred
$\frac{1}{7}$	0.2	Very strongly not preferrred
$\frac{1}{5}$	0.3	Strongly not preferrred
$\frac{1}{3}$	0.4	Moderately not preferrred
1	0.5	Equally preferrred
3	0.6	Moderately preferrred
5	0.7	Strongly preferrred
7	0.8	Very strongly preferrred
9	0.9	Extremely preferrred
Other values between $\frac{1}{9}$ and 9	Other values between 0 and 1	Intermediate values used to present compromise

The IMSs and IMPRs were studied widely [14,15,20,44]. In 2019, Köseoğlu et al. [26] proposed the idea of simplified neutrosophic multiplicative sets (SNMSs) generalizing the IMSs and studied the simplified neutrosophic multiplicative preference relations (SNMPRs).

In 1999, Molodtsov [28] initiated the theory of soft set (SS) which classifies objects according to parameters or attributes. Çağman and Enginoğlu [9] revisited the concept of soft set to make Molodtsov's soft set operations more functional. Many authors studied the theory [6, 8, 9, 27, 38, 41] and applications [22–25, 33–35] of soft sets. In 2016, Deli and Çağman [10] gave an application of soft sets in decision making based on game theory, and thus pioneered the idea of soft games. Moreover, Deli et al. [12] studied several expected impact functions and algorithms modelling games under the soft sets. In recent years, several game schemes based on the fuzzy soft sets, intuitionistic fuzzy soft sets and neutrosophic soft sets have been proposed [11, 29, 40]. The motivation of this paper is to propose a new extension of soft sets and revisit soft games from a different perspective. Relatedly, this paper introduces the simplified neutrosophic multiplicative soft sets (SNMSSs) fusing SNMSs and SSs, and proposes a new game framework based on the SNMSSs called simplified neutrosophic multiplicative soft game (SNM soft game).

The rest of this article is arranged as follows: Section 2 reviews some definitions and results related to the NSs, SNSs, SNMSs and SSs. Section 3 presents the concept of SNMSSs and their fundamental operations with structural properties. Section 4 is devoted to the four different solution methods of two person SNM soft games and and their efficiency in dealing with real-world issues. In Section 5, two person SNM soft games are extended to n-person SNM soft games. In Section 6, the concluding remarks are given.

2. **Preliminaries**

In this section, some basic concepts about the neutrosophic sets, simplified neutrosophic sets, simplified neutrosophic multiplicative sets and soft sets are given.

Let $\mathcal{H}$ be a space of points (object) with a generic element denoted by $\hbar$.

Definition 2.1. ([39]) A neutrosophic set (NS) $\mathfrak{N}$ in $\mathcal{H}$ is characterized by a truth-membership function $t_{\mathfrak{N}} : \mathcal{H} \to]0^-, 1^+[$, an indeterminacy-membership function $i_{\mathfrak{N}} : \mathcal{H} \to]0^-, 1^+[$, and a falsity-membership function $f_{\mathfrak{N}} : \mathcal{H} \to]0^-, 1^+[$. $t_{\mathfrak{N}}(\hbar)$, $i_{\mathfrak{N}}(\hbar)$ and $f_{\mathfrak{N}}(\hbar)$ are real standard or non-standard subsets of $]0^-, 1^+[$. There is no restriction on the sum of $t_{\mathfrak{N}}(\hbar)$, $i_{\mathfrak{N}}(\hbar)$ and $f_{\mathfrak{N}}(\hbar)$, so $0^- \leq \sup t_{\mathfrak{N}}(\hbar) + \sup i_{\mathfrak{N}}(\hbar) + \sup f_{\mathfrak{N}}(\hbar) \leq 3^+$ for $\hbar \in \mathcal{H}$.

However, Wang et al. [43] and Ye [46] stated the difficulty of employing the NSs of non-standard intervals in practice, and proposed the simplified neutrosophic sets.

Definition 2.2. ([46]) An NS $\mathfrak{N}$ is characterized by a truth-membership function $t_{\mathfrak{N}} : \mathcal{H} \to [0, 1]$, an indeterminacy-membership function $i_{\mathfrak{N}} : \mathcal{H} \to [0, 1]$, and a falsity-membership function $f_{\mathfrak{N}} : \mathcal{H} \to [0, 1]$. $t_{\mathfrak{N}}(\hbar)$, $i_{\mathfrak{N}}(\hbar)$ and $f_{\mathfrak{N}}(\hbar)$ are singleton subintervals/subsets in the standard interval $[0, 1]$, then it is said to be a simplified neutrosophic set (SNS) and described by

$$\mathfrak{N} = \{\langle \hbar, (t_{\mathfrak{N}}(\hbar), i_{\mathfrak{N}}(\hbar), f_{\mathfrak{N}}(\hbar))\rangle : \ \hbar \in \mathcal{H}\}. \tag{1}$$

This kind of NS is is termed to be a single-valued neutrosophic set (SVNS) by Wang et al. [43]. Throughout this paper, we will use the term "simplified neutrosophic set (SNS)".

Definition 2.3. ([26]) A simplified neutrosophic multiplicative set (SNMS) $\mathfrak{M}$ in $\mathcal{H}$ is defined as

$$\mathfrak{M} = \{\langle \hbar, (\rho_{\mathfrak{M}}(\hbar), \tau_{\mathfrak{M}}(\hbar), \sigma_{\mathfrak{M}}(\hbar))\rangle : \ \hbar \in \mathcal{H}\}, \tag{2}$$

which assigns to each element $\hbar$ a truth-membership information $\rho_{\mathfrak{M}}(\hbar)$, an indeterminacy-membership information $\tau_{\mathfrak{M}}(\hbar)$, and a falsity-membership information $\sigma_{\mathfrak{M}}(\hbar)$ with conditions

$$\frac{1}{9} \leq \rho_{\mathfrak{M}}(\hbar), \tau_{\mathfrak{M}}(\hbar), \sigma_{\mathfrak{M}}(\hbar) \leq 9 \text{ and } 0 < \rho_{\mathfrak{M}}(\hbar)\sigma_{\mathfrak{M}}(\hbar) \leq 1. \tag{3}$$

for each $\hbar \in \mathcal{H}$.

The set of all SNMSs in $\mathcal{H}$ is denoted by $\mathfrak{P}(\mathcal{H})$.

Definition 2.4. ([26]) Let $\mathfrak{M}$, $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be the SNMSs. Then, some operational rules on SNMSs are given as follows.

(a): $\mathfrak{M}_1 \subseteq \mathfrak{M}_2 \Leftrightarrow \rho_{\mathfrak{M}_1}(\hbar) \leq \rho_{\mathfrak{M}_2}(\hbar)$, $\tau_{\mathfrak{M}_1}(\hbar) \geq \tau_{\mathfrak{M}_2}(\hbar)$ and $\sigma_{\mathfrak{M}_1}(\hbar) \geq \sigma_{\mathfrak{M}_2}(\hbar)$ for all $\hbar \in \mathcal{H}$.
(b): $\mathfrak{M}_1 = \mathfrak{M}_2 \Leftrightarrow \rho_{\mathfrak{M}_1}(\hbar) = \rho_{\mathfrak{M}_2}(\hbar)$, $\tau_{\mathfrak{M}_1}(\hbar) = \tau_{\mathfrak{M}_2}(\hbar)$ and $\sigma_{\mathfrak{M}_1}(\hbar) = \sigma_{\mathfrak{M}_2}(\hbar)$ for all $\hbar \in \mathcal{H}$.
(c): $\mathfrak{M}^c = \{\langle \hbar, (\sigma_{\mathfrak{M}}(\hbar), \frac{1}{\tau_{\mathfrak{M}}(\hbar)}, \rho_{\mathfrak{M}}(\hbar)\}\rangle : \ \hbar \in \mathcal{H}\}$.
(d):

$$\mathfrak{M}_1 \cap \mathfrak{M}_2 = \left\{ \left\langle \ \hbar, \begin{pmatrix} \min\{\rho_{\mathfrak{M}_1}(\hbar), \rho_{\mathfrak{M}_2}(\hbar)\}, \\ \max\{\tau_{\mathfrak{M}_1}(\hbar), \tau_{\mathfrak{M}_2}(\hbar)\}, \\ \max\{\sigma_{\mathfrak{M}_1}(\hbar), \sigma_{\mathfrak{M}_2}(\hbar)\} \end{pmatrix} \ \right\rangle : \hbar \in \mathcal{H} \ \right\}.$$

(e):

$$\mathfrak{M}_1 \cup \mathfrak{M}_2 = \left\{ \left\langle \ \hbar, \begin{pmatrix} \max\{\rho_{\mathfrak{M}_1}(\hbar), \rho_{\mathfrak{M}_2}(\hbar)\}, \\ \min\{\tau_{\mathfrak{M}_1}(\hbar), \tau_{\mathfrak{M}_2}(\hbar)\}, \\ \min\{\sigma_{\mathfrak{M}_1}(\hbar), \sigma_{\mathfrak{M}_2}(\hbar)\} \end{pmatrix} \ \right\rangle : \hbar \in \mathcal{H} \ \right\}.$$

Definition 2.5. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be two SNMSs in $\mathcal{H}$. The cartesian product of $\mathfrak{M}_1$ and $\mathfrak{M}_2$, denoted by $\mathfrak{M}_1 \times \mathfrak{M}_2$, is an SNMS in $\mathcal{H} \times \mathcal{H}$ and defined as

$$\mathfrak{M}_1 \times \mathfrak{M}_2 = \left\{ \left\langle \ (\hbar, \hbar'), \begin{pmatrix} \min\{\rho_{\mathfrak{M}_1}(\hbar), \rho_{\mathfrak{M}_2}(\hbar')\}, \\ \max\{\tau_{\mathfrak{M}_1}(\hbar), \tau_{\mathfrak{M}_2}(\hbar')\}, \\ \max\{\sigma_{\mathfrak{M}_1}(\hbar), \sigma_{\mathfrak{M}_2}(\hbar')\} \end{pmatrix} \ \right\rangle : (\hbar, \hbar') \in \mathcal{H} \times \mathcal{H} \ \right\}.$$

In 1999, Molodtsov [28] introduced the notion of soft set as an effective mathematical model for dealing with uncertainty. In 2010, Çağman and Enginoğlu [9] revisited the concept of soft set to make Molodtsov's soft set operations more functional, and presented the following definition.

Definition 2.6. ([9, 28]) Let $\mathcal{H}$ be a set of alternatives, and $P(\mathcal{H})$ be a power set of $\mathcal{H}$. Also, let $\mathcal{S}$ be a set of parameters (or attributes) and $\mathcal{X} \subseteq \mathcal{S}$. The pair $\Gamma_{\mathcal{X}} = (\gamma_{\mathcal{X}}, \mathcal{S})$ is called a soft set (SS) over $\mathcal{H}$ and described as

$$\Gamma_{\mathcal{X}} = (\gamma_{\mathcal{X}}, \mathcal{S}) = \{(x, \gamma_{\mathcal{X}}(x)) : \ x \in \mathcal{S}, \ \gamma_{\mathcal{X}}(x) \in P(\mathcal{H})\}, \tag{4}$$

where $\gamma_{\mathcal{X}} : \mathcal{S} \to P(\mathcal{H})$, called an approximate function, such that $\gamma_{\mathcal{X}}(x) = \emptyset$ if $x \notin \mathcal{X}$.

3. Simplified Neutrosophic Multiplicative Soft Sets

In this section, we introduce the concept of simplified neutrosophic multiplicative soft set by combining SS and SNMS. Also, we study some simplified neutrosophic multiplicative soft set operations and their remarkable properties.

Definition 3.1. Let $\mathcal{H}$ be a set of alternatives, $\mathcal{S}$ be a set of parameters (or attributes) and $\mathcal{X} \subseteq \mathcal{S}$. Also, $\mathfrak{P}(\mathcal{H})$ denotes the set of all SNMSs in $\mathcal{H}$. The pair $\Theta_{\mathcal{X}} = (\theta_{\mathcal{X}}, \mathcal{S})$ is said to be a simplified neutrosophic multiplicative soft set (SNMSS) over $\mathcal{H}$ and described as

$$\Theta_{\mathcal{X}} = (\theta_{\mathcal{X}}, \mathcal{S}) = \{(x, \theta_{\mathcal{X}}(x)) : \ x \in \mathcal{S}, \ \theta_{\mathcal{X}}(x) \in \mathfrak{P}(\mathcal{H})\}, \tag{5}$$

where $\theta_{\mathcal{X}} : \mathcal{S} \to \mathfrak{P}(\mathcal{H})$, called an approximate function, such that $\theta_{\mathcal{X}}(x) = \emptyset$ if $x \notin \mathcal{X}$.

The set of all SNMSSs over $\mathcal{H}$ for the parameter set $\mathcal{S}$ is denoted by $\Im(\mathcal{H}, \mathcal{S})$.

Example 3.2. With the popularity of computers in daily life, more and more people prefer to buy well-equipped computers. Sometimes, instead of buying a new computer, the well-performing parts of poorly-equipped computer(s) are mounted to the other computer, and thus having a fairly well-equipped computer with more utility than the former. For this purpose, it is necessary to determine the performance of the main accessories (parts) for each computer, and the $\frac{1}{9} - 9$ scale is more suitable for this. Assume that $\mathcal{H} = \{\hbar_1, \hbar_2, \hbar_3, \hbar_4, \hbar_5, \hbar_6\}$ is a set of computers. Also, the CPU (x_1), memory (x_2), hard disk (x_3), motherboard (x_4) and graphics card (x_5) are the main accessories (parts) whose the performance will be tested. By classifying the computers with the $\frac{1}{9} - 9$ scale according to the performance criteria of main accessories (parts) in $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$, the following SNMSS is given.

$$\Theta_{\mathcal{X}} = \left\{ \begin{array}{l} (x_1, \{\langle \hbar_1, (3, 2, \frac{1}{6})\rangle, \langle \hbar_2, (1, \frac{1}{4}, 1)\rangle, \langle \hbar_3, (\frac{3}{7}, 2, 2)\rangle, \langle \hbar_4, (1, 1, 1)\rangle, \langle \hbar_5, (\frac{2}{5}, \frac{1}{9}, 1)\rangle, \langle \hbar_6, (8, 9, \frac{1}{9})\rangle\}) \\ (x_2, \{\langle \hbar_1, (\frac{1}{4}, \frac{3}{5}, \frac{1}{5})\rangle, \langle \hbar_2, (1, \frac{1}{9}, \frac{1}{9})\rangle, \langle \hbar_3, (\frac{2}{7}, 4, 2)\rangle, \langle \hbar_4, (\frac{1}{9}, 3, 5)\rangle, \langle \hbar_5, (4, 2, \frac{1}{4})\rangle, \langle \hbar_6, (\frac{7}{5}, 3, \frac{1}{5})\rangle\}) \\ (x_3, \{\langle \hbar_1, (2, 2, \frac{1}{2})\rangle, \langle \hbar_2, (\frac{1}{8}, \frac{1}{7}, \frac{1}{6})\rangle, \langle \hbar_3, (3, 1, \frac{2}{7})\rangle, \langle \hbar_4, (3, 4, \frac{1}{6})\rangle, \langle \hbar_5, (1, 4, 1)\rangle, \langle \hbar_6, (\frac{1}{9}, \frac{1}{2}, 9)\rangle\}) \\ (x_4, \{\langle \hbar_1, (2, \frac{1}{3}, \frac{1}{2})\rangle, \langle \hbar_2, (\frac{1}{6}, 1, 4)\rangle, \langle \hbar_3, (\frac{2}{9}, \frac{4}{5}, \frac{3}{2})\rangle, \langle \hbar_4, (\frac{1}{9}, 1, \frac{5}{6})\rangle, \langle \hbar_5, (3, 3, \frac{1}{4})\rangle, \langle \hbar_6, (2, \frac{1}{5}, \frac{1}{7})\rangle\}) \end{array} \right\}.$$

Here, $\langle \hbar_1, (3, 2, \frac{1}{6})\rangle \in \theta_{\mathcal{X}}(x_1)$ means that computer $\hbar_1$ has membership information of truth, indeterminacy and falsity as $(3, 2, \frac{1}{6})$ according to the performance of CPU. Other components can be interpreted similarly.

Definition 3.3. Let $\Theta_{\mathcal{X}}, \Theta_{\mathcal{Y}} \in \Im(\mathcal{H}, \mathcal{S})$.

- **(a):** $\Theta_{\mathcal{X}}$ is termed to be an SNMS subset of $\Theta_{\mathcal{Y}}$ if $\theta_{\mathcal{X}}(x) \subseteq \theta_{\mathcal{Y}}(x)$ for all $x \in S$. It is denoted by $\Theta_{\mathcal{X}} \widetilde{\subseteq} \Theta_{\mathcal{Y}}$.
- **(b):** The SNMSSs $\Theta_{\mathcal{X}}$ and $\Theta_{\mathcal{Y}}$ are equal if $\theta_{\mathcal{X}}(x) = \theta_{\mathcal{Y}}(x)$ for all $x \in S$. It is denoted by $\Theta_{\mathcal{X}} = \Theta_{\mathcal{Y}}$.
- **(c):** The complement of $\Theta_{\mathcal{X}}$, denoted by $\Theta_{\mathcal{X}}^{\tilde{c}}$, is an SNMSS defined by the approximate function $\theta_{\mathcal{X}}^{\tilde{c}} : \mathcal{S} \to \mathfrak{P}(\mathcal{H})$ such that

$$\theta_{\mathcal{X}^{\tilde{c}}}(x) = (\theta_{\mathcal{X}}(x))^c$$

  for all $x \in \mathcal{S}$.
- **(d):** The intersection of $\Theta_{\mathcal{X}}$ and $\Theta_{\mathcal{Y}}$, denoted by $\Theta_{\mathcal{X}} \widetilde{\cap} \Theta_{\mathcal{Y}}$, is an SNMSS defined by the SNM approximate function $\theta_{\mathcal{X} \widetilde{\cap} \mathcal{Y}} : \mathcal{S} \to \mathfrak{P}(\mathcal{H})$ such that

$$\theta_{\mathcal{X} \widetilde{\cap} \mathcal{Y}}(x) = \theta_{\mathcal{X}}(x) \cap \theta_{\mathcal{Y}}(x)$$

  for all $x \in \mathcal{S}$.

(e): The union of $\Theta_{\mathcal{X}}$ and $\Theta_{\mathcal{Y}}$, denoted by $\Theta_{\mathcal{X}}\widetilde{\cup}\Theta_{\mathcal{Y}}$, is an SNMSS defined by the SNM approximate function $\theta_{\mathcal{X}\widetilde{\cup}\mathcal{Y}} : \mathcal{S} \to \mathfrak{P}(\mathcal{H})$ such that

$$\theta_{\mathcal{X}\widetilde{\cup}\mathcal{Y}}(x) = \theta_{\mathcal{X}}(x) \cup \theta_{\mathcal{Y}}(x)$$

for all $x \in \mathcal{S}$.

Proposition 3.4. *Let $\Theta_{\mathcal{X}}, \Theta_{\mathcal{Y}}, \Theta_{\mathcal{Z}} \in \Im(\mathcal{H}, \mathcal{S})$. Then,*

(i): $\Theta_{\mathcal{X}}\alpha\Theta_{\mathcal{Y}}$ *and* $\Theta_{\mathcal{Y}}\alpha\Theta_{\mathcal{Z}} \Rightarrow \Theta_{\mathcal{X}}\alpha\Theta_{\mathcal{Z}}$ *for each* $\alpha \in \{\widetilde{\subseteq}, =\}$.
(ii): $\Theta_{\mathcal{X}}\beta\Theta_{\mathcal{X}} = \Theta_{\mathcal{X}}$ *for each* $\beta \in \{\widetilde{\cap}, \widetilde{\cup}\}$.
(iii): $\Theta_{\mathcal{X}}\beta\Theta_{\mathcal{Y}} = \Theta_{\mathcal{Y}}\beta\Theta_{\mathcal{X}}$ *for each* $\beta \in \{\widetilde{\cap}, \widetilde{\cup}\}$.
(iv): $\Theta_{\mathcal{X}}\beta(\Theta_{\mathcal{Y}}\beta\Theta_{\mathcal{Z}}) = (\Theta_{\mathcal{X}}\beta\Theta_{\mathcal{Y}})\beta\Theta_{\mathcal{Z}}$ *for each* $\beta \in \{\widetilde{\cap}, \widetilde{\cup}\}$.
(v): $\Theta_{\mathcal{X}}\beta(\Theta_{\mathcal{Y}}\delta\Theta_{\mathcal{Z}}) = (\Theta_{\mathcal{X}}\beta\Theta_{\mathcal{Y}})\delta(\Theta_{\mathcal{X}}\beta\Theta_{\mathcal{Z}})$ *for each* $\beta, \delta \in \{\widetilde{\cap}, \widetilde{\cup}\}$.
(vi): $(\Theta_{\mathcal{X}}\beta\Theta_{\mathcal{Y}})^{\widetilde{c}} = \Theta_{\mathcal{X}}^{\widetilde{c}}\delta\Theta_{\mathcal{Y}}^{\widetilde{c}}$ *for each* $\beta, \delta \in \{\widetilde{\cap}, \widetilde{\cup}\}$ *and* $\beta \neq \delta$.

Proof. The proofs are straightforward, so they are omitted. □

Definition 3.5. Let $\Theta_{\mathcal{X}}, \Theta_{\mathcal{Y}} \in \Im(\mathcal{H}, \mathcal{S})$.

(a): The And-product of $\Theta_{\mathcal{X}}$ and $\Theta_{\mathcal{Y}}$, denoted by $\Theta_{\mathcal{X}}\widetilde{\wedge}\Theta_{\mathcal{Y}}$, is an SNMSS defined by the SNM approximate function $\theta_{\mathcal{X}\widetilde{\wedge}\mathcal{Y}} : \mathcal{S} \times \mathcal{S} \to \mathfrak{P}(\mathcal{H})$ such that

$$\theta_{\mathcal{X}\widetilde{\wedge}\mathcal{Y}}(x, y) = \theta_{\mathcal{X}}(x) \cap \theta_{\mathcal{Y}}(y)$$

for all $(x, y) \in \mathcal{S} \times \mathcal{S}$.

(b): The Or-product of $\Theta_{\mathcal{X}}$ and $\Theta_{\mathcal{Y}}$, denoted by $\Theta_{\mathcal{X}}\widetilde{\vee}\Theta_{\mathcal{Y}}$, is an SNMSS defined by the SNM approximate function $\theta_{\mathcal{X}\widetilde{\vee}\mathcal{Y}} : \mathcal{S} \times \mathcal{S} \to \mathfrak{P}(\mathcal{H})$ such that

$$\theta_{\mathcal{X}\widetilde{\vee}\mathcal{Y}}(x, y) = \theta_{\mathcal{X}}(x) \cup \theta_{\mathcal{Y}}(y)$$

for all $(x, y) \in \mathcal{S} \times \mathcal{S}$.

(c): The cartesian product of $\Theta_{\mathcal{X}}$ and $\Theta_{\mathcal{Y}}$, denoted by $\Theta_{\mathcal{X}}\widetilde{\times}\Theta_{\mathcal{Y}}$, is an SNMSS defined by the SNM approximate function $\theta_{\mathcal{X}\widetilde{\times}\mathcal{Y}} : \mathcal{S} \times \mathcal{S} \to \mathfrak{P}(\mathcal{H} \times \mathcal{H})$ such that

$$\theta_{\mathcal{X}\widetilde{\times}\mathcal{Y}}(x, y) = \theta_{\mathcal{X}}(x) \times \theta_{\mathcal{Y}}(y)$$

for all $(x, y) \in \mathcal{S} \times \mathcal{S}$.

Proposition 3.6. *Let $\Theta_{\mathcal{X}}, \Theta_{\mathcal{Y}}, \Theta_{\mathcal{Z}}, \Theta_{\mathcal{T}}$ be the SNMSSs over $\mathcal{H}$. Then,*

(i): $\Theta_{\mathcal{X}}\alpha\Theta_{\mathcal{Y}}$ *and* $\Theta_{\mathcal{Z}}\alpha\Theta_{\mathcal{T}} \Rightarrow (\Theta_{\mathcal{X}}\beta\Theta_{\mathcal{Z}})\alpha(\Theta_{\mathcal{Y}}\beta\Theta_{\mathcal{T}})$ *for each* $\alpha \in \{\widetilde{\subseteq}, =\}$ *and* $\beta \in \{\widetilde{\wedge}, \widetilde{\vee}, \widetilde{\times}\}$.
(ii): $\Theta_{\mathcal{X}}\alpha\Theta_{\mathcal{Y}} \Rightarrow (\Theta_{\mathcal{X}}\beta\Theta_{\mathcal{Z}})\alpha(\Theta_{\mathcal{Y}}\beta\Theta_{\mathcal{Z}})$ *for each* $\alpha \in \{\widetilde{\subseteq}, =\}$ *and* $\beta \in \{\widetilde{\wedge}, \widetilde{\vee}, \widetilde{\times}\}$.
(iii): $\Theta_{\mathcal{X}}\beta(\Theta_{\mathcal{Y}}\beta\Theta_{\mathcal{Z}}) = (\Theta_{\mathcal{X}}\beta\Theta_{\mathcal{Y}})\beta\Theta_{\mathcal{Z}}$ *for each* $\beta \in \{\widetilde{\wedge}, \widetilde{\vee}, \widetilde{\times}\}$.
(vi): $(\Theta_{\mathcal{X}}\beta\Theta_{\mathcal{Y}})^{\widetilde{c}} = \Theta_{\mathcal{X}}^{\widetilde{c}}\delta\Theta_{\mathcal{Y}}^{\widetilde{c}}$ *for each* $\beta, \delta \in \{\widetilde{\wedge}, \widetilde{\vee}\}$ *and* $\beta \neq \delta$.

Proof. The proofs are straightforward, hence they are omitted. □

The emerged SNMSS operations are generalized for the family of SNMSSs as follows.

Definition 3.7. Let $\Theta_{\mathcal{X}_p}$ be the SNMSS for each $p \in I = \{1, 2, ..., q\}$.

(a): The intersection of SNMSSs $\Theta_{\mathcal{X}_p}$ $(p = 1, 2, ..., q)$, denoted by $\widetilde{\bigcap}_{p\in I}\Theta_{\mathcal{X}_p}$, is an SNMSS defined by the SNM approximate function $\theta_{\widetilde{\bigcap}_{p\in I}\mathcal{X}_p} : \mathcal{S} \to \mathfrak{P}(\mathcal{H})$ such that

$$\theta_{\widetilde{\bigcap}_{p\in I}\mathcal{X}_p}(x) = \bigcap_{p\in I} \theta_{\mathcal{X}_p}(x)$$

for all $x \in \mathcal{S}$.

(b): The union of SNMSSs $\Theta_{\mathcal{X}_p}$ $(p = 1, 2, ..., q)$, denoted by $\widetilde{\bigcup}_{p\in I}\Theta_{\mathcal{X}_p}$, is an SNMSS defined by the SNM approximate function $\theta_{\widetilde{\bigcup}_{p\in I}\mathcal{X}_p} : \mathcal{S} \to \mathfrak{P}(\mathcal{H})$ such that

$$\theta_{\widetilde{\bigcup}_{p\in I}\mathcal{X}_p}(x) = \bigcup_{p\in I} \theta_{\mathcal{X}_p}(x)$$

for all $x \in \mathcal{S}$.

(c): The And-product of SNMSSs $\Theta_{\mathcal{X}_p}$ $(p = 1, 2, ..., q)$, denoted by $\widetilde{\bigwedge}_{p\in I}\Theta_{\mathcal{X}_p}$, is an SNMSS defined by the SNM approximate function $\theta_{\widetilde{\bigwedge}_{p\in I}\mathcal{X}_p} : \prod_{p\in I}\mathcal{S} \to \mathfrak{P}(\mathcal{H})$ such that

$$\theta_{\widetilde{\bigwedge}_{p\in I}\mathcal{X}_p}((x^p)_{p\in I}) = \bigcap_{p\in I} \theta_{\mathcal{X}_p}(x^p)$$

for all $(x^p)_{p\in I} = (x^1, x^2, ..., x^q) \in \prod_{p\in I}\mathcal{S} = \mathcal{S}^q$.

(d): The Or-product of SNMSSs $\Theta_{\mathcal{X}_p}$ $(p = 1, 2, ..., q)$, denoted by $\widetilde{\bigvee}_{p\in I}\Theta_{\mathcal{X}_p}$, is an SNMSS defined by the SNM approximate function $\theta_{\widetilde{\bigvee}_{p\in I}\mathcal{X}_p} : \prod_{p\in I}\mathcal{S} \to \mathfrak{P}(\mathcal{H})$ such that

$$\theta_{\widetilde{\bigvee}_{p\in I}\mathcal{X}_p}((x^p)_{p\in I}) = \bigcup_{p\in I} \theta_{\mathcal{X}_p}(x^p)$$

for all $(x^p)_{p\in I} = (x^1, x^2, ..., x^q) \in \prod_{p\in I}\mathcal{S} = \mathcal{S}^q$.

(e): The cartesian product of SNMSSs $\Theta_{\mathcal{X}_p}$ $(p = 1, 2, ..., q)$, denoted by $\widetilde{\prod}_{p\in I}\Theta_{\mathcal{X}_p}$, is an SNMSS defined by the SNM approximate function $\theta_{\widetilde{\prod}_{p\in I}\mathcal{X}_p} : \prod_{p\in I}\mathcal{S} \to \mathfrak{P}(\mathcal{H}^q)$ such that

$$\theta_{\widetilde{\prod}_{p\in I}\mathcal{X}_p}((x^p)_{p\in I}) = \prod_{p\in I} \theta_{\mathcal{X}_p}(x^p)$$

for all $(x^p)_{p\in I} = (x^1, x^2, ..., x^q) \in \prod_{p\in I}\mathcal{S} = \mathcal{S}^q$.

4. Two Person Simplified Neutrosophic Multiplicative Soft Games and Their Applications

4.1. *Two Person Simplified Neutrosophic Multiplicative Soft Games*

In this part, we create two person simplified neutrosophic multiplicative soft games with simplified neutrosophic multiplicative soft payoffs. Moreover, we propose the solution models for the simplified neutrosophic multiplicative soft games. For some fundamental notions (such as game, strategy, payoff, saddle point, Nash equilibrium) on game theory, we refer to [13, 30, 31].

In the following, we revisit some concepts and results on game theory given in [13,30,31] and thus adapt them to the simplified neutrosophic multiplicative soft games (SNM soft games) by using SNMSSs.

Definition 4.1. Let $\mathcal{S}$ be a set of strategies and $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{S}$. A choice of behaviour in an SNM soft game is called an action. Each element of $\mathcal{X} \times \mathcal{Y}$ is called action pair. That is, $\mathcal{X} \times \mathcal{Y}$ is the set of available actions.

Definition 4.2. Let $\mathcal{H}$ be a set of alternatives and $\mathfrak{P}(\mathcal{H})$ be set of all SNMSs in $\mathcal{H}$. Also, $\mathcal{S}$ be a set of strategies and $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{S}$. Then, a set-valued function

$$\theta_{\mathcal{X}\times\mathcal{Y}} : \mathcal{X} \times \mathcal{Y} \to \mathfrak{P}(\mathcal{H}) \tag{6}$$

is said to be a simplified neutrosophic multiplicative soft payoff function (SNM soft payoff function). For each $(x, y) \in \mathcal{X} \times \mathcal{Y}$, the value $\theta_{\mathcal{X}\times\mathcal{Y}}(x, y)$ is named a simplified neutrosophic multiplicative soft payoff (SNM soft payoff).

Definition 4.3. Let $\mathcal{X} \times \mathcal{Y}$ be a set of action pairs. Then, an action $(x^*, y^*) \in \mathcal{X} \times \mathcal{Y}$ is said to be an optimal action if

$$\theta_{\mathcal{X}\times\mathcal{Y}}(x, y) \subseteq \theta_{\mathcal{X}\times\mathcal{Y}}(x^*, y^*) \tag{7}$$

for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$.

Definition 4.4. Let $\mathcal{X} \times \mathcal{Y}$ be a set of action pairs and $(x_i, y_j), (x_k, y_l) \in \mathcal{X} \times \mathcal{Y}$.

- **(a):** If $\theta_{\mathcal{X}\times\mathcal{Y}}(x_k, y_l) \subset \theta_{\mathcal{X}\times\mathcal{Y}}(x_i, y_j)$ then it can be said that a player strictly prefers action pair (x_i, y_j) over action pair (x_k, y_l),
- **(b):** If $\theta_{\mathcal{X}\times\mathcal{Y}}(x_k, y_l) = \theta_{\mathcal{X}\times\mathcal{Y}}(x_i, y_j)$ then it can be said that a player is indifferent between the action pairs (x_i, y_j) and (x_k, y_l),
- **(c):** If $\theta_{\mathcal{X}\times\mathcal{Y}}(x_k, y_l) \subseteq \theta_{\mathcal{X}\times\mathcal{Y}}(x_i, y_j)$ then it can be said that a player either prefers action pair (x_i, y_j) to action pair (x_k, y_l) or is indifferent between the action pairs (x_i, y_j) and (x_k, y_l).

Definition 4.5. Let $\theta^r_{\mathcal{X}\times\mathcal{Y}}$ be an SNM soft payoff for Player r and $(x_i, y_j), (x_k, y_l) \in \mathcal{X} \times \mathcal{Y}$. Then, Player r is named rational if the player's SNM soft payoff satisfies the following properties.

- **(1):** Either $\theta^r_{\mathcal{X}\times\mathcal{Y}}(x_k, y_l) \subseteq \theta^r_{\mathcal{X}\times\mathcal{Y}}(x_i, y_j)$ or $\theta^r_{\mathcal{X}\times\mathcal{Y}}(x_k, y_l) \supseteq \theta^r_{\mathcal{X}\times\mathcal{Y}}(x_i, y_j)$.
- **(2):** If $\theta^r_{\mathcal{X}\times\mathcal{Y}}(x_k, y_l) \subseteq \theta^r_{\mathcal{X}\times\mathcal{Y}}(x_i, y_j)$ and $\theta^r_{\mathcal{X}\times\mathcal{Y}}(x_k, y_l) \supseteq \theta^r_{\mathcal{X}\times\mathcal{Y}}(x_i, y_j)$, then $\theta^r_{\mathcal{X}\times\mathcal{Y}}(x_k, y_l) = \theta^r_{\mathcal{X}\times\mathcal{Y}}(x_i, y_j)$.

Definition 4.6. Let $\mathcal{X}$ and $\mathcal{Y}$ be the sets of strategies of Player 1 and Player 2, respectively. Also, $\theta^r_{\mathcal{X}\times\mathcal{Y}} : \mathcal{X} \times \mathcal{Y} \to \mathfrak{P}(\mathcal{H})$ is an SNM soft payoff function for Player r $(r = 1, 2)$. Then, for each Player r, a two person simplified neutrosophic multiplicative soft game (tpSNM soft game) is defined by an SNMSS over $\mathcal{H}$ as

$$\Theta^r_{\mathcal{X}\times\mathcal{Y}} = \{((x, y), \theta^r_{\mathcal{X}\times\mathcal{Y}}(x, y)) : (x, y) \in \mathcal{X} \times \mathcal{Y},\ \theta^r_{\mathcal{X}\times\mathcal{Y}}(x, y) \in \mathfrak{P}(\mathcal{H})\} \tag{8}$$

where $\theta^r_{\mathcal{X}\times\mathcal{Y}}(x, y) = \{\langle \hbar, (\rho^r_{\mathcal{X}\times\mathcal{Y}}(x, y), \tau^r_{\mathcal{X}\times\mathcal{Y}}(x, y), \sigma^r_{\mathcal{X}\times\mathcal{Y}}(x, y))\rangle : \hbar \in \mathcal{H}\}$ and for the triplet $(\rho^r_{\mathcal{X}\times\mathcal{Y}}(x, y), \tau^r_{\mathcal{X}\times\mathcal{Y}}(x, y), \sigma^r_{\mathcal{X}\times\mathcal{Y}}(x, y))$, 1^{st} component is truth-membership information, 2^{nd} component is indeterminacy-membership information and 3^{rd} component is falsity-membership information of $\hbar \in \mathcal{H}$ with respect to the action pair (x, y) for Player r.

The tpSNM soft game is played as follows. At a certain time Player 1 selects a strategy $x_i \in \mathcal{X}$, simultaneously Player 2 selects a strategy $y_i \in \mathcal{Y}$ and once this done each Player r $(r = 1, 2)$ receives the SNM soft payoff $\theta^r_{\mathcal{X}\times\mathcal{Y}}(x_i, y_j)$.
If $\mathcal{X} = \{x_1, x_2, ..., x_t\}$ and $\mathcal{Y} = \{y_1, y_2, ..., y_v\}$ then the SNM soft payoffs of $\Theta^r_{\mathcal{X}\times\mathcal{Y}}$ can be represented in the following form (see Table 2). To illustrate the tpSNM soft game, we present the following example.

Table 2. Two person simplified neutrosophic multiplicative soft game

$\Theta^r_{\mathcal{X}\times\mathcal{Y}}$	y_1	y_2	.	.	.	y_v
x_1	$\theta^r_{\mathcal{X}\times\mathcal{Y}}(x_1, y_1)$	$\theta^r_{\mathcal{X}\times\mathcal{Y}}(x_1, y_2)$	.	.	.	$\theta^r_{\mathcal{X}\times\mathcal{Y}}(x_1, y_v)$
x_2	$\theta^r_{\mathcal{X}\times\mathcal{Y}}(x_2, y_1)$	$\theta^r_{\mathcal{X}\times\mathcal{Y}}(x_2, y_2)$	.	.	.	$\theta^r_{\mathcal{X}\times\mathcal{Y}}(x_2, y_v)$
.	.	.	.			.
.	.	.		.		.
.	.	.			.	.
x_t	$\theta^r_{\mathcal{X}\times\mathcal{Y}}(x_t, y_1)$	$\theta^r_{\mathcal{X}\times\mathcal{Y}}(x_t, y_2)$	.	.	.	$\theta^r_{\mathcal{X}\times\mathcal{Y}}(x_t, y_v)$

Example 4.7. Let $\mathcal{H} = \{\hbar_1, \hbar_2\}$ be a set of alternatives and $\mathcal{S} = \{x_1, x_2, x_3, x_4, x_5\}$ be a set of strategies. Assume that $\mathcal{X} = \{x_1, x_3\}$ and $\mathcal{Y} = \{x_1, x_4, x_5\}$ are the sets of the strategies Player 1 and Player 2, respectively.
If Player 1 creates the following tpSNM soft game

$$\Theta^1_{\mathcal{X}\times\mathcal{Y}} = \left\{ \begin{array}{l} ((x_1, x_1), \{\langle\hbar_1, (7, \frac{1}{3}, \frac{1}{8})\rangle, \langle\hbar_2, (\frac{1}{2}, 5, 2)\rangle\}), ((x_1, x_4), \{\langle\hbar_1, (5, 5, \frac{1}{7})\rangle, \langle\hbar_2, (1, 1, 1)\rangle\}), \\ ((x_1, x_5), \{\langle\hbar_1, (\frac{1}{2}, 3, 1)\rangle, \langle\hbar_2, (9, 3, \frac{1}{9})\rangle\}), ((x_3, x_1), \{\langle\hbar_1, (3, \frac{4}{3}, \frac{1}{4})\rangle, \langle\hbar_2, (\frac{1}{3}, 3, 1)\rangle\}), \\ ((x_3, x_4), \{\langle\hbar_1, (7, \frac{1}{9}, \frac{1}{9})\rangle, \langle\hbar_2, (3, 3, \frac{1}{3})\rangle\}), ((x_3, x_5), \{\langle\hbar_1, (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})\rangle, \langle\hbar_2, (2, 5, \frac{1}{9})\rangle\}) \end{array} \right\}.$$

the SNM soft payoffs of the game can be illustrated as in Table 3.

Table 3. The tpSNM soft payoffs of Player 1

$\Theta^1_{\mathcal{X}\times\mathcal{Y}}$	x_1	x_4	x_5
x_1	$\{\langle\hbar_1, (7, \frac{1}{3}, \frac{1}{8})\rangle, \langle\hbar_2, (\frac{1}{2}, 5, 2)\rangle\}$	$\{\langle\hbar_1, (5, 5, \frac{1}{7})\rangle, \langle\hbar_2, (1, 1, 1)\rangle\}$	$\{\langle\hbar_1, (\frac{1}{2}, 3, 1)\rangle, \langle\hbar_2, (9, 3, \frac{1}{9})\rangle\}$
x_3	$\{\langle\hbar_1, (3, \frac{4}{3}, \frac{1}{4})\rangle, \langle\hbar_2, (\frac{1}{3}, 3, 1)\rangle\}$	$\{\langle\hbar_1, (7, \frac{1}{9}, \frac{1}{9})\rangle, \langle\hbar_2, (3, 3, \frac{1}{3})\rangle\}$	$\{\langle\hbar_1, (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})\rangle, \langle\hbar_2, (2, 5, \frac{1}{9})\rangle\}$

Let us explain any component of this game. If Player 1 chooses x_1 and Player 2 chooses x_4 then the value of game will be an SNM soft payoff $\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_1, x_4) = \{\langle\hbar_1, (5, 5, \frac{1}{7})\rangle, \langle\hbar_2, (1, 1, 1)\rangle\}$. Then, Player 1 wins the set (of alternatives) $\{\langle\hbar_1, (5, 5, \frac{1}{7})\rangle, \langle\hbar_2, (1, 1, 1)\rangle\}$ and Player 2 loses the same set.
Similarly, if Player 2 creates the following tpSNM soft game

$$\Theta^2_{\mathcal{X}\times\mathcal{Y}} = \left\{ \begin{array}{l} ((x_1, x_1), \{\langle\hbar_1, (8, \frac{1}{8}, \frac{1}{8})\rangle, \langle\hbar_2, (7, 9, \frac{1}{9})\rangle\}), ((x_1, x_4), \{\langle\hbar_1, (\frac{1}{4}, 5, 4)\rangle, \langle\hbar_2, (2, \frac{1}{4}, \frac{1}{6})\rangle\}), \\ ((x_1, x_5), \{\langle\hbar_1, (3, \frac{1}{2}, \frac{1}{5})\rangle, \langle\hbar_2, (4, \frac{1}{5}, \frac{1}{5})\rangle\}), ((x_3, x_1), \{\langle\hbar_1, (3, \frac{1}{4}, \frac{1}{3})\rangle, \langle\hbar_2, (8, 8, \frac{1}{9})\rangle\}), \\ ((x_3, x_4), \{\langle\hbar_1, (1, 5, 1)\rangle, \langle\hbar_2, (4, 3, \frac{1}{7})\rangle\}), ((x_3, x_5), \{\langle\hbar_1, (2, 2, \frac{1}{4})\rangle, \langle\hbar_2, (1, \frac{1}{4}, \frac{1}{9})\rangle\}) \end{array} \right\}.$$

TABLE 4. The tpSNM soft payoffs of Player 2

$\Theta^2_{\mathcal{X}\times\mathcal{Y}}$	x_1	x_4	x_5
x_1	$\{\langle\hbar_1,(8,\frac{1}{8},\frac{1}{8})\rangle,\langle\hbar_2,(7,9,\frac{1}{9})\rangle\}$	$\{\langle\hbar_1,(\frac{1}{4},5,4)\rangle,\langle\hbar_2,(2,\frac{1}{4},\frac{1}{6})\rangle\}$	$\{\langle\hbar_1,(3,\frac{1}{2},\frac{1}{5})\rangle,\langle\hbar_2,(4,\frac{1}{5},\frac{1}{5})\rangle\}$
x_3	$\{\langle\hbar_1,(3,\frac{1}{4},\frac{1}{3})\rangle,\langle\hbar_2,(8,8,\frac{1}{9})\rangle\}$	$\{\langle\hbar_1,(1,5,1)\rangle,\langle\hbar_2,(4,3,\frac{1}{7})\rangle\}$	$\{\langle\hbar_1,(2,2,\frac{1}{4})\rangle,\langle\hbar_2,(1,\frac{1}{4},\frac{1}{9})\rangle\}$

the SNM soft payoffs of the game can be tabularized as in Table 4.

The component (x_1, x_4) in this game (or the component in first row-second column of Table 4) can be interpreted that the value of game will be an SNM soft payoff $\theta^2_{\mathcal{X}\times\mathcal{Y}}(x_1,x_4) = \{\langle\hbar_1,(\frac{1}{4},5,4)\rangle,\langle\hbar_2,(2,\frac{1}{4},\frac{1}{6})\rangle\}$ when Player 1 chooses x_1 and Player 2 chooses x_4. Then, Player 1 wins the set (of alternatives) $\{\langle\hbar_1,(\frac{1}{4},5,4)\rangle,\langle\hbar_2,(2,\frac{1}{4},\frac{1}{6})\rangle\}$ and Player 2 loses the same set.

Definition 4.8. Let $\theta^r_{\mathcal{X}\times\mathcal{Y}}$ be an SNM soft payoff function of a tpSNM soft game $\Theta^r_{\mathcal{X}\times\mathcal{Y}}$. If the following properties are satisfied

(1):

$$\widetilde{\bigcup}_{i=1}^{t}\theta^r_{\mathcal{X}\times\mathcal{Y}}(x_i,y_j) = \left\{ \left\langle \hbar, \begin{pmatrix} \max\limits_{i\in\{1,2,..,t\}}\{\rho^r_{\mathcal{X}\times\mathcal{Y}}(x_i,y_j)\}, \\ \min\limits_{i\in\{1,2,..,t\}}\{\tau^r_{\mathcal{X}\times\mathcal{Y}}(x_i,y_j)\}, \\ \min\limits_{i\in\{1,2,..,t\}}\{\sigma^r_{\mathcal{X}\times\mathcal{Y}}(x_i,y_j)\} \end{pmatrix} \right\rangle : \hbar\in\mathcal{H} \right\} = \theta^r_{\mathcal{X}\times\mathcal{Y}}(x,y),$$

(2):

$$\widetilde{\bigcap}_{j=1}^{v}\theta^r_{\mathcal{X}\times\mathcal{Y}}(x_i,y_j) = \left\{ \left\langle \hbar, \begin{pmatrix} \min\limits_{j\in\{1,2,..,v\}}\{\rho^r_{\mathcal{X}\times\mathcal{Y}}(x_i,y_j)\}, \\ \max\limits_{j\in\{1,2,..,v\}}\{\tau^r_{\mathcal{X}\times\mathcal{Y}}(x_i,y_j)\}, \\ \max\limits_{j\in\{1,2,..,v\}}\{\sigma^r_{\mathcal{X}\times\mathcal{Y}}(x_i,y_j)\} \end{pmatrix} \right\rangle : \hbar\in\mathcal{H} \right\} = \theta^r_{\mathcal{X}\times\mathcal{Y}}(x,y),$$

then $\theta^r_{\mathcal{X}\times\mathcal{Y}}(x,y)$ is named a simplified neutrosophic multiplicative soft saddle point value (SNM soft saddle point value) and (x,y) is called an SNM soft saddle point of Player r in the tpSNM soft game.

Note that if (x,y) is an SNM soft saddle point of a tpSNM soft game $\Theta^1_{\mathcal{X}\times\mathcal{Y}}$ then Player 1 can win at least by selecting the strategy $x\in\mathcal{X}$ and Player 2 can keep her/his loss to at most $\theta^1_{\mathcal{X}\times\mathcal{Y}}(x,y)$ by selecting the strategy $y\in\mathcal{Y}$. Hence the tpSNM soft saddle point is a value of the tpSNM soft game.

Example 4.9. Let $\mathcal{H}=\{\hbar_1,\hbar_2,\hbar_3,\hbar_4\}$ be a set of alternatives and $\mathcal{X}=\{x_1,x_2,x_3\}$ and $\mathcal{Y}=\{y_1,y_2\}$ be the sets of the strategies Player 1 and Player 2, respectively. Then, tpSNM soft game of Player 1 is presented as in Table 5.

TABLE 5. The tpSNM soft game $\Theta^1_{\mathcal{X}\times\mathcal{Y}}$ of Player 1

$\Theta^1_{\mathcal{X}\times\mathcal{Y}}$	y_1	y_2
x_1	$\{\langle\hbar_1,(5,4,\frac{1}{6})\rangle,\langle\hbar_2,(\frac{1}{2},1,1)\rangle,\langle\hbar_3,(3,\frac{1}{7},\frac{1}{9})\rangle,\langle\hbar_4,(\frac{1}{5},4,3)\rangle\}$	$\{\langle\hbar_1,(3,4,\frac{1}{5})\rangle,\langle\hbar_2,(\frac{1}{3},3,\frac{1}{3})\rangle,\langle\hbar_3,(\frac{1}{8},\frac{1}{2},\frac{1}{2})\rangle,\langle\hbar_4,(1,4,1)\rangle\}$
x_2	$\{\langle\hbar_1,(5,2,\frac{1}{8})\rangle,\langle\hbar_2,(\frac{1}{2},1,\frac{1}{5})\rangle,\langle\hbar_3,(3,\frac{1}{9},\frac{1}{9})\rangle,\langle\hbar_4,(4,\frac{1}{5},\frac{1}{4})\rangle\}$	$\{\langle\hbar_1,(3,3,\frac{1}{5})\rangle,\langle\hbar_2,(\frac{1}{3},3,\frac{1}{4})\rangle,\langle\hbar_3,(2,\frac{1}{3},\frac{1}{5})\rangle,\langle\hbar_4,(3,\frac{1}{4},\frac{1}{3})\rangle\}$
x_3	$\{\langle\hbar_1,(2,2,\frac{1}{5})\rangle,\langle\hbar_2,(\frac{1}{4},4,\frac{1}{5})\rangle,\langle\hbar_3,(3,\frac{1}{9},\frac{1}{3})\rangle,\langle\hbar_4,(\frac{1}{5},\frac{1}{2},3)\rangle\}$	$\{\langle\hbar_1,(\frac{1}{9},3,\frac{1}{3})\rangle,\langle\hbar_2,(\frac{1}{8},4,\frac{1}{3})\rangle,\langle\hbar_3,(\frac{1}{4},5,\frac{1}{3})\rangle,\langle\hbar_4,(1,4,1)\rangle\}$

Then, we have
$\bigcup_{i=1}^{3}\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_i,y_1)=\{\langle\hbar_1,(5,2,\frac{1}{8})\rangle,\langle\hbar_2,(\frac{1}{2},1,\frac{1}{5})\rangle,\langle\hbar_3,(3,\frac{1}{9},\frac{1}{9})\rangle,\langle\hbar_4,(4,\frac{1}{5},\frac{1}{4})\rangle\}$,
$\bigcup_{i=1}^{3}\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_i,y_2)=\{\langle\hbar_1,(3,3,\frac{1}{5})\rangle,\langle\hbar_2,(\frac{1}{3},3,\frac{1}{4})\rangle,\langle\hbar_3,(2,\frac{1}{3},\frac{1}{5})\rangle,\langle\hbar_4,(3,\frac{1}{4},\frac{1}{3})\rangle\}$,
and
$\bigcap_{j=1}^{2}\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_1,y_j)=\{\langle\hbar_1,(3,4,\frac{1}{5})\rangle,\langle\hbar_2,(\frac{1}{3},3,1)\rangle,\langle\hbar_3,(\frac{1}{8},\frac{1}{2},\frac{1}{2})\rangle,\langle\hbar_4,(\frac{1}{5},4,3)\rangle\}$,
$\bigcap_{j=1}^{2}\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_2,y_j)=\{\langle\hbar_1,(3,3,\frac{1}{5})\rangle,\langle\hbar_2,(\frac{1}{3},3,\frac{1}{4})\rangle,\langle\hbar_3,(2,\frac{1}{3},\frac{1}{5})\rangle,\langle\hbar_4,(3,\frac{1}{4},\frac{1}{3})\rangle\}$,
$\bigcap_{j=1}^{2}\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_3,y_j)=\{\langle\hbar_1,(\frac{1}{9},3,\frac{1}{3})\rangle,\langle\hbar_2,(\frac{1}{8},4,\frac{1}{3})\rangle,\langle\hbar_3,(\frac{1}{4},5,\frac{1}{3})\rangle,\langle\hbar_4,(\frac{1}{5},4,1)\rangle\}$.
Since $\bigcup_{i=1}^{3}\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_i,y_2)=\bigcap_{j=1}^{2}\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_2,y_j)=\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_2,y_2)$, we say that $\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_2,y_2)=\{\langle\hbar_1,(3,3,\frac{1}{5})\rangle,\langle\hbar_2,(\frac{1}{3},3,\frac{1}{4})\rangle,\langle\hbar_3,(2,\frac{1}{3},\frac{1}{5})\rangle,\langle\hbar_4,(3,\frac{1}{4},\frac{1}{3})\rangle\}$ is an SNM soft saddle point value of the tpSNM soft game. Hence, the value of the tpSNM soft game is $\{\langle\hbar_1,(3,3,\frac{1}{5})\rangle,\langle\hbar_2,(\frac{1}{3},3,\frac{1}{4})\rangle,\langle\hbar_3,(2,\frac{1}{3},\frac{1}{5})\rangle,\langle\hbar_4,(3,\frac{1}{4},\frac{1}{3})\rangle\}$.

Note that every tpSNM soft game has not an SNM soft saddle point value. For instance, in Example 4.9, if $\{\langle\hbar_1,(3,3,\frac{1}{5})\rangle,\langle\hbar_2,(\frac{1}{3},3,\frac{1}{4})\rangle,\langle\hbar_3,(2,\frac{1}{3},\frac{1}{5})\rangle,\langle\hbar_4,(5,5,\frac{1}{5})\rangle\}$ is taken instead of $\{\langle\hbar_1,(3,3,\frac{1}{5})\rangle,\langle\hbar_2,(\frac{1}{3},3,\frac{1}{4})\rangle,\langle\hbar_3,(2,\frac{1}{3},\frac{1}{5})\rangle,\langle\hbar_4,(3,\frac{1}{4},\frac{1}{3})\rangle\}$ in the SNM soft payoff $\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_2,y_2)$ then this tpSNM soft game has not an SNM soft saddle point value. If the saddle point cannot found for a tpSNM soft game then simplified neutrosophic multiplicative soft upper value and simplified neutrosophic multiplicative soft lower value of tpSNM soft game may be used. These concepts are given in the following definition.

Definition 4.10. Let $\Theta_{\mathcal{X}\times\mathcal{Y}}$ be a tpSNM soft game with its SNM soft payoff function $\theta_{\mathcal{X}\times\mathcal{Y}}$, where $\mathcal{X}=\{x_i : i=1,2,...,t\}$ and $\mathcal{Y}=\{y_j : j=1,2,...,t\}$. Then,

(a): SNM soft upper value of the tpSNM soft game, symbolized by V_U, is defined by

$$V_U=\widetilde{\bigcap}_{j=1}^{v}(\widetilde{\bigcup}_{i=1}^{t}(\theta_{\mathcal{X}\times\mathcal{Y}}(x_i,y_j))) \tag{9}$$

(b): SNM soft lower value of the tpSNM soft game, symbolized by V_L, is defined by

$$V_L=\widetilde{\bigcup}_{i=1}^{t}(\widetilde{\bigcap}_{j=1}^{v}(\theta_{\mathcal{X}\times\mathcal{Y}}(x_i,y_j))) \tag{10}$$

(c): If the SNM soft upper value and SNM soft lower value of the tpSNM soft game are equal then these are called value of the tpSNM soft game, symbolized by V. That is, $V=V_U=V_L$.

Example 4.11. Let us consider Table 5 in Example 4.9. Then, we have that the SNM soft upper value V_U and SNM soft lower value V_L are equal, i.e.,

$$V_U=V_L=\{\langle\hbar_1,(3,3,\frac{1}{5})\rangle,\langle\hbar_2,(\frac{1}{3},3,\frac{1}{4})\rangle,\langle\hbar_3,(2,\frac{1}{3},\frac{1}{5})\rangle,\langle\hbar_4,(3,\frac{1}{4},\frac{1}{3})\rangle\}.$$

Therefore, we can say that the value of the tpSNM soft game is $V=V_U=V_L$.
On the other hand, for the SNM soft payoff $\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_2,y_2)$ in Table 5, if

$$\{\langle\hbar_1,(3,3,\frac{1}{5})\rangle,\langle\hbar_2,(\frac{1}{3},3,\frac{1}{4})\rangle,\langle\hbar_3,(2,\frac{1}{3},\frac{1}{5})\rangle,\langle\hbar_4,(3,\frac{1}{4},\frac{1}{3})\rangle\}$$

is replaced by $\{\langle \hbar_1, (3, 3, \frac{1}{5})\rangle, \langle \hbar_2, (\frac{1}{3}, 3, \frac{1}{4})\rangle, \langle \hbar_3, (2, \frac{1}{3}, \frac{1}{5})\rangle, \langle \hbar_4, (5, 5, \frac{1}{5})\rangle\}$ then we calculate the SNM soft upper value V_U and the SNM soft lower value V_L as

$$V_U = \widetilde{\bigcap}_{j=1}^{2}(\widetilde{\bigcup}_{i=1}^{3}(\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_i, y_j))) = \{\langle \hbar_1, (3, 3, \frac{1}{5})\rangle, \langle \hbar_2, (\frac{1}{3}, 3, \frac{1}{4})\rangle, \langle \hbar_3, (2, \frac{1}{3}, \frac{1}{5})\rangle, \langle \hbar_4, (4, 4, \frac{1}{4})\rangle\},$$

$$V_L = \widetilde{\bigcup}_{i=1}^{3}(\widetilde{\bigcap}_{j=1}^{2}(\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_i, y_j))) = \{\langle \hbar_1, (3, 3, \frac{1}{5})\rangle, \langle \hbar_2, (\frac{1}{3}, 3, \frac{1}{4})\rangle, \langle \hbar_3, (2, \frac{1}{3}, \frac{1}{5})\rangle, \langle \hbar_4, (4, 4, \frac{1}{4})\rangle\}.$$

Thus, since $V_U = V_L$, we deduce that the value of the tpSNM soft game $\Theta^1_{\mathcal{X}\times\mathcal{Y}}$ is $\{\langle \hbar_1, (3, 3, \frac{1}{5})\rangle, \langle \hbar_2, (\frac{1}{3}, 3, \frac{1}{4})\rangle, \langle \hbar_3, (2, \frac{1}{3}, \frac{1}{5})\rangle, \langle \hbar_4, (4, 4, \frac{1}{4})\rangle\}$.

Theorem 4.12. *Let V_U and V_L be the values of SNM soft upper and SNM soft lower of a tpSNM soft game, respectively. Then,*

$$V_L \subseteq V_U. \tag{11}$$

Proof. Suppose that V_U and V_L are the SNM soft upper and lower values a tpSNM soft game, respectively. Also, $\mathcal{X} = \{x_i : i = 1, 2, ..., t\}$ and $\mathcal{Y} = \{y_j : j = 1, 2, ..., t\}$ are sets of strategies for Player 1 and Player 2, respectively. Then, we calculate

$$\begin{aligned}
V_L = \widetilde{\bigcup}_{i=1}^{t}(\widetilde{\bigcap}_{j=1}^{v}(\theta_{\mathcal{X}\times\mathcal{Y}}(x_i, y_j))) &= \left\{ \left\langle \hbar, \begin{pmatrix} \max\limits_{i\in\{1,2,...,t\}}(\min\limits_{j\in\{1,2,...,v\}}\{\rho_{\mathcal{X}\times\mathcal{Y}}(x_i, y_j)\}), \\ \min\limits_{i\in\{1,2,...,t\}}(\max\limits_{j\in\{1,2,...,v\}}\{\tau_{\mathcal{X}\times\mathcal{Y}}(x_i, y_j)\}), \\ \min\limits_{i\in\{1,2,...,t\}}(\max\limits_{j\in\{1,2,...,v\}}\{\sigma_{\mathcal{X}\times\mathcal{Y}}(x_i, y_j)\}) \end{pmatrix} \right\rangle : \hbar \in \mathcal{H} \right\} \\
&\subseteq \left\{ \left\langle \hbar, \begin{pmatrix} \min\limits_{j\in\{1,2,...,v\}}\{\rho_{\mathcal{X}\times\mathcal{Y}}(x_{i_{p_1}}, y_j)\}), \\ \max\limits_{j\in\{1,2,...,v\}}\{\tau_{\mathcal{X}\times\mathcal{Y}}(x_{i_{p_2}}, y_j)\}), \\ \max\limits_{j\in\{1,2,...,v\}}\{\sigma_{\mathcal{X}\times\mathcal{Y}}(x_{i_{p_3}}, y_j)\}) \end{pmatrix} \right\rangle : \hbar \in \mathcal{H} \right\} \\
&\subseteq \left\{ \left\langle \hbar, \begin{pmatrix} \rho_{\mathcal{X}\times\mathcal{Y}}(x_{i_{p_1}}, y_{j_{q_1}}), \\ \tau_{\mathcal{X}\times\mathcal{Y}}(x_{i_{p_2}}, y_{j_{q_2}}), \\ \sigma_{\mathcal{X}\times\mathcal{Y}}(x_{i_{p_3}}, y_{j_{q_3}}) \end{pmatrix} \right\rangle : \hbar \in \mathcal{H} \right\} \\
&\subseteq \left\{ \left\langle \hbar, \begin{pmatrix} \max\limits_{i\in\{1,2,...,t\}}\{\rho_{\mathcal{X}\times\mathcal{Y}}(x_i, x_{j_{q_1}})\}), \\ \min\limits_{i\in\{1,2,...,t\}}\{\tau_{\mathcal{X}\times\mathcal{Y}}(x_i, x_{j_{q_2}})\}), \\ \min\limits_{i\in\{1,2,...,t\}}\{\sigma_{\mathcal{X}\times\mathcal{Y}}(x_i, x_{j_{q_3}})\}) \end{pmatrix} \right\rangle : \hbar \in \mathcal{H} \right\} \\
&= \left\{ \left\langle \hbar, \begin{pmatrix} \min\limits_{j\in\{1,2,...,v\}}(\max\limits_{i\in\{1,2,...,t\}}\{\rho_{\mathcal{X}\times\mathcal{Y}}(x_i, y_j)\}), \\ \max\limits_{j\in\{1,2,...,v\}}(\min\limits_{i\in\{1,2,...,t\}}\{\tau_{\mathcal{X}\times\mathcal{Y}}(x_i, y_j)\}), \\ \max\limits_{j\in\{1,2,...,v\}}(\min\limits_{i\in\{1,2,...,t\}}\{\sigma_{\mathcal{X}\times\mathcal{Y}}(x_i, y_j)\}) \end{pmatrix} \right\rangle : \hbar \in \mathcal{H} \right\} \\
&= \widetilde{\bigcap}_{j=1}^{v}(\widetilde{\bigcup}_{i=1}^{t}(\theta_{\mathcal{X}\times\mathcal{Y}}(x_i, y_j)))
\end{aligned}$$

where $i_{p_1}, i_{p_2}, i_{p_3} \in \{1, 2, ..., t\}$ and $j_{p_1}, j_{p_2}, j_{p_3} \in \{1, 2, ..., v\}$. Hence, we have $V_L \subseteq V_U$. □

Example 4.13. For the SNM soft payoff $\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_2, y_2)$ in Table 5, we take

$$\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_2, y_2) = \{\langle \hbar_1, (3, 3, \frac{1}{5})\rangle, \langle \hbar_2, (\frac{1}{3}, 3, \frac{1}{4})\rangle, \langle \hbar_3, (2, \frac{1}{3}, \frac{1}{5})\rangle, \langle \hbar_4, (\frac{1}{6}, 5, \frac{1}{6})\rangle\}.$$

Then, we obtain the SNM soft upper value V_U and the SNM soft lower value V_L as

$$V_U = \widetilde{\bigcap}_{j=1}^{2}(\widetilde{\bigcup}_{i=1}^{3}(\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_i, y_j))) = \{\langle \hbar_1, (3, 3, \frac{1}{5})\rangle, \langle \hbar_2, (\frac{1}{3}, 3, \frac{1}{4})\rangle, \langle \hbar_3, (2, \frac{1}{3}, \frac{1}{5})\rangle, \langle \hbar_4, (1, 4, \frac{1}{4})\rangle\}$$

and

$$V_L = \widetilde{\bigcup}_{i=1}^{3}(\widetilde{\bigcap}_{j=1}^{2}(\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_i, y_j))) = \{\langle \hbar_1, (3, 3, \frac{1}{5})\rangle, \langle \hbar_2, (\frac{1}{3}, 3, \frac{1}{4})\rangle, \langle \hbar_3, (2, \frac{1}{3}, \frac{1}{5})\rangle, \langle \hbar_4, (\frac{1}{5}, 4, \frac{1}{4})\rangle\}.$$

It is clear that $V_L \subseteq V_U$.

Theorem 4.14. *Let $\theta_{\mathcal{X}\times\mathcal{Y}}(x, y)$ be an SNM soft saddle point value, and V_U and V_L be the values of SNM soft upper and SNM soft lower of a tpSNM soft game, respectively. Then,*

$$V_L \subseteq \theta_{\mathcal{X}\times\mathcal{Y}}(x, y) \subseteq V_U. \tag{12}$$

Proof. It can be demonstrated using techniques similar to those in the proof of Theorem 4.12. □

Corollary 4.15. *Let (x, y) be an SNM soft saddle point, and V_U and V_L be the values of SNM soft upper and SNM soft lower of a tpSNM soft game, respectively. If $V_U = V_L = V$ then $\theta_{\mathcal{X}\times\mathcal{Y}}(x, y)$ is exactly V.*

Example 4.16. Consider the SNM soft saddle point value in Example 4.9, and SNM soft upper value V_U and SNM soft upper value V_L in Example 4.11. It is obvious that the SNM soft saddle point value $\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_2, y_2)$ is exactly $V = V_U = V_L$.

Note that in every tpSNM soft game, the SNM soft upper value V_U and SNM soft lower value V_L cannot be equals. If $V_U \neq V_L$ in a tpSNM soft game then we achieve the solution of game by using the following simplified neutrosophic multiplicative soft dominated strategy (SNM soft dominated strategy).

Definition 4.17. Let $\Theta_{\mathcal{X}\times\mathcal{Y}}$ be a tpSNM soft game with its SNM soft payoff function $\theta_{\mathcal{X}\times\mathcal{Y}}$. Then,

- **(a):** a strategy $x_i \in \mathcal{X}$ is termed to be an SNM soft dominated to another strategy $x_k \in \mathcal{X}$ if $\theta_{\mathcal{X}\times\mathcal{Y}}(x_k, y) \subseteq \theta_{\mathcal{X}\times\mathcal{Y}}(x_i, y)$ for all $y \in \mathcal{Y}$,
- **(b):** a strategy $y_j \in \mathcal{Y}$ is termed to be an SNM soft dominated to another strategy $y_l \in \mathcal{Y}$ if $\theta_{\mathcal{X}\times\mathcal{Y}}(x, y_j) \subseteq \theta_{\mathcal{X}\times\mathcal{Y}}(x, y_l)$ for all $x \in \mathcal{X}$.

By using the SNM soft dominated strategy, tpSNM soft games may be reduced by deleting columns and rows, which are obviously bad for the player of game. This process of eliminating SNM soft dominated strategies sometimes leads us to a solution of a tpSNM soft game. This method of solving tpSNM soft game is named a simplified neutrosophic multiplicative soft elimination method (SNM soft elimination method).

Now, let us solve the following tpSNM soft game by using the SNM soft elimination method.

Example 4.18. We consider Table 5 in Example 4.9. Since $\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_1, y_j) \subseteq \theta^1_{\mathcal{X}\times\mathcal{Y}}(x_2, y_j)$ and $\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_3, y_j) \subseteq \theta^1_{\mathcal{X}\times\mathcal{Y}}(x_2, y_j)$ for all $y_j \in \mathcal{Y}$, we can say that the strategy x_2 dominates to the strategies x_1 and x_3. That is, the first row and third row are deleted from Table 5, and so Table 6 are created.

Table 6. The reduced tpSNM soft game $\Theta^1_{\mathcal{X}\times\mathcal{Y}}$ for dominated strategy x_i

$\Theta^1_{\mathcal{X}\times\mathcal{Y}}$	y_1	y_2
x_2	$\{\langle\hbar_1,(5,2,\frac{1}{8})\rangle,\langle\hbar_2,(\frac{1}{2},1,\frac{1}{5})\rangle,\langle\hbar_3,(3,\frac{1}{9},\frac{1}{9})\rangle,\langle\hbar_4,(4,\frac{1}{5},\frac{1}{4})\rangle\}$	$\{\langle\hbar_1,(3,3,\frac{1}{5})\rangle,\langle\hbar_2,(\frac{1}{3},3,\frac{1}{4})\rangle,\langle\hbar_3,(2,\frac{1}{3},\frac{1}{5})\rangle,\langle\hbar_4,(3,\frac{1}{4},\frac{1}{3})\rangle\}$

Now, we consider Table 6. Since $\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_2, y_2) \subseteq \theta^1_{\mathcal{X}\times\mathcal{Y}}(x_2, y_1)$ for all $x_2 \in \mathcal{X}$, we can say that the strategy y_1 is dominated by the strategy y_2. Player 1 has SNM soft dominated strategy y_2 so that the strategy y_1 is eliminated. Thus, we delete the first column from Table 6 and present Table 7.

Table 7. The reduced tpSNM soft game $\Theta^1_{\mathcal{X}\times\mathcal{Y}}$ for dominated strategies x_i and y_j

$\Theta^1_{\mathcal{X}\times\mathcal{Y}}$	y_2
x_2	$\{\langle\hbar_1,(3,3,\frac{1}{5})\rangle,\langle\hbar_2,(\frac{1}{3},3,\frac{1}{4})\rangle,\langle\hbar_3,(2,\frac{1}{3},\frac{1}{5})\rangle,\langle\hbar_4,(3,\frac{1}{4},\frac{1}{3})\rangle\}$

Consequently, the solution using tpSNM soft elimination method is (x_2, y_2), that is, the value of tpSNM soft game is $\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_2, y_2) = \{\langle\hbar_1,(3,3,\frac{1}{5})\rangle,\langle\hbar_2,(\frac{1}{3},3,\frac{1}{4})\rangle,\langle\hbar_3,(2,\frac{1}{3},\frac{1}{5})\rangle,\langle\hbar_4,(3,\frac{1}{4},\frac{1}{3})\rangle\}$.

Note that the tpSNM soft elimination method cannot achieve the solutions for some tpSNM soft games that do not have an SNM soft dominated strategies. In such cases, we can utilize simplified neutrosophic multiplicative soft Nash equilibrium (SNM soft Nash equilibrium) described in the following.

Definition 4.19. Let $\Theta^r_{\mathcal{X}\times\mathcal{Y}}$ be a tpSNM soft game with its SNM soft payoff function $\theta^r_{\mathcal{X}\times\mathcal{Y}}$ $(r = 1, 2)$. If the following properties are satisfied then $(x^*, y^*) \in \mathcal{X}\times\mathcal{Y}$ is called an SNM soft Nash equilibrium of a tpSNM soft game.

(1): $\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_i, y^*) \subseteq \theta^1_{\mathcal{X}\times\mathcal{Y}}(x^*, y^*)$ for all $x_i \in \mathcal{X}$.
(2): $\theta^2_{\mathcal{X}\times\mathcal{Y}}(x^*, y_j) \subseteq \theta^2_{\mathcal{X}\times\mathcal{Y}}(x^*, y^*)$ for all $y_j \in \mathcal{Y}$.

Note that if $(x^*, y^*) \in \mathcal{X}\times\mathcal{Y}$ is an SNM soft Nash equilibrium of a tpSNM soft game, then Player 1 can win at least $\theta^1_{\mathcal{X}\times\mathcal{Y}}(x^*, y^*)$ by selecting strategy $x^* \in \mathcal{X}$, and Player 2 can win at least $\theta^2_{\mathcal{X}\times\mathcal{Y}}(x^*, y^*)$ by selecting strategy $y^* \in \mathcal{Y}$. Therefore, the SNM soft Nash equilibrium is an optimal action for tpSNM soft game, and so $\theta^r_{\mathcal{X}\times\mathcal{Y}}(x^*, y^*)$ is the solution of the tpSNM soft game for Player r $(r = 1, 2)$.

Example 4.20. Assume that the tpSNM soft games of Player 1 and Player 2 are given as in Tables 8 and 9, respectively.

Each of tpSNM soft games $\Theta^1_{\mathcal{X}\times\mathcal{Y}}$ and $\Theta^2_{\mathcal{X}\times\mathcal{Y}}$ has not an SNM soft saddle point value and $V_U \neq V_L$. Also, it is obvious that the tpSNM soft elimination method cannot be used for the solutions of these tpSNM soft games.

Table 8. The tpSNM soft game of Player 1

$\Theta^1_{\mathcal{X}\times\mathcal{Y}}$	y_1	y_2
x_1	$\{\langle\hbar_1,(3,\frac{1}{9},\frac{1}{3})\rangle,\langle\hbar_2,(4,\frac{1}{2},\frac{1}{6})\rangle,\langle\hbar_3,(2,2,\frac{1}{5})\rangle,\langle\hbar_4,(\frac{1}{4},4,\frac{1}{5})\rangle\}$	$\{\langle\hbar_1,(\frac{1}{4},5,\frac{1}{3})\rangle,\langle\hbar_2,(1,4,1)\rangle,\langle\hbar_3,(\frac{1}{9},3,\frac{1}{3})\rangle,\langle\hbar_4,(\frac{1}{8},4,\frac{1}{3})\rangle\}$
x_2	$\{\langle\hbar_1,(3,\frac{1}{7},\frac{1}{9})\rangle,\langle\hbar_2,(\frac{1}{5},4,3)\rangle,\langle\hbar_3,(5,4,\frac{1}{6})\rangle,\langle\hbar_4,(\frac{1}{2},1,1)\rangle\}$	$\{\langle\hbar_1,(\frac{1}{8},\frac{1}{2},\frac{1}{2})\rangle,\langle\hbar_2,(1,4,1)\rangle,\langle\hbar_3,(3,4,\frac{1}{5})\rangle,\langle\hbar_4,(\frac{1}{3},3,\frac{1}{3})\rangle\}$
x_3	$\{\langle\hbar_1,(3,\frac{1}{7},\frac{1}{7})\rangle,\langle\hbar_2,(2,\frac{1}{5},\frac{1}{4})\rangle,\langle\hbar_3,(5,2,\frac{1}{8})\rangle,\langle\hbar_4,(\frac{1}{2},1,\frac{1}{5})\rangle\}$	$\{\langle\hbar_1,(2,\frac{1}{3},\frac{1}{5})\rangle,\langle\hbar_2,(3,\frac{1}{4},\frac{1}{3})\rangle,\langle\hbar_3,(3,3,\frac{1}{5})\rangle,\langle\hbar_4,(\frac{1}{3},3,\frac{1}{4})\rangle\}$

Table 9. The tpSNM soft game of Player 2

$\Theta^2_{\mathcal{X}\times\mathcal{Y}}$	y_1	y_2
x_1	$\{\langle\hbar_1,(4,\frac{1}{4},\frac{1}{9})\rangle,\langle\hbar_2,(\frac{1}{5},4,3)\rangle,\langle\hbar_3,(\frac{1}{4},4,\frac{1}{5})\rangle,\langle\hbar_4,(5,2,\frac{1}{8})\rangle\}$	$\{\langle\hbar_1,(3,\frac{1}{9},\frac{1}{4})\rangle,\langle\hbar_2,(4,2,\frac{1}{4})\rangle,\langle\hbar_3,(2,\frac{1}{2},\frac{1}{2})\rangle,\langle\hbar_4,(2,\frac{1}{2},\frac{1}{4})\rangle\}$
x_2	$\{\langle\hbar_1,(\frac{1}{2},2,1)\rangle,\langle\hbar_2,(2,\frac{1}{4},\frac{1}{4})\rangle,\langle\hbar_3,(1,\frac{1}{2},\frac{1}{3})\rangle,\langle\hbar_4,(2,\frac{1}{4},\frac{1}{2})\rangle\}$	$\{\langle\hbar_1,(1,1,\frac{1}{2})\rangle,\langle\hbar_2,(3,2,\frac{1}{4})\rangle,\langle\hbar_3,(3,\frac{1}{2},\frac{1}{3})\rangle,\langle\hbar_4,(2,\frac{1}{2},\frac{1}{4})\rangle\}$
x_3	$\{\langle\hbar_1,(2,\frac{1}{4},\frac{1}{2})\rangle,\langle\hbar_2,(5,1,\frac{1}{6})\rangle,\langle\hbar_3,(1,1,1)\rangle,\langle\hbar_4,(\frac{1}{2},1,1)\rangle\}$	$\{\langle\hbar_1,(4,\frac{1}{5},\frac{1}{5})\rangle,\langle\hbar_2,(5,1,\frac{1}{9})\rangle,\langle\hbar_3,(3,\frac{1}{3},\frac{1}{3})\rangle,\langle\hbar_4,(2,\frac{1}{2},\frac{1}{4})\rangle\}$

From Tables 8 and 9, we have

(1): $\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_i,y_2)\subseteq\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_3,y_2)$ for all $x_i\in\mathcal{X}$.
(2): $\theta^2_{\mathcal{X}\times\mathcal{Y}}(x_3,y_j)\subseteq\theta^2_{\mathcal{X}\times\mathcal{Y}}(x_3,y_2)$ for all $y_j\in\mathcal{Y}$.

Then, $(x_3,y_2)\in\mathcal{X}\times\mathcal{Y}$ is an SNM soft Nash equilibrium. Hence,

$$\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_3,y_2)=\{\langle\hbar_1,(2,\frac{1}{3},\frac{1}{5})\rangle,\langle\hbar_2,(3,\frac{1}{4},\frac{1}{3})\rangle,\langle\hbar_3,(3,3,\frac{1}{5})\rangle,\langle\hbar_4,(\frac{1}{3},3,\frac{1}{4})\rangle\} \tag{13}$$

and

$$\theta^2_{\mathcal{X}\times\mathcal{Y}}(x_3,y_2)=\{\langle\hbar_1,(4,\frac{1}{5},\frac{1}{5})\rangle,\langle\hbar_2,(5,1,\frac{1}{9})\rangle,\langle\hbar_3,(3,\frac{1}{3},\frac{1}{3})\rangle,\langle\hbar_4,(2,\frac{1}{2},\frac{1}{4})\rangle\} \tag{14}$$

are the solutions of the above tpSNM soft games for Player 1 and Player 2, respectively.

4.2. *Applications of Two Person Simplified Neutrosophic Multiplicative Soft Games*

This part presents an example to illustrate the solution procedures (SNM soft saddle point method and SNM soft elimination method) of a tpSNM soft game and also gives comparison implementations.

Example 4.21. Assuming that the demand for beverages in the market is essentially the same, Beverage Company I (Player 1) and Beverage Company II (Player 2) want to increase their market share. These companies have a set of different beverages as $\mathcal{H}=\{\hbar_1=coke,\hbar_2=lemonade,\hbar_3=concentrated\ drink\}$. To achieve their goal, they come up with three alternative marketing strategies: reducing-price (x_1), advertising investment (x_2) and lagnappe (x_3).
Suppose that Beverage Company I (Player 1) chooses the strategies x_1, x_2 and x_3, i.e., $\mathcal{X}=\{x_1,x_2,x_3\}$, and Beverage Company II (Player 2) chooses the strategies x_1 and x_2, i.e., $\mathcal{Y}=\{x_1,x_2\}$. Due to the vagueness and indeterminacy of information, Beverage Company I and II can use simplified neutrosophic multiplicative values to represent the payoff for any one of the marketing strategies. The SNM soft game of Beverage Company I are considered in Table 10.

Table 10. The tpSNM soft game of Beverage Company I

$\Theta^1_{\mathcal{X}\times\mathcal{Y}}$	x_1	x_2
x_1	$\{\langle \hbar_1,(4,\frac{1}{2},\frac{1}{6})\rangle,\langle \hbar_2,(2,2,\frac{1}{5})\rangle,\langle \hbar_3,(7,\frac{1}{5},\frac{1}{9})\rangle\}$	$\{\langle \hbar_1,(4,\frac{1}{2},\frac{1}{6})\rangle,\langle \hbar_2,(2,2,\frac{1}{5})\rangle,\langle \hbar_3,(7,1,\frac{1}{8})\rangle\}$
x_2	$\{\langle \hbar_1,(4,2,\frac{1}{4})\rangle,\langle \hbar_2,(1,4,\frac{1}{3})\rangle,\langle \hbar_3,(3,\frac{1}{5},\frac{1}{3})\rangle\}$	$\{\langle \hbar_1,(3,2,\frac{1}{4})\rangle,\langle \hbar_2,(\frac{1}{5},3,\frac{1}{2})\rangle,\langle \hbar_3,(4,1,\frac{1}{5})\rangle\}$
x_3	$\{\langle \hbar_1,(1,\frac{1}{2},\frac{1}{9})\rangle,\langle \hbar_2,(2,4,\frac{1}{2})\rangle,\langle \hbar_3,(\frac{1}{9},\frac{1}{9},9)\rangle\}$	$\{\langle \hbar_1,(2,\frac{1}{2},\frac{1}{3})\rangle,\langle \hbar_2,(2,2,\frac{1}{2})\rangle,\langle \hbar_3,(\frac{1}{5},1,4)\rangle\}$

In Table 10, we can explain the action pair (x_1, x_2), if Beverage Company I (Player 1) selects the strategy reducing-price (x_1) when Beverage Company II (Player 2) selects the strategy advertising investment (x_2) then the SNM soft payoff of Beverage Company I is a set $\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_1,x_2) = \{\langle \hbar_1,(4,\frac{1}{2},\frac{1}{6})\rangle,\langle \hbar_2,(2,2,\frac{1}{5})\rangle,\langle \hbar_3,(7,1,\frac{1}{8})\rangle\}$. In such case, Beverage Company I increases sale of $\{\langle \hbar_1,(4,\frac{1}{2},\frac{1}{6})\rangle,\langle \hbar_2,(2,2,\frac{1}{5})\rangle,\langle \hbar_3,(7,1,\frac{1}{8})\rangle\}$ and Beverage Company II decreases sale of $\{\langle \hbar_1,(4,\frac{1}{2},\frac{1}{6})\rangle,\langle \hbar_2,(2,2,\frac{1}{5})\rangle,\langle \hbar_3,(7,1,\frac{1}{8})\rangle\}$.

Now, we ready to solve this tpSNM soft game.

It is easily seen from Table 10 that the strategy x_1 dominates to the strategy x_2 since $\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_2,x_j) \subseteq \theta^1_{\mathcal{X}\times\mathcal{Y}}(x_1,x_j)$ for all $x_j \in \mathcal{Y}$. That is, the second row is dominated by the first row. Deleting the second row from Table 10, we obtain Table 11.

Table 11. The reduced tpSNM soft game of Beverage Company I

$\Theta^1_{\mathcal{X}\times\mathcal{Y}}$	x_1	x_2
x_1	$\{\langle \hbar_1,(4,\frac{1}{2},\frac{1}{6})\rangle,\langle \hbar_2,(2,2,\frac{1}{5})\rangle,\langle \hbar_3,(7,\frac{1}{5},\frac{1}{9})\rangle\}$	$\{\langle \hbar_1,(4,\frac{1}{2},\frac{1}{6})\rangle,\langle \hbar_2,(2,2,\frac{1}{5})\rangle,\langle \hbar_3,(7,1,\frac{1}{8})\rangle\}$
x_3	$\{\langle \hbar_1,(1,\frac{1}{2},\frac{1}{9})\rangle,\langle \hbar_2,(2,4,\frac{1}{2})\rangle,\langle \hbar_3,(\frac{1}{9},\frac{1}{9},9)\rangle\}$	$\{\langle \hbar_1,(2,\frac{1}{2},\frac{1}{3})\rangle,\langle \hbar_2,(2,2,\frac{1}{2})\rangle,\langle \hbar_3,(\frac{1}{5},1,4)\rangle\}$

In Table 11, there is no another SNM soft dominated strategy. Now, we try to find the SNM soft saddle point value by using the SNM soft saddle point method.

$\widetilde{\bigcup}_{i\in\{1,3\}}\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_i,x_1) = \{\langle \hbar_1,(4,\frac{1}{2},\frac{1}{9})\rangle,\langle \hbar_2,(2,2,\frac{1}{5})\rangle,\langle \hbar_3,(7,\frac{1}{9},\frac{1}{9})\rangle\}$,

$\widetilde{\bigcup}_{i\in\{1,3\}}\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_i,x_2) = \{\langle \hbar_1,(4,\frac{1}{2},\frac{1}{6})\rangle,\langle \hbar_2,(2,2,\frac{1}{5})\rangle,\langle \hbar_3,(7,1,\frac{1}{8})\rangle\}$,

and

$\widetilde{\bigcap}_{j\in\{1,2\}}\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_1,x_j) = \{\langle \hbar_1,(4,\frac{1}{2},\frac{1}{6})\rangle,\langle \hbar_2,(2,2,\frac{1}{5})\rangle,\langle \hbar_3,(7,1,\frac{1}{8})\rangle\}$,

$\widetilde{\bigcap}_{j\in\{1,2\}}\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_3,x_j) = \{\langle \hbar_1,(1,\frac{1}{2},\frac{1}{3})\rangle,\langle \hbar_2,(2,4,\frac{1}{2})\rangle,\langle \hbar_3,(\frac{1}{9},1,9)\rangle\}$.

Since $\widetilde{\bigcup}_{i\in\{1,3\}}\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_i,x_2) = \widetilde{\bigcap}_{j\in\{1,2\}}\theta^1_{\mathcal{X}\times\mathcal{Y}}(x_1,x_j) = \theta^1_{\mathcal{X}\times\mathcal{Y}}(x_1,x_2)$, the optimal strategy of the game is (x_1,x_2). Hence, the value of tpSNM soft game is $\{\langle \hbar_1,(4,\frac{1}{2},\frac{1}{6})\rangle,\langle \hbar_2,(2,2,\frac{1}{5})\rangle,\langle \hbar_3,(7,1,\frac{1}{8})\rangle\}$.

Comparison and Discussion: In 2016, Deli and Çağman [10] published a seminal paper on soft games and thus took the first step to the application of soft sets in decision making based on game theory. Now, we consider the application (Table 10) in Section 4 of [10]. If the calculations are made by respectively corresponding to $\theta_{\mathcal{X}\times\mathcal{Y}}(x_i,y_j) = \langle u,(9,\frac{1}{9},\frac{1}{9})\rangle$ and $\theta_{\mathcal{X}\times\mathcal{Y}}(x_i,y_j) = \langle u,(\frac{1}{9},9,9)\rangle$ when $u \in f_{S_1}(x_i,y_j)$ and $u \notin f_{S_1}(x_i,y_j)$, then we obtain that the optimal strategy of game (described in [10]) is (x_3,y_3) and the value of game is $\{\langle u_1,(9,\frac{1}{9},\frac{1}{9})\rangle,\langle u_2,(9,\frac{1}{9},\frac{1}{9})\rangle,\langle u_3,(9,\frac{1}{9},\frac{1}{9})\rangle,\langle u_4,(\frac{1}{9},9,9)\rangle,\langle u_5,(\frac{1}{9},9,9)\rangle,\langle u_6,(\frac{1}{9},9,9)\rangle,\langle u_7,(\frac{1}{9},9,9)\rangle,\langle u_8,(\frac{1}{9},9,9)\rangle\}$.

Thus, it is obvious that similar results are obtained. Also, the applications of fuzzy soft games can be adapted by deriving new comparison methods between $0-1$ and $\frac{1}{9}-9$ scales similar to matches between $0-1$ and $\frac{1}{9}-9$ scales given in Table 1 in the Introduction section. The tpSNM soft games proposed in this study use the $\frac{1}{9}-9$ scale instead of the $0-1$ scale used for fuzzy (intuitionistic fuzzy/neutrosophic) soft games, and therefore may be advantageous in some cases. Consequently, we can say that the tpSNM soft games present the solutions to the soft games where alternatives are evaluated with truth, indeterminacy, falsity values scaled between $\frac{1}{9}-9$ with respect to the strategies.

5. n-Person Simplified Neutrosophic Multiplicative Soft Games

In this section, we introduce some fundamental concepts of n-person simplified neutrosophic multiplicative soft games.

In many stages of the real-world, the SNM soft games can also be played between more than two players. To propose the solution procedures for these games, we describe n-person SNM soft games by extending the tpSNM soft games as follows.

From now on, $\prod_{r=1}^{n} \mathcal{X}_r = \mathcal{X}_1 \times \mathcal{X}_2 \times ... \times \mathcal{X}_n$.

Definition 5.1. Let $\mathcal{S}$ be a set of strategies and $\mathcal{X}_1, \mathcal{X}_2, ..., \mathcal{X}_n \subseteq \mathcal{S}$ where $\mathcal{X}_r$ is the set of strategies of Player r $(r = 1, 2, ..., n)$. Then, for each Player r, an n-person SNM soft game (npSNM soft game) can be defined by an SNMSS over $\mathcal{H}$ as follows.

$$\Theta^r_{\prod_{r=1}^{n} \mathcal{X}_r} = \{((x_1, x_2, ..., x_n), \theta^r_{\prod_{r=1}^{n} \mathcal{X}_r}(x_1, x_2, ..., x_n)) : (x_1, x_2, ..., x_n) \in \prod_{r=1}^{n} \mathcal{X}_r, \theta^r_{\prod_{r=1}^{n} \mathcal{X}_r}(x_1, x_2, ..., x_n) \in \mathfrak{P}(\mathcal{H})\}$$

where $\theta^r_{\prod_{r=1}^{n} \mathcal{X}_r}$ is a SNM soft payoff function of Player r.

The npSNM soft game is played as below: at a certain Player 1 selects a strategy $x_1 \in \mathcal{X}_1$ and simultaneously each Player r $(r = 1, 2, ..., s)$ selects a strategy $x_r \in \mathcal{X}_r$ and once this is done each Player r receives the SNM soft payoff $\theta^r_{\prod_{r=1}^{n} \mathcal{X}_r}(x_1, x_2, ..., x_n)$.

Definition 5.2. Let $\Theta^r_{\prod_{r=1}^{n} \mathcal{X}_r}$ be an npSNM soft game with its SNM soft payoff function $\theta^r_{\prod_{r=1}^{n} \mathcal{X}_r}$ for $r = 1, 2, ..., n$. Then, a strategy $x_r \in \mathcal{X}_r$ is said to be an SNM soft dominated to another strategy $x \in \mathcal{X}_r$, if

$$\theta^r_{\prod_{r=1}^{n} \mathcal{X}_r}(x_1, x_2, ..., x_{r-1}, x, x_{r+1}, ..., x_n) \subseteq \theta^r_{\prod_{r=1}^{n} \mathcal{X}_r}(x_1, x_2, ..., x_{r-1}, x_r, x_{r+1}, ..., x_n)$$

for each $x_q \in \mathcal{X}_q$ of Player q $(q = 1, 2, ..., r-1, r+1, ..., n)$, respectively.

Definition 5.3. Let $\theta^{r}_{\prod_{r=1}^{n}\mathcal{X}_r}$ be an SNM soft payoff function of an npSNM soft game $\Theta^{r}_{\prod_{r=1}^{n}\mathcal{X}_r}$. If for each Player r $(r = 1, 2, ..., n)$ the following property are provided

$$\theta^{r}_{\prod_{r=1}^{n}\mathcal{X}_r}(x_1^*, x_2^*, ..., x_{r-1}^*, x, x_{r+1}^*, ..., x_n^*) \subseteq \theta^{r}_{\prod_{r=1}^{n}\mathcal{X}_r}(x_1^*, x_2^*, ..., x_{r-1}^*, x_r^*, x_{r+1}^*, ..., x_n^*)$$

for each $x \in \mathcal{X}_r$, then $(x_1^*, x_2^*, ..., x_n^*) \in \prod_{r=1}^{n}\mathcal{X}_r$ is termed to be an npSNM soft Nash equilibrium of an npSNM soft game.

6. Conclusions

In this paper, the concept of SNMSS was introduced and their fundamental operations such as intersection, union, complement, And-product, Or-product and cartesian product were presented. The desirable properties of the emerged operations of SNMSSs were investigated in detail. By using SNMSS operations, the fundamentals of SNM soft games were studied. The proposed SNM soft game schemes were illustrated by an example regarding the strategy problem. In the near future, it is expected that the approach of SNMSS will advance in several directions such as new operations, measures of similarity, distance and entropy, correlation coefficients, algebraic and topological structures, and thus contribute to many research areas both theoretically and practically. By applying SNM soft games to problems in different fields, their success in practice may be illustrated.

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Properties on Topologized Domination in Neutrosophic Graphs

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Abstract: In this paper, the new concept Topologized domination on Neutrosophic Graphs is introduced. The idea of N-Top domination is discussed in cycle, path, complete graph, star graph. The basic properties of N-Top dom set, N-Top minimum dom set, N-Top minimal dom set are introduced and N-Top dom number is also established with some necessary examples.

Keywords: N-Top dom set, N-Top minimum dom set, N-Top minimal dom set, N-Top dom number.

1 Introduction

The concept of topologized graph was introduced by Antoine Vella in 2005 [1]. Antoine Vella extended topology to the topologized graph by the S_1 space and the boundary of every vertex and edges of a graph G. The space is called S_1 space if every singleton in the topological space either open or closed. Chang [5] introduced the concept of the notion of fuzzy topology.In 2017,topologizd graph extended to Topologized bipartitie graph Topologized Hamiltonian and complete graph by vimala.s et al [13,14]. Ore [9] introduced the concept of theory of domination of graph.In 1997 T.Heynes , S. Hedetniemi and P. Slater published the book, ”Fundamentals of domination in graphs” [6]. After this publication there has been a rapid growth of research on this area and a wide variety of domination parameters have been introduced.

Bhuvaneswari et. al [4] handled the concept of topologized domination in graph and explained some of its properties.Smarandache[10] was first person introduced the idea of neutrosophic theory. He discussed some types of neutrosophic sets like Over, Under/Off sets etc.,[12]. He extended work on HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra[11].Smarandache has introduced in 2020 the n-SuperHyperGraph, with super-vertices [that are groups of vertices] and hyper-edges defined on power-set of power-set... that is the most general form of graph as today, and n-HyperAlgebra. A SuperHyperGraph, is a HyperGraph (where a group of Edges form a HyperEdge) such that a group of vertices are united all together into a SuperVertex like a group of people (=vertices) that are united all together into an organization (=SuperVertex) ;and further on the n-SuperHyperGraph where many groups (=SuperVertices) are united all together to form a group-of-groups (called 2-SuperVertex, or Type-2 SuperVertex), then a group of Type-2 SuperVertices forms a Type-3 SuperVertex, . . . , and so on up to Type-n SuperVertex, for any n 1, which better reflects our reality. Later Narmada Devi[7,8] worked on new type of neutrosophic off graph and minimal domination via neutrosophic over graph. In this article, the novel of topologized domination of N-graphs are developed and some of its interesting properties are established.

2 Preliminiaries

Definition 2.1. [4] A topologized graph is a topological space $\mathscr{H}$ such that

(i) every singleton is open or closed

(ii) $\forall h \in \mathscr{H}$, $|\eth(h)| \leq 2$, since $\eth(h)$ is denoted by the boundary of a point h.

Definition 2.2. [8] A set $\mathscr{S}$ of vertices of $\mathscr{G}$ is said to be a top domination set $\mathscr{S}$ if $\mathscr{G}$ is a top graph and every vertex in $\mathscr{V}(\mathscr{G})) - \mathscr{S}$ is adjacent to atleast one vertex of in $\mathscr{S}$.

Definition 2.3. [8] The minimum cardinality among all the top dom set of $\mathscr{G}$ is called the top dom number of G and it is denoted by $\tau\gamma(\mathscr{G})$.

Definition 2.4. [8] A Ngraph is a pair $\mathscr{G} = (P, Q)$ of a crisp graph $\mathscr{G}^* = (V, E)$ where P is Nvertex set in V and Q is a Nedge set in E such that

(i) $\mathscr{T}_Q\left(m_i m_j\right) \leq \mathscr{T}_P\left(m_i\right) \wedge \mathscr{T}_P\left(m_j\right)$

(ii) $\mathscr{I}_Q\left(m_i m_j\right) \leq \mathscr{I}_P\left(m_i\right) \wedge \mathscr{I}_P\left(m_j\right)$

(iii) $\mathscr{F}_Q\left(m_i m_j\right) \geq \mathscr{T}_P\left(m_i\right) \vee \mathscr{T}_P\left(m_j\right)$ $\left(m_i, m_j\right) \in E$

3 Neutrosophic Topologized Domination Graphs

An important concept of N-Top dom in graphs with suitable examples are discussion this section. Throughout this paper $\mathscr{G}^* = (V, E)$ denotes a crisp graph and $\mathscr{G} = (P, Q)$ a Ngraph.

Definition 3.1. A Ngraph $\mathscr{G}$ is called N-Top graph if $\mathscr{G}^*$ satisfy the following condition

(i) every singleton is open or closed

(ii) $\forall h \in \mathscr{H}$, $|\eth(h)| \leq 2$, since $\eth(h)$ is denoted by the boundary of a point h.

Definition 3.2. A set $\mathscr{S}$ of vertices of $\mathscr{G}$ is said to be N-Top dom set in $\mathscr{G}$ if $\mathscr{G}$ is a N-Top graph and every vertex in $\mathscr{V}(\mathscr{G})) - \mathscr{S}$ is adjacent to atleast one vertex in $\mathscr{S}$at the degree of truth, indeterminacy and falsity-membership belongs to $[0, 1]$ such that $0 \leq \mathscr{T}_P(m) + \mathscr{I}_P(m) + \mathscr{F}_P(m) \leq 3,\ \forall m \in V$

Example 3.1.

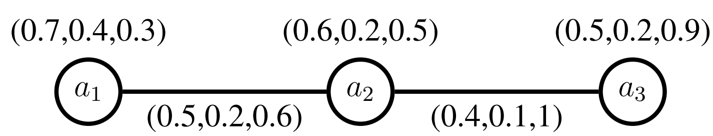


Figure 1: S_3-star graph

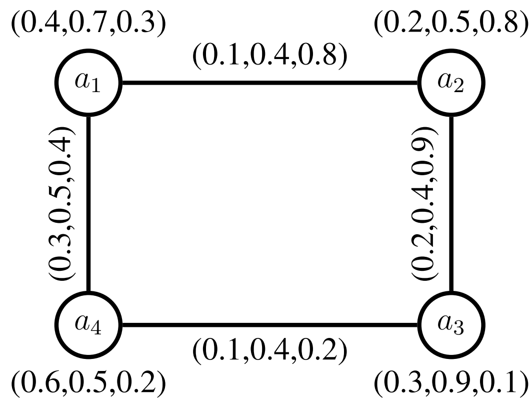
Let $\mathscr{H} = \{a_1, a_2, a_3, (0.5, 0.2, 0.6), (0.4, 0.1, 1)\}$ be a topological space defined by the topology $\tau = \{\mathscr{H}, \emptyset, \{a_1\}, \{a_2\}, \{a_3\}, \{a_1, a_2\}, \{a_1, a_3\}, \{a_2, a_3\}\}$. Here for every $\{h\} \in \mathscr{H}$ is open or closed and $|\eth(h)| \leq 2$. By the definition of 3.1, $\mathscr{G} = (P, Q)$ is N-Top graph. Also N-top dom sets by $\mathscr{D}_1 = \{a_2\}$ and $\mathscr{D}_2 = \{a_1, a_3\}$.

Theorem 3.1. Let $\mathscr{G}$ be a N-Top graph with atmost degree two. If $\mathscr{S}$ is a top dom set of $\mathscr{G}$, then it is a N-Top dom set of $\mathscr{G}$.

Proof:

Let $\mathscr{H}$ be a topological space with topology τ defined by $V \cup E$. Since every singleton set is open or closed and $\mathscr{G}$ is a N-graph where the truth, indeterminacy and falsity membership function with unit interval $[0,1]$ such that $0 \leq \mathscr{T}_P(m) + \mathscr{I}_P(m) + \mathscr{F}_P(m) \leq 3$, $\forall m \in V$ and atmost degree two. Hence $|\eth(h)| \leq 2$. This implies that $\mathscr{G}$ is N-Top graph. Let $\mathscr{S}$ be top dom set then every vertex in $\mathscr{V}(\mathscr{G})) - \mathscr{S}$ is adjacent to atleast one vertex of $\mathscr{S}$ thus implies that $\mathscr{S}$ is a N-Top dom set of $\mathscr{G}$.

Example 3.2.



Let a_1, a_2, a_3 and a_4 denote the vertices and $(0.1, 0.4, 0.8)$, $(0.2, 0.4, 0.9)$, $(0.1, 0.4, 0.2)$, $(0.3, 0.5, 0.4)$ denote the edges which are labelled $f(0.1, 0.4, 0.8) = \{a_1, a_2\}$, $f(0.2, 0.4, 0.9) = \{a_2, a_3\}$, $f(0.1, 0.4, 0.2) = \{a_3, a_4\}$, $f(0.3, 0.5, 0.4) = \{a_4, a_1\}$,

Let a_1, a_2, a_3 and a_4 denote the vertices and $(0.1, 0.4, 0.8)$, $(0.2, 0.4, 0.9)$, $(0.1, 0.4, 0.2)$, $(0.3, 0.5, 0.4)$ denote the edges.

Let $\mathscr{H} = \{a_1, a_2, a_3, a_4, (0.1, 0.4, 0.2), (0.3, 0.5, 0.4), (0.1, 0.4, 0.8), (0.2, 0.4, 0.9)\}$ be a topological space defined by the topology

$$\tau = \Big\{\mathscr{H}, \emptyset, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}, \{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}, \{a_2, a_3\}, \{a_2, a_4\}, \{a_3, a_4\}, \{a_1, a_2, a_3\}, \{a_1, a_2, a_4\}, \{a_2, a_3, a_4\}, \{a_1, a_3, a_4\}\Big\}$$

Here for every $\{h\} \in \mathscr{H}$ is open and $|\partial(h)| \leq 2$. We have $\partial(a_1) = \{a_2, a_4\}$, $\partial(a_2) = \{a_1, a_3\}$, $\partial(a_3) = \{a_2, a_4\}$ and $\partial(a_4) = \{a_1, a_3\}$ with $|\eth(h_i)| = 2$ where $i = 1, 2, 3, 4$.

Hence this graph is a N-Top graph.

Then $D = \{a_1, a_3\}$ and $\{a_2, a_4\}$ is a N-Top dom set in V whose maximum and minimum degrees of $\mathscr{T}, \mathscr{I}, \mathscr{F}$ respectively.

	$\mathscr{T}_A$	$\mathscr{I}_A$	$\mathscr{F}_A$
a_1	0.4	0.7	0.3
a_2	0.2	0.5	0.8
a_3	0.3	0.9	0.1
a_4	0.6	0.5	0.2

	a_1a_2	a_2a_3	a_3a_4	a_4a_1
$\mathscr{T}_{m_i \cup m_j}$ = max $\langle m_i, m_j \rangle$	0.4	0.3	0.6	0.6
$\mathscr{I}_{m_i \cup m_j}$ = max $\langle m_i, m_j \rangle$	0.7	0.9	0.9	0.7
$\mathscr{F}_{m_i \cap m_j}$ = min $\langle m_i, m_j \rangle$	0.3	0.1	0.1	0.2

	a_1a_2	a_2a_3	a_3a_4	a_4a_1
$\mathscr{T}_{m_i \cap m_j} = \min \langle m_i, m_j \rangle$	0.2	0.2	0.3	0.4
$\mathscr{I}_{m_i \cap m_j} = \min \langle m_i, m_j \rangle$	0.5	0.5	0.5	0.5
$\mathscr{F}_{m_i \cup m_j} = \max \langle m_i, m_j \rangle$	0.8	0.8	0.2	0.3

Therefore this graph is N-TOP dom graph.

Definition 3.3. A dominating set $\mathscr{S}$ of the N-Top graph $\mathscr{G}$ is said to be a minimal N-Top dom set if for every vertex v in $\mathscr{S}$, $\mathscr{S} - \{v\}$ is not a of $\mathscr{S}$ is a N-Top dom set. i.e., no proper subset of $\mathscr{S}$ is a N-Top dom set.

Example 3.3. From Example 1, $\{a_1, a_3\}$ is N-Top minimal dom set but which is not a N-Top dom set.

Theorem 3.2. Let $\mathscr{G}$ be a N-Top graph with atmost degree two. If $\mathscr{S}$ is a N-TOP minimum dom set, then D is a N-Top minimal dom set.

Proof:

Let $\mathscr{H}$ be a topological space with topology τ defined by $V \cup E$.

Since every singleton set is open or closed and $\mathscr{G}$ is a N-graph where the truth, indeterminacy and falsity membership function with unit interval $[0, 1]$ such that $0 \leq \mathscr{T}_P(m) + \mathscr{I}_P(m) + \mathscr{F}_P(m) \leq 3$, $\forall m \in V$ and atmost degree two. Hence $|\eth(h)| \leq 2$. This implies that $\mathscr{G}$ is N-Top graph.

Let $\mathscr{S}$ be a top minimum dom set. Then every $v \in \mathscr{S}, \mathscr{S} - \{v\}$ is not a top dom set which implies that $\mathscr{S}$ is a N-Top minimal dom set.

Remark 3.1. The converse of the above theorem need not by true. Since every graph need not be a N-Top graph. Consider the following example.

Example 3.4. Let $\mathscr{G}$ be a N complete graph K_4 with 4 vertices.

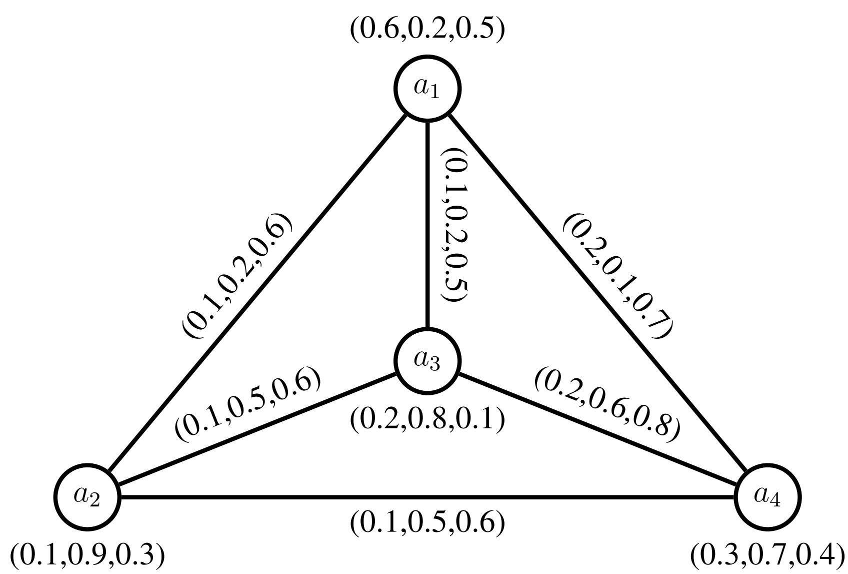
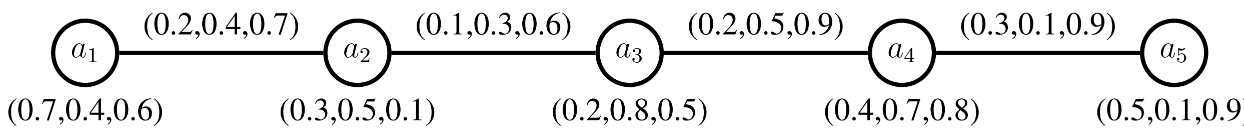


Figure 2: K_4 N-complete graph

Here every singleton sets are minimum dominating sets. Clearly the complete graph K_4 is not a N-Top graph, since $n \geq 4$. Then the a N-Top dom set does exits.Then the dom sets need not be a N-Top dom set.

Lemma .1. Let P_n be a N-path with n vertices which is a N-Top graph. Then the N-Top dom number is $\tau_\gamma(P_n) \geq \lceil n/3 \rceil$.

Example 3.5.



Let a_1, a_2, a_3, a_4 and a_5 denote the vertices and $(0.2, 0.4, 0.7), (0.1, 0.3, 0.6), (0.2, 0.5, 0.9), (0.3, 0.1, 0.9)$ denote the edges which are labelled $f(0.2, 0.4, 0.7) = \{a_1, a_2\}$, $f(0.1, 0.3, 0.6) = \{a_2, a_3\}$, $f(0.2, 0.5, 0.9) = \{a_3, a_4\}$, $f(0.3, 0.1, 0.9) = \{a_4, a_5\}$,

Let $\mathscr{H} = \{a_1, a_2, a_3, a_4, a_5, (0.2, 0.4, 0.7), (0.1, 0.3, 0.6), (0.2, 0.5, 0.9), (0.3, 0.1, 0.9)\}$ be a topological space defined by the topology

$$\tau = \Big\{\mathscr{H}, \emptyset, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}, \{a_5\}, \{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}, \{a_1, a_5\}, \{a_2, a_3\}, \{a_2, a_4\}, \{a_2, a_5\}, \{a_3, a_4\}, \{a_3, a_5\}, \{a_4, a_5\}, \{a_1, a_2, a_3\}, \{a_1, a_2, a_4\}, \{a_1, a_2, a_5\}, \{a_2, a_3, a_4\}, \{a_2, a_3, a_5\}, \{a_3, a_4, a_5\}, \{a_1, a_2, a_3, a_4\}, \{a_1, a_2, a_3, a_5\}\Big\}$$

Here for every $\{h\} \in \mathscr{H}$ is open and $|\eth(h)| \leq 2$. By the definition 3.1,it is a N-Top graph.
Then $\mathscr{S} = \{a_2, a_5\}$ is a N-Top dom set in V whose maximum and minimum degrees of $\mathscr{T}, \mathscr{I}, \mathscr{F}$ respectively.

	a_1	a_2	a_3	a_4	a_5
$\mathscr{T}_A$	0.7	0.3	0.2	0.4	0.5
$\mathscr{I}_A$	0.4	0.5	0.8	0.7	0.1
$\mathscr{F}_A$	0.6	0.1	0.5	0.8	0.9

	a_1a_2	a_2a_3	a_3a_4	a_4a_5
$\mathscr{T}_B$(min)	0.2	0.1	0.2	0.3
$\mathscr{I}_B$(min)	0.4	0.3	0.5	0.1
$\mathscr{F}_B$(max)	0.7	0.6	0.9	0.9

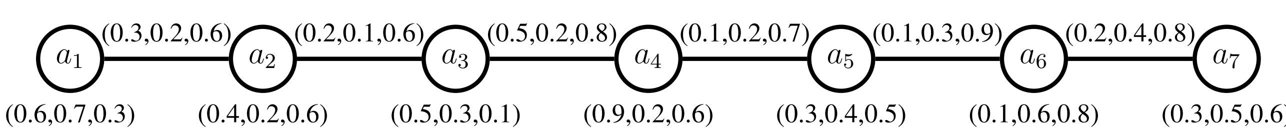
	a_1a_2	a_2a_3	a_3a_4	a_4a_5
$\mathscr{T}_B$(max)	0.7	0.3	0.4	0.5
$\mathscr{I}_B$(max)	0.5	0.8	0.8	0.7
$\mathscr{F}_B$(min)	0.6	0.5	0.8	0.9

The N-Top dom set is $\mathscr{S} = \{a_2, a_5\}$.

$$\tau_\gamma(P_5) = \lceil 5/3 \rceil = \lceil 1.666 \rceil = 2.$$

Therefore this graph is N-Top dom graph.

Example 3.6.



Let $a_1, a_2, a_3, a_4, a_5, a_6$ and a_7 denote the vertices and $(0.3, 0.2, 0.6)$, $(0.2, 0.1, 0.6)$, $(0.5, 0.2, 0.8)$, $(0.1, 0.2, 0.7)$, $(0.1, 0.3, 0.9)$, $(0.2, 0.4, 0.8)$ denote the edges which are labelled $f(0.3, 0.2, 0.6) = \{a_1, a_2\}$, $f(0.2, 0.1, 0.6 = \{a_2, a_3\}$, $f(0.5, 0.2, 0.8) = \{a_3, a_4\}$, $f(0.1, 0.2, 0.7) = \{a_4, a_5\}$, $f(0.1, 0.3, 0.9) = \{a_5, a_6\}$, $f(0.2, 0.4, 0.8) = \{a_6, a_7\}$,

Let $\mathscr{H} = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, (0.3, 0.2, 0.6), (0.2, 0.1, 0.6), (0.5, 0.2, 0.8), (0.1, 0.2, 0.7), (0.1, 0.3, 0.9), (0.2, 0.4, 0.8)\}$ be a topological space defined by the topology

$$\begin{aligned}\tau = \Big\{&\mathscr{H}, \emptyset, \{a_1\}, \{a_2, a_3\}, \{a_4\}, \{a_5, a_6\}, \{a_7\}, \{a_1, a_2, a_3\}, \{a_1, a_4\}, \{a_1, a_5, a_6\}, \{a_1, a_7\},\\
&\{a_2, a_3, a_4\}, \{a_2, a_3, a_5, a_6\}, \{a_2, a_3, a_7\}, \{a_4, a_5, a_6\}, \{a_4, a_7\}, \{a_5, a_6, a_7\}, \{a_1, a_2, a_3, a_4\},\\
&\{a_1, a_2, a_3, a_5, a_6\}, \{a_1, a_2, a_3, a_7\}, \{a_1, a_2, a_3, a_5, a_6\}, \{a_1, a_2, a_3, a_4, a_5, a_6\}, \{a_1, a_2, a_3, a_4, a_7\},\\
&\{a_1, a_2, a_3, a_5, a_6, a_7\}, \{a_1, a_4, a_5, a_6\}, \{a_1, a_4, a_7\}, \{a_1, a_4, a_7\}, \{a_1, a_4, a_5, a_6, a_7\},\\
&\{a_1, a_2, a_3, a_4, a_5, a_6\}, \{a_1, a_2, a_3, a_4, a_7\}, \{a_2, a_3, a_4, a_5, a_6, a_7\}, \{a_2, a_3, a_4, a_7\},\\
&\{a_1, a_2, a_3, a_5, a_6, a_7\}, \{a_4, a_5, a_6, a_7\}, \{a_2, a_3, a_5, a_6, a_7\}, \{a_2, a_3, a_4, a_7\}, \{a_2, a_3, a_4, a_5, a_6\},\\
&\{a_1, a_5, a_6, a_7\}, \{a_1, a_4, a_7\}, \{a_1, a_4, a_5, a_6\}, \{a_1, a_3, a_4, a_5, a_6, a_7\}, \{a_1, a_2, a_4, a_5, a_6, a_7\},\\
&\{a_1, a_2, a_3, a_4, a_6, a_7\}, \{a_1, a_2, a_3, a_4, a_5, a_7\}, \Big\}\end{aligned}$$

Here for every $\{h\} \in \mathscr{H}$ is open or closed and $|\eth(h)| \leq 2$. By the definition of 3.1,it is a N-Top graph.

Then $\mathscr{S} = \{a_2, a_5, a_7\}$ is a N-Top dom set in V whose maximum and minimum degrees of $\mathscr{T}, \mathscr{I}, \mathscr{F}$ respectively.

	a_1	a_2	a_3	a_4	a_5	a_6	a_7
$\mathscr{T}_P$	0.6	0.4	0.5	0.9	0.3	0.1	0.3
$\mathscr{I}_P$	0.7	0.2	0.3	0.2	0.4	0.6	0.5
$\mathscr{F}_P$	0.3	0.6	0.1	0.6	0.5	0.8	0.6

	a_1a_2	a_2a_3	a_3a_4	a_4a_5	a_5a_6	a_6a_7
$\mathscr{T}_Q$(max)	0.6	0.5	0.9	0.9	0.3	0.3
$\mathscr{I}_Q$(max)	0.7	0.3	0.3	0.4	0.6	0.6
$\mathscr{F}_Q$(min)	0.3	0.1	0.1	0.5	0.5	0.6

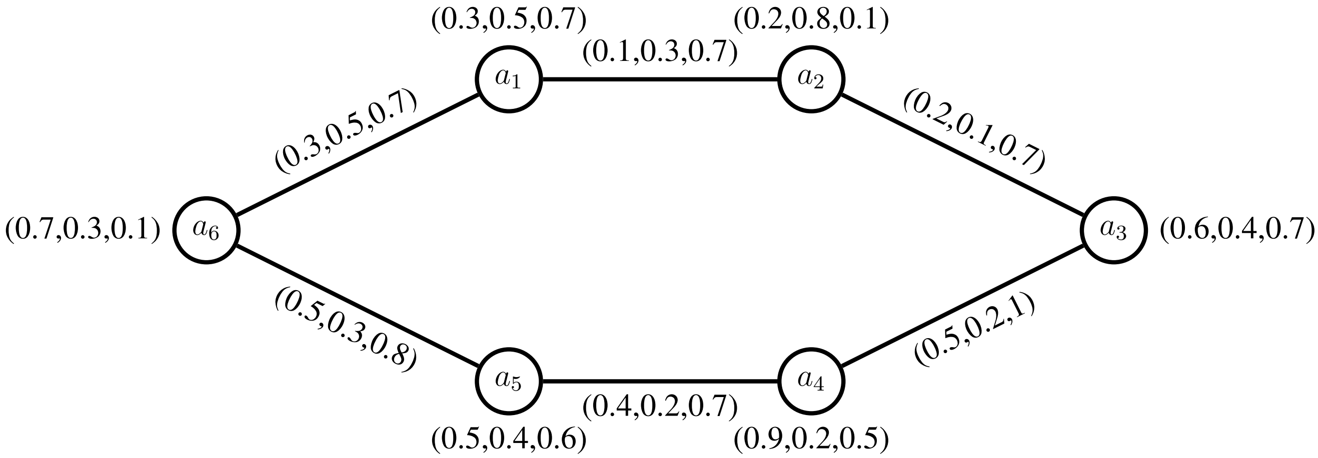
	a_1a_2	a_2a_3	a_3a_4	a_4a_5	a_5a_6	a_6a_7
$\mathscr{T}_Q$(min)	0.4	0.3	0.5	0.3	0.1	0.1
$\mathscr{I}_Q$(min)	0.2	0.1	0.2	0.2	0.4	0.5
$\mathscr{F}_Q$(max)	0.6	0.7	0.8	0.6	0.8	0.9

Thererfore N-top dom set is $\mathscr{S} = \{a_2, a_5, a_7\}$.

$$\tau_\gamma(P_7) = \lceil 7/3 \rceil = \lceil 2.333 \rceil > 2.$$

Lemma .2. Let $\mathscr{C}_n$ be a N-cycle with $n-$vertices which is a N-Top graph. Then the N-Top dom number $\tau_\gamma(\mathscr{C}_n) \geq \lceil n/3 \rceil$.

Example 3.7.



Let a_1, a_2, a_3, a_4, a_5 and a_6 denote the vertices and $(0.1, 0.3, 0.9)$, $(0.2, 0.1, 0.7)$, $(0.5, 0.2, 0.1)$, $(0.4, 0.2, 0.7)$, $(0.5, 0.3, 0.8)$, $(0.3, 0.5, 0.7)$ denote the edges which are labelled $f(0.1, 0.3, 0.9) = \{a_1, a_2\}$, $f(0.2, 0.1, 0.7) = \{a_2, a_3\}$, $f(0.5, 0.2, 0.1) = \{a_3, a_4\}$, $f(0.4, 0.2, 0.7) = \{a_4, a_5\}$, $f(0.5, 0.3, 0.8) = \{a_5, a_6\}$, $f(0.3, 0.5, 0.7) = \{a_6, a_1\}$,

Let $\mathscr{H} = \{a_1, a_2, a_3, a_4, a_5, a_6, (0.1, 0.3, 0.9), (0.2, 0.1, 0.7), (0.5, 0.2, 0.1), (0.4, 0.2, 0.7), (0.5, 0.3, 0.8), (0.3, 0.5, 0.7)\}$ be a topological space defined by the topology

$$\begin{aligned}\tau = \Big\{ & \mathscr{H}, \emptyset, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}, \{a_5\}, \{a_6\}, \\ & \{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}, \{a_1, a_5\}, \{a_1, a_6\}, \{a_2, a_3\}, \{a_2, a_4\}, \{a_2, a_5\}, \{a_2, a_6\}, \{a_3, a_4\}, \\ & \{a_3, a_5\}, \{a_3, a_6\}, \{a_4, a_5\}, \{a_4, a_6\}, \{a_5, a_6\}, \{a_1, a_2, a_3\}, \{a_1, a_2, a_4\}, \{a_1, a_2, a_5\}, \{a_1, a_2, a_6\}, \\ & \{a_1, a_2, a_3, a_4\}, \{a_1, a_2, a_3, a_5\}, \{a_1, a_2, a_3, a_6\}, \{a_1, a_2, a_3, a_4, a_5\}, \{a_1, a_2, a_3, a_4, a_6\}, \\ & \{a_2, a_3, a_4\}, \{a_2, a_3, a_5\}, \{a_2, a_3, a_6\}, \{a_3, a_4, a_5\}, \{a_3, a_4, a_6\}, \{a_4, a_5, a_6\}, \{a_2, a_3, a_4, a_5\}, \\ & \{a_2, a_3, a_4, a_6\}, \{a_2, a_3, a_4, a_5, a_6\}, \{a_1, a_3, a_4, a_5, a_6\}, \{a_1, a_2, a_4, a_5, a_6\}, \{a_1, a_2, a_3, a_5, a_6\}, \\ & \{a_1, a_2, a_3, a_4, a_5\}, \{a_3, a_4, a_5, a_6\}, \{a_2, a_4, a_5, a_6\}, \{a_2, a_3, a_4, a_6\}, \{a_2, a_3, a_4, a_5\}, \{a_3, a_5, a_6\} \Big\}\end{aligned}$$

Here for every $\{h\} \in \mathscr{H}$ is open and $|\eth(h)| \leq 2$. By the definition of 3.1, it is a N-Top graph.

Then $\mathscr{S} = \{a_3, a_6\}$ is a N-Top dom set in V whose maximum and minimum degrees of $\mathscr{T}, \mathscr{I}, \mathscr{F}$ respectively.

	a_1	a_2	a_3	a_4	a_5	a_6
$\mathscr{T}_P$	0.3	0.2	0.6	0.9	0.5	0.7
$\mathscr{I}_P$	0.5	0.8	0.4	0.2	0.4	0.3
$\mathscr{F}_P$	0.7	0.1	0.7	0.5	0.6	0.1

	a_1a_2	a_2a_3	a_3a_4	a_4a_5	a_5a_6	a_6a_1
$\mathscr{T}_Q$(max)	0.3	0.6	0.9	0.9	0.8	0.7
$\mathscr{I}_Q$(max)	0.8	0.8	0.3	0.5	0.4	0.5
$\mathscr{F}_Q$(min)	0.1	0.1	0.5	0.5	0.1	0.1

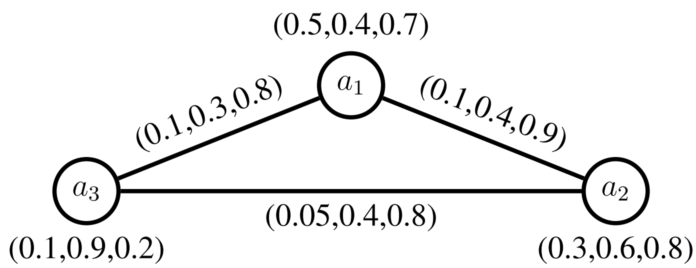
	a_1a_2	a_2a_3	a_3a_4	a_4a_5	a_5a_6	a_6a_1
$\mathscr{T}_Q$(min)	0.1	0.2	0.5	0.4	0.5	0.3
$\mathscr{I}_Q$(min)	0.3	0.1	0.2	0.2	0.3	0.5
$\mathscr{F}_Q$(max)	0.7	0.7	0.8	0.7	0.8	0.7

Therefore N-Top dom set is $\mathscr{S} = \{a_3, a_6\}$.

$$\tau_\gamma(C_6) = \lceil 6/3 \rceil = 2.$$

Lemma .3. Let $(\mathscr{K}_n)$ be a N-complete graph with $n-$vertices $(n = 2, 3)$ which is a N-Top graph. Then the N-Top dom number $\tau_\gamma(\mathscr{K}_n) = 1$.

Example 3.8.



Let a_1, a_2 and a_3 denote the vertices and $(0.1, 0.4, 0.9), (0.05, 0.4, 0.8), (0.1, 0.3, 0.8)$ denote the edges which are labelled $f(0.1, 0.4, 0.9) = \{a_1, a_2\}$, $f(0.05, 0.4, 0.8) = \{a_2, a_3\}$, $f(0.1, 0.3, 0.8) = \{a_3, a_1\}$,

Let $\mathscr{H} = \{a_1, a_2, a_3, (0.1, 0.4, 0.9), (0.05, 0.4, 0.8), (0.1, 0.3, 0.8)\}$ be a topological space defined by the topology

$$\tau = \Big\{\mathscr{H}, \emptyset, \{a_1\}, \{a_2\}, \{a_3\}, \{a_1, a_2\}, \{a_1, a_3\}, \{a_2, a_3\}\Big\}$$

Here for every $\{h\} \in \mathscr{H}$ is open and $|\eth(h)| \leq 2$. By the definition 3.1, it is a N-Top graph.

Then $\mathscr{S} = \{a_1\}$ and $\{a_2a_3\}$ is a N-Top dom set in V whose maximum and minimum degrees of $\mathscr{T}, \mathscr{I}, \mathscr{F}$ respectively.

	a_1	a_2	a_3
T_A	0.5	0.3	0.1
I_A	0.4	0.6	0.9
F_A	0.7	0.8	0.2

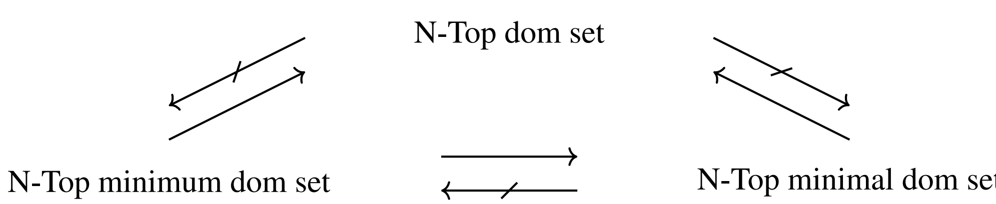
	a_1a_2	a_2a_3	a_3a_1
$\mathscr{T}_P$(max)	0.5	0.3	0.5
$\mathscr{I}_P$(max)	0.6	0.9	0.9
$\mathscr{F}_P$(min)	0.7	0.2	0.2

	a_1a_2	a_2a_3	a_3a_1
$\mathscr{T}_Q$(min)	0.3	0.1	0.1
$\mathscr{I}_Q$(min)	0.4	0.6	0.4
$\mathscr{F}_Q$(max)	0.8	0.8	0.7

Therefore N-Top dom set is $\mathscr{S} = \{a_1\}$, whose top dom number is given by

$$\tau_\gamma(\mathscr{K}_3) = 1.$$

Remark 3.2. The interrelationship among N-Top dom set as given below



4 Conclusion

This paper has focused on calculating the dominating number of N-Top graph G by using top domination conditions.The Top dom condition is introduced in new method to find the domination number. The N-Top domination for some standard N-graphs such as a path,cycle are specified.The future study can be continued by forming different types of N-Top domination set with various applications.

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View On Neutrosophic Over Topologized Domination Graphs

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Abstract: Neutrosophic over set can deal with the uncertainties related to the information of any decision making problem in real life scenaries, where fuzzy set may fail to handle those uncertainties properly. The study on presented in this "Neutrosophic Over topologized domination graphs" and also classification in different frame are discussed.The idea of $NOver\ Top$-dom set, $NOver\ Top$-minimum dom set, $NOver\ Top$-minimal dom set and $NOver\ Top$-dom number are introduced and necessary examples are established. In any neutrosophic over decision-making problem, the decision maker use the comparision of neutrosophic over number to choose alternative solutions.

Keywords: $NOver\ Top$-dom set, $NOver\ Top$-minimum dom set, $NOver\ Top$-minimal dom set, $NOver\ Top$-dom number.

1 Introduction

The uncertainty theory plays an influential role in handling various real-life models in the field of science and engineering, In the current era, the multi-criteria decision making (MCDM) process has gained much attention by several researchers as it can nicely handle many real-life challenging problems in many front line areas like a financial investment,recruitment polices, clinical diagnosis of disease, design of the complex circuit etc., It is not an overstated fact that the fuzzy set theory plays a very crucial role in decision-making problems, especially when decision-makers work in an uncertain environment. The theories of uncertainty have geared up dramatically after the introduction of the fuzzy set by zadeh[20] and intuitionstic fuzzy set where he introduced the concept of membership function of belongingness. Smarandache[19] manifests the idea of a neutrosophic set. Neutrosophic set considers the truth membership,the indeterminacy membership function, and the falsity membership function simultaneously. Invention of neutrosophic set plays an important impact in science and engineering research domain. In this current epoch, it is generally used in decision making (DM) problem and mathematical modelling. As researchers developed single valued neutrosophic set[18],some types of neutrosophic sets like Over, Under/Off sets etc.,[11], Narmada Devi[14,15] worked on new type of neutrosophic off graph and minimal domination via neutrosophic over graph. Recently chakraborty[5]constructed the theory of pentagonal neutrosophic set.Few otherr research work[6,7,10,11,12,13,17] also published in this field. The concept of topologized graph was introduced by Antoine Vella in 2005 [1]. Antoine Vella extended topology to the topologized graph by the S_1 space and the boundary of every vertex and edges of a graph G. The space is called S_1 space if every singleton in the topological space either open or closed.Chang [8] introduced the concept of the notion of fuzzy topology. Ore [16] introduced the concept of theory of domination of graph. Heynes et. al [9] published the book regarding the concept of fundamental domination on graphs and they worked more concepts on domination on graphs. Annadurai and Bhuvaneswari et. al [3,4] handled the concept of topologized domination in graph and explained some of its properties. In this article, the novel of topologized domination on NOver top graphs are developed and some of its interesting properties are established.

2 Preliminiaries

Definition 2.1. [4] A set $\mathscr{D}$ of vertices of $\mathscr{G}$ is said to be a *topologized domination set* $\mathscr{D}$ if $\mathscr{G}$ is a *topologized graph* and every vertex in $\mathscr{V} - \mathscr{D}$ is adjacent to atleast one vertex of in $\mathscr{D}$.

Definition 2.2. [8] A single values *neutrosophic over set* P is defined as $P = (f, \langle \alpha(\text{f}),\beta(\text{f}),\gamma(\text{f})\rangle, f \in F$ such that there exist some element in P that have atleast one neutrosophic component that is > 1 and no element has Neutrosophic component that are < 0 and $\alpha(\text{f}),\beta(\text{f}),\gamma(\text{f}) \in [0, \Omega]$ where Ω is called *overlimit* such that $0 < 1 < \Omega$.

Definition 2.3. [8] A *NOver graph* $\mathscr{G} = (P, Q)$ is a *Ngraph* on a crisp graph G^* where P is an *neutrosophic vertex over set* on $\mathscr{V}$ and Q is a *neutrosophic edge over set* on $\mathscr{E}$ respectively such that

(i) $\alpha_Q(mn) \leq [\alpha_P(m) \wedge \alpha_P(n)]$

(ii) $\beta_Q(mn) \leq [\beta_P(m) \wedge \beta_P(n)]$

(iii) $\gamma_Q(mn) \geq [\gamma_P(m) \vee \gamma_P(n)]$ for every $mn \in \mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$.

3 Neutrosophic Over Topologized Domination Graphs

Definition 3.1. A *Nover graph* $\mathscr{G} = (P, Q)$ is called *NOver Top graph* if $\mathscr{G}^*$ satisfy the following condition

(i) every singleton is open or closed in $\mathscr{V}$.

(ii) $\forall f \in \mathscr{F}$, $|\partial(f)| \leq 2$ where $\partial(f)$ is denoted by the boundary of a point x

Definition 3.2. A set $\mathscr{D}$ of nodes of $\mathscr{G}$ is said to be a *NOver Top dom set* in $\mathscr{G}$ if $\mathscr{G}$ is a *NOver Top* graph and every vertex $\mathscr{V} - \mathscr{D}$ is adjacent to atleast one vertex in $\mathscr{D}$ at the degree of truth membership , indeterminacy membership and falsity membership respectively which is belongs to $[0, \Omega]$.

Example 3.1.

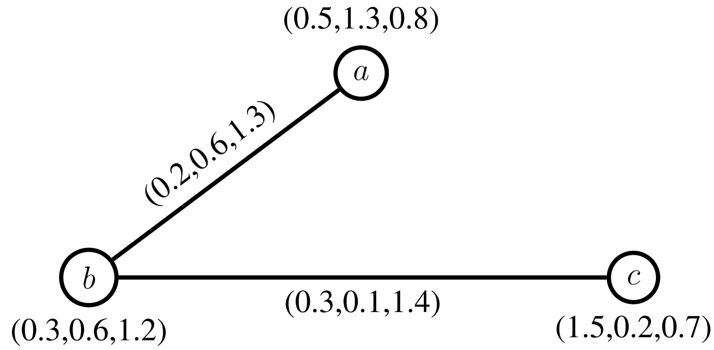


Figure 1: S_3-star graph

Let $\mathscr{F} = \{a, b, c, (0.2, 0.6, 1.3), (0.3, 0.1, 1.4)\}$ be a topological space on $V \cup E$ defined by the topology $\tau = \{\mathscr{F}, \emptyset, \{a\}, \{b, c\}\}$. Here for every $\{f\} \in \mathscr{F}$ is open or closed and $|\partial(v)| \leq 2$.
By the definition of 3.1, $\mathscr{G} = (P, Q)$ is NOver Top graph. Also NOver top dom sets by $\mathscr{D}_1 = \{b\}$ and $\mathscr{D}_2 = \{a, c\}$.

Theorem 3.1. Let $\mathscr{G}$ be a *NOver Top* graph with atmost degree two. If $\mathscr{D}$ is a top dominating set of $\mathscr{G}$, then it is a *NOver Top dominating set* of $\mathscr{G}$.
Proof:
Let $(\mathscr{F}, \tau)$ be a topology on $\mathscr{F}= V \cup E$. Then every singleton set is open or closed and G is a *NOver graph* with atmost degree two. Hence $|\partial(v)| \leq 2$. This implies that $\mathscr{G}$ is *NOver Top graph*. Let $\mathscr{D}$ be top dom set. Then every node in $\mathscr{V} - \mathscr{D}$ is adjacent to atleast one node of $\mathscr{D}$ thus implies that $\mathscr{D}$ is a *NOver Top dom set* of $\mathscr{G}$.

Definition 3.3. A dominating set $\mathscr{D}$ of the *NOver Top graph* $\mathscr{G}$ is said to be a *NOver Top minimal dom set* if for every node in $\mathscr{D}$, $\mathscr{D} - \{v\}$ is not a *NOver Top dom set*. i.e., no proper subset of $\mathscr{D}$ is a *NOver Top dom set*.

Example 3.2. From Example 1, $\{a, c\}$ is *NOver Top minimal dom set* but which is not a *NOver Top dom set*.

Theorem 3.2. Let $\mathscr{G}$ be a *NOver Top graph* with atmost degree two. If $\mathscr{D}$ is a *Top minimum dom set*, then $\mathscr{D}$ is a *Top minimal dom set*.
Proof:
Let $(\mathscr{F}, \tau)$ be a topology on $\mathscr{F}= V \cup E$. Then every singleton set *open or closed* and $\mathscr{G}$ is *NOver graph* with atmost degree two. Hence $|\partial(v)| \leq 2$. Thus implies that $\mathscr{G}$ is *NOver Top graph*. Let $\mathscr{D}$ be a *topologized minimum dom set*. Then every $v \in \mathscr{D}$, $\mathscr{D} - \{v\}$ is not a top dom set which implies that $\mathscr{D}$ is a *NOver Top dom set* of $\mathscr{G}$.

Remark 3.1. The converse of the above theorem need not be true. Every graph need not be a *NOver Top graph*.

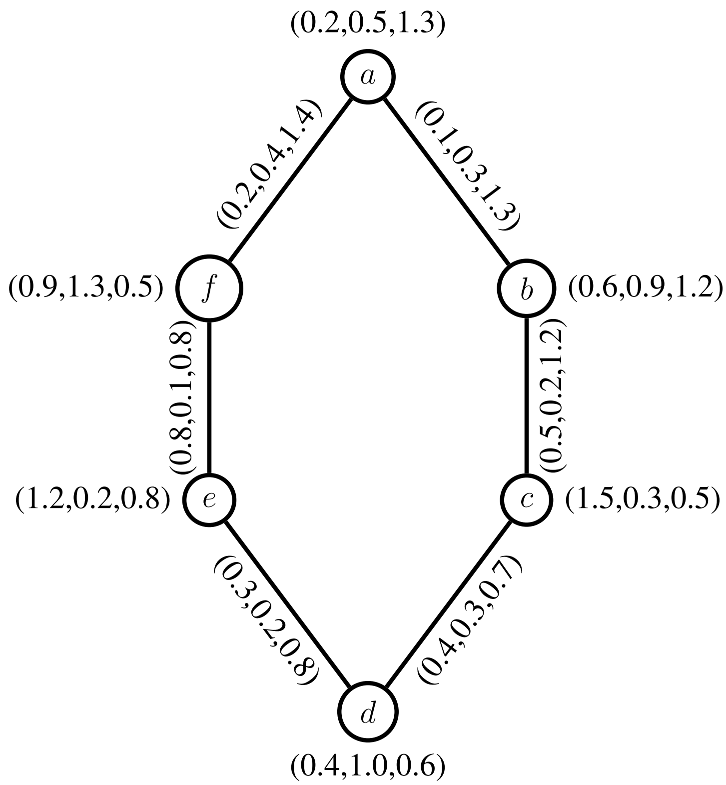
Lemma .1. Let $\mathscr{P}_n$ be a *NOver path* with n nodes which is a *NOver Top graph*. Then the *NOver Top dom number* is $\tau_\gamma(\mathscr{P}_n) \geq \lceil n/3 \rceil$.

Lemma .2. Let $\mathscr{G}$ be a *NOver cycle* with $n-$nodes which is a *NOver Top graph*. Then the *NOver Top dom number* $\tau_\gamma(\mathscr{C}_n) \geq \lceil n/3 \rceil$.

Lemma .3. Let $(\mathscr{K}_n)$ be a *NOver complete graph* with $n-$nodes $(n = 2, 3)$ which is a *NOver Top graph*. Then the *NOver Top dom number* $\tau_\gamma(\mathscr{K}_n) = 1$

4 Some Example For Neutrosophic Over Topologized Domination Graphs

Example 4.1.



Let a, b, c, d, e and f denote the nodes and $(0.1, 0.3, 1.3)$, $(0.5, 0.2, 1.2)$, $(0.4, 0.3, 0.7)$, $(0.3, 0.2, 0.8)$, $(0.8, 0.1, 0.8)$, $(0.2, 0.4, 1.4)$ denote the edges which are labelled $h(0.1, 0.3, 1.3) = \{a, b\}$, $h(0.5, 0.2, 1.2) = \{b, c\}$, $h(0.4, 0.3, 0.7) = \{c, d\}$, $h(0.3, 0.2, 0.8) = \{d, e\}$, $h(0.8, 0.1, 0.8) = \{e, f\}$, $h(0.2, 0.4, 1.4) = \{f, a\}$,

Let $\mathscr{F} = \{a, b, c, d, e, f, (0.1, 0.3, 1.3), (0.5, 0.2, 1.2), (0.4, 0.3, 0.7), (0.3, 0.2, 0.8), (0.8, 0.1, 0.8), (0.2, 0.4, 1.4)\}$ be a topological space defined by the topology

$$\tau = \Big\{\mathscr{F}, \emptyset, \{a\}, \{b, c\}, \{d\}, \{e, f\}, \{a, b, c\}, \{a, d\}, \{a, e, f\}, \{b, c, d\}, \{b, c, e, f\}, \{d, e, f\}, \{a, b, c, d\}, \{a, b, c, e, f\}, \{a, d, e, f\}, \{b, c, d, e, f\}\Big\}$$

Here for every $\{f\} \in \mathscr{F}$ is *open or closed*.

We have $\partial(a) = \{b, f\}$, $\partial(b) = \{a, c\}$, $\partial(c) = \{b, d\}$, $\partial(d) = \{c, e\}$, $\partial(e) = \{f, d\}$ and $\partial(f) = \{a, e\}$ with $|\partial(v_i)| = 2$ where $i = 1, 2, 3, 4, 5, 6$. By the definition of 3.1, $|\partial(v)| \leq 2$.

Hence this graph is a *NOver Top graph*.

Moreover, the *NOver topologized dominating sets* are given by $\mathscr{D} = \{a, d\}$ and the corresponding *NOver sets* for maximum and minimum are given by

	a	b	c	d	e	f
T_A	0.2	0.6	1.5	0.4	1.2	0.9
I_A	0.5	0.9	0.3	1.0	0.2	1.3
F_A	1.3	1.2	0.5	0.6	0.8	0.5

$\cup$	ab	bc	cd	de	ef	fa
T_B	0.6	1.5	1.5	1.2	1.2	0.9
I_B	0.9	0.9	1.0	1.0	1.3	1.3
F_B	1.2	0.5	0.5	0.6	0.5	0.5

$\cap$	ab	bc	cd	de	ef	fa
T_B	0.2	0.6	0.4	0.4	0.9	0.2
I_B	0.5	0.3	0.3	0.2	0.2	0.5
F_B	1.2	1.2	0.6	0.8	0.8	1.3

Therefore this graph is *NOver Top* domination graph.

Example 4.2.

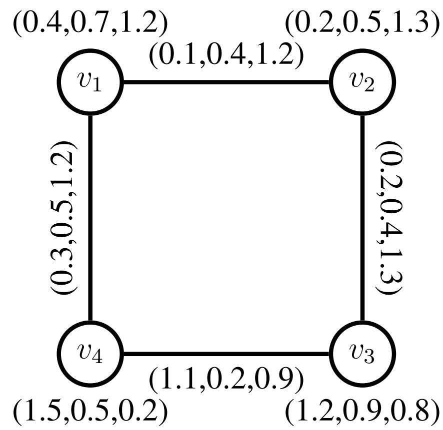


Figure 2: S_3-star graph

Let v_1, v_2, v_3 and v_4 denote the vertices and $(0.1, 0.4, 1.2)$, $(0.2, 0.4, 1.3)$, $(1.1, 0.2, 0.9)$, $(0.3, 0.5, 1.2)$ denote the edges.

Let $\mathscr{F} = \{v_1, v_2, v_3, v_4, (.1, .4, .2), (.3, .5, .4), (.1, .4, .8), (.2, .4, .9)\}$ be a topological space defined by the topology

$$\tau = \Big\{\mathscr{F}, \emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_2, v_3, v_4\}, \{v_1, v_3, v_4\}\Big\}$$

Here for every$\{f\} \in \mathscr{F}$ is open.

By the definition of Nover Top graph $|\partial(v)| \leq 2$.

We have $\partial(v_1) = \{v_2, v_4\}$, $\partial(v_2) = \{v_1, v_3\}$, $\partial(v_3) = \{v_2, v_4\}$ and $\partial(v_4) = \{v_1, v_3\}$ with $|\partial(v_i)| = 2$ where $i = 1, 2, 3, 4$.

Hence this graph is a Nover Top graph.

Moreover, the Nover topologized dominating sets are given by $D = \{v_1, v_3\}$ and $\{v_2, v_4\}$ and the corresponding neutrosophic over sets for maximum and minimum are given by

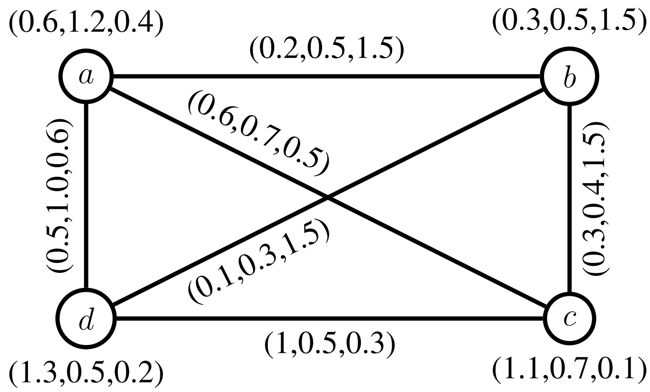
	T_A	I_A	F_A
v_1	0.4	0.7	1.2
v_2	0.2	0.5	1.3
v_3	1.2	0.9	0.8
v_4	1.5	0.5	.2

	v_1v_2	v_2v_3	3 4	4 1	
$T_{v_i \cup v_j} = \max \langle v_i, v_j \rangle$	0.4	1.2	1.5	1.5	$\forall x \in X$
$I_{v_i \cup v_j} = \max \langle v_i, v_j \rangle$	0.7	0.9	0.9	0.7	
$F_{v_i \cup v_j} = \min \langle v_i, v_j \rangle$	1.2	0.8	0.2	0.2	

	v_1v_2	v_2v_3	v_3v_4	v_4v_1	
$T_{v_i \cap v_j} = \min \langle v_i, v_j \rangle$	0.2	0.2	1.2	0.4	$\forall x \in X$
$I_{v_i \cap v_j} = \min \langle v_i, v_j \rangle$	0.5	0.5	0.5	0.5	
$F_{v_i \cap v_j} = \max \langle v_i, v_j \rangle$	1.3	1.3	0.8	1.2	

Hence it is clear that this graph is Nover Top domination graph.

Example 4.3.



Here every singleton sets are minimum dom sets. Clearly the complete graph $\mathscr{K}_4$ is not a *NOver Top graph*. Since $n \geq 4$. Then the dom set need not be a *NOver Top dom set*.

Example 4.4. Let G be a N complete graph K_4 with 4 vertices.

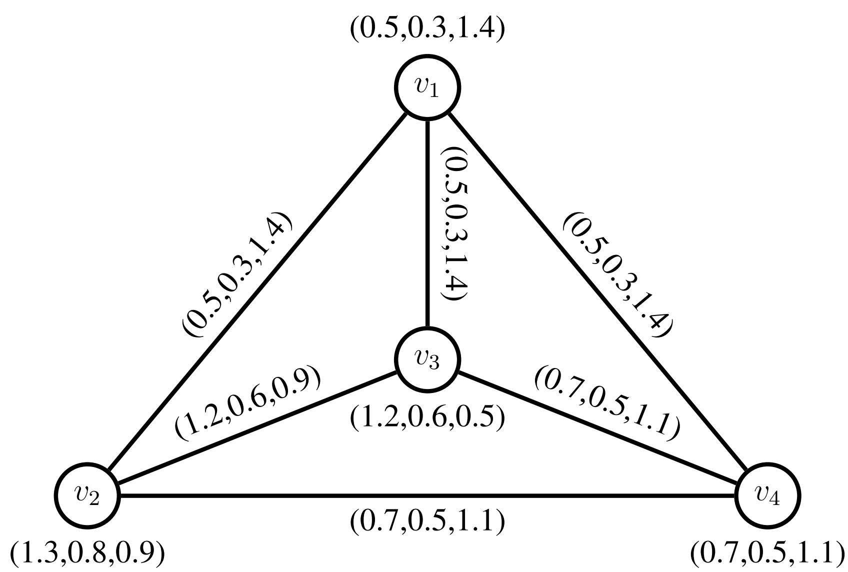


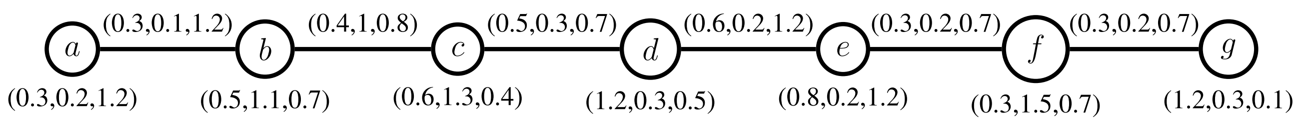
Figure 3: K_4 complete graph

Here every singleton sets are minimum dominating sets. Clearly the complete graph K_4 is not a Nover Top graph, since $n \geq 4$.

Then the a Nover Top dominating set does exits.

Then the dominating sets need not be a Nover Top dominating set.

Example 4.5.



Let a, b, c, d, e, f and g denote the nodes and $(0.3, 0.1, 1.2), (0.4, 1.0, 0.8), (0.5, 0.3, 0.7), (0.6, 0.2, 1.2),$ $, (0.3, 0.2, 0.7)$ denote the edges which are labelled $h(0.3, 0.1, 1.2) = \{a, b\}$, $h(0.4, 1.0, 0.8) = \{b, c\}$, $h(0.5, 0.3, 0.7) = \{c, d\}$, $h(0.6, 0.2, 1.2) = \{d, e\}$, $h(0.3, 0.2, 0.7) = \{e, f\}$, $h(0.3, 0.2, 0.7) = \{f, g\}$,

Let $\mathscr{F} = \{a, b, c, d, e, f, g, (0.3, 0.1, 1.2), (0.4, 1.0, 0.8), (0.5, 0.3, 0.7)(0.6, 0.2, 1.2)$,$(0.3, 0.2, 0.7)$, $(0.3, 0.2, 0.7)\}$ be a topological space defined by the topology

$\tau = \Big\{\mathscr{F}, \emptyset, \{a\}, \{b, c\}, \{d\}, \{e, f\}, \{g\}, \{a, b, c\}, \{a, d\}, \{a, e, f\}, \{a, g\}, \{b, c, d\}, \{b, c, e, f\}, \{b, c, g\},$ $\{d, e, f\}, \{d, g\}, \{e, f, g\}, \{a, b, c, d\}, \{a, b, c, e, f\}, \{a, b, c, g\}, \{a, b, c, e, f\}, \{a, b, c, d, e, f\}, \{a, b, c, d, f\},$ $\{a, b, c, e, f, g\}, \{a, d, e, f\}, \{a, d, f\}, \{a, d, e, f, g\}, \{a, b, c, d, e, f\}, \{a, b, c, d, g\}, \{b, c, d, e, f, g\},$ $\{b, c, d, g\}, \{a, b, c, e, f, g\}, \{d, e, f, g\}, \{b, c, e, f, g\}, \{b, c, d, g\}, \{b, c, d, e, f\}, \{a, e, f, g\},$ $\{a, d, g\}, \{a, d, e, f\}\Big\}$

Here for every $\{f\} \in \mathscr{F}$ is *open or closed* and $|\partial(v)| \leq 2$. So it is a *NOver Top graph.*

	a	b	c	d	e	f	g
T_A	0.3	0.5	0.6	1.2	0.8	0.3	1.2
I_A	0.2	1.1	1.3	0.3	0.2	1.5	0.3
F_A	1.2	0.7	0.4	0.5	1.2	0.7	0.1

Moreover *NOver Top dom sets* are given by $\mathscr{D} = \{a, d, f\}$ and and the corresponding *Nover sets* for maximum and minimum is given by

$\cup$	ab	bc	cd	de	ef	fg
T_B	0.5	0.6	1.2	1.2	0.8	1.2
I_B	1.1	1.3	1.3	0.3	1.5	1.5
F_B	0.7	0.4	0.4	0.5	0.7	0.1

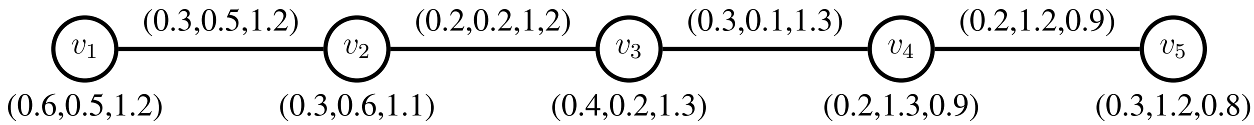
$\cap$	ab	bc	cd	de	ef	fg
T_B	0.3	0.5	0.6	0.8	0.3	0.3
I_B	0.2	1.1	0.3	0.2	0.2	0.3
F_B	1.2	0.7	0.5	1.2	1.2	0.7

The *NOver Top* dom set is $\mathscr{D} = \{a, d, f\}$.

$$\tau_\gamma(\mathscr{P}_7) = \lceil 7/3 \rceil = \lceil 2.333 \rceil > 2.$$

Therefore this graph is *NOver Top dom graph.*

Example 4.6.



Let $\mathscr{F} = \{v_1, v_2, v_3, v_4, v_5, (0.3, 0.5, 1.2), (0.2, 0.2, 1, 2), (0.3, 0.1, 1.3), (0.2, 1.2, 0.9)\}$ be a topological space defined by the topology

$$\tau = \Big\{\mathscr{F}, \emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_2, v_5\}, \{v_2, v_3, v_4\}, \{v_2, v_3, v_5\}, \{v_3, v_4, v_5\}, \{v_1, v_2, v_3, v_4\}, \{v_1, v_2, v_3, v_5\}\Big\}$$

Here for every $\{f\} \in \mathscr{F}$ is open.

Moreover, the Nover Top dominating sets are given by $D = \{v_2, v_5\}$ and the corresponding neutrosophic over sets for maximum and minimum are given by

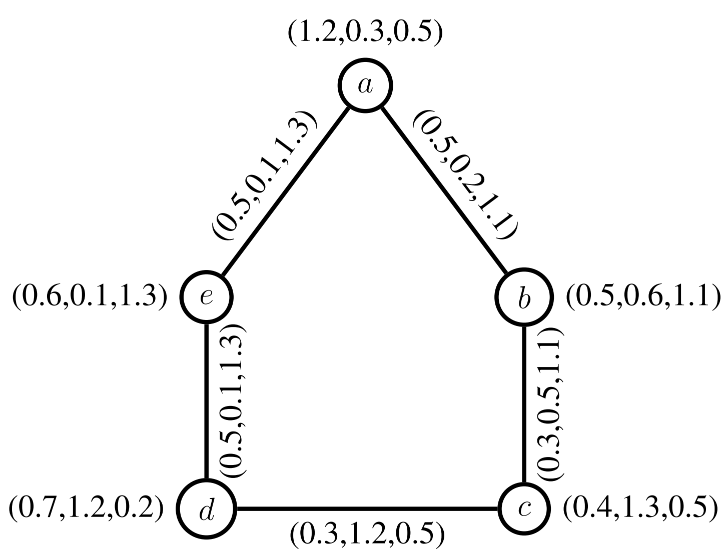
	v_1	v_2	v_3	v_4	v_5
T_A	0.6	0.3	0.4	0.2	0.3
I_A	0.5	0.6	0.2	1.3	1.2
F_A	1.2	1.1	1.3	0.9	0.8

	v_1v_2	v_2v_3	v_3v_4	v_4v_5
T_B(min)	0.3	0.2	0.3	0.2
I_B(min)	0.5	0.2	0.1	1.2
F_B(max)	1.2	1.3	1.3	.9

Hence the dominating set is $D = \{v_2, v_5\}$.

$$\tau_\gamma(P_5) = \lceil 5/3 \rceil = \lceil 1.666 \rceil = 2.$$

Example 4.7.



Let a, b, c, d and e denote the nodes and $(0.5, 0.2, 1.1), (0.3, 0.5, 1.1), (0.3, 1.2, 0.5)$, $(0.5, 0.1, 1.3), (0.5, 0.1, 1.3)$ denote the edges which are labelled $h(0.5, 0.2, 1.1) = \{a, b\}$, $h(0.3, 0.5, 1.1) = \{b, c\}$, $h(0.3, 1.2, 0.5) = \{c, d\}$, $h(0.5, 0.1, 1.3) = \{d, e\}$, $h(0.5, 0.1, 1.3) = \{e, a\}$,

Let $\mathscr{F} = a, b, c, d, e, (0.5, 0.2, 1.1), (0.3, 0.5, 1.1), (0.3, 1.2, 0.5), (0.5, 0.1, 1.3), (0.5, 0.1, 1.3)$ be a topological space defined by the topology

$$\tau = \Big\{\mathscr{F}, \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{a, b\}, \{a, c\}, d\}, \{a, e\}, \{b, c\}, \{b, d\}, \{b, e\}, \{c, d\},$$
$$\{c, e\}, \{d, e\}, \{a, b, c\}, \{a, b, d\}, \{a, b, e\}, \{b, c, d\}, \{b, c, e\}, \{c, d, e\}, \{a, b, c, d\}, \{a, b, c, e\}\Big\}$$

Here for every $\{f\} \in \mathscr{F}$ is *open* and $|\partial(v)| \leq 2$. We have $\partial(a) = \{b, e\}$, $\partial(b) = \{a, c\}$, $\partial(c) = \{b, d\}$, $\partial(d) = \{c, e\}$, and $\partial(e) = \{a, d\}$ with $|\partial(v_i)| = 2$ where $i = 1, 2, 3, 4, 5,$.

By the definition of 3.1 this graph is a *NOver Top graph.*

Moreover *NOver Top dom set* is given by $\mathscr{D} = \{b, d\}$ and the corresponding *NOver sets* for maximum and minimum are given by

	a	b	c	d	e
T_A	1.2	0.5	0.4	0.3	0.6
I_A	0.3	0.6	1.3	1.2	0.1
F_A	0.5	1.1	0.5	0.5	1.3

$\cup$	ab	bc	cd	de	ea
T_B	1.2	0.5	0.4	0.6	1.2
I_B	0.6	1.3	1.3	1.2	0.3
F_B	0.5	0.5	0.5	0.5	0.5

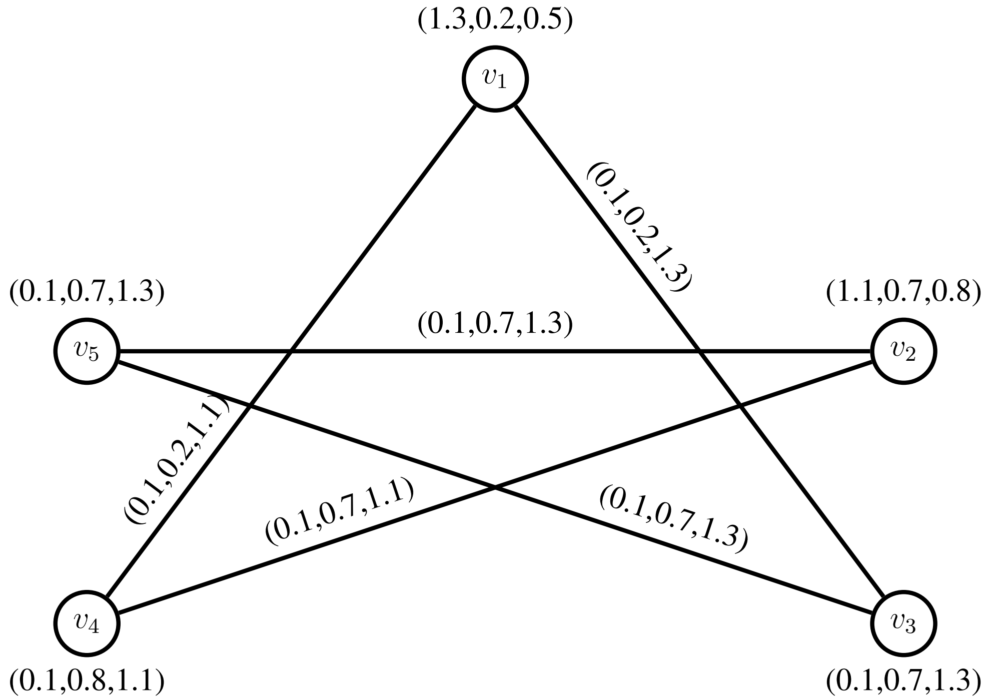
$\cap$	ab	bc	cd	de	ea
T_B	0.5	0.4	0.3	0.3	0.6
I_B	0.3	0.6	1.2	0.1	0.1
F_B	1.1	1.1	0.5	1.3	1.3

The *NOver Top dom set* is $\mathscr{D} = \{b, d\}$.

$$\tau_\gamma(\mathscr{C}_5) = \lceil 5/3 \rceil = \lceil 1.666 \rceil = 2$$

Therefore this graph is *NOver Top dom graph.*

Example 4.8.



Let v_1, v_2, v_3, v_4 and v_5 denote the vertices and $(0.1, 0.2, 0.8)$, $(0.1, 0.7, 1.3)$, $(0.1, 0.7, 1.3)$, $(0.1, 0.7, 1.3)$ and $(0.1, 0.2, 1.3)$ denote the edges which are labelled $h_u(0.1, 0.2, 1.3) = (v_1, v_3)$, $h_u(0.1, 0.7, 1.3) = (v_3, v_5)$, $h_u(0.1, 0.7, 1.3) = (v_5, v_2)$, $h_u(0.1, 0.7, 1.1) = (v_2, v_4)$, $h_u(0.1, 0.2, 1.1) = (v_4, v_1)$.

Let $\mathscr{F} = \{v_1, v_2, v_3, v_4, v_5, (0.1, 0.2, 0.8), (0.1, 0.7, 1.3), (0.1, 0.7, 1.3), (0.1, 0.7, 1.3), (0.1, 0.2, 1.3)\}$ be a topology $\tau = \{\mathscr{F}, \emptyset, \{v_1\}, \{v_2, v_3\}, \{v_1, v_2, v_3\}, \{v_1, v_4, v_5\}, \{v_2, v_3, v_4, v_5\}, \{v_4, v_5\}, \{v_1, v_3, v_4, v_5\}, \{v_3\},$ $\{, v_1, v_4\}, \{v_3, v_4, v_5\}, \{v_4\}, \{v_2\}, \{v_5\}, \{v_2, v_4, v_5\}, \{v_1, v_4\}, \{v_3, v_5\}, \{v_3, v_4, \}, \{v_2, v_3, v_4\},$ $\{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_1, v_2, v_4, v_5\}, \{v_2, v_3, v_5\}, \{v_1, v_2, v_3, v_5\}, \{v_2, v_3, v_4, v_1\}, \{v_1, v_5\}$ $, \{v_1, v_3\}, \{v_1, v_2, v_5\}, \{v_2, v_5\}, \}$.

Here for every $\{f\} \in \mathscr{F}$ is open or closed.

By the definition of Nover Top graph, we have $|\partial(V)| \leq 2$ and $\partial(v_1) = (v_3, v_4)$, $\partial(v_2) = (v_5, v_4)$, $\partial(v_3) = (v_1, v_5)$, $\partial(v_4) = (v_1, v_2)$, $\partial(v_5) = (v_2, v_3)$ with $\partial(v_i) = 2$. Hence this graph is Nover Top graph. Moreover *NOver Top dom set* is given by $\mathscr{D} = \{v_1, v_5\}$ and the corresponding *NOver sets* for maximum and minimum are given by

	v_1	v_2	v_3	v_4	v_5
T_A	1.3	1.1	0.1	0.1	0.1
I_A	0.2	0.7	0.7	0.8	0.7
F_A	0.5	0.8	1.3	1.1	1.3

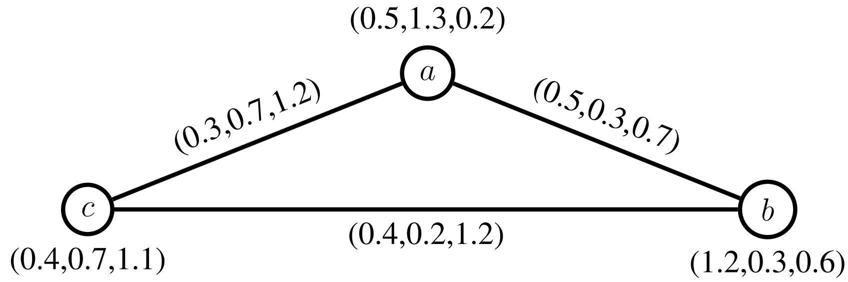
$\cup$	v_1v_3	v_3v_5	v_5v_2	v_2v_4	v_4v_1
T_B	0.1	0.1	0.1	0.1	0.1
I_B	0.2	0.7	0.7	0.7	0.2
F_B	1.3	1.3	1.3	1.3	1.3

The *NOver Top dom set* is $\mathscr{D} = \{v_1, v_5\}$.

$$\tau_\gamma(\mathscr{C}_5) = \lceil 5/3 \rceil = \lceil 1.666 \rceil = 2$$

Therefore this graph is *NOver Top dom graph.*

Example 4.9.

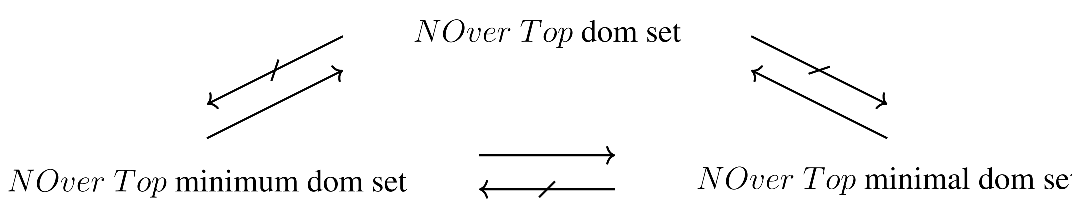


Let a, b and c denote the nodes and $(0.5, 0.3, 0.7), (0.4, 0.2, 1.2), (0.3, 0.7, 1.2)$ denote the edges which are labelled $h(0.5, 0.3, 0.7) = \{a, b\}$, $h(0.4, 0.2, 1.2) = \{b, c\}$, $h(0.3, 0.7, 1.2) = \{c, a\}$,
Let $\mathscr{F} = \{a, b, c, (0.5, 0.3, 0.7), (0.4, 0.2, 1.2), (0.3, 0.7, 1.2)\}$ be a topological space defined by the topology $\tau = \left\{ \mathscr{F}, \emptyset, \{a, b\}, \{c\} \right\}$. Here for every $\{f\} \in \mathscr{F}$ is *open or closed.*

The definition of *NOver Top graph* $|\partial(v)| \leq 2$. We have $\partial(a) = \{b, c\}$, $\partial(b) = \{c, a\}$, and$\partial(c) = \{a, b\}$, with $|\partial(v_i)| = 2$ where $i = 1, 2, 3$. So this graph is a *NOver Top graph.* The dominating set is $\mathscr{D} = \{b\}$,whose top dom number is given by

$$\tau_\gamma(\mathscr{K}_3) = 1.$$

Remark 4.1. The interrelationship among $NOver\ Top$ dom set as given below



5 Comparision

A neutrosophic set and neutrosophic over set are generalised of Atanassov's intuitionstic fuzzy set which consists of three membership grades: truth membership.indeterminancy membership and false membership. The neutrosophic network is an extension of a vague graph and intuitionstic fuzzy graph which provides more flexibility, more effectively precision, and compatibility to design the real-life problem when compared with the intuitionstic fuzzy graphs. Neutrosophic graph and neutrosophic over graph models are recently using model to many real-life problems in several different areas of engineering and science.

6 Conclusion

This paper has focused on calculating the dominating number of Nover top graph G by using top domination conditions.The top dom condition is introduced in new method to find the domination number.The Nover top domination for some standard Nover graphs such as a path,cycle are specified.The future study can be continued by forming different types of Nover top domination set with various applications.

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Generalized Neutrosophic Semirings

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Abstract: In this paper we are going to give the idea of Neutrosophic Semirings and study their algebraic structure. To understand Neutrosophic Semirings, we provide number of examples. Through ideals and congruence relations, we discuss the structural behavior and properties of this notion by establishing number of results.

Keywords: Neutrosophic Sets; Neutrosophic Semirings; Neutrosophic Ideals and Congruence Relations.

1. Introduction

Different researchers have defined algebraic structures which were based on the crisp set. But actually, the real-life problems could not be solved by crisp set theory. The crisp set deals with yes or no only and it never tells about in between yes and no. In 1965, Zadeh [1] introduced a fuzzy set theory to address the vagueness of various real-life problems. The fuzzy sets deal with membership in between 0 and 1. Later, in 1986, Atanassov [2], initiated the idea of intuitionistic fuzzy set. In intuitionistic fuzzy set focused on the degree of membership and non-membership. In fact, these theories have remained unsuccessful in finding a solution to many real-life mathematical challenges.

In 1999, Smarandache [3] gave the notion of Neutrosophic set. Neutrosophic set is a generalized form of fuzzy set as well as intuitionistic fuzzy set. Nowadays, Neutrosophic set attains more attention of researchers due to its characteristic behavior to solve the indeterminate situations in the different fields of life. In 2006, Smarandache et al., [4] were the first ones who applied the concept of Neutrosophic sets on some algebraic structure and in their work, they introduced the Neutrosophic rings. Later, in 2011 Agboola et al., [5], discussed Neutrosophic Rings-I. Neutrosophic groups and Neutrosophic sub-groups were introduced in 2012 by Agboola et al., [6]. Ali et al., [7,8,9,10] have used Neutrosophic set approach for different algebraic structures. In 2016, Khan et al., [11] briefly discussed the characterization of Neutrosophic left almost semigroups. On the other hand, the application of Neutrosophic sets is getting more attention of researchers. For more study about the application aspect of Neutrosophic sets, the readers are referred to [12,13,14,15,16,17,18,19].

In 1992, Golan [20], has done a worth full work and defined a new structure named semiring. A detailed discussion has been done in this paper. We used Neutrosophic set approach to semiring and gave the idea of Neutrosophic semiring. We discussed the characteristic properties of Neutrosophic semiring and its substructures. For that we have given various examples. We talked about the ideals of Neutrosophic semirings. In last section, congruence relations are characterized in the Neutrosophic semirings. We established number of results. For instance: Suppose $\{J_k : k \in K\}$ is a collection of left ideals in neutrosophic semiring N. Then $\cup J_k$ is a left ideal if $J_k \subseteq J_{k+1}$ for all $k \in K$. Another main result: A relation ρ on a neutrosophic semiring N is congruence relation if and only if it is left congruence and right as well.

2. Neutrosophic Semirings

We generalize the theory of semrings to neutrosophic semirings. We also discuss about ideals and congruences of neutrosophic semirings with some properties.

Definition 2.1. An algebraic structure $(S\cup I, *_1, *_2)$ is called neutrosophic semiring if $*_1$ and $*_2$ are the closed and associative binary operations and $*_2$ is distributive over $*_1$ where S is semiring with respect to $*_1$ and $*_2$ and I is the neutrosophic element ($I = I^2$) and $< S\cup I > = \{a + bI ; a, b \in S\}$.

Let us give some examples to understand this notion completely.

Example 2.2. The set $S = \{\overline{0}, \overline{1}, \overline{2}\}$ is the semiring under the operation of addition $*_1$ and multiplication $*_2$(mod3) which can be verified.

Then $< S\cup I > = \{0 + 0I, 0 + 1I, 0 + 2I, 1 + 0I, 1 + 1I, 1 + 2I, 2 + 0I, 2 +1 I, 2 + 2I\}$ is the neutrosophic semiring under the same operations as that of S i. e for all $a + bI, c + dI \in S\cup I$,

$(a + bI) *_1 (c + dI) = (a + c) + (b + d)I$, $(a + bI) *_2 (c + dI) = (ac) + (bd)I$.

Closure property: Let $(a + bI)$ and $(c + dI) \in S\cup I$ then $(a + bI) *_1 (c + dI) = (a + c) + (b + d)I \in S\cup I$ and $(a + bI) *_2 (c + dI) = (ac) + (bd)I \in S\cup I$.Which shows that closure property is satisfied.

Distributive law: Let $(a + bI)$, $(c + dI)$ and $(e + f)I \in S\cup I$ then $(a + bI) *_2 [(c + dI) *_1 (e + f)I]$
$= (a + bI) *_2 ((c + e) + (d + f)I) = (ac + ae) + (bd + bf)I$.

and $[(a + bI) *_2 (c + dI)] *_1[(a + bI) *_2 (e + f)I] = ((ac) + (bd)I) *_1((ae) + (bf)I)$
$= (ac + ae) + (bd + bf)I$.

Associative law: Let $(a + bI)$, $(c + dI)$ and $(e + f)I \in S\cup I$. Then

$[(a + bI) *_1 (c + dI)] *_1 (e + f)I = ((a +c) + (b + d)I) *_1(e + f)I = (a + c + e) + (b +d + f)I$.

$(a + bI) *_1 [(c + dI) *_1 (e + f)I]= (a + bI) *_1 ((c + e) + (d + f)I) = (a + c + e) + (b + d + f)I$.

Again

$[(a + bI) *_2 (c + dI)] *_2 (e + f)I = ((ac) + (bd)I) *_2(e + f)I = (ace) + (bdf)I.$

$(a + bI) *_2 [(c + dI) *_2 (e + f)I]= (a + bI) *_2 ((ce) + (df)I) = (ace) + (bdf)I.$

Example 2.3. The structure $(Z, *_1, *_2)$ is the semiring under the operations of usual multiplication and addition where Z is the set of integers and $*_1$ and $*_2$ represent the usual multiplication and addition. Then $< Z\cup I > = \{a + bI;\ a, b \in Z\}$ is the neutrosophic semiring and for all $a + bI, c + dI\in Z\cup I$ $(a + bI) *_1 (c + dI) = (ac) + (bd)I$ and $(a + bI) *_2 (c + dI) = (a + c) + (b + d)I$.

Definition 2.4. A proper subset P of a neutrosophic semiring $(S\cup I, *_1, *_2)$ is called neutrosophic subsemiring if P itself is a neutrosophic semiring under the same binary operation as that of $S\cup I$.

Let us give some examples of neutrosophic subsemiring.

Example 2.5. Let Wand N are the sets of whole and natural numbers respectively then $W\cup I =\{a + bI;\ a, b \in W\}$ is the neutrosophic semiring under $*_1$ and $*_2$ are of example number **2.3** Then $N\cup I =\{ a + bI;\ a, b \in N \}$ is the subset of $W\cup I$ and hence under the operations of $W\cup I$ the subset $N\cup I$ becomes the neutrosophic subsemiring. One can check it easily.

Example 2.6. Let $N = \{1, 2, 3,\ldots\}$ and $S = \{ 2, 3, 4,\ldots\}$ then algebraic structure $(N\cup I, *_1, *_2)$ is the neutrosophic semiring where $*_1$ and $*_2$ are the operations of example 3. Then $S\cup I$ is obviously subset of $N\cup I$ and hence is the neutrosophic subsemiring of $N\cup I$ under the operations of $N\cup I$.

Lemma 2.7. A subset P of neutrosophic semiring $(S\cup I, *_1, *_2)$ is neutrosophic subsemiring if and only if it satisfies the following:

(i) $(a + bI) *_1 (c + dI) \in P$ and

(ii) $(a + bI) *_2 (c + dI) \in P \quad \forall\ (a + bI), (c + dI) \in P$.

Proof. Let $(P, *_1, *_2)$ is a neutrosophic subsemiring then by closure property

$\forall\ (a + bI), (c + dI) \in P$, $(a + bI) *_1 (c + dI) \in P$ and $(a + bI) *_2 (c + dI) \in P$.

Conversely,

let $\forall\ (a + bI), (c + dI) \in P$,

(i) $(a + bI) *_1 (c + dI) \in P$

(ii) $(a + bI) *_2 (c + dI) \in P$.

Then from (i) and (ii) closure property is satisfied. And since elements of P are of neutrosophic semiring $(S\cup I, *_1, *_2)$ and operations are also same as that of $S\cup I$ and since associative and distributive laws are satisfied therein so both laws are satisfied for P with respect to $*_1$ and $*_2$. Hence this makes $(P, *_1, *_2)$ into neutrosophic subsemiring.

Proposition 2.8. The intersection of any number of neutrosophic subsemiring is either empty or neutrosophic subsemiring.

Proof. Let $\{B_i: i\in J\}$ is the collection of neutrosophic subsemirings of $(S\cup I, *_1, *_2)$. Let $\cap B_i \neq \phi$ then we show that the subset $\cap B_i$ of $S\cup I$ is the neutrosophic subsemiring. Let $(a + bI)$ and $(c + dI) \in \cap B_i \Longrightarrow (a + bI)$ and $(c + dI) \in B_i \quad \forall\ i\in J$. This implies $(a + bI) *_1 (c + dI)$ and $(a + bI) *_2 (c + dI) \in B_i \quad \forall\ i\in J$ because each B_i is neutrosophic subsemiring of $(S\cup I, *_1 , *_2)$. So, this implies $(a + bI) *_1 (c + dI)$ and $(a + bI) *_2 (c + dI) \in\cap B_i \quad \forall\ i\in J$. Thus by the **Lemma 2.7** $\cap B_i$ is the neutrosophic subsemiring.

Remark 2.9. Union of neutrosophic subsemirings may or may not be the neutrosophic subsemiring.

Example 2.10. If X = {a, b, c, d, e} then (P(X), ∪, ∩) is a semiring. Now let P(X)=S, then <S∪I> = {A + BI : A , B ∈ P(X)} is the neutrosophic semiring. Suppose S_1 = {{}, {a}, {a, b}, X} and S_2 = {{}, {c}, {b, c}, X} are the subsets of S. Then < S_1∪I > = {C + DI; C, D ∈ S_1} and < S_2∪I > = {E + FI: E, F ∈ S_2}. In tabular form: < S_1∪I > = {{} + { }I , { } + {a}I, { } + {a, b }I, { } + X I , { a } + { }I , { a} + {a}I, { a} + {a, b }I, {a } + X I, { a, b} + {a, b }I , {a, b } + {a}I, {a, b } + {a, b }I, {a, b } + X I, X + { }I , X + {a}I, X + {a, b }I, X + X I} and < S_2∪I > = {{ } + { }I ,{ } + {c}I, { } + {b, c }I, { } + X I , { c } + { }I , { c} + {c}I, { c} + {b, c }I, {c } + X I, { b, c} + {b, c }I , {b, c } + {c}I, {b, c } + {b, c }I, {b, c } + X I, X + { }I , X + {c}I, X + {b, c }I, X + X I } are the neutrosophic semirings but their union < S_1∪I > ∪ < S_2∪I > = {{ } + { }I , { } + {a}I, { } + {a, b }I, { } + X I , { a } + { }I , { a} + {a}I, { a} + {a, b }I, {a } + X I, { a, b} + {a, b }I , {a, b } + {a}I, {a, b } + {a, b }I, {a, b } + X I, X + { }I , X + {a}I, X + {a, b }I, X + X I , { } + { }I , { } + {c}I, { } + {b, c }I, { } + X I , { c } + { }I , { c} + {c}I, { c} + {b, c }I, {c } + X I, { b, c} + {b, c }I , {b, c } + {c}I, {b, c } + {b, c }I, {b, c } + X I, X + { }I , X + {c}I, X + {b, c }I, X + X I } is not the neutrosophic semiring because the operation of union is not closed i.e. { a, b} + {a, b }I ∪{ b, c} + {b, c }I = ({ a, b}∪{ b, c}) + ({ a, b}∪{ b, c}) I = {a, b, c} + {a, b, c}I ∉ < S_1∪I > ∪ < S_2∪I >.

Now we will show that under what condition union of neutrosophic subsemiring is again a neutrosophic subsemiring.

Proposition 2.11. Let {B_k: k ∈ K} be the family of neutrosophic subsemiring. If $B_k \subseteq B_{k+1}$ then $\bigcup B_k$ is neutrosophic subsemiring.

Proof. Let a + bI, c + dI ∈ $\bigcup B_k$ then this implies that there exist some p and q∈ K such that a + bI ∈ B_p and c + dI ∈ B_q. Let q > p then a + bI, c + dI ∈ B_q. Now since B_q is neutrosophic subsemiring so (a + bI) $*_1$ (c + dI) and (a + bI) $*_2$ (c + dI) ∈ B_q so, this implies that (a + bI) $*_1$ (c + dI) and (a + bI) $*_2$ (c + dI) ∈$\bigcup B_k$ thus by the **Lemma 2.7** $\bigcup B_k$ is neutrosophic subsemiring.

Corollary 2.12. Union of two neutrosophic subsemirings N_1 and N_2 of N is again neutrosophic subsemiring of N if either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$.

Proof. Let $N_1 \subseteq N_2$ then obviously $N_1 \bigcup N_2 = N_2$. It follows that $N_1 \bigcup N_2$ is a neutrosophic subsemiring. Again let $N_2 \subseteq N_1$ then clearly $N_1 \bigcup N_2 = N_1$. Now as N_1 is neutrosophic subsemiring so, $N_1 \bigcup N_2$ is by transitive property.

We discuss the ideals of neutrosophic semirings and describe its properties.

Definition 2.13. Let J be the nonempty subset of neutrosophic semiring S, then J is called left ideal of S if ∀ ($a_1 + b_1$I), ($a_2 + b_2$I) ∈ J, ($a_1 + b_1$I) $*_1$ ($a_2 + b_2$I) ∈ J and (a + bI) $*_2$ (c + dI) ∈ J ∀ (a + bI) ∈ S and (c + dI) ∈ J.

Example 2.14. Let N = {1, 2, 3,…} is the set of natural numbers and $*_1$, $*_2$ represent the operations of multiplication and addition as defined in **Example 2.3** then N∪I ={a + bI: a, b∈ N}becomes the neutrosophic semiring and for set S = {2, 3, 4,…} the structure S∪I ={c + dI: c, d ∈ S } becomes the left ideal because ∀ e + f I ∈ N , a + bI ∈ S and c + dI ∈ S, (a + bI) $*_1$ (c + dI) ∈ S and (e + f I) $*_2$ (c + dI) ∈ S .

Theorem 2.15. The intersection of any number of left ideals of neutrosophic semiring N is again a left ideal.

Proof. We suppose that $\{J_k : k \in K\}$ be the family of left ideals of neutrosophic semiring N. We have to show that $\bigcap J_k$ is the left ideal of N. Let (a + bI), (c + dI) $\in \bigcap J_k$ and (e + f I) $\in$ N this implies that (a + bI), (c + dI) $\in J_k \; \forall \; k \in K$, and (e + f I) $\in$ N this implies that (a + bI) $*_1$ (c + dI) $\in J_k$, and (e + f I) $*_2$ (a + bI) $\in J_k \; \forall \; k \in K$ because each J_k is the left ideal so, this shows that (e + f I) $*_2$ (a + bI) $\in \bigcap J_k \; \forall \; k \in K$ so from hence we say that $\bigcap J_k$ is the left ideal.

Note: The case is same for right ideals and ideals of neutrosophic semiring.

Remark 2.16. Union of left ideals of neutrosophic semiring may or may not be the left ideal again.

In order to be more clear with the preceding remark we give an example.

Example 2.17. The set N∪I ={a + bI :a, b ∈N } under the operations (a + bI) $*_1$ (c + dI) = gcd (a, c) + (gcd (b, d))I and (a + bI) $*_2$ (c + dI) = lcm (a , c) + (lcm (b, d))I is the neutrosophic semiring and 2N∪I = { a + bI : a , b ∈ 2N } and 3N∪I = { a + bI : a , b ∈ 3N }are the left ideals of N∪I but (2N∪I) ∪ (3N∪I) is not the left ideal of N∪I because 2 + 2I and 9 + 9I ∈ (2N∪I) ∪ (3N∪I) and (2 + 2I) $*_1$ (9 + 9I) = gcd(2, 9) + (gcd(2, 9))I = 1 + 1I which does not belong to (2N∪I) ∪ (3N∪I), hence because of failure of closure property (2N∪I) ∪ (3N∪I) does not become ideal again.

Now we write the condition for union of left ideals of neutrosophic semiring to become ideal so, here we go.

Theorem 2.18. Let $\{J_k : k \in K\}$ be the family of left ideals of neutrosophic semiring N. Then $\bigcup J_k$ is left ideal if $J_k \subseteq J_{k+1}$ for all $k \in K$.

Proof. Let (a + bI), (c + dI) $\in \bigcup J_k$ and (e + f I) ∈ N then there exist p and q ∈ K such that (a + bI) $\in J_p$ and (c + dI) $\in J_q$. Let q > p then (a + bI), (c + dI) $\in J_q$. Now as J_q is an ideal so it follows that (a + bI) $*_1$ (c + dI) $\in J_q$ and (e + f I) $*_2$ (a + bI) $\in J_q$. Thus (e + f I) $*_2$ (a + bI) $\in \bigcup J_k$. so it follows that $\bigcup J_k$ is the left ideal. In similar fashion we can show that $\bigcup J_k$ is right ideal. This completes the proof.

Corollary 2.19. Union of two left ideals J_1 and J_2 of neutrosophic semiring N is an ideal if $J_1 \subseteq J_2$ or $J_2 \subseteq J_1$.

Proof. If $J_1 \subseteq J_2$ then $J_1 \bigcup J_2 = J_2$ and result is obvious. Again if $J_2 \subseteq J_1$ then $J_1 \bigcup J_2 = J_1$ and since J_1 is the left ideal of neutrosophic semiring N. So, by transitive property $J_1 \bigcup J_2$ is an ideal.

Theorem 2.20. Every left ideal of a neutrosophic semiring N is neutrosophic subsemiring of N.

Proof. Let J be the left ideal of N and for all (a + bI), (c + dI) ∈ J, (a + bI) $*_1$ (c + dI) ∈ J. Since J⊆ N, so (a + bI) ∈ N. Now as J is the left ideal of N so, (a + bI) $*_2$ (c + dI) ∈ J and associative and distributive laws are surely satisfied because J⊆ N and N is neutrosophic semiring. Similar result can be proved for every right ideal.

Remark.2.21. Converse of the above theorem is not true in general and it can be seen from following example.

Example 2.22. Let N = {1, 2, 3, …} be the set of natural numbers. Then N∪I ={a + bI: a, b ∈ N } under the operations defined as (a + bI) $*_1$ (c + dI) = min(a , c) + (min(b , d))I and (a + bI) $*_2$ (c + dI) = max(a , c) + (max(b, d))I, becomes the neutrosophic semiring and 2N∪I ={a + bI :a, b ∈ 2N }being the subset of N∪I becomes the neutrosophic subsemiring. But 2N∪I ={a+ bI: a, b ∈ 2N } is not a left ideal because if we take 5 + 5I from N∪I and 4 + 4I from 2N∪I and apply operation $*_2$ upon the selected members, we will see that resultant element is not of $2N \cup I$ i.e. $(5+5I) *_2 (4 + 4I) = \max(5, 4) + (\max(5, 4))I = 5 + 5I \notin 2N \cup I$. Hence it is clear now that the converse part of above theorem is not true in general.

3. Congruence Relations in Neutrosophic Semirings

We now study the concept of congruences in neutrosophic semirings.

Definition 3.1. Let $N \cup I = \{a + bI : a, b \in N\}$ be the neutrosophic semiring under the operations $*_1$ and $*_2$. A relation ρ on $N \cup I$ is called left compatible if for all $(a + bI)$, $(c + dI)$ and $(e + fI) \in N \cup I$ and $((a + bI), (c + dI)) \in \rho$, $((e + fI) *_1 (a + bI), (e + fI) *_1 (c + dI)) \in \rho$ and $((e + fI) *_2 (a + bI), (e + fI) *_2 (c + dI)) \in \rho$.

The relation is called right compatible if for all $(a + bI)$, $(c + dI)$ and $(e + fI) \in N \cup I$ and $((a+bI),(c+dI)) \in \rho, ((a+bI)*_1(e+fI),(c+dI)*_1(e+fI)) \in \rho$

and $((a + bI) *_2 (e + fI), (c + dI) *_2 (e + fI)) \in \rho$.

It is called compatible relation if for all $(a + bI)$, $(c + dI)$, $(e + fI)$ and $(g + hI) \in N \cup I$ and $((a + bI), (c + dI))$ and $((e + fI), (g + hI)) \in \rho$, $((a + bI) *_1 (e + fI), (c + dI) *_1 (g + hI)) \in \rho$ and $((a + bI) *_2 (e + fI), (c + dI) *_2 (g + hI)) \in \rho$.

A left (right) compatible equivalence relation is called left (right) congruence relation. A compatible equivalence relation is called congruence relation.

To understand the above concept, we consider the following example.

Example 3.2. Consider the neutrosophic semiring $(Z_5 \cup I, *_1, *_2)$ where for all $(a + bI)$, $(c + dI) \in Z_5 \cup I$, $(a + bI)*_1(c + dI) = \min((a + bI), (c + dI))$ and $(a + bI) *_2 (c + dI) = \max((a + bI), (c + dI))$ and consider $\rho = \{((a + bI), (c + dI)): (a + bI) = (c + dI)\}$ then this relation is left compatible (left congruence), right compatible (right congruence) and compatible(congruence) and one can verify it easily.

Proposition 3.3. A relation ρ on a neutrosophic semiring N is congruence relation if and only if it is both left and right congruence relation.

Proof. Let be a congruence relation on a neutrosophic semiring N. Let $(a + bI)$, $(c + dI)$ and $(e + fI) \in N$ and $((a + bI), (c + dI)) \in \rho$**.** Then $((e + fI)*_1(a + bI), (e + fI)*_1(c + dI))$ and $((e + fI)*_2(a + bI), (e + fI)*_2(c + dI)) \in \rho$. This shows that ρ is a left congruence relation and similarly it can be shown that ρ is a right congruence relation.

Conversely, assume that ρ is both left and right congruence relation. Let $(a + bI)$, $(c + dI)$, $(e + fI)$ and $(g + hI) \in N$ such that $((a + bI), (c + dI))$ and $((e + fI), (g + hI)) \in \rho$. Therefore $((a + bI)*_1(e + fI), (c + dI)*_1(e + fI)) \in \rho$ and $((a + bI)*_2(e + fI), (c + dI)*_2(e + fI)) \in \rho$ as ρ is a right compatible. Also $((c + dI)*_1(e + fI), (c + dI) *_1 (g + hI)) \in \rho$ and $((c + dI) *_2 (e + fI), (c + dI)*_2(g + hI)) \in \rho$ because ρ is a left compatible relation. Therefore $((a + bI)*_1(e + fI), (c + dI)*_1(g + hI)) \in \rho$ and $((a + bI)*_2(e + fI), (c + dI)*_2(g + hI)) \in \rho$. Thus ρ is a congruence relation.

4. Conclusions

In this article, we presented the theme of Neutrosophic Semiring to analyze its algebraic structural behavior. We studied the characteristic properties of Neutrosophic Semirings through their ideals and congruence relations. In the luminosity of our findings, we may conclude that our study is going to be a good addition to the list of algebraic structures based on Neutrosophic sets. Further we are planning to work out the structural study of Neutrosophic Semiring by extending it to Neutrosophic bi-Semiring and Neutrosophic N-semiring and also some theoretical applications of fuzzy sets and soft sets can be studied in these Neutrosophic algebraic structures.

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On Independence Neutrosophic Random Variables

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Abstract. In this article we study independence neutrosophic random variables and conditioned expectation, we prove that conditional variance is equal to neutrosophic conditional variance. These notions are important for neutrosophic probability due to neutrosophic random variables, neutrosophic central limit theorem and neutrosophic laws of large numbers can be studied.

Keywords: Neutrosophic random vector; neutrosophic expectation; independence neutrosophic random variables; Neutrosophic Logic.

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1. **Introduction**

F. Smarandache introduced the notion of neutrosophic probability measure as a function $\mathcal{NP} : Y \rightarrow [0,1]^3$ where Y is a neutrosophic sample space, and introduced the probability mapping to take the form $\mathcal{NP}(B) = (ch(B), ch(neutB), ch(antiB)) = (\alpha, \beta, \gamma)$ where $0 \leq \alpha, \beta, \gamma \leq 1$ and $0 \leq \alpha + \beta + \gamma \leq 3$ [33]. Moreover, many researchers have investigated many neutrosophic probability distributions like Poisson, exponential, binomial, normal, uniform, Weibull,...etc. (See [32], [2], [18], [26]). Furthermore, researchers have introduced the notion of neutrosophic queueing theory in [35], [36] this is one branch of neutrosophic stochastic modelling. Besides, researchers have also investigated neutrosophic time series prediction and modelling in many cases like neutrosophic moving averages, neutrosophic logarithmic models, neutrosophic linear models and so on. [3], [4], [12].

On the other hand, neutrosophic logic is an extension of intuitionistic fuzzy logic by adding indeterminacy component (I) where $I^2 = I, ..., I^n = I, 0.I = 0; n \in \mathbb{N}$ and I^{-1} is undefined (see [20], [32]). Neutrosophic logic has a huge brand of applications in many fields including decision making [29], [19], [25], machine learning [6], [27], intelligent disease diagnosis [30], [11], communication services [8], pattern recognition [28], social network analysis and e-learning systems [21], physics [34], sequences spaces [14] and so on. Neutrosophic logic has solved many decision-making problems efficiently like evaluating green credit rating, personnel selection, etc. [22], [23], [24], [1]. For more notions related to neutrosophic theory, we refer the reader to [9, 10, 14–17].

The study of neutrosophic random variables has become one of the fundamental pillars in neutrosophic theory and probability. Recent results of great importance can be seen in [37] and [13]. Taking into account mentioned above, in this article we study independent neutrosophic random variables and conditioned expectation.

2. Preliminaries

In this section, we show some well-known definitions and properties of neutrosophic logic and neutrosophic probability which are useful for the development of this paper.

Definition 2.1. (see [31]) Let X be a non-empty fixed set. A neutrosophic set A is an object having the form $\{x, (\mu A(x), \delta A(x), \gamma A(x)) : x \in X\}$, where $\mu A(x), \delta A(x)$ and $\gamma A(x)$ represent the degree of membership, the degree of indeterminacy , and the degree of non-membership respectively of each element $x \in X$to the set A.

Definition 2.2. (see [5]) Let K be a field, the neutrosophic filed generated by K and I is denoted by $\langle K \cup I \rangle$ under the operations of K, where I is the neutrosophic element with the property $I^2 = I$.

Definition 2.3. (see [32]) Classical neutrosophic number has the form $a + bI$ where a, b are real or complex numbers and I is the indeterminacy such that $0.I = 0$ and $I^2 = I$ which results that $I^n = I$ for all positive integers n.

Definition 2.4. (see [33]) The neutrosophic probability of event A occurrence is $NP(A) = (ch(A), ch(neutA), ch(antiA)) = (T, I, F)$ where T, I, F are standard or non-standard subsets of the non-standard unitary interval $]^{-}0, 1^{+}[$.

Recently, Bisher and Hatip [37] introduced and studied the notions of neutrosophic random variables by using the concepts presented by [33], these notions were defined as follows:

Definition 2.5. Consider the real valued crisp random variable X which is defined as follows:

$$X : \Omega \to \mathbb{R}$$

where Ω is the events space. Now, they defined a neutrosophic random variable X_N as follows:

$$X_N : \Omega \to \mathbb{R}(I)$$

and

$$X_N = X + I$$

where I is indeterminacy.

Theorem 2.6. *Consider the neutrosophic random variable $X_N = X + I$ where cumulative distribution function of X is $F_X(x) = P(X \leq x)$. Then, the following statements hold:*

(1) $F_{X_N}(x) = F_X(x - I)$,
(2) $f_{X_N}(x) = f_X(x - I)$.

Where F_{X_N} and f_{X_N} are cumulative distribution function and probability density function of X_N, respectively.

Theorem 2.7. *Consider the neutrosophic random variable $X_N = X + I$, expected value can be found as follows:*

$$E(X_N) = E(X) + I.$$

Proposition 2.8 (Properties of expected value of a neutrosophic random variable)**.** *Let X_N and Y_N be neutrosophic random variables, then the following properties holds:*

(1) $E(aX_N + b + cI) = aE(X_N) + b + cI; a, b, c \in \mathbb{R}$,
(2) *If X_N and Y_N are neutrosophic random variables, then $E(X_N \pm E_(Y_N) = E(X_N) \pm E(Y_N)$,*
(3) $E[(a + bI)X_N] = aE(X_N) + bIE(X_N); a, b \in \mathbb{R}$,
(4) $|E(X_N)| \leq E|X_N|$.

Theorem 2.9. *Consider the neutrosophic random variable $X_N = X + I$, variance of X_N is equal to variance of X, i.e. $V(X_N) = V(X)$.*

Now, Granados [13] studied the notions of neutrosophic random vector and joint neutrosophic random variable, these notions were defined as follows:

Definition 2.10. A neutrosophic random vector of two dimension is a vector (X_N, Y_N) in which each coordinate is a neutrosophic random variable. Analogously, we can define a neutrosophic random vector multidimensional as follows $(X_{N_1}, X_{N_2}, ..., X_{N_n})$ in which $X_{N_1}, X_{N_2}, ..., X_{N_n}$ are neutrosophic random variables for each $n = 1, 2, ...$

Definition 2.11. Let (X_N, Y_N) be a neutrosophic random vector, we define probability function of a neutrosophic continuous random vector (X_N, Y_N). Then, joint probability neutrosophic function of a discrete random vector (X_N, Y_N) $f_N(x, y) : \mathbb{R}^2 \to [0, \infty)$ in which is non-negative and integrable, and for any $(x, y) \in \mathbb{R}^2$, it is defined as follows

$$P(X_N \leq x, Y_N \leq y) = P(X \leq x - I, Y \leq y - I) = \int_{-\infty}^{y-I} \int_{-\infty}^{x-I} f_{(X_N,Y_N)}(u, v)dvdu$$

Similarly, probability function of a neutrosophic discrete random vector (X_N, Y_N) is defined similar by using sum.

Definition 2.12. Let (X_N, Y_N) be a neutrosophic random vetor, we define neutrosophic joint distribution function which will be denoted by $F_{(X_N,Y_N)}(x,y) = P(X_N \leq x, Y_N \leq y) = P(X \leq x - I, Y \leq y - I)$.

Definition 2.13. Let $f_{(X_N,Y_N)}(x,y)$ be a joint probability neutrosophic function of a continuous random variable (X_N, Y_N). We define neutrosophic marginal function of X_N as follows:

$$f_{X_N}(x) = \int_{-\infty}^{+\infty} f_{(X_N,Y_N)}(x,y)dy$$

and we define neutrosophic marginal function of Y_N as follows:

$$f_{Y_N}(y) = \int_{-\infty}^{+\infty} f_{(X_N,Y_N)}(x,y)dx$$

Similarly, joint probability neutrosophic function of a discrete random variable is defined similar by using sum.

Definition 2.14. Expected value of a neutrosophic random vector (X_N, Y_N) in which expected value of X_N and Y_N exist, we define $E(X_N, Y_N) = (E(X_N), E(Y_N))$.

Next, we will show some new notions on neutrosophic random variables which have not bee studied so far and are needed.

Let (X_N, Y_N) be neutrosophic vector and $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a function, then $\phi(X_N, Y_N)$ is a neutrosophic random variable and its expectation is defined as follows:

$$E[\phi(X_N, Y_N)] = \int_{-\infty}^{+\infty} (x - I)dF_{\phi(X_N,Y_N)}(x),$$

as well as one dimensional case, we are required to find out the distribution $\phi(X_N, Y_n)$ by which can be difficult in some cases. Next, we establish an alternative way in which we can calculate expectation of $\phi(X_N, Y_N)$ without known its distribution, but we must know (X_N, Y_N) distribution. Let (X_N, Y_N) be neutrosophic vector and $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $\phi(X_N, Y_N)$ has finite expected, then:

$$E[\phi(X_N, Y_N)] = \int_{\mathbb{R}^2} \phi(x - I, y - I)dF_{\phi(X_N,Y_N)}(x,y),$$

This can be proved easily due to this is a Riemmann-Stieltjes integral in two dimensional.

In case of X_N and Y_N be independence (see section 3), this increment is

$$\begin{aligned} &F_X(x_i - I)F_Y(y_j - I) - F_X(x_i - I)F_Y(y_{j-1} - I) - F_X(x_{i-1} - I)F_Y(y_j - I) \\ &+ F_X(x_i - I)F_Y(y_j - I) = \Delta F_X(x_i - I)\Delta F_Y(y_j - I), \end{aligned}$$

i.e, bidimensional integral can be separated in two integral and can be written as follows

$$E[\phi(X_N, Y_N)] = \int_{\mathbb{R}^2} \phi(x - I, y - I) dF_{X_N}(x) dF_{Y_N}(y).$$

When (X_N, Y_N) is a discrete vector, we have

$$E[\phi(X_N, Y_N)] = \sum_{x-I, y-I} \phi(x - I, y - I) P(X = x - I, Y = y - I),$$

in which the sum is applied over all possible valued $(x - I, y - I)$ on (X_N, Y_N).

Theorem 2.15. *Let X_N and Y_N be two neutrosophic random variable with finite expectation, then*

$$E(X_N + Y_N) = E(X_N) + E(Y_N).$$

Proof. Let $\phi(x-I, y-I) = x-I+y-I$, $\phi_1(x-I, y-I) = x-I$ and $\phi_2(x-I, y-I) = y-I$. Then,

$$\begin{aligned} E(X_N + Y_N) &= E(\phi(X_N, Y_N)) \\ &= \int_{\mathbb{R}^2} (x - I + y - I) dF_{X_N, Y_N}(x, y) \\ &= \int_{\mathbb{R}^2} (x - I) dF_{X_N, Y_N}(x, y) + \int_{\mathbb{R}^2} (y - I) dF_{X_N, Y_N}(x, y) \\ &= E(\phi_1(X_N, Y_N)) + E(\phi_2(X_N, Y_N)) \\ &= E(X_N) + E(Y_N) \end{aligned}$$

□

Theorem 2.16. *Let X_N and Y_N be two independence neutrosophic random variable and g and h be two functions such that $g(X_N)$ and $h(Y_N)$ have finite expected, then*

$$E[g(X_N)h(Y_N)] = E[g(X_N)]E[h(Y_N)].$$

Proof.

$$\begin{aligned} E[g(X_N)h(Y_N)] &= \int_{\mathbb{R}^2} g(x - I)h(y - I) dF_{X_N, Y_N}(x, y) \\ &= \int_{\mathbb{R}^2} g(x - I)h(y - I) dF_{X_N}(x) dF_{Y_N}(y) \\ &= \int_{\mathbb{R}} g(x - I) dF_{X_N}(x) \int_{\mathbb{R}} g(y - I) dF_{Y_N}(y) \\ &= E[g(X_N)]E[h(Y_N)] \end{aligned}$$

□

3. Independence neutrosophic random variables

Let X_N and Y_N be two neutrosophic random variables. X_N and Y_N are independence if the events $(X \leq x - I)$ and $(Y \leq y - I)$ are independence for any real-value x and y, i.e., if the following equality is satisfied

$$P[(X \leq x - I) \cap (Y \leq y - I)] = P(X \leq x - I)P(Y \leq y - I). \tag{1}$$

The left side of the equality (1) can be written as $P(X \leq x-I, Y \leq y-I)$ or $F_{X,Y}(x-I, y-I)$, and it is said to be the joint distribution function of X_N and Y_N evaluate in the point (x, y). Therefore, note that (1) can be expressed as follows

$$F_{X,Y}(x - I, y - I) = F_X(x - I)F_Y(y - I),\ for\ x, y \in \mathbb{R}. \tag{2}$$

In this way, in order to determine whether two neutrosophic random variables are independent, it is necessary to know both the joint probabilities $P(X \leq x-I, Y \leq y-I)$ as individual probabilities $P(X \leq x-I)$ and $P(Y \leq y-I)$, and verify the identity (2) for each real numbers x and y. Hence, it is enough that there exists a pair $(x - I, y - I)$ for which the equality (2) does not hold to be able to conclude that X and Y are not independent. Granados [13] studied on random vectors and explained how to obtain the individual distributions from the joint distribution of two random variables.

Example 3.1. Let (X_N, Y_N) be a neutrosophic random vector with density function $f(x - I, y - I) = 4(x - I)(y - I)$ for $I \leq x, y \leq 1 - I$.

The marginal density function of X_N is calculated as follows for $I \leq x \leq 1 - I$

$$f_X(x - I) = \int_I^{1-I} 4(x - I)(y - I)dy = 2(x - I).$$

Analogously, we can prove that $f_{Y_N}(y) = 2(y - I)$ for $I \leq x \leq 1 - I$. Therefore, X_N and Y_N are independence, due to $F_{X,Y}(x - I, y - I) = F_X(x - I)F_Y(y - I)$.

Proposition 3.2. *Let X_N and Y_N be two independence neutrosophic random variables , and let g and h be two functions of $\mathbb{R} \to \mathbb{R}$. Then, the neutrosophic random variables $g(X_N)$ and $h(Y_N)$ are independence neutrosophic random variables.*

Proof: Let $\mathcal{A} = (-\infty, x - I]$ and $\mathcal{B} = (-\infty, y - I]$, then

$$\begin{aligned}
P(g(X_N) \leq x, h(Y_N) \leq y) &= P(g(X_N) \in \mathcal{A}, h(Y_N) \in \mathcal{B}) \\
&= P(X_N \in g^{-1}(\mathcal{A}), Y_N \in h^{-1}(\mathcal{B})) \\
&= P(X_N \in g^{-1}(\mathcal{A}))P(Y_N \in h^{-1}(\mathcal{B})) \\
&= P(g(X_N) \in \mathcal{A})P(h(Y_N) \in \mathcal{B}) \\
&= P(g(X_N) \leq x)P(h(Y_N) \leq y)
\end{aligned}$$

Theorem 3.3. *Let X_N and Y_N two discrete neutrosophic random variables, then*

(1) *Using $P(X \leq x-I, Y \leq y-I) = \sum_{u \leq x-I} \sum_{v \leq y-I} P(X = u, Y = v)$, we have*

$$\begin{aligned} P(X = x-I, Y = y-I) = P(X &\leq x-I, Y \leq y-I) \\ &- P(X \leq x-I-1, Y \leq y-I) \\ &- P(X \leq x-I, Y \leq y-I-1) \\ &+ P(X \leq x-I-1, Y \leq y-I-1). \end{aligned}$$

(2) *Using (1), independence condition $P(X \leq x-I, Y \leq y-I) = P(X \leq x-I)P(Y \leq y-I)$ is equivalent to $P(X = x-I, Y = y-I) = P(X = x-I)P(Y = y-I)$.*

Proof:

(1) This result can be obtained by using the following equality

$$\begin{aligned} (X = x-I, Y = y-I) = (X \leq x-I, Y \leq y-I) &- (X \leq x-I-1, Y \leq y-I) \\ &- (X \leq x-I, Y \leq y-I-1) \\ &+ (X \leq x-I-1, Y \leq y-I-1). \end{aligned}$$

(2) Using (1) and by hypothesis of independence, we have

$$\begin{aligned} P(X = x-I, Y = y-I) &= P(X \leq x-I)P(X \leq y-I) - \\ &P(X \leq x-I-1)P(Y \leq y-I) \\ &- P(X \leq x-I)P(Y \leq y-I) \\ &+ P(X \leq x-I-1)P(Y \leq y-I-1) \\ &= [P(X \leq x-I) - P(X \leq x-I-1)] \\ &\times [P(Y \leq y-I) - P(Y \leq y-I-1)] \\ &= P(X = x-I)P(Y = y-I). \end{aligned}$$

Theorem 3.4. *Let X_N and Y_N be two independence neutrosophic random variables which have finite expectation. Then,*

$$E(X_N Y_N) = E(X_N)E(Y_N). \tag{3}$$

Proof: We prove the case when X_N and Y_N are discrete neutrosophic random variables, the case for continuous neutrosophic random variables are proved similarly.

Let

$$\begin{aligned} E(X_N Y_N) &= \sum_{x,y}(x-I)(y-I)P(X_N = x, Y_N = y) \\ &= \sum_{x}\sum_{y}(x-I)(y-I)P(X_N = x, Y_N = y) \\ &= (\sum_{x}(x-I)P(X_N = x))(\sum_{y}(y-I)P(Y_N = y)) \\ &= E(X_N)E(Y_N). \end{aligned}$$

Definition 3.5. Let $X_{N_1}, X_{N_2}, ..., X_{N_n}$ be a collection of neutrosophic random variables with joint function distribution $F_X(x_1-I, x_2-I, ..., x_n-I)$, and consider marginals functions distribution $F_{X_{N_1}}(x_1), F_{X_{N_2}}(x_2), ..., F_{X_{N_n}}(x_n)$, respectively. Then, we say that $X_{N_1}, X_{N_2}, ..., X_{N_n}$ are independence if for any real numbers $x_1 - I, x_2 - I, ..., x_n - I$ the following equality holds

$$F_X(x_1 - I, x_2 - I, ..., x_n - I) = F_{X_{N_1}}(x_1)F_{X_{N_2}}(x_2)...F_{X_{N_n}}(x_n).$$

Analogously, we can define it in terms of neutrosophic density function $f(x_1-I, x_2-I, ..., x_n-I)$ if the following equality holds

$$f(x_1 - I, x_2 - I, ..., x_n - I) = f_{X_{N_1}}(x_1)f_{X_{N_2}}(x_2)...f_{X_{N_n}}(x_n).$$

Example 3.6. Let X_N and Y_N be two continuous neutrosophic random variables with joint density function

$$f_{(X,Y)}(x - I, y - I) = \begin{cases} e^{-x-y+2I} & if\ x, y > I, \\ \\ 0 & otherwise. \end{cases}$$

Neutrosophic marginals probability functions are defined as follows

$$f_X(x - I) = \begin{cases} e^{-x+I} & if\ x > I, \\ \\ 0 & otherwise. \end{cases}$$

and

$$f_Y(y - I) = \begin{cases} e^{-y+I} & if\ y > I, \\ \\ 0 & otherwise. \end{cases}$$

Therefore, $f_{(X,Y)}(x - I, y - I) = f_X(x - I)f_Y(y - I)$ for any real numbers $x - I$ and $y - I$, and hence we conclude X_N and Y_N are independence.

Example 3.7. Let X_N and Y_N be two discrete neutrosophic random variables with joint density function

$$f_{(X,Y)}(x-I,y-I)=\begin{cases}\frac{1}{4} & if\ x,y\in\{I,1+I\},\\ \\ 0 & otherwise.\end{cases}$$

Neutrosophic marginals probability functions are defined as follows

$$f_X(x-I)=\begin{cases}\frac{1}{2} & if\ x\in\{I,1+I\},\\ \\ 0 & otherwise.\end{cases}$$

and

$$f_Y(y-I)=\begin{cases}\frac{1}{2} & if\ y\in\{I,1+I\},\\ \\ 0 & otherwise.\end{cases}$$

Therefore, $f_{(X,Y)}(x-I,y-I)=f_X(x-I)f_Y(y-I)$ for any real numbers $x-I$ and $y-I$, and hence we conclude X_N and Y_N are independence.

Remark 3.8. It can be said that an infinite set of neutrosophic random variables is independence if any finite subset is independence.

This statement can be useful due to for future work the concepts of neutrosophic central limit theorem and neutrosophic laws of large numbers can be studied.

4. **Conditional expectation**

In this section, we introduce the concept of conditional expectation of a neutrosophic random variable with respect to a σ-algebra, and some of its elemental properties are studied. We will consider that has a base probability space $(\Omega,\mathcal{F},P)$, and $\mathcal{G}$ is a sub-algebra of $\mathcal{F}$. We have defined expectation of a neutrosophic random variable as a Riemann-Stieltjes integral as follows

$$E(X_N)=\int_{-\infty}^{+\infty}(x-I)dF_{X_N}(x).$$

however, to make the notation simpler in this section, it is sometimes convenient to adopt the notation of measure theory and denote the expectation of a netrusophic random variable X_N as follows

$$E(X_N)=\int_\Omega X_N dP.$$

We shall recall that if we know distribution of neutrosophic vector (X_N,Y_N) and we take valued $y-I$ such that $f_Y(y-I)=f_{Y_N}(y)\neq 0$, conditional expectation of X_N known $Y=y-I$

is the function

$$y - I \mapsto E(X_N|Y_N = y) = \int_{-\infty}^{+\infty} (x - I) dF_{X_N|Y_N}(x|y), \tag{4}$$

when $f_{Y_N}(y) \neq 0$. (4) is equivalent to write

$$E(X_N|Y_N = y) = \int_{-\infty}^{+\infty} (x - I) f_{X_N|Y_N}(x|y) dx.$$

If we make a change in order of integration, we can see that

$$E(X_N) = \int_{-\infty}^{+\infty} E(X_N|Y_N = y) f_{Y_N}(y) dy,$$

if we apply this expression by using total probability theorem in terms of expectation. In the case when (X_N, Y_N) is a discrete neutrosophic vector, we have

$$\begin{aligned} E(X_N|Y_N = y) &= \sum_x (x - I) f_{X_N|Y_N}(x|y) \\ &= \sum_x (x - I) P(X_N = x|Y_N = y), \end{aligned}$$

considering $f_{Y_N}(y) \neq 0$ and sum is absolutely convergent. Again, applying a change in order of sum, we have

$$E(X_N) = \sum_y E(X_N|Y_N = y) P(Y_N = y).$$

In any cases, we can also see, when Y_N and X_N are independence, we have

$$E(X_N|Y_N = y) = E(X_N).$$

Example 4.1. We will find expectation of $E(X_N|Y_N = y)$ for each $y \in (I, 1 + I)$ when X_N and Y_N have the following joint neutrosophic density function.

$$f_{X|Y}(x - I, y - I) = \begin{cases} 12(x - I)^2 & if \ I < x < y < 1 + I, \\ \\ 0 & otherwise. \end{cases}$$

For $I < y < 1 + I$,

$$
\begin{aligned}
E(X_N|Y_N = y) &= \int_0^{y-I} (x-I)\frac{12(x-I)^2}{4(y-I)^3}dx \\
&= \frac{3}{(y-I)^3}\int_0^{y-I}(x-I)^3dx \\
&= \frac{3}{4}\frac{(y-2I)^4 - I}{(y-I)^3}
\end{aligned}
$$

Analogously to (4), if Q is an event with positive probability and X_N is a integrable neutrosophic random variable, conditional expectation of X_N known Q is

$$E(X_N|Q) = \int_{-\infty}^{+\infty}(x-I)dF_{X_N|Q}(x),$$

where $F_{X_N|Q}(x) = P(X_N \leq x|A) = P(X \leq x-I|A) = \dfrac{P(X \leq x-I, Q)}{P(Q)}$. Next, we will show a more generally definition which generalized concepts showed so far.

Definition 4.2. Let X_N be a neutrosophic random variable with finite expectation, and let $\mathcal{G}$ be a sub-algebra of $\mathcal{F}$. Conditional expectation of X_N known $\mathcal{G}$, it is a neutrosophic random variable which will be denoted by $E(X_N|\mathcal{G})$ which satisfies the following conditions:

(1) It is $\mathcal{G}$-measurable,
(2) It has finite expectation,
(3) For any event $G \in \mathcal{G}$, $\displaystyle\int_G E(X_N|\mathcal{G})dP = \int_G X_N dP$.

Remark 4.3. Just as it happens in a random variable, part of the difficulty in understanding this general definition is that an explicit formula is not provided for this neutrosophic random variable but only the properties it satisfies. The objective of this section is to find the meaning of this neutrosophic random variable, interpret its meaning, and explain its relationship with the concept of elementary conditional expectation.

Remark 4.4. When $\mathcal{G} = \sigma(Y_N)$ for any neutrosophic random variable Y_N, conditional expected will be written by $E(X_N|Y_N)$ instead of $E(X_N|\sigma(Y_N))$.

Remark 4.5. Let Q be any event, then conditional expected $E(1_Q|\mathcal{G})$ will be denoted by $P(Q|\mathcal{G})$.

Now, we will show some properties on $E(X_N|Y_N)$ when Y_N is a discrete neutrosophic random variable.

Let X_N and Y_N be two neutrosophic random variables. Now, consider that X_N has finite expected and Y_N is discrete with possible values $y_1 - I, y_2 - I, ...$ Conditional expectation of

X_N known event $(Y_N = y_j) = (Y = y_j - I)$ is $E(X_N|Y_N = y_j)$, this value depends of the event $(Y_N = y_j)$, and we can consider that we have a function defined over Ω as follows: If ω is such that $Y(\omega) = y_j - I$, then

$$\omega \mapsto E(X_N|Y_N)(\omega) = E(X_N|Y_N = y_j).$$

We can see that $E(X_N|Y_N)$ takes at much different values as Y_N does. Generally, function can be rewritten in terms of indicator function as follows

$$E(X_N|Y_N)(\omega) = \sum_{j=1}^{\infty} E(X_N|Y_N = y_j)1_{(Y_N=y_j)}(\omega).$$

In this way, we can define the function $E(X_N|Y_N) : \Omega \to \mathbb{R}$ which is denoted by

$$\omega \mapsto E(X_N|Y_N)(\omega) = E(X_N|Y_N = y_j) \ if \ Y(\omega) = y_j - I$$

is a neutrosophic random variable.

Theorem 4.6. *Let X_N be a integrable neutrosophic random variable, and let Y_N discrete with possible values $y_1 - I, y_2 - I, ...$ $E(X_N|Y_N) : \Omega \to \mathbb{R}$ is a neutrosophic random variable which satisfies the following conditions:*

(1) *It is $\sigma(Y_N)$-measurable,*
(2) *It has finite expected,*
(3) *For any event $G \in \sigma(Y_N)$, $\int_G E(X_N|Y_N)dP = \int_G X_N dP$.*

Proof:

(1) Through possibles values, the neutrosophic random variables Y_N divides in different part Ω, i.e $(Y_N = y_1), (Y_N = y_2), ...$ are disjunct events. $\sigma(Y_N) = \sigma\{(Y_N = y_1), (Y_N = y_2), ...\} \subset \mathcal{F}$. Since $E(X_N|Y_N)$ is constant in each element of partition, implies that $E(X_N|Y_N)$ is $\sigma(Y_N)$-measurable, and hence it is a neutrosophic random variable.
(2) Taking the event G as Ω in the third property, we get that X_N and $E(X_N|Y_N)$ have the same finite expectation.
(3) Since each element of $\sigma(Y_N)$ is union of disjunct elements of $(Y_N = y_j)$, by properties of integral it is enough to show that $\int_G E(X_N|Y_N)dP = \int_G X_N dP$ for these simple

events. Then, we have

$$\begin{aligned}\int_{(Y_N=y_j)} E(X_N|Y_N)(\omega)dP(\omega) &= E(X_N|Y_N=y_j)P(Y_N=y_j)\\ &= \int_\Omega X_N(\omega)dP(\omega|Y_N=y_j)P(Y_N=y_j)\\ &= \int_\Omega X_N(\omega)dP(\omega, Y_N=y_j)\\ &= \int_{(Y_N=y_j)} X_N(\omega)dP(\omega).\end{aligned}$$

Remark 4.7. We have to see the different between $E(X_N|Y_N=y_j)$ and $E(X_N|Y_N)$. First term is a possible numerical value, second term is a neutrosophic random variable. However, both expressions are called conditional expectation. We will see next a particular case of this neutrosophic random variable. Besides, we will show that the conditional expectation can be seen as a generalization of the basic concept of conditional probability, and it can also be considered as a generalization of the concept of expectation.

Proposition 4.8. *Let X_N be a neutrosophic random variable with finite expectation, then $E(X_N|\{\emptyset,\Omega\}) = E(X_N) = E(X)+I$.*

Proof: This proofs follows since $E(X_N|\mathcal{G})$ is measurable respected to $\mathcal{G}$, and any measurable function respected to $\{\emptyset,\Omega\}$ is constant. Now, for any event $G\in\{\emptyset,\Omega\}$, $\int_G E(X_N|\{\emptyset,\Omega\})dP = \int_G X_N dP = \int_\Omega X_N dP = E(X_N) = E(X)+I$.

Theorem 4.9. *Let X_N and Y_N be two neutrosophic random variables with finite expectation and $b\in\mathbb{R}$. Then, the following statements hold:*

(1) *If $X\geq I$, then $E(X_N|\mathcal{G})\geq 0$,*
(2) *$E(bX_N+Y_N|\mathcal{G}) = bE(X_N|\mathcal{G})+E(Y_N|\mathcal{G})$,*
(3) *If $X_N\leq Y_N$, then $E(X_N|\mathcal{G})\leq E(Y_N|\mathcal{G})$,*
(4) *$E(E(X_N|\mathcal{G})) = E(X_N)$,*
(5) *If X_N is $\mathcal{G}$-measurable, then $E(X_N|\mathcal{G}) = X_N$ a.s. In particular, $E(b|\mathcal{G}) = b$,*
(6) *If $\mathcal{G}_1\subset\mathcal{G}_2$, then*

$$E(E(X_N|\mathcal{G}_1)|\mathcal{G}_2) = E(E(X_N|\mathcal{G}_2)|\mathcal{G}_1) = E(X_N|\mathcal{G}_1),$$

(7) *$|E(X_N|\mathcal{G})|\leq E(|X_N||\mathcal{G})$,*
(8) *$E|E(X_N|\mathcal{G})|\leq E(|X_N|)$.*

Proof:

(1) Follows directly from definition and the fact that $X\geq I\equiv X_N\geq 0$.

(2) For all $G \in \mathcal{G}$, we have

$$\int_G E(bX_N + Y_N|\mathcal{G})dP = \int_G (bX_N + Y_N)dP$$
$$= b\int_G X_N dP + \int_G Y_N dP.$$

Now,

$$\int_G [bE(X_N|\mathcal{G}) + E(Y_N|\mathcal{G})]dP = b\int_G E(X_N|\mathcal{G})dP + \int_G E(Y_N|\mathcal{G})dP$$
$$= b\int_G X_N dP + \int_G Y_N dP.$$

(3) Follows directly from definition and the fact that $X_N \leq Y_N$.

(4) Taking $G = \Omega$, we get the equality.

(5) Since X_N is $\mathcal{G}$-measurable, three condition of the definition hold. Now, $\int_G E(X_N)dP = \int_G E(X_N)dP$. Therefore, $X_N = E(X_N|\mathcal{G})$ a.s.

(6) For all $G \in \mathcal{G}_1 \subset \mathcal{G}_2$, we have

$$\int_G E(E(X_N|\mathcal{G}_1)|\mathcal{G}_2)dP = \int_G E(X_N|\mathcal{G}_1)dP$$
$$= \int_G X_N dP.$$

Analogously,

$$\int_G E(E(X_N|\mathcal{G}_2)|\mathcal{G}_1)dP = \int_G E(X_N|\mathcal{G}_2)dP$$
$$= \int_G X_N dP.$$

(7) Consider that $\int_G |E(X_N|\mathcal{G})|dP = |\int_G E(X_N|\mathcal{G})dP|$. Then,

$$|\int_G E(X_N|\mathcal{G})dP| = |\int_G X_N dP|$$
$$\leq \int_G |X_N|dP$$
$$= \int_G E(|X_N||\mathcal{G})dP.$$

(8) Proof follows from parts (4) and (7) of this theorem.

Definition 4.10. Let X_N be a neutrosophic random variable with finite second moment, and $\mathcal{G}$ be a sub-algebra of $\mathcal{F}$. Conditional variance of X_N known $\mathcal{G}$ which will be denoted by $Var(X_N|\mathcal{G})$, it is defined as a neutrosophic random variable as follows

$$Var(X_N|\mathcal{G}) = E[(X_N - E(X_N|\mathcal{G}))^2|\mathcal{G}].$$

We shall recall that neutrosophic conditional variance is not a number, it is a neutrosophic random variable. Therefore, the only way that we have the neutrosophic variance of a random variable from conditional variance is $Var(X_N) = Var(X) = Var(X_N|\{\emptyset, \Omega\})$.

On the other hand, we have that

$$\begin{aligned} Var(X_N|\mathcal{G}) &= Var(X + I|\mathcal{G}) \\ &= Var(X|\mathcal{G}) \\ &= E[(X - E(X|\mathcal{G}))^2|\mathcal{G}], \end{aligned}$$

this shows that conditional covariance is equal to neutrosophic conditional covariance.

5. Conclusion

In this article we study the notion of neutrosophic random variable taking into account the notions previously studied by [37] and [13]. These results are of great importance because convergence on neutrosophic random variables, neutrosophic central limit theorem and neutrosophic laws of large numbers can be studied. Secondly, this results can be applied in quality control, stochastic modeling, reliability theory, queueing theory, electrical engineering and so on.

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Conflicts of Interest

The author declares no conflict of interest.

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Practical Applications of the Independent Neutrosophic Components and of the Neutrosophic Offset Components

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Abstract: The newly introduced theories, proposed as extensions of the fuzzy theory, such as the Neutrosophic, Pythagorean, Spherical, Picture, Cubic theories, and their numerous hybrid forms, are criticized by the authors of [1]. In this paper we respond to their critics with respect to the neutrosophic theories and show that the DST, that they want to replace the A-IFS with, has many flaws.

Their misunderstanding, with respect to the partial and total independence of the neutrosophic components, is that in the framework of the neutrosophic theories we deal with a *MultiVariate Truth-Value* (truth upon many independent random variables) as in our real-life world, not with a *UniVariate Truth-Value* (truth upon only one random variable) as they believe.

About the membership degrees outside of the interval [0, 1], which are now in the arXiv and HAL mainstream, it is normal that somebody who over-works (works overtime) to have an over-membership (i.e., membership degree above 1) to be distinguished from those who do not work overtime (whose membership degree is between 0 and 1). And, similarly, a negative employee (that who does only damages to the company) to have a negative membership (i.e., membership degree below 0) in order to distinguish him from the positive employees (those whose membership degree is above 0). There are elementary practical applications in this paper that allow us to think out of box (in this case the box is the interval [0, 1]).

Keywords: Neutrosophy; Neutrosophic Components; Neutrosophic Offset Components; TriVariate Truth-Value; MultiVariate Truth-Value; UniVariate Truth-Value.

1. Independence and Dependence of the Neutrosophic Components

The introduction should briefly place the study in a broad context and highlight why it is important. It should define the purpose of the work and its significance. The current state of the research field should be reviewed carefully and key publications cited. Please highlight controversial and diverging hypotheses when necessary. Finally, briefly mention the main aim of the work and highlight the principal conclusions. As far as possible, please keep the introduction comprehensible to scientists outside your particular field of research. References should be numbered in order of appearance and indicated by a numeral or numerals in square brackets, e.g., [1] or [2,3], or [4–6]. See the end of the document for further details on references.

1.1 TriVariate Truth-Value

Neutrosophy [15], as new branch of philosophy, started from the practical principle that everything (E) should be evaluated from *three independent points of view (or sources of information, or* *random variables)*: two opposite ones (positive and negative), and a third one the neutral in between them, for a fear evaluation. Thus, a neutrosophic triplet has been constructed, <positive, neutral, negative>, for studying especially contrary philosophical concepts, ideas, and schools. Therefore, one deals with a TriVariate Truth-Value because one uses three independent random variables (sources of information): one that presents the degree of positive side of E, another that presents the degree of negative side of E, and a third one that presents the degree of neutral (indeterminate) side of E.

That's what happens in our everyday life, and the most known one is in the court of law (defender, persecutor, jury). Also, everything has good, bad, and common features.

{Surely, more generally, everything may be evaluated from *n* points of view (*n* random variables, or *n* sources of information), for any integer $2 \le n \le \infty$, as such dealing with a *MultiVariate Truth-Value*, where the random variables may have degrees of positiveness, or negativeness, or neutrality (indeterminacy), but this case falls under the Refined Neutrosophic Logic [13], or under the Plithogenic Logic as generalization of MultiValued Logic [14], or under the Plithogenic Probability & Statistics as generalizations of MultiVariate Probability & Statistics [30], which are different stories.}

For example, in general you are evaluated by a friend in a positive way, by an enemy in a negative way, and by a neutral person in a neutral way.

Surely, in the Refined Neutrosophic Set and Logic and Probability , you may be evaluated by many friends in positive ways, and by many enemies in negative ways, and by many neutral persons in neutral ways. That's life, as in neutrosophy.

This ThreeVariate way of thinking (neutrosophic evaluation) was transferred to the scientific disciplines that resulted from neutrosophy:

Neutrosophic Set (degree of membership, degree of indeterminate-membership, degree of nonmembership);

Neutrosophic Logic (degree of truth (T), degree of indeterminate-truth (I), degree of falsehood (F));

Neutrosophic Probability (chance of an event to occur, indeterminate-chance of the event to occur or not, chance of the event not to occur); etc.

For simplicity, we preferred to use the descriptive notation (T, I, F) for all neutrosophic triplets.

Let's consider the single-valued neutrosophic components, where all $T, I, F \in [0,1]$.

Depending on each application, in the neutrosophic theories one may encounter three (or more) possibilities:

a. UniVariate Truth-Value, when only one source assigns values to the neutrosophic components, and thus the neutrosophic components are totally dependent as in the other fuzzy theories, whence $0 \le T+I+F \le 1$.
b. BiVariate Truth-Value, when two independent sources assign values to the neutrosophic components, for example one source assigns values to two neutrosophic components (let's assume to T and F, thus $0 \le T+F \le 1$) and the second one to the other neutrosophic component (which is I, thus $0 \le I \le 1$), and therefore the neutrosophic components are partially dependent and partially independent {or T and F are totally dependent of each other, while I is totally independent from both of them}, whence $0 \le T+I+F \le 2$.
c. TriVariate Truth-Value, when three independent sources assign values to the neutrosophic components, each source to one distinct neutrosophic component, thus $0 \le T+I+F \le 3$) and all three neutrosophic components are totally independent.
d. TriVariate Truth-Value, when the three sources are partially dependent and partially independent. For example, John's work is evaluated by three sources: a friend, an enemy, and a neutral person, which communicate with each other and arrive to some agreement about John's work that is interpreted as *degree of dependence* (*d*) between these three sources,

and to some disagreement about John's work that is interpreted as *degree of independence* (*i*) between the three sources, where $d, i \in [0,1], d + i = 1$.

e. MultiVariate Truth-Value, in general, for Refined Neutrosophic Set/Logic/Probability [13], and for Plithogenic Logic/Probability/Statistics [14, 30].

1.2 "Unfortunately, this fact [*independence of components – our note*] is not usually taken into account in the works, where NST was applied."

Their assertion is untrue, the independence of components was used in most of the neutrosophic applications.

The independence of the neutrosophic components comes from the unrestricted summation T + I + F that can get any value between 0 and 3. The independence comes from the fact that if a neutrosophic component gets a value, it does not affect in no way the other two neutrosophic components' values. Not restricting the value of the sum T + I + F means from the start the existence of degrees of independence and dependence between the components.

In many neutrosophic applications that presented numerical examples, looking at the neutrosophic triplets (T, I, F), you would see: some whose sum is < 1, others whose sum is > 1, and others whose sum is = 1. For example (0.1, 0.3, 0.5), or (0.9, 0.8, 0.6), or (0.7, 0.1, 0.2), etc.

Also, in all neutrosophic papers the neutrosophic operators were employed, which means that the Indeterminacy (I) was used independently from T and F into the operators' formulas, which is not the case for the previous classical, fuzzy (especially A-IFS) set and logic, and probability theories.

Unlike in other previous theories (for example in DST), no normalization is done in the neutrosophic theories, therefore, after aggregation, the resulted neutrosophic components sum may be any number between 0 and 3.

Yet, the situation is more complex, since the neutrosophic theories comprises all possibilities of the neutrosophic components, i.e.: to be totally independent, partially independent and partially dependent, and totally dependent. Not only the case of the totally independent components - as they have written in their equation (6).

1.3. In their paper [1], their equation (6):

"$0 \le T + I + F \le 3$ for the completely independent components"

is partially wrong.

The correct one is only:

" $0 \le T + I + F \le 3$ "

which means that the summation T + I + F can be any number in [0, 3], with $T, I, F \in [0,1]$,

and consequently, it comprises all possibilities, i.e. the components may be:

either totally independent, or partially independent and partially dependent, or totally dependent.

The independence and dependence of the components depend on each application and on the experts. Practical examples will follow below.

It is obvious that if $T, I, F \in [0,1]$, then of course $0 \le T + I + F \le 3$, but we emphasized this double inequality to make sure the readers would not take for granted that $0 \le T + I + F \le 1$ as in the previous classical, fuzzy set and logic, and probability theories. Therefore $0 \le T + I + F \le 3$ is no restriction at all!

1.4. "We have deep doubts about the validity of this hypothesis of the components mutual independence from its practical applicability point of view" (p. 3).

Ironically, just the practical applications have inspired us to consider the independence of the components, and very simple ones, as these authors will see below.

Their misunderstanding is that these authors are considering only the UniVariate Truth-Value {truth that depends on a single parameter (or point of view, or random variable), which enforces the sum of the neutrosophic components to be up to 1, and they are totally dependent}. But, in our everyday life, we almost always deal with a MultiVariate Truth-Value {truth that depends on many independent parameters (or random variables, or sources of information), and the neutrosophic components may be: partially dependent and partially independent, or they may be totally independent}.

Practical Examples will follow below.

In general,

UniVariate Truth-Value $\neq$ *MultiVariate Truth-Value*.

Complete Independence of the neutrosophic components means that there are different (and independent) sources of information that provide estimations on each of T, I, and F respectively.

This happens in our everyday life: an item (person, object, event, action, proposition, theory, etc.) is evaluated from many points of view (or many random variables).

1.5 "According to the independence hypothesis, the event $T = 1$, $F = 1$ and $I = 1$ is allowed in the NST and in this case, the constraint (6) is fulfilled. Suppose T, F and I are the degrees of truth, false and indeterminacy, respectively (this is the notation used in the NST). Thus, if we deal with a complete truth ($T = 1$), then in compliance with the formal logic and common sense, the measure of false is 0 ($F = 0$) without any indeterminacy ($I = 0$)."

{We used the notations T, I, F because they are more descriptive for the Truth, Indeterminacy (or Neutrality), and Falsehood respectively, instead of the Greek letters μ, π, ν that are not descriptive and were used in their paper [1].}

Here it is their confusion, these authors consider only the UniVariate Truth-Value of a proposition.

As we showed before, from a point of view a proposition may be true, from other point of view it may be false, or may be neutral (or indeterminate).

When these authors talk about "common sense" they are automatically / stereotypically referring to a single source of information that provides information about all three neutrosophic components of a proposition (therefore the components are all totally dependent). When a single source provides information about an event, it knows and adjusts the sum of the components to be 1. See the below practical examples.

1.6 "It is interesting that the events $T = 1$, $F = 1$, $I = 1$ and $T = 0$, $F = 0$, $I = 0$ are interpreted in [9] as a paradox, and its definition is treated as a merit of the NST. In our opinion, generally, it seems to be more reliable to use theories, which have no paradoxes" (p. 4).

We agree to these authors with the fact that the theories that have paradoxes are not reliable, but the Neutrosophic Logic was not designed for the theories with paradoxes.

We only proved that a proposition (not theory) P, which is a paradox (totally true and false in the same time, and totally uncertain as well), can be represented in the Neutrosophic Logic as P(1, 1, 1), while in other classical or fuzzy and fuzzy extension set, logic or probability theories the proposition P cannot be represented, since the sum of the components is not allowed to be greater than 1.

2. Practical Examples of Independent or Dependent Neutrosophic Components

Let's see several practical examples, as these authors have required:

2.1 Practical Example 1

The following event E takes place:

E = {There is a street protest in Minneapolis}.

a. From the point of view of the Human Rights Activists the protest is positive, because people have the right to express their view, and consequently the CNN television station (reflecting the *left politics*) joys it. Let's say T(positiveness) = 0.8.
b. But, from the point view of the Police, the protest is negative, since the protesters are violent and destroy and burn houses and injure people; then the Fox News television station (reflecting the *right politics*) presents the negative side of the protests: violence, destruction, arson, chaos. Let's say F(negativeness) = 0.9.
c. Let's consider an unbiased (neutral) Media that reports on the event. This is the neutral source, it evaluates the event in general as, for example, I(positiveness and negativeness) = 0.4.

As seen, T + I + F > 1, and the three neutrosophic components T, I, and F are totally independently assessed, since the Human Right Activists, the Police, and Media are three different and independent entities.

The authors wrote: "Therefore, we can say, e.g., that the high degree of truth is obligatory accompanied by the low degrees of false and indeterminacy." (p. 3).

This is true ONLY for the UniVariate Truth-Value of the Classical and Fuzzy Logic. This is false for the MultiVariate Truth-Value of the Neutrosophic Logic as we previously proved with several elementary practical examples.

To contradict these authors, let's assume, in this practical example, that the Human Rights Activists reassess their evaluation of the event, and they reassign T(positiveness) = 0.7. But this has nothing to do with the Police or Media to reassess their evaluations of F(negativeness) and I(positiveness and negativeness) respectively. Since all three sources, and thus the T, I, F, are totally independent. If a neutrosophic component increases or decreases, it may have no effect on the other neutrosophic components.

This is a TriVariate Truth-Value, since the event E is evaluated by three independent parameters (from three different points of view): Human Rights Activists, Police, and Media.

As seen, it's not fair to analyze something from only one point of view (from only one parameter).

This is a TriVariate Truth-Value.

2.2 Practical Example 2

A murderer John Doe is being tried in the court of law for having committed a crime. There are three player parts in the court:

the Persecutor team, which presents the suspect in a negative way, for example F(Doe) = 0.9;

the Defense team, that presents the suspect in a positive way, for example T(Doe) = 0.4;

and the Jury, that is neutral, where $I(Doe) \in [0,1]$.

Herein, the Persecutor and the Defense are totally independent sources (since they are opposite). Therefore, T and F are totally independent.

But the Jury is dependent on the evidences provided by both the Persecutor and the Defense.

Therefore, the neutrosophic component I is totally dependent on both T and F.

Let's assume I = 0 means not guilty, I = 1 means guilty, while $I \in (0,1)$ means a hung-jury (i.e. some jurors say he is guilty, while others say he is not guilty) or unable to reach a verdict.

This is a TriVariate Truth-Value.

2.3 Practical Example 3 that refutes their assertation

Proposition: G = George is a good student.

George is evaluated by three different independent professors.

The math prof: George is excellent in mathematics and he gets only A's. Hence T(G) = 1.

The sport prof: George is the worst athlete in the team since he cannot run, cannot play baseball. Hence F(G) = 1.

The literature prof: I am totally uncertain about George's ability to write a literary composition since he never turned in any of them. Hence I(G) = 1.

Therefore we got G(1, 1, 1).

This is a TriVariate Truth-Value.

2.4 Example 4 that refutes their assertation

A paradox is a proposition that is true and false at the same time (hence T = F = 1), and completely unclear/indeterminate (hence I = 1).

2.5 Example 5 from mathematics that refutes their assertation

Assume the proposition M is "1 + 1 = 10".

If the base of numeration is 2, then proposition M is true: T(M) = 1.

If the base of numeration is 10, the proposition M is false: F(M) = 1.

This is a proposition that is totally true and totally false, without being a paradox.

Herein one has a BiVariate Truth-Value (i.e. with respect to two parameters: Base 2, and respectively Base 10).

If the base of numeration is unknown (let's denote it by *b*), then the truth-value of M is also unknown (indeterminate): I(M) = 1.

Now one has a TriVariate Truth-Value (i.e. with respect to three parameters: Base 2, Base 5, and unknown Base *b*).

2.6 Example 6 of independent and dependent neutrosophic components

There will be a football match between Poland and Belarus. For each country there are three possibilities: to win, to draw, or to lose. Therefore, as in neutrosophic theories.

a) Totally independent neutrosophic components

Asking a Polish person what is Poland's chance to win, he may say T(Poland) = 0.8.

But a Belarusian person may say that Belarus will win, let's say F(Poland) = 0.7.

Another person, from another country (Romania), may answer that it is a chance of a tie game: I(Poland) = 0.4.

It is supposed that the three sources, the Polish, Belarusian, and Romanian persons do not communicate nor know the evaluations of the others. They are totally independent and consequently are the components T, I, F.

Herein there is a TriVariate Truth-Value.

b) Totally Dependent Neutrosophic Components

Let's assume that a Polish mathematician evaluates all three possibilities of Poland. Being a mathematician, he knows that the sum of the component has to be 1, as in the classical and fuzzy set theories, logic, or probability.

He then may say: T = 0.7, F = 0.1, I = 0.2.

The neutrosophic components are totally dependent, since all three depend on a single source.

Herein there is a UniVariate Truth-Value.

c) Partially Dependent and Partially Independent Neutrosophic Components

Another situation. Assume that a scientist George has to evaluate both chances of Poland, to win or to lose.

If he choses $T = 0.6$, for example, he knows that $0 \le F \le 1 - 0.6 = 0.4$. Suppose he takes $F = 0.3$.

A second source Marcel has to evaluate the possibility of tie-game, without nothing anything about George's. Let's suppose that he says: $I = 0.8$.

In this case, T and F are totally dependent of each other, while I is totally independent from both T and F. Herein $0 \le T + I + F \le 2$.

Herein there is a BiVariate Truth-Value.

3. Neutrosophic Overset/Underset/Offset

"In our opinion, the most daring theory was proposed in [*18]. This theory allows negative and greater than 1 values of membership degrees. There are some basic definitions introduced in [*18], but here, we analyze only the most general one:

Definition 4. For T(x), I(x) and F(x) being the degrees of truth, indeterminacy and false, respectively, a Single-Valued Neutrosophic Offset A is defined as follows:

A = {(x, <T(x), I(x), F(x)>), x U},

such that there exist some elements in A that have at least one neutrosophic component that is > 1, and at least another neutrosophic component that is < 0.

For example: A = {(x1, <1.3, 0.3, 0.2>), (x2, <0.1, 0.4, -0.8>)}, since T(x1) = 1.3 > 1 and F(x2) = -0.8 < 0." (p. 6)

{We took the liberty of updating the reference citation to be adjusted to our paper. Instead of [16] as in these authors' reference, we wrote [*18]. See more papers on Neutrosophic Overset/Underset/Offset: [27-29].}

These neutrosophic overset (degree > 1), neutrosophic underset (degree < 0), and neutrosophic offset (some degree > 1 and other degree < 0) were well understood by the prestigious Cornell University arXiv (New York City) mainstream Archives that approved our book:

https://arxiv.org/ftp/arxiv/papers/1607/1607.00234.pdf

and by the mainstream French Hal Archives as well:

https://hal.archives-ouvertes.fr/hal-01340830 .

These concepts were inspired from our real life [*18, 27, 28, 29].

The authors continue with the below citation from our book:

"There is a crucial example in [*18], which clarifies the author's reasoning that we critically analyze: "In a given company a full-time employer works 40 h per week. Let's consider the last week period. Helen worked part-time, only 30 h, and the other 10 h she was absent without payment; hence, her membership degree was 30/40 = 0.75 < 1.

John worked full-time, 40 h, so he had the membership degree 40/40 = 1, with respect to this company. But George worked overtime 5 h, so his membership degree was (40+5)/40 = 45/40 = 1.125 > 1. Thus, we need to make distinction between employees who work overtime, and those who work full-time or part-time. That's why we need to associate a degree of membership strictly greater than 1 to the overtime workers." (p. 6)

The above was our practical example.

The authors reject it:

"The crucial drawback of this reasoning is the lack of the clear definition of fuzzy classes, which memberships are estimated. We can see here two distinct fuzzy classes: the class of employees working at least no more than 40 h a week and the class of employees that works more than 40 h. The first class is presented by the membership function rising from 0 to 1 in the interval [0, 40] of worked hours and equal to 0 if the sum of worked hours is greater than 40. The second class is defined by the membership function increasing from 0 to 1 in the interval of worked hours from 40 to Hmax, where Hmax is the maximal allowed by government (and trade unions) value of worked hours. We can see that such an obvious reasoning does not allow membership degrees greater than 1. The incorrect reasoning of the author of [*18] is also based on the implicit mechanical conjunction of two different classes with not intersected supports. Of course, such a conjunction can be made, but the resulting fuzzy class and the corresponding membership function should have a new sense reflecting a synthetic nature of a new class. In the considered case, we can introduce the class of "hard working employees" with the membership function rising from 0 to 1 in the interval [0, Hmax]."

There are people who invent theories and then try to squeeze the reality into them.

But, we did the opposite, we started from the real-world problems (over-work, negative work) and tried to make the theories that model / approximate the reality as accurate as possible. Late on, we improved our models little by little.

First, we do not work with fuzzy classes, but with a neutrosophic approach.

Also, we see no reason to make two classes where the membership, in both of them, starts from 0 and ends to 1. What about if one gets the same value, for example the membership degree T = 0.3 in both classes [or in the three classes, as they added one more similar class for the negative membership]? It's a confusion. On the other hand, these two classes cannot catch the employees with negative membership (those who produce damages to the company, T < 0).

These authors belong to the category of people that try to squeeze the reality (the membership degree of overtime workers which overpasses 1, or T > 1) to the narrow classical set theory, where the membership degree has to be T ≤ 1. The classical set theory is not written in stone, so we may enlarge it if the reality requires it.

When Zadeh founded in 1965 the Fuzzy Set and allowed the membership degree to be any number between 0 and 1 (not only 0 or 1 as in classical Set Theory) he was criticized at that time by several scientists (as he told me in 2003 at an international conference at the University of Berkeley, California, where we met). But he prevailed, because in the real world there exist many partial memberships.

About the membership degrees that are outside of the interval [0, 1], it is normal that somebody who over-works (works overtime) to have an over-membership (i.e., membership degree above 1) to be distinguished from those who do not work overtime (whose membership degree is between 0 and 1).

Our example of negative employee who deserves a negative membership (T < 0), is cited by these authors:

"Let us turn to the example: "Yet, Richard, who was also hired as a full-time, not only did not come to work last week at all (0 worked hours), but he produced, by accidentally starting a devastating fire, much damage to the company, which was estimated at a value half of his salary (i.e., as he would have gotten for working 20 h that week). Therefore, his membership degree has to be less that Jane's (since Jane produced no damage). Whence, Richard's degree of membership, with respect to this company, was -20/40 = -0.50 < 0." " (p. 6)

The authors continue:

"As we are analyzing only the last week, we can see that Richard does not belong to any of the classes described above. It is a member of a practically unlimited class of those who do not work for a given company. We can significantly narrow this class by considering only those people who, by their actions or inaction, cause damage to the company (the most harmful are the top managers of competing firms). This way we can estimate the maximum damage Dmax (it does not matter in money or equivalent worked hours), which can be inflicted on the company by an external detractor. Thus, the class of external (nonworking for the company) people who bring company damages can be presented by the membership function varying from 0 to 1 in the interval of damages [0, Dmax]. There is no place for any negative membership degree."

These authors did not read/understand exactly: Richard is indeed a full-time employee, he works for the company, as we have written into our book: "Richard, who was also hired as a full-time" it is certainly an employee. The authors make a false statement for Richard as "nonworking for the company".

Even so, it is not clear, why did they make a third class varying from 0 to 1 for the negative employees? As such, we'd like to return the ancient Occam's wisdom back to themselves: "Entities should not be multiplied unnecessarily."

If you have a negative person in your group, for example, which creates only problems to the group, you cannot assign him a membership degree equals to zero (as for people that do neither positive nor negative things to the group), but you should assign him a negative membership degree. It is very logical this way.

A negative employee (that who does only damages to the company) has to have a negative membership (i.e., membership degree below 0) in order to distinguish him from the positive employees (those whose membership degree is above 0).

We see no reason to complicate the problem by creating three classes of membership degrees in order to avoid membership degree values greater than 1 or less than 0, instead of keeping a single class, but enlarging it to the left-hand side of 0 and respectively to the right-hand side of 1.

Because neutrosophic set has 3 components, they would need 9 classes, not talking of the refined neutrosophic set, that may have any number $2 \leq n \leq \infty$ of refined neutrosophic components, therefore they would need 3n classes! Better they should think out of box (in this case the box is the interval [0, 1]).

4. Applicability

The authors wrote: "there is no need for such somewhat artificial and heuristic theories as the Neutrosophic, Pythagorean and Spherical sets and their derivatives" (p. 5).

We disagree. The neutrosophic theories are not artificial, they started from our real-world practicability, where there are so many neutrosophic triplets (<A>, <neutA>, <antiA), where <A> is an item (concept, proposition, idea, etc.), formed by two opposites <A> and <antiA>, together with their neutrality (indeterminacy) <neutA>.

For examples: (friend, neutral, enemy), (positive particle, neutral particle, negative particle), (masculine, transgender, feminine), (true, indeterminate, false), (win, tie-game, defeat), (yes, uncertain, no), (take a decision, pending, not taking a decision), etc.

The neutrosophic theories have many applications [25] in various fields such as: Artificial Intelligence, Information Systems, Computer Science, Cybernetics, Theory Methods, Mathematical Algebraic Structures, Applied Mathematics, Automation, Control Systems, Big Data, Engineering, Electrical, Electronic, Philosophy, Social Science, Psychology, Biology, Genetics, Biomedical, Engineering, Medical Informatics, Operational Research, Management Science, Imaging Science, Photographic Technology, Instruments, Instrumentation, Physics, Chemistry, Optics, Economics, Mechanics, Neurosciences, Radiology Nuclear, Medicine, Medical Imaging, Interdisciplinary Applications, Multidisciplinary Sciences, etc. and there were published over 2,000 papers, books, conference presentations, MSc and PhD theses by researchers from 82 countries around the world.

With respect to what the neutrosophic theories brought new, we invite these authors to read our 2019 paper, so we do not repeat the things [26], whose weblink is provided.

Rather, these authors' transformation/substitution of the Atanassov-Intuitionistic Fuzzy Set (A-IFS) into the Dempster-Shafer Theory (DST) framework is artificial, since their transformation is not quite equivalent with the A-IFS, while practically their transformation is useless because of the very large intervals they use that supposed to catch the solution.

5. Publications

They say that the "caution of editors and reviewers of solid old journals is not caused by their conservatism at all, but by the desire to see, in addition to formal definitions of these theories and numerous theorems, the solution of real methodological and practical problems" (p. 1).

In general, in any field of knowledge, when a Theory1 is generalized by the Theory2, the proponents of Theory1 are reluctant to publish and even to admit Theory2, and the first reason is the rivalry between theories, the conservatism is only an excuse. But each theory has its flavor.

The authors are less informed, since in the last years there have been books published by prestigious publishing houses such as Springer [19, 21], Elsevier [20], IGI Global [22-24] (we cite the last ones), etc. and many high rank journals by Springer, Elsevier, IOS Press, Tayler & Francis, MDPI, Hindawi, Emerald Publishing, IGI Global, World Academy of Science Engineering and Technology, IEEE, Wiley, etc. have published papers on the neutrosophic environment, such as: Complex & Intelligent Systems, Cognitive Computation, Artificial Intelligence Review, International Journal of Fuzzy Systems, Evolving Systems, Complex & Intelligent Systems, Soft Computing, Journal of Machine Learning & Cybernetics, Multiple-Valued Logic, Design Automation for Embedded Systems, Granular Computing, Neural Computing and Applications, Journal of Systems Architecture, Applied Soft Computing, Measurement, Symmetry, Mathematics, Information, Axioms, Entropy, Computational and Applied Mathematics, BMC Medical Research Methodology, International Journal of Aerospace and Mechanical, Cognitive Systems Research, Theoretical and Applied Climatology, Journal of Metrology Society of India, Journal of King Saud University – Science, Journal of Intelligent & Fuzzy Systems, IEEE Access, Expert Systems, etc.

Further on, they will see in this paper many solutions using the neutrosophic theories to practical problems.

6. Critics of the DST

These authors [1] want to destroy the fuzzy extension theories just to promote the Dempster-Shafer Theory (DST) that they support, but from the beginning they are going on an uncertain way, since DST is a flawed theory which gives many counter-intuitive results [2-8; weblinks provided; download the papers and respond to the DST problems], as we'll show below.

They assert that all fuzzy extension theories can be substituted by the DST, which is not true.

6.1 The DST fails in the Zadeh's Counter-Example

Zadeh's Counter-Example [2], as know by all fusion community, is the following:

Two doctors examine a patient and agree that he suffers from either meningitis (M), contusion (C), or brain tumor (T). Thus $\Theta = \{M, C, T\}$ is the frame of discernment. Assume that the doctors agree in their low expectation of a tumor, but disagree in likely cause and provide the following diagnosis:

$m_1(M) = 0.99$, $m_1(T) = 0.01$, and $m_2(C) = 0.99$ $m_2(T) = 0.01$, where $m_1(.)$ represents the diagnoses provided by the first doctor, while $m_2(.)$ the diagnoses by the second doctor. If we combine the two basic belief functions using the DST (first doing the conjunctive rule, then the Dempster's rule of combination), one gets the unexpected conclusion:

$$m(T) = \frac{0.0001}{1 - 0.0099 - 0.0099 - 0.9801} = 1$$

which means that the patient suffers <u>with certainty</u> from brain tumor, which is wrong.

Zadeh [2] has clearly written down: "there is a serious flaw in Dempster's rule which restricts its use in many applications".

Similarly, P. M. Williams questioned the validity of Dempster's Rule [31].

6.2 The A-IFS gives a better solution to Zadeh's Counter-Example than DST

After criticizing Atanassov's Intuitionistic Fuzzy Set (A-IFS), the authors proposed "redefining the A-IFS in the framework of the more general Dempster–Shafer theory of evidence (DST)" (p. 2).

Okay, then let's set and analyze the Zadeh's Counter-Example in the frame of the A-IFS, and we show that A-IFS gives better result than DST.

Let:

D_1 = {M(0.99, 0), C(0, 0), T(0.01, 0)},

D_2 = {M(0, 0), C(0.99, 0), T(0.01, 0)},

where D_1 represents the diagnoses provided by the first doctor, i.e.

M(0.99, 0) means that the degree of membership (truth) of the patient to have meningitis is 0.99, and the degree of nonmembership (falsehood) of the patient not to have meningitis is 0;

And similarly for the other diseases.

And where D_2 represents the diagnoses provided by the second doctor.

Let's use the A-IFS min/max intersection operator ($\wedge_{A-IFS}$) for the two doctors' diagnoses:

$D_1 \wedge_{A-IFS} D_2 = \{(0.99,\ 0) \wedge_{A-IFS} (0,\ 0), (0,\ 0) \wedge_{A-IFS} (0.99,\ 0), (0.01,\ 0) \wedge_{A-IFS} (0.01,\ 0)\} =$

$\{(\min\{0.99, 0\}, \max\{0, 0\}), (\min\{0, 0.99\}, \max\{0, 0\}), (\min\{0.01, 0.01\}, \max\{0, 0\})\} =$

$\{(0,0), (0,0), (0.01,0)\} \equiv \{M(0,0), C(0,0), T(0.01,0)\}$.

A-IFS shows that the chance of the patient of having tumor is 0.01, which is more realistic with respect to the chance of tumor of the patient, than DST's.

More counter-examples to the Dempster's rule have been published in the literature [3-8].

After these failures of the DST, new theories have been proposed, such as TBM, DSmT [9], etc. and many quantitative and qualitative fusion rules [10-12] in order to overcome the Dempster's rule counter-intuitive results.

7. Conversion from A-IFS to DST

The authors [1] propose the conversion from the framework of the A-IFS to the DST in the following way (pp. 7-8).

Let *U* be a universe of discourse, and:

$B_{A-IFS} = \{(x, <T(x), F(x)>), T(x), F(x) \in [0,1], 0 \le T(x) + F(x) \le 1, x \in U\}$ be a non-empty subset of it, that is called an Atanassov-Intuitionistic Fuzzy Set (A-IFS).

Let's $x(T(x), F(x))$ be a generic element that belongs to B_{A-IFS}, with $T(x), F(x) \in [0,1], 0 \le T(x) + F(x) \le 1,$ whence the indeterminacy (hesitancy) is $I(x) = 1 - T(x) - F(x) \in [0,1]$.

From the fusion theory, and especially from Dempster-Shafer Theory, the Basic Believe Assignment (*bba*), denoted by *m(.)*, is defined as:

$m: 2^{B_{A-IFS}} \to [0,1]$, such that $m(\phi) = 0,$ where ϕ is the empty-set, and $\sum_{x \in 2^{B_{A-IFS}}} m(x) = 1.$

And the Believe Function *Bel* and the Plausible Function *Pl* are defined as follows:

$$Bel: 2^{B_{A-IFS}} \to [0,1], Bel(x) = \sum_{y \in 2^{B_{A-IFS}}, y \subseteq x} m(y),$$

$$Pl: 2^{B_{A-IFS}} \to [0,1], Pl(x) = \sum_{y \in 2^{B_{A-IFS}}, y \cap x \ne \phi} m(y).$$

Afterwards, they approximate the above B_{A-IFS} to an interval-valued fuzzy set (IVFS), denoted as $C_{IVFS} = \{(x, [Bel(x), Pl(x)]), x \in 2^{B_{A-IFS}}\} = \{(x, [T(x), T(x) + I(x)]); T(x), I(x) \in [0,1], T(x) + I(x) \le 1; x \in 2^{B_{A-IFS}}\}$ which is not equal to B_{A-IFS}. Their approach is similar to that of a Vague Set.

The interval $[Bel(x), Pl(x)] \square BI(x)$ was called Believe Interval (BI).

Mathematically, this is beautiful, but practically it is useless. When converting from an approach to another one, it is supposed to diminish the indeterminacy (hesitancy) and get better results. But, it is not the case. The higher is indeterminacy (I) the larger is the believe interval that suppose to catch the solution.

As counter-examples, let's consider the following A-IFS triplets (their components' sums are equal to 1):

(T, I, F) = (0.2, 0.5, 0.3) produces the BI = [0.2, 0.7];

(T, I, F) = (0.3, 0.6, 0.1) produces the BI = [0.3, 0.9];

(T, I, F) = (0.2, 0.8, 0.0) produces the BI = [0.2, 1], etc.

There are pretty large intervals to deal with, that make the result vaguer. To say that the solution lies inside of the interval, for example [0.2, 1], means almost nothing towards solving the problem whose solution is always between 0 and 1.

Another drawback is the fact that computing with intervals is more complicated than computing with crisp numbers.

8. Differences between A-IFS and NST

"The conceptual difference between the NST and the A - IFS is the introduction of the hypothesis of complete independence of the components" (p. 3).

By NST they meant Neutrosophic Set Theory.

This is not the only difference, another big distinction is with respect to the construction of the neutrosophic operators (negation, intersection, union, implication, equivalence, etc.), since within the frame of neutrosophic environment the Indeterminacy (I) is getting full consideration and "I" is involved in the neutrosophic operators' formulas, while in the A-IFS operators the indeterminacy (called hesitancy) is completely ignored and does not appear in none of their operators' formulas.

Even for the case when the sum of the neutrosophic components is equal to 1, as occurs for the A-IFS components, the results after applying the neutrosophic operators are different from those obtained by the A-IFS operators.

A simple example is below, for the neutrosophic conjunction ($\wedge_{NS}$) vs. A-IFS conjunction ($\wedge_{A-IFS}$).

Let's denote by $\wedge_{FS}$ the fuzzy set t-norm, and by $\vee_{FS}$ the fuzzy set co-norm.

Let (T_1, I_1, F_1) and (T_2, I_2, F_2) be two neutrosophic triplets, where $T_1, I_1, F_1, T_2, I_2, F_2 \in [0,1]$, and there is no restriction on the sums of the two neutrosophic triplets.

Then, the neutrosophic conjunction is:

$$(T_1, I_1, F_1) \wedge_{NS} (T_2, I_2, F_2) = (T_1 \wedge_{FS} T_2, I_1 \vee_{FS} I_2, F_1 \vee_{FS} F_2),$$

where we clearly see that the indeterminacy/hesitancy (I) is involved in the above formula on the right-hand side: $I_1 \vee_{FS} I_2$.

But, for the A-IFS conjunction formula the indeterminacy/hesitancy is completely ignored, which makes the operator less accurate. If $T_1 + F_1 \leq 1$, $T_2 + F_2 \leq 1$, and $T_1 + I_1 + F_1 = 1, T_2 + I_2 + F_2 = 1$, in order to comply with the A-IFS constrains, one gets:

$$(T_1, F_1) \wedge_{A-IFS} (T_2, F_2) = (T_1 \wedge_{FS} T_2, F_1 \vee_{FS} F_2),$$

unfortunately, no indeterminacy/hesitancy (*I*) is involved into the formula.

Even when the sum pf the neutrosophic components is 1, as in A-IFS, the results of the neutrosophic and respectively A-IFS operators are different. Let's see this numerical example:

$$(0.6, 0.1, 0.3) \wedge_{NS} (0.5, 0.4, 0.1) = (\min\{0.6, 0.5\}, \max\{0.1, 0.4\}, \max\{0.3, 0.1\}) = (0.5, 0.4, 0.3)$$

while

$$(0.6, 0.3) \wedge_{A-IFS} (0.5, 0.1) = (\min\{0.6, 0.5\}, \max\{0.3, 0.1\}) = (0.5, 0.3)$$

whence the indeterminacy/hesitancy = 1 – 0.5 – 0.3 = 0.2 $\neq$ 0.4.

In this case these authors agree with us:

"In the case of mutually dependent components, the main constraint $0 \leq T + F + I \leq 1$ in the NST seems to be more fruitful than that in the A - IFS ($T + F + I = 1$). This was quickly discovered and the so-called Picture fuzzy sets theory (PFS) was proposed" (p. 4).

Thanks to the indeterminacy (I), that plays an important role in the neutrosophic environment and in the real world that is full of indeterminate (vague, unclear, conflicting, incomplete, etc.) data, more fields were developed within the field of neutrosophy, such as: Neutrosophic Algebraic Structures (based on neutrosophic numbers of the form $a + bI$, where I = literal indeterminacy, and a, *b* are real or complex numbers), Neutrosophic Statistics (using classical statistical procedures and inference methods but on indeterminate data), Neutrosophic Probability (chance of an event to occur, indeterminate-chance of the event to occurring or not, and chance of the event not to occur), etc.

Therefore, there are many distinctions between the neutrosophic theories and the A-IFS.

9. Conclusions (authors also should add some future directions points related to her/his research)

Many practical applications have been given in this paper about the independence and dependence of the neutrosophic components in our every-day life.

The misunderstanding of some authors, with respect to the partial and total independence of the neutrosophic components, is that in the framework of the neutrosophic theories we deal with a *MultiVariate Truth-Value* (truth upon many independent random variables) as in our real-life world, not with a *UniVariate Truth-Value* (truth upon only one random variable) as they believe.

Similarly with respect to the degrees of memberships greater than 1 or less than 0, which are now mainstream subjects. The neutrosophic theories were inspired from the practical applications..

Conflicts of Interest: The authors declare no conflict of interest.

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Generalized Euclid Measures Based on Generalized Set Valued Neutrosophic Quadruple Numbers and Multi Criteria Decision Making Applications

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Abstract: In this article, we define firstly generalized Euclid distance measure and generalized Euclid similarity measure based on generalized set valued neutrosophic quadruple numbers. Also, we show that generalized Euclid distance measure and generalized Euclid similarity measure satisfy the distance measure conditions and similarity measure conditions, respectively. Furthermore, we define a score function for generalized Euclid similarity measure. In addition, we generalize an algorithm, for single valued neutrosophic set, based on generalized Euclid similarity measure and generalized set valued neutrosophic quadruple numbers. Also, we give a multi criteria decision making applications for this generalized algorithm. This application based on patients, diseases, drugs and this application is different from previous applications because of generalized Euclid similarity measure and generalized set valued neutrosophic quadruple numbers. Furthermore, this application has different result according to some previous similarity measure. Thus, this application can used for covid-19 treatment due to structure of generalized Euclid similarity measure and generalized set valued neutrosophic quadruple numbers

Keywords: Distance measure, similarity measure, Euclid measures, generalized set valued neutrosophic quadruple numbers, generalized Euclid distance measure, generalized Euclid similarity measure

1 Introduction

Many uncertainties arise in daily life. Most of the time, Aristotle logic (classical logic) is insufficient to explain these uncertainties mathematically. Fuzzy logic [1] and intuitionistic fuzzy logic [2] were defined to deal with uncertainties. However, in these structures, membership functions were defined as dependent on each other. Finally, neutrosophic logic and sets [3], (T, I, F) membership functions are independent of each other, were defined by Smarandache. Thus, uncertainties are taken into calculations more precisely. Due to this advantage, many studies have been carried out in both algebra and application areas by using neutrosophic sets [4-11]. In particular, decision making applications have found more application areas with the definition of neutrosophic sets and more precise results have been obtained. Therefore, many decision-making applications have been obtained. Recently, Hashmi et al. studied multi-criteria decision-making in medical diagnosis for m-Polar neutrosophic topology [12]; Khalil et al. introduced decision making applications for the single-valued neutrosophic fuzzy set and the soft set [13]; Olgun et al. studied neutrosophic logic on the decision tree [14]. Also, similarity measures defined for neutrosophic sets have an important place in these applications, and these similarity measures have been used in many studies [15-18]. Recently, Mukherjee et al. obtained several similarity measures for neutrosophic soft sets [19]; Saqlain et al. studied tangent similarity measure of single valued neutrosophic hypersoft sets [20]; Şahin and Kargın introduced decision making applications in professional proficiencies based on new similarity measure for single valued neutrosophic sets [21]; Saqlain et al. studied distance and similarity measures for neutrosophic HyperSoft Set with construction of NHSS-TOPSIS and applications [46]. Also, hybrid of neutrosophic numbers and methods have very important place in decision making applications [40-43]. Recently, Abdel-Monem and Gawad studied a hybrid model using MCDM Methods and bipolar neutrosophic sets for select optimal wind turbine: case study in Egypt [44]; Fahmi obtained group decision based on trapezoidal neutrosophic Dombi fuzzy hybrid operator [45].

Neutrosophic quadruple sets [22], which are a generalized form of neutrosophic sets, were defined by Smarandache in 2015. Unlike neutrosophic sets, neutrosophic quadruple set contain a known part and an unknown part. However, known membership functions (T, I, F) are located in sets of neutrosophic quadruple set. A neutrosophic quadruple set is denoted by

$$\{(k, lT, mI, nF): k, l, m, n \in \mathbb{R} \text{ or } \mathbb{C}\}$$

Here, k is referred to as the known part, (lT, mI, nF) as the unknown part. With the help of this definition, many algebraic structures are reconsidered in the neutrosophic quadruple theory [23-32].

Also, set valued neutrosophic quadruple sets [33] and generalized set valued neutrosophic quadruple sets [34] have been defined in order to use neutrosophic quadruple sets in application studies. A generalized set valued neutrosophic quadruple set denoted by

$$G_{s_i} = \{(K_{s_i}, L_{s_i}T_{s_i}, M_{s_i}I_{s_i}, N_{s_i}F_{s_i}): K_{s_i}, L_{s_i}, M_{s_i}, N_{s_i} \in \mathrm{P(X)};\ \mathrm{i} = 1, 2, 3, \dots, \mathrm{n}\}.$$

Where T_i, I_i and F_i have their usual neutrosophic logic; K_{s_i} is called the known part and $(L_{s_i}T_{s_i}, M_{s_i}I_{s_i}, N_{s_i}F_{s_i})$ is called the unknown part. Thanks to this definition, neutrosophic quadruple sets have become available in the field of application. Most importantly, this definition, which has a more general structure than neutrosophic sets, will find more application areas and will give more objective results to many problems with the help of the known part and unknown part. Recently, Kandasamy et al. studied neutrosophic quadruple algebraic codes over Z2 [35]; Ma et al. obtained neutrosophic quadruple rings [36]; Mohseni et al. introduced commutative neutrosophic quadruple ideals [37]; Rezaei et al. studied neutrosophic quadruple a-ideals [38]; Kargın et al. obtained generalized Hamming similarity measure based on neutrosophic quadruple numbers [39]; Şahin et al. studied Hausdorff Measures on generalized set valued neutrosophic quadruple numbers [40].

While the treatment of many diseases is known in the field of medicine, there are still diseases whose treatment is not fully known. It is also clear that new drugs can be found for the treatment of unknown diseases based on the treatment of known diseases. However, treating patients struggling with more than one disease can become even more complex. Because, in addition to the unknown treatment, they will have separate medications for other diseases. It is clear that in solving such problems there is a need for a structure in which the known part is the unknown part and (T, I, F) known neutrosophic membership functions. Since each known disease will have separate medications and it will be investigated which results (true, indeterminate, false) these drugs will give in unknown diseases, a structure such as $(L_{s_i}T_{s_i}, M_{s_i}I_{s_i}, N_{s_i}F_{s_i})$, containing both cluster and T, I, F, will be needed. For this reason, using generalized Euclid measures based on generalized set valued neutrosophic quadruple numbers in solving such problems can give better results. So, we give a multi criteria decision making application based on generalized algorithm for solving such problems. Also, this application is different from previous applications because of generalized Euclid similarity measure and generalized set valued neutrosophic quadruple numbers. Furthermore, this application has different result according to some previous similarity measure. Thus, this application can used for covid-19 treatment due to structure of generalized Euclid similarity measure and generalized set valued neutrosophic quadruple numbers.

In this paper; in 2. Section, we give some information for neutrosophic sets, some similarity measures, generalized set valued neutrosophic quadruple sets and numbers, In 3. Section, we define firstly generalized Euclid distance measure and generalized Euclid similarity measure based on generalized set valued neutrosophic quadruple numbers. Also, we show that generalized Euclid distance measure and generalized Euclid similarity measure satisfy the distance measure conditions and similarity measure conditions, respectively. Furthermore, we define a score function for generalized Euclid similarity measure. Then we give examples for generalized Euclid distance measure, generalized Euclid similarity measure and score function. In 4. Section, we define a generalized algorithm and multi criteria decision making application based on generalized Euclid similarity measure and generalized set valued neutrosophic quadruple numbers. In 5. Section, we give conclusions.

2 Preliminaries

In this chapter, we give some information for neutrosophic sets, some similarity measures, generalized set valued neutrosophic quadruple sets and numbers.

Definition 2.1: [3] Let E be the universal set. For $\forall x \in E$,

$$0^{-} \leq T_A(x) + I_A(x) + F_A(x) \leq 3^{+}$$

by the help of the functions $T_A: E \rightarrow\ ]^{-}0, 1^{+}[$, $I_A: E \rightarrow\ ]^{-}0, 1^{+}[$ and $F_A: E \rightarrow 0\ ]^{-}0, 1^{+}[$ a neutrosophic set A on E is defined by

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in E\}.$$

Here, $T_A(x), I_A(x)$ and $F_A(x)$ are the degrees of truth, indeterminacy and falsity of $x \in E$ respectively. Also, for $\varepsilon > 0$, $^{-}0 = 0 - \varepsilon$ and $1^{+} = 1 + \varepsilon$.

Definition 2.2: [4] Let E be the universal set. For $\forall x \in E$,

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$$

using the functions $T_A: E \rightarrow [0,1]$, $I_A: E \rightarrow [0,1]$ and $F_A: E \rightarrow [0,1]$, a single-valued neutrosophic set A on E is defined by

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in E\} .$$

Here, $T_A(x), I_A(x)$ and $F_A(x)$ are the degrees of truth, indeterminacy and falsity of $x \in E$ respectively.

Also, a single valued neutrosophic number is denoted by

$$B = \langle T_B(x), I_B(x), F_B(x)\rangle.$$

Definition 2.3: [17] Let E be an universal set,

$A_1 = \{x_i, <T_{A_1}(x_i), I_{A_1}(x_i), F_{A_1}(x_i)>: x_i \in E\}$ and

$A_2 = \{x_i, <T_{A_2}(x_i), I_{A_2}(x_i), F_{A_2}(x_i)>: x_i \in E\}$ be two single – valued neutrosophic sets. The Euclid similarity measure between A_1 and A_2, is defined by

$$S_E(A_1, A_2) = 1 - \sqrt{\sum_{i=1}^{n} \frac{\left(T_{A_1}(x_i) - T_{A_2}(x_i)\right)^2 + \left(I_{A_1}(x_i) - I_{A_2}(x_i)\right)^2 + \left(F_{A_1}(x_i) - F_{A_2}(x_i)\right)^2}{3}}.$$

Also, The Euclid distance measure between A_1 and A_2, is defined by

$$d_E(A_1, A_2) = \sqrt{\sum_{i=1}^{n} \frac{\left(T_{A_1}(x_i) - T_{A_2}(x_i)\right)^2 + \left(I_{A_1}(x_i) - I_{A_2}(x_i)\right)^2 + \left(F_{A_1}(x_i) - F_{A_2}(x_i)\right)^2}{3}}.$$

Definition 2.4: [17] Let E be an universal set,

$A_1 = \{x_i, <T_{A_1}(x_i), I_{A_1}(x_i), F_{A_1}(x_i)>: x_i \in E\}$ and

$A_2 = \{x_i, <T_{A_2}(x_i), I_{A_2}(x_i), F_{A_2}(x_i)>: x_i \in E\}$ be two single – valued neutrosophic sets. The Hamming similarity measure between A_1 and A_2, is defined by

$$S_{h_d} = (A_1, A_2) = 1 - \frac{1}{3}[\sum_{i=1}^{n} |T_{A_1}(x_j) - T_{A_2}(x_j)| + |I_{A_1}(x_j) - I_{A_2}(x_j)| + |F_{A_1}(x_j) - F_{A_2}(x_j)|]$$

Theorem 2.5: [17] Let X_1, X_2 and X_3 be three single – valued neutrosophic sets, d_E be a distance measure. Then the following properties hold.

i. $0 \leq d_E(X_1, X_2) \leq 1$
ii. $X_1 = X_2$ if and only if $d_E(X_1, X_2) = 0$
iii. $d_E(X_1, X_2) = d_E(X_2, X_1)$
iv. If $X_1 \subseteq X_2 \subseteq X_3$, then $d_E(X_1, X_2) \leq d_E(X_1, X_3)$ and $d_E(X_2, X_3) \leq d_E(X_1, X_3)$.

Theorem 2.6: [17] Let X_1, X_2 and X_3 be three single – valued neutrosophic sets, S be a similarity measure. Then the following properties hold.

i. $0 \leq S(A_1, A_2) \leq 1$
ii. $S(A_1, A_2) = 1 \Leftrightarrow A_1 = A_2$
iii. $S(A_1, A_2) = S(A_2, A_1)$
iv. If $A_1 \subseteq A_2 \subseteq A_3 \in E$, then $(A_1, A_3) \leq S(A_1, A_2)$ and $S(A_1, A_3) \leq S(A_2, A_3)$.

Definition 2.7: [5] Let $A_1 = <T_1, I_1, F_1>$ and $A_2 = <T_2, I_2, F_2>$ be two single-valued neutrosophic numbers. Let's define the measure of similarity between A_1 and A_2 as follows

$$S_N(A_1, A_2) = 1\text{-}2/3.\ \{ \frac{min\{\sqrt{3(T_1-T_2)^2 + (I_1-I_2)^2},\ |2(T_1-T_2)-(I_1-I_2)|/3\ \}}{\{max\{\sqrt{3(T_1-T_2)^2 + (I_1-I_2)^2},\ |2(T_1-T_2)-(I_1-I_2)|/3\}/2\}+1}$$

$$+ \frac{min\{\sqrt{3(T_1-T_2)^2 + (F_1-F_2)^2},\ |2(T_1-T_2)-(F_1-F_2)|/3\ \}}{\{max\{\sqrt{3(T_1-T_2)^2 + (F_1-F_2)^2},\ |2(T_1-T_2)-(F_1-F_2)|/3\}/2\}+1}$$

$$+ \frac{min\{\sqrt{2(T_1-T_2)^2 + (I_1-I_2)^2 + (F_1-F_2)^2},\ |3(T_1-T_2)-(I_1-I_2)-(F_1-F_2)|/5\ \}}{\{max\{\sqrt{2(T_1-T_2)^2 + (I_1-I_2)^2 + (F_1-F_2)^2},\ |3(T_1-T_2)-(I_1-I_2)-(F_1-F_2)|/5\}/2\}+1}\}$$

Definition 2.8: [34] Let X be a set and P(X) be power set of X. A generalized set – valued neutrosophic quadruple set is a set of the form

$G_{s_i} = \{(A_{s_i}, B_{s_i}T_{s_i}, C_{s_i}I_{s_i}, D_{s_i}F_{s_i}): A_{s_i}, B_{s_i}, C_{s_i}, D_{s_i} \in$ P(X); i = 1, 2, 3, … , n}.

Where T_i, I_i and F_i have their usual neutrosophic logic means and generalized set – valued neutrosophic quadruple number defined by

$$G_{N_i} = (A_{s_i}, B_{s_i}T_{s_i}, C_{s_i}I_{s_i}, D_{s_i}F_{s_i}).$$

As in neutrosophic quadruple number, for a generalized set – valued neutrosophic quadruple number $(A_{s_i}, B_{s_i}T_{s_i}, C_{s_i}I_{s_i}, D_{s_i}F_{s_i})$ representing any entity which may be a number, an idea, an object, etc.; A_{s_i} is called the known part and $(B_{s_i}T_{s_i}, C_{s_i}I_{s_i}, D_{s_i}F_{s_i})$ is called the unknown part.

Definition 2.9: [34] Let $G_{N_i} = (A_{s_i}, B_{s_i}T_{s_i}, C_{s_i}I_{s_i}, D_{s_i}F_{s_i})$ and $G_{N_j} = (A_{s_j}, B_{s_j}T_{s_j}, C_{s_j}I_{s_j}, D_{s_j}F_{s_j})$ be two generalized set – valued neutrosophic quadruple numbers. $A_{s_i} \subseteq A_{s_j}$, $A_{s_i} \subseteq A_{s_j}$, $A_{s_i} \subseteq A_{s_j}$, $A_{s_i} \subseteq A_{s_j}$ and $T_{s_i} \leq T_{s_j}, I_{s_i} \leq I_{s_j}, F_{s_i} \leq F_{s_j}$ if and only if we say G_{N_i} is a subset of G_{N_j} and denote it by $G_{N_i} \subseteq G_{N_j}$.

3 Generalized Euclid Measures Based on Generalized Set Valued Neutrosophic Quadruple Numbers

In this chapter, we define firstly generalized Euclid distance measure and generalized Euclid similarity measure based on generalized set-valued neutrosophic quadruple numbers.

Also, in this paper we assume that T, I, F ∈ [0, 1] as single valued neutrosophic numbers.

Definition 3.1: Let **:** $X \neq \emptyset$ be a non-empty set and $P(X)$ be the power set of X.

Let $G_{N_i^1} = (A_{S_i^1}, B_{S_i^1}T_{S_i^1}, C_{S_i^1}I_{S_i^1}, D_{S_i^1}F_{S_i^1})$ and $G_{N_i^2} = (A_{S_i^2}, B_{S_i^2}T_{S_i^2}, C_{S_i^2}I_{S_i^2}, D_{S_i^2}F_{S_i^2})$ be two generalized set-valued neutrosophic quadruple numbers.

Define a function $d_E: G_{N_i^1} \times G_{N_i^2} \to [0,1]$ such that

$$d_G\left(G_{N_i^1}, G_{N_i^2}\right) = \frac{1}{2}\Bigg[\frac{\sqrt{\left(T_{S_i^1}-T_{S_i^2}\right)^2}+\sqrt{\left(I_{S_i^1}-I_{S_i^2}\right)^2}+\sqrt{\left(F_{S_i^1}-F_{S_i^2}\right)^2}}{3}$$

$$+\frac{\sqrt{\frac{s\left(A_{S_i^1}\backslash A_{S_i^2}\right)+s\left(A_{S_i^2}\backslash A_{S_i^1}\right)}{\max\{s\left(A_{S_i^1}\cup A_{S_i^2}\right),1\}}+\frac{s\left(B_{S_i^1}\backslash B_{S_i^2}\right)+s\left(B_{S_i^2}\backslash B_{S_i^1}\right)}{\max\{s\left(B_{S_i^1}\cup B_{S_i^2}\right),1\}}+\frac{s\left(C_{S_i^1}\backslash C_{S_i^2}\right)+s\left(C_{S_i^2}\backslash C_{S_i^1}\right)}{\max\{s\left(C_{S_i^1}\cup C_{S_i^2}\right),1\}}+\frac{s\left(D_{S_i^1}\backslash D_{S_i^2}\right)+s\left(D_{S_i^2}\backslash D_{S_i^1}\right)}{\max\{s\left(D_{S_i^1}\cup D_{S_i^2}\right),1\}}}}{2}\Bigg]$$

Then, $d_G\left(G_{N_i^1}, G_{N_i^2}\right)$ is called generalized Euclid distance measure for generalized set valued neutrosophic quadruple numbers.

Where, s(A) is the number of element of set A. Also, we generalize Euclid distance measure in Definition 2.3.

Theorem 3.2: Let

$G_{N_i^1} = (A_{S_i^1}, B_{S_i^1}T_{S_i^1}, C_{S_i^1}I_{S_i^1}, D_{S_i^1}F_{S_i^1})$,

$G_{N_i^2} = (A_{S_i^2}, B_{S_i^2}T_{S_i^2}, C_{S_i^2}I_{S_i^2}, D_{S_i^2}F_{S_i^2})$,

$G_{N_i^3} = (A_{S_i^3}, B_{S_i^3}T_{S_i^3}, C_{S_i^3}I_{S_i^3}, D_{S_i^3}F_{S_i^3})$

be three generalized set valued neutrosophic quadruple numbers. The generalized Euclid distance measure in Definition 3.1 satisfies the following conditions.

i) $d_G\left(G_{N_i^1}, G_{N_i^2}\right) \in [0,1]$

ii) $d_G\left(G_{N_i^1}, G_{N_i^2}\right) = 0 \Leftrightarrow G_{N_i^1} = G_{N_i^2}$

iii) $d_G\left(G_{N_i^1}, G_{N_i^2}\right) = d_G\left(G_{N_i^2}, G_{N_i^1}\right)$

iv) If $G_{N_i^1} \subset G_{N_i^2} \subset G_{N_i^3}$, then

$d_G\left(G_{N_i^1}, G_{N_i^2}\right) \leq d_G\left(G_{N_i^1}, G_{N_i^3}\right)$ and $d_G\left(G_{N_i^2}, G_{N_i^3}\right) \leq d_G\left(G_{N_i^1}, G_{N_i^3}\right)$.

Proof:

i) We assume that $G_{N_i^1} = G_{N_i^2}$. From Definition 2.9, we obtain that

$A_{S_i^1} = A_{S_i^2}$, $B_{S_i^1} = B_{S_i^2}$, $C_{S_i^1} = C_{S_i^2}$, $D_{S_i^1} = D_{S_i^2}$.

Thus,

$$d_G\left(G_{N_i^1}, G_{N_i^1}\right) = \frac{1}{2}\Bigg[\frac{\sqrt{\left(T_{S_i^1}-T_{S_i^1}\right)^2}+\sqrt{\left(I_{S_i^1}-I_{S_i^1}\right)^2}+\sqrt{\left(F_{S_i^1}-F_{S_i^1}\right)^2}}{3}$$

$$+\frac{\sqrt{\frac{s\left(A_{S_i^1}\backslash A_{S_i^1}\right)+s\left(A_{S_i^1}\backslash A_{S_i^1}\right)}{\max\{s\left(A_{S_i^1}\cup A_{S_i^1}\right),1\}}+\frac{s\left(B_{S_i^1}\backslash B_{S_i^1}\right)+s\left(B_{S_i^1}\backslash B_{S_i^1}\right)}{\max\{s\left(B_{S_i^1}\cup B_{S_i^1}\right),1\}}+\frac{s\left(C_{S_i^1}\backslash C_{S_i^1}\right)+s\left(C_{S_i^1}\backslash C_{S_i^1}\right)}{\max\{s\left(C_{S_i^1}\cup C_{S_i^1}\right),1\}}+\frac{s\left(D_{S_i^1}\backslash D_{S_i^1}\right)+s\left(D_{S_i^1}\backslash D_{S_i^1}\right)}{\max\{s\left(D_{S_i^1}\cup D_{S_i^1}\right),1\}}}}{2}\Bigg]$$

$$= \frac{1}{2}\left[\frac{0+0+0}{3} + \frac{\sqrt{0+0+0+0}}{2}\right] = \frac{1}{2}.0 = 0. \tag{1}$$

We assume that

$$A_{S_i^1} \neq A_{S_i^2} \text{ , } B_{S_i^1} \neq B_{S_i^2} \text{ , } C_{S_i^1} \neq C_{S_i^2} \text{ , } D_{S_i^1} \neq D_{S_i^2} \text{ .}$$

In this case,

$$d_G\left(G_{N_i^1}, G_{N_i^2}\right) > 0. \tag{2}$$

We assume that

$$G_{N_i^1} \neq \emptyset \text{ and } G_{N_i^2} = (\emptyset, \emptyset\, T_{S_i^1}, \emptyset\, I_{S_i^1}, \emptyset\, F_{S_i^1}).$$

Thus,

$$d_G\left(G_{N_i^1}, G_{N_i^2}\right) = \frac{1}{2}\Bigg[\frac{\sqrt{\left(T_{S_i^1}-T_{S_i^2}\right)^2}+\sqrt{\left(I_{S_i^1}-I_{S_i^2}\right)^2}+\sqrt{\left(F_{S_i^1}-F_{S_i^2}\right)^2}}{3}$$

$$+\frac{\sqrt{\frac{s\left(A_{S_i^1}\backslash\emptyset\right)+s\left(\emptyset\backslash A_{S_i^1}\right)}{\max\{s\left(A_{S_i^1}\cup\emptyset\right),1\}}+\frac{s\left(B_{S_i^1}\backslash\emptyset\right)+s\left(\emptyset\backslash B_{S_i^1}\right)}{\max\{s\left(B_{S_i^1}\cup\emptyset\right),1\}}+\frac{s\left(C_{S_i^1}\backslash\emptyset\right)+s\left(\emptyset\backslash C_{S_i^1}\right)}{\max\{s\left(C_{S_i^1}\cup\emptyset\right),1\}}+\frac{s\left(D_{S_i^1}\backslash\emptyset\right)+s\left(\emptyset\backslash D_{S_i^1}\right)}{\max\{s\left(D_{S_i^1}\cup\emptyset\right),1\}}}}{2}$$

$$= \frac{1}{2}\left[\frac{1+1+1}{3} + \frac{\sqrt{1+1+1+1}}{2}\right].$$

$$= \frac{1}{2}.[1 + 1] = 1 \tag{3}$$

Hence, from (1), (2) and (3) we obtain

$$0 \leq d_G\left(G_{N_i^1}, G_{N_i^2}\right) \leq 1.$$

ii) ⇒: We assume that

$$d_G\left(G_{N_i^1}, G_{N_i^2}\right) = \frac{1}{2}\Bigg[\frac{\sqrt{\left(T_{S_i^1}-T_{S_i^2}\right)^2}+\sqrt{\left(I_{S_i^1}-I_{S_i^2}\right)^2}+\sqrt{\left(F_{S_i^1}-F_{S_i^2}\right)^2}}{3}$$

$$+\frac{\sqrt{\frac{s\left(A_{S_i^1}\backslash A_{S_i^2}\right)+s\left(A_{S_i^2}\backslash A_{S_i^1}\right)}{\max\{s\left(A_{S_i^1}\cup A_{S_i^2}\right),1\}}+\frac{s\left(B_{S_i^1}\backslash B_{S_i^2}\right)+s\left(B_{S_i^2}\backslash B_{S_i^1}\right)}{\max\{s\left(B_{S_i^1}\cup B_{S_i^2}\right),1\}}+\frac{s\left(C_{S_i^1}\backslash C_{S_i^2}\right)+s\left(C_{S_i^2}\backslash C_{S_i^1}\right)}{\max\{s\left(C_{S_i^1}\cup C_{S_i^2}\right),1\}}+\frac{s\left(D_{S_i^1}\backslash D_{S_i^2}\right)+s\left(D_{S_i^2}\backslash D_{S_i^1}\right)}{\max\{s\left(D_{S_i^1}\cup D_{S_i^2}\right),1\}}}}{2}\Bigg]$$

$$= 0.$$

Thus,

$$\frac{\sqrt{\left(T_{S_i^1}-T_{S_i^2}\right)^2}+\sqrt{\left(I_{S_i^1}-I_{S_i^2}\right)^2}+\sqrt{\left(F_{S_i^1}-F_{S_i^2}\right)^2}}{3} = 0 \tag{4}$$

and

$$\frac{\sqrt{\frac{s\left(A_{S_i^1}\backslash A_{S_i^2}\right)+s\left(A_{S_i^2}\backslash A_{S_i^1}\right)}{\max\{s\left(A_{S_i^1}\cup A_{S_i^2}\right),1\}}+\frac{s\left(B_{S_i^1}\backslash B_{S_i^2}\right)+s\left(B_{S_i^2}\backslash B_{S_i^1}\right)}{\max\{s\left(B_{S_i^1}\cup B_{S_i^2}\right),1\}}+\frac{s\left(C_{S_i^1}\backslash C_{S_i^2}\right)+s\left(C_{S_i^2}\backslash C_{S_i^1}\right)}{\max\{s\left(C_{S_i^1}\cup C_{S_i^2}\right),1\}}+\frac{s\left(D_{S_i^1}\backslash D_{S_i^2}\right)+s\left(D_{S_i^2}\backslash D_{S_i^1}\right)}{\max\{s\left(D_{S_i^1}\cup D_{S_i^2}\right),1\}}}}{2} = 0 \tag{5}$$

From (4), we obtain

$$\sqrt{\left(T_{S_i^1}-T_{S_i^2}\right)^2}+\sqrt{\left(I_{S_i^1}-I_{S_i^2}\right)^2}+\sqrt{\left(F_{S_i^1}-F_{S_i^2}\right)^2} = 0.$$

Hence, we obtain that

$$\sqrt{\left(T_{S_i^1}-T_{S_i^2}\right)^2} = 0 \text{ and } T_{S_i^1} = T_{S_i^2}$$

$$\sqrt{\left(I_{S_i^1}-I_{S_i^2}\right)^2} = 0 \text{ and } I_{S_i^1} = I_{S_i^2}$$

$$\sqrt{\left(F_{S_i^1}-F_{S_i^2}\right)^2} = 0 \text{ and } F_{S_i^1} = F_{S_i^2} \tag{6}$$

Also, From (5), we obtain

$$\frac{s\left(A_{S_i^1}\backslash A_{S_i^2}\right)+s\left(A_{S_i^2}\backslash A_{S_i^1}\right)}{\max\left\{s\left(A_{S_i^1}\cup A_{S_i^2}\right),1\right\}}+\frac{s\left(B_{S_i^1}\backslash B_{S_i^2}\right)+s\left(B_{S_i^2}\backslash B_{S_i^1}\right)}{\max\left\{s\left(B_{S_i^1}\cup B_{S_i^2}\right),1\right\}}$$

$$+\frac{s\left(C_{S_i^1}\backslash C_{S_i^2}\right)+s\left(C_{S_i^2}\backslash C_{S_i^1}\right)}{\max\left\{s\left(C_{S_i^1}\cup C_{S_i^2}\right),1\right\}}+\frac{s\left(D_{S_i^1}\backslash D_{S_i^2}\right)+s\left(D_{S_i^2}\backslash D_{S_i^1}\right)}{\max\left\{s\left(D_{S_i^1}\cup D_{S_i^2}\right),1\right\}}$$

$= 0.$

Hence,

$$s\left(A_{S_i^1} \backslash A_{S_i^2}\right) + s\left(A_{S_i^2} \backslash A_{S_i^1}\right) = 0$$

$$s\left(B_{S_i^1} \backslash B_{S_i^2}\right) + s\left(B_{S_i^2} \backslash B_{S_i^1}\right) = 0$$

$$s\left(C_{S_i^1} \backslash C_{S_i^2}\right) + s\left(C_{S_i^2} \backslash C_{S_i^1}\right) = 0$$

$$s\left(D_{S_i^1} \backslash D_{S_i^2}\right) + s\left(D_{S_i^2} \backslash D_{S_i^1}\right) = 0 \quad (7)$$

From (7), we obtain

$$A_{S_i^1} = A_{S_i^2}\ , B_{S_i^1} = B_{S_i^2}\ , C_{S_i^1} = C_{S_i^2} \text{ and } D_{S_i^1} = D_{S_i^2} \quad (8)$$

Therefore, from (6), (9) and Definition 2.9 we obtain

$$G_{N_i^1} = G_{N_i^2}$$

$\Leftarrow$: We assume that $G_{N_i^1} = G_{N_i^2}$. It is clear that from (1),

$$d_G\left(G_{N_i^1}, G_{N_i^1}\right) = 0.$$

Hence, we obtain

$$d_G\left(G_{N_i^1}, G_{N_i^2}\right) = 0 \Leftrightarrow G_{N_i^1} = G_{N_i^2}.$$

iii)

$$d_G\left(G_{N_i^1}, G_{N_i^2}\right) = \frac{1}{2}\Bigg[\frac{\sqrt{\left(T_{S_i^1}-T_{S_i^2}\right)^2}+\sqrt{\left(I_{S_i^1}-I_{S_i^2}\right)^2}+\sqrt{\left(F_{S_i^1}-F_{S_i^2}\right)^2}}{3}$$

$$+\frac{\sqrt{\frac{s\left(A_{S_i^1}\backslash A_{S_i^2}\right)+s\left(A_{S_i^2}\backslash A_{S_i^1}\right)}{\max\{s\left(A_{S_i^1}\cup A_{S_i^2}\right),1\}}+\frac{s\left(B_{S_i^1}\backslash B_{S_i^2}\right)+s\left(B_{S_i^2}\backslash B_{S_i^1}\right)}{\max\{s\left(B_{S_i^1}\cup B_{S_i^2}\right),1\}}+\frac{s\left(C_{S_i^1}\backslash C_{S_i^2}\right)+s\left(C_{S_i^2}\backslash C_{S_i^1}\right)}{\max\{s\left(C_{S_i^1}\cup C_{S_i^2}\right),1\}}+\frac{s\left(D_{S_i^1}\backslash D_{S_i^2}\right)+s\left(D_{S_i^2}\backslash D_{S_i^1}\right)}{\max\{s\left(D_{S_i^1}\cup D_{S_i^2}\right),1\}}}}{2}\Bigg]$$

$$= \frac{1}{2}\Bigg[\frac{\sqrt{\left(T_{S_i^2}-T_{S_i^1}\right)^2}+\sqrt{\left(I_{S_i^2}-I_{S_i^1}\right)^2}+\sqrt{\left(F_{S_i^2}-F_{S_i^1}\right)^2}}{3}$$

$$+\frac{\sqrt{\frac{s\left(A_{S_i^2}\backslash A_{S_i^1}\right)+s\left(A_{S_i^1}\backslash A_{S_i^2}\right)}{\max\{s\left(A_{S_i^2}\cup A_{S_i^1}\right),1\}}+\frac{s\left(B_{S_i^2}\backslash B_{S_i^1}\right)+s\left(B_{S_i^1}\backslash B_{S_i^2}\right)}{\max\{s\left(B_{S_i^2}\cup B_{S_i^1}\right),1\}}+\frac{s\left(C_{S_i^2}\backslash C_{S_i^1}\right)+s\left(C_{S_i^1}\backslash C_{S_i^2}\right)}{\max\{s\left(C_{S_i^2}\cup C_{S_i^1}\right),1\}}+\frac{s\left(D_{S_i^2}\backslash D_{S_i^1}\right)+s\left(D_{S_i^1}\backslash D_{S_i^2}\right)}{\max\{s\left(D_{S_i^2}\cup D_{S_i^1}\right),1\}}}}{2}$$

$$= d_G\left(G_{N_i^2}, G_{N_i^1}\right)$$

iv) We assume that

$$G_{N_i^1} \subseteq G_{N_i^2} \subseteq G_{N_i^3}.$$

From Definition 2.9, we obtain

$$s(A_{S_i^1}) \leq s(A_{S_i^2}) \leq s(A_{S_i^3})$$

$$s(B_{S_i^1}) \leq s(B_{S_i^2}) \leq s(B_{S_i^3})$$

$$s(C_{S_i^1}) \leq s(C_{S_i^2}) \leq s(C_{S_i^3})$$

$$s(D_{S_i^1}) \leq s(D_{S_i^2}) \leq s(D_{S_i^3}) \tag{9}$$

$$s\left(A_{S_i^1} \backslash A_{S_i^2}\right) = s\left(A_{S_i^1} \backslash A_{S_i^3}\right) = s\left(A_{S_i^2} \backslash A_{S_i^3}\right) = \emptyset$$

$$s\left(B_{S_i^1} \backslash B_{S_i^2}\right) = s\left(B_{S_i^1} \backslash B_{S_i^3}\right) = s\left(B_{S_i^2} \backslash B_{S_i^3}\right) = \emptyset$$

$$s\left(C_{S_i^1} \backslash C_{S_i^2}\right) = s\left(C_{S_i^1} \backslash C_{S_i^3}\right) = s\left(C_{S_i^2} \backslash C_{S_i^3}\right) = \emptyset$$

$$s\left(D_{S_i^1} \backslash D_{S_i^2}\right) = s\left(D_{S_i^1} \backslash D_{S_i^3}\right) = s\left(D_{S_i^2} \backslash D_{S_i^3}\right) = \emptyset \tag{10}$$

$$s\left(A_{S_i^2} \backslash A_{S_i^1}\right) \leq s\left(A_{S_i^3} \backslash A_{S_i^1}\right),$$

$$s\left(B_{S_i^2} \backslash B_{S_i^1}\right) \leq s\left(B_{S_i^3} \backslash B_{S_i^1}\right),$$

$$s\left(C_{S_i^2} \backslash C_{S_i^1}\right) \leq s\left(C_{S_i^3} \backslash C_{S_i^1}\right),$$

$$s\left(D_{S_i^2} \backslash D_{S_i^1}\right) \leq s\left(D_{S_i^3} \backslash D_{S_i^1}\right),$$

$$s\left(A_{S_i^3} \backslash A_{S_i^2}\right) \leq s\left(A_{S_i^3} \backslash A_{S_i^1}\right),$$

$$s\left(B_{S_i^3} \backslash B_{S_i^2}\right) \leq s\left(B_{S_i^3} \backslash B_{S_i^1}\right),$$

$$s\left(C_{S_i^3} \backslash C_{S_i^2}\right) \leq s\left(C_{S_i^3} \backslash C_{S_i^1}\right),$$

$$s\left(D_{S_i^3} \backslash D_{S_i^2}\right) \leq s\left(D_{S_i^3} \backslash D_{S_i^1}\right) \tag{11}$$

$$\max\left\{s\left(A_{S_i^1} \cup A_{S_i^2}\right), 1\right\} = \max\{s\left(A_{S_i^2}\right), 1\},$$

$$\max\left\{s\left(B_{S_i^1} \cup B_{S_i^2}\right), 1\right\} = \max\{s\left(B_{S_i^2}\right), 1\},$$

$$\max\left\{s\left(C_{S_i^1} \cup C_{S_i^2}\right), 1\right\} = \max\{s\left(C_{S_i^2}\right), 1\},$$

$$\max\left\{s\left(D_{S_i^1} \cup D_{S_i^2}\right), 1\right\} = \max\{s\left(D_{S_i^2}\right), 1\},$$

$$\max\left\{s\left(A_{S_i^2} \cup A_{S_i^3}\right), 1\right\} = \max\{s\left(A_{S_i^3}\right), 1\},$$

$$\max\left\{s\left(B_{S_i^2} \cup B_{S_i^3}\right), 1\right\} = \max\{s\left(B_{S_i^3}\right), 1\},$$

$$\max\left\{s\left(C_{S_i^2} \cup C_{S_i^3}\right), 1\right\} = \max\{s\left(C_{S_i^3}\right), 1\},$$

$$\max\left\{s\left(D_{S_i^2} \cup D_{S_i^3}\right), 1\right\} = \max\{s\left(D_{S_i^3}\right), 1\},$$

$$\max\left\{s\left(A_{S_i^1} \cup A_{S_i^3}\right), 1\right\} = \max\{s\left(A_{S_i^3}\right), 1\},$$

$$\max\left\{s\left(B_{S_i^1} \cup B_{S_i^3}\right), 1\right\} = \max\{s\left(B_{S_i^3}\right), 1\},$$

$$\max\left\{s\left(C_{S_i^1} \cup C_{S_i^3}\right), 1\right\} = \max\{s\left(C_{S_i^3}\right), 1\},$$

$$\max\left\{s\left(D_{S_i^1} \cup D_{S_i^3}\right), 1\right\} = \max\{s\left(D_{S_i^3}\right), 1\} \tag{12}$$

$$\sqrt{\left(T_{S_i^1} - T_{S_i^2}\right)^2} + \sqrt{\left(I_{S_i^1} - I_{S_i^2}\right)^2} + \sqrt{\left(F_{S_i^1} - F_{S_i^2}\right)^2} \le \sqrt{\left(T_{S_i^1} - T_{S_i^3}\right)^2} + \sqrt{\left(I_{S_i^1} - I_{S_i^3}\right)^2} + \sqrt{\left(F_{S_i^1} - F_{S_i^3}\right)^2}$$

$$\sqrt{\left(T_{S_i^2} - T_{S_i^3}\right)^2} + \sqrt{\left(I_{S_i^2} - I_{S_i^3}\right)^2} + \sqrt{\left(F_{S_i^2} - F_{S_i^3}\right)^2} \le \sqrt{\left(T_{S_i^1} - T_{S_i^3}\right)^2} + \sqrt{\left(I_{S_i^1} - I_{S_i^3}\right)^2} + \sqrt{\left(F_{S_i^1} - F_{S_i^3}\right)^2} \tag{13}$$

Hence, from (9), (10), (11), (12), (13);

$$d_G\left(G_{N_i^1}, G_{N_i^2}\right) = \frac{1}{2}\left[\frac{\sqrt{\left(T_{S_i^1}-T_{S_i^2}\right)^2}+\sqrt{\left(I_{S_i^1}-I_{S_i^2}\right)^2}+\sqrt{\left(F_{S_i^1}-F_{S_i^2}\right)^2}}{3}\right.$$

$$\left.+\frac{\sqrt{\frac{s\left(A_{S_i^1}\backslash A_{S_i^2}\right)+s\left(A_{S_i^2}\backslash A_{S_i^1}\right)}{\max\{s\left(A_{S_i^1}\cup A_{S_i^2}\right),1\}}+\frac{s\left(B_{S_i^1}\backslash B_{S_i^2}\right)+s\left(B_{S_i^2}\backslash B_{S_i^1}\right)}{\max\{s\left(B_{S_i^1}\cup B_{S_i^2}\right),1\}}+\frac{s\left(C_{S_i^1}\backslash C_{S_i^2}\right)+s\left(C_{S_i^2}\backslash C_{S_i^1}\right)}{\max\{s\left(C_{S_i^1}\cup C_{S_i^2}\right),1\}}+\frac{s\left(D_{S_i^1}\backslash D_{S_i^2}\right)+s\left(D_{S_i^2}\backslash D_{S_i^1}\right)}{\max\{s\left(D_{S_i^1}\cup D_{S_i^2}\right),1\}}}}{2}\right]$$

$$\le \frac{1}{2}\left[\frac{\sqrt{\left(T_{S_i^1}-T_{S_i^3}\right)^2}+\sqrt{\left(I_{S_i^1}-I_{S_i^3}\right)^2}+\sqrt{\left(F_{S_i^1}-F_{S_i^3}\right)^2}}{3}\right.$$

$$\left.+\frac{\sqrt{\frac{s\left(A_{S_i^1}\backslash A_{S_i^3}\right)+s\left(A_{S_i^3}\backslash A_{S_i^1}\right)}{\max\{s\left(A_{S_i^1}\cup A_{S_i^3}\right),1\}}+\frac{s\left(B_{S_i^1}\backslash B_{S_i^3}\right)+s\left(B_{S_i^3}\backslash B_{S_i^1}\right)}{\max\{s\left(B_{S_i^1}\cup B_{S_i^3}\right),1\}}+\frac{s\left(C_{S_i^1}\backslash C_{S_i^3}\right)+s\left(C_{S_i^3}\backslash C_{S_i^1}\right)}{\max\{s\left(C_{S_i^1}\cup C_{S_i^3}\right),1\}}+\frac{s\left(D_{S_i^1}\backslash D_{S_i^3}\right)+s\left(D_{S_i^3}\backslash D_{S_i^1}\right)}{\max\{s\left(D_{S_i^1}\cup D_{S_i^3}\right),1\}}}}{2}\right]$$

$$= d_G\left(G_{N_i^1}, G_{N_i^3}\right).$$

Therefore,

$$d_G\left(G_{N_i^1}, G_{N_i^2}\right) \le d_G\left(G_{N_i^1}, G_{N_i^3}\right)$$

Also, from (9), (10), (11), (12), (13); we obtain

$$d_G\left(G_{N_i^2}, G_{N_i^3}\right) \le d_G\left(G_{N_i^1}, G_{N_i^3}\right)$$

Definition 3.3: Let **:** $X \neq \emptyset$ be a non-empty set and $P(X)$ be the power set of X.

Let $G_{N_i^1} = (A_{S_i^1}, B_{S_i^1}T_{S_i^1}, C_{S_i^1}I_{S_i^1}, D_{S_i^1}F_{S_i^1})$ and $G_{N_i^2} = (A_{S_i^2}, B_{S_i^2}T_{S_i^2}, C_{S_i^2}I_{S_i^2}, D_{S_i^2}F_{S_i^2})$ be two generalized set-valued neutrosophic quadruple numbers.

Define a function $S_E: G_{N_i^1} \times G_{N_i^2} \rightarrow [0,1]$ such that

$$S_G\left(G_{N_i^1}, G_{N_i^2}\right) = 1 - \frac{1}{2}\Bigg[\frac{\sqrt{\left(T_{S_i^1} - T_{S_i^2}\right)^2} + \sqrt{\left(I_{S_i^1} - I_{S_i^2}\right)^2} + \sqrt{\left(F_{S_i^1} - F_{S_i^2}\right)^2}}{3}$$

$$+\frac{\sqrt{\frac{s\left(A_{S_i^1}\backslash A_{S_i^2}\right)+s\left(A_{S_i^2}\backslash A_{S_i^1}\right)}{\max\{s\left(A_{S_i^1}\cup A_{S_i^2}\right),1\}}+\frac{s\left(B_{S_i^1}\backslash B_{S_i^2}\right)+s\left(B_{S_i^2}\backslash B_{S_i^1}\right)}{\max\{s\left(B_{S_i^1}\cup B_{S_i^2}\right),1\}}+\frac{s\left(C_{S_i^1}\backslash C_{S_i^2}\right)+s\left(C_{S_i^2}\backslash C_{S_i^1}\right)}{\max\{s\left(C_{S_i^1}\cup C_{S_i^2}\right),1\}}+\frac{s\left(D_{S_i^1}\backslash D_{S_i^2}\right)+s\left(D_{S_i^2}\backslash D_{S_i^1}\right)}{\max\{s\left(D_{S_i^1}\cup D_{S_i^2}\right),1\}}}}{2}\Bigg]$$

Then, $S_G\left(G_{N_i^1}, G_{N_i^2}\right)$ is called generalized Euclid similarity measure for generalized set valued neutrosophic quadruple numbers.

Where, s(A) is the number of element of set A. Also, we generalize Euclid similarity measure in Definition 2.3.

Corollary 3.4: Let $S_G\left(G_{N_i^1}, G_{N_i^2}\right)$ be Euclid similarity measure for generalized set valued neutrosophic quadruple numbers in Definition 3.3 and $d_G\left(G_{N_i^1}, G_{N_i^2}\right)$ be Euclid distance measure for generalized set valued neutrosophic quadruple numbers in Definition 3.1. Then,

$$S_G\left(G_{N_i^1}, G_{N_i^2}\right) = 1 - d_G\left(G_{N_i^1}, G_{N_i^2}\right)$$

Theorem 3.5: Let

$G_{N_i^1} = (A_{S_i^1}, B_{S_i^1}T_{S_i^1}, C_{S_i^1}I_{S_i^1}, D_{S_i^1}F_{S_i^1})$,

$G_{N_i^2} = (A_{S_i^2}, B_{S_i^2}T_{S_i^2}, C_{S_i^2}I_{S_i^2}, D_{S_i^2}F_{S_i^2})$,

$G_{N_i^3} = (A_{S_i^3}, B_{S_i^3}T_{S_i^3}, C_{S_i^3}I_{S_i^3}, D_{S_i^3}F_{S_i^3})$

be three generalized set valued neutrosophic quadruple numbers. the generalized Euclid similarity measure in Definition 3.3 satisfies the following conditions.

i) $S_G\left(G_{N_i^1}, G_{N_i^2}\right) \in [0,1]$

ii) $S_G\left(G_{N_i^1}, G_{N_i^2}\right) = 1 \Leftrightarrow G_{N_i^1} = G_{N_i^2}$

iii) $S_G\left(G_{N_i^1}, G_{N_i^2}\right) = S_G\left(G_{N_i^2}, G_{N_i^1}\right)$

iv) If $G_{N_i^1} \subset G_{N_i^2} \subset G_{N_i^3}$, then

$S_G\left(G_{N_i^1}, G_{N_i^3}\right) \leq S_G\left(G_{N_i^1}, G_{N_i^2}\right)$ and $S_G\left(G_{N_i^1}, G_{N_i^3}\right) \leq S_G\left(G_{N_i^2}, G_{N_i^3}\right)$.

Proof: From Corollary 3.4 and Theorem 3.2, it is clear that $S_G\left(G_{N_i^1}, G_{N_i^2}\right)$ satisfies the conditions of Theorem 3.5.

Example 3.6: Let

$X_1 = (\{\varpi_4, \varpi_1\}, \{\varpi_7, \varpi_6\}(0{,}3), \{\varpi_8, \varpi_9\}(0{,}4), \{\varpi_{10}\}(0{,}1))$

$X_2 = (\{\varpi_1, \varpi_2\}, \{\varpi_6\}(1), \emptyset(0), \emptyset(0))$

be two generalized set valued neutrosophic quadruple numbers, $S_G\left(G_{N_i^1}, G_{N_i^2}\right)$ be Euclid similarity measure for generalized set valued neutrosophic quadruple numbers in Definition 3.3 and $d_G\left(G_{N_i^1}, G_{N_i^2}\right)$ be Euclid distance measure for generalized set valued neutrosophic quadruple numbers in Definition 3.1. Then,

$$d_G(X_1, X_2) = \frac{1}{2}\left[\frac{\sqrt{(0{,}3-1)^2}+\sqrt{(0{,}4-0)^2}+\sqrt{(0{,}1-0)^2}}{3} + \frac{\sqrt{\begin{array}{c}\frac{s(\{\varpi_4,\varpi_1\}\backslash\{\varpi_1,\varpi_2\})+s(\{\varpi_1,\varpi_2\}\backslash\{\varpi_4,\varpi_1\})}{\max\{s(\{\varpi_4,\varpi_1,\varpi_2\}),1\}} \\ +\frac{s(\{\varpi_7,\varpi_6\}\backslash\{\varpi_6\})+s(\{\varpi_6\}\backslash\{\varpi_7,\varpi_6\})}{\max\{s(\{\varpi_7,\varpi_6\}),1\}} \\ +\frac{s(\{\varpi_8,\varpi_9\}\backslash\emptyset)+s(\emptyset\backslash\{\varpi_8,\varpi_9\})}{\max\{s(\{\varpi_8,\varpi_9\}),1\}} \\ +\frac{s(\{\varpi_{10}\}\backslash\emptyset)+s(\emptyset\backslash\{\varpi_{10}\})}{\max\{s(\{\varpi_{10}\}),1\}}\end{array}}}{2}\right]$$

$$= \frac{1}{2}\left[\left(\frac{(0{,}7)+(0{,}4)+(0{,}1)}{3}\right) + \left(\frac{\sqrt{\frac{1+1}{3}+\frac{1+0}{2}+\frac{2+0}{2}+\frac{1+0}{1}}}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1{,}2}{3} + \frac{\sqrt{\frac{21}{12}}}{2}\right]$$

$$= 0.53$$

and

$S_G(X_1,\ X_2) = 1 - d_G(X_1,\ X_2) = 1 - 0{,}53 = 0{,}47.$

Definition 3.7: (Score Function) Let

$X_1 = (A_1, B_1T_1, C_1I_1, D_1F_1)$

$X_2 = (A_2, B_2T_2, C_2I_2, D_2F_2)$

be two generalized set valued neutrosophic number, X_i be a generalized set valued neutrosophic number representing a sample and S_G be Euclid similarity measure for generalized set valued neutrosophic quadruple numbers in Definition 3.3. It is unclear which number is more similar to the sample in case

$$S_G(X_i, X_1) = S_G(X_i, X_2)$$

In these cases, we will define a score function to determine which number is more similar to the sample.

a)

If $s(A_1) + s(B_1) > s(A_2) + s(B_2)$, then choose X_1.

If $s(A_1) + s(B_1) < s(A_2) + s(B_2)$, then choose X_2.

b) We assume that $s(A_1) + s(B_1) = s(A_2) + s(B_2)$.

If $T_1 > T_2$, then choose X_1.

If $T_1 < T_2$, then choose X_2.

c) We assume that $s(A_1) + s(B_1) = s(A_2) + s(B_2)$ and $T_1 = T_2$.

If $F_1 > F_2$, then choose X_2.

If $F_1 < F_2$, then choose X_1.

d) We assume that $s(A_1) + s(B_1) = s(A_2) + s(B_2)$, $T_1 = T_2$ and $F_1 = F_2$.

If $I_1 > I_2$, then choose X_2.

If $I_1 < I_2$, then choose X_1.

e) We assume that $s(A_1) + s(B_1) = s(A_2) + s(B_2)$, $T_1 = T_2$ and $I_1 = I_2$.

If $F_1 > F_2$, then choose X_2.

If $F_1 < F_2$, then choose X_1.

f) We assume that $(A_1) + s(B_1) = s(A_2) + s(B_2)$, $T_1 = T_2$, $I_1 = I_2$ and $F_1 = F_2$.

If $s(C_1) + s(D_1) > s(C_2) + s(D_2)$, then choose X_2.

If $s(C_1) + s(D_1) < s(C_2) + s(D_2)$, then choose X_1.

g) We assume that $(A_1) + s(B_1) = s(A_2) + s(B_2)$, $T_1 = T_2$, $I_1 = I_2$ and $F_1 = F_2$ and $s(C_1) + s(D_1) = s(C_2) + s(D_2)$.

Then, we choose X_1 or X_2.

Example 3.8: Let

$X_1 = (\{\omega_4, \omega_1, \omega_{12}, \omega_{13}\}, \{\omega_7, \omega_6\}(0,3), \{\omega_8, \omega_9, \omega_{14}\}(0,2), \{\omega_{15}, \omega_{16}\}(0,1))$

$X_2 = (\{\omega_2, \omega_3, \omega_4, \omega_{12}\}, \{\omega_8, \omega_6\}(0,4), \{\omega_9, \omega_{15}\}(0,3), \{\omega_8, \omega_{10}, \omega_{16}\}(0,1))$

be two generalized set valued neutrosophic quadruple numbers and

$X = (\{\omega_1, \omega_2\}, \{\omega_6\}(1), \emptyset(0), \emptyset(0))$ be a generalized set valued neutrosophic number representing a sample. We choose X_1 or X_2 according to S_G in Definition 3.3.

$$S_G(X_1, X) = 1 - \frac{1}{2}\left[\frac{\sqrt{(0,3-1)^2}+\sqrt{(0,2-0)^2}+\sqrt{(0,1-0)^2}}{3} + \frac{\sqrt{\begin{array}{c}\frac{s(\{\omega_4,\omega_1,\omega_{12},\omega_{13}\}\setminus\{\omega_1,\omega_2\})+s(\{\omega_1,\omega_2\}\setminus\{\omega_4,\omega_1,\omega_{12},\omega_{13}\})}{\max\{s(\{\omega_4,\omega_1,\omega_2,\omega_{12},\omega_{13}\}),1\}} \\ +\frac{s(\{\omega_7,\omega_6\}\setminus\{\omega_6\})+s(\{\omega_6\}\setminus\{\omega_7,\omega_6\})}{\max\{s(\{\omega_7,\omega_6\}),1\}} \\ +\frac{s(\{\omega_8,\omega_9,\omega_{14}\}\setminus\emptyset)+s(\emptyset\setminus\{\omega_8,\omega_9,\omega_{14}\})}{\max\{s(\{\omega_8,\omega_9,\omega_{14}\}),1\}} \\ +\frac{s(\{\omega_{15},\omega_{16}\}\setminus\emptyset)+s(\emptyset\setminus\{\omega_{15},\omega_{16}\})}{\max\{s(\{\omega_{15},\omega_{16}\}),1\}}\end{array}}}{2}\right]$$

$$= 1 – 0,62 = 0,38.$$

$$S_G(X_2, \mathrm{X}) = 1 - \frac{1}{2}\left[\frac{\sqrt{(0{,}4-1)^2}+\sqrt{(0{,}3-0)^2}+\sqrt{(0{,}1-0)^2}}{3} + \frac{\sqrt{\frac{s(\{\varpi_2,\varpi_3,\varpi_4,\varpi_{12}\}\setminus\{\varpi_1,\varpi_2\})+s(\{\varpi_1,\varpi_2\}\setminus\{\varpi_2,\varpi_3,\varpi_4,\varpi_{12}\})}{\max\{s(\{\varpi_2,\varpi_3,\varpi_4,\varpi_{12},\varpi_1\}),1\}} + \frac{s(\{\varpi_8,\varpi_6\}\setminus\{\varpi_6\})+s(\{\varpi_6\}\setminus\{\varpi_8,\varpi_6\})}{\max\{s(\{\varpi_8,,\varpi_6\}),1\}} + \frac{s(\{\varpi_9,\varpi_{15}\}\setminus\emptyset)+s(\emptyset\setminus\{\varpi_9,\varpi_{15}\})}{\max\{s(\{\varpi_9,\varpi_{15}\}),1\}} + \frac{s(\{\varpi_8,\varpi_{10},\varpi_{16}\}\setminus\emptyset)+s(\emptyset\setminus\{\varpi_8,\varpi_{10},\varpi_{16}\})}{\max\{s(\{\varpi_8,\varpi_{10},\varpi_{16}\}),1\}}}}{2}\right]$$

$= 1 – 0{,}62 = 0{,}38.$

Thus,

$$S_G(X_1, \mathrm{X}) = S_G(X_2, \mathrm{X}).$$

In this case, by the Score Function in Definition 3.7,

$s(A_1) + s(B_1) = s(\{\varpi_4, \varpi_1, \varpi_{12}, \varpi_{13}\}) + s(\{\varpi_7, \varpi_6\}) = 6$

and

$s(A_2) + s(B_2) = s(\{\varpi_2, \varpi_3, \varpi_4, \varpi_{12}\}) + s(\{\varpi_8, \varpi_6\}) = 6.$

As $s(A_1) + s(B_1) = s(A_1) + s(B_1)$, we compare T_1 with T_2.

Since,

$$T_1 = 0.3 \text{ and } T_2 = 0.4$$

by the Score Function, we choose X_2.

4 MULTI CRITERIA DECISION MAKING APPLICATIONS WITH GENERALIZED SET VALUED NEUTROSOPHIC QUADRUPLE NUMBERS AND GENERALIZED EUCLID SIMILARITY MEASURE

In this section, we use the generalized algorithm used in [39, 40] using neutrosophic sets and give it again for the generalized set valued neutrosophic quadruple numbers. We will also use the generalized Euclid similarity measure (in Definition 3.3) in this algorithm.

Using this algorithm [39, 40], we will give an example of individuals with more than one disease to determine which of the known disease medications will be good for their unknown disease. We compared the results we obtained in this example with the results obtained in neutrosophic numbers and showed that we obtained different results.

This example will be especially useful for healthcare professionals in determining which drugs to use in corona (covid-19) treatment of an individual with various diseases.

4.1 Multi Critarias Decision Making Algorithm with Generalized Set Valued Neutrosophic Quadruple Numbers and Generalized Euclid Similarity Measure

Step 1: Let $H = \{h_1, h_2, \dots, h_n\}$ be set of criterias.

Step 2: Let $W = \{w_1, w_2, \dots, w_n\}$ be weighted value set of criterias such that

w_1 is weighted value of h_1,

w_2 is weighted value of h_2,

.

.

.

w_n is weighted value of h_n

Also, $\sum_{i=1}^{n} w_i = 1$ and $w_i \in \mathbb{R}^+$.

Step 3:

Let us express an ideal object K that we can compare as a generalized set-valued neutrophic quadruple set

$$K = \{h_1: (P(A), P(A)T_{1i}, \emptyset I_{1i}, \emptyset F_{1i}), h_2: (P(B), P(B)T_{2i}, \emptyset I_{2i}, \emptyset F_{2i}), \ldots,$$
$$h_n: (P(Y), P(Y)T_{ni}, \emptyset I_{ni}, \emptyset F_{ni})\}$$

Where,

$(P(A)T_{1i}, \emptyset I_{1i}, \emptyset F_{1i})$ is ideal set for h_1,

$(P(A)T_{2i}, \emptyset I_{2i}, \emptyset F_{2i})$ is ideal set for h_2,

.

.

.

$(P(A)T_{ni}, \emptyset I_{ni}, \emptyset F_{ni})$ is ideal set for h_n,

Step 4: Let $X = \{X_1, X_2, \ldots, X_n\}$ be the set of objects that we will choose the best according to their ideal object similarity values. Now, we give each object as a generalized set valued neutrophic quadruple set.

$X_1 = \{h_1: (A_{11}, A_{12}T_{11}, A_{13}I_{11}, A_{14}F_{11}), h_2: (B_{11}, B_{12}T_{12}, B_{13}I_{12}, B_{14}F_{12}), \ldots,$

$h_n: (Y_{11}, Y_{12}T_{1n}, Y_{13}I_{1n}, Y_{14}F_{1n})\}$

$X_2 = \{h_1: (A_{21}, A_{22}T_{21}, A_{23}I_{21}, A_{24}F_{21}), h_2: (B_{21}, B_{22}T_{22}, B_{23}I_{22}, B_{24}F_{22}), \ldots,$

$h_n: (Y_{21}, Y_{22}T_{2n}, Y_{23}I_{2n}, Y_{24}F_{2n})\}$

.

.

.

$X_i = \{h_1: (A_{i1}, A_{i2}T_{i1}, A_{i3}I_{i1}, A_{i4}F_{i1}), h_2: (B_{i1}, B_{i2}T_{i2}, B_{i3}I_{i2}, B_{i4}F_{i2}), \ldots,$

$h_n: (Y_{i1}, Y_{i2}T_{in}, Y_{i3}I_{in}, Y_{i4}F_{in})\}, \quad i = 1,2, \ldots, n.$

Where,

$A_{11}, A_{12}, A_{13}, A_{14}, A_{21}, A_{22}, A_{23}, A_{24}, \ldots, A_{i1}, A_{i2}, A_{i3}, A_{i4} \in P(A)$

$B_{11}, B_{12}, B_{13}, B_{14}, B_{21}, B_{22}, B_{23}, B_{24}, \dots, B_{i1}, B_{i2}, B_{i3}, B_{i4} \in P(B)$

.

.

.

$Y_{11}, Y_{12}, Y_{13}, Y_{14}, Y_{21}, Y_{22}, Y_{23}, Y_{24}, \dots, Y_{i1}, Y_{i2}, Y_{i3}, Y_{i4} \in P(Y)$.

Step 5: Let's show the objects in step 4 in Table 1.

Table 1. Table of Objects

	$\boldsymbol{h_1}$	$\boldsymbol{h_2}$	$\dots$	$\boldsymbol{h_n}$
$\boldsymbol{X_1}$	$(A_{11}, A_{12}T_{11}, A_{13}I_{11}, A_{14}F_{11})$	$(B_{11}, B_{12}T_{12}, B_{13}I_{12}, B_{14}F_{12})$	$\dots$	$(Y_{11}, Y_{12}T_{1n}, Y_{13}I_{1n}, Y_{14}F_{1n})$
$\boldsymbol{X_2}$	$(A_{21}, A_{22}T_{21}, A_{23}I_{21}, A_{24}F_{21})$	$(B_{21}, B_{22}T_{22}, B_{23}I_{22}, B_{24}F_{22})$	$\dots$	$(Y_{21}, Y_{22}T_{2n}, Y_{23}I_{2n}, Y_{24}F_{2n})$
. . .	. . .	. . .	$\dots$ $\dots$ $\dots$	. . .
$\boldsymbol{X_i}$	$(A_{i1}, A_{i2}T_{i1}, A_{i3}I_{i1}, A_{i4}F_{i1})$	$(B_{i1}, B_{i2}T_{i2}, B_{i3}I_{i2}, B_{i4}F_{i2})$	$\dots$	$(Y_{i1}, Y_{i2}T_{in}, Y_{i3}I_{in}, Y_{i4}F_{in})$

Step 6: We find the similarity value of the criteria value of each object in Table 1 and the criteria values of the ideal object with the generalized Euclidean similarity measure. Thus, we obtain Table 2.

Table 2. Criteria Similarity Table

	$\boldsymbol{h_1}$	$\boldsymbol{h_2}$	$\dots$	$\boldsymbol{h_n}$
$\boldsymbol{X_1}$	$S_G(\mathrm{K}(h_1), X_1(h_1))$	$S_G(\mathrm{K}(h_2), X_1(h_2))$	$\dots$	$S_G(\mathrm{K}(h_n), X_1(h_n))$
$\boldsymbol{X_2}$	$S_G(\mathrm{K}(h_1), X_2(h_1))$	$S_G(\mathrm{K}(h_2), X_2(h_2))$	$\dots$	$S_G(\mathrm{K}(h_n), X_2(h_n))$
. . .	. . .	. . .	$\dots$ . .	. . .
$\boldsymbol{X_i}$	$S_G(\mathrm{K}(h_1), X_i(h_1))$	$S_G(\mathrm{K}(h_2), X_i(h_2))$	$\dots$	$S_G(\mathrm{K}(h_n), X_i(h_n))$

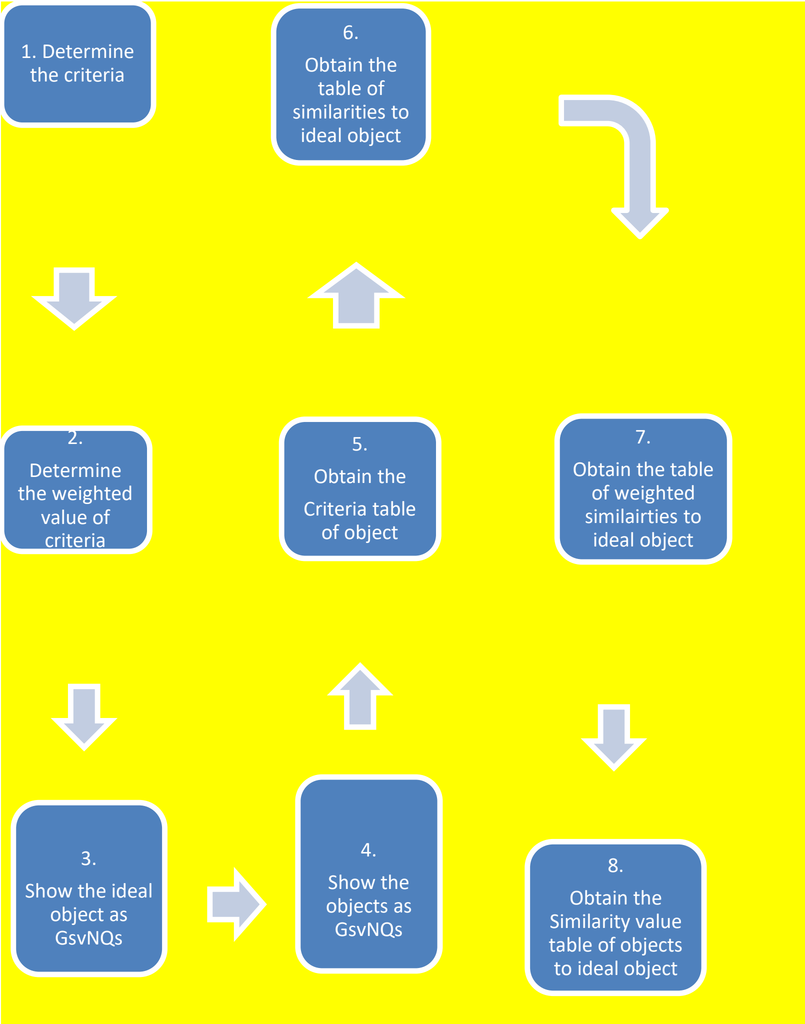
Step 7: Each criteria similarity value in Table 2 is multiplied by its own criteria weight value, and by adding the weighted similarity values for each object, the ideal object similarity values are obtained. Thus, we obtain Table 3.

Where, i = 1,2,…,n and $S_{Gi}(K, X_i) = \sum_{k=1}^{n} w_k \,.S_G(\text{K}(h_k), X_i(h_k))$.

Table 3. Weighted Similarity Table of Objects with Ideal Object

	$w_1 h_1$	$w_2 h_2$	…	$w_n h_n$	$\sum_{k=1}^{n} w_k \,. S_G(\text{K}(h_k), X_i(h_k))$
X_1	$w_1 . S_G(\text{K}(h_1), X_1(h_1))$	$w_2 . S_G(\text{K}(h_2), X_1(h_2))$	…	$w_n . S_G(\text{K}(h_n), X_1(h_n))$	$S_{G1}(K, X_1)$
X_2	$w_1 . S_G(\text{K}(h_1), X_2(h_1))$	$w_2 . S_G(\text{K}(h_2), X_2(h_2))$	…	$w_n . S_G(\text{K}(h_n), X_2(h_n))$	$S_{G2}(K, X_2)$
. . .	. . .	. . .	… . .	. . .	. . .
X_i	$w_1 . S_G(\text{K}(h_1), X_i(h_1))$	$w_2 . S_G(\text{K}(h_2), X_i(h_2))$	…	$w_n . S_G(\text{K}(h_n), X_i(h_n))$	$S_{Gi}(K, X_i)$

According to the values of S_{Gi} in Table 3, the objects closest to the ideal object are determined.



Graph 1: Diagram of the algorithm. [39]

4.2 Multi Criteria Decision Making Applications with Generalized Set Valued Neutrosophic Quadruple Numbers and Generalized Euclid Similarity Measure

In this section, we give an application of individuals with more than one disease to determine which of the known disease medications be good for their unknown disease using to algorithm in 4.2.

In this application, we find out which drugs used in 10 patients with 4 different known diseases are the most ideal treatment for an unknown disease**.** Where, diseases are taken as criteria and patients as objects for algorithm 4.2. It is clear that in solving such problems there is a need for a structure in which the known part is the unknown part and (T, I, F) known neutrosophic membership functions. Since each known disease will have separate medications and it will be investigated which results (true, indeterminate, false) these drugs will give in unknown diseases, a structure such as $(L_{s_i}T_{s_i}, M_{s_i}I_{s_i}, N_{s_i}F_{s_i})$, containing both cluster and T, I, F, will be needed.

Step 1: Let $H = \{h_1, h_2, \dots, h_n\}$ be set of diseases.

Step 2: Let $W = \{w_1, w_2, \dots, w_n\}$ be weighted value set of diseases such that

$w_1 = 0.2$ is weighted value of h_1,

w_2 = 0.3 is weighted value of h_2,

w_3 = 0.4 is weighted value of h_3,

w_4 = 0.1 is weighted value of h_4,

Step 3: We choose the ideal patient K such that

$$K = \{h_1: (\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\}, \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\}1, \emptyset 0, \emptyset 0),$$
$$h_2: (\{b_1, b_2, b_3, b_4, b_5, b_6, b_8, b_9\}, \{b_1, b_2, b_3, b_4, b_5, b_6, b_8, b_9\}1, \emptyset 0, \emptyset 0),$$
$$h_3: (\{c_1, c_2, c_3, c_5, c_6, c_7, c_8, c_9, c_{10}\}, \{c_1, c_2, c_3, c_5, c_6, c_7, c_8, c_9, c_{10}\}1, \emptyset 0, \emptyset 0),$$
$$h_4: (\{d_1, d_2, d_4, d_6, d_7, d_8, d_9, d_{10}\}, \{d_1, d_2, d_4, d_6, d_7, d_8, d_9, d_{10}\}1, \emptyset 0, \emptyset 0)\}.$$

Since K is ideal patient, the truth set of the known part of the criteria and the unknown part must be equal, and the truth value of the unknown part must be 1. Also, other sets must be empty and other values must be 0. For example, at ideal patient K;

for h_1,
the set $\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\}$ is regarded as a set of drugs that are good for disease h_1 and $(\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\}1, \emptyset 0, \emptyset 0)$ is regarded as a set of drugs that are good for unknown disease. Aslo, this applies to h_1 and other diseases.

Step 4: Let $X = \{X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9, X_{10}\}$ be set of patients such that

$$X_1 = \{h_1: (\{a_1, a_2, a_4, a_7\}, \{a_1, a_2\}(0.5), \{a_1, a_4\}(0.2), \{a_7\}(0.1)),$$
$$h_2: (\{b_3, b_4, b_5, b_6\}, \{b_4, b_5, b_6\}(0.6), \{b_4, b_6\}(0.1), \{b_4, b_5\}(0.1)),$$
$$h_3: (\{c_1, c_3, c_5, c_{10}\}, \{c_3\}(0.3), \{c_1, c_5\}(0.4), \{c_{10}\}(0.2)),$$
$$h_4: (\{d_1, d_2, d_8\}, \{d_1, d_8\}(0.4), \{d_8\}(0.1), \{d_2\}(0.2))\}$$
$$X_2 = \{h_1: (\{a_7, a_8, a_9\}, \{a_7\}(0.2), \{a_8, a_9\}(0.3), \{a_9\}(0.2)),$$
$$h_2: (\{b_1, b_3, b_8\}, \{b_1, b_3\}(0.5), \{b_3, b_8\}(0.2), \{b_8\}(0.1)),$$
$$h_3: (\{c_7, c_9, c_{10}\}, \{c_7\}(0.3), \{c_9\}(0.4), \{c_{10}\}(0.2)),$$
$$h_4: (\{d_6, d_8\}, \{d_8\}(0.4), \{d_6, d_8\}(0), \{d_6\}(0.6))\}$$
$$X_3 = \{h_1: (\{a_3, a_5, a_6\}, \{a_5, a_6\}(0.3), \{a_5\}(0.2), \{a_3\}(0.2)),$$
$$h_2: (\{b_1, b_2\}, \{b_1\}(0.7), \{b_1\}(0.1), \{b_2\}(0.1)),$$
$$h_3: (\{c_6, c_7, c_8\}, \{c_7\}(0.1), \{c_6, c_8\}(0.2), \{c_6\}(0.5)),$$
$$h_4: (\{d_7, d_{10}\}, \{d_{10}\}(0.4), \{d_7\}(0.2), \{d_7, d_{10}\}(0))\}$$
$$X_4 = \{h_1: (\{a_5, a_9\}, \{a_9\}(0.2), \{a_5\}(0.5), \{a_5\}(0.2)),$$
$$h_2: (\{b_3, b_6, b_9\}, \{b_3, b_9\}(0.1), \{b_6, b_9\}(0.2), \{b_3\}(0.5)),$$
$$h_3: (\{c_2, c_5, c_7, c_{10}\}, \{c_5, c_{10}\}(0.3), \{c_7, c_{10}\}(0.4), \{c_2\}(0.3)),$$
$$h_4: (\{d_2, d_8, d_9\}, \{d_8, d_9\}(0.6), \{d_2, d_9\}(0.1), \{d_2, d_8\}(0.1))\}$$
$$X_5 = \{h_1: (\{a_2, a_3, a_5, a_7, a_9\}, \{a_2, a_3, a_9\}(0.7), \{a_5, a_7, a_9\}(0.1), \{a_7\}(0.2)),$$
$$h_2: (\{b_2, b_3, b_5, b_8\}, \{b_2, b_5\}(0.4), \{b_3\}(0.2), \{b_8\}(0.3)),$$
$$h_3: (\{c_1, c_7, c_9, c_{10}\}, \{c_7, c_9, c_{10}\}(0.6), \{c_1\}(0.2), \{c_1, c_7\}(0.2)),$$
$$h_4: (\{d_2, d_4\}, \{d_2\}(0.3), \{d_4\}(0.3), \{d_4\}(0.2))\}$$
$$X_6 = \{h_1: (\{a_5, a_6, a_9\}, \{a_5, a_6\}(0.4), \{a_5\}(0.2), \{a_9\}(0.3)),$$
$$h_2: (\{b_3, b_4, b_8\}, \{b_8\}(0.2), \{b_3, b_8\}(0.1), \{b_4\}(0.5)),$$
$$h_3: (\{c_3, c_6, c_9\}, \{c_6, c_9\}(0.5), \{c_3\}(0.3), \{c_9\}(0.2)),$$
$$h_4: (\{d_1, d_4, d_7, d_9\}, \{d_1, d_4, d_9\}(0.6), \{d_1, d_9\}(0.1), \{d_7\}(0.2))\}$$
$$X_7 = \{h_1: (\{a_1, a_7\}, \{a_7\}(0.2), \{a_7\}(0.5), \{a_1\}(0.2)),$$
$$h_2: (\{b_2, b_4, b_6, b_8, b_9\}, \{b_4, b_6\}(0.2), \{b_3, b_8, b_9\}(0.3), \{b_2, b_6\}(0.4)),$$
$$h_3: (\{c_2, c_3, c_6, c_7\}, \{c_3, c_6, c_7\}(0.6), \{c_2\}(0.2), \{c_2, c_6, c_3\}(0.1)),$$
$$h_4: (\{d_4, d_8, d_{10}\}, \{d_8, d_{10}\}(0.7), \{d_4\}(0.1), \{d_4, d_{10}, \}(0.1))\}$$
$$X_8 = \{h_1: (\{a_2, a_4, a_6, a_7, a_8\}, \{a_4, a_6, a_7, a_8\}(0.7), \{a_2, a_4, a_6, a_7\}(0.1), \{a_4, a_6, a_7\}(0.1)),$$
$$h_2: (\{b_2, b_3, b_5, b_6, b_9\}, \{b_2, b_3\}(0.3), \{b_3, b_5, b_6, b_9\}(0.1), \{b_2, b_5, b_7\}(0.5)),$$

$h_3: \big(\{c_3, c_7, c_9\}, \{c_7, c_9\}(0.4), \{c_3, c_7\}(0.5), \{c_7\}(0.1)\big),$
$h_4: (\{d_8, d_9, d_{10}\}, \{d_8, d_{10}\}(0.3), \{d_8\}(0.3), \{d_9\}(0.2))\}$
$X_9 = \{h_1: \big(\{a_6, a_7, a_8\}, \{a_7, a_8\}(0.5), \{a_6, a_7\}(0.2), \{a_6, a_8\}(0.3)\big),$
$h_2: (\{b_3, b_4, b_6, b_8, b_9\}, \{b_4, b_6, b_8, b_9\}(0.7), \{b_3, b_4, b_6\}(0.1), \{b_3, b_4, b_6, b_8\}(0.1)),$
$h_3: \big(\{c_3, c_6, c_7\}, \{c_6, c_7\}(0.4), \{c_3, c_6\}(0.3), \{c_6\}(0.1)\big),$
$h_4: (\{d_4, d_7, d_{10}\}, \{d_7, d_{10}\}(0.5), \{d_4\}(0.3), \{d_4, d_{10}\}(0.2))\}$
$X_{10} = \{h_1: \big(\{a_2, a_3, a_7, a_8, a_9\}, \{a_7, a_8, a_9\}(0.4), \{a_2, a_7, a_8, a_9\}(0.1), \{a_3\}(0.4)\big),$
$h_2: (\{b_2, b_3, b_5, b_6, b_8, b_9\}, \{b_3, b_6, b_8, b_9\}(0.6), \{b_2, b_6, b_8, b_9\}(0.2), \{b_3, b_5\}(0.2)),$
$h_3: \big(\{c_3, c_7, c_{10}\}, \{c_7, c_{10}\}(0.6), \{c_3, c_7\}(0.1), \{c_3\}(0.2)\big),$
$h_4: (\{d_1, d_7, d_9, d_{10}\}, \{d_1, d_{10}\}(0.3), \{d_7, d_9, d_{10}\}(0.1), \{d_1, d_9\}(0.4))\}$

Step 5: Let's show the diseases according to patients in step 4 in Table 4.

Table 4. Table of Diseases

	h_1	h_2	h_3	h_4
X_1	$(\{a_1, a_2, a_4, a_7\}$, $\{a_1, a_2\}(0.5)$, $\{a_1, a_4\}(0.2)$, $\{a_7\}(0.1))$	$(\{b_3, b_4, b_5, b_6\}$, $\{b_4, b_5, b_6\}(0.6)$, $\{b_4, b_6\}(0.1)$, $\{b_4, b_5\}(0.1))$	$(\{c_1, c_3, c_5, c_{10}\}$, $\{c_3\}(0.3)$, $\{c_1, c_5\}(0.4)$, $\{c_{10}\}(0.2))$	$(\{d_1, d_2, d_8\}$, $\{d_1, d_8\}(0.4)$, $\{d_8\}(0.1)$, $\{d_2\}(0.2))$
X_2	$(\{a_7, a_8, a_9\}$, $\{a_7\}(0.2)$, $\{a_8, a_9\}(0.3)$, $\{a_9\}(0.2))$	$(\{b_1, b_3, b_8\}$, $\{b_1, b_3\}(0.5)$, $\{b_3, b_8\}(0.2)$, $\{b_8\}(0.1))$	$(\{c_7, c_9, c_{10}\}$, $\{c_7\}(0.3)$, $\{c_9\}(0.4)$, $\{c_{10}\}(0.2))$	$(\{d_6, d_8\}$, $\{d_8\}(0.4)$, $\{d_6, d_8\}(0)$, $\{d_6\}(0.6))$
X_3	$(\{a_3, a_5, a_6\}$, $\{a_5, a_6\}(0.3)$, $\{a_5\}(0.2)$, $\{a_3\}(0.2))$	$(\{b_1, b_2\}$, $\{b_1\}(0.7)$, $\{b_1\}(0.1)$, $\{b_2\}(0.1))$	$(\{c_6, c_7, c_8\}$, $\{c_7\}(0.1)$, $\{c_6, c_8\}(0.2)$, $\{c_6\}(0.5))$	$(\{d_7, d_{10}\}$, $\{d_{10}\}(0.4)$, $\{d_7\}(0.2)$, $\{d_7, d_{10}\}(0))$
X_4	$(\{a_5, a_9\}$, $\{a_9\}(0.2)$, $\{a_5\}(0.5)$, $\{a_5\}(0.2))$	$(\{b_3, b_6, b_9\}$, $\{b_3, b_9\}(0.1)$, $\{b_6, b_9\}(0.2)$, $\{b_3\}(0.5))$	$(\{c_2, c_5, c_7, c_{10}\}$, $\{c_5, c_{10}\}(0.3)$, $\{c_7, c_{10}\}(0.4)$, $\{c_2\}(0.3))$	$(\{d_2, d_8, d_9\}$, $\{d_8, d_9\}(0.6)$, $\{d_2, d_9\}(0.1)$, $\{d_2, d_8\}(0.1))$

X_5	$(\{a_2, a_3, a_5, a_7, a_9\}$, $\{a_2, a_3, a_9\}(0.7)$, $\{a_5, a_7, a_9\}(0.1)$, $\{a_7\}(0.2))$	$(\{b_2, b_3, b_5, b_8\}$, $\{b_2, b_5\}(0.4)$, $\{b_3\}(0.2)$, $\{b_8\}(0.3))$	$(\{c_1, c_7, c_9, c_{10}\}$, $\{c_7, c_9, c_{10}\}(0.6)$, $\{c_1\}(0.2)$, $\{c_1, c_7\}(0.2))$	$(\{d_2, d_4\}$, $\{d_2\}(0.3)$, $\{d_4\}(0.3)$, $\{d_4\}(0.2))$
X_6	$(\{a_5, a_6, a_9\}$, $\{a_5, a_6\}(0.4)$, $\{a_5\}(0.2)$, $\{a_9\}(0.3))$	$(\{b_3, b_4, b_8\}$, $\{b_8\}(0.2)$, $\{b_3, b_8\}(0.1)$, $\{b_4\}(0.5))$	$(\{c_3, c_6, c_9\}$, $\{c_6, c_9\}(0.5)$, $\{c_3\}(0.3)$, $\{c_9\}(0.2))$	$(\{d_1, d_4, d_7, d_9\}$, $\{d_1, d_4, d_9\}(0.6)$, $\{d_1, d_9\}(0.1)$, $\{d_7\}(0.2))$
X_7	$(\{a_1, a_7\}$, $\{a_7\}(0.2)$, $\{a_7\}(0.5)$, $\{a_1\}(0.2))$	$(\{b_2, b_4, b_6, b_8, b_9\}$, $\{b_4, b_6\}(0.2)$, $\{b_3, b_8, b_9\}(0.3)$, $\{b_2, b_6\}(0.4))$	$(\{c_2, c_3, c_6, c_7\}$, $\{c_3, c_6, c_7\}(0.6)$, $\{c_2\}(0.2)$, $\{c_2, c_6, c_3\}(0.1))$	$(\{d_4, d_8, d_{10}\}$, $\{d_8, d_{10}\}(0.7)$, $\{d_4\}(0.1)$, $\{d_4, d_{10}, \}(0.1))$
X_8	$(\{a_2, a_4, a_6, a_7, a_8\}$, $\{a_4, a_6, a_7, a_8\}(0.7)$, $\{a_2, a_4, a_6, a_7\}(0.1)$, $\{a_4, a_6, a_7\}(0.1))$	$(\{b_2, b_3, b_5, b_6, b_9\}$, $\{b_2, b_3\}(0.3)$, $\{b_3, b_5, b_6, b_9\}(0.1)$, $\{b_2, b_5, b_7\}(0.5))$	$(\{c_3, c_7, c_9\}$, $\{c_7, c_9\}(0.4)$, $\{c_3, c_7\}(0.5)$, $\{c_7\}(0.1))$	$(\{d_8, d_9, d_{10}\}$, $\{d_8, d_{10}\}(0.3)$, $\{d_8\}(0.3)$, $\{d_9\}(0.2))$
X_9	$(\{a_6, a_7, a_8\}$, $\{a_7, a_8\}(0.5)$, $\{a_6, a_7\}(0.2)$, $\{a_6, a_8\}(0.3))$	$(\{b_3, b_4, b_6, b_8, b_9\}$, $\{b_4, b_6, b_8, b_9\}(0.7)$, $\{b_3, b_4, b_6\}(0.1)$, $\{b_3, b_4, b_6, b_8\}(0.1))$	$(\{c_3, c_6, c_7\}$, $\{c_6, c_7\}(0.4)$, $\{c_3, c_6\}(0.3)$, $\{c_6\}(0.1))$	$(\{d_4, d_7, d_{10}\}$, $\{d_7, d_{10}\}(0.5)$, $\{d_4\}(0.3)$, $\{d_4, d_{10}\}(0.2))$
X_{10}	$(\{a_2, a_3, a_7, a_8, a_9\}$, $\{a_7, a_8, a_9\}(0.4)$, $\{a_2, a_7, a_8, a_9\}(0.1)$, $\{a_3\}(0.4))$	$(\{b_2, b_3, b_5, b_6, b_8, b_9\}$, $\{b_3, b_6, b_8, b_9\}(0.6)$, $\{b_2, b_6, b_8, b_9\}(0.2)$, $\{b_3, b_5\}(0.2))$	$(\{c_3, c_7, c_{10}\}$, $\{c_7, c_{10}\}(0.6)$, $\{c_3, c_7\}(0.1)$, $\{c_3\}(0.2))$	$(\{d_1, d_7, d_9, d_{10}\}$, $\{d_1, d_{10}\}(0.3)$, $\{d_7, d_9, d_{10}\}(0.1)$, $\{d_1, d_9\}(0.4))$

Step 6: We obtain diseases similarity in Table 5.

Table 5. Diseases Similarity Table

	h_1	h_2	h_3	h_4
X_1	0.4102	0.4580	0.3193	0.3907
X_2	0.3119	0.4073	0.3119	0.3906

X_3	0.3526	0.4073	0.2619	0.3906
X_4	0.2712	0.2740	0.3102	0.4407
X_5	0.4590	0.3659	0.4179	0.3240
X_6	0.3526	0.2989	0.3693	0.4413
X_7	0.2712	0.3080	0.4345	0.4573
X_8	0.4836	0.3413	0.3360	0.3407
X_9	0.3693	0.4927	0.3693	0.3740
X_{10}	0.3757	0.4520	0.4193	0.3854

Step 7: We obtain weighted similarity of patients with ideal patient in Table 6.

Table 6. Weighted Similarity Table of Patients with Ideal Patient

	$(0,2).h_1$	$(0,3).h_2$	$(0,4).h_3$	$(0,1).h_4$	$\sum_{k=1}^{4} w_k \,.\, S_G(\mathrm{K}(h_k),X_i(h_k))$
X_1	0,0820	0,1374	0,1277	0,0390	$S_{G1}(K,X_1) = 0{,}3861$
X_2	0,0623	0,1221	0,1247	0,0390	$S_{G2}(K,X_2) =0{,}3481$
X_3	0,0705	0,1221	0,1047	0,0390	$S_{G3}(K,X_3) = 0{,}3363$
X_4	0,0542	0,0822	0,1240	0,0440	$S_{G4}(K,X_4) = 0{,}3044$
X_5	0,0918	0,1097	0,1671	0,0324	$S_{G5}(K,X_5) = 0{,}4010$
X_6	0,0705	0,0896	0,1477	0,0441	$S_{G6}(K,X_6) = 0{,}3519$
X_7	0,0542	0,0924	0,1738	0,0457	$S_{G7}(K,X_7) = 0{,}3661$
X_8	0,0967	0,1023	0,1344	0,0340	$S_{G8}(K,X_8) = 0{,}3674$
X_9	0,0738	0,1478	0,1477	0,0374	$S_{G9}(K,X_9) = 0{,}4067$

X_{10}	0,0751	0,1356	0,1677	0,0385	$S_{G10}(K, X_{10}) = 0{,}4169$

According to the values in the Table 6, the patients with the best treatment are

$$X_{10}, X_9, X_5, X_1, X_8, X_7, X_6, X_2, X_3, X_4$$

respectively.

4.3 Comparison Analysis

In this section, we compare the results of the 4.2 Application with the results of some similarity measures previously defined for neutrosophic sets. However, since generalized set valued neutrosophic numbers are used in 5.2 Application, only the components in the unknown parts (T, I, F) of the generalized set valued neutrosophic numbers be taken for neutrosophic similarity measures in this comparison.

a)

In 4.2 Application,

if we use the Euclid similarity measure in Definition 2.3 [17], we obtain weighted similarity of patients with ideal patient in Table 7.

Table 7. Weighted Similarity Table of Patients with Ideal Patient according to Euclid similarity measure

$\boldsymbol{X_1}$	0,7887
$\boldsymbol{X_2}$	0,7476
$\boldsymbol{X_3}$	0,7554
$\boldsymbol{X_4}$	0,7017
$\boldsymbol{X_5}$	0,8132
$\boldsymbol{X_6}$	0,7609
$\boldsymbol{X_7}$	0,7448
$\boldsymbol{X_8}$	0,7598
$\boldsymbol{X_9}$	0,8146
$\boldsymbol{X_{10}}$	0,8141

According to the values in the Table 7, the patients with the best treatment are

$$X_9, X_{10}, X_5, X_1, X_6, X_8, X_3, X_2, X_7, X_4$$

respectively.

b) In 4.2 Application,

if we use the Hamming similarity measure in Definition 2.4 [17], we obtain weighted similarity of patients with ideal patient in Table 8.

Table 8. Weighted Similarity Table of Patients with Ideal Patient according to Hamming similarity measure

X_1	0,6832
X_2	0,6198
X_3	0,6364
X_4	0,5332
X_5	0,7032
X_6	0,6297
X_7	0,6399
X_8	0,6365
X_9	0,7164
X_{10}	0,7131

According to the values in the Table 8, the patients with the best treatment are

$$X_9, X_{10}, X_5, X_1, X_7, X_8, X_3, X_6, X_2, X_4$$

respectively.

c)) In 4.2 Application,

if we use the similarity measure in Definition 2.7 [5], we obtain weighted similarity of patients with ideal patient in Table 9.

Table 9. Weighted Similarity Table of Patients with Ideal Patient according to similarity measure in Definition 2.5

X_1	0,4238
X_2	0,3532
X_3	0,3822
X_4	0,2779
X_5	0,4691
X_6	0,3764
X_7	0,3974
X_8	0,3933
X_9	0,4747
X_{10}	0,4741

According to the values in the Table 9, the patients with the best treatment are

$$X_9, X_{10}, X_5, X_1, X_7, X_8, X_3, X_6, X_2, X_4$$

respectively.

We give Comparison of similarity measures in Table 10.

Table 10. Comparison of similarity measures

Similarity Measure	Result
Generalized Euclid Similarity Measure (proposed method)	$X_{10}, X_9, X_5, X_1, X_8, X_7, X_6, X_2, X_3, X_4$
Euclid Similarity Measure [17]	$X_9, X_{10}, X_5, X_1, X_6, X_8, X_3, X_2, X_7, X_4$
Hamming Similarity Measure [17]	$X_9, X_{10}, X_5, X_1, X_7, X_8, X_3, X_6, X_2, X_4$
Similarity Measure in Definition 2.5 [5]	$X_9, X_{10}, X_5, X_1, X_7, X_8, X_3, X_6, X_2, X_4$

From Table 10, we obtain different result from Euclid similarity measure [17], Hamming similarity measure [17] and similarity measure in Definition 2.7 [5].

5 Discussion and Conclusions

In this article, we define firstly generalized Euclid distance measure and generalized Euclid similarity measure based on generalized set valued neutrosophic quadruple numbers. Also, we show that generalized Euclid distance measure and generalized Euclid similarity measure satisfy the distance measure conditions and similarity measure conditions, respectively. Furthermore, we define a score function for generalized Euclid similarity measure.

In addition, we generalized algorithm, for single valued neutrosophic set, based on generalized Euclid similarity measure and generalized set valued neutrosophic quadruple numbers. Using this algorithm, we give an example of individuals with more than one disease to determine which of the known disease medications will be good for their unknown disease (for example covid-19). We compared the results we obtained in this example with the results obtained in neutrosophic numbers and showed that we obtained different results. For example, in Table 10, we obtain different result from Euclid similarity measure [17], Hamming similarity measure [17] and similarity measure in Definition 2.7 [5]. It is clear that in solving such problems there is a need for a structure in which the known part is the unknown part and (T, I, F) known neutrosophic membership functions. Since each known disease will have separate medications and it will be investigated which results (true, indeterminate, false) these drugs will give in unknown diseases, a structure such as $(L_{s_i}T_{s_i}, M_{s_i}I_{s_i}, N_{s_i}F_{s_i})$, containing both cluster and T, I, F, will be needed. For this reason, using generalized Euclid measures based on generalized set valued neutrosophic quadruple numbers in solving such problems can give better results.

Also, using the similarity measures and algorithm in this article, solutions can be found to other problems in the medical field. In addition, decision making applications can be obtained with the help of these similarity measures and algorithms for other branches of science.

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An Introduction to Neutrosophic Vague Refined set

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Abstract. The aim of this paper is to introduce a new set namely neutrosophic vague refined set to handle inconsistent and indeterminate information. Set theoretic operations depend on neutrosophic vague refined set are discussed along with desired properties. Also, algebraic operations on neutrosophic vague refined set are studied with suitable examples.

Keywords: Neutrosophic set, Vague set, Neutrosophic refined set, Neutrosophic Vague set, Neutrosophic vague refined set.

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1. **Introduction**

Many of the philosophers make an attempt to model unsure information using classical mathematics might not achieve success manner. This is because the uncertainty that is just too convoluted and in addition not understandably outlined. Several scientists place an effort to search out applicable solutions to some mathematical issues that can't be resolved by ancient strategies. These issues have an effect truth that traditional strategies cannot work out the issues of uncertainty in engineering, economy, decision-making, medicine etc. Therefore, many alternative theories was developed to clear up uncertainty and vagueness together with the fuzzy set theory[28], vague set theory[6] and rough set theory [9]. But, these theories cannot deal with uncertainty and consistent info.

In the theory of classical set, the membership of the elements in a non-empty set is of two conditions. As per that conditions, an element either belongs to the set or doesn't belong to the set. In fuzzy set theory, an element is assigned by the grade of membership value within the closed unit interval [0,1]. Therefore, a fuzzy set A in the universal set of discourse X is a function f: A $\rightarrow$ [0,1] and often this function is mentioned as the membership function.

Later, because of the generalisation of fuzzy set Atanassov in 1986 introduced the intuitionistic fuzzy set[1] where an element is assigned by values of truth membership and false membership ranges from [0,1] individually. Vague set theory[6] in 1993 was initially found by

Gau et.al. which is an extension of theory of fuzzy set. Vague sets are taken into account as an efficient tool to influence uncertainty since it provides additional info as compared to fuzzy sets[28]. Many studies have unconcealed that, several researchers have combined vague set with other theories.

Neutrosophic set was developed by Smarandache[21] in 1998 considered as the generalization of probability set, fuzzy set[28] and intuitionistic fuzzy set[1]. The neutrosophic set has 3 independent membership functions. Not like fuzzy set and intuitionistic fuzzy set, the membership functions in neutrosophic sets are truth, indeterminate and falsity.

As a generalisation of set theory, Yager[25] first of all introduced new theory named as theory of bags which in turn gives the concept of multiset (i.e) the element may take place more than once. Subsequently, the idea of multiset was originally initiated by Blizard[2] and Calude et al.[3]. Many authors now and then created variety of generalization of theory of set. As the generalisation of multiset, many researchers [4, 8, 16, 17, 18, 22, 24] discussed additional properties on fuzzy multiset which takes place more than once with possibly same or different membership values. Shinoj et.al.[20] created associate extension of combining the concept of fuzzy multiset by an intuitionstic fuzzy set, which is known as intuitionstic fuzzy multiset. By the study on intuitionstic fuzzy multiset, loads of excellent results are achieved by researchers [5, 10, 11, 12, 13, 14, 15]. The ideas of fuzzy multiset and intuitionistic fuzzy multiset fails to influence uncertainty. Chatterjee et. al.[3] initiated single valued neutrosophic multiset very well.

In 2013, by extending classical neutrosophic logic Smarandache[21] gave n-valued refined neutrosophic logic and its applications in which the neutrosophic components T,I,F are refined to n number of components. Neutrosophic Refined set are defined by the generalisation of fuzzy multi set, intuitionistic fuzzy multi set. Irfan Deli[7] in 2016 extended refined neutrosophic set to refined neutrosophic soft set. Ye and Ye[26] developed single valued neutrosophic set and operations with laws. Ye et. al.[27] originated generalized distance measure and its similarity measure between single valued neutrosophic multi sets. Additionally, they applied the measures to a diagnosing medical side with incomplete, indeterminate and inconsistent info.

This paper is organized as follows: The essential definitions of Fuzzy set, Intuitionistic fuzzy set, Vague set, Neutrosophic set, Neutrosophic vague set and few of its properties that are helpful for the discussion have been presented. We have established a new concept by extending neutrosophic vague set namely neutrosophic vague refined(multi) set and few of its properties and operations on neutrosophic vague refined(multi) set are discussed. Additionally, algebraic operations of neutrosophic vague refined(multi) set along with some examples are given.

2. Preliminaries

In this section, we recall the useful basic definitions and related results for developing the desired set.

Definition 2.1. [28](Fuzzy Set)
Let X be a non-empty set in the universe of discourse. A fuzzy set A_F on X is defined as follows:

$$A_F = \{\langle x, \alpha_A(x)\rangle : x \in X\}$$

where $\alpha_A(x)$: $X \to [0,1]$ is represented as a grade of membership function of the fuzzy set A_F. It is denoted as A_F.

Definition 2.2. [1](Intuitionistic Fuzzy Set)
Let X be a non-empty set in the universe of discourse. A intuitionistic fuzzy set A_{IF} on X is an object defined as follows:

$$A_{IF} = \{\langle x, \alpha_A(x), \beta_A(x)\rangle : x \in X\}$$

where $\alpha_A(x) : X \to [0,1]$, $\beta_A(x) : X \to [0,1]$ represents the degree of membership function and degree of non-membership function respectively of the element x∈X to the set A_{IF}, which is a subset of X. And also for every element x∈X, membership functions $\alpha_A(x)$, $\beta_A(x)$ ranges $0 \leq \alpha_A(x) + \beta_A(x) \leq 1$. It is denoted as A_{IF}.

Definition 2.3. [6](Vague Set)
Let X be a non-empty set in the universe of discourse. A Vague set A_V in X is defined as follows:

$$A_V = \{\langle x, \alpha_A(x), 1 - \beta_A(x)\rangle : x \in X\}$$

where α_A: X→[0,1], β_A: X→[0,1] denotes truth membership function and false membership function respectively and $\alpha_A(x) + \beta_A(x) \leq 1$. Here $\alpha_A(x)$ is a lower bound on the grade of membership of x derived from the evidence for x and $\beta_A(x)$ is a lower bound on the grade of membership of the negation of x derived from the evidence against x.

Definition 2.4. [21](Neutrosophic set)
Let X be a non-empty set in the universe of discourse. A neutrosophic set A_N in X is defined as follows:

$$A_N = \{\langle x, \mu_A(x), \eta_A(x), \nu_A(x)\rangle : x \in X\}$$

where $\mu_A(x), \eta_A(x), \nu_A(x)$ represents truth membership function, indeterminate membership function and a falsity membership function of the element x∈ X to the set A_N respectively and the condition $^{-}0 \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 3^{+}$ holds.

Definition 2.5. [19](Neutrosophic Vague set)

Let X be the non-empty set in universe of discourse. A Neutrosophic vague set (M_{NV}in short) is of the form,

$$M_{NV} = \{\langle x, (\hat{T}_{M_{NV}}(x), \hat{I}_{M_{NV}}(x), \hat{F}_{M_{NV}}(x)\rangle : x \in X\}$$

whose truth membership, indeterminacy membership and false membership functions is defined as

$$\hat{T}_{M_{NV}}(x) = [T^-, T^+], \hat{I}_{M_{NV}}(x) = [I^-, I^+], \hat{F}_{M_{NV}}(x) = [F^-, F^+]$$

where,

$$T^+ = 1 - F^-, F^+ = 1 - T^-$$

and the condition $^-0 \leq T^- + I^- + F^- \leq 2^+$, $^-0 \leq T^+ + I^+ + F^+ \leq 2^+$ holds.

Definition 2.6. [19]

Let M_{NV} and N_{NV} be two neutrosophic vague set of the universe U. For x∈X, M_{NV} is included by N_{NV} if $\hat{T}_{M_{NV}}(x) \leq \hat{T}_{N_{NV}}(x); \hat{I}_{M_{NV}}(x) \geq \hat{I}_{N_{NV}}(x); \hat{F}_{M_{NV}}(x) \geq \hat{F}_{N_{NV}}(x)$. It is denoted as $M_{NV} \subseteq N_{NV}$.

Definition 2.7. [19]

The complement of neutrosophic vague set M_{NV} is defined by $\hat{T}^c_{M_{NV}}(x)= [1 - T^+, 1 - T^-]$, $\hat{I}^c_{M_{NV}}(x)= [1 - I^+, 1 - I^-]$, $\hat{F}^c_{M_{NV}}(x)= [1 - F^+, 1 - F^-]$.It is denoted as M^c_{NV},

Definition 2.8. [19]

The union of two neutrosophic vague sets M_{NV} and N_{NV} is given by P_{NV} where P_{NV}= $M_{NV} \cup N_{NV}$, whose truth-membership, indeterminacy-membership and false-membership functions are

$$\hat{T}_{P_{NV}}(\mathrm{x})=\left[max(T^-_{M_{NV_x}}, T^-_{N_{NV_x}}), max(T^+_{M_{NV_x}}, T^+_{N_{NV_x}})\right]$$
$$\hat{I}_{P_{NV}}(\mathrm{x})=\left[min(I^-_{M_{NV_x}}, I^-_{N_{NV_x}}), min(I^+_{M_{NV_x}}, I^+_{N_{NV_x}})\right]$$
$$\hat{F}_{P_{NV}}(\mathrm{x})=\left[min(F^-_{M_{NV_x}}, F^-_{N_{NV_x}}), min(F^+_{M_{NV_x}}, F^+_{N_{NV_x}})\right]$$

Definition 2.9. [19]

The intersection of two neutrosophic vague sets M_{NV} and N_{NV} is given by P_{NV} where P_{NV}= $M_{NV} \cap N_{NV}$, whose truth-membership, indeterminacy-membership and false-membership functions are

$$\hat{T}_{P_{NV}}(\mathrm{x})=\left[min(T^-_{M_{NV_x}}, T^-_{N_{NV_x}}), min(T^+_{M_{NV_x}}, T^+_{N_{NV_x}})\right]$$
$$\hat{I}_{P_{NV}}(\mathrm{x})=\left[max(I^-_{M_{NV_x}}, I^-_{N_{NV_x}}), max(I^+_{M_{NV_x}}, I^+_{N_{NV_x}})\right]$$
$$\hat{F}_{P_{NV}}(\mathrm{x})=\left[max(F^-_{M_{NV_x}}, F^-_{N_{NV_x}}), max(F^+_{M_{NV_x}}, F^+_{N_{NV_x}})\right]$$

Definition 2.10. [22](Neutrosophic Refined set)

Let X be a non empty set of universe of discourse. A neutrosophic refined set (briefly NRS) on X can be defined as follows:

$$A_{NR}=\{\langle x,(T^1_{A_{NR}}(x),T^2_{A_{NR}}(x),...T^p_{A_{NR}}(x)),(I^1_{A_{NR}}(x),I^2_{A_{NR}}(x),...I^p_{A_{NR}}(x)),(F^1_{A_{NR}}(x),F^2_{A_{NR}}(x),...F^p_{A_{NR}}(x))\rangle : x\in X\}$$

where

$$T^1_{A_{NR}}(x),T^2_{A_{NR}}(x),...T^p_{A_{NR}}(x):X\to[0,1],$$
$$I^1_{A_{NR}}(x),I^2_{A_{NR}}(x),...I^p_{A_{NR}}(x):X\to[0,1],$$
$$F^1_{A_{NR}}(x),F^2_{A_{NR}}(x),...F^p_{A_{NR}}(x):X\to[0,1]$$

such that 0$\leq T^j_{A_{NR}}(x)+I^j_{A_{NR}}(x)+F^j_{A_{NR}}(x)\leq 3$ for j=1,2,...,p for any x∈X, $T^1_{A_{NR}}(x),T^2_{A_{NR}}(x),...T^p_{A_{NR}}(x)$, $I^1_{A_{NR}}(x),I^2_{A_{NR}}(x),...I^p_{A_{NR}}(x)$ and $F^1_{A_{NR}}(x),F^2_{A_{NR}}(x),...F^p_{A_{NR}}(x)$ is the truth-membership sequence, indeterminate-membership sequence and falsity-membership sequence of the element x, respectively. Also, p is called the dimension of neutrosophic refined set A_{NR}.

3. Neutrosophic Vague Refined sets

In this section, we introduce the new set namely neutrosophic vague refined set with suitable example.

Definition 3.1. (Neutrosophic vague refined set)

Let X be a non empty set in the universe of discourse. A neutrosophic vague refined set (in short NVRS)D_{NVR} on X can be defined by the form

$$D_{NVR}=\{\langle x,(\hat{T}^1_{D_{NVR}}(x),\hat{T}^2_{D_{NVR}}(x),....\hat{T}^u_{D_{NVR}}(x)),(\hat{I}^1_{D_{NVR}}(x),\hat{I}^2_{D_{NVR}}(x),....\hat{I}^u_{D_{NVR}}(x)),(\hat{F}^1_{D_{NVR}}(x),\hat{F}^2_{D_{NVR}}(x),....\hat{F}^u_{D_{NVR}}(x))\rangle : x\in X\}$$

The truth-membership, indeterminacy-membership and falsity-membership functions of neutrosophic vague refined set is defined as,

$$\hat{T}^j_{D_{NVR}}=[T^-_j,T^+_j],\hat{I}^j_{D_{NVR}}=[I^-_j,I^+_j],\hat{F}^j_{D_{NVR}}=[F^-_j,F^+_j]$$

and

$$T^+_j=1-F^-_j, F^+_j=1-T^-_j$$

where

$\hat{T}^1_{D_{NVR}}(x),\hat{T}^2_{D_{NVR}}(x),...,\hat{T}^u_{D_{NVR}}(x):X\longrightarrow P[0,1],\hat{I}^1_{D_{NVR}}(x),\hat{I}^2_{D_{NVR}}(x),...,\hat{I}^u_{D_{NVR}}(x):X\longrightarrow P[0,1],\hat{F}^1_{D_{NVR}}(x),\hat{F}^2_{D_{NVR}}(x),...,\hat{F}^u_{D_{NVR}}(x):X\longrightarrow P[0,1]$ such that $^-0\leq\hat{T}^j_{D_{NVR}}(x)+\hat{I}^j_{D_{NVR}}(x)+\hat{F}^j_{D_{NVR}}(x)\leq 2^+$ for j=1,2,...,u for any element x ∈ X and P[0,1] is the power set of [0,1].

Here, $\hat{T}^1_{D_{NVR}}(x),\hat{T}^2_{D_{NVR}}(x),...,\hat{T}^u_{D_{NVR}}(x),\hat{I}^1_{D_{NVR}}(x),\hat{I}^2_{D_{NVR}}(x),...,\hat{I}^u_{D_{NVR}}(x)$, $\hat{F}^1_{D_{NVR}}(x),\hat{F}^2_{D_{NVR}}(x),...,\hat{F}^u_{D_{NVR}}(x)$ is represented as truth membership sequence, indeterminate membership sequence and falsity membership sequence of the element x respectively. And also, u is called the dimension of neutrosophic vague refined set D_{NVR}.

Remark 3.2.

We arrange the membership sequence in decreasing order but the corresponding indeterminate sequence and non-membership sequence may not be in increasing or decreasing order.

Example 3.3.

Let X={a,b} be any non empty set. Then

D_{NVR}=$\{\langle a, ([0.1, 0.6], [0.5, 0.8], [0.1, 0.8]), ([0.3, 0.5], [0.8, 0.9], [0.6, 0.8]), ([0.4, 0.9], [0.2, 0.5],$
$[0.2, 0.9]), b, ([0.5, 0.8], [0.3, 0.9], [0.2, 0.5]), ([0.7, 0.8], [0.3, 0.4], [0.2, 0.3]), ([0.2, 0.5],$
$[0.1, 0.7], [0.5, 0.8])\rangle\}$

is said to be a neutrosophic vague refined subset of X.

Definition 3.4. (Null set)

A neutrosophic vague refined sets on the universe X is said to be a null neutrosophic vague refined set denoted by 0_{NVR} if

$$\hat{T}^j_{P_{NVR}}(x) = [0, 0], \hat{I}^j_{P_{NVR}}(x) = [1, 1], \hat{F}^j_{P_{NVR}}(x) = [1, 1]$$

for j=1,2,3,...u and x∈ X.

Definition 3.5. (Absolute set)

A neutrosophic vague refined sets on the universe X is said to be a absolute neutrosophic vague refined set denoted by 1_{NVR} if

$$\hat{T}^j_{P_{NVR}}(x) = [1, 1], \hat{I}^j_{P_{NVR}}(x) = [0, 0], \hat{F}^j_{P_{NVR}}(x) = [0, 0]$$

for j=1,2,3,...u and x∈ X.

4. Set Theoretic Operations on Neutrosophic Vague Refined Set

In this section, we study some of the set theoretic operations on neutrosophic vague refined set with suitable examples.

Definition 4.1. (Union)

Let P_{NVR} and Q_{NVR} be two neutrosophic vague refined sets in the universal set X, where

$$P_{NVR} = \{\langle(\hat{T}^1_{P_{NVR}}(x), \hat{T}^2_{P_{NVR}}(x), ...\hat{T}^u_{P_{NVR}}(x)), (\hat{I}^1_{P_{NVR}}(x), \hat{I}^2_{P_{NVR}}(x), ...\hat{I}^u_{P_{NVR}}(x)),$$
$$(\hat{F}^1_{P_{NVR}}(x), \hat{F}^2_{P_{NVR}}(x), ...\hat{F}^u_{P_{NVR}}(x))\rangle : x \in X\ \}$$
$$Q_{NVR} = \{\langle(\hat{T}^1_{Q_{NVR}}(x), \hat{T}^2_{Q_{NVR}}(x), ...\hat{T}^v_{Q_{NVR}}(x)), (\hat{I}^1_{Q_{NVR}}(x), \hat{I}^2_{Q_{NVR}}(x), ...\hat{I}^v_{Q_{NVR}}(x)),$$
$$(\hat{F}^1_{Q_{NVR}}(x), \hat{F}^2_{Q_{NVR}}(x), ...\hat{F}^v_{Q_{NVR}}(x))\rangle : x \in X\ \}$$

Then, the neutrosophic vague refined union of two sets P_{NVR} and Q_{NVR} denoted by S_{NVR}

where $S_{NVR} = P_{NVR} \cup Q_{NVR}$, whose truth membership function, indeterminate membership function and false membership function is

$$\hat{T}^j_{S_{NVR}}(x) = max\{\hat{T}^j_{P_{NVR}}(x), \hat{T}^j_{Q_{NVR}}(x)\},$$
$$\hat{I}^j_{S_{NVR}}(x) = min\{\hat{I}^j_{P_{NVR}}(x), \hat{I}_{Q_{NVR}}(x)\},$$
$$\hat{F}^j_{S_{NVR}}(x) = min\{\hat{F}^j_{P_{NVR}}(x), \hat{F}^j_{Q_{NVR}}(x)\}$$

∀ x∈X, j=1,2,...,r.

Example 4.2.

Let X= {a,b} be the non empty set. Let the sets P_{NVR} and Q_{NVR} be defined as:

$$P_{NVR} = \{\langle a, ([0.3, 0.4], [0.4, 0.5], [0.4, 0.6]), ([0.5, 0.6], [0.4, 0.5], [0.1, 0.2]), ([0.6, 0.7], [0.5, 0.6], [0.4, 0.6]), b, ([0.4, 0.6], [0.4, 0.5], [0.4, 0.7]), ([0.5, 0.6], [0.4, 0.5], [0.1, 0.3]), ([0.4, 0.6], [0.5, 0.6], [0.3, 0.6])\rangle\},$$
$$Q_{NVR} = \{\langle a, ([0.1, 0.2], [0.3, 0.5], [0.1, 0.3]), ([0.7, 0.8], [0.6, 0.8], [0.2, 0.3]), ([0.8, 0.9], [0.5, 0.7], [0.7, 0.9]), b, ([0.3, 0.6], [0.7, 0.8], [0.2, 0.4]), ([0.6, 0.8], [0.8, 0.9], [0.2, 0.3]), ([0.4, 0.7], [0.2, 0.3], [0.6, 0.8])\rangle\}$$

Then, the union of two neutrosophic vague refined sets P_{NVR} and Q_{NVR} is

$$S_{NVR} = \{\langle a, ([0.3, 0.4], [0.4, 0.5], [0.4, 0.6]), ([0.5, 0.6], [0.4, 0.5], [0.1, 0.2]), ([0.6, 0.7], [0.5, 0.6], [0.4, 0.6]), b, ([0.4, 0.6], [0.7, 0.8], [0.4, 0.7]), ([0.5, 0.6], [0.4, 0.5], [0.1, 0.3]), ([0.4, 0.6], [0.2, 0.3], [0.3, 0.6])\rangle\}$$

Definition 4.3. (Intersection)

Let P_{NVR} and Q_{NVR} be two neutrosophic vague refined sets in the universal set X, where

$$P_{NVR} = \{\langle(\hat{T}^1_{P_{NVR}}(x), \hat{T}^2_{P_{NVR}}(x), ...\hat{T}^u_{P_{NVR}}(x)), (\hat{I}^1_{P_{NVR}}(x), \hat{I}^2_{P_{NVR}}(x), ...\hat{I}^u_{P_{NVR}}(x)), (\hat{F}^1_{P_{NVR}}(x), \hat{F}^2_{P_{NVR}}(x), ...\hat{F}^u_{P_{NVR}}(x))\rangle : x \in X\ \}.$$
$$Q_{NVR} = \{\langle(\hat{T}^1_{Q_{NVR}}(x), \hat{T}^2_{Q_{NVR}}(x), ...\hat{T}^v_{Q_{NVR}}(x)), (\hat{I}^1_{Q_{NVR}}(x), \hat{I}^2_{Q_{NVR}}(x), ...\hat{I}^v_{Q_{NVR}}(x)), (\hat{F}^1_{Q_{NVR}}(x), \hat{F}^2_{Q_{NVR}}(x), ...\hat{F}^v_{Q_{NVR}}(x))\rangle : x \in X\ \}.$$

Then, the neutrosophic vague refined intersection of two sets P_{NVR} and Q_{NVR} denoted by S_{NVR} where $S_{NVR} = P_{NVR} \cap Q_{NVR}$, whose truth membership function, indeterminate membership function and false membership function is

$$\hat{T}^j_{S_{NVR}}(x) = min\{\hat{T}^j_{P_{NVR}}(x), \hat{T}^j_{Q_{NVR}}(x)\},$$
$$\hat{I}^j_{S_{NVR}}(x) = max\{\hat{I}^j_{P_{NVR}}(x), \hat{I}^j_{Q_{NVR}}(x)\},$$
$$\hat{F}^j_{S_{NVR}}(x) = max\{\hat{F}^j_{P_{NVR}}(x), \hat{F}^j_{Q_{NVR}}(x)\}$$

∀ x∈X, j=1,2,...,r.

Example 4.4.

Let X= {a,b} be the non empty set. Let the sets P_{NVR} and Q_{NVR} be defined as:

$$P_{NVR} = \{\langle a, ([0.1, 0.5], [0.4, 0.7], [0.1, 0.8]), ([0.7, 0.8], [0.4, 0.8], [0.2, 0.7]), ([0.5, 0.9], [0.3, 0.6], [0.2, 0.9]), b, ([0.1, 0.6], [0.3, 0.9], [0.4, 0.7]), ([0.8, 0.9], [0.8, 0.9], [0.5, 0.8]), ([0.4, 0.9], [0.1, 0.7], [0.3, 0.6])\rangle\},$$
$$Q_{NVR} = \{\langle a, ([0.3, 0.8], [0.4, 0.7], [0.2, 0.6]), ([0.5, 0.6], [0.6, 0.8], [0.1, 0.2]), ([0.5, 0.7], [0.4, 0.8], [0.2, 0.7]), b, ([0.2, 0.7], [0.3, 0.6], [0.4, 0.8]), ([0.5, 0.6], [0.4, 0.5], [0.2, 0.3]), ([0.3, 0.5], [0.2, 0.6], [0.3, 0.8])\rangle\}$$

Then intersection of two neutrosophic vague refined sets P_{NVR} and Q_{NVR} is

$$S_{NVR} = \{\langle a, ([0.1, 0.5], [0.4, 0.7], [0.1, 0.6]), ([0.7, 0.8], [0.6, 0.8], [0.2, 0.7]), ([0.5, 0.9], [0.4, 0.8], [0.2, 0.9]), b, ([0.1, 0.6], [0.3, 0.6], [0.4, 0.7]), ([0.8, 0.9], [0.8, 0.9], [0.5, 0.8]), ([0.4, 0.9], [0.2, 0.7], [0.3, 0.8])\rangle\}$$

Definition 4.5. (Inclusion)

A neutrosophic vague refined set P_{NVR} is a subset of another neutrosophic vague refined set Q_{NVR} denoted by $P_{NVR} \subseteq Q_{NVR}$,

$$P_{NVR} = \{\langle(\hat{T}^1_{P_{NVR}}(x), \hat{T}^2_{P_{NVR}}(x), ...\hat{T}^u_{P_{NVR}}(x)), (\hat{I}^1_{P_{NVR}}(x), \hat{I}^2_{P_{NVR}}(x), ...\hat{I}^u_{P_{NVR}}(x)), (\hat{F}^1_{P_{NVR}}(x), \hat{F}^2_{P_{NVR}}(x), ...\hat{F}^u_{P_{NVR}}(x))\rangle : x \in X\ \}.$$
$$Q_{NVR} = \{\langle(\hat{T}^1_{Q_{NVR}}(x), \hat{T}^2_{Q_{NVR}}(x), ...\hat{T}^v_{Q_{NVR}}(x)), (\hat{I}^1_{Q_{NVR}}(x), \hat{I}^2_{Q_{NVR}}(x), ...\hat{I}^v_{Q_{NVR}}(x)), (\hat{F}^1_{Q_{NVR}}(x), \hat{F}^2_{Q_{NVR}}(x), ...\hat{F}^v_{Q_{NVR}}(x))\rangle : x \in X\ \}.$$

$P_{NVR} \subseteq Q_{NVR} \Rightarrow \hat{T}^j_{P_{NVR}}(x) \leq \hat{T}^j_{Q_{NVR}}(x), \hat{I}^j_{P_{NVR}}(x) \geq \hat{I}^j_{Q_{NVR}}(x), \hat{F}^j_{P_{NVR}}(x) \geq \hat{F}^j_{Q_{NVR}}(x)\ \forall$ x∈X, j=1,2,...,r.

Definition 4.6. (Equality)

Let P_{NVR} and Q_{NVR} be two neutrosophic vague refined sets. These sets are said to be neutrosophic vague refined equal if P_{NVR} is neutrosophic vague refined subset of Q_{NVR} and Q_{NVR} is neutrosophic vague refined subset of P_{NVR}. It is denoted by $P_{NVR} = Q_{NVR}$.

Definition 4.7. (Complement)

Let P_{NVR} be the neutrosophic vague refined set in the universal set X.

$$P_{NVR} = \{\langle(\hat{T}^1_{P_{NVR}}(x), \hat{T}^2_{P_{NVR}}(x), ...\hat{T}^u_{P_{NVR}}(x)), (\hat{I}^1_{P_{NVR}}(x), \hat{I}^2_{P_{NVR}}(x), ...\hat{I}^u_{P_{NVR}}(x)), (\hat{F}^1_{P_{NVR}}(x), \hat{F}^2_{P_{NVR}}(x), ...\hat{F}^u_{P_{NVR}}(x))\rangle : x \in X\ \}.$$

Then the complement of the neutrosophic vague refined set denoted by P^c_{NVR} is defined as

$$\hat{T}^{j}_{P_{NVR}}(x)^{c} = 1 - \hat{T}^{j}_{P_{NVR}}(x)$$
$$\hat{I}^{j}_{P_{NVR}}(x)^{c} = 1 - \hat{I}^{j}_{P_{NVR}}(x)$$
$$\hat{F}^{j}_{P_{NVR}}(x)^{c} = 1 - \hat{F}^{j}_{P_{NVR}}(x)$$

∀ x∈X, j=1,2,...,r.

Example 4.8.

Let P_{NVR} be the neutrosophic vague refined set in the universal set X,

$P_{NVR} = \{\langle a, ([0.2, 0.4], [0.6, 0.7], [0.4, 0.9]), ([0.2, 0.5], [0.5, 0.7], [0.8, 0.9]), ([0.6, 0.8], [0.3, 0.4], [0.1, 0.6])\rangle\}$

Then the complement of Neutrosophic vague refined set A is as follows:

$P^{c}_{NVR} = \{\langle a, ([0.8, 0.6], [0.4, 0.3], [0.6, 0.1]), ([0.8, 0.5], [0.5, 0.3], [0.2, 0.1]), ([0.4, 0.2], [0.7, 0.6], [0.9, 0.4])\rangle\}$

Definition 4.9. (Addition)

Let P_{NVR} and Q_{NVR} be two neutrosophic vague refined sets in the universal set X, where

$$P_{NVR} = \{\langle(\hat{T}^{1}_{P_{NVR}}(x), \hat{T}^{2}_{P_{NVR}}(x), ...\hat{T}^{u}_{P_{NVR}}(x)), (\hat{I}^{1}_{P_{NVR}}(x), \hat{I}^{2}_{P_{NVR}}(x), ...\hat{I}^{u}_{P_{NVR}}(x)), (\hat{F}^{1}_{P_{NVR}}(x), \hat{F}^{2}_{P_{NVR}}(x), ...\hat{F}^{u}_{P_{NVR}}(x))\rangle : x \in X\ \}.$$
$$Q_{NVR} = \{\langle(\hat{T}^{1}_{Q_{NVR}}(x), \hat{T}^{2}_{Q_{NVR}}(x), ...\hat{T}^{v}_{Q_{NVR}}(x)), (\hat{I}^{1}_{Q_{NVR}}(x), \hat{I}^{2}_{Q_{NVR}}(x), ...\hat{I}^{v}_{Q_{NVR}}(x)), (\hat{F}^{1}_{Q_{NVR}}(x), \hat{F}^{2}_{Q_{NVR}}(x), ...\hat{F}^{v}_{Q_{NVR}}(x))\rangle : x \in X\ \}.$$

Then the neutrosophic vague refined addition of two sets P_{NVR} and Q_{NVR} denoted by $P_{NVR} \oplus Q_{NVR}$, whose truth membership function, indeterminate membership function and false membership function is

$$\hat{T}^{j}_{P_{NVR} \bigoplus Q_{NVR}}(x) = \hat{T}^{j}_{P_{NVR}}(x) + \hat{T}^{j}_{Q_{NVR}}(x) - \hat{T}^{j}_{P_{NVR}}(x).\hat{T}^{j}_{Q_{NVR}}(x)$$
$$\hat{I}^{j}_{P_{NVR} \bigoplus Q_{NVR}}(x) = \hat{I}^{j}_{P_{NVR}}(x) + \hat{I}^{j}_{Q_{NVR}}(x) - \hat{I}^{j}_{P_{NVR}}(x).\hat{I}^{j}_{Q_{NVR}}(x)$$
$$\hat{F}^{j}_{P_{NVR} \bigoplus Q_{NVR}}(x) = \hat{F}^{j}_{P_{NVR}}(x).\hat{F}^{j}_{Q_{NVR}}(x)$$

∀ x∈X, j=1,2,...,r.

Example 4.10.

Let X= {a,b} be the set in universal set X. Let the sets P_{NVR} and Q_{NVR} be defined as:

$P_{NVR} = \{\langle a, ([0.2, 0.5], [0.8, 0.9], [0.4, 0.7]), ([0.2, 0.3], [0.5, 0.8], [0.4, 0.6]), ([0.5, 0.8], [0.1, 0.2], [0.3, 0.6]), b, ([0.3, 0.8], [0.5, 0.9], [0.2, 0.6]), ([0.3, 0.5], [0.7, 0.9], [0.1, 0.5]), ([0.2, 0.7], [0.1, 0.5], [0.4, 0.8])\rangle\}$,

$Q_{NVR} = \{\langle a, ([0.3, 0.7], [0.2, 0.4], [0.5, 0.7]), ([0.3, 0.6], [0.2, 0.7], [0.5, 0.8]), ([0.3, 0.7], [0.6, 0.8], [0.3, 0.5]), b, ([0.4, 0.8], [0.6, 0.7], [0.1, 0.6]), ([0.5, 0.5], [0.7, 0.9], [0.8, 0.9]), ([0.2, 0.6], [0.3, 0.4], [0.4, 0.9])\rangle\}$

Then the addition of two neutrosophic vague refined sets P_{NVR} and Q_{NVR} is
$P_{NVR} \oplus Q_{NVR}$=

$\{\langle a, ([0.44, 0.85], [0.84, 0.94], [0.70, 0.91]), ([0.44, 0.72], [0.6, 0.94], [0.70, 0.92]), ([0.15, 0.56], [0.06, 0.16], ([0.09, 0.30]), b, ([0.58, 0.96], [0.80, 0.97], [0.28, 0.84]), ([0.51, 0.75], [0.91, 0.99], [0.82, 0.95]), ([0.04, 0.42], [0.03, 0.20], [0.16, 0.72])\rangle\}$

Definition 4.11. (Multiplication)
Let P_{NVR} and Q_{NVR} be two neutrosophic vague refined sets in the universal set X, where

$$P_{NVR} = \{\langle(\hat{T}^1_{P_{NVR}}(x), \hat{T}^2_{P_{NVR}}(x), ...\hat{T}^u_{P_{NVR}}(x)), (\hat{I}^1_{P_{NVR}}(x), \hat{I}^2_{P_{NVR}}(x), ...\hat{I}^u_{P_{NVR}}(x)), (\hat{F}^1_{P_{NVR}}(x), \hat{F}^2_{P_{NVR}}(x), ...\hat{F}^u_{P_{NVR}}(x))\rangle : x \in X\ \}.$$
$$Q_{NVR} = \{\langle(\hat{T}^1_{Q_{NVR}}(x), \hat{T}^2_{Q_{NVR}}(x), ...\hat{T}^v_{Q_{NVR}}(x)), (\hat{I}^1_{Q_{NVR}}(x), \hat{I}^2_{Q_{NVR}}(x), ...\hat{I}^v_{Q_{NVR}}(x)), (\hat{F}^1_{Q_{NVR}}(x), \hat{F}^2_{Q_{NVR}}(x), ...\hat{F}^v_{Q_{NVR}}(x))\rangle : x \in X\ \}.$$

Then the neutrosophic vague refined multiplication of two sets P_{NVR} and Q_{NVR} denoted by $P_{NVR} \otimes Q_{NVR}$, whose truth membership function, indeterminate membership function and false membership function is

$$\hat{T}^j_{P_{NVR} \bigotimes Q_{NVR}}(x) = \hat{T}^j_{P_{NVR}}(x).\hat{T}^j_{Q_{NVR}}(x)$$
$$\hat{I}^j_{P_{NVR} \bigotimes Q_{NVR}}(x) = \hat{I}^j_{P_{NVR}}(x).\hat{I}^j_{Q_{NVR}}(x)$$
$$\hat{F}^j_{P_{NVR} \bigotimes Q_{NVR}}(x) = \hat{F}^j_{P_{NVR}}(x) + \hat{F}^j_{Q_{NVR}}(x) - \hat{F}^j_{P_{NVR}}(x).\hat{F}^j_{Q_{NVR}}(x)$$

$\forall$ x$\in$X, j=1,2,...,r.

Example 4.12.
Let X= {a,b} be the set in universal set X. Let the sets P_{NVR} and Q_{NVR} be defined as:

$P_{NVR} = \{\langle a, ([0.2, 0.4], [0.8, 0.9], [0.4, 0.5]), ([0.2, 0.4], [0.5, 0.8], [0.3, 0.6]), ([0.6, 0.8], [0.1, 0.2], [0.5, 0.6]), b, ([0.3, 0.5], [0.5, 0.7], [0.1, 0.6]), ([0.3, 0.5], [0.4, 0.9], [0.1, 0.7]), ([0.5, 0.7], [0.3, 0.5], [0.4, 0.9])\rangle\}$,

$Q_{NVR} = \{\langle a, ([0.3, 0.9], [0.2, 0.7], [0.5, 0.8]), ([0.1, 0.6], [0.5, 0.7], [0.3, 0.8]), ([0.1, 0.7], [0.3, 0.8], [0.2, 0.5]), b, ([0.2, 0.8], [0.6, 0.8], [0.3, 0.6]), ([0.4, 0.5], [0.5, 0.9], [0.8, 0.8]), ([0.2, 0.8], [0.2, 0.4], [0.4, 0.7])\rangle\}$

Then the multiplication of two neutrosophic vague refined sets P_{NVR} and Q_{NVR} is
$P_{NVR} \otimes Q_{NVR}$=

$\{\langle a, ([0.06, 0.36], [0.16, 0.63], [0.20, 0.40]), ([0.02, 0.24], [0.25, 0.56], [0.09, 0.40]), ([0.64, 0.94],$

$[0.37, 0.84], ([0.60, 0.80]), b, ([0.06, 0.40], [0.30, 0.56], [0.03, 0.36]), ([0.12, 0.25], [0.20, 0.81], [0.08, 0.56]), ([0.60, 0.94], [0.44, 0.70], [0.64, 0.97])\rangle\}$

Definition 4.13. (AND operation)

Let P_{NVR} and Q_{NVR} be two neutrosophic vague refined sets in the universal set X, where

$$P_{NVR} = \{\langle(\hat{T}^1_{P_{NVR}}(x), \hat{T}^2_{P_{NVR}}(x), ...\hat{T}^u_{P_{NVR}}(x)), (\hat{I}^1_{P_{NVR}}(x), \hat{I}^2_{P_{NVR}}(x), ...\hat{I}^u_{P_{NVR}}(x)),$$
$$(\hat{F}^1_{P_{NVR}}(x), \hat{F}^2_{P_{NVR}}(x), ...\hat{F}^u_{P_{NVR}}(x))\rangle : x \in X\ \}.$$
$$Q_{NVR} = \{\langle(\hat{T}^1_{Q_{NVR}}(x), \hat{T}^2_{Q_{NVR}}(x), ...\hat{T}^v_{Q_{NVR}}(x)), (\hat{I}^1_{Q_{NVR}}(x), \hat{I}^2_{Q_{NVR}}(x), ...\hat{I}^v_{Q_{NVR}}(x)),$$
$$(\hat{F}^1_{Q_{NVR}}(x), \hat{F}^2_{Q_{NVR}}(x), ...\hat{F}^v_{Q_{NVR}}(x))\rangle : x \in X\ \}.$$

AND operation between these two neutrosophic vague refined sets is denoted by $P_{NVR} \wedge Q_{NVR}$ is the intersection of two neutrosophic vague refined sets P_{NVR} and Q_{NVR}.
$\forall$ x$\in$X, j=1,2,...,r.

Definition 4.14. (OR operation)

Let P_{NVR} and Q_{NVR} be two neutrosophic vague refined sets in the universal set X, where

$$P_{NVR} = \{\langle(\hat{T}^1_{P_{NVR}}(x), \hat{T}^2_{P_{NVR}}(x), ...\hat{T}^u_{P_{NVR}}(x)), (\hat{I}^1_{P_{NVR}}(x), \hat{I}^2_{P_{NVR}}(x), ...\hat{I}^u_{P_{NVR}}(x)),$$
$$(\hat{F}^1_{P_{NVR}}(x), \hat{F}^2_{P_{NVR}}(x), ...\hat{F}^u_{P_{NVR}}(x))\rangle : x \in X\ \}.$$
$$Q_{NVR} = \{\langle(\hat{T}^1_{Q_{NVR}}(x), \hat{T}^2_{Q_{NVR}}(x), ...\hat{T}^v_{Q_{NVR}}(x)), (\hat{I}^1_{Q_{NVR}}(x), \hat{I}^2_{Q_{NVR}}(x), ...\hat{I}^v_{Q_{NVR}}(x)),$$
$$(\hat{F}^1_{Q_{NVR}}(x), \hat{F}^2_{Q_{NVR}}(x), ...\hat{F}^v_{Q_{NVR}}(x))\rangle : x \in X\ \}.$$

OR operation between these two neutrosophic vague refined sets is denoted by $P_{NVR} \vee Q_{NVR}$ is the intersection of two neutrosophic vague refined sets P_{NVR} and Q_{NVR}.
$\forall$ x$\in$X, j=1,2,...,r.

Definition 4.15. (Cartesian Product)

Let P_{NVR} and Q_{NVR} be the two neutrosophic vague refined sets.

$$P_{NVR} = \{\langle(\hat{T}^1_{P_{NVR}}(x), \hat{T}^2_{P_{NVR}}(x), ...\hat{T}^u_{P_{NVR}}(x)), (\hat{I}^1_{P_{NVR}}(x), \hat{I}^2_{P_{NVR}}(x), ...\hat{I}^u_{P_{NVR}}(x)),$$
$$(\hat{F}^1_{P_{NVR}}(x), \hat{F}^2_{P_{NVR}}(x), ...\hat{F}^u_{P_{NVR}}(x))\rangle : x \in X\ \}.$$
$$Q_{NVR} = \{\langle(\hat{T}^1_{Q_{NVR}}(y), \hat{T}^2_{Q_{NVR}}(y), ...\hat{T}^v_{Q_{NVR}}(y)), (\hat{I}^1_{Q_{NVR}}(y), \hat{I}^2_{Q_{NVR}}(y), ...\hat{I}^v_{Q_{NVR}}(y)),$$
$$(\hat{F}^1_{Q_{NVR}}(x), \hat{F}^2_{Q_{NVR}}(y), ...\hat{F}^v_{Q_{NVR}}(y))\rangle : y \in X\ \}.$$

Then the cartesian product of two sets $P_{NVR} \times Q_{NVR}$ is defined as

$$P_{NVR} \times Q_{NVR} = \{\langle(\hat{T}^1_{P_{NVR} \times Q_{NVR}}(x, y), \hat{T}^2_{P_{NVR} \times Q_{NVR}}(x, y), ...\hat{T}^P_{P_{NVR} \times Q_{NVR}}(x, y)),$$

$(\hat{I}^1_{P_{NVR}\times Q_{NVR}}(x,y), \hat{I}^2_{P_{NVR}\times Q_{NVR}}(x,y), ...\hat{I}^P_{P_{NVR}\times Q_{NVR}}(x,y))$
$(\hat{F}^1_{P_{NVR}\times Q_{NVR}}(x,y), \hat{F}^2_{P_{NVR}\times Q_{NVR}}(x,y), ...\hat{F}^P_{P_{NVR}\times Q_{NVR}}(x,y))\}$

where

$$\hat{T}^j_{P_{NVR}\times Q_{NVR}}, \hat{I}^j_{P_{NVR}\times Q_{NVR}}, \hat{F}^j_{P_{NVR}\times Q_{NVR}} : X \longrightarrow [0,1]$$

and also,

$$\hat{T}^j_{P_{NVR}\times Q_{NVR}}(x,y) = min\{\hat{T}^j_{P_{NVR}\times Q_{NVR}}(x), \hat{T}^j_{P_{NVR}\times Q_{NVR}}(y)\}$$
$$\hat{I}^j_{P_{NVR}\times Q_{NVR}}(x,y) = max\{\hat{I}^j_{P_{NVR}\times Q_{NVR}}(x), \hat{I}^j_{P_{NVR}\times Q_{NVR}}(y)\}$$
$$\hat{F}^j_{P_{NVR}\times Q_{NVR}}(x,y) = max\{\hat{F}^j_{P_{NVR}\times Q_{NVR}}(x), \hat{F}^j_{P_{NVR}\times Q_{NVR}}(y)\}$$

for all x,y $\in$ X and j= $\{1, 2, ..., r\}$.

Example 4.16.

Let X= {a,b} be the set in universal set X. Let the sets P_{NVR} and Q_{NVR} be defined as:

$P_{NVR} = \{\langle a, ([0.3, 0.7], [0.4, 0.5], [0.4, 0.7]), ([0.2, 0.5], [0.4, 0.9], [0.3, 0.8]), ([0.3, 0.7], [0.5, 0.9],$
$[0.3, 0.6]), b, ([0.4, 0.6], [0.3, 0.8], [0.1, 0.5]), ([0.3, 0.7], [0.5, 0.9], [0.2, 0.6]), ([0.4, 0.6],$
$[0.2, 0.7], [0.5, 0.9])\rangle\}$,
$Q_{NVR} = \{\langle a, ([0.1, 0.5], [0.5, 0.7], [0.5, 0.8]), ([0.2, 0.7], [0.5, 0.8], [0.4, 0.8]), ([0.5, 0.9], [0.3, 0.5],$
$[0.2, 0.5]), b, ([0.4, 0.9], [0.1, 0.5], [0.5, 0.6]), ([0.3, 0.6], [0.4, 0.9], [0.5, 0.8]), ([0.1, 0.6],$
$[0.5, 0.9], [0.4, 0.5])\rangle\}$

Then the cartesian product of two neutrosophic vague refined sets P_{NVR} and Q_{NVR} is as follows

$P_{NVR} \times Q_{NVR} =$
$\{\langle(a, a)([0.1, 0.5], [0.1, 0.5], [0.4, 0.7]), ([0.2, 0.7], [0.5, 0.9], [0.4, 0.8]), ([0.5, 0.9], [0.5, 0.9], [0.3, 0.6])\rangle,$
$\langle(a, b)([0.3, 0.7], [0.1, 0.5], [0.4, 0.6]), ([0.3, 0.6], [0.4, 0.9], [0.5, 0.8]), ([0.1, 0.6], [0.5, 0.9], [0.4, 0.6])\rangle,$
$\langle(b, a)([0.1, 0.5], [0.3, 0.7], [0.1, 0.5]), ([0.3, 0.7], [0.5, 0.9], [0.4, 0.8]), ([0.5, 0.9], [0.3, 0.7], [0.5, 0.9])\rangle,$
$\langle(b, b)([0.4, 0.6], [0.1, 0.5], [0.1, 0.5]), ([0.3, 0.7], [0.5, 0.9], [0.5, 0.8]), ([0.4, 0.6], [0.5, 0.9], [0.5, 0.9])\rangle\}$

5. Algebraic properties of neutrosophic vague refined set operations

In this section, we study algebraic properties of above operations in neutrosophic vague refined set with examples.

Proposition 5.1. (Identity Law)

For any neutrosophic vague refined set P_{NVR} defined on the absolute neutrosophic vague refined set X.

(1)$P_{NVR} \cup \emptyset_{NVR}$= P_{NVR}

(2)$P_{NVR} \cap X_{NVR}$= P_{NVR}

Proof.

(1)Let P_{NVR} be the two neutrosophic vague refined set.

$$P_{NVR} = \{\langle(\hat{T}^1_{P_{NVR}}(x), \hat{T}^2_{P_{NVR}}(x), ...\hat{T}^u_{P_{NVR}}(x)), (\hat{I}^1_{P_{NVR}}(x), \hat{I}^2_{P_{NVR}}(x), ...\hat{I}^u_{P_{NVR}}(x)), (\hat{F}^1_{P_{NVR}}(x), \hat{F}^2_{P_{NVR}}(x), ...\hat{F}^u_{P_{NVR}}(x))\rangle : x \in X\ \}.$$

$\emptyset_{NVR}$ is defined as follows

$$\emptyset_{NVR}=\{\langle a, ([0,0], [0,0], ...[0,0]), ([1,1], [1,1], ...[1,1]), ([1,1], [1,1], ..., [1,1]) \\ b, ([0,0], [0,0], ...[0,0]), ([1,1], [1,1], ...[1,1]), ([1,1], [1,1], ..., [1,1])\rangle\}.$$

So, $P_{NVR} \cup \emptyset_{NVR}=$

$$\{< (max\{\hat{T}^1_{P_{NVR}}(x), [0,0]\}, min\{\hat{T}^2_{P_{NVR}}(x), [0,0]\}, ..., min\{\hat{T}^u_{P_{NVR}}(x), [0,0]\}), \\ (max\{\hat{I}^1_{P_{NVR}}(x), [1,1]\}, min\{\hat{I}^2_{P_{NVR}}(x), [1,1]\}, ..., min\{\hat{T}^u_{P_{NVR}}(x), [1,1]\}), \\ (max\{\hat{F}^1_{P_{NVR}}(x), [1,1]\}, min\{\hat{F}^2_{P_{NVR}}(x), [1,1]\}, ..., min\{\hat{u}^2_{P_{NVR}}(x), [1,1]\}) >\}$$

Therefore,

$$P_{NVR} \cup \emptyset_{NVR} = \{\langle(\hat{T}^1_{P_{NVR}}(x), \hat{T}^2_{P_{NVR}}(x), ...\hat{T}^u_{P_{NVR}}(x)), (\hat{I}^1_{P_{NVR}}(x), \hat{I}^2_{P_{NVR}}(x), ...\hat{I}^u_{P_{NVR}}(x)), (\hat{F}^1_{P_{NVR}}(x), \hat{F}^2_{P_{NVR}}(x), ...\hat{F}^u_{P_{NVR}}(x))\rangle : x \in X\}.$$

(2)Proof is simitar to (1). □

Example 5.2.

(1)Let P_{NVR} be neutrosophic vague refined set in the universal set X.

$$P_{NVR} = \{\langle a, ([0.2, 0.4], [0.8, 0.9], [0.4, 0.5]), ([0.2, 0.4], [0.5, 0.8], [0.3, 0.6]), ([0.6, 0.8], [0.1, 0.2], [0.5, 0.6]), b, ([0.3, 0.5], [0.5, 0.7], [0.1, 0.6]), ([0.3, 0.5], [0.4, 0.9], [0.1, 0.7]), ([0.5, 0.7], [0.3, 0.5], [0.4, 0.9])\rangle\},$$

$$\emptyset_{NVR} = \{\langle a, ([0,0], [1,1], [1,1]), ([0,0], [1,1], [1,1]), ([0,0], [1,1], [1,1])\rangle\}.$$

Then,

$$P_{NVR} \cup \emptyset_{NVR} =\{\langle a, ([0.2, 0.4], [0.8, 0.9], [0.4, 0.5]), ([0.2, 0.4], [0.5, 0.8], [0.3, 0.6]), ([0.6, 0.8], [0.1, 0.2], [0.5, 0.6]), b, ([0.3, 0.5], [0.5, 0.7], [0.1, 0.6]), ([0.3, 0.5], [0.4, 0.9], [0.1, 0.7]), ([0.5, 0.7], [0.3, 0.5], [0.4, 0.9])\rangle\},$$

(2) Obvious.

Proposition 5.3. (Domination Law)

For any neutrosophic vague refined set P_{NVR} defined on absolute neutrosophic vague refined set X

(1) $P_{NVR} \cap \emptyset_{NVR}= \emptyset_{NVR}$

(2) $P_{NVR} \cup X_{NVR}= X_{NVR}$

Proof.

(1)Let P_{NVR} be the two neutrosophic vague refined set.

$$P_{NVR} = \{\langle(\hat{T}^1_{P_{NVR}}(x), \hat{T}^2_{P_{NVR}}(x), ...\hat{T}^u_{P_{NVR}}(x)), (\hat{I}^1_{P_{NVR}}(x), \hat{I}^2_{P_{NVR}}(x), ...\hat{I}^u_{P_{NVR}}(x)),$$

$$(\hat{F}^1_{P_{NVR}}(x), \hat{F}^2_{P_{NVR}}(x), ...\hat{F}^u_{P_{NVR}}(x))\rangle : x \in X \ \}.$$

$\emptyset_{NVR}$ is defined as follows

$\emptyset_{NVR}$=$\{\langle a, ([0,0],[0,0],....[0,0]),([1,1],[1,1],....[1,1]),([1,1],[1,1],...,[1,1])$
$b, ([0,0],[0,0],....[0,0]),([1,1],[1,1],....[1,1]),([1,1],[1,1],...,[1,1])\rangle\}$.

So, $P_{NVR} \cap \emptyset_{NVR}$=

$$\{< (min\{\hat{T}^1_{P_{NVR}}(x),[0,0]\}, max\{\hat{T}^2_{P_{NVR}}(x),[0,0]\}, ..., max\{\hat{T}^u_{P_{NVR}}(x),[0,0]\}),$$
$$(min\{\hat{I}^1_{P_{NVR}}(x),[1,1]\}, max\{\hat{I}^2_{P_{NVR}}(x),[1,1]\}, ..., max\{\hat{T}^u_{P_{NVR}}(x),[1,1]\}),$$
$$(min\{\hat{F}^1_{P_{NVR}}(x),[1,1]\}, max\{\hat{F}^2_{P_{NVR}}(x),[1,1]\}, ..., max\{\hat{F}^u_{P_{NVR}}(x),[1,1]\}) >\}$$

Therefore,

$P_{NVR} \cap \emptyset_{NVR} = \{\langle a, ([0,0],[0,0],....[0,0]),([1,1],[1,1],....[1,1]),([1,1],[1,1],...,[1,1])\rangle\}$.

(2)Proof is similar to (1). □

Example 5.4.

(1) Let P_{NVR} be neutrosophic vague refined set in the universal set X.

$P_{NVR} = \{\langle a, ([0.1,0.2],[0.3,0.5],[0.1,0.3]),([0.7,0.8],[0.6,0.8],[0.2,0.3]),([0.8,0.9],[0.5,0.7],$
$[0.7,0.9]), b, ([0.3,0.6],[0.7,0.8],[0.2,0.4]),([0.6,0.8],[0.8,0.9],[0.2,0.3]),([0.4,0.7],$
$[0.2,0.3],[0.6,0.8])\rangle\}$

$\emptyset_{NVR}$=$\{\langle a, ([0,0],[0,0],....[0,0]),([1,1],[1,1],....[1,1]),([1,1],[1,1],...,[1,1])$
$b, ([0,0],[0,0],....[0,0]),([1,1],[1,1],....[1,1]),([1,1],[1,1],...,[1,1])\rangle\}$.

Then,

$P_{NVR} \cap \emptyset_{NVR}$=

$\{\langle a, ([0,0],[0,0],....[0,0]),([1,1],[1,1],....[1,1]),([1,1],[1,1],...,[1,1])$
$b, ([0,0],[0,0],....[0,0]),([1,1],[1,1],....[1,1]),([1,1],[1,1],...,[1,1])\rangle\}$.

(2) Obvious.

Proposition 5.5. (Idempotent Law)

For any neutrosophic vague refined set P_{NVR} defined on absolute neutrosophic vague refined set X.

(1) $P_{NVR} \cap P_{NVR}$= P_{NVR}

(2) $P_{NVR} \cup P_{NVR}$= P_{NVR}

Proof.

The proof is obvious. □

Example 5.6.

(1)Let P_{NVR} be neutrosophic vague refined set in the universal set X.

$P_{NVR} = \{\langle a, ([0.1, 0.5], [0.5, 0.7], [0.5, 0.8]), ([0.2, 0.7], [0.5, 0.8], [0.4, 0.8]), ([0.5, 0.9], [0.3, 0.5], [0.2, 0.5]), b, ([0.4, 0.9], [0.1, 0.5], [0.5, 0.6]), ([0.3, 0.6], [0.4, 0.9], [0.5, 0.8]), ([0.1, 0.6], [0.5, 0.9], [0.4, 0.5])\rangle\}$

Then,

$P_{NVR} \cap P_{NVR} = \{\langle a, ([0.1, 0.5], [0.5, 0.7], [0.5, 0.8]), ([0.2, 0.7], [0.5, 0.8], [0.4, 0.8]), ([0.5, 0.9], [0.3, 0.5], [0.2, 0.5]), b, ([0.4, 0.9], [0.1, 0.5], [0.5, 0.6]), ([0.3, 0.6], [0.4, 0.9], [0.5, 0.8]), ([0.1, 0.6], [0.5, 0.9], [0.4, 0.5])\rangle\}$

Hence, $P_{NVR} \cap P_{NVR} = P_{NVR}$.

(2) Obvious.

Proposition 5.7. (Commutative Law)

For any neutrosophic vague refined set P_{NVR} defined on absolute neutrosophic vague refined set X

(1) $P_{NVR} \cup Q_{NVR} = Q_{NVR} \cup P_{NVR}$

(2) $P_{NVR} \cap Q_{NVR} = Q_{NVR} \cap P_{NVR}$

Proof. Proof is obvious. □

Proposition 5.8. (Associative Law)

For any neutrosophic vague refined sets P_{NVR}, Q_{NVR} and R_{NVR} defined on absolute neutrosophic vague refined set X.

(1) $(P_{NVR} \cup Q_{NVR}) \cup R_{NVR} = P_{NVR} \cup (Q_{NVR} \cup R_{NVR})$

(2) $(P_{NVR} \cap Q_{NVR}) \cap R_{NVR} = P_{NVR} \cap (Q_{NVR} \cap R_{NVR})$

Proof.

(1) Let P_{NVR}, Q_{NVR} and R_{NVR} be three neutrosophic vague refined sets defined as follows:

$$P_{NVR} = \{\langle(\hat{T}^1_{P_{NVR}}(x), \hat{T}^2_{P_{NVR}}(x), ...\hat{T}^u_{P_{NVR}}(x)), (\hat{I}^1_{P_{NVR}}(x), \hat{I}^2_{P_{NVR}}(x), ...\hat{I}^u_{P_{NVR}}(x)), (\hat{F}^1_{P_{NVR}}(x), \hat{F}^2_{P_{NVR}}(x), ...\hat{F}^u_{P_{NVR}}(x))\rangle : x \in X\}.$$

$$Q_{NVR} = \{\langle(\hat{T}^1_{Q_{NVR}}(x), \hat{T}^2_{Q_{NVR}}(x), ...\hat{T}^v_{Q_{NVR}}(x)), (\hat{I}^1_{Q_{NVR}}(x), \hat{I}^2_{Q_{NVR}}(x), ...\hat{I}^v_{Q_{NVR}}(x)), (\hat{F}^1_{Q_{NVR}}(x), \hat{F}^2_{Q_{NVR}}(x), ...\hat{F}^v_{Q_{NVR}}(x))\rangle : x \in X\}.$$

$$R_{NVR} = \{\langle(\hat{T}^1_{R_{NVR}}(x), \hat{T}^2_{R_{NVR}}(x), ...\hat{T}^w_{R_{NVR}}(x)), (\hat{I}^1_{R_{NVR}}(x), \hat{I}^2_{R_{NVR}}(x), ...\hat{I}^w_{R_{NVR}}(x)), (\hat{F}^1_{R_{NVR}}(x), \hat{F}^2_{R_{NVR}}(x), ...\hat{F}^w_{R_{NVR}}(x))\rangle : x \in X\}.$$

Then, $(P_{NVR} \cup Q_{NVR}) \cup R_{NVR}$

$$=\{\langle(\hat{T}^1_{P_{NVR}}(x), \vee\hat{T}^1_{Q_{NVR}}(x)), ..., (\hat{T}^u_{P_{NVR}}(x) \vee \hat{T}^v_{Q_{NVR}}(x)), (\hat{I}^1_{P_{NVR}}(x) \vee \hat{I}^1_{Q_{NVR}}(x)), ..., (\hat{I}^u_{P_{NVR}}(x) \vee \hat{I}^v_{Q_{NVR}}(x)), (\hat{F}^1_{P_{NVR}}(x) \vee \hat{F}^1_{Q_{NVR}}(x)), ..., (\hat{F}^u_{P_{NVR}}(x) \vee \hat{F}^v_{Q_{NVR}}(x))\rangle : x \in X\} \cap$$

$\{\langle(\hat{T}^{1}_{R_{NVR}}(x),...,\hat{T}^{w}_{R_{NVR}}(x)),(\hat{I}^{1}_{R_{NVR}}(x),...,\hat{I}^{w}_{R_{NVR}}(x)),(\hat{F}^{1}_{R_{NVR}}(x),..,\hat{F}^{w}_{R_{NVR}}(x))\rangle : x \in X\}$

$=\{\langle(((\hat{T}^{1}_{P_{NVR}}(x)\vee\hat{T}^{1}_{Q_{NVR}}(x))\vee\hat{T}^{1}_{R_{NVR}}(x)....,((\hat{T}^{P}_{P_{NVR}}(x)\vee\hat{T}^{P}_{Q_{NVR}}(x))\vee\hat{T}^{P}_{R_{NVR}}(x)),((\hat{I}^{1}_{P_{NVR}}(x) \vee (\hat{I}^{1}_{Q_{NVR}}(x))\vee\hat{I}^{1}_{R_{NVR}}(x))....,((\hat{I}^{P}_{P_{NVR}}(x)\vee\hat{I}^{P}_{Q_{NVR}}(x))\vee\hat{I}^{P}_{R_{NVR}}(x)),((\hat{F}^{1}_{P_{NVR}}(x)\vee (\hat{F}^{1}_{Q_{NVR}}(x))\vee\hat{F}^{1}_{R_{NVR}}(x))....,((\hat{F}^{u}_{P_{NVR}}(x)\vee\hat{F}^{v}_{Q_{NVR}}(x))\vee\hat{F}^{w}_{R_{NVR}}(x))\rangle : x\in X\}$

$=\{\langle(\hat{T}^{1}_{P_{NVR}}(x)\vee(\hat{T}^{1}_{Q_{NVR}}(x)\vee\hat{T}^{1}_{R_{NVR}}(x)))....,(\hat{T}^{u}_{P_{NVR}}(x)\vee(\hat{T}^{v}_{Q_{NVR}}(x)\vee\hat{T}^{w}_{R_{NVR}}(x))), (\hat{I}^{1}_{P_{NVR}}(x)\vee(\hat{I}^{1}_{Q_{NVR}}(x)\vee\hat{I}^{1}_{R_{NVR}}(x)))....,(\hat{I}^{u}_{P_{NVR}}(x)\vee(\hat{I}^{v}_{Q_{NVR}}(x)\vee\hat{I}^{w}_{R_{NVR}}(x))),(\hat{F}^{1}_{P_{NVR}}(x)\vee (\hat{F}^{1}_{Q_{NVR}}(x)\vee\hat{F}^{1}_{R_{NVR}}(x)))....,(\hat{F}^{u}_{P_{NVR}}(x)\vee(\hat{F}^{v}_{Q_{NVR}}(x)\vee\hat{F}^{w}_{R_{NVR}}(x)))\rangle : x\in X\}$

$=P_{NVR}\cup(Q_{NVR}\cup R_{NVR})$

(2) The proof is similar to (1) □

Proposition 5.9. (Distributive Law)

For any neutrosophic vague refined sets P_{NVR}, Q_{NVR} and R_{NVR} defined on absolute neutrosophic vague refined set X.

(1) $P_{NVR}\cup(Q_{NVR}\cap R_{NVR}) = (P_{NVR}\cup Q_{NVR})\cap(P_{NVR}\cup R_{NVR})$

(2) $P_{NVR}\cap(Q_{NVR}\cup R_{NVR}) = (P_{NVR}\cap Q_{NVR})\cup(P_{NVR}\cap R_{NVR})$

Proof.

(1) Let P_{NVR}, Q_{NVR} and R_{NVR} be three neutrosophic vague refined sets defined as follows:

$$P_{NVR} = \{\langle(\hat{T}^{1}_{P_{NVR}}(x),\hat{T}^{2}_{P_{NVR}}(x),....\hat{T}^{u}_{P_{NVR}}(x)),(\hat{I}^{1}_{P_{NVR}}(x),\hat{I}^{2}_{P_{NVR}}(x),....\hat{I}^{u}_{P_{NVR}}(x)), (\hat{F}^{1}_{P_{NVR}}(x),\hat{F}^{2}_{P_{NVR}}(x),....\hat{F}^{u}_{P_{NVR}}(x))\rangle : x\in X\ \}.$$

$$Q_{NVR} = \{\langle(\hat{T}^{1}_{Q_{NVR}}(x),\hat{T}^{2}_{Q_{NVR}}(x),....\hat{T}^{v}_{Q_{NVR}}(x)),(\hat{I}^{1}_{Q_{NVR}}(x),\hat{I}^{2}_{Q_{NVR}}(x),....\hat{I}^{v}_{Q_{NVR}}(x)), (\hat{F}^{1}_{Q_{NVR}}(x),\hat{F}^{2}_{Q_{NVR}}(x),....\hat{F}^{v}_{Q_{NVR}}(x))\rangle : x\in X\ \}.$$

$$R_{NVR} = \{\langle(\hat{T}^{1}_{R_{NVR}}(x),\hat{T}^{2}_{R_{NVR}}(x),....\hat{T}^{w}_{R_{NVR}}(x)),(\hat{I}^{1}_{R_{NVR}}(x),\hat{I}^{2}_{R_{NVR}}(x),....\hat{I}^{w}_{R_{NVR}}(x)), (\hat{F}^{1}_{R_{NVR}}(x),\hat{F}^{2}_{R_{NVR}}(x),....\hat{F}^{w}_{R_{NVR}}(x))\rangle : x\in X\}.$$

Then, $P_{NVR}\cup(Q_{NVR}\cap R_{NVR})$

$= \{\langle(\hat{T}^{1}_{P_{NVR}}(x),....\hat{T}^{u}_{P_{NVR}}(x)),(\hat{I}^{1}_{P_{NVR}}(x),....\hat{I}^{u}_{P_{NVR}}(x)),\ (\hat{F}^{1}_{P_{NVR}}(x),....\hat{F}^{u}_{P_{NVR}}(x))\cup ((\hat{T}^{1}_{Q_{NVR}}(x)\wedge\hat{T}^{1}_{R_{NVR}}(x)),....,(\hat{T}^{v}_{Q_{NVR}}(x)\wedge\hat{T}^{w}_{R_{NVR}}(x)))\rangle : x\in X\}$

$=\{\langle(\hat{T}^{1}_{P_{NVR}}(x)\vee(\hat{T}^{1}_{Q_{NVR}}(x)\wedge\hat{T}^{1}_{R_{NVR}}(x)),....,(\hat{T}^{u}_{P_{NVR}}(x)\vee(\hat{T}^{v}_{Q_{NVR}}(x)\wedge(\hat{T}^{w}_{R_{NVR}}(x))), (\hat{I}^{1}_{P_{NVR}}(x)\wedge(\hat{I}^{1}_{Q_{NVR}}(x)\vee\hat{I}^{1}_{R_{NVR}}(x)),...,(\hat{I}^{u}_{P_{NVR}}(x)\wedge(\hat{I}^{v}_{Q_{NVR}}(x)\vee(\hat{I}^{w}_{R_{NVR}}(x))), (\hat{F}^{1}_{P_{NVR}}(x)\wedge(\hat{F}^{1}_{Q_{NVR}}(x)\vee\hat{F}^{1}_{R_{NVR}}(x)),...,(\hat{F}^{u}_{P_{NVR}}(x)\wedge(\hat{F}^{v}_{Q_{NVR}}(x)\vee(\hat{F}^{w}_{R_{NVR}}(x)))\rangle : x\in X\}$

$=\{\langle((((\hat{T}^1_{P_{NVR}}(x) \vee (\hat{T}^1_{Q_{NVR}}(x)) \wedge (\hat{T}^1_{P_{NVR}}(x) \vee \hat{T}^1_{R_{NVR}}(x)), ..., ((\hat{T}^u_{P_{NVR}}(x) \vee (\hat{T}^v_{Q_{NVR}}(x)) \wedge (\hat{T}^u_{P_{NVR}}(x) \vee \hat{T}^w_{R_{NVR}}(x)))), (((\hat{I}^1_{P_{NVR}}(x) \wedge \hat{I}^1_{Q_{NVR}}(x)) \vee (\hat{I}^1_{P_{NVR}}(x) \wedge \hat{I}^1_{R_{NVR}}(x)), ..., ((\hat{I}^u_{P_{NVR}}(x) \wedge \hat{I}^v_{Q_{NVR}}(x)) \vee (\hat{I}^u_{P_{NVR}}(x) \wedge \hat{I}^w_{R_{NVR}}(x)))), (((\hat{F}^1_{P_{NVR}}(x) \wedge \hat{F}^1_{Q_{NVR}}(x)) \vee (\hat{F}^1_{P_{NVR}}(x) \wedge \hat{F}^1_{R_{NVR}}(x)), ..., ((\hat{F}^u_{P_{NVR}}(x) \wedge \hat{F}^v_{Q_{NVR}}(x)) \vee (\hat{F}^u_{P_{NVR}}(x) \wedge \hat{F}^w_{R_{NVR}}(x))))\rangle : x \in X\}$

$= (P_{NVR} \cup Q_{NVR}) \cap (P_{NVR} \cup R_{NVR})$

(2) The proof is similar to (1) □

Proposition 5.10. (Double Complement Law)

For any neutrosophic vague refined set P_{NVR} defined on absolute neutrosophic vague refined set X

$(P^c_{NVR})^c = P_{NVR}$

Proof.

Let P_{NVR} be neutrosophic vague refined set defined as follows:

$$P_{NVR} = \{\langle(\hat{T}^1_{P_{NVR}}(x), \hat{T}^2_{P_{NVR}}(x), ...\hat{T}^u_{P_{NVR}}(x)), (\hat{I}^1_{P_{NVR}}(x), \hat{I}^2_{P_{NVR}}(x), ...\hat{I}^u_{P_{NVR}}(x)), (\hat{F}^1_{P_{NVR}}(x), \hat{F}^2_{P_{NVR}}(x), ...\hat{F}^u_{P_{NVR}}(x))\rangle : x \in X\ \}.$$

Then,

$$P^c_{NVR}=\{\langle(1-\hat{T}^1{}_{P_{NVR}}(x), 1-\hat{T}^2{}_{P_{NVR}}(x), ...1-\hat{T}^u{}_{P_{NVR}}(x)), (1-\hat{I}^1{}_{P_{NVR}}(x), 1-\hat{I}^2{}_{P_{NVR}}(x), ..., 1-\hat{I}^u{}_{P_{NVR}}(x)), (1-\hat{F}^1{}_{P_{NVR}}(x), 1-\hat{F}^2{}_{P_{NVR}}(x), ...1-\hat{F}^u{}_{P_{NVR}}(x))\rangle : x \in X\}.$$

So,

$$(P^c_{NVR})^c = \{\langle(\hat{T}^1_{P_{NVR}}(x), \hat{T}^2_{P_{NVR}}(x), ...\hat{T}^u_{P_{NVR}}(x)), (\hat{I}^1_{P_{NVR}}(x), \hat{I}^2_{P_{NVR}}(x), ...\hat{I}^u_{P_{NVR}}(x)), (\hat{F}^1_{P_{NVR}}(x), \hat{F}^2_{P_{NVR}}(x), ..., \hat{F}^u_{P_{NVR}}(x))\rangle : x \in X\}.$$

Therefore, $(P^c_{NVR})^c = P_{NVR}$ □

Proposition 5.11. (Absorption Law)

For any neutrosophic vague refined set P_{NVR} and Q_{NVR} defined on absolute neutrosophic vague refined set X

(1) $P_{NVR} \cup (P_{NVR} \cap Q_{NVR}) = P_{NVR}$

(2) $P_{NVR} \cap (P_{NVR} \cup Q_{NVR}) = P_{NVR}$

Proof. Proof is obvious. □

6. Conclusions

This paper ensures the work of introducing the new set namely neutrosophic vague refined set by the combination of neutrosophic vague set and neutrosophic refined(multi) set. Several operations and laws have been discussed along with some examples. In future, neutrosophic vague refined topological spaces can be introduced. And also, decision making problems on neutrosophic vague refined sets can be introduced.

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Neutrosophic implicative UP-filters, neutrosophic comparative UP-filters, and neutrosophic shift UP-filters of UP-algebras

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Abstract. The notions of neutrosophic UP-subalgebras, neutrosophic near UP-filters, neutrosophic UP-filters, neutrosophic UP-ideals, and neutrosophic strong UP-ideals were introduced by Songsaeng and Iampan [M. Songsaeng, A. Iampan, Neutrosophic set theory applied to UP-algebras, Eur. J. Pure Appl. Math., 12 (2019), 1382-1409]. In this paper, we introduce the notions of neutrosophic implicative UP-filters, neutrosophic comparative UP-filters, and neutrosophic shift UP-filters of UP-algebras by applying the notions of implicative UP-filters, comparative UP-filters, and shift UP-filters of UP-algebras to neutrosophic set, and investigate some of their important properties. Relations between neutrosophic implicative UP-filters (resp., neutrosophic comparative UP-filters, neutrosophic shift UP-filters) and their level subsets are considered.

Keywords: UP-algebra; neutrosophic implicative UP-filter; neutrosophic comparative UP-filter; neutrosophic shift UP-filter

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1. **Introduction**

A fuzzy set f in a nonempty set S is a function from S to the closed interval $[0, 1]$. The concept of a fuzzy set in a nonempty set was first considered by Zadeh [27]. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere. The notion of neutrosophic sets was introduced by Smarandache [19] in 1999 which is a more general platform that extends the notions of classic sets, (intuitionistic) fuzzy sets and interval valued (intuitionistic) fuzzy sets (see [19, 20]). Neutrosophic set theory is applied to various part which is referred to the site

http://fs.unm.edu/neutrosophy.htm.

The above-mentioned section has been derived from [24]. Wang et al. [26] introduced the notion of interval neutrosophic sets in 2005. The notion of neutrosophic $\mathcal{N}$-structures and their applications in semigroups was introduced by Khan et al. [12] in 2017. The notion of neutrosophic sets was applied to many logical algebras (see [7, 11–13, 16]).

The notions of neutrosophic UP-subalgebras, neutrosophic near UP-filters, neutrosophic UP-filters, neutrosophic UP-ideals, and neutrosophic strong UP-ideals were introduced by Songsaeng and Iampan [22] in 2019. In this paper, we introduce the notions of neutrosophic implicative UP-filters, neutrosophic comparative UP-filters, and neutrosophic shift UP-filters of UP-algebras by applying the notions of implicative UP-filters, comparative UP-filters, and shift UP-filters of UP-algebras to neutrosophic set, and investigated some of their important properties. Relations between neutrosophic implicative UP-filters (resp., neutrosophic comparative UP-filters, neutrosophic shift UP-filters) and their level subsets are considered.

2. Basic results on UP-algebras

Before we begin our study, we will give the definition and useful properties of UP-algebras.

Definition 2.1. [4] An algebra $X = (X, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra*, where X is a nonempty set, $\cdot$ is a binary operation on X, and 0 is a fixed element of X (i.e., a nullary operation) if it satisfies the following axioms:

$$(\forall x, y, z \in X)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0), \tag{1}$$

$$(\forall x \in X)(0 \cdot x = x), \tag{2}$$

$$(\forall x \in X)(x \cdot 0 = 0), \text{ and} \tag{3}$$

$$(\forall x, y \in X)(x \cdot y = 0, y \cdot x = 0 \Rightarrow x = y). \tag{4}$$

From [4], we know that the notion of UP-algebras is a generalization of KU-algebras (see [15]).

For examples of UP-algebras, see [1, 2, 5, 14, 17, 18].

The binary relation $\leq$ on a UP-algebra $X = (X, \cdot, 0)$ is defined as follows:

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x \cdot y = 0) \tag{5}$$

and the following assertions are valid (see [4,5]).

$$(\forall x \in X)(x \leq x), \tag{6}$$

$$(\forall x, y, z \in X)(x \leq y, y \leq z \Rightarrow x \leq z), \tag{7}$$

$$(\forall x, y, z \in X)(x \leq y \Rightarrow z \cdot x \leq z \cdot y), \tag{8}$$

$$(\forall x, y, z \in X)(x \leq y \Rightarrow y \cdot z \leq x \cdot z), \tag{9}$$

$$(\forall x, y, z \in X)(x \leq y \cdot x, \text{ in particular, } y \cdot z \leq x \cdot (y \cdot z)), \tag{10}$$

$$(\forall x, y \in X)(y \cdot x \leq x \Leftrightarrow x = y \cdot x), \tag{11}$$

$$(\forall x, y \in X)(x \leq y \cdot y), \tag{12}$$

$$(\forall a, x, y, z \in X)(x \cdot (y \cdot z) \leq x \cdot ((a \cdot y) \cdot (a \cdot z))), \tag{13}$$

$$(\forall a, x, y, z \in X)(((a \cdot x) \cdot (a \cdot y)) \cdot z \leq (x \cdot y) \cdot z), \tag{14}$$

$$(\forall x, y, z \in X)((x \cdot y) \cdot z \leq y \cdot z), \tag{15}$$

$$(\forall x, y, z \in X)(x \leq y \Rightarrow x \leq z \cdot y), \tag{16}$$

$$(\forall x, y, z \in X)((x \cdot y) \cdot z \leq x \cdot (y \cdot z)), \text{ and} \tag{17}$$

$$(\forall a, x, y, z \in X)((x \cdot y) \cdot z \leq y \cdot (a \cdot z)). \tag{18}$$

Definition 2.2. [3,4,6,8–10,21] A nonempty subset S of a UP-algebra $X = (X, \cdot, 0)$ is called

(1) a *UP-subalgebra* of X if it satisfies the following condition:

$$(\forall x, y \in S)(x \cdot y \in S), \tag{19}$$

(2) a *near UP-filter* of X if it satisfies the following condition:

$$(\forall x, y \in X)(y \in S \Rightarrow x \cdot y \in S). \tag{20}$$

(3) a *UP-filter* of X if it satisfies the following conditions:

$$\text{the constant } 0 \text{ of } X \text{ is in } S, \tag{21}$$

$$(\forall x, y \in X)(x \cdot y \in S, x \in S \Rightarrow y \in S), \tag{22}$$

(4) an *implicative UP-filter* of X if it satisfies the condition (21) and the following condition:

$$(\forall x, y, z \in X)(x \cdot (y \cdot z) \in S, x \cdot y \in S \Rightarrow x \cdot z \in S), \tag{23}$$

(5) a *comparative UP-filter* of X if it satisfies the condition (21) and the following condition:

$$(\forall x, y, z \in X)(x \cdot ((y \cdot z) \cdot y) \in S, x \in S \Rightarrow y \in S), \tag{24}$$

(6) a *shift UP-filter* of X if it satisfies the condition (21) and the following condition:

$$(\forall x, y, z \in X)(x \cdot (y \cdot z) \in S, x \in S \Rightarrow ((z \cdot y) \cdot y) \cdot z \in S), \tag{25}$$

(7) a *UP-ideal* of X if it satisfies the condition (21) and the following condition:

$$(\forall x, y, z \in X)(x \cdot (y \cdot z) \in S, y \in S \Rightarrow x \cdot z \in S), \tag{26}$$

(8) a *strong UP-ideal* of X if it satisfies the condition (21) and the following condition:

$$(\forall x, y, z \in X)((z \cdot y) \cdot (z \cdot x) \in S, y \in S \Rightarrow x \in S). \tag{27}$$

Guntasow et al. [3] proved that the only strong UP-ideal of a UP-algebra X is X.

3. NSs in UP-algebras

In 1999, Smarandache [19] introduced the notion of neutrosophic sets as the following definition.

A *neutrosophic set* (briefly, NS) in a nonempty set X is a structure of the form:

$$\Lambda = \{(x, \lambda_T(x), \lambda_I(x), \lambda_F(x)) \mid x \in X\} \tag{28}$$

where $\lambda_T : X \to [0, 1]$ is a *truth membership function*, $\lambda_I : X \to [0, 1]$ is an *indeterminate membership function*, and $\lambda_F : X \to [0, 1]$ is a *false membership function.*

For our convenience, we will denote a NS as $\Lambda = (X, \lambda_T, \lambda_I, \lambda_F) = (X, \lambda_{T,I,F}) = \{(x, \lambda_T(x), \lambda_I(x), \lambda_F(x)) \mid x \in X\}$.

Definition 3.1. [19] Let Λ be a NS in a nonempty set X. The NS $\overline{\Lambda} = (X, \overline{\lambda}_{T,I,F})$ in X defined by

$$(\forall x \in X) \begin{pmatrix} \overline{\lambda}_T(x) = 1 - \lambda_T(x) \\ \overline{\lambda}_I(x) = 1 - \lambda_I(x) \\ \overline{\lambda}_F(x) = 1 - \lambda_F(x) \end{pmatrix} \tag{29}$$

is called the *complement* of Λ in X. For all NS Λ in a nonempty set X, we have $\Lambda = \overline{\overline{\Lambda}}$.

In what follows, let X denote a UP-algebra $(X, \cdot, 0)$ unless otherwise specified.

Songsaeng and Iampan [23] introduced the new concepts of neutrosophic sets in UP-algebras: neutrosophic UP-subalgebras, neutrosophic near UP-filters, neutrosophic UP-filters, neutrosophic UP-ideals, and neutrosophic strong UP-ideals.

Definition 3.2. A NS Λ in X is called

(1) a *neutrosophic UP-subalgebra* of X if it satisfies the following conditions:

$$(\forall x, y \in X)(\lambda_T(x \cdot y) \geq \min\{\lambda_T(x), \lambda_T(y)\}), \tag{30}$$

$$(\forall x, y \in X)(\lambda_I(x \cdot y) \leq \max\{\lambda_I(x), \lambda_I(y)\}), \tag{31}$$

$$(\forall x, y \in X)(\lambda_F(x \cdot y) \geq \min\{\lambda_F(x), \lambda_F(y)\}), \tag{32}$$

(2) a *neutrosophic near UP-filter* of X if it satisfies the following conditions:

$$(\forall x, y \in X)(\lambda_T(x \cdot y) \geq \lambda_T(y)), \tag{33}$$

$$(\forall x, y \in X)(\lambda_I(x \cdot y) \leq \lambda_I(y)), \tag{34}$$

$$(\forall x, y \in X)(\lambda_F(x \cdot y) \geq \lambda_F(y)), \tag{35}$$

(3) a *neutrosophic UP-filter* of X if it satisfies the following conditions:

$$(\forall x \in X)(\lambda_T(0) \geq \lambda_T(x)), \tag{36}$$

$$(\forall x \in X)(\lambda_I(0) \leq \lambda_I(x)), \tag{37}$$

$$(\forall x \in X)(\lambda_F(0) \geq \lambda_F(x)), \tag{38}$$

$$(\forall x, y \in X)(\lambda_T(y) \geq \min\{\lambda_T(x \cdot y), \lambda_T(x)\}), \tag{39}$$

$$(\forall x, y \in X)(\lambda_I(y) \leq \max\{\lambda_I(x \cdot y), \lambda_I(x)\}), \tag{40}$$

$$(\forall x, y \in X)(\lambda_F(y) \geq \min\{\lambda_F(x \cdot y), \lambda_F(x)\}), \tag{41}$$

(4) a *neutrosophic UP-ideal* of X if it satisfies the following conditions: (36), (37), (38), and

$$(\forall x, y, z \in X)(\lambda_T(x \cdot z) \geq \min\{\lambda_T(x \cdot (y \cdot z)), \lambda_T(y)\}), \tag{42}$$

$$(\forall x, y, z \in X)(\lambda_I(x \cdot z) \leq \max\{\lambda_I(x \cdot (y \cdot z)), \lambda_I(y)\}), \tag{43}$$

$$(\forall x, y, z \in X)(\lambda_F(x \cdot z) \geq \min\{\lambda_F(x \cdot (y \cdot z)), \lambda_F(y)\}), \tag{44}$$

(5) a *neutrosophic strong UP-ideal* of X if it satisfies the following conditions: (36), (37), (38), and

$$(\forall x, y, z \in X)(\lambda_T(x) \geq \min\{\lambda_T((z \cdot y) \cdot (z \cdot x)), \lambda_T(y)\}), \tag{45}$$

$$(\forall x, y, z \in X)(\lambda_I(x) \leq \max\{\lambda_I((z \cdot y) \cdot (z \cdot x)), \lambda_I(y)\}), \tag{46}$$

$$(\forall x, y, z \in X)(\lambda_F(x) \geq \min\{\lambda_F((z \cdot y) \cdot (z \cdot x)), \lambda_F(y)\}). \tag{47}$$

Definition 3.3. [23] A NS Λ in X is said to be *constant* if Λ is a constant function from X to $[0,1]^3$. That is, λ_T, λ_I, and λ_F are constant functions from X to $[0,1]$.

Songsaeng and Iampan [23] proved the generalization that the concept of neutrosophic UP-subalgebras is a generalization of neutrosophic near UP-filters, neutrosophic near UP-filters is a generalization of neutrosophic UP-filters, neutrosophic UP-filters is a generalization of neutrosophic UP-ideals, and neutrosophic UP-ideals is a generalization of neutrosophic strong UP-ideals. Moreover, they proved that neutrosophic strong UP-ideals and constant neutrosophic sets coincide.

Definition 3.4. A NS Λ in X is called a *neutrosophic implicative UP-filter* of X if it satisfies the following conditions: (36), (37), (38), and

$$(\forall x, y, z \in X)(\lambda(x \cdot z) \geq \min\{\lambda(x \cdot (y \cdot z)), \lambda(x \cdot y)\}), \tag{48}$$

$$(\forall x, y, z \in X)(\lambda(x \cdot z) \leq \max\{\lambda(x \cdot (y \cdot z)), \lambda(x \cdot y)\}), \tag{49}$$

$$(\forall x, y, z \in X)(\lambda(x \cdot z) \geq \min\{\lambda(x \cdot (y \cdot z)), \lambda(x \cdot y)\}). \tag{50}$$

Example 3.5. Let $X = \{0, 1, 2, 3, 4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	0	4
3	0	1	2	0	4
4	0	0	0	0	0

We define a NS Λ in X as follows:

$$\lambda_T = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.8 & 0.8 & 0.6 & 0.6 & 0.6 \end{pmatrix}, \lambda_I = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.4 & 0.4 & 0.4 & 0.4 & 0.8 \end{pmatrix}, \lambda_F = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.7 & 0.7 & 0.5 & 0.7 & 0.5 \end{pmatrix}.$$

Hence, Λ is a neutrosophic implicative UP-filter of X.

Definition 3.6. A NS Λ in X is called a *neutrosophic comparative UP-filter* of X if it satisfies the following conditions: (36), (37), (38), and

$$(\forall x, y, z \in X)(\lambda(y) \geq \min\{\lambda(x \cdot ((y \cdot z) \cdot y)), \lambda(x)\}), \tag{51}$$

$$(\forall x, y, z \in X)(\lambda(y) \leq \max\{\lambda(x \cdot ((y \cdot z) \cdot y)), \lambda(x)\}), \tag{52}$$

$$(\forall x, y, z \in X)(\lambda(y) \geq \min\{\lambda(x \cdot ((y \cdot z) \cdot y)), \lambda(x)\}). \tag{53}$$

Example 3.7. Let $X = \{0, 1, 2, 3, 4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	2	4
2	0	0	0	1	4
3	0	0	0	0	4
4	0	1	2	3	0

We define a NS Λ in X as follows:

$$\lambda_T = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.6 & 0.6 & 0.6 & 0.6 & 0.4 \end{pmatrix}, \lambda_I = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.4 & 0.4 & 0.4 & 0.4 & 0.8 \end{pmatrix}, \lambda_F = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.5 \end{pmatrix}.$$

Hence, Λ is a neutrosophic comparative UP-filter of X.

Definition 3.8. A NS Λ in X is called a *neutrosophic shift UP-filter* of X if it satisfies the following conditions: (36), (37), (38), and

$$(\forall x, y, z \in X)(\lambda(((z \cdot y) \cdot y) \cdot z) \geq \min\{\lambda(x \cdot (y \cdot z)), \lambda(x)\}), \tag{54}$$

$$(\forall x, y, z \in X(\lambda(((z \cdot y) \cdot y) \cdot z) \leq \max\{\lambda(x \cdot (y \cdot z)), \lambda(x)\}), \tag{55}$$

$$(\forall x, y, z \in X)(\lambda(((z \cdot y) \cdot y) \cdot z) \geq \min\{\lambda(x \cdot (y \cdot z)), \lambda(x)\}). \tag{56}$$

Example 3.9. Let $X = \{0, 1, 2, 3, 4\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	2	4
3	0	0	0	0	4
4	0	1	2	3	0

We define a NS Λ in X as follows:

$$\lambda_T = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.6 & 0.6 & 0.6 & 0.6 & 0.4 \end{pmatrix}, \lambda_I = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.4 & 0.4 & 0.8 & 0.8 & 0.8 \end{pmatrix}, \lambda_F = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.9 & 0.9 & 0.7 & 0.7 & 0.7 \end{pmatrix}.$$

Hence, Λ is a neutrosophic shift UP-filter of X.

Theorem 3.10. *[23] A NS Λ in X is constant if and only if it is a neutrosophic strong UP-ideal of X.*

Theorem 3.11. *Every neutrosophic implicative UP-filter of X is a neutrosophic UP-ideal.*

Proof. Assume that Λ is a neutrosophic implicative UP-filter of X. Then Λ satisfies the conditions (36), (37), and (38).

Let $x, y, z \in X$. Then $\lambda_T(x \cdot z) \geq \min\{\lambda_T(x \cdot (y \cdot z)), \lambda_T(x \cdot y)\}$ By generalization of neutrosophic near UP-filter, neutrosophic UP-filter, and the condition 33, we have $\lambda_T(x \cdot z) \geq \min\{\lambda_T(x \cdot (y \cdot z)), \lambda_T(y)\}$,

Let $x, y, z \in X$. Then $\lambda_I(x \cdot z) \leq \max\{\lambda_I(x \cdot (y \cdot z)), \lambda_I(x \cdot y)\}$ By generalization of neutrosophic near UP-filter, neutrosophic UP-filter, and the condition 34, we have $\lambda_I(x \cdot z) \leq \max\{\lambda_I(x \cdot (y \cdot z)), \lambda_I(y)\}$, Let $x, y, z \in X$. Then $\lambda_F(x \cdot z) \geq \min\{\lambda_F(x \cdot (y \cdot z)), \lambda_F(x \cdot y)\}$ By generalization of neutrosophic near UP-filter, neutrosophic UP-filter, and the condition 35, we have $\lambda_F(x \cdot z) \geq \min\{\lambda_F(x \cdot (y \cdot z)), \lambda_F(y)\}$,

Hence, Λ is a neutrosophic UP-ideal of X. □

Example 3.12. From the Cayley table in Example 3.7, we define a NS Λ in X as follows:

$$\lambda_T = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0.7 & 0.6 & 0.6 & 0.4 \end{pmatrix}, \lambda_I = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0.3 & 0.5 & 0.5 & 0.7 \end{pmatrix}, \lambda_F = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.9 & 0.8 & 0.7 & 0.7 & 0.5 \end{pmatrix}.$$

Then Λ is a neutrosophic UP-ideal of X. Since $\lambda_I(2 \cdot 3) = 0.3 > 0 = \max\{\lambda_I(2 \cdot (1 \cdot 3)), \lambda_I(2 \cdot 1)\}$, we have Λ is not a neutrosophic implicative UP-filter of X.

Theorem 3.13. *Every neutrosophic comparative UP-filter of X is a neutrosophic UP-filter.*

Proof. Assume that Λ is a neutrosophic comparative UP-filter of X. Then Λ satisfies the conditions (36), (37), and (38). Next, let $x, y \in X$. Then

$$\begin{aligned}
\lambda_T(y) &\geq \min\{\lambda_T(x \cdot ((y \cdot y) \cdot y)), \lambda_T(x)\} && \text{by (51)} \\
&= \min\{\lambda_T(x \cdot (0 \cdot y)), \lambda_T(x)\} && \text{by (6)} \\
&= \min\{\lambda_T(x \cdot y), \lambda_T(x)\}, && \text{by (2)} \\
\lambda_I(y) &\leq \max\{\lambda_I(x \cdot ((y \cdot y) \cdot y)), \lambda_I(x)\} && \text{by (52)} \\
&= \max\{\lambda_I(x \cdot (0 \cdot y)), \lambda_I(x)\} && \text{by (6)} \\
&= \max\{\lambda_I(x \cdot y), \lambda_I(x)\}, && \text{by (2)} \\
\lambda_F(y) &\geq \min\{\lambda_F(x \cdot ((y \cdot y) \cdot y)), \lambda_F(x)\} && \text{by (53)} \\
&= \min\{\lambda_F(x \cdot (0 \cdot y)), \lambda_F(x)\} && \text{by (6)} \\
&= \min\{\lambda_F(x \cdot y), \lambda_F(x)\}. && \text{by (2)}
\end{aligned}$$

Hence, Λ is a neutrosophic UP-filter of X. □

Example 3.14. From Example 3.12, we have Λ is a neutrosophic UP-ideal of X and so Λ is a neutrosophic UP-filter of X. Since $\lambda_T(1) = 0.7 < 1 = \min\{\lambda_T(0 \cdot ((1 \cdot 3) \cdot 1)), \lambda_T(0)\}$, we have Λ is not a neutrosophic comparative UP-filter of X.

Theorem 3.15. *Every neutrosophic shift UP-filter of X is a neutrosophic UP-filter.*

Proof. Assume that Λ is a neutrosophic shift UP-filter of X. Then Λ satisfies the conditions (36), (37), and (38). Next, let $x, y \in X$. Then

$$\begin{aligned}
\lambda_T(y) &= \lambda_T(((y \cdot 0) \cdot 0) \cdot y) && \text{by (2) and (3)}\\
&\geq \min\{\lambda_T(x \cdot (0 \cdot y)), \lambda_T(x)\} && \text{by (54)}\\
&= \min\{\lambda_T(x \cdot y), \lambda_T(x)\}, && \text{by (2)}\\
\lambda_I(y) &= \lambda_I(((y \cdot 0) \cdot 0) \cdot y) && \text{by (2) and (3)}\\
&\leq \max\{\lambda_I(x \cdot (0 \cdot y)), \lambda_I(x)\} && \text{by (55)}\\
&= \max\{\lambda_I(x \cdot y), \lambda_I(x)\}, && \text{by (2)}\\
\lambda_F(y) &= \lambda_F(((y \cdot 0) \cdot 0) \cdot y) && \text{by (2) and (3)}\\
&\geq \min\{\lambda_F(x \cdot (0 \cdot y)), \lambda_F(x)\} && \text{by (56)}\\
&= \min\{\lambda_F(x \cdot y), \lambda_F(x)\}. && \text{by (2)}
\end{aligned}$$

Hence, Λ is a neutrosophic UP-filter of X. $\square$

Example 3.16. From Example 3.12, we have Λ is a neutrosophic UP-ideal of X and so Λ is a neutrosophic UP-filter of X. Since $\lambda_T(((1 \cdot 2) \cdot 2) \cdot 1) = 0.7 < 1 = \min\{\lambda_T(0 \cdot (2 \cdot 1)), \lambda_T(0)\}$, we have Λ is not a neutrosophic shift UP-filter of X.

Theorem 3.17. *Every neutrosophic strong UP-ideal of X is a neutrosophic implicative UP-filter.*

Proof. Assume that Λ is a neutrosophic strong UP-ideal of X. Then Λ satisfies the conditions (36), (37), and (38). By Theorem 3.10, we have Λ is constant. Then for all $x \in X, \lambda_T(x) = \lambda_T(0), \lambda_I(x) = \lambda_I(0)$, and $\lambda_F(x) = \lambda_F(0)$. Next, let $x, y, z \in X$. Then

$$\begin{aligned}
\lambda_T(x \cdot z) &= \lambda_T(x \cdot y) && \text{by } \lambda_T \text{ is constant}\\
&\geq \min\{\lambda_T(x \cdot (y \cdot z)), \lambda_T(x \cdot y)\},\\
\lambda_I(x \cdot z) &= \lambda_I(x \cdot y) && \text{by } \lambda_I \text{ is constant}\\
&\leq \max\{\lambda_I(x \cdot (y \cdot z)), \lambda_I(x \cdot y)\},\\
\lambda_F(x \cdot z) &= \lambda_F(x \cdot y) && \text{by } \lambda_F \text{ is constant}\\
&\geq \min\{\lambda_F(x \cdot (y \cdot z)), \lambda_F(x \cdot y)\}.
\end{aligned}$$

Hence, Λ is a neutrosophic implicative UP-filter of X. $\square$

Example 3.18. From Example 3.5, we have Λ is a neutrosophic implicative UP-filter of X. Since Λ is not constant, it follows from Theorem 3.10 that it is not a neutrosophic strong UP-ideal of X.

Theorem 3.19. *Every neutrosophic strong UP-ideal of X is a neutrosophic comparative UP-filter.*

Proof. Assume that Λ is a neutrosophic strong UP-ideal of X. Then Λ satisfies the conditions (36), (37), and (38). By Theorem 3.10, we have Λ is constant. Then for all $x \in X, \lambda_T(x) = \lambda_T(0), \lambda_I(x) = \lambda_I(0)$, and $\lambda_F(x) = \lambda_F(0)$. Next, let $x, y, z \in X$. Then

$$\begin{aligned}
\lambda_T(y) &= \lambda_T(x) && \text{by } \lambda_T \text{ is constant} \\
&\geq \min\{\lambda_T(x \cdot ((y \cdot z) \cdot y)), \lambda_T(x)\}, \\
\lambda_I(y) &= \lambda_I(x) && \text{by } \lambda_I \text{ is constant} \\
&\leq \max\{\lambda_I(x \cdot ((y \cdot z) \cdot y)), \lambda_I(x)\}, \\
\lambda_F(y) &= \lambda_F(x) && \text{by } \lambda_F \text{ is constant} \\
&\geq \min\{\lambda_F(x \cdot ((y \cdot z) \cdot y)), \lambda_F(x)\}.
\end{aligned}$$

Hence, Λ is a neutrosophic comparative UP-filter of X. □

Example 3.20. From Example 3.7, we have Λ is a neutrosophic comparative UP-filter of X. Since Λ is not constant, it follows from Theorem 3.10 that it is not a neutrosophic strong UP-ideal of X.

Theorem 3.21. *Every neutrosophic strong UP-ideal of X is a neutrosophic shift UP-filter.*

Proof. Assume that Λ is a neutrosophic strong UP-ideal of X. Then Λ satisfies the conditions (36), (37), and (38). By Theorem 3.10, we have Λ is constant. Then for all $x \in X, \lambda_T(x) = \lambda_T(0), \lambda_I(x) = \lambda_I(0)$, and $\lambda_F(x) = \lambda_F(0)$. Next, let $x, y, z \in X$. Then

$$\begin{aligned}
\lambda_T(((z \cdot y) \cdot y) \cdot z) &= \lambda_T(x) && \text{by } \lambda_T \text{ is constant} \\
&\geq \min\{\lambda_T(x \cdot (y \cdot z)), \lambda_T(x)\}, \\
\lambda_I(((z \cdot y) \cdot y) \cdot z) &= \lambda_I(x) && \text{by } \lambda_I \text{ is constant} \\
&\leq \max\{\lambda_I(x \cdot (y \cdot z)), \lambda_I(x)\}, \\
\lambda_F(((z \cdot y) \cdot y) \cdot z) &= \lambda_F(x) && \text{by } \lambda_F \text{ is constant} \\
&\geq \min\{\lambda_F(x \cdot (y \cdot z)), \lambda_F(x)\}.
\end{aligned}$$

Hence, Λ is a neutrosophic shift UP-filter of X. □

Example 3.22. From Example 3.9, we have Λ is a neutrosophic shift UP-filter of X. Since Λ is not constant, it follows from Theorem 3.10 that it is not a neutrosophic strong UP-ideal of X.

Example 3.23. From Example 3.5, we have Λ is a neutrosophic implicative UP-filter of X. Since $\lambda_T(((3\cdot2)\cdot2)\cdot3) = 0.6 < 0.8 = \min\{\lambda_T(0\cdot(2\cdot3)), \lambda_T(0)\}$, we have Λ is not a neutrosophic shift UP-filter of X.

Example 3.24. From Example 3.9, we have Λ is a neutrosophic shift UP-filter of X. Since $\lambda_F(2\cdot3) = 0.7 < 0.9 = \min\{\lambda_F(2\cdot(2\cdot3)), \lambda_F(2\cdot2)\}$, we have Λ is not a neutrosophic implicative UP-filter of X.

By Theorems 3.11, 3.13, 3.15, 3.17, 3.19, and 3.21 and Examples 3.12, 3.14, 3.16, 3.18, 3.20, and 3.22, we have that the notion of neutrosophic UP-ideals is a generalization of neutrosophic implicative UP-filters, the notion of neutrosophic UP-filters is a generalization of neutrosophic comparative UP-filters, the notion of neutrosophic UP-filters is a generalization of neutrosophic shift UP-filters, and the notions of neutrosophic implicative UP-filters, neutrosophic comparative UP-filters, neutrosophic shift UP-filters is a generalization of neutrosophic strong UP-ideals.

Theorem 3.25. *If Λ is a neutrosophic UP-ideal of X satisfying the following condition:*

$$(\forall x, y, z \in X)\begin{pmatrix}\lambda_T(x\cdot(y\cdot z)) \geq \lambda_T(y) \Rightarrow \lambda_T(y) \geq \lambda_T(x\cdot y)\\ \lambda_I(x\cdot(y\cdot z)) \leq \lambda_I(y) \Rightarrow \lambda_I(y) \leq \lambda_I(x\cdot y)\\ \lambda_F(x\cdot(y\cdot z)) \geq \lambda_F(y) \Rightarrow \lambda_F(y) \geq \lambda_F(x\cdot y)\end{pmatrix}, \tag{57}$$

then Λ is a neutrosophic implicative UP-filter of X.

Proof. Assume that Λ is a neutrosophic UP-ideal of X satisfying the condition (57). Then Λ satisfies the conditions (36), (37), and (38). Next, let $x, y, z \in X$. Then

$$\begin{aligned}\lambda_T(x\cdot z) &\geq \min\{\lambda_T(x\cdot(y\cdot z)), \lambda_T(y)\} && \text{by (42)}\\ &\geq \min\{\lambda_T(x\cdot(y\cdot z)), \lambda_T(x\cdot y)\}, && \text{by (57) for } \lambda_T\\ \lambda_I(x\cdot z) &\leq \max\{\lambda_I(x\cdot(y\cdot z)), \lambda_I(y)\} && \text{by (43)}\\ &\leq \max\{\lambda_I(x\cdot(y\cdot z)), \lambda_I(x\cdot y)\}, && \text{by (57) for } \lambda_I\\ \lambda_F(x\cdot z) &\geq \min\{\lambda_F(x\cdot(y\cdot z)), \lambda_F(y)\} && \text{by (44)}\\ &\geq \min\{\lambda_F(x\cdot(y\cdot z)), \lambda_F(x\cdot y)\}. && \text{by (57) for } \lambda_F\end{aligned}$$

Hence, Λ is a neutrosophic implicative UP-filter of X. □

Theorem 3.26. *If Λ is a neutrosophic UP-filter of X satisfying the following condition:*

$$(\forall x, y, z \in X) \begin{pmatrix} \lambda_T(x) \geq \lambda_T(x \cdot y) \Rightarrow \lambda_T(x \cdot y) \geq \lambda_T(x \cdot ((y \cdot z) \cdot y)) \\ \lambda_I(x) \leq \lambda_I(x \cdot y) \Rightarrow \lambda_I(x \cdot y) \leq \lambda_I(x \cdot ((y \cdot z) \cdot y)) \\ \lambda_F(x) \geq \lambda_F(x \cdot y) \Rightarrow \lambda_F(x \cdot y) \geq \lambda_F(x \cdot ((y \cdot z) \cdot y)) \end{pmatrix}, \tag{58}$$

then Λ is a neutrosophic comparative UP-filter of X.

Proof. Assume that Λ is a neutrosophic UP-filter of X satisfying the condition (58). Then Λ satisfies the conditions (36), (37), and (38). Next, let $x, y, z \in X$. Then

$$\begin{aligned}
\lambda_T(y) &\geq \min\{\lambda_T(x \cdot y), \lambda_T(x)\} && \text{by (39)} \\
&\geq \min\{\lambda_T(x \cdot ((y \cdot z) \cdot y)), \lambda_T(x)\}, && \text{by (58) for } \lambda_T \\
\lambda_I(y) &\leq \max\{\lambda_I(x \cdot y), \lambda_I(x)\} && \text{by (40)} \\
&\leq \max\{\lambda_I(x \cdot ((y \cdot z) \cdot y)), \lambda_I(x)\}, && \text{by (58) for } \lambda_I \\
\lambda_F(y) &\geq \min\{\lambda_F(x \cdot y), \lambda_F(x)\} && \text{by (41)} \\
&\geq \min\{\lambda_F(x \cdot ((y \cdot z) \cdot y)), \lambda_F(x)\}. && \text{by (58) for } \lambda_F
\end{aligned}$$

Hence, Λ is a neutrosophic comparative UP-filter of X. □

Theorem 3.27. *If Λ is a neutrosophic UP-filter of X satisfying the following condition:*

$$(\forall x, y, z \in X) \begin{pmatrix} \lambda_T(x) \geq \lambda_T(x \cdot (((z \cdot y) \cdot y) \cdot z)) \\ \Rightarrow \lambda_T(x \cdot (((z \cdot y) \cdot y) \cdot z)) \geq \lambda_T(x \cdot (y \cdot z)) \\ \lambda_I(x) \leq \lambda_I(x \cdot (((z \cdot y) \cdot y) \cdot z)) \\ \Rightarrow \lambda_I(x \cdot (((z \cdot y) \cdot y) \cdot z)) \leq \lambda_I(x \cdot (y \cdot z)) \\ \lambda_F(x) \geq \lambda_F(x \cdot (((z \cdot y) \cdot y) \cdot z)) \\ \Rightarrow \lambda_F(x \cdot (((z \cdot y) \cdot y) \cdot z)) \geq \lambda_F(x \cdot (y \cdot z)) \end{pmatrix}, \tag{59}$$

then Λ is a neutrosophic shift UP-filter of X.

Proof. Assume that Λ is a neutrosophic UP-filter of X satisfying the condition (59). Then Λ satisfies the conditions (36), (37), and (38). Next, let $x, y, z \in X$. Then

$$\begin{aligned}
\lambda_T(((z \cdot y) \cdot y) \cdot z) &\geq \min\{\lambda_T(x \cdot (((z \cdot y) \cdot y) \cdot z), \lambda_T(x)\} && \text{by (39)} \\
&\geq \min\{\lambda_T(x \cdot (y \cdot z)), \lambda_T(x)\}, && \text{by (59) for } \lambda_T \\
\lambda_I(((z \cdot y) \cdot y) \cdot z) &\leq \max\{\lambda_I(x \cdot (((z \cdot y) \cdot y) \cdot z), \lambda_I(x)\} && \text{by (40)} \\
&\leq \max\{\lambda_I(x \cdot (y \cdot z)), \lambda_I(x)\}, && \text{by (59) for } \lambda_I \\
\lambda_F(((z \cdot y) \cdot y) \cdot z) &\geq \min\{\lambda_F(x \cdot (((z \cdot y) \cdot y) \cdot z)), \lambda_F(x)\} && \text{by (41)} \\
&\geq \min\{\lambda_F(x \cdot (y \cdot z)), \lambda_F(x)\}. && \text{by (59) for } \lambda_F
\end{aligned}$$

Hence, Λ is a neutrosophic shift UP-filter of X. □

Theorem 3.28. *If Λ is a NS in X satisfying the following condition:*

$$(\forall a, x, y, z \in X) \left(a \leq x \cdot (y \cdot z) \Rightarrow \begin{cases} \lambda_T(x \cdot z) \geq \min\{\lambda_T(a), \lambda_T(x \cdot y)\} \\ \lambda_I(x \cdot z) \leq \max\{\lambda_I(a), \lambda_I(x \cdot y)\} \\ \lambda_F(x \cdot z) \geq \min\{\lambda_F(a), \lambda_F(x \cdot y)\} \end{cases} \right), \tag{60}$$

then Λ is a neutrosophic implicative UP-filter of X.

Proof. Assume that Λ is a NS in X satisfying the condition (59). Let $x \in X$. By (3), we have $x \cdot (0 \cdot (x \cdot 0)) = 0$, that is, $x \leq 0 \cdot (x \cdot 0)$. It follows from (60) that

$$\begin{aligned} \lambda_T(0) = \lambda_T(0 \cdot 0) &\geq \min\{\lambda_T(x), \lambda_T(0 \cdot x)\} \\ &= \min\{\lambda_T(x), \lambda_T(x)\} = \lambda_T(x), && \text{by (2)} \\ \lambda_I(0) = \lambda_I(0 \cdot 0) &\leq \max\{\lambda_I(x), \lambda_I(0 \cdot x)\} \\ &= \max\{\lambda_I(x), \lambda_I(x)\} = \lambda_I(x), && \text{by (2)} \\ \lambda_F(0) = \lambda_F(0 \cdot 0) &\geq \min\{\lambda_F(x), \lambda_F(0 \cdot x)\} \\ &= \min\{\lambda_F(x), \lambda_F(x)\} = \lambda_F(x). && \text{by (2)} \end{aligned}$$

Next, let $x, y, z \in X$. By (6), we have $(x \cdot (y \cdot z)) \cdot (x \cdot (y \cdot z)) = 0$, that is, $x \cdot (y \cdot z) \leq x \cdot (y \cdot z)$. It follows from (60) that

$$\lambda_T(x \cdot z) \geq \min\{\lambda_T(x \cdot (y \cdot z)), \lambda_T(x \cdot y)\},$$
$$\lambda_I(x \cdot z) \leq \max\{\lambda_I(x \cdot (y \cdot z)), \lambda_I(x \cdot y)\},$$
$$\lambda_F(x \cdot z) \geq \min\{\lambda_F(x \cdot (y \cdot z)), \lambda_F(x \cdot y)\}.$$

Hence, Λ is a neutrosophic implicative UP-filter of X. □

Theorem 3.29. *If Λ is a NS in X satisfying the following condition:*

$$(\forall a, x, y, z \in X) \left(a \leq x \cdot ((y \cdot z) \cdot y) \Rightarrow \begin{cases} \lambda_T(y) \geq \min\{\lambda_T(a), \lambda_T(x)\} \\ \lambda_I(y) \leq \max\{\lambda_I(a), \lambda_I(x)\} \\ \lambda_F(y) \geq \min\{\lambda_F(a), \lambda_F(x)\} \end{cases} \right), \tag{61}$$

then Λ is a neutrosophic comparative UP-filter of X.

Proof. Assume that Λ is a NS in X satisfying the condition (61). Let $x \in X$. By (3), we have $x \cdot (x \cdot ((0 \cdot x) \cdot 0)) = 0$, that is, $x \leq x \cdot ((0 \cdot x) \cdot 0)$. It follows from (61) that

$$\lambda_T(0) \geq \min\{\lambda_T(x), \lambda_T(x)\} = \lambda_T(x),$$
$$\lambda_I(0) \leq \max\{\lambda_I(x), \lambda_I(x)\} = \lambda_I(x),$$
$$\lambda_F(0) \geq \min\{\lambda_F(x), \lambda_F(x)\} = \lambda_F(x).$$

Next, let $x, y, z \in X$. By (6), we have $(x \cdot ((y \cdot z) \cdot y)) \cdot (x \cdot ((y \cdot z) \cdot y)) = 0$, that is, $x \cdot ((y \cdot z) \cdot y) \leq x \cdot ((y \cdot z) \cdot y)$. It follows from (61) that

$$\lambda_T(y) \geq \min\{\lambda_T(x \cdot ((y \cdot z) \cdot y)), \lambda_T(x)\},$$
$$\lambda_I(y) \leq \max\{\lambda_I(x \cdot ((y \cdot z) \cdot y)), \lambda_I(x)\},$$
$$\lambda_F(y) \geq \min\{\lambda_F(x \cdot ((y \cdot z) \cdot y)), \lambda_F(x)\}.$$

Hence, Λ is a neutrosophic comparative UP-filter of X. $\square$

Theorem 3.30. *If Λ is a NS in X satisfying the following condition:*

$$(\forall a, x, y, z \in X)\left(\begin{array}{c} a \leq x \cdot (y \cdot z) \\ \Rightarrow \begin{cases} \lambda_T(((z \cdot y) \cdot y) \cdot z) \geq \min\{\lambda_T(a), \lambda_T(x)\} \\ \lambda_I(((z \cdot y) \cdot y) \cdot z) \leq \max\{\lambda_I(a), \lambda_I(x)\} \\ \lambda_F(((z \cdot y) \cdot y) \cdot z) \geq \min\{\lambda_F(a), \lambda_F(x)\} \end{cases} \end{array}\right), \tag{62}$$

then Λ is a neutrosophic shift UP-filter of X.

Proof. Assume that Λ is a NS in X satisfying the condition (62). Let $x \in X$. By (3), we have $x \cdot (x \cdot (x \cdot 0)) = 0$, that is, $x \leq x \cdot (x \cdot 0)$. It follows from (62) that

$$\lambda_T(0) = \lambda_T(((0 \cdot x) \cdot x) \cdot 0) \geq \min\{\lambda_T(x), \lambda_T(x)\} = \lambda_T(x), \quad \text{by (3)}$$
$$\lambda_I(0) = \lambda_I(((0 \cdot x) \cdot x) \cdot 0) \leq \max\{\lambda_I(x), \lambda_I(x)\} = \lambda_I(x), \quad \text{by (3)}$$
$$\lambda_F(0) = \lambda_F(((0 \cdot x) \cdot x) \cdot 0) \geq \min\{\lambda_F(x), \lambda_F(x)\} = \lambda_F(x). \quad \text{by (3)}$$

Next, let $x, y, z \in X$. By (6), we have $(x \cdot (y \cdot z)) \cdot (x \cdot (y \cdot z)) = 0$, that is, $x \cdot (y \cdot z) \leq x \cdot (y \cdot z)$. It follows from (62) that

$$\lambda_T(((z \cdot y) \cdot y) \cdot z) \geq \min\{\lambda_T(x \cdot (y \cdot z)), \lambda_T(x)\},$$
$$\lambda_I(((z \cdot y) \cdot y) \cdot z) \leq \max\{\lambda_I(x \cdot (y \cdot z)), \lambda_I(x)\},$$
$$\lambda_F(((z \cdot y) \cdot y) \cdot z) \geq \min\{\lambda_F(x \cdot (y \cdot z)), \lambda_F(x)\}.$$

Hence, Λ is a neutrosophic shift UP-filter of X. $\square$

For any fixed numbers $\alpha^+, \alpha^-, \beta^+, \beta^-, \gamma^+, \gamma^- \in [0,1]$ such that $\alpha^+ > \alpha^-, \beta^+ > \beta^-, \gamma^+ > \gamma^-$ and a nonempty subset G of X, a NS $\Lambda^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}] = (X, \lambda^G_T[^{\alpha^+}_{\alpha^-}], \lambda^G_I[^{\beta^-}_{\beta^+}], \lambda^G_F[^{\gamma^+}_{\gamma^-}])$ in X where $\lambda^G_T[^{\alpha^+}_{\alpha^-}], \lambda^G_I[^{\beta^-}_{\beta^+}]$, and $\lambda^G_F[^{\gamma^+}_{\gamma^-}]$ are functions on X which are given as follows:

$$\lambda^G_T[^{\alpha^+}_{\alpha^-}](x) = \begin{cases} \alpha^+ & \text{if } x \in G, \\ \alpha^- & \text{otherwise,} \end{cases}$$

$$\lambda_I^G[^{\beta^-}_{\beta^+}](x) = \begin{cases} \beta^- & \text{if } x \in G, \\ \beta^+ & \text{otherwise,} \end{cases}$$

$$\lambda_F^G[^{\gamma^+}_{\gamma^-}](x) = \begin{cases} \gamma^+ & \text{if } x \in G, \\ \gamma^- & \text{otherwise.} \end{cases}$$

Lemma 3.31. *[23] If the constant* 0 *of* X *is in a nonempty subset* G *of* X*, then a NS* $\Lambda^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ *in* X *satisfies the conditions* (36)*,* (37)*, and* (38)*.*

Lemma 3.32. *[23] If a NS* $\Lambda^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ *in* X *satisfies the condition* (36) *(resp.,* (37)*,* (38)*), then the constant* 0 *of* X *is in a nonempty subset* G *of* X*.*

Theorem 3.33. *A NS* $\Lambda^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ *in* X *is a neutrosophic implicative UP-filter of* X *if and only if a nonempty subset* G *of* X *is an implicative UP-filter of* X*.*

Proof. Assume that $\Lambda^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is neutrosophic implicative UP-filter of X. Since $\Lambda^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ satisfies the condition (36), it follows from Lemma 3.32 that $0 \in G$. Next, let $x \cdot (y \cdot z), x \cdot y \in G$. Then $\lambda_T^G[^{\alpha^+}_{\alpha^-}](x \cdot (y \cdot z)) = \alpha^+ = \lambda_T^G[^{\alpha^+}_{\alpha^-}](x \cdot y)$. Thus, by (48), we have

$$\lambda_T^G[^{\alpha^+}_{\alpha^-}](x \cdot z) = \min\{\lambda_T^G[^{\alpha^+}_{\alpha^-}](x \cdot (y \cdot z)), \lambda_T^G[^{\alpha^+}_{\alpha^-}](x \cdot y)\} = \alpha^+ \geq \lambda_T^G[^{\alpha^+}_{\alpha^-}](x \cdot z)$$

and so $\lambda_T^G[^{\alpha^+}_{\alpha^-}](x \cdot z) = \alpha^+$. Thus $x \cdot z \in G$. Hence, G is an implicative UP-filter of X.

Conversely, assume that G is an implicative UP-filter of X. Since $0 \in G$, it follows from Lemma 3.31 that $\Lambda^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ satisfies the conditions (36), (37), and (38). Next, let $x, y, z \in X$.

Case 1: $x \cdot (y \cdot z), x \cdot y \in G$. Then $\lambda_T^G[^{\alpha^+}_{\alpha^-}](x \cdot (y \cdot z)) = \lambda_T^G[^{\alpha^+}_{\alpha^-}](x \cdot y) = \alpha^+, \lambda_I^G[^{\beta^-}_{\beta^+}](x \cdot (y \cdot z)) = \lambda_I^G[^{\beta^-}_{\beta^+}](x \cdot y) = \beta^-$, and $\lambda_F^G[^{\gamma^+}_{\gamma^-}](x \cdot (y \cdot z)) = \lambda_F^G[^{\gamma^+}_{\gamma^-}](x \cdot y) = \gamma^+$. Since G is an implicative UP-filter of X, we have $x \cdot z \in G$ and so $\lambda_T^G[^{\alpha^+}_{\alpha^-}](x \cdot z) = \alpha^+, \lambda_I^G[^{\beta^-}_{\beta^+}](x \cdot z) = \beta^-$, and $\lambda_F^G[^{\gamma^+}_{\gamma^-}](x \cdot z) = \gamma^+$. Thus

$$\begin{aligned} \min\{\lambda_T^G[^{\alpha^+}_{\alpha^-}](x \cdot (y \cdot z)), \lambda_T^G[^{\alpha^+}_{\alpha^-}](x \cdot y)\} &= \alpha^+ \geq \alpha^+ = \lambda_T^G[^{\alpha^+}_{\alpha^-}](x \cdot z), \\ \max\{\lambda_I^G[^{\beta^-}_{\beta^+}](x \cdot (y \cdot z)), \lambda_I^G[^{\beta^-}_{\beta^+}](x \cdot y)\} &= \beta^- \leq \beta^- = \lambda_I^G[^{\beta^-}_{\beta^+}](x \cdot z), \\ \min\{\lambda_F^G[^{\gamma^+}_{\gamma^-}](x \cdot (y \cdot z)), \lambda_F^G[^{\gamma^+}_{\gamma^-}](x \cdot y)\} &= \gamma^+ \geq \gamma^+ = \lambda_F^G[^{\gamma^+}_{\gamma^-}](x \cdot z). \end{aligned}$$

Case 2: $x \cdot (y \cdot z) \notin G$ or $x \cdot y \notin G$. Then

$$\begin{aligned} &\lambda_T^G[^{\alpha^+}_{\alpha^-}](x \cdot (y \cdot z)) = \alpha^- \text{ or } \lambda_T^G[^{\alpha^+}_{\alpha^-}](x \cdot y) = \alpha^-, \\ &\lambda_I^G[^{\beta^-}_{\beta^+}](x \cdot (y \cdot z)) = \beta^+ \text{ or } \lambda_I^G[^{\beta^-}_{\beta^+}](x \cdot y) = \beta^+, \\ &\lambda_F^G[^{\gamma^+}_{\gamma^-}](x \cdot (y \cdot z)) = \gamma^- \text{ or } \lambda_F^G[^{\gamma^+}_{\gamma^-}](x \cdot y) = \gamma^-. \end{aligned}$$

Thus

$$\min\{\lambda_T^G[^{\alpha^+}_{\alpha^-}](x\cdot(y\cdot z)),\lambda_T^G[^{\alpha^+}_{\alpha^-}](x\cdot y)\}=\alpha^-,$$
$$\max\{\lambda_I^G[^{\beta^-}_{\beta^+}](x\cdot(y\cdot z)),\lambda_I^G[^{\beta^-}_{\beta^+}](x\cdot y)\}=\beta^+,$$
$$\min\{\lambda_F^G[^{\gamma^+}_{\gamma^-}](x\cdot(y\cdot z)),\lambda_F^G[^{\gamma^+}_{\gamma^-}](x\cdot y)\}=\gamma^-.$$

Therefore,

$$\lambda_T^G[^{\alpha^+}_{\alpha^-}](x\cdot z)\geq\alpha^-=\min\{\lambda_T^G[^{\alpha^+}_{\alpha^-}](x\cdot(y\cdot z)),\lambda_T^G[^{\alpha^+}_{\alpha^-}](x\cdot y)\},$$
$$\lambda_I^G[^{\beta^-}_{\beta^+}](x\cdot z)\leq\beta^+=\max\{\lambda_I^G[^{\beta^-}_{\beta^+}](x\cdot(y\cdot z)),\lambda_I^G[^{\beta^-}_{\beta^+}](x\cdot y)\},$$
$$\lambda_F^G[^{\gamma^+}_{\gamma^-}](x\cdot z)\geq\gamma^-=\min\{\lambda_F^G[^{\gamma^+}_{\gamma^-}](x\cdot(y\cdot z)),\lambda_F^G[^{\gamma^+}_{\gamma^-}](x\cdot y)\}.$$

Hence, $\Lambda^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a neutrosophic implicative UP-filter of X. $\square$

Theorem 3.34. *A NS $\Lambda^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ in X is a neutrosophic comparative UP-filter of X if and only if a nonempty subset G of X is a comparative UP-filter of X.*

Proof. Assume that $\Lambda^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a neutrosophic comparative UP-filter of X. Since $\Lambda^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ satisfies the condition (36), it follows from Lemma 3.32 that $0\in G$. Next, let $x,y\in X$ be such that $x\cdot((y\cdot z)\cdot y),x\in G$. Then $\lambda_T^G[^{\alpha^+}_{\alpha^-}](x\cdot((y\cdot z)\cdot y))=\alpha^+=\lambda_T^G[^{\alpha^+}_{\alpha^-}](x)$. Thus, by (51), we have

$$\lambda_T^G[^{\alpha^+}_{\alpha^-}](y)\geq\min\{\lambda_T^G[^{\alpha^+}_{\alpha^-}](x\cdot((y\cdot z)\cdot y)),\lambda_T^G[^{\alpha^+}_{\alpha^-}](x)\}=\alpha^+\geq\lambda_T^G[^{\alpha^+}_{\alpha^-}](y)$$

and so $\lambda_T^G[^{\alpha^+}_{\alpha^-}](y)=\alpha^+$. Thus $y\in G$. Hence, G is a comparative UP-filter of X.

Conversely, assume that G is a comparative UP-filter of X. Since $0\in G$, it follows from Lemma 3.31 that $\Lambda^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ satisfies the conditions (36), (37), and (38). Next, let $x,y,z\in X$.

Case 1: $x\cdot((y\cdot z)\cdot y)\in G$ and $x\in G$. Then

$$\lambda_T^G[^{\alpha^+}_{\alpha^-}](x\cdot((y\cdot z)\cdot y))=\alpha^+=\lambda_T^G[^{\alpha^+}_{\alpha^-}](x),$$
$$\lambda_I^G[^{\beta^-}_{\beta^+}](x\cdot((y\cdot z)\cdot y))=\beta^-=\lambda_I^G[^{\beta^-}_{\beta^+}](x),$$
$$\lambda_F^G[^{\gamma^+}_{\gamma^-}](x\cdot((y\cdot z)\cdot y))=\gamma^+=\lambda_F^G[^{\gamma^+}_{\gamma^-}](x).$$

Since G is a comparative UP-filter of X, we have $y\in G$ and so $\lambda_T^G[^{\alpha^+}_{\alpha^-}](y)=\alpha^+$, $\lambda_I^G[^{\beta^-}_{\beta^+}](y)=\beta^-$, and $\lambda_F^G[^{\gamma^+}_{\gamma^-}](y)=\gamma^+$. Thus

$$\lambda_T^G[^{\alpha^+}_{\alpha^-}](y)=\alpha^+\geq\alpha^+=\min\{\lambda_T^G[^{\alpha^+}_{\alpha^-}](x\cdot((y\cdot z)\cdot y)),\lambda_T^G[^{\alpha^+}_{\alpha^-}](x)\},$$
$$\lambda_I^G[^{\beta^-}_{\beta^+}](y)=\beta^-\leq\beta^-=\max\{\lambda_I^G[^{\beta^-}_{\beta^+}](x\cdot((y\cdot z)\cdot y)),\lambda_I^G[^{\beta^-}_{\beta^+}](x)\},$$
$$\lambda_F^G[^{\gamma^+}_{\gamma^-}](y)=\gamma^+\geq\gamma^+=\min\{\lambda_F^G[^{\gamma^+}_{\gamma^-}](x\cdot((y\cdot z)\cdot y)),\lambda_F^G[^{\gamma^+}_{\gamma^-}](x)\}.$$

Case 2: $x \cdot ((y \cdot z) \cdot y) \notin G$ or $x \notin G$. Then

$$\lambda_T^G[^{\alpha^+}_{\alpha^-}](x \cdot ((y \cdot z) \cdot y)) = \alpha^- \text{ or } \lambda_T^G[^{\alpha^+}_{\alpha^-}](x) = \alpha^-,$$
$$\lambda_I^G[^{\beta^-}_{\beta^+}](x \cdot ((y \cdot z) \cdot y)) = \beta^+ \text{ or } \lambda_I^G[^{\beta^-}_{\beta^+}](x) = \beta^+,$$
$$\lambda_F^G[^{\gamma^+}_{\gamma^-}](x \cdot ((y \cdot z) \cdot y)) = \gamma^- \text{ or } \lambda_F^G[^{\gamma^+}_{\gamma^-}](x) = \gamma^-.$$

Thus

$$\min\{\lambda_T^G[^{\alpha^+}_{\alpha^-}](x \cdot ((y \cdot z) \cdot y)), \lambda_T^G[^{\alpha^+}_{\alpha^-}](x)\} = \alpha^-,$$
$$\max\{\lambda_I^G[^{\beta^-}_{\beta^+}](x \cdot ((y \cdot z) \cdot y)), \lambda_I^G[^{\beta^-}_{\beta^+}](x)\} = \beta^+,$$
$$\min\{\lambda_F^G[^{\gamma^+}_{\gamma^-}](x \cdot ((y \cdot z) \cdot y)), \lambda_F^G[^{\gamma^+}_{\gamma^-}](x)\} = \gamma^-.$$

Therefore,

$$\lambda_T^G[^{\alpha^+}_{\alpha^-}](y) \geq \alpha^- = \min\{\lambda_T^G[^{\alpha^+}_{\alpha^-}](x \cdot ((y \cdot z) \cdot y)), \lambda_T^G[^{\alpha^+}_{\alpha^-}](x)\},$$
$$\lambda_I^G[^{\beta^-}_{\beta^+}](y) \leq \beta^+ = \max\{\lambda_I^G[^{\beta^-}_{\beta^+}](x \cdot ((y \cdot z) \cdot y)), \lambda_I^G[^{\beta^-}_{\beta^+}](x)\},$$
$$\lambda_F^G[^{\gamma^+}_{\gamma^-}](y) \geq \gamma^- = \min\{\lambda_F^G[^{\gamma^+}_{\gamma^-}](x \cdot ((y \cdot z) \cdot y)), \lambda_F^G[^{\gamma^+}_{\gamma^-}](x)\}.$$

Hence, $\Lambda^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a neutrosophic comparative UP-filter of X. □

Theorem 3.35. *A NS $\Lambda^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ in X is a neutrosophic shift UP-filter of X if and only if a nonempty subset G of X is a shift UP-filter of X.*

Proof. Assume that $\Lambda^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a neutrosophic shift UP-filter of X. Since $\Lambda^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ satisfies the condition (36), it follows from Lemma 3.32 that $0 \in G$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in G$ and $x \in G$. Then $\lambda_T^G[^{\alpha^+}_{\alpha^-}](x \cdot (y \cdot z)) = \alpha^+ = \lambda_T^G[^{\alpha^+}_{\alpha^-}](x)$. Thus, by (54), we have

$$\lambda_T^G[^{\alpha^+}_{\alpha^-}](((z \cdot y) \cdot y) \cdot z) \geq \min\{\lambda_T^G[^{\alpha^+}_{\alpha^-}](x \cdot (y \cdot z)), \lambda_T^G[^{\alpha^+}_{\alpha^-}](y)\} = \alpha^+ \geq \lambda_T^G[^{\alpha^+}_{\alpha^-}](((z \cdot y) \cdot y) \cdot z)$$

and so $\lambda_T^G[^{\alpha^+}_{\alpha^-}](((z \cdot y) \cdot y) \cdot z) = \alpha^+$. Thus $((z \cdot y) \cdot y) \cdot z \in G$. Hence, G is a shift UP-filter of X.

Conversely, assume that G is a shift UP-filter of X. Since $0 \in G$, it follows from Lemma 3.31 that $\Lambda^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ satisfies the conditions (36), (37), and (38). Next, let $x, y, z \in X$.

Case 1: $x \cdot (y \cdot z) \in G$ and $x \in G$. Then

$$\lambda_T^G[^{\alpha^+}_{\alpha^-}](x \cdot (y \cdot z)) = \alpha^+ = \lambda_T^G[^{\alpha^+}_{\alpha^-}](x),$$
$$\lambda_I^G[^{\beta^-}_{\beta^+}](x \cdot (y \cdot z)) = \beta^- = \lambda_I^G[^{\beta^-}_{\beta^+}](x),$$
$$\lambda_F^G[^{\gamma^+}_{\gamma^-}](x \cdot (y \cdot z)) = \gamma^+ = \lambda_F^G[^{\gamma^+}_{\gamma^-}](x).$$

Thus

$$\min\{\lambda_T^G[^{\alpha^+}_{\alpha^-}](x\cdot(y\cdot z)),\lambda_T^G[^{\alpha^+}_{\alpha^-}](x)\}=\alpha^+,$$
$$\max\{\lambda_I^G[^{\beta^-}_{\beta^+}](x\cdot(y\cdot z)),\lambda_I^G[^{\beta^-}_{\beta^+}](x)\}=\beta^-,$$
$$\min\{\lambda_F^G[^{\gamma^+}_{\gamma^-}](x\cdot(y\cdot z)),\lambda_F^G[^{\gamma^+}_{\gamma^-}](x)\}=\gamma^+.$$

Since G is a shift UP-filter of X, we have $((z\cdot y)\cdot y)\cdot z\in G$ and so $\lambda_T^G[^{\alpha^+}_{\alpha^-}](((z\cdot y)\cdot y)\cdot z)=\alpha^+,\lambda_I^G[^{\beta^-}_{\beta^+}](((z\cdot y)\cdot y)\cdot z)=\beta^-$, and $\lambda_F^G[^{\gamma^+}_{\gamma^-}](((z\cdot y)\cdot y)\cdot z)=\gamma^+$. Thus

$$\lambda_T^G[^{\alpha^+}_{\alpha^-}](((z\cdot y)\cdot y)\cdot z)=\alpha^+\geq\alpha^+=\min\{\lambda_T^G[^{\alpha^+}_{\alpha^-}](x\cdot(y\cdot z)),\lambda_T^G[^{\alpha^+}_{\alpha^-}](x)\},$$
$$\lambda_I^G[^{\beta^-}_{\beta^+}](((z\cdot y)\cdot y)\cdot z)=\beta^-\leq\beta^-=\max\{\lambda_I^G[^{\beta^-}_{\beta^+}](x\cdot(y\cdot z)),\lambda_I^G[^{\beta^-}_{\beta^+}](x)\},$$
$$\lambda_F^G[^{\gamma^+}_{\gamma^-}](((z\cdot y)\cdot y)\cdot z)=\gamma^+\geq\gamma^+=\min\{\lambda_F^G[^{\gamma^+}_{\gamma^-}](x\cdot(y\cdot z)),\lambda_F^G[^{\gamma^+}_{\gamma^-}](x)\}.$$

Case 2: $x\cdot(y\cdot z)\notin G$ or $x\notin G$. Then

$$\lambda_T^G[^{\alpha^+}_{\alpha^-}](x\cdot(y\cdot z))=\alpha^-\text{ or }\lambda_T^G[^{\alpha^+}_{\alpha^-}](x)=\alpha^-,$$
$$\lambda_I^G[^{\beta^-}_{\beta^+}](x\cdot(y\cdot z))=\beta^+\text{ or }\lambda_I^G[^{\beta^-}_{\beta^+}](x)=\beta^+,$$
$$\lambda_F^G[^{\gamma^+}_{\gamma^-}](x\cdot(y\cdot z))=\gamma^-\text{ or }\lambda_F^G[^{\gamma^+}_{\gamma^-}](x)=\gamma^-.$$

Thus

$$\min\{\lambda_T^G[^{\alpha^+}_{\alpha^-}](x\cdot(y\cdot z)),\lambda_T^G[^{\alpha^+}_{\alpha^-}](x)\}=\alpha^-,$$
$$\max\{\lambda_I^G[^{\beta^-}_{\beta^+}](x\cdot(y\cdot z)),\lambda_I^G[^{\beta^-}_{\beta^+}](x)\}=\beta^+,$$
$$\min\{\lambda_F^G[^{\gamma^+}_{\gamma^-}](x\cdot(y\cdot z)),\lambda_F^G[^{\gamma^+}_{\gamma^-}](x)\}=\gamma^-.$$

Therefore,

$$\lambda_T^G[^{\alpha^+}_{\alpha^-}](((z\cdot y)\cdot y)\cdot z)\geq\alpha^-=\min\{\lambda_T^G[^{\alpha^+}_{\alpha^-}](x\cdot(y\cdot z)),\lambda_T^G[^{\alpha^+}_{\alpha^-}](x)\},$$
$$\lambda_I^G[^{\beta^-}_{\beta^+}](((z\cdot y)\cdot y)\cdot z)\leq\beta^+=\max\{\lambda_I^G[^{\beta^-}_{\beta^+}](x\cdot(y\cdot z)),\lambda_I^G[^{\beta^-}_{\beta^+}](x)\},$$
$$\lambda_F^G[^{\gamma^+}_{\gamma^-}](((z\cdot y)\cdot y)\cdot z)\geq\gamma^-=\min\{\lambda_F^G[^{\gamma^+}_{\gamma^-}](x\cdot(y\cdot z)),\lambda_F^G[^{\gamma^+}_{\gamma^-}](x)\}.$$

Hence, $\Lambda^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a neutrosophic shift UP-filter of X. □

4. **Level subsets of a NS**

In this section, we discuss the relationships between neutrosophic implicative UP-filters (resp., neutrosophic comparative UP-filters, neutrosophic shift UP-filters) of UP-algebras and their level subsets.

Definition 4.1. [21] Let f be a fuzzy set in A. For any $t \in [0,1]$, the sets

$$U(f;t) = \{x \in X \mid f(x) \geq t\},$$
$$L(f;t) = \{x \in X \mid f(x) \leq t\},$$
$$E(f;t) = \{x \in X \mid f(x) = t\}$$

are called an *upper t-level subset*, a *lower t-level subset*, and an *equal t-level subset* of f, respectively.

Theorem 4.2. *A NS Λ in X is a neutrosophic implicative UP-filter of X if and only if for all $\alpha, \beta, \gamma \in [0,1]$, the sets $U(\lambda_T;\alpha), L(\lambda_I;\beta)$, and $U(\lambda_F;\gamma)$ are implicative UP-filters of X if $U(\lambda_T;\alpha), L(\lambda_I;\beta)$, and $U(\lambda_F;\gamma)$ are nonempty.*

Proof. Assume that Λ is a neutrosophic implicative UP-filter of X. Let $\alpha, \beta, \gamma \in [0,1]$ be such that $U(\lambda_T;\alpha), L(\lambda_I;\beta)$, and $U(\lambda_F;\gamma)$ are nonempty.

Let $x \in U(\lambda_T;\alpha)$. Then $\lambda_T(x) \geq \alpha$. By (36), we have $\lambda_T(0) \geq \lambda_T(x) \geq \alpha$. Thus $0 \in U(\lambda_T;\alpha)$. Next, let $x\cdot(y\cdot z), x\cdot y \in U(\lambda_T;\alpha)$. Then $\lambda_T(x\cdot(y\cdot z)) \geq \alpha$ and $\lambda_T(x\cdot y) \geq \alpha$. By (48), we have $\lambda_F(x\cdot z) \geq \min\{\lambda_F(x\cdot(y\cdot z)), \lambda_F(x\cdot y)\} \geq \alpha$. Thus $x\cdot z \in U(\lambda_T;\alpha)$.

Let $x \in L(\lambda_I;\beta)$. Then $\lambda_I(x) \leq \beta$. By (37), we have $\lambda_I(0) \leq \lambda_I(x) \leq \beta$. Thus $0 \in L(\lambda_I;\beta)$. Next, let $x\cdot(y\cdot z), x\cdot y \in L(\lambda_I;\beta)$. Then $\lambda_I(x\cdot(y\cdot z)) \leq \beta$ and $\lambda_I(x\cdot y) \geq \beta$. By (49), we have $\lambda_I(x\cdot z) \leq \max\{\lambda_I(x\cdot(y\cdot z)), \lambda_I(x\cdot y)\} \leq \beta$. Thus $x\cdot z \in L(\lambda_I;\beta)$.

Let $x \in U(\lambda_F;\gamma)$. Then $\lambda_F(x) \geq \gamma$. By (38), we have $\lambda_F(0) \geq \lambda_F(x) \geq \gamma$. Thus $0 \in U(\lambda_F;\gamma)$. Next, let $x\cdot(y\cdot z), x\cdot y \in U(\lambda_F;\gamma)$. Then $\lambda_F(x\cdot(y\cdot z)) \geq \gamma$ and $\lambda_F(x\cdot y) \geq \gamma$. By (50), we have $\lambda_F(x\cdot z) \geq \min\{\lambda_F(x\cdot(y\cdot z)), \lambda_F(x\cdot y)\} \geq \gamma$. Thus $x\cdot z \in U(\lambda_F;\gamma)$.

Hence, $U(\lambda_T;\alpha), L(\lambda_I;\beta)$, and $U(\lambda_F;\gamma)$ are implicative UP-filters of X.

Conversely, assume that for all $\alpha, \beta, \gamma \in [0,1]$, the sets $U(\lambda_T;\alpha), L(\lambda_I;\beta)$, and $U(\lambda_F;\gamma)$ are implicative UP-filters of X if $U(\lambda_T;\alpha), L(\lambda_I;\beta)$, and $U(\lambda_F;\gamma)$ are nonempty.

Let $x \in X$. Then $\lambda_T(x) \in [0,1]$. Choose $\alpha = \lambda_T(x)$. Thus $\lambda_T(x) \geq \alpha$, so $x \in U(\lambda_T;\alpha) \neq \emptyset$. By assumption, we have $U(\lambda_T;\alpha)$ is an implication UP-filter of X and so $0 \in U(\lambda_T;\alpha)$. Thus $\lambda_T(0) \geq \alpha = \lambda_T(x)$. Next, let $x, y, z \in X$. Then $\lambda_T(x\cdot(y\cdot z)), \lambda_T(x\cdot y) \in [0,1]$. Choose $\alpha = \min\{\lambda_T(x\cdot(y\cdot z)), \lambda_T(x\cdot y)\}$. Thus $\lambda_T(x\cdot(y\cdot z)) \geq \alpha$ and $\lambda_T(x\cdot y) \geq \alpha$, so $x\cdot(y\cdot z), x\cdot y \in U(\lambda_T;\alpha) \neq \emptyset$. By assumption, we have $U(\lambda_T;\alpha)$ is an implication UP-filter of X and so $x\cdot z \in U(\lambda_T;\alpha)$. Thus $\lambda_T(x\cdot z) \geq \alpha = \min\{\lambda_T(x\cdot(y\cdot z)), \lambda_T(x\cdot y)\}$.

Let $x \in X$. Then $\lambda_I(x) \in [0,1]$. Choose $\beta = \lambda_I(x)$. Thus $\lambda_I(x) \leq \beta$, so $x \in L(\lambda_I;\beta) \neq \emptyset$. By assumption, we have $L(\lambda_I;\beta)$ is an implicative UP-filter of X and so $0 \in L(\lambda_I;\beta)$. Thus $\lambda_I(0) \leq \beta = \lambda_I(x)$. Next, let $x, y, z \in X$. Then $\lambda_T(x\cdot(y\cdot z)), \lambda_T(x\cdot y) \in [0,1]$. Choose $\beta = \max\{\lambda_T(x\cdot(y\cdot z)), \lambda_T(x\cdot y)\}$. Thus $\lambda_T(x\cdot(y\cdot z)) \leq \beta$ and $\lambda_T(x\cdot y) \leq \beta$, so $x\cdot(y\cdot z), x\cdot y \in$

$L(\lambda_T;\beta) \neq \emptyset$. By assumption, we have $L(\lambda_T;\beta)$ is an implication UP-filter of X and so $x \cdot z \in L(\lambda_T;\beta)$. Thus $\lambda_T(x \cdot z) \leq \beta = \max\{\lambda_T(x \cdot (y \cdot z)), \lambda_T(x \cdot y)\}$.

Let $x \in X$. Then $\lambda_F(x) \in [0,1]$. Choose $\gamma = \lambda_F(x)$. Thus $\lambda_F(x) \geq \gamma$, so $x \in U(\lambda_F;\gamma) \neq \emptyset$. By assumption, we have $U(\lambda_F;\gamma)$ is an implicative UP-filter of X and so $0 \in U(\lambda_F;\gamma)$. Thus $\lambda_F(0) \geq \gamma = \lambda_F(x)$. Next, let $x, y, z \in X$. Then $\lambda_T(x \cdot (y \cdot z)), \lambda_T(x \cdot y) \in [0,1]$. Choose $\gamma = \min\{\lambda_T(x \cdot (y \cdot z)), \lambda_T(x \cdot y)\}$. Thus $\lambda_T(x \cdot (y \cdot z)) \geq \gamma$ and $\lambda_T(x \cdot y) \geq \gamma$, so $x \cdot (y \cdot z), x \cdot y \in U(\lambda_T;\gamma) \neq \emptyset$. By assumption, we have $U(\lambda_T;\gamma)$ is an implication UP-filter of X and so $x \cdot z \in U(\lambda_T;\gamma)$. Thus $\lambda_T(x \cdot z) \geq \gamma = \min\{\lambda_T(x \cdot (y \cdot z)), \lambda_T(x \cdot y)\}$.

Therefore, Λ is a neutrosophic implicative UP-filter of X. □

Theorem 4.3. *A NS Λ in X is a neutrosophic comparative UP-filter of X if and only if for all $\alpha, \beta, \gamma \in [0,1]$, the sets $U(\lambda_T;\alpha), L(\lambda_I;\beta)$, and $U(\lambda_F;\gamma)$ are comparative UP-filters of X if $U(\lambda_T;\alpha), L(\lambda_I;\beta)$, and $U(\lambda_F;\gamma)$ are nonempty.*

Proof. Assume that Λ is a neutrosophic comparative UP-filter of X. Let $\alpha, \beta, \gamma \in [0,1]$ be such that $U(\lambda_T;\alpha), L(\lambda_I;\beta)$, and $U(\lambda_F;\gamma)$ are nonempty.

Let $x \in U(\lambda_T;\alpha)$. Then $\lambda_T(x) \geq \alpha$. By (36), we have $\lambda_T(0) \geq \lambda_T(x) \geq \alpha$. Thus $0 \in U(\lambda_T;\alpha)$. Next, let $x, y, z \in X$ be such that $x \cdot ((y \cdot z) \cdot y), x \in U(\lambda_T;\alpha)$. Then $\lambda_T(x \cdot ((y \cdot z) \cdot y)) \geq \alpha$ and $\lambda_T(x) \geq \alpha$, so α is an lower bound of $\{\lambda_T(x \cdot ((y \cdot z) \cdot y)), \lambda_T(x)\}$. By (51), we have $\lambda_T(y) \geq \min\{\lambda_T(x \cdot ((y \cdot z) \cdot y)), \lambda_T(x)\} \geq \alpha$. Thus $y \in U(\lambda_T;\alpha)$.

Let $x \in L(\lambda_I;\beta)$. Then $\lambda_I(x) \leq \beta$. By (37), we have $\lambda_I(0) \leq \lambda_I(x) \leq \beta$. Thus $0 \in L(\lambda_I;\beta)$. Next, let $x, y, z \in X$ be such that $x \cdot ((y \cdot z) \cdot y), x \in L(\lambda_I;\beta)$. Then $\lambda_I(x \cdot ((y \cdot z) \cdot y)) \leq \beta$ and $\lambda_I(x) \leq \beta$, so β is a upper bound of $\{\lambda_I(x \cdot ((y \cdot z) \cdot y)), \lambda_I(x)\}$. By (52), we have $\lambda_I(y) \leq \max\{\lambda_I(x \cdot ((y \cdot z) \cdot y)), \lambda_I(x)\} \leq \beta$ Thus $y \in L(\lambda_I;\beta)$.

Let $x \in U(\lambda_F;\gamma)$. Then $\lambda_F(x) \geq \gamma$. By (38), we have $\lambda_F(0) \geq \lambda_F(x) \geq \gamma$. Thus $0 \in U(\lambda_F;\gamma)$. Next, let $x, y, z \in X$ be such that $x \cdot ((y \cdot z) \cdot y), x \in U(\lambda_F;\gamma)$. Then $\lambda_F(x \cdot ((y \cdot z) \cdot y)) \geq \gamma$ and $\lambda_F(x) \geq \gamma$, so γ is an lower bound of $\{\lambda_F(x \cdot ((y \cdot z) \cdot y)), \lambda_F(x)\}$. By (53), we have $\lambda_F(y) \geq \min\{\lambda_F(x \cdot ((y \cdot z) \cdot y)), \lambda_F(x)\} \geq \gamma$. Thus $y \in U(\lambda_F;\gamma)$.

Hence, $U(\lambda_T;\alpha), L(\lambda_I;\beta)$, and $U(\lambda_F;\gamma)$ are comparative UP-filters of X.

Conversely, assume that for all $\alpha, \beta, \gamma \in [0,1]$, the sets $U(\lambda_T;\alpha), L(\lambda_I;\beta)$, and $U(\lambda_F;\gamma)$ are UP-filters of X if $U(\lambda_T;\alpha), L(\lambda_I;\beta)$, and $U(\lambda_F;\gamma)$ are nonempty.

Let $x \in X$. Then $\lambda_T(x) \in [0,1]$. Choose $\alpha = \lambda_T(x)$. Thus $\lambda_T(x) \geq \alpha$, so $x \in U(\lambda_T;\alpha) \neq \emptyset$. By assumption, we have $U(\lambda_T;\alpha)$ is a comparative UP-filter of X and so $0 \in U(\lambda_T;\alpha)$. Thus $\lambda_T(0) \geq \alpha = \lambda_T(x)$. Next, let $x, y, z \in X$. Then $\lambda_T(x \cdot ((y \cdot z) \cdot y)), \lambda_T(x) \in [0,1]$. Choose $\alpha = \min\{\lambda_T(x \cdot ((y \cdot z) \cdot y)), \lambda_T(x)\}$. Thus $\lambda_T(x \cdot ((y \cdot z) \cdot y)) \geq \alpha$ and $\lambda_T(x) \geq \alpha$, so $x \cdot ((y \cdot z) \cdot y), x \in U(\lambda_T;\alpha) \neq \emptyset$. By assumption, we have $U(\lambda_T;\alpha)$ is a comparative UP-filter of X and so $y \in U(\lambda_T;\alpha)$. Thus $\lambda_T(y) \geq \alpha = \min\{\lambda_T(x \cdot ((y \cdot z) \cdot y)), \lambda_T(x)\}$.

Let $x \in X$. Then $\lambda_I(x) \in [0,1]$. Choose $\beta = \lambda_I(x)$. Thus $\lambda_I(x) \leq \beta$, so $x \in L(\lambda_I;\beta) \neq \emptyset$. By assumption, we have $L(\lambda_I;\beta)$ is a comparative UP-filter of X and so $0 \in L(\lambda_I;\beta)$. Thus $\lambda_I(0) \leq \beta = \lambda_I(x)$. Next, let $x,y,z \in X$. Then $\lambda_I(x \cdot ((y \cdot z) \cdot y)), \lambda_I(x) \in [0,1]$. Choose $\beta = \max\{\lambda_I(x \cdot ((y \cdot z) \cdot y)), \lambda_I(x)\}$. Thus $\lambda_I(x \cdot ((y \cdot z) \cdot y)) \leq \beta$ and $\lambda_I(x) \leq \beta$, so $x \cdot ((y \cdot z) \cdot y), x \in L(\lambda_I;\beta) \neq \emptyset$. By assumption, we have $L(\lambda_I;\beta)$ is a comparative UP-filter of X and so $y \in L(\lambda_I;\beta)$. Thus $\lambda_I(y) \leq \beta = \max\{\lambda_I(x \cdot ((y \cdot z) \cdot y)), \lambda_I(x)\}$.

Let $x \in X$. Then $\lambda_F(x) \in [0,1]$. Choose $\gamma = \lambda_F(x)$. Thus $\lambda_F(x) \geq \gamma$, so $x \in U(\lambda_F;\gamma) \neq \emptyset$. By assumption, we have $U(\lambda_F;\gamma)$ is a comparative UP-filter of X and so $0 \in U(\lambda_F;\gamma)$. Thus $\lambda_F(0) \geq \gamma = \lambda_F(x)$. Next, let $x,y,z \in X$. Then $\lambda_F(x \cdot ((y \cdot z) \cdot y)), \lambda_F(x) \in [0,1]$. Choose $\gamma = \min\{\lambda_F(x \cdot ((y \cdot z) \cdot y)), \lambda_F(x)\}$. Thus $\lambda_F(x \cdot ((y \cdot z) \cdot y)) \geq \gamma$ and $\lambda_F(x) \geq \gamma$, so $x \cdot ((y \cdot z) \cdot y), x \in U(\lambda_F;\gamma) \neq \emptyset$. By assumption, we have $U(\lambda_F;\gamma)$ is a comparative UP-filter of X and so $y \in U(\lambda_F;\gamma)$. Thus $\lambda_F(y) \geq \gamma = \min\{\lambda_F(x \cdot ((y \cdot z) \cdot y)), \lambda_F(x)\}$.

Therefore, Λ is a neutrosophic comparative UP-filter of X. $\square$

Theorem 4.4. *A NS Λ in X is a neutrosophic shift UP-filter of X if and only if for all $\alpha,\beta,\gamma \in [0,1]$, the sets $U(\lambda_T;\alpha), L(\lambda_I;\beta)$, and $U(\lambda_F;\gamma)$ are shift UP-filters of X if $U(\lambda_T;\alpha), L(\lambda_I;\beta)$, and $U(\lambda_F;\gamma)$ are nonempty.*

Proof. Assume that Λ is a neutrosophic shift UP-filter of X. Let $\alpha,\beta,\gamma \in [0,1]$ be such that $U(\lambda_T;\alpha), L(\lambda_I;\beta)$, and $U(\lambda_F;\gamma)$ are nonempty.

Let $x \in U(\lambda_T;\alpha)$. Then $\lambda_T(x) \geq \alpha$. By (36), we have $\lambda_T(0) \geq \lambda_T(x) \geq \alpha$. Thus $0 \in U(\lambda_T;\alpha)$. Next, let $x,y,z \in X$ be such that $x \cdot (y \cdot z) \in U(\lambda_T;\alpha)$ and $x \in U(\lambda_T;\alpha)$. Then $\lambda_T(x \cdot (y \cdot z)) \geq \alpha$ and $\lambda_T(x) \geq \alpha$, so α is an lower bound of $\{\lambda_T(x \cdot (y \cdot z)), \lambda_T(x)\}$. By (54), we have $\lambda_T(((z \cdot y) \cdot y) \cdot z) \geq \min\{\lambda_T(x \cdot (y \cdot z)), \lambda_T(x)\} \geq \alpha$. Thus $((z \cdot y) \cdot y) \cdot z \in U(\lambda_T;\alpha)$.

Let $x \in L(\lambda_I;\alpha)$. Then $\lambda_I(x) \leq \beta$. By (37), we have $\lambda_I(0) \leq \lambda_I(x) \leq \beta$. Thus $0 \in L(\lambda_I;\beta)$. Next, let $x,y,z \in X$ be such that $x \cdot (y \cdot z) \in L(\lambda_I;\beta)$ and $x \in L(\lambda_I;\beta)$. Then $\lambda_I(x \cdot (y \cdot z)) \leq \beta$ and $\lambda_I(x) \leq \beta$, so β is a upper bound of $\{\lambda_I(x \cdot (y \cdot z)), \lambda_I(x)\}$. By (55), we have $\lambda_I(((z \cdot y) \cdot y) \cdot z) \leq \max\{\lambda_I(x \cdot (y \cdot z)), \lambda_I(x)\} \leq \beta$. Thus $((z \cdot y) \cdot y) \cdot z \in L(\lambda_I;\beta)$.

Let $x \in U(\lambda_F;\gamma)$. Then $\lambda_F(x) \geq \gamma$. By (38), we have $\lambda_F(0) \geq \lambda_F(x) \geq \gamma$. Thus $0 \in U(\lambda_F;\gamma)$. Next, let $x,y,z \in X$ be such that $x \cdot (y \cdot z) \in U(\lambda_F;\gamma)$ and $x \in U(\lambda_F;\gamma)$. Then $\lambda_F(x \cdot (y \cdot z)) \geq \gamma$ and $\lambda_F(y) \geq \gamma$, so γ is an lower bound of $\{\lambda_F(x \cdot (y \cdot z)), \lambda_F(x)\}$. By (56), we have $\lambda_F(((z \cdot y) \cdot y) \cdot z) \geq \min\{\lambda_F(x \cdot (y \cdot z)), \lambda_F(x)\} \geq \gamma$. Thus $((z \cdot y) \cdot y) \cdot z \in U(\lambda_F;\gamma)$.

Hence, $U(\lambda_T;\alpha), L(\lambda_I;\beta)$, and $U(\lambda_F;\gamma)$ are shift UP-filters of X.

Conversely, assume that for all $\alpha,\beta,\gamma \in [0,1]$, the sets $U(\lambda_T;\alpha), L(\lambda_I;\beta)$, and $U(\lambda_F;\gamma)$ are shift UP-filters of X if $U(\lambda_T;\alpha), L(\lambda_I;\beta)$, and $U(\lambda_F;\gamma)$ are nonempty.

Let $x \in X$. Then $\lambda_T(x) \in [0,1]$. Choose $\alpha = \lambda_T(x)$. Thus $\lambda_T(x) \geq \alpha$, so $x \in U(\lambda_T;\alpha) \neq \emptyset$. By assumption, we have $U(\lambda_T;\alpha)$ is a shift UP-filter of X and so $0 \in U(\lambda_T;\alpha)$. Thus

$\lambda_T(0) \geq \alpha = \lambda_T(x)$. Next, let $x, y, z \in X$. Then $\lambda_T(x \cdot (y \cdot z)), \lambda_T(x) \in [0,1]$. Choose $\alpha = \min\{\lambda_T(x \cdot (y \cdot z)), \lambda_T(x)\}$. Thus $\lambda_T(x \cdot (y \cdot z)) \geq \alpha$ and $\lambda_T(x) \geq \alpha$, so $x \cdot (y \cdot z), x \in U(\lambda_T; \alpha) \neq \emptyset$. By assumption, we have $U(\lambda_T; \alpha)$ is a shift UP-filter of X and so $((z \cdot y) \cdot y) \cdot z \in U(\lambda_T; \alpha)$. Thus $\lambda_T(((z \cdot y) \cdot y) \cdot z) \geq \alpha = \min\{\lambda_T(x \cdot (y \cdot z)), \lambda_T(x)\}$.

Let $x \in X$. Then $\lambda_I(x) \in [0,1]$. Choose $\beta = \lambda_I(x)$. Thus $\lambda_I(x) \leq \beta$, so $x \in L(\lambda_I; \beta) \neq \emptyset$. By assumption, we have $L(\lambda_I; \beta)$ is a shift UP-filter of X and so $0 \in L(\lambda_I; \beta)$. Thus $\lambda_I(0) \leq \beta = \lambda_I(x)$. Next, let $x, y, z \in X$. Then $\lambda_I(x \cdot (y \cdot z)), \lambda_I(x) \in [0,1]$. Choose $\beta = \max\{\lambda_I(x \cdot (y \cdot z)), \lambda_I(x)\}$. Thus $\lambda_I(x \cdot (y \cdot z)) \leq \beta$ and $\lambda_I(x) \leq \beta$, so $x \cdot (y \cdot z), x \in L(\lambda_I; \beta) \neq \emptyset$. By assumption, we have $L(\lambda_I; \beta)$ is a shift UP-filter of X and so $((z \cdot y) \cdot y) \cdot z \in L(\lambda_I; \beta)$. Thus $\lambda_I(((z \cdot y) \cdot y) \cdot z) \leq \beta = \max\{\lambda_I(x \cdot (y \cdot z)), \lambda_I(x)\}$.

Let $x \in X$. Then $\lambda_F(x) \in [0,1]$. Choose $\gamma = \lambda_F(x)$. Thus $\lambda_F(x) \geq \gamma$, so $x \in U(\lambda_F; \gamma) \neq \emptyset$. By assumption, we have $U(\lambda_F; \gamma)$ is a shift UP-filter of X and so $0 \in U(\lambda_F; \gamma)$. Thus $\lambda_F(0) \geq \gamma = \lambda_F(x)$. Next, let $x, y, z \in X$. Then $\lambda_F(x \cdot (y \cdot z)), \lambda_F(y) \in [0,1]$. Choose $\gamma = \min\{\lambda_F(x \cdot (y \cdot z)), \lambda_F(x)\}$. Thus $\lambda_F(x \cdot (y \cdot z)) \geq \gamma$ and $\lambda_F(x) \geq \gamma$, so $x \cdot (y \cdot z), x \in U(\lambda_F; \gamma) \neq \emptyset$. By assumption, we have $U(\lambda_F; \gamma)$ is a shift UP-filter of X and so $((z \cdot y) \cdot y) \cdot z \in U(\lambda_F; \gamma)$. Thus $\lambda_F(((z \cdot y) \cdot y) \cdot z) \geq \gamma = \min\{\lambda_F(x \cdot (y \cdot z)), \lambda_F(x)\}$.

Therefore, Λ is a neutrosophic shift UP-filter of X. $\square$

Definition 4.5. [23] Let Λ be a NS in X. For $\alpha, \beta, \gamma \in [0,1]$, the sets

$$ULU_\Lambda(\alpha, \beta, \gamma) = \{x \in X \mid \lambda_T \geq \alpha, \lambda_I \leq \beta, \lambda_F \geq \gamma\},$$

$$LUL_\Lambda(\alpha, \beta, \gamma) = \{x \in X \mid \lambda_T \leq \alpha, \lambda_I \geq \beta, \lambda_F \leq \gamma\},$$

$$E_\Lambda(\alpha, \beta, \gamma) = \{x \in X \mid \lambda_T = \alpha, \lambda_I = \beta, \lambda_F = \gamma\}$$

are called a *ULU-(α, β, γ)-level subset*, a *LUL-(α, β, γ)-level subset*, and an *E-(α, β, γ)-level subset* of Λ, respectively.

The following corollary is straightforward by Theorems 4.2, 4.3, and 4.4.

Corollary 4.6. *A NS Λ in X is a neutrosophic implicative UP-filter (resp., neutrosophic comparative UP-filter, neutrosophic shift UP-filter) of X if and only if for all $\alpha, \beta, \gamma \in [0,1]$, $ULU_\Lambda(\alpha, \beta, \gamma)$ is a implicative UP-filter (resp., comparative UP-filter, shift UP-filter) of X where $ULU_\Lambda(\alpha, \beta, \gamma)$ is nonempty.*

5. **Conclusions**

In this paper, we have introduced the notions of neutrosophic implicative UP-filters, neutrosophic comparative UP-filters, and neutrosophic shift UP-filters of UP-algebras and investigated some of their important properties. Then, we get the diagram of generalization of NSs in UP-algebras as shown in Figure 1.

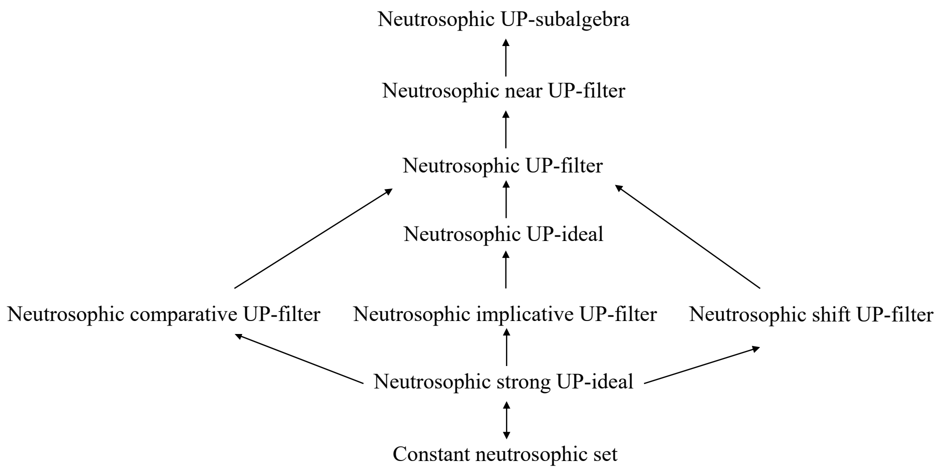

FIGURE 1. NSs in UP-algebras

In our future study, we will study the soft set theory/cubic set theory of neutrosophic implicative UP-filters, neutrosophic comparative UP-filters, and neutrosophic shift UP-filters of UP-algebras.

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