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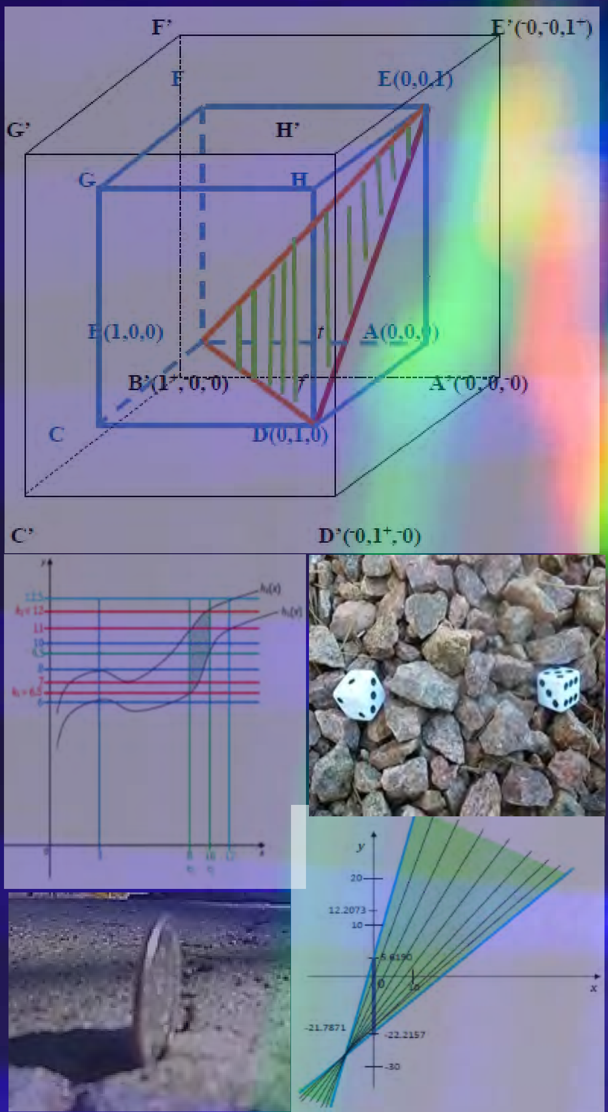

<A> <neutA> <antiA>

Florentin Smarandache . Mohamed Abdel-Basset . Said Broumi
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Information for Authors and Subscribers

*"*Neutrosophic Sets and Systems*"* has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Google Dictionaries have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
FOLDOC Dictionary of Computing (**1, 2**), Webster Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

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Contents

Division of refined neutrosophic numbers

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Abstract: Previously, the mathematical operations on refined neutrosophic numbers were studied by researchers, but these studies did not address the division of refined neutrosophic numbers. The aim of this research was how to find the division, in addition to discussing special cases of dividing refined neutrosophic numbers.

Keywords: division; indeterminacy; refined neutrosophic numbers; division conditions of refined neutrosophic numbers.

1. Introduction and Preliminaries

To describe a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, and contradiction, Smarandache suggested the neutrosophic Logic as an alternative to the current logics. Smarandache made refined neutrosophic numbers available in the following form: $(a, b_1 I_1, b_2 I_2, \ldots, b_n I_n)$ where $a, b_1, b_2, \ldots, b_n \in R$ or C [1]

Agboola introduced the concept of refined neutrosophic algebraic structures [2]. Also, the refined neutrosophic rings I was studied in paper [3], where it assumed that I splits into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. It then follows logically that: [3]

$$I_1 I_1 = {I_1}^2 = I_1 \qquad (1)$$

$$I_2 I_2 = {I_2}^2 = I_2 \qquad (2)$$

$$I_1 I_2 = I_2 I_1 = I_1 \qquad (3)$$

In addition, there are many papers presenting studies on refined neutrosophic numbers [4-5-6-7-8].

This paper dealt with several topics, in the first part of which introduction and preliminaries were presented, and in the main discussion part the division of refined neutrosophic numbers and the conditions related to them were studied. In the last part, the conclusion was presented.

Main Discussion

Division of refined neutrosophic numbers

Let $\dot{w}_1, \dot{w}_2$ are two refined neutrosophic numbers, where:

$$\dot{w}_1 = \dot{a}_1 + \dot{b}_1 I_1 + \dot{c}_1 I_2 \quad , \qquad \dot{w}_2 = \dot{a}_2 + \dot{b}_2 I_1 + \dot{c}_2 I_2$$

To find $\dot{w}_1 \div \dot{w}_2$, we can write:

$$\frac{\dot{a}_1 + \dot{b}_1 I_1 + \dot{c}_1 I_2}{\dot{a}_2 + \dot{b}_2 I_1 + \dot{c}_2 I_2} \equiv x + yI_1 + zI_2$$

where x , y and z are real unknowns.

$$\dot{a}_1 + \dot{b}_1 I_1 + \dot{c}_1 I_2 \equiv (\dot{a}_2 + \dot{b}_2 I_1 + \dot{c}_2 I_2)(x + yI_1 + zI_2)$$

$$\dot{a}_1 + \dot{b}_1 I_1 + \dot{c}_1 I_2 \equiv \dot{a}_2 x + \dot{a}_2 y I_1 + \dot{a}_2 z I_2 + \dot{b}_2 I_1 x + \dot{b}_2 I_1 y I_1 + \dot{b}_2 z I_1 + \dot{c}_2 I_2 x + \dot{c}_2 y I_1 + \dot{c}_2 z I_2$$

$$\dot{a}_1 + \dot{b}_1 I_1 + \dot{c}_1 I_2 \equiv \dot{a}_2 x + [\dot{b}_2 x + (\dot{a}_2 + \dot{b}_2 + \dot{c}_2) y + \dot{b}_2 z] I_1 + [\dot{c}_2 x + (\dot{a}_2 + \dot{c}_2) z] I_2$$

where: $I_1 I_2 = I_2 I_1 = I_1$

by identifying the coefficients, we get:

$$\dot{a}_2 x = \dot{a}_1$$
$$\dot{b}_2 x + (\dot{a}_2 + \dot{b}_2 + \dot{c}_2) y + \dot{b}_2 z = \dot{b}_1$$
$$\dot{c}_2 x + (\dot{a}_2 + \dot{c}_2) z = \dot{c}_1$$

We obtain unique one solution only, provided that:

$$\begin{vmatrix} \dot{a}_2 & 0 & 0 \\ \dot{b}_2 & \dot{a}_2 + \dot{b}_2 + \dot{c}_2 & \dot{b}_2 \\ \dot{c}_2 & 0 & \dot{a}_2 + \dot{c}_2 \end{vmatrix} \neq 0 \qquad \Rightarrow \qquad \dot{a}_2(\dot{a}_2 + \dot{c}_2)(\dot{a}_2 + \dot{b}_2 + \dot{c}_2) \neq 0$$

From this, we get on the conditions for the division of two refined neutrosophic numbers to exist:

$$\dot{a}_2 \neq 0 \quad , \ \dot{a}_2 \neq -\dot{c}_2 \quad \text{and} \ \dot{a}_2 \neq -\dot{b}_2 - \dot{c}_2$$

then:

$$x = \frac{\dot{a}_1}{\dot{a}_2}$$

$$z = \frac{\dot{a}_2 \dot{c}_1 - \dot{a}_1 \dot{c}_2}{\dot{a}_2(\dot{a}_2 + \dot{c}_2)}$$

$$\frac{\dot{a}_1 \dot{b}_2}{\dot{a}_2} + (\dot{a}_2 + \dot{b}_2 + \dot{c}_2) y + \frac{\dot{a}_2 \dot{b}_2 \dot{c}_1 - \dot{a}_1 \dot{b}_2 \dot{c}_2}{\dot{a}_2(\dot{a}_2 + \dot{c}_2)} = \dot{b}_1$$

$$(\dot{a}_2 + \dot{b}_2 + \dot{c}_2) y = \dot{b}_1 - \frac{\dot{a}_1 \dot{b}_2}{\dot{a}_2} - \left(\frac{\dot{a}_2 \dot{b}_2 \dot{c}_1 - \dot{a}_1 \dot{b}_2 \dot{c}_2}{\dot{a}_2(\dot{a}_2 + \dot{c}_2)}\right)$$

$$(\dot{a}_2+\dot{b}_2+\dot{c}_2)y=\frac{\dot{b}_1\dot{a}_2(\dot{a}_2+\dot{c}_2)-\dot{a}_1\dot{b}_2(\dot{a}_2+\dot{c}_2)-\dot{a}_2\dot{b}_2\dot{c}_1+\dot{a}_1\dot{b}_2\dot{c}_2}{\dot{a}_2(\dot{a}_2+\dot{c}_2)}$$

$$(\dot{a}_2+\dot{b}_2+\dot{c}_2)y=\frac{\dot{a}_2^{\,2}\dot{b}_1+\dot{a}_2\dot{b}_1\dot{c}_2-\dot{a}_1\dot{a}_2\dot{b}_2-\dot{a}_1\dot{b}_2\dot{c}_2-\dot{a}_2\dot{b}_2\dot{c}_1+\dot{a}_1\dot{b}_2\dot{c}_2}{\dot{a}_2(\dot{a}_2+\dot{c}_2)}$$

$$y=\frac{\dot{a}_2^{\,2}\dot{b}_1+\dot{a}_2\dot{b}_1\dot{c}_2-\dot{a}_1\dot{a}_2\dot{b}_2-\dot{a}_2\dot{b}_2\dot{c}_1}{\dot{a}_2(\dot{a}_2+\dot{c}_2)(\dot{a}_2+\dot{b}_2+\dot{c}_2)}$$

hence:

$$\frac{\dot{a}_1+\dot{b}_1I_1+\dot{c}_1I_2}{\dot{a}_2+\dot{b}_2I_1+\dot{c}_2I_2}\equiv\frac{\dot{a}_1}{\dot{a}_2}+\left[\frac{\dot{a}_2^{\,2}\dot{b}_1+\dot{a}_2\dot{b}_1\dot{c}_2-\dot{a}_1\dot{a}_2\dot{b}_2-\dot{a}_2\dot{b}_2\dot{c}_1}{\dot{a}_2(\dot{a}_2+\dot{c}_2)(\dot{a}_2+\dot{b}_2+\dot{c}_2)}\right]I_1+\left[\frac{\dot{a}_2\dot{c}_1-\dot{a}_1\dot{c}_2}{\dot{a}_2(\dot{a}_2+\dot{c}_2)}\right]I_2$$

Example1:

$$\frac{4+I_1+I_2}{1+2I_1+3I_2}=4-\frac{1}{4}I_1-\frac{11}{4}I_2$$

Let's check the answer:

$$(1+2I_1+3I_2)\left(4-\tfrac{1}{4}I_1-\tfrac{11}{4}I_2\right)=4+I_1+I_2 \quad \text{(True)}$$

As consequences, we have:

1) $\dfrac{\dot{a}_1+\dot{b}_1I_1+\dot{c}_1I_2}{k(\dot{a}_1+\dot{b}_1I_1+\dot{c}_1I_2)}=\dfrac{1}{k}$

$where\ \ k\neq 0\ ,\dot{a}_1\neq 0\ ,\ \dot{a}_1\neq -\dot{b}_1$ and $\dot{a}_1\neq -\dot{b}_1-\dot{c}_1$

2) $\dfrac{I_1}{\dot{a}_2+\dot{b}_2I_1+\dot{c}_2I_2}=\dfrac{\dot{a}_2\dot{b}_1+\dot{b}_1\dot{c}_2}{(\dot{a}_2+\dot{c}_2)(\dot{a}_2+\dot{b}_2+\dot{c}_2)}I_1$

Example2:

$$\frac{I_1}{2-4I_1+3I_2}=I_1$$

Let's check the answer:

$$(2-4I_1+3I_2)(I_1)=I_1 \quad \text{(True)}$$

3) $\dfrac{I_2}{\dot{a}_2+\dot{b}_2I_1+\dot{c}_2I_2}=\left[\dfrac{-\dot{b}_2\dot{c}_1}{(\dot{a}_2+\dot{c}_2)(\dot{a}_2+\dot{b}_2+\dot{c}_2)}\right]I_1+\left[\dfrac{\dot{c}_1}{\dot{a}_2+\dot{c}_2}\right]I_2$

Example3:

$$\frac{I_2}{1+3I_1-5I_2}=-\frac{3}{4}I_1-\frac{1}{4}I_2$$

Let's check the answer:

$$(1+3I_1-5I_2)\left(-\tfrac{3}{4}I_1-\tfrac{1}{4}I_2\right)=I_2 \quad \text{(True)}$$

4) $\dfrac{I_1+I_2}{\dot{a}_2+\dot{b}_2 I_1+\dot{c}_2 I_2}=\left[\dfrac{\dot{a}_2\dot{b}_1+\dot{b}_1\dot{c}_2-\dot{b}_2\dot{c}_1}{(\dot{a}_2+\dot{c}_2)(\dot{a}_2+\dot{b}_2+\dot{c}_2)}\right]I_1+\left[\dfrac{\dot{c}_1}{\dot{a}_2+\dot{c}_2}\right]I_2$

Example4:

$$\frac{I_1+I_2}{2+I_1+2I_2}=\frac{3}{20}I_1+\frac{1}{4}I_2$$

Let's check the answer:

$$(2+I_1+2I_2)\left(\frac{3}{20}I_1+\frac{1}{4}I_2\right)=I_1+I_2 \qquad \text{(True)}$$

5) $\dfrac{\dot{a}_1+\dot{b}_1 I_1+\dot{c}_1 I_2}{k(I_1+I_2)}=undefined$

$where\ \ k,\dot{a}_1,\dot{b}_1$ and $\dot{c}_1$ any real number.

In particular:

$i)$ $\dfrac{\dot{a}_1+\dot{b}_1 I_1+\dot{c}_1 I_2}{I_1+I_2}=undefined$

$ii)$ $\dfrac{\dot{a}_1+\dot{b}_1 I_1+\dot{c}_1 I_2}{I_1}=undefined$

$iii)$ $\dfrac{\dot{a}_1+\dot{b}_1 I_1+\dot{c}_1 I_2}{I_2}=undefined$

6) $\dfrac{\dot{a}_1+\dot{b}_1 I_1+\dot{c}_1 I_2}{k}=\dfrac{\dot{a}_1}{k}+\dfrac{\dot{a}_1}{k}I_1+\dfrac{\dot{c}_1}{k}I_2\ \ ;k\neq 0$

7) $\dfrac{k}{\dot{a}_2+\dot{b}_2 I_1+\dot{c}_2 I_2}\equiv\dfrac{k}{\dot{a}_2}+k\left[\dfrac{-\dot{b}_2}{(\dot{a}_2+\dot{c}_2)(\dot{a}_2+\dot{b}_2+\dot{c}_2)}\right]I_1-\left[\dfrac{k\dot{c}_2}{\dot{a}_2(\dot{a}_2+\dot{c}_2)}\right]I_2$

Where $\dot{a}_2\neq 0$, $\dot{a}_2\neq -\dot{c}_2$ and $\dot{a}_2\neq -\dot{b}_2-\dot{c}_2$

8) $\dfrac{k(I_1+I_2)}{\dot{a}_2+\dot{b}_2 I_1+\dot{c}_2 I_2}=k\ \left[\dfrac{\dot{a}_2\dot{b}_1+\dot{b}_1\dot{c}_2-\dot{b}_2\dot{c}_1}{(\dot{a}_2+\dot{c}_2)(\dot{a}_2+\dot{b}_2+\dot{c}_2)}\right]I_1+\left[\dfrac{k\dot{c}_1}{\dot{a}_2+\dot{c}_2}\right]I_2$

Example5:

$$\frac{4I_1+4I_2}{2+I_1+2I_2}=\frac{3}{5}I_1+I_2$$

Let's check the answer:

$$(2+I_1+2I_2)\left(\frac{3}{5}I_1+I_2\right)=4(I_1+I_2) \qquad \text{(True)}$$

Conclusions

In this work, we conclusion formula to evaluate division of refined neutrosophic numbers, also, we get on the conditions for the division of two refined neutrosophic numbers to exist. In addition to providing direct special cases for finding the result of the division.

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Solving Non-linear Neutrosophic Linear Programming Problems

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Abstract: Non-linear neutrosophic numbers (NLNNs) are different kinds of neutrosophic numbers with at least one non-linear membership function (either of truthiness, falsity or indeterminacy part) of the information. Furthermore, a linear programming problem with non-linear neutrosophic numbers as coefficients/parameters is a special type of programming problem known as a non-linear linear programming problem (NLN-LPP). This paper elaborates on the concepts of non-linear neutrosophic number (NLNN) sets, different forms of non-linear neutrosophic numbers (NLNNs), α, β, γ cuts on non-linear neutrosophic numbers (NLNNs), possibility mean, possibility standard deviation, and possibility variance of non-linear neutrosophic numbers (NLNNs). In this paper, we also propose the solution technique for non-linear neutrosophic linear programming problems (NLN-LPPs) in which all coefficients/parameters are non-linear neutrosophic numbers (NLNNs). In this continuation, we suggest a new modified possibility score function for non-linear NNs in terms of possibility means and possibility standard deviations of non-linear neutrosophic numbers (NLNNs) for better use of all parts of information. This modified score function is used to convert non-linear neutrosophic number (NLNN) coefficients/parameters of non-linear neutrosophic linear programming problem (NLN-LPP) into equivalent crisp values. Thereafter, the equivalent crisp problem is solved with the usual method to obtain the optimal solution of non-linear neutrosophic linear programming problem (NLN-LPP). The proposed solution algorithm is unique and new for solving non-linear neutrosophic linear programming problems. A numerical example is solved with the proposed algorithm to legitimate the research output. A case study is also discussed to show its applicability in solving real-life problems.

Keywords: Linear programming problem; Non-linear neutrosophic numbers (NLNNs), Possibility score function of Non-linear neutrosophic numbers (NLNNs), Possibility mean of Non-linear neutrosophic numbers (NLNNs), Possibility standard deviation of Non-linear neutrosophic numbers (NLNNs).

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1. Background of the Problem and Motivation – An Introduction

In 1965, Prof. Zadeh [1] introduced the concept of fuzzy set theory to deal with the uncertainty and ambiguity in information due to human language error and human perceptions. Prof. Zadeh [1] defined a set $A = \left\{ x : \left| \mu_A^T(x), 0 \le \mu_A^T(x) \le 1; x \in X \right. \right\}$ with objects x having $\mu_A^T(x)$ degree of acceptance of particular characteristic. This set A is called as fuzzy set with membership function $\mu_A^T(x)$. Since 1965, many researchers have contributed in the area of fuzzy set, fuzzy logic, and its application in solving real-world problems.

Fuzzy sets (FSs) are further classified into two major types – (i) Linear fuzzy set - FS with linear membership function e.g. triangular, trapezoidal, pentagonal (Chakraborty et al. [2]), hexagonal

(Chakraborty et al. [3]), heptagonal (Maity et al. [4]), etc. (ii) Non-linear Fuzzy set – FS with non-linear membership functions e.g. logarithmic, exponential membership function, etc. In general, fuzzy set (FS) theory avoids the involvement of other parts of information. Later, Atanassov [5-6] proposed the intuitionistic fuzzy set (IFS) theory and properties of IFS. Intuitionistic fuzzy set (IFS) – a more generalized FS theory that considers two parts of information i.e. acceptance (truthiness of information) and non-acceptance (falsity of information). Liu and Yuan [7] combined intuitionistic fuzzy set (IFS) and triangular fuzzy number (TFN) to introduce the intuitionistic triangular fuzzy set (ITFS) theory which has triangular membership functions for the truthiness and falsity part of information. Ye [8] extended the TIFS to trapezoidal form to introduce intuitionistic trapezoidal fuzzy set (ITrFS). On the other hand, a linear programming problem is one of the simplest problems of MPPs (Mathematical programming problems) which has linear objective function and linear constraints. LPPs play a vital role in formulating simple real-life problems that arise in Business, Govt. policies, industries, etc. LP problems are easy to solve with the Graphical and Simplex method depending upon the number of decision variables involved. The simplex method is a generalized method for solving any LPP with some manual computational efforts. In contrast with the past, LPPs and NLPPs (non-linear programming problems) are solved quickly and efficiently with the help of computational tools like LINGO©, MATLAB©, etc.

1.1. Fuzzy and neutrosophic programming problems – Literature Review

With time, fuzzy set theory and fuzzy numbers were incorporated in MPPs (LPPs, multiobjective programming problems (MOPPs), Bi-level/Multi-level programming problems (BLPPs/MLPPs), other extension problems, etc.) and many new solution techniques have been developed by researchers for solving MPPs with fuzzy parameters/coefficients. Some of the notable contributions are: Luhandjula [9] developed a new solution technique for fuzzy linear programming problem (FLPP). Arikan and Gunjar [10] proposed a new solution algorithm known as a two-phase approach for MOPPs with fuzzy coefficients. Wu [11] proposed to solve MOPP with fuzzy coefficients using the scalarization technique. For BLPPs/MLPPs, Shih et al. [12] suggested a general solution approach to solve fuzzy multi-level programming problems (FMLPPs). Baky [13] proposed an algorithm for ML-MOPPs through fuzzy goal programming approach. Osman et al. [14] suggested an interactive solution approach for ML-MOPPs with fractional objective functions and fuzzy parameters. Fuzzy set theory is based on only one aspect of information i.e. truthiness and avoids the other two parts of information which are indeterminacy and falsity. On the other hand, intuitionistic fuzzy set (IFS) theory considers two parts of information i.e. truthiness and falsity but ignores a third important part of information i.e. indeterminacy. To disseminate these shortcomings of FS and IFS, Samarandche [15] introduced a new theory known as Neutrosophic set theory (NN set theory) dealing with the object along with three parts of information - truthiness, falsity, and indeterminacy. Later, Samarandche [16-17] specified some properties of neutrosophic set (NS) theory and linear neutrosophic numbers (NNs) including Addition and subtraction of linear NNs, α, β, γ cuts on linear NNs, etc. Wang et al. [27] discovered a new type of NS – Single valued neutrosophic set (SVNS) to apply in real-life problems. Ye [18] introduced trapezoidal interval-valued NNs (IV TrNNs) by combining triangular neutrosophic numbers (TrNNs) and trapezoidal fuzzy numbers (TrFNs). Since the past few years, the combination of neutrosophic set theory (linear NNs) and MPPs (specifically for LPPs) has become a prominent area of research. This is exhibited in a literature survey of recent years, e.g. Hussian et al. [19] used properties of NNs to convert neutrosophic LPP into an equivalent crisp LPP. Abdel-Basset et al. [20] suggested a new ranking function for the solution of neutrosophic LPP. Bera and Mahapatra [21] suggested a real-life application of neutrosophic LPP and developed a simplex method to solve it. Darehmiraki [22] proposed a new parametric ranking function to solve neutrosophic LPPs. Khatter [23] used properties of possibility mean of NNs to solve neutrosophic LPPs. Tamilarasi and Paulraj [24] developed a solution technique for neutrosophic LPPs with triangular NNs and de-neutrosophication of NNs with Melin transform. Similar to FS theory, neutrosophic sets (NN sets) are classified as (i) linear NN sets and (ii) Non-linear NN sets. A linear NN set is a neutrosophic set with all linear membership functions (membership for truthiness, falsity, and indeterminacy) whereas a non-linear NN set is a neutrosophic set with at least one non-linear membership function (either of truthiness, falsity or indeterminacy).On the non-linear neutrosophic numbers (NLNNs), Chakraborty et al. [25] and Javier and Francisco [26] discussed about properties of NLNNs and their applications. Recently, Rabie A et al. [28] suggested a dual artificial variable-free simplex algorithm for neutrosophic linear programming problems. Badr El-Sayed et al. [29] discovered the exterior point simplex method for solving neutrosophic linear programming problems. Badr El-Sayed et al. [30] proposed two phase method approach for solving neutrosophic linear programming problems. Badr El-Sayed et al. [31] proposed an application part of neutrosophic goal programming in the context of sustainable development of Egypt.

1.2. Novelty and Major Contributions

Neutrosophic set theory plays a vital role in dealing with the uncertain and vague information that arises in real-world industrial problems. Many researchers have contributed on neutrosophic set theory and applied new techniques for solving real problems. Some of recent contributions are: Abdel-Basset M et al. [32] suggested important neutrosophic techniques for solving problems in various smart environments. Maissam Jdid and Smarandache [33] described the use of neutrosophic technique in solving two important operation research problems of 'optimal design of warehouses' and 'capital budget allocation'. Abduallah Gamal et al. [34] proposed the use of type -2 neutrosophic number to obtain optimal solution of multi-criteria decision-making problems of autonomous vehicles and distributed resources. During the literature survey on NNs, it is disclosed that contrary to research on linear NNs, only a few researchers contributed on properties of non-linear neutrosophic numbers (NLNNs), arithmetic operations on NLNNs, its application in formulating real problems, etc. These are: Chakraborty et al. [25] discussed different types of Non-linear trapezoidal NNs and their properties. Javier and Francisco [26] discovered the basic properties of NLNNs, a new scoring function, and demonstrated its application to multiple criteria assessment problems of industry. Some typo errors have been pointed out in the work of Javier and Francisco [26] in defining different properties of NLNNs which are rectified in this manuscript. Further, the involvement of non-linear NNs in MPPs (LPPs or other complex MPPs) as coefficients/parameters is hardly ever been researched to date due to the computational complexities of Non-linear NNs, and therefore, no solution methodology has been developed for non-linear neutrosophic linear programming problem (NLN-LPP) till date. This motivates us to extend the use of NLNNs in LPPs, propose a modified score function of NLNNs, and propose a solution technique for NLN-LPPs. In this view, the main contribution of this paper can be summarized:

(i) Proposed a new modified possibility score function for non-linear neutrosophic numbers (NLNNs) with the concept of normal approximation.

(ii) Proposed a novel and unique solution technique for non-linear neutrosophic linear programming problem (NLN-LPP) using a modified possibility score function.

(iii) Elaborated different properties of non-linear neutrosophic numbers (NLNNs) in the corrected form in a systematic manner for future researchers.

In nutshell, this paper elaborates on the concepts of non-linear neutrosophic number (NLNN) sets, different forms of NLNNs, α,β,γ cuts on non-linear neutrosophic numbers (NLNNs), possibility mean, possibility standard deviation, and possibility variance of NLNNs. In this paper, we propose the solution technique for non-linear neutrosophic linear programming problems (NLN-LPPs) in which all coefficients/parameters are NLNNs. In this continuation, we suggest a new modified possibility score function for non-linear NNs in terms of possibility means and possibility standard deviations of NLNNs for better use of all parts of information. This modified score function is used to convert NLNN coefficients/parameters of NLN-LPP into equivalent crisp values. Thereafter, the equivalent crisp problem is solved with the usual method to obtain the optimal solution of NLN-LPP. The proposed solution algorithm is unique and new for solving non-linear neutrosophic linear programming problems. A numerical example is solved with the proposed algorithm to legitimate the research output. A case study is also discussed to show its applicability in solving real-life problems.

This paper is organized in a section-wise format: This first section of the paper gives a systematic introduction of the current research problem and focused literature review from the beginning. In sub-section 1.1, literature review on fuzzy and neutrosophic programming problems are presented. Sub-section 1.2 discloses the causes of motivation for proposing this research work, novelty of proposed work and major contributions. Some preliminaries on the neutrosophic set (NN set) are presented in next section 2 and its subsections. α,β,γ cut sets of NLNNs are defined and derived in subsection 2.1. Possibility mean, possibility variance, and possibility standard deviations are defined and derived in subsection 2.2. The modified possibility score function for NL-NNs is proposed in section 3. The formulation of non-linear neutrosophic linear programming problem (NLN-LPP) and suggested solution technique for NLN-LPPs are described and explained in section 4. To better understand the proposed algorithm, one numerical example and a case study of an industrial decision-making problem based on NLN-LPP are illustrated in section 5. Conclusions and research directions for future researchers are proposed in the last section.

2. Neutrosophic Set: Preliminaries

In this section, we shall discuss some generic preliminaries on neutrosophic set (NN set) related to the research area under study. As we know that neutrosophic set (introduced by Smarandche [15]) is a set of objects with membership function values of truthiness, indeterminacy, and falsity of information of objects of concern set. Later, Wang et al. [27] gave the concept of single-valued NN which is NN set with values of membership functions lying within the interval [0, 1]. In continuation of this context, a single-valued neutrosophic set is mathematically defined by the following generic definition:

Definition 1. (Wang et al. [27]): A neutrosophic set A in X is characterized as $A=\{x:\mu_A^T(x),\sigma_A^T(x),\upsilon_A^T(x),x\in X\}$ where $T_A(x), I_A(x), F_A(x)\in[0,1]$ represents degree of membership for truthiness, indeterminacy and falsity parts of information respectively along with condition $0\le T_A(x)+I_A(x)+F_A(x)\le 3$. If all membership functions of defined SVNNs are linear, then it is called as linear SVNNs. It is being reiterated that NNs are further classified as linear NNs and Non-linear NNs (NLNNs) on the linearity of all membership functions and non-linearity of at least one membership function of NN. Chakrabort et al. [25] presented the definition of non-linear trapezoidal type NN as:

Definition 2. (Chakraborty et al. [25]): A single valued non-linear trapezoidal NN is defined as:

$$A=\{x:(a_1,a_2,a_3,a_4;b_1,b_2,b_3,b_4;c_1,c_2,c_3,c_4;p_1,p_2;q_1,q_2;r_1,r_2;\omega,\rho,\lambda);T_A(x), I_A(x), F_A(x), x\in X\} \quad (1)$$

where $T_A(x), I_A(x), F_A(x)\in[0,1]$ represents membership for truthiness, indeterminacy and falsity of information respectively are given as:

$$T_A(x)=\begin{cases}\omega\left(\dfrac{x-a_1}{a_2-a_1}\right)^{p_1}, & a_1\le x<a_2,\\ \omega, & a_2<x\le a_3,\\ \omega\left(\dfrac{a_4-x}{a_4-a_3}\right)^{p_2}, & a_3<x\le a_4,\\ 0, & \text{Otherwise,}\end{cases} \tag{2}$$

$$I_A(x)=\begin{cases}\rho\left(\dfrac{b_2-x}{b_2-b_1}\right)^{q_1}, & b_1\le x<b_2,\\ 0, & b_2<x\le b_3,\\ \rho\left(\dfrac{x-b_3}{b_4-b_3}\right)^{p_2}, & b_3<x\le b_4,\\ \rho, & \text{Otherwise,}\end{cases} \tag{3}$$

$$F_A(x)=\begin{cases}\lambda\left(\dfrac{c_2-x}{c_2-c_1}\right)^{q_1}, & \mathrm{c}_1\le x<c_2,\\ 0, & \mathrm{c}_2<x\le c_3,\\ \lambda\left(\dfrac{x-c_3}{c_4-c_3}\right)^{p_2}, & \mathrm{c}_3<x\le c_4,\\ \lambda, & \text{Otherwise,}\end{cases} \tag{4}$$

along with conditions $0\le T_A(x)+I_A(x)+F_A(x)\le 3$ and $p_1,p_2;q_1,q_2;r_1,r\ne 1$.

Thereafter, Javier and Francisco [26] proposed an alternate definition of non-linear NN for triangular values in view of mapping of parameter values ρ and λ with their minimum and maximum values. According to Javier and Francisco [26], NLNNs are defined as follows:

Definition 3. (Javier and Francisco [26]):

$$A_{(m,n)}=\left\{x:(\underline{a},a,\overline{a};\ w,y,u);T(x),I(x),F(x);\ x\in X\right\} \tag{5}$$

is a single valued non-linear NN (SVNN) whose respective membership function $T(x),I(x)$ and $F(x)$ are defined as:

$$T_A(x)=\begin{cases}\left[1-\left(\dfrac{a-x}{a-\underline{a}}\right)^{m_T}\right]w, & \underline{a}\le x<a,\\ w, & x=a,\\ \left[1-\left(\dfrac{a-x}{a-\overline{a}}\right)^{n_T}\right]w, & a\le x<\overline{a},\\ 0, & \text{Otherwise,}\end{cases} \tag{6}$$

$$I(x)=\begin{cases}\left[y+(1-y)\left(\dfrac{a-x}{\underline{a}}\right)^{m_I}\right], & \underline{a}\le x<a,\\ y, & x=a,\\ \left[y+(1-y)\left(\dfrac{a-x}{a-\overline{a}}\right)^{n_I}\right], & a<x\le\overline{a},\\ 1, & \text{Otherwise,}\end{cases} \tag{7}$$

$$F(x)=\begin{cases}\left[u+(1-u)\left(\dfrac{a-x}{\underline{a}}\right)^{m_F}\right], & \underline{a}\le x<a,\\ u, & x=a,\\ \left[u+(1-u)\left(\dfrac{a-x}{a-\bar{a}}\right)^{n_F}\right], & a<x\le\bar{a},\\ 1, & \text{Otherwise,}\end{cases} \tag{8}$$

Where parameters $m=(m_T,m_I,m_F)\in[1,\infty]$; $n=(n_T,n_I,n_F)\in[1,\infty]$. It can be observed from definition (6)-(8), when $m=(1,1,1)$; $n=(1,1,1)$, then NLNN reduces to a triangular linear NN.

2.1. α,β,γ cut- sets of non-linear neutrosophic numbers (NLNNs)

Definition 4. : The α,β,γ cut sets of NLNN $A_{(m,n)}=\{x:(\underline{a},a,\bar{a};\ w,y,u);T(x),I(x),F(x);\ x\in X\}$ are defined as:

$$A_{(\alpha,\beta,\gamma)}=\{\langle x,T(x)\ge\alpha,I(x)\le\beta,F(x)\le\gamma\rangle:x\in X\} \tag{9}$$

With the conditions $0\le\alpha\le w;\ y\le\beta\le1;\ u\le\gamma\le1$ and $\alpha+\beta+\gamma\le3$. Using the definition (5) - (8) of NLNN, we can obtain α,β,γ cut sets as:

For α cut set $T(x)\ge\alpha\Rightarrow\left[1-\left(\dfrac{a-x}{a-\underline{a}}\right)^{m_T}\right]w\ge\alpha\Rightarrow x\ge a-\left(1-\dfrac{\alpha}{w}\right)^{1/m_T}(a-\underline{a})$

And $T(x)\ge\alpha$ gives $x\ge a-\left(1-\dfrac{\alpha}{w}\right)^{1/m_T}(a-\underline{a})$

Thus α cut set of NLNN $A_{(m,n;\alpha,\beta,\gamma)}$ is a closed interval described as:

$$A_{(m_T,n_T;\alpha)}=\left[T^L(\alpha,m_T),T^U(\alpha,n_T)\right] \tag{10}$$

Where
$$T^L(\alpha,m_T)=a-\left(1-\frac{\alpha}{w}\right)^{1/m_T}(a-\underline{a}) \tag{11}$$

And
$$T^U(\alpha,n_T)=a-\left(1-\frac{\alpha}{w}\right)^{1/n_T}(a-\bar{a}) \tag{12}$$

In similar manner, β and cut set of NLNN $A_{(m,n;\alpha,\beta,\gamma)}$ are closed interval sets described as:

$$A_{(m_I,n_I;\beta)}=\left[I^L(\beta,m_T),\ I^U(\beta,n_T)\right] \tag{13}$$

Where $I^L(\beta,m_I)=a-\left(\dfrac{\beta-y}{1-y}\right)^{1/m_I}(a-\underline{a})$; $\quad I^U(\beta,n_T)=a-\left(\dfrac{\beta-y}{1-y}\right)^{1/n_I}(a-\bar{a})$

$$A_{(m_F,n_F;\gamma)}=\left[F^L(\gamma,m_F),\ F^U(\gamma,n_F)\right] \tag{14}$$

Where $F^L(\gamma,m_F)=a-\left(\dfrac{\gamma-u}{1-u}\right)^{1/m_F}(a-\underline{a})$; $\quad F^U(\gamma,n_F)=a-\left(\dfrac{\gamma-u}{1-u}\right)^{1/n_F}(a-\bar{a})$

2.2 Possibility mean, possibility variance and possibility standard deviation of non-linear neutrosophic numbers (NLNNs)

Definition 5. (Possibility mean of a NLNN): (Javier and Francisco [26]): For a NLNN as defined in (5) – (8), $A_{(m,n)} = \left\{x : (\underline{a}, a, \overline{a};\ w, y, u);\ x \in X\right\}$ and its α cut set i.e. $A_{(m_T, n_T;\alpha)} = \left[T^L(\alpha, m_T), T^U(\alpha, n_T)\right]$, then *f*-weighted possibility mean of truth membership function is defined as:

$$M_T(A_{(m,n)}) = \frac{w}{2}\left(\frac{2m_T^2\underline{a} + 3m_T a + a}{(1+2m_T)(1+m_T)} + \frac{2n_T^2\overline{a} + 3n_T a + a}{(1+2n_T)(1+n_T)}\right) \tag{15}$$

Where *f*-weight is considered as $f = {}^{2\alpha}\!/_{w}$ as suggested by Chakraborty et al. [25]. Similarly, *g*-weighted ($g = {}^{2(1-\beta)}\!/_{(1-y)}$) possibility mean of indeterminacy membership function is defined as:

$$M_I(A_{(m,n)}) = \frac{(1-y)}{2}\left(\frac{2m_I^2\underline{a} + 3m_I a + a}{(1+2m_I)(1+m_I)} + \frac{2n_I^2\overline{a} + 3n_I a + a}{(1+2n_I)(1+n_I)}\right) \tag{16}$$

Also, *h*- weighted ($h = {}^{2(1-\gamma)}\!/_{(1-u)}$) possibility mean of indeterminacy membership function is defined as:

$$M_F(A_{(m,n)}) = \frac{(1-u)}{2}\left(\frac{2m_F^2\underline{a} + 3m_F a + a}{(1+2m_F)(1+m_F)} + \frac{2n_F^2\overline{a} + 3n_F a + a}{(1+2n_F)(1+n_F)}\right) \tag{17}$$

Definition 6. (Possibility variance of a NLNN): (Javier and Francisco [26]): For a NLNN as defined in (5)–(8), $A_{(m,n)} = \left\{x : (\underline{a}, a, \overline{a};\ w, y, u);\ x \in X\right\}$ and its α cut set i.e. $A_{(m_T, n_T;\alpha)} = \left[T^L(\alpha, m_T), T^U(\alpha, n_T)\right]$, then *f*- weighted possibility variance of truth membership function is defined as:

$$V_T(A_{(m,n)}) = w\left[\frac{m_T^2(a-\underline{a})^2}{4(1+m_T)(2+m_T)} + \frac{n_T^2(\overline{a}-a)^2}{4(1+n_T)(2+n_T)} - \frac{n_T^2 m_T^2(a^2 - \underline{a}a - a\overline{a} + \underline{a}\overline{a})}{(m_T + n_T + 2m_T n_T)(m_T + n_T + m_T n_T)}\right] \tag{18}$$

Where *f*-weight is considered as $f = {}^{2\alpha}\!/_{w}$ as suggested by Chakraborty et al. [25]. Similarly, *g*-weighted ($g = {}^{2(1-\beta)}\!/_{(1-y)}$) possibility variance of indeterminacy membership function is defined as:

$$V_I(A_{(m,n)}) = (1-y)\left[\frac{m_I^2(a-\underline{a})^2}{4(1+m_I)(2+m_I)} + \frac{n_I^2(\overline{a}-a)^2}{4(1+n_I)(2+n_I)} - \frac{n_I^2 m_I^2(a^2 - \underline{a}a - a\overline{a} + \underline{a}\overline{a})}{(m_I + n_I + 2m_I n_I)(m_I + n_I + m_I n_I)}\right] \tag{19}$$

Also, *h*- weighted ($h = {}^{2(1-\gamma)}\!/_{(1-u)}$) possibility mean of indeterminacy membership function is defined as:

$$V_F(A_{(m,n)}) = (1-u)\left[\frac{m_F^2(a-\underline{a})^2}{4(1+m_F)(2+m_F)} + \frac{n_F^2(\overline{a}-a)^2}{4(1+n_F)(2+n_F)} - \frac{n_F^2 m_F^2(a^2 - \underline{a}a - a\overline{a} + \underline{a}\overline{a})}{(m_F + n_F + 2m_F n_F)(m_F + n_F + m_F n_F)}\right] \tag{20}$$

Definition 7. (Possibility standard deviation of a NLNN): (Javier and Francisco [26]): For a NLNN $A_{(m,n)} = \left\{x : (\underline{a}, a, \overline{a};\ w, y, u);\ x \in X\right\}$ and its α cut set i.e. $A_{(m_T, n_T;\alpha)} = \left[T^L(\alpha, m_T), T^U(\alpha, n_T)\right]$, then possibility standard deviation of its membership functions are defined as:

Possibility S.D = $\sqrt{\text{Possibility variance}}$

$\Rightarrow$ Possibility standard deviation of truth membership $D_T(A_{(m,n)}) = \sqrt{V_T(A_{(m,n)})}$

$$i.e.\quad D_T(A_{(m,n)}) = \left[w \left[\frac{m_T^2(a-\underline{a})^2}{4(1+m_T)(2+m_T)} + \frac{n_T^2(\overline{a}-a)^2}{4(1+n_T)(2+n_T)} - \frac{n_T^2 m_T^2(a^2 - \underline{a}a - a\overline{a} + \underline{a}\overline{a})}{(m_T+n_T+2m_Tn_T)(m_T+n_T+m_Tn_T)} \right] \right]^{1/2} \quad (21)$$

Possibility standard deviation of indeterminacy membership $D_I(A_{(m,n)}) = \sqrt{V_I(A_{(m,n)})}$

$$i.e\quad D_I(A_{(m,n)}) = \left[(1-y) \left[\frac{m_I^2(a-\underline{a})^2}{4(1+m_I)(2+m_I)} + \frac{n_I^2(\overline{a}-a)^2}{4(1+n_I)(2+n_I)} - \frac{n_I^2 m_I^2(a^2 - \underline{a}a - a\overline{a} + \underline{a}\overline{a})}{(m_I+n_I+2m_In_I)(m_I+n_I+m_In_I)} \right] \right]^{1/2} \quad (22)$$

and possibility standard deviation of falsity membership function $D_F(A_{(m,n)}) = \sqrt{V_F(A_{(m,n)})}$

$$i.e\quad D_F(A_{(m,n)}) = \left[(1-u) \left[\frac{m_F^2(a-\underline{a})^2}{4(1+m_F)(2+m_F)} + \frac{n_F^2(\overline{a}-a)^2}{4(1+n_F)(2+n_F)} - \frac{n_F^2 m_F^2(a^2 - \underline{a}a - a\overline{a} + \underline{a}\overline{a})}{(m_F+n_F+2m_Fn_F)(m_F+n_F+m_Fn_F)} \right] \right]^{1/2} \quad (23)$$

Remark 1. It is to be noted that there were some typo errors in definitions of possibility mean, possibility variance and possibility standard deviations of NLNNs given by Javier and Francisco [26] which are respectively corrected here and presented definitions (5) – (8) are in corrected form.

Remark 2. It can also be observed from definition (5) – (8), when $m = (1,1,1)$; $n = (1,1,1)$, then NLNN reduces to a single valued triangular NN (SVTNN) and accordingly their characteristics as: by definitions (15) – (23),

Possibility means $M_T(A_{(1,1)}) = \frac{(\underline{a}+4a+\overline{a})w}{6}$ $M_I(A_{(1,1)}) = \frac{(\underline{a}+4a+\overline{a})(1-y)}{6}$; $M_F(A_{(1,1)}) = \frac{(\underline{a}+4a+\overline{a})(1-u)}{6}$

Possibility variance $V_T(A_{(1,1)}) = \frac{(\overline{a}-\underline{a})^2 w}{24}$; $V_I(A_{(1,1)}) = \frac{(\overline{a}-\underline{a})^2(1-y)}{24}$; $V_F(A_{(1,1)}) = \frac{(\overline{a}-\underline{a})^2(1-u)}{24}$

Possibility SD $D_T(A_{(1,1)}) = (\overline{a}-\underline{a})\sqrt{\frac{w}{24}}$; $D_I(A_{(1,1)}) = (\overline{a}-\underline{a})\sqrt{\frac{(1-y)}{24}}$; $D_F(A_{(1,1)}) = (\overline{a}-\underline{a})\sqrt{\frac{(1-u)}{24}}$

3. Proposed modified possibility score function for non-linear neutrosophic numbers (NLNNs)

Possibility score functions are used for ranking purposes and conversion of NNs into their equivalent crisp values. Javier and Francisco [26] proposed a possibility score function for NLNNs as a simple addition of the average of possibility means and possibility standard deviations related to truthiness. Indeterminacy and falsity membership values of NLNNs. Here we argue that this score function is a limitation to express all x values of the domain set in decision-making context. Therefore, to better characterize the role of all range values x in expressing the possibility score function, we propose a modified form of possibility score function for NLNNs:

$$PS(A_{(m,n)}) = \frac{PS_T(A_{(m,n)}) + PS_I(A_{(m,n)}) + PS_F(A_{(m,n)})}{3} \quad (24)$$

Where $PS_T(A_{(m,n)}), PS_I(A_{(m,n)}), PS_F(A_{(m,n)})$ are respective possibility score functions for truth, indeterminacy and falsity membership functions which are defined as:

$$PS_T(A_{(m,n)}) = M_T(A_{(m,n)}) + 2.58 D_T(A_{(m,n)}) \quad (25)$$

$$PS_I(A_{(m,n)}) = M_I(A_{(m,n)}) + 2.58 D_I(A_{(m,n)}) \quad (26)$$

$$PS_F(A_{(m,n)}) = M_F(A_{(m,n)}) + 2.58 D_F(A_{(m,n)}) \quad (27)$$

As the normal curve is the best fitted curve for all membership values in general conditions and this curve covers values with 99% confidence in interval $Mean \pm 2.58\, S.D.$ This is the main reason of proposing the possibility score function ((23) – (26)) in the modified form so that membership functions *T(x), I(x)* and *F(x)* graphs can be well approximated to normal curve with statistical parameters - possibility means and possibility standard deviation. The proposed modified possibility score function displays the contribution of all x values as in the normal curve.

4. Formulation of Non-linear Neutrosophic Linear Programming Problem (NLN-LPP) and Proposed Solution Technique

During the literature review, it has already been disclosed in particular that non-linear neutrosophic linear programming problems have not been discussed so far due to the non-linear complexities of functions. So, now we propose non-linear neutrosophic linear programming problems (NLN –LPP) as linear programming problems with non-linear neutrosophic numbers (NLNNs) as parameters/ coefficients of LPPs. In mathematical format, a single objective NLN-LPP with N decision variables can be described as:

$$Maximize\,/\,Minimize\ Z = c_{1,(m,n)}x_1 + c_{2,(m,n)}x_2 + ... + c_{N,(m,n)}x_N \quad \text{(Objective function)}$$

Subject to the set of constraints,

$$a_{11,(m,n)}x_1 + a_{12,(m,n)}x_2 + ... + a_{1N,(m,n)}x_N\ (\le = \ge)\, b_{1,(m,n)}$$
$$a_{21,(m,n)}x_1 + a_{22,(m,n)}x_2 + ... + a_{2N,(m,n)}x_N\ (\le = \ge)\, b_{2,(m,n)}$$
$$\vdots$$
$$a_{M1,(m,n)}x_1 + a_{M2,(m,n)}x_2 + ... + a_{MN,(m,n)}x_N\ (\le = \ge)\, b_{M,(m,n)}$$

And Non-negativity restrictions $\quad x_1, x_2, ..., x_N \ge 0 \qquad (28)$

Where superscript $\tilde{}$ on coefficients indicates that concern coefficients are single valued NLNNs with the set of values $A_{(m,n)} = \left\{ x : (\underline{a}, a, \overline{a};\ w, y, u); (m,n) = (m_T, m_I, m_F; n_T, n_I, n_F);\ x \in X \right\}$. The other notations have usual meaning in respect of LPPs. Such problems (27) have incomplete, vague and uncertain information on coefficients in terms of NLNNs are defined as NLN-LPPs. In the real world, such decision-making problems are expected to have a crisp optimal solution. Thus, we here propose a solution methodology for NLN-LPPs in which firstly all NLNNs are converted into equivalent crisp values using respective modified possibility score functions. Mathematically, converted equivalent crisp LPP with modified possibility score functions can be described as:

$$Maximize\,/\,Minimize\ Z = PS(c_{1,(m,n)})x_1 + PS(c_{2,(m,n)})x_2 + ... + PS(c_{N,(m,n)})x_N \quad \text{(Objective function)}$$

Subject to,

$$PS(a_{11,(m,n)})x_1 + PS(a_{12,(m,n)})x_2 + ... + PS(a_{1N,(m,n)})x_N\ (\le = \ge)\, PS(b_{1,(m,n)})$$
$$PS(a_{21,(m,n)})x_1 + PS(a_{22,(m,n)})x_2 + ... + PS(a_{2N,(m,n)})x_N\ (\le = \ge)\, PS(b_{2,(m,n)})$$
$$\vdots$$
$$PS(a_{M1,(m,n)})x_1 + PS(a_{M2,(m,n)})x_2 + ... + PS(a_{MN,(m,n)})x_N\ (\le = \ge)\, PS(b_{M,(m,n)})$$

And $\quad x_1, x_2, ..., x_N \ge 0 \qquad (29)$

Where $PS(A_{ij,(m,n)})$ indicates the corresponding possibility score function values as defined in (24) – (27). The satisfactory solution to original NLN-LPP is the optimal solution of equivalent crisp LPP (29).

5. Numerical illustration and case study

To describe the proposed algorithm, we shall consider the following numerical example and a case study of industrial problem based on NLN-LPP as:

Numerical Example

$$Maximize\ Z = c_{1,(m,n)}x_1 + c_{2,(m,n)}x_2 + c_{3,(m,n)}x_3$$

Subject to, $a_{11,(m,n)}x_1 + a_{12,(m,n)}x_2 + a_{13,(m,n)}x_3 \le b_{1,(m,n)}$

$$a_{21,(m,n)}x_1 + a_{22,(m,n)}x_2 + a_{23,(m,n)}x_3 \le b_{2,(m,n)}$$

and non-negativity restrictions $x_1, x_2, ..., x_N \ge 0$

where neutrosophic coefficients are given as:

$c_1 = ((2,3,4); 0.5, 0.25, 0.25, (2,2,2); (2,2,2))$; $c_2 = ((3,4,5); 0.5, 0.25, 0.25, (2,2,2); (2,2,2))$

$c_3 = ((4,5,6); 0.5, 0.25, 0.25, (2,2,2); (2,2,2))$; $a_{11} = ((3,4,5); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$

$a_{12} = ((4,5,6); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$; $a_{13} = ((4,5,6); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$

$b_1 = ((6,7,8); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$; $a_{21} = ((1,2,3); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$

$a_{22} = ((3,4,5); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$; $a_{23} = ((2.5,3.5,4.5); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$

$b_2 = ((5,6,7); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$

For the proposed solution technique, we calculate possibility means (by equations (15) – (17)), possibility SD (by equations (21) – (23)) and modified possibility score function (by equations (24) – (27)) corresponding to each NN coefficient of the problem. These values are described in tabular format (Table 1, Appendix A) in correspondence to the neutrosophic coefficients of the problem. Using these values, the given NLN-LPP is converted into equivalent crisp problem as:

$$Maximize\ Z = 3.211x_1 + 3.8192x_2 + 4.2109x_3$$

Subject to, $3.5229x_1 + 4.1896x_2 + 4.1896x_3 \le 5.2313$

$$2.2729x_1 + 3.5229x_2 + 3.2313x_3 \le 4.8559$$

and non-negativity restrictions $x_1, x_2, x_3 \ge 0$

Solving with the Simplex method, the optimal solution obtained is as: $x_1 = 0$, $x_2 = 0$, $x_3 = 1.2486$, $Z = 5.2578$ which is also the solution to original NLN-LPP. If we use possibility score function as $PS^*(A_{(m,n)}) = M(A_{(m,n)}) + D(A_{(m,n)})$ (as suggested by Javier and Francisco [26]) to convert NLNNs into corresponding equivalent crisp values and solve the converted crisp LPP, we obtain the optimal solution of the problem as: $x_1 = 0$, $x_2 = 0$, $x_3 = 1.2885$, $Z = 4.9001$. On comparison, it is clear that the modified possibility score function gives better values of objective function.

Case Study

Let us consider a case study of 'XYZ' company manufacturing certain fashion items in different production slots. The production variables of these items as well as demands are decided with the help of information gathered via social media networks, reviews, customer comments, etc. It is known to decision-makers of production units that this information is not fully true and reliable. Decision makers assume that related information on social media is in NNs format *i.e.* truthiness, falsity, and indeterminacy also their degree of memberships varies mostly in a non-linear way. For sake of simplicity in this case study, it is assumed that production and demand coefficients are in SVTNN. The profit maximization LP problem of this company with NLNNs can be presented as:

$$Maximize\ Z = c_{1,(m,n)}x_1 + c_{2,(m,n)}x_2 \quad \text{(Profit)}$$

Subject to, $a_{11,(m,n)}x_1 + a_{12,(m,n)}x_2 \le b_{1,(m,n)}$

$$a_{21,(m,n)}x_1 + a_{22,(m,n)}x_2 \le b_{2,(m,n)}$$

$$a_{31,(m,n)}x_1 + a_{32,(m,n)}x_2 \le b_{3,(m,n)}$$

$$a_{41,(m,n)}x_1 + a_{42,(m,n)}x_2 \le b_{4,(m,n)}$$

and non-negativity restrictions $x_1, x_2 \ge 0$

where neutrosophic coefficients are given as:

$c_1 = ((4,5,6); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$; $c_2 = ((6,7,8); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$;

$a_{11} = ((4,5,6); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$; $a_{12} = ((1,2,3); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$;

$a_{21} = ((2.5,3.5,4.5); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$; $a_{22} = ((3.5,4.5,5.5); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$;

$a_{31} = ((5,6,7); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$; $a_{32} = ((5,6,7); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$;

$a_{41} = ((5.5,6.5,7.5); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$; $a_{42} = ((3,4,5); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$;

$b_1 = ((1,2,3); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$; $b_2 = ((8,9,10); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$;

$b_3 = ((5,6,7); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$; $b_4 = ((1.5,2.5,3.5); 0.5, 0.25, 0.25, (1,1,1); (1,1,1))$

with the proposed solution technique and tabulated values (Table 2, Appendix A), this problem is converted into equivalent crisp problem as:

$$Maximize\ Z = 4.1758x_1 + 5.4305x_2 \qquad \text{(profit)}$$

Subject to,

$$4.2457x_1 + 2.9679x_2 \leq 3.3989$$
$$3.0282x_1 + 3.9957x_2 \leq 5.1853$$
$$4.6322x_1 + 5.4120x_2 \leq 5.1853$$
$$5.7870x_1 + 3.9120x_2 \leq 2.1478$$

and non-negativity restrictions $x_1, x_2 \geq 0$

With the help of simplex method, the optimal solution to this crisp problem is obtained as: $x_1 = 0$, $x_2 = 0.5490$, $x_3 = 0$, $Z = 2.9815$ which is also the solution to original NLN-LPP. This is too better solution to the problem than solution by technique based on possibility score function by Javier and Francisco [26] which is $x_1 = 0$, $x_2 = 0.5414$, $x_3 = 0$, $Z = 2.7063$.

6. Conclusions and future research directions

Non-linear neutrosophic numbers (NLNNs) are different kinds of neutrosophic numbers (NNs) with at least one non-linear membership function (either of truthiness, falsity or indeterminacy part) of the information. Furthermore, a non-linear neutrosophic linear programming problem (NLN-LPP) is a special type of linear programming problem in which coefficients/parameters are non-linear neutrosophic numbers. This paper presents comprehensive research on non-linear neutrosophic numbers (NLNNs) and non-linear neutrosophic linear programming problems (NLN-LPPs). Here, the author proposed a novel solution technique for NLN-LPPs based on the proposed modified possibility score function. This proposed modified possibility score function covers the almost entire range of values of NNs. Besides this, this paper elaborates on the concepts of non-linear neutrosophic (NLNN) sets, different forms of NLNNs, α, β , γ - cuts on NLNNs, possibility mean, possibility standard deviation, and possibility variance of NLNNs in corrected forms for clear understanding to future researchers. As future research, this work can be extended to solve non-linear neutrosophic non-linear programming problems (NLN-NLPPs), non-linear neutrosophic multiobjective programming problems (NLN-MOPPs), non-linear neutrosophic bi-level and multi-level programming problems (NLN-BL/MLPPs), etc. There is a scope of research investigations on basic operations on NLNNs – addition, substation, multiplication, and division of two or more NLNNs.

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Appendix A.

Table 1. Possibility score function values for NLNN coefficients of numerical example

NLNN Coefficient	$M_T(A_{(m,n)})$	$M_F(A_{(m,n)})$	$M_I(A_{(m,n)})$	$D_T(A_{(m,n)})$	$D_F(A_{(m,n)})$	$D_I(A_{(m,n)})$	$PS_T(A)$	$PS_T(A)^*$	$PS_F(A)$	$PS_F(A)^*$	$PS_I(A)$	$PS_I(A)^*$	$PS(A)$	$PS(A)^*$
c_1	1.5	2.25	2.25	0.4081	0.5	0.5	2.553	1.9081	3.54	2.75	3.54	2.75	3.211	2.4693
c_2	2	3	2.825	0.4081	0.5	0.5	3.053	2.4081	4.2897	3.5	4.115	3.325	3.8192333	3.0777
c_3	2.5	3.75	3.75	0.4081	0.5	0.5	3.553	2.9081	4.03974	4.25	5.04	4.25	4.2109133	3.8027
a_{11}	2	3	3.75	0.2886	0.3535	0.3535	2.7447	2.2886	3.9121	3.3535	3.9121	4.1035	3.5229667	3.2485
a_{12}	2.5	3.75	3.75	0.2886	0.3535	0.3535	3.2447	2.7886	4.6621	4.1035	4.6621	4.1035	4.1896333	3.6652
a_{13}	2.3333	3.75	3.75	0.2886	0.3535	0.3535	3.2447	2.6219	4.6621	4.1035	4.6621	4.1035	4.1896333	3.6096
a_{21}	0.8333	1.5	1.5	0.2886	0.3535	0.3535	1.7447	1.1219	2.6621	1.8535	2.4121	1.8535	2.2729667	1.6096
a_{22}	1.8333	3	3	0.2886	0.3535	0.3535	2.7447	2.1219	3.9121	3.3535	3.9121	3.3535	3.5229667	2.9429
a_{23}	1.5833	2.75	2.625	0.2886	0.3535	0.3535	2.4947	1.8719	3.6621	3.1035	3.5371	2.9785	3.2313	2.6513
b_1	3.3333	4.375	5.25	0.2886	0.3535	0.3535	4.2447	3.6219	5.2871	4.7285	6.1621	5.6035	5.2313	4.6513
b_2	2.8333	4.5	4.5	0.2886	0.3535	0.3535	3.7437	3.1219	5.4121	4.8535	5.4121	4.8535	4.8559667	4.2763

*Possibility score function $PS^*(A_{(m,n)}) = M(A_{(m,n)}) + D(A_{(m,n)})$ suggested by Javier and Francisco [26]

Table 2. Possibility score function values for NLNN coefficients of case study

NLNN Coefficient	$M_T(A_{(m,n)})$	$M_F(A_{(m,n)})$	$M_I(A_{(m,n)})$	$D_T(A_{(m,n)})$	$D_F(A_{(m,n)})$	$D_I(A_{(m,n)})$	$PS_T(A)$	$PS_T(A)^*$	$PS_F(A)$	$PS_F(A)$ *	$PS_I(A)$	$PS_I(A)$ *	$PS(A)$	$PS(A)$ *
c_1	2.5	3.75	3.75	0.2886	0.3535	0.3535	3.2034	2.7886	4.66203	4.1035	4.66203	4.1035	4.17582	3.6652
c_2	3.5	5.25	5.25	0.2886	0.3535	0.3535	3.9676	3.7886	6.16203	5.6035	6.16203	5.6035	5.430553	4.9985333
a_{11}	2.5	3.75	3.75	0.2886	0.3535	0.3535	3.4132	2.7886	4.66203	4.1035	4.66203	4.1035	4.245753	3.6652
a_{12}	1	1.5	1.5	0.2886	0.3535	0.3535	4.0799	1.2886	2.41203	1.8535	2.41203	1.8535	2.967987	1.6652
a_{21}	1.75	2.625	2.625	0.2886	0.3535	0.3535	2.0108	2.0386	3.53703	2.9785	3.53703	2.9785	3.028287	2.6652
a_{22}	2.25	3.375	3.375	0.2886	0.3535	0.3535	3.4132	2.5386	4.28703	3.7285	4.28703	3.7285	3.995753	3.3318667
a_{31}	3	4.5	4.5	0.2886	0.3535	0.3535	3.07278	3.2886	5.41203	4.8535	5.41203	4.8535	4.63228	4.3318667
a_{32}	3	4.5	4.5	0.2886	0.3535	0.3535	3.7445	3.2886	5.41203	4.8535	5.41203	4.8535	5.41203	4.3318667
a_{41}	3.25	4.875	4.875	0.2886	0.3535	0.3535	3.9945	3.5386	5.78703	5.2285	5.78703	5.2285	5.78703	4.6652
a_{42}	2	3	3	0.2886	0.3535	0.3535	2.7445	2.2886	3.91203	3.3535	3.91203	3.3535	3.91203	2.9985333
b_1	1	1.5	1.5	0.2886	0.3535	0.3535	5.3729	1.2886	2.41203	1.8535	2.41203	1.8535	3.398987	1.6652
b_2	3	4.5	4.5	0.2886	0.3535	0.3535	4.732	3.2886	5.41203	4.8535	5.41203	4.8535	5.185353	4.3318667
b_3	3	4.5	4.5	0.2886	0.3535	0.3535	4.732	3.2886	5.41203	4.8535	5.41203	4.8535	5.185353	4.3318667
b_4	0.125	1.875	1.875	0.2886	0.3535	0.3535	0.8695	0.4136	2.7870	2.2285	2.7870	2.2285	2.1478	1.6235

*Possibility score function $PS^*(A_{(m,n)}) = M(A_{(m,n)}) + D(A_{(m,n)})$ suggested by Javier and Francisco [26]

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Study of the Pampay Mass (burial of offering) at the summit of the Andean snow-capped mountain as an example of *(t,i,f)*-Neutrosophic social structure

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Abstract. The purpose of this research is to analyze the characteristics of the Mass Pampay ritual (burying an offering with special objects) of the peasants on the summit of the snow-capped Apu Razuhuillca located in the Andes of Peru at 4,800 meters above sea level as (t,i,f)-Neutrosophic social structure. The essence is to explain the syncretic relationship between the Catholic religion and the ancestral Andean philosophy that has survived over time for generations. The method of study is a survey of the natives and tourists, believer or not in the sacred power of the Apu (Andean God) that provides spiritual security to the pilgrims on special dates of the Andean calendar, which is July 31 of each year, the ceremony on the eve of the Andean custom to the branding of cattle in honor of the feast of "Taita Santiago". We measure approximately the degree of certainty, ignorance, and contradiction which are present in this tradition as a social phenomenon, which is a pillar in the local culture. For processing the data we used Neutrosophic Statistics.

Keywords: Peruvian Cultural traditions, *(t,i,f)*-structure, *(t,i,f)*- Neutrosophic social structure, Refined I-Neutrosophic Structure, Neutrosophic Statistics.

1 Introduction

The present research is related to the syncretic relationship between the Catholic religion and the ancestral Andean philosophy called Pampay Mass (burial of blessed offering) in the province of Huanta, Ayacucho region of the Peruvian highlands. This activity has positive effects on the pilgrims, granting good health, work, economy, social welfare, and multiplication of their livestock: "Similarly, rituals are approached primarily as expressions of thoughts and feelings of those who participate in them" [1]. The Apu (hill, Andean god) of Razuhuillca (the name of the snow-capped mountain) has an altitude of 4,800 meters above sea level. The peculiar characteristic that makes it different from other mountains of similar altitude, is its particularity of being an enchanted hill (bewitched) considered the Andean God of the Huantina population.

Independently of the ancestral activity of Mass Pampay, the Andean ceremony of the pilgrims that climb the summit of the snowy Razuhuillca; there are other similar activities about the Andean calendar in Peru. The activity of Samikuy (offering to the Apu with agricultural species) is registered and explained: "Samikuy of Quechua root, means an Andean ritual with natural products produced with the blessing of the Apu that is deposited in the hills, in this case in the apu Razuhuillca" [2]. This activity is in addition to the Pampay Mass that is celebrated every

July 31st with the participation of pilgrims from different parts of the Peruvian highlands. The pilgrims who climb every year to the summit of Razuhuillca, have the same common interests in getting a reward for the effort made in climbing and carrying the ancestral offering as an interpretation of their Andean conception of: "exchange of offerings with the apus is the essence of the Andean ritual. Colonization fused Samikuy rituals with European products, the mixture of Quechua and Spanish in their prayers" [2].

The veneration of the highest snow-capped mountains in the Peruvian highlands is a common denominator of the rituals that syncretize the conception of Western philosophy that merged with the Christian religion and the survival in time of the ancestral knowledge interpreted in the Andean philosophy that survives to the present day.

According to information gathered at the site of the activity on the summit of Razuhuillca at 4,800 meters above sea level, the Pampay Mass ritual has a positive effect on all pilgrims and their families that keeps them in economic prosperity, social, family, health, work, vigilance of the Apu against rustling, multiplication of their livestock throughout the year.

Our hypothesis is that the ritual is an example of the existence of Neutrosophic Social Structures [3-6]. According to F. Smarandache, this is defined as a *(t,i,f)*-structure, where there is a component of truth, another of indeterminacy, and a third one of falsehood [7-12]. This is a ritual practice that involves all members of the community and tourists from inside and outside the country, so there is an epistemological contradiction between the objective and the subjective meanings of the ritual. Some individuals from other cultures or who come from other realities of the country perceive this ritual as an exotic folkloric manifestation, which has an attractive cultural value, but they do not feel spiritually connected with them. Certain families within the community prepare throughout the year to perform the ritual, therefore this is part of their social and cultural achievement.

On the other hand, it is a syncretic festivity, that is, the religious symbol has a double meaning, representing one or another religious significance depending on the person who consumes it [6]. A Catholic image is revered by Catholics according to the Christian Catholic tradition, however for the indigenous, it can have a meaning according to their polytheistic traditions in the representation of their gods, and this is the way that the conquered and colonized peoples used to maintain their traditions without disturbing to its conquerors.

This phenomenon becomes more complex when ethnic and cultural miscegenation appears, where the mestizo may have a degree of belief in the Catholic God and another degree of belief in the pre-Hispanic God. This is added to the historical perception of the ritual, young people may feel alien to this activity because they have influences from other modern cultures consumed by social networks, by tourist visits, among others. However, older people, who were raised in love with their traditions, may feel much more tied to this type of ceremony.

This article aims to measure approximately the degree of certainty, ignorance, and contradiction that the ritual called Pampay Mass presents, as a social phenomenon among the visitors and the inhabitants of the community in Huanta dedicated to this ceremonial. To do this, the researchers surveyed community members and visitors, on the days before, during, and after the ceremony [13]. The collected data was processed with the help of Neutrosophic Statistics and refined neutrosophic numbers [14]. Neutrosophic Statistics uses the methods of classical statistics applied to data in the form of intervals, or where the size of the sample or that of the population is indeterminate [15-18]. Additionally, there was the opinion of 3 anthropologists who studied the subject, their opinions were aggregated to the results obtained from the survey of the participants.

The article is divided into a section on Materials and Methods where the basic notions of neutrosophy, *(t,i,f)*-structures, their refined variant, and neutrosophic statistics are explained. The section called Results summarizes the results obtained from this research. In the last section, the Conclusions are given.

2 Materials and Methods

A *(t,i,f)*-structure is composed of one space *S* endowed with a set of axioms (or laws) acting (governing) on it, such that the space or at least one of its axioms has an indeterminacy. *t* represents the degree of truthfulness, *i* represents the degree of indeterminacy, and *f* represents the degree of falseness [7].

Originally, this theory was designed for applications in Algebra, Geometry, etc. However, later F. Smarandache recognized its applicability in other sciences like sociology. Thus, he said that the different points of view of all the individuals in society have as a consequence complex social relationships, which causes indeterminacy. One example is syncretism in religion, many people can believe in a Christian God, and however, they practice this “pagan” ritual. Maybe they cannot explain why, so contradiction is part of this phenomenon. Additionally, some persons who are part of the ceremony could not explain their motivation to participate in this mass, thus there is an indeterminacy due to ignorance or lack of information.

Specifically, we are dealing with Refined I- Neutrosophic Structures, as we explain further.

In the following, there are some important concepts to develop this study:

Definition 1: ([8]) A *neutrosophic number* N is defined as a number as follows:

$$N = a + bI \quad (1)$$

Where a is called the *determined part* and bI is called the *indeterminate part*.

Given $N_1 = a_1 + b_1 I$ and $N_2 = a_2 + b_2 I$ are two neutrosophic numbers, some operations between them are defined as follows:

$N_1 + N_2 = a_1 + a_1 + (b_1 + b_2)I$ (Addition);

$N_1 - N_2 = a_1 - a_1 + (b_1 - b_2)I$ (Difference),

$N_1 \times N_2 = a_1 a_2 + (a_1 b_2 + b_1 a_2 + b_1 b_2)I$ (Product),

$\frac{N_1}{N_2} = \frac{a_1 + b_1 I}{a_2 + b_2 I} = \frac{a_1}{a_2} + \frac{a_2 b_1 - a_1 b_2}{a_2(a_2 + b_2)} I$ (Division).

Professor Smarandache also defined types of truthfulness, indeterminacy, and falsity symbolically beyond T, I, and F, respectively. This is what he called refinement, where T is split into $T_1, T_2, \dots, T_p$; I into $I_1, I_2, \dots, I_q$; and F into $F_1, F_2, \dots, F_r$, which depend on the problem we are treating [19]. Specifically, he generalized the neutrosophic numbers in Equation 1 to represent the Refined Neutrosophic Numbers like in Definition 2 [14].

Definition 2: ([8]) Given $I_1, I_2, \dots, I_q$, with $q \geq 1$, a *Refined Neutrosophic Number* is obtained as $N_q = a + b_1 I_1 + b_2 I_2 + \cdots + b_q I_q$, where a is the *determined part* and $b_j I_j (j = 1,2, \dots, q)$ are the *indeterminate parts*, such that $a, b_1, b_2, \dots, b_q$ are real or complex numbers.

Some of the properties that hold are the following:

- $mI_k + nI_k = (m + n)I_k$,
- $0I_k = 0$,
- $I_k^n = I_k$,
- I_k / I_k = undefined,
- $I_j \cdot I_k$ with $j \neq k$ is defined depending on the problem being addressed.

Specifically, we will use the following type of Refined Neutrosophic numbers [14, 19]:

$N_q = a + b_1 I_1 + b_2 I_2$, where I_1 denotes contradiction (simultaneously true and false proposition), while I_2 denotes ignorance (true or false proposition without being able to determine which of the two it is) [8].

Neutrosophic statistics refers to a set of data, such that the data or a part of it is indeterminate to some degree, and to the methods used for analyzing them [15-17, 20].

In classical statistics, all data is determined. This is the distinction between neutrosophic statistics and classical statistics. In many cases, when the indeterminacy is zero, the neutrosophic statistics coincide with the classical statistics. *Neutrosophic measurement* can be used to measure indeterminate data. *Neutrosophic statistical methods* will allow us to interpret and organize neutrosophic data (data that may have some indeterminacies) to reveal underlying patterns. Many approaches can be used in neutrosophic statistics.

In *neutrosophic probability*, indeterminacy is different from randomness. While classical statistics is concerned solely with randomness, neutrosophic statistics is concerned with both randomness and especially indeterminacy.

Neutrosophic descriptive statistics consists of all the techniques for summarizing and describing the characteristics of neutrosophic numerical data. Since neutrosophic numerical data contain indeterminacies, *neutrosophic line plots*, and *neutrosophic histograms* are plotted in 3D space, rather than 2D space as in classical statistics. The third dimension, in addition to the Cartesian XOY system, is that of indeterminacy (*I*). From unclear graphical data, we can extract (unclear) neutrosophic information.

Neutrosophic data are data containing some indeterminacy. In a similar way to classical statistics, it can be classified as:

- *Discrete neutrosophic data*, if the values are isolated points; for example $3 + I_1$, where $I_1 \in [0,1]$, $27 + I_2$, where $I_2 \in [2.3, 5.5]$;

- and *Continuous neutrosophic data*, if the values form one or more intervals, for example [0.01, 0.9] or [0.12, 1.0] (i.e., not sure which).

Other classification:

- *Quantitative* (*numerical*) *neutrosophic data*;

For example a number in the interval [1, 4] (we don't know exactly), or; 60, 62, 67, or 70 (we don't know exactly);

- and *Qualitative* (*categorical*) *neutrosophic data*; for example: black or blue (we don't know exactly), white, orange or green or gray (we don't know exactly). Also, we can have:

-*Univariate neutrosophic data*, that is, neutrosophic data consisting of observations on a single neutrosophic attribute;

- and *Multivariate neutrosophic data*, that is neutrosophic data consisting of observations on two or more attributes. In particular cases, we mention *bivariate neutrosophic data* and *trivariate neutrosophic data*.

A *neutrosophic sample* is a chosen subset of a population, a subset that contains some indeterminacy: either concerning several of its individuals (who may not belong to the population we are studying or may only partially belong to it) or for the subset as a whole.

While classical samples provide precise information, neutrosophic samples provide vague or incomplete information. By abuse of language, it can be said that any sample is a neutrosophic sample since it can be considered that its indetermination is equal to zero.

Neutrosophic survey results are survey results containing some indeterminacy. A *neutrosophic population* is a population not well determined at the membership level (i.e., it is not sure whether some individuals do or do not belong to the population). For example, as in the neutrosophic set, a generic element x belongs to the neutrosophic population M as follows, $x(t,i,f) \in M$ which means: x is $t\%$ in the population M, $f\%$ of x is not in the population M, while $i\%$ membership of x in M is indeterminate (unknown, unclear, neutral: neither in the population nor outside).

3 Results

A survey was prepared for the participants of the Mass Pampay. Sampling was non-probabilistic for convenience, that is, we interviewed all the participants in this ritual who appeared on the way to mass and who were willing to give their opinion. The questions were simple and recorded on a tape to avoid the use of pencil and paper or the investment of inconvenient time to give the answers. Interviews were practically carried out with the participants with questions that had short answers, with the idea of not disturbing the celebration of the participants.

On the other hand, the interviewers kept in mind to identify the broadest possible variety of types of participants, whether they are national or foreign tourists or natives. Thus, the survey questions were as follows:

Survey on the Pampay Mass
1. Please say your name: 2. Are you a tourist or a native? 3. You participate in the Pampay Mass for these reasons: a) Religious b) Cultural c) Economic d) By family tradition 4. What religion or religions do you practice? a) Catholic b) Protestant Christian c) Peruvian indigenous native d) From other origins 5. Do you know what the meaning of the Pampay Mass is? a) Yes b) No c) I don't know 6. Do you believe that Pampay Mass will bring you economic prosperity, health, family happiness, etc.? a) Yes b) No c) I don't know

For each respondent, let us call him/her s_k in $S = \{s_1, s_2, \dots, s_n\}$, a value $a_k + b_k I_1 + c_k I_2$ is associated, where I_1 denotes the indeterminacy due to contradiction, and I_2 is the indeterminacy due to ignorance [8].

The following variables were used to process the responses:

The variable $T = \{Tourist, Native\}$ is defined as an answer to Question 2.

The variable $M = \{Religion,\ Cultural,\ Economic, Family\ tradition\}$ is defined as responses to Question 3.

The variable $R = \{Catholic,\ Protestant,\ Native, Others\}$ is defined as an answer to Question 4.

The variable $m = \{Yes,\ No,\ I\ don't\ know\}$, is defined as an answer to Question 5.

The variable $P = \{Yes,\ No,\ I\ don't\ know\}$, is defined as an answer to Question 6.

Let us start with $a_k = b_k = c_k = 0$

The following IF-THEN rules below are used for processing the data:

R1: IF $T = Tourist$ AND $M = Cultural,\ Economic$ THEN add 1 to a_k. **R2**: IF $T = Tourist$ AND $m = No$ THEN add 1 to c_k. **R3**: IF $T = Tourist$ AND $M = Religion,\ Tradition$ THEN add 1 to b_k. **R4**: IF $T = Native$ AND $M = Cultural,\ Economic$ THEN add 1 to b_k. **R5**: IF $T = Native$ AND $M = Religion,\ Family\ tradition$ THEN add 1 to a_k. **R6**: IF There is more than one answer of R THEN add 1 to b_k. **R7**: IF $R = Catholic,\ Protestant$ AND $P = Yes$ THEN add 1 to b_k. **R8**: IF $m = I\ don't\ know$ THEN add 1 to c_k. **R9**: IF $P = I\ don't\ know$ THEN add 1 to c_k. **R10**: IF $M \neq Religion$ AND $P = Yes$ THEN add 1 to b_k.

The explanation of the rules above is as follows:

R1: It makes sense that tourists go to the Pampay Mass for cultural or economic reasons.

R2: The tourist who goes to the ritual and does not know its meaning denotes ignorance.

R3: Complementing R1, the tourist who goes to the Pampay Mass for religious or traditional reasons contradicts himself/herself because this is a local ceremony, which the tourist would not appreciate beyond the cultural motivation.

R4: This is the rule that complements R1.

R5: This is the rule that complements R3.

R6: The religions that are shown as an answer are contradictory to each other in their foundations, if more than one of them is marked it denotes a contradiction.

R7: If the person professes a Christian religion (Catholic or Protestant) and thinks that this pre-Hispanic rite will bring prosperity thanks to a non-Christian God, then this is contradictory.

In the case of the natives, we do not consider it contradictory because these are syncretized gods for them.

R8 and **R9**: Obviously a response of I don't know denotes ignorance.

R10: If the person claims to have come for reasons unrelated to religion and thinks that the ritual will have welfare effects on his/her life due to the powers of the god of the mountain, then this contradicts that he/her did not come for religious reasons.

These values were calculated for all respondents in S and aggregated. In practice, we obtained 184 opinions from those surveyed [13]. The following Refined Neutrosophic Numbers were obtained for each of them:

$\bar{N}_k = \bar{a}_k + \bar{b}_k I_1 + \bar{c}_k I_2$, where $\bar{a}_k = \frac{a_k}{2}$, $\bar{b}_k = \frac{b_k}{5}$, and $\bar{c}_k = \frac{c_k}{3}$, this guarantees that $\bar{a}_k, \bar{b}_k, \bar{c}_k \in [0, 1]$.

Neutrosophic Numbers were obtained with the help of the following formula:

$\bar{\bar{N}} = \bar{\bar{a}} + \bar{\bar{b}} I_1 + \bar{\bar{c}} I_2$, where $\bar{\bar{a}}$ is the mean of the $\bar{a}_k$s, $\bar{\bar{b}}$ is the mean of the $\bar{b}_k$s, and $\bar{\bar{c}}$ is the mean of the $\bar{c}_k$s.

The result was $\bar{\bar{N}} = 0.64809 + 0.29428 I_1 + 0.082376 I_2$.

See Figure 1, where this result is graphically represented [18].

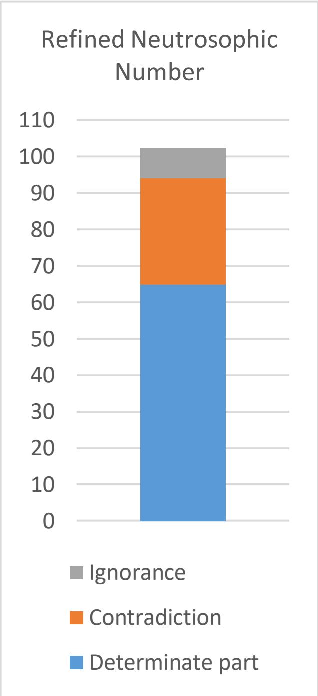

Figure 1: Neutrosophic Chart Graph representing the Refined Neutrosophic Number obtained from the survey in percent. The determined part is represented in blue, in orange it is the indeterminate part due to contradiction, and in gray it is the indeterminate part due to ignorance.

Let us note that in Figure 1 it was considered $I_1 = I_2 = [0, 1]$ for the graphic representation.

Additionally, three anthropologists, familiar with this ceremony and with scientific knowledge of its meanings, were surveyed. This guarantees having expert opinions on the subject to complement the results obtained previously. We believe that the results will be more indeterminate, but more accurate.

They were asked on a scale of 0 to 100 the following three questions:

1. In what percentage do you consider that the people who participate in the Pampay Mass are coherent with their philosophy of life?
2. In what percentage do you consider that the people who participate in the Pampay Mass contradict their own religious beliefs above all?
3. In what percentage do you consider that the people who participate in the Pampay Mass do not know at all the characteristics and history of this ritual?

The results obtained were the following, after dividing the Refined Neutrosophic Numbers by 100:

Expert 1: $N_1 = 0.75 + 0.24I_1 + 0.1I_2$,

Expert 2: $N_2 = 0.7 + 0.2I_1 + 0.1I_2$,

Expert 3: $N_3 = 0.7 + 0.25I_1 + 0.05I_2$,

We find the average of the results of the 3 experts with the results of the interviews and we have the following Refined Neutrosophic Number:

$N_m = 0.71667 + 0.21333I_1 + 0.083333I_2$

This Refined Neutrosophic Number is graphed with a Neutrosophic Chart Graphic in Figure 2 [18].

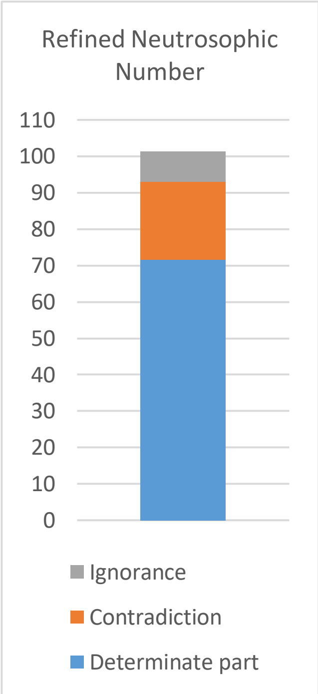

Figure 2: Neutrosophic Chart Graphic of the Refined Neutrosophic Number N_m in percent. The determined part is represented in blue, the indeterminate part due to contradiction is represented in orange, and in gray it is the indeterminate part due to ignorance.

Conclusion

Pampay Mass is a traditional religious ritual on the top of the Snow-Capped Mountain in Peru. This receives national and foreign tourists, as well as the natives of the region who wait all a year to celebrate this important event. In this anthropological article, we show that from a sociological point of view, this is an example that reflects the existence of *(t,i,f)*-Neutrosophic social structures in real-life. Specifically, we use the concept of Refined I-Neutrosophic Structures with the structure $N = a + bI_1 + cI_2$, where a is the determined part, bI_1 is the indeterminate part that means contradiction, and cI_2 is the indeterminate part that means ignorance or unknowing. To calculate the values of a, b, c we used a survey based on a non-probabilistic convenience sampling having 184 participants in the ritual, we designed rules to determine the coherence, contradiction, and ignorance of the respondents about the ritual. The results were that 64.809% are coherent, 29.428% have had contradictory beliefs, whereas 8.2376% are ignorant about the ritual. We also surveyed 3 expert anthropologists familiar with this ceremony. We calculated the mean of all the Refined Neutrosophic Numbers and we arrived at that approximately up to 69.952% of the participants in the Pampay Mass are consistent about their beliefs and this rite, up to 24.607% are not consistent, while up to 8.3094% participate without knowing anything about this ritual. This is a type of data processing with the help of Neutrosophic Statistics.

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The indefinite symbolic plithogenic integrals

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Abstract: In order to calculate the indefinite integrals of the symbolic plithogenic field, we used the substitution method, which was provided in this article. We also established a theorem that allowed us to locate the majority of the integrals for the symbolic plithogenic functions, in addition to the condition that must be met for the integration operation to be possible.

Keywords: symbolic plithogenic; division of symbolic plithogenic numbers; indefinite; integrals; substitution.

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1. Introduction and Preliminaries

To The genesis, origination, formation, development, and evolution of new entities through dynamics of contradictory and/or neutral and/or noncontradictory multiple old entities is known as plithogenic. Plithogeny advocates for the integration of theories from several fields.

We use numerous "knowledges" from domains like soft sciences, hard sciences, arts and literature theories, etc. as "entities" in this study, this is what Smarandache introduced, as he presented a study on plithogeny, plithogenic set, logic, probability, and statistics [2], in addition to presenting introduction to the symbolic plithogenic algebraic structures (revisited), through which he discussed several ideas, including mathematical operations on plithogenic numbers [1]. Also, an overview of plithogenic set and symbolic plithogenic algebraic structures was discussed by him [3]. It is thought that the symbolic n-plithogenic sets are a good place to start when developing algebraic extensions for other classical structures including rings, vector spaces, modules, and equations [4-5-6-7].

Alhasan also presented several papers on calculus, in which he discussed neutrosophic definite and indefinite integrals. He also presented the most important applications of definite integrals in neutrosophic logic [8-9].

Integration is important in human life, and one of its most important applications is the calculation of area, size and arc length. In our reality we find things that cannot be precisely defined, and that contain an indeterminacy part. This is the reason for studying neutrosophic integration and methods of its integration in this paper.

Smarandache presented the division operation in the symbolic plithogenic field as follows [1]:

Division of Symbolic Plithogenic Components

$$\frac{P_i}{P_j} = \begin{cases} x_0 + x_1P_1 + x_2P_2 + \cdots + x_jP_j + P_i & x_0 + x_1 + x_2 + \cdots + x_j = 0 \quad i > j \\ x_0 + x_1P_1 + x_2P_2 + \cdots + x_iP_i & x_0 + x_1 + x_2 + \cdots + x_i = 1 \quad i = j \\ \emptyset & i < j \end{cases}$$

where all coefficients $x_0, x_1, x_2, \dots, x_i, \dots \in$ SPS.

Division of Symbolic Plithogenic Numbers

Let consider two symbolic plithogenic numbers as below:

$$PN_r = a_0 + a_1P_1 + a_2P_2 + \cdots + a_rP_r$$

$$PN_s = b_0 + b_1P_1 + b_2P_2 + \cdots + b_sP_s$$

$$\frac{PN_r}{PN_s} = \begin{cases} none, one \;\; many & r \geq s \\ \emptyset & r < s \end{cases}$$

This paper dealt with several topics, in the first part of which introduction and preliminaries were presented, and in the main discussion part the indefinite symbolic plithogenic integrals. In the last part, a conclusion to the paper is given.

Main Discussion

The indefinite symbolic plithogenic integrals

Definition 1

Let $f: SPS \to SPS$ to evaluate $\int f(x, PN)dx$

where $PN = d_0 + d_1P_1 + d_2P_2 + \cdots + d_nP_n$

put: $x = g(u) \quad \Rightarrow \quad dx = \acute{g}(u)du$

by substitution, we get:

$$\int f(x, PN)dx = \int f(u)\acute{g}(u)du$$

then we can directly integral it.

Theorem 1

If $\int f(x, PN)dx = \varphi(x, PN)$, then:

$$\int PN_r f(PN_s x + PN_n)\, dx = \frac{PN_r}{PN_s}\varphi(PN_s x + PN_n) + PC$$

provided that $\frac{PN_r}{PN_s}$ is divisible.

where $PN_r = a_0 + a_1P_1 + a_2P_2 + \cdots + a_rP_r$, $PN_s = b_0 + b_1P_1 + b_2P_2 + \cdots + b_sP_s$, $PN_n = c_0 + c_1P_1 + c_2P_2 + \cdots + c_nP_n$ and $PC = c_0 + c_1P_1 + c_2P_2 + \cdots + c_rP_r \in SPS$ is symbolic plithogenic constant.

Proof:

put: $PN_s x + PN_n = u \quad \Rightarrow \quad PN_s dx = du$

$$\Rightarrow \quad dx = \frac{1}{PN_s} \; du$$

$$\int PN_r f(PN_s x + PN_n)\, dx = \int PN_r f(u) \frac{1}{PN_s} \; du$$

$$= \int \frac{PN_r}{PN_s} f(u)\, du$$

$$= \frac{PN_r}{PN_s} \varphi(u) + PC$$

back to the variable x, we get:

$$\int PN_r f(PN_s x + PN_n)\, dx = \frac{PN_r}{PN_s} \varphi(PN_s x + PN_n) + PC$$

Using the previous theorem, we get on:

1) $\int PN_r (PN_s x + PN_n)^n dx = \frac{PN_r}{PN_s} \frac{(PN_s x + PN_n)^{n+1}}{n+1} + PC$

2) $\int \frac{PN_r}{PN_s x + PN_n} dx = \frac{PN_r}{PN_s} ln|PN_s x + PN_n| + PC$

3) $\int PN_r e^{PN_s x + PN_n}\, dx = \frac{PN_r}{PN_s} e^{PN_s x + PN_n} + PC$

4) $\int \frac{PN_r}{\sqrt{PN_s x + PN_n}} dx = 2 \frac{PN_r}{PN_s} \sqrt{PN_s x + PN_n} + PC$

5) $\int PN_r cos(PN_s x + PN_n)\, dx = \frac{PN_r}{PN_s} sin(PN_s x + PN_n) + PC$

6) $\int PN_r sin(PN_s x + PN_n)\, dx = \frac{PN_r}{PN_s} cos(PN_s x + PN_n) + PC$

7) $\int PN_r sec^2(PN_s x + PN_n)\, dx = \frac{PN_r}{PN_s} tan(PN_s x + PN_n) + PC$

8) $\int PN_r csc^2(PN_s x + PN_n)\, dx = \frac{PN_r}{PN_s} cot(PN_s x + PN_n) + PC$

9) $\int PN_r sec(PN_s x + PN_n) \;\; tan(PN_s x + PN_n)\, dx = \frac{PN_r}{PN_s} sec(PN_s x + PN_n) + PC$

10) $\int PN_r csc(PN_s x + PN_n) \;\; cot(PN_s x + PN_n)\, dx = \frac{PN_r}{PN_s} csc(PN_s x + PN_n) + PC$

Example 1

1) $\int P_2 (P_1 x + 5 - 3P_1 + 4P_2)^5 \;\; dx = \frac{P_2}{P_1} \frac{(P_1 x + 5 - 3P_1 + 4P_2)^6}{6} + PC$

$$= (x_0 + x_1P_1 + P_2)\frac{(P_1x + 5 - 3P_1 + 4P_2)^6}{6} + PC$$

where:

$$\frac{P_2}{P_1} = x_0 + x_1P_1 + x_2P_2 \quad \Rightarrow \quad P_2 = x_0P_1 + x_1P_1 + x_2P_2$$

$$\Rightarrow \quad P_2 = (x_0 + x_1)P_1 + x_2P_2 \text{ , then:}$$

$$x_0 + x_1 = 0 \text{ and } x_2 = 1$$

hence: $\frac{P_2}{P_1} = x_0 + x_1P_1 + P_2$, where: $x_0 + x_1 = 0$

let's check the answer:

$$\frac{d}{dx}\left[(x_0 + x_1P_1 + P_2)\frac{(P_1x + 5 - 3P_1 + 4P_2)^6}{6} + PC\right] = 6P_1(x_0 + x_1P_1 + P_2)\frac{(P_1x + 5 - 3P_1 + 4P_2)^5}{6}$$

$$= (x_0P_1 + x_1P_1 + P_2)(P_1x + 5 - 3P_1 + 4P_2)^5$$

$$= \big((x_0 + x_1)P_1 + P_2\big)(P_1x + 5 - 3P_1 + 4P_2)^5$$

but we have: $x_0 + x_1 = 0$, then:

$$\frac{d}{dx}\left[(x_0 + x_1P_1 + P_2)\frac{(P_1x + 5 - 3P_1 + 4P_2)^6}{6} + PC\right] = P_2(P_1x + 5 - 3P_1 + 4P_2)^5$$

$= (The \ \ same \ \ integral \ \ function)$

2) $\int \frac{2P_1 + 3}{P_1x + 1 + 2P_1 - 7P_2 + 4P_5} dx = does \ \ not \ \ exist$

because:

$$\frac{2P_1 + 3}{P_1} = x_0 + x_1P_1$$

$$2P_1 + 3 = x_0P_1 + x_1P_1$$

$$2P_1 + 3 = (x_0 + x_1)P_1$$

then: $x_0 + x_1 = 2$, but we are not able to catch the free coefficient 1 from the left-hand side so:

$$\frac{2P_1 + 3}{P_1} = (does \ \ not \ \ exist)$$

3) $\int \boldsymbol{e}^{P_3x-3+P_2} dx = does \ \ not \ \ exist$

because: $\frac{1}{P_3} = (does \ \ not \ \ exist)$

4) $\int 6P_4cos\big((P_4 - 3P_2)x + 2 - P_3 + P_4\big) dx = \frac{6P_4}{P_4 - 3P_2} sin\big((P_4 - 3P_2)x + 2 - P_3 + P_4\big) + PC$

$$= (x_0 + x_1P_1 + x_2P_2 - 3P_4)sin\big((P_4 - 3P_2)x + 2 - P_3 + P_4\big) + PC$$

where:

$$\frac{6P_4}{P_4-3P_2}=x_0+x_1P_1+x_2P_2+x_3P_3+x_4P_4$$

$$6P_4=(P_4-3P_2)(x_0+x_1P_1+x_2P_2+x_3P_3+x_4P_4)$$

$$6P_4=x_0P_4+x_1P_4+x_2P_4+x_3P_4+x_4P_4-3x_0P_2-3x_1P_2-3x_2P_2-3x_3P_3-3x_4P_4$$

$$6P_4=-3(x_0+x_1+x_2)P_2-3x_3P_3+(x_0+x_1+x_2+x_3-2x_4)P_4$$, then:

$$x_0+x_1+x_2=0\ ,\ x_3=0 \text{ and } x_4=-3$$

hence: $\frac{6P_4}{P_4-3P_2}=x_0+x_1P_1+x_2P_2-3P_4$, where: $x_0+x_1+x_2=0$

5) $\int (P_3+5P_2-5P_1+6)sec^2\big((P_3+5)x-8+4P_1-5P_2+3P_3\big)\,dx$

$$=\frac{P_3+5P_2-5P_1+6}{P_3+5}tan\big((P_3+5)x-8+4P_1-5P_2+3P_3\big)+PC$$

$$=\left(\frac{6}{5}+P_1+P_2-\frac{1}{30}P_3\right)tan\big((P_3+5)x-8+4P_1-5P_2+3P_3\big)+PC$$

where:

$$\frac{P_3+5P_2-5P_1+6}{P_3+5}=x_0+x_1P_1+x_2P_2+x_3P_3$$

$$P_3+5P_2-5P_1+6=(P_3+5)(x_0+x_1P_1+x_2P_2+x_3P_3)$$

$$P_3+5P_2-5P_1+6=x_0P_3+x_1P_3+x_2P_3+x_3P_3+5x_0+5x_1P_1+5x_2P_2+5x_3P_3$$

$P_3+5P_2-5P_1+6=5x_0+5x_1P_1+5x_2P_2+(x_0+x_1+x_2+6x_3)P_3$, then:

$$x_0=\frac{6}{5}\ ,\ x_1=1\ ,\ x_2=-1 \text{ and } x_3=-\frac{1}{30}$$

hence: $\frac{P_3+5P_2-5P_1+6}{P_3+5}=\frac{6}{5}+P_1+P_2-\frac{1}{30}P_3$

6) $\int P_2 csc(P_4x)\ cot(P_4x)\,dx = does\ \ not\ \ exist$

because: $\frac{P_2}{P_4}=(does\ \ not\ \ exist)$

7) $\int \frac{P_2}{\sqrt{P_2x+3-P_1}}dx$

$$=\frac{2P_2}{P_2}\sqrt{P_2x+3-P_1}+PC$$

$$=2(x_0+x_1P_1+x_2P_2)\sqrt{P_2x+3-P_1}+PC$$

where:

$$\frac{2P_2}{P_2}=x_0+x_1P_1+x_2P_2$$

$$2P_2 = x_0P_2 + x_1P_2 + x_2P_2$$

$$2P_2 = (x_0 + x_1 + x_2)P_2$$

then: $x_0 + x_1 + x_2 = 2$

hence: $\frac{2P_2}{P_2} = x_0 + x_1P_1 + x_2P_2$, where: $x_0 + x_1 + x_2 = 2$

Theorem 2

Let $f: SPS \to SPS$, then:

$$\int \frac{\acute{f}(x,PN)}{f(x,PN)} dx = ln|f(x,PN)| + PC$$

Proof:

put: $f(x,PN) = u \quad \Rightarrow \acute{f}(x,PN)dx = du$

$$\Rightarrow dx = \frac{1}{\acute{f}(x,PN)} du$$

$$\Rightarrow dx = \frac{1}{\acute{u}} du$$

$$\int \frac{\acute{f}(x,PN)}{f(x,PN)} dx = \int \frac{\acute{u}}{u}\frac{1}{\acute{u}} du = \int \frac{1}{u} du = ln|u| + PC$$

back to the $f(x,PN)$, we get:

$$\int \frac{\acute{f}(x,PN)}{f(x,PN)} dx = ln|f(x,PN)| + PC$$

Example 2

1) $\int \frac{(3 + 2P_1 - 7P_2 + P_3 - 5P_4)x^7}{(3 + 2P_1 - 7P_2 + P_3 - 5P_4)x^8 + 8 - P_1} dx = \frac{1}{8} ln|(3 + 2P_1 - 7P_2 + P_3 - 5P_4)x^8 + 8 - P_1| + PC$

2) $\int \frac{(1 + 4P_1 - P_2)e^{(1+4P_1-P_2)x+2P_3}}{e^{(1+4P_1-P_2)x+2P_3} + 5P_4} dx = ln\left|e^{(1+4P_1-P_2)x+2P_3} + 5P_4\right| + PC$

3) $\int (P_5 + 3)\, tan(P_4 + 1)x\, dx = (P_5 + 3)\int \frac{sin(P_5 + 1)x}{cos(P_5 + 1)x} dx = \frac{P_5 + 3}{P_4 + 1} ln|(P_4 + 1)x| + PC$

$$= (3 - P_5) ln|cos(P_4 + 1)x| + PC$$

where:

$$\frac{P_5 + 3}{P_4 + 1} = x_0 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4 + x_5P_5$$

$$P_5 + 3 = (P_4 + 1)(x_0 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4 + x_5P_5)$$

$$P_5 + 3 = x_0P_4 + x_1P_4 + x_2P_4 + x_3P_4 + x_4P_4 + x_5P_5 + x_0 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4 + x_5P_5$$

$P_5 + 3 = x_0 + x_1P_1 + x_2P_2 + x_3P_3 + (x_0 + x_1 + x_2 + x_3 + 2x_4)P_4 + (2x_5)P_5$, then:

$$x_0 = 3 \ , x_1 = 0 \ , x_2 = 0 \ , x_3 = 0 \ , x_4 = \frac{-3}{2} \ , x_5 = \frac{1}{2}$$

hence: $\frac{P_5+3}{P_4+1} = 3 - \frac{3}{2}P_4 + \frac{1}{2}P_5$

4) $\int \frac{-7P_4}{1+tan(P_3x)}dx = \int \frac{-7P_4}{1+\frac{sin(P_3x)}{cos(P_3x)}}dx$

$$= \frac{1}{2}\int \frac{-14P_4\, cos(P_3x)}{cos(P_3x)+sin(P_3x)}dx$$

$$= \frac{-7P_4}{2P_3}\int \frac{cos(P_3x)+sin(P_3x)+cos(P_3x)-sin(P_3x)}{cos(P_3x)+sin(P_3x)}dx$$

$$= \frac{-7P_4}{2P_3}\int dx + \frac{-7P_4}{2P_3}\int \frac{cos(P_3x)-sin(P_3x)}{cos(P_3x)+sin(P_3x)}dx$$

$$= \left(x_0 + x_1P_1 + x_2P_2 + x_3P_3 - \frac{7}{2}P_4\right)x + \left(x_0 + x_1P_1 + x_2P_2 + x_3P_3 - \frac{7}{2}P_4\right)ln|cos(P_3x)+sin(P_3x)| + PC$$

where:

$$\frac{-7P_4}{2P_3} = x_0 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4$$

$$-7P_4 = (2P_3)(x_0 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4)$$

$$-7P_4 = 2x_0P_3 + 2x_1P_3 + 2x_2P_3 + 2x_3P_3 + 2x_4P_4$$

$-7P_4 = 2(x_0 + x_1 + x_2 + x_3)P_3 + 2x_4P_4$, then:

$x_0 + x_1 + x_2 + x_3 = 0$ and $x_4 = -\frac{7}{2}$

hence: $\frac{6P_4}{P_4-3P_2} = x_0 + x_1P_1 + x_2P_2 + x_3P_3 - \frac{7}{2}P_4$, where: $x_0 + x_1 + x_2 + x_3 = 0$

Theorem 3

Let $f: SPS \rightarrow SPS$, then:

$$\int \frac{\acute{f}(x,PN)}{\sqrt{f(x,PN)}}dx = 2\sqrt{f(x,PN)} + PC$$

Proof:

put: $f(x,PN) = u$ $\Rightarrow \acute{f}(x,PN)dx = du$

$$\Rightarrow dx = \frac{1}{\acute{f}(x,PN)}\ du$$

$$\Rightarrow dx = \frac{1}{\acute{u}}\ du$$

$$\int \frac{\acute{f}(x,PN)}{\sqrt{f(x,PN)}}dx = \int \frac{\acute{u}}{\sqrt{u}}\frac{1}{\acute{u}}du = \int \frac{1}{\sqrt{u}}du = 2\sqrt{u} + PC$$

back to $f(x,PN)$, we get:

$$\int \frac{\acute{f}(x,PN)}{\sqrt{f(x,PN)}}dx = 2\sqrt{f(x,PN)} + PC$$

Example 3

1) $$\int \frac{(5+P_1-4P_2+2P_3)x-7P_2}{\sqrt{(10+2P_1-8P_2+4P_3)x^2-28P_2x}}dx = \frac{-1}{2}\sqrt{(10+2P_1-8P_2+4P_3)x^2-28P_2x} + PC$$

2) $$\int \frac{(1+P_1)x^9}{\sqrt{(1+P_1)x^{10}-P_1+9P_2}}dx = \frac{2}{9}\sqrt{(1+P_1)x^{10}-P_1+9P_2} + PC$$

Theorem 4

$f: SPS \to SPS$, then:

$$\int [f(x,PN)]^n \acute{f}(x,PN)\ \ dx = \frac{[f(x,PN)]^{n+1}}{n+1} + PC$$

Proof:

put: $f(x,PN) = u$ $\quad\quad \Rightarrow \acute{f}(x,PN)dx = \ \ du$

$$\Rightarrow dx = \frac{1}{\acute{f}(x,PN)}\ \ du$$

$$\Rightarrow dx = \frac{1}{\acute{u}}\ \ du$$

$$\int [f(x,PN)]^n \acute{f}(x,PN)\ \ dx = \int u^n \acute{u}\frac{1}{\acute{u}}du = \int u^n du = \frac{u^{n+1}}{n+1} + PC$$

back to $f(x,PN)$, we get:

$$\int [f(x,PN)]^n \acute{f}(x,PN)\ \ dx = \frac{[f(x,PN)]^{n+1}}{n+1} + PC$$

Example 5

1) $$\int P_3x^3[(P_2+1)x^3]^4dx = \frac{1}{4}\int 4P_3x^3[(P_2+1)x^3]^4dx$$

$$= \frac{P_3}{P_2+1}\frac{[(3+2I_1+2I_2)x^3]^5}{5} + PC$$

$$= \frac{1}{2}P_3\frac{[(3+2I_1+2I_2)x^3]^5}{5} + PC$$

$$= \frac{1}{10}P_3[(3+2I_1+2I_2)x^3]^5 + PC$$

where:

$$\frac{P_3}{P_2+1} = x_0 + x_1P_1 + x_2P_2 + x_3P_3$$

$$P_3 = (P_2+1)(x_0 + x_1P_1 + x_2P_2 + x_3P_3)$$

$$P_3 = x_0P_3 + x_1P_3 + x_2P_3 + x_3P_3 + x_0 + x_1P_1 + x_2P_2 + x_3P_3$$

$$P_3 = x_0 + x_1P_1 + x_2P_2 + (x_0 + x_1 + x_2 + 2x_3)P_3 \text{ , then:}$$

$$x_0 = 0 \ \ , x_1 = 0 \ \ , x_2 = 0 \ \ , x_3 = \frac{1}{2}$$

hence: $\frac{P_3}{P_2+1} = \frac{1}{2}P_3$

$$2) \int \frac{P_2}{\sqrt{P_1x - 5P_1 + 8P_2}}\left(\sqrt{P_1x - 5P_1 + 8P_2}\right)^{11} dx$$

$$= \frac{P_2}{P_1}\frac{\left(\sqrt{(2 + I_1 + I_2)x - I_1 + 2I_2}\right)^{12}}{12} + PC$$

$$= (x_0 + x_1P_1 + P_2)\frac{\left(\sqrt{(2 + I_1 + I_2)x - I_1 + 2I_2}\right)^{12}}{12} + PC$$

where:

$$\frac{P_2}{P_1} = x_0 + x_1P_1 + x_2P_2 \quad \Rightarrow \quad P_2 = x_0P_1 + x_1P_1 + x_2P_2$$

$$\Rightarrow \quad P_2 = (x_0 + x_1)P_1 + x_2P_2 \text{ , then:}$$

$$x_0 + x_1 = 0 \text{ and } P_2 = 1$$

hence: $\frac{P_2}{P_1} = x_0 + x_1P_1 + P_2$, where: $x_0 + x_1 = 0$

5. Conclusions

In this paper, we discussed integrations in the symbolic plithogenic field, where we presented direct methods for solving most integrations of symbolic plithogenic functions, and we arrived at the condition that must be met in order for integration to be possible.

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Extraction of Knowledge from Uncertain Data Employing Weighted Bipolar and Neutrosophic Soft Sets

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Abstract: The discovery of soft sets is accredited to Molodtsov. This theory can cope with difficult circumstances with a lot of ambiguity, like those where deciding is hard. The bipolar soft set (BSS) and neutrosophic soft set (NSS) are algebraic models that can be viewed as soft set expansions. The BSS theory states that we weigh the pros and cons when deciding and NSS theory can handle belief system ambiguity, contradiction, and lack of knowledge due to its truth and falsity membership values. The concept of BSS and NSS are explained in comprehensive detail in this article. This article examined the weighted bipolar soft set (WBSS) and the weighted neutrosophic soft set (WNSS), as well as how to make accurate decisions under uncertain or inadequate information. A detailed comparison of information extraction approaches using weighted bipolar and neutrosophic soft sets may be lacking in the literature. These strategies may have been studied separately, but there may be little research comparing their performance under different settings and with diverse data. Filling this gap with a thorough and rigorous comparison study would help comprehend these techniques' practical benefits and drawbacks.

Keywords: Decision making problem, Soft set, Neutrosophic soft set, Bipolar soft set, Weighted Neutrosophic Soft Set, Weighted bipolar soft set, Uncertain data.

1. Introduction

This research is motivated by the increasing prevalence of indeterminate, imprecise, and uncertain data in our data-driven society. In disciplines varying from healthcare and finance to environmental science and decision support, traditional approaches to data analysis frequently fall short of handling these complexities effectively. Weighted bipolar and neutrosophic soft sets can explicitly model and extract knowledge from dual viewpoint and indeterminate data, meeting a critical need for advanced tools to empower decision-makers with more comprehensive insights and support interdisciplinary research. The goal of this effort is to close the knowledge gap between theoretical developments and real-world applications, which will eventually improve our capacity to make intelligent decisions and gather insightful information from the ambiguous data environments of the modern world.

The novelty of this work is in its detailed comparison of WBSS and WNSS, both of which are employed for extracting information from uncertain data. This work provides new insights into the relative effectiveness of these two frameworks by systematically evaluating their strengths and weaknesses, allowing researchers and practitioners to choose the best methodology based on data uncertainty. As a result of this study, we now have a better knowledge of how these soft computing methods can be used in real life. This will help people make better decisions based on data in many different areas.

This study is necessary due to the ubiquitous prevalence of uncertainty in modern data-driven decision-making processes across multiple domains. Now more than ever, advanced techniques are needed to model and extract knowledge from uncertain data in the era of big data, when information frequently comprises inaccurate, inconsistent, and incomplete parts. Weighted bipolar soft sets and weighted neutrosophic soft sets offer intriguing paths for addressing this difficulty since they enable the explicit consideration of both positive and negative elements of uncertainty and indeterminacy. This study is vital for expanding our capacity to make informed decisions, manage risks, and extract valuable insights in settings where standard data analysis approaches fail to cope with the complexities of uncertain data. This research is crucial for advancing our ability to make informed decisions, manage risks, and extract valuable insights in context.

Through reading this article, we gained an understanding of the fundamental concepts and algorithms behind WBSS and WNSS, such as how these methodologies contribute to the process of decision-making in the face of ambiguous data and an example of this process. This paper demonstrates how we may obtain an accurate ranking order of items by assigning weights to each parameter in the ranking criteria. Within the scope of this research, a comparison study is carried out between WBSS and WNSS.

The area that needs more exploration is how to evaluate and quantify the uncertainty in the knowledge that has been retrieved. In what ways can we effectively express this uncertainty and what level of confidence can we place in the knowledge that is derived utilizing these soft set models. We might research how these soft set models can be modified for streaming or real-time data environments, in which data is continually incoming. What kinds of methods can be used for online learning.

This research is needed to handle today's data-driven world's growing uncertainty and imprecision. Diverse decision-makers struggle to choose acceptable methods to extract insight from such data. To clarify their benefits and applicability in diverse settings, weighted bipolar soft sets and neutrosophic soft sets must be systematically compared. This study helps decision support, risk assessment, and insights production in complicated, uncertain data by guiding uncertainty management.

The efficiency of WBSS and WNSS sets may rely on data features, hence this study might not be applicable to all uncertain data circumstances. The study might not have looked at all uncertainty modeling techniques, so it might not have included other useful methods for comparison. This research can improve decision-making by systematically comparing two prominent uncertainty modeling techniques, helping practitioners and researchers navigate uncertain data landscapes across domains.

The main objective of this work is to investigate knowledge extraction methodologies using WBSS and WNSS, conduct a comprehensive comparative analysis to assess their performance across diverse datasets and scenarios, identify their strengths and weaknesses in handling uncertainty, ambiguity, and imprecision in data, and evaluate their applicability in real-world decision-making and data analysis tasks.

While individual studies have explored these methodologies separately, there is a notable dearth of systematic and rigorous comparative analyses that assess their performance under varying conditions and with diverse datasets. Such a gap hinders a clear understanding of when and where each method excels, potentially limiting their practical utility. Addressing this gap is essential to provide researchers and practitioners with valuable insights into the relative strengths and weaknesses of these techniques, enabling informed choices for knowledge extraction in scenarios characterized by uncertainty, ambiguity, and imprecision in data.

A review of the literature on WBSS and WNSS is presented in section 2 of this article. The core principle, method, and decision-making example employing WBSS are described in Section 3. In

Section 4, we'll learn about the idea behind WNSS and the algorithm it employs to make decisions. The hypothesis for this investigation is presented in Section 5. In Section 6, the comparative research between the WBSS and the WNSS is discussed. The sensitivity analysis for each method is discussed in detail in Section 7. In Section 8, we present the results discussion, and in Section 9, we provide the summary and final thoughts.

2. Literature Review

Uncertainty management is a challenge for researchers and decision makers across all fields and scientific disciplines, from the fundamental to the managerial, social, and technological. To solve this issue, a great number of different initiatives have been started. Even though each method has its own set of advantages and has demonstrated its usefulness, the theory of soft sets, which was developed by Molodtsov generalizes fuzzy set and rough set techniques [1]. This makes it a significant development in this field. Soft sets have been provided with some procedures in [2]. Newly specified operations on soft sets are discussed in [3], along with some algebraic structures were considered related to these operations. Soft rings were introduced by Bera and Mahapatra [4], soft vector spaces by Faried et al. [5], soft graph representations by Ali et al. [6], soft topological spaces by Asaad et al. [7], soft intersection semigroups by Elavarasan et al. [8], soft lattice ordered sets by Kashif et al. [9], and a novel method to soft sets by Cagman and Eraslan [10]. Maji et al. were the ones who first began applying soft sets in the context of decision making [11]. Numerous writers have since added to the body of literature on the topic, such as extensive work regarding the implementations in the decision-making problem was conducted in [12].

Fuzzy soft set concept was discussed by El-Atik et al. [13]. The object parameter methodology was recommended in this article for use in the process of forecasting unseen data in imprecise fuzzy soft sets [14]. Yiarayong put forward the notion of bipolar-valued fuzzy sets [15]. Alqaraleh et al. discussed the bipolar fuzzy soft sets and use this recognition in a decision-making scenario [16]. Different approaches to introducing BSS were proposed by Deli and Karaaslan in 2020 [17], and subsequent work on bipolar soft groups was done by Karaaslan et al. [18]. You can look at these articles to learn more about the bipolarity in soft sets and related subjects, as well as see some examples of its practical applications [19-21].

Philosophically, Smarandache introduced the concept of a neutrosophic set (NS) for the first time [22]. A NS can be defined in terms of its truth-membership degree, indeterminacy-membership degree, or falsity-membership degree. This broadens the applicability of concepts like fuzzy set and interval-valued fuzzy set. The NS and the set theoretic operators need to be described to satisfy the requirements of a scientific or engineering investigation. Otherwise, it will be challenging to implement in the situations that occur. Thus, Smarandache proposed the SVNS concept. The set-theoretic operators and many different features of SVNS have been discussed in [23]. In a SVNS setting, these papers suggested a multi-attribute decision making (MADM) approach based on the correlation coefficient [24, 25]. By utilizing SVNS similarity measures, authors refined and expanded upon previous clustering and decision-making techniques [26, 27]. A novel SVNS similarity measure has been introduced and used to aid in decision-making [28].

TOPSIS technique to solve decision making problems on multi-attribute SVNS was expanded here [29]. To evaluate its subsethood [30], this paper developed a measure that was applied to MADM. Its relations were proposed by Latreche et al. [31], and their properties were explored. For the purposes of cluster analysis and MADM, Luo et al. devised a novel distance measure of SVNSs [32]. SVNS aggregation operators based on t-conorm, and t-norm were proposed by Rong et al. and applied in MADM [33]. Simplified neutrosophic sets and a cross-entropy aggregation algorithm were suggested in [34]. Broumi et al. offer single valued neutrosophic graphs in [35], while suggest bipolar single valued neutrosophic graphs in [36].

SVNS are suggested by [37], which combines the benefits of NS with those of soft sets. Based on SVNS, a few novel operators and a soft matrix have been specified by Broumi et al. [38]. Evaluation of Q-Neutrosophic soft expert set has been defined by Al-Hijjawi et al. [39]. Neutrosophic vague soft expert set theory was described in [40]. Currently, researchers are concentrating on developing and presenting theories for coping with ambiguity [41-42], elaborating those theories with relevant examples. Numerous researchers today are hard at work debating the veracity of Neutrosophy in decision issues, as the TOPSIS method and NSS are commonly used in finding solutions in the decision-making problems [43–44].

3. Weighted Bipolar Soft Set Theory

3.1. Soft Set Theory

Let Ġ represents the initial universe set and X represents the parameters that have been defined. Power set of Ġ is denoted by Ṕ(Ġ). A pair (L, X) is called a soft set over Ġ, where L is a mapping given by [1],

L: X → Ṕ (Ġ).

Here, L(û)(ϑ) = Ø if ϑ ∉ Ġ. As ϑ(û) is approximate function of the soft set (L, X) and the value is a set called ϑ-element of the soft set for all ϑ ∈ Ġ.

3.2. Bipolar Soft Set Theory

3.2.1. Definition

Let R_1 and R_2 are two nonempty subsets of R, as $R_1 \cup R_2 = R$ and $R_1 \cap R_2 = \emptyset$. Then, (Ƴ, Ɲ, R) is BSS over Ġ, where Ƴ and Ɲ are set valued mappings, where Ƴ: $R_1 \longrightarrow$P(Ġ), Ɲ: $R_2 \longrightarrow$P(Ġ) and Ƴ(û) ∩ Ɲ(Ƴ(û)) = ∅, where Ƴ: $R_1 \longrightarrow R_2$ is a bijective function [15].

3.2.2. Properties

1) Let ($Ƴ_1$, $Ɲ_1$, Ʀ) and ($Ƴ_2$, $Ɲ_2$, Ƙ) are two BSS. ($Ƴ_1$, $Ɲ_1$, Ʀ) is a bipolar soft subset of ($Ƴ_2$, $Ɲ_2$, Ƙ) if, Ʀ ⊆ Ƙ, along with ∀û ∈ ƶ, $Ƴ_1$(û) ⊆ $Ƴ_2$(û) and $Ɲ_2$(¬û) ⊆ $Ɲ_1$(¬û). We can write it as, ($Ƴ_1$, $Ɲ_1$, Ʀ) ⊆ ($Ƴ_2$, $Ɲ_2$, Ƙ) [16].
2) ($Ƴ_1$, $Ɲ_1$, Ʀ) and ($Ƴ_2$, $Ɲ_2$, Ƙ) are said to be equal if and only if ($Ƴ_1$, $Ɲ_1$, Ʀ) ⊆ ($Ƴ_2$, $Ɲ_2$, Ƙ) and ($Ƴ_2$, $Ɲ_2$, Ƙ) ⊆ ($Ƴ_1$, $Ɲ_1$, Ʀ). We can write it as, ($Ƴ_1$, $Ɲ_1$, Ʀ) = ($Ƴ_2$, $Ɲ_2$, Ƙ)
3) Let (Ƴ, Ɲ, Ʀ) is a BSS. Then, $(Ƴ, Ɲ, Ʀ)^c$ = ($Ƴ^c$, $Ɲ^c$, Ʀ) = {⟨ û, $Ƴ^c$(û) = X− Ƴ(û), $Ɲ^c$(û) = Ƴ− Ɲ(û)⟩}.
4) (Ƴ, Ɲ, Ʀ) is null, if ∀û ∈ ƶ, Ƴ(û) = ∅ and Ɲ(û) = Ġ. Defined as {⟨∅, Ġ, Ʀ⟩}.
5) (Ƴ, Ɲ, Ʀ) is absolute, if ∀û ∈ ƶ, Ƴ(û) = Ġ and Ɲ(û) = ∅. Defined as {⟨Ġ, ∅, Ʀ⟩}.

3.2.3. Tabular Representation of BSS

Let, Ġ = Universal set = {$ϑ_1$, $ϑ_2$, $ϑ_3$, $ϑ_4$, $ϑ_5$}

W = Set of parameters = {$û_1$, $û_2$, $û_3$, $û_4$}

Then, ¬W = {¬$û_1$, ¬$û_2$, ¬$û_3$, ¬$û_4$}

Ƴ($û_1$) = {$ϑ_1$, $ϑ_5$} Ɲ(¬$û_1$) = {$ϑ_2$, $ϑ_3$, $ϑ_4$}

Ƴ($û_2$) = {$ϑ_2$, $ϑ_4$} Ɲ(¬$û_2$) = {$ϑ_1$, $ϑ_3$, $ϑ_5$}

Ƴ($û_3$) = {$ϑ_3$, $ϑ_4$, $ϑ_5$} Ɲ(¬$û_3$) = {$ϑ_1$, $ϑ_2$}

Ƴ($û_4$) = {$ϑ_5$} Ɲ(¬$û_4$) = {$ϑ_1$, $ϑ_2$, $ϑ_3$, $ϑ_4$}

Here, BSS (Ƴ, Ɲ, Ʀ) represented by this table 1.

Table 1. BSS (Ƴ, Ɲ, Ɍ)

(Ƴ, Ɲ, Ɍ)	$\hat{u}_1$	$\hat{u}_2$	$\hat{u}_3$	$\hat{u}_4$
ϑ_1	1	-1	-1	-1
ϑ_2	-1	1	-1	-1
ϑ_3	-1	-1	1	-1
ϑ_4	-1	1	1	-1
ϑ_5	1	-1	1	1

Here, Table [1] represents BSS using equation (1). Where, $\xi_{\delta\tau}$ is the δ-th entry of the τ-th column of the table.

$$\xi_{\delta\tau} = \begin{cases} 1 & \text{if} \quad \vartheta_\delta \in Ƴ(\hat{u}_\tau) \\ 0 & \text{if} \quad \vartheta_\delta \in \dot{G} - \{Ƴ(\hat{u}_\tau) \cup Ɲ(\neg\hat{u}_\tau)\} \\ -1 & \text{if} \quad \vartheta_\delta \in Ɲ(\neg\hat{u}_\tau) \end{cases} \tag{1}$$

3.2.4. Algorithm

1) The BSS (Ƴ, Ɲ, W).
2) Enter the parameters that have been chosen. Ɍ ⊆ W.
3) Decision parameter $Ɗ_\delta$ calculated considering all the selected parameters for each row.

$$Ɗ_\delta = \sum_\tau \xi_{\delta\tau} \tag{2}$$

4) Find out φ, where; $Ɗ_\varphi = \max(Ɗ_\delta)$.
5) The best option available is the item denoted by ϑ_φ, if φ might take on more than one value, then the value of φ that is selected can be any one of them.

3.2.5. Example-1

Let's say a new client interested in purchasing a car from a selection of available cars. It's possible that he would choose the car that suits his requirements the best based on a set of criteria.
Let, $\dot{G} = \{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5, \vartheta_6\}$ a set of cars.

$W = \{\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4, \hat{u}_5, \hat{u}_6, \hat{u}_7, \hat{u}_8\}$ set of parameters.

As, $\hat{u}_1$= automated
$\hat{u}_2$= petrol car
$\hat{u}_3$= cheap
$\hat{u}_4$= comfortable seat
$\hat{u}_5$= air conditioning
$\hat{u}_6$= power windows
$\hat{u}_7$= remote start
$\hat{u}_8$= air bag

$\neg W = \{\neg\hat{u}_1, \neg\hat{u}_2, \neg\hat{u}_3, \neg\hat{u}_4, \neg\hat{u}_5, \neg\hat{u}_6, \neg\hat{u}_7, \neg\hat{u}_8\}$ = {Not automated, Not a patrol car, Not cheap, No comfortable seat, No air conditioning, No power windows, No remote start, No air bag}.

Let, $Ƴ(\hat{u}_1) = \{\vartheta_1, \vartheta_2, \vartheta_3\}$ $Ɲ(\neg\hat{u}_1) = \{\vartheta_4, \vartheta_5\}$
$Ƴ(\hat{u}_2) = \{\vartheta_3, \vartheta_4, \vartheta_5\}$ $Ɲ(\neg\hat{u}_2) = \{\vartheta_1\}$
$Ƴ(\hat{u}_3) = \{\vartheta_1, \vartheta_5\}$ $Ɲ(\neg\hat{u}_3) = \{\vartheta_2, \vartheta_3\}$

Ƴ(û$_4$) = {ϑ$_1$, ϑ$_3$, ϑ$_5$} Ɲ(¬û$_4$) = {ϑ$_2$, ϑ$_6$}

Ƴ(û$_5$) = {ϑ$_2$, ϑ$_4$, ϑ$_5$} Ɲ(¬û$_5$) = {ϑ$_3$}

Ƴ(û$_6$) = {ϑ$_3$, ϑ$_5$, ϑ$_6$} Ɲ(¬û$_6$) = {ϑ$_1$}

Ƴ(û$_7$) = {ϑ$_2$, ϑ$_3$} Ɲ(¬û$_7$) = {ϑ$_5$, ϑ$_6$}

Ƴ(û$_8$) = {ϑ$_4$, ϑ$_5$, ϑ$_6$} Ɲ(¬û$_8$) = {ϑ$_3$}

1) Data entry for the BSS (Ƴ, Ɲ, W) should follow the table 2.
2) Assume, set of selected parameters by the client; R= {û$_1$, û$_2$, û$_4$, û$_5$, û$_6$, û$_8$}.
3) After determining the parameters to use, we can determine the value of the decision parameter Ɗ and then describe the BSS using those parameters in the manner given in table 3.
4) The value of Ɗ; Ɗ$_5$ = max Ɗ$_\delta$ = 4 and hence φ = 5.
5) According to the criteria that the client had chosen, the ϑ$_5$ or fifth car is the ideal one to recommend to the customer. If ϑ$_5$ is not accessible, then the client has the option of selecting either ϑ$_3$ or ϑ$_4$ as their replacement. The customer can choose any one of these two cars between the third and fourth car. In the situation that ϑ$_3$ and ϑ$_4$ are not available, then the choice will be made between ϑ$_2$ and ϑ$_6$.

Table 2. BSS (Ƴ, Ɲ, W)

(Ƴ, Ɲ, W)	û$_1$	û$_2$	û$_3$	û$_4$	û$_5$	û$_6$	û$_7$	û$_8$
ϑ$_1$	1	-1	1	1	0	-1	0	0
ϑ$_2$	1	0	-1	-1	1	0	1	0
ϑ$_3$	1	1	-1	1	-1	1	1	-1
ϑ$_4$	-1	1	0	0	1	0	0	1
ϑ$_5$	-1	1	1	1	1	1	-1	1
ϑ$_6$	0	0	0	-1	0	1	-1	1

Table 3. BSS (Ƴ, Ɲ, R)

(Ƴ, Ɲ, W)	û$_1$	û$_2$	û$_4$	û$_5$	û$_6$	û$_8$	Ɗ
ϑ$_1$	1	-1	1	0	-1	0	0
ϑ$_2$	1	0	-1	1	0	0	1
ϑ$_3$	1	1	1	-1	1	-1	2
ϑ$_4$	-1	1	0	1	0	1	2
ϑ$_5$	-1	1	1	1	1	1	4
ϑ$_6$	0	0	-1	0	1	1	1

This table reveals that some objects have the same decision value, making it impossible to rank them based on expert's values given to each parameter. ϑ$_5$ received the highest decision value, resulting in first position. ϑ$_3$ and ϑ$_4$ both had the same decision value of 2, making it impossible to decide which object is best. Similarly, ϑ$_2$ and ϑ$_6$ also had the same decision value of 1, making it impossible to determine which object is better. Here the ranking order of object is, ϑ$_5$ > ϑ$_3$ = ϑ$_4$ > ϑ$_2$ = ϑ$_6$ > ϑ$_1$.

3.3. Weighted Bipolar Soft Set Theory

3.3.1. Definition

The idea of WBSS is a hybridization of soft sets and weighted parameters of BSS. In the WBSS, certain weightages are assigned to parameters that are required for the decision-making process or that are selected for it. Because some of the features are more significant than others, it is necessary to provide higher priority to those characteristics while giving lower importance to the other criteria. When applied to a decision-making challenge, this strategy yields more precise results. These weights are assigned by the people who make decisions and vary from person to person. As a result, the decision that is made by each decision maker will be unique because not everyone's priorities are the same. For WBSS, the entries are determined by.

$$\Pi_{\delta\tau} = \begin{cases} \xi_{\delta\tau} \times \eta_{\tau} & \text{if } \xi_{\delta\tau} = 1 \\ 0 & \text{if } \xi_{\delta\tau} = 0 \\ \xi_{\delta\tau} \times (1 - \eta_{\tau}) & \text{if } \xi_{\delta\tau} = -1 \end{cases} \quad (3)$$

Where, $\xi_{\delta\tau}$ = entries in BSS (Ƴ, Ɲ, Ʀ).

The formula that is used to determine an object's weighted decision value is as follows:

$$Ɗ_{\delta} = \sum_{\tau} \Pi_{\delta\tau} \quad (4)$$

3.3.2. Algorithm

1) Enter Weighted Bipolar Soft Set (Ƴ, Ɲ, W).
2) Enter the parameters that have been chosen. Ʀ ⊆ W.
3) Based on the selected parameters, construct the WBSS (Ƴ, Ɲ, Ʀ) weighted table.
4) Weighted Decision parameter $Ɗ_{\delta}$, has been calculated considering all the selected parameters for each row.
5) Find out φ, where; $Ɗ_{\varphi} = \max(Ɗ_{\delta})$.
6) The best option available is the item denoted by ϑ_{φ}, if φ might take on more than one value, then the value of φ that is selected can be any one of them.

3.3.3. Flowchart of WBSS

Figure 1 shows the flowchart diagram of Weighted Bipolar Soft Set.

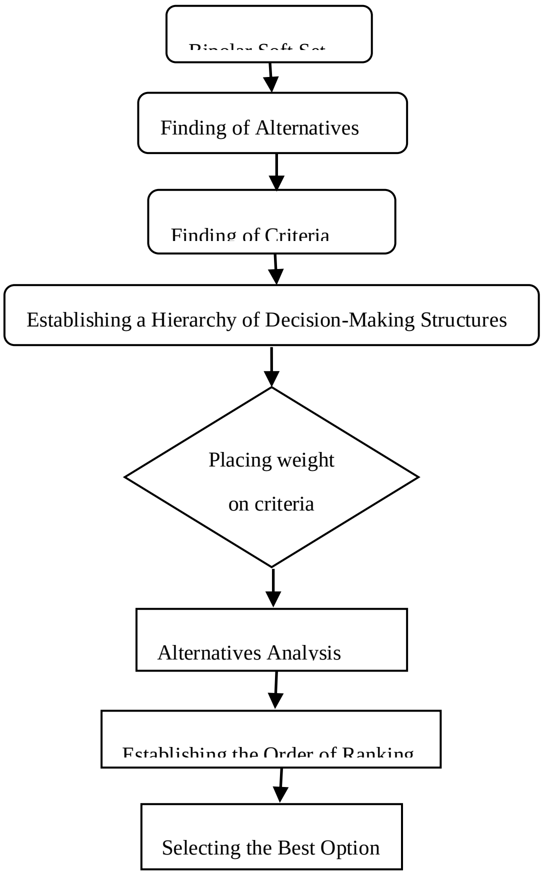

Figure 1. Flowchart of WBSS

3.3.4. Example

Let us assume example 1 to explain this algorithm for WBSS. Now employ this revised strategy to address the initial issue. Start the updated algorithm's third step after giving the parameters weights based on priority.

R = {$\hat{u}_1$, $\hat{u}_2$, $\hat{u}_4$, $\hat{u}_5$, $\hat{u}_6$, $\hat{u}_8$}

Weight of $\hat{u}_1$: $\eta_1 = 0.9$

Weight of $\hat{u}_2$: $\eta_2 = 0.7$

Weight of $\hat{u}_4$: $\eta_4 = 0.8$

Weight of $\hat{u}_5$: $\eta_5 = 0.7$

Weight of $\hat{u}_6$: $\eta_6 = 0.5$

Weight of $\hat{u}_8$: $\eta_8 = 0.9$

Table 4. WBSS (Ƴ, Ɲ, R)

(Ƴ, Ɲ, R)	**$\hat{u}_1$**	**$\hat{u}_2$**	**$\hat{u}_4$**	**$\hat{u}_5$**	**$\hat{u}_6$**	**$\hat{u}_8$**	**Ɗ**
ϑ_1	0.9	0.3	0.8	0	0.5	0	2.5
ϑ_2	0.9	0	0.2	0.7	0	0	1.8
ϑ_3	0.9	0.7	0.8	0.3	0.5	0.1	3.3
ϑ_4	0.1	0.7	0	0.7	0	0.9	2.4
ϑ_5	0.1	0.7	0.8	0.7	0.5	0.9	3.7
ϑ_6	0	0	0.2	0	0.5	0.9	1.6

Table 4 represents WBSS (Y, Ɲ, Ɍ) including weightage of each parameter and calculated the decision parameter $Ɗ_δ$. Max($Ɗ_δ$) = $Ɗ_5$ = 3.7 and hence φ = 5. From the table, $ϑ_5$ or the fifth car is the greatest possible selection object, that car is the best option for the consumer according to his priorities. In the event, if the fifth vehicle is not accessible, then the third one $ϑ_3$ will be selected as the alternative. If option 3 is unavailable, the customer will select $ϑ_1$ followed by $ϑ_4$. The ranking order of object is, $ϑ_5 > ϑ_3 > ϑ_1 > ϑ_4 > ϑ_2 > ϑ_6$.

After considering these two options side by side, the fifth car is the best one to buy for that client. According to BSS, if there is no fifth car available, the customer has the option of selecting either the third or the fourth car. However, according to WBSS, if the fifth one is not available, the customer should purchase the third one instead. From the WBSS table, we were able to determine the ranking order of items based on the values that experts had assigned to each parameter, and now we can choose which one is the most suitable.

4. Weighted Neutrosophic Soft Set Theory

4.1. Neutrosophic Soft Set Theory

4.1.1. Definition

Neutrosophic soft set (NSS) (L, X) over Ġ is defined by a mapping [23], L: X → P(Ġ);

Here, L = Approximate function of the NSS(L, X).

(L, X) = {û, ⟨ ϑ, T_L(ϑ), I_L(ϑ), F_L(ϑ)⟩: ϑ ∈ Ġ and û ∈ X}

And, Power set of Ġ is denoted by Ṕ(Ġ).

T_L(ϑ), I_L(ϑ), F_L(ϑ) ∈ [0, 1], are the truth-membership, indeterminacy-membership, and falsity-membership function respectively. Supremum of each T, I, F is 1 so, $0 \le T_L(ϑ) + I_L(ϑ) + F_L(ϑ) \le 3$. A statement or a neutrosophic term describes each of the parameters.

4.1.2. Properties

1) Let (L, X) and (V, N) be two NSS. (L, X) is neutrosophic soft subset of (V, N) if
   (i) X ⊂ N
   (ii) $T_{L(û)}(ϑ) \le T_{V(û)}(ϑ)$, $I_{L(û)}(ϑ) \le I_{V(û)}(ϑ)$, $F_{L(û)}(ϑ) \ge F_{V(û)}(ϑ)$, ∀û ∈ X, ϑ ∈ Ġ.
   Symbolized (L, X) ⊂ (V, N).

   (L, X) is neutrosophic soft super set of (V, N) if (V, N) is neutrosophic soft subset of (L, X). Denoted (L, X) ⊃ (V, N) [24].
2) Equality of two NSSs can be written as, (L, X) = (V, N). If (L, X) ⊆ (V, N) and (L, X) ⊇ (V, N).
3) Let W = {$û_1$, $û_2$, $û_3$, $û_4$} set of parameters. The NOT set of W = ꓶW = {ꓶ$û_1$, ꓶ$û_2$, · · · ꓶ$û_π$}, where ꓶ$û_τ$ = not $û_τ$, ∀τ.
4) Complement of NSS = $(L, X)^c$ = (L^c, ꓶX), Where L^c : ꓶX → P(Ġ), with $T^c_{L(ϑ)} = F^c_{L(ϑ)}$, $I^c_{L(ϑ)} = I^c_{L(ϑ)}$, $F^c_{L(ϑ)} = T^c_{L(ϑ)}$.
5) A neutrosophic soft set (L, X) defined as empty or null, If $T_{L(û)}(ϑ) = 0$, $F_{L(û)}(ϑ) = 0$ and $I_{L(û)}(ϑ) = 0$, ∀ϑ ∈ Ġ, ∀û ∈ X [32].

4.1.3. Comparison Table

It is a table whose rows are objects $ϑ_1$, $ϑ_2$, …, $ϑ_ω$ and columns are parameters $û_1$, $û_2$, …, $û_π$. The entries $ϱ_{δτ}$ are calculated by, $ϱ_{δτ}$ = m + q – b. Where; m = Count of instances where $T_{ϑ(δ)}(û_τ)$ is greater than or equivalent to $T_{\vartheta(\varphi)}(\hat{u}_\tau)$, for $\vartheta_\delta \neq \vartheta_\varphi$, $\forall \vartheta_\varphi \in \dot{G}$, q = Count of instances where $I_{\vartheta(\delta)}(\hat{u}_\tau)$ is greater than or equivalent to $I_{\vartheta(\varphi)}(\hat{u}_\tau)$, for $\vartheta_\delta \neq \vartheta_\varphi$, $\forall \vartheta_\varphi \in \dot{G}$, b = Count of instances where $F_{\vartheta(\delta)}(\hat{u}_\tau)$ is greater than or equivalent to $F_{\vartheta(\varphi)}(\hat{u}_\tau)$, for $\vartheta_\delta \neq \vartheta_\varphi$, $\forall \vartheta_\varphi \in \dot{G}$ [45].

Decision value of an Object ϑ_δ, δ = {1, 2, …, ω} is $Ɗ_\delta$, where; $Ɗ_\delta = \sum_\tau \rho_{\delta\tau}$

4.1.4. Algorithm

1) The Neutrosophic Soft Set (L, X) should be entered.
2) Using the NSS (L, X), calculate the comparative matrix.
3) Analyze the value of $Ɗ_\delta$, ∀δ.
4) Calculate φ, where $Ɗ_\varphi = \max(Ɗ_\delta)$.
5) If φ has more than one value, then any one of ϑ_δ could be the preferable choice.

4.1.5. Example-2

Suppose there were five applicants for the teaching position who walked in for an interview. There are certain requirements or characteristics that must be fulfilled for a candidate to be considered for the position of teacher. The person responsible for making the decision or conducting the interview assigned a score to each criterion based on the candidate's performance. The top applicant was selected for the teaching position based on their score from the interview. In order to address the challenge of making decisions regarding NSS, the one above has been taken into consideration.

Let $\dot{G}$ is the universal set of candidates for teacher, $\dot{G} = \{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5\}$ and W is the set of parameters, $W = \{\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4, \hat{u}_5\}$

Where, $\hat{u}_1$ = Experience
$\hat{u}_2$ = Technical Skill
$\hat{u}_3$ = Behaviour
$\hat{u}_4$ = Communication skill
$\hat{u}_5$ = Punctuality

And, NSS(L, X) = {Experience = {<ϑ_1, 0.9, 0.3, 0.2>, <ϑ_2, 0.7, 0.5, 0.1>, <ϑ_3, 0.4, 0.1, 0.8>, <ϑ_4, 0.7, 0.5, 0.9>, <ϑ_5, 0.5, 0.4, 0.3>}, Technical Skill = {<ϑ_1, 0.8, 0.5, 0.3>, <ϑ_2, 0.5, 0.4, 0.7>, <ϑ_3, 0.9, 0.6, 0.3>, <ϑ_4, 0.4, 0.3, 0.4>, <ϑ_5, 0.6, 0.8, 0.2>}, Behavior = {<ϑ_1, 0.5, 0.7, 0.1>, <ϑ_2, 0.8, 0.6, 0.4>, <ϑ_3, 0.3, 0.1, 0.8>, <ϑ_4, 0.7, 0.5, 0.6>, <ϑ_5, 0.4, 0.3, 0.4>}, Communication skill = {<ϑ_1, 0.7, 0.3, 0.2>, <ϑ_2, 0.6, 0.8, 0.3>, <ϑ_3, 0.8, 0.4, 0.5>, <ϑ_4, 0.5, 0.3, 0.6>, <ϑ_5, 0.7, 0.4, 0.3>}, Punctuality = {<ϑ_1, 0.6, 0.4, 0.2>, <ϑ_2, 0.4, 0.5, 0.3>, <ϑ_3, 0.7, 0.4, 0.1>, <ϑ_4, 0.8, 0.7, 0.2>, <ϑ_5, 0.5, 0.6, 0.4>}}.

The tabular representation of NSS (L, X) is given in Table 5.

Table 5. NSS (L, X)

$\dot{G}$	$\hat{u}_1$	$\hat{u}_2$	$\hat{u}_3$	$\hat{u}_4$	$\hat{u}_5$
ϑ_1	(0.9, 0.3, 0.2)	(0.8, 0.5, 0.3)	(0.5, 0.7, 0.1)	(0.7, 0.3, 0.2)	(0.6, 0.4, 0.2)
ϑ_2	(0.7, 0.5, 0.1)	(0.5, 0.4, 0.7)	(0.8, 0.6, 0.4)	(0.6, 0.8, 0.3)	(0.4, 0.5, 0.3)
ϑ_3	(0.4, 0.1, 0.8)	(0.9, 0.6, 0.3)	(0.3, 0.1, 0.8)	(0.8, 0.4, 0.5)	(0.7, 0.4, 0.1)
ϑ_4	(0.7, 0.5, 0.9)	(0.4, 0.3, 0.4)	(0.7, 0.5, 0.6)	(0.5, 0.3, 0.6)	(0.8, 0.7, 0.2)
ϑ_5	(0.5, 0.4, 0.3)	(0.6, 0.8, 0.2)	(0.4, 0.3, 0.4)	(0.7, 0.4, 0.3)	(0.5, 0.6, 0.4)

Table 6 shows the comparative table for the above NSS (L, X), after calculating comparative value and decision value for each object.

Table 6: Comparative table of the NSS (L, X)

Ġ	û1	û2	û3	û4	û5	Decision value
ϑ_1	4	3	6	4	1	18
ϑ_2	7	-2	5	3	-1	12
ϑ_3	-3	5	-4	4	4	6
ϑ_4	4	-3	2	-3	6	6
ϑ_5	1	6	0	4	0	11

It is visible from the above table that the first applicant ϑ_1 received the highest decision value or score, which is 18. This is the primary reason why the first applicant is the most qualified individual to be appointed as a teacher. If applicant ϑ_1 is not present, the position will be given to candidate ϑ_2, who received the second highest score in the interview. Similarly, if the second applicant is absent, the fifth option, ϑ_5, will be selected.

According to the NSS table, some objects share the same decision value, hence a ranking based on the values assigned by experts to each attribute is impossible. Due to the limitations of the NSS table, we are unable to determine the ranking order of each object. The ranking order of object is, $\vartheta_1 > \vartheta_2 > \vartheta_5 > \vartheta_3 = \vartheta_4$.

4.2. Weighted Neutrosophic Soft Set Theory

4.2.1. Definition

The idea of WNSS is a hybridization of soft sets and weighted parameters of NSS. If a weight, which is a real positive integer greater than 1, is applied on the parameter of a NSS, then the set is referred to as being WNSS. The entries of WNSS [45];

$Å_{\eth\tau} = \eta_{\eth\tau} \times \varrho_{\eth\tau}$;

Where, $\eta_{\eth\tau}$ = Weight of each parameter.

$\varrho_{\eth\tau}$ = $\eth\tau$-th entry in the table of NSS.

We refer to (L, X^{η}) as the WNSS for the NSS (L, X) with weights η associated with the parameter û.

4.2.3. Algorithm

1) Enter Weighted Neutrosophic Soft Set (L, X^{η}).
2) Using the WNSS (L, X^{η}), calculate the comparative matrix.
3) Decision parameter $Ɗ_{\eth}$, has been calculated considering all the parameters for each row.
4) Find out φ, where; $Ɗ_{\varphi} = \max(Ɗ_{\eth})$.
5) The best option available is the item denoted by $Ɗ_{\varphi}$, if φ might take on more than one value, then the value of ð that is selected can be any one of them.

4.2.4. Flowchart of WNSS

Figure 2 shows the flowchart diagram of Weighted Neutrosophic Soft Set.

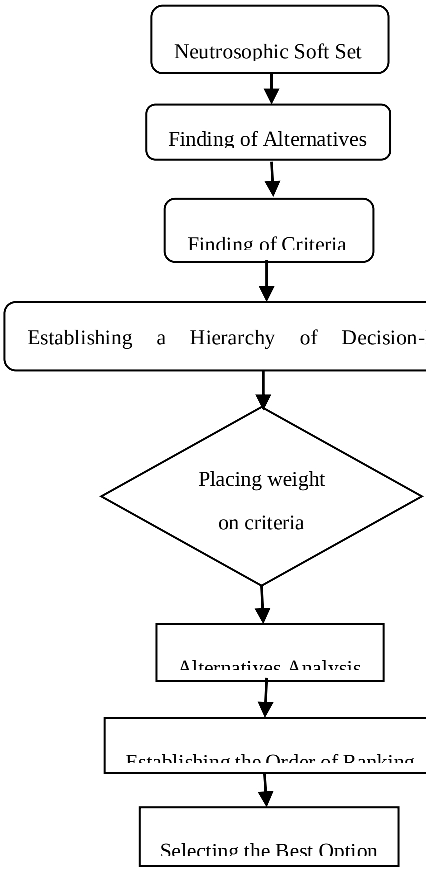

Figure 2. Flowchart of WNSS

4.2.5. Example

Let us consider example 2. Putting the weights on the parameters Experience, Technical Skill, Behavior, Communication skill, Punctuality the WNSS corresponding to the NSS (L, X) denoted by (L, X^{η}) and is given in the following table 7.

According to the decision maker or interviewer, each criterion or parameter was assigned a weight $(\eta)_j$; weight of parameters, for j= {1, 2, 3, 4, 5}.

Where, $(\eta)_1$ = Weight of Experience = 0.7

$(\eta)_2$ = Weight of Technical Skill = 0.9

$(\eta)_3$ = Weight of Behavior = 0.4

$(\eta)_4$ = Weight of Communication skill = 0.6

$(\eta)_5$ = Weight of Punctuality = 0.5

Table 7. WNSS (L, X^{η})

$\dot{G}$	$\hat{u}_1$	$\hat{u}_2$	$\hat{u}_3$	$\hat{u}_4$	$\hat{u}_5$
ϑ_1	(0.63, 0.21, 0.14)	(0.72, 0.45, 0.27)	(0.20, 0.28, 0.04)	(0.42, 0.18, 0.12)	(0.30, 0.20, 0.10)
ϑ_2	(0.49, 0.35, 0.07)	(0.45, 0.36, 0.63)	(0.32, 0.24, 0.16)	(0.36, 0.48, 0.18)	(0.20, 0.25, 0.15)
ϑ_3	(0.28, 0.07, 0.56)	(0.81, 0.54, 0.27)	(0.12, 0.04, 0.32)	(0.48, 0.24, 0.30)	(0.35, 0.20, 0.05)
ϑ_4	(0.49, 0.35, 0.63)	(0.36, 0.27, 0.36)	(0.28, 0.20, 0.24)	(0.30, 0.18, 0.36)	(0.40, 0.35, 0.10)
ϑ_5	(0.35, 0.28, 0.21)	(0.54, 0.72, 0.18)	(0.16, 0.12, 0.16)	(0.42, 0.24, 0.18)	(0.25, 0.30, 0.20)

Table 8 shows the comparative table for the above WNSS.

Table 8: Comparative table of WNSS (L, Xᶯ)

Ġ	$\hat{u}_1$	$\hat{u}_2$	$\hat{u}_3$	$\hat{u}_4$	$\hat{u}_5$	Decision value
ϑ_1	4	3	6	4	1	18
ϑ_2	7	-2	5	3	-1	12
ϑ_3	-3	5	-4	4	4	6
ϑ_4	3	-3	2	-3	6	5
ϑ_5	1	6	0	4	0	11

It is clear from the data presented in the chart that the first candidate, ϑ_1, was given the maximum possible score of 18, representing the best decision value. The position will be offered to candidate ϑ_2, who obtained the second highest score in the interview if applicant ϑ_1 is not present for the selection process. In a similar fashion, the fifth choice, which is designated by the letter ϑ_5, will be chosen if the second candidate is not present. In the NSS, we do not know the candidate ϑ_3 and ϑ_4's position or the number that the interviewer gave them. However, with the help of this WNSS, we were able to obtain precise information regarding the applicants ϑ_3 and ϑ_4 and their respective rankings. Therefore, if candidate ϑ_5 is not available, applicant ϑ_3 can be selected, and then ϑ_4 comes next.

Based on the values and weightage supplied to each parameter by the experts, we were able to establish the ranking order of items in the WNSS table and select the best option. The ranking order of object is, $\vartheta 1 > \vartheta 2 > \vartheta 5 > \vartheta 3 > \vartheta 4$.

5. Hypothesis

The incorporation of weighted attributes in bipolar soft sets enhances the accuracy and flexibility of knowledge representation and extraction in uncertain and imprecise data environments, leading to improved decision-making outcomes when compared to traditional bipolar soft sets that do not consider attribute weighting.

The introduction of attribute weighting in neutrosophic soft sets enhances the adaptability and effectiveness of knowledge extraction in contexts characterized by uncertainty and indeterminacy, resulting in superior decision support capabilities compared to traditional neutrosophic soft sets without attribute weighting.

6. Comparison study

6.1. Comparison of WBSS and WNSS

This comparative research presents an overview of the most important aspects, strengths, and problems of WBSS and WNSS. Table 9 shows the comparative analysis between WBSS and WNSS.

Table 9: Comparison analysis between WBSS and WNSS

Aspect	**WBSS**	**WNSS**
Definition	In WBSS, each element is associated with both a positive and a negative membership degree, along with a weight that indicates the strength or significance of that element.	In WNSS, each element is characterized by a degree of membership, non-membership, and indeterminacy, along with a weight that signifies the importance of that element.
Membership Interpretation	The positive and negative membership degrees represent the levels of acceptance and rejection of an element with respect to a certain property or concept. The weights provide a measure of the element's influence in the decision-making process.	The membership, non- membership and indeterminacy degrees capture the ambiguity and uncertainty in an element's classification into a particular category. The weight reflects the relative importance of the element's attributes.
Handling Uncertainty	This framework is effective in capturing uncertainty when there are conflicting opinions about an element's affiliation with a particular property. It accounts for both favorable and unfavorable viewpoints.	This framework is suitable for handling uncertainty in a scenario where the information about an element's membership, non-membership, or indeterminacy is incomplete or vague.
Decision-Making	The use of positive and negative membership degrees, along with weights, enables a comprehensive evaluation of elements considering both supportive and opposing characteristics. Elements with higher weights might have a stronger impact on decision outcomes.	The incorporation of weights can allow certain elements to carry more significance in decision-making processes. This prioritization can be based on the relative importance of elements in a specific context.

6.2. Comparison of BSS and WBSS

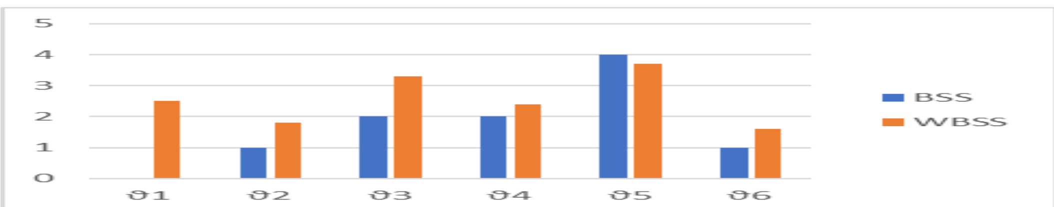

Figure 3. Ranking of objects' orders from our example using BSS and WBSS

Figure 3 shows a graph, that compares the results of BSS and WBSS approach in our example to rank the same set of items and demonstrate how their rankings change. Here, the x-axis represents the objects and the y-axis represents the decision values, with the graph displaying the decision values for each object. As can be seen, ϑ_5 is the superior option for both strategies, earning it number 1 in our rankings. However, ϑ_3 and ϑ_4 are ranked the same as rank 2. If for some reason ϑ_5 is not available, then we will have to settle with either ϑ_3 or ϑ_4 as our alternative. In a similar manner, ϑ_2 and ϑ_6 are placed in the same order inside the ranking. We are unable to conclude which option is preferable. However, with the help of WBSS, we were able to determine that ϑ_3 has higher priority than ϑ_4, and that ϑ_2 has higher priority than ϑ_6 by giving weightage to each parameter.

6.3. Comparison of NSS and WNSS

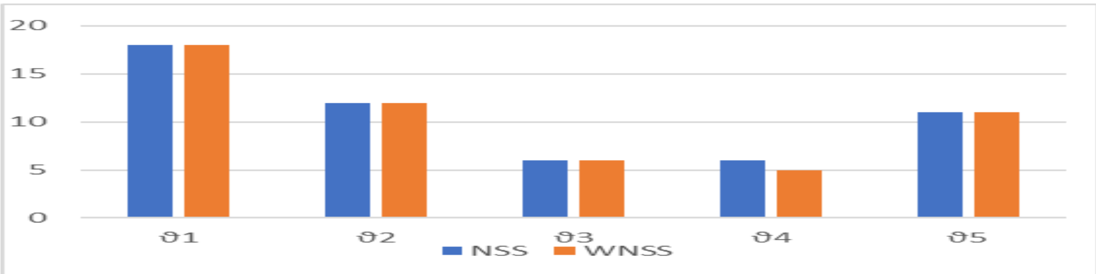

Figure 4. Ranking of objects' orders from our example using NSS and WNSS

Figure 4 shows the differences in ranking order of objects that we got from our example by applying NSS and WNSS approaches. The graph displays the choice values for each object, with the x-axis representing the objects and the y-axis representing the decision values. As we can see that, for both the approaches ϑ_1 is the best choice and got rank 1. Then ϑ_2 got the rank 2 and ϑ_5 got the rank 3. If in any situation ϑ_1 is not available, then we can go for ϑ_2 followed by ϑ_5. But ϑ_3 and ϑ_4 are in same ranking as rank 4. We can't find out which one is best out of ϑ_3 and ϑ_4. By using WNSS, we got that in between ϑ_3 and ϑ_4, ϑ_3 has more priority.

7. Sensitivity Analysis

The results were subjected to sensitivity analysis in order to verify their dependability and validity as well as to look at how they changed when certain inputs and parameters were changed. Approaches to decision-making sometimes involve defining certain criteria in a manner that is open to interpretation and is dependent on the decision-makers' perceptions of the situation as well as the degree to which environmental hazards are present. So, these factors change depending on the situation where the system for making decisions is being modeled. Here, WBSS and WNSS have undergone sensitivity analysis from the standpoint of parameter modifications. The sensitivity analysis that is going to be performed on WBSS is going to assess the effect that a change in parameters $\hat{u}_1$, $\hat{u}_2$, $\hat{u}_4$, $\hat{u}_5$, $\hat{u}_6$, and $\hat{u}_8$ will have on the evaluation of ranking orders of objects. The impact of a modification in parameters $\hat{u}_1$, $\hat{u}_2$, $\hat{u}_3$, $\hat{u}_4$, $\hat{u}_5$ on the assessment of object ranking orders will be examined through sensitivity analysis on WNSS. In this case, each variable is set according to the preferences of the experts. Therefore, multiple experiments were conducted with different values for these factors to demonstrate their significant impact on the final ranking order using WBSS and WNSS approaches.

7.1. Sensitivity Analysis of WBSS

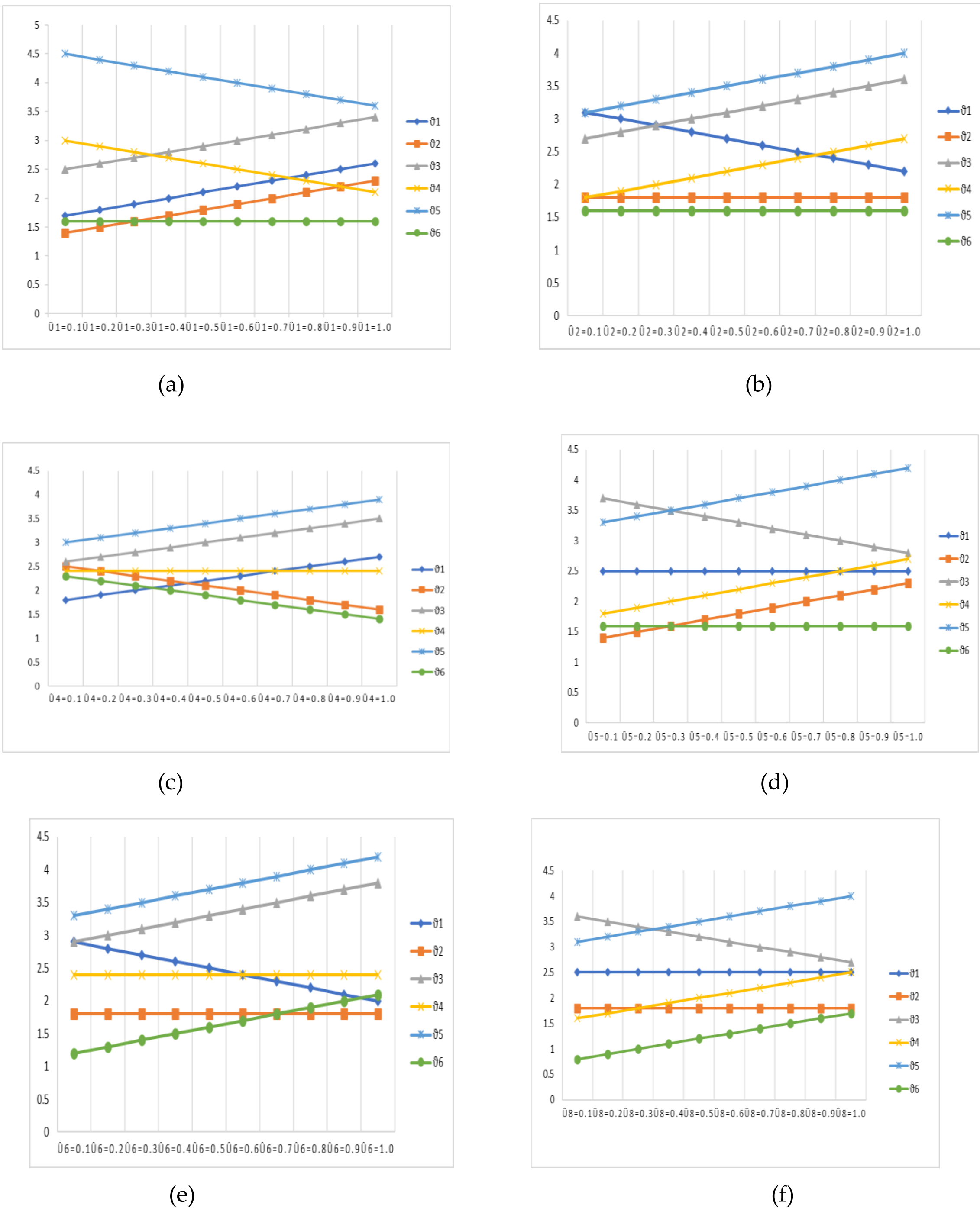

Figure 5. This figure shows the evaluation of ranking orders of object by changing the values of parameters during WBSS approach. Effect of each parameter on decision making result has been shown here; (a) Influence of parameter $\hat{u}_1$ on ranking order evaluation; (b) Influence of parameter $\hat{u}_2$ on ranking order evaluation; (c) Influence of parameter $\hat{u}_4$ on ranking order evaluation; (d) Influence of parameter $\hat{u}_5$ on ranking order evaluation; (e) Influence of parameter $\hat{u}_6$ on ranking order evaluation; (f) Influence of parameter $\hat{u}_8$ on ranking order evaluation.

By altering the values of parameters in our example, the effect on the evaluation of ranking orders of items through WBSS approach is illustrated in this graph. Specifically, the effect shows how changing these values in WBSS approach affects the ranking of the objects.

As can be seen in Figure 5(a), the significance of the parameter $\hat{u}_1$'s influence on the ranking evaluation was investigated by experimenting with a variety of different values for it, ranging from $\hat{u}_1 = 0.1$ to $\hat{u}_1 = 1.0$. The order of the ranking has not been affected in any way, even though the value of the $\hat{u}1$ parameter has been modified multiple times. Throughout the entirety of the sensitivity study and parameter value change $\hat{u}1$, option ϑ_5 has remained the most favorable choice, followed by option ϑ_3. ϑ_6, on the other hand, maintains its position as the lowest in the order despite the modification in the value of the parameter $\hat{u}_1$.

In a similar manner, the impact on the ranking orders of objects has been illustrated in figures 5(b), 5(c), 5(d), 5(e) and 5(f) by modifying the values of the parameters $\hat{u}_1$, $\hat{u}_2$, $\hat{u}_4$, $\hat{u}_5$, $\hat{u}_6$ and $\hat{u}_8$ accordingly to highlight the sensitivity analysis of the parameters on the ranking orders. As can be seen, ϑ_5 is the greatest option to select out of all the other possible things to go with, and ϑ_3 comes in second place. By modifying the parameter values of $\hat{u}_5$ and $\hat{u}_8$, we can observe that ϑ_3 is greater than ϑ_5 on occasion, but ϑ_5 is usually greater. In a similar vein, if we examine the least one, then we find that ϑ_6 is the one that has less decision worth in every circumstance. The parameter values used in WBSS's sensitivity study had no influence on the final rankings.

7.2. Sensitivity Analysis of WNSS

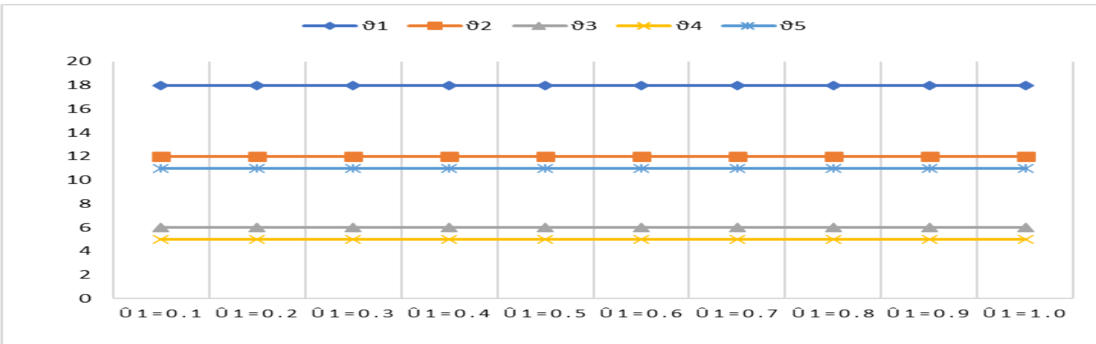

Figure 6. Evaluation of ranking orders of object by changing the values of parameters during WNSS approach.

This graph illustrates the effect that changing the values of the parameters in our example has on the evaluation of ranking orders of items using the WNSS technique. More specifically, the result shows how changing these values in the WNSS method changes how the items are ranked.

The relevance of the parameter $\hat{u}_1$'s impact on the ranking evaluation was studied by testing with a number of various values for it, ranging from $\hat{u}_1 = 0.1$ to $\hat{u}_1 = 1.0$. This was done using Figure 6, which displays the results of the experiment. In spite of the fact that the value of the $\hat{u}_1$ parameter has been altered on several occasions, the sequence in which the rankings are presented has not been altered in any manner at all. Option ϑ_1 has been determined to be the optimal selection throughout the whole of the sensitivity analysis and parameter value change $\hat{u}_1$, with option ϑ_2 coming in a close second. ϑ_4, on the other hand, remains in the position of having the lowest value in the order despite the fact that the value of the parameter $\hat{u}_1$ has been changed.

Changing the values of each parameter in the WNSS model from 0.1 to 1.0, as we did in the previous example, maintains the same order of ranking for the items in the model. The order of the rankings has not been altered in any way as a result of changing the values of any parameters. According to the results of the sensitivity study performed on WNSS, changing the values of the parameters does not affect the ranking orders in any way.

8. Discussion

We observed in BSS database that some items have the same decision value, making it difficult to rank them by expert parameter values. ϑ_5 placed top due to its highest decision value. It is impossible to choose between objects 3 and 4 because they both have a decision value of 2, making it impossible to choose which is the better option. It was also impossible to tell which item is better because ϑ_2 and ϑ_6 both had the same decision value of 1. WBSS recommends buying the third one if the fifth is unavailable. The WBSS table showed the ranking order of things based on experts' parameter values, so we may choose the best one.

The NSS table shows that applicant ϑ_1 had the highest decision value. In the absence of application ϑ_1, the position will be awarded to candidate ϑ_2, who scored second in the interview. The fifth option, ϑ_5, will be chosen if the second candidate is absent. NSS table restrictions prevent us from rating objects. The interviewer's number and position of candidates ϑ_3 and ϑ_4 are unknown in the NSS. Through WNSS, we were able to gather accurate information on candidates ϑ_3 and ϑ_4 and their rankings. Thus, if ϑ_5 is unavailable, ϑ_3 can be chosen, followed by ϑ_4. Based on the experts' parameter values and weightages, we ranked the WNSS table elements and chose the optimal choice.

In WBSS, the combination of positive and negative membership degrees with weights permits a full evaluation of items that takes into account both supportive and opposing qualities. It's possible that factors with higher weights will have a greater bearing on how the decision turns out. Because the WNSS incorporates weights, certain components of the decision-making process may be given the ability to have a greater bearing on the final outcome. This ranking might be done on the basis of the relative value of the components within a particular context.

9. Conclusion

In conclusion, our study has revealed that both weighted bipolar soft sets and weighted neutrosophic soft sets exhibit strengths and applicability in knowledge extraction from uncertain data, with their comparative performance contingent on specific data characteristics and task objectives. While weighted bipolar soft sets excel in scenarios necessitating strict consideration of positive and negative attributes, weighted neutrosophic soft sets offer flexibility to handle inherent data uncertainty. These findings provide valuable insights for decision support, pattern recognition, and data mining applications. Our research contributes to the field of soft computing by illuminating the strengths and weaknesses of these techniques, paving the way for future research on hybrid approaches and domain-specific refinements. Overall, these methodologies serve as versatile tools for navigating the intricacies of uncertain information, offering practitioners informed choices for knowledge extraction in uncertain environments.

In the future, researchers will investigate how deep learning and neural network models can be combined with WBSS and WNSS approaches to improve the ability to retrieve information. In complex datasets, detailed patterns may be easier to discern with the assistance of deep learning. Researchers may use this to tackle scalability problems and use knowledge extraction techniques in situations with massive datasets. They will also be able to develop strategies that efficiently manage enormous amounts of uncertain data. The Human Computer Interaction (HCI) field may be utilized in the future of research to create practitioner-friendly interfaces.

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Equitable Domination in Neutrosophic Graph Using Strong Arc

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Abstract: In this paper, we have initiated the study of domination and equitable domination of Neutrosophic graphs using strong arcs. Strong arcs represent the optimal (minimum) degree of truth membership value, the optimal (minimum) degree of indeterminacy membership value, and the non-optimal (maximum) degree of falsity membership value. Hence, the studies of domination and equitable domination using strong arcs have been explored. Upper bounds and minimality conditions for the existence of the introduced parameters were discussed. Extend the studies on strong and weak equitable domination of Neutrosophic graphs and obtain the relationship between domination and the equitable domination parameter. Furthermore, we have provided some theorems based on equitable domination of Neutrosophic graphs and discussed the upper and lower bounds of the strong and weak equitable domination in terms of order and size with other existing domination parameters of Neutrosophic graphs.

Keywords: Domination, Equitable Domination, Strong arc, Strong and weak

1. Introduction

In 1965[1], L.A. Zadeh put forth a mathematical framework to describe the occurrence of uncertainty in real-world circumstances. Rosenfeld[2] was the first to propose the concept of fuzzy graphs and different fuzzy analogues of connectedness in graph theory concepts. Berge and Ore[3] began studying the domination sets of graphs. Studies on paired domination were started by Teresa et al. [4]. Biggs [5] and V.R. Kulli [6] both contributed to the development of efficient domination, and he [7] also developed the theory of domination in graphs. Cockayne[8] employed the independent domination number for the first time in graphs. Swaminathan and Dharmalingam introduced equitable domination [9]. A. Meenakshi developed and explored paired equitable domination [10], and it was continued in an inflated graph and its graph complement [11, 12].

Nagoor Gani and M. Basheer Ahmed developed and investigated the concepts of Strong and Weak Domination of Fuzzy Graph[13]. K.T. Atanassov created intuitionistic fuzzy relations and intuitionistic fuzzy graphs (IFGS)[14]. A.Shannon and Atanassov of [15] and M.G. Karunambigai et al. [16] identified IFGS as a particular instance of IFG. A. Nagoor Gani and Shajitha Begum developed the words "order," "degree," and "magnitude" of IFG[17]. A.Nagoor Gani and S. Anu Priya developed split domination in intuitionistic fuzzy graphs [18] and the author studied Dombi Fuzzy Graphs [19].Mullai et al.[20] studied equitable domination parameter in neutrosophic graphs. In this paper we have developed the equitable domination parameter using strong arcs.

The motivation of this research is to study domination and equitable domination in neutrosophic graphs using strong arcs. In [18], the vertex cardinality and edge cardinality of the intuitionistic graphs in the study of split dominations were focused. This study motivates us to define the order, size, and degree of the vertex of a neutrosophic graph that is optimal while initiating studies of another domination parameter, named equitable domination of a neutrosophic graph using a strong arc. The study of weak and strong domination in fuzzy graphs [13] motivated us to focus our research on the strong and weak equitable domination of neutrosophic graphs using the score function. Section 2 focused on the preliminary work related to our studies. Section 3 explored the studies of domination in the neutrosophic graph using a strong arc, and the upper bounds are given in terms of the order and degree of the neutrosophic graph. Sections 4 and 5 focused on the domination parameter equitable domination using a strong arc; weak and strong equitable domination using a score function are illustrated with an example.

The existence of equitable domination in a neutrosophic graph is guaranteed on the degree of the vertex of the neutrosophic graph

2. Preliminaries

Definition 2.1[11].

An intuitionistic fuzzy graph(IFG) is of the form $G_{IF} = (A_{IF}, B_{IF})$ where A_{IF} is a finite vertex set such that (i) $\mu_1 : A_{IF} \to [0,1]\,; \gamma_1 : A_{IF} \to [0,1]$ denote the degree of truth membership value and degree of falsity membership value respectively and $0 \le \mu_1(v_s) + \gamma_1(v_s) \le 1$ for every $v_s \in V$.

(ii) $B_{IF} \subseteq A_{IF} \times A_{IF}$ *where* $\mu_2 : A_{IF} \times A_{IF} \to [0,1]$; $\gamma_2 : A_{IF} \times A_{IF} \to [0,1]$ are such that $\mu_2\{(a_i, a_j)\} \le \min\{\mu_1(a_i), \mu_1(a_j)\}$; $\gamma_2\{(a_i, a_j)\} \ge \max\{\gamma_1(a_i), \gamma_1(a_j)\}$ and where

$0 \le \mu_2\{(a_i, a_j)\}\} + \mu_2\{(a_i, a_j)\} \le 1 \ \forall (a_i, a_j) \in B_{IF}$.

Definition 2.2[11]. An arc (u_d, v_d) is said to be strong arc if

$\mu_2(a_i, a_j) = \min\{\mu_1(a_1), \mu_1(a_2)\}$ and $\gamma_2(a_i, a_j) = \max\{\gamma_1(a_1), \gamma_1(a_2)\}$

Definition 2.3[11]. The degree of a vertex u_d in an IFG,

$G_{IF} = (A_{IF}, B_{IF})$ is defined as the sum of the weight of the strong arcs incident at u_d and is denoted by deg(u_d). The neiborhood of u_d is denoted by

$N(u_d) = \{v_d \in A_{IF} / (u_d, v_d)$ is an strong arc}

The minimum degree of G_{IF} is $\delta(G_{IF}) = \min\{d_{G_{IF}}(u_d) / u_d \in A_{IF}\}$

The maximum degree of G_{IF} is $\Delta(G_{IF}) = \max\{d_{G_{IIF}}(u_d) / u_d \in A_{IF}\}$

Definition 2.4[11]. A vertex $u_d \in A_{IF}$ in an IFG, $G_{IF} = (A_{IF}, B_{IF})$ is said to be an isolate vertex if

$\mu_2(a_i, a_j) = 0$ and $\gamma_2(a_i, a_j) = 0$

Definition 2.5[11]. Let $G_{IF} = (A_{IF}, B_{IF})$ be a intuitionistic fuzzy graph. Then the cardinality of G_{NS} is defined

$$|G| = \left| \sum_{a_i \in A_{NS}} \frac{1 + T_{A_{NS}} - F_{A_{NS}}}{2} + \sum_{a_i a_j \in B_{NS}} \frac{1 + T_{B_{NS}} - F_{B_{NS}}}{2} \right|$$

Definition 2.6 [11]. Let $G_{IF} = (A_{IF}, B_{IF})$ be an intuitionistic fuzzy graph and let u_{if} and $v_{if} \in A_{IF}$, we say that u_{if} dominates u_{if} in G_{IF} if there exists a strong arc between them. A subset $D_d \subseteq A_{IF}$ is said to be dominating set in G_{IF} if for every $v_{if} \in A_{IF}$ - D_d, there exists $u_{if} \in D_d$ dominates v_{if}.

Definition 2.7[4].

Let X_{sp} be a space of points (objects) with generic elements in X_{sp} is denoted by X_{sp}. A single valued neutrosophic set A_{NS} (SVNS) is characterized by truth membership function $T_{A_{NS}}(X_{sp})$, an indeterminacy membership function $I_{A_{NS}}(X_{sp})$, and a falsity membership function $F_{A_{NS}}(X_{sp})$. For each point x' in X_{sp}, $T_{A_{NS}}(X_{sp})$, $I_{A_{NS}}(X_{sp})$, and $F_{A_{NS}}(X_{sp}) \in [0, 1]$. A SVNS A can be written as $A_{NS} = \{< x' : T_{A_{NS}}(x'), I_{A_{NS}}(x'), F_{A_{NS}}(x') >, x' \in X_{sp}\}$.

Definition 2.8[4]

Let $A_{NS} = (T_{A_{NS}}, I_{A_{NS}}, F_{A_{NS}})$ and $B_{NS} = (T_{B_{NS}}, I_{B_{NS}}, F_{B_{NS}})$ be single valued neutrosophic sets on a set X_{sp}. If $A_{NS} = (T_{A_{NS}}, I_{A_{NS}}, F_{A_{NS}})$ is a single valued neutrosophic relation on a set X_{sp}, then $A_{NS} = (T_{A_{NS}}, I_{A_{NS}}, F_{A_{NS}})$ is called a single valued neutrosophic relation on $B_{NS} = (T_{B_{NS}}, I_{B_{NS}}, F_{B_{NS}})$, if $T_{B_{NS}}(x', y') \le \min\{T_{A_{NS}}(x'), T_{A_{NS}}(y')\}$

$I_{B_{NS}}(x', y') \le \min\{I_{A_{NS}}(x'), I_{A_{NS}}(y')\}$, $F_{B_{NS}}(x', y') \ge \max\{F_{A_{NS}}(x'), F_{A_{NS}}(y')\}$ for all x', y' in X_{sp}.

Definition 2.9[4]. An arc (a_i, a_j) of a neutrosophic graph, $G_{NS} = (A_{NS}, B_{NS})$ is said to be strong if

$T_{E_s}\{(a_i, a_j)\} = \min\{T_{V_s}(a_i), T_{V_s}(a_j)\}$; $I_{E_s}\{(a_i, a_j)\} = \min\{I_{V_s}(a_i), I_{V_s}(a_j)\}$;

$F_{E_s}\{(a_i, a_j)\} = \max\{F_{V_s}(a_i), F_{V_s}(a_j)\}$ where $(a_i, a_j) \in E_s$ *and* $a_i \,\&\, a_j \in V_s$.

3. Domination of a Neutrosophic graph(NSG) Using Strong Arc

Definition 3.1. Let u_d be a vertex in a NSG, $G_{NS} = (A_{NS}, B_{NS})$. The degree of a vertex u_d is defined as the sum of the weight of the strong arcs incident at u_d and is denoted by deg (u_d). The neighbourhood of u_d is denoted by $N(u_d) = \{v_d \in A_{NS} / (u_d, v_d)$ is a strong arc$\}$

The minimum degree of G_{NS} is $\delta(G_{NS}) = \min\{d_{G_{NS}}(u_d) / u_d \in A_{NS}\}$

The maximum degree of G_{NS} is $\Delta(G_{NS}) = \max\{d_{G_{INS}}(u_d) / u_d \in A_{NS}\}$

- ❖ The degree of indeterminancy membership value (I) is not a complement of degree of truth membership value (T) and degree of falsity membership value (F) and the values of T, I, and F are independent of one another, value (I) does not depend on either the Truth (T) or Falsity (F) value.
- ❖ Despite the fact that the value of indeterminacy is unknown, we presume it by using 0.5 for both the possibilities of truth and falsity. This truthness makes our study more significant .Hence to attain feasibility, the order of G_{NS} is defined as follows

Definition 3.2. Let $G_{NS} = (A_{NS}, B_{NS})$ be a neutrosophic graph. Then order of G_{NS} is defined

$$|A_{NS}| = \left| \sum_{a_i \in A_{NS}} \frac{2 + T_{A_{NS}} - (0.5) I_{A_{NS}} - F_{A_{NS}}}{3} \right|$$

Definition 3.3. Let $G_{NS} = (A_{NS}, B_{NS})$ be a neutrosophic graph. Then size of G_{NS} is defined

$$|B_{NS}| = \left| \sum_{a_i \in A_{NS}} \frac{2 + T_{B_{NS}} - (0.5) I_{B_{NS}} - F_{B_{NS}}}{3} \right|$$

Definition 3.4. Let $G_{NS} = (A_{NS}, B_{NS})$ be a neutrosophic graph and let u_d, $v_d \in A_{NS}$, we say that u_d dominates v_d in G_{NS} if there exists a strong arc between them. A subset $D_{NS} \subseteq A_{NS}$ is said to be dominating set if for every $v_d \in A_{IF} - D_{NS}$ there exists at least one $u_d \in D_{NS}$ dominates v_d. The minimum cardinality of a dominating set is called a domination number and is denoted by $\gamma_{NG}(G_{NS})$.

Definition 3.6. A dominating set D_{NS} of A_{NS} is said to be a minimal if no proper subset of D_{NS} is a dominating set of G_{NS}.

Example 3.7: Let $G_{NS} = (A_{NS}, B_{NS})$ be the NSG shown in figure (1)

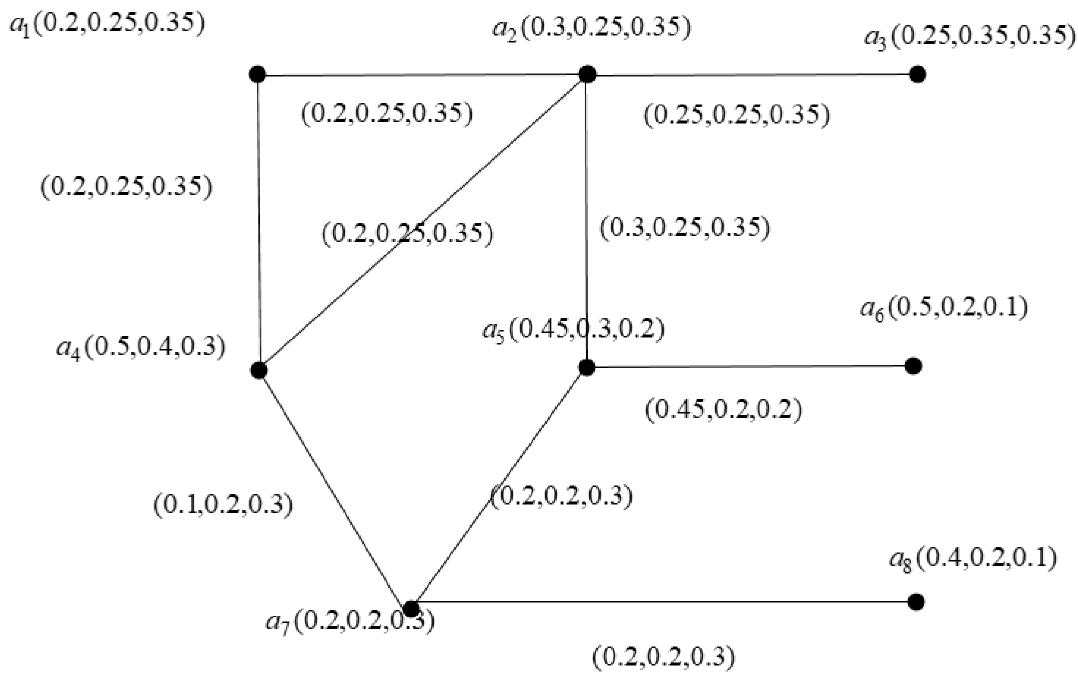

Figure 1. Example of domination of NSG using strong arc

The arcs a_2a_4, a_4a_7 are not strong arcs.

deg(a_1) = (0.4,0.5,0.7), deg(a_2) = (0.75,0.75, 1.05), deg(a_3) = (0.25,0.25,0.35) , deg(a_4) = (0.2,0.25,0.35) , deg(a_5) =(0.95,0.65,0.85) , deg(a_6) =(0.45,0.2,0.2) , deg(a_7) =(0.4,0.4,0.6) , deg(a_8) = (0.2,0.2,0.3).

The minimum degree of truth membership value of G_{NS} is

$$\delta(G_{NS}) = \min\{d_{G_{NS}}(u_d)/u_d \in A_{NS}\} = 0.2$$

The minimum degree of indeterminacy membership value of G_{NS} is

$$\delta(G_{NS}) = \min\{d_{G_{NS}}(u_d)/u_d \in A_{NS}\} = 0.2$$

The minimum degree of falsity membership value of G_{NS} is

$$\delta(G_{NS}) = \min\{d_{G_{NS}}(u_d)/u_d \in A_{NS}\} = 0.2$$

The minimum degree of G_{NS} is $\delta(G_{NS}) = \min\{d_{G_{NS}}(u_d)/u_d \in A_{NS}\} = (0.2,0.2,0.2)$

The maximum degree of truth membership value of G_{NS} is

$$\delta(G_{NS}) = \max\{d_{G_{NS}}(u_d)/u_d \in A_{NS}\} = 0.95$$

The maximum degree of indeterminacy membership value of G_{NS} is

$$\delta(G_{NS}) = \max\{d_{G_{NS}}(u_d)/u_d \in A_{NS}\} = 0.75$$

The maximum degree of falsity membership value of G_{NS} is

$$\delta(G_{NS}) = \max\{d_{G_{NS}}(u_d) / u_d \in A_{NS}\} = 1.05$$

The maximum degree of G_{NS} is $\Delta(G_{NS}) = \max\{d_{G_{INS}}(u_d) / u_d \in A_{NS}\}$ =(0.95,0.75,1.05)

Order of G_{NS} = 5.2248

One of the dominating set is $D_{NS}{}^1$ = { a2, a5 ,a7}, since for every vertex a_i in A_{NS} - $D_{NS}{}^1$ dominated by at least one $a_j \in D_{NS}{}^1$. Hence $\gamma_{NG}(D_{NS}{}^1)$ = 0.6083+0.7+0.6 = 1.9083

Other dominating sets are

(ii) $D_{NS}{}^2$ = {a1, a3, a6, a8} (iii) $D_{NS}{}^3$ = { a3, a4, a5, a7} (iv)$D_{NS}{}^4$ = {a1, a3, a5, a7}

Hence $\gamma_{NG}(D_{NS}{}^2)$ = 2.6499 , $\gamma_{NG}(D_{NS}{}^3)$ = 2.5416 and $\gamma_{NG}(D_{NS}{}^4)$ = 2.45

Domination number of G_{NS} is $\gamma_{NG}(G_{NS})$=1.9083.

Theorem 3.7: A dominating set D_{NS} of a neutrosophic graph G_{NS} = (A_{NS}, B_{NS}) is minimal if and only if for each vertex $v_d \in D_{NS}$ one of the following conditions holds

(i)There exists a vertex $u_d \in A_{NS} - D_{NS}$ such that $N(u_d) \cap D_{NS} = \{v_d\}$

(ii)v_d is an isolate in $\langle D_{NS} \rangle$

Proof: Suppose D_{NS} is a minimal dominating set of G_{NS} there exists a vertex v_d of D_{NS} which does not satisfy any of the above conditions. Hence there exists a vertex $u_d \in A_{NS} - D_{NS}$ such that $N(u_d) \cap D_{NS} \neq \{v_d\}$. Furthermore by condition(ii) v_d is not an isolate in $\langle D_{NS} \rangle$, then $D_{NS} - v_d$ will be a minimal dominating set of G_{NS} which is a contradiction to the assumption.

Theorem 3.8: A subset D_{NS} of A_{NS} of a NSG, G_{NS} = (A_{NS}, B_{NS}) is a dominating then there exists two vertices u_d , $v_d \in A_{NS} - D_{NS}$ such that every u_d - v_d path contains at least one vertex of D_{NS}.

Proof: Suppose D_{NS} is a dominating set of G_{NS}. Since every vertex in A_{NS} - D_{NS} is dominated by at least one vertex of D_{NS}, there exists a u_d - v_d path contains at least one vertex of D_{NS}.

Theorem 3.9: For any NSG, G_{NS} = (A_{NS}, B_{NS}) with order p

(i) $\gamma_{NG}(G_{NS}) \le p - \Delta_T(G_{NS})$

(ii) $\gamma_{NG}(G_{NS}) \le p - \Delta_I(G_{NS})$

(iii) $\gamma_{NG}(G_{NS}) \le p - \Delta_F(G_{NS})$

Proof: Let G_{NS} = (A_{NS}, B_{NS}) be a NSG with order p.

Since $\sum_{i=1}^{n} d(v_i) \le p$, where v_i represents the vertices present in the strong arc, the no of vertices present in a dominating set is less than n. Furthermore, by the definition of $\Delta_T(G_{NS})$, the maximum truth membership values of v_i among all the vertices of G_{NS} and by the definition of minimal dominating set we have

$$\gamma_{NG}(G_{NS}) \le \sum_{i=1}^{n} d(v_i) \le p - \Delta_T(G_{NS})$$

Similarly, we prove $\gamma_{NG}(G_{NS}) \le p - \Delta_I(G_{NS})$ and $\gamma_{NG}(G_{NS}) \le p - \Delta_F(G_{NS})$

4. Equitable domination of a Neutrosophic graph(NSG)

Definition 4.1. A dominating set D_{NS} of A_{NS} of a neutrosophic graph $G_{NS} = (A_{NS}, B_{NS})$ is a equitable dominating set if for every $u_d \in A_{NS} - D_{NS}$ there exists $u_d\ v_d \in B_{NS}$ such that $|\deg(u_d) - \deg(v_d)| \le 1$. The minimum cardinality of an equitable dominating set is called an equitable domination number and is denoted by $\gamma_{ef-NG}(G_{NS})$.

Definition 4.2. An equitable dominating set D_{NS} of A_{NS} is said to be a minimal if no proper subset of D_{NS} is a equitable dominating set of G_{NS}.

Example 4.3. Let $G_{NS} = (A_{NS}, B_{NS})$ be the NSG shown in figure (2)

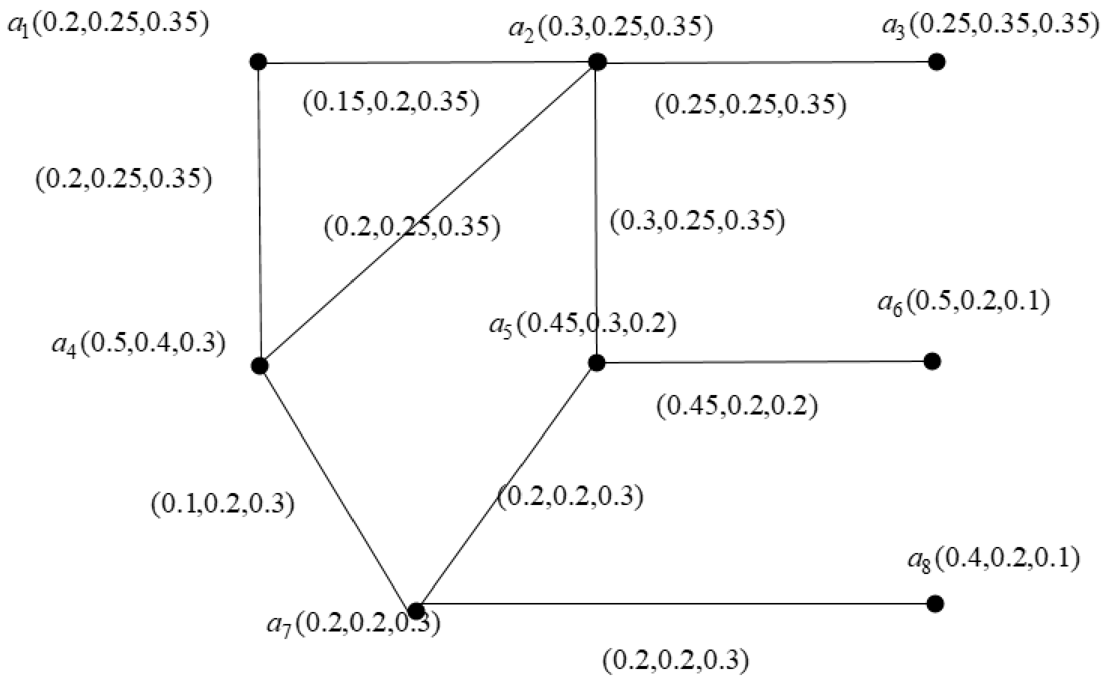

Figure 2. Example of Equitable domination of NSG Using Strong arc

The arcs a_1a_2, a_2a_4, a_4a_7 are not strong arcs.

deg(a_1) = (0.2,0.25,0.35), deg(a_2) = (0.55,0.5,0.7), deg(a_3) =(0.25,0.25,0.35) , deg(a_4) = (0.2,0.25,0.35) , deg(a_5) =(0.95,0.65,0.85) , deg(a_6) =(0.45,0.2,0.2) , deg(a_7) =(0.4,0.4,0.6) , deg(a_8) = (0.2,0.2,0.3).

One of the equitable dominating set is $D_{ed\text{-}NS}{}^{1}$ = { a_2, a_4 , a_5 , a_7}, since $\left|\deg(a_i)-\deg(a_j)\right| \le 1$ for every vertex a_i in A_{NS} - $D_{ed\text{-}NS}{}^{1}$ there exists $a_j \in D_{ed\text{-}NS}{}^{1}$ such that $a_ia_j \in A_{NS}$. Hence $\gamma_{ef-NG}(D_{ed-NS}{}^{1})$ = 0.6083+0.6666+0.7+0.6 = 2.5749

Other equitable dominating sets are

(ii) $D_{NS}{}^{2}$ = {a_1, a_3, a_6, a_8} (iii) $D_{NS}{}^{3}$ = { a_3, a_4, a_5, a_7}

Hence $\gamma_{ef-NG}(D_{ed-NS}{}^{2})$ = 2.6499 and $\gamma_{ef-NG}(D_{ed-NS}{}^{3})$ = 2.5416.

Equitable domination number of G_{NS} is $\gamma_{ef-NG}(G_{ed-NS}) = \gamma_{ef-NG}(D_{ed-NS}{}^{3})$ = 2.5416.

Domination number of G_{NS} is $\gamma_{NG}(D_{ed-NS}{}^{3})$ = 2.5416.

Theorem 4.4. An equitable dominating set D_{NS} of a neutrosophic graph G_{NS} = (A_{NS}, B_{NS}) is minimal if and only if for each vertex $v_d \in D_{NS}$ one of the following conditions holds

(i) There exists a vertex $u_d \in A_{NS}$ - D_{NS} such that $N(u_d) \cap D_{NS} = \{v_d\}$

(ii) v_d is an isolate in $\langle D_{NS} \rangle$

Proof: Suppose D_{NS} is a minimal equitable dominating set of G_{NS} there exists a vertex v_d of D_{NS} which does not satisfy any of the above conditions. Hence there exists a vertex $u_d \in A_{NS}$ - D_{NS} such that $N(u_d) \cap D_{NS} \ne \{v_d\}$. Furthermore by condition(ii) v_d is not an isolate in $\langle D_{NS} \rangle$, then D_{NS} - v_d will be a minimal equitable dominating set of G_{NS} which is a contradiction to the assumption.

Theorem 4.5. For any NSG, G_{NS} = (A_{NS}, B_{NS})

$\gamma_{NG}(G_{NS}) \le \gamma_{ed-NG}(G_{NS})$

Proof: Let D_d and D_{NS} be the minimal dominating set and minimal equitable dominating set of G_{NS} respectively. Let $v_d \in D_{NS}$ be a vertex which is adjacent to r- number of vertices such that deg(v_d) = t, where r > t and rest of the vertices in A_{NS}, A_{NS} - v_d is adjacent to exactly one vertex say v_d . By the definition of equitable domination, A_{NS} - v_d will be the members of D_{NS}. But $v_d \in D_d$ is the only member of dominating set of G_{NS}. Hence the inequality holds. In the case of proving equality, Let H_{NS} be a neutrosophic path P_4. Clearly $\gamma_{NG}(G_{NS}) = \gamma_{ed-NG}(G_{NS}) = 2$.

Example 4.6

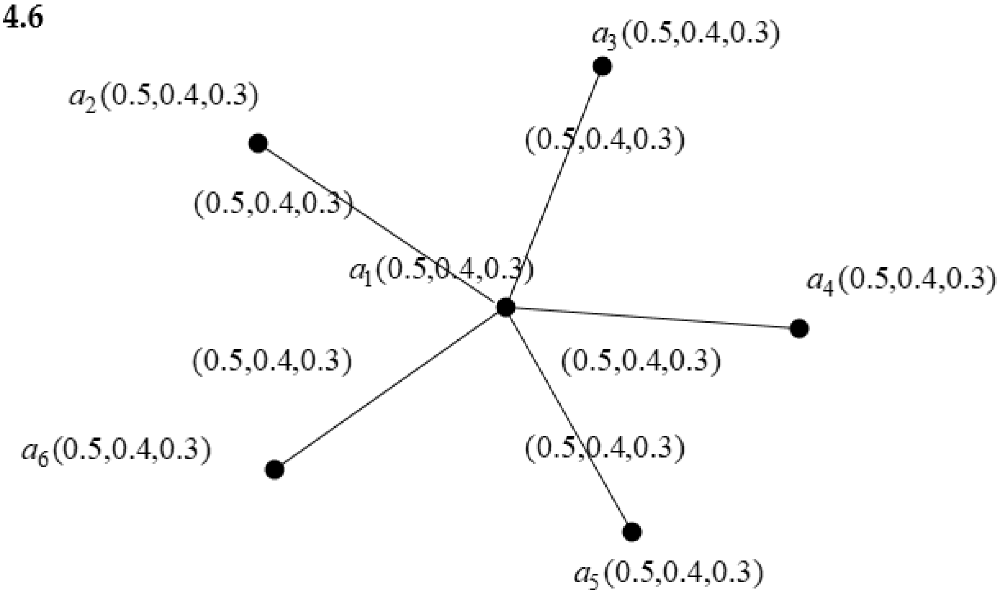

Figure 3. Example of Equitable domination

Let $G_{NS} = (A_{NS}, B_{NS})$ be the NSG shown in figure (3)
All arcs are strong.

Only possible equitable dominating set is $D_{ed\text{-}NS}$ = { a_1, a_2, a_3, a_4, a_5, a_6}, since $|\deg(a_1) - \deg(a_j)| \geq 1$ for every j = 2,3,4&5. Hence $\gamma_{ef-NG}(D_{ed-NS}) = 4$

But the dominating set is D_{NS} = { a_1 }, hence $\gamma_{NG}(D_{NS}) = 0.6666$

By example 4.3, $\gamma_{ef-NG}(G_{ed-NS}) = \gamma_{NG}(D_{NS}) = 2.5416$.

By example 4.6, $\gamma_{ef-NG}(G_{ed-NS}) > \gamma_{NG}(D_{NS})$.

Theorem 4.7. If a dominating set $D_{ed\text{-}NS}$ of a NSG, $G_{NS} = (A_{NS}, B_{NS})$ is a equitable dominating then there exists two vertices u_d, $v_d \in A_{NS} - D_{ed\text{-}NS}$ such that every u_d - v_d path contains at least one vertex of $D_{ed\text{-}NS}$.

Proof: Suppose $D_{ed\text{-}NS}$ is equitable dominating set of G_{NS}. Since every vertex in $A_{NS} - D_{NS}$ is equitably dominated by at least one vertex of D_{NS}, there exists a u_d - v_d path contains at least one vertex of $D_{ed\text{-}NS}$.

Theorem 4.8. For any NSG, $G_{NS} = (A_{NS}, B_{NS})$ with order p

(i) $\gamma_{ef-NG}(G_{NS}) \leq p - \Delta_T(G_{NS})$

(ii) $\gamma_{ef-NG}(G_{NS}) \leq p - \Delta_I(G_{NS})$

(iii) $\gamma_{ef-NG}(G_{NS}) \leq p - \Delta_F(G_{NS})$

Proof: Let $G_{NS} = (A_{NS}, B_{NS})$ be a NSG with order p.

Since $\sum_{i=1}^{n} d(v_i) \leq p$, where v_i represents the vertices present in the strong arc, the no of vertices present in an equitable dominating set is less than n. Furthermore, by the definition of $\Delta_T(G_{NS})$, the maximum truth membership values of v_i among all the vertices of G_{NS} and by the definition of minimal equitable dominating set we have

$$\gamma_{ef-NG}(G_{NS}) \leq \sum_{i=1}^{n} d(v_i) \leq p - \Delta_T(G_{NS})$$

Similarly, we prove $\gamma_{ef-NG}(G_{NS}) \leq p - \Delta_I(G_{NS})$ and $\gamma_{ef-NG}(G_{NS}) \leq p - \Delta_F(G_{NS})$

5. Strong and Weak Equitable Domination in NSG

The concept strong and weak domination in neutrosophic graph is more difficult to handle the values on degree of truth membership, indeterminacy membership and falsity membership, as the degree of edge membership values follows from the degree of incident vertex membership values as in the order of minimum of degree of truth membership values, minimum of degree of indeterminacy values and maximum of degree of falsity membership values respectively. To overcome this difficulty as in the concept of strong and weak equitable domination, we use the score function of vertex cardinality for each vertex and edge cardinality for each edge. The existence of strong and weak equitable domination in a neutrosophic graph is guaranteed on the degree of the neutrosophic graph

Definition 5.1. Let $G_{NS} = (A_{NS}, B_{NS})$ be a neutrosophic graph. Then the vertex score function v_{sf} of G_{NS} is defined

$$v_{sf} = \frac{2 + T_{A_{NS}} - (0.5)I_{A_{NS}} - F_{A_{NS}}}{3}$$

Definition 5.2. Let $G_{NS} = (A_{NS}, B_{NS})$ be a neutrosophic graph. Then vertex score function e_{sf} of G_{NS} is defined

$$e_{sf} = \frac{2 + T_{B_{NS}} - (0.5)I_{B_{NS}} - F_{B_{NS}}}{3}$$

Definition 5.3. Let u_d be a vertex in a NSG, $G_{NS} = (A_{NS}, B_{NS})$. The degree of a vertex u_d is defined as the sum of the weight of the score function of strong arcs incident at u_d and is denoted by deg(u_d). The neighbourhood of u_d is denoted by

$N(u_d) = \{v_d \in A_{NS} / (u_d, v_d)$ is a strong arc}

The minimum degree of G_{NS} is $\delta(G_{NS}) = \min\{d_{G_{NS}}(u_d) / u_d \in A_{NS}\}$

The maximum degree of G_{NS} is $\Delta(G_{NS}) = \max\{d_{G_{INS}}(u_d) / u_d \in A_{NS}\}$

Definition 5.4. Order of $G_{NS} = (A_{NS}, B_{NS})$ is the sum of the score function of vertex cardinality of each vertex and is denoted by $O(G_{NS})$ and size of $G_{NS} = (A_{NS}, B_{NS})$ is the sum of the score function of edge cardinality of each edge.

Definition 5.5. Let $G_{NS} = (A_{NS}, B_{NS})$ be a neutrosophic graph. For any u_d , $v_d \in A_{NS}$, we say u_d strongly equitable dominates v_d if $\deg(u_d) \geq \deg(v_d)$ and u_d is a member of equitable dominating set

.**Definition 5.6.** Let $G_{NS} = (A_{NS}, B_{NS})$ be a neutrosophic graph. For any u_d , $v_d \in A_{NS}$, we say u_d weakly equitable dominates v_d if $\deg(u_d) \leq \deg(v_d)$ and u_d is a member of equitable dominating set .

Definition 5.7. A dominating set $D_{ed\text{-}NS}{}^{S}$ of A_{NS} of a neutrosophic graph $G_{NS} = (A_{NS}, B_{NS})$ is a strong equitable dominating set if for every $v_d \in A_{NS} - D_{ed\text{-}NS}{}^{S}$ there exists at least one $u_d \in D_{ed\text{-}NS}$ such that u_d strongly equitable dominates v_d .The minimum cardinality of a strong equitable dominating set is called a strong equitable domination number and is denoted by $\gamma_{ed-NG}{}^{S}(G_{NS})$.

Definition 5.8. A dominating set $D_{ed\text{-}NS}{}^{W}$ of A_{NS} of a neutrosophic graph $G_{NS} = (A_{NS}, B_{NS})$ is a weak equitable dominating set if for every $v_d \in A_{NS} - D_{ed\text{-}NS}{}^{W}$ there exists at least one $u_d \in D_{ed\text{-}NS}{}^{W}$ such that u_d weakly equitable dominates v_d .The minimum cardinality of a weak equitable dominating set is called a weak equitable domination number and is denoted by $\gamma_{ef-NG}{}^{W}(G_{NS})$.

Example 5.9. Let $G_{NS} = (A_{NS}, B_{NS})$ be the NSG represented in figure1. The following figure 4 is the Neutrosophic graph with score function of vertex cardinality and edge cardinality.

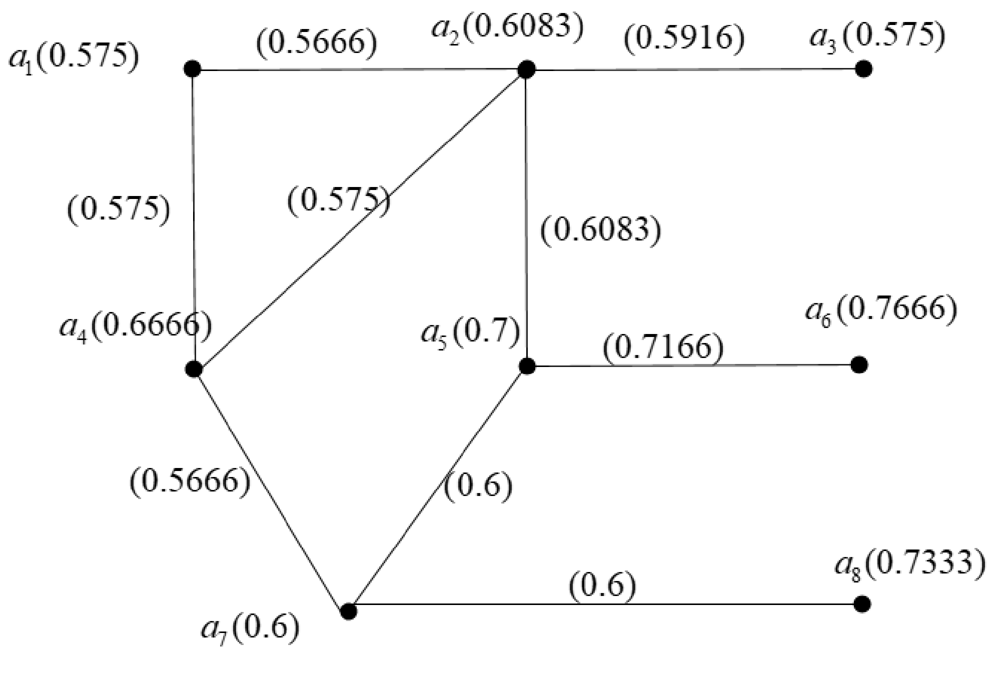

Figure 4. Neutrosophic graph with score function

The arcs a_1a_2, a_2a_5 , a_4a_7 are not strong arcs.

deg(a_1) = 0.575, deg(a_2) = 1.1999, deg(a_3) = 0.5916, deg(a_4) = 0.575, deg(a_5) = 1.9249, deg(a_6) = 0.7166, deg(a_7) = 1.2 , deg(a_8) = 0.6.

Strong equitable dominating set is $D_{ed\text{-}NS}{}^{S}$ = { a_2, a_4 ,a_6 ,a_8}, since $|\deg(u_d)-\deg(v_d)|\le 1$, for every vertex v_d in A_{NS} - $D_{ed\text{-}NS}{}^{S}$ there exists at least one $u_d \in D_{NS}$ such that u_d strongly equitable dominates v_d . Hence $\gamma_{ef-NG}(D_{ed-NS}{}^{S})$ = 0.6083+0.7666+0.7333+0.6666= 2.7748

Weak equitable dominating set is $D_{ed\text{-}NS}{}^{W}$ = { a_1, a_3 ,a_5 ,a_7}, since $|\deg(u_d)-\deg(v_d)|\le 1$, for every vertex v_d in A_{NS} - $D_{ed\text{-}NS}{}^{W}$ there exists at least one $u_d \in D_{NS}$ such that u_d weakly equitable dominates v_d . Hence $\gamma_{ef-NG}(D_{ed-NS}{}^{W})$ = 0.575+0.575+0.7+0.6= 2.45

Theorem 5.10. For any NSG, $G_{NS} = (A_{NS}, B_{NS})$

$\gamma_{ed-NG}{}^{W}(G_{NS}) \le \gamma_{ed-NG}{}^{S}(G_{NS})$ or $\gamma_{ed-NG}{}^{W}(G_{NS}) \ge \gamma_{ed-NG}{}^{S}(G_{NS})$

Proof. Let $D_{NS}{}^{w}$ and $D_{NS}{}^{S}$ be the weak and strong equitable dominating set of G_{NS}.
Case(i) Let the number of vertices present in the strong and weak domination is same then by the definition of strong and weak equitable domination, we have $\gamma_{ef-NG}{}^{W}(G_{NS}) < \gamma_{ef-NG}{}^{S}(G_{NS})$.

Example 5.8 shows that $\gamma_{ef-NG}{}^{W}(G_{NS}) < \gamma_{ef-NG}{}^{S}(G_{NS})$

Case(ii) Let the number of vertices present in the weak domination is more than by strong (with nearly equal membership values) then we have $\gamma_{ef-NG}{}^{W}(G_{NS}) > \gamma_{ef-NG}{}^{S}(G_{NS})$.

Case(iii) Let the arcs present in the given neutrosophic graph is strong with equal number of vertices present in strong and weak equitable dominating set then we have $\gamma_{ef-NG}{}^{W}(G_{NS}) = \gamma_{ef-NG}{}^{S}(G_{NS})$

Theorem 5.9. For any NSG, $G_{NS} = (A_{NS}, B_{NS})$

(i) $\gamma_{ed-NG}{}^{S}(G_{NS}) \le O(G_{NS}) - \Delta(G_{NS})$

(ii) $\gamma_{ed-NG}{}^{S}(G_{NS}) \le O(G_{NS}) - \delta(G_{NS})$

Proof: Let $G_{NS} = (A_{NS}, B_{NS})$ be a NSG.

Since $\sum_{i=1}^{n} d(v_i) \le p$, where v_i represents the vertices present in the strong arc, the number of vertices present in an equitable dominating set is less than *n*. Furthermore, by the definition of $\Delta_T(G_{NS})$, the maximum truth membership values of v_i among all the vertices of G_{NS} and by the definition of minimal strong equitable dominating set we have

$$\gamma_{ed-NG}(G_{NS}{}^{S}) \leq \sum_{i=1}^{n} d(v_i) \leq p - \Delta(G_{NS})$$

Similarly, we prove $\gamma_{ef-NG}(G_{NS}{}^{S}) \leq p - \delta(G_{NS})$.

Theorem 5.10. For any NSG, $G_{NS} = (A_{NS}, B_{NS})$

(i) $\gamma_{ed-NG}{}^{W}(G_{NS}) \leq O(G_{NS}) - \Delta(G_{NS})$

(ii) $\gamma_{ed-NG}{}^{W}(G_{NS}) \leq O(G_{NS}) - \delta(G_{NS})$

Proof: Theorem 5.10. follows from Theorem 5.9.

Theorem 5.11. For any NSG, $G_{NS} = (A_{NS}, B_{NS})$

(i) $\gamma_{ed-NG}{}^{S}(G_{NS}) \geq O(G_{NS})/(\Delta(G_{NS})+1)$

(ii) $\gamma_{ed-NG}{}^{W}(G_{NS}) \geq O(G_{NS})/(\Delta(G_{NS})+1)$

Proof: Let $G_{NS} = (A_{NS}, B_{NS})$ be a NSG and its strong equitable domination number be $\gamma_{ef-NG}{}^{S}(G_{NS})$

$$\left| O(G_{NS}) - \gamma_{ef-NS}{}^{S}(G_{NS}) \right| \leq \sum_{i=1}^{n} d(v_i) \leq \gamma_{ed-NS}{}^{S}(G_{NS})\Delta(G_{NS})$$

$$\leq \gamma_{ed-NS}{}^{S}(G_{NS})\Delta(G_{NS})$$

$$O(G_{NS}) \leq \gamma_{ed-NS}{}^{S}(G_{NS})\Delta(G_{NS}) + \gamma_{ed-NS}{}^{S}(G_{NS})$$

$$\leq \gamma_{ed-NS}{}^{S}(G_{NS})(\Delta(G_{NS})+1)$$

Hence $\gamma_{ed-NG}{}^{S}(G_{NS}) \geq O(G_{NS})/(\Delta(G_{NS})+1)$

Similarly prove $\gamma_{ed-NG}{}^{W}(G_{NS}) \geq O(G_{NS})/(\Delta(G_{NS})+1)$

6.Conclusions Strong and weak equitable domination in a neutrosophic graph is difficult to initiate, as the NSG has degrees of truth membership value, degrees of indeterminacy membership value, and degrees of truth membership value. Comparing these three types of degrees of membership values of one vertex to another will help us identify strong and weak equitable dominating vertices. But in implementing this, the research focus is very narrowly focused on strong and weak equitable domination. Hence, we conclude that, using the vertex cardinality score function, we can convert all these three degree of membership values into a single value and then proceed with the concept of strong and weak equitable domination. In future, we have planned to continue the work on paired equitable domination of NSG using strong arcs and furthermore to find relationships between domination, equitable domination and paired equitable domination of neutrosophic graphs.

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A New Notion of Neighbourhood and Continuity in Neutrosophic Topological Spaces

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Abstract: Owing to a wide range of applications in various fields, the neutrosophic theory initiated by Smarandache has been highly featured in research. This concept led to the evolution of neutrosophic topological spaces which is being explored extensively. The focus of this paper is to introduce and study the concept of neutrosophic $\Upsilon-$neighbourhood and neutrosophic $\Upsilon-$continuity in neutrosophic topological spaces. Further, we define the notion of neutrosophic $\Upsilon-$irresolute functions. We also observe their attributes and relationship with functions existing in literature. Moreover, we present some equivalent conditions for the existence of these functions in which the concept of neighbourhood has been wielded.

Keywords: neutrosophic $\Upsilon-$open, neutrosophic $\Upsilon-$closed, neutrosophic $\Upsilon-$neighbourhood, neutrosophic $\Upsilon-$continuous, neutrosophic $\Upsilon-$irrseolute.

1. Introduction

Several theories were developed as mathematical approaches to rectify the difficulties pertained to uncertainty. Accordingly, the concept of neutrosophy initiated by Florentine Smarandache[1] evolved as a branch of philosophy to study the scope and nature of neutralities. This induced the concept of neutrosophic logic which further led to the conceptualization of neutrosophic sets as a generalization of fuzzy sets and intuitionistic fuzzy sets. A neutrosophic set is characterized by three independent components namely membership, indeterminacy and non-membership functions defined on the non-standard unit interval. Later, Salama and Albowi[3] in 2012 induced the concept of neutrosophic sets in topological spaces which originated as neutrosophic topological spaces. In addition, some basic notions and properties of topological structures such as interior, closure, subspaces and separation axioms have been presented in [4-8]. G. C. Ray and Sudeep[9] proposed the definitions of neutrosophic point and neighbourhood structure. They have also explored the relation of quasi coincidence between neutrosophic sets and characterized the neutrosophic topological spaces by means of quasi-neighbourhood. Meanwhile, Salama et.al[10] in 2014, studied the concept of continuous functions in neutrosophic topological spaces. Further, P. Iswarya and

K. Bageerathi[11], in 2016 introduced the concept of semi-open sets in neutrosophic topological spaces and later the notion of semi-continuous functions[12,13] were also studied. Dhavaseelan and Saeid Jafari[14], in 2017 established the idea of generalized closed sets and continuous functions in neutrosophic topological spaces. C. Maheshwari and S. Chandrasekar[15] defined the notion of gb-closed sets and continuous functions in 2019. Moreover, some novel concepts of continuous functions and other topological structures have been defined and studied by various authors[16-18] in the subsequent years. Recently, the authors[19] of this paper introduced and analyzed a new class of neutrosophic sets namely neutrosophic $\Upsilon-$open sets and neutrosophic $\Upsilon-$closed sets. The main objective of this paper is to introduce and study the concepts of neutrosophic $\Upsilon-$neighbourhood, neutrosophic $\Upsilon-$continuous and irresolute functions in neutrosophic topological spaces. The characterization and composition of these functions have been presented through results and counter examples. Further, various equivalent conditions for the existence of these concepts have also been observed.

The structure of the paper is as follows: section 2 comprises of the prerequisites essential for this work. Section 3 establishes a novel concept of neighbourhood namely neutrosophic $\Upsilon-$neighbourhood and $\Upsilon-$quasi neighbourhood. Section 4 imparts the notion of neutrosophic $\Upsilon-$continuous functions and its attributes. Further, section 5 presents the idea of neutrosophic $\Upsilon-$irresolute functions and the article is ceased with a conclusion in section 6.

2. Preliminaries

In this section, we have presented some basic notions and results required for the progression of this work.

Definition 2.1[3]**:** Let U be a non-empty fixed set. A **neutrosophic set** L is an object having the form $L = \{< u, \mu_L(u), \sigma_L(u), \gamma_L(u) >: u \in U\}$ where $\mu_L(u), \sigma_L(u)$ and $\gamma_L(u)$ represent the degree of membership, the degree of indeterminacy and the degree of non-membership respectively of each element $u \in U$. A neutrosophic set $L = \{< u, \mu_L(u), \sigma_L(u), \gamma_L(u) >: u \in U\}$ can be identified to an ordered triple $< \mu_L, \sigma_L, \gamma_L >$ in $]^-0,1^+[$ on U.

Definition 2.2[3]: Let U be a non-empty set and $L = \{< u, \mu_L(u), \sigma_L(u), \gamma_L(u) >: u \in U\}$, $M = \{< u, \mu_M(u), \sigma_M(u), \gamma_M(u) >: u \in U\}$ be neutrosophic sets in U. Then

(i) $L \subseteq M$ if $\mu_L(u) \leq \mu_M(u), \sigma_L(u) \leq \sigma_M(u)$ and $\gamma_L(u) \geq \gamma_M(u)$ for all $u \in U$.

(ii) $L \cup M = \{< u,\ \max\{\mu_L(u), \mu_M(u)\},\ \max\{\sigma_L(u), \sigma_M(u)\},\ \min\{\gamma_L(u), \gamma_M(u)\} >: u \in U\}$

(iii) $L \cap M = \{< u,\ \min\{\mu_L(u), \mu_M(u)\},\ \min\{\sigma_L(u), \sigma_M(u)\},\ \max\{\gamma_L(u), \gamma_M(u)\} >: u \in U\}$

(iv) $L^c = \{< u, \gamma_L(u), 1 - \sigma_L(u), \mu_L(u) >: u \in U\}$

(v) $0_{N_{tr}} = \{< u, 0,0,1 >: u \in U\}$ and $1_{N_{tr}} = \{< u, 1,1,0 >: u \in U\}$

Definition 2.3[3]: A **neutrosophic topology** on a non-empty set U is a family $\tau_{N_{tr}}$ of neutrosophic sets in U satisfying the following axioms:

(i) $0_{N_{tr}}, 1_{N_{tr}} \in \tau_{N_{tr}}$

(ii) $\cup L_i \in \tau_{N_{tr}} \forall \{L_i : i \in I\} \subseteq \tau_{N_{tr}}$

(iii) $L_1 \cap L_2 \in \tau_{N_{tr}}$ for any $L_1, L_2 \in \tau_{N_{tr}}$

The pair $(U, \tau_{N_{tr}})$ is called a neutrosophic topological space. The members of $\tau_{N_{tr}}$ are called neutrosophic open and its complements are called neutrosophic closed.

Definition 2.4[5]: A neutrosophic set $L = \{< u, \mu_L(u), \sigma_L(u), \gamma_L(u) >: u \in U\}$ is called a **neutrosophic point** if for any element $v \in U, \mu_L(v) = a, \sigma_L(v) = b, \gamma_L(v) = c$ for $u = v$ and $\mu_L(v) = 0, \sigma_L(v) = 0, \gamma_L(v) = 1$ for $u \neq v$, where a, b, c are real standard or non standard subsets of $]^-0,1^+[$. A neutrosophic point is denoted by $u_{a,b,c}$. For the neutrosophic point $u_{a,b,c}, u$ will be called its support.

Definition 2.5[4]: Let $(U, \tau_{N_{tr}})$ be a neutrosophic topological space and S be a non-empty subset of U. Then, a neutrosophic relative topology on S is defined by

$$\tau^S_{N_{tr}} = \{L \cap 1^S_{N_{tr}} : L \in \tau_{N_{tr}}\}$$

where

$$1^S_{N_{tr}} = \begin{cases} < 1,1,0 >, & if s \in S \\ < 0,0,1 >, & otherwise \end{cases}$$

Thus, $(S, \tau^S_{N_{tr}})$ is called a **neutrosophic subspace** of $(U, \tau_{N_{tr}})$.

Definition 2.6[14]: Let U and V be two non-empty sets and $f_{N_{tr}}: U \longrightarrow V$ be a function. If $M = \{< v, \mu_M(v), \sigma_M(v), \gamma_M(v) >: v \in V\}$ is a neutrosophic set in V, then the preimage of M under $f_{N_{tr}}$, denoted by $f^{-1}_{N_{tr}}(M)$, is the neutrosophic set in U defined by

$$f^{-1}_{N_{tr}}(M) = \{< u, f^{-1}_{N_{tr}}(\mu_M)(u), f^{-1}_{N_{tr}}(\sigma_M)(u), f^{-1}_{N_{tr}}(\gamma_M)(u) >: u \in U\}$$

If $L = \{< u, \mu_L(u), \sigma_L(u), \gamma_L(u) >: u \in U\}$ is a neutrosophic set in U, then the image of L under $f_{N_{tr}}$, denoted by $f_{N_{tr}}(L)$, is the neutrosophic set in V defined by

$$f_{N_{tr}}(L) = \{< v, f_{N_{tr}}(\mu_L)(v), f_{N_{tr}}(\sigma_L)(v), \left(1 - f_{N_{tr}}(1 - \gamma_L)\right)(v) > \ : v \in V\} \text{ where}$$

$$f_{N_{tr}}(\mu_L)(v) = \begin{cases} sup_{u \in f^{-1}_{N_{tr}}(v)} \mu_L(u), & \text{if } f^{-1}_{N_{tr}}(v) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

$$f_{N_{tr}}(\sigma_L)(v) = \begin{cases} sup_{u \in f^{-1}_{N_{tr}}(v)} \sigma_L(u), & \text{if } f^{-1}_{N_{tr}}(v) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

$$\left(1 - f_{N_{tr}}(1 - \gamma_L)\right)(v) = \begin{cases} inf_{u \in f^{-1}_{N_{tr}}(v)} \gamma_L(u), & \text{if } f^{-1}_{N_{tr}}(v) \neq \emptyset \\ 1, & \text{otherwise} \end{cases}$$

Definition 2.7: Let $(U, \tau_{N_{tr}})$ and $(V, \rho_{N_{tr}})$ be neutrosophic topological spaces. Then the function $f_{N_{tr}}: (U, \tau_{N_{tr}}) \longrightarrow (V, \rho_{N_{tr}})$ is said to be **neutrosophic continuous**[10] (respectively, neutrosophic semi−continuous[12], neutrosophic α −continuous[14], neutrosophic β −continuous, neutrosophic $gs-$ continuous, neutrosophic $gb-$ continuous[15]) if $f^{-1}_{N_{tr}}(M)$ is N_{tr} open(respectively N_{tr} semi − open, $N_{tr}\alpha$ − open, $N_{tr}\beta$ − open, $N_{tr}gs$ − open, $N_{tr}gb$ − open) in $(U, \tau_{N_{tr}})$ for every N_{tr}open set M in $(V, \rho_{N_{tr}})$.

Definition 2.8[7]: Let $u_{a,b,c}$ be a neutrosophic point in a neutrosophic topological space $(U, \tau_{N_{tr}})$. Then a neutrosophic set N in U is said to be **neutrosophic neighbourhood**($N_{tr}nbhd$) of $u_{a,b,c}$ if there exists a N_{tr}open set M such that $u_{a,b,c} \in M \subseteq N$.

Definition 2.9[6]: A neutrosophic point $u_{a,b,c}$ is said to be **neutrosophic quasi – coincident** with a neutrosophic set L, denoted by $u_{a,b,c}qL$ if $u_{a,b,c} \notin L^c$. If $u_{a,b,c}$ is not neutrosophic quasi – coincident with L, we denote it by $u_{a,b,c}\hat{q}L$.

Definition 2.10[6]: A neutrosophic set M is said to be neutrosophic quasi – coincident with a neutrosophic set L, denoted by MqL if $M \nsubseteq L^c$. If M is not neutrosophic quasi – coincident with L, we denote it by $M\hat{q}L$.

Definition 2.11[6]: A neutrosophic set N in U is said to be **neutrosophic quasi-neighbourhood**($N_{tr}Qnbhd$) of $u_{a,b,c}$ if there exists a N_{tr}open set M such that $u_{a,b,c}qM \subseteq N$.

Definition 2.12[19]:A neutrosophic set L of a neutrosophic topological space $(U,\tau_{N_{tr}})$ is said to be **neutrosophic Υ −open** if for every non-empty N_{tr}closed set $F\neq 1_{N_{tr}}, L\subseteq N_{tr}cl(N_{tr}int(L\cup F))$. The complement of neutrosophic Υ −open set is neutrosophic Υ −closed. The class of neutrosophic Υ −open sets in $(U,\tau_{N_{tr}})$ is denoted by $N_{tr}\Upsilon O(U,\tau_{N_{tr}})$.

Theorem 2.13[19]: The union of an arbitrary collection of $N_{tr}\Upsilon$ −open sets is also $N_{tr}\Upsilon$ −open.

Theorem 2.14[19]: In any neutrosophic topological space $(U,\tau_{N_{tr}})$,

(i) Every N_{tr}open set is $N_{tr}\Upsilon$ −open.

(ii) Every N_{tr}semi−open set is $N_{tr}\Upsilon$ −open.

(iii) Every $N_{tr}\alpha-$ open set is $N_{tr}\Upsilon$ −open.

(iv) Every $N_{tr}\Upsilon$ −open set is $N_{tr}\beta-$ open.

(v) Every $N_{tr}\Upsilon$ −open set is $N_{tr}gs-$ open.

(vi) Every $N_{tr}\Upsilon$ −open set is $N_{tr}gb-$ open.

Remark 2.15[19]: The above theorem is also true for $N_{tr}\Upsilon$ −closed sets.

Theorem 2.16[19]: A neutrosophic set L in a neutrosophic topological space $(U,\tau_{N_{tr}})$ is $N_{tr}\Upsilon-$ open if and only if for every neutrosophic point $u_{a,b,c}\in L$, there exists a $N_{tr}\Upsilon-$ open set $M_{u_{a,b,c}}$ such that $u_{a,b,c}\in M_{u_{a,b,c}}\subseteq L$.

Definition 2.17[19]: Let be a neutrosophic topological space and L be a neutrosophic set in U.

(i) The **neutrosophic Υ −interior** of L is the union of all $N_{tr}\Upsilon$ −open sets contained in L. It is denoted by $N_{tr}\Upsilon int(L)$.

(ii) The **neutrosophic Υ − closure** of L is the intersection of all $N_{tr}\Upsilon-$ closed sets containing L. It is denoted by $N_{tr}\Upsilon cl(L)$.

3. Neutrosophic Υ −neighbourhood

This section conceptualizes the idea of neutrosophic Υ −neighbourhood and neutrosophic Υ −quasi neighbourhood. Moreover, their characterizations have been depicted through results and illustrations.

Definition 3.1: Let $u_{a,b,c}$ be a neutrosophic point in a neutrosophic topological space $(U,\tau_{N_{tr}})$. Then a neutrosophic set N in U is said to be a

(i) **neutrosophic Υ −neighbourhood**($N_{tr}\Upsilon-nbhd$) of $u_{a,b,c}$ if there exists a $N_{tr}\Upsilon-$ open set M such that $u_{a,b,c}\in M\subseteq N$.

(ii) **neutrosophic Υ −quasi neighbourhood**($N_{tr}\Upsilon-Qnbhd$) of $u_{a,b,c}$ if there exists a $N_{tr}\Upsilon-$ open set M such that $u_{a,b,c}qM\subseteq N$.

Example 3.2: Let $U=\{a,b\}$ and $\tau_{N_{tr}}=\{0_{N_{tr}},1_{N_{tr}},L\}$ where$L=\{<a,0.7,0.5,0.3><b,0.2,0.7,0.1>\}$. Now, let us consider a neutrosophic point $a_{0.1,0.2,0.5}$ in U. Then, there is a $N_{tr}\Upsilon-$ open set $M=\{<a,0.8,0.8,0.1><b,0.5,0.9,0.1>\}$ such that $a_{0.1,0.2,0.5}\in M\subseteq N$ where$N=\{<a,0.9,0.8,0.1><b,0.6,0.9,0.1>\}$. Hence N is a $N_{tr}\Upsilon-nbhd$ of $a_{0.1,0.2,0.5}$.

Example 3.3: Let $U=\{a,b\}$ and $\tau_{N_{tr}}=\{0_{N_{tr}},1_{N_{tr}},L\}$ where$L=\{<a,0.8,0.7,0.1><b,0.4,0.9,0.1>\}$. Now, let us consider a neutrosophic point $a_{0.2,0.9,0.7}$ in U. Then, there is a $N_{tr}\Upsilon-$ open set $M=\{<a,0.9,0.8,0.1><b,0.7,0.9,0.1>\}$ such that $a_{0.2,0.9,0.7}qM\subseteq N$ where $N=\{<a,0.9,0.9,0.1><b,0.8,0.9,0.1>\}$. Hence N is a $N_{tr}\Upsilon-Qnbhd$ of $a_{0.2,0.9,0.7}$.

Theorem 3.4: Every $N_{tr}nbhd$ (resp. $N_{tr}Qnbhd$) of a neutrosophic point $u_{a,b,c}$ in a neutrosophic topological space $(U,\tau_{N_{tr}})$ is a $N_{tr}\Upsilon-nbhd$(resp.$N_{tr}\Upsilon-Qnbhd$) of $u_{a,b,c}$.

Proof: Let N be a $N_{tr}nbhd$ (resp. $N_{tr}Qnbhd$) of a neutrosophic point $u_{a,b,c}$ in U. Then, there exists a N_{tr}open set M in U such that $u_{a,b,c} \in M \subseteq N$ (resp. $u_{a,b,c}qM \subseteq N$). Now, by theorem 2.14, M is $N_{tr}\Upsilon-$open in U. Hence there exists a $N_{tr}\Upsilon-$open set M in U such that $u_{a,b,c} \in M \subseteq N$ (resp. $u_{a,b,c}qM \subseteq N$). Therefore N is a $N_{tr}\Upsilon - nbhd$(resp. $N_{tr}\Upsilon - Qnbhd$) of $u_{a,b,c}$.

The following example substantiates that the converse of the above-stated theorem need not be true.

Example 3.5: (i) Let $U = \{a, b\}$ and $\tau_{N_{tr}} = \{0_{N_{tr}}, 1_{N_{tr}}, L\}$ where $L = \{< a, 0.6,0.6,0.2 >< b, 0.2,0.9, 0.1 >\}$. Now, let us consider a neutrosophic point $a_{0.7,0.1,0.5}$ in U. Then there is a $N_{tr}\Upsilon-$open set $M = \{< a, 0.8,0.7,0.2 >< b, 0.3,0.9,0.1 >\}$ such that $a_{0.7,0.1,0.5} \in M \subseteq N$ where $N = \{< a, 0.8,0.9, 0.1 >< b, 0.4,0.9,0.1 >\}$. This implies N is a $N_{tr}\Upsilon - nbhd$ of $a_{0.7,0.1,0.5}$. However, N is not a $N_{tr}nbhd$ of $a_{0.7,0.1,0.5}$.

(ii) Let $U = \{a, b\}$ and $\tau_{N_{tr}} = \{0_{N_{tr}}, 1_{N_{tr}}, L\}$ where $L = \{< a, 0.7,0.9,0.1 >< b, 0.5,0.7,0.4 >\}$. Now, let us consider a neutrosophic point $a_{0.1,0.1,0.7}$ in U. Then there is a $N_{tr}\Upsilon-$open set $M = \{< a, 0.8,0.9,0.1 >< b, 0.7,0.7,0.2 >\}$ such that $a_{0.1,0.1,0.7}qM \subseteq N$ where $N = \{< a, 0.9,0.9,0.1 > < b, 0.9,0.8,0.1 >\}$. This implies N is a $N_{tr}\Upsilon - Qnbhd$ of $a_{0.1,0.1,0.7}$. However, N is not a $N_{tr}nbhd$ of $a_{0.1,0.1,0.7}$.

Theorem 3.6: A neutrosophic set L in a neutrosophic topological space $(U, \tau_{N_{tr}})$ is $N_{tr}\Upsilon-$open if and only if for every neutrosophic point $u_{a,b,c} \in L, L$ is a $N_{tr}\Upsilon - nbhd$ of $u_{a,b,c}$.

Proof: Let L be $N_{tr}\Upsilon-$open in U. Also, for each $u_{a,b,c} \in L, L \subseteq L$. Then, by definition 3.1(i), it follows that L is a $N_{tr}\Upsilon - nbhd$ of $u_{a,b,c}$. Conversely, assume that for every $u_{a,b,c} \in L, L$ is a $N_{tr}\Upsilon - nbhd$ of $u_{a,b,c}$. Then, there exists a $N_{tr}\Upsilon-$open set M in U such that $u_{a,b,c} \in M \subseteq L$. Therefore, by theorem 2.16, L is $N_{tr}\Upsilon-$open.

Theorem 3.7: Every $N_{tr}\Upsilon-$open set L in a neutrosophic topological space $(U, \tau_{N_{tr}})$ is a $N_{tr}\Upsilon - Qnbhd$ of every neutrosophic point quasi−coincident with L.

Proof: The proof is obvious since for every neutrosophic point $u_{a,b,c}qL$, we have $u_{a,b,c}qL \subseteq L$.

Theorem 3.8: Let L be a $N_{tr}\Upsilon-$closed set in a neutrosophic topological space $(U, \tau_{N_{tr}})$ and $u_{a,b,c}qL^c$. Then, there exists a $N_{tr}\Upsilon - Qnbhd$ M of $u_{a,b,c}$ such that $L\hat{q}M$.

Proof: Since L is $N_{tr}\Upsilon-$closed in U, L^c is $N_{tr}\Upsilon-$open in U such that $u_{a,b,c}qL^c$. Then, by theorem 3.7, L^c is a $N_{tr}\Upsilon - Qnbhd$ of $u_{a,b,c}$. Hence there exists a $N_{tr}\Upsilon-$open set M in U such that $u_{a,b,c}qM \subseteq L^c$. Again, by theorem 3.7, M is a $N_{tr}\Upsilon - Qnbhd$ of $u_{a,b,c}$. Also, since $M \subseteq L^c, L\hat{q}M$. Hence there exists a $N_{tr}\Upsilon - Qnbhd$ M of $u_{a,b,c}$ such that $L\hat{q}M$.

Theorem 3.9: Let L be a neutrosophic set in a neutrosophic topological space $(U, \tau_{N_{tr}})$. Then a neutrosophic point $u_{a,b,c} \in N_{tr}\Upsilon cl(L)$ if and only if every $N_{tr}\Upsilon - Qnbhd$ of $u_{a,b,c}$ is quasi-coincident with L.

Proof: Let $u_{a,b,c} \in N_{tr}\Upsilon cl(L)$ and N be a $N_{tr}\Upsilon - Qnbhd$ of $u_{a,b,c}$ such that $N\hat{q}L$. Then, there exists a $N_{tr}\Upsilon-$open set M such that $u_{a,b,c}qM \subseteq N$. Since $N\hat{q}L, N \subseteq L^c$ and therefore $M \subseteq L^c$ which implies $L \subseteq M^c$. Now, M^c is a $N_{tr}\Upsilon-$closed set containing L and $N_{tr}\Upsilon cl(L)$ is the smallest $N_{tr}\Upsilon-$closed set containing L. Hence $N_{tr}\Upsilon cl(L) \subseteq M^c$. Also, since $u_{a,b,c}qM, u_{a,b,c} \notin M^c$. Therefore $u_{a,b,c} \notin N_{tr}\Upsilon cl(L)$ which is a contradiction. Conversely, suppose every $N_{tr}\Upsilon - Qnbhd$ of $u_{a,b,c}$ is quasi-coincident with L. If $u_{a,b,c} \notin N_{tr}\Upsilon cl(L)$, then there exists a $N_{tr}\Upsilon-$closed set M such that $L \subseteq M$ and $u_{a,b,c} \notin M$. This implies that $u_{a,b,c}qM^c$, where M^c is a $N_{tr}\Upsilon-$open set in U. Now, by theorem 3.7, M^c is a $N_{tr}\Upsilon - Qnbhd$ of $u_{a,b,c}$ such that $M^c\hat{q}L$ which is a contradiction.

4. Neutrosophic Υ −continuous functions

Topology is constantly intrigued by issues that are either directly or indirectly related to continuity. Accordingly, continuity plays a prominent role in the characterization of topological spaces. This section deals with the origination of neutrosophic $\Upsilon -$ continuous functions in neutrosophic topological spaces. Further, we have observed their properties and discussed the composition of functions.

Definition 4.1: Let $(U,\tau_{N_{tr}})$ and $(V,\rho_{N_{tr}})$ be neutrosophic topological spaces. Then the function $f_{N_{tr}}:(U,\tau_{N_{tr}}) \longrightarrow (V,\rho_{N_{tr}})$ is said to be **neutrosophic $\Upsilon -$ continuous** if $f_{N_{tr}}^{-1}(M)$ is $N_{tr}\Upsilon -$ open in $(U,\tau_{N_{tr}})$ for every N_{tr}open set M in $(V,\rho_{N_{tr}})$.

Example 4.2: Let $U = \{a,b\}, V = \{x,y\}, \tau_{N_{tr}} = \{0_{N_{tr}}, 1_{N_{tr}}, L, M\}$ and $\rho_{N_{tr}} = \{0_{N_{tr}}, 1_{N_{tr}}, N\}$ where $L = \{< a, 0.6,0.3,0.5 >< b, 0.5,0.8,0.4 >\}$, $M = \{< a, 0.5,0.2,0.7 >< b, 0.2,0.7,0.9 >\}$ and $N = \{< x, 0.9, 0.9,0.1 >< y, 0.8,0.9,0.2 >\}$. Consider the collections $\mathcal{P} = \{P : L \subset P, M^c \subset P\}$ and $\mathcal{Q} = \{Q : L \subset Q; Q \not\subset M^c;\ M^c \not\subset Q\}$ of neutrosophic sets in U. Then $N_{tr}\Upsilon O(U,\tau_{N_{tr}}) = \{0_{N_{tr}}, L, M, \mathcal{P}, \mathcal{Q}, 1_{N_{tr}}\}$. Define $f_{N_{tr}}:(U,\tau_{N_{tr}}) \longrightarrow (V,\rho_{N_{tr}})$ as $f_{N_{tr}}(a) = y$ and $f_{N_{tr}}(b) = x$. Then, $f_{N_{tr}}^{-1}(N) = \{< a, 0.8,0.9,0.2 > < b, 0.9,0.9,0.1 >\} \in \mathcal{P}$ which implies $f_{N_{tr}}^{-1}(N)$ is $N_{tr}\Upsilon -$ open in U. Hence $f_{N_{tr}}$ is $N_{tr}\Upsilon$ −continuous.

Theorem 4.3: Every N_{tr}continuous function is $N_{tr}\Upsilon -$ continuous.

Proof: Let $f_{N_{tr}}:(U,\tau_{N_{tr}}) \longrightarrow (V,\rho_{N_{tr}})$ be a N_{tr}continuous function. Let M be a N_{tr}open set in V. Since $f_{N_{tr}}$ is N_{tr}continuous, $f_{N_{tr}}^{-1}(M)$ is N_{tr}open in U. By theorem 2.14, $f_{N_{tr}}^{-1}(M)$ is $N_{tr}\Upsilon -$ open in U. Hence $f_{N_{tr}}$ is $N_{tr}\Upsilon -$ continuous.

The following example substantiates that the converse of the above-stated theorem need not be true.

Example 4.4: Let $U = \{a,b\}, V = \{x,y\}, \tau_{N_{tr}} = \{0_{N_{tr}}, 1_{N_{tr}}, L, M\}$ and $\rho_{N_{tr}} = \{0_{N_{tr}}, 1_{N_{tr}}, N\}$ where $L = \{< a, 0.6,0.4,0.9 >< b, 0.5,0.7,1 >\}, M = \{< a, 0.7,0.6,0.8 >< b, 0.6,0.8,0.9 >\}$ and $N = \{< x, 0.6, 0.9,0.3 >< y, 0.7,0.6,0.2 >\}$. Consider the collections $\mathcal{P} = \{P : M \subset P, L^c \subset P\}$ and $\mathcal{Q} = \{Q : M \subset Q; Q \not\subset L^c;\ L^c \not\subset Q\}$ of neutrosophic sets in U.Then $N_{tr}\Upsilon O(U,\tau_{N_{tr}}) = \{0_{N_{tr}}, L, M, \mathcal{P}, \mathcal{Q}, 1_{N_{tr}}\}$. Define $f_{N_{tr}}:(U,\tau_{N_{tr}}) \longrightarrow (V,\rho_{N_{tr}})$ as $f_{N_{tr}}(a) = y$ and $f_{N_{tr}}(b) = x$. Then, $f_{N_{tr}}^{-1}(N) = \{< a, 0.7,0.6,0.2 > < b, 0.6,0.9,0.3 >\} \in \mathcal{Q}$ which implies $f_{N_{tr}}^{-1}(N)$ is $N_{tr}\Upsilon -$ open but not N_{tr}open in U. Hence $f_{N_{tr}}$ is $N_{tr}\Upsilon -$ continuous but not N_{tr}continuous.

Theorem 4.5: Let $f_{N_{tr}}:(U,\tau_{N_{tr}}) \longrightarrow (V,\rho_{N_{tr}})$ be a function between two neutrosophic topological spaces.

(i) If $f_{N_{tr}}$ is N_{tr}semi − continuous, then $f_{N_{tr}}$ is $N_{tr}\Upsilon -$ continuous.

(ii) If $f_{N_{tr}}$ is $N_{tr}\alpha -$ continuous, then $f_{N_{tr}}$ is $N_{tr}\Upsilon -$ continuous.

(iii) If $f_{N_{tr}}$ is $N_{tr}\Upsilon -$ continuous, then $f_{N_{tr}}$ is $N_{tr}\beta -$ continuous.

(iv) If $f_{N_{tr}}$ is $N_{tr}\Upsilon -$ continuous, then $f_{N_{tr}}$ is $N_{tr}gs -$ continuous.

(v) If $f_{N_{tr}}$ is $N_{tr}\Upsilon -$ continuous, then $f_{N_{tr}}$ is $N_{tr}gb -$ continuous.

Proof: Proof is obvious.

However, the ensuing examples reveal that the converse of these implications is not necessarily true in general.

Example 4.6: Let $U = \{a,b\}, V = \{x,y\}, \tau_{N_{tr}} = \{0_{N_{tr}}, 1_{N_{tr}}, L\}$ and $\rho_{N_{tr}} = \{0_{N_{tr}}, 1_{N_{tr}}, M\}$ where $L = \{< a, 0.2,0.4,0.7 >< b, 0.1,0.2,0.3 >\}$, $M = \{< x, 0,0.1,0.6 >< y, 0.1,0.2,0.9 >\}$. Consider the collections $\mathcal{P} = \{P :\ 0_{N_{tr}} \subset P \subset L\}$, $\mathcal{Q} = \{Q :\ L \not\subset Q\ ; Q \not\subset L\ ; Q \subset L^c\}$ and $\mathcal{R} = \{R : L \subset R \subset L^c\}$ of neutrosophic sets in U. Then, $N_{tr}\alpha O(U,\tau_{N_{tr}}) = \{0_{N_{tr}}, 1_{N_{tr}}, L\}, N_{tr}SO(U,\tau_{N_{tr}}) = \{0_{N_{tr}}, L, L^c, \mathcal{R}, 1_{N_{tr}}\}$

and $N_{tr}\Upsilon O(U,\tau_{N_{tr}}) = \{0_{N_{tr}}, L, L^c, \mathcal{P}, \mathcal{Q}, \mathcal{R}, 1_{N_{tr}}\}$. Define $f_{N_{tr}}: (U,\tau_{N_{tr}}) \longrightarrow (V,\rho_{N_{tr}})$ as $f_{N_{tr}}(a) = y$ and $f_{N_{tr}}(b) = x$. Then, $f_{N_{tr}}^{-1}(M) = \{< a, 0.1,0.2,0.9 >< b, 0,0.1,0.6 >\} \in \mathcal{P}$ which implies $f_{N_{tr}}^{-1}(M)$is $N_{tr}\Upsilon-$open. However, it is neither N_{tr}semi$-$open nor $N_{tr}\alpha-$open in U. Hence $f_{N_{tr}}$ is $N_{tr}\Upsilon-$continuous but not N_{tr}semi$-$continuous and $N_{tr}\alpha-$continuous.

Example 4.7: Let $U = \{a,b\}, V = \{x,y\}, \tau_{N_{tr}} = \{0_{N_{tr}}, 1_{N_{tr}}, L\}$ and $\rho_{N_{tr}} = \{0_{N_{tr}}, 1_{N_{tr}}, M\}$ where $L = \{< a, 0.7,0.8,0.6 >< b, 0.7,0.7,0.5 >\}$ and $M = \{< x, 0.5,0.7,0.2 >< y, 0.6,0.9,0.1 >\}$. Consider the collections $\mathcal{P} = \{P : L^c \subset P \subset L\}$, $\mathcal{Q} = \{Q : L \subset Q \subset 1_{N_{tr}}\}$, $\mathcal{R} = \{R : L^c \not\subset R ; R \not\subset L^c ; R \subset L\}$, $\mathcal{S} = \{S : L^c \not\subset S ; S \not\subset L^c ; S \not\subset L\}$, $\mathcal{T} = \{T : L^c \subset T \not\subset L\}$ and $\mathcal{W} = \{W : 0_{N_{tr}} \subset W \subset L^c\}$ of neutrosophic sets in U. Then, $N_{tr}\beta O(U,\tau_{N_{tr}}) = \{0_{N_{tr}}, L, \mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}, \mathcal{T}, 1_{N_{tr}}\}$, $N_{tr}gsO(U,\tau_{N_{tr}}) = \{0_{N_{tr}}, L, \mathcal{Q}, \mathcal{R}, \mathcal{S}, \mathcal{W}, 1_{N_{tr}}\}$, $N_{tr}gbO(U,\tau_{N_{tr}}) = \{0_{N_{tr}}, L, \mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{W}, 1_{N_{tr}}\}$ and $N_{tr}\Upsilon O(U,\tau_{N_{tr}}) = \{0_{N_{tr}}, L, \mathcal{Q}, 1_{N_{tr}}\}$. Define $f_{N_{tr}}: (U,\tau_{N_{tr}}) \longrightarrow (V,\rho_{N_{tr}})$ as $f_{N_{tr}}(a) = x$ and $f_{N_{tr}}(b) = y$. Then, $f_{N_{tr}}^{-1}(M) = \{< a, 0.5,0.7,0.2 >< b, 0.6,0.9,0.1 >\} \in \mathcal{S}$ which implies $f_{N_{tr}}^{-1}(M)$ is $N_{tr}\beta-$open, $N_{tr}gs-$open and $N_{tr}gb-$open but not $N_{tr}\Upsilon-$open. Hence $f_{N_{tr}}$ is $N_{tr}\beta-$continuous, $N_{tr}gs-$continuous and $N_{tr}gb-$continuous but not $N_{tr}\Upsilon-$continuous.

Theorem 4.8: Let $f_{N_{tr}}: (U,\tau_{N_{tr}}) \longrightarrow (V,\rho_{N_{tr}})$ be a function between two neutrosophic topological spaces. Then the following statements are equivalent:

(i) $f_{N_{tr}}$ is $N_{tr}\Upsilon-$continuous.

(ii) The inverse image of every N_{tr}closed set in $(V,\rho_{N_{tr}})$ is $N_{tr}\Upsilon-$closed in $(U,\tau_{N_{tr}})$.

(iii) $f_{N_{tr}}(N_{tr}\Upsilon cl(L)) \subseteq N_{tr}cl\left(f_{N_{tr}}(L)\right)$ for every neutrosophic set L in U.

(iv) $N_{tr}\Upsilon cl\left(f_{N_{tr}}^{-1}(M)\right) \subseteq f_{N_{tr}}^{-1}(N_{tr}cl(M))$ for every neutrosophic set M in V.

Proof:

(i)⟹(ii) Let $f_{N_{tr}}$ be a $N_{tr}\Upsilon-$continuous function and N be a N_{tr}closed set in V. Then N^c is N_{tr}open in V. Since $f_{N_{tr}}$ is $N_{tr}\Upsilon-$continuous, $f_{N_{tr}}^{-1}(N^c)$ is $N_{tr}\Upsilon-$open in U. That is, $(f_{N_{tr}}^{-1}(N))^c$ is $N_{tr}\Upsilon-$open in U. Hence $f_{N_{tr}}^{-1}(N)$ is $N_{tr}\Upsilon-$closed in U.

(ii)⟹(i) Let M be N_{tr}open in V. Then M^c is N_{tr}closed in V. By assumption, $f_{N_{tr}}^{-1}(M^c)$ is $N_{tr}\Upsilon-$closed in U. That is, $(f_{N_{tr}}^{-1}(M))^c$is $N_{tr}\Upsilon-$closed in U. Hence $f_{N_{tr}}^{-1}(M)$ is $N_{tr}\Upsilon-$open in U. Therefore, $f_{N_{tr}}$ is $N_{tr}\Upsilon-$continuous.

(ii)⟹(iii) Let L be a neutrosophic set in U. Now, $L \subseteq f_{N_{tr}}^{-1}\left(f_{N_{tr}}(L)\right)$ implies $L \subseteq f_{N_{tr}}^{-1}(N_{tr}cl\left(f_{N_{tr}}(L)\right)$

Since $N_{tr}cl\left(f_{N_{tr}}(L)\right)$ is N_{tr}closed in V, by assumption $f_{N_{tr}}^{-1}\left(N_{tr}cl\left(f_{N_{tr}}(L)\right)\right)$ is a $N_{tr}\Upsilon-$closed set containing L. Also, $N_{tr}\Upsilon cl(L)$ is the smallest $N_{tr}\Upsilon-$closed set containing L. Hence, $N_{tr}\Upsilon cl(L) \subseteq f_{N_{tr}}^{-1}(N_{tr}cl\left(f_{N_{tr}}(L)\right)$. Therefore, $f_{N_{tr}}(N_{tr}\Upsilon cl(L)) \subseteq N_{tr}cl\left(f_{N_{tr}}(L)\right)$.

(iii)⟹(ii) Let N be a N_{tr}closed set in V. Then, by assumption

$f_{N_{tr}}\left(N_{tr}\Upsilon cl\left(f_{N_{tr}}^{-1}(N)\right)\right) \subseteq N_{tr}cl\left(f_{N_{tr}}\left(f_{N_{tr}}^{-1}(N)\right)\right) \subseteq N_{tr}cl(N) = N$ implies $N_{tr}\Upsilon cl\left(f_{N_{tr}}^{-1}(N)\right) \subseteq f_{N_{tr}}^{-1}(N)$.

Also, $f_{N_{tr}}^{-1}(N) \subseteq N_{tr}\Upsilon cl\left(f_{N_{tr}}^{-1}(N)\right)$. Hence $f_{N_{tr}}^{-1}(N)$ is $N_{tr}\Upsilon-$closed in U.

(iii)⟹(iv) Let M be a neutrosophic set in V and let $L = f_{N_{tr}}^{-1}(M)$. By assumption, $f_{N_{tr}}(N_{tr}\Upsilon cl(L)) \subseteq N_{tr}cl\left(f_{N_{tr}}(L)\right) = N_{tr}cl(M)$.This implies $N_{tr}\Upsilon cl\left(f_{N_{tr}}^{-1}(M)\right) \subseteq f_{N_{tr}}^{-1}(N_{tr}cl(M))$.

(iv)⟹(iii) Let $M = f_{N_{tr}}(L)$. Then, by assumption, $N_{tr}\Upsilon cl(L) = N_{tr}\Upsilon cl\left(f_{N_{tr}}^{-1}(M)\right) \subseteq f_{N_{tr}}^{-1}(N_{tr}cl(M)) \subseteq f_{N_{tr}}^{-1}\left(N_{tr}cl\left(f_{N_{tr}}(L)\right)\right)$. This implies $f_{N_{tr}}(N_{tr}\Upsilon cl(L)) \subseteq N_{tr}cl\left(f_{N_{tr}}(L)\right)$.

(iv) ⟹ (i) Let M be N_{tr}open in V. Then M^c is N_{tr}closed in V. By assumption, $f_{N_{tr}}^{-1}(M^c) = f_{N_{tr}}^{-1}(N_{tr}cl(M^c)) \supseteq N_{tr}\Upsilon cl\left(f_{N_{tr}}^{-1}(M^c)\right)$. Also, we know that $f_{N_{tr}}^{-1}(M^c) \subseteq N_{tr}\Upsilon cl\left(f_{N_{tr}}^{-1}(M^c)\right)$. Hence $f_{N_{tr}}^{-1}(M^c) = N_{tr}\Upsilon cl\left(f_{N_{tr}}^{-1}(M^c)\right)$. Therefore, $f_{N_{tr}}^{-1}(M^c)$ is $N_{tr}\Upsilon -$closed in U. That is, $\left(f_{N_{tr}}^{-1}(M)\right)^c$ is $N_{tr}\Upsilon -$closed in U. Hence $f_{N_{tr}}^{-1}(M)$ is $N_{tr}\Upsilon -$open in U. Therefore $f_{N_{tr}}$ is $N_{tr}\Upsilon -$continuous.

Example 4.9: (i) Consider the topological spaces and the functions defined in example 4.2. Here $f_{N_{tr}}$ is $N_{tr}\Upsilon -$continuous and $N_{tr}\Upsilon C(U, \tau_{N_{tr}}) = \{0_{N_{tr}}, L^c, M^c, \mathcal{P}', \mathcal{Q}', 1_{N_{tr}}\}$ where $\mathcal{P}' = \{P^c : P \in \mathcal{P}\}$ and $\mathcal{Q}' = \{Q^c : Q \in \mathcal{Q}\}$. Now, $f_{N_{tr}}^{-1}(N^c) = \{< a, 0.2\ \ 0.1, 0.8 >< b, 0.1, 0.1, 0.9 >\} \in \mathcal{P}'$. Hence the inverse image of every N_{tr}closed set in $(V, \rho_{N_{tr}})$ is $N_{tr}\Upsilon -$closed in $(U, \tau_{N_{tr}})$ if $f_{N_{tr}}$ is $N_{tr}\Upsilon -$continuous.
(ii) Let $U = \{a, b\}, V = \{x, y\}, \tau_{N_{tr}} = \{0_{N_{tr}}, 1_{N_{tr}}, L\}$ and $\rho_{N_{tr}} = \{0_{N_{tr}}, 1_{N_{tr}}, M\}$ where $L = \{< a, 0.2, 0.4, 0.9 >< b, 0.3, 0.8, 0.7 >\}$ and $M = \{< x, 0.9, 0.7, 0.1 >< y, 0.8, 0.9, 0.2 >\}$. Consider the collections $\mathcal{P} = \{P : P \subset L, P \subset L^c\}$ and $\mathcal{Q} = \{Q : Q \subset L^c; Q \not\subset L;\ L \not\subset Q\}$ of neutrosophic sets in U. Then $N_{tr}\Upsilon C(U, \tau_{N_{tr}}) = \{0_{N_{tr}}, L^c, \mathcal{P}, \mathcal{Q}, 1_{N_{tr}}\}$. Now, define $f_{N_{tr}} : (U, \tau_{N_{tr}}) \longrightarrow (V, \rho_{N_{tr}})$ as $f_{N_{tr}}(a) = x$ and $f_{N_{tr}}(b) = y$. Then, $f_{N_{tr}}^{-1}(M^c) = \{< a, 0.1, 0.3, 0.9 >< b, 0.2, 0.1, 0.8 >\} \in \mathcal{P}$. Now, $f_{N_{tr}}^{-1}(M^c) = (f_{N_{tr}}^{-1}(M))^c$ is $N_{tr}\Upsilon -$closed implies $f_{N_{tr}}^{-1}(M)$ is $N_{tr}\Upsilon -$open. Hence $f_{N_{tr}}$ is $N_{tr}\Upsilon -$continuous if the inverse image of every N_{tr}closed set in $(V, \rho_{N_{tr}})$ is $N_{tr}\Upsilon -$closed in $(U, \tau_{N_{tr}})$.

Theorem 4.10: A function $f_{N_{tr}} : (U, \tau_{N_{tr}}) \longrightarrow (V, \rho_{N_{tr}})$ is $N_{tr}\Upsilon -$continuous if and only if $f_{N_{tr}}^{-1}(N_{tr}int(M)) \subseteq N_{tr}\Upsilon int\left(f_{N_{tr}}^{-1}(M)\right)$ for every neutrosophic set M in V.

Proof: Let $f_{N_{tr}}$ be a $N_{tr}\Upsilon -$continuous function and M be a neutrosophic set in V. Then $N_{tr}int(M)$ is N_{tr}open in V. By assumption, $f_{N_{tr}}^{-1}(N_{tr}int(M))$ is $N_{tr}\Upsilon -$open in U. Now, $f_{N_{tr}}^{-1}(N_{tr}int(M)) \subseteq f_{N_{tr}}^{-1}(M)$ and $N_{tr}\Upsilon int\left(f_{N_{tr}}^{-1}(M)\right)$ is the largest $N_{tr}\Upsilon -$open set contained in $f_{N_{tr}}^{-1}(M)$. Hence $f_{N_{tr}}^{-1}(N_{tr}int(M)) \subseteq N_{tr}\Upsilon int\left(f_{N_{tr}}^{-1}(M)\right)$. Conversely, let M be a N_{tr}open set in V. Then $f_{N_{tr}}^{-1}(M) = f_{N_{tr}}^{-1}(N_{tr}int(M)) \subseteq N_{tr}\Upsilon int\left(f_{N_{tr}}^{-1}(M)\right)$. Also, $N_{tr}\Upsilon int\left(f_{N_{tr}}^{-1}(M)\right) \subseteq f_{N_{tr}}^{-1}(M)$. This implies $f_{N_{tr}}^{-1}(M)$ is $N_{tr}\Upsilon -$open in U. Hence $f_{N_{tr}}$ is $N_{tr}\Upsilon -$continuous.

Theorem 4.11: Let $f_{N_{tr}} : (U, \tau_{N_{tr}}) \longrightarrow (V, \rho_{N_{tr}})$ be a function between two neutrosophic topological spaces. Then the following statements are equivalent:
(i) $f_{N_{tr}}$ is $N_{tr}\Upsilon -$continuous.
(ii) For each neutrosophic point $u_{a,b,c}$, the inverse image of every $N_{tr}nbhd$ of $f_{N_{tr}}(u_{a,b,c})$ is $N_{tr}\Upsilon - nbhd$ of $u_{a,b,c}$.
(iii) For each neutrosophic point $u_{a,b,c}$in U and every $N_{tr}nbhd$ N of $f_{N_{tr}}(u_{a,b,c})$, there exists a $N_{tr}\Upsilon -$open set L in U such that $u_{a,b,c} \in L$ and $f_{N_{tr}}(L) \subseteq N$.

Proof:

(i)⟹(ii) Let $u_{a,b,c}$ be a neutrosophic point in U and let N be a $N_{tr}nbhd$ of $f_{N_{tr}}(u_{a,b,c})$. Then there exists a N_{tr} open set M in V such that $f_{N_{tr}}(u_{a,b,c}) \in M \subseteq N$. Since $f_{N_{tr}}$ is $N_{tr}\Upsilon-$continuous, $f_{N_{tr}}^{-1}(M)$ is $N_{tr}\Upsilon-$open in U. Also, $u_{a,b,c} \in f_{N_{tr}}^{-1}\left(f_{N_{tr}}(u_{a,b,c})\right) \in f_{N_{tr}}^{-1}(M) \subseteq f_{N_{tr}}^{-1}(N)$. Hence there exists a $N_{tr}\Upsilon-$open set $f_{N_{tr}}^{-1}(M)$ such that $u_{a,b,c} \in f_{N_{tr}}^{-1}(M) \subseteq f_{N_{tr}}^{-1}(N)$. This implies $f_{N_{tr}}^{-1}(N)$ is $N_{tr}\Upsilon-nbhd$ of $u_{a,b,c}$.

(ii)⟹(iii) Let $u_{a,b,c}$ be a neutrosophic point in U and let N be a $N_{tr}nbhd$ of $f_{N_{tr}}(u_{a,b,c})$. Then by assumption, $f_{N_{tr}}^{-1}(N)$ is $N_{tr}\Upsilon-nbhd$ of $u_{a,b,c}$. Then there exists a $N_{tr}\Upsilon-$open set L in U such that $u_{a,b,c} \in L \subseteq f_{N_{tr}}^{-1}(N)$. Thus $u_{a,b,c} \in L$ and $f_{N_{tr}}(L) \subseteq f_{N_{tr}}\left(f_{N_{tr}}^{-1}(N)\right) \subseteq N$.

(iii) ⟹ (i) Let M be a N_{tr} open set in V and let $u_{a,b,c} \in f_{N_{tr}}^{-1}(M)$. Since M is N_{tr} open and $f_{N_{tr}}(u_{a,b,c}) \in M, M$ is a $N_{tr}nbhd$ of $f_{N_{tr}}(u_{a,b,c})$. Hence it follows (iii) that there exists a $N_{tr}\Upsilon-$open set L in U such that $u_{a,b,c} \in L$ and $f_{N_{tr}}(L) \subseteq M$. This implies $u_{a,b,c} \in L \subseteq f_{N_{tr}}^{-1}\left(f_{N_{tr}}(L)\right) \subseteq f_{N_{tr}}^{-1}(M)$. By theorem 2.16, $f_{N_{tr}}^{-1}(M)$ is $N_{tr}\Upsilon-$open in U. Therefore $f_{N_{tr}}$ is $N_{tr}\Upsilon-$continuous.

Remark 4.12: The statements of theorem 4.8, 4.10 and 4.11 are all equivalent.

Definition 4.13: A neutrosophic topological space $(U, \tau_{N_{tr}})$ is said to be $\boldsymbol{N_{tr}T_{\Upsilon}}$ **−space** if every $N_{tr}\Upsilon-$open set in $(U, \tau_{N_{tr}})$ is N_{tr}open.

Remark 4.14: The composition of two $N_{tr}\Upsilon-$continuous functions need not be $N_{tr}\Upsilon-$continuous.

Example 4.15: Let $U = \{a, b\}, V = \{x, y\}$and $W = \{p, q\}$. Consider the neutrosophic topologies $\tau_{N_{tr}} = \{0_{N_{tr}}, 1_{N_{tr}}, L\}, \rho_{N_{tr}} = \{0_{N_{tr}}, 1_{N_{tr}}, M\}$ and $\xi_{N_{tr}} = \{0_{N_{tr}}, 1_{N_{tr}}, N\}$ where $L = \{< a, 0.3,0.4,0.9 > < b, 0.4,0.5,0.8 >\}$, $M = \{< x, 0.9,0.6,0.3 >< y, 0.8,0.5,0.4 >\}$ and $N = \{< p, 0.9,0.6,0.1 >< q, 0.9,0.7 0.2 >\}$. Consider the collections $\mathcal{P} = \{P : 0_{N_{tr}} \subset P \subset L\}, \mathcal{Q} = \{Q : L \subset Q \subset L^c\}, \mathcal{R} = \{R : R \not\subset L; L \not\subset R; R \subset L^c\}$ of neutrosophic sets in U and $\mathcal{S} = \{S : M \subset S \subset 1_{N_{tr}}\}$, the collection of neutrosophic sets in V. Then, $N_{tr}\Upsilon O(U, \tau_{N_{tr}}) = \{0_{N_{tr}}, L, L^c, \mathcal{P}, \mathcal{Q}, \mathcal{R}, 1_{N_{tr}}\}$ and $N_{tr}\Upsilon O(V, \rho_{N_{tr}}) = \{0_{N_{tr}}, M, \mathcal{S}, 1_{N_{tr}}\}$. Define $f_{N_{tr}}: (U, \tau_{N_{tr}}) \longrightarrow (V, \rho_{N_{tr}})$ as $f_{N_{tr}}(a) = x$ and $f_{N_{tr}}(b) = y$. Then $f_{N_{tr}}^{-1}(M) = \{< a, 0.9,0.6,0.3 > < b, 0.8,0.9,0.4 >\}$ is $N_{tr}\Upsilon-$open in $(U, \tau_{N_{tr}})$. Also, define $g_{N_{tr}}: (V, \rho_{N_{tr}}) \longrightarrow (W, \xi_{N_{tr}})$ as $g_{N_{tr}}(x) = q$ and $g_{N_{tr}}(y) = p$. Then $g_{N_{tr}}^{-1}(N) = \{< x, 0.9,0.7,0.2 >< y, 0.9,0.6,0.1 >\} \in \mathcal{S}$ which implies $g_{N_{tr}}^{-1}(N)$ is $N_{tr}\Upsilon-$open in $(V, \rho_{N_{tr}})$. This implies that both $f_{N_{tr}}$ and $g_{N_{tr}}$ are $N_{tr}\Upsilon-$continuous. Now, let $g_{N_{tr}} \circ f_{N_{tr}}: (U, \tau_{N_{tr}}) \longrightarrow (W, \xi_{N_{tr}})$ be the composition of two $N_{tr}\Upsilon-$continuous functions. Then, $g_{N_{tr}} \circ f_{N_{tr}}$ is not $N_{tr}\Upsilon-$continuous since $\left(g_{N_{tr}} \circ f_{N_{tr}}\right)^{-1}(N) = f_{N_{tr}}^{-1}\left(g_{N_{tr}}^{-1}(N)\right) = \{< a, 0.9,0.7,0.2 >< b, 0.9,0.6,0.1 >\}$ is not $N_{tr}\Upsilon-$open in $(U, \tau_{N_{tr}})$.

Theorem 4.16: Let $(U, \tau_{N_{tr}}), (V, \rho_{N_{tr}})$ and $(W, \xi_{N_{tr}})$ be neutrosophic topological space and let $(V, \rho_{N_{tr}})$ be $N_{tr}T_{\Upsilon}$ −space. Then the composition $g_{N_{tr}} \circ f_{N_{tr}}: (U, \tau_{N_{tr}}) \longrightarrow (W, \xi_{N_{tr}})$ of two $N_{tr}\Upsilon-$continuous functions $f_{N_{tr}}: (U, \tau_{N_{tr}}) \longrightarrow (V, \rho_{N_{tr}})$ and $g_{N_{tr}}: (V, \rho_{N_{tr}}) \longrightarrow (W, \xi_{N_{tr}})$ is $N_{tr}\Upsilon-$continuous.

Proof: Let N be any N_{tr}open set in W. Since $g_{N_{tr}}$ is $N_{tr}\Upsilon-$continuous, $g_{N_{tr}}^{-1}(N)$ is $N_{tr}\Upsilon-$open in V. Then, by assumption $g_{N_{tr}}^{-1}(N)$ is N_{tr} open in V. Also, since $f_{N_{tr}}$ is $N_{tr}\Upsilon-$continuous, $f_{N_{tr}}^{-1}\left(g_{N_{tr}}^{-1}(N)\right) = \left(g_{N_{tr}} \circ f_{N_{tr}}\right)^{-1}(N)$ is $N_{tr}\Upsilon-$open in U. Hence $g_{N_{tr}} \circ f_{N_{tr}}$ is $N_{tr}\Upsilon-$continuous.

Theorem 4.17: Let $f_{N_{tr}}:(U,\tau_{N_{tr}}) \longrightarrow (V,\rho_{N_{tr}})$ be a $N_{tr}\Upsilon-$ continuous function and $g_{N_{tr}}:(V,\rho_{N_{tr}}) \longrightarrow (W,\xi_{N_{tr}})$ be a N_{tr}continuous function. Then their composition $g_{N_{tr}} \circ f_{N_{tr}}:(U,\tau_{N_{tr}}) \longrightarrow (W,\xi_{N_{tr}})$is $N_{tr}\Upsilon-$ continuous.

Proof: Let N be any N_{tr}open set in W. Since $g_{N_{tr}}$ is N_{tr}continuous, $g_{N_{tr}}^{-1}(N)$ is N_{tr}open in V.Also, since $g_{N_{tr}}$ is $N_{tr}\Upsilon-$ continuous,$f_{N_{tr}}^{-1}\left(g_{N_{tr}}^{-1}(N)\right) = \left(g_{N_{tr}} \circ f_{N_{tr}}\right)^{-1}(N)$ is $N_{tr}\Upsilon-$ open in U. Hence $g_{N_{tr}} \circ f_{N_{tr}}$ is $N_{tr}\Upsilon-$ continuous.

Theorem 4.18: Let $f_{N_{tr}}:(U,\tau_{N_{tr}}) \longrightarrow (V,\rho_{N_{tr}})$ be a $N_{tr}\Upsilon-$ continuous function where $(U,\tau_{N_{tr}})$ is a $N_{tr}T_{\Upsilon}$ −space. If S is a subset of U, then the restriction $f_{N_{tr}}|_S : (S,\tau_{N_{tr}}^S) \longrightarrow (V,\rho_{N_{tr}})$ is also $N_{tr}\Upsilon-$ continuous.

Proof: Let M be a N_{tr}open set in V. Since $f_{N_{tr}}$ is $N_{tr}\Upsilon-$ continuous, $f_{N_{tr}}^{-1}(M)$ is $N_{tr}\Upsilon-$open in U. Now, since U is a $N_{tr}T_{\Upsilon}$ −space, $f_{N_{tr}}^{-1}(M)$ is N_{tr}open in U. Hence $f_{N_{tr}}|_S{}^{-1}(M) = f_{N_{tr}}^{-1}(M) \cap 1_{N_{tr}}^S$ is N_{tr} open in S. By theorem 2.14, $f_{N_{tr}}|_S{}^{-1}(M)$ is $N_{tr}\Upsilon-$ open in S. Hence $f_{N_{tr}}|_S$ is $N_{tr}\Upsilon-$ continuous.

5. Neutrosophic Υ −irresolute functions

Analogous to the previous section, this segment deals with the concept of neutrosophic $\Upsilon-$ irresolute functions and its behavior.

Definition 5.1: Let $(U,\tau_{N_{tr}})$ and $(V,\rho_{N_{tr}})$ be neutrosophic topological spaces. Then the function $f_{N_{tr}}:(U,\tau_{N_{tr}}) \longrightarrow (V,\rho_{N_{tr}})$ is said to be neutrosophic $\Upsilon-$irresolute if $f_{N_{tr}}^{-1}(M)$ is $N_{tr}\Upsilon-$ open in $(U,\tau_{N_{tr}})$ for every $N_{tr}\Upsilon-$ open set M in $(V,\rho_{N_{tr}})$.

Example 5.2: Let $U = \{a,b\}, V = \{x,y\}, \tau_{N_{tr}} = \{0_{N_{tr}}, 1_{N_{tr}}, L\}$ and $\rho_{N_{tr}} = \{0_{N_{tr}}, 1_{N_{tr}}, M\}$ where $L = \{< a, 0.5,0.6,0.3 >< b, 0.6,0.7,0.2 >\}$ and $M = \{< x, 0.5,0.7,0.3 >< y, 0.8,0.7,0.2 >\}$. Also, consider the collections $\mathcal{P} = \{P : L \subset P \subset 1_{N_{tr}}\}$ and $\mathcal{Q} = \{Q : M \subset Q \subset 1_{N_{tr}}\}$ of neutrosophic sets in U and V respectively. Then, $N_{tr}\Upsilon O(U,\tau_{N_{tr}}) = \{0_{N_{tr}}, L, \mathcal{P}, 1_{N_{tr}}\}$ and $N_{tr}\Upsilon O(V,\rho_{N_{tr}}) = \{< 0_{N_{tr}}, M, \mathcal{Q}, 1_{N_{tr}}\}$. Now, let us define $f_{N_{tr}}:(U,\tau_{N_{tr}}) \longrightarrow (V,\rho_{N_{tr}})$ as $f_{N_{tr}}(a) = x$ and $f_{N_{tr}}(b) = y$. Then,$f_{N_{tr}}^{-1}(M) = \{< a, 0.5,0.7,0.3 >< b, 0.8,0.7,0.2 >\} \in \mathcal{P}$ and for each $Q \in \mathcal{Q}$, there exists some $P \in \mathcal{P}$ such that $f_{N_{tr}}^{-1}(Q) = P$. Hence the inverse image of every $N_{tr}\Upsilon-$ open set in V is $N_{tr}\Upsilon-$ open in U. Therefore $f_{N_{tr}}$ is $N_{tr}\Upsilon$ −irresolute.

Theorem 5.3: Every $N_{tr}\Upsilon$ −irresolute function is $N_{tr}\Upsilon-$ continuous.

Proof: Let $f_{N_{tr}}:(U,\tau_{N_{tr}}) \longrightarrow (V,\rho_{N_{tr}})$ be a $N_{tr}\Upsilon$ −irresolute function and M be a N_{tr}open set in V. Then, by theorem 2.14, M is $N_{tr}\Upsilon-$ open in V. Since $f_{N_{tr}}$ is $N_{tr}\Upsilon$ −irreolsute, $f_{N_{tr}}^{-1}(M)$ is $N_{tr}\Upsilon-$ open in U. Hence $f_{N_{tr}}$ is $N_{tr}\Upsilon-$ continuous.

The following example substantiates that the converse of the above-stated theorem need not be true.

Example 5.4: Let $U = \{a,b\}$, $V = \{x,y\}$, $\tau_{N_{tr}} = \{0_{N_{tr}}, 1_{N_{tr}}, L\}$ and $\rho_{N_{tr}} = \{0_{N_{tr}}, 1_{N_{tr}}, M\}$ where $L = \{< a, 0.1,0.3,0.7 >< b, 0.3,0.2,0.8 >\}$ and $M = \{< x, 0.7,0.7,0.1 >< y, 0.8,0.8,0.3 >\}$. Consider the collections $\mathcal{P} = \{P : 0_{N_{tr}} \subset P \subset L\}$, $\mathcal{Q} = \{Q : L \not\subset Q \ ; Q \not\subset L \ ; Q \subset L^c\}$ and $\mathcal{R} = \{R : L \subset R \subset L^c\}$ of neutrosophic sets in U. Then, $N_{tr}\Upsilon O(U,\tau_{N_{tr}}) = \{0_{N_{tr}}, L, L^c, \mathcal{P}, \mathcal{Q}, \mathcal{R}, 1_{N_{tr}}\}$. Now, let us define $f_{N_{tr}}:(U,\tau_{N_{tr}}) \longrightarrow (V,\rho_{N_{tr}})$ as $f_{N_{tr}}(a) = x$ and $f_{N_{tr}}(b) = y$. Then,$f_{N_{tr}}^{-1}(M) = \{< a, 0.7,0.7,0.1 >< b, 0.8, 0.8,0.3 >\} = L^c$ which implies $f_{N_{tr}}$ is $N_{tr}\Upsilon$ −continuous. However, the inverse image of a $N_{tr}\Upsilon-$ open set $S = \{< x, 0.8,0.7,0.1 >< y, 0.9,0.8,0.2 >\}$ in V is not $N_{tr}\Upsilon-$ open in U. Hence $f_{N_{tr}}$ is $N_{tr}\Upsilon-$ continuous but not $N_{tr}\Upsilon$ −irresolute.

Theorem 5.5: Let $f_{N_{tr}}: (U, \tau_{N_{tr}}) \longrightarrow (V, \rho_{N_{tr}})$ be a $N_{tr}\Upsilon-$continuous function where $(V, \rho_{N_{tr}})$ is a $N_{tr}T_{\Upsilon}$ $-$space. Then $f_{N_{tr}}$ is $N_{tr}\Upsilon$ $-$irresolute.

Proof: Let M be $N_{tr}\Upsilon-$ open in V. Then, by assumption M is N_{tr}open in V. Since $f_{N_{tr}}$ is $N_{tr}\Upsilon-$ continuous, $f_{N_{tr}}^{-1}(M)$ is $N_{tr}\Upsilon-$ open in U. Hence $f_{N_{tr}}$ is $N_{tr}\Upsilon$ $-$irresolute.

Theorem 5.6: Let $f_{N_{tr}}: (U, \tau_{N_{tr}}) \longrightarrow (V, \rho_{N_{tr}})$ be a function between two neutrosophic topological spaces. Then the following statements are equivalent:

(i) $f_{N_{tr}}$ is $N_{tr}\Upsilon$ $-$irresolute.

(ii) The inverse image of every $N_{tr}\Upsilon-$ closed set in $(V, \rho_{N_{tr}})$ is $N_{tr}\Upsilon-$ closed in $(U, \tau_{N_{tr}})$.

(iii) $f_{N_{tr}}(N_{tr}\Upsilon cl(L)) \subseteq N_{tr}\Upsilon cl\left(f_{N_{tr}}(L)\right)$ for every neutrosophic set L in U.

(iv) $N_{tr}\Upsilon cl\left(f_{N_{tr}}^{-1}(M)\right) \subseteq f_{N_{tr}}^{-1}(N_{tr}\Upsilon cl(M))$ for every neutrosophic set M in V.

(v) $f_{N_{tr}}^{-1}(N_{tr}\Upsilon int(M)) \subseteq N_{tr}\Upsilon int\left(f_{N_{tr}}^{-1}(M)\right)$ for every neutrosophic set M in V.

(vi) For each neutrosophic point $u_{a,b,c}$, the inverse image of every $N_{tr}\Upsilon - nbhd$ of $f_{N_{tr}}(u_{a,b,c})$ is $N_{tr}\Upsilon - nbhd$ of $u_{a,b,c}$.

(vii) For each neutrosophic point $u_{a,b,c}$in U and every $N_{tr}\Upsilon - nbhdN$ of $f_{N_{tr}}(u_{a,b,c})$, there exists a $N_{tr}\Upsilon$ $-$open set L in U such that $u_{a,b,c} \in L$ and $f_{N_{tr}}(L) \subseteq N$.

Proof:

(i)⟹(ii) Let $f_{N_{tr}}$ be a $N_{tr}\Upsilon$ $-$irresolute function and N be a $N_{tr}\Upsilon-$ closed set in V. Then N^c is $N_{tr}\Upsilon-$ open in V. Since $f_{N_{tr}}$ is $N_{tr}\Upsilon-$ irresolute, $f_{N_{tr}}^{-1}(N^c)$ is $N_{tr}\Upsilon-$ open in U. That is, $(f_{N_{tr}}^{-1}(N))^c$ is $N_{tr}\Upsilon-$ open in U. Hence $f_{N_{tr}}^{-1}(N)$ is $N_{tr}\Upsilon-$ closed in U.

(ii)⟹(i) Let M be $N_{tr}\Upsilon-$ open in V. Then M^c is $N_{tr}\Upsilon-$ closed in V. By assumption, $f_{N_{tr}}^{-1}(M^c)$ is $N_{tr}\Upsilon-$ closed in U. That is, $(f_{N_{tr}}^{-1}(M))^c$is $N_{tr}\Upsilon-$ closed in U. Hence $f_{N_{tr}}^{-1}(M)$ is $N_{tr}\Upsilon-$ open in U. Therefore, $f_{N_{tr}}$ is $N_{tr}\Upsilon$ $-$irresolute.

(ii)⟹(iii) Let L be a neutrosophic set in U. Now, $L \subseteq f_{N_{tr}}^{-1}\left(f_{N_{tr}}(L)\right) \Rightarrow L \subseteq f_{N_{tr}}^{-1}\left(N_{tr}\Upsilon cl\left(f_{N_{tr}}(L)\right)\right)$.

Since $N_{tr}\Upsilon cl\left(f_{N_{tr}}(L)\right)$ is $N_{tr}\Upsilon-$ closed in V, by assumption $f_{N_{tr}}^{-1}\left(N_{tr}\Upsilon cl\left(f_{N_{tr}}(L)\right)\right)$ is a $N_{tr}\Upsilon-$ closed set containing L. Also, $N_{tr}\Upsilon cl(L)$ is the smallest $N_{tr}\Upsilon-$ closed set containing L. Hence, $N_{tr}\Upsilon cl(L) \subseteq f_{N_{tr}}^{-1}\left(N_{tr}\Upsilon cl\left(f_{N_{tr}}(L)\right)\right)$. Therefore, $f_{N_{tr}}(N_{tr}\Upsilon cl(L)) \subseteq N_{tr}\Upsilon cl\left(f_{N_{tr}}(L)\right)$.

(iii)⟹(ii) Let N be a $N_{tr}\Upsilon-$ closed set in V. Then, by assumption $f_{N_{tr}}\left(N_{tr}\Upsilon cl\left(f_{N_{tr}}^{-1}(N)\right)\right) \subseteq$ $N_{tr}\Upsilon cl\left(f_{N_{tr}}\left(f_{N_{tr}}^{-1}(N)\right)\right) \subseteq N_{tr}\Upsilon cl(N) = N$ implies $N_{tr}\Upsilon cl\left(f_{N_{tr}}^{-1}(N)\right) \subseteq f_{N_{tr}}^{-1}(N)$. Also, $f_{N_{tr}}^{-1}(N) \subseteq$ $N_{tr}\Upsilon cl\left(f_{N_{tr}}^{-1}(N)\right)$. Hence $f_{N_{tr}}^{-1}(N)$ is $N_{tr}\Upsilon-$ closed in U.

(iii)⟹(iv) Let M be a neutrosophic set in V and let $L = f_{N_{tr}}^{-1}(M)$. By assumption, $f_{N_{tr}}(N_{tr}\Upsilon cl(L)) \subseteq$ $N_{tr}\Upsilon cl\left(f_{N_{tr}}(L)\right) = N_{tr}\Upsilon cl(M)$.This implies $N_{tr}\Upsilon cl\left(f_{N_{tr}}^{-1}(M)\right) \subseteq f_{N_{tr}}^{-1}(N_{tr}\Upsilon cl(M))$.

(iv)⟹(iii) Let $M = f_{N_{tr}}(L)$. Then, by assumption, $N_{tr}\Upsilon cl(L) = N_{tr}\Upsilon cl\left(f_{N_{tr}}^{-1}(M)\right) \subseteq f_{N_{tr}}^{-1}\left(N_{tr}\Upsilon cl(M)\right)$

$= f_{N_{tr}}^{-1}\left(N_{tr}\Upsilon cl\left(f_{N_{tr}}(L)\right)\right)$. This implies $f_{N_{tr}}\left(N_{tr}\Upsilon cl(L)\right) \subseteq N_{tr}\Upsilon cl\left(f_{N_{tr}}(L)\right)$.

(iv)⟺(v) This can be proved by taking complements.

(v)⟹(i) Let M be a $N_{tr}\Upsilon -$ open set in V. Then $f_{N_{tr}}^{-1}(M) = f_{N_{tr}}^{-1}\left(N_{tr}\Upsilon int(M)\right) \subseteq N_{tr}\Upsilon int\left(f_{N_{tr}}^{-1}(M)\right)$.

Also, $N_{tr}\Upsilon int\left(f_{N_{tr}}^{-1}(M)\right) \subseteq f_{N_{tr}}^{-1}(M)$. This implies $f_{N_{tr}}^{-1}(M)$ is $N_{tr}\Upsilon -$ open in U. Hence $f_{N_{tr}}$ is $N_{tr}\Upsilon -$irresolute.

(i)⟹(vi) Let $u_{a,b,c}$ be a neutrosophic point in U and let N be a $N_{tr}\Upsilon - nbhd$ of $f_{N_{tr}}(u_{a,b,c})$. Then there exists a $N_{tr}\Upsilon -$ open set M in V such that $f_{N_{tr}}\left(u_{a,b,c}\right) \in M \subseteq N$. Since $f_{N_{tr}}$ is $N_{tr}\Upsilon -$irresolute, $f_{N_{tr}}^{-1}(M)$ is $N_{tr}\Upsilon -$ open in U. Also, $u_{a,b,c} \in f_{N_{tr}}^{-1}\left(f_{N_{tr}}\left(u_{a,b,c}\right)\right) \in f_{N_{tr}}^{-1}(M) \subseteq f_{N_{tr}}^{-1}(N)$. Hence there exists a $N_{tr}\Upsilon -$ open set $f_{N_{tr}}^{-1}(M)$ such that $u_{a,b,c} \in f_{N_{tr}}^{-1}(M) \subseteq f_{N_{tr}}^{-1}(N)$. This implies $f_{N_{tr}}^{-1}(N)$ is $N_{tr}\Upsilon - nbhd$ of $u_{a,b,c}$.

(vi)⟹(vii) Let $u_{a,b,c}$ be a neutrosophic point in U and let N be a $N_{tr}\Upsilon - nbhd$ of $f_{N_{tr}}(u_{a,b,c})$. Then by assumption, $f_{N_{tr}}^{-1}(N)$ is $N_{tr}\Upsilon - nbhd$ of $u_{a,b,c}$. Then there exists a $N_{tr}\Upsilon -$ open set L in U such that $u_{a,b,c} \in L \subseteq f_{N_{tr}}^{-1}(N)$. Thus $u_{a,b,c} \in L$ and $f_{N_{tr}}(L) \subseteq f_{N_{tr}}\left(f_{N_{tr}}^{-1}(N)\right) \subseteq N$.

(vii)⟹(i) Let M be a $N_{tr}\Upsilon -$ open set in V and let $u_{a,b,c} \in f_{N_{tr}}^{-1}(M)$. Since M is $N_{tr}\Upsilon -$ open and $f_{N_{tr}}\left(u_{a,b,c}\right) \in M, M$ is a $N_{tr}\Upsilon - nbhd$ of $f_{N_{tr}}\left(u_{a,b,c}\right)$. Hence it follows from (vii) that there exists a $N_{tr}\Upsilon -$ open set L in U such that $u_{a,b,c} \in L$ and $f_{N_{tr}}(L) \subseteq M$. This implies $u_{a,b,c} \in L \subseteq f_{N_{tr}}^{-1}\left(f_{N_{tr}}(L)\right) \subseteq f_{N_{tr}}^{-1}(M)$. By theorem 3.6, $f_{N_{tr}}^{-1}(M)$ is $N_{tr}\Upsilon -$ open in U. Therefore $f_{N_{tr}}$ is $N_{tr}\Upsilon -$irresolute.

Example 5.7: (i) Consider the topological spaces and the function $f_{N_{tr}}$ defined in example 5.2. Here $f_{N_{tr}}$ is $N_{tr}\Upsilon -$irresolute and $N_{tr}\Upsilon C\left(U, \tau_{N_{tr}}\right) = \left\{0_{N_{tr}}, L^c, \mathcal{P}', 1_{N_{tr}}\right\}$, $N_{tr}\Upsilon C\left(V, \rho_{N_{tr}}\right) = \left\{0_{N_{tr}}, M^c, \mathcal{Q}', 1_{N_{tr}}\right\}$ where $\mathcal{P}' = \{P^c : P \in \mathcal{P}\}$ and $\mathcal{Q}' = \{Q^c : Q \in \mathcal{Q}\}$. Now, $f_{N_{tr}}^{-1}(M^c) = \{< a, 0.3, 0.3, 0.5 >< b, 0.2, 0.3, 0.8 >\} \in \mathcal{P}'$ and for each $Q \in \mathcal{Q}'$, there exists some $P \in \mathcal{P}'$ such that $f_{N_{tr}}^{-1}(Q) = P$. Hence the inverse image of every $N_{tr}\Upsilon -$ closed set in $(V, \rho_{N_{tr}})$ is $N_{tr}\Upsilon -$ closed in $(U, \tau_{N_{tr}})$ if $f_{N_{tr}}$ is $N_{tr}\Upsilon -$irresolute.

(ii) Let $U = \{a, b\}, V = \{x, y\}, \tau_{N_{tr}} = \left\{0_{N_{tr}}, 1_{N_{tr}}, L\right\}$ and $\rho_{N_{tr}} = \left\{0_{N_{tr}}, 1_{N_{tr}}, M\right\}$ where $L = \{< a, 0.7, 0.5, 0.5 >< b, 0.8, 0.6, 0.4 >\}$ and $M = \{< x, 0.8, 0.6, 0.4 >< y, 0.9, 0.7, 0.1 >\}$. Consider the collections $\mathcal{P} = \{P : 0_{N_{tr}} \subset P \subset L^c\}$ and $\mathcal{Q} = \{Q : 0_{N_{tr}} \subset Q \subset M^c\}$ of neutrosophic sets in U and V respectively. Then, $N_{tr}\Upsilon C\left(U, \tau_{N_{tr}}\right) = \left\{0_{N_{tr}}, L^c, \mathcal{P}, 1_{N_{tr}}\right\}$ and $N_{tr}\Upsilon C\left(V, \rho_{N_{tr}}\right) = \left\{0_{N_{tr}}, M^c, \mathcal{Q}, 1_{N_{tr}}\right\}$. Now, define $f_{N_{tr}}: (U, \tau_{N_{tr}}) \longrightarrow (V, \rho_{N_{tr}})$ as $f_{N_{tr}}(a) = x$ and $f_{N_{tr}}(b) = y$. Then, $f_{N_{tr}}^{-1}(M^c) = \{< a, 0.1, 0.3, 0.9 >< b, 0.2, 0.1, 0.8 >\} \in \mathcal{P}$ and for each $Q \in \mathcal{Q}$, there exists some $P \in \mathcal{P}$ such that $f_{N_{tr}}^{-1}(Q) = P$. Now, $f_{N_{tr}}^{-1}(M^c) = (f_{N_{tr}}^{-1}(M))^c$ is $N_{tr}\Upsilon -$ closed implies $f_{N_{tr}}^{-1}(M)$ is $N_{tr}\Upsilon -$ open . Similarly, we can prove that the inverse image of every $N_{tr}\Upsilon -$ open set in $(V, \rho_{N_{tr}})$ is $N_{tr}\Upsilon -$ open in $\left(U, \tau_{N_{tr}}\right)$. Hence $f_{N_{tr}}$ is $N_{tr}\Upsilon -$irresolute if the inverse image of every $N_{tr}\Upsilon -$ closed set in $(V, \rho_{N_{tr}})$ is $N_{tr}\Upsilon -$ closed in $\left(U, \tau_{N_{tr}}\right)$.

Theorem 5.8: If $f_{N_{tr}}:(U,\tau_{N_{tr}}) \longrightarrow (V,\rho_{N_{tr}})$ and $g_{N_{tr}}:(V,\rho_{N_{tr}}) \longrightarrow (W,\xi_{N_{tr}})$ are $N_{tr}\Upsilon-$ irresolute functions, then their composition $g_{N_{tr}} \circ f_{N_{tr}}:(U,\tau_{N_{tr}}) \longrightarrow (W,\xi_{N_{tr}})$ is also $N_{tr}\Upsilon-$irresolute.

Proof: Let N be $N_{tr}\Upsilon-$open in W. Since $g_{N_{tr}}$ is $N_{tr}\Upsilon-$irresolute, $g_{N_{tr}}^{-1}(N)$ is $N_{tr}\Upsilon-$open in V. Again, since $f_{N_{tr}}$ is $N_{tr}\Upsilon-$ irresolute, $f_{N_{tr}}^{-1}\left(g_{N_{tr}}^{-1}(N)\right) = \left(g_{N_{tr}} \circ f_{N_{tr}}\right)^{-1}(N)$ is $N_{tr}\Upsilon-$open in U. Hence $g_{N_{tr}} \circ f_{N_{tr}}$ is $N_{tr}\Upsilon-$irresolute.

Theorem 5.9: If $f_{N_{tr}}:(U,\tau_{N_{tr}}) \longrightarrow (V,\rho_{N_{tr}})$ is $N_{tr}\Upsilon-$irresolute and $g_{N_{tr}}:(V,\rho_{N_{tr}}) \longrightarrow (W,\xi_{N_{tr}})$ is $N_{tr}\Upsilon-$ continuous, then $g_{N_{tr}} \circ f_{N_{tr}}:(U,\tau_{N_{tr}}) \longrightarrow (W,\xi_{N_{tr}})$ is $N_{tr}\Upsilon-$ continuous.

Proof: Let N be N_{tr}open in W. Since $g_{N_{tr}}$ is $N_{tr}\Upsilon-$continuous, $g_{N_{tr}}^{-1}(N)$ is $N_{tr}\Upsilon-$open in V. Also, since $f_{N_{tr}}$ is $N_{tr}\Upsilon-$irresolute, $f_{N_{tr}}^{-1}\left(f_{N_{tr}}^{-1}(N)\right) = \left(g_{N_{tr}} \circ f_{N_{tr}}\right)^{-1}(N)$ is $N_{tr}\Upsilon-$open in U. Hence $g_{N_{tr}} \circ f_{N_{tr}}$ is $N_{tr}\Upsilon-$continuous.

6. Conclusions

The theory of neutrosophic sets is essential in many application areas since indeterminacy is ubiquitous and these membership functions are crucial. In this paper, we have introduced and analyzed the concepts of neutrosophic $\Upsilon-$neighbourhood and neutrosophic $\Upsilon-$continuity. In addition, we have also defined neutrosophic $\Upsilon-$irreolute functions in neutrosophic topological spaces. As mentioned earlier, continuity features a prominent position in the characterization of topological spaces. Accordingly, this concept can be wielded in the description of various topological structures in future. Moreover, several other topological concepts such as homeomorphisms, connectedness and separation axioms could be explored by means of neutrosphic $\Upsilon-$open sets and neutrosophic $\Upsilon-$continuity.

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Neutrosophic Fuzzy Magic Labeling Graph with its Application in Academic Performance of the Students

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Abstract: Graph labelling is the assignment of labels to the edges, vertices, or both. The issue of limiting the spread of non-interfering frequencies allotted to radio transmitters serves as the motivation for the research on labelling graphs according to different limitations. However, compared to classical models, fuzzy labelling models provide the system with greater accuracy, adaptability, and compatibility. The definition of the neutrosophic fuzzy magic labelling graph and a detailed discussion of its properties using numerical examples for the path, cycle, and star graphs in a neutrosophic environment are presented in this study. The proposed work has also been used in decision-making situations to choose the optimal subject combinations based on student interests for the best academic performance. In order to demonstrate the validity of the suggested work, a comparative analysis with the current methodology has also been conducted.

Keywords: Fuzzy magic labeling, Neutrosophic star graph, Neutrosophic cycle graph, Neutrosophic path graph, Neutrosophic fuzzy magic labeling.

1. Introduction

In order to address the issues of uncertainty and ambiguity in real-world settings, fuzzy relations were first introduced by Zadeh [1]. Fuzzy relations have a wide range of applications in pattern recognition. By substituting Zadeh's fuzzy sets for traditional sets, one can improve theoretical validity and reliability, in addition to application productivity and system connectivity. Numerous mathematical examples exist for the fuzzy graph. Nageswara Rao et al. [2] exhibited several forms of dominance, such as edge, total, strong, regular, linked, split, and, in practical applications, inverse dominance in fuzzy graphs. For visualising data on the connections between items, a graph is a valuable tool. Vertices identify the object, whereas edges highlight relationships. The use of graph theory is essential for illuminating numerous practical problems. Graphs no longer accurately represent every system because of the haziness or uncertainty of the system parameters. In general, when characterising the objects, their relationships, or both are uncertain, a fuzzy graph model needs to be created.

In order to assess the relationships between accounts as good or poor based on how frequently they interact, fuzziness must also be added to the representation. Fuzzy graphs were developed as a result of these and numerous other problems. The concepts of essential blocks and t-components of fuzzy graphs, as well as the creation of t-connected, uniformly connected, and average fuzzy graphs, were developed by John and Sunil Mathew [3]. The principle of the fuzzy equitable association graph was described by Rani and Dharmalingam [4]. The highly irregular and highly total irregular fuzzy graphs, as well as the neighbourly irregular and neighbourly total irregular fuzzy graphs, were all introduced by Huda Mutab [5].

The characteristics of Cartesian multiplication operations in full fuzzy graphs, effective fuzzy graphs, and complement fuzzy graphs were first introduced by Yulianto et al. in [6]. A hesitant fuzzy hypergraph model was suggested by Junhu Wang and Zengtai Gong, based on hesitant fuzzy sets and fuzzy hypergraphs [7]. Using fuzzy graphs in cubic Pythagorean fuzzy sets, Muhiuddin et al. [8] investigated the concept and utilised it to solve a problem involving decision-making. Crisp and fuzzy graphs have equivalent structural characteristics. However, fuzzy graphs emphasise the ambiguity surrounding vertices and edges more. Furthermore, the fuzzy graph is frequently seen in real-life scenarios since there is uncertainty in the world. Building fuzzy graphs draws on a variety of scientific disciplines, including those in mathematics, physics, chemistry, and computer science.

It was suggested that the intuitionistic fuzzy set by Attanassov [9–10] The concept of an intuitionistic fuzzy graph (IFG) was introduced by Atanassov and Shanon [11]. Several variations of the IFG concept were created, such as the very irregular and neighbourly irregular IFG by Nagoor Gani [12]. In [50], Garai developed a ranking technique based on generalised intuitionistic fuzzy numbers. [51] Giri et al. designed the mathematical operations of the generalised non-linear intuitionistic fuzzy number using the alpha-beta cut technique applied in the multi-item inventory model. Mathematics and its applications have seen a sharp increase in research on intuitionistic fuzzy sets. Information sciences and classical mathematics differ from one another. This makes me consider IFGs and how they might be used. Increased issue accuracy, reduced implementation costs, and improved efficacy are all advantages of intuitionistic fuzzy sets and graphs.

The concept has been examined, as have the IFG's properties and structure, according to Karunambigai [13]. The IFS operations were identified by John and Sunil investigators [14], and the suggested strategy was applied to trafficking channels. The concept of effective colouring was developed by Revathy et al. [15] of IFG. The colouring concept for IFG was described by Rifayathali et al. [16]. Akmaland Akram (2017 developed the organisational structure and layout of IFG [17]. In 2022, Amsaveni and Nandhini suggested using IFG in a bipolar complex intuitionistic fuzzy set [18]. The three further IFG activities of product, semi-strong product, and strong product were proposed to be added by Talal and Bayan [19]. The concept and attributes of IFG were first presented by Muhammad et al. in [20].

The neutrosophic graph, which is a fuzzy logic extension with indeterminacy, was proposed by Florentin Smarandache [21]. It has become imperative that the idea of a neutrosophic graph play a significant role in a number of real-world challenges, including computer technology, communication, genetics, economics, sociology, linguistics, legal, medical, finance, engineering IT, networking, and so forth. A fresh aspect of graph theory was introduced by Florentin Smarandache et al. [22]. The graph notion was invented by Euler. The phrase "fuzzy graph" was first used by Rosenfield [23]. Graph theory is very effective in simulating the characteristics of finite-component systems. A graph is a visual representation of information that shows how things are connected, and its vertices and edges show the objects and their connections. Graphical models are employed to describe a variety of networks, including telephone networks, railroad networks, communication networks, traffic networks, and other networks. Data mining, image segmentation, categorization, laser scanners, communication, preparation, and programming all make growing use of fuzzy graphs.

The novel idea of a Pythagorean neutrosophic fuzzy graph (NFG) was introduced by Ajay and Chellamani [24], who also examined its characteristics. Broumi et al. [25] described the many types of single-valued neutrosophic graphs (SVNG) and examined some of their characteristics in relevant scenarios. The effectiveness of the bipartite, regular, and irregular neutrosophic graphs was demonstrated by Huang [26]. In addition to a number of SVNG operations, such as rejection, symmetric difference, maximum product, and residue product, Mohanta [27] provided numerous additional SVNG concepts.

The graph labeling method was introduced by Rosa [28]. A mapping from a collection of edges, vertices, or both to a number of tags is known as graph labelling. Graph labelling has proven useful in many areas. Multiple labels are obtained depending on the demands made of the labelling. Among the most common labels are those that are graceful and attractive. In graphs, there are numerous forms of labelling, including beautiful, friendly, and mean labelling. We want the total number of labels associated with a vertex or edge to be constant across the graph when we apply the "magic" idea to graphs. Magic graph labelling is a logical continuation of the well-known magic squares and magic rectangles. Magic-type labelling is useful when avoiding a look-up table or when a check total is required. A straightforward graph can be used to depict a network that consists of nodes, links, and addresses (labels) assigned to both the links and the nodes. [48] Jafar et al. employed the notion in site selection for solid waste management when they provided length and identity measurements utilizing max-min operators under neutrosophic hypersoft sets. [49] Muhammed elaborates on the principle of neutrosophic hypersoft set to the neutrosophic hypersoft matrices applied in decision-making problems. [52] Garai created a unique ranking method that uses single-valued neutrosophic numbers for multi-attribute decision-making. Garai [53] developed a ranking system using single-valued bipolar neutrosophic numbers to address the challenge of managing water resources in a bipolar neutrosophic environment.

The notion of a magic graph was developed by Sedlack [29]. Magic labelling is a sort of graph labelling that has received a lot of attention and development. The labelling of the whole magic point, the labelling of the super magic point, the labelling of the magic side, and the labelling of the super magic side are also well known in the development of magic labelling. In this work, a neutrosophic number may be computed using various graph types. The neutrosophic fuzzy magic labelling graph has been proposed in the neutrosophic environment using this notion.

The rest of the paper is structured as follows: The literature review for the proposed theory is found in Section 2, and it demonstrates the originality of the methods provided in this work. Section 3 has presented fundamental ideas. In Section 4, the idea of a neutrosophic magic labelling graph was put forth. As an example, Section 5 defines neutrosophic fuzzy magic route graph labelling. In Section 6, an illustration of neutrosophic fuzzy magic labelling of a cycle is given. Neutosophic fuzzy magic labelling of star graphs is described in Section 7 along with an illustration. To choose the best combination of subjects based on the student's interests for the best academic performance, Section 8 applied the indicated strategy in decision-making situations. In Section 9, a comparison analysis using the current methodology is covered, and in Section 10, the current work is concluded with a look towards the future.

2. **Review of Literature**

The authors [1–8] introduced and developed graphs and fuzzy graphs in different types of fuzzy environments. The authors [10–14] proposed graphs and irregular graphs in the IFS environment. The authors [20–25] developed a neutrosophic graph in different types of neutrosophic environments. The authors Fathalian et al. [30] examined whether simple graphs are fuzzy magic labels, as well as whether every linked network is a fuzzy magic labelling network. New concepts for the labelling and calculation of Pythagorean fuzzy magic constants and Pythagorean fuzzy magic graphs have been presented by Rani and Ashwin [31], and the notion of magic labelling in hesitancy fuzzy graphs was proposed, and outcomes in hesitancy fuzzy graphs such as the route, cycle, and star graphs were obtained. Fathalian et al. [32] are credited with introducing the concept of Hesitancy fuzzy magic labelling for basic graphs. Fuzzy magic and bimagic labelling of neutrosophic route graphs were studied to see whether they included magic value in intuitionistic fuzzy graphs and to further understand bi-magic labelling on intuitionistic fuzzy graphs. Krishnaraj et al. [33] looked at how it differs from traditional labelling methods on the graphs. Fuzzy sequential vertex magic labelling with z-index in trees was studied along with numerous extensions by the authors Nishanthini et al. [34]. It was observed that magic labelling may be used for intuitively fuzzy graphs such as routes, cycles, and stars. The bridge management problem was solved using new neutrosophic labelling graph connection concepts suggested by Seemaand Majeet [35]. A relationship between strongly c-elegant labelling, super-edge magic total labelling, edge antimagic labelling, and super-t-1 magical labelling was proposed by Wang and Bing Ya [36], and it was investigated. Farida et al. [37] investigated the magic covering and edge magic labelling on a simple graph, and Krishnaraj and Vikramaprasad [38] extended the Bi-Magic concepts. There was an introduction to image fuzzy labelling of graphs and the notions of strong arc, partial cut node, and bridge of picture fuzzy labelling graphs, as well as their properties, explained by Ajay and Chellamani [38]. According to a proposal made by Jeyanthi and Jeya [39], Zk-magic graphs also contain the flower, double wheel, shell, cylinder, gear, generalised Jahangir, lotus inside a circle, wheel, and closed helm graphs. The authors, Wasim Hani and Muhamad [40], proposed that the direct product of a directed graph might be labelled using orientable group distance magic labelling. Maheswar et al. [41] introduced anti-Magic labelling, which involves assigning distinct values to various vertices in a network such that the total of the labels has different restrictions.

However. Fuzzy labelling models offer the system higher accuracy, adaptability, and compatibility when compared to classical methods. But in fuzzy, the magic values are discussed only for the membership grades, which should be constant. Whereas in the Intuitionistic Fuzzy Magic Labelling Graph (IFMLG), the magic values are discussed in both membership and non-membership grades. The magic values for membership grades and non-membership grades are both constant in IFMLG. As of the above research and findings, there is less contribution in the neutrosophic fuzzy magic labelling graph (NFMLG), which also shows that the magic labelling graph has not yet been properly proposed and that there has been very little progress in that direction in a neutrosophic environment. This study is inspired by that fact.

The NFMLG discusses magic values in both membership and non-membership grades, as well as indeterminacy grades. This is one of the main advantages that FMLG and IFMLG fail to prove.

3. Preliminaries

In this part, we will go over some fundamental terminology as well as the findings of our study.

Definition 3.1: Fuzzy Graph [42]

A Fuzzy graph denoted by $G(\sigma,\mu)$ is a couple of functions $\sigma: V \to [0,1]$ and $\mu: V \times V \to [0,1]$, where $\forall\, u,v \in V,\ \mu(u,v) = \mu(uv) \le \sigma(u) \wedge \sigma(v)$ is satisfied.

Definition 3.2: Fuzzy Labeling Graph [43]

If $\delta(\sigma): V \to [0,1]$ and $\delta(\mu): V \times V \to [0,1]$ are bijective such that the membership value of the edges and vertices are distinct and $\mu(\delta(u),\delta(v)) \le \sigma(\delta(u)) \wedge \sigma(\delta(v)), \forall\, \delta(u),\delta(v) \in V$,then the graph $\mathrm{G} = (\delta(\sigma),\delta(\mu))$ is said to be a fuzzy labeling graph.

Definition 3.3: Fuzzy Magic Labeling Graph (FMLG)[44]

A fuzzy labeling graph $G = (\sigma,\mu)$ is called a FMLG if there exist an 'm' such that $\sigma(\delta(u)) + \sigma(\delta(v)) + \mu(\delta(uv)) = constant \quad \forall uv \in E$ and $\delta(u),\delta(v) \in V$.

Definition 3.4: Intuitionistic Fuzzy Graph (IFG) [34]

A IFG of the form $G_g = (V_v, E_e)$ where $V_v = \{\delta(v_1),\delta(v_2),\delta(v_3)..........\delta(v_n)\}$ such that $\delta(\mu_1): V_v \to [0,1],\ \delta(\vartheta_1): V_v \to [0,1]$ represent the order of membership function, and non-membership function of the element $\delta(v_i) \in V$ respectively, and $0 \le \mu_1(\delta(v_i)) + \vartheta_1(\delta(v_i)) \le 1$ for every $v_i \in V_v\ (i = 1,2,3,..........n),\ E_e \subseteq V_v \times V_v$ where $\mu_2 : V_v \times V_v \to [0,1], \vartheta_2 : V_v \times V_v \to [0,1]$ are such that

$$\mu_2\left(\delta(\mathrm{v_i}),\delta(\mathrm{v_j})\right) \le \min\mathrm{ima}\left[\mu_1\left(\delta(\mathrm{v_i})\right),\mu_1\left(\delta(\mathrm{v_j})\right)\right],$$
$$\vartheta_2\left(\delta(\mathrm{v_i}),\delta(\mathrm{v_j})\right) \le \max\mathrm{ima}\left[\vartheta_1\left(\delta(\mathrm{v_i})\right),\vartheta_1\left(\delta(\mathrm{v_j})\right)\right]$$

fulfills the condition $0 \le \mu_1\left(\delta(v_i),\delta(v_j)\right) + \vartheta_1\left(\delta(v_i),\delta(v_j)\right) \le 1\ \forall\ v_i, v_j \in E(i, j = 1,2,3.............n)$.

Definition 3.5:Intuitionistic Fuzzy Labeling Graph (IFLG) [35]

A IFLG is of the form $G_g = (V_v, E_e)$ is called an IFLGif $\delta(\mu_1): V_v \to [0,1], \delta(\vartheta_1): V_v \to [0,1]$ & $\delta(\mu_2): V_v \times V_v \to [0,1], \delta(\vartheta_2): V_v \times V_v \to [0,1]$ are bijective in order forif $\delta\{\mu_1(m)\},\delta\{\vartheta_1(m)\},\delta\{\mu_2(m)\},\delta\{\vartheta_2(m)\} \in [0,1]$ all are unique ∀ vertices and edges, where

$\delta\{\mu_V(m)\},\delta\{\vartheta_V(m)\}$ is order of membership function and $\delta\{\mu_E(m)\},\delta(\vartheta_E(m))$ degree of non-membership function.

Definition 3.6: Intuitionistic Fuzzy Magic Labeling Graph (IFMLG) [35]

A IFLG is an IFMLG if thedegree of membership value $\delta\{\mu_1(m)\}+\delta\{\mu_2(m,n)\}+\delta\{\mu_1(n)\}$ remains equal $\forall m,n\in V$ and degree of non-membership value $\delta(\vartheta_1(m))+\delta(\vartheta_2(m,n))+\delta(\vartheta_1(n))$ remain equal $\forall m,n\in V$. The magic membership value denoted M, therefore $M=\{\delta(\mu_1(m))+\delta(\mu_2(m,n))+\delta(\mu_1(n)),\delta\{\vartheta_1(m)\}+\delta(\vartheta_2(m,n))+\delta(\vartheta_1(n))\}$.

Definition 3.7: Single–valued Neutrosophic Fuzzy Graph (SVNFG)[45]

A SVNFG is of the form $G=(V_v,\sigma,\mu)$ where $\sigma=(Tr_1,Ind_1,Fal_1)\,\&\,(Tr_2,Ind_2,Fal_2)$

$V_v=\{v_1,v_2,v_3..........v_n\}$ such that $Tr_1:V_v\to[0,1], Ind_1:V_v\to[0,1]\,\&\,Fal_1:V_v\to[0,1]$ denote the degree of truth-membership function, indeterminacy and falsity-membership function of the element $v_i\in V_v$ respectively, and $0\le Tr_1(\delta(v_i))+Ind_1(\delta(v_i))+Fal_1(\delta(v_i))\le 3\quad\forall\quad\delta(v_i)\in V(i=1,2,3,...........n)$, where $Tr_2:V_v\times V_v\to[0,1], Ind_2:V_v\times V_v\to[0,1]\,\&\,Fal_2:V_v\times V_v\to[0,1]$ of the edge

$$\begin{aligned}&\mathrm{Tr}_2(\delta(\mathrm{v_i}),\delta(\mathrm{v_j}))\le \min\mathrm{ima}\left[\mathrm{Tr}_1(\delta(\mathrm{v_i})),\mathrm{Tr}_1(\delta(\mathrm{v_j}))\right],\\&\mathrm{Ind}_2(\delta(\mathrm{v_i}),\delta(\mathrm{v_j}))\le \min\mathrm{ima}\left[\mathrm{Ind}_1(\delta(\mathrm{v_i})),\mathrm{Ind}_1(\delta(\mathrm{v_j}))\right],\\&\mathrm{Fal}_2(\delta(\mathrm{v_i}),\delta(\mathrm{v_j}))\le \max\mathrm{ima}\left[\mathrm{Fal}_1(\delta(\mathrm{v_i})),\mathrm{Fal}_1(\delta(\mathrm{v_j}))\right]\end{aligned}$$

satisfies the condition $0\le Tr_1(\delta(v_i),\delta(v_j))+Ind_1(\delta(v_i),\delta(v_j))+Fal_1(\delta(v_i),\delta(v_j))\le 3$ for every $v_i,v_j\in E(i,j=1,2,3.............n)$.

Definition 3.8: Score function SVNS [46]

Let (Tr_N,Ind_N,Fal_N) be a single-valued neutrosophic number. Then the score function is classified by $S(\alpha)=\frac{1+Tr-2Ind-Fal}{2}$ where $S(\alpha)\in[-1,1]$.

4. Proposed definition for Neutrosophic Fuzzy Magic Labeling Graph

The definition of neutrosophic fuzzy magic labeling graph has been proposed in this section.

Definition 4.1: Neutrosophic Fuzzy Labeling Graph (NFLG).

A NFG is of the form $G=\left(\delta\left(V_v\right),\delta\left(E_e\right),\delta\left(\sigma\right)\right)$ is called aNFLGif $Tr_1:\delta\left(V_v\right)\to\left[0,1\right], Ind_1:\delta\left(V_v\right)\to\left[0,1\right]$ & $Fal_1:\delta\left(V_v\right)\to\left[0,1\right]$ & $Tr_2:\delta\left(V_v\right)\times\delta\left(V_v\right)\to\left[0,1\right]$, $Ind_2:\delta\left(V_v\right)\times\delta\left(V_v\right)\to\left[0,1\right]$ & $Fal_2:\delta\left(V_v\right)\times\delta\left(V_v\right)\to\left[0,1\right]$ is bijective such that vertices and edges each have a separate degree of truth, indeterminacy and falsity-membership function for all $\left(\delta\left(v_i\right),\delta\left(v_j\right)\right)$,

$$\begin{aligned}
&\mathrm{Tr}_2\left(\delta\left(\mathrm{v_i}\right),\delta\left(\mathrm{v_j}\right)\right)\le \min\mathrm{ima}\left[\mathrm{Tr}_1\left(\delta\left(\mathrm{v_i}\right)\right),\mathrm{Tr}_1\left(\delta\left(\mathrm{v_j}\right)\right)\right],\\
&\mathrm{Ind}_2\left(\delta\left(\mathrm{v_i}\right),\delta\left(\mathrm{v_j}\right)\right)\le \min\mathrm{ima}\left[\mathrm{Ind}_1\left(\delta\left(\mathrm{v_i}\right)\right),\mathrm{Ind}_1\left(\delta\left(\mathrm{v_j}\right)\right)\right],\\
&\mathrm{Fal}_2\left(\delta\left(\mathrm{v_i}\right),\delta\left(\mathrm{v_j}\right)\right)\le \max\mathrm{ima}\left[\mathrm{Fal}_1\left(\delta\left(\mathrm{v_i}\right)\right),\mathrm{Fal}_1\left(\delta\left(\mathrm{v_j}\right)\right)\right]
\end{aligned}$$

and $0\le Tr_1\left(\delta\left(v_i\right),\delta\left(v_j\right)\right)+Ind_1\left(\delta\left(v_i\right),\delta\left(v_j\right)\right)+Fal_1\left(\delta\left(v_i\right),\delta\left(v_j\right)\right)\le 3$.

Definition 4.2:Neutrosophic Fuzzy Magic Labeling Graph (NFMLG)

A NFLG is a NFMLG if there exists amagic graph 'M' such that

degree of truth-membership function equals $Tr_1\left(\delta\left(v_i\right)\right)+Tr_2\left(\delta\left(v_i\right),\delta\left(v_j\right)\right)+Tr_1\left(\delta\left(v_j\right)\right)\ \forall\delta\left(v_i\right),\delta\left(v_j\right)\in E$

degree of indeterminacy function equals $Ind_1\left(\delta\left(v_i\right)\right)+Ind_2\left(\delta\left(v_i\right),\delta\left(v_j\right)\right)+Ind_1\left(\delta\left(v_j\right)\right)\ \forall\delta\left(v_i\right),\delta\left(v_j\right)\in E$

anddegree of falsity function equals $Fal_1\left(\delta\left(v_i\right)\right)+Fal_2\left(\delta\left(v_i\right),\delta\left(v_j\right)\right)+Fal_1\left(\delta\left(v_j\right)\right)\quad\forall\delta\left(v_i\right),\delta\left(v_j\right)\in E$

That is

$$M=\left\{Tr_1\left(\delta\left(v_i\right)\right)+Tr_2\left(\delta\left(v_i\right),\delta\left(v_j\right)\right)+Tr_1\left(\delta\left(v_j\right)\right),Ind_1\left(\delta\left(v_i\right)\right)+Ind_2\left(\delta\left(v_i\right),\delta\left(v_j\right)\right)+Ind_1\left(\delta\left(v_j\right)\right),Fal_1\left(\delta\left(v_i\right)\right)+Fal_2\left(\delta\left(v_i\right),\delta\left(v_j\right)\right)+Fal_1\left(\delta\left(v_j\right)\right)\right\}=cons\tan t$$

The magic truth membership value represented by $m_{Tr}(G)$

The magic indeterminacy value represented by $m_{Ind}(G)$

The magic falsity membership value represented by $m_{Fal}(G)$

We represent NFMLG by $M_{m(G)}\left(G\right)=\left(m_{Tr}\left(G\right),m_{Ind}\left(G\right),m_{Fal}\left(G\right)\right)$.

Definition 4.3:Difference between Fuzzy Magic Labeling and NFMLG

Fuzzy Magic Labeling Graph	**NMFG**
Fuzzy magic labeling graph contains only membership function.	NMFG depends on membership, non-membership also indeterminacy.

Membership value alone constant	Membership, non-membership and indeterminacy values are equal to constant.
Fuzzy number is of the form: Example: 0.5	Neutrosphic Number is of the form: Example: (0.8,0.6,0.2)

5. NFMLG of Path Graph

The magic value of a neutrosophic route graph is examined in this section because it satisfies the requirements for a neutrosophic magic labeling graph.

Theorem 5.1: For all $\rho \geq 1(\rho \in \square)$ theneutrosophic path P_ρ admits fuzzy magic labeling.

Proof. Let 'P' be any path with distance $n \geq 1(n \in N)$ and $v_1, v_2, v_3 .... v_\rho$ and $v_1, v_2, v_3 .... v_{\rho-1}v_\rho$ are vertices and edges of P. Let $\delta_1, \delta_2, \delta_3 \in [0,1]$ such that we choose $\delta_1 = 0.001, \delta_2 = 0.01 \& \delta_2 = 0.1$ if $\rho \leq 3$ and $\delta_1 = 0.0001, \delta_2 = 0.001 \& \delta_3 = 0.01$ if $\rho \geq 4$ and $\delta_1 = 0.00001, \delta_2 = 0.0001 \& \delta_3 = 0.001$ if $\rho \geq 5$. Where δ_1, δ_2 & δ_3 choose for truth, indeterminacy, and falsity are all degrees of NFMLG membership.

Therefore, NFMLG is given as:

When length is odd:

$$Tr_V(v_{2i-1}) = (2\rho + 2 - i)\delta_1, 1 \leq i \leq \frac{\rho+1}{2}$$
$$Ind_V(v_{2i-1}) = (2\rho + 2 - i)\delta_2, 1 \leq i \leq \frac{\rho+1}{2}$$
$$Fal_V(v_{2i-1}) = (2\rho + 2 - i)\delta_3, 1 \leq i \leq \frac{\rho+1}{2}$$

$$\mathrm{Tr_V}(\mathrm{v_{2i}}) = \min\mathrm{ima}\left\{\mathrm{Tr}(\mathrm{v_{2i-1}}) / 1 \leq \mathrm{i} \leq \frac{\rho+1}{2}\right\} - \mathrm{i}\delta_1, 1 \leq \mathrm{i} \leq \frac{\rho+1}{2}$$
$$\mathrm{Ind_V}(\mathrm{v_{2i}}) = \min\mathrm{ima}\left\{\mathrm{Ind}(\mathrm{v_{2i-1}}) / 1 \leq \mathrm{i} \leq \frac{\rho+1}{2}\right\} - \mathrm{i}\delta_2, 1 \leq \mathrm{i} \leq \frac{\rho+1}{2}$$
$$\mathrm{Fal_V}(\mathrm{v_{2i}}) = \min\mathrm{ima}\left\{\mathrm{Fal}(\mathrm{v_{2i-1}}) / 1 \leq \mathrm{i} \leq \frac{\rho+1}{2}\right\} - \mathrm{i}\delta_3, 1 \leq \mathrm{i} \leq \frac{\rho+1}{2}$$

$$\mathrm{Tr_E}(\mathrm{v}_{\rho-\mathrm{i}+2}, \mathrm{v}_{\rho+1-\mathrm{i}}) = \max\mathrm{ima}\{\mathrm{Tr_V}(\mathrm{v_i}) / 1 \leq \mathrm{i} \leq \rho+1\} - \min\mathrm{ima}\{\mathrm{Tr_V}(\mathrm{v_i}) / 1 \leq \mathrm{i} \leq \rho+1\} - (\mathrm{i}-1)\delta_1, 1 \leq \mathrm{i} \leq \rho.$$
$$\mathrm{Ind_E}(\mathrm{v}_{\rho-\mathrm{i}+2}, \mathrm{v}_{\rho+1-\mathrm{i}}) = \max\mathrm{ima}\{\mathrm{Ind_V}(\mathrm{v_i}) / 1 \leq \mathrm{i} \leq \rho+1\} - \min\mathrm{ima}\{\mathrm{Ind_V}(\mathrm{v_i}) / 1 \leq \mathrm{i} \leq \rho+1\} - (\mathrm{i}-1)\delta_2, 1 \leq \mathrm{i} \leq \rho.$$
$$\mathrm{Fal_E}(\mathrm{v}_{\rho-\mathrm{i}+2}, \mathrm{v}_{\rho+1-\mathrm{i}}) = \max\mathrm{ima}\{\mathrm{Fal_V}(\mathrm{v_i}) / 1 \leq \mathrm{i} \leq \rho+1\} - \min\mathrm{ima}\{\mathrm{Fal_V}(\mathrm{v_i}) / 1 \leq \mathrm{i} \leq \rho+1\} - (\mathrm{i}-1)\delta_3, 1 \leq \mathrm{i} \leq \rho.$$

Case (i) 'i' is even

Then i=2m, where $m \in Z^+$ and for each edge v_i, v_{i+1}

$$\begin{aligned}
m_{Tr}(P_\rho) &= Tr_V(v_i)+Tr_E(v_i,v_{i+1})+Tr_V(v_{i+1}) \\
&= Tr_V(v_{2m})+Tr_E(v_{2m},v_{2m+1})+Tr_V(v_{2m+1}) \\
&= \min ima\left\{Tr_V(v_{2i-1})/1\le i\le \frac{\rho+1}{2}\right\}-m\delta_1+\max ima\{Tr_V(v_i)/1\le i\le \rho+1\} \\
&\quad -\min ima\{Tr_V(v_i)/1\le i\le \rho+1\}-(\rho-2m)\delta_1+(2\rho-m+1)\delta_1
\end{aligned}$$

$$\begin{aligned}
m_{Tr}(P_\rho) &= \min ima\left\{Tr_V(v_{2i-1})/1\le i\le \frac{\rho+1}{2}\right\}-m\delta_1+\max ima\{Tr_V(v_i)/1\le i\le \rho+1\} \\
&\quad -\min ima\{Tr_V(v_i)/1\le i\le \rho+1\}+(\rho+1)\delta_1
\end{aligned} \tag{1}$$

$$\begin{aligned}
m_{Ind}(P_\rho) &= Ind_V(v_i)+Ind_E(v_i,v_{i+1})+Ind_V(v_{i+1}) \\
&= Ind_V(v_{2m})+Ind_E(v_{2m},v_{2m+1})+Ind_V(v_{2m+1}) \\
&= \min ima\left\{Ind_V(v_{2m-1})/1\le i\le \frac{\rho+1}{2}\right\}-a\delta_2+\max ima\{Ind_V(v_i)/1\le i\le \rho+1\} \\
&\quad -\min ima\{Ind_V(v_i)/1\le i\le \rho+1\}-(\rho-2m)\delta_2+(2\rho-m+1)\delta_2
\end{aligned}$$

$$\begin{aligned}
m_{Ind}(P_\rho) &= \min ima\left\{Ind_V(v_{2i-1})/1\le i\le \frac{\rho+1}{2}\right\}-m\delta_2+\max ima\{Ind_V(v_i)/1\le i\le \rho+1\} \\
&\quad -\min ima\{Ind_V(v_k)/1\le i\le \rho+1\}+(\rho+1)\delta_2
\end{aligned} \tag{2}$$

$$\begin{aligned}
m_{Fal}(P_\rho) &= Fal_V(v_i)+Fal_E(v_i,v_{i+1})+Fal_V(v_{i+1}) \\
&= Fal_V(v_{2m})+Fal_E(v_{2m},v_{2m+1})+Fal_V(v_{2m+1}) \\
&= \min ima\left\{Fal_V(v_{2i-1})/1\le i\le \frac{\rho+1}{2}\right\}-m\delta_3+\max ima\{Fal_V(v_i)/1\le i\le \rho+1\} \\
&\quad -\min ima\{Fal_V(v_i)/1\le i\le \rho+1\}-(\rho-2m)\delta_3+(2\rho-m+1)\delta_3
\end{aligned}$$

$$\begin{aligned}
m_{Fal}(P_\rho) &= \min\left\{Fal_V(v_{2i-1})/1\le i\le \frac{\rho+1}{2}\right\}-m\delta_3+\max ima\{Fal_V(v_k)/1\le i\le \rho+1\} \\
&\quad -\min ima\{Fal_V(v_k)/1\le i\le \rho+1\}+(\rho+1)\delta_3
\end{aligned} \tag{3}$$

so that $M_{m(G)}(P_\rho)=(m_{Tr}(G),m_{Ind}(G),m_{Fal}(G))$ = constant.

When 'i' is even, Equations (1), (2), and (3) satisfy the requirement for NFMLG.

Case (i) 'i' is odd

Then i=2m+1, where $m\in Z^+$ and for each edge v_i,v_{i+1}

$$\begin{aligned}
m_{Tr}(P_\rho) &= Tr_V(v_i)+Tr_E(v_i,v_{i+1})+Tr_V(v_{i+1}) \\
&= Tr_V(v_{2m+1})+Tr_E(v_{2m+1},v_{2m+2})+Tr_V(v_{2m+2}) \\
&= (2\rho-m+1)\delta_1+\max ima\{Tr_V(v_i)/1\le i\le \rho+1\}-\min ima\{Tr_V(v_i)/1\le i\le \rho+1\} \\
&\quad -(\rho-2m-1)\delta_1+\min ima\left\{Tr_V(v_{2i-1})/1\le i\le \frac{\rho+1}{2}\right\}-(m+1)\delta_1
\end{aligned}$$

$$\begin{aligned}
m_{Tr}(P_\rho) &= \min ima\left\{Tr_V(v_{2i-1})/1\le i\le \frac{\rho+1}{2}\right\}+\max ima\{Tr_V(v_i)/1\le i\le n+1\} \\
&\quad -\min ima\{Tr_V(v_i)/1\le i\le \rho+1\}+(\rho+1)\delta_1
\end{aligned} \tag{4}$$

$$\begin{aligned}
m_{Ind}(P_\rho) &= Ind_V(v_i) + Ind_E(v_i, v_{i+1}) + Ind_V(v_{i+1}) \\
&= Ind_V(v_{2m+1}) + Ind_E(v_{2m+1}, v_{2m+2}) + Ind_V(v_{2m+2}) \\
&= (2\rho - m + 1)\delta_2 + \max ima\{Tr_V(v_i)/1 \le i \le \rho+1\} - \min ima\{Tr_V(v_i)/1 \le i \le \rho+1\} \\
&\quad -(\rho - 2m - 1)\delta_2 + \min ima\left\{Tr_V(v_{2i-1})/1 \le i \le \frac{\rho+1}{2}\right\} - (m+1)\delta_2
\end{aligned}$$

$$\begin{aligned}
m_{Ind}(P_\rho) &= \min ima\left\{Ind_V(v_{2i-1})/1 \le i \le \frac{\rho+1}{2}\right\} + \max ima\{Ind_V(v_i)/1 \le i \le \rho+1\} \\
&\quad - \min ima\{Ind_V(v_k)/1 \le i \le \rho+1\} + (\rho+1)\delta_2
\end{aligned} \qquad (5)$$

$$\begin{aligned}
m_{Fal}(P_\rho) &= Fal_V(v_i) + Fal_E(v_i, v_{i+1}) + Fal_V(v_{i+1}) \\
&= Fal_V(v_{2m+1}) + Fal_E(v_{2m+1}, v_{2m+2}) + Fal_V(v_{2m+2}) \\
&= (2\rho - m + 1)\delta_3 + \max ima\{Tr_V(v_i)/1 \le i \le \rho+1\} - \min ima\{Tr_V(v_i)/1 \le i \le \rho+1\} \\
&\quad -(\rho - 2m - 1)\delta_3 + \min ima\left\{Fal_V(v_{2i-1})/1 \le i \le \frac{\rho+1}{2}\right\} - (m+1)\delta_3
\end{aligned}$$

$$\begin{aligned}
m_{Fal}(P_\rho) &= \min ima\left\{Fal_V(v_{2i-1})/1 \le i \le \frac{\rho+1}{2}\right\} + \max ima\{Fal_V(v_i)/1 \le i \le n+1\} \\
&\quad - \min ima\{Fal_V(v_i)/1 \le i \le \rho+1\} + (\rho+1)\delta_3
\end{aligned} \qquad (6)$$

so that $M_{m(G)}(P_\rho) = (m_{Tr}(G), m_{Ind}(G), m_{Fal}(G))$ = constant.

When k is odd, Equations (4), (5), and (6) satisfy the requirement for NFMLG.

When the length is even:

$$Tr_V(v_{2i}) = (2\rho + 2 - i)\delta_1, 1 \le i \le \frac{\rho}{2}$$

$$Ind_V(v_{2i}) = (2\rho + 2 - i)\delta_2, 1 \le i \le \frac{\rho}{2}$$

$$Fal_V(v_{2i}) = (2\rho + 2 - i)\delta_3, 1 \le i \le \frac{\rho}{2}$$

$$Tr_V(v_{2i-1}) = \min ima\left\{Tr(v_{2i})/1 \le i \le \frac{\rho}{2}\right\} - i\delta_1, 1 \le i \le \frac{\rho+2}{2}$$

$$Ind_V(v_{2i-1}) = \min ima\left\{Tr(v_{2i})/1 \le i \le \frac{\rho}{2}\right\} - i\delta_2, 1 \le i \le \frac{\rho+2}{2}$$

$$Fal_V(v_{2i-1}) = \min ima\left\{Tr(v_{2i})/1 \le i \le \frac{\rho}{2}\right\} - i\delta_3, 1 \le i \le \frac{\rho+2}{2}$$

$$Tr_E(v_{\rho-i+2}, v_{n+1-i}) = \max ima\{Tr_V(v_i)/1 \le i \le \rho+1\} - \min ima\{Tr_V(v_i)/1 \le i \le \rho+1\} - (i-1)\delta_1, 1 \le i \le \rho.$$

$$Ind_E(v_{\rho-i+2}, v_{n+1-i}) = \max\{Ind_V(v_i)/1 \le i \le \rho+1\} - \min ima\{Ind_V(v_i)/1 \le i \le \rho+1\} - (i-1)\delta_2, 1 \le i \le \rho.$$ Case

$$Fal_E(v_{\rho-i+2}, v_{n+1-i}) = \max ima\{Fal_V(v_i)/1 \le i \le \rho+1\} - \min ima\{Fal_V(v_i)/1 \le i \le \rho+1\} - (i-1)\delta_3, 1 \le i \le \rho.$$

(i) ‘i’ is even then i=2m, where $m \in Z^+$ and for each edge v_i, v_{i+1}

$$\begin{aligned} m_{Tr}(P_\rho) &= Tr_V(v_i) + Tr_E(v_i, v_{i+1}) + Tr_V(v_{i+1}) \\ &= Tr_V(v_{2m}) + Tr_E(v_{2m}, v_{2m+1}) + Tr_V(v_{2m+1}) \\ &= (2\rho - m + 2)\delta_1 + \max ima\{Tr_V(v_i)/1 \le i \le n+1\} - \min ima\{Tr_V(v_i)/1 \le i \le \rho+1\} - (\rho - 2m)\delta_1 \\ &\quad + \min ima\left\{Tr_V(v_{2i})/1 \le i \le \frac{\rho}{2}\right\} - (m+1)\delta_1 \end{aligned}$$

$$\begin{aligned} m_{Tr}(P_\rho) &= \min ima\left\{Tr_V(v_{2i-1})/1 \le i \le \frac{\rho}{2}\right\} + \max ima\{Tr_V(v_i)/1 \le i \le \rho+1\} \\ &\quad - \min ima\{Tr_V(v_i)/1 \le i \le \rho+1\} + (\rho+1)\delta_1 \end{aligned} \tag{7}$$

$$\begin{aligned} m_{Ind}(P_\rho) &= Ind_V(v_i) + Ind_E(v_i, v_{i+1}) + Ind_V(v_{i+1}) \\ &= Ind_V(v_{2m}) + Ind_E(v_{2m}, v_{2m+1}) + Ind_V(v_{2m+1}) \\ &= (2\rho - m + 2)\delta_2 + \max ima\{Tr_V(v_i)/1 \le i \le \rho+1\} - \min ima\{Tr_V(v_i)/1 \le i \le \rho+1\} \\ &\quad - (\rho - 2i)\delta_2 + \min ima\left\{Tr_V(v_{2i})/1 \le i \le \frac{\rho}{2}\right\} - (m+1)\delta_2 \end{aligned}$$

$$\begin{aligned} m_{Ind}(P_\rho) &= \min ima\left\{Tr_V(v_{2i-1})/1 \le i \le \frac{\rho}{2}\right\} + \max ima\{Tr_V(v_i)/1 \le i \le \rho+1\} \\ &\quad - \min ima\{Tr_V(v_i)/1 \le i \le \rho+1\} + (\rho+1)\delta_2 \end{aligned} \tag{8}$$

$$\begin{aligned} m_{Fal}(P_\rho) &= Fal_V(v_i) + Fal_E(v_i, v_{i+1}) + Fal_V(v_{i+1}) \\ &= Fal_V(v_{2m}) + Fal_E(v_{2m}, v_{2m+1}) + Fal_V(v_{2m+1}) \\ &= (2\rho - m + 2)\delta_3 + \max ima\{Fal_V(v_i)/1 \le i \le \rho+1\} - \min ima\{Fal_V(v_i)/1 \le i \le \rho+1\} \\ &\quad - (\rho - 2m)\delta_3 + \min ima\left\{Fal_V(v_{2i})/1 \le i \le \frac{\rho}{2}\right\} - (m+1)\delta_3 \end{aligned}$$

$$\begin{aligned} m_{Fal}(P_\rho) &= \min ima\left\{Fal_V(v_{2i-1})/1 \le i \le \frac{\rho}{2}\right\} + \max ima\{Fal_V(v_i)/1 \le i \le \rho+1\} \\ &\quad - \min ima\{Fal_V(v_i)/1 \le i \le \rho+1\} + (\rho+1)\delta_3 \end{aligned} \tag{9}$$

so that $M_{m(G)}(P_\rho) = (m_{Tr}(G), m_{Ind}(G), m_{Fal}(G))$ = constant.

When i=2m, equations (7), (8), and (9) satisfy the requirement for NFMLG.

Case (ii) 'i' is odd

Then i=2m+1, where $m \in Z^+$ and for each edge v_i, v_{i+1}

$$\begin{aligned}
m_{Tr}(P_\rho) &= Tr_V(v_i) + Tr_E(v_i, v_{i+1}) + Tr_V(v_{i+1}) \\
&= Tr_V(v_{2m+1}) + Tr_E(v_{2m+1}, v_{2m+2}) + Tr_V(v_{2m+2}) \\
&= \min ima\left\{Tr_V(v_{2i}) / 1 \le i \le \frac{\rho}{2}\right\} - (m+1)\delta_1 + \max ima\left\{Tr_V(v_i) / 1 \le i \le \rho+1\right\} \\
&\quad - \min ima\left\{Tr_V(v_i) / 1 \le i \le \rho+1\right\} - (\rho - 2m - 1)\delta_1 - (2\rho - m + 1)\delta_1
\end{aligned}$$

$$\begin{aligned}
m_{Tr}(P_\rho) &= \min ima\left\{Tr_V(v_{2i}) / 1 \le i \le \frac{\rho}{2}\right\} + \max ima\left\{Tr_V(v_i) / 1 \le i \le \rho+1\right\} \\
&\quad - \min ima\left\{Tr_V(v_i) / 1 \le i \le \rho+1\right\} + (\rho+1)\delta_1
\end{aligned} \tag{10}$$

$$\begin{aligned}
m_{Ind}(P_\rho) &= Ind_V(v_i) + Ind_E(v_i, v_{i+1}) + Ind_V(v_{i+1}) \\
&= Ind_V(v_{2m+1}) + Ind_E(v_{2m+1}, v_{2m+2}) + Ind_V(v_{2m+2}) \\
&= \min ima\left\{Ind_V(v_{2i}) / 1 \le i \le \frac{\rho}{2}\right\} - (m+1)\delta_2 + \max ima\left\{Ind_V(v_i) / 1 \le i \le \rho+1\right\} \\
&\quad - \min ima\left\{Ind_V(v_i) / 1 \le i \le \rho+1\right\} - (\rho - 2m - 1)\delta_1 - (2\rho - m + 1)\delta_2
\end{aligned}$$

$$\begin{aligned}
m_{Ind}(P_\rho) &= \min ima\left\{Ind_V(v_{2i}) / 1 \le i \le \frac{\rho}{2}\right\} + \max ima\left\{Ind_V(v_k) / 1 \le i \le \rho+1\right\} \\
&\quad - \min ima\left\{Ind_V(v_i) / 1 \le i \le \rho+1\right\} + (\rho+1)\delta_2
\end{aligned} \tag{11}$$

$$\begin{aligned}
m_{Fal}(P_n) &= Fal_V(v_i) + Fal_E(v_i, v_{i+1}) + Fal_V(v_{i+1}) \\
&= Fal_V(v_{2m+1}) + Fal_E(v_{2m+1}, v_{2m+2}) + Fal_V(v_{2m+2}) \\
&= \min ima\left\{Fal_V(v_{2i}) / 1 \le i \le \frac{\rho}{2}\right\} - (m+1)\delta_2 + \max ima\left\{Fal_V(v_i) / 1 \le i \le \rho+1\right\} \\
&\quad - \min ima\left\{Fal_V(v_i) / 1 \le i \le \rho+1\right\} - (\rho - 2m - 1)\delta_1 - (2\rho - m + 1)\delta_3
\end{aligned}$$

$$\begin{aligned}
m_{Fal}(P_n) &= \min ima\left\{Fal_V(v_{2i}) / 1 \le i \le \frac{\rho}{2}\right\} + \max ima\left\{Fal_V(v_i) / 1 \le i \le \rho+1\right\} \\
&\quad - \min ima\left\{Fal_V(v_i) / 1 \le i \le \rho+1\right\} + (\rho+1)\delta_3
\end{aligned} \tag{12}$$

so that $M_{m(G)}(P_\rho) = (m_{Tr}(G), m_{Ind}(G), m_{Fal}(G))$ = constant.

Equations (10), (11), (12) satisfies the condition for NFMLG when i=2m+1.

Example. 5.2:

Figure 1 represents NFMLG path graph with eight nodes and seven edges.

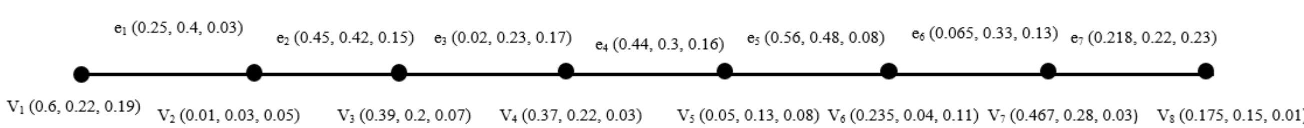

Figure1: NFMLG of path graph

The aforementionedNeutrosophic path graph P_8's magic value is $(0.86, 0.65, 0.27)$.

The value 0.86 indicates truth membership

The value 0.65 indicates indeterminacy

The value 0.27 indicates falsity

Using definition 3.8, the scorevalue of the magic value of the NFMLG of path graph is given by $S(\alpha)=0.145$ .Here in NFMLG-path graph the score value indicates that our result fits the requirement for a neutrosophic set because it falls with the [-1, -1] score limit.

6. NFMGL of Cycle Graph

In this part, we examine the magic value of a neutrosophic cycle graph, which satisfies the requirements for a neutrosophic magic labeling graph.

Theorem 6.1: If ' ρ ' is odd, then the cycle C_ρ is an NFMLG.

Proof: Let C_ρ be any cycle with odd integers $v_1, v_2, v_3................v_n$ and $v_1, v_2, v_3................v_{n-1}v_1$ are vertices and edges of C_ρ. Let $\delta_1, \delta_2, \delta_3 \in [0,1]$ such that we choose $\delta_1 = 0.001, \delta_2 = 0.01 \& \delta_2 = 0.1$ if $\rho \le 3$ and $\delta_1 = 0.0001, \delta_2 = 0.001 \& \delta_3 = 0.01$ if $\rho \ge 4$ and $\delta_1 = 0.00001, \delta_2 = 0.0001 \& \delta_3 = 0.001$ if $\rho \ge 5$. Where δ_1, δ_2 & δ_3 choose for collection of truth, indeterminacy and falsity membership degree in NFMLG.

Therefore, NFMLG is given as:

When length is odd:

$$Tr_V(v_{2i}) = (2\rho+1-i)\delta_1, 1 \le i \le \frac{\rho-1}{2}$$
$$Ind_V(v_{2i}) = (2\rho+1-i)\delta_2, 1 \le i \le \frac{\rho-1}{2}$$
$$Fal_V(v_{2i}) = (2\rho+1-i)\delta_3, 1 \le i \le \frac{\rho-1}{2}$$

$$\mathrm{Tr}_V(\mathrm{v}_{2i}) = \min\mathrm{ima}\left\{\mathrm{Tr}(\mathrm{v}_{2i}) / 1 \le \mathrm{i} \le \frac{\rho-1}{2}\right\} - \mathrm{i}\delta_1, 1 \le \mathrm{i} \le \frac{\rho+1}{2}$$
$$\mathrm{Ind}_V(\mathrm{v}_{2i}) = \min\mathrm{ima}\left\{\mathrm{Tr}(\mathrm{v}_{2i}) / 1 \le \mathrm{i} \le \frac{\rho-1}{2}\right\} - \mathrm{i}\delta_2, 1 \le \mathrm{i} \le \frac{\rho+1}{2}$$
$$\mathrm{Fal}_V(\mathrm{v}_{2i}) = \min\mathrm{ima}\left\{\mathrm{Tr}(\mathrm{v}_{2i}) / 1 \le \mathrm{i} \le \frac{\rho-1}{2}\right\} - \mathrm{i}\delta_3, 1 \le \mathrm{i} \le \frac{\rho+1}{2}$$

$$Tr_E(v_1,v_\rho)=\frac{1}{2}\max ima\{Tr_V(v_i)/1\le i\le\rho\}$$

$$Fal_E(v_1,v_\rho)=\frac{1}{2}\max ima\{Fal_V(v_i)/1\le i\le\rho\}$$

$$Ind_E(v_1,v_\rho)=\frac{1}{2}\max ima\{Ind_V(v_i)/1\le i\le\rho\}$$

$$Tr_E(v_{\rho-i+1},v_{\rho-i})=Tr_E(v_1,v_\rho)-i\delta_1,\quad 1\le i\le\rho-1$$
$$Ind_E(v_{\rho-i+1},v_{\rho-i})=Fal_E(v_1,v_\rho)-i\delta_2,\quad 1\le i\le\rho-1$$
$$Tr_E(v_{\rho-i+1},v_{\rho-i})=Ind_E(v_1,v_\rho)-i\delta_3,\quad 1\le i\le\rho-1.$$

Case (i) 'i' is even

Then i=2m, where $m\in Z^+$ and for each edge v_i,v_{i+1}

$$\begin{aligned}m_{Tr}(C_\rho)&=Tr_V(v_i)+Tr_E(v_i,v_{i+1})+Tr_V(v_{i+1})\\&=Tr_V(v_{2m})+Tr_E(v_{2m},v_{2m+1})+Tr_V(v_{2m+1})\\&=(2\rho-m+1)\delta_1+\frac{1}{2}\max ima\{Tr_V(v_i)/1\le i\le\rho\}-(\rho-2m)\delta_1\\&\quad+\min ima\left\{Tr_V(v_{2i})/1\le i\le\frac{\rho-1}{2}\right\}-(m+1)\delta_1\end{aligned}$$

$$m_{Tr}(C_\rho)=\frac{1}{2}\max ima\{Tr_V(v_i)/1\le i\le\rho\}+\min ima\left\{Tr_V(v_{2i})/1\le i\le\frac{\rho-1}{2}\right\}+\rho\delta_1 \tag{13}$$

$$\begin{aligned}m_{Ind}(C_\rho)&=Ind_V(v_i)+Ind_E(v_i,v_{i+1})+Ind_V(v_{i+1})\\&=Ind_V(v_{2m})+Ind_E(v_{2m},v_{2m+1})+Ind_V(v_{2m+1})\\&=(2\rho-m+1)\delta_2+\frac{1}{2}\max ima\{Ind_V(v_i)/1\le i\le\rho\}-(\rho-2m)\delta_2\\&\quad+\min ima\left\{Ind_V(v_{2i})/1\le i\le\frac{\rho-1}{2}\right\}-(m+1)\delta_2\end{aligned}$$

$$m_{Ind}(C_\rho)=\frac{1}{2}\max ima\{Ind_V(v_i)/1\le i\le\rho\}+\min ima\left\{Ind_V(v_{2i})/1\le i\le\frac{\rho-1}{2}\right\}+\rho\delta_2 \tag{14}$$

$$\begin{aligned}m_{Fal}(C_\rho)&=Fal_V(v_i)+Fal_E(v_i,v_{i+1})+Fal_V(v_{i+1})\\&=Fal_V(v_{2m})+Fal_E(v_{2m},v_{2m+1})+Fal_V(v_{2m+1})\\&=(2\rho-m+1)\delta_3+\frac{1}{2}\max\{Fal_V(v_i)/1\le i\le\rho\}-(\rho-2m)\delta_3\\&\quad+\min ima\left\{Ind_V(v_{2i})/1\le i\le\frac{\rho-1}{2}\right\}-(m+1)\delta_3\end{aligned}$$

$$m_{Fal}(C_\rho)=\frac{1}{2}\max ima\{Fal_V(v_i)/1\le i\le\rho\}+\min ima\left\{Fal_V(v_{2i})/1\le i\le\frac{\rho-1}{2}\right\}+\rho\delta_3 \tag{15}$$

so that $M_{m(G)}(P_\rho)=(m_{Tr}(G),m_{Ind}(G),m_{Fal}(G))$ = constant.

Equations (13), (14), (15) satisfy the condition for NFMLG when k is even.

Case (ii) 'i' is odd

Then i=2m+1, where $m \in Z^{+}$ and for each edge v_i, v_{i+1}

$$\begin{aligned} m_{Tr}(C_\rho) &= Tr_V(v_i) + Tr_E(v_i, v_{i+1}) + Tr_V(v_{i+1}) \\ &= Tr_V(v_{2m+1}) + Tr_E(v_{2m+1}, v_{2m+2}) + Tr_V(v_{2m+2}) \\ &= \min ima\left\{Tr_V(v_{2m}) / 1 \le i \le \frac{\rho-1}{2}\right\} - (m+1)\delta_1 + \frac{1}{2}\max ima\{Tr_V(v_i)/1 \le i \le \rho\} \\ &\quad -(\rho - 2m - 1)\delta_1 - (2\rho - m)\delta_1 \end{aligned}$$

$$m_{Tr}(C_\rho) = \frac{1}{2}\max ima\{Tr_V(v_i)/1 \le i \le \rho\} + \min ima\left\{Tr_V(v_{2i})/1 \le i \le \frac{\rho-1}{2}\right\} + \rho\delta_1 \quad (16)$$

$$\begin{aligned} m_{Ind}(C_\rho) &= Ind_V(v_i) + Ind_E(v_k, v_{i+1}) + Ind_V(v_{i+1}) \\ &= Ind_V(v_{2m+1}) + Ind_E(v_{2m+1}, v_{2m+2}) + Ind_V(v_{2m+2}) \\ &= \min ima\left\{Ind_V(v_{2i}) / 1 \le i \le \frac{\rho-1}{2}\right\} - (m+1)\delta_2 + \frac{1}{2}\max ima\{Ind_V(v_i)/1 \le i \le \rho\} \\ &\quad -(\rho - 2m - 1)\delta_2 - (2\rho - m)\delta_2 \end{aligned}$$

$$m_{Ind}(C_\rho) = \frac{1}{2}\max ima\{Ind_V(v_i)/1 \le i \le \rho\} + \min ima\left\{Ind_V(v_{2i})/1 \le i \le \frac{\rho-1}{2}\right\} + v\delta_2 \quad (17)$$

$$\begin{aligned} m_{Fal}(C_\rho) &= Fal_V(v_i) + Fal_E(v_i, v_{i+1}) + Fal_V(v_{i+1}) \\ &= Fal_V(v_{2m+1}) + Fal_E(v_{2m+1}, v_{2m+2}) + Fal_V(v_{2m+2}) \\ &= \min ima\left\{Fal_V(v_{2i}) / 1 \le i \le \frac{\rho-1}{2}\right\} - (m+1)\delta_3 + \frac{1}{2}\max ima\{Fal_V(v_i)/1 \le i \le \rho\} \\ &\quad -(\rho - 2m - 1)\delta_3 - (2\rho - m)\delta_3 \end{aligned}$$

$$m_{Fal}(C_\rho) = \frac{1}{2}\max ima\{Fal_V(v_i)/1 \le i \le \rho\} + \min ima\left\{Fal_V(v_{2i})/1 \le i \le \frac{\rho-1}{2}\right\} + \rho\delta_3 \quad (18)$$

Hence $M_{m(G)}(C_\rho) = (m_{Tr}(G), m_{Ind}(G), m_{Fal}(G))$ = constant.

When 'i' is odd, equations (16), (17), and (18) satisfy the requirement for NFMLG.

The magic value $M_{m(G)}(C_\rho)$ is same and unique in above cases. Thus C_ρ is an NFMLG.

Example 6.2:

Figure 2 represents NFMLG cycle Graph with five nodes and five edges.

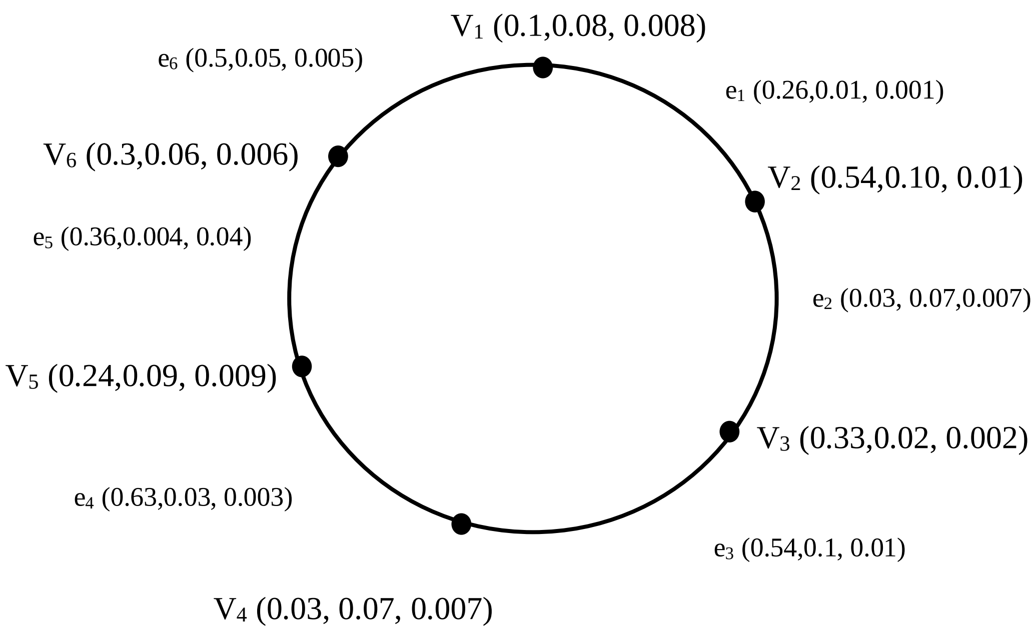

Figure2: NFMLG of cycle graph

The magic value of the aforementionedneutrosophic cycle graph $C_6 \quad is\left(0.9, 0.19, 0.019\right)$

The values are 0.9,0.19,0.019 indicates the truthmembership,indeterminacy and falsity function for NFMLG-cycle graph.

Using definition 3.8 we find the score value of the magic value of NFMLG of cycle graph is given by $S\left(\alpha\right) = 0.7505$. Because our result falls under the [-1,1] score limit, the NFMLG-cycle graph's score value here indicates that out result satisfies the criteria for a neutrosophic set.

7. NFMLG of Star Graph

The magic value of a neutrosophic star graph is examined in this section because it satisfies the requirements for a neutrosophic magic labeling graph.

Theorem 7.1: For any $\rho \geq 2$, star graph $S_{1,\rho}$ **is an NFMLG.**

Proof: Let $S_{1,\rho}$ be any star graph having $v, u_1, u_2, u_3 .... u_\rho$ as vertices and $vu_1, vu_2, vu_3 .... vuv_\rho$ as edges. Let $\delta_1, \delta_2, \delta_3 \in [0,1]$ such that we choose $\delta_1 = 0.001$, $\delta_2 = 0.01$ & $\delta_2 = 0.1$ if $\rho \leq 3$ and $\delta_1 = 0.0001$, $\delta_2 = 0.001$ & $\delta_3 = 0.01$ if $\rho \geq 4$ and $\delta_1 = 0.00001, \delta_2 = 0.0001$ & $\delta_3 = 0.001$ if $\rho \geq 5$. Where δ_1, δ_2 & δ_3 choose for collection of truth membership degree, indeterminacy and falsity membership degree in NFMLG.

Therefore, NFMLG is given as:

$$Tr_V(u_i) = (2(\rho+1) - i)\delta_1, 1 \leq i \leq \rho$$
$$Ind_V(u_i) = (2(\rho+1) - i)\delta_2, 1 \leq i \leq \rho$$
$$Fal_V(u_i) = (2(\rho+1) - i)\delta_3, 1 \leq i \leq \rho$$

$$Tr_V(v_i) = \min\{Tr(u_i) / 1 \leq i \leq \rho\} - \delta_1$$
$$Ind_V(v_i) = \min\{Tr(u_i) / 1 \leq k \leq \rho\} - \delta_2$$
$$Fal_V(v_i) = \min\{Tr(u_i) / 1 \leq i \leq \rho\} - \delta_3$$

$$Tr_E(v, u_{\rho-i}) = \max\{Tr_V(v_i), Tr_V(v) / 1 \leq i \leq \rho\} - \min\{Tr_V(v_i), Tr_V(v) / 1 \leq i \leq \rho\} - i\delta_1, 0 \leq i \leq \rho - 1$$
$$Fal_E(v, u_{\rho-i}) = \max\{Ind_V(v_i), Ind_V(v) / 1 \leq i \leq \rho\} - \min\{Ind_V(v_i), Ind_V(v) / 1 \leq i \leq \rho\} - i\delta_2, 0 \leq i \leq \rho - 1$$
$$Ind_E(v, u_{\rho-i}) = \max\{Fal_V(v_i), Fal_V(v) / 1 \leq i \leq \rho\} - \min\{Fal_V(v_i), Fal_V(v) / 1 \leq i \leq \rho\} - k\delta_3, 0 \leq i \leq \rho - 1$$

Case (i) 'i' is even.

Then i=2m, where $m \in Z^+$ and for each edge v, u_i.

$$\begin{aligned}
m_{Tr}(S_{1,\rho}) &= Tr_V(v) + Tr_E(v, u_i) + Tr_V(u_i) \\
&= Tr_V(v) + Tr_E(v, u_{2m}) + Tr_V(u_{2m}) \\
&= \min ima\{Tr_V(u_i) / 1 \leq i \leq \rho\} - \delta_1 + \max ima\{Tr_V(v_i), Tr_V(v) / 1 \leq i \leq \rho\} \\
&\quad - \min ima\{Tr_V(v_i), Tr_V(v) / 1 \leq i \leq \rho\} - (v - 2m)\delta_1 + [2(\rho+1) - 2m]\delta_1 \\
m_{Tr}(S_{1,\rho}) &= \min ima\{Tr_V(u_i) / 1 \leq i \leq \rho\} + \max ima\{Tr_V(v_i), Tr_V(v) / 1 \leq i \leq \rho\} \\
&\quad - \min ima\{Tr_V(v_i), Tr_V(v) / 1 \leq i \leq \rho\} + (\rho+1)\delta_1
\end{aligned} \tag{19}$$

$$\begin{aligned}
m_{Ind}(S_{1,\rho}) &= Ind_V(v) + Ind_E(v,u_i) + Fal_V(u_i) \\
&= Ind_V(v) + Ind_E(v,u_{2m}) + Ind_V(u_{2m}) \\
&= \min ima\{Ind_V(u_i)/1 \le i \le \rho\} - \delta_2 + \max ima\{Ind_V(v_i), Ind_V(v)/1 \le i \le \rho\} \\
&\quad - \min ima\{Ind_V(v_i), Ind_V(v)/1 \le i \le \rho\} - (\rho - 2m)\delta_2 + [2(\rho+1) - 2m]\delta_2 \\
m_{Ind}(S_{1,\rho}) &= \min ima\{Ind_V(u_i)/1 \le i \le \rho\} + \max ima\{Ind_V(v_i), Ind_V(v)/1 \le i \le \rho\} \\
&\quad - \min ima\{Ind_V(v_i), Ind_V(v)/1 \le i \le \rho\}(\rho+1)\delta_2
\end{aligned} \tag{20}$$

$$\begin{aligned}
m_{Fal}(S_{1,\rho}) &= Fal_V(v) + Fal_E(v,u_i) + Fal_V(u_i) \\
&= Fal_V(v) + Fal_E(v,u_{2m}) + Fal_V(u_{2m}) \\
&= \min\{Fal_V(u_i)/1 \le i \le \rho\} - \delta_3 + \max\{Fal_V(v_i), Fal_V(v)/1 \le i \le \rho\} \\
&\quad - \min\{Fal_V(v_i), Fal_V(v)/1 \le i \le \rho\} - (\rho - 2m)\delta_3 + [2(\rho+1) - 2m]\delta_3 \\
m_{Fal}(S_{1,n}) &= \min\{Fal_V(u_i)/1 \le i \le \rho\} + \max\{Fal_V(v_i), Fal_V(v)/1 \le i \le \rho\} \\
&\quad - \min\{Fal_V(v_i), Fal_V(v)/1 \le i \le \rho\} + (\rho+1)\delta_3
\end{aligned} \tag{21}$$

so that $M_{m(G)}(S_{1,\rho}) = (m_{Tr}(G), m_{Ind}(G), m_{Fal}(G))$ = constant.

Equations (19), (20), (21) satisfy the condition for NFMLG when 'i' is even.

Case (ii) 'i' is odd

Then i=2m+1, where $m \in Z^+$ and for each edge v, u_i

$$\begin{aligned}
m_{Tr}(S_{1,\rho}) &= Tr_V(v) + Tr_E(v,u_i) + Tr_V(u_i) \\
&= Tr_V(v) + Tr_E(v,u_{2m+1}) + Tr_V(u_{2m+1}) \\
&= \min ima\{Tr_V(u_i)/1 \le i \le \rho\} - \delta_1 + \max ima\{Tr_V(v_i), Tr_V(v)/1 \le i \le \rho\} \\
&\quad - \min\{Tr_V(v_i), Tr_V(v)/1 \le i \le \rho\} - (\rho - 2m - 1)\delta_1 + (2(\rho+1) - 2m + 1)\delta_1 \\
m_{Tr}(S_{1,\rho}) &= \min ima\{Tr_V(u_i)/1 \le i \le \rho\} + \max ima\{Tr_V(v_i), Tr_V(v)/1 \le i \le \rho\} \\
&\quad - \min ima\{Tr_V(v_i), Tr_V(v)/1 \le i \le \rho\} + (\rho+1)\delta_1
\end{aligned} \tag{22}$$

$$\begin{aligned}
m_{Ind}(S_{1,\rho}) &= Ind_V(v) + Ind_E(v,u_i) + Ind_V(u_i) \\
&= Ind_V(v) + Ind_E(v,u_{2m+1}) + Ind_V(u_{2m+1}) \\
&= \min ima\{Ind_V(u_i)/1 \le i \le \rho\} - \delta_1 + \max ima\{Ind_V(v_i), Ind_V(v)/1 \le i \le \rho\} \\
&\quad - \min\{Ind_V(v_i), Ind_V(v)/1 \le i \le \rho\} - (\rho - 2m - 1)\delta_2 + (2(\rho+1) - 2m + 1)\delta_2 \\
m_{Ind}(S_{1,\rho}) &= \min ima\{Ind_V(u_i)/1 \le i \le \rho\} + \max ima\{Ind_V(v_i), Ind_V(v)/1 \le i \le n\} \\
&\quad - \min ima\{Ind_V(v_i), Ind_V(v)/1 \le i \le \rho\} + (\rho+1)\delta_2
\end{aligned} \tag{23}$$

$$\begin{aligned}
m_{Fal}(S_{1,\rho}) &= Fal_V(v) + Fal_E(v,u_i) + Fal_V(u_i) \\
&= Fal_V(v) + Fal_E(v,u_{2m+1}) + Fal_V(u_{2m+1}) \\
&= \min ima\{Fal_V(u_i)/1 \le i \le \rho\} - \delta_3 + \max ima\{Fal_V(v_i), Fal_V(v)/1 \le i \le \rho\} \\
&\quad - \min ima\{Fal_V(v_i), Fal_V(v)/1 \le i \le \rho\} - (\rho - 2m - 1)\delta_3 + (2(\rho+1) - 2m + 1)\delta_3 \\
m_{Fal}(S_{1,\rho}) &= \min ima\{Fal_V(u_i)/1 \le i \le \rho\} + \max ima\{Fal_V(v_k), Fal_V(v)/1 \le i \le \rho\} \\
&\quad - \min ima\{Fal_V(v_i), Fal_V(v)/1 \le i \le \rho\} + (\rho+1)\delta_3
\end{aligned} \tag{24}$$

So that, $M_{m(G)}\left(S_{1,\rho}\right)=\left(m_{Tr}(G),m_{Ind}(G),m_{Fal}(G)\right)$ = constant.

Equations (16), (17), and (18) satisfy the condition for NFMLG when 'i' is odd.

The magic value is the same and distinct in all of the cases before it. The star graph is therefore NFMLG.

Example 7.2:

The NFMLG of star graph shown in Figure 3 has seven nodes and seven edges.

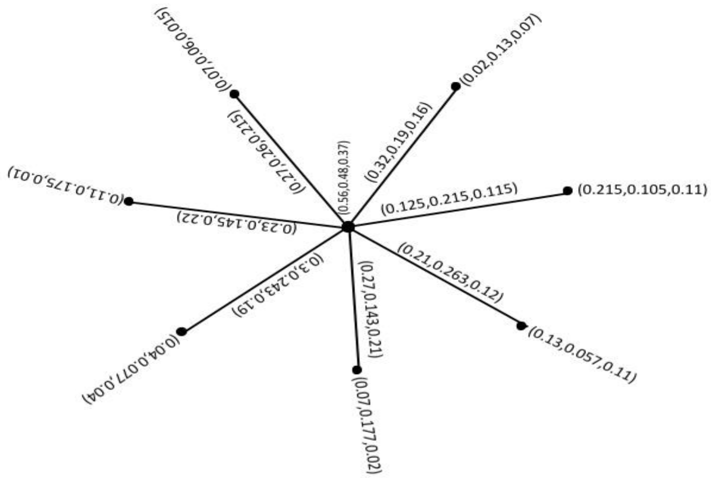

Figure 3:NFMLG of star graph

The magic value of the aforementionedNeutrosophic star graph $S_{1,7}$ is $(0.9, 0.8, 0.6)$

The truth membership,indeterminacy and falsity value of NFMLG-star graph as follows:

Truthmembership value-0.9

Indeterminacy-0.8

Falsity-0.6

Using definition 3.8,the score value of the magic value of NFMLG ofa star graph is calculated andis given by $S(\alpha)=-0.15$. The NFMLG-star graph's score value here indicates that our solution meets the requirements for a neutrosophic set because it is between [-1,1] score limit.

8. Application of Neutrosophicfuzzy magic labelingpathgraphs

In this section, the advantage of NFMLG and why we recommended NFMLG concept to a problem involving decision-making has been elaborated.

Advantage, Uses and Limitation of Neutrosophic fuzzy magic labeling:

The neutrosophic becomes appreared and found their place in research since the world is full of uncertainty (indeterminacy) so we used neutrosophic in our research. These ideas, while applicable to a variety of real-world problems, cannot deal with all forms of uncertainty, such as ambiguous and inconsistent information.

Uses of Neutrosphic magic labeling:

In many field like medical image processing applications, neutrosophic sets (NS) play a vital role in denoising, clustering, segmentation, and classification. In order to reduce uncertainty for effective diagnosis, clustering techniques have been integrated with NS for the efficient creation of computer-aided diagnosis systems.

When it comes to fuzzy magic labelling graphs, we can only talk about the criteria in one category, but when it comes to intuitionistic fuzzy magic labelling graphs, we can discuss the criteria in two different scenarios. However, in the case of NFMLG, we are talking about the criteria in three different scenarios. This is one of the key benefits of NFMLG because it offers more options, greater flexibility, and greater compatibility. Therefore, we are using the NFMLG-path graph in this student's subject selection decision-making problem in the following way:

The subject of education is divisive and has generated numerous arguments. One of them deals with topic choice. Some people think that students should be free to select the subjects they want to learn about, while others think that all subjects should be obligatory. In our opinion, kids need to have the option to select topics based on their interests. Students today are very well educated, and they pick their classes based on what they want to do for a living in the future.

Despite a few very popular courses, students who have a strong interest in a certain subject or career will regard all options available as the best option. Gaining curiosity and confidence is enough to become a master in any field. It is crucial for students whose secondary education is about to end to choose their career path as soon as they graduate. At this time, it is crucial to stress how important it is to give children enough information about career alternatives that are related to their interests.

This section displays the interest, confusion, or lack thereof among students in a subject or a group of topics based on their replies, which were provided by 100 students in class "X" [47]. The data show that NFMLG can be used as a tool because it considers three different membership functions, including membership with indeterminacy (the conundrum that a certain proportion of students in a particular subject or pair face) and non-membership (the extent to which students do not belong to a particular subject or pair) (the disinterest of a percentage of students in a subject or pair of subjects). Using NFMLG, we can determine which courses will likely benefit the most students and result in the highest levels of learning when taken together.

Let $Subject\,(S) =$ {English (ENG), Language (LANG), Mathematics (MAT),Science (SC),Social Science (SSC)} be the collection of vertices. The table below depicts the proportion of students that are interested, undecided, or disinterested in picking a subject or pair of topics.

Table 1: Indicated the subject/subject combination

Subject/Subject Combination	Interest	Dilemma	Disinterest
	*Here 0.72 represents 72%		
ENG	0.16	0.30	0.49
LANG	0.12	0.23	0.13
MAT	0.72	0.15	0.46
SC	0.15	0.63	0.41
SSC	0.63	0.19	0.21
ENG-MAT	0.12	0.55	0.05
ENG-LANG	0.72	0.47	0.38
ENG-SC	0.69	0.07	0.10
ENG-SSC	0.12	0.51	0.30
MAT-SC	0.13	0.22	0.13
LANG-MAT	0.16	0.62	0.41
LANG-SC	0.73	0.14	0.43
SC-SSC	0.22	0.18	0.38

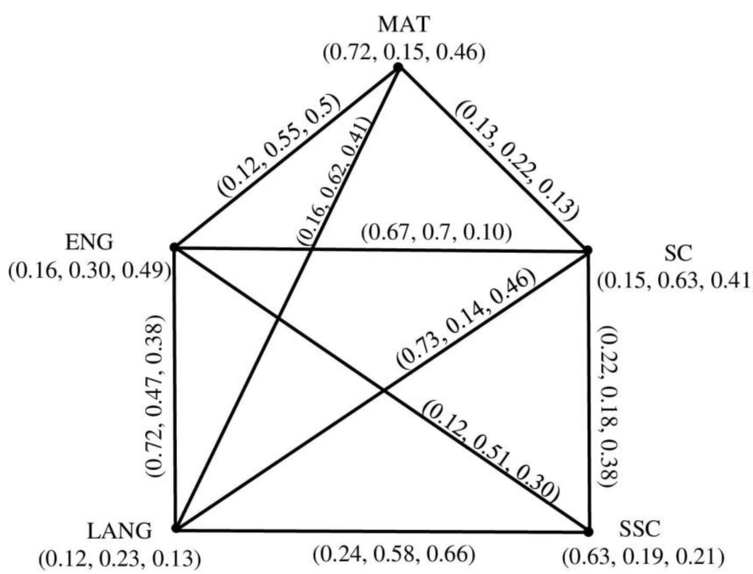

Figure 4.Graph representation the subject/subject combination

Figure 4 displays the percentage of students who are passionate about a certain subject, the percentage of students who are undecided, and the percentage of students who have no interest in a subject.The students' interest in, perplexity over, and lack of interest in integrating any two upper-secondary courses may be seen in the graph edges' membership, indeterminacy, and non-membership.

Figure 4 shows that the majority of students who study both language and science are interested in pursuing a career in medicine, which is shown by an edge (LANG, SC) with the highest degree of membership function. According to research, students who are afraid of math lessons might choose this option. Most students are pulled between studying Language and Math, according to the edge (LANG, MATHS). Language and social science are not subjecting that students who have strong non-membership functions on the edge (LANG, SSC) desire to study together.

Research limitations:

It has limitations for Bipolar neutrosophic set, complex intuitionistic fuzzy hypersoft set, complex neutrosophic hypersoft set and other complex neutrosophic hypersoft set-like models.

10. Comparative analysis

Comparative analysis and the current methodology have been addressed throughout this section.

Table 2: Comparative analysis

Method	Results
Intuitionistic Fuzzy Graph	Membership Function-Interest: (MAT,SC)
	Non-Membership Function-Disinterest: (LANG,SC)
Neutrosophic Fuzzy Magic Labeling of Cycle graph	Membership Function-Interest: (LANG, SC)
	Indeterminacy Membership Function- Dilemma: (LANG, MATHS)
	Non-Membership Function-Disinterest: (LANG,SSC)

The results for the present approach of neutrosophic fuzzy magic labelling of simple graphs are proven in Table 2. The membership function's output reveals that the majority of students are drawn to the idea of merging mathematics and science. The majority of students do not want to study a mix of language and social science courses, according to the results of the non-membership function. This NFMLG-recommended approach reveals that students are interested in choosing Language and Science topics based on the membership function and that they are interested in Math and Language based on the indeterminacy result. Additionally, the non-membership function demonstrates that the majority of students detest the combination of the social science and language fields. In the NFMLG context, we are debating how to divide the subjects into three groups to choose the best selection for the students. Students will instinctively select the better option if there are more options available. This multiple option and different subject combination facility is possible while using only NFMLG; this is the main advantage of the neutrosophic fuzzy magic labelling graph.

11. Conclusion

A Neutrosophic network is an extension of an intuitionistic fuzzy network that offers greater precision, compatibility, and flexibility when organising the modelling in many real-world applications than an intuitionistic fuzzy graph. In neutrosophic graph problems, connectivity principles are the main solution strategy. Especially the magic labelling model offers the system higher accuracy, adaptability, and compatibility when compared to classical methods. Hence, in this paper, we propose the definition of the neutrosophic fuzzy magic labelling graph and a detailed discussion of its properties using numerical examples for the path, cycle, and star graphs in a neutrosophic environment. The proposed work has also been used in decision-making situations to choose the optimal subject combinations based on student interests for the best academic success. In order to demonstrate the validity of the proposed work, a comparative analysis with the current methodology has also been conducted. The current research may be in future expanded into a neutrosophic superhypergraph environment.

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Neutrosophic Statistical Analysis on Gold Rate

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Abstract: Gold ornaments are always been seen as a metal of pride and happiness. It is also considered to be one of the auspicious costliest metals. Gold rate is found at an increasing rate from an early period. This paper analyzes the gold rate at six different cities in India. For each of the cities, neutrosophic mean and coefficient of variation have been calculated. The analysis results have been presented. In addition to that the neutrosophic analysis based on each month also has been calculated. The results help to know the beneficial month and city for the purchase of gold.

Keywords: Gold Price; Neutrosophic mean; Coefficient of variation; Neutrosophic statistics;

1. Introduction

Gold is a naturally occurring chemical element. It is one of the elements which is less reactive than other metals. The luster of the metal is not lost since it does not react with oxygen, acids etc. Being malleable it can be easily molded into any favorable design as we require. From ancient times to modern times, gold possesses its own tradition and it is one of the favorite metals liked by women. Gold ornaments are worn by people to look pretty well, as a prestigious thing to show off their status and so on. In many of the middle class families, golden ornaments are bought as savings to make use for pledging when there is a need for money. Apart from these, wearing of gold also helps us in regulate our body temperature. Gold is also a useful metal.

In the ancient period, ornaments were worn by both men and women to beautify themselves and look elegant. Women even used jewelry from their head to toe. In the early period beads and shells were used to make ornaments. But these ornaments did not have a good life. Then people of the Indus valley region introduced jewels made of metal. Gold has been regarded as a precious metal that occupies the first position as mangal sutra at the beginning of marriage life. Diamonds are the next adorable stones which were also first introduced by Indians. Diamonds tie up with gold in making pretty ornaments.

There are many kinds of ornaments which are worn in different parts of the body. Maangtika, paasa, veni are the names of the ornaments that are used in hair plaiting. These are the ornaments that are worn by women occasionally to give a special look during functions like marriage, baby shower ceremony and so on. Jhumkas, chandbalis, kanvelis are worn as earrings. Nath is an ornament worn as nose pins. Gulbandh, rani haar and satlada are a few ornaments worn on neck similar to necklace. Kamarbandh is worn around the waist. Hands are decorated with bracelets, bangles. Hathpool ornament is worn as a connector of rings worn in fingers, which looks like a spider web. Paizeb, ghungroo payals, toe ring payals are worn to beautify feet. Most popular models of jewellery are termed as bead jewels, bridal jewels, antique jewels, kundan jewels, ivory jewels, temple jewellery, jadau jewels.

Most of the people invest in gold savings. People also look for auspicious days to turn their money into gold. It would be helpful if the people were aware of the month and place where the price of gold could be an optimized state for the buyers. With a motive to make people beneficial from the purchase, this study is conducted to predict the rate of gold considering 6 popular cities in Tamilnadu.

Many researchers have done their work on neutrosophic statistical analysis. A few of them are listed as literature survey as given below. Muhammad Aslam in 2019 proposed a plan to identify a plan based on attributes using the method of interval in neutrosophic statistics [4]. The same researcher in 2019 performed a neutrosophic analysis on identifying the applications used by university students [6]. R. Dhavaseelan et al in 2019 discussed about neutrosophic continuity [5]. Alexandra Dolores Molina Manzano et al in 2020 conducted an analysis for people aged between 16 to 18 regarding the voting systems by applying neutrosophic statistical analysis [7]. Broumi said et al in 2020 introduced trapezoidal fuzzy numbers in obtaining a new distance measure with the usage of centroids [8]. Muhammad Naveed Jafar et al in 2020 discussed about the neutrosophic environment concerning the similarity measures in trigonometric functions [9]. Carlos Acosta Mayorga et al in 2021 analyzed on surgical site infection after the procedures of vascular surgery. Concerning the field of medical sciences, statistical approaches for management with indeterminacy were involved in their work [10]. Ishmal Shahzadi et al in 2021 underwent neutrosophic statistical analysis to report on the income of YouTube channels [11]. Fernando Castro Sánchez et al in 2021 processed their work on developing education by applying neutrosophic and plithogenic statistical analysis [12]. Lysbeth Kruscthalia Álvarez Gómez et al in 2021 involved themselves in making an analysis on E-commerce [13].

Lester Wong Vázquez et al in 2021 Rehabilitation of Arterial Hypertension using neutrosophic statistical analysis [14]. Sara Guerrón Enríquez et al in 2021 considered Arthrofibrosis of the Knee Rehabilitation for their study. They applied neutrosophic statistical analysis for their study [16]. Abdullah Gamal et al in 2022 have framed an intelligent model related to the manufacturing sector in overcoming the barriers during COVID pandemic [18]. Elizabeth Cristina Mayorga Aldaiz et al in 2022 conducted an assessment of university students for rehabilitation using neutrosophic statistical analysis [19]. Kenia Mariela Peñafiel Jaramillo et al in 2022 performed an analysis to know the behavioral medical knowledge among university students with the help of neutrosophic statistical analysis [20]. Florentin Smarandache in 2022 performed a comparative study on neutrosophic statistics and plithogenic statistics [22].

Muhammad Rafiq et al in 2022 proposed a statistical analysis for formulating a trend in the temperature of Baluchistan at Pakistan [24]. Rushikesh Ghule, Abhijeet Gadhave in 2022 used a machine learning approach in the method of forecasting the rate of gold [26]. Said Broumi et al in 2022 constructed a survey process on identifying the problems in medical diagnosis. They used neutrosophic sets along with their hybrid structures for their research work. [27]. Ishmal Shahzadi in 2023 performed a statistical analysis on the temperature of various cities of Pakistan [30]. Muhammad Aslam, and Muhammad Saleem in 2023 privileged to used F test neturosophic statistics to make analysis on linear applications [32]. Regan Murugesan et al in 2023 have conducted a study on variants of covid applying Neutrosophic cognitive maps and Fuzzy cognitive maps [33]. Aral et al in 2023 have discussed normed linear spaces. They have considered difference sequence of fractional order and their strongly lacunary convergence with order β in their work [28]. Kandemir et al in 2023 have done a work similar to previously mentioned work of Aral et al with order α [31]. Mohamed Abdel-Basset et al in 2023 did a network security communication with the help of their optimization model [23]. Nada A. Nabeeh et al in 2022 discussed their twin type block chain technology and its production [25]. Ayman H. Abdel-aziem et al in 2023 presented on decision making algorithm with respect to bank sector in bringing about optimization in investment [29]. Uma G and Nandhitha S in 2023 performed their study Neutrosophic Poisson distribution with a quick switching system [34].

Waleed Tawfiq Al-Nami in 2023 discussed the strategies for the safety of people in the crowd applying their ranks and analysis [35].

The work done by a few other researchers in predicting the price of gold applying various methods is listed below. M. Khalid et al in 2014 performed forecasting on the price of gold collected evidences from the Pakistan gold market [1] Iftikhar ul Sami and Khurum Nazir Junejo in 2017 presented their work on prediction of gold rate using machine language approach [2]. Naliniprava Tripathy in 2017 applied moving average model which integrates with auto regression for their analysis in foretelling the price of gold [3]. Mustafa Yurtsever in 2021 made an analysis on forecasting the price of gold using the methods of LSTM, Bi-LSTM and GRU [15]. Sultan Salem et al in 2021 performed a life time data analysis in finding the properties and their applications applying neutrosophic lognormal method [17]. Laor Boongasame in 2022 made use of the association rule and memory of long and short term in the forecasting of the price of gold [21].

This paper aims at bringing out an analysis on the rate of gold comparison at various regions of India using neutrosophic statistics.

2. Methodology

The prediction of gold rate involved various steps. Initially the period on which the data is to be collected is finalized. Then the required data is collected with the help of the application related to it. The collected data are tabulated in required format. Then the neutrosophic statistical measures such as mean and standard deviation are evaluated with the help of MATLAB software. With the calculated data. Neutrosophic statistical analysis is performed and the conclusion is given. which are given as diagrammatic representation beneath given as Figure 1.

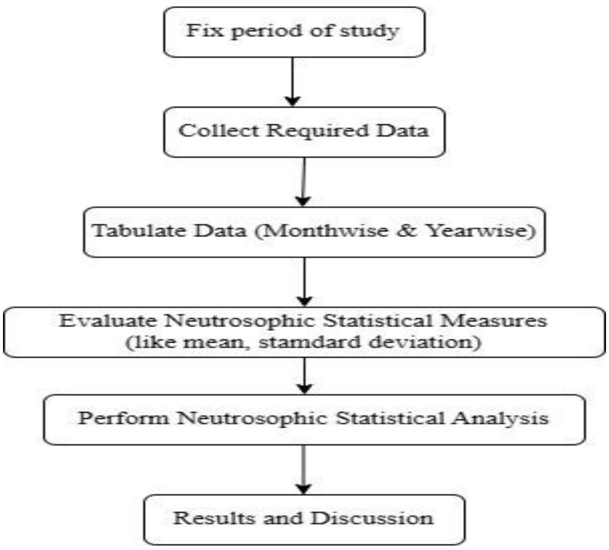

Figure 1: Schema of the proposed procedure

Various definitions applied related to the work performed are listed below.

Consider $\boldsymbol{X_N = X_L + X_U I_N, I_N} \in [\boldsymbol{I_L}, \mathbf{I_U}]$ to be a neutrosophic random variable representing the rate of gold at various month and at various cities. Her $\boldsymbol{X_L}$ denotes the lower rate of gold and $\boldsymbol{X_U}$ denotes the upper rate of gold. $\boldsymbol{I_N}$ is the interval of indeterminacy.

The neutrosophic average of gold data $\boldsymbol{X_N} \in [\boldsymbol{X_L}, \mathbf{X_U}]$ is $\mathbf{\bar{X}_N = \bar{X}_L + \bar{X}_U \ I_N; I_N} \in [\mathbf{I_L}, \mathbf{I_U}]$

for which $\mathbf{\bar{X}_L} = \sum_{\mathbf{i=1}}^{\mathbf{n_N}} \mathbf{\bar{X}_{iL}}$, $\mathbf{\bar{X}_U} = \sum_{\mathbf{i=1}}^{\mathbf{n_N}} \mathbf{\bar{X}_{iU}}$ and $\boldsymbol{n_N} = [\boldsymbol{n_L}, \boldsymbol{n_U}]$

The neutrosophic standard deviation is calculated as given below.

$$\sum_{i=1}^{n_N}(\mathbf{X_i}-\bar{\mathbf{X}}_{iU})^2=\sum_{i-1}^{n_N}\begin{bmatrix}\min\begin{pmatrix}(a_i+b_iI_L)(\bar{a}+\bar{b}I_L),(a_i+b_iI_L)(\bar{a}+\bar{b}I_U)\\(a_i+b_iI_U)(\bar{a}+\bar{b}I_L),(a_i+b_iI_L)(\bar{a}+\bar{b}I_L)\end{pmatrix}\\\max\begin{pmatrix}(a_i+b_iI_L)(\bar{a}+\bar{b}I_L),(a_i+b_iI_L)(\bar{a}+\bar{b}I_U)\\(a_i+b_iI_U)(\bar{a}+\bar{b}I_L),(a_i+b_iI_L)(\bar{a}+\bar{b}I_L)\end{pmatrix}\end{bmatrix}$$

where $\mathbf{I}\ \epsilon\ [\mathbf{I_L},\mathbf{I_U}]$ and $a_i=X_L;\quad b_i=X_U$

The neutrosophic sample variance is $\mathbf{S}_N^2=\frac{\sum_{i=1}^{n_N}(X_i-\bar{X}_{iN})^2}{n_N};\quad \mathbf{S}_N^2\ \in\ [\mathbf{S}_L^2,\mathbf{S}_U^2]$

The neutrosophic form of $\mathbf{S}_N^2\ \in\ [\mathbf{S}_L^2,\mathbf{S}_U^2]$ is given as $a_S+b_S\ I_{NS}\ \in\ [I_{LS},I_{US}]$

The consistency on the rate of gold can be known from the coefficient of variation given as

$$CV_N=\frac{\sqrt{s_N^2}}{\bar{X}_N}\times 100;\quad C_{VN}\ \in\ [C_{VL},C_{VU}]$$

The neutrosophic form of CV_N is $a_v\ +b_vI_{Nv};I_{Nv}\ \in\ [I_{Lv},I_{Uv}]$

3. Data Collection

The rate of gold remains different even in a same day. It also remains different in different places of the state. In such case of uncertainty, the data to be collected from reliable source. It is proposed to consider six major cities in India for the analysis on gold rates. The cities considered for analysis are Chennai, Kolkatta, Bangalore, Madurai, Hyderabad and Delhi. The everyday price of 22 carat gold in each of these cities are collected from the application named 'Indian Daily Gold Prices Android App' which can be downloaded using the link https://goo.gl/KoCNzt/. The collected rate are for one gram of gold which is mentioned in rupees. The data ranging from February 1 2022 to January 31, 2023 are collected. From the collected data, maximum and minimum rate corresponding to each month is listed out and presented in Tables as given below. Table 1 represents the low and high gold price rate corresponding the cities Chennai, Kolkatta, Bangalore. Table 2 corresponds to the low and high gold price rate corresponding to the cities Madurai, Hyderabad and Delhi.

Table 1. Gold price in rupees for the first three cities

	Chennai		**Kolkatta**		**Bangalore**	
	Low	**High**	**Low**	**High**	**Low**	**High**
Feb	5099.86	5330.14	5033.4	5283.5	5117.4	5283.5
Mar	5245.9	5549.19	5278.5	5659.4	5278.5	5659.4
Apr	5369.46	5599.74	5287.5	5496.1	5287.5	5496.1
May	5178.5	5436.86	5179.4	5361.2	5179.4	5361.2
Jun	5240.28	5434.28	4897.65	5434.28	5240.28	5434.28
Jul	5167.26	5397.54	5157.49	5393.45	5167.26	5397.54
Aug	5279.59	5408.78	5281.09	5337.26	5290.83	5341.38
Sep	5144.8	5268.36	5146.25	5269.85	5144.8	5268.36
Oct	5195.35	5375.08	5196.81	5376.6	5195.35	5375.08
Nov	5234.66	5526.72	5179.96	5498.64	5178.5	5526.72
Dec	5532.34	5740.15	5477.73	5685.6	5526.72	5740.15
Jan	5717.69	5998.52	5668.74	5959.2	5717.69	5998.52

Table 2. Gold price in rupees for the second three cities

	Madurai		Hyderabad		Delhi	
	Low	**High**	**Low**	**High**	**Low**	**High**
Feb	5116.71	5330.14	5033.4	5283.5	5117.4	5283.5
Mar	5245.9	5594.12	5278.5	5659.6	5278.5	5659.4
Apr	5369.46	5599.74	5287.4	5496	5287.5	5496.1
May	5178.5	5436.86	5179.4	5361.2	5179.4	5361.2
Jun	5240.28	5434.28	4897.65	5434.28	4897.65	5434.28
Jul	5167.26	5397.54	5157.49	5393.45	5157.49	5393.45
Aug	5290.83	5341.38	5281.09	5337.26	5281.09	5337.26
Sep	5144.8	5268.36	5146.25	5269.85	5146.25	5269.85
Oct	5195.35	5375.08	5196.81	5376.6	5196.81	5376.6
Nov	5234.66	5526.72	5179.96	5498.64	5179.96	5498.64
Dec	5532.34	5740.15	5477.73	5685.6	5477.73	5685.6
Jan	5717.69	5998.52	5668.74	5959.2	5668.74	5959.2

Using the data presented in Table 1 and 2, neutrosophic statistical analysis is conducted whose outcome is presented in section 4.

4. Results and Interpretation

The neutrosophic statistical analysis is performed to the data on gold price. The city wise data analysis are presented from Table 3 to Table 5. The month wise data analysis are presented form Table 6 to Table 9. Table 3 represents the city wise neutrosophic mean of gold data. Table 4 represents the city wise neutrosophic standard deviation of gold data. Table 5 represents the city wise neutrosophic coefficient of variation of gold data. Table 6 represents the month wise neutrosophic mean of gold data. Table 7 represents the month wise neutrosophic standard deviation of gold data. Table 8 represents the month wise neutrosophic coefficient of variation of gold data.

Table 3. The Neutrosophic mean of gold price at different cities

Cities	$\bar{X}_N$	$a_{\bar{X}} + b_{\bar{X}} I_{N\bar{X}};\ I_{N\bar{X}};\ \in [I_{L\bar{X}}, I_{U\bar{X}}]$
$Chennai$	[5283.81, 5505.45]	5283.81+5505.45IN, IN ∈ [0, 0.04]
$Kolkatta$	[5232.04, 5479.59]	5232.04+5479.59IN, IN ∈ [0, 0.04]
$Bangalore$	[5277.02, 5490.19]	5277.02+5490.19IN, IN ∈ [0, 0.038]
$Madurai$	[5286.15, 5503.57]	5286.15+5503.57IN, IN ∈ [0, 0.039]
$Hyderabad$	[5232.04, 5479.60]	5232.04+5479.60IN, IN ∈ [0, 0.044]
$Delhi$	[5239.04, 5479.59]	5239.04+5479.59IN, IN ∈ [0, 0.043]

The neutrosophic average rate of gold at Chennai city lies between 5283.81 and 5505.45 and at Kolkatta city lies between 5232.04, 5479.59. Both these cities are with a measure of indeterminacy being 0.04. Bangalore city has a neutrosophic average rate of gold lying between 5277.02 and 5490.19 with a measure of indeterminacy 0.038. The neutrosophic average rate of gold at Madurai lies between 5286.15, 5503.57 with a indeterminacy rate of 0.039. The neutrosophic average rate of gold at

Hyderabad lies between 5232.04, 5479.60 with a indeterminacy rate of 0.044. The neutrosophic average rate of gold at Delhi lies between 5239.04, 5479.59 with a indeterminacy rate of 0.043. It is seen that the neutrosophic average at low rate of gold is minimum at Kolkatta city which can be considered for purchase.

Table 4. Gold price as its Neutrosophic standard deviation at preferred cities

City	S_N	$a_S + b_S I_{NS;}\ I_{NS;}\ \in [I_{LS}\ ,I_{US}]$
Chennai	[178.07, 200.94]	178.07+200.94IN, IN ∈ [0, 0.1]
Kolkatta	[197.63, 200.51]	197.63+200.51IN, IN ∈ [0, 0.015]
Bangalore	[175.58, 214.59]	175.58+214.59IN, IN ∈ [0, 0.18]
Madurai	[176.54, 206.19]	176.54+206.19IN, IN ∈ [0, 0.14]
Hyderabad	[197.63, 200.52]	197.63+200.52IN, IN ∈ [0, 0.01]
Delhi	[191.34, 200.51]	191.34+200.51IN, IN ∈ [0, 0.04]

The neutrosophic standard deviation at Chennai city lies between 178.07 and 200.94 with a measure of indeterminacy 0.01. The neutrosophic standard deviation at Kolkatta city lies between 197.63 and 200.51 with a measure of indeterminacy 0.015. The neutrosophic standard deviation at Bangalore city lies between 175.58 and 214.59 with a measure of indeterminacy 0.018. The neutrosophic standard deviation at Madurai city lies between 176.54 and 206.19 with a measure of indeterminacy 0.014. The neutrosophic standard deviation at Hyderabad city lies between 197.63 and 200.52 with a measure of indeterminacy 0.01. The neutrosophic standard deviation at Delhi city lies between 191.3 and 200.51 with a measure of indeterminacy 0.014. The least standard deviation among the low gold rate is found at Bangalore city. Among the high gold rate minimum is found equal at both Kolkotta and Delhi city.

Table 5. Gold price as its neutrosophic coefficient of variation at different cities

City	CV_N	$a_V + b_V I_{NV;}\ I_{NV;}\ \in [I_{LV}\ ,I_{UV}]$
Chennai	[3.37, 3.65]	3.37+3.65IN, IN ∈ [0, 0.08]
Kolkatta	[3.78, 3.66]	3.78-3.66IN, IN ∈ [0, 0.032]
Bangalore	[3.33, 3.91]	3.33+3.91IN, IN ∈ [0,0.14]
Madurai	[3.34, 3.75]	3.34+3.75IN, IN ∈ [0, 0.109]
Hyderabad	[3.78, 3.66]	3.78-3.66IN, IN ∈ [0, 0.032]
Delhi	[3.65, 3.66]	3.65+3.66IN, IN ∈ [0, 0.001]

The neutrosophic coefficient of variation at Chennai city lies between 3.37 and 3.65 with a measure of indeterminacy 0.06. The neutrosophic coefficient of variation at Kolkatta city lies between 3.78 and 3.66 with a measure of indeterminacy 0.0305. The neutrosophic coefficient of variation at Bangalore city lies between 3.33 and 3.91 with a measure of indeterminacy 0.14. The neutrosophic coefficient of variation at Madurai city lies between 3.34 and 3.75 with a measure of indeterminacy 0.09. The neutrosophic coefficient of variation at Hyderabad city lies between 3.78 and 3.66 with a measure of indeterminacy 0.033. The neutrosophic coefficient of variation at Delhi city lies between 3.65 and 3.66 with a measure of indeterminacy 0.001. The minimum coefficient of variation is found among low gold rate at Madurai city and the minimum among the high gold rate if found at Chennai.

Table 6. The Neutrosophic mean of gold price taken monthwise

Month	$\overline{X}_N$	$a_{\overline{X}} + b_{\overline{X}} I_{N\overline{X};}\ I_{N\overline{X};}\ \in [I_{L\overline{X}}\ ,I_{U\overline{X}}\ ]$
Feb	[5086.36, 5299.05]	5086.36+5299.05IN, IN ∈ [0, 0.04]
Mar	[5267.63, 5630.19]	5267.63+5630.19IN, IN ∈ [0, 0.06]

Apr	[5314.80, 5530.63]	5314.80+5530.63IN, IN ∈ [0, 0.04]
May	[5179.1, 5386.42]	5179.1+5386.42IN, IN ∈ [0, 0.03]
Jun	[5068.97, 5434.28]	5068.97+5434.28IN, IN ∈ [0, 0.06]
Jul	[5162.38, 5395.495]	5162.38+5395.495IN, IN ∈ [0, 0.04]
Aug	[5284.09, 5350.56]	5284.09+5350.56IN, IN ∈ [0, 0.01]
Sep	[5145.53, 5269.11]	5145.53+5269.11IN, IN ∈ [0, 0.023]
Oct	[5196.08, 5375.84]	5196.08+5375.84IN, IN ∈ [0, 0.033]
Nov	[5197.95, 5512.68]	5197.95+5512.68IN, IN ∈ [0, 0.057]
Dec	[5504.10, 5712.88]	5504.10+5712.88IN, IN ∈ [0, 0.036]
Jan	[5693.22, 5978.86]	5693.22+5978.86IN, IN ∈ [0, 0.047]

The neutrosophic average rate of gold at Febraury month lies between 5086.36 and 5299.05 and with a measure of indeterminacy being 0.1. The neutrosophic average rate of gold at March month lies between 5267.63 and 5630.19 and with a measure of indeterminacy being 0.07. The neutrosophic average rate of gold at April month lies between 5314.80 and 5530.63 and with a measure of indeterminacy being 0.03. The neutrosophic average rate of gold at May month lies between 5179.1 and 5386.42 and with a measure of indeterminacy being 0.03. The neutrosophic average rate of gold at June month lies between 5068.97 and 5434.28 and with a measure of indeterminacy being 0.06. The neutrosophic average rate of gold at July month lies between 5162.38 and 5395.495 and with a measure of indeterminacy being 0.04. The neutrosophic average rate of gold at August month lies between 5284.09 and 5350.56 and with a measure of indeterminacy being 0.01. The neutrosophic average rate of gold at September month lies between 5145.53 and 5269.11 and with a measure of indeterminacy being 0.023. The neutrosophic average rate of gold at October month lies between 5196.08 and 5375.84 and with a measure of indeterminacy being 0.033. The neutrosophic average rate of gold at November month lies between 5197.95 and 5512.68 and with a measure of indeterminacy being 0.057. The neutrosophic average rate of gold at December month lies between 5504.10 and 5712.88 and with a measure of indeterminacy being 0.036. The neutrosophic average rate of gold at January month lies between 5693.22 and 5978.86 and with a measure of indeterminacy being 0.047. The neutrosophic average is found to be minimum at the month of September which can be considered to be the favorite moth for the purchase of gold. The minimum mean among the low rate of gold is found at June month and the minimum mean among the high gold rate is found at September month.

Table 7. The Neutrosophic standard deviation of gold price taken monthwise

Month	S_N	$a_S + b_S I_{NS};\ I_{NS};\ \in [I_{LS}\ ,I_{US}]$
$\mathcal{F}eb$	[41.57, 24.08]	41.57-24.08IN, IN ∈ [0, 0.7564]
Mar	[16.84, 47.51]	16.84+47.51IN, IN ∈ [0, 0.64]
Apr	[42.34, 53.53]	42.34+53.53IN, IN ∈ [0, 0.209]
May	[0.46, 39.07]	0.46+39.07IN, IN ∈ [0, 0.98]
Jun	[187.67, 0]	187.67+0IN, IN ∈ [0, 0.1]
Jul	[5.35, 2.24]	5.35-2.24IN, IN ∈ [0, 1.388]
Aug	[5.26, 28.60]	5.26+28.60IN, IN ∈ [0, 0.81]
Sep	[0.79, 0.82]	0.79+0.82IN, IN ∈ [0, 0.037]
Oct	[0.80, 0.83]	0.80+0.83IN, IN ∈ [0, 0.03]
Nov	[28.44, 15.38]	28.44-15.38IN, IN ∈ [0, 0.849]
Dec	[28.96, 29.88]	28.96-29.88IN, IN ∈ [0, 0.03079]
Jan	[26.81, 21.5]	26.81-21.5IN, IN ∈ [0, 0.2469]

The neutrosophic standard deviation at February month lies between 41.57 and 24.08 with a measure of indeterminacy 0.7564. The neutrosophic standard deviation at March month lies between 16.84 and 47.51 with a measure of indeterminacy 0.64. The neutrosophic standard deviation at April month lies between 42.34 and 53.53 with a measure of indeterminacy 0.209. The neutrosophic standard deviation at May month lies between 0.46 and 39.07 with a measure of indeterminacy 0.98. The neutrosophic standard deviation at June month lies between 187.67 and 0 with a measure of indeterminacy 0, 0.1. The neutrosophic standard deviation at July month lies between 5.35 and 2.24 with a measure of indeterminacy 1.388. The neutrosophic standard deviation at August month lies between 5.26 and 28.60with a measure of indeterminacy 0.81. The neutrosophic standard deviation at September month lies between 0.79 and 0.82 with a measure of indeterminacy 0.037. The neutrosophic standard deviation at October month lies between 0.80 and 0.83 with a measure of indeterminacy 0.03. The neutrosophic standard deviation at November month lies between 28.44 and 15.38 with a measure of indeterminacy 0.849. The neutrosophic standard deviation at December month lies between 28.96 and 29.88 with a measure of indeterminacy 0.03079. The neutrosophic standard deviation at January month lies between 26.81 and 21.5 with a measure of indeterminacy 0.2469. The minimum standard deviation among low rate of gold is found at May month and minimum standard deviation among high gold rate is found at June month.

Table 8. The Neutrosophic coefficient of variation of gold price taken monthwise

Month	CV_N	$a_V + b_V I_{NV;}\ I_{NV;}\ \in [I_{LV}\ ,I_{UV}]$
$\mathcal{F}eb$	[0.82, 0.45]	0.82-0.45IN, IN ∈ [0, 0.822]
Mar	[0.32, 0.84]	0.32+0.84IN, IN ∈ [0, 0.61]
Apr	[0.80, 0.97]	0.80+0.97IN, IN ∈ [0, 0.17]
May	[0.01, 0.73]	0.01+0.73IN, IN ∈ [0, 0.98]
Jun	[3.70, 0]	3.70+0IN, IN ∈ [0, 0.1]
Jul	[0.10, 0.04]	0.10-0.04IN, IN ∈ [0, 1.5]
Aug	[0.10, 0.53]	0.10+0.53IN, IN ∈ [0, 0.81]
Sep	[0.02, 0.02]	0.02+0.02IN, IN ∈ [0, 0]
Oct	[0.02, 0.02]	0.02+0.02IN, IN ∈ [0, 0]
Nov	[0.55, 0.28]	0.55-0.28IN, IN ∈ [0, 0.96]
Dec	[0.53, 0.52]	0.53-0.52IN, IN ∈ [0, 0.019]
Jan	[0.47, 0.36]	0.47-0.36IN, IN ∈ [0, 0.3056]

The neutrosophic coefficient of variation of rate of gold at Febraury month lies between 0.82, 0.45 and with a measure of indeterminacy being 0.822. The neutrosophic coefficient of variation of rate of gold at March month lies between 0.32, 0.84 and with a measure of indeterminacy being 0.61. The neutrosophic coefficient of variation of rate of gold at April month lies between 0.80, 0.97 and with a measure of indeterminacy being 0.17. The neutrosophic coefficient of variation of rate of gold at May month lies between 0.01, 0.73 and with a measure of indeterminacy being 0.98. The neutrosophic coefficient of variation of rate of gold at June month lies between 3.70, 0 and with a measure of indeterminacy being 0.1. The neutrosophic coefficient of variation of rate of gold at July month lies between 0.10, 0.04 and with a measure of indeterminacy being 1.5. The neutrosophic coefficient of variation of rate of gold at August month lies between 0.10, 0.53 and with a measure of indeterminacy being 0.81. The neutrosophic coefficient of variation of rate of gold at September month lies between 0.02, 0.02 and with a measure of indeterminacy being 0. The neutrosophic coefficient of variation of rate of gold at October month lies between 0.02, 0.02 and with a measure of indeterminacy being 0. The neutrosophic coefficient of variation of rate of gold at November month lies between 0.55, 0.28 and with a measure of indeterminacy being 0.96. The neutrosophic coefficient of variation of rate of gold at December month lies between 0.53, 0.52 and with a measure of indeterminacy being 0.019. The neutrosophic coefficient of variation of rate of gold at January month lies between 0.47, 0.36 and with a measure of indeterminacy being 0.3056. The minimum coefficient of variation among low gold rate is found at September and October month. The minimum coefficient of variation among high gold rate is found at June month.

4. Comparative study

The gold rate of six different cities are analysed using neutrosophic statistics. The analysis is performed in two ways. In the first method, the gold rates of the cities are considered. In the second method, the changes in gold rate in every month are considered. For both the methods, the neutrosophic mean, neutrosophic standard deviation, neutrosophic coefficient of variation are calculated. This comparison helps to check the beneficial month and city for the purchase of gold.

5. Future Work

Though this study has been made targeting gold as is basic element, the same type of prediction is also required in many other situations also. Hence as a future work it is proposed to develop an user friendly application which is suitable for any kind of prediction which applies neutrosophic statistical analysis. The application is to be developed with a motive to get an updated rate prediction which helps any of the user to know the current scenario on which they need. Various machine learning techniques to evaluated neutrosophic statistical measures can also be performed as a future work.

6. Conclusion

In this article, the neutrosophic analysis on cost of gold for various six cities are collected. Their rate has been analyzed both city wise and month wise. The results obtained among the city wise data indicates the favourable city to purchase gold is kolkatta at first. Then it can be Delhi, Madurai and Chennai. Month wise analysis of gold rate indicates that it is beneficial to purchase gold during June month. Thus according to this study, gold lovers can make use of June month at Kolkatta for their purchase to make this shopping beneficial. Most of the peoples who demand for gold wait for a better period and cost to make their wish possible. This study has brought about an analysis which has provided with a suggestion on which month and place it is beneficiary for the buyers of gold. This gives a prediction of gold rate, which provides an optimized situation to buy gold, even in uncertainty condition.

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Neutrosophic Static Model without Deficit and Variable Prices

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Abstract:

Inventory management plays an important role in production and marketing processes, especially in production facilities and commercial institutions that have warehouses in which they store their equipment and goods. Inventory management is considered one of the most important management functions in terms of determining the ideal volume of inventory and calculating its costs, as this affects the facility's efficiency and achieves either large profits or it causes huge losses, so warehouse managers in production facilities or institutions must determine the appropriate and ideal volume of inventory, especially when they are presented with price offers from companies seeking to market their products. These offers are directly related to the volume of the order, and in this case, they must make an ideal decision through which determine the volume of the order by taking into account the following matters:

1. Securing the quantities required for production or sale so that there is no deficit.
2. The storage cost should be as low as possible.
3. Benefit as much as possible from the discounts offered by companies.

In this research, we present a complementary study to what we did in researching the neutrosophic treatment of static inventory models without deficit, through which we arrived at mathematical relationships through which we can calculate the ideal volume of the order at the lowest possible cost. We will use these relationships to determine the ideal volume of the order so that we achieve the greatest benefit from price offers and discounts. Provided by companies.

Keywords: Static inventory models without deficits - Static inventory models without neutrosophic deficits - Variable price (discounts) models without neutrosophic deficits.

Introduction:

Before the emergence of the science of operations research, decision makers relied on experience gained through the profits they obtained for correct decisions and losses for wrong decisions. With the great development witnessed by our contemporary world, we find that experience alone is not sufficient to make decisions on the scale of this development, and a comprehensive study of the reality must be conducted. The work of the system is a study that relies on modern scientific methods, such as operations research methods. Even operations research methods are not sufficient in the face of these changes that the work environment is witnessing because they depend on restricted and specific classical data. Therefore, these methods had to be reformulated using non-specific data, completely specific Neutrosophical data that gives decision makers have a margin of freedom that enables them to face all circumstances, and this is what has been done by researchers and scholars interested in science and scientific development. In the following research, we find many studies presented using the concepts of neutrosophic science [1-15]. In this research, we will present a study using complementary neutrosophic concepts. As we have done previously, we know that most companies provide price offers and these offers are related to the volume of the order in order to market their goods, which are manufactured materials or raw materials used in the manufacturing process. This matter requires warehouse managers in production facilities or institutions that need these goods to determine the appropriate and ideal volume of the stock of each material to secure the demand at one time. We know that if the volume of the stock is very large, this guarantees the provision of the material, but in return, it may cause the organization to suffer losses because the value of the stock is frozen capital, and this large quantity requires a marketing period that depends on the rate of demand. If the quantity of inventory is small, this may lead to a bottleneck in securing materials and to various disturbances such as rising prices and others. In all cases, we find that the rate of demand for inventory is the primary control over the volume of the order. Therefore, in previous research [16], we studied the static model without a deficit using the concepts of neutrosophic science and we reached relationships through which we can calculate the ideal volume of the order and the corresponding cost was a neutrosophic value that takes into account all circumstances. In another research we also calculated a set of neutrosophic indicators for the static inventory model without a deficit [17]. In this research we will present A study based on what was presented in previous research, the purpose of which is to determine the ideal volume for students and to make the most of the price offers provided for the materials that we need to store so that no shortage occurs and the cost of storage is as low as possible.

Discussion:

Inventory models were studied in classical logic, and this study was presented on the basis that the rate of demand for inventory is a fixed value throughout the duration of the storage cycle. Therefore, the rate of demand for inventory is subject to a uniform probability distribution. This issue was addressed by relying on studies presented according to classical logic in the field of operations research, which relies on based on basic concepts in mathematics such as calculus of integration and others [21-32], we presented, in previous research [16-20], static inventory models using the concepts of neutrosophic science. We took in the study the rate of demand for inventory during the duration of the storage cycle. neutrosophic value.

Undetermined values. Completely determined. It is subject to a regular neutrosophic distribution, and we found the extent to which this value affects the ideal volume of the order. Among the research was the neutrosophic treatment of inventory models without a deficit, which is based on the following hypotheses:

Basic hypotheses of the study:

1 – Order volume Q.

2 - The rate of demand for inventory at one time λ_N (unspecified), where $\lambda_N \in \{\lambda_1, \lambda_2\}$ or $\lambda_N \in [\lambda_1, \lambda_2]$ or... so that λ_1 is the minimum rate of demand for inventory and λ_2 is the upper limit of the rate of demand for inventory.

3 – The fixed cost of preparing the order $C_1 = K$.

4 – The cost of purchase, delivery and receipt $C_2 = C.Q$.

5 - The cost of storage for the remaining quantity in the warehouse during one time C_3.

6 - The duration of running out of the stored quantity is $\frac{Q}{\lambda_N}$ (or the duration of the storage cycle).

Using the previous assumptions, we were able to build a non-linear mathematical model, and the optimal solution for it, i.e., the ideal volume of the order, is given by the following relationship:

$$Q_N^* = \sqrt{\frac{2K\lambda_N}{h}} \qquad (1)$$

The ideal total cost is calculated from the relationship:

$$C(Q_N^*) = \frac{K\lambda_N}{Q_N^*} + C\lambda_N + \frac{hQ_N^*}{2} \qquad (2)$$

In this research, we present a study of static inventory models without shortages after adding a new hypothesis to the basic hypotheses imposed by the reality of the market situation through the offers made by the producing companies. The content of these offers is to provide a discount whose value is determined according to the quantity that is purchased. Here the official must determine the size of the order. So that it suits the system's demand and is sufficient for the duration of the storage cycle without causing a shortage and with the lowest possible storage cost and at the same time benefiting from the companies' offer. According to the above, we present the following formulation of the issue:

Text of the issue:

An insurance company needs a certain material, so if it knows that the rate of demand for this material is λ_N during the storage cycle, and that the cost of purchasing one unit is C monetary units, the cost of storing one unit is h, the cost of preparing the order is K monetary units, and the offer provided by the company the producers of this material are:

$$Price\ \ for\ \ one\ \ item\begin{cases} C_1 & If \quad 0 \le Q < Q_1 \\ C_2 & If \quad Q_1 \le Q < Q_2 \\ C_3 & If \quad Q_2 \le Q < Q_3 \\ C_4 & If \quad Q_3 \le Q < \infty \end{cases}$$

Where $C_1 > C_2 > C_3 > C_4$

What is required is to determine the ideal volume of the order so that the storage cost is as low as possible.

From the above, the basic hypotheses of this model are written as follows:

Basic assumptions of the model of variable prices without neutrosophic deficit:

1 – Order volume Q.

2 - The rate of demand for inventory at one time λ_N (unspecified), where $\lambda_N \in \{\lambda_1, \lambda_2\}$ or $\lambda_N \in [\lambda_1, \lambda_2]$ or... so that λ_1 is the minimum rate of demand for inventory and λ_2 is the upper limit of the rate of demand for inventory.

3 – The fixed cost of preparing the order $C_1 = K$.

4 – The cost of purchase, delivery and receipt $C_2 = C.Q$.

5 - The cost of storage for the remaining quantity in the warehouse during one time C_3.

6 - The duration of running out of the stored quantity is $\frac{Q}{\lambda_N}$ (or the duration of the storage cycle).

7- One of the companies producing materials to be stored provided four price levels that are inversely proportional to the volume of the order:

$$Price\ \ for\ \ one\ \ item\begin{cases} C_1 & If \quad 0 \le Q < Q_1 \\ C_2 & If \quad Q_1 \le Q < Q_2 \\ C_3 & If \quad Q_2 \le Q < Q_3 \\ C_4 & If \quad Q_3 \le Q < \infty \end{cases}$$

Neutrosophic treatment of the issue:

We follow the following steps:

1 -We build a mathematical model for this issue within hypotheses 1-6, which in themselves are the basic hypotheses of the static model without neutrosophic deficit that was studied previously, and we arrived at the following:

Find

$$C(Q) = \frac{K\lambda_N}{Q} + C\lambda_N + \frac{hQ}{2} \quad \longrightarrow Min$$

Condition:

$$Q \ge 0$$

It is a nonlinear neutrosophic model whose optimal solution, i.e., the ideal volume of the students, is given by the relationship (1) and the minimum storage cost is calculated from the relationship (2).

To address the issue at hand and choose the optimal volume of the order, taking into account hypothesis No. (7) of the offers presented, we calculate the following storage costs:

$$C_i(Q_N) = \frac{K\lambda_N}{Q_N} + C_i\lambda_N + \frac{hQ_N}{2} \qquad (3)$$

Where $i = 1,2,3,4,\ldots$ for all price cases

Since $C_1 > C_2 > C_3 > C_4$the function $C_i(Q)$ satisfies the following inequality:

$$C_1(Q) > C_2(Q) > C_3(Q) > C_4(Q)$$

For $i = 1,2,3,4$.

Each of the previous functions has a minimum limit, which is the optimal solution to the nonlinear model that we arrived at in the first step, corresponding to the inventory volume, which we symbolize as Q_{0N}, calculated from the following relationship:

$$Q_{0N} = \sqrt{\frac{2K\lambda_N}{h}}$$

After calculating Q_{N0}we compare it with the given offers.

We assume that $Q_1 \leq \quad Q_{N0} < Q_2$. Then we calculate the cost corresponding to this volume from the relationship (3). We get $C_2(Q_{0N})$. To determine the ideal volume of the order Q_N^*, we calculate the cost functions for the minimum limits of the quantity ranges specified in the discounts table. Then we choose the smallest of these costs, which is the volume. The corresponding ideal is the volume that secures inventory for the system during the storage cycle period at the lowest possible cost and taking advantage of the offers provided.

We explain the above through the following example:

Example:

A production institution wants to secure its need for a certain material. If it knows that the rate of demand for this material is $[250,330]$,units per year, the cost of purchasing one unit in the market is 400 monetary units, the cost of storing one unit per year is 10% of its price, and the cost of preparing the order is equal to 150 units. In cash, the company producing this material offers the following offers:

2% discount if quantity $50 \leq Q \leq 100$.

3% discount if quantity $100 \leq Q \leq 200$.

5% discount if quantity is $200 \leq Q$.

Required: Find the optimal quantity Q_N^*,that makes the total costs of storage as small as possible.

The solution:

Data:

$\lambda \in [250,350] \quad ,K = 150 \ \ ,C = 400$

h is the storage cost and is 10% of the price of one unit of stock in the market. Therefore:

$$h = \frac{10}{100}.400 = 40$$

1- We determine price levels by discounts:

a. When $Q \leq 50$ then $C_1 = C = 400$ there is no discount.

b. When $50 \leq Q \leq 100$ the discount is 2% and the purchase price is equal to:

$$C_2 = 400\left(1 - \frac{2}{100}\right) = 392$$

c. When $100 \leq Q \leq 200$ the discount is 3% and the purchase price equals to:

$$C_3 = 400\left(1 - \frac{3}{100}\right) = 388$$

d. When $Q \geq 200$ the discount is 5% and the purchase price is equal to:

$$C_4 = 400\left(1 - \frac{5}{100}\right) = 380$$

2- We calculate the initial quantity of inventory:

We study the issue based on hypotheses 1-6, and here we are faced with a storage model without a neutrosophic deficit. We calculate the ideal volume of the order through the following relationship:

$$Q_{0N} = \sqrt{\frac{2K\lambda_N}{h}} = \sqrt{\frac{2.[250,350].150}{20}} \in [61,72]$$

This means that in order for the company to provide a safe work environment without shortages and with the lowest storage cost, the volume of the order must be greater than Q_{0N}. To calculate the cost, we compare the initial quantity with the offers presented by the producing company. We find that $Q_{0N} \in [50,100]$, meaning that this quantity deserves a 2% discount. The purchase price per unit is $C_2 = 392$, and then the total storage cost is calculated from the relationship:

$$C_2(Q_{0N}) = \frac{K\lambda_N}{Q_{0N}} + \frac{hQ_{0N}}{2} + C_2\lambda_N$$

$$C_2([61,72]) = \frac{150.[250,350]}{[61,72]} + \frac{40.[61,72]}{2} + 392.[250,350] \in [99834,139369]$$

To benefit more from the offers presented.

3- We calculate costs for the minimum offer areas:

For the range $100 \leq Q \leq 200$we find:

$$C_3(100) = \frac{150.[250,350]}{100} + \frac{40.100}{2} + 392.[250,350] \in [100375,139725]$$

$$C_4(200) = \frac{150.[250,350]}{200} + \frac{40.200}{2} + 380.[250,350] \in [99187.5,137262.5]$$

After obtaining the costs corresponding to the offers, we find that:
As for the cost $C_1 = C = 400$, it is offset by the order volume $Q \leq 50$, and for this volume there is no discount. In addition, this volume is less than the minimum required, meaning that it does not suit the company because the company works on the basis of not having a deficit, and this order quantity will cause a deficit to be paid. The company will face fines, which will be reflected in the total cost, so we rule out this solution.

4- We choose the lowest cost from the remaining costs:

That is, we take:

$$Min\{C_2(Q), C_3(Q), C_3(Q)\}$$

We find:

$$Min\{[99834,139369], [100375,139725], [99187.5,137262.5]\}$$
$$= [99187.5,137262.5] \quad (*)$$

It corresponds to an order size $Q = 200$. This volume is appropriate for the company's workflow.
In order to achieve the maximum benefit from the offers presented, we calculate the costs corresponding to the largest volume that the company can adopt if the storage cost is appropriate, which corresponds to an order volume equal to the rate of demand for inventory, that is:
We calculate storage costs if the order volume equals the inventory demand rate. That is, when $Q \in [250,350]$, we find that the price of one unit will be $C_4 = 380$, and the total storage costs are equal to:

$$C([250,350]) = \frac{150[250,350]}{[250,350]} + \frac{40.[250,350]}{2} + [250,350].380 \in [100150,140150]$$

We compare this cost with the cost we obtained through the comparison (*). We find that the cost is greater and there is no interest for the company in requesting this size of the order because it can ensure a safe work flow and benefit from the offers provided at a lower cost when the order volume is $Q = 200$.

From the above, we note that the minimum value of storage costs is:

$$[99187.5,137262.5]$$

And corresponds to the ideal volume of the order, which is equal to $Q^* = 200$.

Conclusion and Results:

The storage model with variable prices is considered one of the important models in inventory models because we encounter it frequently in practical life and it requires careful study from us so that we do not fall into the temptation of the offers presented. Through the offers we may be able to obtain lower prices for large quantities, but these quantities may become a burden on the company. During the costs that must be paid for the storage process, on the other hand, we find that using the neutrosophic value of the demand rate for inventory gave a careful study of the model, as we obtained the neutrosophic storage cost from which we can determine the lowest cost and the largest cost if the ideal volume of the order is adopted $Q^* = 200$.

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Evaluation Strategies of International Business Administration in the E-Commerce Sector Using Single-Valued Neutrosophic Sets

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Abstract
In today's digitally connected and globally interconnected corporate world, the assessment of International Corporate Administration in E-commerce is of the utmost importance. To help with the evaluation of such a program, this paper summarizes important topics. The evaluation covers a wide range of topics, such as strategies for entering and selecting markets, digital marketing for acquiring customers, logistics and supply chain management, payment, and security systems, customer experience and retention, data analytics for measuring performance, communication across cultures, and continuous learning. When considering an International Business Administration program in E-commerce, prospective students should take these factors into account. This will help them make informed decisions and ensure that they gain the knowledge and skills needed to succeed in the complex world of international e-commerce and make a positive impact on global businesses. So, we proposed a single-valued neutrosophic framework for dealing with uncertain information in the evaluation process. We used multi-criteria decision-making methods named the AHP and WSM methods. The AHP method is used to compute the weights of criteria. The WSM method is used to rank the alternatives. We used 15 criteria and 10 strategies in this study. We performed a sensitivity analysis to show the stability of the results.

Keywords: Single Valued Neutrosophic Sets; International Business Administration; E-Commerce; Strategies; AHP Method; WSM Method.

1. Introduction

The ever-changing subject of international business administration in e-commerce integrates the tenets of traditional international company management with the complexities of online trade. Companies are looking beyond local markets to take advantage of the enormous opportunities presented by global e-commerce as a result of the world's growing interconnection and digitization. Because of this, there is a need for experts in the field of e-commerce who also have a solid grasp of global business tactics[1,2].

Students majoring in International Business Administration with a concentration in E-commerce learn to deal with the challenges of doing business online on a worldwide basis. Digital marketing tactics, supply chain management, payment and security systems, analytics, customer experience optimization, cross-border legal and regulatory issues, and market research and selection are all part of the extensive study that is required[3,4].

Professionals who can see possibilities in global markets, create solid plans for online sales, and deal with the inevitable problems that arise from doing business across borders are in great demand in today's fast-paced, cutthroat business environment. If a company wants to expand, gain market share, and solidify its position on the international stage, it needs these experts[3,5].

International business administration with a focus on e-commerce provides students with a well-rounded education in both traditional company management and innovative approaches to online sales and marketing. Because of this, they will be prepared to take advantage of the possibilities and overcome the obstacles that come with international e-commerce[6,7].

Career opportunities in international business administration with a focus on electronic commerce are also quite promising. A wide variety of positions are available to graduates, such as executives in company development, supply chain management, digital marketing, e-commerce management, and worldwide market analysis. They have a lot of options for where to work, from software and logistics firms to consumer goods stores, or they may start their international e-commerce businesses[8-10].

People who are enthusiastic about integrating their knowledge of global business with the rapidly developing field of Internet commerce will find an ideal fit with the International Business Administration in E-commerce programed. As a result of their extensive training in e-commerce best practices and strong academic background in business administration, individuals working in this area are well-positioned to succeed in today's interconnected world[10,11]. The evolution of international business administration in E-Commerce is a multi-criteria decision-making model (MCDM)[12,13].

Uncertainty, imprecision, inconsistency, and vagueness are given a fresh perspective in Florantin Smarandache's neutrosophic sets, which build upon the intuitionistic fuzzy sets (IFSs) proposed by Atanassov. Smarandache described a neutrosophic set as having three elements—truth membership, indeterminacy membership, and falsity membership—and added the degree of indeterminacy/neutrality as a distinct, separate component of fuzzy sets. Neutrosophic sets may improve decision-making because the indeterminacy parameter allows for a more precise formulation of membership functions[14–17]. However, a neutrosophic set presents greater challenges in practical scientific and technical contexts[18-19].

To differentiate between absolute truth and relative truth, absolute falsity and relative falsehood in logic, absolute membership and relative membership, absolute non-membership, relative non-membership, and so on[20-26], neutrosophic logic is a great tool. The requirement that the total of a membership function's components for a given event not exceed 1 is not satisfied when neutrosophic sets are favored. The total may go as high as three if those parts are unrelated.

2. Related work

In this section, we introduce some related studies in the single-valued neutrosophic set with the AHP method and the WSM method. Naderi et al. [27] proposed an adaptive candidate rely set based on the SVN-AHP method. They used the AHP method to compute the criteria weights. They proposed a model for adaption in vehicular networks. Karasan et al. [28] proposed a decision-making model for the design of a car seat. They integrated the neutrosophic set with the AHP method. The neutrosophic AHP was used for computing the weights of customers' requirements.

Gulum et al. [29]proposed a neutrosophic framework for post-earthquake fire risk evaluation. They proposed a neutrosophic AHP and neutrosophic TOPSIS for this evaluation. The neutrosophic AHP was used to compute the weights of criteria for post-earthquake fire risk problems. Kavus et al. [30] proposed a three-level framework for assessing airline service quality. They proposed a neutrosophic framework for overcoming the uncertainty of information. They used the neutrosophic AHP in their evaluation. They used the neutrosophic AHP for computing the criteria weights and sub-criteria. Fatih Yiğit [31] proposed a decision-making model for human resource decisions. He used the neutrosophic set in his process. He has used the neutrosophic AHP method. The neutrosophic AHP method was used to compute the weights of the criteria.

3. Single Valued Neutrosophic Framework

In this section, we introduce the mathematical equation based on operations of the single values neutrosophic numbers (SVNNs) and we introduce the SVN-AHP and the SVN-WSM. The SVN-AHP method is used to compute the weights of criteria[32-33]. The SVN-WSM is used to rank the strategies of IBA in the E-Commerce field as shown in Figure 1.

3.1 Single Valued Neutrosophic Sets

In this part, we introduce some mathematical operations of SVNSs.

The SVNSs can be defined by three membership degrees such as truth, indeterminacy, and falsity degrees as:

$$NS = \{A_{NS}(x), B_{NS}(x), C_{NS}(x)\} \tag{1}$$

$$0 \leq A_{NS}(x) + B_{NS}(x) + C_{NS}(x) \leq 3 \tag{2}$$

Let $y_1 = \left(A_{NS_1}(x), B_{NS_1}(x), C_{NS_1}(x)\right)$ and $y_2 = \left(A_{NS_2}(x), B_{NS_2}(x), C_{NS_3}(x)\right)$ be two neutrosophic numbers and the operation can be computed as:

$$y_1 \cup y_2 = \left(\max\{A_{NS_1}(x), A_{NS_2}(x)\}, \min\{B_{NS_1}(x), B_{NS_2}(x)\}, \min\{C_{NS_1}(x), C_{NS_2}(x)\}\right) \tag{3}$$

$$y_1 \cap y_2 = \left(\min\{A_{NS_1}(x), A_{NS_2}(x)\}, \max\{B_{NS_1}(x), B_{NS_2}(x)\}, \max\{C_{NS_1}(x), C_{NS_2}(x)\}\right) \tag{4}$$

$$y_1^c = \left(C_{NS_1}(x), 1 - B_{NS_1}(x), A_{NS_1}(x)\right) \tag{5}$$

$$y_1 \oplus y_2 = \left(A_{NS_1}(x) + A_{NS_2}(x), A_{NS_1}(x)A_{NS_2}(x), B_{NS_1}(x)B_{NS_2}(x), C_{NS_1}(x)C_{NS_2}(x)\right) \tag{6}$$

$$y_1 \otimes y_2 = \left(A_{NS_1}(x)A_{NS_2}(x), B_{NS_1}(x) + B_{NS_2}(x) - B_{NS_1}(x)B_{NS_2}(x), C_{NS_1}(x) + C_{NS_2}(x) - C_{NS_1}(x)C_{NS_2}(x)\right) \tag{7}$$

Figure 1. The steps of the single-valued neutrosophic model.

3.2 SVN-AHP Method

In this part, we introduce the steps of the SVN-AHP method to compute the criteria weights.

Step 1. Determine the single-valued neutrosophic scale.

Step 2. Determine the goal, criteria, and alternatives.

Step 3. Build the pairwise comparison matrix

The pairwise comparison matrix is built based on the single-valued neutrosophic numbers. This matrix is changed to the crisp valued as:

$$X_{ij} = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix} \tag{8}$$

Step 4. Compute the normalized matrix

The normalized pairwise matrix is computed by dividing the value in the pairwise matrix into a sum of each column as:

$$R_{ij} = \frac{x_{ij}}{\sum_{j=1}^{n} x_j} \tag{9}$$

Where j refers to the number of criteria and $j = 1,2, \dots n$

Step 5. Compute the mean of every row to compute the weights of the criteria.

Step 6. Test the consistency ratio (CR)

$$CR = \frac{CI}{RI} \tag{10}$$

Where CI refers to the consistency index and RI refers to the random index.

3.3 SVN-WSM Method

In this part, we introduce the steps of the SVN-WSM. The weighted sum method is used to compute the weights of criteria. The steps of the SVN-WSM are introduced as:

Step 7. Build the decision matrix

$$Z_{ij} = \begin{bmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{m1} & \cdots & z_{mn} \end{bmatrix} \tag{11}$$

Step 8. Normalize the decision matrix

$$E_{ij} = \frac{z_{ij}}{\sum_{j=1}^{m} z_{ij}} \tag{12}$$

Step 9. Compute the weighted normalized decision matrix

$$WE_{ij} = w_j * z_{ij} \tag{13}$$

Then we rank the strategies of IBA based on the largest value in the sum of each row of the weighted normalized decision matrix.

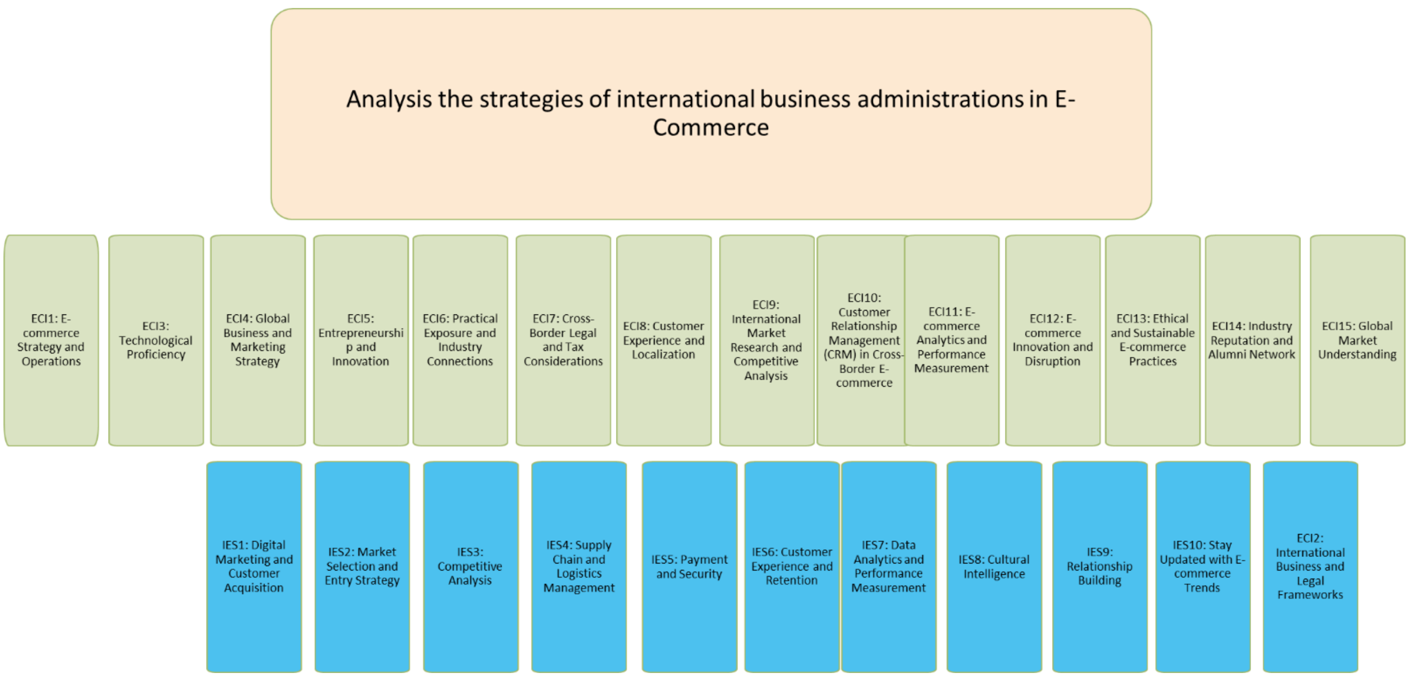

Figure 2. The goal, criteria, and alternatives in this study.

4. Results

In this part, we introduce the results of the applied proposed model and we discuss it by the criteria weights of the rank of alternatives.

Step 1. We identify the scale of the SVNSs.

Step 2. We identify the set of criteria and alternatives to rank and analyze the strategies of IBA in the E-Commerce sector as shown in Figure 2.

Step 3. Build the pairwise comparison matrix between criteria by Eq. (8).

Step 4. Compute the normalized matrix by Eq. (9) see Table 1.

Table 1. The normalized pairwise comparison matrix.

	ECI_1	ECI_2	ECI_3	ECI_4	ECI_5	ECI_6	ECI_7	ECI_8	ECI_9	ECI_{10}	ECI_{11}	ECI_{12}	ECI_{13}	ECI_{14}	ECI_{15}
ECI_1	0.031359	0.007987	0.009287	0.016512	0.023702	0.015514	0.015031	0.026563	0.037697	0.044678	0.02802	0.033482	0.052711	0.039236	0.044009
ECI_2	0.084985	0.021646	0.018935	0.023139	0.00474	0.017957	0.021426	0.048631	0.032994	0.044678	0.032321	0.007829	0.01959	0.021909	0.017341
ECI_3	0.121548	0.041151	0.035997	0.034871	0.009708	0.003552	0.021426	0.032067	0.03256	0.060725	0.059296	0.054488	0.062756	0.056332	0.104522
ECI_4	0.068771	0.033874	0.03738	0.036211	0.009708	0.017957	0.034876	0.023028	0.038059	0.016683	0.032321	0.041948	0.035279	0.030944	0.115404
ECI_5	0.050175	0.173164	0.140615	0.141448	0.037923	0.010286	0.021426	0.031916	0.017076	0.032236	0.059296	0.033482	0.027061	0.021883	0.07869
ECI_6	0.057435	0.034249	0.287979	0.057295	0.10476	0.028414	0.021426	0.016463	0.037842	0.029125	0.007558	0.016296	0.01959	0.035474	0.030615
ECI_7	0.084985	0.041151	0.068436	0.042292	0.072098	0.054018	0.040733	0.031916	0.0089	0.037184	0.032321	0.040675	0.043464	0.082318	0.054533
ECI_8	0.059619	0.022477	0.056689	0.079409	0.060005	0.087158	0.064451	0.0505	0.069678	0.060513	0.022674	0.016296	0.030299	0.044963	0.102369
ECI_9	0.060191	0.047468	0.079994	0.068842	0.160692	0.054328	0.331164	0.05244	0.072355	0.068219	0.045778	0.054488	0.048893	0.082318	0.04365
ECI_{10}	0.049619	0.034249	0.041906	0.153435	0.083165	0.068965	0.077439	0.058995	0.074979	0.070693	0.032321	0.061427	0.053043	0.044963	0.102369
ECI_{11}	0.068771	0.041151	0.037303	0.068842	0.039299	0.231005	0.077439	0.136856	0.097121	0.134397	0.061447	0.061299	0.021251	0.038979	0.04963
ECI_{12}	0.059619	0.17598	0.042053	0.054948	0.072098	0.110991	0.063745	0.197265	0.084527	0.073257	0.063807	0.063654	0.021251	0.059409	0.04365

ECI_{13}	0.049385	0.091718	0.047616	0.085202	0.116329	0.120397	0.077794	0.138356	0.122844	0.11063	0.240026	0.24865	0.08301	0.021883	0.063012
ECI_{14}	0.068321	0.084454	0.054624	0.10003	0.148138	0.068467	0.042298	0.096007	0.075135	0.134397	0.134751	0.091589	0.324258	0.085481	0.030615
ECI_{15}	0.085216	0.149279	0.041187	0.037524	0.057634	0.110991	0.089327	0.058995	0.198233	0.082585	0.148064	0.174396	0.157544	0.333909	0.11959

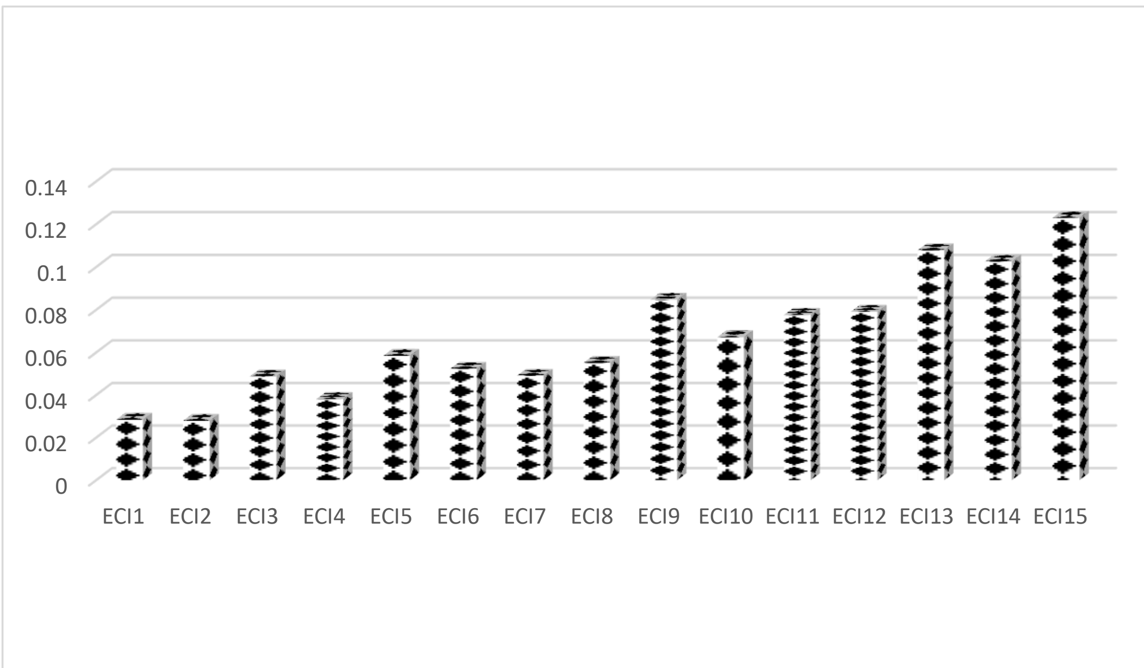

Figure 3. The criteria weights of international business administrations in E-Commerce.

Step 5. Compute the mean of every row to compute the weights of criteria as shown in Figure 3.

Step 6. Test the consistency ratio (CR). The CR is less than 0.1. This indicates the pairwise comparison is consistent.

3.3 SVN-WSM Method Results

Step 7. Build the decision matrix by Eq. (11).

Step 8. Normalize the decision matrix by Eq. (12) see Table 2.

Table 2. The normalized decision matrix.

	IES_1	IES_2	IES_3	IES_4	IES_5	IES_6	IES_7	IES_8	IES_9	IES_{10}
ECI_1	0.049486	0.133152	0.110296	0.132523	0.110296	0.146362	0.05368	0.132523	0.083456	0.048228
ECI_2	0.021481	0.16853	0.110898	0.113867	0.16853	0.044708	0.114914	0.091862	0.044708	0.120503
ECI_3	0.067414	0.156758	0.105913	0.151559	0.040611	0.160006	0.019981	0.156758	0.038337	0.102664
ECI_4	0.098034	0.118465	0.069167	0.184633	0.124651	0.061107	0.061107	0.098765	0.085475	0.098596
ECI_5	0.090487	0.101325	0.077757	0.113367	0.169448	0.090487	0.147256	0.109926	0.044039	0.055909
ECI_6	0.124679	0.08187	0.124679	0.103768	0.103768	0.128625	0.190373	0.052279	0.064312	0.025646
ECI_7	0.094468	0.1338	0.09393	0.113506	0.113506	0.058549	0.047593	0.113506	0.172953	0.05819
ECI_8	0.065943	0.100901	0.110912	0.083581	0.153338	0.099312	0.153338	0.083581	0.083581	0.065514
ECI_9	0.134551	0.134551	0.044757	0.134272	0.073341	0.134133	0.08812	0.097323	0.129671	0.029281
ECI_{10}	0.045985	0.154302	0.094485	0.153763	0.113526	0.113526	0.074546	0.117119	0.094485	0.038261
ECI_{11}	0.079724	0.11172	0.11172	0.170585	0.174121	0.045254	0.11172	0.131695	0.021743	0.041718
ECI_{12}	0.024169	0.113382	0.11587	0.110716	0.073218	0.110716	0.093478	0.171495	0.093478	0.093478
ECI_{13}	0.159848	0.074872	0.039092	0.039092	0.10916	0.108001	0.159848	0.141792	0.054	0.114295
ECI_{14}	0.127052	0.145689	0.101908	0.077799	0.093921	0.077799	0.126313	0.093921	0.077799	0.077799
ECI_{15}	0.137568	0.178192	0.120395	0.097129	0.04801	0.060013	0.076632	0.121688	0.116794	0.043579

Step 9. Compute the weighted normalized decision matrix by Eq. (13) see Table 3. The rank of strategies is shown in Figure 4.

Table 3. The weighted normalized decision matrix.

	ECI_1	ECI_2	ECI_3	ECI_4	ECI_5	ECI_6	ECI_7	ECI_8	ECI_9	ECI_{10}	ECI_{11}	ECI_{12}	ECI_{13}	ECI_{14}	ECI_{15}
IES_1	0.001405	0.000599	0.003285	0.003741	0.005289	0.006521	0.004629	0.003636	0.011399	0.003089	0.006191	0.001911	0.01723	0.013032	0.016916
IES_2	0.00378	0.004698	0.007639	0.004521	0.005922	0.004282	0.006556	0.005564	0.011399	0.010365	0.008675	0.008967	0.00807	0.014943	0.021911
IES_3	0.003131	0.003091	0.005161	0.00264	0.004545	0.006521	0.004603	0.006116	0.003792	0.006347	0.008675	0.009163	0.004214	0.010453	0.014804
IES_4	0.003762	0.003174	0.007386	0.007046	0.006626	0.005427	0.005562	0.004609	0.011376	0.010328	0.013246	0.008756	0.004214	0.00798	0.011943
IES_5	0.003131	0.004698	0.001979	0.004757	0.009903	0.005427	0.005562	0.008455	0.006214	0.007626	0.013521	0.00579	0.011766	0.009634	0.005904
IES_6	0.004155	0.001246	0.007798	0.002332	0.005289	0.006727	0.002869	0.005476	0.011364	0.007626	0.003514	0.008756	0.011641	0.00798	0.007379
IES_7	0.001524	0.003203	0.000974	0.002332	0.008606	0.009957	0.002332	0.008455	0.007466	0.005007	0.008675	0.007393	0.01723	0.012956	0.009423
IES_8	0.003762	0.002561	0.007639	0.003769	0.006425	0.002734	0.005562	0.004609	0.008245	0.007867	0.010226	0.013562	0.015284	0.009634	0.014963
IES_9	0.002369	0.001246	0.001868	0.003262	0.002574	0.003364	0.008475	0.004609	0.010986	0.006347	0.001688	0.007393	0.005821	0.00798	0.014362
IES_{10}	0.001369	0.003359	0.005003	0.003763	0.003268	0.001341	0.002851	0.003612	0.002481	0.00257	0.00324	0.007393	0.01232	0.00798	0.005359

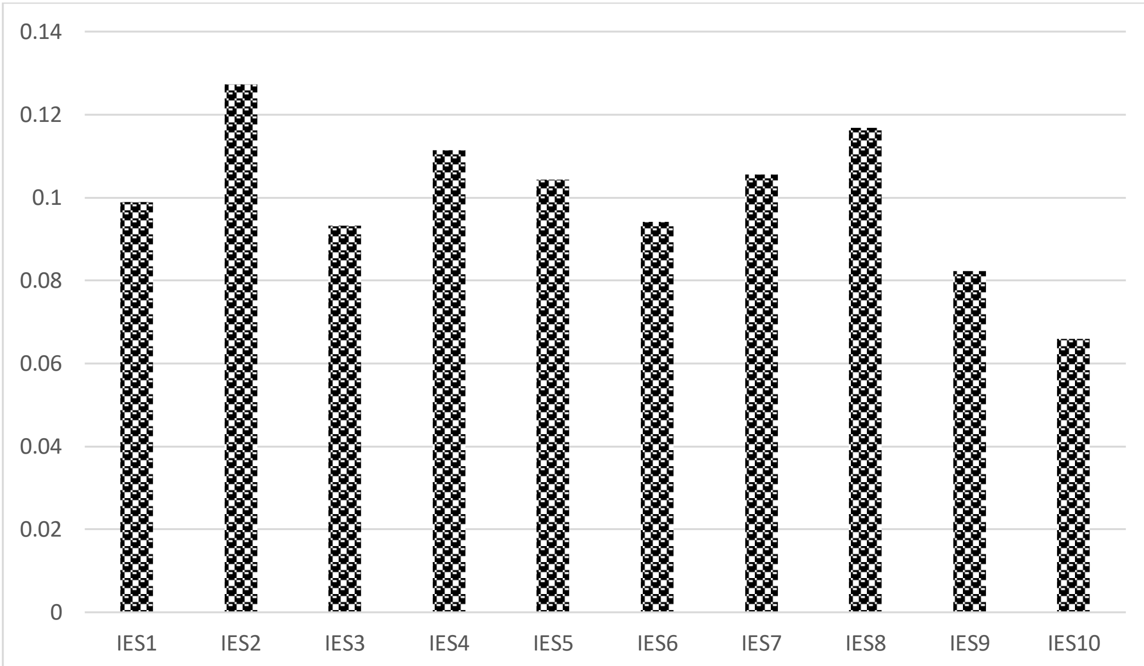

Figure 4. The ranking of strategies in international business administration in E-Commerce.

We applied the single-valued neutrosophic set framework for ranking and analyzing the strategies of international business administrations in E-Commerce. We applied to MCDM method named AHP and WSM method. We let the experts evaluate the criteria by building the pairwise comparison matrix between the criteria. We used 15 criteria and 10 alternatives in this study. The criteria and alternatives are collected based on the literature review and opinions of experts who have experience of more than 10 years in international business administration. The SVN-AHP is used to compute the criteria weights. The results show the ECI15 (Global Market Understanding) is the best criterion followed by ECI13 (Ethical and Sustainable E-commerce Practices) and ECI14 (Industry Reputation and Alumni Network) and the least weight is ECI2 (International Business and Legal Frameworks).

Then we applied the WSM method to the 15 criteria and 10 alternatives by building the decision matrix. Then we rank the strategies by the WSM method. The results show that IES2 (Market Selection and Entry Strategy) is the best strategy, followed by IES8 (Cultural Intelligence), and IES4 (Supply Chain and Logistics Management), and the lowest strategy is IES10 (Stay Updated with E-commerce Trends).

5. Sensitivity Analysis

We change the weights of the criteria and then rank the alternatives with the WSM method under different cases to show the stability of the results. We change the weights of criteria by 15 cases as shown in Figure 5. We put one criterion with 0.07 weight and all weights are equal. We show the rank of alternatives under different cases is stable as shown in Figure 5.

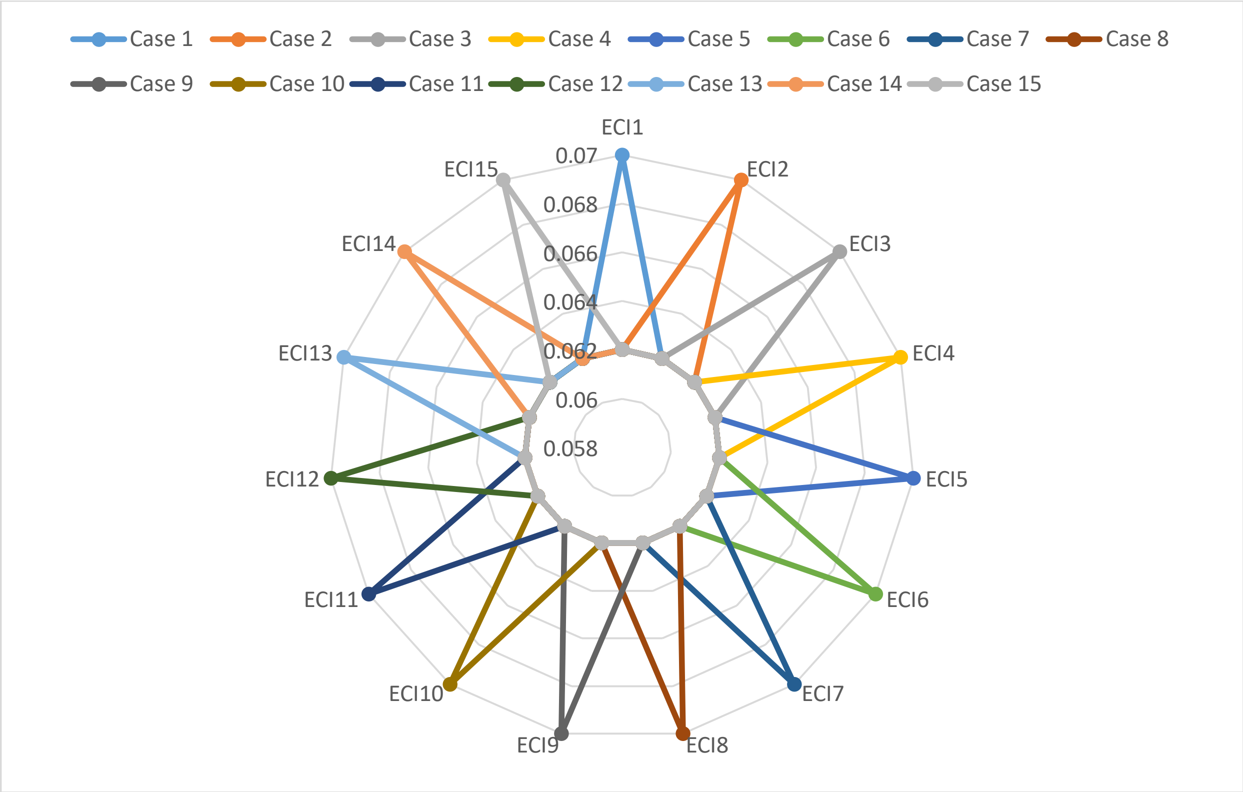

Figure 5. The 15 cases in criteria weights.

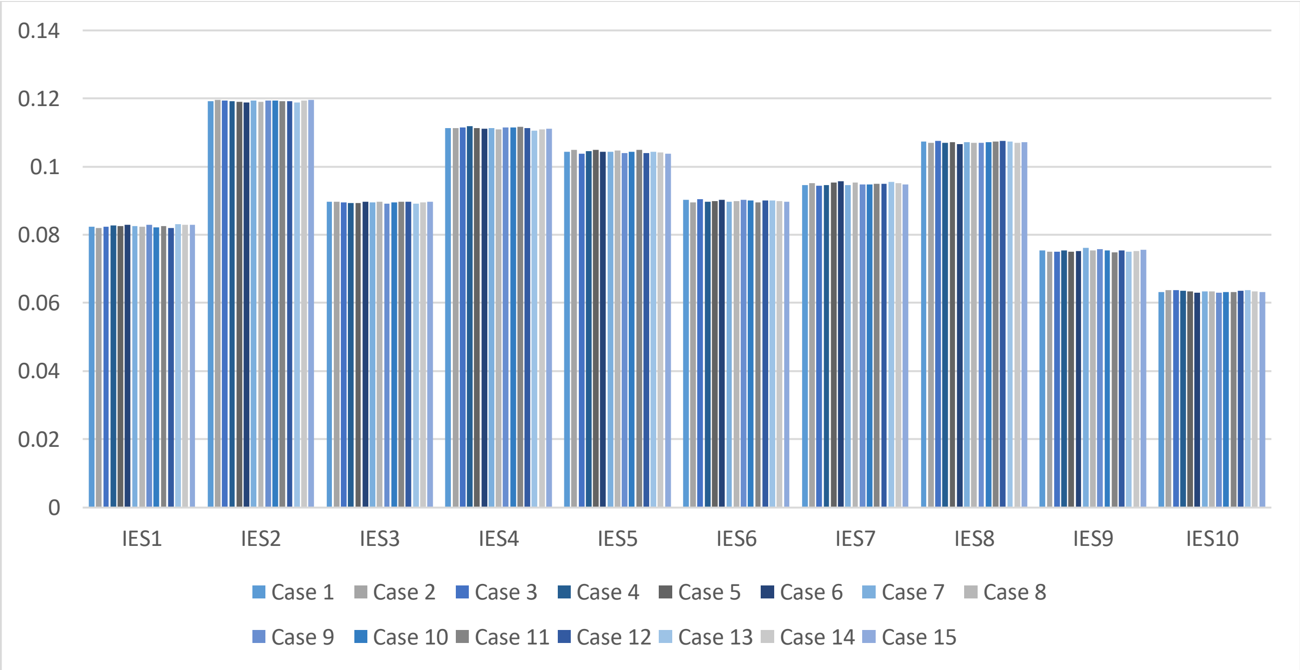

Figure 6. The rank of strategies under 15 cases.

6. Conclusions

The evaluation of International Business Administration in E-commerce is essential for individuals seeking a comprehensive education in the field. By considering market selection and entry strategies, digital marketing techniques, supply chain management, customer experience optimization, and other relevant factors, students can assess the program's suitability for their career goals. The integration of specialized e-commerce knowledge with international business principles equips graduates with the skills required to excel in the global e-commerce landscape. Moreover, the continuous learning mindset, cross-cultural understanding, and adaptation to emerging trends ensure that professionals remain competitive and contribute to the growth of businesses operating in the digital realm. Evaluating International Business Administration in E-commerce empowers students to make informed decisions and embark on a rewarding career path in this dynamic and rapidly evolving field.

We proposed a single-valued framework with the MCDM method to analyze the strategies in the international business administration in E-Commerce. The single-valued neutrosophic was used to deal with vague information. The two MCDM methods used in this study are the AHP and WSM methods. The SVN-AHP was used to compute the weights of the criteria. Then the SVN-WSM was used to rank the strategies. We used the 15 criteria and 10 strategies.

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Interval Valued Pentapartitioned Neutrosophic Set

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Abstract: The notions of interval valued pentapartitioned neutrosophic sets ($IVPNS$s), where the membership values of truth, contradiction, ignorance, unknown, and falsity always fall inside a closed interval [0,1] are introduced in this paper. Also an example of COVID-19 has been discussed using $IVPNS$. Later we have established some basic operations between $IVPNSs$ and useful features of $IVPNS$s have also been presented and discussed.

Keywords: Neutrosophic set, pentapartitioned neutrosophic set, interval neutrosophic set, interval valued pentapartitioned neutrosophic set.

1. Introduction

There are numerous common issues in the disciplines of economics, engineering, environmental research, social science etc in everyday life that can't be solved with classical mathematics. To handle such a circumstance Fuzzy set (FS) [12], rough set (RS) [7] and intuitionistic fuzzy set (IFS) concepts [1] have all been introduced. Traditional FStheory only considers membership values and due to this IFS theory which includes both membership values as well as non-membership values, serves a crucial function in the study of uncertainties. Though the indeterminacy and inconsistent observation that exist in belief systems are not addressed by intuitionistic fuzzy set theory. In order to address this type of indeterminacy, Smarandache developed neutrosophic set (NS) [8] as an addition to IFS theory. Single valued neutrosophic sets were first established by Wang and others [10] in 2010 and this idea is expanded to establish quadripartition single valued neutrosophic sets by Chatterjee et al. [2]. Smarandache [9] classified indeterminacy into three functions in 2013 as the unknown, contradiction and ignorance membership functions and proposed five symbol valued neutrosophic logic using these functions. And he further on extended it to: p types of Truths, $T_1, T_2, \ldots, T_p$ and r types of Indeterminacies $I_1, I_2, \ldots, I_r$ also s types of falsities $F_1, F_2, \ldots, F_s$ where $\mathrm{p+r+s} = \mathrm{n}$ greater than 4, which is the most general form of fuzzy extension of today [9]. Later, adopting this idea, Mallick introduced the pentapartitioned neutrosophic set (PNS) [5], where membership functions of truthiness, disagreement, lack of understanding i.e. ignorance, unknowability, and falsehood were taken into consideration. Das established the concept of single valued pentapartitioned neutrosophic graphs, sub graph, and complete graphs [3] to address graph theoretic challenges including indeterminacy in the form of three distinct elements viz contradictions, ignorances and unknowability. Das et al. has also proposed the Hamming distance in pentapartitioned neutrosophic sets and developed a GRA-based Single valued pentapartitioned neutrosophic sets in MADM method [4]. In a decision-making dilemma, they further validate their findings by choosing a supplier to purchase electronic items for an organization.

In practical scenario to deal with societal uncertainty there are various situations where different membership values belong to some interval. So to overcome from such type of scenario we develop $IVPNS$.

The structure of this article is as follows: Introduction is included in 1st Section, preliminary notions are included in 2nd Section, the concept of $IVPNSs$ is included in 3rd Section, along with certain operations and outcomes and Section 4 concludes and outlines the research's next directions.

2. Preliminaries

Definition 2.1 [8] A NS $\widehat{£}$ on W (the universe), defined as $\widehat{£} = \left\{\left(\hbar, T_{\widehat{£}}(\hbar), I_{\widehat{£}}(\hbar), F_{\widehat{£}}(\hbar)\right) : \hbar \in W\right\}$,

where $T_{\widehat{£}}, I_{\widehat{£}}, F_{\widehat{£}} : W \to]^{-}0,1^{+}[$ satisfying $\forall \hbar \in W, ^{-}0 \leq T_{\widehat{£}}(\hbar) + I_{\widehat{£}}(\hbar) + F_{\widehat{£}}(\hbar) \leq 3^{+}$.

Here $T_{\widehat{£}}(\hbar), I_{\widehat{£}}(\hbar)$ and $F_{\widehat{£}}(\hbar)$ represent the truth, indeterminacy and falsity membership values respectively of $\hbar \in W$.

Definition 2.2 [5] A PNS $\widehat{£}$ on the universe W is defined as $\widehat{£} = \left\{\left(\hbar, T_{\widehat{£}}(\hbar), C_{\widehat{£}}(\hbar), G_{\widehat{£}}(\hbar), U_{\widehat{£}}(\hbar), F_{\widehat{£}}(\hbar)\right) : \hbar \in W\right\}$, where $T_{\widehat{£}}, C_{\widehat{£}}, G_{\widehat{£}}, U_{\widehat{£}}, F_{\widehat{£}} : W \to [0,1]$ satisfying $\forall \hbar \in W$, $0 \leq T_{\widehat{£}}(\hbar) + C_{\widehat{£}}(\hbar) + G_{\widehat{£}}(\hbar) + U_{\widehat{£}}(\hbar) + F_{\widehat{£}}(\hbar) \leq 5$.

Here $T_{\widehat{£}}(\hbar), C_{\widehat{£}}(\hbar), G_{\widehat{£}}(\hbar), U_{\widehat{£}}(\hbar)$ and $F_{\widehat{£}}(\hbar)$ represent the truthiness, disagreement, lack of understanding i.e. ignorance, unknowability, and falsehood membership values respectively of $\hbar \in W$.

Definition 2.3 [11] An interval neutrosophic set (INS) $\widehat{£}$ on the universe W is defined as $\widehat{£} = \left\{\left(\hbar, T_{\widehat{£}}(\hbar), I_{\widehat{£}}(\hbar), F_{\widehat{£}}(\hbar)\right) : \hbar \in W\right\}$, where $T_{\widehat{£}}, I_{\widehat{£}}, F_{\widehat{£}} : W \to]^{-}0,1^{+}[$ satisfying $\forall \hbar \in W, ^{-}0 \leq T_{\widehat{£}}(\hbar) + I_{\widehat{£}}(\hbar) + F_{\widehat{£}}(\hbar) \leq 3^{+}$.

Here $T_{\widehat{£}}(\hbar), I_{\widehat{£}}(\hbar)$ and $F_{\widehat{£}}(\hbar)$ represent the truthiness, indeterminacy and falsehood membership values respectively of the element $\hbar \in W$.

Definition 2.4 [11] Let $\widehat{£}$, $\widehat{¥}$ are two INSs on W defined by $\widehat{£} = \left\{\left(\hbar, T_{\widehat{£}}(\hbar), I_{\widehat{£}}(\hbar), F_{\widehat{£}}(\hbar)\right) : \hbar \in W\right\}$

and $\widehat{¥} = \left\{\left(\hbar, T_{\widehat{¥}}(\hbar), I_{\widehat{¥}}(\hbar), F_{\widehat{¥}}(\hbar)\right) : \hbar \in W\right\}$. Then for all $\hbar \in W$

i. $\widehat{£}$ is contained in $\widehat{¥}$ if and only if

$$glbT_{\widehat{£}}(\hbar) \leq glbT_{\widehat{¥}}(\hbar),\ lubT_{\widehat{£}}(\hbar) \leq lubT_{\widehat{¥}}(\hbar),$$
$$glbI_{\widehat{£}}(\hbar) \geq glbI_{\widehat{¥}}(\hbar),\ lubI_{\widehat{£}}(\hbar) \geq lubI_{\widehat{¥}}(\hbar),$$
$$glbF_{\widehat{£}}(\hbar) \geq glbF_{\widehat{¥}}(\hbar),\ lubF_{\widehat{£}}(\hbar) \geq lubF_{\widehat{¥}}(\hbar).$$

ii. The union of $\widehat{£}$ and $\widehat{¥}$ is an INS $\widehat{\omega}$, defined by

$$\widehat{\omega} = \widehat{£} \cup \widehat{¥} = \{(\hbar, T_{\widehat{\omega}}(\hbar), I_{\widehat{\omega}}(\hbar), F_{\widehat{\omega}}(\hbar)) : \hbar \in W\}$$

where, $glbT_{\widehat{\omega}}(\hbar) = \vee\left(glbT_{\widehat{£}}(\hbar), glbT_{\widehat{¥}}(\hbar)\right)$, $lubT_{\widehat{\omega}}(\hbar) = \vee\left(lubT_{\widehat{£}}(\hbar), lubT_{\widehat{¥}}(\hbar)\right)$

$$glbI_{\widetilde{\omega}}(\hbar) = \wedge\left(glbI_{\widetilde{£}}(\hbar), glbI_{\widetilde{¥}}(\hbar)\right),\ lubI_{\widetilde{\omega}}(\hbar) = \wedge\left(lubI_{\widetilde{£}}(\hbar), lubI_{\widetilde{¥}}(\hbar)\right)$$

$$glbF_{\widetilde{\omega}}(\hbar) = \wedge\left(glbF_{\widetilde{£}}(\hbar), glbF_{\widetilde{¥}}(\hbar)\right),\ lubF_{\widetilde{\omega}}(\hbar) = \wedge\left(lubF_{\widetilde{£}}(\hbar), lubF_{\widetilde{¥}}(\hbar)\right).$$

iii. The intersection of $\widetilde{£}$ and $\widetilde{¥}$ is an INS $\widetilde{\omega}$, defined by

$$\widetilde{\omega} = \widetilde{£} \cap \widetilde{¥} = \{(\hbar, T_{\widetilde{\omega}}(\hbar), I_{\widetilde{\omega}}(\hbar), F_{\widetilde{\omega}}(\hbar)) : \hbar \in W\}$$

where, $$glbT_{\widetilde{\omega}}(\hbar) = \wedge\left(glbT_{\widetilde{£}}(\hbar), glbT_{\widetilde{¥}}(\hbar)\right),\ lubT_{\widetilde{\omega}}(\hbar) = \wedge\left(lubT_{\widetilde{£}}(\hbar), lubT_{\widetilde{¥}}(\hbar)\right)$$

$$glbI_{\widetilde{\omega}}(\hbar) = \vee\left(glbI_{\widetilde{£}}(\hbar), glbI_{\widetilde{¥}}(\hbar)\right),\ lubI_{\widetilde{\omega}}(\hbar) = \vee\left(lubI_{\widetilde{£}}(\hbar), lubI_{\widetilde{¥}}(\hbar)\right)$$

$$glbF_{\widetilde{\omega}}(\hbar) = \vee\left(glbF_{\widetilde{£}}(\hbar), glbF_{\widetilde{¥}}(\hbar)\right),\ lubF_{\widetilde{\omega}}(\hbar) = \vee\left(lubF_{\widetilde{£}}(\hbar), lubF_{\widetilde{¥}}(\hbar)\right).$$

iv. The complement of $\widetilde{£}$ is $\widetilde{£}^{c}$, defined by $\widetilde{£}^{c} = \left\{\left(\hbar, T_{\widetilde{£}^c}(\hbar), I_{\widetilde{£}^c}(\hbar), F_{\widetilde{£}^c}(\hbar)\right) : \hbar \in W\right\}$ where $T_{\widetilde{£}^c}(\hbar) = F_{\widetilde{£}}(\hbar)$, $I_{\widetilde{£}^c}(\hbar) = [1 - lubI_{\widetilde{£}}(\hbar), 1 - glbI_{\widetilde{£}}(\hbar)]$ and $F_{\widetilde{£}^c}(\hbar) = T_{\widetilde{£}}(\hbar)$.

3. Interval Valued Pentapartitioned Neutrosophic Sets

Here, we provide a novel idea of interval valued pentapartitioned neutrosophic sets and examine some fundamental characteristics.

In Neutrosophic sets there are three characteristic aspects including membership, non membership and indeterminacy whereas in Pentapartitioned Neutrosophic sets, the indeterminacy membership function has been subdivided into three parts: contradictory membership, ignorance membership and unknown membership. However, it has been observed that in issues involving group decision-making, the expert's opinion values differ from individual to individual and as a consequence, it is essential to present the idea of interval valued neutrosophic sets, where each characteristic aspect values are subsets of [0, 1] as opposed to single valued pentapartitioned neutrosophic sets.

Definition 3.1 An interval valued pentapartitioned neutrosophic set ($IVPNS$) $\widetilde{£}$ on the universe W is defined as $\widetilde{£} = \left\{\left(\hbar, T_{\widetilde{£}}(\hbar), C_{\widetilde{£}}(\hbar), G_{\widetilde{£}}(\hbar), U_{\widetilde{£}}(\hbar), F_{\widetilde{£}}(\hbar)\right) : \hbar \in W\right\}$, where $T_{\widetilde{£}}, C_{\widetilde{£}}, G_{\widetilde{£}}, U_{\widetilde{£}}, F_{\widetilde{£}} : W \to Int([0,1])$ satisfying $\forall \hbar \in W$, $0 \le lubT_{\widetilde{£}}(\hbar) + lubC_{\widetilde{£}}(\hbar) + lubG_{\widetilde{£}}(\hbar) + lubU_{\widetilde{£}}(\hbar) + lubF_{\widetilde{£}}(\hbar) \le 5$.

Here $T_{\widetilde{£}}(\hbar), C_{\widetilde{£}}(\hbar), G_{\widetilde{£}}(\hbar), U_{\widetilde{£}}$ and $F_{\widetilde{£}}(\hbar)$ represent the truthiness, disagreement, lack of understanding i.e. ignorance, unknowability and falsehood membership values respectively of $\hbar \in W$.

Example 3.2 Consider the statement, "Does humans are immune to COVID-19 infection after vaccination?"

Suppose the statement is given to two groups of peoples for their personal views where each group consists of five peoples, say, $M = \{m_{11}, m_{12}, m_{13}, m_{14}, m_{15}, m_{21}, m_{22}, m_{23}, m_{24}, m_{25}\}$. Now it is obvious that different perspective will be observed regarding the statement with distinct membership value. The possible perspective may be expressed as degrees of "agreement (T)", "both agreement and disagreement (C)", "ignorance (G)", "neither agreement not disagreement (U)", "disagreement (F)".

Now from the group of peoples, suppose m_{i1}, $(i = 1,2)$ make agreement (T) with distinct membership value which may lie in an interval$\in Int([0,1])$. Similarly $m_{i2}, m_{i3}, m_{i4}, m_{i5}$, $(i = 1,2)$ make their perspectives C, G, U, F respectively which may also lie in an interval$\in Int([0,1])$.

Here some fundamental operators are defined in interval valued pentapartitioned neutrosophic sets ($IVPNS$s) which are further utilized to examine various $IVPNS$ features.

Definition 3.3 Let $\widetilde{\text{£}}$ and $\widetilde{\text{¥}}$ are two $IVPNS$s on W defined by $\widetilde{\text{£}} = \left\{\left(\hbar, T_{\widetilde{\text{£}}}(\hbar), C_{\widetilde{\text{£}}}(\hbar), G_{\widetilde{\text{£}}}(\hbar), U_{\widetilde{\text{£}}}(\hbar), F_{\widetilde{\text{£}}}(\hbar)\right) : \hbar \in W\right\}$ and $\widetilde{\text{¥}} = \left\{\left(\hbar, T_{\widetilde{\text{¥}}}(\hbar), C_{\widetilde{\text{¥}}}(\hbar), G_{\widetilde{\text{¥}}}(\hbar), U_{\widetilde{\text{¥}}}(\hbar), F_{\widetilde{\text{¥}}}(\hbar)\right) : \hbar \in W\right\}$. Then for every $\hbar \in W$

i. $\widetilde{\text{£}}$ is contained in $\widetilde{\text{¥}}$ iff

$$glbT_{\widetilde{\text{£}}}(\hbar) \le glbT_{\widetilde{\text{¥}}}(\hbar), \quad lubT_{\widetilde{\text{£}}}(\hbar) \le lubT_{\widetilde{\text{¥}}}(\hbar),$$
$$glbC_{\widetilde{\text{£}}}(\hbar) \le glbC_{\widetilde{\text{¥}}}(\hbar), \quad lubC_{\widetilde{\text{£}}}(\hbar) \le lubC_{\widetilde{\text{¥}}}(\hbar),$$
$$glbG_{\widetilde{\text{£}}}(\hbar) \ge glbG_{\widetilde{\text{¥}}}(\hbar), \quad lubG_{\widetilde{\text{£}}}(\hbar) \ge lubG_{\widetilde{\text{¥}}}(\hbar),$$
$$glbU_{\widetilde{\text{£}}}(\hbar) \ge glbU_{\widetilde{\text{¥}}}(\hbar), \quad lubU_{\widetilde{\text{£}}}(\hbar) \ge lubU_{\widetilde{\text{¥}}}(\hbar),$$
$$glbF_{\widetilde{\text{£}}}(\hbar) \ge glbF_{\widetilde{\text{¥}}}(\hbar), \quad lubF_{\widetilde{\text{£}}}(\hbar) \ge lubF_{\widetilde{\text{¥}}}(\hbar).$$

ii. The union of $\widetilde{\text{£}}$ and $\widetilde{\text{¥}}$ is an $IVPNS$ $\widetilde{\omega}$, defined by

$$\widetilde{\omega} = \widetilde{\text{£}} \cup \widetilde{\text{¥}} = \{(\hbar, T_{\widetilde{\omega}}(\hbar), C_{\widetilde{\omega}}(\hbar), G_{\widetilde{\omega}}(\hbar), U_{\widetilde{\omega}}(\hbar), F_{\widetilde{\omega}}(\hbar)) : \hbar \in W\}$$

where, $glbT_{\widetilde{\omega}}(\hbar) = \vee\left(glbT_{\widetilde{\text{£}}}(\hbar), glbT_{\widetilde{\text{¥}}}(\hbar)\right)$, $lubT_{\widetilde{\omega}}(\hbar) = \vee\left(lubT_{\widetilde{\text{£}}}(\hbar), lubT_{\widetilde{\text{¥}}}(\hbar)\right)$

$glbC_{\widetilde{\omega}}(\hbar) = \vee\left(glbC_{\widetilde{\text{£}}}(\hbar), glbC_{\widetilde{\text{¥}}}(\hbar)\right)$, $lubC_{\widetilde{\omega}}(\hbar) = \vee\left(lubC_{\widetilde{\text{£}}}(\hbar), lubC_{\widetilde{\text{¥}}}(\hbar)\right)$

$glbG_{\widetilde{\omega}}(\hbar) = \wedge\left(glbG_{\widetilde{\text{£}}}(\hbar), glbG_{\widetilde{\text{¥}}}(\hbar)\right)$, $lubG_{\widetilde{\omega}}(\hbar) = \wedge\left(lubG_{\widetilde{\text{£}}}(\hbar), lubG_{\widetilde{\text{¥}}}(\hbar)\right)$

$glbU_{\widetilde{\omega}}(\hbar) = \wedge\left(glbU_{\widetilde{\text{£}}}(\hbar), glbU_{\widetilde{\text{¥}}}(\hbar)\right)$, $lubU_{\widetilde{\omega}}(\hbar) = \wedge\left(lubU_{\widetilde{\text{£}}}(\hbar), lubU_{\widetilde{\text{¥}}}(\hbar)\right)$

$glbF_{\widetilde{\omega}}(\hbar) = \wedge\left(glbF_{\widetilde{\text{£}}}(\hbar), glbF_{\widetilde{\text{¥}}}(\hbar)\right)$, $lubF_{\widetilde{\omega}}(\hbar) = \wedge\left(lubF_{\widetilde{\text{£}}}(\hbar), lubF_{\widetilde{\text{¥}}}(\hbar)\right)$

or simply we can write

$$\widetilde{\text{£}} \cup \widetilde{\text{¥}} = \Big\{\hbar, \Big[\vee\left(glbT_{\widetilde{\text{£}}}(\hbar), glbT_{\widetilde{\text{¥}}}(\hbar)\right), \vee\left(lubT_{\widetilde{\text{£}}}(\hbar), lubT_{\widetilde{\text{¥}}}(\hbar)\right)\Big], \Big[\vee\left(glbC_{\widetilde{\text{£}}}(\hbar), glbC_{\widetilde{\text{¥}}}(\hbar)\right), \vee\left(lubC_{\widetilde{\text{£}}}(\hbar), lubC_{\widetilde{\text{¥}}}(\hbar)\right)\Big], \Big[\wedge\left(glbG_{\widetilde{\text{£}}}(\hbar), glbG_{\widetilde{\text{¥}}}(\hbar)\right), \wedge\left(lubG_{\widetilde{\text{£}}}(\hbar), lubG_{\widetilde{\text{¥}}}(\hbar)\right)\Big], \Big[\wedge\left(glbU_{\widetilde{\text{£}}}(\hbar), glbU_{\widetilde{\text{¥}}}(\hbar)\right), \wedge\left(lubU_{\widetilde{\text{£}}}(\hbar), lubU_{\widetilde{\text{¥}}}(\hbar)\right)\Big], \Big[\wedge\left(glbF_{\widetilde{\text{£}}}(\hbar), glbF_{\widetilde{\text{¥}}}(\hbar)\right), \wedge\left(lubF_{\widetilde{\text{£}}}(\hbar), lubF_{\widetilde{\text{¥}}}(\hbar)\right)\Big]\Big\}$$

iii. The intersection is an $IVPNS$ $\widetilde{\omega}$, defined by

$$\widetilde{\omega} = \widetilde{\text{£}} \cap \widetilde{\text{¥}} = \{(\hbar, T_{\widetilde{\omega}}(\hbar), C_{\widetilde{\omega}}(\hbar), G_{\widetilde{\omega}}(\hbar), U_{\widetilde{\omega}}(\hbar), F_{\widetilde{\omega}}(\hbar)) : \hbar \in W\}$$

where, $glbT_{\widetilde{\omega}}(\hbar) = \wedge\left(glbT_{\widetilde{\text{£}}}(\hbar), glbT_{\widetilde{\text{¥}}}(\hbar)\right)$, $lubT_{\widetilde{\omega}}(\hbar) = \wedge\left(lubT_{\widetilde{\text{£}}}(\hbar), lubT_{\widetilde{\text{¥}}}(\hbar)\right)$

$$glbC_{\widetilde{\omega}}(\hbar) = \wedge\left(glbC_{\widetilde{£}}(\hbar), glbC_{\widetilde{¥}}(\hbar)\right),\ lubC_{\widetilde{\omega}}(\hbar) = \wedge\left(lubC_{\widetilde{£}}(\hbar), lubC_{\widetilde{¥}}(\hbar)\right)$$

$$glbG_{\widetilde{\omega}}(\hbar) = \vee\left(glbG_{\widetilde{£}}(\hbar), glbG_{\widetilde{¥}}(\hbar)\right),\ lubG_{\widetilde{\omega}}(\hbar) = \vee\left(lubG_{\widetilde{£}}(\hbar), lubG_{\widetilde{¥}}(\hbar)\right)$$

$$glbU_{\widetilde{\omega}}(\hbar) = \vee\left(glbU_{\widetilde{£}}(\hbar), glbU_{\widetilde{¥}}(\hbar)\right),\ lubU_{\widetilde{\omega}}(\hbar) = \vee\left(lubU_{\widetilde{£}}(\hbar), lubU_{\widetilde{¥}}(\hbar)\right)$$

$$glbF_{\widetilde{\omega}}(\hbar) = \vee\left(glbF_{\widetilde{£}}(\hbar), glbF_{\widetilde{¥}}(\hbar)\right),\ lubF_{\widetilde{\omega}}(\hbar) = \vee\left(lubF_{\widetilde{£}}(\hbar), lubF_{\widetilde{¥}}(\hbar)\right)$$

or simply we can write $\widetilde{£} \cap \widetilde{¥} = \Big\{\hbar, \Big[\wedge\left(glbT_{\widetilde{£}}(\hbar), glbT_{\widetilde{¥}}(\hbar)\right), \wedge\left(lubT_{\widetilde{£}}(\hbar), lubT_{\widetilde{¥}}(\hbar)\right)\Big], \Big[\wedge\left(glbC_{\widetilde{£}}(\hbar), glbC_{\widetilde{¥}}(\hbar)\right), \wedge\left(lubC_{\widetilde{£}}(\hbar), lubC_{\widetilde{¥}}(\hbar)\right)\Big], \Big[\vee\left(glbG_{\widetilde{£}}(\hbar), glbG_{\widetilde{¥}}(\hbar)\right), \vee\left(lubG_{\widetilde{£}}(\hbar), lubG_{\widetilde{¥}}(\hbar)\right)\Big], \Big[\vee\left(glbU_{\widetilde{£}}(\hbar), glbU_{\widetilde{¥}}(\hbar)\right), \vee\left(lubU_{\widetilde{£}}(\hbar), lubU_{\widetilde{¥}}(\hbar)\right)\Big], \Big[\vee\left(glbF_{\widetilde{£}}(\hbar), glbF_{\widetilde{¥}}(\hbar)\right), \vee\left(lubF_{\widetilde{£}}(\hbar), lubF_{\widetilde{¥}}(\hbar)\right)\Big]\Big\}$

iv. The complement of $\widetilde{£}$ is $\widetilde{£}^c$, defined by $\widetilde{£}^c = \left\{\left(\hbar, T_{\widetilde{£}^c}(\hbar), C_{\widetilde{£}^c}(\hbar), G_{\widetilde{£}^c}(\hbar), U_{\widetilde{£}^c}(\hbar), F_{\widetilde{£}^c}(\hbar)\right) : \hbar \in W\right\}$ where $T_{\widetilde{£}^c}(\hbar) = F_{\widetilde{£}}(\hbar)$, $C_{\widetilde{£}^c}(\hbar) = U_{\widetilde{£}}(\hbar)$ $G_{\widetilde{£}^c}(\hbar) = [1 - lubG_{\widetilde{£}}(\hbar), 1 - glbG_{\widetilde{£}}(\hbar)]$, $U_{\widetilde{£}^c}(\hbar) = C_{\widetilde{£}}(\hbar)$ and $F_{\widetilde{£}^c}(\hbar) = T_{\widetilde{£}}(\hbar)$.

or simply we can write $\widetilde{£}^c = \left\{\left(\hbar, F_{\widetilde{£}}(\hbar), U_{\widetilde{£}}(\hbar), [1 - lubG_{\widetilde{£}}(\hbar), 1 - glbG_{\widetilde{£}}(\hbar)], C_{\widetilde{£}}(\hbar), T_{\widetilde{£}}(\hbar)\right) : \hbar \in W\right\}$.

Example 3.4 Consider two $IVPNS$s $\widetilde{£}$ and $\widetilde{¥}$ defined over W as

$\widetilde{£} = \{(\hbar_1, [0.32,0.54], [0.23,0.65], [0.56,0.79], [0.32,0.43], [0.85,0.96]), (\hbar_2, [0.67,0.78], [0.55,0.78], [0.11,0.32], [0.23,0.84], [0.15,0.38]), [(\hbar_3, [0.24,0.56], [0.17,0.52], [0.25,0.75], [0.21,0.63], [0.31,0.56])\}$

$\widetilde{¥} = \{(\hbar_1, [0.57,0.91], [0.52,0.83], [0.57,0.78], [0.23,0.39], [0.61,0.84]), (\hbar_2, [0.52,0.71], [0.24,0.56], [0.20,0.52], [0.75,0.80], [0.41,0.62]), [(\hbar_3, [0.12,0.31], [0.38,0.56], [0.55,0.74], [0.19,0.86], [0.16,0.83])\}$

Then

$\widetilde{£} \cup \widetilde{¥} = \{(\hbar_1, [0.57,0.91], [0.52,0.83], [0.56,0.78], [0.23,0.39], [0.61,0.84]), (\hbar_2, [0.67,0.78], [0.55,0.78], [0.11,0.32], [0.23,0.80], [0.15,0.38]), [(\hbar_3, [0.24,0.56], [0.38,0.56], [0.25,0.74], [0.19,0.63], [0.16,0.56])\}$

$\widetilde{£} \cap \widetilde{¥} = \{(\hbar_1, [0.32,0.54], [0.23,0.65], [0.57,0.79], [0.32,0.43], [0.85,0.96]), (\hbar_2, [0.52,0.71], [0.24,0.56], [0.20,0.52], [0.75,0.84], [0.41,0.62]), [(\hbar_3, [0.24,0.56], [0.38,0.56], [0.25,0.74], [0.19,0.63], [0.16,0.56])\}$

$\widetilde{£}^c = \{(\hbar_1, [0.85,0.96], [0.32,0.43], [0.21,0.34], [0.23,0.65], [0.32,0.54]), (\hbar_2, [0.15,0.38], [0.23,0.84], [0.68,0.91], [0.55,0.78], [0.67,0.78]), [(\hbar_3, [0.31,0.56], [0.21,0.63], [0.25,0.75], [0.17,0.52], [0.24,0.56])\}$

Theorem 3.5 For any three $IVPNS$s $\widetilde{£}$, $\widetilde{¥}$ and $\widetilde{\omega}$

i. $\widetilde{£} \cup \widetilde{£} = \widetilde{£}$, $\widetilde{£} \cap \widetilde{£} = \widetilde{£}$ (Idempotent Law)

ii. $\widetilde{£} \cup \widetilde{¥} = \widetilde{¥} \cup \widetilde{£}$, $\widetilde{£} \cap \widetilde{¥} = \widetilde{¥} \cap \widetilde{£}$ (Commutative Law)

iii. $\left(\widetilde{£} \cup \widetilde{¥}\right) \cup \widetilde{\omega} = \widetilde{£} \cup \left(\widetilde{¥} \cup \widetilde{\omega}\right)$, $\left(\widetilde{£} \cap \widetilde{¥}\right) \cap \widetilde{\omega} = \widetilde{£} \cap \left(\widetilde{¥} \cap \widetilde{\omega}\right)$ (Associative Law)

iv. $\widetilde{£} \cup \left(\widetilde{¥} \cap \widetilde{\omega}\right) = \left(\widetilde{£} \cup \widetilde{¥}\right) \cap \left(\widetilde{£} \cup \widetilde{\omega}\right)$, $\widetilde{£} \cap \left(\widetilde{¥} \cup \widetilde{\omega}\right) = \left(\widetilde{£} \cap \widetilde{¥}\right) \cup \left(\widetilde{£} \cap \widetilde{\omega}\right)$ (Distributive Law)

v. $\left(\widehat{£}\cup\widehat{¥}\right)^c=\widehat{£}^c\cap\widehat{¥}^c$, $\left(\widehat{£}\cap\widehat{¥}\right)^c=\widehat{£}^c\cup\widehat{¥}^c$ (De Morgan's Law)

vi. $\widehat{£}\cup\left(\widehat{£}\cap\widehat{¥}\right)=\widehat{£}$, $\widehat{£}\cap\left(\widehat{£}\cup\widehat{¥}\right)=\widehat{£}$ (Absorption Law)

vii. $\left(\widehat{£}^c\right)^c=\widehat{£}$ (Involution Law)

Proof: Let $\widehat{£}$, $\widehat{¥}$ and $\widehat{\omega}$ are two $IVPNS$s on W defined by

$\widehat{£}=\left\{\left(\hbar,T_{\widehat{£}}(\hbar),C_{\widehat{£}}(\hbar),G_{\widehat{£}}(\hbar),U_{\widehat{£}}(\hbar),F_{\widehat{£}}(\hbar)\right):\hbar\in W\right\}$, $\widehat{¥}=\left\{\left(\hbar,T_{\widehat{¥}}(\hbar),C_{\widehat{¥}}(\hbar),G_{\widehat{¥}}(\hbar),U_{\widehat{¥}}(\hbar),F_{\widehat{¥}}(\hbar)\right):\hbar\in W\right\}$ and $\widehat{\omega}=\left\{\left(\hbar,T_{\widehat{\omega}}(\hbar),C_{\widehat{\omega}}(\hbar),G_{\widehat{\omega}}(\hbar),U_{\widehat{\omega}}(\hbar),F_{\widehat{\omega}}(\hbar)\right):\hbar\in W\right\}$ respectively. Then for every $\hbar\in W$

(i) Straight forward.

(ii) We know that,

$$\begin{aligned}\widehat{£}\cup\widehat{¥}=\Big\{\hbar,&\left[\vee\left(glbT_{\widehat{£}}(\hbar),glbT_{\widehat{¥}}(\hbar)\right),\vee\left(lubT_{\widehat{£}}(\hbar),lubT_{\widehat{¥}}(\hbar)\right)\right],\\&\left[\vee\left(glbC_{\widehat{£}}(\hbar),glbC_{\widehat{¥}}(\hbar)\right),\vee\left(lubC_{\widehat{£}}(\hbar),lubC_{\widehat{¥}}(\hbar)\right)\right],\\&\left[\wedge\left(glbG_{\widehat{£}}(\hbar),glbG_{\widehat{¥}}(\hbar)\right),\wedge\left(lubG_{\widehat{£}}(\hbar),lubG_{\widehat{¥}}(\hbar)\right)\right],\\&\left[\wedge\left(glbU_{\widehat{£}}(\hbar),glbU_{\widehat{¥}}(\hbar)\right),\wedge\left(lubU_{\widehat{£}}(\hbar),lubU_{\widehat{¥}}(\hbar)\right)\right],\\&\left[\wedge\left(glbF_{\widehat{£}}(\hbar),glbF_{\widehat{¥}}(\hbar)\right),\wedge\left(lubF_{\widehat{£}}(\hbar),lubF_{\widehat{¥}}(\hbar)\right)\right]:\hbar\in W\Big\}\end{aligned}$$

$$\begin{aligned}=\Big\{\hbar,&\left[\vee\left(glbT_{\widehat{¥}}(\hbar),glbT_{\widehat{£}}(\hbar)\right),\vee\left(lubT_{\widehat{¥}}(\hbar),lubT_{\widehat{£}}(\hbar)\right)\right],\\&\left[\vee\left(glbC_{\widehat{¥}}(\hbar),glbC_{\widehat{£}}(\hbar)\right),\vee\left(lubC_{\widehat{¥}}(\hbar),lubC_{\widehat{£}}(\hbar)\right)\right],\\&\left[\wedge\left(glbG_{\widehat{¥}}(\hbar),glbG_{\widehat{£}}(\hbar)\right),\wedge\left(lubG_{\widehat{¥}}(\hbar),lubG_{\widehat{£}}(\hbar)\right)\right],\\&\left[\wedge\left(glbU_{\widehat{¥}}(\hbar),glbU_{\widehat{£}}(\hbar)\right),\wedge\left(lubU_{\widehat{¥}}(\hbar),lubU_{\widehat{£}}(\hbar)\right)\right],\\&\left[\wedge\left(glbF_{\widehat{¥}}(\hbar),glbF_{\widehat{£}}(\hbar)\right),\wedge\left(lubF_{\widehat{¥}}(\hbar),lubF_{\widehat{£}}(\hbar)\right)\right]:\hbar\in W\Big\}\end{aligned}$$

$$=\widehat{¥}\cup\widehat{£}$$

$$\therefore\widehat{£}\cup\widehat{¥}=\widehat{¥}\cup\widehat{£}$$

Similarly, $\widehat{£}\cap\widehat{¥}=\widehat{¥}\cap\widehat{£}$.

(iii) We know that,

$$\left(\widetilde{\widehat{£}} \cup \widetilde{\widehat{¥}}\right) \cup \widetilde{\widehat{\omega}} = \Big\{\hbar, \Big[\vee\left(glbT_{\widehat{£}}(\hbar), glbT_{\widehat{¥}}(\hbar)\right), \vee\left(lubT_{\widehat{£}}(\hbar), lubT_{\widehat{¥}}(\hbar)\right)\Big],$$

$$\Big[\vee\left(glbC_{\widehat{£}}(\hbar), glbC_{\widehat{¥}}(\hbar)\right),$$

$$\vee\left(lubC_{\widehat{£}}(\hbar), lubC_{\widehat{¥}}(\hbar)\right)\Big],$$

$$\Big[\wedge\left(glbG_{\widehat{£}}(\hbar), glbG_{\widehat{¥}}(\hbar)\right),$$

$$\wedge\left(lubG_{\widehat{£}}(\hbar), lubG_{\widehat{¥}}(\hbar)\right)\Big],$$

$$\Big[\wedge\left(glbU_{\widehat{£}}(\hbar), glbU_{\widehat{¥}}(\hbar)\right),$$

$$\wedge\left(lubU_{\widehat{£}}(\hbar), lubU_{\widehat{¥}}(\hbar)\right)\Big],$$

$$\Big[\wedge\left(glbF_{\widehat{£}}(\hbar), glbF_{\widehat{¥}}(\hbar)\right),$$

$$\wedge\left(lubF_{\widehat{£}}(\hbar), lubF_{\widehat{¥}}(\hbar)\right)\Big] : \hbar \in W\Big\} \cup \widetilde{\widehat{\omega}}$$

$$= \Big\{\hbar, \Big[\vee\left(glbT_{\widehat{£}}(\hbar), glbT_{\widehat{¥}}(\hbar), glbT_{\widehat{\omega}}(\hbar)\right), \vee\left(lubT_{\widehat{£}}(\hbar), lubT_{\widehat{¥}}(\hbar), lubT_{\widehat{\omega}}(\hbar)\right)\Big],$$

$$\Big[\vee\left(glbC_{\widehat{£}}(\hbar), glbC_{\widehat{¥}}(\hbar), glbC_{\widehat{\omega}}(\hbar)\right),$$

$$\vee\left(lubC_{\widehat{£}}(\hbar), lubC_{\widehat{¥}}(\hbar), lubC_{\widehat{\omega}}(\hbar)\right)\Big],$$

$$\Big[\wedge\left(glbG_{\widehat{£}}(\hbar), glbG_{\widehat{¥}}(\hbar), glbG_{\widehat{\omega}}(\hbar)\right),$$

$$\wedge\left(lubG_{\widehat{£}}(\hbar), lubG_{\widehat{¥}}(\hbar), lubG_{\widehat{\omega}}(\hbar)\right)\Big],$$

$$\Big[\wedge\left(glbU_{\widehat{£}}(\hbar), glbU_{\widehat{¥}}(\hbar), glbU_{\widehat{\omega}}(\hbar)\right),$$

$$\wedge\left(lubU_{\widehat{£}}(\hbar), lubU_{\widehat{¥}}(\hbar), lubU_{\widehat{\omega}}(\hbar)\right)\Big],$$

$$\Big[\wedge\left(glbF_{\widehat{£}}(\hbar), glbF_{\widehat{¥}}(\hbar), glbF_{\widehat{\omega}}(\hbar)\right),$$

$$\wedge\left(lubF_{\widehat{£}}(\hbar), lubF_{\widehat{¥}}(\hbar), lubF_{\widehat{\omega}}(\hbar)\right)\Big] : \hbar \in W\Big\}$$

$$= \widetilde{\widehat{£}} \cup \Big\{\hbar, \Big[\vee\left(glbT_{\widehat{¥}}(\hbar), glbT_{\widehat{\omega}}(\hbar)\right), \vee\left(lubT_{\widehat{¥}}(\hbar), lubT_{\widehat{\omega}}(\hbar)\right)\Big],$$

$$\Big[\vee\left(glbC_{\widehat{¥}}(\hbar), glbC_{\widehat{\omega}}(\hbar)\right), \vee\left(lubC_{\widehat{¥}}(\hbar), lubC_{\widehat{\omega}}(\hbar)\right)\Big],$$

$$\Big[\wedge\left(glbG_{\widehat{¥}}(\hbar), glbG_{\widehat{\omega}}(\hbar)\right), \wedge\left(lubG_{\widehat{¥}}(\hbar), lubG_{\widehat{\omega}}(\hbar)\right)\Big],$$

$$\left[\wedge\left(glbU_{\widetilde{¥}}(\hbar), glbU_{\widetilde{\omega}}(\hbar)\right), \wedge\left(lubU_{\widetilde{¥}}(\hbar), lubU_{\widetilde{\omega}}(\hbar)\right)\right],$$

$$\left[\wedge\left(glbF_{\widetilde{¥}}(\hbar), glbF_{\widetilde{\omega}}(\hbar)\right), \wedge\left(lubF_{\widetilde{¥}}(\hbar), lubF_{\widetilde{\omega}}(\hbar)\right)\right] : \hbar \in W\right\}$$

$$= \widetilde{£} \cup \left(\widetilde{¥} \cup \widetilde{\omega}\right)$$

$$\therefore \left(\widetilde{£} \cup \widetilde{¥}\right) \cup \widetilde{\omega} = \widetilde{£} \cup \left(\widetilde{¥} \cup \widetilde{\omega}\right)$$

Similarly, $\left(\widetilde{£} \cap \widetilde{¥}\right) \cap \widetilde{\omega} = \widetilde{£} \cap \left(\widetilde{¥} \cap \widetilde{\omega}\right)$.

(iv) We know that,

$$\widetilde{£} \cup \left(\widetilde{¥} \cap \widetilde{\omega}\right)$$

$$= \widetilde{£} \cup \left\{\hbar, \left[\wedge\left(glbT_{\widetilde{¥}}(\hbar), glbT_{\widetilde{\omega}}(\hbar)\right), \wedge\left(lubT_{\widetilde{¥}}(\hbar), lubT_{\widetilde{\omega}}(\hbar)\right)\right],\right.$$

$$\left[\wedge\left(glbC_{\widetilde{¥}}(\hbar), glbC_{\widetilde{\omega}}(\hbar)\right),\right.$$

$$\left.\wedge\left(lubC_{\widetilde{¥}}(\hbar), lubC_{\widetilde{\omega}}(\hbar)\right)\right],$$

$$\left[\vee\left(glbG_{\widetilde{¥}}(\hbar), glbG_{\widetilde{\omega}}(\hbar)\right),\right.$$

$$\left.\vee\left(lubG_{\widetilde{¥}}(\hbar), lubG_{\widetilde{\omega}}(\hbar)\right)\right],$$

$$\left[\vee\left(glbU_{\widetilde{¥}}(\hbar), glbU_{\widetilde{\omega}}(\hbar)\right),\right.$$

$$\left.\vee\left(lubU_{\widetilde{¥}}(\hbar), lubU_{\widetilde{\omega}}(\hbar)\right)\right],$$

$$\left[\vee\left(glbF_{\widetilde{¥}}(\hbar), glbF_{\widetilde{\omega}}(\hbar)\right),\right.$$

$$\left.\left.\vee\left(lubF_{\widetilde{¥}}(\hbar), lubF_{\widetilde{\omega}}(\hbar)\right)\right] : \hbar \in W\right\}$$

$$= \left\{\hbar, \left[\vee\left(glbT_{\widetilde{£}}(\hbar), \wedge\left(glbT_{\widetilde{¥}}(\hbar), glbT_{\widetilde{\omega}}(\hbar)\right)\right), \vee\left(lubT_{\widetilde{£}}(\hbar), \wedge\left(lubT_{\widetilde{¥}}(\hbar), lubT_{\widetilde{\omega}}(\hbar)\right)\right)\right],\right.$$

$$\left[\vee\left(glbC_{\widetilde{£}}(\hbar), \wedge\left(glbC_{\widetilde{¥}}(\hbar), glbC_{\widetilde{\omega}}(\hbar)\right)\right), \vee\left(lubC_{\widetilde{£}}(\hbar), \wedge\left(lubC_{\widetilde{¥}}(\hbar), lubC_{\widetilde{\omega}}(\hbar)\right)\right)\right],$$

$$\left[\wedge\left(glbG_{\widetilde{£}}(\hbar), \vee\left(glbG_{\widetilde{¥}}(\hbar), glbG_{\widetilde{\omega}}(\hbar)\right)\right), \wedge\left(lubG_{\widetilde{£}}(\hbar), \vee\left(lubG_{\widetilde{¥}}(\hbar), lubG_{\widetilde{\omega}}(\hbar)\right)\right)\right],$$

$$\left[\wedge\left(glbU_{\widetilde{£}}(\hbar), \vee\left(glbU_{\widetilde{¥}}(\hbar), glbU_{\widetilde{\omega}}(\hbar)\right)\right), \wedge\left(lubU_{\widetilde{£}}(\hbar), \vee\left(lubU_{\widetilde{¥}}(\hbar), lubU_{\widetilde{\omega}}(\hbar)\right)\right)\right],$$

$$\left[\wedge\left(glbF_{\widetilde{£}}(\hbar),\vee\left(glbF_{\widetilde{¥}}(\hbar), glbF_{\widetilde{\varpi}}(\hbar)\right)\right),\wedge\left(lubF_{\widetilde{£}}(\hbar),\vee\left(lubF_{\widetilde{¥}}(\hbar), lubF_{\widetilde{\varpi}}(\hbar)\right)\right)\right]:\hbar \in W\right\}$$

$$=\left\{\hbar,\left[\vee\left(glbT_{\widetilde{£}}(\hbar), glbT_{\widetilde{¥}}(\hbar)\right),\vee\left(lubT_{\widetilde{£}}(\hbar), lubT_{\widetilde{¥}}(\hbar)\right)\right],\right.$$

$$\left[\vee\left(glbC_{\widetilde{£}}(\hbar), glbC_{\widetilde{¥}}(\hbar)\right),\vee\left(lubC_{\widetilde{£}}(\hbar), lubC_{\widetilde{¥}}(\hbar)\right)\right],$$

$$\left[\wedge\left(glbG_{\widetilde{£}}(\hbar), glbG_{\widetilde{¥}}(\hbar)\right),\wedge\left(lubG_{\widetilde{£}}(\hbar),\vee\, lubG_{\widetilde{¥}}(\hbar)\right)\right],$$

$$\left[\wedge\left(glbU_{\widetilde{£}}(\hbar), glbU_{\widetilde{¥}}(\hbar)\right),\wedge\left(lubU_{\widetilde{£}}(\hbar), lubU_{\widetilde{¥}}(\hbar)\right)\right],$$

$$\left.\left[\wedge\left(glbF_{\widetilde{£}}(\hbar), glbF_{\widetilde{¥}}(\hbar)\right),\wedge\left(lubF_{\widetilde{£}}(\hbar), lubF_{\widetilde{¥}}(\hbar)\right)\right]:\hbar \in W\right\}$$

$$\cap\left\{\hbar,\left[\vee\left(glbT_{\widetilde{£}}(\hbar), glbT_{\widetilde{\varpi}}(\hbar)\right),\vee\left(lubT_{\widetilde{£}}(\hbar), lubT_{\widetilde{\varpi}}(\hbar)\right)\right],\right.$$

$$\left[\vee\left(glbC_{\widetilde{£}}(\hbar), glbC_{\widetilde{\varpi}}(\hbar)\right),\vee\left(lubC_{\widetilde{£}}(\hbar), lubC_{\widetilde{\varpi}}(\hbar)\right)\right],$$

$$\left[\wedge\left(glbG_{\widetilde{£}}(\hbar), glbG_{\widetilde{\varpi}}(\hbar)\right),\wedge\left(lubG_{\widetilde{£}}(\hbar),\vee\, lubG_{\widetilde{\varpi}}(\hbar)\right)\right],$$

$$\left[\wedge\left(glbU_{\widetilde{£}}(\hbar), glbU_{\widetilde{\varpi}}(\hbar)\right),\wedge\left(lubU_{\widetilde{£}}(\hbar), lubU_{\widetilde{\varpi}}(\hbar)\right)\right],$$

$$\left.\left[\wedge\left(glbF_{\widetilde{£}}(\hbar), glbF_{\widetilde{\varpi}}(\hbar)\right),\wedge\left(lubF_{\widetilde{£}}(\hbar), lubF_{\widetilde{\varpi}}(\hbar)\right)\right]:\hbar \in W\right\}$$

$$=\left(\widetilde{£}\cup\widetilde{¥}\right)\cap\left(\widetilde{£}\cup\widetilde{\varpi}\right)$$

$$\therefore \widetilde{£}\cup\left(\widetilde{¥}\cap\widetilde{\varpi}\right)=\left(\widetilde{£}\cup\widetilde{¥}\right)\cap\left(\widetilde{£}\cup\widetilde{\varpi}\right)$$

Similarly, $\widetilde{£}\cap\left(\widetilde{¥}\cup\widetilde{\varpi}\right)=\left(\widetilde{£}\cap\widetilde{¥}\right)\cup\left(\widetilde{£}\cap\widetilde{\varpi}\right)$.

(v) We know that,

$$\left(\widetilde{£}\cup\widetilde{¥}\right)^c=\left\{\hbar,\left[\vee\left(glbT_{\widetilde{£}}(\hbar), glbT_{\widetilde{¥}}(\hbar)\right),\vee\left(lubT_{\widetilde{£}}(\hbar), lubT_{\widetilde{¥}}(\hbar)\right)\right],\right.$$

$$\left[\vee\left(glbC_{\widetilde{£}}(\hbar), glbC_{\widetilde{¥}}(\hbar)\right),\vee\left(lubC_{\widetilde{£}}(\hbar), lubC_{\widetilde{¥}}(\hbar)\right)\right],$$

$$\left[\wedge\left(glbG_{\widetilde{£}}(\hbar), glbG_{\widetilde{¥}}(\hbar)\right),\wedge\left(lubG_{\widetilde{£}}(\hbar), lubG_{\widetilde{¥}}(\hbar)\right)\right],$$

$$\left[\wedge\left(glbU_{\widetilde{£}}(\hbar), glbU_{\widetilde{¥}}(\hbar)\right),\wedge\left(lubU_{\widetilde{£}}(\hbar), lubU_{\widetilde{¥}}(\hbar)\right)\right],$$

$$\left.\left[\wedge\left(glbF_{\widetilde{£}}(\hbar), glbF_{\widetilde{¥}}(\hbar)\right),\wedge\left(lubF_{\widetilde{£}}(\hbar), lubF_{\widetilde{¥}}(\hbar)\right)\right]:\hbar \in W\right\}^c$$

$$= \Big\{\hbar, \Big[\wedge\Big(glbF_{\widehat{£}}(\hbar), glbF_{\widehat{¥}}(\hbar)\Big), \wedge\Big(lubF_{\widehat{£}}(\hbar), lubF_{\widehat{¥}}(\hbar)\Big)\Big],$$

$$\Big[\wedge\Big(glbU_{\widehat{£}}(\hbar), glbU_{\widehat{¥}}(\hbar)\Big), \wedge\Big(lubU_{\widehat{£}}(\hbar), lubU_{\widehat{¥}}(\hbar)\Big)\Big],$$

$$\Big[1 - \wedge\Big(lubG_{\widehat{£}}(\hbar), lubG_{\widehat{¥}}(\hbar)\Big), 1 -$$

$$\wedge\Big(glbG_{\widehat{£}}(\hbar), glbG_{\widehat{¥}}(\hbar)\Big)\Big],$$

$$\Big[\vee\Big(glbC_{\widehat{£}}(\hbar), glbC_{\widehat{¥}}(\hbar)\Big), \vee\Big(lubC_{\widehat{£}}(\hbar), lubC_{\widehat{¥}}(\hbar)\Big)\Big],$$

$$\Big[\vee\Big(glbT_{\widehat{£}}(\hbar), glbT_{\widehat{¥}}(\hbar)\Big), \vee\Big(lubT_{\widehat{£}}(\hbar), lubT_{\widehat{¥}}(\hbar)\Big)\Big] : \hbar$$

$$\in W\Big\}$$

$$= \Big\{\hbar, \Big[\wedge\Big(glbF_{\widehat{£}}(\hbar), glbF_{\widehat{¥}}(\hbar)\Big), \wedge\Big(lubF_{\widehat{£}}(\hbar), lubF_{\widehat{¥}}(\hbar)\Big)\Big],$$

$$\Big[\wedge\Big(glbU_{\widehat{£}}(\hbar), glbU_{\widehat{¥}}(\hbar)\Big), \wedge\Big(lubU_{\widehat{£}}(\hbar), lubU_{\widehat{¥}}(\hbar)\Big)\Big],$$

$$\Big[\vee\Big(1 - lubG_{\widehat{£}}(\hbar), 1 - lubG_{\widehat{¥}}(\hbar)\Big),$$

$$\vee\Big(1 - glbG_{\widehat{£}}(\hbar), 1 - glbG_{\widehat{¥}}(\hbar)\Big)\Big],$$

$$\Big[\vee\Big(glbC_{\widehat{£}}(\hbar), glbC_{\widehat{¥}}(\hbar)\Big), \vee\Big(lubC_{\widehat{£}}(\hbar), lubC_{\widehat{¥}}(\hbar)\Big)\Big],$$

$$\Big[\vee\Big(glbT_{\widehat{£}}(\hbar), glbT_{\widehat{¥}}(\hbar)\Big), \vee\Big(lubT_{\widehat{£}}(\hbar), lubT_{\widehat{¥}}(\hbar)\Big)\Big] : \hbar$$

$$\in W\Big\}$$

$$= \Big\{\Big(\hbar, F_{\widehat{£}}(\hbar), U_{\widehat{£}}(\hbar), [1 - lubG_{\widehat{£}}(\hbar), 1 - glbG_{\widehat{£}}(\hbar)], C_{\widehat{£}}(\hbar), T_{\widehat{£}}(\hbar)\Big) : \hbar \in W\Big\}$$

$$\cap \Big\{\Big(\hbar, F_{\widehat{¥}}(\hbar), U_{\widehat{¥}}(\hbar), [1 - lubG_{\widehat{¥}}(\hbar), 1 - glbG_{\widehat{¥}}(\hbar)], C_{\widehat{¥}}(\hbar), T_{\widehat{¥}}(\hbar)\Big) : \hbar \in W\Big\}$$

$$= \widetilde{\widehat{£}}^{c} \cap \widetilde{\widehat{¥}}^{c}$$

$$\therefore \Big(\widetilde{\widehat{£}} \cup \widetilde{\widehat{¥}}\Big)^{c} = \widetilde{\widehat{£}}^{c} \cap \widetilde{\widehat{¥}}^{c}$$

Similarly, $\Big(\widetilde{\widehat{£}} \cap \widetilde{\widehat{¥}}\Big)^{c} = \widetilde{\widehat{£}}^{c} \cup \widetilde{\widehat{¥}}^{c}$.

(vi) We know that

$$\widetilde{\widehat{£}} \cup \Big(\widetilde{\widehat{£}} \cap \widetilde{\widehat{¥}}\Big)$$

$$= \widetilde{\widehat{£}} \cup \Big\{\hbar, \Big[\wedge\Big(glbT_{\widehat{£}}(\hbar), glbT_{\widehat{¥}}(\hbar)\Big), \wedge\Big(lubT_{\widehat{£}}(\hbar), lubT_{\widehat{¥}}(\hbar)\Big)\Big],$$

$$\left[\wedge\left(glbC_{\widehat{£}}(\hbar), glbC_{\widehat{¥}}(\hbar)\right),\right.$$

$$\left.\wedge\left(lubC_{\widehat{£}}(\hbar), lubC_{\widehat{¥}}(\hbar)\right)\right],$$

$$\left[\vee\left(glbG_{\widehat{£}}(\hbar), glbG_{\widehat{¥}}(\hbar)\right),\right.$$

$$\left.\vee\left(lubG_{\widehat{£}}(\hbar), \vee\, lubG_{\widehat{¥}}(\hbar)\right)\right],$$

$$\left[\vee\left(glbU_{\widehat{£}}(\hbar), glbU_{\widehat{¥}}(\hbar)\right),\right.$$

$$\left.\vee\left(lubU_{\widehat{£}}(\hbar), lubU_{\widehat{¥}}(\hbar)\right)\right],$$

$$\left[\vee\left(glbF_{\widehat{£}}(\hbar), glbF_{\widehat{¥}}(\hbar\right.\right.$$

$$\left.\left.\vee\left(lubF_{\widehat{£}}(\hbar), lubF_{\widehat{¥}}(\hbar)\right)\right] : \hbar \in W\right\}$$

$$= \left\{\hbar, \left[\vee\left(glbT_{\widehat{£}}(\hbar), \wedge\left(glbT_{\widehat{£}}(\hbar), glbT_{\widehat{¥}}(\hbar)\right)\right), \vee\left(lubT_{\widehat{£}}(\hbar), \wedge\left(lubT_{\widehat{£}}(\hbar), lubT_{\widehat{¥}}(\hbar)\right)\right)\right],\right.$$

$$\left[\vee\left(glbC_{\widehat{£}}(\hbar), \wedge\left(glbC_{\widehat{£}}(\hbar), glbC_{\widehat{¥}}(\hbar)\right)\right), \vee\left(lubC_{\widehat{£}}(\hbar), \wedge\left(lubC_{\widehat{£}}(\hbar), lubC_{\widehat{¥}}(\hbar)\right)\right)\right],$$

$$\left[\wedge\left(glbG_{\widehat{£}}(\hbar), \vee\left(glbG_{\widehat{£}}(\hbar), glbG_{\widehat{¥}}(\hbar)\right)\right), \wedge\left(lubG_{\widehat{£}}(\hbar), \vee\left(lubG_{\widehat{£}}(\hbar), \vee\, lubG_{\widehat{¥}}(\hbar)\right)\right)\right],$$

$$\left[\wedge\left(glbU_{\widehat{£}}(\hbar), \vee\left(glbU_{\widehat{£}}(\hbar), glbU_{\widehat{¥}}(\hbar)\right)\right), \wedge\ lubU_{\widehat{£}}(\hbar), \vee\left(lubU_{\widehat{£}}(\hbar), lubU_{\widehat{¥}}(\hbar)\right)\right],$$

$$\left[\wedge\left(glbF_{\widehat{£}}(\hbar), \vee\left(glbF_{\widehat{£}}(\hbar), glbF_{\widehat{¥}}(\hbar)\right)\right),\right.$$

$$\left.\left.\wedge\left(lubF_{\widehat{£}}(\hbar), \vee\left(lubF_{\widehat{£}}(\hbar), lubF_{\widehat{¥}}(\hbar)\right)\right)\right] : \hbar \in W\right\}$$

$$= \left\{\hbar, \left[glbT_{\widehat{£}}(\hbar), lubT_{\widehat{£}}(\hbar)\right], \left[glbC_{\widehat{£}}(\hbar), lubC_{\widehat{£}}(\hbar)\right], \left[glbG_{\widehat{£}}(\hbar), lubG_{\widehat{£}}(\hbar)\right],\right.$$

$$\left.\left[glbU_{\widehat{£}}(\hbar), lubU_{\widehat{£}}(\hbar)\right], \left[glbF_{\widehat{£}}(\hbar), lubF_{\widehat{£}}(\hbar)\right] : \hbar \in W\right\}$$

$$= \left\{\left(\hbar, T_{\widehat{£}}(\hbar), C_{\widehat{£}}(\hbar), G_{\widehat{£}}(\hbar), U_{\widehat{£}}(\hbar), F_{\widehat{£}}(\hbar)\right) : \hbar \in W\right\}$$

$$= \widehat{£}$$

$$\therefore \widehat{£} \cup \left(\widehat{£} \cap \widehat{¥}\right) = \widehat{£}$$

Similarly, $\widehat{£} \cap \left(\widehat{£} \cup \widehat{¥}\right) = \widehat{£}$.

(vii) $\left(\widehat{£}^{c}\right)^{c}$

$$= \left\{\left(\hbar, F_{\widehat{£}}(\hbar), U_{\widehat{£}}(\hbar), \left[1 - lubG_{\widehat{£}}(\hbar), 1 - glbG_{\widehat{£}}(\hbar)\right], C_{\widehat{£}}(\hbar), T_{\widehat{£}}(\hbar)\right) : \hbar \in W\right\}^{c}$$

$$= \left\{ \left(\hbar, T_{\widehat{£}}\ (\hbar), C_{\widehat{£}}\ (\hbar), \left[1 - \left(1 - glbG_{\widehat{£}}(\hbar)\right), 1 - \left(1 - lubG_{\widehat{£}}(\hbar)\right)\right], U_{\widehat{£}}\ (\hbar), F_{\widehat{£}}\ (\hbar) \right) : \hbar \in W \right\}$$

$$= \left\{ \left(\hbar, T_{\widehat{£}}\ (\hbar), C_{\widehat{£}}\ (\hbar), \left[glbG_{\widehat{£}}(\hbar), lubG_{\widehat{£}}(\hbar)\right], U(\hbar), F_{\widehat{£}}\ (\hbar) \right) : \hbar \in W \right\}$$

$$= \left\{ \left(\hbar, T_{\widehat{£}}\ (\hbar), C(\hbar), G_{\widehat{£}}(\hbar), U_{\widehat{£}}\ (\hbar), F_{\widehat{£}}\ (\hbar) \right) : \hbar \in W \right\}$$

$$= \widehat{£}^{c}$$

$$\therefore \left(\widehat{£}^{c}\right)^{c} = \widehat{£}$$

4. Conclusion

This research includes the idea of $IVPNS$s. Also some important properties of $IVPNS$s have been studied along with examples. A real life example of COVID-19 has been discussed in the paper using $IVPNS$. Some more operations along with aggregation operators on $IVPNS$s can be studied in future with the help of important results obtained here. Further while making decision like MCDM [6], $IVPNS$s also applicable to deal with uncertain observation.

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A New Approach to Neutrosophic Soft Sets and their Application in Decision Making

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Abstract: In literature, several models which can handle uncertainty in datasets have been introduced. Fuzzy set introduced by Zadeh in 1965, is one of the earliest such models and Atanassov generalised it by introducing the notion of Intuitionistic fuzzy sets(IFS) in 1986. However, these models are handicaped due to their inadequacy as parameterization tools. The notion of Soft sets (SS) was introduced by Molodtsov in 1999 to solve this problem. Almost at the same time, Neutrosophic set (NS) model was introduced by Smarandache, which is a huge generalisation of IFS. As has been the practice, the hybrid model of SS and NS was proposed to frame the notion of Neutrosophic Soft Set (NSS) by Ali and Smaranche in 2015 and studied their properties. Since its inception, one of the major areas of application of Soft Sets has been that of Multi-criterian Decision Making (MCDM). Many problems in MCDM were solved by using hybrid models of SS. Following this trend, in this paper, we develop an algorithm basing upon NSS to handle the problem of MCDM in the selection of faculty through an interview prcess. For this, we had to introduce an improved score function which is used to rank the candidates basing upon several of their characteristics including interview perfromances. This application is better handled by the NSS model as is evident from the results. We illustrated the superiority of our proposed algorithm by providing a comparative analysis with many exieting algorithms in the literature.

Keywords: Fuzzy Set; Intuitionistic fuzzy set; Soft Set; Neutrosophic Set; Neutrosophic soft set; Multicriteria Decision Making.

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1. Introduction

The notion of FSs [1] is one of the most popular mathematical model to handle uncertainty and vagueness. In contrast to classical notion of sets where the elements of the set are characterised by either “does not belongs” or “belongs” to a set; notion of FSs provides a grade of membership to each element through a membership function. Sometimes, in real-life situations, it is not easy to define membership functions. So, to capture more uncertainty, Zadeh proposed the notion of interval valued fuzzy sets [2]. However, there are situations exist, where grade of membership is not complement to non-membership values. Both the notions of fuzzy set and the notions of interval valued fuzzy sets can not capture such kind of uncertainty. To handle such kind of scenarios, Atanassov [3, 4] introduced the notions of IFS where hesitation function comes into picture, if the membership and non-mebership are not complement of eachother. An IFS becomes a fuzzy set when the hesitation becomes zero. Similarly a fuzzy set becomes a classical set, if the membership value is restricted to either one or zero. Atanassov [5] further generalised the concept of IFS and introduced interval valued IFS.

In all the uncertainty based models discussed above, we need to define a membership function. However, by using a single mebership function is not enough to handle all kind of uncertainties involve in some situations. It will create sutuations like adding weight parameter to a length parameter. It happens due to lack of parameterization tools in the previous models. To handle such issues, Moldtsov [6] introduced the notion of soft set in 1999. It adds topological features to the notions of set theory. In soft sets, each one of the parmeters from a parameter set are associated with a subset of the universe of discourse. Recently, soft sets and its hybrid models are gaining popularity for their ability to handle uncertainty in multicriteria decision making. Due to its topological nature and availability of parameterization tool, its easier and convinient to capture uncertainty in decision making problems using soft sets or any of its hybrid models.

Tripathy et al [7] redefined the notions soft set using characteristic functions approach which seems to be more convenient and easy to understand incomparison to its previous models. Later, there are many more hybrids models were proposed using the same concept [8-17]. There are several papers published on hybrid models of soft set and their application in MCDM.

One more contemporaries of the notions of soft set is the concept NSs [18] proposed by smarandache. It is a generalization of IFSs. In contrast to other generalisations or hybrid models of IFSs; in NSs, the membership, non-mebership and hesitation functions are independent of each other. So, the sum of the grades of Truthness, Falsity and Indeterminacy can vary in the interval $[0^-, 3^+]$. There are several articles on NSs and its hybrid models in literature to solve multicriteria decision-making problem [19-30].

Maji [31] introduced the concept of NSSs.

This paper provides a new approach to redefine the notions of NSS. It redefines some operations of NSS using characterastic function approach [7, 8]. An application in decision making using NSS is also discussed in this article.

2. Definitions and Notions

To understand the proposed model, we need to understand some prerequisite models which are discussed in this section.

Let U be a universal set and E be a set of parameters.

2.1. Soft Set

A soft set is a collection of parameterized family of subsets. A soft set over U is denoted by (F, E) and is defined as

$$F : E \to P(U)$$

where $P(U)$ is the power set of U .

2.2 Fuzzy Set

A fuzzy set A drawn from U is given by its membership function μ_A where $\mu_A : U \to [0,1]$ such that $\forall x \in U$, $\mu_A(x)$ is the grade of membership of x in A . A fuzzy set reduces to a crisp set when $\mu_A : U \to \{0,1\}$.

2.3 Fuzzy Soft Set

A fuzzy soft set over U is denoted by (F_m, E) and is defined as

$$F_m : E \to FP(U)$$

where $FP(U)$ is the set of all fuzzy subsets of U.

2.4 Intuitionistic fuzzy set

An IFS A over a universe of discourse U is a pair (m_A, n_A), where $m_A : U \to [0,1]$ and $n_A : U \to [0,1]$, called the membership and non-membership functions of A respectively are such that for any $x \in U$, $0 \le m_A(x) + n_A(x) \le 1$.

The function given by $\pi_A(x) = 1 - m_A(x) - n_A(x)$ is called the hesitation function associated with A.

2.5 Intuitionistic fuzzy soft set

An IFSS (F, E) over a universal discourse U is defined as

$$F : E \to IFP(U)$$

Where, $IFP(U)$ is the powerset of all IFSs in U.

3. Neutrosophic Sets

A neutrosophic set B over a universe of discourse U is defined as $B = \{\langle x, T_B(x), I_B(x), F_B(x)\rangle, \forall x \in U\}$, where $(T_B, I_B, F_B) : U \to]0^-, 1^+[$. T_B, I_B, F_B are called as the Truthness, Indeterminacy and Falsity membership functions of B respectively.

In real life engineering applications, it is difficult to use non-standard real values. Hence in this article, the value range for the NSs are restricted to the subsets of $[0,1]$.

3.1 Neutrosophic subset

A neutrosophic set B is said to be a neutrosophic subset of A denoted by $B \subseteq A$ iff $\forall x \in U$, $T_B(x) \le T_A(x), I_B(x) \ge I_A(x), F_B(x) \ge F_A(x)$.

3.2 Union of two NS

Union of two NSs A and B denoted by $A \bigcup B$ is defined as

$$A \bigcup B = \{\max(T_A, T_B), \min(I_A, I_B), \min(F_A, F_B)\}$$

3.3 Intersection of two NS

Intersection of two NSs A and B denoted by $A \bigcup B$ is defined as

$$A \bigcap B = \{\min(T_A, T_B), \max(I_A, I_B), \max(F_A, F_B)\}$$

4. Neutrosophic Soft Set

A neutrosophic set (F_B, E) over a universe of discourse U is defined as

$$F : E \to NPow(U)$$

Where, $NPow(U)$ is the neutrosophic powerset of U .

A NSS (F_B, E) can also be defined using membership function approach (Tripathy et. al, 2016) as follows.

The set of parametric membership functions of NSS (F_B, E) defined over (U, E) as shown below.

$(F_B, E) = \{(F_B, E)(e) : e \in E\}$ such that $\forall e \in E$, $\left\{T_{(F_B,E)(e)}, I_{(F_B,E)(e)}, F_{(F_B,E)(e)}\right\} : U \to]0^-, 1^+[$

and $\forall x \in U$, the membership function is defined as

$$T_{(F_B,E)(e)}(x) = \alpha,\ \alpha \in]0^-, 1^+[\ ;$$

$$I_{(F_B,E)(e)}(x) = \beta,\ \beta \in]0^-, 1^+[\ ; \text{ and}$$

$$F_{(F_B,E)(e)}(x) = \lambda,\ \lambda \in]0^-, 1^+[\ .$$

4.1 Neutrosophic soft subset

A NSS (F_B, E) is said to be a neutrosophic soft subset of (F_A, E) denoted by $(F_B, E) \subseteq (F_A, E)$ if $\forall x \in U$, $T_{(F_B,E)(e)}(x) \le T_{(F_A,E)(e)}(x)$, $I_{(F_B,E)(e)}(x) \ge I_{(F_A,E)(e)}(x)$, $F_{(F_B,E)(e)}(x) \ge F_{(F_A,E)(e)}(x)$.

4.2 Union of two Neutrosophic soft Sets

Union of two NSSs (F_A, E) and (F_B, E) denoted by $(F_{A\cup B}, E)$ is defined as

$$(F_{A\cup B}, E) = \left\{ \begin{array}{l} \max\left(T_{(F_A,E)(e)}(x), T_{(F_B,E)(e)}(x)\right), \min\left(I_{(F_A,E)(e)}(x), I_{(F_B,E)(e)}(x)\right), \\ \min\left(F_{(F_A,E)(e)}(x), F_{(F_B,E)(e)}(x)\right) \end{array} \right\}$$

4.3 Intersection of two Neutrosophic soft Sets

Intersection of two NSSs A and B denoted by $A \bigcup B$ is defined as

$$(F_{A\cap B}, E) = \left\{ \begin{array}{l} \min\left(T_{(F_A,E)(e)}(x), T_{(F_B,E)(e)}(x)\right), \max\left(I_{(F_A,E)(e)}(x), I_{(F_B,E)(e)}(x)\right), \\ \max\left(F_{(F_A,E)(e)}(x), F_{(F_B,E)(e)}(x)\right) \end{array} \right\}$$

4. Application of Neutrosophic Soft Sets in Decision Making

An application of NSSs in multicriteria decision making is provided in this section. As a generalised model of IFS [3], NS [18] inherently a good mathematical model to handle uncertainty. Molodtsov [6] has given many applications of soft sets in the introductory article. Recently, hybrid models of soft sets are among the popular models to handle multicriteria decision making problems. There are everal articles in literature using NSS model to handle multicriteria decision making problems. This article provides a new approach for decision making using notions of NSS.

There are two types of parameters (Tripathy et al. 2016),
i) Positive Parameters and
ii) Negative Parameters.
A parameter which is having positive impact on decision making is called as positive parameter and if the parameter is having negative impact on decision-making, then that is called as negative parameter.

A priority value is expressed through a real number lies in [-1, 1] and is attached to each parameter as per the degree of impact of the parameter on the user's decision-making. For positive parameters the priority value lies in [0,1] and for the negative parameter the priority value lies in [-1,0].

Sometimes we may have a parameter which is given zero priority by the user irrespective of the type of the category of the parameter that can be either positive or negative. Though these parameters would not affect user's decision-making usually, but the effect comes into picture during close comparisons. For example, one can say, if everything is good, price does not matter. But, if two same things are available with different prices, everyone will choose the thing with lower price. These kinds of situations are ignored in the existing approaches. In this paper, these kinds of situations managed by giving a very low priority value which won't affect the decision choices until there is a close comparison.

A small user defined value d is used in the application, which helps to maintain better precision in results. In this application, value of d is taken as 0.001. To manage parameters with zero priority, a small priority value is attached to the parameter instead of zero. The formula to compute the priority value to be attached with a parameter having zero priority is given below.

$$p_0 = \frac{sign(\text{User's Priority}(P_n)) \times d}{\sum_{1}^{n} Abs(\text{User's Priority}(P_n))} \tag{4.1}$$

Where, Abs $\rightarrow$ Absolute value
Sign $\rightarrow$ Signum Function

To make comparisons among different sets values, the values need to be normalized. In this paper, the formula used for normalizing values is given below.

$$\text{Normalized priority} = \frac{P_n}{\sum_{1}^{n} \text{Priority}(P_n)} \tag{4.2}$$

To compare a series of values $V_i, i = 1, 2, ..., n$ and get a comparison value; the following formula can be used.

$$Comparison\ Score(V_i) = nV_i - \sum_{j=1}^{n} V_j \quad (4.3)$$

for $i, j = 1, 2, ..., n$.

To use NSs in decision making, a score function is needed to compute the score and order the neutrosophic values. The formula given in Equation (4.4) is used to compute the score from a neutrosophic value.

$$NS\_Score(T, I, F) = ((T*(1+2d)) - F + dI\left[1+\min\{1, T+(I/2)\} - \min\{1, F+(I/2)\}\right]$$

$$Score = \begin{cases} \dfrac{NS\_Score(T, I, F)}{1+\frac{7}{2}d}, & if\ NS\_Score(T, I, F) > 0; \\ NS\_Score(T, I, F), & Otherwiese. \end{cases} \quad (4.4)$$

where, d $\rightarrow$ very small positive real number. (In this paper d =0.01)
T, I, F $\rightarrow$ Represents Truthness, Indeterminacy and Falsity values, respectively.
$Score(T, F, I)$ $\rightarrow$ Score function for the Neutrosophic value (T, F, I).

The formula in (4.4) provides a real number from a particular neutrosophic value. This formula will be extremely helpful to resolve neutrosophic decision making problems. The formula will reduce a neutrosophic set problem to a bipolar fuzzy set problem. The basic structure of the formula is $T - F + I*(\min(1, T+\frac{I}{2}) - \min(1, F+\frac{I}{2}))$. The formula is based on optimistic approach. So, the truth value is boosted by a small margin to tackle the problem when $T = F$. To reduce the effect of $I*(\min(1, T+\frac{I}{2}) - \min(1, F+\frac{I}{2}))$ value, so that , it would not overshadow the T value which may lead to wrong decision making, it is multiplied by a small positive real number d. Value of d is taken as 0.01 in this article.

T	I	F	Score
1	1	0	1
1	0.8	0	0.99787
1	0.6	0	0.99536
1	0.4	0	0.99246
1	0.2	0	0.98918
1	0	0	0.98551
1	0.2	0.2	0.8234
1	0.4	0.4	0.6256
1	0.6	0.6	0.4266
1	0.8	0.8	0.228
1	1	1	0.03
1	0.8	1	0.028
1	0.6	1	0.026
1	0.4	1	0.024
1	0.2	1	0.022
1	0	1	0.02
0.8	0.2	0.8	0.018
0.6	0.4	0.6	0.016
0.4	0.6	0.4	0.014
0.2	0.8	0.2	0.012
0	1	0	0.01
0	0.8	0	0.008
0	0.6	0	0.006
0	0.4	0	0.004
0	0.2	0	0.002
0	0	0	0
0	0.2	0.2	-0.1984
0	0.4	0.4	-0.3976
0	0.6	0.6	-0.5976
0	0.8	0.8	-0.7968
0	1	1	-0.995
0	0.8	1	-0.9968
0	0.6	1	-0.9982
0	0.4	1	-0.9992
0	0.2	1	-0.9998
0	0	1	-1

Figure 1. Score from a neutrosophic value

In Figure 1, we can see that the score function is working fine and giving an intuitive score for each of the value. The graph in Figure 1 seems inverted "Z" shape due to the limited number of data points in that region. If we provide a greater number of data points in the graph, it gives a smoother line.

Graphs in Figure 2 provide a perspective in the change of score value with a constant value of either T or F.

T	I	F	NS Score (T Constant)
0.6	0	0	0.6
0.6	0.2	0	0.6
0.6	0.4	0	0.6
0.6	0.6	0	0.6
0.6	0.8	0	0.6
0.6	1	0	0.6
0.6	0	0.2	0.4
0.6	0.2	0.2	0.4
0.6	0.4	0.2	0.4
0.6	0.6	0.2	0.4
0.6	0.8	0.2	0.4
0.6	1	0.2	0.4
0.6	0	0.4	0.2
0.6	0.2	0.4	0.2
0.6	0.4	0.4	0.2
0.6	0.6	0.4	0.2
0.6	0.8	0.4	0.2
0.6	1	0.4	0.2
0.6	0	0.6	0
0.6	0.2	0.6	0
0.6	0.4	0.6	0
0.6	0.6	0.6	0
0.6	0.8	0.6	0
0.6	1	0.6	0
0.6	0	0.8	-0.2
0.6	0.2	0.8	-0.2
0.6	0.4	0.8	-0.2
0.6	0.6	0.8	-0.2
0.6	0.8	0.8	-0.2
0.6	1	0.8	-0.2
0.6	0	1	-0.4
0.6	0.2	1	-0.4
0.6	0.4	1	-0.4
0.6	0.6	1	-0.4
0.6	0.8	1	-0.4
0.6	1	1	-0.4

T	I	F	NS Score (F Constant)
0	0	0.6	-0.6
0.2	0	0.6	-0.6
0.4	0	0.6	-0.6
0.6	0	0.6	-0.6
0.8	0	0.6	-0.6
1	0	0.6	-0.6
0	0.2	0.6	-0.4
0.2	0.2	0.6	-0.4
0.4	0.2	0.6	-0.4
0.6	0.2	0.6	-0.4
0.8	0.2	0.6	-0.4
1	0.2	0.6	-0.4
0	0.4	0.6	-0.2
0.2	0.4	0.6	-0.2
0.4	0.4	0.6	-0.2
0.6	0.4	0.6	-0.2
0.8	0.4	0.6	-0.2
1	0.4	0.6	-0.2
0	0.6	0.6	0
0.2	0.6	0.6	0
0.4	0.6	0.6	0
0.6	0.6	0.6	0
0.8	0.6	0.6	0
1	0.6	0.6	0
0	0.8	0.6	0.2
0.2	0.8	0.6	0.2
0.4	0.8	0.6	0.2
0.6	0.8	0.6	0.2
0.8	0.8	0.6	0.2
1	0.8	0.6	0.2
0	1	0.6	0.4
0.2	1	0.6	0.4
0.4	1	0.6	0.4
0.6	1	0.6	0.4
0.8	1	0.6	0.4
1	1	0.6	0.4

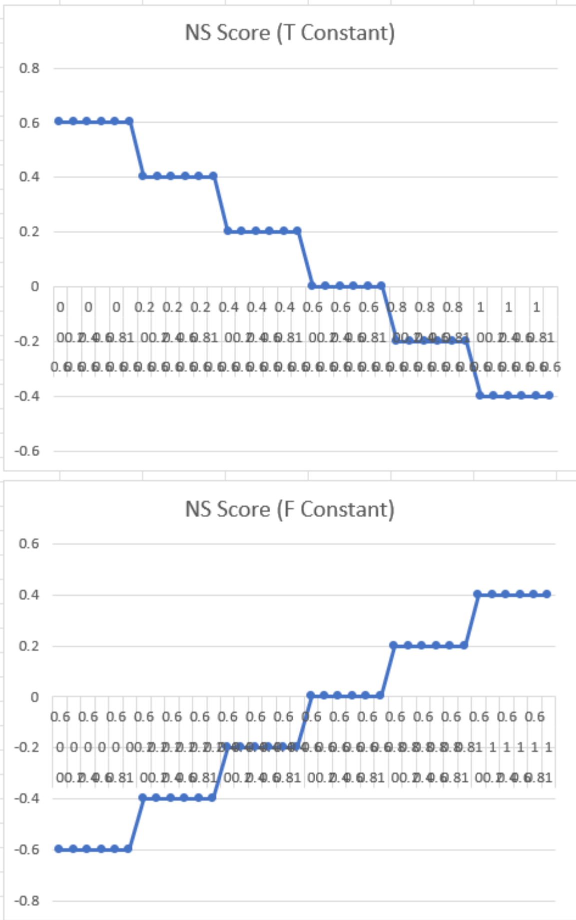

Figure 2. Score comparison when either T or F value is constant.

4.1 Algorithm

Step 1: Get the priority values of parameters from the user.

Step 1.1: Compute the priority for the parameters having zero priority using the formula in Equation (4.1).

Step 1.2: Compute the normalized priority using the formula in Equation (4.2).

Step 1.3: Rank the parameters as per their absolute priority values.

Step 2: Get the data in neutrosophic format.

Step 2.1: Construct the Truthness Table, Indeterminacy Table and Falsity Table by Segregating the columns of Truthness, Indeterminacy and Falsity values for each parameter.

Step3: Construct the Truthness Priority Table, Indeterminacy Priority Table and Falsity Priority Table by multiplying the priority values to their corresponding Truthness, Indeterminacy and Falsity values.

Step 4: Construct the Comparison Tables for Truthness, Indeterminacy and Falsity values by using the formula given in Equation (4.3) for each column.

Step 4.1: Get the comparison score, by computing the sum of the comparison scores for each competitor.

Step 4.2: Normalize comparison scores of all comparison tables using the formula in Equation (4.2).

Step 5: Construct the decision table by taking the normalized scores from comparison tables for Truthness, Indeterminacy and Falsity values.

Step 5.1: Compute the neutrosophic score by using the formula in the Equation (4.4)

Step 5.2: Rank the competitors according to their final score (neutrosophic score).

Step 5.3: If multiple participants are getting same score, for those participants with same score, repeat all the previous steps ignoring the parameter having lowest rank. Continue the process until all participants getting a distinct rank or reaching the comparison with only the values for the highest ranked parameter.

4.2 Application

The application provided here is selection of faculties in an interview process.

The parameters considered for the selection are Teaching, Research, Academic, Presentation, Subject Knowledge, Communication Skill, Gaps, Body Language, Nativity. The parameters are represented respectively as a set of parameters $\rho=\{\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6,\rho_7,\rho_9\}$.

Let us assume that there are 10 participants given by $U=\{\square_1,\square_2,\square_3,\square_4,\square_5,\square_6,\square_7,\square_9,\square_{10}\}$ shortlisted after the interview. The authorities assign the priority for each parameter as per their requirements. If there are any parameters having zero priority, a small priority needs to be assigned that can be computed using formula in Equation (4.1). Normalize the priority values using formula in Equation (4.2). Rank the parameters by their absolute priority values. Because, the priority becomes negative due to the negative parameter, but the effect of the priority value remains same irrespective of the type of the parameter. Table 1 shows all the data about the parameters and the priorities assigned to those parameters.

Parameter Rank: Parameters are ranked as per their absolute priority value. In Table 1, it can be noticed that the User's priority for the parameter Gaps (ρ_7) is a negative number, because the parameter Gaps is a negative parameter. But the significance of a parameter in decision making is depends on its absolute priority value. So, the parameters ρ_6,ρ_7,ρ_8 are having same parameter rank. Parameter ranks plays a vital role to resolve the problem when two participants are having same final score. In that case, we can ignore one or more less significant parameters.

Table 1. Parameter Table

Parameters	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8	ρ_9
User's Priority	.20	.40	.30	.20	.30	.10	-0.10	.10	0
Handling Zero Priority	.20	.40	.30	.20	.30	.10	-0.10	.10	.01
Normalized Priority	.117	.234	.176	.117	.176	.059	-0.059	.059	.003
Parameter Rank	4	1	2	4	2	6	6	6	9

The quality of all the participants evaluated and can be represented in a NSS as shown in Table 2. Construct the Truthness Table, Indeterminacy Table and Falsity Table (Tables 3, 4, 5) by segregating the columns of Truthness, Indeterminacy and Falsity values for every parameter.

Construct the Truthness Priority Table, Indeterminacy Priority Table and Falsity Priority Table (Tables 6, 7, 8) by multiplying the priority values to their corresponding Truthness, Indeterminacy and Falsity values. Priority Tables are having both -ve and +ve real numbers. Because, after multiplying with priority values, the data are not necessarily positive. It can be any real number.

Construct the Comparison Tables for Truthness, Indeterminacy and Falsity values by using the formula given in Equation (4.3) for each column (Tables 9,10,11). Get the comparison score, by computing the sum of the comparison scores for each competitor. Normalize comparison scores of all comparison tables using the formula in Equation (4.2).

Table 2. Data in Neutrosophic soft set model

ρ	ρ_1			ρ_2			ρ_3			ρ_4			ρ_5			ρ_6			ρ_7			ρ_8			ρ_9		
U	T	I	F	T	I	F	T	I	F	T	I	F	T	I	F	T	I	F	T	I	F	T	I	F	T	I	F
□ 1	.09	.7	.28	.25	.25	.07	.48	.19	.75	.62	.64	.07	.18	.97	.25	.86	.28	.51	.85	.16	.9	.12	.79	.71	.56	.71	.97
□ 2	.68	.33	.13	.94	.98	.93	.41	.79	.88	.69	.69	.95	.48	.87	.81	.25	.75	.43	.19	.7	.74	.83	.23	.22	.81	.06	.08
□ 3	.79	.57	.25	.84	.45	.96	.06	.4	.41	.5	.47	.56	.18	.82	.23	.62	.28	.9	.09	.24	.63	.02	.33	.22	.38	.84	.02
□ 4	.01	.57	.27	.44	.3	.66	.85	.59	.32	.99	.98	.88	.55	.11	.85	.68	.88	.95	.49	.38	.24	.67	.83	.85	.17	.36	.89
□ 5	.87	.02	.91	.66	.97	.86	.62	.68	.24	.71	.45	.35	.97	.14	.96	.39	.76	.74	.76	.22	.53	.69	.29	.7	.26	.13	.9
□ 6	.61	.27	.31	.55	.6	.3	.95	.17	.5	.36	.37	.51	.14	.7	.19	.38	.73	.04	.57	.69	.53	.22	.77	.47	.37	.88	.24
□ 7	.96	.82	.33	.49	.49	.34	.65	.02	.71	.61	.07	.66	.71	.68	.17	.46	.84	.12	.83	.14	.73	.16	.24	.29	.83	.86	.83
□ 8	.04	.34	.02	.39	.56	.76	.07	.12	.48	.85	.73	.28	.53	.38	.32	.23	.98	.32	.78	.11	.2	.49	.23	.55	.09	.35	.29
□ 9	.19	.55	.1	.97	.67	.88	.37	.81	.37	.53	.24	.84	.84	.19	.99	.5	.4	.45	.6	.05	.95	.9	.72	.74	.01	.52	.59
□ 10	.05	.79	.01	.82	.05	.8	.48	.56	.38	.13	.11	.71	.96	.7	.26	.81	.5	.72	.2	.56	.47	.18	.85	.55	.53	.24	.18

Table 3. Truthness Table

(U,ρ)	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8	ρ_9
□ 1	.09	.25	.48	.62	.18	.86	.85	.12	.56
□ 2	.68	.94	.41	.69	.48	.25	.19	.83	.81
□ 3	.79	.84	.06	.5	.18	.62	.09	.02	.38
□ 4	.01	.44	.85	.99	.55	.68	.49	.67	.17
□ 5	.87	.66	.62	.71	.97	.39	.76	.69	.26
□ 6	.61	.55	.95	.36	.14	.38	.57	.22	.37
□ 7	.96	.49	.65	.61	.71	.46	.83	.16	.83
□ 8	.04	.39	.07	.85	.53	.23	.78	.49	.09
□ 9	.19	.97	.37	.53	.84	.5	.6	.9	.01
□ 10	.05	.82	.48	.13	.96	.81	.2	.18	.53

Table 4. Indeterminacy Table

(U,ρ)	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8	ρ_9
□ 1	.7	.25	.19	.64	.97	.28	.16	.79	.71
□ 2	.33	.98	.79	.69	.87	.75	.7	.23	.06
□ 3	.57	.45	.4	.47	.82	.28	.24	.33	.84
□ 4	.57	.3	.59	.98	.11	.88	.38	.83	.36
□ 5	.02	.97	.68	.45	.14	.76	.22	.29	.13
□ 6	.27	.6	.17	.37	.7	.73	.69	.77	.88
□ 7	.82	.49	.02	.07	.68	.84	.14	.24	.86
□ 8	.34	.56	.12	.73	.38	.98	.11	.23	.35
□ 9	.55	.67	.81	.24	.19	.4	.05	.72	.52
□ 10	.79	.05	.56	.11	.7	.5	.56	.85	.24

Table 5. Falsity Table

(U,ρ)	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8	ρ_9
□ 1	.28	.07	.75	.07	.25	.51	.9	.71	.97
□ 2	.13	.93	.88	.95	.81	.43	.74	.22	.08
□ 3	.25	.96	.41	.56	.23	.9	.63	.22	.02
□ 4	.27	.66	.32	.88	.85	.95	.24	.85	.89
□ 5	.91	.86	.24	.35	.96	.74	.53	.7	.9
□ 6	.31	.03	.5	.51	.19	.04	.53	.47	.24
□ 7	.33	.34	.71	.66	.17	.12	.73	.29	.83
□ 8	.02	.76	.48	.28	.32	.32	.2	.55	.29
□ 9	.1	.88	.37	.84	.99	.45	.95	.74	.59
□ 10	.01	.8	.38	.71	.26	.72	.47	.55	.18

Table 6. Priority Table for Truthness

(U,ρ)	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8	ρ_9
□ 1	.011	.059	.084	.073	.032	.050	-0.050	.007	.002
□ 2	.080	.220	.072	.081	.084	.015	-0.011	.049	.003
□ 3	.093	.197	.011	.059	.032	.036	-0.005	.001	.001
□ 4	.001	.103	.149	.116	.097	.040	-0.029	.039	.001
□ 5	.102	.155	.109	.083	.171	.023	-0.045	.040	.001
□ 6	.072	.129	.167	.042	.025	.022	-0.033	.013	.001
□ 7	.113	.115	.114	.072	.125	.027	-0.049	.009	.003
□ 8	.005	.091	.012	.100	.093	.013	-0.046	.029	.000
□ 9	.022	.227	.065	.062	.148	.029	-0.035	.053	.000
□ 10	.006	.192	.084	.015	.169	.047	-0.012	.011	.002

Table 7. Priority Table for Indeterminacy

(U,ρ)	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8	ρ_9
□ 1	.082	.059	.033	.075	.171	.016	-0.009	.046	.002
□ 2	.039	.230	.139	.081	.153	.044	-0.041	.013	.000
□ 3	.067	.106	.070	.055	.144	.016	-0.014	.019	.003
□ 4	.067	.070	.104	.115	.019	.052	-0.022	.049	.001
□ 5	.002	.227	.120	.053	.025	.045	-0.013	.017	.000
□ 6	.032	.141	.030	.043	.123	.043	-0.040	.045	.003
□ 7	.096	.115	.004	.008	.120	.049	-0.008	.014	.003
□ 8	.040	.131	.021	.086	.067	.057	-0.006	.013	.001
□ 9	.064	.157	.142	.028	.033	.023	-0.003	.042	.002
□ 10	.093	.012	.098	.013	.123	.029	-0.033	.050	.001

Table 8. Priority Table for Falsity

(U,ρ)	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8	ρ_9
□ 1	.033	.016	.132	.008	.044	.030	-0.053	.042	.003
□ 2	.015	.218	.155	.111	.142	.025	-0.043	.013	.000
□ 3	.029	.225	.072	.066	.040	.053	-0.037	.013	.000
□ 4	.032	.155	.056	.103	.149	.056	-0.014	.050	.003
□ 5	.107	.202	.042	.041	.169	.043	-0.031	.041	.003
□ 6	.036	.007	.088	.060	.033	.002	-0.031	.028	.001
□ 7	.039	.080	.125	.077	.030	.007	-0.043	.017	.003
□ 8	.002	.178	.084	.033	.056	.019	-0.012	.032	.001
□ 9	.012	.206	.065	.098	.174	.026	-0.056	.043	.002
□ 10	.001	.188	.067	.083	.046	.042	-0.028	.032	.001

Table 9. Comparison Table for Truthness

(U,ρ)	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8	ρ_9	**Score**	**Normalized Score**
□ 1	-0.397	-0.903	-0.025	0.025	-0.658	0.200	-0.184	-0.181	0.005	-2.117	0.038
□ 2	0.294	0.715	-0.148	0.107	-0.130	-0.157	0.203	0.236	0.014	1.134	0.808
□ 3	0.423	0.481	-0.763	-0.116	-0.658	0.060	0.261	-0.239	-0.001	-0.552	0.409
□ 4	-0.491	-0.457	0.626	0.458	-0.007	0.095	0.027	0.142	-0.008	0.385	0.631
□ 5	0.517	0.059	0.222	0.130	0.732	-0.075	-0.131	0.154	-0.005	1.601	0.919
□ 6	0.212	-0.199	0.802	-0.280	-0.728	-0.081	-0.020	-0.122	-0.001	-0.417	0.441
□ 7	0.623	-0.340	0.274	0.013	0.274	-0.034	-0.172	-0.157	0.015	0.495	0.657
□ 8	-0.456	-0.574	-0.746	0.294	-0.042	-0.169	-0.143	0.036	-0.011	-1.810	0.110
□ 9	-0.280	0.786	-0.218	-0.081	0.503	-0.011	-0.038	0.277	-0.013	0.924	0.759
□ 10	-0.444	0.434	-0.025	-0.550	0.714	0.171	0.197	-0.145	0.004	0.356	0.624

Table 10. Comparison Table for Indeterminacy

(U,ρ)	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8	ρ_9	**Score**	**Normalized Score**
□ 1	0.239	-0.661	-0.427	0.193	0.728	-0.211	0.097	0.154	0.007	0.119	0.568
□ 2	-0.195	1.050	0.628	0.252	0.552	0.064	-0.220	-0.175	-0.015	1.943	1.000
□ 3	0.087	-0.192	-0.058	-0.006	0.464	-0.211	0.050	-0.116	0.012	0.029	0.546
□ 4	0.087	-0.544	0.276	0.592	-0.784	0.141	-0.032	0.177	-0.005	-0.093	0.518
□ 5	-0.558	1.027	0.434	-0.029	-0.732	0.070	0.062	-0.140	-0.013	0.122	0.568
□ 6	-0.265	0.159	-0.463	-0.123	0.253	0.053	-0.214	0.142	0.013	-0.444	0.434
□ 7	0.380	-0.098	-0.726	-0.475	0.218	0.117	0.108	-0.169	0.013	-0.632	0.390
□ 8	-0.183	0.066	-0.550	0.299	-0.310	0.199	0.126	-0.175	-0.005	-0.533	0.413
□ 9	0.063	0.324	0.663	-0.276	-0.644	-0.141	0.161	0.113	0.001	0.265	0.602
□ 10	0.345	-1.130	0.223	-0.428	0.253	-0.082	-0.138	0.189	-0.009	-0.777	0.355

Table 11. Comparison Table for Falsity

(U,ρ)	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8	ρ_9	**Score**	**Normalized Score**
□ 1	0.022	-1.311	0.433	-0.599	-0.445	-0.005	-0.181	0.106	0.016	-1.963	0.074
□ 2	-0.154	0.706	0.661	0.433	0.540	-0.052	-0.087	-0.182	-0.014	1.851	0.978
□ 3	-0.013	0.776	-0.165	-0.025	-0.480	0.224	-0.022	-0.182	-0.017	0.097	0.562
□ 4	0.011	0.073	-0.324	0.351	0.610	0.253	0.206	0.188	0.013	1.381	0.867
□ 5	0.761	0.542	-0.464	-0.271	0.804	0.130	0.036	0.100	0.014	1.651	0.931
□ 6	0.057	-1.405	-0.007	-0.083	-0.550	-0.280	0.036	-0.035	-0.009	-2.276	0.000
□ 7	0.081	-0.678	0.362	0.093	-0.586	-0.233	-0.081	-0.141	0.011	-1.171	0.262
□ 8	-0.283	0.307	-0.042	-0.353	-0.322	-0.116	0.230	0.012	-0.007	-0.574	0.403
□ 9	-0.189	0.589	-0.236	0.304	0.856	-0.040	-0.210	0.123	0.003	1.201	0.824
□ 10	-0.294	0.401	-0.218	0.151	-0.427	0.118	0.072	0.012	-0.011	-0.197	0.493

Construct the decision table by taking the normalized scores from comparison tables for Truthness, Indeterminacy and Falsity values (Table 12). Compute the neutrosophic score by using the formula in the Equation (4.4). Rank the competitors according to their final score (neutrosophic score). If multiple participants are getting same score, for those participants with same score, repeat all the previous steps ignoring the parameter having lowest rank. Continue the process until all participants getting a distinct rank or reaching the comparison with only the values for the highest ranked parameter.

Table 12. Decision Table

Candidates	Truthness Score	Indeterminacy Score	Falsity Score	Neutrosophic Score	Rank
□ 1	0.0377	0.5676	0.0741	-0.0301	5
□ 2	0.8082	1.0000	0.9783	-0.1440	8
□ 3	0.4087	0.5464	0.5624	-0.1409	7
□ 4	0.6307	0.5175	0.8668	-0.2189	9
□ 5	0.9190	0.5684	0.9308	0.0123	4
□ 6	0.4405	0.4342	0.0000	0.4556	1
□ 7	0.6569	0.3896	0.2619	0.4136	2
□ 8	0.1103	0.4132	0.4034	-0.2879	10
□ 9	0.7586	0.6022	0.8241	-0.0443	6
□ 10	0.6239	0.3553	0.4928	0.1476	3

In this application, participant C6 is the best fit candidate as per requirements. If multiple candidates need to be selected, it can be selected as per their rankings.

4.3 Comparative Analysis

This section provides a comparison analysis with other existing decision-making approaches by using the common neutrosophic data as given in Table 13. Table 14 provides the result of the comparative analysis that establishes the correctness of the approach used in this article.

Table 13. Neutrosophic data for comparison

	e_1	e_2	e_3
x_1	(0.5, 0.4, 0.7)	(0.7, 0.5, 0.1)	(0.6, 0.6, 0.3)
x_2	(0.6, 0.5, 0.6)	(0.6, 0.2, 0.2)	(0.5, 0.4, 0.4)
x_3	(0.7, 0.3, 0.5)	(0.7, 0.2, 0.1)	(0.7, 0.5, 0.4)
x_4	(0.6, 0.4, 0.5)	(0.7, 0.4, 0.2)	(0.5, 0.6, 0.4)

Table 14. Comparison study with some existing methods

Method	The final ranking	The optimal alternative
Peng and Liu [32] Algorithm 1	$x_3 \succ x_2 \succ x_4 \succ x_1$	x_3
Peng and Liu [32] Algorithm 2	$x_3 \succ x_4 \succ x_1 \succ x_2$	x_3
Peng and Liu [32] Algorithm 3	$x_3 \succ x_4 \succ x_2 \succ x_1$	x_3
Deli and Broumi [33]	$x_3 \succ x_4 \succ x_1 \succ x_2$	x_3
Maji [34]	$x_3 \succ x_4 \succ x_1 \succ x_2$	x_3
Karaaslan [35]	$x_3 \succ x_4 \succ x_1 \succ x_2$	x_3
Deli and Broumi [36]	$x_3 \succ x_4 \succ x_1 \succ x_2$	x_3
Proposed Algorithm	$x_3 \succ x_1 \succ x_4 \succ x_2$	x_3

Reason behind the obtained result: x_3 is best of all in both T (higher) and F (Lower) aspects. Similarly, x_2 is worst of all in both T (higher) and F (Lower) values. $x_1 \succ x_4$ because x_1 in 2nd and 3rd parameter is having lower F value. Other values are just cancelling out each other as F and T both are increasing or decreasing. It seems, the ordering is logical, which matches with the outcome of our algorithm.

5. Conclusions

In this article an MCDM algorithm based on NSS is introduced to model an interview process and rank the candidates. A general score function is introduced, in the algorithm by taking into account the three parameters of a NS (namely Truth, falsity and Indeterminancy). It is capable of ordering the neutrosophic values efficiently. To show the adequateness of the approach and establish its superiority, the results are compared with those of many of the existing algorithms in this direction It is to note that the outcome of the algorithm is natural and matches with the anticipations. Further extensions of our algorithm can be carried out by considering the generalisations of the soft set model in thr fom of Hypersoftset, IndermSoftset, IndetermHyperSoftset, Tree Softset and PlithogenicHyperSoftset models.

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An Integrated Maple Package for Algebraic Interval Neutrosophic Matrices

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Abstract: In this paper, a maple code is presented to do many operations on interval valued neutrosophic matrices including entering the elements of the matrix, checking whether a given matrix is an interval valued neutrosophic matrix or not, finding complement of an interval valued neutrosophic matrix, finding score, accuracy, and certainty measures, union and intersection of two interval valued neutrosophic matrices, sum and product of two interval valued neutrosophic matrices and finding the transpose of a given interval valued of neutrosophic matrix. What distinguishes this code is its simplicity to understand and to call the functions. Many examples are presented and solved successfully.

Keywords: Maple; Neutrosophic Set; Single Valued Neutrosophic Set; Interval Valued Neutrosophic Set; Operations on Matrices.

1. Introduction

Fuzzy Sets were presented by Zadeh [1] to expand the concept of crisp sets allowing elements to belong to the sets partially (with membership degree between 0 and 1), then the last concept expanded by Atanassov [2] to what is known by intuitionistic fuzzy sets adding nonmebmbership component to describe elements of sets. In 1995, Smarandache [3] presented neutrosophic sets as an extension of fuzzy sets and intuitionistic fuzzy sets in which each element is described by three independent componentsl; truth, indeterminacy and false memberships. Many other extensions to neutrosophic sets were presented including refined neutrosophic sets, interval valued neutrosophic sets, bipolar neutrosophic sets, generalized neutrosophic sets, neutrosophic vague soft expert set, fermatean neutrosophic sets, etc.

Many mathematical studies were done on neutrosophic sets and many branches of mathematics were extended to the new concept of logic including probabiliy theory, operations research, statistics, linear algebra, abstract algebra, queueuing theory, artificial intelligence and data minig.[4-14]

Since delaing with neutrosophic sets is very complex and operations on it take long time , then many researchers wrote programming packages and codes to make dealing with it more simple.

Salama et al.[15] presented and introduction to develop programming softwares to deal with neutrosophic sets. Bakro et al.[16] wrote a matlab code to neutrosophication functions and their implementation. Broumi et al.[17] wrote a matlab code to implement neutrosophic membership functions and graphing it. Bisher Zeina et al.[18] presented a maple package to do operations on single valued neutrosophic sets using α,β,γ-Cuts, Broumi et al.[19] wrote a maple package to perform operations on single valued neutrosophic matrices. In this paper we generalize the code presented in [19] to deal with interval valued neutrosphic matrices and do operations on it.

2. Background on Neutrosophic Sets

Definition 2.1[20]

Let Ω be a universe, we call $A \subseteq \Omega$ a neutrosophic set if elements of A are described by their membership degree $T_A(x)$, nonmembership degree $F_A(x)$ and indeterminacy degree $I_A(x)$ and we denote that by:

$$A = \{< T_A(x), I_A(x), F_A(x) > \ ; x \in \Omega\}$$

Where:

$$T_A(x), \mathrm{I_A}(x), F_A(x) \in]^-0, 1^+[\ \& \ ^-0 \ \leq T_A(x) + I_A(x) + F_A(x) \leq \ 3^+$$

Definition 2.2 [20]

Let Ω be a universe, we call $A \subseteq \Omega$ a single valued neutrosophic set if:

$$A = \{< T_A(x), I_A(x), F_A(x) > \ ; x \in \Omega\}$$

Where:

$$T_A(x), \mathrm{I_A}(x), F_A(x) \in [0,1] \ \& \ 0 \ \leq T_A(x) + I_A(x) + F_A(x) \leq \ 3$$

Definition 2.3 [21]

Let Ω be a universe, we call $A \subseteq \Omega$ an interval valued neutrosophic set if:

$$A = \{< [T_A^L(x), T_A^U(x)], [I_A^L(x), I_A^U(x)], [F_A^L(x), F_A^U(x)] > \ ; \ \mathrm{x} \in \Omega\}$$

Where:

$$T_A^L(x), T_A^U(x), I_A^L(x), I_A^U(x), F_A^L(x), F_A^U(x) \in [0,1]$$

Definition 2.4

Interval valued neutrosophic matrix of order $m \times n$ is defined as follows:

$$A = \left[< a_{ij}, \left[Ta_{ij}^L, Ta_{ij}^U\right], \left[Ia_{ij}^L, Ia_{ij}^U\right], \left[Fa_{ij}^L, Fa_{ij}^U\right] >\right]_{m\times n}$$

3. Maple Package to Do Operations on Interval Valued Neutrosophic Matrices

3.1. Entering Interval Valued Neutrosophic Matrices

To enter interval valued neutrosophic matrix we call the function IVNIInput(m,n) where m, n are numbers of rows and columns respectively and the written function is as follows:

```
restart;interface(warnlevel=0);
```

```
with(Maplets[Elements]):

with(Maplets):
IVNIInput:=proc(m::integer,n::integer)

local mat:=Matrix(m,n);

for i from 1 to m by 1 do

for j from 1 to n by 1 do

truthL:=Maplet(InputDialog['x'](cat("Enter lower truth of element
",i,",",j),'onapprove'=Shutdown(['x']),'oncancel'=Shutdown()));

truthL:=Display(truthL);

truthL:=parse(op(truthL));

truthU:=Maplet(InputDialog['x'](cat("Enter upper truth of element
",i,",",j),'onapprove'=Shutdown(['x']),'oncancel'=Shutdown()));

truthU:=Display(truthU);

truthU:=parse(op(truthU));

indeterminacyL:=Maplet(InputDialog['x'](cat("Enter lower indeterminacy of element
",i,",",j),'onapprove'=Shutdown(['x']),'oncancel'=Shutdown()));

indeterminacyL:=Display(indeterminacyL);

indeterminacyL:=parse(op(indeterminacyL));

indeterminacyU:=Maplet(InputDialog['x'](cat("Enter upper indeterminacy of element
",i,",",j),'onapprove'=Shutdown(['x']),'oncancel'=Shutdown()));

indeterminacyU:=Display(indeterminacyU);

indeterminacyU:=parse(op(indeterminacyU));

falsityL:=Maplet(InputDialog['x'](cat("Enter lower falsity of element
",i,",",j),'onapprove'=Shutdown(['x']),'oncancel'=Shutdown()));

falsityL:=Display(falsityL);

falsityL:=parse(op(falsityL));

falsityU:=Maplet(InputDialog['x'](cat("Enter upper falsity of element
",i,",",j),'onapprove'=Shutdown(['x']),'oncancel'=Shutdown()));

falsityU:=Display(falsityU);

falsityU:=parse(op(falsityU));

mat(i,j):=convert([[truthL,truthU],[indeterminacyL,indeterminacyU],[falsityL,falsityU]],string);

end do;
```

```
end do;
mat;
end proc:
```

3.2. Checking whether the matrix is IVNM or not

We can call the function **IVNCheck (mat)** defined below to check whether matrix mat is interval valued neutrosophic matrix or not:

```
IVNCheck:=proc(mat)
IsMembership:=proc(num)
if num<0 or num>1 then return false else return true end if;
end proc:
m,n:=LinearAlgebra[Dimension](mat);
result:=true;
for i from 1 to m by 1 do
for j from 1 to n by 1 do
x:=parse(mat(i,j));
truth:=x[1];
indeterminacy:=x[2];
falsity:=x[3];
truthL:=truth[1];
truthU:=truth[2];
indeterminacyL:=indeterminacy[1];
indeterminacyU:=indeterminacy[2];
falsityL:=falsity[1];
falsityU:=falsity[2];
result:= IsMembership(truthL) and IsMembership(truthU) and IsMembership(indeterminacyL) and
IsMembership(indeterminacyU) and IsMembership(falsityL) and IsMembership(falsityU);
if not result then break; end if;
end do;
if not result then break; end if;
```

```
end do;

if result then cat("your matrix is an interval valued neutrosophic matrix") else cat("your matrix is not a single valued neutrosophic matrix") end if;

end proc:
```

Example 1. In this example we define an interval valued neutrosophic matrix E and check whether it is right defined or not where:

E=

$$\begin{pmatrix} < [1,1],[.7,.8],[.1,.6] > & < [.2,.4],[.2,.8],[.1,.9] > \\ < [.8,.9],[.3,.5],[.1,.2] > & < [.1,.2],[.5,.7],[.2,.5] > \end{pmatrix}$$

The interval valued neutrosophic matrix E can be inputted in Maple like this:

E := Matrix(2, 2, [["[[1, 1], [.7, .8], [.1, .6]]", "[[.2, .4], [.2, .8], [.1, .9]]"], ["[[.8, .9], [.3, .5], [.1, .2]]", "[[.1, .2], [.5, .7], [.2, .5]]"]]);

Or like this:

x:= IVNIInput (2,2);

Then an input box dialogue is going to appear and lead you how to input elements.

Result of checking whether matrix E is Interval-Valued Neutrosophic Matrix or not can be obtained by calling the command IVNCheck(E);

And the result will be:

"your matrix is an interval valued neutrosophic matrix"

3.3. Finding complement of interval valued neutrosophic matrix

For a given IVNM $A = \left[< a_{ij}, \left[Ta_{ij}^{L}, Ta_{ij}^{U}\right], \left[Ia_{ij}^{L}, Ia_{ij}^{U}\right], \left[Fa_{ij}^{L}, Fa_{ij}^{U}\right] >\right]_{m\times n}$, the complement of A is defined as follow:

$$A^{c}= A = \left[< a_{ij}, \left[Fa_{ij}^{L}, Fa_{ij}^{U}\right], \left[1 - Ia_{ij}^{U}, 1 - Ia_{ij}^{L}\right], \left[Ta_{ij}^{L}, Ta_{ij}^{U}\right] >\right]_{m\times n} \quad (10)$$

To find the complement of interval valued neutrosophic matrix we can call the function **IVNMCompelementOf(mat)** which is defined as follow:

```
IVNMCompelementOf:=proc(mat::Matrix)

temp:=LinearAlgebra[Copy](mat);

m,n:=LinearAlgebra[Dimension](temp);
```

```
for i from 1 to m by 1 do
for j from 1 to n by 1 do
x:=parse(mat(i,j));
falsity:=x[1];
indeterminacy:=x[2];
truth:=x[3];
indeterminacyL:=1-indeterminacy[2];
indeterminacyU:=1-indeterminacy[1];
temp(i,j):=convert([truth,[indeterminacyL,indeterminacyU],falsity],string);
end do;
end do;
temp;
end proc:
```

Example 2. find the complement of matrix E in example 1.

the complement of matrix E is:

$$E^c = \begin{pmatrix} < [.1,.6],[.2,.3],[1,1] > & < [.1,.9],[.2,.8],[.2,.4] > \\ < [.1,.2],[.5,.7],[.8,.9] > & < [.2,.5],[.3,.5],[.1,.2] > \end{pmatrix}$$

By calling the function
SVNMCompelementOf1(E);

Same results appear:

$$\begin{bmatrix} \text{"[[.1, .6], [.2, .3], [1, 1]]"} & \text{"[[.1, .9], [.2, .8], [.2, .4]]"} \\ \text{"[[.1, .2], [.5, .7], [.8, .9]]"} & \text{"[[.2, .5], [.3, .5], [.1, .2]]"} \end{bmatrix}$$

3.4. Finding score, accuracy and certainty matrices of interval valued neutrosophic matrices

Suppose A is an interval neutrosophic matrix, then score, accuracy and certainty measures are defined as follows: [22]

$$\tilde{S}_{IVNN}(x)=\frac{T_A^L(x)+T_A^U(x)+4-I_A^L(x)-I_A^U(x)-F_A^L(x)-F_A^U(x)}{6}$$

$\tilde{S}_{SVNN}(x)=\frac{2+T_A(x)-I_A(x)-F_A(x)}{3}$

$\tilde{A}_{IVNN}(x)=\frac{T_A^L(x)+T_A^U(x)-F_A^L(x)-F_A^U(x)}{2}$

$\tilde{A}_{SVNN}(x)$=$T_A(x) - F_A(x)$

$\tilde{C}_{IVNN}(x)$=$\frac{T_A^L(x)+T_A^U(x)}{2}$

$\tilde{C}_{SVNN}(x)$=$T_A(x)$

Three maple functions **ScoreMatrix( )**, **AccuracyMatrix ( )** and **CertaintyMatrix ( )** are defined as follows:

```
ScoreMatrix:=proc(mat::Matrix)
m,n:=LinearAlgebra[Dimension](mat);
scoreMat:=Matrix(m,n);
for i from 1 to m by 1 do
for j from 1 to n by 1 do
x:=parse(mat(i,j));
truth:=x[1];
indeterminacy:=x[2];
falsity:=x[3];
truthL:=truth[1];
truthU:=truth[2];
indeterminacyL:=indeterminacy[1];
indeterminacyU:=indeterminacy[2];
falsityL:=falsity[1];
falsityU:=falsity[2];
score:=(4+truthL+truthU-indeterminacyL-indeterminacyU-falsityL-falsityU)/6;
scoreMat(i,j):=score;
end do;
end do;
scoreMat;
end proc:
AccuracyMatrix:=proc(mat::Matrix)
m,n:=LinearAlgebra[Dimension](mat);
aMat:=Matrix(m,n);
```

```
for i from 1 to m by 1 do
for j from 1 to n by 1 do
x:=parse(mat(i,j));
truth:=x[1];
indeterminacy:=x[2];
falsity:=x[3];
truthL:=truth[1];
truthU:=truth[2];
indeterminacyL:=indeterminacy[1];
indeterminacyU:=indeterminacy[2];
falsityL:=falsity[1];
falsityU:=falsity[2];
a:=(truthL+truthU-falsityL-falsityU)/2;
aMat(i,j):=a;
end do;
end do;
aMat;
end proc:
CertaintyMatrix:=proc(mat::Matrix)
m,n:=LinearAlgebra[Dimension](mat);
cMat:=Matrix(m,n);
for i from 1 to m by 1 do
for j from 1 to n by 1 do
x:=parse(mat(i,j));
truth:=x[1];
indeterminacy:=x[2];
falsity:=x[3];
truthL:=truth[1];
truthU:=truth[2];
indeterminacyL:=indeterminacy[1];
```

```
indeterminacyU:=indeterminacy[2];
falsityL:=falsity[1];
falsityU:=falsity[2];
c:=(truthL+truthU)/2;
cMat(i,j):=c;
end do;
end do;
cMat;
end proc:
```

and by calling the previous three functions we get:

ScoreMatrix(E);

$$\begin{bmatrix} 0.6333333333 & 0.4333333334 \\ 0.7666666667 & 0.4000000001 \end{bmatrix}$$

AccuracyMatrix(E);

$$\begin{bmatrix} 0.6500000000 & -0.2000000000 \\ 0.7000000000 & -0.2000000000 \end{bmatrix}$$

CertaintyMatrix(E);

$$\begin{bmatrix} 1 & 0.3000000000 \\ 0.8500000000 & 0.1500000000 \end{bmatrix}$$

3.5. Computing union of two interval valued neutrosophic matrices

Union of two interval valued neutrosophic matrices A and B is defined as follow:

$A \cup B = C = \left[< c_{ij_T}, c_{ij_I}, c_{ij_F} >\right]_{m\times\ n}$

where

$c_{ij_T} = \left[a_{ij_{TL}} \vee b_{ij_{TL}}, a_{ij_{TU}} \vee b_{ij_{TU}}\ \right],$

$c_{ij_I} = \left[a_{ij_{IL}} \vee b_{ij_{IL}}, a_{ij_{IU}} \vee b_{ij_{IU}}\ \right],$

$c_{ij_F} = \left[a_{ij_{FL}} \vee b_{ij_{FL}}, a_{ij_{FU}} \vee b_{ij_{FU}}\ \right]$

And it can be evaluated by calling the function **Union(A, B)** described as follows:

```
Union:=proc(mat1::Matrix,mat2::Matrix)

m1,n1:=LinearAlgebra[Dimension](mat1);

m2,n2:=LinearAlgebra[Dimension](mat2);

if (n1=n2) and (m1=m2) then

m:=m1;n:=n1;

unionMat:=Matrix(m,n);

for i from 1 to m by 1 do

for j from 1 to n by 1 do

x:=parse(mat1(i,j));

y:=parse(mat2(i,j));

truth1:=x[1];

indeterminacy1:=x[2];

falsity1:=x[3];

truthL1:=truth1[1];

truthU1:=truth1[2];

indeterminacyL1:=indeterminacy1[1];

indeterminacyU1:=indeterminacy1[2];

falsityL1:=falsity1[1];

falsityU1:=falsity1[2];

truth2:=y[1];

indeterminacy2:=y[2];

falsity2:=y[3];

truthL2:=truth2[1];

truthU2:=truth2[2];

indeterminacyL2:=indeterminacy2[1];

indeterminacyU2:=indeterminacy2[2];

falsityL2:=falsity2[1];
```

```
falsityU2:=falsity2[2];

truthL:=max(truthL1,truthL2);

truthU:=max(truthU1,truthU2);

indeterminacyL:=max(indeterminacyL1,indeterminacyL2);

indeterminacyU:=max(indeterminacyU1,indeterminacyU2);

falsityL:=max(falsityL1,falsityL2);

falsityU:=max(falsityU1,falsityU2);

unionMat(i,j):=convert([[truthL,truthU],[indeterminacyL,indeterminacyU],[falsityL,falsityU]],string)

;

end do;

end do;

unionMat;

else

print("dimension of given matrices must be equal!");

end if;

end proc:
```

Example 3. Say that:

$$E=\begin{bmatrix} "[[1, 1], [.7, .8], [.1, .6]]" & "[[.2, .4], [.2, .8], [.1, .9]]" \\ "[[.8, .9], [.3, .5], [.1, .2]]" & "[[.1, .2], [.5, .7], [.2, .5]]" \end{bmatrix}$$

$$F=\begin{bmatrix} "[[0.7, 1], [.7, .8], [.4, .6]]" & "[[.2, .3], [.2, .6], [.1, .3]]" \\ "[[.2, .4], [.3, .5], [.1, .3]]" & "[[.1, .2], [.5, .7], [.3, .5]]" \end{bmatrix}$$

So, the union of previous matrices is done by calling the function:

Union(E, F);

And the result is:

$$E_{IVNM} \cup F_{IVNM} = \begin{bmatrix} "[[1, 1], [.7, .8], [.4, .6]]" & "[[.2, .4], [.2, .8], [.1, .9]]" \\ "[[.8, .9], [.3, .5], [.1, .3]]" & "[[.1, .2], [.5, .7], [.3, .5]]" \end{bmatrix}$$

3.6. Computing intersection of two interval valued neutrosophic matrices

The intersection of two interval valued neutrosophic matrices A and B is defined as follow:

$A \cap B = C = [< d_{ij_T}, d_{ij_I}, d_{ij_F} >]_{m\times n}$

Where:

$c_{ij_T} = \left[a_{ij_{TL}} \wedge b_{ij_{TL}}, a_{ij_{TU}} \wedge b_{ij_{TU}}\right],$

$c_{ij_I} = \left[a_{ij_{IL}} \vee b_{ij_{IL}}, a_{ij_{IU}} \vee b_{ij_{IU}}\right],$

$c_{ij_F} = \left[a_{ij_{FL}} \vee b_{ij_{FL}}, a_{ij_{FU}} \vee b_{ij_{FU}}\right]$

And this is done calling the function Intersection(A,B) is defined in the following manner.

```
Intersection:=proc(mat1::Matrix,mat2::Matrix)
m1,n1:=LinearAlgebra[Dimension](mat1);
m2,n2:=LinearAlgebra[Dimension](mat2);
if (n1=n2) and (m1=m2) then
m:=m1;n:=n1;
intersectMat:=Matrix(m,n);
for i from 1 to m by 1 do
for j from 1 to n by 1 do
x:=parse(mat1(i,j));
y:=parse(mat2(i,j));
truth1:=x[1];
indeterminacy1:=x[2];
falsity1:=x[3];
truthL1:=truth1[1];
truthU1:=truth1[2];
indeterminacyL1:=indeterminacy1[1];
indeterminacyU1:=indeterminacy1[2];
falsityL1:=falsity1[1];
falsityU1:=falsity1[2];
truth2:=y[1];
```

```
indeterminacy2:=y[2];
falsity2:=y[3];
truthL2:=truth2[1];
truthU2:=truth2[2];
indeterminacyL2:=indeterminacy2[1];
indeterminacyU2:=indeterminacy2[2];
falsityL2:=falsity2[1];
falsityU2:=falsity2[2];
truthL:=min(truthL1,truthL2);
truthU:=min(truthU1,truthU2);
indeterminacyL:=max(indeterminacyL1,indeterminacyL2);
indeterminacyU:=max(indeterminacyU1,indeterminacyU2);
falsityL:=max(falsityL1,falsityL2);
falsityU:=max(falsityU1,falsityU2);
intersectMat(i,j):=convert([[truthL,truthU],[indeterminacyL,indeterminacyU],[falsityL,falsityU]],stri
ng);
end do;
end do;
intersectMat;
else
print("dimension of given matrices must be equal!");
end if;
end proc:
```

Example 4. Find intersection of interval valued neutrosophic matrices E and F presented in example 3.

Solution:

Calling the function Intersection (E, F); yields to the solution:

$$\begin{bmatrix} \text{"[[.7, 1], [.7, .8], [.4, .6]]"} & \text{"[[.2, .3], [.2, .8], [.1, .9]]"} \\ \text{"[[.2, .4], [.3, .5], [.1, .3]]"} & \text{"[[.1, .2], [.5, .7], [.3, .5]]"} \end{bmatrix}$$

3.7. Addition of two interval valued neutrosophic matrices.

The Addition of two interval valued neutrosophic matrices A and B is defined as follow:

$A \oplus B = S = [< s_{ij_T}, s_{ij_I}, s_{ij_F} >]_{m\times n}$

Where:

$s_{ij_T} = \left[a_{ij_{TL}} + b_{ij_{TL}} - a_{ij_{TL}}.b_{ij_{TL}}, a_{ij_{TU}} + b_{ij_{TU}} - a_{ij_{TU}}.b_{ij_{TU}}\right],$

$s_{ij_I} = \left[a_{ij_{IL}}.b_{ij_{IL}}, a_{ij_{IU}}.b_{ij_{IU}}\right],$

$s_{ij_F} = \left[a_{ij_{FL}}.b_{ij_{FL}}, a_{ij_{FU}}.b_{ij_{FU}}\right],$

And can be done calling the function **Addition (A, B)** which is defined as follow:

```
Addition:=proc(mat1::Matrix,mat2::Matrix)
m1,n1:=LinearAlgebra[Dimension](mat1);
m2,n2:=LinearAlgebra[Dimension](mat2);
if (n1=n2) and (m1=m2) then
m:=m1;n:=n1;
addMat:=Matrix(m,n);
for i from 1 to m by 1 do
for j from 1 to n by 1 do
x:=parse(mat1(i,j));
y:=parse(mat2(i,j));
truth1:=x[1];
indeterminacy1:=x[2];
falsity1:=x[3];
truthL1:=truth1[1];
truthU1:=truth1[2];
indeterminacyL1:=indeterminacy1[1];
indeterminacyU1:=indeterminacy1[2];
falsityL1:=falsity1[1];
falsityU1:=falsity1[2];
truth2:=y[1];
```

```
indeterminacy2:=y[2];
falsity2:=y[3];
truthL2:=truth2[1];
truthU2:=truth2[2];
indeterminacyL2:=indeterminacy2[1];
indeterminacyU2:=indeterminacy2[2];
falsityL2:=falsity2[1];
falsityU2:=falsity2[2];
truthL:=truthL1+truthL2-truthL1*truthL2;
truthU:=truthU1+truthU2-truthU1*truthU2;
indeterminacyL:=indeterminacyL1*indeterminacyL2;
indeterminacyU:=indeterminacyU1*indeterminacyU2;
falsityL:=falsityL1*falsityL2;
falsityU:=falsityU1*falsityU2;
addMat(i,j):=convert([[truthL,truthU],[indeterminacyL,indeterminacyU],[falsityL,falsityU]],string);
end do;
end do;
addMat;
else
print("dimension of given matrices must be equal!");
end if;
end proc:
```

Example 5. In this example we find the addition of two interval valued neutrosophic matrices E and F presented in example 3 calling the function:

Addition(E,F);

$$\begin{bmatrix} \text{"[[1.0, 1], [.49, .64], [.4e-1, .36]]"} & \text{"[[.36, .58], [.4e-1, .48], [.1e-1, .27]]"} \\ \text{"[[.84, .94], [.9e-1, .25], [.1e-1, .6e-1]]"} & \text{"[[.19, .36], [.25, .49], [.6e-1, .25]]"} \end{bmatrix}$$

3.8. Product of two interval valued neutrosophic matrices

The product of two interval valued neutrosophic matrices A and B is defined as follow:

$A \odot B = R = [< r_{ij_T}, r_{ij_I}, r_{ij_F} >]_{m \times n}$

where

$r_{ij_T} = \left[a_{ij_{TL}} . b_{ij_{TL}}, a_{ij_{TU}} . b_{ij_{TU}}\right],$

$r_{ij_I} = \left[a_{ij_{IL}} + b_{ij_{IL}} - a_{ij_{IL}} . b_{ij_{IL}}, a_{ij_{IU}} + b_{ij_{IU}} - a_{ij_{IU}} . b_{ij_{IU}}\right],$

$r_{ij_F} = \left[a_{ij_{FL}} + b_{ij_{FL}} - a_{ij_{FL}} . b_{ij_{FL}}, a_{ij_{FU}} + b_{ij_{FU}} - a_{ij_{FU}} . b_{ij_{FU}}\right]$

Which is simply done by the call of the function **Product (A, B)** defined as follow:

```
Prod:=proc(mat1::Matrix,mat2::Matrix)
m1,n1:=LinearAlgebra[Dimension](mat1);
m2,n2:=LinearAlgebra[Dimension](mat2);
if (n1=n2) and (m1=m2) then
m:=m1;n:=n1;
prodMat:=Matrix(m,n);
for i from 1 to m by 1 do
for j from 1 to n by 1 do
x:=parse(mat1(i,j));
y:=parse(mat2(i,j));
truth1:=x[1];
indeterminacy1:=x[2];
falsity1:=x[3];
truthL1:=truth1[1];
truthU1:=truth1[2];
indeterminacyL1:=indeterminacy1[1];
indeterminacyU1:=indeterminacy1[2];
falsityL1:=falsity1[1];
falsityU1:=falsity1[2];
truth2:=y[1];
```

```
indeterminacy2:=y[2];
falsity2:=y[3];
truthL2:=truth2[1];
truthU2:=truth2[2];
indeterminacyL2:=indeterminacy2[1];
indeterminacyU2:=indeterminacy2[2];
falsityL2:=falsity2[1];
falsityU2:=falsity2[2];
truthL:=truthL1*truthL2;
truthU:=truthU1*truthU2;
indeterminacyL:=indeterminacyL1+indeterminacyL2-indeterminacyL1*indeterminacyL2;
indeterminacyU:=indeterminacyU1+indeterminacyU2-indeterminacyU1*indeterminacyU2;
falsityL:=falsityL1+falsityL2-falsityL1*falsityL2;
falsityU:=falsityU1+falsityU2-falsityU1*falsityU2;
prodMat(i,j):=convert([[truthL,truthU],[indeterminacyL,indeterminacyU],[falsityL,falsityU]],string);
end do;
end do;
prodMat;
else
print("dimension of given matrices must be equal!");
end if;
end proc:
```

Example 6. In this example we evaluate the product of the two interval valued neutrosophic matrices E and F presented in example 3 by calling of the command:

Product(E, F);

$$\begin{bmatrix} \text{"[[.7, 1], [.91, .96], [.46, .84]]"} & \text{"[[.4e-1, .12], [.36, .92], [.19, .93]]"} \\ \text{"[[.16, .36], [.51, .75], [.19, .44]]"} & \text{"[[.1e-1, .4e-1], [.75, .91], [.44, .75]]"} \end{bmatrix}$$

3.9. Transpose of interval valued neutrosophic matrix

Transpose of interval valued neutrosophic matrix simply done by calling of the function **Transpose(A)** defined as follow:

```
Transpose:=proc(mat::Matrix)
m,n:=LinearAlgebra[Dimension](mat);
temp:=Matrix(n,m);
for i from 1 to n by 1 do
for j from 1 to m by 1 do
temp(i,j):=mat(j,i);
end do;
end do;
temp;
end proc:
```

Example 7. In this example we evaluate the transpose of the interval valued neutrosophic matrix E presented in example 3:

Transpose(E);

$$\begin{bmatrix} \text{"[[1, 1], [.7, .8], [.1, .6]]"} & \text{"[[.8, .9], [.3, .5], [.1, .2]]"} \\ \text{"[[.2, .4], [.2, .8], [.1, .9]]"} & \text{"[[.1, .2], [.5, .7], [.2, .5]]"} \end{bmatrix}$$

4. Conclusions

This paper proposed new Maple package to do operations on interval valued neutrosophic matrices including complement, transpose, union, intersection, addition, product, sum and product of interval valued neutrosophic matrices. This package is very useful in neutrosophic decision making operations and on neutrosophic events simulation. In future work we are looking forward to generalize this package to other neutrosophic sets like fermatean neutrosophic sets and refined neutrosophic sets.

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New Observation on Cesaro Summability in Neutrosophic $\mathcal{n}$-Normed Linear Spaces

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Abstract: We define Cesaro summability in a Neutrosophic $\mathcal{n}$ -Normed Linear Space in this article. In a Neutrosophic $\mathcal{n}$-Normed Linear Space, we show the Cesaro summability method to be regular, albeit this does not imply typical convergence in general. We also look for more circumstances in which the opposite is true.

Keywords: Convergence, Cesaro summability, Normed Space, Neutrosophic Normed Linear Space.

1. Introduction

In 1965, Zadeh [20] was the first to present the idea of fuzzy sets, which was an expansion of classical set theoretical concept. This theory has been used in numerous areas of mathematics, including the theory of functions, metric spaces, topological spaces and approximation theory, in addition to numerous branches of engineering, such as population dynamics, nonlinear dynamic systems, and quantum physics. Gunawan and Mashadi [5], Kim and Cho [8] and Malceski [9] and several researchers have studied$\mathcal{n}$-normed linear spaces. Vijayabalaji and Narayanan [18] defined fuzzy $\mathcal{n}$-normalized linear space. Following the definition of Intuitionistic Fuzzy $\mathcal{n}$-Normed Space [$\mathcal{IFnNS}$] given by Vijayabalaji et al. [19], Saadati and Park [11] proposed the idea of Intuitionistic Fuzzy Normed Space [$\mathcal{IFNS}$].

The Neutrosophic Set [$\mathcal{NS}$] is a fresh interpretation of Smarandache's definition of the classical set [14,15]. A neutrosophic set's elements are made up of the triplets true- membership function (T), indeterminacy membership function (I) and falsity membership function (F). When all elements of the universe have a certain degree of T, F, and I, a set is said to be neutrosophic. Some findings on fixed-points were demonstrated in the context of these spaces by Sowndrarajan et. al. [16]. Approximate Fixed Point Theorems for Weak Contractions on Neutrosophic normed Spaces were proved in 2022 by Jeyaraman et. al. [7].

Our goal in this study is to introduce summability theory in a Neutrosophic $\mathcal{n}$-Normed Linear Space [$\mathcal{NnNLS}$]. We introduce the idea of Cesaro in this context. The definition of convergence for a sequence in $\mathcal{NnNLS}$ affects our findings. This new definition is the foundation for the development of our current findings. Pertaining to the conventional analogs of the findings reported in this work.

2. Preliminaries

Definition 2.1:

The following axioms define a continuous t-norm as a binary operation $* : [0,1] \times [0,1] \rightarrow [0,1]$

1. $*$ is continuous, commutative and associative,
2. $\mathfrak{a} * 1 = \mathfrak{a}$ for every $\mathfrak{a} \in [0,1]$,
3. If $\mathfrak{a} \leq \mathfrak{c}$ and $\mathfrak{b} \leq \mathfrak{d}$ then $\mathfrak{a} * \mathfrak{b} \leq \mathfrak{c} * \mathfrak{d}$, for each $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in [0,1]$.

Definition 2.2:

The following axioms define a continuous t-conorm as a binary operation $\diamond : [0,1] \times [0,1] \rightarrow [0,1]$

1. $\diamond$ is continuous, commutative and associative,
2. $\mathfrak{a} \diamond 0 = \mathfrak{a}$ for every $\mathfrak{a} \in [0,1]$,
3. If $\mathfrak{a} \leq \mathfrak{c}$ and $\mathfrak{b} \leq \mathfrak{d}$ then $\mathfrak{a} \diamond \mathfrak{b} \leq \mathfrak{c} \diamond \mathfrak{d}$, for each $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in [0,1]$.

Definition 2.3:

A $\mathcal{NnNLS}$ is the 7-tuple $(\mathfrak{H}, \zeta, \vartheta, \varpi, *, \diamond, \odot)$ where $\mathfrak{H}$ is a linear space over a field F, $*$ is a continuous t-norm, $\diamond$ and $\odot$ continuous t-conorm, μ, ϑ *and* ω are fuzzy sets on $\mathfrak{H}^n \times (0,\infty)$, μ denotes the degree of membership, ϑ denotes the indeterminacy and ω denotes the non –membership of $(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t) \in \mathfrak{H}^n \times (0,\infty)$ satisfying the following conditions for every $(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n) \in \mathfrak{H}^n$ and $s, t > 0$.

a) $0 \leq \zeta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t) \leq 1$; $0 \leq \vartheta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t) \leq 1$; $0 \leq \varpi(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t) \leq 1$;

b) $\zeta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t) + \vartheta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t) + \varpi(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t) \leq 3$,

c) $\zeta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t) > 0$,

d) $\zeta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t) = 1$ if and only if $\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n$ are linearly dependent,

e) $\zeta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t)$ is constant for any combination of$\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n$,

f) $\zeta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \propto \mathfrak{h}_n, t) = \zeta\left(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, \frac{t}{|\propto|}\right)$ for each $\propto \neq 0, \propto \in$ F,

g) $\zeta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, s) * \zeta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}'_n, t) \leq \zeta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n + \mathfrak{h}'_n, s + t)$,

h) $\zeta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t) : (0,\infty) \rightarrow [0,1]$ is continuous,

i) $\lim_{t\to\infty} \zeta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t) = 1$ and $\lim_{t\to 0} \zeta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t) = 0$,

j) $\vartheta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t) < 1$,

k) $\vartheta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t) = 0$ if and only if $\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n$ are linearly dependent,

l) $\vartheta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t)$ is constant for any combination of$\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n$,

m) $\vartheta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \propto \mathfrak{h}_n, t) = \vartheta\left(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, \frac{t}{|\propto|}\right)$ for each $\propto \neq 0, \propto \in$ F,

n) $\vartheta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, s) \diamond \vartheta(\mathfrak{x}_1, \mathfrak{x}_2, \dots, \mathfrak{x}'_n, t) \geq \vartheta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n + \mathfrak{x}'_n, s + t)$,

o) $\vartheta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t) : (0,\infty) \rightarrow [0,1]$ is continuous,

p) $\lim_{t\to\infty} \vartheta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t) = 0$ and $\lim_{t\to 0} \vartheta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t) = 1$,

q) $\varpi(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t) < 1$,

r) $\varpi(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t) = 0$ if and only if $\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n$ are linearly dependent,

s) $\varpi(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t)$ is constant for any combination of$\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n$,

t) $\varpi(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \propto \mathfrak{h}_n, t) = \varpi\left(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, \frac{t}{|\propto|}\right)$ for each $\propto \neq 0, \propto \in$ F,

u) $\varpi(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, s) \odot \varpi(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}'_n, t) \geq \varpi(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n + \mathfrak{h}'_n, s + t)$,

v) $\varpi(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t) : (0,\infty) \rightarrow [0,1]$ is continuous,

w) $\lim_{t\to\infty} \varpi(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t) = 0$ and $\lim_{t\to 0} \varpi(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t) = 1$.

Example 2.4:

Let $(\mathfrak{H}, \|\cdot, \dots, \cdot\|)$ be a linear space with n norms. Also, let $\mathfrak{a} * \mathfrak{b} = \mathfrak{a}b$, $\mathfrak{a} \diamond \mathfrak{b} = min\{\mathfrak{a} + \mathfrak{b}, 1\}$ and $\mathfrak{a} \odot \mathfrak{b} = min\{\mathfrak{a} + \mathfrak{b}, 1\}$, for every $\mathfrak{a}, \mathfrak{b} \in [0,1]$, $\zeta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t) = \frac{t}{t + \|\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n\|}$, $\vartheta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t) = \frac{\|\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n\|}{t + \|\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n\|}$ and $\varpi(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n, t) = \frac{\|\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_n\|}{t}$. Then $(\mathfrak{H}, \zeta, \xi, \varpi, *, \diamond, \odot)$ is a $\mathcal{NnNLS}$.

Lemma 2.5.

We define $<\varrho>=\varrho-[\varrho]$, for every $\varrho>0$, where $[\cdot]$ stands for the largest integer function. What follows is accurate:

(i) If $\varrho > 1$, then $\varrho_n > n$ for every $n \in \mathbb{N}\backslash\{0\}$ with $n \geq \frac{1}{\langle\varrho\rangle}$,

(ii) If $0 < \varrho < 1$, then $\varrho_n < n$ for every $n \in \mathbb{N}\backslash\{0\}$, where $\varrho_n = [n\varrho]$.

Lemma 2.6.

The following claims are accurate:

(i) If $\varrho > 1$, then for every $n \in \mathbb{N}\backslash\{0\}$ with $n \geq \frac{3\varrho-1}{\varrho(\varrho-1)}$, we have $\frac{\varrho}{\varrho-1} < \frac{\varrho_n+1}{\varrho_n-n} < \frac{2\varrho}{\varrho-1}$,

(ii) If $0 < \varrho < 1$, then for every $n \in \mathbb{N}\backslash\{0\}$ with $n > \frac{1}{\varrho}$, we have $0 < \frac{\varrho_n+1}{n-\varrho_n} < \frac{2\varrho}{\varrho-1}$.

3. In $\mathcal{NnNLS}$ Cesaro Summability

Definition 3.1.

In $\mathcal{NnNLS}$ $(\mathfrak{H}, \zeta, \vartheta, \varpi, *, \diamond, \odot)$, choose the sequence to be $\{\mathfrak{a}_n\}$. The Arithmetic means of $\{\mathfrak{a}_n\}$, is defined and denoted by

$$\chi_n = \frac{1}{n+1}\sum_{k=0}^{n} \mathfrak{a}_k.$$

If $\lim_{n\to\infty} \chi_n = \mathfrak{a}$, then $\{\mathfrak{a}_n\}$ is said to be Cesaro summable to $\mathfrak{a} \in \mathfrak{H}$.

Theorem 3.2.

In $\mathcal{NnNLS}$ $(\mathfrak{H}, \zeta, \vartheta, \varpi, *, \diamond, \odot)$, if the sequence to be $\{\mathfrak{a}_n\}$converges to $\mathfrak{a} \in \mathfrak{H}$, then $\{\mathfrak{a}_n\}$ is Cesaro summable to $\mathfrak{a}$.

Proof.

Choose $\mathfrak{a} \in \mathfrak{H}$ be the converging point of the sequence $\{\mathfrak{a}_n\}$.

Fix $r > 0$ and $\mathfrak{h}_1, \mathfrak{h}_2, \ldots, \mathfrak{h}_{n-1} \in \mathfrak{H}$.

Then for given $\varepsilon > 0$, there exists $n_0 \in \mathfrak{N}$ such that

$\zeta\left(\mathfrak{h}_1, \mathfrak{h}_2, \ldots, \mathfrak{h}_{n-1}, \mathfrak{a}_n - \mathfrak{a}, \frac{r}{2}\right) > 1-\varepsilon, \vartheta\left(\mathfrak{h}_1, \mathfrak{h}_2, \ldots, \mathfrak{h}_{n-1}, \mathfrak{a}_n - \mathfrak{a}, \frac{r}{2}\right) < \varepsilon$ and

$\varpi\left(\mathfrak{h}_1, \mathfrak{h}_2, \ldots, \mathfrak{h}_{n-1}, \mathfrak{a}_n - \mathfrak{a}, \frac{r}{2}\right) < \varepsilon$, for all $n > n_0$.

Also, from Definition (2.3), we have that

$\lim_{n\to\infty} \zeta\left(\mathfrak{h}_1, \mathfrak{h}_2, \ldots, \mathfrak{h}_{n-1}, \sum_{k=0}^{n_0}(\mathfrak{a}_k - \mathfrak{a}), \frac{(n+1)r}{2}\right) = 1, \lim_{n\to\infty} \vartheta\left(\mathfrak{h}_1, \mathfrak{h}_2, \ldots, \mathfrak{h}_{n-1}, \sum_{k=0}^{n_0}(\mathfrak{a}_k - \mathfrak{a}), \frac{(n+1)r}{2}\right) = 0$ and

$\lim_{n\to\infty} \varpi\left(\mathfrak{h}_1, \mathfrak{h}_2, \ldots, \mathfrak{h}_{n-1}, \sum_{k=0}^{n_0}(\mathfrak{a}_k - \mathfrak{a}), \frac{(n+1)r}{2}\right) = 0.$

Consequently, there are $n_1 \in \mathfrak{N}$such that

$$\zeta\left(\mathfrak{h}_1, \mathfrak{h}_2, \ldots, \mathfrak{h}_{n-1}, \sum_{k=0}^{n_0}(\mathfrak{a}_k - \mathfrak{a}), \frac{(n+1)r}{2}\right) > 1-\varepsilon,$$

$\vartheta\left(\mathfrak{h}_1, \mathfrak{h}_2, \ldots, \mathfrak{h}_{n-1}, \sum_{k=0}^{n_0}(\mathfrak{a}_k - \mathfrak{a}), \frac{(n+1)r}{2}\right) < \varepsilon$ and

$$\varpi\left(\mathfrak{h}_1, \mathfrak{h}_2, \ldots, \mathfrak{h}_{n-1}, \sum_{k=0}^{n_0}(\mathfrak{a}_k - \mathfrak{a}), \frac{(n+1)r}{2}\right) < \varepsilon, \text{for all } n > n_1.$$

Now, we have that

$$\zeta\left(\mathfrak{h}_1, \mathfrak{h}_2, \ldots, \mathfrak{h}_{n-1}, \frac{1}{n+1}\sum_{k=0}^{n} \mathfrak{a}_k - \mathfrak{a}, r\right) = \zeta\left(\mathfrak{h}_1, \mathfrak{h}_2, \ldots, \mathfrak{h}_{n-1}, \frac{1}{n+1}\sum_{k=0}^{n} \mathfrak{a}_k - \mathfrak{a}, r\right)$$

$$\geq \min\left\{\begin{array}{c}\zeta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\sum_{k=0}^{n_0}(\mathfrak{a}_k-\mathfrak{a}),\frac{(n+1)r}{2}\right),\\ \zeta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\sum_{k=n_0+1}^{n}(\mathfrak{a}_k-\mathfrak{a}),\frac{(n+1)r}{2}\right)\end{array}\right\}$$

$$\geq \min\left\{\begin{array}{c}\zeta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\sum_{k=0}^{n_0}(\mathfrak{a}_k-\mathfrak{a}),\frac{(n+1)r}{2}\right),\\ \zeta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\sum_{k=n_0+1}^{n}(\mathfrak{a}_k-\mathfrak{a}),\frac{(n-n_0)r}{2}\right)\end{array}\right\}$$

$$\geq \min\left\{\begin{array}{c}\zeta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\sum_{k=0}^{n_0}(\mathfrak{a}_k-\mathfrak{a}),\frac{(n+1)r}{2}\right),\\ \zeta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\mathfrak{a}_{n_0+1}-\mathfrak{a}\frac{r}{2}\right),\\ \zeta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\mathfrak{a}_{n_0+2}-\mathfrak{a},\frac{r}{2}\right),\\ \dots,\zeta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\mathfrak{a}_n-\mathfrak{a},\frac{r}{2}\right)\end{array}\right\} > 1-\varepsilon,$$

$$\vartheta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\frac{1}{n+1}\sum_{k=0}^{n}\mathfrak{a}_k-\mathfrak{a},r\right)=\vartheta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\frac{1}{n+1}\sum_{k=0}^{n}\mathfrak{a}_k-\mathfrak{a},r\right)$$

$$\leq \max\left\{\begin{array}{c}\vartheta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\sum_{k=0}^{n_0}(\mathfrak{a}_k-\mathfrak{a}),\frac{(n+1)r}{2}\right),\\ \vartheta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\sum_{k=n_0+1}^{n}(\mathfrak{a}_k-\mathfrak{a}),\frac{(n+1)r}{2}\right)\end{array}\right\}$$

$$\leq \max\left\{\begin{array}{c}\vartheta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\sum_{k=0}^{n_0}(\mathfrak{a}_k-\mathfrak{a}),\frac{(n+1)r}{2}\right),\\ \vartheta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\sum_{k=n_0+1}^{n}(\mathfrak{a}_k-\mathfrak{a}),\frac{(n-n_0)r}{2}\right)\end{array}\right\}$$

$$\leq \max\left\{\begin{array}{c}\vartheta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\sum_{k=0}^{n_0}(\mathfrak{a}_k-\mathfrak{a}),\frac{(n+1)r}{2}\right),\\ \vartheta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\mathfrak{a}_{n_0+1}-\mathfrak{a},\frac{r}{2}\right),\\ \vartheta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\mathfrak{a}_{n_0+2}-\mathfrak{a},\frac{r}{2}\right),\dots,\\ \vartheta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\mathfrak{a}_n-\mathfrak{a},\frac{r}{2}\right)\end{array}\right\} < \varepsilon,$$

and in a similar manner, we also have that

$$\varpi\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\frac{1}{n+1}\sum_{k=0}^{n}\mathfrak{a}_k-\mathfrak{a},r\right)=\varpi\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\frac{1}{n+1}\sum_{k=0}^{n}\mathfrak{a}_k-\mathfrak{a},r\right)$$

$$\leq \max\left\{\begin{array}{l}\varpi\left(\mathfrak{y}_1,\mathfrak{y}_2,\dots,\mathfrak{y}_{n-1},\sum_{k=0}^{n_0}(\mathfrak{a}_k-\mathfrak{a}),\frac{(n+1)r}{2}\right),\\ \varpi\left(\mathfrak{y}_1,\mathfrak{y}_2,\dots,\mathfrak{y}_{n-1},\sum_{k=n_0+1}^{n}(\mathfrak{a}_k-\mathfrak{a}),\frac{(n+1)r}{2}\right)\end{array}\right\}$$

$$\leq \max\left\{\begin{array}{l}\varpi\left(\mathfrak{y}_1,\mathfrak{y}_2,\dots,\mathfrak{y}_{n-1},\sum_{k=0}^{n_0}(\mathfrak{a}_k-\mathfrak{a}),\frac{(n+1)r}{2}\right),\\ \varpi\left(\mathfrak{y}_1,\mathfrak{y}_2,\dots,\mathfrak{y}_{n-1},\sum_{k=n_0+1}^{n}(\mathfrak{a}_k-\mathfrak{a}),\frac{(n-n_0)r}{2}\right)\end{array}\right.$$

$$\leq \max\left\{\begin{array}{c}\varpi\left(\mathfrak{y}_1,\mathfrak{y}_2,\dots,\mathfrak{y}_{n-1},\sum_{k=0}^{n_0}(\mathfrak{a}_k-\mathfrak{a}),\frac{(n+1)r}{2}\right),\\ \varpi\left(\mathfrak{y}_1,\mathfrak{y}_2,\dots,\mathfrak{y}_{n-1},\mathfrak{a}_{n_0+1}-\mathfrak{a},\frac{r}{2}\right),\\ \varpi\left(\mathfrak{y}_1,\mathfrak{y}_2,\dots,\mathfrak{y}_{n-1},\mathfrak{a}_{n_0+2}-\mathfrak{a},\frac{r}{2}\right),\dots,\\ \varpi\left(\mathfrak{y}_1,\mathfrak{y}_2,\dots,\mathfrak{y}_{n-1},\mathfrak{a}_{n}-\mathfrak{a},\frac{r}{2}\right)\end{array}\right\}<\varepsilon,$$

for all $n > \max\{n_0, n_1\}$. Thus, the proof is completed.

Example 3.3.

Let $\mathfrak{H} = \mathfrak{R}^n$ with

$$\|\mathfrak{y}_1,\mathfrak{y}_2,\dots,\mathfrak{y}_{n-1}\| = abs\left(\begin{vmatrix}\mathfrak{y}_{11} & \cdots & \mathfrak{y}_{1n}\\ \vdots & \ddots & \vdots\\ \mathfrak{y}_{n1} & \cdots & \mathfrak{y}_{nn}\end{vmatrix}\right)$$

where $\mathfrak{y}_i = (\mathfrak{y}_{i1},\mathfrak{y}_{i2},\dots,\mathfrak{y}_{in}) \in \mathfrak{R}^n$ for every$i = 1,2,\dots,n$ and $\mathfrak{a} * \mathfrak{b} = \mathfrak{a}\,\mathfrak{b}, \mathfrak{a} \diamond \mathfrak{b} = \min\{\mathfrak{a}+\mathfrak{b},1\}$ and $a \odot b = \min\{a+b,1\}$ for all $\mathfrak{a},\mathfrak{b} \in [0,1]$.
Now for all $v_1, v_2, \dots, v_n \in \mathfrak{R}^n$ and $r > 0$, define
$\zeta(y_1,y_2,\dots,y_n,r) = \frac{r}{r+\|y_1,y_2,\dots,y_n\|}$, $\vartheta(y_1,y_2,\dots,y_n,r) = \frac{\|y_1,y_2,\dots,y_n\|}{r+\|y_1,y_2,\dots,y_n\|}$ and

$$\varpi(y_1,y_2,\dots,\quad y_n,t) = \frac{\|y_1,y_2,\dots,y_n\|}{r}.$$

Then $(\mathfrak{R}^n,\zeta,\vartheta,\varpi,*,\diamond,\odot)$ is a $\mathcal{N}n\mathcal{NLS}$.
Choose the sequence $\{\mathfrak{a}_k\} = \left((-1)^{k+1},0,0,\dots,0\right) \in \mathfrak{R}^n$.
Then $\lim_{n\to\infty}\zeta(\mathfrak{y}_1,\mathfrak{y}_2,\dots,\chi_{2n},r) = \lim_{n\to\infty}\frac{r}{r+\left\|\mathfrak{y}_1,\mathfrak{y}_2,\dots,\mathfrak{y}_{n-1},-\frac{1}{2n+1}\right\|} = \lim_{n\to\infty}\frac{r}{r+\left|-\frac{1}{2n+1}\right|\mathfrak{A}} = 1,$
where the value of $\mathfrak{A}$, which is always a finite number, relies on the selection of $\mathfrak{y}_1,\mathfrak{y}_2,\dots,\mathfrak{y}_{n-1}$.

$$\lim_{n\to\infty}\vartheta(\mathfrak{y}_1,\mathfrak{y}_2,\dots,\chi_{2n},r) = \lim_{n\to\infty}\frac{\left\|\mathfrak{y}_1,\mathfrak{y}_2,\dots,\mathfrak{y}_{n-1},-\frac{1}{2n+1}\right\|}{r+\left\|\mathfrak{y}_1,\mathfrak{y}_2,\dots,\mathfrak{y}_{n-1},-\frac{1}{2n+1}\right\|} = \lim_{n\to\infty}\frac{\left|-\frac{1}{2n+1}\right|\mathfrak{B}}{r+\left|-\frac{1}{2n+1}\right|\mathfrak{B}} = 0,$$

where the value of $\mathfrak{B}$, which is always a finite number, relies on the selection of $\mathfrak{y}_1,\mathfrak{y}_2,\dots,\mathfrak{y}_{n-1}$.
and $\lim_{n\to\infty}\varpi(\mathfrak{y}_1,\mathfrak{y}_2,\dots,\chi_{2n},r) = \lim_{n\to\infty}\frac{\left\|\mathfrak{y}_1,\mathfrak{y}_2,\dots,\mathfrak{y}_{n-1},-\frac{1}{2n+1}\right\|}{r} = \lim_{n\to\infty}\frac{\left|-\frac{1}{2n+1}\right|\mathfrak{C}}{r} = 0,$
where the value of $\mathfrak{C}$, which is always a finite number, relies on the selection of $\mathfrak{y}_1,\mathfrak{y}_2,\dots,\mathfrak{y}_{n-1}$.
Therefore, we have that $\chi_{2n} \to \bar{0} = (0,0,\dots,0) \in \mathfrak{R}^n$.
Also, $\lim_{n\to\infty}\zeta(\mathfrak{y}_1,\mathfrak{y}_2,\dots,\mathfrak{y}_{n-1},\chi_{2n+1},r) = \lim_{n\to\infty}\zeta(\mathfrak{y}_1,\mathfrak{y}_2,\dots,\bar{0},r) = 1,$
$\lim_{n\to\infty}\vartheta(\mathfrak{y}_1,\mathfrak{y}_2,\dots,\mathfrak{y}_{n-1},\chi_{2n+1},r) = \lim_{n\to\infty}\xi(\mathfrak{y}_1,\mathfrak{y}_2,\dots,\bar{0},r) = 0$ and
$\lim_{n\to\infty}\varpi(\mathfrak{y}_1,\mathfrak{y}_2,\dots,\mathfrak{y}_{n-1},\chi_{2n+1},r) = \lim_{n\to\infty}\varpi(\mathfrak{y}_1,\mathfrak{y}_2,\dots,\bar{0},r) = 0.$
Thus, we have $\chi_{2n+1} \to \bar{0}$. From the reasoning listed above, we conclude that $\chi_n \to \bar{0}$, i.e., the sequence $\{\mathfrak{a}_k\}$ is Cesaro summable to $\bar{0}$. However, it is clear that $\{\mathfrak{a}_k\}$ is not convergent because $\{\mathfrak{a}_{2k}\} \to (-1,0,0,\dots,0)$ and $\{\mathfrak{a}_{2k+1}\} \to (1,0,0,\dots,0)$.

Theorem 3.4.

Let $\{\mathfrak{a}_n\}$ be a sequence in a $\mathcal{LS}(\mathfrak{H},\zeta,\vartheta,\varpi,*,\diamond,\odot)$. If $\{\mathfrak{a}_n\}$ is Cesaro summable to $\mathfrak{a}$, then it is convergent to $\mathfrak{a}$ if and only if for any $\mathfrak{x}_1,\mathfrak{x}_2,\dots,\mathfrak{x}_{n-1}\in\mathfrak{H}$ and $r>0$ the following conditions are met:

(3.1) $\sup_{\lambda>1}\liminf_{n\to\infty}\zeta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\frac{1}{\lambda_n-n}\sum_{k=n+1}^{\lambda_n}(\mathfrak{a}_k-\mathfrak{a}_n),r\right)=1,$

(3.2) $\inf_{\lambda>1}\limsup_{n\to\infty}\vartheta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\frac{1}{\lambda_n-n}\sum_{k=n+1}^{\lambda_n}(\mathfrak{a}_k-\mathfrak{a}_n),r\right)=0$ and

(3.3) $\inf_{\lambda>1}\limsup_{n\to\infty}\varpi\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\frac{1}{\lambda_n-n}\sum_{k=n+1}^{\lambda_n}(\mathfrak{a}_k-\mathfrak{a}_n),r\right)=0.$

Proof.

Assume that $\{\mathfrak{a}_n\}$ is summable to Cesaro. Assume that $\{\mathfrak{a}_n\}$ converges to $\mathfrak{a}$.
Fix $\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1}\in\mathfrak{H}$ and $r>0$. For any $\lambda>1$,utilising Lemma (2.5), for each $n\in\mathbb{N}\setminus\{0\}$ with $n\geq\langle\lambda\rangle^{-1}$, we have

(3.4) $\mathfrak{a}_n-\chi_n=\frac{\lambda_n+1}{\lambda_n-n}(\chi\lambda_n-\chi_n)-\frac{1}{\lambda_n-n}\sum_{k=n+1}^{\lambda_n}(\mathfrak{a}_k-\mathfrak{a}_n).$

Again, by Lemma (2.6), for $n\geq\frac{3\lambda-1}{\lambda(\lambda-1)}$, we have

$$\zeta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\frac{\lambda_n+1}{\lambda_n-n}(\chi\lambda_n-\chi_n),r\right)=\zeta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},(\chi\lambda_n-\chi_n),\frac{r}{\frac{\lambda_n+1}{\lambda_n-n}}\right)$$
$$\geq\zeta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},(\chi\lambda_n-\chi_n),\frac{r}{\frac{2\lambda}{\lambda-1}}\right),$$

$$\vartheta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\frac{\lambda_n+1}{\lambda_n-n}(\chi\lambda_n-\chi_n),r\right)=\vartheta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},(\chi\lambda_n-\chi_n),\frac{r}{\frac{\lambda_n+1}{\lambda_n-n}}\right)$$
$$\leq\vartheta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},(\chi\lambda_n-\chi_n),\frac{r}{\frac{2\lambda}{\lambda-1}}\right)\text{ and}$$

$$\varpi\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\frac{\lambda_n+1}{\lambda_n-n}(\chi\lambda_n-\chi_n),r\right)=\varpi\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},(\chi\lambda_n-\chi_n),\frac{r}{\frac{\lambda_n+1}{\lambda_n-n}}\right)$$
$$\leq\varpi\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},(\chi\lambda_n-\chi_n),\frac{r}{\frac{2\lambda}{\lambda-1}}\right)$$

Since $\{\chi_n\}$ is a Cauchy sequence, we have

$\lim_{n\to\infty}\zeta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\frac{\lambda_n+1}{\lambda_n-n}(\chi\lambda_n-\chi_n),r\right)=1$,

$\lim_{n\to\infty}\vartheta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\frac{\lambda_n+1}{\lambda_n-n}(\chi\lambda_n-\chi_n),r\right)=0$ and

$\lim_{n\to\infty}\varpi\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\frac{\lambda_n+1}{\lambda_n-n}(\chi\lambda_n-\chi_n),r\right)=0$, and therefore $\lim_{n\to\infty}\frac{\lambda_n+1}{\lambda_n-n}(\chi\lambda_n-\chi_n)=0$.

Hence using (3.4), we have

$\lim_{n\to\infty}\zeta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\frac{1}{\lambda_n-n}\sum_{k=n+1}^{\lambda_n}(\mathfrak{a}_k-\mathfrak{a}_n),r\right)=1,$

$\lim_{n\to\infty}\vartheta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\frac{1}{\lambda_n-n}\sum_{k=n+1}^{\lambda_n}(\mathfrak{a}_k-\mathfrak{a}_n),r\right)=0$ and

$$\lim_{n\to\infty}\varpi\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\frac{1}{\lambda_n-n}\sum_{k=n+1}^{\lambda_n}(\mathfrak{a}_k-\mathfrak{a}_n),r\right)=0.$$

As a result, (3.1), (3.2), and (3.3). We presume that (3.1), (3.2), and (3.3) are true in order to demonstrate the converse. Fix $\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1}\in\mathfrak{H}$ and $r>0$. Then for given $\varepsilon>0$, we have the following:

(i) a thing exists $\lambda>1$ and $n_0\in\mathbb{N}$ such that
$\zeta\left(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_{n-1},\frac{1}{\lambda_n-n}\sum_{k=n+1}^{\lambda_n}(\mathfrak{a}_k-\mathfrak{a}_n),\frac{r}{3}\right)>1-\varepsilon$,

$\vartheta\left(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_{n-1}, \frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(\mathfrak{a}_k - \mathfrak{a}_n), r\right) < \varepsilon$ and

$\varpi\left(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_{n-1}, \frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(\mathfrak{a}_k - \mathfrak{a}_n), r\right) < \varepsilon$, for every $n > n_0$.

(ii) a thing exists $n_0 \in \mathbb{N}$ such that $\zeta\left(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_{n-1}, \chi_n - \mathfrak{a}, \frac{r}{3}\right) > 1 - \varepsilon$,

$\vartheta\left(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_{n-1}, \chi_n - \mathfrak{a}, \frac{r}{3}\right) < \varepsilon$ and $\varpi\left(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_{n-1}, \chi_n - \mathfrak{a}, \frac{r}{3}\right) < \varepsilon$, for all $n > n_1$.

(iii) Also, since $\lim_{n\to\infty}\frac{\lambda_n+1}{\lambda_n - n}(\chi\lambda_n - \chi_n) = 0$, there exists $n_2 \in \mathbb{N}$ such that

$\zeta\left(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_{n-1}, \frac{\lambda_n+1}{\lambda_n - n}(\chi\lambda_n - \chi_n), \frac{r}{3}\right) > 1 - \varepsilon$,

$\vartheta\left(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_{n-1}, \frac{\lambda_n+1}{\lambda_n - n}(\chi\lambda_n - \chi_n), \frac{r}{3}\right) < \varepsilon$ and

$\varpi\left(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_{n-1}, \frac{\lambda_n+1}{\lambda_n - n}(\chi\lambda_n - \chi_n), \frac{r}{3}\right) < \varepsilon$, for all $n > n_2$.

Therefore, we have

$$\zeta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_{n-1}, \mathfrak{a}_n - \mathfrak{a}, r) = \zeta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_{n-1}, \mathfrak{a}_n - \chi_n + \chi_n - \mathfrak{a}, r)$$

$$= \zeta\left(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_{n-1}, \frac{\lambda_n+1}{\lambda_n - n}(\chi\lambda_n - \chi_n) - \frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(\mathfrak{a}_k - \mathfrak{a}_n) + \chi_n - \mathfrak{a}, r\right)$$

$$\geq \min\left\{\begin{array}{c}\zeta\left(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_{n-1}, \frac{\lambda_n+1}{\lambda_n - n}(\chi\lambda_n - \chi_n), \frac{r}{3}\right), \\ \zeta\left(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_{n-1}, \frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(\mathfrak{a}_k - \mathfrak{a}_n), \frac{r}{3}\right), \\ \zeta\left(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_{n-1}, \chi_n - \mathfrak{a}, \frac{r}{3}\right)\end{array}\right\} > 1 - \varepsilon,$$

$$\vartheta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_{n-1}, \mathfrak{a}_n - \mathfrak{a}, r) = \vartheta(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_{n-1}, \mathfrak{a}_n - \chi_n + \chi_n - \mathfrak{a}, r)$$

$$= \vartheta\left(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_{n-1}, \frac{\lambda_n+1}{\lambda_n - n}(\chi\lambda_n - \chi_n) - \frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(\mathfrak{a}_k - \mathfrak{a}_n) + \chi_n - \mathfrak{a}, r\right)$$

$$\leq \max\left\{\begin{array}{c}\vartheta\left(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_{n-1}, \frac{\lambda_n+1}{\lambda_n - n}(\chi\lambda_n - \chi_n), \frac{r}{3}\right), \\ \vartheta\left(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_{n-1}, \frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(\mathfrak{a}_k - \mathfrak{a}_n), \frac{r}{3}\right), \\ \vartheta\left(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_{n-1}, \chi_n - \mathfrak{a}, \frac{r}{3}\right)\end{array}\right\} < \varepsilon \text{ and}$$

$$\varpi(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_{n-1}, \mathfrak{a}_n - \mathfrak{a}, r) = \varpi(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_{n-1}, \mathfrak{a}_n - \chi_n + \chi_n - \mathfrak{a}, r)$$

$$= \varpi\left(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_{n-1}, \frac{\lambda_n+1}{\lambda_n - n}(\chi\lambda_n - \chi_n) - \frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(\mathfrak{a}_k - \mathfrak{a}_n) + \chi_n - \mathfrak{a}, r\right)$$

$$\leq \max\left\{\begin{array}{c}\varpi\left(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_{n-1}, \frac{\lambda_n+1}{\lambda_n - n}(\chi\lambda_n - \chi_n), \frac{r}{3}\right), \\ \varpi\left(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_{n-1}, \frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(\mathfrak{a}_k - \mathfrak{a}_n), \frac{r}{3}\right), \\ \varpi\left(\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_{n-1}, \chi_n - \mathfrak{a}, \frac{r}{3}\right)\end{array}\right\} < \varepsilon,$$

for all $n > \max\{n_0, n_1, n_2\}$. This completes the proof.

4. Conclusion

The idea of Cesaro summability in a $\mathcal{N}n\mathcal{NLS}$, one of the most general mathematical structures with both algebraic and analytic features, is discussed in this study. As a result, many current theorems are extended and generalized by the current results in Cesaro summability. Future work on this topic might result in the expansion of neutrosophic normed spaces and finite-dimensional $\mathcal{N}n\mathcal{NLS}$.

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Gateaux And Frechet Derivative In Neutrosophic Normed Linear Spaces

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Abstract: In this study, we present the neutrosophic derivatives, the neutrosophic Gateaux derivative, and the neutrosophic Frechet derivative, and we examine some of their features. The relationship between the neutrophilic Frechet derivative and the neutrophilic Gateaux derivative is examined.

Keywords: Neutrosophic differentiation; Neutrosophic continuity; Neutrosophic Gateaux derivative; Neutrosophic Frechet derivative.

1. Introduction

The notion of normed linear space is essential to functional analysis. Dimension in normed linear space is catching the attention of researchers more and more. In recent years, many researchers have worked to expand the idea of n-normed linear space. The fuzzy set is a great theory for handling uncertainty that was invented by Zadeh [26]. This idea served as the cornerstone for a broad range of mathematical applications, as well as a large number of situations in everyday life.

In 1986, Atanassov [2] investigated intuitionistic fuzzy sets, which are distinguished by a membership function and non-membership function for each in the universe. Smarandache [23–25] later developed another concept known as neutrosophic set by introducing an intermediate membership function. Katsaras [16] presented the idea of a fuzzy norm in 1984. The fuzzy norm on a linear space was first proposed by Felbin [10] in 1992. Cheng Moderson [5] proposed another fuzzy norm idea for a linear space. After Cheng and Moderson's fuzzy norm formulation was refined by Bag and Samanta [3], they created the concepts of continuity and boundedness of a linear operator with respect to their fuzzy norm. Frechet and Gateaux Bivas Dinda, Samanta, and Bera [4] are the ones who originally introduced derivative in intuitionistic fuzzy normed linear spaces. Neutrosophic norm in a linear space was proposed by Dass Sarath Kumar and Prakasam Muralikrishna [19].

In this article, we define neutrosophic derivative in R, neutrosophic Gateaux derivative, and neutrosophic Frechet derivative on a linear space and examine some of their features. The relationships between the neutrosophic Gateaux derivative and the neutrosophic Frechet derivative are also discussed.

2. Preliminaries

Definition 2.1. [19] A 7-tuple $(\Xi, \mathfrak{A}, \mathfrak{B}, \mathfrak{W}, *, \diamond, \otimes)$ is said to be a Neutrosophic Normed Space [NNS], if Ξ be a linear space over the field $F = (\mathbb{R}$ or C), Let $*$ be a continuous t-norm,$\diamond$, $\otimes$ be a continuous t-conorm and $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{W}$ are functions from $\Xi \times \mathbb{R}^+ \to [0, 1]$, fulfilling the following conditions for every $\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}} \in \mathbb{R}^+$ and $\mathfrak{d}, \tau \in \mathbb{R}$.
(i) $0 \le \mathfrak{A}(\tilde{\mathfrak{p}}, \tau) \le 1; 0 \le \mathfrak{B}(\tilde{\mathfrak{p}}, \tau) \le 1; 0 \le \mathfrak{W}(\tilde{\mathfrak{p}}, \tau) \le 1;$
(ii) $0 \le \mathfrak{A}(\tilde{\mathfrak{p}}, \tau) + \mathfrak{B}(\tilde{\mathfrak{p}}, \tau) + \mathfrak{W}(\tilde{\mathfrak{p}}, \tau) \le 3;$
(iii) $\mathfrak{A}(\tilde{\mathfrak{p}}, \tau) > 0;$
(iv) $\mathfrak{A}(\tilde{\mathfrak{p}}, \tau) = 1 \Leftrightarrow \tilde{\mathfrak{p}} = \theta$, θ is null vector;
(v) $\mathfrak{A}(c\tilde{\mathfrak{p}}, \tau) = \mathfrak{A}\left(\tilde{\mathfrak{p}}, \frac{\tau}{|c|}\right), \forall c \in F$ and $c \neq 0;$
(vi) $\mathfrak{A}(\tilde{\mathfrak{p}}, \mathfrak{d}) * \mathfrak{A}(\tilde{\mathfrak{q}}, \tau) \le \mathfrak{A}(\tilde{\mathfrak{p}} + \tilde{\mathfrak{q}}, \mathfrak{d} + \tau);$
(vii) $\mathfrak{A}(\tilde{\mathfrak{p}}, \cdot)$ is non-decreasing function of $\mathbb{R}^+$ and $\lim_{\tau \to \infty} \mathfrak{A}(\tilde{\mathfrak{p}}, \tau) = 1;$
(viii) $\mathfrak{B}(\tilde{\mathfrak{p}}, \tau) < 1;$
(ix) $\mathfrak{B}(\tilde{\mathfrak{p}}, \tau) = 0 \Leftrightarrow \tilde{\mathfrak{p}} = \theta;$
(x) $\mathfrak{B}(c\tilde{\mathfrak{p}}, \tau) = \mathfrak{B}\left(\tilde{\mathfrak{p}}, \frac{\tau}{|c|}\right) \forall c \in F$ and $c \neq 0;$
(xi) $\mathfrak{B}(\tilde{\mathfrak{p}}, \mathfrak{d}) \diamond \mathfrak{B}(\tilde{\mathfrak{q}}, \tau) \ge \mathfrak{B}(\tilde{\mathfrak{p}} + \tilde{\mathfrak{q}}, \mathfrak{d} + \tau);$
(xii) $\mathfrak{B}(\tilde{\mathfrak{p}}, \cdot)$ is non-increasing function of $\mathbb{R}^+$and $\lim_{\tau \to \infty} \mathfrak{B}(\tilde{\mathfrak{p}}, \tau) = 0.$
(xiii) $\mathfrak{W}(\tilde{\mathfrak{p}}, \tau) < 1;$
(xiv) $\mathfrak{W}(\tilde{\mathfrak{p}}, \tau) = 0 \Leftrightarrow \tilde{\mathfrak{p}} = \theta;$
(xv)$\mathfrak{W}(c\tilde{\mathfrak{p}}, \tau) = \mathfrak{W}\left(\tilde{\mathfrak{p}}, \frac{\tau}{|c|}\right) \forall c \in F$ and $c \neq 0;$
(xvi) $\mathfrak{W}(\tilde{\mathfrak{p}}, \mathfrak{d}) \otimes \mathfrak{W}(\tilde{\mathfrak{q}}, \tau) \ge \mathfrak{W}(\tilde{\mathfrak{p}} + \tilde{\mathfrak{q}}, \mathfrak{d} + \tau);$
(xvii) $\mathfrak{W}(\tilde{\mathfrak{p}}, \cdot)$ is non-increasing function of $\mathbb{R}^+$and $\lim_{\tau \to \infty} \mathfrak{W}(\tilde{\mathfrak{p}}, \tau) = 0.$

Definition 2.2. [19] The pair (Ξ, A) is called a Neutrosophic Normed Linear Space [NNLS], If A is a Neutrosophic norm on a linear space Ξ.
For the NNLS (Ξ, A), We also suppose that $\mathfrak{A}, \mathfrak{B}, \mathfrak{W}, *, \diamond, \otimes$ fulfilling the axioms listed below:
(xviii) $\left\{\begin{matrix} \dot{a} * \dot{a} = \dot{a} \\ \dot{a} \diamond \dot{a} = \dot{a} \\ \dot{a} \otimes \dot{a} = \dot{a} \end{matrix}\right\}$, for all $\dot{a} \in [0,1]$.
(xix) $\mathfrak{A}(\tilde{\mathfrak{p}}, \tau) > 0$, for every $\tau > 0 \Rightarrow \tilde{\mathfrak{p}} = \theta$.
(xx) $\mathfrak{B}(\tilde{\mathfrak{p}}, \tau) < 1$, for every $\tau > 0 \Rightarrow \tilde{\mathfrak{p}} = \theta$.
(xi) $\mathfrak{W}(\tilde{\mathfrak{p}}, \tau) < 1$, for every $\tau > 0 \Rightarrow \tilde{\mathfrak{p}} = \theta$.
(xii) For $\tilde{\mathfrak{p}} \neq \theta, \mathfrak{A}(\tilde{\mathfrak{p}}, \cdot)$ is strictly increasing on the subset $\{\tau : \mathfrak{A}(\tilde{\mathfrak{p}}, \tau) \in (0, 1)\}$ of $\mathbb{R}$ and continuous function of $\mathbb{R}$.
(xiii) For $\tilde{\mathfrak{p}} \neq \theta, \mathfrak{B}(\tilde{\mathfrak{p}}, \cdot)$ is strictly decreasing on the subset $\{\tau : \mathfrak{B}(\tilde{\mathfrak{p}}, \tau) \in (0, 1)\}$ of $\mathbb{R}$ and continuous function of $\mathbb{R}$.
(xiv) For $\tilde{\mathfrak{p}} \neq \theta, \mathfrak{W}(\tilde{\mathfrak{p}}, \cdot)$ is strictly decreasing on the subset $\{\tau : \mathfrak{W}(\tilde{\mathfrak{p}}, \tau) \in (0, 1)\}$ of $\mathbb{R}$ and continuous function of $\mathbb{R}$.

Definition 2.3. Let $\{\tilde{\mathfrak{p}}_n\}_n$ be a sequence in a NNLS $(\Xi, \mathfrak{I})$, if for given $\dot{r} > 0$; $\tau > 0$; $0 < \dot{r} < 1$, there exist an integer $n_0 \in \mathbb{N}$ such that $\mathfrak{A}(\tilde{\mathfrak{p}}_n - \tilde{\mathfrak{p}}, \tau) > 1 - \dot{r}$, $\mathfrak{B}(\tilde{\mathfrak{p}}_n - \tilde{\mathfrak{p}}, \tau) < \dot{r}$ and $\mathfrak{W}(\tilde{\mathfrak{p}}_n - \tilde{\mathfrak{p}}, \tau) < \dot{r}$ for all $n \ge n_0$ then the sequence is named to be converge to $\tilde{\mathfrak{p}} \in \Xi$.

Definition 2.4. A mapping $\zeta : (\widetilde{\Theta}, \mathfrak{I}) \rightarrow (\Xi, \mathcal{J})$ is named to be Neutrosophic continuous at $\tilde{\mathfrak{p}}_0 \in \widetilde{\Theta}$, where $(\widetilde{\Theta}, \mathfrak{I})$ and $(\Xi, \mathcal{J})$ are NNLS over the same field F, if for any given $\epsilon > 0$, $\varrho \in (0,1)$, there exists $\sigma = \sigma(\varrho, \epsilon) > 0, \dot{\eta} = \dot{\eta}(\varrho, \epsilon) \in (0,1)$ such that for every $\tilde{\mathfrak{p}} \in \widetilde{\Theta}$,

$$\mathfrak{A}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma > 1 - \dot{\eta}) \Rightarrow \mathfrak{A}_{\Xi}(\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0), \epsilon) > 1 - \varrho,$$
$$\mathfrak{B}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma) < \dot{\eta} \Rightarrow \mathfrak{B}_{\Xi}(\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0), \epsilon) < \varrho,$$
$$\mathfrak{W}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma) < \dot{\eta} \Rightarrow \mathfrak{W}_{\Xi}(\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0), \epsilon) < \varrho.$$

3. Neutrosophic Gateaux Derivative

Definition 3.1. A function $\zeta : (\mathbb{R}, \mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{W}_1 *, \diamond, \otimes) \rightarrow (\mathbb{R}, \mathfrak{A}_2, \mathfrak{B}_2, \mathfrak{W}_2, *, \diamond, \otimes)$ is named to be Neutrosophic Differentiable [ND] at $\tilde{\mathfrak{p}} \in \mathbb{R}$, where $(\mathbb{R}, \mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{W}_1 *, \diamond, \otimes)$ and $(\mathbb{R}, \mathfrak{A}_2, \mathfrak{B}_2, \mathfrak{W}_2, *, \diamond, \otimes)$ are NNLS over the same field F, if for any given $\epsilon > 0, \varrho \in (0,1)$, there exists $\sigma = \sigma(\varrho, \epsilon) > 0$, $\dot{\eta} = \dot{\eta}(\varrho, \epsilon) \in (0,1)$ such that for every $\tilde{\mathfrak{p}}(\neq \tilde{\mathfrak{p}}_0) \in \mathbb{R}$,

$$\mathfrak{A}_1(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma) > 1 - \dot{\eta} \Rightarrow \mathfrak{A}_2\left(\frac{\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0} - \zeta'(\tilde{\mathfrak{p}}_0), \epsilon\right) > 1 - \varrho,$$
$$\mathfrak{B}_1(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma) < \dot{\eta} \Rightarrow \mathfrak{B}_2\left(\frac{\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0} - \zeta'(\tilde{\mathfrak{p}}_0), \epsilon\right) < \varrho,$$
$$\mathfrak{W}_1(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma) < \dot{\eta} \Rightarrow \mathfrak{W}_2\left(\frac{\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0} - \zeta'(\tilde{\mathfrak{p}}_0), \epsilon\right) < \varrho.$$

We denote ND of ζ at $\tilde{\mathfrak{p}}_0$ by $\zeta'(\tilde{\mathfrak{p}}_0)$.

Alternative definition: A mapping $\zeta : (\mathbb{R}, \mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{W}_1 *, \diamond, \otimes) \rightarrow (\mathbb{R}, \mathfrak{A}_2, \mathfrak{B}_2, \mathfrak{W}_2, *, \diamond, \otimes)$ is named to be ND at $\tilde{\mathfrak{p}} \in \mathbb{R}$, where $(\mathbb{R}, \mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{W}_1 *, \diamond, \otimes)$ and $(\mathbb{R}, \mathfrak{A}_2, \mathfrak{B}_2, \mathfrak{W}_2, *, \diamond, \otimes)$ are NNLS over the same field F, if for every $\tau > 0$.

$$\lim_{\mathfrak{A}_1(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau) \rightarrow 1} \mathfrak{A}_2\left(\frac{\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0} - \zeta'(\tilde{\mathfrak{p}}_0), \tau\right) = 1,$$
$$\lim_{\mathfrak{B}_1(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau) \rightarrow 0} \mathfrak{B}_2\left(\frac{\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0} - \zeta'(\tilde{\mathfrak{p}}_0), \tau\right) = 0,$$
$$\lim_{\mathfrak{W}_1(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau) \rightarrow 0} \mathfrak{W}_2\left(\frac{\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0} - \zeta'(\tilde{\mathfrak{p}}_0), \tau\right) = 0,$$

$\zeta'(\tilde{\mathfrak{p}}_0)$ is called ND of ζ at $\tilde{\mathfrak{p}}_0$.

Example 3.2. Let $(\mathbb{R}, \mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{W}_1 *, \diamond, \otimes)$ and $(\mathbb{R}, \mathfrak{A}_2, \mathfrak{B}_2, \mathfrak{W}_2, *, \diamond, \otimes)$ be two NNLS over the same field $\mathbb{R}$. Let $\mathfrak{A}_1(\tilde{\mathfrak{p}}, \tau) = \mathfrak{A}_2(\tilde{\mathfrak{p}}, \tau) = \frac{\tau}{\tau + |\tilde{\mathfrak{p}}|'}$, $\mathfrak{B}_1(\tilde{\mathfrak{p}}, \tau) = \mathfrak{B}_2(\tilde{\mathfrak{p}}, \tau) = \frac{|\tilde{\mathfrak{p}}|}{\tau + |\tilde{\mathfrak{p}}|}$ and $\mathfrak{W}_1(\tilde{\mathfrak{p}}, \tau) = \mathfrak{W}_2(\tilde{\mathfrak{p}}, \tau) = \frac{|\tilde{\mathfrak{p}}|}{\tau}$. Let $\dot{\mathfrak{a}} * \dot{\mathfrak{b}} = \dot{\mathfrak{a}}\dot{\mathfrak{b}}$ and $\dot{\mathfrak{a}} \diamond \dot{\mathfrak{b}} = \dot{\mathfrak{a}} \otimes \dot{\mathfrak{b}} = \dot{\mathfrak{a}} + \dot{\mathfrak{b}} - \dot{\mathfrak{a}}\dot{\mathfrak{b}}$. A mapping $\zeta : (\mathbb{R}, \mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{W}_1 *, \diamond, \otimes) \rightarrow (\mathbb{R}, \mathfrak{A}_2, \mathfrak{B}_2, \mathfrak{W}_2, *, \diamond, \otimes)$ defined by $\zeta(\tilde{\mathfrak{p}}) = \tilde{\mathfrak{p}}^2$. Let $\tilde{\mathfrak{p}}_0 \in \mathbb{R}$ be any point. Clearly, $\lim_{\mathfrak{A}_1(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau) \rightarrow 1} \mathfrak{A}_2\left(\frac{\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0} - \zeta'(\tilde{\mathfrak{p}}_0), \tau\right) = 1$,
$\lim_{\mathfrak{B}_1(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau) \rightarrow 0} \mathfrak{B}_2\left(\frac{\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0} - \zeta'(\tilde{\mathfrak{p}}_0), \tau\right) = 0$,
$\lim_{\mathfrak{W}_1(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau) \rightarrow 0} \mathfrak{W}_2\left(\frac{\zeta(\tilde{\mathfrak{p}}) - f(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0} - \zeta'(\tilde{\mathfrak{p}}_0), \tau\right) = 0$. Therefore ζ is ND at $\tilde{\mathfrak{p}}_0$.

Theorem 3.3. Let $\zeta: (\mathbb{R}, \mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{W}_1 *, \diamond, \otimes) \rightarrow (\mathbb{R}, \mathfrak{A}_2, \mathfrak{B}_2, \mathfrak{W}_2, *, \diamond, \otimes)$ and $g: (\mathbb{R}, \mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{W}_1 *, \diamond, \otimes) \rightarrow (\mathbb{R}, \mathfrak{A}_2, \mathfrak{B}_2, \mathfrak{W}_2, *, \diamond, \otimes)$ are two Neutrosophic differentiable functions differentiable at $\tilde{\mathfrak{p}}_0$ and $(\mathbb{R}, \mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{W}_1 *, \diamond, \otimes)$ and $(\mathbb{R}, \mathfrak{A}_2, \mathfrak{B}_2, \mathfrak{W}_2, *, \diamond, \otimes)$ fulfilling the condition (xviii). Then for $K \in F, K\zeta + g$ is ND at $\tilde{\mathfrak{p}}_0$ and $(K\zeta + g)'(\tilde{\mathfrak{p}}_0) = K\zeta'(\tilde{\mathfrak{p}}_0) + g'(\tilde{\mathfrak{p}}_0)$.

Proof. Since ζ and g are ND at $\tilde{\mathfrak{p}}_0$. So that, for any given $\epsilon > 0, \varrho \in (0,1)$, there exists $\sigma = \sigma(\varrho, \epsilon) > 0, \dot{\eta} = \dot{\eta}(\varrho, \epsilon) \in (0,1)$ such that for every $\tilde{\mathfrak{p}} \in \mathbb{R}$,

$$\mathfrak{A}_1(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma) > 1 - \dot{\eta} \Rightarrow \mathfrak{A}_2\left(\frac{\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0} - \zeta'(\tilde{\mathfrak{p}}_0), \epsilon\right) > 1 - \varrho$$
$$\mathfrak{B}_1(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma) < \dot{\eta} \Rightarrow \mathfrak{B}_2\left(\frac{\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0} - \zeta'(\tilde{\mathfrak{p}}_0), \epsilon\right) < \varrho,$$
$$\mathfrak{W}_1(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma) < \dot{\eta} \Rightarrow \mathfrak{W}_2\left(\frac{\zeta(\tilde{\mathfrak{p}}) - \zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}} - \zeta'(\tilde{\mathfrak{p}}_0), \epsilon\right) < \varrho.$$

$$\mathfrak{A}_1(\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0,\sigma)>1-\dot{\eta}\Rightarrow\mathfrak{A}_2\left(\frac{g(\tilde{\mathfrak{p}})-g(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0}-g'(\tilde{\mathfrak{p}}_0),\epsilon\right)>1-\varrho,$$
$$\mathfrak{B}_1(\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0,\sigma)<\dot{\eta}\Rightarrow\mathfrak{B}_2\left(\frac{g(\tilde{\mathfrak{p}})-g(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0}-g'(\tilde{\mathfrak{p}}_0),\epsilon\right)<\varrho,$$
$$\mathfrak{W}_1(\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0,\sigma)<\dot{\eta}\Rightarrow\mathfrak{W}_2\left(\frac{g(\tilde{\mathfrak{p}})-g(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0}-g'(\tilde{\mathfrak{p}}_0),\epsilon\right)<\varrho.$$

Now,

$$\mathfrak{A}_2\left(\frac{(K\zeta+g)(\tilde{\mathfrak{p}})-(K\zeta+g)(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0}-\left(K\zeta'(\tilde{\mathfrak{p}}_0)+g'(\tilde{\mathfrak{p}}_0)\right),\epsilon\right)$$
$$=\mathfrak{A}_2\left(\frac{K\zeta(\tilde{\mathfrak{p}})+g(\tilde{\mathfrak{p}})-K\zeta(\tilde{\mathfrak{p}}_0)-g(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0}-K\zeta'(\tilde{\mathfrak{p}}_0)-g'(\tilde{\mathfrak{p}}_0),\epsilon\right)$$
$$\geq\mathfrak{A}_2\left(\frac{K\zeta(\tilde{\mathfrak{p}})-K\zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0}-K\zeta'(\tilde{\mathfrak{p}}_0),\frac{\epsilon}{2}\right)*\mathfrak{A}_2\left(\frac{g(\tilde{\mathfrak{p}})-g(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0}-g'(\tilde{\mathfrak{p}}_0),\frac{\epsilon}{2}\right)$$
$$=\mathfrak{A}_2\left(\frac{\zeta(\tilde{\mathfrak{p}})-\zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0}-\zeta'(\tilde{\mathfrak{p}}_0),\frac{\epsilon}{2|K|}\right)*\mathfrak{A}_2\left(\frac{g(\tilde{\mathfrak{p}})-g(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0}-g'(\tilde{\mathfrak{p}}_0),\frac{\epsilon}{2}\right)$$
$$>(1-\varrho)*(1-\varrho)=(1-\varrho),\text{ whenever }\mathfrak{A}_1(\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0,\sigma)>1-\dot{\eta},$$

$$\mathfrak{B}_2\left(\frac{(K\zeta+g)(\tilde{\mathfrak{p}})-(K\zeta+g)(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0}-\left(K\zeta'(\tilde{\mathfrak{p}}_0)+g'(\tilde{\mathfrak{p}}_0)\right),\epsilon\right)$$
$$=\mathfrak{B}_2\left(\frac{K\zeta(\tilde{\mathfrak{p}})+g(\tilde{\mathfrak{p}})-K\zeta(\tilde{\mathfrak{p}}_0)-g(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0}-K\zeta'(\tilde{\mathfrak{p}}_0)-g'(\tilde{\mathfrak{p}}_0),\epsilon\right)$$
$$\geq\mathfrak{B}_2\left(\frac{K\zeta(\tilde{\mathfrak{p}})-K\zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0}-K\zeta'(\tilde{\mathfrak{p}}_0),\frac{\epsilon}{2}\right)\diamond\mathfrak{B}_2\left(\frac{g(\tilde{\mathfrak{p}})-g(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0}-g'(\tilde{\mathfrak{p}}_0),\frac{\epsilon}{2}\right)$$
$$=\mathfrak{B}_2\left(\frac{\zeta(\tilde{\mathfrak{p}})-\zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0}-\zeta'(\tilde{\mathfrak{p}}_0),\frac{\epsilon}{2|K|}\right)\diamond\mathfrak{B}_2\left(\frac{g(\tilde{\mathfrak{p}})-g(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0}-g'(\tilde{\mathfrak{p}}_0),\frac{\epsilon}{2}\right)$$
$$<\varrho\diamond\varrho=\varrho,\text{ whenever }\mathfrak{B}_1(\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0,\sigma)<\dot{\eta}\text{ and}$$

$$\mathfrak{W}_2\left(\frac{(K\zeta+g)(\tilde{\mathfrak{p}})-(K\zeta+g)(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0}-\left(K\zeta'(\tilde{\mathfrak{p}}_0)+g'(\tilde{\mathfrak{p}}_0)\right),\epsilon\right)$$
$$=\mathfrak{W}_2\left(\frac{K\zeta(\tilde{\mathfrak{p}})+g(\tilde{\mathfrak{p}})-K\zeta(\tilde{\mathfrak{p}}_0)-g(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0}-K\zeta'(\tilde{\mathfrak{p}}_0)-g'(\tilde{\mathfrak{p}}_0),\epsilon\right)$$
$$\geq\mathfrak{W}_2\left(\frac{K\zeta(\tilde{\mathfrak{p}})-K\zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0}-K\zeta'(\tilde{\mathfrak{p}}_0),\frac{\epsilon}{2}\right)\otimes\mathfrak{W}_2\left(\frac{g(\tilde{\mathfrak{p}})-g(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0}-g'(\tilde{\mathfrak{p}}_0),\frac{\epsilon}{2}\right)$$
$$=\mathfrak{W}_2\left(\frac{\zeta(\tilde{\mathfrak{p}})-\zeta(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0}-\zeta'(\tilde{\mathfrak{p}}_0),\frac{\epsilon}{2|K|}\right)\otimes\mathfrak{W}_2\left(\frac{g(\tilde{\mathfrak{p}})-g(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0}-g'(\tilde{\mathfrak{p}}_0),\frac{\epsilon}{2}\right)$$
$$<\varrho\otimes\varrho=\varrho,\text{ whenever }\mathfrak{W}_1(\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0,\sigma)<\dot{\eta}.$$

So, $K\zeta+g$ is ND at $\tilde{\mathfrak{p}}_0\in\mathbb{R}$ and $(K\zeta+g)'(\tilde{\mathfrak{p}}_0)=K\zeta'(\tilde{\mathfrak{p}}_0)+g'(\tilde{\mathfrak{p}}_0)$.

Definition 3.4. Let $(\widetilde{\Theta},\mathfrak{I})$ and $(\Xi,\mathcal{J})$ be two NNLS over the same field F. An operator Υ from $(\widetilde{\Theta},\mathfrak{I})$ to $(\Xi,\mathcal{J})$ is named to be Neutrosophic Gateaux differentiable [NGD] at $\tilde{\mathfrak{p}}_0\in\widetilde{\Theta}$, where, $(\widetilde{\Theta},\mathfrak{I})$ and $(\Xi,\mathcal{J})$ are NNLS over the same field F, if there exists a Neutrosophic continuous linear operator $G:(\widetilde{\Theta},\mathfrak{I})\to(\Xi,\mathcal{J})$ (generally depends upon $\tilde{\mathfrak{p}}_0$) and for any given $\epsilon>0,\varrho\in(0,1)$, there exists $\sigma=\sigma(\varrho,\epsilon)>0$, $\dot{\eta}=\dot{\eta}(\varrho,\epsilon)\in(0,1)$ such that for every $\tilde{\mathfrak{p}}\in\widetilde{\Theta}$ and $s(\neq 0)\in\mathbb{R}$,

$$\mathfrak{A}_{\widetilde{\Theta}}(\mathfrak{d},\sigma)>1-\dot{\eta}\Rightarrow\mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G(\tilde{\mathfrak{p}}),\epsilon\right)>1-\epsilon,$$
$$\mathfrak{B}_{\widetilde{\Theta}}(\mathfrak{d},\delta)<\dot{\eta}\Rightarrow\mathfrak{B}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G(\tilde{\mathfrak{p}}),\epsilon\right)<\sigma,$$
$$\mathfrak{W}_{\widetilde{\Theta}}(\mathfrak{d},\delta)<\dot{\eta}\Rightarrow\mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G(\tilde{\mathfrak{p}}),\epsilon\right)<\sigma,$$

The operator G becomes known to as NGD of Υ at $\tilde{\mathfrak{p}}_0$. and it is represented by $D_{\zeta(\tilde{\mathfrak{p}}_0)}$.

Alternative definition: An operator Υ from $(\widetilde{\Theta},\mathfrak{I})$ to $(\Xi,\mathcal{J})$ is said to be Neutrosophic Gateaux and Frechet Derivative differentiable at $\tilde{\mathfrak{p}}_0\in\widetilde{\Theta}$, where, $(\widetilde{\Theta},\mathfrak{I})$ and $(\Xi,\mathcal{J})$ are NNLS over the same field F,if there exists a Neutrosophic continuous linear operator $G:(\widetilde{\Theta},\mathfrak{I})\to(\Xi,\mathcal{J})$such that for every $\tilde{\mathfrak{p}}\in\widetilde{\Theta},\tau>0$ and $\mathfrak{d}(\neq 0)\in\mathbb{R}$

$$\lim_{\mathfrak{A}_{\widetilde{\Theta}}(\mathfrak{d},\tau)\to 1}\mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G(\tilde{\mathfrak{p}}),\tau\right)=1,$$

$$\lim_{\mathfrak{V}_{\widetilde{\Theta}}(\mathfrak{d},\tau)\to 0} \mathfrak{V}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}} - G(\tilde{\mathfrak{p}}),\tau\right) = 0,$$
$$\lim_{\mathfrak{W}_{\widetilde{\Theta}}(\mathfrak{d},\tau)\to 0} \mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}} - G(\tilde{\mathfrak{p}}),\tau\right) = 0.$$

Here, the operator G is called NGD of Υ at $\tilde{\mathfrak{p}}_0$ and it is denoted by $D_{\zeta(\tilde{\mathfrak{p}}_0)}$.

Example 3.5. Let $\widetilde{\Theta} = \Xi = F = \mathbb{R}$ and let $\tilde{\mathfrak{p}}_0 \in \mathbb{R}$ be any point. Let $(\mathbb{R}, \mathfrak{U}_1, \mathfrak{V}_1, \mathfrak{W}_1 *, \diamond, \otimes)$ and $(\mathbb{R}, \mathfrak{U}_2, \mathfrak{V}_2, \mathfrak{W}_2, *, \diamond, \otimes)$ are NNLS over the same field $\mathbb{R}$. Let $\mathfrak{U}_1(\tilde{\mathfrak{p}},\tau) = \mathfrak{U}_2(\tilde{\mathfrak{p}},\tau) = \frac{\tau}{\tau+|\tilde{\mathfrak{p}}|}$, $\mathfrak{V}_1(\tilde{\mathfrak{p}},\tau) = \mathfrak{V}_2(\tilde{\mathfrak{p}},\tau) = \frac{|\tilde{\mathfrak{p}}|}{\tau+|\tilde{\mathfrak{p}}|}$ and $\mathfrak{W}_1(\tilde{\mathfrak{p}},\tau) = \mathfrak{W}_2(\tilde{\mathfrak{p}},\tau) = \frac{|\tilde{\mathfrak{p}}|}{\tau}$. Let $\dot{a} * \dot{b} = \dot{a}\dot{b}$ and $\dot{a} \diamond \dot{b} = \dot{a} \otimes \dot{b} = \dot{a} + \dot{b} - \dot{a}\dot{b}$. An operator $\Upsilon: (\widetilde{\Theta}, \mathfrak{I}) \to (\Xi, \mathcal{J})$ be defined by $\Upsilon(\tilde{\mathfrak{p}}) = \tilde{\mathfrak{p}}$. There exist a neutrosophic continuous linear operator $G: (\widetilde{\Theta}, \mathfrak{I}) \to (\Xi, \mathcal{J})$ be defined by $G(\tilde{\mathfrak{p}}) = \frac{\tilde{\mathfrak{p}}}{2}$ such that

$$\lim_{\mathfrak{U}_{\widetilde{\Theta}}(\mathfrak{d},\tau)\to 1} \mathfrak{U}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}} - G(\tilde{\mathfrak{p}}),\tau\right) = 1,$$
$$\lim_{\mathfrak{V}_{\widetilde{\Theta}}(\mathfrak{d},\tau)\to 0} \mathfrak{V}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{s} - G(\tilde{\mathfrak{p}}),\tau\right) = 0,$$

$\lim_{\mathfrak{W}_{\widetilde{\Theta}}(\mathfrak{d},\tau)\to 0} \mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}} - G(\tilde{\mathfrak{p}}),\tau\right) = 0$. Hence Υ is Neutrosophic Gateaux and Frechet differentiable at $\tilde{\mathfrak{p}}_0$.

Theorem 3.6. Let $\Upsilon: (\widetilde{\Theta}, \mathfrak{I}) \to (\Xi, \mathcal{J})$ be a linear operator, where $(\widetilde{\Theta}, \mathfrak{I})$ and $(\Xi, \mathcal{J})$ are two NNLS satisfying (xviii), (xix), (xx) and (xxi). If Υ is NGD at $\tilde{\mathfrak{p}}_0$ then it is unique at $\tilde{\mathfrak{p}}_0$.

Proof. Let G_1, G_2 be two NGD of Υ at $\tilde{\mathfrak{p}}_0$. Then for any given $\epsilon > 0, \varrho \in (0,1), \exists\, \sigma = \sigma(\varrho,\epsilon) > 0, \dot{\eta} = \dot{\eta}(\varrho,\epsilon) \in (0,1)$ such that for every $\tilde{\mathfrak{p}} \in U$ and
$\mathfrak{d}(\neq 0) \in \mathbb{R}$,

$$\mathfrak{U}_{\widetilde{\Theta}}(\mathfrak{d},\sigma) > 1-\dot{\eta} \Rightarrow \mathfrak{U}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}} - G_1(\tilde{\mathfrak{p}}),\epsilon\right) > 1-\varrho,$$
$$\mathfrak{V}_{\widetilde{\Theta}}(\mathfrak{d},\sigma) < \dot{\eta} \Rightarrow \mathfrak{V}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}} - G_1(\tilde{\mathfrak{p}}),\epsilon\right) < \varrho,$$
$$\mathfrak{W}_{\widetilde{\Theta}}(\mathfrak{d},\sigma) < \dot{\eta} \Rightarrow \mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}} - G_1(\tilde{\mathfrak{p}}),\epsilon\right) < \varrho \text{ and}$$
$$\mathfrak{U}_{\widetilde{\Theta}}(\mathfrak{d},\sigma) > 1-\dot{\eta} \Rightarrow \mathfrak{U}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}} - G_2(\tilde{\mathfrak{p}}),\epsilon\right) > 1-\varrho,$$
$$\mathfrak{V}_{\widetilde{\Theta}}(\mathfrak{d},\sigma) < \dot{\eta} \Rightarrow \mathfrak{V}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}} - G_2(\tilde{\mathfrak{p}}),\epsilon\right) < \varrho,$$
$$\mathfrak{W}_{\widetilde{\Theta}}(\mathfrak{d},\sigma) < \dot{\eta} \Rightarrow \mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}} - G_2(\tilde{\mathfrak{p}}),\epsilon\right) < \varrho.$$

$$\mathfrak{U}_{\Xi}(G_1(\tilde{\mathfrak{p}}) - G_2(\tilde{\mathfrak{p}}),\tau) = \mathfrak{U}_{\Xi}\left(\left\{\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}} - G_1(\tilde{\mathfrak{p}})\right\} - \left\{\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}} - G_2(\tilde{\mathfrak{p}})\right\},\tau\right)$$
$$= \mathfrak{U}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}} - G_1(\tilde{\mathfrak{p}}),\frac{\tau}{2}\right) * \mathfrak{U}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}} - G_2(\tilde{\mathfrak{p}}),\frac{\tau}{2}\right)$$
$$> (1-\varrho) * (1-\varrho) = (1-\varrho), \ \forall \varrho \in (0,1).$$

Therefore, $\mathfrak{U}_{\Xi}(G_1(\tilde{\mathfrak{p}}) - G_2(\tilde{\mathfrak{p}}),\tau) > 0$, $\forall t > 0$, (3.1)

$$\mathfrak{V}_{\Xi}(G_1(\tilde{\mathfrak{p}}) - G_2(\tilde{\mathfrak{p}}),\tau) \le \mathfrak{V}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}} - G_1(\tilde{\mathfrak{p}}),\frac{\tau}{2}\right) \diamond \mathfrak{V}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}} - G_2(\tilde{\mathfrak{p}}),\frac{\tau}{2}\right)$$
$$< \varrho \diamond \varrho = \varrho \ \ \forall \varrho \in (0,1).$$

$\mathfrak{V}_{\Xi}(G_1(\tilde{\mathfrak{p}}) - G_2(\tilde{\mathfrak{p}}),\tau) < 1 \ \ \forall \tau > 0,$ (3.2)

$$\mathfrak{W}_{\Xi}(G_1(\tilde{\mathfrak{p}}) - G_2(\tilde{\mathfrak{p}}),\tau) \le \mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}} - G_1(\tilde{\mathfrak{p}}),\frac{\tau}{2}\right) \otimes \mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}} - G_2(\tilde{\mathfrak{p}}),\frac{\tau}{2}\right)$$
$$< \varrho \otimes \varrho = \varrho \ \ \forall \varrho \in (0,1).$$

$\mathfrak{W}_{\Xi}(G_1(\tilde{\mathfrak{p}}) - G_2(\tilde{\mathfrak{p}}),\tau) < 1$, $\forall \tau > 0.$ (3.3)

From (3.1) , (3.2) and (3.3) we get $G_1(\tilde{\mathfrak{p}}) - G_2(\tilde{\mathfrak{p}}) = \theta$. Thus, $G_1(\tilde{\mathfrak{p}}) - G_2(\tilde{\mathfrak{p}})$.

Theorem 3.7. If Υ_1 and Υ_2 have NGD at $\tilde{\mathfrak{p}}_0$ then $\Upsilon = c\Upsilon_1 + \Upsilon_2$ has NGD at $\tilde{\mathfrak{p}}_0$, where c is a scalar.
Proof. Straight forward.

4. Neutrosophic Frechet Derivative

Definition 4.1. An operator $\Upsilon : (\widetilde{\Theta}, \mathfrak{I}) \to (\Xi, \mathcal{J})$ is named to be Neutrosophic Frechet Differentiable [NFD] at an interior $\tilde{\mathfrak{p}}_0 \in \widetilde{\Theta}$, where, $(\widetilde{\Theta}, \mathfrak{I})$ and $(\Xi, \mathcal{J})$ be two NNLS over the same field F, if there exists a continuous linear operator $\zeta: (\widetilde{\Theta}, \mathfrak{I}) \to (\Xi, \mathcal{J})$(in general depends on $\tilde{\mathfrak{p}}_0$) and if for any given $\epsilon > 0$, $\varrho \in (0,1)$, there exists $\sigma = \sigma(\varrho, \epsilon) > 0, \dot{\eta} = \dot{\eta}(\varrho, \epsilon) \in (0,1)$ such that for all $\tilde{\mathfrak{p}} \in \widetilde{\Theta}$,

$$\mathfrak{A}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma) > 1 - \dot{\eta} \Rightarrow \mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}) - \Upsilon(\tilde{\mathfrak{p}}_0) - (\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0)\zeta}{1 - \mathfrak{A}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau)}, \epsilon\right) > 1 - \varrho,$$

$$\mathfrak{B}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma) < \dot{\eta} \Rightarrow \mathfrak{B}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}) - \Upsilon(\tilde{\mathfrak{p}}_0) - (\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0)\zeta}{\mathfrak{B}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau)}, \epsilon\right) < \varrho \text{ and}$$

$$\mathfrak{W}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma) < \dot{\eta} \Rightarrow \mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}) - \Upsilon(\tilde{\mathfrak{p}}_0) - (\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0)\zeta}{\mathfrak{W}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau)}, \epsilon\right) < \varrho$$

Here, ζ is called NFD of Υ at $\tilde{\mathfrak{p}}_0$ and is represented by $D\Upsilon(\tilde{\mathfrak{p}}_0)$.

Alternative definition: An operator $\Upsilon: (\widetilde{\Theta}, \mathfrak{I}) \to (\Xi, \mathcal{J})$ is said to be NFD at an interior $\tilde{\mathfrak{p}}_0 \in U$, where, $(\widetilde{\Theta}, \mathfrak{I})$ and $(\Xi, \mathcal{J})$ be two NNLS over the same field F, if there exists a continuous linear operator $\zeta: (\widetilde{\Theta}, \mathfrak{I}) \to (\Xi, \mathcal{J})$ (in general depends on $\tilde{\mathfrak{p}}_0$) such that for every $\tau > 0$

$$\lim_{\mathfrak{A}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau) \to 1} \mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}) - \Upsilon(\tilde{\mathfrak{p}}_0) - (\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0)\zeta}{1 - \mathfrak{A}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau)}, \tau\right) = 1,$$

$$\lim_{\mathfrak{B}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau) \to 0} \mathfrak{B}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}) - \Upsilon(\tilde{\mathfrak{p}}_0) - (\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0)\zeta}{\mathfrak{B}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau)}, \tau\right) = 0 \text{ and}$$

$$\lim_{\mathfrak{W}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau) \to 0} \mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}) - \Upsilon(\tilde{\mathfrak{p}}_0) - (\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0)\zeta}{\mathfrak{W}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau)}, \tau\right) = 0.$$

Here, ζ is called NFD of Υ at $\tilde{\mathfrak{p}}_0$ and is represented by $D\Upsilon(\tilde{\mathfrak{p}}_0)$.

Theorem 4.2. Let $\Upsilon: (\widetilde{\Theta}, \mathfrak{I}) \to (\Xi, \mathcal{J})$ be a linear operator, where $(\widetilde{\Theta}, \mathfrak{I})$ and $(\Xi, \mathcal{J})$ are two NNLS satisfying (xix), (xx) and (xxi). If Υ is NFD at $\tilde{\mathfrak{p}}_0$ then it is unique at $\tilde{\mathfrak{p}}_0$.
Proof. Straight forward.

Example 4.3. Let $\widetilde{\Theta} = \Xi = \mathbb{R}$ and $[a, b]$ be an interval of $\mathbb{R}$ and $\Upsilon: [a, b] \to \mathbb{R}$. For every $\tau > 0$ define $\mathfrak{A}(\tilde{\mathfrak{p}}, \tau) = \frac{\tau}{\tau + |\tilde{\mathfrak{p}}|}$, $\mathfrak{B}(x, \tau) = \frac{|\tilde{\mathfrak{p}}|}{\tau + |\tilde{\mathfrak{p}}|}$ and $\mathfrak{W}(\tilde{\mathfrak{p}}, \tau) = \frac{|\tilde{\mathfrak{p}}|}{\tau}$, then the NFD of Υ at $\tilde{\mathfrak{p}}_0$ is ND.

Proof. If Υ is NFD at $\tilde{\mathfrak{p}}_0$ then for any given $\epsilon > 0$, $\varrho \in (0,1)$, there exists $\sigma = \sigma(\varrho, \epsilon) > 0$, $\dot{\eta} = \dot{\eta}(\varrho, \epsilon) \in (0,1)$ such that for all $\tilde{\mathfrak{p}} \in \widetilde{\Theta}$,

$$\mathfrak{A}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma) > 1 - \dot{\eta} \Rightarrow \mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}) - \Upsilon(\tilde{\mathfrak{p}}_0) - (\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0)\zeta}{1 - \mathfrak{A}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau)}, \epsilon\right) > 1 - \varrho$$

$$\mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}) - \Upsilon(\tilde{\mathfrak{p}}_0) - (\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0)\zeta}{|\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0|}, \frac{\epsilon}{\tau + |\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0|}\right) > 1 - \varrho$$

$$\mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}) - \Upsilon(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0} - \zeta, \frac{\epsilon}{\tau + |\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0|}\right) > 1 - \varrho,$$

$$\mathfrak{B}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \sigma) < \dot{\eta} \Rightarrow \mathfrak{B}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}) - \Upsilon(\tilde{\mathfrak{p}}_0) - (\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0)\zeta}{\mathfrak{B}_{\widetilde{\Theta}}(\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0, \tau)}, \epsilon\right) < \varrho$$

$$\Rightarrow \mathfrak{B}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}) - \Upsilon(\tilde{\mathfrak{p}}_0) - (\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0)\zeta}{|\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0|}, \frac{\epsilon}{\tau + |\tilde{\mathfrak{p}} - \tilde{\mathfrak{p}}_0|}\right) < \varrho$$

$$\mathfrak{V}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0}-\zeta,\frac{\epsilon}{\tau+|\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0|}\right)<\varrho \text{ and}$$

$$\mathfrak{W}_{\tilde{\Theta}}(\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0,\sigma)<\dot{\eta}\Rightarrow\mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)-(\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0)\zeta}{\mathfrak{W}_{\tilde{\Theta}}(\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0,\tau)},\epsilon\right)<\varrho$$

$$\Rightarrow\mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)-(\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0)\zeta}{|\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0|},\frac{\epsilon}{\tau+|\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0|}\right)<\varrho$$

$$\mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}})-\Upsilon(\tilde{\mathfrak{p}}_0)}{\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0}-\zeta,\frac{\epsilon}{\tau+|\tilde{\mathfrak{p}}-\tilde{\mathfrak{p}}_0|}\right)<\varrho$$

Hence, NFD of Υ at $\tilde{\mathfrak{p}}_0$ implies ND Υ at $\tilde{\mathfrak{p}}_0$ and $\Upsilon'^{(\tilde{\mathfrak{p}}_0)}=D\Upsilon(\tilde{\mathfrak{p}}_0)$.

Theorem 4.4. An operator $\Upsilon:(\tilde{\Theta},\mathfrak{I})\to(\Xi,\mathcal{J})$ is NFD at $\tilde{\mathfrak{p}}_0\in U$ then Υ is NGD at $\tilde{\mathfrak{p}}_0$.
Proof. Since Υ is NFD at $\tilde{\mathfrak{p}}_0$, therefore, for $\tau>0$ wehave

$$\mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+h)-\Upsilon(\tilde{\mathfrak{p}}_0)-D\Upsilon(\tilde{\mathfrak{p}}_0)h}{1-\mathfrak{A}_{\tilde{\Theta}}(h,\tau)},\tau\right)>1-\varrho,$$

$$\mathfrak{V}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+h)-\Upsilon(\tilde{\mathfrak{p}}_0)-D\Upsilon(\tilde{\mathfrak{p}}_0)h}{\mathfrak{V}_{\tilde{\Theta}}(h,\tau)},\tau\right)<\varrho \text{ and}$$

$$\mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+h)-\Upsilon(\tilde{\mathfrak{p}}_0)-D\Upsilon(\tilde{\mathfrak{p}}_0)h}{\mathfrak{W}_{\tilde{\Theta}}(h,\tau)},\tau\right)<\varrho.$$

Now, $\mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+h)-\Upsilon(\tilde{\mathfrak{p}}_0)-D\Upsilon(\tilde{\mathfrak{p}}_0)h}{1-\mathfrak{A}_{\tilde{\Theta}}(h,\tau)},\tau\right)>1-\varrho\Rightarrow\mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}h)-\Upsilon(\tilde{\mathfrak{p}}_0)-\mathfrak{d}D\Upsilon(\tilde{\mathfrak{p}}_0)h}{1-\mathfrak{A}_{\tilde{\Theta}}(\mathfrak{d}h,\tau)},\tau\right)>1-\varrho.$
Putting $h=\mathfrak{d}h,\mathfrak{d}\neq0$

$$\Rightarrow\mathfrak{A}_{\Xi}\left(\frac{\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}h)-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-D\Upsilon(\tilde{\mathfrak{p}}_0)h}{\frac{1}{\mathfrak{d}}\left(1-\mathfrak{A}\left(h,\frac{\tau}{|\mathfrak{d}|}\right)\right)},\tau\right)>1-\varrho,$$

$$\Rightarrow\mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}h)-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-D\Upsilon(\tilde{\mathfrak{p}}_0)h,\frac{\tau}{|\mathfrak{d}|}\left(1-\mathfrak{A}_{\tilde{\Theta}}\left(h,\frac{\tau}{|\mathfrak{d}|}\right)\right)\right)>1-\varrho,$$

$\Rightarrow\mathfrak{A}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}h)-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-D\Upsilon(\tilde{\mathfrak{p}}_0)h,\tau_1\right)>1-\varrho,$ where $\tau_1=\frac{\tau}{|\mathfrak{d}|}\left(1-\mathfrak{A}_{\tilde{\Theta}}\left(h,\frac{\tau}{|\mathfrak{d}|}\right)\right),$

$\mathfrak{V}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+h)-\Upsilon(\tilde{\mathfrak{p}}_0)-D\Upsilon(\tilde{\mathfrak{p}}_0)h}{\mathfrak{V}_{\tilde{\Theta}}(h,\tau)},\tau\right)<\varrho$

$\Rightarrow\mathfrak{V}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}h)-\Upsilon(\tilde{\mathfrak{p}}_0)-\mathfrak{d}D\Upsilon(\tilde{\mathfrak{p}}_0)h}{\mathfrak{V}_{\tilde{\Theta}}(\mathfrak{d}h,\tau)},\tau\right)<\varrho.$

Putting $h=\mathfrak{d}h,\mathfrak{d}\neq0.$

$$\Rightarrow\mathfrak{V}_{\Xi}\left(\frac{\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}h)-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-D\Upsilon(\tilde{\mathfrak{p}}_0)h}{\frac{1}{\mathfrak{d}}\mathfrak{V}_{\tilde{\Theta}}\left(h,\frac{\tau}{|\mathfrak{d}|}\right)},\tau\right)<\varrho,$$

$$\Rightarrow\mathfrak{V}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}h)-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-D\Upsilon(\tilde{\mathfrak{p}}_0)h,\quad\frac{\tau}{|\mathfrak{d}|}\mathfrak{V}_{\tilde{\Theta}}\left(h,\frac{\tau}{|\mathfrak{d}|}\right)\right)<\varrho,$$

$\Rightarrow\mathfrak{V}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}h)-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-D\Upsilon(\tilde{\mathfrak{p}}_0)h,\tau_2\right)<\varrho,$ where $\tau_2=\frac{\tau}{|\mathfrak{d}|}\mathfrak{V}_{\tilde{\Theta}}\left(h,\frac{\tau}{\mathfrak{d}}\right)$ and

$\mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+h)-\Upsilon(\tilde{\mathfrak{p}}_0)-D\Upsilon(\tilde{\mathfrak{p}}_0)h}{\mathfrak{W}_{\tilde{\Theta}}(h,\tau)},\tau\right)<\varrho,$

$\Rightarrow\mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}h)-\Upsilon(\tilde{\mathfrak{p}}_0)-\mathfrak{d}D\Upsilon(\tilde{\mathfrak{p}}_0)h}{\mathfrak{W}_{\tilde{\Theta}}(\mathfrak{d}h,\tau)},\tau\right)<\varrho.$

Putting $h=\mathfrak{d}h,\mathfrak{d}\neq0$

$$\Rightarrow\mathfrak{W}_{\Xi}\left(\frac{\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}h)-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-D\Upsilon(\tilde{\mathfrak{p}}_0)h}{\frac{1}{\mathfrak{d}}\mathfrak{W}_{\tilde{\Theta}}\left(h,\frac{\tau}{|\mathfrak{d}|}\right)},\tau\right)<\varrho,$$

$$\Rightarrow \mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}h)-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-D\Upsilon(\tilde{\mathfrak{p}}_0)h, \quad \frac{\tau}{|\mathfrak{d}|}\mathfrak{W}_{\widetilde{\Theta}}\left(h,\frac{\tau}{|\mathfrak{d}|}\right)\right)<\varrho,$$

$\Rightarrow \mathfrak{W}_{\Xi}\left(\frac{\Upsilon(\tilde{\mathfrak{p}}_0+\mathfrak{d}h)-\Upsilon(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-D\Upsilon(\tilde{\mathfrak{p}}_0)h,\tau_3\right)<\varrho$, where $\tau_3=\frac{\tau}{|\mathfrak{d}|}\mathfrak{W}_{\widetilde{\Theta}}\left(h,\frac{\tau}{|\mathfrak{d}|}\right)$.

Hence, Υ is NGD at $\tilde{\mathfrak{p}}_0$ and $D_{\Upsilon(\tilde{\mathfrak{p}}_0)h}=D\Upsilon(\tilde{\mathfrak{p}}_0)h$.

Theorem 4.5. Let $P:\widetilde{\Theta}\subset X\to\Xi\subset Y$ and $Q:\Xi\to Z$ be two linear operator satisfying (xviii). Suppose P is Neutrosophic continuous and has NGD at $\tilde{\mathfrak{p}}_0\in\widetilde{\Theta}$ and Q has NFD at Neutrosophic Gateaux and Frechet Derivative $y_0=P(\tilde{\mathfrak{p}}_0)$. Then $R=QP$ has NGD at $\tilde{\mathfrak{p}}_0$ and $D_{R(\tilde{\mathfrak{p}}_0)}=DQ(\tilde{\mathfrak{q}}_0)D_{P(\tilde{\mathfrak{p}}_0)}$.

Proof. For convenience, we write $G=D_{P(\tilde{\mathfrak{p}}_0)}$ and $\zeta=DQ(\tilde{\mathfrak{q}}_0)$.

Let $\tilde{\mathfrak{p}}\in X$ and we further write $\Delta\tilde{\mathfrak{q}}=P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0)$. Then

$\mathfrak{A}\left(\frac{R(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-R(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-\zeta G,\tau\right)=\mathfrak{A}\left(\frac{QP(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-QP(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-\zeta G,\tau\right)=\mathfrak{A}\left(\frac{\zeta(\Delta\tilde{\mathfrak{q}})+A(\Delta\tilde{\mathfrak{q}})}{\mathfrak{d}}-\zeta G,\tau\right)$,

where $A(\Delta\tilde{\mathfrak{q}})=Q(\tilde{\mathfrak{q}}_0+\Delta\tilde{\mathfrak{q}})-Q(\tilde{\mathfrak{q}}_0)-\zeta(\Delta\tilde{\mathfrak{q}})$

$$=\mathfrak{A}\left(\zeta\frac{P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}+\frac{A(\Delta\tilde{\mathfrak{q}})}{\mathfrak{d}}-\zeta G,\tau\right)$$

$$\geq\mathfrak{A}\left(\zeta\frac{P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-\zeta G,\frac{\tau}{2}\right)*\mathfrak{A}\left(\frac{A(\Delta\tilde{\mathfrak{q}})}{\mathfrak{A}(\Delta\tilde{\mathfrak{q}},\tau)}\frac{\mathfrak{A}(P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0),\tau_1)}{\mathfrak{d}},\frac{\tau}{2}\right)$$

$$=\mathfrak{A}\left(\frac{P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G,\frac{\tau}{2\mathfrak{A}(\zeta,\tau_2)}\right)*\mathfrak{A}\left(\frac{A(\Delta\tilde{\mathfrak{q}})}{\mathfrak{A}(\Delta\tilde{\mathfrak{q}},\tau_1)}\frac{\tau s}{2\mathfrak{A}(P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0),\tau_1)}\right)$$

$$=\mathfrak{A}\left(\frac{P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G,\frac{\tau}{2\mathfrak{A}(\zeta,\tau_2)}\right)*\mathfrak{A}\left(\frac{Q(\tilde{\mathfrak{q}}_0+\Delta\tilde{\mathfrak{q}})-Q(\tilde{\mathfrak{q}}_0)-\zeta(\Delta\tilde{\mathfrak{q}})}{\mathfrak{A}(\Delta\tilde{\mathfrak{q}},\tau_1)},\frac{\tau\mathfrak{d}}{2\mathfrak{A}(P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0),\tau_1)}\right)$$

$$>(1-\varrho)*(1-\varrho)=(1-\varrho).$$

Since P has NGD and Q has NFD.

$$\mathfrak{B}\left(\frac{R(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-R(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-\zeta G,\tau\right)=\mathfrak{B}\left(\frac{QP(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-QP(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-\zeta G,\tau\right)=\mathfrak{B}\left(\frac{\zeta(\Delta\tilde{\mathfrak{q}})+A(\Delta\tilde{\mathfrak{q}})}{\mathfrak{d}}-\zeta G,\tau\right)$$

where $A(\Delta\tilde{\mathfrak{q}})=Q(\tilde{\mathfrak{q}}_0+\Delta\tilde{\mathfrak{q}})-Q(\tilde{\mathfrak{q}}_0)-\zeta(\Delta\tilde{\mathfrak{q}})$

$$=\mathfrak{B}\left(\zeta\frac{P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}+\frac{A(\Delta\tilde{\mathfrak{q}})}{\mathfrak{d}}-\zeta G,\tau\right)$$

$$\leq\mathfrak{B}\left(\zeta\frac{P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-\zeta G,\frac{\tau}{2}\right)\diamond\mathfrak{B}\left(\frac{A(\Delta\tilde{\mathfrak{q}})}{1-\mathfrak{B}(\Delta\tilde{\mathfrak{q}},\tau_1)}\frac{1-\mathfrak{B}(P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0),\tau_1)}{\mathfrak{d}},\frac{\tau}{2}\right)$$

$$=\mathfrak{B}\left(\frac{P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G,\frac{\tau}{2\mathfrak{B}(\zeta,\tau_2)}\right)$$

$$\diamond\,\mathfrak{B}\left(\frac{Q(\tilde{\mathfrak{q}}_0+\Delta\tilde{\mathfrak{q}})-Q(\tilde{\mathfrak{q}}_0)-\zeta(\Delta\tilde{\mathfrak{q}})}{1-\mathfrak{B}(\Delta\tilde{\mathfrak{q}},\tau_1)},\frac{\tau\mathfrak{d}}{2\left(1-\mathfrak{B}(P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0),\tau_1)\right)}\right)$$

$<\varrho\diamond\varrho=\varrho$ and

$\mathfrak{W}\left(\frac{R(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-R(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-\zeta G,\tau\right)=\mathfrak{W}\left(\frac{QP(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-QP(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-\zeta G,\tau\right)=\mathfrak{W}\left(\frac{\zeta(\Delta\tilde{\mathfrak{q}})+A(\Delta\tilde{\mathfrak{q}})}{\mathfrak{d}}-\zeta G,\tau\right)$,

where $A(\Delta\tilde{\mathfrak{q}})=Q(\tilde{\mathfrak{q}}_0+\Delta\tilde{\mathfrak{q}})-Q(\tilde{\mathfrak{q}}_0)-\zeta(\Delta\tilde{\mathfrak{q}})$

$$=\mathfrak{W}\left(\zeta\frac{P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}+\frac{A(\Delta\tilde{\mathfrak{q}})}{\mathfrak{d}}-\zeta G,\tau\right)$$

$$\leq\mathfrak{W}\left(\zeta\frac{P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-\zeta G,\frac{\tau}{2}\right)\otimes\mathfrak{W}\left(\frac{A(\Delta\tilde{\mathfrak{q}})}{1-\mathfrak{W}(\Delta\tilde{\mathfrak{q}},\tau_1)}\frac{1-\mathfrak{W}(P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0),\tau_1)}{\mathfrak{d}},\frac{\tau}{2}\right)$$

$$=\mathfrak{W}\left(\frac{P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0)}{\mathfrak{d}}-G,\frac{\tau}{2\mathfrak{W}(\zeta,\tau_2)}\right)$$

$$\otimes\,\mathfrak{W}\left(\frac{Q(\tilde{\mathfrak{q}}_0+\Delta\tilde{\mathfrak{q}})-Q(\tilde{\mathfrak{q}}_0)-\zeta(\Delta\tilde{\mathfrak{q}})}{1-\mathfrak{W}(\Delta\tilde{\mathfrak{q}},\tau_1)},\frac{\tau\mathfrak{d}}{2\left(1-\mathfrak{W}(P(\tilde{\mathfrak{p}}_0+\mathfrak{d}\tilde{\mathfrak{p}})-P(\tilde{\mathfrak{p}}_0),\tau_1)\right)}\right)$$

$<\varrho\otimes\varrho=\varrho$.

Since P has NGD and Q has NFD. Hence $\boldsymbol{R}=\boldsymbol{QP}$ has NGD at $\tilde{\mathfrak{p}}_0$ and $\boldsymbol{D}_{\boldsymbol{R}(\tilde{\mathfrak{p}}_0)}=\boldsymbol{DQ}(\tilde{\mathfrak{q}}_0)\boldsymbol{D}_{\boldsymbol{P}(\tilde{\mathfrak{p}}_0)}$.

Conclusion: In this article we present the idea of Neutrosophic derivative, Neutrosophic Gateaux derivative and Neutrosophic Frechet derivative and we explore some of the properties of this concepts . Moreover, we provide non-trivial examples. We have discussed about the relation between Neutrosophic Gateaux derivative and Neutrosophic Frechet derivative.

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On Treating Input Oriented Data Envelopment Analysis Model under Neutrosophic Environment

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Abstract: Data Envelopment Analysis (DEA) stands out as the most commonly employed approach for assessing the overall performance of a group of similar Decision-Making Units (DMUs) that utilize similar resources to produce comparable outputs. Nonetheless, the observed characteristics of symmetry or asymmetry in various types of data in real-world applications can often be imprecise, unclear, insufficient, or contradictory. Neglecting these conditions can potentially result in erroneous decision-making. Certain models take a more restrictive approach by assuming that inputs and outputs possess the same level of determinism. Regrettably, such constraints don't hold true for the majority of real-world scenarios. In actual situations, however, the observed input and output data may sometimes be neutrosophic numbers. So, the primary purpose of this study is to construct a Neutrosophic Input Oriented DEA (NIODEA) Model that incorporates both neutrosophic and deterministic output and/or input variables, handled in accordance with the scoring function. The model we have developed has broad applicability across diverse organizations, aiding decision-makers in making informed choices and optimizing resource allocation, a particularly valuable asset in today's intensely competitive business environment. To underscore the practical utility of the model, we provide an illustrative example that demonstrates its effectiveness and relevance for decision-makers.

Keywords: Optimization, Data Envelopment Analysis; Neutrosophic Variables; Single Valued Neutrosophic; Neutrosophic Score Function; Performance Measure; Efficiency Analysis; Decision Making.

1. Introduction

The concepts of efficiency are utilized to determine whether restrictions have an impact and if so, how substantial. Efficiency is primarily defined as an organization's capacity to produce the greatest amount of output from a given set of inputs [1]. Data Envelopment Analysis (DEA) firstly developed by Charnes et al. [2], a method to assess the effectiveness of decision-making units (DMUs), is a potent analytical instrument for effective and benchmarking evaluation. Almost all applications, including healthcare, banking, transportation, and education, use DEA for various factors, as Golany and Roll [3] observed that it may be utilized to determine the reasons of inefficiency, DMUs ranking, and measure the effectiveness of programs. About 30 years after the landmark study [2] was published, the application field of DEA It has grown to the point that virtually none of the researchers in DEA field can keep up with its progress, especially in terms of how frequently DEA is employed in practical applications.

The DEA approach has various strengths, one of them is that it doesn't need a preference weight or a particular link between the multiple inputs and outputs. Nevertheless, one of the major significant shortcomings of standard DEA issues is that they do not permit for vague variance in the multiple inputs and outputs, even though that many crucial real-world situations may be of the fuzzy form. Consequently, the DEA model's efficiency ratings may be susceptible to various fluctuations in factors. An efficient DMU that is relative efficient to other comparable DMUs may become inefficient if such ambiguous, confusing, inconsistent, and incomplete variance in variables including inputs, outputs, or perhaps both. In other words, because efficiency scores are highly sensitive to the actual levels of inputs or outputs, they will be inaccurate and misleading if the data gained is not displayed in the proper form.

The DEA models have made great attempts in recent years to address the ambiguity in variables, whether fuzzy input or output. Commonly, the applicability of the fuzzy DEA model is split into four categories α-cut, tolerance, possibility, and fuzzy ranking approaches [4 - 8]. The α-cut approach is regarded as the most common fuzzy DEA issue [9 – 17]. Fuzzy sets, however, only take into account the membership function (MF) and are unable to set further vagueness parameters. As a result, Pythagorean fuzzy sets have also been introduced in [18], along with intuitionistic fuzzy sets. Smarandache [19] proposed neutrosophic set theory; it is an extension of fuzzy set, since each element has a truth, indeterminacy, and falsity membership function. Neutrosophic set has been employed for solving models including indeterminacy, uncertainty, imprecision, ambiguity, inconsistency, and incompleteness, among others. Moreover, there are multiple approaches exist for addressing different issues within a neutrosophic environment such that Haque et al. [20], Pal et al. [21], Haque et al. [22], Chakraborty et al. [23], Singh et al. [24], Jdid and Smarandache [25], Singh et al. [26], Sasikala and Divya [27], Gamal and Mohamed [28].

Recent attempts have been made to include neutrosophic data into the DEA model, either as neutrosophic input or neutrosophic output. Edalatpanah [29] devolved a new form of

DEA involved neutrosophic input and output. Abdelfattah [30] provided a DEA model using triangular neutrosophic for both inputs and outputs variables that takes the truth, indeterminacy, and falsity degrees of each data value into consideration. Kahraman et al. [31] introduced a novel Neutrosophic Analytic Hierarchy Process that was subsequently combined with neutrosophic DEA to be employed in performance evaluation. All inputs and outputs in the DEA model suggested by Yang et al. [32] are single-valued neutrosophic triangular numbers. Mao, et al. [33] proposed a neutrosophic DEA model with undesirable outputs, it is constructed simply and is based on the aggregation operator.

Motivation and contribution

In real-world scenarios, it's not uncommon for observed values of inputs and/or outputs to exhibit neutrosophic characteristics. However, one of the significant limitations of the traditional DEA model is its inability to account for uncertainty or variations in input and output data. It assumes that all data are precisely known or represented as crisp values. Consequently, DEA efficiency measurements can be highly sensitive to such variations. A DMU that appears efficient relative to others may become inefficient when these uncertainties are considered, or vice versa. In other words, if the collected data for a variable are not accurately represented in their true neutrosophic nature, the resulting efficiency scores can be inaccurate and misleading due to their sensitivity to the actual levels of inputs or outputs. Additionally, like any empirical technique, DEA relies on simplifying assumptions that researchers must acknowledge when interpreting the results. Recent research in DEA has aimed to address these limitations, but certain challenges persist. Firstly, the developed DEA models are not universally applicable for handling both deterministic variables and variables with neutrosophic variations. Secondly, the DEA models designed to accommodate neutrosophic variables often assume that all variables (whether inputs or outputs) share the same neutrosophic nature.

Our primary focus is on assessing the performance of comparable DMUs with the goal of ensuring quality, identifying areas of deficiency, and ultimately enhancing their efficiency. Given this problem context, the principal objective of this research is to develop a Neutrosophic Input-oriented Data Envelopment Analysis (NIODEA) model. This model will account for a blend of neutrosophic and deterministic input and/or output variables, effectively addressing the complexities of real-world scenarios.

The remaining sections are categorized as follows. Some definitions pertaining to the triangular neutrosophic fuzzy number are introduced in the next section. The third section talks over the conventional DEA models. The suggested NIODEA model is presented in the fourth section. The next section includes an illustrative example. The study concludes with the customary findings and the future implications.

2. Preliminaries

This section gives some brief overview for essential definitions of triangular neutrosophic concept to help in understanding the proposed model.

Neutrosophic theory, a groundbreaking branch of mathematics and philosophy, ventures into the heart of uncertainty, ambiguity, and imprecision. It grapples with the fundamental notion that in the real world, many phenomena are not entirely true or false, but rather possess shades of truth, falsity, and indeterminacy. Traditional mathematics, rooted in classical logic, often struggles to capture the complexity of such situations. At the core of Neutrosophic theory are three fundamental components: neutrosophic sets, neutrosophic logic, and neutrosophic probability. These concepts provide a powerful framework for dealing with indeterminate and contradictory information, opening doors to a deeper understanding of complex systems and uncertain data.

Neutrosophic sets allow us to represent elements with imprecise or conflicting attributes, offering a flexible alternative to the crisp sets of classical mathematics. Neutrosophic logic extends this flexibility by embracing degrees of truth, falsity, and indeterminacy in reasoning, enabling more realistic and nuanced decision-making. Neutrosophic probability, in turn, quantifies the likelihood of neutrosophic events, offering a richer perspective on uncertainty compared to traditional probability theory. Let W to be a set of positive real numbers coupled with a variable, and w to be a general element of W. A fuzzy set A in W is defined mathematically as the collection of ordered pairs: "$A = \{(w, \mu_A(w)) \mid w \epsilon W\}$, where μ_A: is the MF and usually assumed to vary in the interval [0,1]".

A MF is a mapping that allocates $\forall\ w \in W$ a number, $\mu_A(w) \in [0,1]$ and represents the membership degree of w in A. The closer value of $\mu_A(w)$ is to one, the largest membership of w in A. Hence, a fuzzy set A may be accurately described by associating a number ranging from 0 to 1 with each element w, which indicates its membership degree in A. The MF of a fuzzy set A may also be denoted by $A(w)$ [34].

Definition 1: [13] A fuzzy number $\widetilde{w}_i = (w^1, w^2, w^3)$, where $w^1 \leq w^2 \leq w^3$ on $\mathbb{R}$ is a triangular fuzzy number if its MF define as follows:

$$\mu_{\widetilde{w}_i} = \begin{cases} 0 & , y \leq w^1 \\ \frac{y-w^1}{w^2-w^1} & , w^1 < y \leq w^2 \\ 1 & , x = w^2 \\ \frac{w^3-y}{w^3-w^2} & , w^2 < y \leq w^3 \\ 0 & , y \geq w^3. \end{cases}$$

(1)

Definition 2: [35] Let us denote the space of objects by Y and its generic element as y, $y \in Y$. The neutrosophic set of $\tilde{Q}^M$ has the form $\tilde{Q}^M = \{\langle y: T_{\tilde{Q}^M}(y), I_{\tilde{Q}^M}(y), F_{\tilde{Q}^M}(y)\rangle, y \in Y, T_{\tilde{Q}^M}(y), I_{\tilde{Q}^M}(y), F_{\tilde{Q}^M}(y) \in]0^-, 1^+[\}$, where $T_{\tilde{Q}^M}(y), I_{\tilde{Q}^M}(y), F_{\tilde{Q}^M}(y)$ are truth, indeterminacy, falsity MFs with the no restriction condition on their sum, $0^- \leq T_{\tilde{Q}^M}(y) + I_{\tilde{Q}^M}(y) + F_{\tilde{Q}^M}(y) \leq 3^+$, and $]0^-, 1^+[$ is an irregular unit interval.

Definition 3: [35] A single valued neutrosophic set $\tilde{Q}^{SVN}$ of a nonempty set y is constructed as $\tilde{Q}^{SVN} = \{\langle y, T_{\tilde{Q}^M}(y), I_{\tilde{Q}^M}(y), F_{\tilde{Q}^M}(y)\rangle, y \in Y\}$, where $T_{\tilde{Q}^M}(y), I_{\tilde{Q}^M}(y)$, and $F_{\tilde{Q}^M}(y) \in [0,1]$ and $0 \leq T_{\tilde{Q}^M}(y) + I_{\tilde{Q}^M}(y) + F_{\tilde{Q}^M}(y) \leq 3$.

Definition 4: [36] Let $\mathcal{L}_{\tilde{s}}, \delta_{\tilde{s}}, \mathcal{F}_{\tilde{s}} \in [0,1]$ $w^1, w^2, w^3 \in \mathbb{R}$ such that $w^1 \leq w^2 \leq w^3$. Then a single valued triangular fuzzy neutrosophic set (SVTFN), $\tilde{s}^{TN} = \langle (w^1, w^2, w^3); \mathcal{L}_{\tilde{s}}, \delta_{\tilde{s}}, \mathcal{F}_{\tilde{s}}\rangle$ is a special neutrosophic set on $\mathbb{R}$, whose truth, indeterminacy, falsity MFs are:

$$T_{\tilde{Q}^M}(y) = \begin{cases} 0 & , y < w^1 \\ \frac{(y-w^1)\mathcal{L}_{\tilde{s}^{TN}}}{w^2-w^1} & , w^1 \leq y \leq w^2 \\ \frac{(c-y)\mathcal{L}_{\tilde{s}^{TN}}}{w^3-w^2} & , w^2 \leq y \leq w^3 \\ 0 & , y > w^3 \end{cases} \quad (2)$$

$$I_{\tilde{Q}^M}(y) = \begin{cases} 0 & , y < w^1 \\ \frac{(w^2-y)+(y-w^1)\delta_{\tilde{s}^{TN}}}{w^2-w^1} & , w^1 < y \leq w^2 \\ \frac{(y-w^2)+(w^3-y)\delta_{\tilde{s}^{TN}}}{w^3-w^2} & , w^2 < y \leq w^3 \\ 0 & , w^3 > c \end{cases} \quad (3)$$

$$F_{\tilde{Q}^M}(y) = \begin{cases} 0 & , y < w^1 \\ \frac{(w^2-y)+(y-w^1)\mathcal{F}_{\tilde{s}^{TN}}}{w^2-w^1} & , w^1 < y \leq w^2 \\ \frac{(y-w^2)+(w^3-y)\mathcal{F}_{\tilde{s}^{TN}}}{w^3-w^2} & , w^2 < y \leq w^3 \\ 0 & , y > w^3 \end{cases} \quad (4)$$

***Definition 5*:** [36] let $\tilde{v}^{TN} = \langle (a,b,c); \mathcal{L}_{\tilde{v}^{TN}}, \delta_{\tilde{v}^{TN}}, \mathcal{F}_{\tilde{v}^{TN}}\rangle$ be SVTFN, then

1. Score function $SF(\tilde{v}^{TN}) = \left(\frac{1}{4}(a+2b+c)\right)\left(\frac{1}{3}(2+\mathcal{L}_{\tilde{v}^{TN}} - \delta_{\tilde{v}^{TN}} - \mathcal{F}_{\tilde{v}^{TN}})\right)$ (5)

2. Accuracy function $AF(\tilde{v}^{TN}) = \left(\frac{1}{4}(a+2b+c)\right)\left(\frac{1}{3}(2+\mathcal{L}_{\tilde{v}^{TN}} - \delta_{\tilde{v}^{TN}} + \mathcal{F}_{\tilde{v}^{TN}})\right)$ (6)

***Definition 6*:** [35] let $\tilde{v}^{TN} = \langle (a,b,c); \mathcal{L}_{\tilde{v}^{TN}}, \delta_{\tilde{v}^{TN}}, \mathcal{F}_{\tilde{v}^{TN}}\rangle$ and $\tilde{u}^{TN} = \langle (a_1,b_1,c_1); \mathcal{L}_{\tilde{u}^{TN}}, \delta_{\tilde{u}^{TN}}, \mathcal{F}_{\tilde{u}^{TN}}\rangle$ be two SVTFN, the arithmetic operations on $\tilde{v}^{TN}$ and $\tilde{u}^{TN}$ as follows:

1. $\tilde{v}^{TN} \oplus \tilde{u}^{TN} = \langle (a+a_1, b+b_1, c+c_1,); \mathcal{L}_{\tilde{v}^{TN}} \wedge \mathcal{L}_{\tilde{u}^{TN}}, \delta_{\tilde{v}^{TN}} \vee \delta_{\tilde{u}^{TN}}, \mathcal{F}_{\tilde{v}^{TN}} \vee \mathcal{F}_{\tilde{u}^{TN}}\rangle$ (7)
2. $\tilde{v}^{TN} \ominus \tilde{u}^{TN} = \langle (a-a_1, b-b_1, c-c_1,); \mathcal{L}_{\tilde{v}^{TN}} \wedge \mathcal{L}_{\tilde{u}^{TN}}, \delta_{\tilde{v}^{TN}} \vee \delta_{\tilde{u}^{TN}}, \mathcal{F}_{\tilde{v}^{TN}} \vee \mathcal{F}_{\tilde{u}^{TN}}\rangle$ (8)
3. $n\tilde{v}^{TN} = \begin{cases} \langle (na, nb, nc); \mathcal{L}_{\tilde{v}^{TN}}, \delta_{\tilde{v}^{TN}}, \mathcal{F}_{\tilde{v}^{TN}}\rangle, n > 0 \\ \langle (nc, nb, na); \mathcal{L}_{\tilde{v}^{TN}}, \delta_{\tilde{v}^{TN}}, \mathcal{F}_{\tilde{v}^{TN}}\rangle, n < 0 \end{cases}$ (9)
4. ${\tilde{v}^{TN}}^{-1} = \langle (a^{-1}, b^{-1}, c^{-1}); \mathcal{L}_{\tilde{v}^{TN}}, \delta_{\tilde{v}^{TN}}, \mathcal{F}_{\tilde{v}^{TN}}\rangle, \tilde{v}^{TN} \neq 0$ (10)

Definition 7: [35] the order relation between $\tilde{v}^{TN}$ and $\tilde{u}^{TN}$ based on score and accuracy functions are:

1. If $SF(\tilde{v}^{TN}) > SF(\tilde{u}^{TN})$, then $\tilde{v} > \tilde{u}$
2. If $SF(\tilde{v}^{TN}) < SF(\tilde{u}^{TN})$, then $\tilde{v} < \tilde{u}$
3. If $SF(\tilde{v}^{TN}) = SF(\tilde{u}^{TN})$, then
   a) If $AF(\tilde{v}^{TN}) > AF(\tilde{u}^{TN})$, then $\tilde{v} > \tilde{u}$
   b) If $AF(\tilde{v}^{TN}) < AF(\tilde{u}^{TN})$, then $\tilde{v} < \tilde{u}$
   c) If $AF(\tilde{v}^{TN}) = AF(\tilde{u}^{TN})$, then $\tilde{v} = \tilde{u}$

3. DEA Mathematical Model

DEA's essential model with 'n' DMUs, 'J' inputs and 'S' outputs was first introduced in [2]. The model provides the relative efficiency scores for all DMUs and it hinges on optimizing a DEA-estimated production function, it is a deterministic frontier function. The DEA estimate value for all inputs provides the maximum output that can be achieved from inputs under all conditions. Conversely, for any outputs, the DEA value estimate the minimum input achieving a certain output under all scenarios. In this regard, it resembles the parametric frontier with one-sided deviations determined utilizing mathematical programming techniques.

The DEA model may be categorized as either having constant returns to scale (CRS) or variable returns to scale (VRS) based on the assumptions connecting the change in outputs to the change in inputs (VRS). In CRS models, the outputs are not impacted by the size of the DMU; rather, they vary directly proportional to the change in inputs, assuming that the scale of operation has no effect on efficiency; hence, output and input oriented measures of efficiency are equivalent. In VRS models, changes in outputs are not always proportionate to changes in inputs; hence, output and input oriented measures of efficiency scores for inefficient units are not equivalent [37]. This work focuses on the input-oriented VRS model, which may be described as follows:

$$
\begin{aligned}
\text{Min} \quad & Z_p = \theta \\
\text{s.t.} \quad & \\
& \sum_{i=1}^{n} \lambda_i x_{ij} \le \theta x_{pj} \qquad ,\forall\ j = 1, \dots J \\
& \sum_{i=1}^{n} \lambda_i y_{is} \ge y_{ps} \qquad ,\forall\ s = 1, \dots S \\
& \sum_{i=1}^{n} \lambda_i = 1 \\
& \lambda_i \ge 0 \qquad ,\forall\ i = 1, \dots n
\end{aligned} \tag{11}
$$

Where θ is the efficiency score of DMU p; s is the no. of outputs, $1 \le s \le S$; j is the no. of inputs, $1 \le j \le J$; i is the no. of DMUs, $1 \le i \le n$; y_{is} is the amount of outputs produced by the

i^{th} DMU; x_{ij} is the amount of the j^{th}input utilized by the i^{th}DMU; and λ_i is the weight of the i^{th}DMU.

4. Developed neutrosophic input-oriented Data Envelopment Analysis Model

Now we are going to formulate a NIODEA model in order to evaluate quality by comparing the performance of similar organizations, assuming that some of the input and/or output variables may be in neutrosophic settings. Here, we introduce our modification to the conventional DEA model in order to evaluate relative efficiency in the case of neutrosophic variation in a portion of the outputs and/or inputs. The constructed NIODEA model relies on the score function. The restriction affecting some of the input and/or output values in the DEA model will be a neutrosophic inequality that may sometimes be violated. Since an inequality incorporating several neutrosophic variables may never be established with crisp. The suggested model consists of three stages. First, the MF for neutrosophic input and output variables is specified. Finding the score and accuracy function for neutrosophic variables based on the MF is the second stage. In the third step, each DMU's relative efficiency score is determined. The NIODEA model for evaluating the efficiency level of p^{th} DMU is as follows:

$$\text{Min} \quad \tilde{Z}_p{}^{TN} = \theta$$

$s.t.$

$$\sum_{i=1}^{n} \lambda_i x_{ij} \leq \theta x_{pj} \quad ,\forall j \in J_D$$

$$\sum_{i=1}^{n} \lambda_i \tilde{x}_{ij}{}^{TN} \leq \theta \tilde{x}_{pj}{}^{TN} \quad ,\forall j \in J_N$$

$$\sum_{i=1}^{n} \lambda_i y_{is} \geq y_{ps} \quad ,\forall s \in S_D$$

$$\sum_{i=1}^{n} \lambda_i \tilde{y}_{is}{}^{TN} \geq \tilde{y}_{ps}{}^{TN} \quad ,\forall s \in S_N$$

$$\sum_{i=1}^{n} \lambda_i = 1$$

$$\lambda_i \geq 0, \quad (i = 1{:}n)$$

where J_D is the deterministic inputs set, J_N is the neutrosophic inputs set, J is the total inputs set, i.e., $J_D \cup J_N = J$. S_D is the deterministic outputs set, S_N is the neutrosophic outputs set, and S is total outputs set, $S_D \cup S_N = S$.

Comparing model (11) to model (12), it is clear that each of the two constraints controlling the inputs and outputs is split into two constraints in order to manage the deterministic variables separately from the neutrosophic variables.

In the suggested model, it is assumed that the neutrosophic variables have triangle MFs. Depending on the score function described in Section 2, the triangular NIODEA model was transformed to a standard DEA model that can be solved easily.

Min $Z_p = \theta$

$s.t.$

$$\sum_{i=1}^{n} \lambda_i x_{ij} \leq \theta x_{pj} \quad ,\forall j \in J_D$$

$$\sum_{i=1}^{n} \lambda_i SF(\tilde{x}_{ij}{}^{TN}) \leq \theta SF(\tilde{x}_{pj}{}^{TN}) \quad ,\forall j \in J_N$$

$$\sum_{i=1}^{n} \lambda_i y_{is} \geq y_{ps} \quad ,\forall s \in S_D$$

$$\sum_{i=1}^{n} \lambda_i SF(\tilde{y}_{is}{}^{TN}) \geq SF(\tilde{y}_{ps}{}^{TN}) \quad ,\forall s \in S_N$$

$$\sum_{i=1}^{n} \lambda_i = 1$$

$$\lambda_i \geq 0, \quad (i = 1{:}n)$$

5. Illustrative Example

In this section, a numerical model is employed to demonstrate the application of the improved model. Seven DMUs (D1, D2, …., D7) with three inputs (N1, N2 and N3) two are deterministic (N1 and N2) and (N3) is neutrosophic. The outputs are O1 and O2, where (O1) is deterministic and (O2) is neutrosophic. The values are considered in the following hypothetical example. The data for deterministic variables are presented in Table 1, while those for neutrosophic variables are given in Table 2. The objective of this problem is to evaluate the relative efficiency of the DMUs using the NIODEA model that we have developed.

Before formulating and solving the problem, we must compute the score function for each neutrosophic variable (input or output). Table 3 presents the computed values.

Table 1 Hypothetical data for the DMUs' deterministic variables

DMUs	Inputs		Output
	N 1	N 2	O 1
D1	6.11	4.36	0.21
D2	3.66	2.54	0.12

D3	1.44	0.48	0.14
D4	1.21	0.23	0.10
D5	2.75	1.40	0.10
D6	4.18	2.74	0.06
D7	6.39	3.36	0.18

Table 2 Hypothetical neutrosophic variables data for DMUs

DMUs	Input	Output
	N 3	O 2
D1	$\langle(1.76, 7.27, 12.27); 0.9,0.4,0.1\rangle$	$\langle(0.12, 0.19, 0.27); 1.0, 0.0,0.0\rangle$
D2	$\langle(3.85, 4.65, 5.53); 0.9,0.7,0.1\rangle$	$\langle(0.00, 0.10, 0.24); 1.0,0.0,0.0\rangle$
D3	$\langle(1.33, 1.88, 3.38); 0.9, 0.4, 0.1\rangle$	$\langle(0.05, 0.10, 0.16); 1.0,0.0,0.0\rangle$
D4	$\langle(0.78, 1.48, 2.06); 0.8, 0.5, 0.1\rangle$	$\langle(0.00, 0.06, 0.16); 1.0,0.0,0.0\rangle$
D5	$\langle(3.22, 3.63, 4.61); 0.8, 0.5, 0.2\rangle$	$\langle(0.02, 0.07, 0.17); 1.0,0.0,0.0\rangle$
D6	$\langle(4.30, 6.13, 8.03); 0.9, 0.5,0.2\rangle$	$\langle(0.00, 0.06, 0.15); 1.0,0.0,0.0\rangle$
D7	$\langle(4.40, 8.00, 10.68); 0.9, 0.4, 0.1\rangle$	$\langle(0.06, 0.17, 0.30); 1.0,0.0,0.0\rangle$

Table 3 Score functions of N3, O2

DMUs	Input	Output
	N 3	O 2
D1	5.71	0.19
D2	3.27	0.11
D3	1.69	0.10
D4	1.06	0.07
D5	2.64	0.08
D6	4.51	0.07
D7	6.22	0.18

Each DMU requires a linear programming formulation to evaluate its relative efficiency. Below is D1's linear programming model.

$$\text{Min } Z_A = \theta$$

$s.t.$

$$6.11\lambda_{D1} + 3.66\lambda_{D2} + 1.44\lambda_{D3} + 1.21\lambda_{D4} + 2.75\lambda_{D5} + 4.18\lambda_{D6} + 6.39\lambda_{D7} \leq 6.11\theta$$
$$4.36\lambda_{D1} + 2.54\lambda_{D2} + 0.48\lambda_{D3} + 0.23\lambda_{D4} + 1.04\lambda_{D5} + 2.74\lambda_{D6} + 3.36\lambda_{D7} \leq 4.36\theta$$
$$5.716\lambda_{D1} + 3.27\lambda_{D2} + 1.69\lambda_{D3} + 1.06\lambda_{D4} + 2.64\lambda_{D5} + 4.51\lambda_{D6} + 6.22\lambda_{D7} \leq 5.71\theta$$
$$0.21\lambda_{D1} + 0.12\lambda_{D2} + 0.14\lambda_{D3} + 0.10\lambda_{D4} + 0.10\lambda_{D5} + 0.06\lambda_{D6} + 0.18\lambda_{D7} \geq 0.21$$
$$0.19\lambda_{D1} + 0.11\lambda_{D2} + 0.10\lambda_{D3} + 0.07\lambda_{D4} + 0.08\lambda_{D5} + 0.07\lambda_{D6} + 0.18\lambda_{D7} \geq 0.19$$
$$\lambda_{D1} + \lambda_{D2} + \lambda_{D3} + \lambda_{D4} + \lambda_{D5} + \lambda_{D6} + \lambda_{D7} = 1$$
$$\lambda_{Di} \geq 0, \ (i = 1{:}7). \quad (14)$$

Furthermore, relative efficiency models are formulated for DMUs D2 to D7. The models are then solved using GAMS software. The relative efficiency of each DMU is listed in Table 4.

Table 4 Relative efficiency

DMUs	NIODEA	Fuzzy IODEA	Stochastic IODEA	Deterministic IODEA
D1	1	1	0.59	1

D2	0.65	0.36	1	0.40
D3	1	1	1	1
D4	1	1	1	1
D5	0.48	0.44	1	0.46
D6	0.29	0.29	0.99	0.29
D7	1	0.80	0.84	1

The provided table displays the efficiency scores for each DMUs obtained from the NIODEA model. A careful look at the efficiency scores for the seven DMUs reveals that four are efficient (DMUs D1, D3, D4, and D7), while the other three are inefficient (DMUs D2, D5, and D6). Two of the three ineffective DMUs are quite unproductive (D5 and D6). We provided suggestions for improving the inefficient DMUs to enhance its performance by conduct a comprehensive analysis with efficient DMUs to identify the factors causing inefficiency and then explore ways to optimize resource utilization or improve the quality of outputs.

Comparatively, we also designed relative efficiency models for three distinct cases: the first with the neutrosophic variables considered fuzzy, the second with the neutrosophic variables considered stochastic, and the third with all variables considered deterministic. The models established by Tharwat et al. [17] and El-Demerdash et al. [38] were applied to the first and second cases, respectively. In the first scenario, we assume the three values for the triangular neutrosophic variable to represent the values for the triangle fuzzy variable so that the fuzzy IODEA model may be executed. To run the stochastic IODEA model for the stochastic variables in the second scenario, we averaged the three values for the neutrosophic function to represent the mean and assumed the variance and covariance between DMUs. In the last scenario, the neutrosophic variables' average values were used as the deterministic values. Table 4 also displays the relative efficacy of these three cases.

Table 4 demonstrates that the nature of the variables may have a significant impact on the relative efficiency of the DMUs. As seen by the data in Table 4, several DMUs have altered their status from efficient to inefficient and conversely. DMUs (D_1, D_3, D_4, and D_7) consistently exhibit high efficiency scores (close to or equal to 1) across all models. This suggests that they are consistently efficient regardless of the modeling approach used. DMU D_2 displays a variable level of efficiency across different models. It is efficient in the Stochastic IODEA model but less so in the Fuzzy IODEA and Deterministic IODEA models. This highlights the sensitivity of its efficiency to the modeling methodology. DMUs (D_5 and D_6) demonstrate consistently lower efficiency scores across all models, indicating a need for improvement in their performance. Therefore, to get accurate findings about the efficacy and inefficacy of the investigated DMUs, it is essential to identify the precise nature of the variables. In addition, the results indicate that the NIODEA model, due to its integration of various uncertainty dimensions, may offer a more comprehensive yet complex view of efficiency. The Fuzzy IODEA model tends to provide lower efficiency scores and may be less suitable for these DMUs. The Stochastic IODEA model appears to be sensitive to variations, assigning high efficiency scores even for DMUs that are less efficient than other models. Finally, the NIODEA model yields more accurate and reliable results than the classic DEA model and its variations, such as the fuzzy and stochastic DEA models.

6. Conclusion and Future Work

In this research, we introduce a novel approach, the NIODEA model, designed to handle both deterministic and neutrosophic variables. This innovative model, utilizing a specified scoring function and triangular Membership Functions (MFs) for neutrosophic variables, enables us to effectively assess the relative efficiency of Decision-Making Units (DMUs). The illustrative example highlights the profound impact of the inherent characteristics of variables on the determination of relative efficiency. It demonstrates how variables can shift the status of DMUs from efficient to inefficient, and vice versa. This underscores the critical importance of precisely defining variable structures and selecting the appropriate DEA model to ensure the generation of dependable results.

Our research emphasizes the sensitivity of DEA efficiency measurements to changes in variable nature. An initially efficient DMU, relative to others, can become inefficient when uncertainty variations are considered, and conversely, due to the high sensitivity of efficiency scores to variable levels of inputs or outputs. Hence, it is imperative to discern the nature of variables from the outset and apply the most suitable DEA model to attain accurate and reliable outcomes. By implementing the four different models in our illustrative example, we observed similarities in efficient DMUs and disparities in inefficient DMUs regarding their efficiency levels. As part of our future research agenda, we intend to apply the developed NIODEA model to real-world scenarios, thereby enhancing its practicality and relevance. Additionally, we aim to augment the model's versatility by exploring alternative MFs for neutrosophic variables. Our ongoing work will concentrate on the development of an integrated IODEA model capable of handling deterministic, neutrosophic, and stochastic variables, further contributing to the field of decision analysis.

Data Availability

In this article, no data were used.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Study of the Circular Economy Model as a Strategy for the Development of Peru Based on Neutrosophic Cognitive Maps

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Abstract. The circular economy has become a crucial approach to address the environmental and economic challenges associated with sustainable production and consumption. Instead of the traditional linear model of "take, make, use, and dispose of", the circular economy seeks to close material and resource cycles, minimize waste and maximize added value throughout the supply chain. In this context, the different strata of the Peruvian economy play an important role in the transition toward a circular economy. However, several indicators have been identified that show poor performance in this form of development. The purpose of this study is to identify the opportunities for the implementation of the circular economy model in the Peruvian economy. The investigation results provide a comprehensive vision of the pitfalls and congruences for implementing this strategy. Neutrosophic cognitive maps were selected as a study tool, which represents relationships between concepts; in this case, indeterminacy is also included to represent unknown, neutral, imprecise relationships, etc. between concepts. To design the dynamic neutrosophic cognitive map, 14 circular economy specialists were surveyed, based on their individual and independent criteria and so the model was obtained. Then, from the run of all the possible cases according to the algorithm of seeking the hidden patterns, the absolute and relative frequencies for the convergence to each one of the possible values were obtained.

Keywords: Circular Economy, Sustainable Development, Neutrosophic Cognitive Maps, Neutrosophic Number.

1 Introduction

The circular economy is a strategy that aims to reduce both the input of materials and the production of virgin waste, closing the "loops" or economic and ecological flows of resources. It encompasses much more than the production and consumption of goods and services, as it includes, among other aspects, the switch from fossil fuels to the use of renewable energy, and diversification as a means of achieving resilience. As part of the debate, it should also include a deep discussion on the role and use of money and finance, and some of its pioneers have also called for renewing the tools for measuring economic performance.

Peru has not developed coordinated actions for the establishment of circular economy processes, so it is necessary to use international indicators that are utilized to measure the scope and optimal use of this form of management. The authors, based on the thematic analysis of twenty authors, set as references the following:

a) Recycling rate: It is the percentage of materials recycled of the total materials generated. Measures the effectiveness of the material recycling and reuse system.
b) Consumption of natural resources: A measure of the total consumption of natural resources, such as water, minerals, wood, and energy, compared to the country's economic output.
c) Added value per material: This indicator shows how much economic value is generated per unit of material used. Greater efficiency in the use of materials may indicate a more circular economy.

d) Carbon footprint: It is the total amount of greenhouse gas emissions produced by a specific nation, company, or activity. The circular economy seeks to reduce these emissions throughout the life cycle of products.
e) Municipal Waste Recycling Rate: Measures the percentage of recycled waste instead of being sent to landfills or incinerated.
f) Number of companies adopting circular practices: Counts the number of companies that have implemented circular economy strategies, such as design for recycling, use of recycled materials, and service-based business models.
g) Number of Refurbished Products: Measures the number of products that are repaired or refurbished rather than scrapped and replaced.
h) Investment in research and development in the circular economy: Measures the level of investment in research and development of technologies and practices related to the circular economy.

Neutrosophic Cognitive Maps are used as a study tool [1]. These are defined as directed graphs, where the nodes represent concepts and the edges represent causal relationships between the concepts. In the theory of Cognitive Maps, each edge is associated with a numerical value in the set $\{-1, 0, 1\}$, where -1 represents that the cause-effect relationship is negative (if the value of one concept increases, the value of the other decreases and vice versa), 0 represents the absence of a relationship between the concepts, while 1 represents a positive relationship (when the presence of one concept increases, the other also increases and vice versa) [2]. In the case of the Dynamic Neutrosophic Cognitive Maps, the symbol "I" also appear to indicate that the relationship between the concepts is not exactly known. This tool has been applied in many problems [1, 3-8].

The difficult with designing a circular economy is that it is a complex phenomenon, where there are non-linear relationships between variables, in addition to the fact that some of the relationships may be conflicting or contradictory, for example, a successful economy in the short or medium term can be more efficient if it ignores the care of the environment, however in the long term it will fail. For all this, this work is also in tune with the theory of Neutrosophic Systems and Neutrosophic Dynamic Systems introduced by F. Smarandache in [9].

Dynamic Systems is a methodology for analyzing and modeling temporal behavior in complex environments. It is based on the identification of feedback loops between elements, and also on information and material delays within the system. What makes this approach different from others used to study complex systems is the analysis of the effects of loops or feedback loops, in terms of flows and adjacent deposits. In this way, the dynamics of the behavior of these systems can be structured through mathematical models. The simulation of these models can currently be performed with the help of specific computer programs.

To obtain the elements of the cognitive map, 14 Peruvian specialists in Circular Economy were surveyed, who gave their opinion on the subject, each one individually. In this way, by simulating all possible cases, the absolute frequency was calculated for the system to converge to one of the three possible values, that is, for the system to converge to an activated (1), deactivated (0), and indeterminate value (I), for each of the possible states. This problem is treated as a non-linear system, within the framework of Neutrosophy

This study aims to determine the elements of the circular economy that are opportunities and those that are challenges, for the current economy of Peru.

This paper is divided into sections, the next one is called Materials and Methods, where the fundamental concepts of Dynamic Neutrosophic Cognitive Maps are explained. The Results section contains the elements used for the study and the results obtained. Last section is dedicated to give the conclusions.

2 Materials and Methods

Neutrosophic Cognitive Maps will be used in this study, so we explain them in the following.

2.1 Neutrosophic Cognitive Maps

Definition 1: ([10, 11]) Let X be a universe of discourse. A *Neutrosophic Set* (NS) is characterized by three membership functions, $u_A(x), r_A(x), v_A(x) : X \rightarrow \,]^-0, 1^+[$, which satisfy the condition $^-0 \leq \inf u_A(x) + \inf r_A(x) + \inf v_A(x) \leq \sup u_A(x) + \sup r_A(x) + \sup v_A(x) \leq 3^+$ for all $x \in X$. $u_A(x), r_A(x)$ and $v_A(x)$ are the membership functions of truthfulness, indeterminacy and falseness of x in A, respectively, and their images are standard or non-standard subsets of $]^-0, 1^+[$.

Definition 2: ([12, 13]) Let X be a universe of discourse. A *Single-Valued Neutrosophic Set* (SVNS) A on X is a set of the form:

$$A = \{\langle x, u_A(x), r_A(x), v_A(x)\rangle : x \in X\} \quad (1)$$

Where $u_A, r_A, v_A : X \rightarrow [0,1]$, satisfy the condition $0 \leq u_A(x) + r_A(x) + v_A(x) \leq 3$ for all $x \in X$. $u_A(x), r_A(x)$ and $v_A(x)$ are the membership functions of truthfulness, indeterminate and falseness of x in A, respectively. For convenience, a *Single-Valued Neutrosophic Number* (SVNN) will be expressed as $A = (a, b, c)$, where $a, b, c \in [0,1]$ and satisfy $0 \leq a + b + c \leq 3$.

Other important definitions are related to the graphs.

Definition 3: ([12, 14]) A *Neutrosophic graph* contains at least one indeterminate edge, represented by dotted lines.

Definition 4: ([12, 14]) A *Neutrosophic directed graph* is a directed graph containing at least one indeterminate edge, which is represented by dotted lines.

Definition 5: ([5, 7]) A *Neutrosophic Cognitive Map* (NCM) is a neutrosophic directed graph, whose nodes represent concepts and whose edges represent causal relationships among the edges.

Let $C_1, C_2, \dots, C_k$ be k nodes, each of the C_i $(i = 1, 2, \dots, k)$ can be represented by a vector $(x_1, x_2, \dots, x_k)$ where $x_i \in \{0, 1, I\}$. $x_i = 1$ means that the node C_i is in an activated state, $x_i = 0$ means that the node C_i is in a deactivated state and $x_i = I$ means that the node C_i is in an indeterminate state, in a specific time or a specific situation [15].

If C_m and C_n are two nodes of the NCM, a directed edge from C_m to C_n is called a *connection* and represents the causality from C_m to C_n. Each node in the NCM is associated with a weight within the set $\{-1, 0, 1, I\}$. If α_{mn} denote the weight of the edge C_mC_n, $\alpha_{mn} \in \{-1, 0, 1, I\}$, then we have the following:

$\alpha_{mn} = 0$ if C_m does not affect C_n,

$\alpha_{mn} = 1$ if an increase (decrease) in C_m produces an increase (decrease) in C_n,

$\alpha_{mn} = -1$ if an increase (decrease) in C_m produces a decrease (increase) in C_n,

$\alpha_{mn} = I$ if the effect of C_m on C_n is indeterminate.

Definition 6: ([5, 7]) A NCM having edges with weights in $\{-1, 0, 1, I\}$ is called *Simple Neutrosophic Cognitive Map*.

Definition 7: ([5, 7]) If $C_1, C_2, \dots, C_k$ are the nodes of an NCM. The *neutrosophic matrix* N(E) is defined as $N(E) = (\alpha_{mn})$, where α_{mn} denotes the weight of the directed edge C_mC_n, such that $\alpha_{mn} \in \{-1, 0, 1, I\}$. N(E) is called the *neutrosophic adjacency matrix* of the NCM.

Definition 8: ([5, 7]) Let $C_1, C_2, \dots, C_k$ be the nodes of an NCM. Let $A = (a_1, a_2, \dots, a_k)$, where $a_m \in \{-1, 0, 1, I\}$. A is called an *instantaneous state neutrosophic vector* and means a position on-off-indeterminate state of the node in a given instant.

$a_m = 0$ if C_m is deactivated (has no effect),

$a_m = 1$ if C_m is activated (has an effect),

and $a_m = I$ if it is indeterminate (its effect cannot be determined).

Definition 9: ([5, 7]) Let $C_1, C_2, \dots, C_k$ be the nodes of an NCM. Let $\overrightarrow{C_1C_2}, \overrightarrow{C_2C_3}, \overrightarrow{C_3C_4}, \dots, \overrightarrow{C_mC_n}$ be the edges of the NCM, then the edges constitute a *directed cycle*.

The NCM is called *cyclic* if it has a directed cycle. It is said *acyclic* if it has not a directed cycle.

Definition 10: ([5, 7]) A NCM containing cycles is said to have *feedback*. When there is feedback in the NCM, it is said that it is a *dynamic system*.

Definition 11: ([5, 7]) Let $\overrightarrow{C_1C_2}, \overrightarrow{C_2C_3}, \overrightarrow{C_3C_4}, \dots, \overrightarrow{C_{k-1}C_k}$ be a cycle. When C_m is activated and its causality flows through the edges of the cycle and then it is the cause of C_m itself, then the dynamic system circulates. This is fulfilled for each node C_m with $m = 1, 2, \dots, k$. The equilibrium state for this dynamic system is called the *hidden pattern*.

Definition 12: ([5, 7]) If the equilibrium state of a dynamic system is a single state, then it is called a *fixed point*.

An example of a fixed point is when a dynamic system starts for being activated by C_l. If it is assumed that the NCM sits on C_l and C_k, i.e. the state remains as $(1, 0, \dots, 0, 1)$, then this vector of the neutrosophic state is called a *fixed point*.

Definition 13: ([5, 7]) If the NCM is established with a neutrosophic state-vector that repeats itself in the form:

$A_1 \to A_2 \to \cdots \to A_m \to A_1$, then the equilibrium is called a *limit cycle* of the NCM.

<u>Method for Determining the Hidden Patterns</u>

Let $C_1, C_2, \dots, C_k$ be the nodes of the NCM with feedback. Assume that E is the associated adjacency matrix. A hidden pattern is found when C_l is activated and a vector input $A_1 = (1, 0, 0, \dots, 0)$ is given. The data must pass through the neutrosophic matrix $N(E)$, which is obtained by multiplying A_1 by the matrix $N(E)$.

Let $A_1N(E) = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ with the threshold operation of replacing α_m by 1 if $\alpha_m > p$ and α_m by 0 if $\alpha_m < p$ (p is a suitable positive integer) and α_m is replaced by I if this is not an integer. The resulting concept is updated; the vector C_1 is included in the updated vector by transforming the first coordinate of the resulting vector into 1.

If $A_1N(E) \rightarrow A_2$ is assumed then $A_2N(E)$ is considered and the same procedure is repeated. This procedure is repeated until a limit cycle or fixed point is reached.

Definition 14: ([16]) A *neutrosophic number* N is defined as a number as follows:

$$N = d + I \qquad (2)$$

Where d is called the *determined part* and I is called the *indeterminate part.*

Given $N_1 = a_1 + b_1I$ and $N_2 = a_2 + b_2I$ two neutrosophic numbers, some operations between them are defined as follows:

$N_1 + N_2 = a_1 + a_1 + (b_1 + b_2)I$ (Addition);

$N_1 - N_2 = a_1 - a_1 + (b_1 - b_2)I$ (Difference),

$N_1 \times N_2 = a_1a_2 + (a_1b_2 + b_1a_2 + b_1b_2)I$ (Product),

$\frac{N_1}{N_2} = \frac{a_1+b_1I}{a_2+b_2I} = \frac{a_1}{a_2} + \frac{a_2b_1-a_1b_2}{a_2(a_2+b_2)}I$ (Division).

3 Results

To reach the expected results, the following procedure was followed:

1. 14 Circular Economy specialists were asked to give their opinion on a scale from -10 to 10, including 0, in addition to being asked to use the symbol I as an indicator that they do not know, about the possible relationship between each pair of the following variables:
   V1: Recycling rate.
   V2: Consumption of natural resources.
   V3: Added value by material.
   V4: Carbon footprint.
   V5: Municipal waste recycling rate.
   V6: Number of companies adopting circular practices.
   V7: Number of retrofitted products.
   V8: Investment in research and development in circular economy.
   This is justified because it is easier for specialists to evaluate on this scale than on a more restrictive one in the range of $\{-1,0,1,I\}$. -10 means a complete inverse relationship, 10 means a complete direct relationship, and 0 means that there is no relationship between the variables. Values between -9 and -1 or between 1 and 9 represent intermediate opinions between the three previous values.
   Each specialist was surveyed individually and independently from the rest to avoid influencing the answers.

In other words, formally if we call $E = \{e_1, e_2, \cdots, e_{14}\}$ to the set of 14 experts. R_{ijk} symbolizes the relationship between the jth and kth criteria ($j, k \in \{1,2, \ldots, 8\}$, $j \neq k$) according to the expert e_i ($i = 1,2, \ldots, 14$) such that $R_{ijk} \in \{-10, -9, \cdots, -1, 0, 1, \cdots, 9, 10, I\}$.

2. The numerical values of R_{ijk} are calculated $\hat{R}_{ijk} = round\left(\frac{R_{ijk}}{10}\right)$ and $R_{ijk} = I$ if $\hat{R}_{ijk} = I$.

   It approximates -0.5 to -1 and 0.5 to 1.
3. For each fixed pair $j, k \in \{1,2, \ldots, 8\}$, it is calculated $\bar{R}_{jk}$ as follows:
   - If the mode of $\hat{R}_{ijk}$ for $i = 1,2, \ldots, 14$ is unimodal, take $\bar{R}_{jk} = mode_i(\hat{R}_{ijk})$ and $\bar{R}_{kj} = 0$.
   - If the mode of $\hat{R}_{ijk}$ for $i = 1,2, \ldots, 14$ is not unimodal, it is defined as follows:
     - If $\hat{R}_{ikj}$ for $i = 1,2, \ldots, 14$ is unimodal, take $\bar{R}_{kj} = mode_i(\hat{R}_{ikj})$ and $\bar{R}_{jk} = 0$.
     - If $\hat{R}_{ikj}$ for $i = 1,2, \ldots, 14$ is not unimodal, take $\bar{R}_{jk} = \bar{R}_{kj} = I$.
4. In this way, the adjacency matrix is formed with the elements $\bar{R}_{jk}$ obtained from this algorithm.

After applying the surveys to the 14 specialists and processing the data obtained with the help of the previous algorithm, we arrive at the following adjacency matrix:

$$N(E) = \begin{pmatrix} 0 & -1 & 1 & -1 & 0 & 1 & 1 & I \\ 0 & 0 & I & 1 & 0 & 0 & 0 & I \\ 0 & I & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ I & I & 1 & -1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Figure 1 contains the graphical representation of the graph obtained from the previous adjacency matrix.

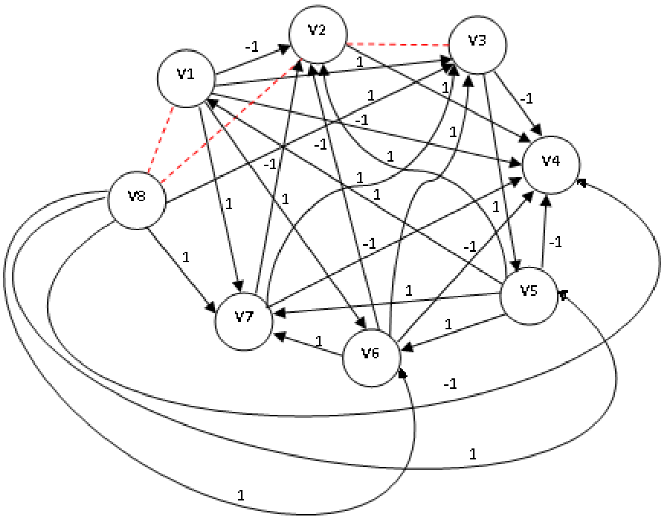

Figure 1: neutrosophic cognitive Map obtained from the experts.

We ran the system for every possible value of the initial state with the help of the Hidden Patterns algorithm. This is a quantity of $2^8 - 1 = 255$, excluding the case where no node is activated. The absolute frequency of convergence of each variable to each of the possible values within the set $\{0, 1, I\}$ was calculated, in addition to the relative frequencies. The results are shown in Table 1.

	Convergence to the value		
Variable	0	1	I
V1	1 (0.0039216)	252 (0.98824)	2 (0.0078431)
V2	1 (0.0039216)	254 (0.99608)	0 (0)
V3	1 (0.0039216)	252 (0.98824)	2 (0.0078431)
V4	0 (0)	255 (1)	0 (0)
V5	1 (0.0039216)	252 (0.98824)	2 (0.0078431)
V6	1 (0.0039216)	252 (0.98824)	2 (0.0078431)
V7	1 (0.0039216)	252 (0.98824)	2 (0.0078431)
V8	1 (0.0039216)	128 (0.50196)	126 (0.49412)

Table 1: Absolute frequency of convergence of the system to each of the possible values $\{0, 1, I\}$. The relative frequencies appear into parentheses.

All variables are activated most of the time. At least 252 times, excluding the variable V8 which is activated 128 times. 128 is the number of times that a variable appears activated as an initial value, not as a consequence of the activation of the rest of the variables. That is why V8 is activated only if there is the political will to invest in Research and Development on circular economy, it will never be activated as a consequence of the activation of any other variable. The rest of the variables are activated at least 124 times as a consequence of the activation of the other variables. On the other hand, the V4 carbon footprint variable is always activated, even as activation of the rest, this is because it is a variable that is a consequence of the others. There are few cases in which a variable remains inactive, which is due to the great interrelation that exists between the elements of the system. On the other hand, there is rarely indetermination, excluding V4, which remains indeterminate 126 times.

Below we delve into the results obtained from the surveys and the experts' assessments of the status of each of the concepts. The experts were asked to give a rating between 0 and 10 on the status of each of the variables in Peru today and the results were as follows:

V1 (Recycling rate): 3 expresses that the recycling rate is poor, which represents the insufficient quantity of materials that are recycled.

V2 (Consumption of natural resources): 7 Value that means that the available natural resources are being over-exploited.

V3 (Added Value): 2 Value that formulates that insufficient economic value is being generated from the efficient use of materials and resources.

V4 (Carbon Footprint): 8 Amount from which it can be inferred that the carbon footprint is very high, which is interpreted as high greenhouse gas emissions related to economic and production activities.

V5 (Recycling Index): 3 which indicates that the use of municipal waste is insufficient, a difference that is sent to landfills or incinerated.

V6 (Number of Companies with Circular Economy practices): 2 indicates that a low number of companies have adopted circular economy practices, which shows little importance given by companies to this activity.

V7 (Quantity of refurbished product): 4 that expresses an insufficient culture and practice of refurbishing or reusing products.

V8 (Investment in Research in Recycling): 2 A value that indicates that there is inadequate investment in R&D investment processes in research and development of technologies for recycling as a means of obtaining raw materials.

The obtained values were divided by 10 and the initial vector was obtained $S = (0.3, 0.7,0.2,0.8,0.3,0.2,0.4,0.2)$. The algorithm for seeking Hidden Patterns is designed for initial values of 0 (inactivated) or 1 (activated), therefore we converted S into an initial vector closer to the integer values 0 or 1, leaving $X_0 = (0, 1,0,1,0,0,0,0)$, the result obtained from applying the algorithm of seeking Hidden Patterns was: $(I, 1,0,1, I, I, I, I)$, that is, there are not encouraging results, in this case, significant added values will never be obtained and the rest of the variables remain undetermined. Although the negative results of high consumption of natural resources and high carbon footprint remain active.

We can carry out simulations with the model to determine which strategies to follow, for example, suppose that we considerably reduce the high consumption of natural resources and the carbon footprint so that they are practically inactive, and at the same time we activate the recycling rate, that is, we start from the initial vector $X_0 = (1, 0,0,0,0,0,0,0)$, the result is $(1, 1,1,1,1,1,1, I)$. In other words, all the variables are activated except the investment in R&D, this means that by activating only recycling the results are encouraging. We checked that V2 and V4 were activated in a negative sense, i.e. they were reduced. See below, the complete run of the algorithm:

$X_0 = (1, 0,0,0,0,0,0,0) \hookrightarrow X_0M = (0, -1,1, -1, 0 ,1,1 , I)$

$X_1 = (1, 1,1,1,0,1,1,0) \hookrightarrow X_1M = (I, -2 + 2I, 2 + 2I, -2 - I,\ 1 + I, I,\ 1 + I,\ I)$

$X_2 = (1,1, 1,1,1, I, 1, I) \hookrightarrow X_2M = (1 + I,\ -2,\ 1 + 4I,\ -2 - 3I,\ 1 + I,\ 1 + 2I, 1 + 3I, 2I)$

$X_3 = (1,1, 1,1,1, 1,1, I) \hookrightarrow X_3M = (1 + I,\ -4 + 2I,\ 3 + 2I,\ -4 - I,\ 1 + I,\ 2 + I,\ 3 + I , 2I)$

$X_4 = X_3 = (1,1, 1,1,1, 1,1, I)$.

Now suppose that investment in R&D is also activated with $X_0 = (1, 0,0,0,0,0,0,1\ )$ the result is $(1, 1,1,1,1,1,1,1)$.

Suppose that only investment in R&D is activated, with the negative values of high consumption of natural resources and high carbon footprint, that is, in $X_0 = (0, 1,0,1,0,0,0,1\ )$ this case we have the final result $(1, 1,1,1,1,1,1,1)$, that is, investment in technologies is essential to obtain an effective circular economy.

Conclusion

In this article, the possibility of obtaining an effective circular economy in Peru was studied. This is a theme that responds to the theory of nonlinear systems, which is why dynamic cognitive maps were used to process the results, specifically, we rely on neutrosophic cognitive maps because we include the possibility that there is not enough knowledge about the relationship between two concepts, which is natural in any system with non-linear dynamics. We resorted to the opinion of 14 Peruvian experts on the subject, from which the NCM was designed, concerning 8 variables determined by bibliographic research, where we reached the following conclusions:

1. Starting from the current state of the variables, the results in the future are unknown. In other words, it is necessary to make economic policy changes to improve the current situation of the circular economy in Peru. The experts assessed the current situation as inadequate.

2. Encouraging results will only be achieved by considerably increasing the recycling rate.
3. Investment in R&D on circular economy is not activated alone from the other variables, a State and government will is needed to achieve this. Once this is substantially achieved the rest of the indicators will be activated.

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On Certain Operations on Strong Interval Valued Neutrosophic Graph with Application in the Cardiac Functioning of the Human Heart

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Abstract. The neutrosophic theory is known for its prominent application in real life, possessing unclear, indeterminate information. Interval valued neutrosophic theory is even more flexible to handle indeterminacy effectively since the membership functions are depicted as intervals that lie in [0, 1]. In this article, the operations on Strong Interval Valued Neutrosophic graph have been newly defined along with their related theorems. The Strong Interval Valued Neutrosophic Digraph has been newly introduced for evaluating the blood pressure that fluctuates during the blood flow of the human heart. By considering the hemodynamic parameters of a healthy adult of age above 35 years without any cardiac malfunction, we model the cardiac functioning of the human heart during the Systolic and Diastolic phases as Strong Interval Valued Neutrosophic Digraph and its evaluation observed to be analogous to the conventional biological approach.

Keywords: Interval Valued Neutrosophic Graph; Strong Interval Valued Neutrosophic Digraph; Cardiac Cycle of the Human Heart; Wright table.

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1. **Introduction**

To address the ambiguity and imprecision on crisp sets, Zadeh, L., [1] in the year 1965, described the fuzzy set (FS) theory and consequently fuzzy logic. This proposed theory is identified with a membership function assigning all the members of a given Universal set X, a degree of membership m_A in a FS A.

Interval-valued fuzzy sets (IVFS) were initially analyzed by Sambuc [2] who termed as ϕ - floues functions, to identify the features of unpredictability by attributing the membership degrees. Atanassov, K., [3] defined the intuitionistic fuzzy sets (IFS) assigning to each member of the Universal set both a degree of membership and one of the non-membership n_A such that $0 \leq m_A(x) + n_A(x) \leq 1$ which relaxes the duality in the fuzzy set and as a consequence, it allows to address the positive and negative side of an imprecise concept.

The IFS by K. Atanassov [3] corresponds with the definition of vague sets introduced by Gau and Byehrer [4] accordingly by Bustince and Burillo [5]. Further, he introduced the interval-valued intuitionistic fuzzy set (IVIFS) as an extension of both IFS and IVFS. Inspired by the real-time situations, winning/defeating or tie scores from sports games, and yes/no/NA from decision making, Florentin Smarandache [6] proposed the concept of the neutrosophic set (NS) to understand the standard as well as the non-standard analysis.

Thus NS is a systematic paradigm that generalizes the concepts in [1], [3]. Wang et. al [7] defined the single-valued neutrosophic set (SVNS) by defining T, I and F from a nonempty set A to $[0, 1]$. The interval-valued neutrosophic set (IVNS) [8], is more efficient than the SVNS, in which their membership functions are all independent as well as their values are included in [0,1].

Fuzzy analogous of several graph - theoretic concepts are described by Azriel Rosenfield [9]. and thus fuzzy graphs (FG) have diversified applications in the areas of Science, Engineering, Technology, etc and it is essential to model those problems in comparison to the classical graph. With additional remarks on Fuzzy graphs, Bhattacharya [10], established that the concepts in fuzzy graphs do not match with the graph-theoretical concepts all the time.

The Intuitionistic fuzzy graph (IFG) [11] arises by taking the vertex and edge sets as IFS. K. Atanassov [12] in 2019, introduced eight different types of interval-valued intuitionistic fuzzy graphs (IVIFG) and their representations by index matrices.

In many situations, as the relations between the vertices (nodes) are indeterminate, the fuzzy graphs along with their extensions fail, Smarandache [13], defined neutrosophic graph (NG). Thus the single-valued neutrosophic graph (SVNG) is a NG model that generalizes the FG and IFG and Said Broumi [14], [15] further extended to interval-valued neutrosophic graphs (IVNG) and its stron form which are used to model the real-life problems with uncertain, irreconcilable, non-deterministic, unpredictable information effectively and further Mohammed Akram extended to interval-valued neutrosophic digraph (IVNDG) [16] to analyze the applied network models.

Furthermore, Shouzhen Zeng et. al [17] introduced maximal product, rejection, symmetric difference, residue product on SVNG having application in FAO for finding the most reasonable organization for the farmers to develop more food grains and to increase yearning.

Recently, Haque et. al [18], [19], [20], [21] defined various operational laws, logarithmic operational law and exponential operational law for evaluating Multi-criteria group decision-making (MCGDM), Multi -attribute decision-making(MADM) problems under spherical fuzzy, interval neutrosophic environments.

1.1. *Motivation and Novelty*

Based on the literature survey, we found that SIVNG has not been explored in detail. This motivates us to study the operations such as maximal product, rejection, symmetric difference and residue product of any two SIVNGs. Further SIVNG helps to maintain the optimal minimum value between any two nodes. By considering this, we model the cardiac functioning of the human heart under SIVN environment. Also, the blood flow cannot be reversed and falls within certain range. This necessitates us to model the cardiac cycle of the human heart as the SIVNDG, which is a novel concept. By modelling the cardiac functioning of the human heart as SIVNDG helps to explore the blood flow of the human heart in each phase.

1.2. *Organization of the article*

In this article, Section 2 contains preliminaries. In Section 3, the maximal product $(*)$, rejection $(|)$, symmetric difference $(\oplus)$, residue product $(\bullet)$ of any two SIVNG have been introduced and if G_1 and G_2 are any two SIVNGs, we prove that $G_1 * G_2$, $G_1 \mid G_2$, $G_1 \oplus G_2$ and $G_1 \bullet G_2$ is again a SIVNG. Further, the degree and total degree of these operations and their related theorems are discussed in detail. In Section 4, we propose the Strong Interval-valued Neutrosophic Digraph (SIVNDG) based on a Strong Interval-valued Neutrosophic graph (SIVNG) [6] to explore the cardiac cycle of the human heart.By converting the blood pressure values to SIVN values, we study the blood flow of the human heart. Section 5 contains the Sensitivity analysis and Comparative study. Section 6 and Section 7 deals with Results and discussion. Section 8 possess the need and limitation and impact of the research work and Section 9 contains the conclusion.

2. **Preliminaries**

Definition 2.1. An Interval Valued Neutrosophic Graph (IVNG) [14] of a graph $G' = (P', Q')$, we mean a pair $G = (P, Q)$, where $P = ([t^l_P, t^u_P], [i^l_P, i^u_P], [f^l_P, f^u_P])$ is an IVN - set on P' and $Q = ([t^l_Q, t^u_Q], [i^l_Q, i^u_Q], [f^l_Q, f^u_Q])$ is an IVN - relation on Q' that satisfies the following conditions:

(1) $P' = \{p_1, p_2, ..., p_n\}$ such that $t^l_P : P' \to [0, 1]$, $t^u_P : P' \to [0, 1]$, $i^l_P : P' \to [0, 1]$, $i^u_P : P' \to [0, 1]$, $f^l_P : P' \to [0, 1]$, $f^u_P : P' \to [0, 1]$ represent the corresponding degree of membership functions of T, I and F of $p_i \in P'$ with $0 \le t_P(p_i) + i_P(p_i) + f_P(p_i) \le 3, \forall p_i \in P'(i = 1, 2, ..., n)$.

(2) The mappings $t^l_Q : P' \times P' \to [0,1]$, $t^u_Q : P' \times P' \to [0,1]$, $i^l_Q : P' \times P' \to [0,1]$, $i^u_Q : P' \times P' \to [0,1]$, $f^l_Q : P' \times P' \to [0,1]$, $f^u_Q : P' \times P' \to [0,1]$ are such that

$$\begin{aligned}
&(1)\ t^l_Q(p_i,p_j) \le min(t^l_P(p_i), t^l_P(p_j)),\\
&(2)\ t^u_Q(p_i,p_j) \le min(t^u_P(p_i), t^u_P(p_j)),\\
&(3)\ i^l_Q(p_i,p_j) \ge max(i^l_P(p_i), i^l_P(p_j)),\\
&(4)\ i^u_Q(p_i,p_j) \ge max(i^u_P(p_i), i^u_P(p_j)),\\
&(5)\ f^l_Q(p_i,p_j) \ge max(f^l_P(p_i), f^l_P(p_j)),\\
&(6)\ f^u_Q(p_i,p_j) \ge max(f^u_P(p_i), f^u_P(p_j)),
\end{aligned}$$

where $(p_i,p_j) \in Q'$ and $0 \le t_Q(p_i,p_j) + i_Q(p_i,p_j) + f_Q(p_i,p_j) \le 3, \forall (p_i,p_j) \in Q'(i,j = 1,2,...,n)$.

Definition 2.2. An IVNG $G = (P,Q)$ of $G' = (P',Q')$ is called Strong IVNG (SIVNG) [15] if for any pair $(p_i,p_j) \in Q'$ we have :

$$\begin{aligned}
&(1)\ t^l_Q(p_i,p_j) = min(t^l_P(p_i), t^l_P(p_j)),\\
&(2)\ t^u_Q(p_i,p_j) = min(t^u_P(p_i), t^u_P(p_j)),\\
&(3)\ i^l_Q(p_i,p_j) = max(i^l_P(p_i), i^l_P(p_j)),\\
&(4)\ i^u_Q(p_i,p_j) = max(i^u_P(p_i), i^u_P(p_j)),\\
&(5)\ f^l_Q(p_i,p_j) = max(f^l_P(p_i), f^l_P(p_j)),\\
&(6)\ f^u_Q(p_i,p_j) = max(f^u_P(p_i), f^u_P(p_j)).
\end{aligned}$$

Definition 2.3. A Strong Interval Valued Neutrosophic Digraph (SIVNDG) on a non-empty Universal set X is a pair $G = (P, \overrightarrow{Q})$, where $P = ([t^l_P, t^u_P], [i^l_P, i^u_P], [f^l_P, f^u_P])$ is an IVN - set corresponds to X and $Q = ([t^l_Q, t^u_Q], [i^l_Q, i^u_Q], [f^l_Q, f^u_Q])$ is an IVN - relation corresponds to X such that

$$\begin{aligned}
&(1)\ t^l_Q(\overrightarrow{p_i,p_j}) = t^l_P(p_i) \wedge t^l_P(p_j),\\
&(2)\ t^u_Q(\overrightarrow{p_i,p_j}) = t^u_P(p_i) \wedge t^u_P(p_j),\\
&(3)\ i^l_Q(\overrightarrow{p_i,p_j}) = i^l_P(p_i) \vee i^l_P(p_j),\\
&(4)\ i^u_Q(\overrightarrow{p_i,p_j}) = i^u_P(p_i) \vee i^u_P(p_j),\\
&(5)\ f^l_Q(\overrightarrow{p_i,p_j}) = f^l_P(p_i) \vee f^l_P(p_j),\\
&(6)\ f^u_Q(\overrightarrow{p_i,p_j}) = f^u_P(p_i) \vee f^u_P(p_j),
\end{aligned}$$

$\forall p_i, p_j \in X$.

Example 2.4. Consider a SIVN-digraph $G = (P, \overrightarrow{Q})$ on $X = \{l_1, l_2, l_3, l_4\}$ in Figure 1. The vertices and the edges of G along with their membership functions are given by

$P = \{l_1 < [0.1,0.2],[0.3,0.4],[0.2,0.5] >, l_2 < [0.4,0.5],[0.2,0.3],[0.1,0.5] >, l_3 < [0.2,0.3],[0.3,0.5],[0.6,0.8] >, l_4 < [0.4,0.6],[0.3,0.5],[0.2,0.4] >\}$,

$Q = \{l_1l_2 < [0.1,0.2],[0.3,0.4],[0.2,0.5] >, l_2l_3 < [0.2,0.3],[0.3,0.5],[0.6,0.8] >, l_3l_4 < [0.2,0.3],[0.3,0.5],[0.6,0.8], l_4l_1 < [0.1,0.2],[0.3,0.5],[0.2,0.5] >\}$.

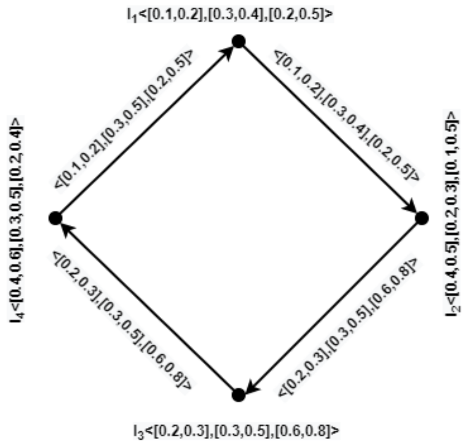

FIGURE 1. Strong Interval Valued Neutrosophic Digraph G

3. Operations on SIVNG

In this section, we introduce two different operations on SIVNG, maximal product and symmetric difference. We prove that for any two SIVNGs, the maximal product and symmetric difference is a SIVNG.

Definition 3.1. The maximal product $G_1 * G_2 = (P_1 * P_2, Q_1 * Q_2)$ of two SIVNGs $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ on the crisp graphs $G_1' = (P_1', Q_1')$ and $G_2' = (P_2', Q_2')$ is defined as

(1) $(t^l_{P_1} * t^l_{P_2})(p_1, p_2) = max\{t^l_{P_1}(p_1), t^l_{P_2}(p_2)\}$,
$(t^u_{P_1} * t^u_{P_2})(p_1, p_2) = max\{t^u_{P_1}(p_1), t^u_{P_2}(p_2)\}$,
$(i^l_{P_1} * i^l_{P_2})(p_1, p_2) = min\{i^l_{P_1}(p_1), i^l_{P_2}(p_2)\}$,
$(i^u_{P_1} * i^u_{P_2})(p_1, p_2) = min\{i^u_{P_1}(p_1), i^u_{P_2}(p_2)\}$,
$(f^l_{P_1} * f^l_{P_2})(p_1, p_2) = min\{f^l_{P_1}(p_1), f^l_{P_2}(p_2)\}$,
$(f^u_{P_1} * f^u_{P_2})(p_1, p_2) = min\{f^u_{P_1}(p_1), f^u_{P_2}(p_2)\}$, $\forall (p_1, p_2) \in (P_1' \times P_2')$.

(2) $(t^l_{Q_1} * t^l_{Q_2})((p, p_2)(p, q_2)) = max\{t^l_{P_1}(p), t^l_{Q_2}(p_2q_2)\}$,
$(t^u_{Q_1} * t^u_{Q_2})((p, p_2)(p, q_2)) = max\{t^u_{P_1}(p), t^u_{Q_2}(p_2q_2)\}$,
$(i^l_{Q_1} * i^l_{Q_2})((p, p_2)(p, q_2)) = min\{i^l_{P_1}(p), i^l_{Q_2}(p_2q_2)\}$,
$(i^u_{Q_1} * i^u_{Q_2})((p, p_2)(p, q_2)) = min\{i^u_{P_1}(p), i^u_{Q_2}(p_2q_2)\}$,
$(f^l_{Q_1} * f^l_{Q_2})((p, p_2)(p, q_2)) = min\{f^l_{P_1}(p), f^l_{Q_2}(p_2q_2)\}$,
$(f^u_{Q_1} * f^u_{Q_2})((p, p_2)(p, q_2)) = min\{f^u_{P_1}(p), f^u_{Q_2}(p_2q_2)\}$, $\forall p \in P_1'$ and $p_2q_2 \in Q_2'$.

(3) $(t^l_{Q_1} * t^l_{Q_2})((p_1, r)(q_1, r)) = max\{t^l_{Q_1}(p_1q_1), t^l_{P_2}(r)\}$,
$(t^u_{Q_1} * t^u_{Q_2})((p_1, r)(q_1, r)) = max\{t^u_{Q_1}(p_1q_1), t^u_{P_2}(r)\}$,
$(i^l_{Q_1} * i^l_{Q_2})((p_1, r)(q_1, r)) = min\{i^l_{Q_1}(p_1q_1), i^l_{P_2}(r)\}$,
$(i^u_{Q_1} * i^u_{Q_2})((p_1, r)(q_1, r)) = min\{i^u_{Q_1}(p_1q_1), i^u_{P_2}(r)\}$,

$(f^l_{Q_1} * f^l_{Q_2})((p_1, r)(q_1, r)) = min\{f^l_{Q_1}(p_1q_1), f^l_{P_2}(r)\}$,
$(f^u_{Q_1} * f^u_{Q_2})((p_1, r)(q_1, r)) = min\{f^u_{Q_1}(p_1q_1), f^u_{P_2}(r)\}$, $\forall p_1q_1 \in Q'_1$ and $r \in P'_2$.

Theorem 3.2. *The maximal product of two SIVNGs G_1 and G_2 is a SIVNG.*

Proof. Let $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ be two SIVNGs on $G'_1 = (P'_1, Q'_1)$ and $G'_2 = (P'_2, Q'_2)$ respectively and $((p_1, p_2), (q_1, q_2)) \in Q'_1 \times Q'_2$. Then, we have,

Case 1. If $p_1 = q_1 = p$,
$(t^l_{Q_1} * t^l_{Q_2})((p, p_2), (p, q_2)) = max\{t^l_{P_1}(p), t^l_{Q_2}(p_2q_2)\}$
$= max\{t^l_{P_1}(p), min\{t^l_{P_2}(p_2), t^l_{P_2}(q_2)\}\}$
$= min\{max\{t^l_{P_1}(p), t^l_{P_2}(p_2)\}, max\{t^l_{P_1}(p), t^l_{P_2}(q_2)\}\}$
$= min\{(t^l_{P_1} * t^l_{P_2})(p, p_2), (t^l_{P_1} * t^l_{P_2})(p, q_2)\}$
In the same way, the other conditions can also be verified.

Case 2. If $p_2 = q_2 = r$,
$(t^l_{Q_1} * t^l_{Q_2})((p_1, r), (q_1, r)) = max\{t^l_{Q_1}(p_1q_1), t^l_{P_2}(r)\}$
$= max\{min\{t^l_{P_1}(p_1), t^l_{P_1}(q_1)\}, t^l_{P_2}(r)\}$
$= min\{max\{t^l_{P_1}(p_1), t^l_{P_2}(r)\}, max\{t^l_{P_1}(q_1), t^l_{P_2}(r)\}\}$
$= min\{(t^l_{P_1} * t^l_{P_2})(p_1, r), (t^l_{P_1} * t^l_{P_2})(q_1, r)\}$
The other conditions can also be verified using the same approach.
Thus, the maximal product $G_1 * G_2$ is a SIVNG. □

Example 3.3. Consider two SIVNGs G_1 and G_2 as represented in Figure 2. Their maximal product $G_1 * G_2$ is represented in Figure 3.

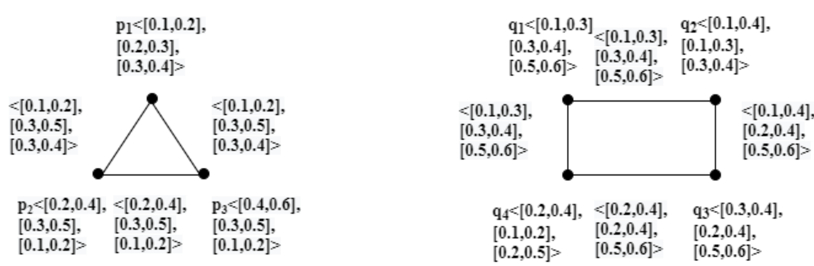

FIGURE 2. Strong Interval Valued Neutrosophic Graphs G_1 and G_2

Definition 3.4. Let $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ be two SIVNGs. The degree for any vertex $(p_1, p_2) \in (P'_1 \times P'_2)$ is,
$(d_{t^l})_{G_1 * G_2}(p_1, p_2) = \sum_{((p_1,p_2),(q_1,q_2)) \in Q'_1 \times Q'_2} (t^l_{Q_1} * t^l_{Q_2})((p_1, p_2), (q_1, q_2))$
$= \sum_{p_1=q_1, p_2q_2 \in Q'_2} max\{t^l_{P_1}(p_1), t^l_{Q_2}(p_2q_2)\} + \sum_{p_1q_1 \in Q'_1, p_2=q_2} max\{t^l_{Q_1}(p_1q_1), t^l_{P_2}(p_2)\}$,
$(d_{t^u})_{G_1 * G_2}(p_1, p_2) = \sum_{((p_1,p_2),(q_1,q_2)) \in Q'_1 \times Q'_2} (t^u_{Q_1} * t^u_{Q_2})((p_1, p_2), (q_1, q_2))$

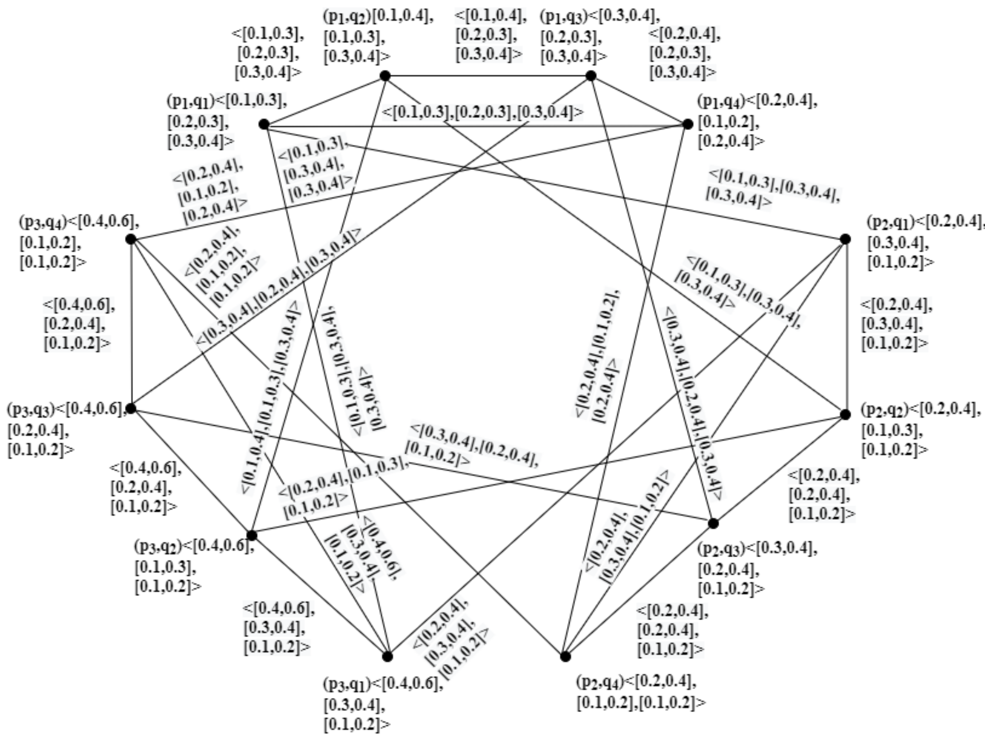

FIGURE 3. Maximal Product $G_1 * G_2$

$= \sum_{p_1=q_1, p_2q_2 \in Q_2'} max\{t^u_{P_1}(p_1), t^u_{Q_2}(p_2q_2)\} + \sum_{p_1q_1 \in Q_1', p_2=q_2} max\{i^u_{Q_1}(p_1q_1), i^u_{P_2}(p_2)\}$,
$(d_{i^l})_{G_1*G_2}(p_1,p_2) = \sum_{((p_1,p_2),(q_1,q_2)) \in Q_1' \times Q_2'} (i^l_{Q_1} * i^l_{Q_2})((p_1,p_2),(q_1,q_2))$
$= \sum_{p_1=q_1, p_2q_2 \in Q_2'} min\{i^l_{P_1}(p_1), i^l_{Q_2}(p_2q_2)\} + \sum_{p_1q_1 \in Q_1', p_2=q_2} min\{i^l_{Q_1}(p_1q_1), i^l_{P_2}(p_2)\}$,
$(d_{i^u})_{G_1*G_2}(p_1,p_2) = \sum_{((p_1,p_2),(q_1,q_2)) \in Q_1' \times Q_2'} (i^u_{Q_1} * i^u_{Q_2})((p_1,p_2),(q_1,q_2))$
$= \sum_{p_1=q_1, p_2q_2 \in Q_2'} min\{i^u_{P_1}(p_1), i^u_{Q_2}(p_2q_2)\} + \sum_{p_1q_1 \in Q_1', p_2=q_2} min\{i^u_{Q_1}(p_1q_1), i^u_{P_2}(p_2)\}$,
$(d_{f^l})_{G_1*G_2}(p_1,p_2) = \sum_{((p_1,p_2),(q_1,q_2)) \in Q_1' \times Q_2'} (f^l_{Q_1} * f^l_{Q_2})((p_1,p_2),(q_1,q_2))$
$= \sum_{p_1=q_1, p_2q_2 \in Q_2'} min\{f^l_{P_1}(p_1), f^l_{Q_2}(p_2q_2)\} + \sum_{p_1q_1 \in Q_1', p_2=q_2} min\{f^l_{Q_1}(p_1q_1), f^l_{P_2}(p_2)\}$,
$(d_{f^u})_{G_1*G_2}(p_1,p_2) = \sum_{((p_1,p_2),(q_1,q_2)) \in Q_1' \times Q_2'} (f^u_{Q_1} * f^u_{Q_2})((p_1,p_2),(q_1,q_2))$
$= \sum_{p_1=q_1, p_2q_2 \in Q_2'} min\{f^u_{P_1}(p_1), f^u_{Q_2}(p_2q_2)\} + \sum_{p_1q_1 \in Q_1', p_2=q_2} min\{f^u_{Q_1}(p_1q_1), f^u_{P_2}(p_2)\}$.

Theorem 3.5. *Let $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ be two SIVNGs. If $t^l_{P_1} \geq t^l_{Q_2}, t^u_{P_1} \geq t^u_{Q_2}, i^l_{P_1} \leq i^l_{Q_2}, i^u_{P_1} \leq i^u_{Q_2}, f^l_{P_1} \leq f^l_{Q_2}, f^u_{P_1} \leq f^u_{Q_2}$ and $t^l_{P_2} \geq t^l_{Q_1}, t^u_{P_2} \geq t^u_{Q_1}, i^l_{P_2} \leq i^l_{Q_1}, i^u_{P_2} \leq i^u_{Q_1}, f^l_{P_2} \leq f^l_{Q_1}, f^u_{P_2} \leq f^u_{Q_1}$, then for every $(p_1, p_2) \in (P_1' \times P_2')$, we have,*
$(d_{t^l})_{(G_1*G_2)}(p_1,p_2) = (d)_{(G_2)}(p_2) t^l_{P_1}(p_1) + (d)_{(G_1)}(p_1) t^l_{(P_2)}(p_2)$
$(d_{t^u})_{(G_1*G_2)}(p_1,p_2) = (d)_{(G_2)}(p_2) t^u_{P_1}(p_1) + (d)_{(G_1)}(p_1) t^u_{(P_2)}(p_2)$
$(d_{i^l})_{(G_1*G_2)}(p_1,p_2) = (d)_{(G_2)}(p_2) i^l_{P_1}(p_1) + (d)_{(G_1)}(p_1) i^l_{(P_2)}(p_2)$
$(d_{i^u})_{(G_1*G_2)}(p_1,p_2) = (d)_{(G_2)}(p_2) i^u_{P_1}(p_1) + (d)_{(G_1)}(p_1) i^u_{(P_2)}(p_2)$
$(d_{f^l})_{(G_1*G_2)}(p_1,p_2) = (d)_{(G_2)}(p_2) f^l_{P_1}(p_1) + (d)_{(G_1)}(p_1) f^l_{(P_2)}(p_2)$
$(d_{f^u})_{(G_1*G_2)}(p_1,p_2) = (d)_{(G_2)}(p_2) f^u_{P_1}(p_1) + (d)_{(G_1)}(p_1) f^u_{(P_2)}(p_2)$

Proof. Consider,
$(d_{t^l})_{G_1*G_2}(p_1,p_2) = \sum_{((p_1,p_2),(q_1,q_2))\in Q_1'\times Q_2'}(t^l_{Q_1} * t^l_{Q_2})((p_1,p_2),(q_1,q_2))$
$= \sum_{p_1=q_1,p_2q_2\in Q_2'} max\{t^l_{P_1}(p_1), t^l_{Q_2}(p_2q_2)\} + \sum_{p_1q_1\in Q_1',p_2=q_2} max\{t^l_{Q_1}(p_1q_1), t^l_{P_2}(p_2)\}$
$= \sum_{p_1=q_1,p_2q_2\in Q_2'} t^l_{Q_2}(p_2q_2) + \sum_{p_1q_1\in Q_1',p_2=q_2} t^l_{Q_1}(p_1q_1)$
$= (d)_{(G_2)}(p_2)t^l_{P_1}(p_1) + (d)_{(G_1)}(p_1)t^l_{(P_2)}(p_2)$
Similarly, the other conditions can also be proved. □

Definition 3.6. Let $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ be two SIVNGs. The total degree for any vertex $(p_1, p_2) \in (P_1' \times P_2')$ is,
$(td_{t^l})_{G_1*G_2}(p_1,p_2) = \sum_{((p_1,p_2),(q_1,q_2))\in Q_1'\times Q_2'}(t^l_{Q_1} * t^l_{Q_2})((p_1,p_2),(q_1,q_2)) + (t^l_{P_1} * t^l_{P_2})(p_1,p_2)$
$= \sum_{p_1=q_1,p_2q_2\in Q_2'} max\{t^l_{P_1}(p_1), t^l_{Q_2}(p_2q_2)\} + \sum_{p_1q_1\in Q_1',p_2=q_2} max\{t^l_{Q_1}(p_1q_1), t^l_{P_2}(p_2)\}$
$+ max\{t^l_{P_1}(p_1), t^l_{P_2}(p_2)\}$,
$(td_{t^u})_{G_1*G_2}(p_1,p_2) = \sum_{((p_1,p_2),(q_1,q_2))\in Q_1'\times Q_2'}(t^u_{Q_1} * t^u_{Q_2})((p_1,p_2),(q_1,q_2)) + (t^u_{P_1} * t^u_{P_2})(p_1,p_2)$
$= \sum_{p_1=q_1,p_2q_2\in Q_2'} max\{t^u_{P_1}(p_1), t^u_{Q_2}(p_2q_2)\} + \sum_{p_1q_1\in Q_1',p_2=q_2} max\{t^u_{Q_1}(p_1q_1), t^u_{P_2}(p_2)\}$
$+ max\{t^u_{P_1}(p_1), t^u_{P_2}(p_2)\}$,
$(td_{i^l})_{G_1*G_2}(p_1,p_2) = \sum_{((p_1,p_2),(q_1,q_2))\in Q_1'\times Q_2'}(i^l_{Q_1} * i^l_{Q_2})((p_1,p_2),(q_1,q_2)) + (i^l_{P_1} * i^l_{P_2})(p_1,p_2)$
$= \sum_{p_1=q_1,p_2q_2\in Q_2'} min\{i^l_{P_1}(p_1), i^l_{Q_2}(p_2q_2)\} + \sum_{p_1q_1\in Q_1',p_2=q_2} min\{i^l_{Q_1}(p_1q_1), i^l_{P_2}(p_2)\}$
$+ min\{i^l_{P_1}(p_1), i^l_{P_2}(p_2)\}$,
$(td_{i^u})_{G_1*G_2}(p_1,p_2) = \sum_{((p_1,p_2),(q_1,q_2))\in Q_1'\times Q_2'}(i^u_{Q_1} * i^u_{Q_2})((p_1,p_2),(q_1,q_2)) + (i^u_{P_1} * i^u_{P_2})(p_1,p_2)$
$= \sum_{p_1=q_1,p_2q_2\in Q_2'} min\{i^u_{P_1}(p_1), i^u_{Q_2}(p_2q_2)\} + \sum_{p_1q_1\in Q_1',p_2=q_2} min\{i^u_{Q_1}(p_1q_1), i^u_{P_2}(p_2)\}$
$+ min\{i^u_{P_1}(p_1), i^u_{P_2}(p_2)\}$,
$(td_{f^l})_{G_1*G_2}(p_1,p_2) = \sum_{((p_1,p_2),(q_1,q_2))\in Q_1'\times Q_2'}(f^l_{Q_1} * f^l_{Q_2})((p_1,p_2),(q_1,q_2)) + (f^l_{P_1} * f^l_{P_2})(p_1,p_2)$
$= \sum_{p_1=q_1,p_2q_2\in Q_2'} min\{f^l_{P_1}(p_1), f^l_{Q_2}(p_2q_2)\} + \sum_{p_1q_1\in Q_1',p_2=q_2} min\{f^l_{Q_1}(p_1q_1), f^l_{P_2}(p_2)\}$
$+ min\{f^l_{P_1}(p_1), f^l_{P_2}(p_2)\}$,
$(td_{f^u})_{G_1*G_2}(p_1,p_2) = \sum_{((p_1,p_2),(q_1,q_2))\in Q_1'\times Q_2'}(f^u_{Q_1} * f^u_{Q_2})((p_1,p_2),(q_1,q_2)) + (f^u_{P_1} * f^u_{P_2})(p_1,p_2)$
$= \sum_{p_1=q_1,p_2q_2\in Q_2'} min\{f^u_{P_1}(p_1), f^u_{Q_2}(p_2q_2)\} + \sum_{p_1q_1\in Q_1',p_2=q_2} min\{f^u_{Q_1}(p_1q_1), f^u_{P_2}(p_2)\}$
$+ min\{f^u_{P_1}(p_1), f^u_{P_2}(p_2)\}$.

Theorem 3.7. *Let $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ be two SIVNGs. If $t^l_{P_1} \geq t^l_{Q_2}, t^u_{P_1} \geq t^u_{Q_2}, i^l_{P_1} \leq i^l_{Q_2}, i^u_{P_1} \leq i^u_{Q_2}, f^l_{P_1} \leq f^l_{Q_2}, f^u_{P_1} \leq f^u_{Q_2}$ and $t^l_{P_2} \geq t^l_{Q_1}, t^u_{P_2} \geq t^u_{Q_1}, i^l_{P_2} \leq i^l_{Q_1}, i^u_{P_2} \leq i^u_{Q_1}, f^l_{P_2} \leq f^l_{Q_1}, f^u_{P_2} \leq f^u_{Q_1}$, then for every $(p_1, p_2) \in (P_1' \times P_2')$, we have,*
$(td_{t^l})_{(G_1*G_2)}(p_1,p_2) = (d)_{(G_2)}(p_2)t^l_{P_1}(p_1) + (d)_{(G_1)}(p_1)t^l_{(P_2)}(p_2) + max\{t^l_{P_1}(p_1), t^l_{P_2}(p_2)\}$
$(td_{t^u})_{(G_1*G_2)}(p_1,p_2) = (d)_{(G_2)}(p_2)t^u_{P_1}(p_1) + (d)_{(G_1)}(p_1)t^u_{(P_2)}(p_2) + max\{t^u_{P_1}(p_1), t^u_{P_2}(p_2)\}$
$(td_{i^l})_{(G_1*G_2)}(p_1,p_2) = (d)_{(G_2)}(p_2)i^l_{P_1}(p_1) + (d)_{(G_1)}(p_1)i^l_{(P_2)}(p_2) + min\{i^l_{P_1}(p_1), i^l_{P_2}(p_2)\}$
$(td_{i^u})_{(G_1*G_2)}(p_1,p_2) = (d)_{(G_2)}(p_2)i^u_{P_1}(p_1) + (d)_{(G_1)}(p_1)i^u_{(P_2)}(p_2) + min\{i^u_{P_1}(p_1), i^u_{P_2}(p_2)\}$
$(td_{f^l})_{(G_1*G_2)}(p_1,p_2) = (d)_{(G_2)}(p_2)f^l_{P_1}(p_1) + (d)_{(G_1)}(p_1)f^l_{(P_2)}(p_2) + min\{f^l_{P_1}(p_1), f^l_{P_2}(p_2)\}$
$(td_{f^u})_{(G_1*G_2)}(p_1,p_2) = (d)_{(G_2)}(p_2)f^u_{P_1}(p_1) + (d)_{(G_1)}(p_1)f^u_{(P_2)}(p_2) + min\{f^u_{P_1}(p_1), f^u_{P_2}(p_2)\}$

Proof. Consider the case of $(td_{f^u})_{(G_1*G_2)}(p_1, p_2)$, we have,

$(td_{f^u})_{G_1*G_2}(p_1, p_2) = \sum_{((p_1,p_2),(q_1,q_2))\in Q_1'\times Q_2'}(f^u_{Q_1} * f^u_{Q_2})((p_1, p_2),(q_1, q_2)) + (f^u_{P_1} * f^u_{P_2})(p_1, p_2)$
$= \sum_{p_1=q_1,p_2q_2\in Q_2'} min\{f^u_{P_1}(p_1), f^u_{Q_2}(p_2q_2)\} + \sum_{p_1q_1\in Q_1',p_2=q_2} min\{f^u_{Q_1}(p_1q_1), f^u_{P_2}(p_2)\}$
$+min\{f^u_{P_1}(p_1), f^u_{P_2}(p_2)\}$,
$= \sum_{p_1=q_1,p_2q_2\in Q_2'} f^u_{Q_2}(p_2q_2) + \sum_{p_1q_1\in Q_1',p_2=q_2} f^u_{Q_1}(p_1q_1) + min\{f^u_{P_1}(p_1), f^u_{P_2}(p_2)\}$,
In the same way, the other conditions can also be verified. □

Example 3.8. Consider two SIVNGs $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ as represented in Figure 4. Their maximal product $G_1 * G_2$ is represented in Figure 5.

From Figures 4 and 5, d_i, td_f for the vertex (p_3, q_2) are calculated below.

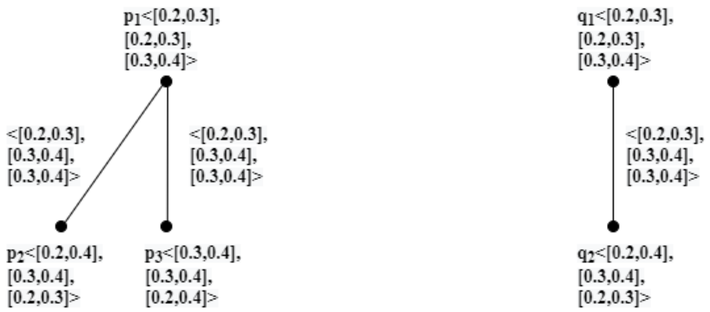

FIGURE 4. Strong Interval Valued Neutrosophic Graphs G_1 and G_2

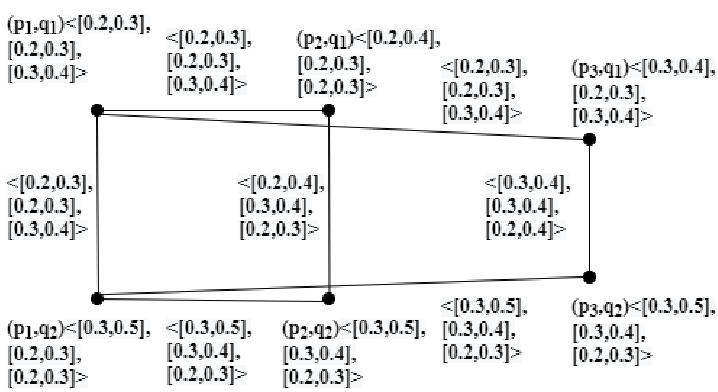

FIGURE 5. Maximal Product $G_1 * G_2$

By direct calculations, $d_{i^l}(p_3, q_2) = 0.3 + 0.3 = 0.6$, $d_{i^u}(p_3, q_2) = 0.4 + 0.4 = 0.8$, $d_i(p_3, q_2) = [0.6, 0.8]$.

$td_{f^l}(p_1, q_1) = 0.2 + 0.2 + 0.2 = 0.6$, $td_{f^u}(p_1, q_1) = 0.4 + 0.3 + 0.3 = 1.0$, $td_f(p_1, q_1) = [0.6, 1]$.

By using theorem, $d_{i^l}(p_3, q_2) = 1(0.3) + 1(0.3) = 0.6$, $d_{i^u}(p_3, q_2) = 1(0.4) + 1(0.4) = 0.8$, $d_i(p_3, q_2) = [0.6, 0.8]$.

$td_{f^l}(p_1, q_1) = 1(0.2) + 1(0.2) + min\{0.2, 0.2\} = 0.6$, $td_{f^u}(p_1, q_1) = 1(0.4) + 1(0.3) +$

$min\{0.4, 0.3\} = 1.0$, $td_f(p_1, q_1) = [0.6, 1]$.

Definition 3.9. The rejection $G_1 \mid G_2 = (P_1 \mid P_2, Q_1 \mid Q_2)$of two SIVNGs $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ is defined as

(1) $(t^l_{P_1} \mid t^l_{P_2})(p_1, p_2) = min\{t^l_{P_1}(p_1), t^l_{P_2}(p_2)\}$,
$(t^u_{P_1} \mid t^u_{P_2})(p_1, p_2) = min\{t^u_{P_1}(p_1), t^u_{P_2}(p_2)\}$,
$(i^l_{P_1} \mid i^l_{P_2})(p_1, p_2) = max\{i^l_{P_1}(p_1), i^l_{P_2}(p_2)\}$,
$(i^u_{P_1} \mid i^u_{P_2})(p_1, p_2) = max\{i^u_{P_1}(p_1), i^u_{P_2}(p_2)\}$,
$(f^l_{P_1} \mid f^l_{P_2})(p_1, p_2) = max\{f^l_{P_1}(p_1), f^l_{P_2}(p_2)\}$,
$(f^u_{P_1} \mid f^u_{P_2})(p_1, p_2) = max\{f^u_{P_1}(p_1), f^u_{P_2}(p_2)\}, \forall (p_1, p_2) \in (P'_1 \times P'_2)$.

(2) $(t^l_{Q_1} \mid t^l_{Q_2})((p, p_2), (p, q_2)) = min\{t^l_{P_1}(p), t^l_{P_2}(p_2), t^l_{P_2}(q_2)\}$,
$(t^u_{Q_1} \mid t^u_{Q_2})((p, p_2), (p, q_2)) = min\{t^u_{P_1}(p), t^u_{P_2}(p_2), t^u_{P_2}(q_2)\}$,
$(i^l_{Q_1} \mid i^l_{Q_2})((p, p_2), (p, q_2)) = max\{i^l_{P_1}(p), i^l_{P_2}(p_2), i^l_{P_2}(q_2)\}$,
$(i^u_{Q_1} \mid i^u_{Q_2})((p, p_2), (p, q_2)) = max\{i^u_{P_1}(p), i^u_{P_2}(p_2), i^u_{P_2}(q_2)\}$,
$(f^l_{Q_1} \mid f^l_{Q_2})((p, p_2), (p, q_2)) = max\{f^l_{P_1}(p), f^l_{P_2}(p_2), f^l_{P_2}(q_2)\}$,
$(f^u_{Q_1} \mid f^u_{Q_2})((p, p_2), (p, q_2)) = max\{f^u_{P_1}(p), f^u_{P_2}(p_2), f^u_{P_2}(q_2)\}, \forall p \in P'_1, p_2q_2 \notin Q'_2$.

(3) $(t^l_{Q_1} \mid t^l_{Q_2})((p_1, r), (q_1, r)) = min\{t^l_{P_1}(p_1), t^l_{P_1}(q_1), t^l_{P_2}(r)\}$,
$(t^u_{Q_1} \mid t^u_{Q_2})((p_1, r), (q_1, r)) = min\{t^u_{P_1}(p_1), t^u_{P_1}(q_1), t^u_{P_2}(r)\}$,
$(i^l_{Q_1} \mid i^l_{Q_2})((p_1, r), (q_1, r)) = max\{i^l_{P_1}(p_1), i^l_{P_1}(q_1), i^l_{P_2}(r)\}$,
$(i^u_{Q_1} \mid i^u_{Q_2})((p_1, r), (q_1, r)) = max\{i^u_{P_1}(p_1), i^u_{P_1}(q_1), i^u_{P_2}(r)\}$,
$(f^l_{Q_1} \mid f^l_{Q_2})((p_1, r), (q_1, r)) = max\{f^l_{P_1}(p_1), f^l_{P_1}(q_1), f^l_{P_2}(r)\}$,
$(f^u_{Q_1} \mid f^u_{Q_2})((p_1, r), (q_1, r)) = max\{f^u_{P_1}(p_1), f^u_{P_1}(q_1), f^u_{P_2}(r)\}, \forall p_1q_1 \notin Q'_1, r \in P'_2$.

(4) $(t^l_{Q_1} \mid t^l_{Q_2})((p_1, p_2), (q_1, q_2)) = min\{t^l_{P_1}(p_1), t^l_{P_1}(q_1), t^l_{P_2}(p_2), t^l_{P_2}(q_2)\}$,
$(t^u_{Q_1} \mid t^u_{Q_2})((p_1, p_2), (q_1, q_2)) = min\{t^u_{P_1}(p_1), t^u_{P_1}(q_1), t^u_{P_2}(p_2), t^u_{P_2}(q_2)\}$,
$(i^l_{Q_1} \mid i^l_{Q_2})((p_1, p_2), (q_1, q_2)) = max\{i^l_{P_1}(p_1), i^l_{P_1}(q_1), i^l_{P_2}(p_2), i^l_{P_2}(q_2)\}$,
$(i^u_{Q_1} \mid i^u_{Q_2})((p_1, p_2), (q_1, q_2)) = max\{i^u_{P_1}(p_1), i^u_{P_1}(q_1), i^u_{P_2}(p_2), i^u_{P_2}(q_2)\}$,
$(f^l_{Q_1} \mid f^l_{Q_2})((p_1, p_2), (q_1, q_2)) = max\{f^l_{P_1}(p_1), f^l_{P_1}(q_1), f^l_{P_2}(p_2), f^l_{P_2}(q_2)\}$,
$(f^u_{Q_1} \mid f^u_{Q_2})((p_1, p_2), (q_1, q_2)) = max\{f^u_{P_1}(p_1), f^u_{P_1}(q_1), f^u_{P_2}(p_2), f^u_{P_2}(q_2)\}$,
$\forall p_1q_1 \notin Q'_1, p_2q_2 \notin Q'_2$.

Example 3.10. Consider two SIVNGs $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ as represented in Figure 6. Their rejection $G_1 \mid G_2$ is represented in Figure 7.

Theorem 3.11. *The rejection of two SIVNGs G_1 and G_2 is a SIVNG.*

Proof. Let $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ be two SIVNGs on $G'_1 = (P'_1, Q'_1)$ and $G'_2 = (P'_2, Q'_2)$ respectively and $((p_1, p_2), (q_1, q_2)) \in Q'_1 \times Q'_2$. Then, we have,

Case 1. If $p_1 = q_1, p_2q_2 \notin Q'_2$

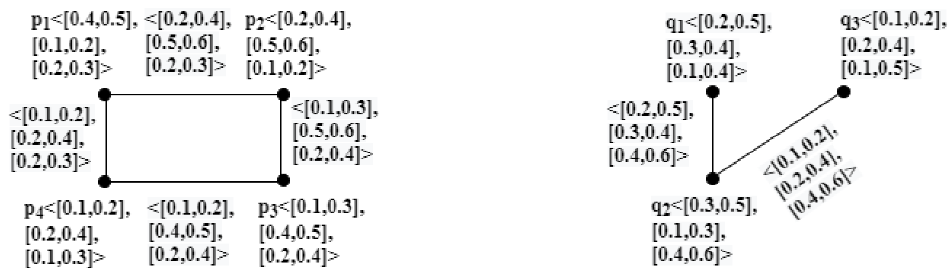

FIGURE 6. Strong Interval Valued Neutrosophic Graphs G_1 and G_2

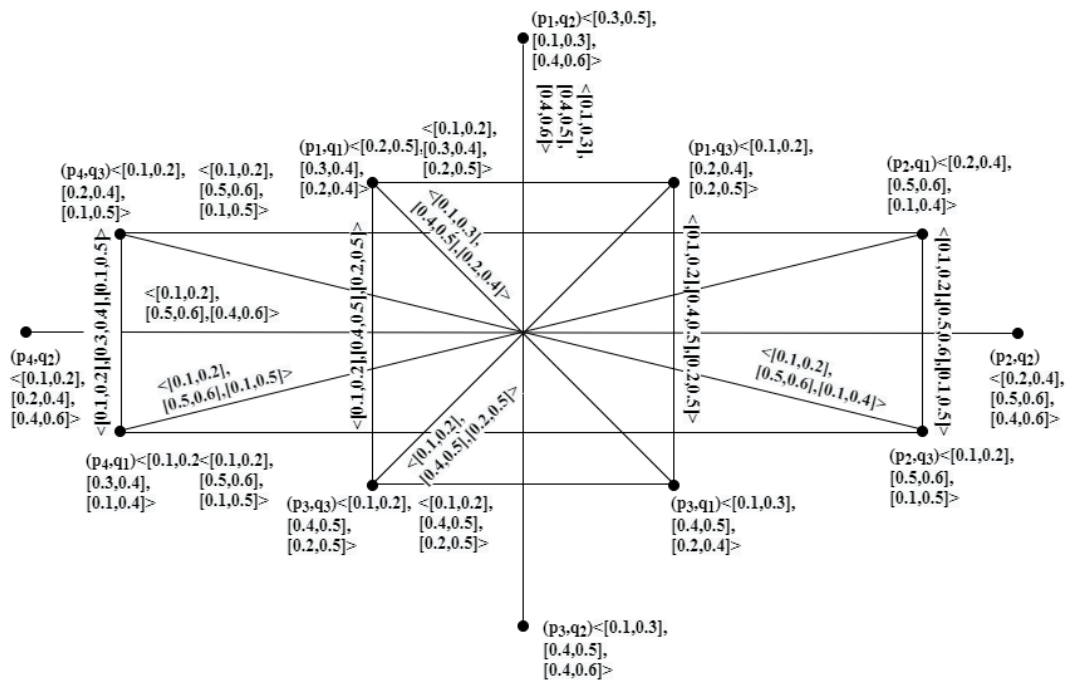

FIGURE 7. Rejection $G_1 \mid G_2$

$(t^l_{Q_1} \mid t^l_{Q_2})((p_1, p_2), (p_1, q_2)) = min\{t^l_{P_1}(p_1), t^l_{P_2}(p_2), t^l_{P_2}(q_2)\}$
$= min\{min\{t^l_{P_1}(p_1), t^l_{P_1}(q_1)\}, min\{t^l_{P_2}(p_2), t^l_{P_2}(q_2)\}\}$
$= min\{(t^l_{P_1} \mid t^l_{P_2})(p_1, p_2), (t^l_{P_1} \mid t^l_{P_2})(q_1, q_2)\}$

In the same way, the other conditions can also be verified.

Case 2. If $p_2 = q_2, p_1q_1 \notin Q_1'$

$(i^l_{Q_1} \mid i^l_{Q_2})((p_1, p_2), (q_1, q_2)) = max\{i^l_{P_1}(p_1), i^l_{P_1}(q_1), i^l_{P_2}(p_2)\}$
$= max\{max\{i^l_{P_1}(p_1), i^l_{P_1}(q_1)\}, max\{i^l_{P_2}(p_2), i^l_{P_2}(q_2)\}\}$
$= max\{(i^l_{P_1} \mid i^l_{P_2})(p_1, p_2), (i^l_{P_1} \mid i^l_{P_2})(q_1, q_2)\}$

Similarly, the other conditions can also be verified.

Case 3. If $p_1q_1 \notin Q_1', p_2q_2 \notin Q_2'$

$(f^l_{Q_1} \mid f^l_{Q_2})((p_1, p_2), (q_1, q_2)) = max\{f^l_{P_1}(p_1), f^l_{P_1}(q_1), f^l_{P_2}(p_2), f^l_{P_2}(q_2)\}$

$= max\{max\{f^l_{P_1}(p_1), f^l_{P_1}(q_1)\}, max\{f^l_{P_2}(p_2), f^l_{P_2}(q_2)\}\}$
$= max\{(f^l_{P_1} \mid f^l_{P_2})(p_1,p_2), (f^l_{P_1} \mid f^l_{P_2})(q_1,q_2)\}$
Similarly, the other conditions can also be verified. □

Definition 3.12. Let $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ be two SIVNGs. The degree for any vertex $(p_1, p_2) \in (P'_1 \times P'_2)$ is,
$(d_{t^l})_{G_1|G_2}(p_1,p_2) = \sum_{((p_1,p_2),(p_1,q_2)) \in Q'_1 \times Q'_2} (t^l_{Q_1} \mid t^l_{Q_2})((p_1,p_2),(p_1,q_2))$
$= \sum_{p_1=q_1, p_2q_2 \notin Q'_2} min\{t^l_{P_1}(p_1), t^l_{P_2}(p_2), t^l_{P_2}(q_2)\} + \sum_{p_1q_1 \notin Q'_1, p_1=q_1} min\{t^l_{P_1}(p_1), t^l_{P_1}(q_1), t^l_{P_2}(p_2)\}$
$+ \sum_{p_1q_1 \notin Q'_1, p_2q_2 \notin Q'_2} min\{t^l_{P_1}(p_1), t^l_{P_1}(q_1), t^l_{P_2}(p_2), t^l_{P_2}(q_2)\}$
$(d_{t^u})_{G_1|G_2}(p_1,p_2) = \sum_{((p_1,p_2),(p_1,q_2)) \in Q'_1 \times Q'_2} (t^u_{Q_1} \mid t^u_{Q_2})((p_1,p_2),(p_1,q_2))$
$= \sum_{p_1=q_1, p_2q_2 \notin Q'_2} min\{t^u_{P_1}(p_1), t^u_{P_2}(p_2), t^u_{P_2}(q_2)\} + \sum_{p_1q_1 \notin Q'_1, p_1=q_1} min\{t^u_{P_1}(p_1), t^u_{P_1}(q_1), t^u_{P_2}(p_2)\}$
$+ \sum_{p_1q_1 \notin Q'_1, p_2q_2 \notin Q'_2} min\{t^u_{P_1}(p_1), t^u_{P_1}(q_1), t^u_{P_2}(p_2), t^u_{P_2}(q_2)\}$
$(d_{i^l})_{G_1|G_2}(p_1,p_2) = \sum_{((p_1,p_2),(p_1,q_2)) \in Q'_1 \times Q'_2} (i^l_{Q_1} \mid i^l_{Q_2})((p_1,p_2),(p_1,q_2))$
$= \sum_{p_1=q_1, p_2q_2 \notin Q'_2} max\{i^l_{P_1}(p_1), i^l_{P_2}(p_2), i^l_{P_2}(q_2)\} + \sum_{p_1q_1 \notin Q'_1, p_1=q_1} max\{i^l_{P_1}(p_1), i^l_{P_1}(q_1), i^l_{P_2}(p_2)\}$
$+ \sum_{p_1q_1 \notin Q'_1, p_2q_2 \notin Q'_2} max\{i^l_{P_1}(p_1), i^l_{P_1}(q_1), i^l_{P_2}(p_2), i^l_{P_2}(q_2)\}$
$(d_{i^u})_{G_1|G_2}(p_1,p_2) = \sum_{((p_1,p_2),(p_1,q_2)) \in Q'_1 \times Q'_2} (i^u_{Q_1} \mid i^u_{Q_2})((p_1,p_2),(p_1,q_2))$
$= \sum_{p_1=q_1, p_2q_2 \notin Q'_2} max\{i^u_{P_1}(p_1), i^u_{P_2}(p_2), i^u_{P_2}(q_2)\} + \sum_{p_1q_1 \notin Q'_1, p_1=q_1} max\{i^u_{P_1}(p_1), i^u_{P_1}(q_1), i^u_{P_2}(p_2)\}$
$+ \sum_{p_1q_1 \notin Q'_1, p_2q_2 \notin Q'_2} max\{i^u_{P_1}(p_1), i^u_{P_1}(q_1), i^u_{P_2}(p_2), i^u_{P_2}(q_2)\}$
$(d_{f^l})_{G_1|G_2}(p_1,p_2) = \sum_{((p_1,p_2),(p_1,q_2)) \in Q'_1 \times Q'_2} (f^l_{Q_1} \mid f^l_{Q_2})((p_1,p_2),(p_1,q_2)) =$
$\sum_{p_1=q_1, p_2q_2 \notin Q'_2} max\{f^l_{P_1}(p_1), f^l_{P_2}(p_2), f^l_{P_2}(q_2)\} + \sum_{p_1q_1 \notin Q'_1, p_1=q_1} max\{f^l_{P_1}(p_1), f^l_{P_1}(q_1), f^l_{P_2}(p_2)\}$
$+ \sum_{p_1q_1 \notin Q'_1, p_2q_2 \notin Q'_2} max\{f^l_{P_1}(p_1), f^l_{P_1}(q_1), f^l_{P_2}(p_2), f^l_{P_2}(q_2)\}$
$(d_{f^u})_{G_1|G_2}(p_1,p_2) = \sum_{((p_1,p_2),(p_1,q_2)) \in Q'_1 \times Q'_2} (f^u_{Q_1} \mid f^u_{Q_2})((p_1,p_2),(p_1,q_2)) =$
$\sum_{p_1=q_1, p_2q_2 \notin Q'_2} max\{f^u_{P_1}(p_1), f^u_{P_2}(p_2), f^u_{P_2}(q_2)\} + \sum_{p_1q_1 \notin Q'_1, p_1=q_1} max\{f^u_{P_1}(p_1), f^u_{P_1}(q_1), f^u_{P_2}(p_2)\}$
$+ \sum_{p_1q_1 \notin Q'_1, p_2q_2 \notin Q'_2} max\{f^u_{P_1}(p_1), f^u_{P_1}(q_1), f^u_{P_2}(p_2), f^u_{P_2}(q_2)\}$

Definition 3.13. Let $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ be two SIVNGs. The total degree for any vertex $(p_1, p_2) \in (P'_1 \times P'_2)$ is,
$(td_{t^l})_{G_1|G_2}(p_1,p_2) = \sum_{((p_1,p_2),(p_1,q_2)) \in Q'_1 \times Q'_2} (t^l_{Q_1} \mid t^l_{Q_2})((p_1,p_2),(p_1,q_2)) + (t^l_{P_1} \mid t^l_{P_2})(p_1,p_2)$
$= \sum_{p_1=q_1, p_2q_2 \notin Q'_2} min\{t^l_{P_1}(p_1), t^l_{P_2}(p_2), t^l_{P_2}(q_2)\} + \sum_{p_1q_1 \notin Q'_1, p_1=q_1} min\{t^l_{P_1}(p_1), t^l_{P_1}(q_1), t^l_{P_2}(p_2)\}$
$+ \sum_{p_1q_1 \notin Q'_1, p_2q_2 \notin Q'_2} min\{t^l_{P_1}(p_1), t^l_{P_1}(q_1), t^l_{P_2}(p_2), t^l_{P_2}(q_2)\} + min\{t^l_{P_1}(p_1), t^l_{P_2}(p_2)\}$
$(td_{t^u})_{G_1|G_2}(p_1,p_2) = \sum_{((p_1,p_2),(p_1,q_2)) \in Q'_1 \times Q'_2} (t^u_{Q_1} \mid t^u_{Q_2})((p_1,p_2),(p_1,q_2)) + (t^u_{P_1} \mid t^u_{P_2})(p_1,p_2)$
$= \sum_{p_1=q_1, p_2q_2 \notin Q'_2} min\{t^u_{P_1}(p_1), t^u_{P_2}(p_2), t^u_{P_2}(q_2)\} + \sum_{p_1q_1 \notin Q'_1, p_1=q_1} min\{t^u_{P_1}(p_1), t^u_{P_1}(q_1), t^u_{P_2}(p_2)\}$
$+ \sum_{p_1q_1 \notin Q'_1, p_2q_2 \notin Q'_2} min\{t^u_{P_1}(p_1), t^u_{P_1}(q_1), t^u_{P_2}(p_2), t^u_{P_2}(q_2)\} + min\{t^u_{P_1}(p_1), t^u_{P_2}(p_2)\}$
$(td_{i^l})_{G_1|G_2}(p_1,p_2) = \sum_{((p_1,p_2),(p_1,q_2)) \in Q'_1 \times Q'_2} (i^l_{Q_1} \mid i^l_{Q_2})((p_1,p_2),(p_1,q_2)) + (i^l_{P_1} \mid i^l_{P_2})(p_1,p_2)$
$= \sum_{p_1=q_1, p_2q_2 \notin Q'_2} max\{i^l_{P_1}(p_1), i^l_{P_2}(p_2), i^l_{P_2}(q_2)\} + \sum_{p_1q_1 \notin Q'_1, p_1=q_1} max\{i^l_{P_1}(p_1), i^l_{P_1}(q_1), i^l_{P_2}(p_2)\}$
$+ \sum_{p_1q_1 \notin Q'_1, p_2q_2 \notin Q'_2} max\{i^l_{P_1}(p_1), i^l_{P_1}(q_1), i^l_{P_2}(p_2), i^l_{P_2}(q_2)\} + max\{i^l_{P_1}(p_1), i^l_{P_2}(p_2)\}$
$(td_{i^u})_{G_1|G_2}(p_1,p_2) = \sum_{((p_1,p_2),(p_1,q_2)) \in Q'_1 \times Q'_2} (i^u_{Q_1} \mid i^u_{Q_2})((p_1,p_2),(p_1,q_2)) + (i^u_{P_1} \mid i^u_{P_2})(p_1,p_2)$

$= \sum_{p_1=q_1, p_2q_2 \notin Q_2'} max\{i^u_{P_1}(p_1), i^u_{P_2}(p_2), i^u_{P_2}(q_2)\} + \sum_{p_1q_1 \notin Q_1', p_1=q_1} max\{i^u_{P_1}(p_1), i^u_{P_1}(q_1), i^u_{P_2}(p_2)\}$
$+ \sum_{p_1q_1 \notin Q_1', p_2q_2 \notin Q_2'} max\{i^u_{P_1}(p_1), i^u_{P_1}(q_1), i^u_{P_2}(p_2), i^u_{P_2}(q_2)\} + max\{i^u_{P_1}(p_1), i^u_{P_2}(p_2)\}$
$(td_{f^l})_{G_1|G_2}(p_1,p_2) = \sum_{((p_1,p_2),(p_1,q_2)) \in Q_1' \times Q_2'} (f^l_{Q_1} \mid f^l_{Q_2})((p_1,p_2),(p_1,q_2)) + (f^l_{P_1} \mid f^l_{P_2})(p_1,p_2) =$
$\sum_{p_1=q_1, p_2q_2 \notin Q_2'} max\{f^l_{P_1}(p_1), f^l_{P_2}(p_2), f^l_{P_2}(q_2)\} + \sum_{p_1q_1 \notin Q_1', p_1=q_1} max\{f^l_{P_1}(p_1), f^l_{P_1}(q_1), f^l_{P_2}(p_2)\}$
$+ \sum_{p_1q_1 \notin Q_1', p_2q_2 \notin Q_2'} max\{f^l_{P_1}(p_1), f^l_{P_1}(q_1), f^l_{P_2}(p_2), f^l_{P_2}(q_2)\} + max\{f^l_{P_1}(p_1), f^l_{P_2}(p_2)\}$
$(td_{f^u})_{G_1|G_2}(p_1,p_2) = \sum_{((p_1,p_2),(p_1,q_2)) \in Q_1' \times Q_2'} (f^u_{Q_1} \mid f^u_{Q_2})((p_1,p_2),(p_1,q_2)) + (f^u_{P_1} \mid f^u_{P_2})(p_1,p_2) =$
$\sum_{p_1=q_1, p_2q_2 \notin Q_2'} max\{f^u_{P_1}(p_1), f^u_{P_2}(p_2), f^u_{P_2}(q_2)\} + \sum_{p_1q_1 \notin Q_1', p_1=q_1} max\{f^u_{P_1}(p_1), f^u_{P_1}(q_1), f^u_{P_2}(p_2)\}$
$+ \sum_{p_1q_1 \notin Q_1', p_2q_2 \notin Q_2'} max\{f^u_{P_1}(p_1), f^u_{P_1}(q_1), f^u_{P_2}(p_2), f^u_{P_2}(q_2)\} + max\{f^u_{P_1}(p_1), f^u_{P_2}(p_2)\}$

From Figure 7, $d_i(p_1,q_3)$ and $td_i(p_1,q_3)$ for the vertex (p_1,q_3) are calculated below.

$d_{i^l}(p_1,q_3) = 0.3 + 0.4 + 0.4 = 1.1, d_{i^u}(p_1,q_3) = 0.4 + 0.5 + 0.5 = 1.4, d_i(p_1,q_3) = [1.1, 1.4].$

$td_{i^l}(p_1,q_3) = 0.3 + 0.4 + 0.4 + 0.2 = 1.3, td_{i^u}(p_1,q_3) = 0.4 + 0.5 + 0.5 + 0.4 = 1.8, td_i(p_1,q_3) = [1.3, 1.8].$

Definition 3.14. The symmetric difference $G_1 \oplus G_2 = (P_1 \oplus P_2, Q_1 \oplus Q_2)$ of two SIVNGs $G_1 = (P_1,Q_1)$ and $G_2 = (P_2,Q_2)$ is defined as

(1) $(t^l_{P_1} \oplus t^l_{P_2})(p_1,p_2) = min\{t^l_{P_1}(p_1), t^l_{P_2}(p_2)\}$,
$(t^u_{P_1} \oplus t^u_{P_2})(p_1,p_2) = min\{t^u_{P_1}(p_1), t^u_{P_2}(p_2)\}$,
$(i^l_{P_1} \oplus i^l_{P_2})(p_1,p_2) = max\{i^l_{P_1}(p_1), i^l_{P_2}(p_2)\}$,
$(i^u_{P_1} \oplus i^u_{P_2})(p_1,p_2) = max\{i^u_{P_1}(p_1), i^u_{P_2}(p_2)\}$,
$(f^l_{P_1} \oplus f^l_{P_2})(p_1,p_2) = max\{f^l_{P_1}(p_1), f^l_{P_2}(p_2)\}$,
$(f^u_{P_1} \oplus f^u_{P_2})(p_1,p_2) = max\{f^u_{P_1}(p_1), f^u_{P_2}(p_2)\}$, $\forall (p_1,p_2) \in (P_1' \times P_2')$.

(2) $(t^l_{Q_1} \oplus t^l_{Q_2})((p,p_2)(p,q_2)) = min\{t^l_{P_1}(p), t^l_{Q_2}(p_2q_2)\}$,
$(t^u_{Q_1} \oplus t^u_{Q_2})((p,p_2)(p,q_2)) = min\{t^u_{P_1}(p), t^u_{Q_2}(p_2q_2)\}$,
$(i^l_{Q_1} \oplus i^l_{Q_2})((p,p_2)(p,q_2)) = max\{i^l_{P_1}(p), i^l_{Q_2}(p_2q_2)\}$,
$(i^u_{Q_1} \oplus i^u_{Q_2})((p,p_2)(p,q_2)) = max\{i^u_{P_1}(p), i^u_{Q_2}(p_2q_2)\}$,
$(f^l_{Q_1} \oplus f^l_{Q_2})((p,p_2)(p,q_2)) = max\{f^l_{P_1}(p), f^l_{Q_2}(p_2q_2)\}$,
$(f^u_{Q_1} \oplus f^u_{Q_2})((p,p_2)(p,q_2)) = max\{f^u_{P_1}(p), f^u_{Q_2}(p_2q_2)\}$, $\forall p \in P_1'$ and $p_2q_2 \in Q_2'$.

(3) $(t^l_{Q_1} \oplus t^l_{Q_2})((p_1,r)(q_1,r)) = min\{t^l_{Q_1}(p_1q_1), t^l_{P_2}(r)\}$
$(t^u_{Q_1} \oplus t^u_{Q_2})((p_1,r)(q_1,r)) = min\{t^u_{Q_1}(p_1q_1), t^u_{P_2}(r)\}$,
$(i^l_{Q_1} \oplus i^l_{Q_2})((p_1,r)(q_1,r)) = max\{i^l_{Q_1}(p_1q_1), i^l_{P_2}(r)\}$,
$(i^u_{Q_1} \oplus i^u_{Q_2})((p_1,r)(q_1,r)) = max\{i^u_{Q_1}(p_1q_1), i^u_{P_2}(r)\}$,
$(f^l_{Q_1} \oplus f^l_{Q_2})((p_1,r)(q_1,r)) = max\{f^l_{Q_1}(p_1q_1), f^l_{P_2}(r)\}$,
$(f^u_{Q_1} \oplus f^u_{Q_2})((p_1,r)(q_1,r)) = max\{f^u_{Q_1}(p_1q_1), f^u_{P_2}(r)\}$, $\forall p_1q_1 \in Q_1'$ and $r \in P_2'$.

(4) $(t^l_{Q_1} \oplus t^l_{Q_2})(p_1,p_2)(q_1,q_2) = min\{t^l_{P_1}(p_1), t^l_{P_1}(q_1), t^l_{Q_2}(p_2q_2)\}$,
$(t^u_{Q_1} \oplus t^u_{Q_2})(p_1,p_2)(q_1,q_2) = min\{t^u_{P_1}(p_1), t^u_{P_1}(q_1), t^u_{Q_2}(p_2q_2)\}$,
$(i^l_{Q_1} \oplus i^l_{Q_2})(p_1,p_2)(q_1,q_2) = max\{i^l_{P_1}(p_1), i^l_{P_1}(q_1), i^l_{Q_2}(p_2q_2)\}$,
$(i^u_{Q_1} \oplus i^u_{Q_2})(p_1,p_2)(q_1,q_2) = max\{i^u_{P_1}(p_1), i^u_{P_1}(q_1), i^u_{Q_2}(p_2q_2)\}$,

$(f^l_{Q_1} \oplus f^l_{Q_2})(p_1, p_2)(q_1, q_2) = max\{f^l_{P_1}(p_1), f^l_{P_1}(q_1), f^l_{Q_2}(p_2q_2)\}$,
$(f^u_{Q_1} \oplus f^u_{Q_2})(p_1, p_2)(q_1, q_2) = max\{f^u_{P_1}(p_1), f^u_{P_1}(q_1), f^u_{Q_2}(p_2q_2)\}$,
$\forall p_1q_1 \notin Q'_1, p_2q_2 \in Q'_2$.

(5) $(t^l_{Q_1} \oplus t^l_{Q_2})(p_1, p_2)(q_1, q_2) = min\{t^l_{Q_1}(p_1q_1), t^l_{P_2}(p_2), t^l_{P_2}(q_2)\}$,
$(t^u_{Q_1} \oplus t^u_{Q_2})(p_1, p_2)(q_1, q_2) = min\{t^u_{Q_1}(p_1q_1), t^u_{P_2}(p_2), t^u_{P_2}(q_2)\}$,
$(i^l_{Q_1} \oplus i^l_{Q_2})(p_1, p_2)(q_1, q_2) = max\{i^l_{Q_1}(p_1q_1), i^l_{P_2}(p_2), i^l_{P_2}(q_2)\}$,
$(i^u_{Q_1} \oplus i^u_{Q_2})(p_1, p_2)(q_1, q_2) = max\{i^u_{Q_1}(p_1q_1), i^u_{P_2}(p_2), i^u_{P_2}(q_2)\}$,
$(f^l_{Q_1} \oplus f^l_{Q_2})(p_1, p_2)(q_1, q_2) = max\{f^l_{Q_1}(p_1q_1), f^l_{P_2}(p_2), f^l_{P_2}(q_2)\}$,
$(f^u_{Q_1} \oplus f^u_{Q_2})(p_1, p_2)(q_1, q_2) = max\{f^u_{Q_1}(p_1q_1), f^u_{P_2}(p_2), f^u_{P_2}(q_2)\}$,
$\forall p_1q_1 \in Q'_1, p_2q_2 \notin Q'_2$.

Example 3.15. Consider two SIVNGs $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ as represented in Figure 8. Their symmetric difference $G_1 \oplus G_2$ is represented in Figure 9. For instance, consider the vertex p_1q_1 in Figure 9. Then from the above definition, $(t^l_{P_1} \oplus t^l_{P_2})(p_1, q_1) = min\{t^l_{P_1}(p_1), t^l_{P_2}(q_1)\} = min\{0.2, 0.1\} = 0.1$ and $(t^u_{P_1} \oplus t^u_{P_2})(p_1, q_1) = min\{t^u_{P_1}(p_1), t^u_{P_2}(q_1)\} = min\{0.4, 0.3\} = 0.3$.The other membership values can be found accordingly. Further, $(t^l_{Q_1} \oplus t^l_{Q_2})(p_1, q_1)(p_1, q_2) = min\{t^l_{P_1}(p_1), t^l_{Q_2}(q_1, q_2)\} = min\{0.2, 0.1\} = 0.1$ and $(t^u_{Q_1} \oplus t^u_{Q_2})(p_1, q_1)(p_1, q_2) = min\{t^u_{P_1}(p_1), t^u_{Q_2}(q_1, q_2)\} = min\{0.4, 0.3\} = 0.3$. Similarly, all the other membership values can be calculated.

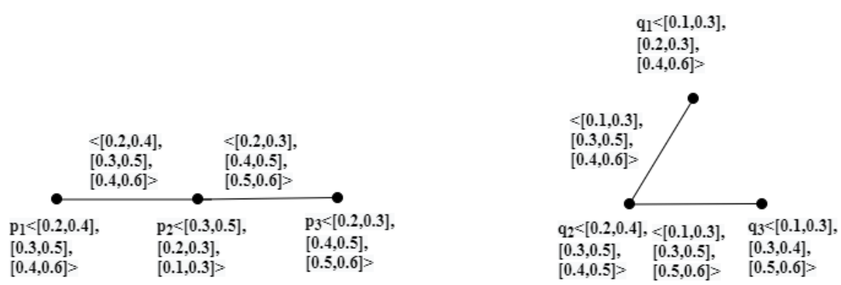

FIGURE 8. Strong Interval Valued Neutrosophic Graphs G_1 and G_2

Theorem 3.16. *The symmetric difference of two SIVNGs G_1 and G_2 is a SIVNG.*

Proof. Let $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ be two SIVNGs on $G'_1 = (P'_1, Q'_1)$ and $G'_2 = (P'_2, Q'_2)$ respectively and $((p_1, p_2), (q_1, q_2)) \in Q'_1 \times Q'_2$. Then, we have,

Case 1. If $p_1 = q_1 = p, p_2q_2 \in Q'_2$,

$(t^u_{Q_1} \oplus t^u_{Q_2})((p, p_2)(p, q_2)) = min\{t^u_{P_1}(p), t^u_{Q_2}(p_2q_2)\}$
$= min\{t^u_{P_1}(p), min\{t^u_{P_2}(p_2), t^u_{P_2}(q_2)\}\}$
$= min\{min\{t^u_{P_1}(p), t^u_{P_2}(p_2)\}, min\{t^u_{P_1}(p), t^u_{P_2}(q_2)\}\}$
$= min\{(t^u_{P_1} \oplus t^u_{P_2})(p, p_2), (t^u_{P_1} \oplus t^u_{P_2})(p, q_2)\}$

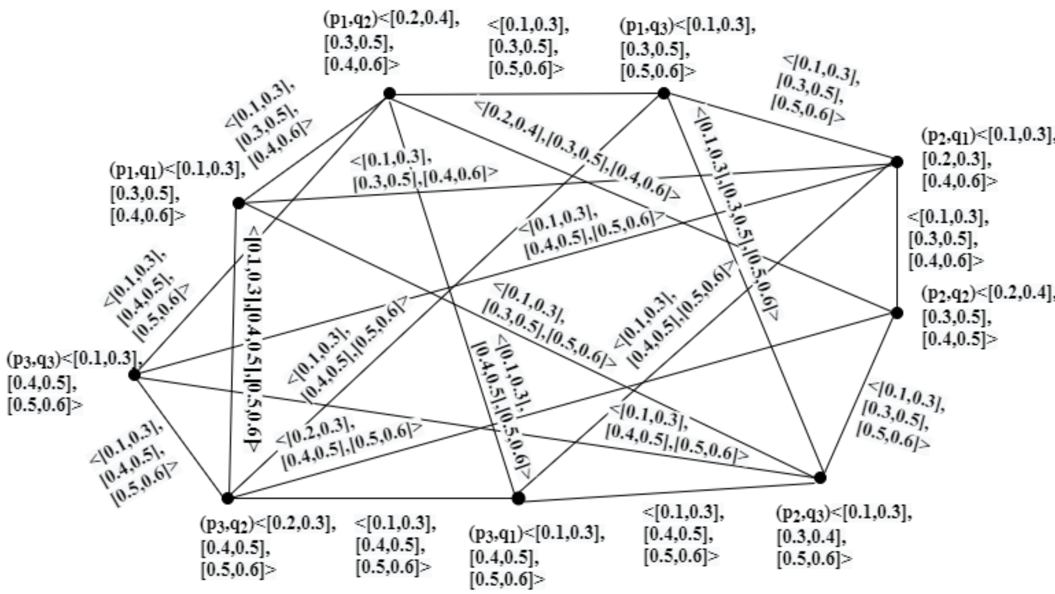

FIGURE 9. Symmetric difference $G_1 \oplus G_2$

Using the same approach, the other conditions can also be evaluated.

Case 2. If $p_2 = q_2 = r, p_1q_1 \in Q_1'$,

$(i^u_{Q_1} \oplus i^u_{Q_2})((p_1, r)(q_1, r)) = max\{i^u_{Q_1}(p_1q_1), i^u_{P_2}(r)\}$

$= max\{max\{i^u_{P_1}(p_1), i^u_{P_1}(q_1)\}, i^u_{P_2}(r)\}$

$= max\{max\{i^u_{P_1}(p_1), i^u_{P_2}(r)\}, max\{i^u_{P_1}(q_1), i^u_{P_2}(r)\}\}$

$= max\{(i^u_{P_1} \oplus i^u_{P_2})(p_1, r), (i^u_{P_1} \oplus i^u_{P_2})(q_1, r)\}$

In the same way, the other conditions can also be verified.

Case 3. If $p_1q_1 \notin Q_1', p_2q_2 \in Q_2'$,

$(f^u_{Q_1} \oplus f^u_{Q_2})((p_1, p_2)(q_1, q_2)) = max\{f^u_{P_1}(p_1), f^u_{P_1}(q_1), f^u_{Q_2}(p_2q_2)\}$

$= max\{f^u_{P_1}(p_1), f^u_{P_1}(q_1), max\{f^u_{P_2}(p_2), f^u_{P_2}(q_2)\}\}$

$= max\{max\{f^u_{P_1}(p_1), f^u_{P_2}(p_2)\}, max\{, f^u_{P_1}(q_1), f^u_{P_2}(q_2)\}\},$

$= max\{(f^u_{P_1} \oplus f^u_{P_2})(p_1, p_2), (f^u_{P_1} \oplus f^u_{P_2})(q_1, q_2)\}$

In the same way, the other conditions can also be verified.

Case 4. If $p_1q_1 \in Q_1', p_2q_2 \notin Q_2'$,

$(f^u_{Q_1} \oplus f^u_{Q_2})((p_1, p_2)(q_1, q_2)) = max\{f^u_{Q_1}(p_1q_1), f^u_{P_2}(p_2), f^u_{P_2}(q_2)\}$

$= max\{max\{f^u_{P_1}(p_1), f^u_{P_1}(q_1)\}, f^u_{P_2}(p_2), f^u_{P_2}(q_2)\}$

$= max\{max\{f^u_{P_1}(p_1), f^u_{P_2}(p_2)\}, max\{f^u_{P_1}(q_1), f^u_{P_2}(q_2)\}\}$

$= max\{(f^u_{P_1} \oplus f^u_{P_2})(p_1, p_2), (f^u_{P_1} \oplus f^u_{P_2})(q_1, q_2)\}$

Similarly, the other conditions can also be verified. □

Definition 3.17. Let $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ be two SIVNGs. The degree for any vertex $(p_1, p_2) \in (P_1' \times P_2')$ is,

$(d_{t^l})_{G_1 \oplus G_2}(p_1, p_2) = \sum_{((p_1,p_2)(q_1,q_2)) \in Q_1' \times Q_2'} (t^l_{Q_1} \oplus t^l_{Q_2})((p_1, p_2)(q_1, q_2))$
$= \sum_{p_1=q_1, p_2q_2 \in Q_2'} min\{t^l_{P_1}(p_1), t^l_{Q_2}(p_2q_2)\} + \sum_{p_1q_1 \in Q_1', p_2=q_2} min\{t^l_{Q_1}(p_1q_1), t^l_{P_2}(p_2)\}$
$+ \sum_{p_1q_1 \notin Q_1', p_2q_2 \in Q_2'} min\{t^l_{P_1}(p_1), t^l_{P_1}(q_1), t^l_{Q_2}(p_2q_2)\}$
$+ \sum_{p_1q_1 \in Q_1', p_2q_2 \notin Q_2'} min\{t^l_{Q_1}(p_1q_1), t^l_{P_2}(p_2), t^l_{P_2}(q_2)\}$
$(d_{t^u})_{G_1 \oplus G_2}(p_1, p_2) = \sum_{((p_1,p_2)(q_1,q_2)) \in Q_1' \times Q_2'} (t^u_{Q_1} \oplus t^u_{Q_2})((p_1, p_2)(q_1, q_2))$
$= \sum_{p_1=q_1, p_2q_2 \in Q_2'} min\{t^u_{P_1}(p_1), t^u_{Q_2}(p_2q_2)\} + \sum_{p_1q_1 \in Q_1', p_2=q_2} min\{t^u_{Q_1}(p_1q_1), t^u_{P_2}(p_2)\}$
$+ \sum_{p_1q_1 \notin Q_1', p_2q_2 \in Q_2'} min\{t^u_{P_1}(p_1), t^u_{P_1}(q_1), t^u_{Q_2}(p_2q_2)\}$
$+ \sum_{p_1q_1 \in Q_1', p_2q_2 \notin Q_2'} min\{t^u_{Q_1}(p_1q_1), t^u_{P_2}(p_2), t^u_{P_2}(q_2)\}$
$(d_{i^l})_{G_1 \oplus G_2}(p_1, p_2) = \sum_{((p_1,p_2)(q_1,q_2)) \in Q_1' \times Q_2'} (i^l_{Q_1} \oplus i^l_{Q_2})((p_1, p_2)(q_1, q_2))$
$= \sum_{p_1=q_1, p_2q_2 \in Q_2'} max\{i^l_{P_1}(p_1), i^l_{Q_2}(p_2q_2)\} + \sum_{p_1q_1 \in Q_1', p_2=q_2} max\{i^l_{Q_1}(p_1q_1), i^l_{P_2}(p_2)\}$
$+ \sum_{p_1q_1 \notin Q_1', p_2q_2 \in Q_2'} max\{i^l_{P_1}(p_1), i^l_{P_1}(q_1), i^l_{Q_2}(p_2q_2)\}$
$+ \sum_{p_1q_1 \in Q_1', p_2q_2 \notin Q_2'} max\{i^l_{Q_1}(p_1q_1), i^l_{P_2}(p_2), i^l_{P_2}(q_2)\}$
$(d_{i^u})_{G_1 \oplus G_2}(p_1, p_2) = \sum_{((p_1,p_2)(q_1,q_2)) \in Q_1' \times Q_2'} (i^u_{Q_1} \oplus i^u_{Q_2})((p_1, p_2)(q_1, q_2))$
$= \sum_{p_1=q_1, p_2q_2 \in Q_2'} max\{i^u_{P_1}(p_1), i^u_{Q_2}(p_2q_2)\} + \sum_{p_1q_1 \in Q_1', p_2=q_2} max\{i^u_{Q_1}(p_1q_1), i^u_{P_2}(p_2)\}$
$+ \sum_{p_1q_1 \notin Q_1', p_2q_2 \in Q_2'} max\{i^u_{P_1}(p_1), i^u_{P_1}(q_1), i^u_{Q_2}(p_2q_2)\}$
$+ \sum_{p_1q_1 \in Q_1', p_2q_2 \notin Q_2'} max\{i^u_{Q_1}(p_1q_1), i^u_{P_2}(p_2), i^u_{P_2}(q_2)\}$
$(d_{f^l})_{G_1 \oplus G_2}(p_1, p_2) = \sum_{((p_1,p_2)(q_1,q_2)) \in Q_1' \times Q_2'} (f^l_{Q_1} \oplus f^l_{Q_2})((p_1, p_2)(q_1, q_2))$
$= \sum_{p_1=q_1, p_2q_2 \in Q_2'} max\{f^l_{P_1}(p_1), f^l_{Q_2}(p_2q_2)\} + \sum_{p_1q_1 \in Q_1', p_2=q_2} max\{f^l_{Q_1}(p_1q_1), f^l_{P_2}(p_2)\}$
$+ \sum_{p_1q_1 \notin Q_1', p_2q_2 \in Q_2'} max\{f^l_{P_1}(p_1), f^l_{P_1}(q_1), f^l_{Q_2}(p_2q_2)\}$
$+ \sum_{p_1q_1 \in Q_1', p_2q_2 \notin Q_2'} max\{f^l_{Q_1}(p_1q_1), f^l_{P_2}(p_2), f^l_{P_2}(q_2)\}$
$(d_{f^u})_{G_1 \oplus G_2}(p_1, p_2) = \sum_{((p_1,p_2)(q_1,q_2)) \in Q_1' \times Q_2'} (f^u_{Q_1} \oplus f^u_{Q_2})((p_1, p_2)(q_1, q_2))$
$= \sum_{p_1=q_1, p_2q_2 \in Q_2'} max\{f^u_{P_1}(p_1), f^u_{Q_2}(p_2q_2)\} + \sum_{p_1q_1 \in Q_1', p_2=q_2} max\{f^u_{Q_1}(p_1q_1), f^u_{P_2}(p_2)\}$
$+ \sum_{p_1q_1 \notin Q_1', p_2q_2 \in Q_2'} max\{f^u_{P_1}(p_1), f^u_{P_1}(q_1), f^u_{Q_2}(p_2q_2)\}$
$+ \sum_{p_1q_1 \in Q_1', p_2q_2 \notin Q_2'} max\{f^u_{Q_1}(p_1q_1), f^u_{P_2}(p_2), f^u_{P_2}(q_2)\}$

Theorem 3.18. *Let $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ be two SIVNGs. If $t^l_{P_1} \geq t^l_{Q_2}, t^u_{P_1} \geq t^u_{Q_2}, i^l_{P_1} \leq i^l_{Q_2}, i^u_{P_1} \leq i^u_{Q_2}, f^l_{P_1} \leq f^l_{Q_2}, f^u_{P_1} \leq f^u_{Q_2}$ and $t^l_{P_2} \geq t^l_{Q_1}, t^u_{P_2} \geq t^u_{Q_1}, i^l_{P_2} \leq i^l_{Q_1}, i^u_{P_2} \leq i^u_{Q_1}, f^l_{P_2} \leq f^l_{Q_1}, f^u_{P_2} \leq f^u_{Q_1}$, then for every $(p_1, p_2) \in (P_1' \times P_2')$,*
$(d)_{G_1 \oplus G_2}(p_1, p_2) = q'(d)_{G_1}(p_1) + s'(d)_{G_2}(p_2)$, where $s' = \mid P_1' \mid -(d)_{G_1}(p_1)$ and $q' = \mid P_2' \mid -(d)_{G_2}(p_2)$.

Proof. Consider,

$(d_{i^l})_{G_1 \oplus G_2}(p_1, p_2) = \sum_{((p_1,p_2)(q_1,q_2)) \in Q_1' \times Q_2'} (i^l_{Q_1} \oplus i^l_{Q_2})((p_1, p_2)(q_1, q_2))$
$= \sum_{p_1=q_1, p_2q_2 \in Q_2'} max\{i^l_{P_1}(p_1), i^l_{Q_2}(p_2q_2)\} + \sum_{p_1q_1 \in Q_1', p_2=q_2} max\{i^l_{Q_1}(p_1q_1), i^l_{P_2}(p_2)\}$
$+ \sum_{p_1q_1 \notin Q_1', p_2q_2 \in Q_2'} max\{i^l_{P_1}(p_1), i^l_{P_1}(q_1), i^l_{Q_2}(p_2q_2)\}$

$+\sum_{p_1q_1\in Q_1',p_2q_2\notin Q_2'} max\{i^l_{Q_1}(p_1q_1), i^l_{P_2}(p_2), i^l_{P_2}(q_2)\}$
$= q^{'}(d^l_i)_{G_1}(p_1) + s^{'}(d^l_i)_{G_2}(p_2)$

In the same way, the other conditions can also be verified. □

Definition 3.19. Let $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ be two SIVNGs. The total degree for any vertex $(p_1, p_2) \in (P_1' \times P_2')$ is,

$(td_{t^l})_{G_1\oplus G_2}(p_1, p_2) = \sum_{((p_1,p_2)(q_1,q_2))\in Q_1'\times Q_2'}(t^l_{Q_1} \oplus t^l_{Q_2})((p_1, p_2)(q_1, q_2)) + (t^l_{P_1} \oplus t^l_{P_2})(p_1, p_2)$
$= \sum_{p_1=q_1,p_2q_2\in Q_2'} min\{t^l_{P_1}(p_1), t^l_{Q_2}(p_2q_2)\} + \sum_{p_1q_1\in Q_1',p_2=q_2} min\{t^l_{Q_1}(p_1q_1), t^l_{P_2}(p_2)\}$
$+\sum_{p_1q_1\notin Q_1',p_2q_2\in Q_2'} min\{t^l_{P_1}(p_1), t^l_{P_1}(q_1), t^l_{Q_2}(p_2q_2)\}$
$+\sum_{p_1q_1\in Q_1',p_2q_2\notin Q_2'} min\{t^l_{Q_1}(p_1q_1), t^l_{P_2}(p_2), t^l_{P_2}(q_2)\} + min\{t^l_{P_1}(p_1), t^l_{P_1}(p_2)\}$
$(td_{t^u})_{G_1\oplus G_2}(p_1, p_2) = \sum_{((p_1,p_2)(q_1,q_2))\in Q_1'\times Q_2'}(t^u_{Q_1} \oplus t^u_{Q_2})((p_1, p_2)(q_1, q_2)) + (t^u_{P_1} \oplus t^u_{P_2})(p_1, p_2)$
$= \sum_{p_1=q_1,p_2q_2\in Q_2'} min\{t^u_{P_1}(p_1), t^u_{Q_2}(p_2q_2)\} + \sum_{p_1q_1\in Q_1',p_2=q_2} min\{t^u_{Q_1}(p_1q_1), t^u_{P_2}(p_2)\}$
$+\sum_{p_1q_1\notin Q_1',p_2q_2\in Q_2'} min\{t^u_{P_1}(p_1), t^u_{P_1}(q_1), t^u_{Q_2}(p_2q_2)\}$
$+\sum_{p_1q_1\in Q_1',p_2q_2\notin Q_2'} min\{t^u_{Q_1}(p_1q_1), t^u_{P_2}(p_2), t^u_{P_2}(q_2)\} + min\{t^u_{P_1}(p_1), t^u_{P_1}(p_2)\}$
$(td_{i^l})_{G_1\oplus G_2}(p_1, p_2) = \sum_{((p_1,p_2)(q_1,q_2))\in Q_1'\times Q_2'}(i^l_{Q_1} \oplus i^l_{Q_2})((p_1, p_2)(q_1, q_2)) + (i^l_{P_1} \oplus i^l_{P_2})(p_1, p_2)$
$= \sum_{p_1=q_1,p_2q_2\in Q_2'} max\{i^l_{P_1}(p_1), i^l_{Q_2}(p_2q_2)\} + \sum_{p_1q_1\in Q_1',p_2=q_2} max\{i^l_{Q_1}(p_1q_1), i^l_{P_2}(p_2)\}$
$+\sum_{p_1q_1\notin Q_1',p_2q_2\in Q_2'} max\{i^l_{P_1}(p_1), i^l_{P_1}(q_1), i^l_{Q_2}(p_2q_2)\}$
$+\sum_{p_1q_1\in Q_1',p_2q_2\notin Q_2'} max\{i^l_{Q_1}(p_1q_1), i^l_{P_2}(p_2), i^l_{P_2}(q_2)\} + max\{i^l_{P_1}(p_1), i^l_{P_1}(p_2)\}$
$(td_{i^u})_{G_1\oplus G_2}(p_1, p_2) = \sum_{((p_1,p_2)(q_1,q_2))\in Q_1'\times Q_2'}(i^u_{Q_1} \oplus i^u_{Q_2})((p_1, p_2)(q_1, q_2)) + (i^u_{P_1} \oplus i^u_{P_2})(p_1, p_2)$
$= \sum_{p_1=q_1,p_2q_2\in Q_2'} max\{i^u_{P_1}(p_1), i^u_{Q_2}(p_2q_2)\} + \sum_{p_1q_1\in Q_1',p_2=q_2} max\{i^u_{Q_1}(p_1q_1), i^u_{P_2}(p_2)\}$
$+\sum_{p_1q_1\notin Q_1',p_2q_2\in Q_2'} max\{i^u_{P_1}(p_1), i^u_{P_1}(q_1), i^u_{Q_2}(p_2q_2)\}$
$+\sum_{p_1q_1\in Q_1',p_2q_2\notin Q_2'} max\{i^u_{Q_1}(p_1q_1), i^u_{P_2}(p_2), i^u_{P_2}(q_2)\} + max\{i^u_{P_1}(p_1), i^u_{P_1}(p_2)\}$
$(td_{f^l})_{G_1\oplus G_2}(p_1, p_2) = \sum_{((p_1,p_2)(q_1,q_2))\in Q_1'\times Q_2'}(f^l_{Q_1} \oplus f^l_{Q_2})((p_1, p_2)(q_1, q_2)) + (f^l_{P_1} \oplus f^l_{P_2})(p_1, p_2)$
$= \sum_{p_1=q_1,p_2q_2\in Q_2'} max\{f^l_{P_1}(p_1), f^l_{Q_2}(p_2q_2)\} + \sum_{p_1q_1\in Q_1',p_2=q_2} max\{f^l_{Q_1}(p_1q_1), f^l_{P_2}(p_2)\}$
$+\sum_{p_1q_1\notin Q_1',p_2q_2\in Q_2'} max\{f^l_{P_1}(p_1), f^l_{P_1}(q_1), f^l_{Q_2}(p_2q_2)\}$
$+\sum_{p_1q_1\in Q_1',p_2q_2\notin Q_2'} max\{f^l_{Q_1}(p_1q_1), f^l_{P_2}(p_2), f^l_{P_2}(q_2)\} + max\{f^l_{P_1}(p_1), f^l_{P_1}(p_2)\}$
$(td_{f^u})_{G_1\oplus G_2}(p_1, p_2) = \sum_{((p_1,p_2)(q_1,q_2))\in Q_1'\times Q_2'}(f^u_{Q_1} \oplus f^u_{Q_2})((p_1, p_2)(q_1, q_2)) + (f^u_{P_1} \oplus f^u_{P_2})(p_1, p_2)$
$= \sum_{p_1=q_1,p_2q_2\in Q_2'} max\{f^u_{P_1}(p_1), f^u_{Q_2}(p_2q_2)\} + \sum_{p_1q_1\in Q_1',p_2=q_2} max\{f^u_{Q_1}(p_1q_1), f^u_{P_2}(p_2)\}$
$+\sum_{p_1q_1\notin Q_1',p_2q_2\in Q_2'} max\{f^u_{P_1}(p_1), f^u_{P_1}(q_1), f^u_{Q_2}(p_2q_2)\}$
$+\sum_{p_1q_1\in Q_1',p_2q_2\notin Q_2'} max\{f^u_{Q_1}(p_1q_1), f^u_{P_2}(p_2), f^u_{P_2}(q_2)\} + max\{f^u_{P_1}(p_1), f^u_{P_1}(p_2)\}$

Theorem 3.20. *Let $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ be two SIVNGs. If $t^l_{P_1} \geq t^l_{Q_2}, t^u_{P_1} \geq t^u_{Q_2}, i^l_{P_1} \leq i^l_{Q_2}, i^u_{P_1} \leq i^u_{Q_2}, f^l_{P_1} \leq f^l_{Q_2}, f^u_{P_1} \leq f^u_{Q_2}$ and $t^l_{P_2} \geq t^l_{Q_1}, t^u_{P_2} \geq t^u_{Q_1}, i^l_{P_2} \leq i^l_{Q_1}, i^u_{P_2} \leq i^u_{Q_1}, f^l_{P_2} \leq f^l_{Q_1}, f^u_{P_2} \leq f^u_{Q_1}$, then for every $(p_1, p_2) \in (P_1' \times P_2')$,*

$(td_{t^l})_{G_1\oplus G_2}(p_1, p_2) = q'(td_{t^l})_{G_1}(p_1) + s'(td_{t^l})_{G_2}(p_2) - (q' - 1)t^l_{G_1}(p_1) - (s' - 1)t^l_{G_2}(p_2) - max\{t^l_{G_1}(p_1), t^l_{G_2}(p_2)\}$
$(td_{t^u})_{G_1\oplus G_2}(p_1, p_2) = q'(td_{t^u})_{G_1}(p_1) + s'(td_{t^u})_{G_2}(p_2) - (q' - 1)t^u_{G_1}(p_1) - (s' - 1)t^u_{G_2}(p_2) -$

$max\{t^u_{G_1}(p_1), t^u_{G_2}(p_2)\}$
$(td_{i^l})_{G_1 \oplus G_2}(p_1, p_2) = q'(td_{i^l})_{G_1}(p_1) + s'(td_{i^l})_{G_2}(p_2) - (q' - 1)i^l_{G_1}(p_1) - (s' - 1)i^l_{G_2}(p_2) - min\{i^l_{G_1}(p_1), i^l_{G_2}(p_2)\}$
$(td_{i^u})_{G_1 \oplus G_2}(p_1, p_2) = q'(td_{i^u})_{G_1}(p_1) + s'(td_{i^u})_{G_2}(p_2) - (q' - 1)i^u_{G_1}(p_1) - (s' - 1)i^u_{G_2}(p_2) - min\{i^u_{G_1}(p_1), i^u_{G_2}(p_2)\}$
$(td_{f^l})_{G_1 \oplus G_2}(p_1, p_2) = q'(td_{f^l})_{G_1}(p_1) + s'(td_{f^l})_{G_2}(p_2) - (q' - 1)f^l_{G_1}(p_1) - (s' - 1)f^l_{G_2}(p_2) - min\{f^l_{G_1}(p_1), f^l_{G_2}(p_2)\}$
$(td_{f^u})_{G_1 \oplus G_2}(p_1, p_2) = q'(td_{f^u})_{G_1}(p_1) + s'(td_{f^u})_{G_2}(p_2) - (q' - 1)f^u_{G_1}(p_1) - (s' - 1)f^u_{G_2}(p_2) - min\{f^u_{G_1}(p_1), f^u_{G_2}(p_2)\}$

Proof. Consider,
$(td_{f^l})_{G_1 \oplus G_2}(p_1, p_2) = \sum_{((p_1,p_2)(q_1,q_2)) \in Q'_1 \times Q'_2}(f^l_{Q_1} \oplus f^l_{Q_2})((p_1, p_2)(q_1, q_2)) + (f^l_{P_1} \oplus f^l_{P_2})(p_1, p_2)$
$= \sum_{p_1=q_1, p_2q_2 \in Q'_2} max\{f^l_{P_1}(p_1), f^l_{Q_2}(p_2q_2)\} + \sum_{p_1q_1 \in Q'_1, p_2=q_2} max\{f^l_{Q_1}(p_1q_1), f^l_{P_2}(p_2)\}$
$+ \sum_{p_1q_1 \notin Q'_1, p_2q_2 \in Q'_2} max\{f^l_{P_1}(p_1), f^l_{P_1}(q_1), f^l_{Q_2}(p_2q_2)\}$
$+ \sum_{p_1q_1 \in Q'_1, p_2q_2 \notin Q'_2} max\{f^l_{Q_1}(p_1q_1), f^l_{P_2}(p_2), f^l_{P_2}(q_2)\} + max\{f^l_{P_1}(p_1), f^l_{P_1}(p_2)\}$
$= \sum_{p_1=q_1, p_2q_2 \in Q'_2}, f^l_{Q_2}(p_2q_2) + \sum_{p_1q_1 \in Q'_1, p_2=q_2} f^l_{Q_1}(p_1q_1) + \sum_{p_1q_1 \notin Q'_1, p_2q_2 \in Q'_2} f^l_{Q_2}(p_2q_2)$
$+ \sum_{p_1q_1 \in Q'_1, p_2q_2 \notin Q'_2} f^l_{Q_1}(p_1q_1) + max\{f^l_{P_1}(p_1), f^l_{P_1}(p_2)\}$
$= \sum_{p_1=q_1, p_2q_2 \in Q'_2}, f^l_{Q_2}(p_2q_2) + \sum_{p_1q_1 \in Q'_1, p_2=q_2} f^l_{Q_1}(p_1q_1) + \sum_{p_1q_1 \notin Q'_1, p_2q_2 \in Q'_2} f^l_{Q_2}(p_2q_2)$
$+ \sum_{p_1q_1 \in Q'_1, p_2q_2 \notin Q'_2} f^l_{Q_1}(p_1q_1) - min\{f^l_{P_1}(p_1), f^l_{P_1}(p_2)\}$
$= q'(td_{f^l})_{G_1}(p_1) + s'(td_{f^l})_{G_2}(p_2) - (q' - 1)f^l_{G_1}(p_1) - (s' - 1)f^l_{G_2}(p_2) - min\{f^l_{G_1}(p_1), f^l_{G_2}(p_2)\}.$
Similarly, the other conditions can also be proved. □

Example 3.21. The symmetric difference $G_1 \oplus G_2$ of two SIVNGs G_1 and $G_2 = (P_2, Q_2)$ is represented in Figure 10 and 11.

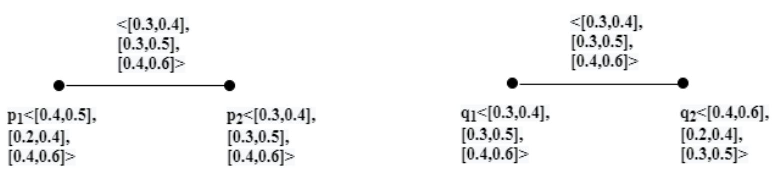

FIGURE 10. Strong Interval Valued Neutrosophic Graphs G_1 and G_2

By direct calculations, $d_{t^l}(p_1, q_1) = 0.3 + 0.3 = 0.6$; $d_{t^u}(p_1, q_1) = 0.4 + 0.4 = 0.8$; $d_t(p_1, q_1) = [0.6, 0.8]$; $td_{t^l}(p_1, q_1) = 0.3 + 0.3 + 0.3 = 0.9$; $td_{t^u}(p_1, q_1) = 0.4 + 0.4 + 0.4 = 1.2$; $td_t(p_1, q_1) = [0.9, 1.2]$.
By using theorem, $s' =| P'_1 | -(d)_{G_1}(p_1) = 2 - 1 = 1$; $q' =| P'_2 | -(d)_{G_2}(p_2) = 2 - 1 = 1$; $d_{t^l}(p_1, q_1) = 1(0.3) + 1(0.3) = 0.6$; $d_{t^u}(p_1, q_1) = 1(0.4) + 1(0.4) = 0.8$; $d_t(p_1, q_1) = [0.6, 0.8]$; $td_{t^l}(p_1, q_1) = 1(0.7) + 1(0.6) - 0(0.4) - 0(0.3) - max\{0.4, 0.3\} = 0.9$; $td_{t^u}(p_1, q_1) = 1(0.9) +$

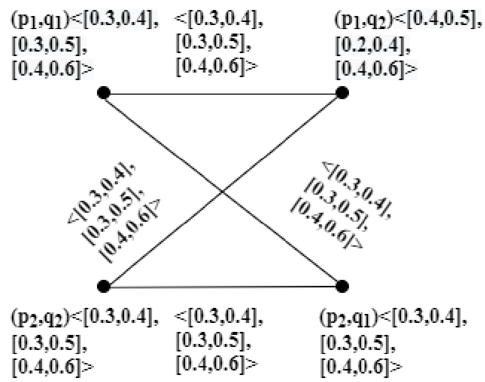

Figure 11. Symmetric difference $G_1 \oplus G_2$

$1(0.8) - 0(0.5) - 0(0.4) - max\{0.5, 0.4\} = 1.2$; $td_t(p_1, q_1) = [0.9, 1.2]$.

Definition 3.22. The residue product $G_1\ G_2 = (P_1 \bullet P_2, Q_1 \bullet Q_2)$ of two SIVNGs $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ is defined as

(1) $(t^l_{P_1} \bullet t^l_{P_2})(p_1, p_2) = max\{t^l_{P_1}(p_1), t^l_{P_2}(p_2)\}$,
$(t^u_{P_1} \bullet t^u_{P_2})(p_1, p_2) = max\{t^u_{P_1}(p_1), t^u_{P_2}(p_2)\}$,
$(i^l_{P_1} \bullet i^l_{P_2})(p_1, p_2) = min\{i^l_{P_1}(p_1), i^l_{P_2}(p_2)\}$,
$(i^u_{P_1} \bullet i^u_{P_2})(p_1, p_2) = min\{i^u_{P_1}(p_1), i^u_{P_2}(p_2)\}$,
$(f^l_{P_1} \bullet f^l_{P_2})(p_1, p_2) = min\{f^l_{P_1}(p_1), f^l_{P_2}(p_2)\}$,
$(f^u_{P_1} \bullet f^u_{P_2})(p_1, p_2) = min\{f^u_{P_1}(p_1), f^u_{P_2}(p_2)\}$.

(2) $(t^l_{Q_1} \bullet t^l_{Q_2})((p_1, p_2)(q_1, q_2)) = t^l_{Q_1}(p_1q_1)$,
$(t^u_{Q_1} \bullet t^u_{Q_2})((p_1, p_2)(q_1, q_2)) = t^u_{Q_1}(p_1q_1)$,
$(i^l_{Q_1} \bullet i^l_{Q_2})((p_1, p_2)(q_1, q_2)) = i^l_{Q_1}(p_1q_1)$,
$(i^u_{Q_1} \bullet i^u_{Q_2})((p_1, p_2)(q_1, q_2)) = i^u_{Q_1}(p_1q_1)$,
$(f^l_{Q_1} \bullet f^l_{Q_2})((p_1, p_2)(q_1, q_2)) = f^l_{Q_1}(p_1q_1)$,
$(f^u_{Q_1} \bullet f^u_{Q_2})((p_1, p_2)(q_1, q_2)) = f^u_{Q_1}(p_1q_1), \forall p_1q_1 \in Q'_1, p_2 \neq q_2$.

Theorem 3.23. *The residue product of two SIVNGs G_1 and G_2, need not be a SIVNG.*

From example 3.24, Figure 13, it is clear that t, i and f values of the vertices and the edges in $G_1 \bullet G_2$ do not satisfy the strong condition and hence it is an IVNG.

Example 3.24. Consider two SIVNGs $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ as represented in Figure 12. Their residue product $G_1 \bullet G_2$ is represented in Figure 13.

Theorem 3.25. *Let $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ be two SIVNGs. If $t^l_{P_1} \geq t^l_{Q_2}, t^u_{P_1} \geq t^u_{Q_2}, i^l_{P_1} \leq i^l_{Q_2}, i^u_{P_1} \leq i^u_{Q_2}, f^l_{P_1} \leq f^l_{Q_2}, f^u_{P_1} \leq f^u_{Q_2}$ and $t^l_{P_2} \geq t^l_{Q_1}, t^u_{P_2} \geq t^u_{Q_1}, i^l_{P_2} \leq i^l_{Q_1}, i^u_{P_2} \leq i^u_{Q_1}, f^l_{P_2} \leq f^l_{Q_1}, f^u_{P_2} \leq f^u_{Q_1}$, then the residue product of two SIVNGs G_1 and G_2 is a SIVNG.*

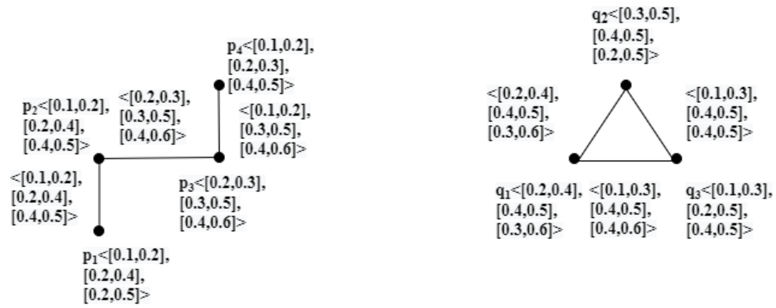

FIGURE 12. Strong Interval Valued Neutrosophic Graphs G_1 and G_2

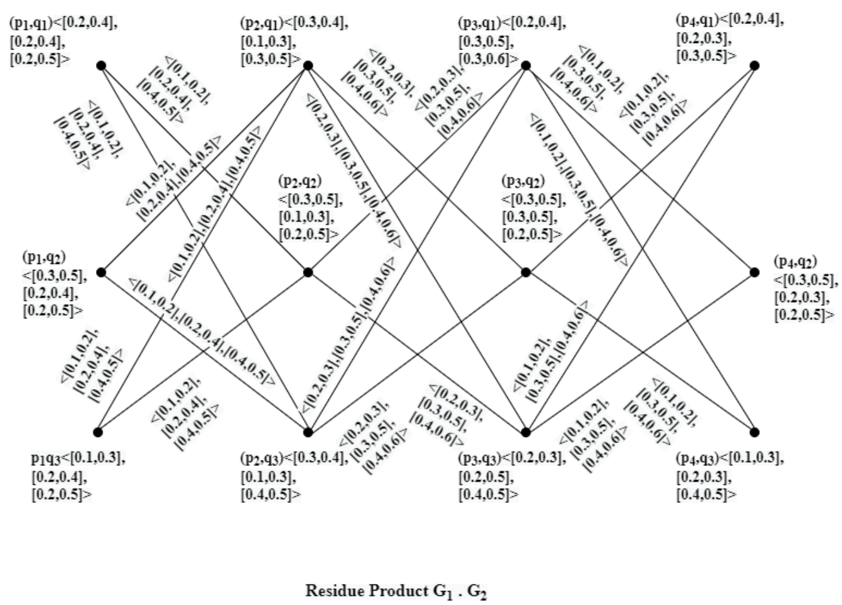

FIGURE 13. Residue product $G_1 \bullet G_2$

Proof. For $p_1q_1 \in Q_1', p_2 \neq q_2$,
$(t^l_{Q_1} \bullet t^l_{Q_2})((p_1,p_2)(q_1,q_2)) = t^l_{Q_1}(p_1q_1), = min\{t^l_{P_1}(p_1), t^l_{P_1}(q_1)\}$,
$= min\{max\{t^l_{P_1}(p_1), t^l_{P_1}(p_2)\}, max\{t^l_{P_1}(q_1), t^l_{P_1}(q_2)\}\}$,
$= min\{(t^l_{P_1} \bullet t^l_{P_2})(p_1,p_2), (t^l_{P_1} \bullet t^l_{P_2})(q_1,q_2)\}$. □

Example 3.26. Consider two SIVNGs $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ as represented in Figure 14. Their residue product $G_1 \bullet G_2$ is represented in Figure 15.

Definition 3.27. Let $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ be two SIVNGs. The degree for any vertex $(p_1, p_2) \in (P_1' \times P_2')$ is,
$(d_{t^l})_{G_1 \bullet G_2}(p_1,p_2) = \sum_{((p_1,p_2)(q_1,q_2)) \in Q_1' \times Q_2'} (t^l_{Q_1} \bullet t^l_{Q_2})((p_1,p_2)(q_1,q_2))$
$= \sum_{p_1q_1 \in Q_1', p_2 \neq q_2} t^l_{Q_1}(p_1q_1) = (d^l_t)_{G_1}(p_1)$
$(d_{t^u})_{G_1 \bullet G_2}(p_1,p_2) = \sum_{((p_1,p_2)(q_1,q_2)) \in Q_1' \times Q_2'} (t^u_{Q_1} \bullet t^u_{Q_2})((p_1,p_2)(q_1,q_2))$

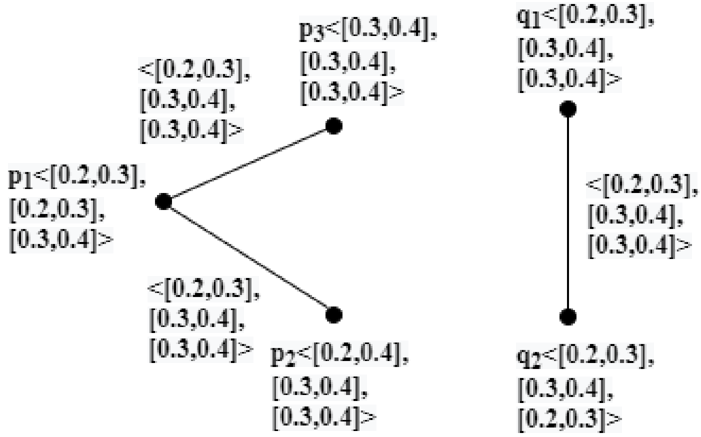

FIGURE 14. Strong Interval Valued Neutrosophic Graphs G_1 and G_2

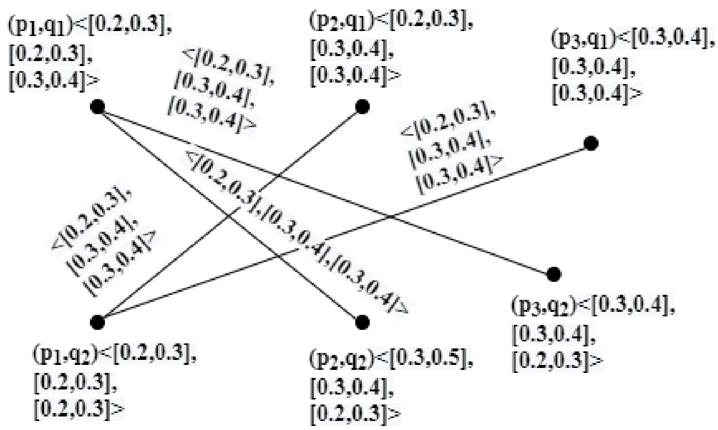

FIGURE 15. Residue product $G_1 \bullet G_2$

$= \sum_{p_1q_1 \in Q_1', p_2 \neq q_2} t^u_{Q_1}(p_1q_1) = (d^u_t)_{G_1}(p_1)$
$(d_{i^l})_{G_1 \bullet G_2}(p_1, p_2) = \sum_{((p_1,p_2)(q_1,q_2)) \in Q_1' \times Q_2'} (i^l_{Q_1} \bullet i^l_{Q_2})((p_1, p_2)(q_1, q_2))$
$= \sum_{p_1q_1 \in Q_1', p_2 \neq q_2} i^l_{Q_1}(p_1q_1) = (d^l_i)_{G_1}(p_1)$
$(d_{i^u})_{G_1 \bullet G_2}(p_1, p_2) = \sum_{((p_1,p_2)(q_1,q_2)) \in Q_1' \times Q_2'} (i^u_{Q_1} \bullet i^u_{Q_2})((p_1, p_2)(q_1, q_2))$
$= \sum_{p_1q_1 \in Q_1', p_2 \neq q_2} i^u_{Q_1}(p_1q_1) = (d^u_i)_{G_1}(p_1)$
$(d_{f^l})_{G_1 \bullet G_2}(p_1, p_2) = \sum_{((p_1,p_2)(q_1,q_2)) \in Q_1' \times Q_2'} (f^l_{Q_1} \bullet f^l_{Q_2})((p_1, p_2)(q_1, q_2))$
$= \sum_{p_1q_1 \in Q_1', p_2 \neq q_2} f^l_{Q_1}(p_1q_1) = (d^l_f)_{G_1}(p_1)$
$(d_{f^u})_{G_1 \bullet G_2}(p_1, p_2) = \sum_{((p_1,p_2)(q_1,q_2)) \in Q_1' \times Q_2'} (f^u_{Q_1} \bullet f^u_{Q_2})((p_1, p_2)(q_1, q_2))$
$= \sum_{p_1q_1 \in Q_1', p_2 \neq q_2} f^u_{Q_1}(p_1q_1) = (d^u_f)_{G_1}(p_1)$

Definition 3.28. Let $G_1 = (P_1, Q_1)$ and $G_2 = (P_2, Q_2)$ be two SIVNGs. The total degree for any vertex $(p_1, p_2) \in (P_1' \times P_2')$ is,

$(td_{t^l})_{G_1 \bullet G_2}(p_1,p_2) = \sum_{((p_1,p_2)(q_1,q_2)) \in Q_1' \times Q_2'} (t^l_{Q_1} \bullet t^l_{Q_2})((p_1,p_2)(q_1,q_2)) + (t^l_{P_1} \bullet t^l_{P_2})(p_1,p_2)$
$= \sum_{p_1q_1 \in Q_1', p_2 \neq q_2} t^l_{Q_1}(p_1q_1) + max\{t^l_{P_1}(p_1), t^l_{P_2}(p_2)\}$
$= \sum_{p_1q_1 \in Q_1', p_2 \neq q_2} t^l_{Q_1}(p_1q_1) + t^l_{P_1}(p_1) + t^l_{P_2}(p_2) - min\{t^l_{P_1}(p_1), t^l_{P_2}(p_2)\}$
$= (td_{t^l})_{G_1}(p_1) + t^l_{P_2}(p_2) - min\{t^l_{P_1}(p_1), t^l_{P_2}(p_2)$
$(td_{t^u})_{G_1 \bullet G_2}(p_1,p_2) = \sum_{((p_1,p_2)(q_1,q_2)) \in Q_1' \times Q_2'} (t^u_{Q_1} \bullet t^u_{Q_2})((p_1,p_2)(q_1,q_2)) + (t^u_{P_1} \bullet t^u_{P_2})(p_1,p_2)$
$= \sum_{p_1q_1 \in Q_1', p_2 \neq q_2} t^u_{Q_1}(p_1q_1) + max\{t^u_{P_1}(p_1), t^l_{P_2}(p_2)\}$
$= \sum_{p_1q_1 \in Q_1', p_2 \neq q_2} t^u_{Q_1}(p_1q_1) + t^u_{P_1}(p_1) + t^u_{P_2}(p_2) - min\{t^u_{P_1}(p_1), t^u_{P_2}(p_2)\}$
$= (td_{t^u})_{G_1}(p_1) + t^u_{P_2}(p_2) - min\{t^u_{P_1}(p_1), t^u_{P_2}(p_2)$
$(td_{i^l})_{G_1 \bullet G_2}(p_1,p_2) = \sum_{((p_1,p_2)(q_1,q_2)) \in Q_1' \times Q_2'} (i^l_{Q_1} \bullet i^l_{Q_2})((p_1,p_2)(q_1,q_2)) + (i^l_{P_1} \bullet i^l_{P_2})(p_1,p_2)$
$= \sum_{p_1q_1 \in Q_1', p_2 \neq q_2} i^l_{Q_1}(p_1q_1) + min\{i^l_{P_1}(p_1), i^l_{P_2}(p_2)\}$
$= \sum_{p_1q_1 \in Q_1', p_2 \neq q_2} i^l_{Q_1}(p_1q_1) + i^l_{P_1}(p_1) + i^l_{P_2}(p_2) - max\{i^l_{P_1}(p_1), i^l_{P_2}(p_2)\}$
$= (td_{i^l})_{G_1}(p_1) + i^l_{P_2}(p_2) - max\{i^l_{P_1}(p_1), i^l_{P_2}(p_2)$
$(td_{i^u})_{G_1 \bullet G_2}(p_1,p_2) = \sum_{((p_1,p_2)(q_1,q_2)) \in Q_1' \times Q_2'} (i^u_{Q_1} \bullet i^u_{Q_2})((p_1,p_2)(q_1,q_2)) + (i^u_{P_1} \bullet i^u_{P_2})(p_1,p_2)$
$= \sum_{p_1q_1 \in Q_1', p_2 \neq q_2} i^u_{Q_1}(p_1q_1) + min\{i^u_{P_1}(p_1), i^u_{P_2}(p_2)\}$
$= \sum_{p_1q_1 \in Q_1', p_2 \neq q_2} i^u_{Q_1}(p_1q_1) + i^u_{P_1}(p_1) + i^u_{P_2}(p_2) - max\{i^u_{P_1}(p_1), i^u_{P_2}(p_2)\}$
$= (td_{i^u})_{G_1}(p_1) + i^u_{P_2}(p_2) - max\{i^u_{P_1}(p_1), i^u_{P_2}(p_2)$
$(td_{f^l})_{G_1 \bullet G_2}(p_1,p_2) = \sum_{((p_1,p_2)(q_1,q_2)) \in Q_1' \times Q_2'} (f^l_{Q_1} \bullet f^l_{Q_2})((p_1,p_2)(q_1,q_2)) + (f^l_{P_1} \bullet f^l_{P_2})(p_1,p_2)$
$= \sum_{p_1q_1 \in Q_1', p_2 \neq q_2} f^l_{Q_1}(p_1q_1) + min\{f^l_{P_1}(p_1), f^l_{P_2}(p_2)\}$
$= \sum_{p_1q_1 \in Q_1', p_2 \neq q_2} f^l_{Q_1}(p_1q_1) + f^l_{P_1}(p_1) + f^l_{P_2}(p_2) - max\{f^l_{P_1}(p_1), f^l_{P_2}(p_2)\}$
$= (td_{f^l})_{G_1}(p_1) + f^l_{P_2}(p_2) - max\{f^l_{P_1}(p_1), f^l_{P_2}(p_2)$
$(td_{f^u})_{G_1 \bullet G_2}(p_1,p_2) = \sum_{((p_1,p_2)(q_1,q_2)) \in Q_1' \times Q_2'} (f^u_{Q_1} \bullet f^u_{Q_2})((p_1,p_2)(q_1,q_2)) + (f^u_{P_1} \bullet f^u_{P_2})(p_1,p_2)$
$= \sum_{p_1q_1 \in Q_1', p_2 \neq q_2} f^u_{Q_1}(p_1q_1) + min\{f^u_{P_1}(p_1), f^u_{P_2}(p_2)\}$
$= \sum_{p_1q_1 \in Q_1', p_2 \neq q_2} f^u_{Q_1}(p_1q_1) + f^u_{P_1}(p_1) + f^u_{P_2}(p_2) - max\{f^u_{P_1}(p_1), f^u_{P_2}(p_2)\}$
$= (td_{f^u})_{G_1}(p_1) + f^u_{P_2}(p_2) - max\{f^u_{P_1}(p_1), f^u_{P_2}(p_2)$

From Figure 15, $d_f(p_1,q_2)$ and $td_f(p_1,q_2)$ for the vertex (p_1,q_2) are calculated below.
$d_{f^l}(p_1,q_2) = 0.3 + 0.3 = 0.6$, $d_{f^u}(p_1,q_2) = 0.4 + 0.4 = 0.8$, $d_f(p_1,q_2) = [0.6, 0.8]$
$td_{f^l}(p_1,q_2) = 0.9 + 0.2 - 0.3 = 0.8$, $td_{f^u}(p_1,q_2) = 1.2 + 0.3 - 0.4 = 1.1$, $td_f(p_1,q_2) = [0.8, 1.1]$.

4. **Application**

4.1. *The Cardiac Cycle of a Human Heart*

The right atrium (RA) of the heart receives deoxygenated blood from both Superior Vena Cava (SVC) and Inferior Vena Cava (IVC). Then, the tricuspid valve (TVL) opens due to the contraction of the right atrium and the deoxygenated blood has directed to the right ventricle (RV). After the ventricular filling, the tricuspid valve (TVL) shuts. Now, the right ventricle (RV) gets contracted, which causes the opening of the pulmonary valve (PVL) and the blood is transferred to the pulmonary artery (PA) and then to the lungs for oxygenation. After the blood gets oxygenated, it enters the left atrium (LA) via pulmonary veins (PV). Now, the left atrium gets contracted and the mitral valve (MVL) opens for transferring the oxygenated blood to the left ventricle. After passing out the blood to the left ventricle (LV), the mitral valve (MVL) closes. Now, the left ventricle contracts for ejecting the blood to the aorta (A) through the aortic valve (AVL). From there, the oxygenated blood passes to all the parts of the human body. The blood flow through the human heart has presented in Figure 16. Biologically during the period of cardiac cycle, it is observed that

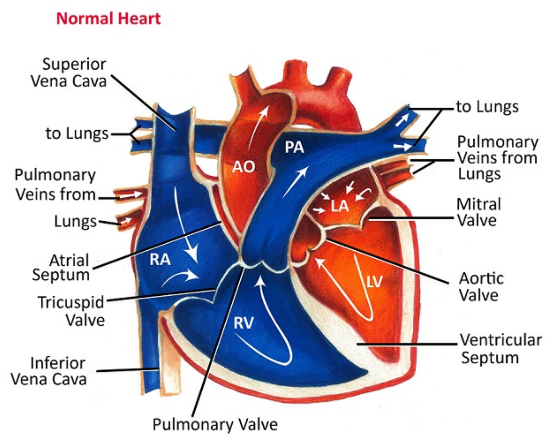

FIGURE 16. The Human Heart

(1) Left ventricular systole and diastole is the most effective phase on the whole.
(2) The Left side of the human heart has comparatively higher pressure than on the right side. i.e., Left Atrial (Ventricular) Systole has higher pressure than Right Atrial (Ventricular) Systole and vice versa.
(3) Systolic (ventricular) pressure is higher than diastolic (ventricular) pressure.

The flowchart given in Figure 17 illustrates the method for evaluating the cardiac functioning of the human heart.

4.2. *The Wright Table - Study of blood flow along with their blood pressure values*

Wright's table [22], a teaching tool to learn and understand the cardiac cycle, has elaborated the path of blood flow with the blood pressure changes. The Wright table explains how the pressures and flows of each compartment fluctuate over time, as well as how the heart functions as a pump, first filling and then emptying the ventricles and thereby transferring blood from low-pressure venous to high-pressure arterial compartments. The Wright's table provided in Table 1 and Table 2 elaborates the path of blood flow along with the blood pressure changes during AS/VD and AD/VS phase of the human heart observed for a healthy adult of age

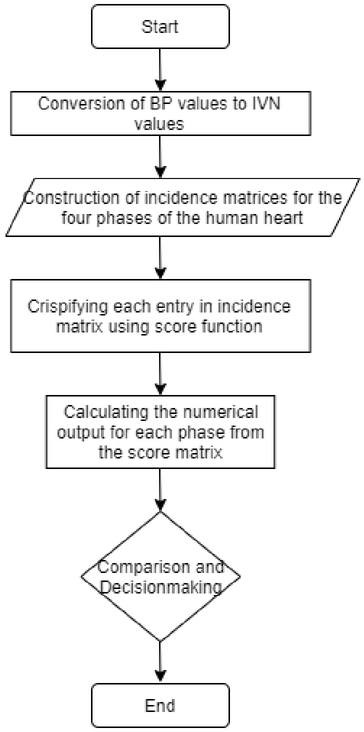

FIGURE 17. Flowchart for evaluating Cardiac Functioning of the Human Heart

above 35 years without any cardiac malfunction along with their corresponding hemodynamic parameters.

TABLE 1. The Wright's table representation of AS/VD and AD/VS phase (Right Side)

	SVC and IVC		RA		TVL		RV		PVL		PA	
AS/VD	2-5	$\longrightarrow$	4-6	$\longrightarrow$	12.6-29.3	$\longrightarrow$	0-8	(closed valve)	0	(closed valve)	8-15	$\longrightarrow$
AD/VS	2-5	$\longrightarrow$	0	(closed valve)	0	(closed valve)	15-25	$\longrightarrow$	15-25	$\longrightarrow$	15-25	$\longrightarrow$

4.3. *Conversion of Blood pressure values into Interval Valued Neutrosophic values (IVN-values)*

As the rate of blood pressure changes from time to time under a certain interval and it is highly impracticable for getting the same blood pressure value in each prediction, a minute level of indeterminacy and falsity have been observed.

TABLE 2. The Wright's table representation of AS/VD and AD/VS phase (Left Side)

	PV		LA		MVL		LV		AVL		A	
AS/VD	2-5	$\longrightarrow$	7-8	$\longrightarrow$	0	$\longrightarrow$	0-12	(closed valve)	0	(closed valve)	60-90	$\longrightarrow$
AD/VS	2-5	$\longrightarrow$	0	(closed valve)	0	(closed valve)	100-140	$\longrightarrow$	8-12	$\longrightarrow$	100-140	$\longrightarrow$

The blood pressure values given in Table 1 and Table 2 are converted to fit under an IVN-environment. The truth-membership values are exactly the blood pressure values taken for consideration and the indeterminacy-membership values and the falsity-membership values are estimated accordingly.

Since the IVN-values lie in the range of [0, 1], the blood pressure values (mm/Hg) given in Table 1 and Table 2 are re-scaled using bar conversion. For instance, the blood pressure value in Superior Vena Cava is 2-5 mm/Hg and its bar conversion becomes $[0.00266645, 0.00666612] \approx [0.003, 0.007]$.

Thus with reference to the bar conversion, Table 3 shows the re-scaled Interval Valued Neutrosophic blood pressure values observed in Table 1 and Table 2.

TABLE 3. Rescaled IVN Blood Pressure Values

	AS/VD	AD/VS
SVC & IVC	$< [0.003, 0.007], [0.001, 0.0015], [0.001, 0.002] >$	$< [0.003, 0.007], [0.001, 0.0015], [0.001, 0.002] >$
RA	$< [0.005, 0.008], [0.001, 0.003], [0.001, 0.002] >$	$< [0, 0], [0.001, 0.001], [0.001, 0.001] >$
TVL	$< [0.017, 0.04], [0.001, 0.002], [0.001, 0.002] >$	$< [0, 0], [0, 0], [0, 0] >$
RV	$< [0, 0.01], [0.001, 0.002], [0.002, 0.004] >$	$< [0.02, 0.03], [0.001, 0.002], [0.001, 0.0015] >$
PVL	$< [0, 0], [0, 0], [0, 0] >$	$< [0.02, 0.03], [0.001, 0.002], [0.001, 0.002] >$
PA	$< [0.01, 0.02], [0.002, 0.004], [0.001, 0.003] >$	$< [0.02, 0.03], [0.002, 0.003], [0.001, 0.003] >$
PV	$< [0.003, 0.007], [0.001, 0.0015], [0.001, 0.002] >$	$< [0.003, 0.007], [0.001, 0.0015], [0.001, 0.002] >$
LA	$< [0.009, 0.01], [0.001, 0.002], [0.001, 0.002] >$	$< [0, 0], [0.001, 0.001], [0.001, 0.001] >$
MVL	$< [0, 0], [0.001, 0.001], [0.001, 0.001] >$	$< [0, 0], [0, 0], [0, 0] >$
LV	$< [0, 0.016], [0.001, 0.0012], [0.001, 0.0015] >$	$< [0.13, 0.19], [0.002, 0.003], [0.001, 0.002] >$
AVL	$< [0, 0], [0, 0], [0, 0] >$	$< [0.01, 0.016], [0.0012, 0.0016], [0.001, 0.0015] >$
A	$< [0.08, 0.12], [0.005, 0.007], [0.003, 0.005] >$	$< [0.13, 0.19], [0.003, 0.005], [0.002, 0.004] >$

4.4. *Modeling of Human Heart as SIVN - Digraph*

The blood flow through right and left heart as given in Figure 16 is represented as a digraph $G = (P, \overrightarrow{Q})$ with the vertex set $X = \{p_1, p_2, p_3, ..., p_{16}\}$ along with the directed edges in Figure 18.

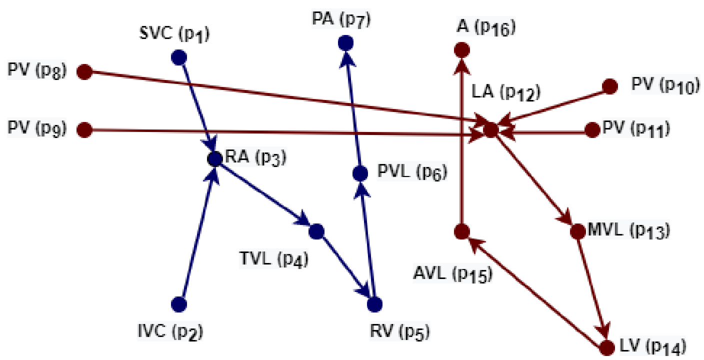

FIGURE 18. The Human Heart digraph

4.5. *SIVNDG representation of the cardiac cycle functioning during AS/VD and AD/VS Phases*

During the cardiac cycle functioning, both the right and the left atria narrow down at first, pumping blood to the right ventricle and the left ventricle, respectively. During this period, both the right and the left atria are in systolic phase and the corresponding right and the left ventricles are in diastolic phase. In response to electrical impulses the right and left ventricles contract instantly, allowing blood to flow to the lungs and to the rest of the body.At this time, the atria remain in diastolic phase and the ventricles are in systolic phase and their corresponding strong interval-valued neutrosophic values during this AS/VD and AD/VS phases are represented in Figure 19 and Figure 20 with reference to Table 3.

During AS/VD phase, the vertices and the edges along with their membership functions for the directed subgraphs $H_1 = (P_{H_1}, \overrightarrow{Q_{H_1}})$ and $H_2 = (P_{H_2}, \overrightarrow{Q_{H_2}})$ for $X = \{p_1, p_2, p_3, ..., p_{16}\}$ are defined by

$P_{H_1} = \{p_1 < [0.003, 0.007], [0.001, 0.0015], [0.001, 0.002] >, p_2 < [0.003, 0.007],$
$[0.001, 0.0015], [0.001, 0.002] >, p_3 < [0.005, 0.008], [0.001, 0.003], [0.001,$
$0.002] >, p_4 < [0.017, 0.04], [0.001, 0.002], [0.001, 0.002] >, p_5 < [0, 0.01],$
$[0.001, 0.002], [0.002, 0.004] >, p_6 < [0, 0], [0, 0], [0, 0] >, p_7 < [0.01, 0.02],$
$[0.002, 0.004], [0.001, 0.003] >\}.$
$\overrightarrow{Q_{H_1}} = \{\overrightarrow{p_1p_3} < [0.003, 0.007], [0.001, 0.003], [0.001, 0.002] >,$
$\overrightarrow{p_2p_3} < [0.003, 0.007], [0.001, 0.003], [0.001, 0.002] >, \overrightarrow{p_3p_4} < [0.005, 0.008],$

$[0.001, 0.003], [0.001, 0.002] >, \overrightarrow{p_4p_5} < [0, 0.01], [0.001, 0.002], [0.002, 0.004] >,$
$\overrightarrow{p_5p_6} < [0, 0], [0.001, 0.002], [0.002, 0.004] >, \overrightarrow{p_6p_7} < [0, 0], [0.002, 0.004],$
$[0.001, 0.003] >\}.$
$P_{H_2} = \{p_8 < [0.003, 0.007], [0.001, 0.0015], [0.001, 0.002] >,$
$p_9 < [0.003, 0.007], [0.001, 0.0015], [0.001, 0.002] >, p_{10} < [0.003, 0.007],$
$[0.001, 0.0015], [0.001, 0.002] >, p_{11} < [0.003, 0.007], [0.001, 0.0015],$
$[0.001, 0.002] >, p_{12} < [0.009, 0.01], [0.001, 0.002], [0.001, 0.002] >, p_{13} < [0, 0],$
$[0.001, 0.001], [0.001, 0.001] >, p_{14} < [0, 0.016], [0.001, 0.0012], [0.001, 0.0015] >, p_{15} <$
$[0, 0], [0, 0], [0, 0] >, p_{16} < [0.08, 0.12], [0.005, 0.007], [0.003, 0.005] >\}.$
$\overrightarrow{Q_{H_2}} = \{\overrightarrow{p_8p_{12}} < [0.003, 0.007], [0.001, 0.002], [0.001, 0.002] >,$
$\overrightarrow{p_9p_{12}} < [0.003, 0.007], [0.001, 0.002], [0.001, 0.002] >,$
$\overrightarrow{p_{10}p_{12}} < [0.003, 0.007], [0.001, 0.002], [0.001, 0.002] >,$
$\overrightarrow{p_{11}p_{12}} < [0.003, 0.007], [0.001, 0.002], [0.001, 0.002] >,$
$\overrightarrow{p_{12}p_{13}} < [0, 0], [0.001, 0.002], [0.001, 0.002] >,$
$\overrightarrow{p_{13}p_{14}} < [0, 0], [0.001, 0.0012], [0.001, 0.0015] >,$
$\overrightarrow{p_{14}p_{15}} < [0, 0], [0.001, 0.0012], [0.001, 0.0015] >,$
$\overrightarrow{p_{15}p_{16}} < [0, 0], [0.005, 0.007], [0.003, 0.005] >\}.$

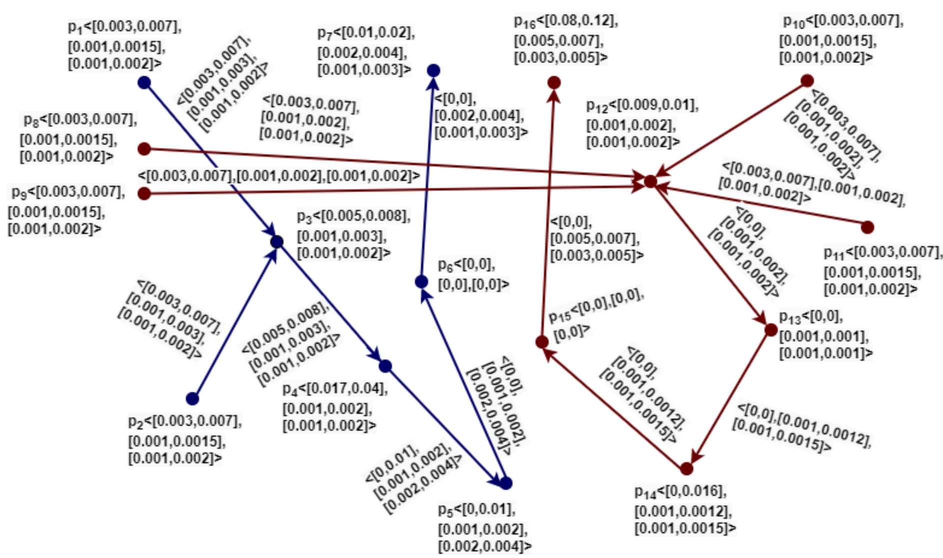

Figure 19. SIVN Digraph $G = H_1 \cup H_2$ during AS/ VD Phase

During AD / VS phase, the vertices and the edges along with their membership functions for the directed subgraphs $H_3 = (P_{H_3}, \overrightarrow{Q_{H_3}})$ and $H_4 = (P_{H_4}, \overrightarrow{Q_{H_4}})$ on X are defined by,
$P_{H_3} = \{p_1 < [0.003, 0.007], [0.001, 0.0015], [0.001, 0.002] >, p_2 < [0.003, 0.007],$
$[0.001, 0.0015], [0.001, 0.002] >, p_3 < [0, 0], [0.001, 0.001], [0.001, 0.001] >,$
$p_4 < [0, 0], [0, 0], [0, 0] >, p_5 < [0.02, 0.03], [0.001, 0.002], [0.001, 0.0015] >,$

$p_6 < [0.02, 0.03], [0.001, 0.002], [0.001, 0.002] >, p_7 < [0.02, 0.03], [0.002, 0.003],$ $[0.001, 0.003] >\}$.

$\overrightarrow{Q_{H_3}} = \{\overrightarrow{p_1p_3} < [0, 0], [0.001, 0.0015], [0.001, 0.002] >, \overrightarrow{p_2p_3} < [0, 0],$ $[0.001, 0.0015], [0.001, 0.002] >, \overrightarrow{p_3p_4} < [0, 0], [0.001, 0.001], [0.001, 0.001] >,$ $\overrightarrow{p_4p_5} < [0, 0], [0.001, 0.002], [0.001, 0.0015] >, \overrightarrow{p_5p_6} < [0.02, 0.03], [0.001,$ $0.002], [0.001, 0.002] >, \overrightarrow{p_6p_7} < [0.02, 0.03], [0.002, 0.003], [0.001, 0.003] >\}$.

$P_{H_4} = \{p_8 < [0.003, 0.007], [0.001, 0.0015], [0.001, 0.002] >, p_9 < [0.003,$ $0.007], [0.001, 0.0015], [0.001, 0.002] >, p_{10} < [0.003, 0.007], [0.001, 0.0015],$ $[0.001, 0.002] >, p_{11} < [0.003, 0.007], [0.001, 0.0015], [0.001, 0.002] >,$ $p_{12} < [0, 0], [0.001, 0.001], [0.001, 0.001] >, p_{13} < [0, 0], [0, 0], [0, 0] >,$ $p_{14} < [0.13, 0.19], [0.002, 0.003], [0.001, 0.002] >, p_{15} < [0.01, 0.016], [0.0012,$ $0.0016], [0.001, 0.0015] >, p_{16} < [0.13, 0.19], [0.003, 0.005], [0.002, 0.004] >\}$.

$\overrightarrow{Q_{H_4}} = \{\overrightarrow{p_8p_{12}} < [0, 0], [0.001, 0.0015], [0.001, 0.002] >, \overrightarrow{p_9p_{12}} < [0, 0],$ $[0.001, 0.0015], [0.001, 0.002] >, \overrightarrow{p_{10}p_{12}} < [0, 0], [0.001, 0.0015], [0.001, 0.002] >, \overrightarrow{p_{11}p_{12}} <$ $[0, 0], [0.001, 0.0015], [0.001, 0.002] >, \overrightarrow{p_{12}p_{13}} < [0, 0], [0.001, 0.001],$ $[0.001, 0.001] >, \overrightarrow{p_{13}p_{14}} < [0, 0], [0.002, 0.003], [0.001, 0.002] >, \overrightarrow{p_{14}p_{15}} < [0.01,$ $0.016], [0.002, 0.003], [0.001, 0.002] >, \overrightarrow{p_{15}p_{16}} < [0.01, 0.016], [0.003, 0.005],$ $[0.002, 0.004] >\}$.

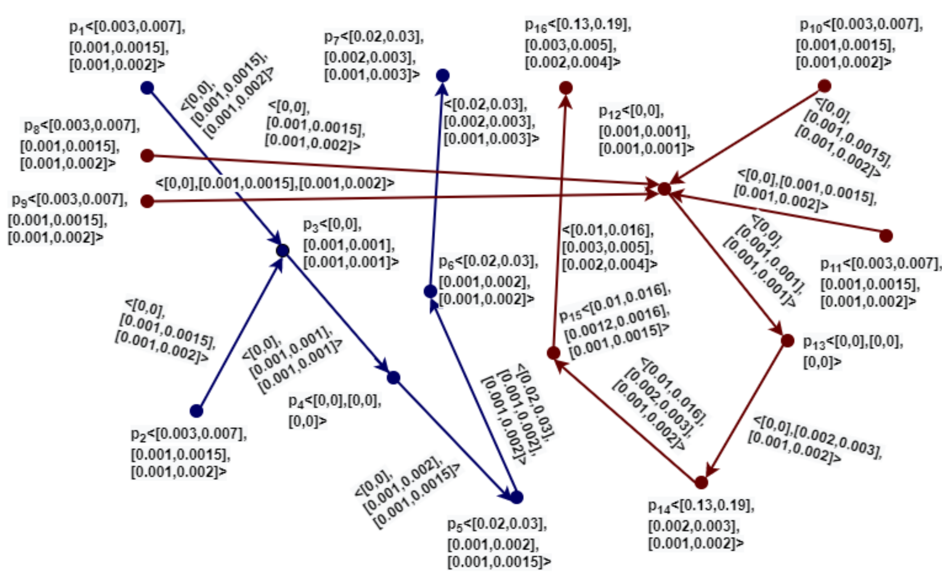

FIGURE 20. SIVN Digraph $G = H_3 \cup H_4$ during AD/ VS Phase

4.6. *Matrix form of SIVNDG during AS/VD and AD/VS phase*

The strong Interval Valued Neutrosophic directed subgraphs $\overrightarrow{H_l}(l = 1, 2, 3, 4)$ during AS/ VD and AD/ VS Phase are represented by the following incident matrices $m_{\overrightarrow{H_l}} = (\overrightarrow{p_ip_j})$ where

$i, j = 1, 2, .., 16$ and $l = 1, 2, 3, 4$.

$$m_{\overrightarrow{H_1}} = \begin{cases} < [0.003, 0.007], [0.001, 0.0015], [0.001, 0.002] >, & i = 1, 2, j = 3 \\ < [0.005, 0.008], [0.001, 0.003], [0.001, 0.002] >, & i = 3, j = 4 \\ < [0, 0.01], [0.001, 0.002], [0.002, 0.004] >, & i = 4, j = 5 \\ < [0, 0], [0.001, 0.002], [0.002, 0.004] >, & i = 5, j = 6 \\ < [0, 0], [0.002, 0.004], [0.001, 0.003] >, & i = 6, j = 7 \\ < [0, 0], [0, 0], [1, 1] > & \text{otherwise} \end{cases}$$

$$m_{\overrightarrow{H_2}} = \begin{cases} < [0.003, 0.007], [0.001, 0.002], [0.001, 0.002] >, & i = 8, 9, 10, 11, j = 12 \\ < [0, 0], [0.001, 0.002], [0.001, 0.002] >, & i = 12, j = 13 \\ < [0, 0], [0.001, 0.0012], [0.001, 0.0015] >, & i = 13, j = 14 \\ < [0, 0], [0.001, 0.0012], [0.001, 0.0015] >, & i = 14, j = 15 \\ < [0, 0], [0.005, 0.007], [0.003, 0.005] >, & i = 15, j = 16 \\ < [0, 0], [0, 0], [1, 1] > & \text{otherwise} \end{cases}$$

$$m_{\overrightarrow{H_3}} = \begin{cases} < [0, 0], [0.001, 0.0015], [0.001, 0.002] >, & i = 1, 2, j = 3 \\ < [0, 0], [0.001, 0.001], [0.001, 0.001] >, & i = 3, j = 4 \\ < [0, 0], [0.001, 0.002], [0.001, 0.0015] >, & i = 4, j = 5 \\ < [0.02, 0.03], [0.001, 0.002], [0.001, 0.002] >, & i = 5, j = 6 \\ < [0.02, 0.03], [0.002, 0.003], [0.001, 0.003] >, & i = 6, j = 7 \\ < [0, 0], [0, 0], [1, 1] > & \text{otherwise} \end{cases}$$

$$m_{\overrightarrow{H_4}} = \begin{cases} < [0, 0], [0.001, 0.0015], [0.001, 0.002] >, & i = 8, 9, 10, 11, j = 12 \\ < [0, 0], [0.001, 0.001], [0.001, 0.001] >, & i = 12, j = 13 \\ < [0, 0], [0.002, 0.003], [0.001, 0.002] >, & i = 13, j = 14 \\ < [0.01, 0.016], [0.002, 0.003], [0.001, 0.002] >, & i = 14, j = 15 \\ < [0.01, 0.016], [0.003, 0.005], [0.002, 0.004] >, & i = 15, j = 16 \\ < [0, 0], [0, 0], [1, 1] > & \text{otherwise} \end{cases}$$

For any given Strong Interval Valued Neutrosophic Number $a_P = ([t^l_P, t^u_P], [i^l_P, i^u_P], [f^l_P, f^u_P])$ with the score function [23]

$$S(a_P) = \left(\frac{2 + (t^l_P + t^u_P) - 2(i^l_P + i^u_P) - (f^l_P + f^u_P)}{4} \right) \tag{1}$$

In order to obtain the crisp values from the corresponding SIVN values from the above incidence matrices $\overrightarrow{H_l}(l = 1, 2, 3, 4)$, the score function is used. The score values of each entry of the corresponding incidence matrices $\overrightarrow{H_l}(l = 1, 2, 3, 4)$ are consolidated in Table 4, 5, 6, 7.

TABLE 4. Score matrix for AS/VD on the Right side of the heart

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	$RowTotal$
p_1	0.0	0.0	0.49975	0.0	0.0	0.0	0.0	0.49975
p_2	0.0	0.0	0.49975	0.0	0.0	0.0	0.0	0.49975
p_3	0.0	0.0	0.0	0.5005	0.0	0.0	0.0	0.5005
p_4	0.0	0.0	0.0	0.0	0.4995	0.0	0.0	0.4995
p_5	0.0	0.0	0.0	0.0	0.0	0.497	0.0	0.497
p_6	0.0	0.0	0.0	0.0	0.0	0.0	0.496	0.496
p_7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
				Total				2.9925

TABLE 5. Score matrix for AD/VS on the Right side of the heart

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	$RowTotal$
p_1	0.0	0.0	0.498	0.0	0.0	0.0	0.0	0.498
p_2	0.0	0.0	0.498	0.0	0.0	0.0	0.0	0.498
p_3	0.0	0.0	0.0	0.4985	0.0	0.0	0.0	0.4985
p_4	0.0	0.0	0.0	0.0	0.497875	0.0	0.0	0.497875
p_5	0.0	0.0	0.0	0.0	0.0	0.51025	0.0	0.51025
p_6	0.0	0.0	0.0	0.0	0.0	0.0	0.509	0.509
p_7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
				Total				3.011625

TABLE 6. Score matrix for AS/VD on the Left side of the heart

	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	$RowTotal$
p_8	0.0	0.0	0.0	0.0	0.50025	0.0	0.0	0.0	0.0	0.50025
p_9	0.0	0.0	0.0	0.0	0.50025	0.0	0.0	0.0	0.0	0.50025
p_{10}	0.0	0.0	0.0	0.0	0.50025	0.0	0.0	0.0	0.0	0.50025
p_{11}	0.0	0.0	0.0	0.0	0.50025	0.0	0.0	0.0	0.0	0.50025
p_{12}	0.0	0.0	0.0	0.0	0.0	0.49775	0.0	0.0	0.0	0.49775
p_{13}	0.0	0.0	0.0	0.0	0.0	0.0	0.498275	0.0	0.0	0.498275
p_{14}	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.498275	0.0	0.498275
p_{15}	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.492	0.492
p_{16}	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
						Total				3.9873

5. Sensitivity Analysis and Comparative Study

The Sensitivity Analysis focuses on the uncertainty analysis of a mathematical model or a system. In decision making problems, it helps to determine the significance of each criterion

Table 7. Score matrix for AD/VS on the Left side of the heart

	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	$RowTotal$
p_8	0.0	0.0	0.0	0.0	0.498	0.0	0.0	0.0	0.0	0.498
p_9	0.0	0.0	0.0	0.0	0.498	0.0	0.0	0.0	0.0	0.498
p_{10}	0.0	0.0	0.0	0.0	0.498	0.0	0.0	0.0	0.0	0.498
p_{11}	0.0	0.0	0.0	0.0	0.498	0.0	0.0	0.0	0.0	0.498
p_{12}	0.0	0.0	0.0	0.0	0.0	0.4985	0.0	0.0	0.0	0.4985
p_{13}	0.0	0.0	0.0	0.0	0.0	0.0	0.49675	0.0	0.0	0.49675
p_{14}	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.50325	0.0	0.50325
p_{15}	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.501	0.501
p_{16}	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
						Total				3.9915

used. Since both SVC and IVC push the deoxygenated blood to the RA with high pressure, the RA remains in Systolic phase at this time. Then the deoxygenated blood passes to the RV which remains in diastolic phase. Table 4 represents the crisp value that depicts the flow of deoxygenated blood during AS/VD phase on the Right side of the human heart. After the ventricular filling, the deoxygenated blood is transferred to PA. During this time, the RV stays in Systolic phase and RA remains in Diastolic phase. Table 5 gives the numerical values of the blood flow of the human heart during AD/VS phase on the Right side. Then, the oxygenated blood passes to LA and then to the LV. At this time, the LA is in Systolic phase and the corresponding LV is in Diastolic phase. Table 6 gives the values of the blood flow during AS/VD phase on the Left side of the human heart. Finally, the LV pushes out the blood to the Aorta and simultaneously the LV is in Systolic phase whereas the LA is in Diastolic phase. Table 7 illustrates the values of the blood flow during AD/VS phase on the Left side of the human heart. Now, by comparing the score values in Table 4, Table 5, Table 6 and Table 7, the most crucial phase during the cardiac cycle is evaluated. From the cumulative numerical values of the score matrices for the AS/VD and AD/VS phases on the Right and the Left side of the human heart, the sensitivity analysis is tabulated in Table 8.

Comparatively, from Table 4, Table 5, Table 6 and Table 7, the AS/VD phase on the Left side of the human heart is highly significant phase.

6. Results

It is evident that AS/VD phase on the Left side of the human heart is the most crucial phase. Also, it is observed that,

Table 8. Sensitivity Analysis

Phases in the Human Heart	Row Total	Ordering
AD/VS Phase(Left Side)	3.9915	1
AS/VD Phase (Left Side)	3.9873	2
AD/VS Phase (Right Side)	3.011625	3
AS/VD Phase (Right Side)	2.9925	4

(1) Atrial Systole / Ventricular Diastole on the left-hand side of the human heart (3.9873) has comparatively higher pressure than Atrial Systole / Ventricular Diastole on the right-hand side of the human heart (2.9925).
(2) Atrial Diastole / Ventricular Systole on the left-hand side of the human heart (3.9915) has comparatively higher pressure than Atrial Diastole / Ventricular Systole on the right-hand side of the human heart (3.011625).

7. Discussion

From Table 8, it is clear that

(1) Ventricular Systole and Diastole on the Left side of the human heart is the most significant process as compared to the Right side.
(2) Systolic ventricular phase is comparatively greater than diastolic ventricular phase.

The above analysis are analogous to the cardiac functioning of a normal and healthy individual.

8. Need, Limitation and Impact

(1) Since the blood flow is uni-directional and the blood pressure values fluctuates within certain range, it is necessary to depict the blood flow under a directed interval valued neutrosophic environment. Also, the blood usually flows from high to low pressure, in order to maintain the optimal level between any two compartments of the human heart, we model the cardiac functioning of the human heart as SIVNDG.
(2) The score function helps to make the deneutrosophication of SIVN values to a crisp value.
(3) Modelling the cardiac cycle of the human heart as SIVNDG helps to evaluate the blood flow in each phase effectively.
(4) The blood pressure is dynamic in nature as it changes while sleeping or doing exercise or a rest etc. The study of blood flow under these circumstances can be studied by our proposed model only if the necessary blood pressure values available.

9. Conclusions

For any two SIVNGs, it is shown that $G_1 * G_2, G_1 \mid G_2, G_1 \oplus G_2$ and $G_1 \bullet G_2$ is again a SIVNG. By modeling the cardiac functioning of the human heart, it is observed that the cardiac cycle is fit under the SIVNDG since the blood flow is unidirectional and the hemodynamic parameters show a varying pattern. Furthermore, the indeterminacy observed in the interval of blood pressure values is limited within and not more or less that range.With the observation of score function, we found that our result is identical to the conventional biological approach. Hence, evaluating the cardiac functioning of the heart by modeling as SIVNDG is the most reasonable choice.

Conflicts of Interest

The authors declare no conflicts of interest.

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SV NM/NM/c Queuing Model with Encouraged Arrival and Heterogeneous servers

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Abstract. This article shows the neutrosophic abstraction of M/M/c Queuing model (QM) with Encouraged Arrival (En. A) and heterogeneous servers. Here we derive the system's performance indicators of NM/NM/c QM with En. A and heterogeneous servers. The numerical example for the above model with its respective graphs are also depicted.

Keywords: NM/NM/c queuing model, Encouraged arrival, Heterogeneous service, Performance measures.

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1. Introduction

A Mathematical study of queues or waiting in lines was given by Erlang in the year 1909 is defined to be queuing which plays a significant role in almost all-fields. Finding the amount of customers waiting in line and system and the waiting lines of customers in both queue and system are the basic components of queuing theory defined by Shortle and Thompson [23]. Applying fuzzy logic to queuing theory makes solution for imprecise cases or uncertainty in the Fuzzy logic was firstly introduced by Zadeh in 1965 [24]. Rather than crisp queues, fuzzy queues are much more sensible in many real circumstances.

Neutrosophic Philosophy in queuing deals with situations in which the queue parameters are inaccurate. Neutrosophic logic which is a generalisation of fuzzy logic and intuitionistic fuzzy logic [17, 18, 19, 20] was introduced by Florentin Smarandache in the year 1995. This deals with indeterminacy data realistic and thereby gives understandable efficient outcomes [14, 15, 16, 22]. If the parameters of the queuing system are neutrosophic numbers, the system is said to be a neutrosophic queue.

The Neutrosophic set is employed to explain the uncertainty and indeterminacy in any information. This set is characterized by a truth 'T', indeterminacy 'I' and false 'F' membership functions, where $T, I, F \in]^{-}0, 1^{+}[$. There is no restriction on the sum and so $0^{-} \leq T + I + F \leq 3^{+}$. Neutrosophic set needs to be specified from a technical point of view. To this effect, we define certain set-theoretic operators on neutrosophic set, which in turn called as Single valued Neutrosophic set [21]. The membership functions for truth (T), indeterminacy (I), and falseness (F) define a single valued neutrosophic set, where $T, I, F \in [0, 1]$, which fulfills the following requirements: $0 \leq T + I + F \leq 3$. We examine single valued neutrosophic encouraged arrival and heterogeneous service rate. Encouraged arrival defines where the customers are drew towards profitable deals or offers. For instances, people rush towards ticket counters in railway station during vacation or special holidays. Also, Heterogeneous server defines the service occurs in a varied service rate. This in combination with Neutrosophy gives precise output.

Patro with Smarandache discussed more problems and solutions on Neutrosophic Statistical Distribution [13]. Bisher Zeina [11,12] studied Erlang Service Queuing Model and Event-Based Queuing Model on Neutrosophic basis in the year 2020. Deepa and Julia Rose Mary studied Heterogeneous Bulk tandem fluid multiple vacations queuing model for encouraged arrival with catastrophe[3]. Krishnakumar and Maheshwari [4] found the transient solution of M/M/2 queue with heterogeneous servers subject to catastrophes. Maissam Jdid, Smarandache and Said Browmi [5] inspected the assignment form of Product quality control using Neutrosophic logic. Jdid and Smarandache [6] explained the use of Neutrosophic Methods of Operations Research in the Management of Corporate work. Manas et. al. [7] found the solution of transportation issues in a neutrosophic setting.

Bisher Zeina [8,9] studied the M/M/1 Queue's Performance measures on fuzzy environment and Neutrosophic concept of M/M/1, M/M/c, M/M/1/b Queuing system utilizing interval-valued neutrosophic sets in the year 2020. Also in the year 2021, Mohamed Bisher Zeina [10] analysed Single Valued Neutrosophic M/M/1 Queue in Linguistic terms. Some operations of Single Valued Neutrosophic numbers mentioned by Bisher Zeina is utilized here. Bhupender Singh Som and Sunny Seth [1, 2] developed M/M/c queuing system with encouraged arrivals with N number of customers, Impatient customers and Retention of Impatient Customers queuing system in the year 2018.

Here, we consider a case where the customer's arrival and service are neutrosophic. We deduce the steady state equations and thereby deriving the system's performance indicators.

Some Procedures on Single Valued Neutrosophic

Here we provide some basic operations on Neutrosophic sets dealing with addition, subtraction, multiplication, division, power and scalar multiplication. This is one of the necessary factor in performing any operation between two different sets.

In this paper, These Neutrosophic processes are used to determine Neutrosophic queuing models' performance metrics, including as L_s, L_q, W_s and W_q respectively.

Suppose that we have two neutrosophic numbers given by $X = (t_1, i_1, f_1)$,
$Y = (t_2, i_2, f_2)$ where:

$$0 \leq t_1, i_1, f_1, t_2, i_2, f_2$$
$$0 \leq t_1 + i_1 + f_1 \leq 3 \text{ and } 0 \leq t_2 + i_2 + f_2 \leq 3$$

Then:

Neutrosophic Summation

$$X \oplus Y = (t_1 + t_2 - t_1 t_2, i_1 i_2, f_1 f_2)$$

Neutrosophic Multiplication

$$X \otimes Y = (t_1 t_2, i_1 + i_2 - i_1 i_2, f_1 + f_2 - f_1 f_2)$$

Neutrosophic Subtraction

$$X \ominus Y = \left(\tfrac{t_1 - t_2}{1 - t_2}, \tfrac{i_1}{i_2}, \tfrac{f_1}{f_2}\right); t_2 \neq 1, i_2 \neq 0, f_2 \neq 0$$

Neutrosophic Division

$$\tfrac{X}{Y} = \left(\tfrac{t_1}{t_2}, \tfrac{i_1 - i_2}{1 - i_2}, \tfrac{f_1 - f_2}{1 - f_2}\right); t_2 \neq 0, i_2 \neq 1, f_2 \neq 1$$

Neutrosophic Scalar Multiplication

$$\lambda X = \left(1 - (1 - t_1)^{\lambda}, i_1^{\lambda}, f_1^{\lambda}\right); \lambda > 0$$

Neutrosophic Power

$$X^{\lambda} = \left(t_1^{\lambda}, 1 - (1 - i_1)^{\lambda}, 1 - (1 - f_1)^{\lambda}\right)$$

With the aid of the above operations, our model is explained.

2. (NM/NM/c):(FIFO/∞/∞) QM with En. A and Heterogeneous service - Model Description:

Here the customers arrive with a mean arrival rate λ_N and a maximum of c customers may be served simultaneously. The encouraged arrival rate=$\lambda_N(1 + \eta)$. The service rate per busy server is equal to $\sum_{i=1}^{n} \mu_{Ni}$. Also we get $\lambda_{Neff} = \lambda_N$. Neutrosophic Philosophy used here is to present precise information dealing with with uncertainty, untruth, and truth (i.,e) The arrival rate λ_N is assumed to be $\lambda_N = (T_\lambda, I_\lambda, F_\lambda)$ and the service rate $\sum_{i=1}^{n} \mu_N = (T_{\mu_i}, I_{\mu_i}, F_{\mu_i})$. If the customer base in the system , n equals or exceed c, the combined departure rate from the facility is $c\sum_{i=1}^{n}\mu_{Ni}$. Else, if $n<c$, the service rate $n\sum_{i=1}^{n}\mu_{Ni}$. Thus in terms of generalized model, $\lambda_n=\lambda$. Here $\lambda_{N_n}=\lambda_N\left(1+\eta\right)$ if $n\geq 0$

$$\sum_{i=1}^{n}\mu_{Nn_i}=\begin{cases} n\sum_{i=1}^{n}\mu_{Ni}; n<c \\ c\sum_{i=1}^{n}\mu_{Ni}; n\geq c\end{cases}$$

Also, the intensity ρ_N is given by,

$$\begin{aligned}\rho_N &= \frac{\lambda_N(1+\eta)}{c\sum_{i=1}^{n}\mu_N} \\ &= \frac{\left(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\right)}{c\sum_{i=1}^{n}\left(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\right)} \\ &= \frac{1}{c}\left(\frac{T_{\lambda(1+\eta)}}{\sum_{i=1}^{n}T_{\mu_i}}, \frac{I_{\lambda(1+\eta)}-\sum_{i=1}^{n}I_{\mu_i}}{1-\sum_{i=1}^{n}I_{\mu_i}}, \frac{F_{\lambda(1+\eta)}-\sum_{i=1}^{n}F_{\mu_i}}{1-\sum_{i=1}^{n}F_{\mu_i}}\right)\end{aligned}$$

where $\rho_N=\frac{\lambda_N}{c\sum_{i=1}^{n}\mu_{Ni}}<1$.

For the Neutrosophic M/M/c QM with En. A and Heterogeneous server, the steady state equation becomes,

Case I:

$$\frac{dNP_0(t)}{dt}=-\lambda_N\left(1+\eta\right)NP_0(t)+\mu_{N_1}NP_1(t); n=0$$

$$\frac{dNP_0(t)}{dt}=-\left(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\right)NP_0(t)+\left(T_{\mu_1}, I_{\mu_1}, F_{\mu_1}\right)NP_1(t); n=0 \tag{1}$$

$$\begin{aligned}\frac{dNP_n(t)}{dt} &= -\left[\left(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\right)+\sum_{i=1}^{n}\left(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\right)\right]NP_n(t) \\ &+\left(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\right)NP_{n-1}(t)+\sum_{i=1}^{n+1}\left(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\right)NP_{n+1}(t) \quad ; n=1,2,...c-1.\end{aligned} \tag{2}$$

$$\begin{aligned}\frac{dNP_c(t)}{dt} &= -\left[\left(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\right)+\sum_{i=1}^{c}\left(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\right)\right]NP_c(t) \\ &+\left(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\right)NP_{c-1}(t)+\sum_{i=1}^{c}\left(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\right)NP_{c+1}(t) \quad ; n=c.\end{aligned} \tag{3}$$

$$\frac{dNP_n(t)}{dt} = -\Bigg[\Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big) + \sum_{i=1}^{n}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)\Bigg]NP_n(t)$$
$$+\Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)NP_{n-1}(t) + \sum_{i=1}^{c}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)NP_{n+1}(t) \quad ; n \geq c+1. \quad (4)$$

In steady state,

$lim_{t\to\infty}\ NP_n(t) = NP_n$

$lim_{t\to\infty}\ \frac{dNP_n(t)}{dt} = 0$

Then,

$$(1) \Rightarrow 0 = -\Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)NP_0 + \Big(T_{\mu_1}, I_{\mu_1}, F_{\mu_1}\Big)NP_1; n = 0 \quad (5)$$

$$(2) \Rightarrow 0 = -\Bigg[\Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big) + \sum_{i=1}^{n}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)\Bigg]NP_n$$
$$+\Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)NP_{n-1} + \sum_{i=1}^{n+1}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)NP_{n+1} \quad ; n = 1, 2, ...c-1. \quad (6)$$

$$(3) \Rightarrow 0 = -\Bigg[\Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big) + \sum_{i=1}^{n}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)\Bigg]NP_c$$
$$+\Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)NP_{c-1} + \sum_{i=1}^{c}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)NP_{c+1} \quad ; n = c. \quad (7)$$

$$(4) \Rightarrow 0 = -\Bigg[\Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big) + \sum_{i=1}^{n}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)\Bigg]NP_n$$
$$+\Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)NP_{n-1} + \sum_{i=1}^{c}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)NP_{n+1} \quad ; n \geq c+1. \quad (8)$$

From (5), we have

$$NP_1 = \frac{\Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)}{\Big(T_{\mu_1}, I_{\mu_1}, F_{\mu_1}\Big)}NP_0$$

$$NP_1 = \frac{\lambda_N\Big(1+\eta\Big)}{\mu_{N_1}}NP_0$$

From (6), we have

$$\sum_{i=1}^{n+1}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)NP_{n+1} = \Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)NP_n + \sum_{i=1}^{n}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)NP_n - \Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)NP_{n-1} \quad (9)$$

Put $n = 1$ in eq (9)

$$\sum_{i=1}^{2}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)NP_2 = \Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)NP_1 + \Big(T_{\mu_1}, I_{\mu_1}, F_{\mu_1}\Big)NP_1 - \Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)NP_0$$

$$NP_2 = \frac{\Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)}{\sum_{i=1}^{2}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)}NP_1$$

$$NP_2 = \left(\frac{\lambda_N\Big(1+\eta\Big)}{\sum_{i=1}^{2}\mu_{Ni}}\right)NP_1$$

$$NP_2 = \left(\frac{\lambda_N\Big(1+\eta\Big)^2}{\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\right)NP_0$$

Putting $n = 2$ in (9), we get

$$\sum_{i=1}^{3}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)NP_3 = \Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)NP_2 + \sum_{i=1}^{2}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)NP_2 - \Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)NP_1$$

$$\sum_{i=1}^{3}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)NP_3 = \Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)NP_2$$

$$NP_3 = \frac{\left(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\right)}{\sum_{i=1}^{3}\left(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\right)} NP_2$$

$$NP_3 = \left(\frac{\lambda_N\left(1+\eta\right)}{\sum_{i=1}^{3}\mu_{Ni}}\right) NP_2$$

$$NP_3 = \left(\frac{\lambda_N\left(1+\eta\right)^3}{\sum_{i=1}^{3}\mu_{Ni}.\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\right) NP_0$$

Similarly,

$$NP_4 = \left(\frac{\lambda_N\left(1+\eta\right)^4}{\sum_{i=1}^{4}\mu_{Ni}.\sum_{i=1}^{3}\mu_{Ni}.\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\right) NP_0$$

$$NP_n = \left(\frac{\lambda_N\left(1+\eta\right)^n}{\sum_{i=1}^{n}\mu_{Ni}.....\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\right) NP_0 \tag{10}$$

If $n = c - 1$, we get from eq (9)

$$\sum_{i=1}^{c}\left(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\right) NP_c = \left(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\right) NP_{c-1}$$

$$NP_c = \frac{\left(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\right)}{\sum_{i=1}^{c}\left(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\right)} NP_{c-1}$$

$$NP_c = \left(\frac{\lambda_N\left(1+\eta\right)}{\sum_{i=1}^{c}\mu_{Ni}}\right) NP_{c-1}$$

$$NP_{c-1} = \left(\frac{\lambda_N\left(1+\eta\right)^{c-1}}{\sum_{i=1}^{c-1}\mu_{Ni}.\sum_{i=1}^{c-2}\mu_{Ni}.....\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\right) NP_0$$

$$NP_c = \left(\frac{\lambda_N\left(1+\eta\right)^{c}}{\sum_{i=1}^{c}\mu_{Ni}.\sum_{i=1}^{c-1}\mu_{Ni}.....\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\right) NP_0$$

Case II: When n=c in eq (9),

$$\sum_{i=1}^{c+1}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)NP_{c+1} = \Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)NP_c$$

$$NP_{c+1} = \left(\frac{\lambda_N\Big(1+\eta\Big)}{\sum_{i=1}^{c+1}\mu_{Ni}}\right)NP_c$$

$$NP_{c+1} = \left(\frac{\lambda_N\Big(1+\eta\Big)^{c+1}}{\sum_{i=1}^{c+1}\mu_{Ni}.\sum_{i=1}^{c}\mu_{Ni}.....\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\right)NP_0$$

Case III: When n=c+1 in eq(9),

$$\sum_{i=1}^{c+2}\Big(T_{\mu_i}, I_{\mu_i}, F_{\mu_i}\Big)NP_{c+2} = \Big(T_{\lambda(1+\eta)}, I_{\lambda(1+\eta)}, F_{\lambda(1+\eta)}\Big)NP_{c+1}$$

$$NP_{c+2} = \left(\frac{\lambda_N\Big(1+\eta\Big)^{c+2}}{\sum_{i=1}^{c+2}\mu_{Ni}.\sum_{i=1}^{c+1}\mu_{Ni}.....\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\right)NP_0$$

$$NP_{c+(n-c)} = \left(\frac{\lambda_N\Big(1+\eta\Big)^{n}}{\sum_{i=1}^{c+(n-c)}\mu_{Ni}.....\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\right)NP_0 \tag{11}$$

Now, to find: NP_0

$$\sum_{n=0}^{\infty}NP_n = 1$$

$$\Rightarrow \sum_{n=0}^{c-1}NP_n + NP_c + \sum_{n=c+1}^{\infty}NP_n = 1$$

$$\Rightarrow \sum_{n=0}^{c-1}\left(\frac{\lambda_N\Big(1+\eta\Big)^{n}}{\sum_{i=1}^{n}\mu_{Ni}.....\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\right)NP_0 + \frac{\lambda_N\Big(1+\eta\Big)^{c}}{\sum_{i=1}^{c}\mu_{Ni}.....\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}NP_0$$
$$+ \sum_{n=c+1}^{\infty}\left(\frac{\lambda_N\Big(1+\eta\Big)^{n}}{\sum_{i=1}^{n}\mu_{Ni}.....\sum_{i=1}^{c+1}\mu_{Ni}.\sum_{i=1}^{c}\mu_{Ni}.....\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\right)NP_0 = 1$$

$$\Rightarrow NP_0[\sum_{n=0}^{c-1}\left(\frac{\lambda_N\left(1+\eta\right)^n}{\sum_{i=1}^{n}\mu_{Ni}.....\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\right)+\frac{\lambda_N\left(1+\eta\right)^c}{\sum_{i=1}^{c}\mu_{Ni}.....\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}$$
$$+\sum_{n=c+1}^{\infty}\left(\frac{\lambda_N\left(1+\eta\right)^n}{\sum_{i=1}^{n}\mu_{Ni}.....\sum_{i=1}^{c+1}\mu_{Ni}.\sum_{i=1}^{c}\mu_{Ni}.....\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\right)]=1$$

$$\Rightarrow NP_0[\sum_{n=0}^{c-1}\left(\frac{\lambda_N\left(1+\eta\right)^n}{\sum_{i=1}^{n}\mu_{Ni}.....\mu_{N_1}}\right)+\frac{\lambda_N\left(1+\eta\right)^c}{\sum_{i=1}^{c}\mu_{Ni}....\mu_{N_1}}$$
$$+\left(\frac{\lambda_N\left(1+\eta\right)^{c+1}}{\sum_{i=1}^{c+1}\mu_{Ni}.....\mu_{N_1}}\right)+\left(\frac{\lambda_N\left(1+\eta\right)^{c+2}}{\sum_{i=1}^{c+2}\mu_{Ni}.....\mu_{N_1}}\right)+........]=1$$

$$\Rightarrow NP_0[\sum_{n=0}^{c-1}\left(\frac{\lambda_N\left(1+\eta\right)^n}{\sum_{i=1}^{n}\mu_{Ni}.....\mu_{N_1}}\right)+\frac{\lambda_N\left(1+\eta\right)^c}{\sum_{i=1}^{c}\mu_{Ni}....\mu_{N_1}}[1+\frac{\lambda_N\left(1+\eta\right)}{\mu_{N_1}}$$
$$+\frac{\lambda_N\left(1+\eta\right)^2}{\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}+........]]=1$$

$$\Rightarrow \quad NP_0 \;=\; \left[\sum_{n=0}^{c-1}\left(\frac{\lambda_N\left(1+\eta\right)^n}{\sum_{i=1}^{n}\mu_{Ni}.....\mu_{N_1}}\right)+\frac{\lambda_N\left(1+\eta\right)^c}{\sum_{i=1}^{c}\mu_{Ni}....\mu_{N_1}}\left[1+\sum_{n=1}^{\infty}\frac{\lambda_N(1+\eta)^n}{\sum_{i=1}^{n}\mu_{Nn_i}}\right]\right]^{-1}$$

Sub NP_0 in eq(10) and (11),

$$(10)\Rightarrow NP_n=\frac{\lambda_N(1+\eta)^n}{\sum_{i=1}^{n}\mu_{Ni}....\mu_{N_1}}[\sum_{n=0}^{c-1}\left(\frac{\lambda_N\left(1+\eta\right)^n}{\sum_{i=1}^{n}\mu_{Ni}.....\mu_{N_1}}\right)$$
$$+\frac{\lambda_N\left(1+\eta\right)^c}{\sum_{i=1}^{c}\mu_{Ni}....\mu_{N_1}}[1+\sum_{n=1}^{\infty}\frac{\lambda_N(1+\eta)^n}{\sum_{i=1}^{n}\mu_{Nn_i}}]]^{-1}$$

$$(11)\Rightarrow NP_n=\frac{\lambda_N(1+\eta)^n}{\sum_{i=1}^{n}\mu_{Ni}....\sum_{i=1}^{c+1}\mu_{Ni}....\mu_{N_1}}[\sum_{n=0}^{c-1}\left(\frac{\lambda_N\left(1+\eta\right)^n}{\sum_{i=1}^{n}\mu_{Ni}.....\mu_{N_1}}\right)$$
$$+\frac{\lambda_N\left(1+\eta\right)^c}{\sum_{i=1}^{c}\mu_{Ni}....\mu_{N_1}}[1+\sum_{n=1}^{\infty}\frac{\lambda_N(1+\eta)^n}{\sum_{i=1}^{n}\mu_{Nn_i}}]]]^{-1}$$

for $\frac{\rho_N}{c} < 1$ or $\frac{\lambda_N\left(1+\eta\right)}{c\sum_{i=1}^{n}\mu_{Ni}} < 1$,

$$NL_q = \sum_{n=c}^{\infty}\left(n-c\right)NP_n \qquad [\text{Take } n-c=k]$$

$$= \sum_{k=0}^{\infty} kNP_{k+c}$$

$$= \sum_{k=0}^{\infty} k\Bigg(\frac{\lambda_N\left(1+\eta\right)^{k+c}}{\sum_{i=1}^{k+c}\mu_{Ni}.....\sum_{i=1}^{2}\mu_{Ni}.\mu_{N_1}}\Bigg)NP_0$$

$$= \frac{\lambda_N\left(1+\eta\right)^{c}}{\sum_{i=1}^{c}\mu_{Ni}.....\mu_{N_1}}NP_0\Bigg[\sum_{k=0}^{\infty} k\Bigg(\frac{\lambda_N\left(1+\eta\right)^{k}}{\sum_{i=1}^{k}\mu_{Ni}.....\sum_{i=1}^{c+1}\mu_{Ni}}\Bigg)\Bigg]$$

$$= \frac{\lambda_N\left(1+\eta\right)^{c}}{\sum_{i=1}^{c}\mu_{Ni}.....\mu_{N_1}}NP_0\Bigg[\frac{\lambda_N\left(1+\eta\right)}{\mu_{N_1}} + 2\frac{\lambda_N\left(1+\eta\right)^{2}}{\sum_{i=1}^{2}\mu_{Ni}} + ...\Bigg]$$

$$= \frac{\lambda_N\left(1+\eta\right)^{c}}{\sum_{i=1}^{c}\mu_{Ni}.....\mu_{N_1}}NP_0\Bigg[\sum_{n=1}^{\infty}\frac{\lambda_N(1+\eta)^n}{\sum_{i=1}^{n}\mu_{N_n}}\Bigg]$$

Also, the system's performance indicators for a neutrosophic M/M/c QM with En. A and Heterogeneous server was given by,

As, $NL_q = \rho_N^c \sum_{n=1}^{\infty}\rho_{N_n}.NP_0$

$$\boxed{NL_q = \sum_{n=1}^{\infty}\rho_{N_n}.NP_c} \tag{12}$$

$NL_s = NL_q + \rho_N$

The neutrosophic form of ρ_N is already defined, using that we get

$$\boxed{Nl_s = NL_q + \frac{\lambda_N(1+\eta)}{\sum_{i=1}^{n}\mu_{N_i}}} \tag{13}$$

Similarly, we can also find NW_q and NW_s.

$$\boxed{NW_q = \frac{NL_q}{\lambda_N(1+\eta)}} \tag{14}$$

$$\boxed{NW_s = NW_q + \frac{1}{\sum_{i=1}^{n} \mu_{N_i}}} \tag{15}$$

3. Numerical Example

In this portion, we take some observed neutrosophic values for $\lambda_N(1+\eta)$ and $\sum_{i=1}^{n} \mu_{N_i}$. The values of $T_{\lambda(1+\eta)}$, $I_{\lambda(1+\eta)}$ and $F_{\lambda(1+\eta)}$ denoting the arrival rate $\lambda_N(1+\eta)$, and $\sum_{i=1}^{n}(T_{\mu_{N_i}}, I_{\mu_{N_i}}, F_{\mu_{N_i}})$ representing the service rate. Also, the observed data takes η to be 0.005 and 0.01 and considering the number of servers as c=1, 2 and 3. By letting the heterogeneous service rate $\sum_{i=1}^{n} \mu_{N_i}$ as $(T_{\mu_{N_1}}, I_{\mu_{N_1}}, F_{\mu_{N_1}}) = (0.5, 0.8, 0.7)$, $(T_{\mu_{N_2}}, I_{\mu_{N_2}}, F_{\mu_{N_2}}) = (0.6, 0.7, 0.6)$, $(T_{\mu_{N_3}}, I_{\mu_{N_3}}, F_{\mu_{N_3}}) = (0.7, 0.6, 0.5)$, $(T_{\mu_{N_4}}, I_{\mu_{N_4}}, F_{\mu_{N_4}}) = (0.8, 0.5, 0.4)$.

$(T_{\mu_{N_1}}, I_{\mu_{N_1}}, F_{\mu_{N_1}})$ values can be viewed as first varied service rate of validity, ambiguity, and fake membership values. the system's performance indicators were calculated using (12), (13), (14) and (15) respectively. By considering all the above data and by using equations (12) and (13), we obtain NL_s as $(0.123, 0.8032, 0.6493)$ which means the anticipated size of customers in the system to be truth, indeterminate and false. We apply all the observed values in the concerned system measures of performance and depict a line graph. Here we use the operations of Single valued neutrosophic numbers. The resulted system measures of performance provides the degree to which the system's reported client count, queue length, and wait time are true, unreliable, or false respectively. The evaluated values are tabulated below:

Various Performance measures by varying λ_N with respect to c and $\eta = 0.005$

λ_N	NL_q	NL_s	NW_q	NW_s
$(0.1, 0.9, 0.8)$	$(0.0237, 0.9135, 0.8317)$	$(0.123, 0.8032, 0.6493)$	$(0.2358, 0.1393, 0.1623)$	$(1.0093, -0.0281, -0.0149)$
$(0.2, 0.8, 0.9)$	$(0.1101, -0.027, 0.9544)$	$(0.291, -0.0205, 0.8497)$	$(0.548, -4.0916, -0.3065)$	$(1.0055, 0.8261, 0.0281)$
$(0.3, 0.7, 0.6)$	$(0.2847, 3.0318, -6.7481)$	$(0.5028, 1.9343, -3.7904)$	$(0.9452, 7.7457, -18.2979)$	$(1.0007, -1.5639, 1.6779)$
$(0.4, 0.6, 0.7)$	$(0.5923, 2.0323, -0.0168)$	$(0.758, 1.0515, -0.0113)$	$(1.4752, 3.5711, -2.3758)$	$(0.9942, -0.721, 0.2179)$

TABLE 1. When c=1, $\eta = 0.005$

λ_N	NL_q	NL_s	NW_q	NW_s
$(0.1, 0.9, 0.8)$	$(0.003, 0.9786, 0.8984)$	$(0.1044, 0.8604, 0.7014)$	$(0.0299, 0.7871, 0.4943)$	$(1.0118, -0.1589, -0.0453)$
$(0.2, 0.8, 0.9)$	$(0.0287, 0.5347, 0.9918)$	$(0.2262, 0.8876, 0.9991)$	$(0.1429, -1.3161, 0.9184)$	$(1.0105, 0.2657, -0.0842)$
$(0.3, 0.7, 0.6)$	$(0.1146, -0.3405, -0.2547)$	$(0.3846, -0.2172 +0.1431)$	$(0.3805, -3.4505, -2.125)$	$(1.0076, 0.6967, 0.1949)$
$(0.4, 0.6, 0.7)$	$(0.3092, 0.3442, 0.4747)$	$(0.5899, 0.1781, 0.3186)$	$(0.7701, -0.6334, -0.744)$	$(1.0028, 0.1279, 0.0682)$

TABLE 2. When c=2, $\eta = 0.005$

λ_N	NL_q	NL_s	NW_q	NW_s
$(0.1, 0.9, 0.8)$	$(0.0003, 0.9968, 0.4369)$	$(0.102, 0.8764, 0.3411)$	$(0.003, 0.9682, -1.8029)$	$(1.0122, -0.1955, 0.1653)$
$(0.2, 0.8, 0.9)$	$(0.0062, 0.8595, 0.999)$	$(0.2082, 0.6519, 0.8894)$	$(0.0309, 0.3007, 0.9901)$	$(1.0118, -0.0607, -0.0908)$
$(0.3, 0.7, 0.6)$	$(0.0373, 0.3749, 0.4626)$	$(0.3308, 0.2392, 0.2598)$	$(0.1238, -1.0754, -0.3385)$	$(1.0107, 0.2171, 0.031)$
$(0.4, 0.6, 0.7)$	$(0.1344, -1.7047, 0.8)$	$(0.4862, -0.882, 0.537)$	$(0.3348, -5.7365, 0.336)$	$(1.0081, 1.1582, -0.0308)$

TABLE 3. When c=3, $\eta = 0.005$

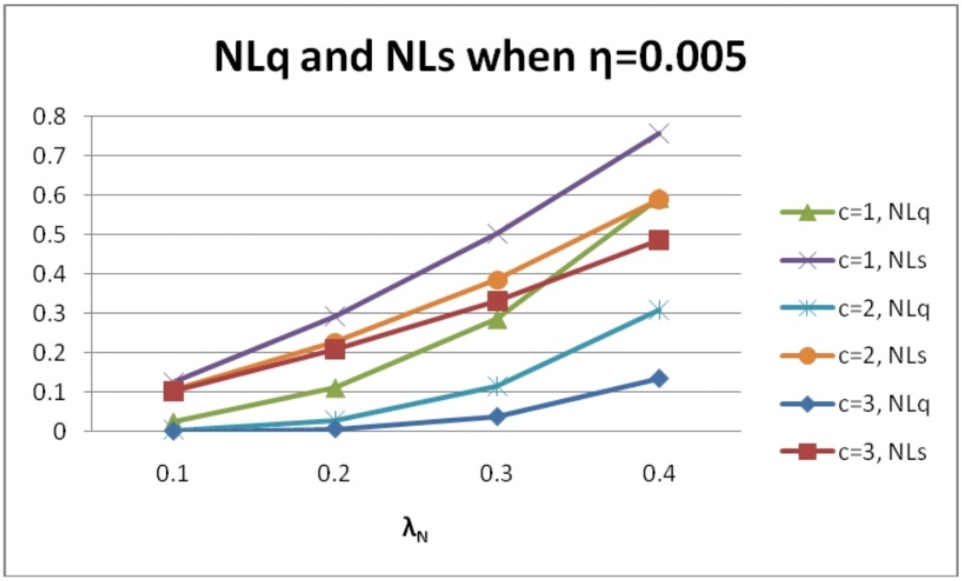

FIGURE 1. NL_q and NL_s when $\eta = 0.005$

In the above line graph, NL_q and NL_s values are plotted with the number of servers as 1, 2 and 3. Here, we consider a parameter η which is considered as encouraged arrival parameter and it takes the value 0.005 and the heterogeneous service rate is taken as mentioned above. At c=1, NL_q and NL_s increases steadily with the increase in λ_N. When c=2, NL_q increases measurably and so NL_s.

Also, when c=3, both NL_q and NL_s increases with λ_N. Now we find that the amount of customers waiting in line and system increases with steady increase of λ_N it demonstrates that it is adequate. for the customers to get served with 2 heterogeneous services. As there is only minute differences between c=2 and c=3, customers can get served with 2 servers as it is unnecessary to increase the server which lead to the loss to the service provider.

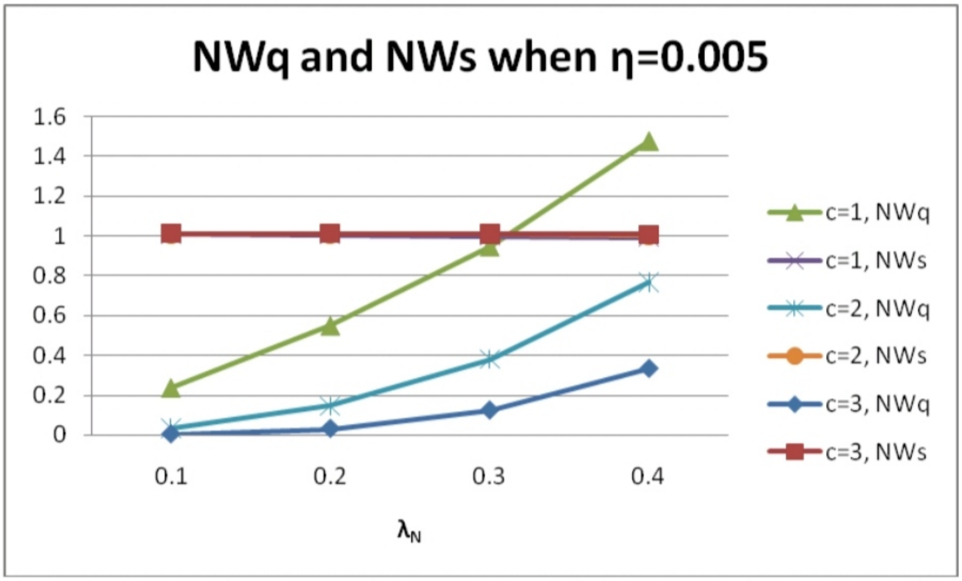

FIGURE 2. NW_q and NW_s when $\eta = 0.005$

In Fig 2, NW_q and NW_s values are plotted with the number of servers to be taken 1, 2 and 3. Here, we consider a parameter η which is considered as encouraged arrival parameter and it takes the value 0.005. At c=1, 2 and 3, the clients' wait times in the system remains almost the same throughout the λ_N. And the waiting time of customers in the queue gradually increasing with increase in λ_N. As the NW_q at c=2 increases step by step, it is sufficient for the customers to get the effective service. Increasing the server is inessential as it leads to loss.

Various Performance measures by varying λ_N with respect to c and $\eta = 0.01$

λ_N	NL_q	NL_s	NW_q	NW_s
$(0.1, 0.9, 0.8)$	$(0.0231, 0.9171, 0.7455)$	$(0.1228, 0.8059, 0.5813)$	$(0.2289, 0.1784, -0.2612)$	$(1.0094, -0.036, 0.024)$
$(0.2, 0.8, 0.9)$	$(0.1112, -0.0543, 0.954)$	$(0.2928, -0.0411, 0.849)$	$(0.551, -4.2245, 0.5441)$	$(1.0055, 0.8529, -0.00499)$
$(0.3, 0.7, 0.6)$	$(0.2877, 2.9952, -7.2821)$	$(0.5058, 1.9062, -4.0773)$	$(0.9511, 7.5957, -19.546)$	$(1.0006, -1.5336, 1.7924)$
$(0.4, 0.6, 0.7)$	$(0.581, 2.0263, -0.0356)$	$(0.752, 1.0446, -0.0239)$	$(1.4413, 3.546, -2.4235)$	$(0.9946, -0.7159, 0.2223)$

TABLE 4. When c=1, $\eta = 0.01$

λ_N	NL_q	NL_s	NW_q	NW_s
$(0.1, 0.9, 0.8)$	$(0.0031, 0.9782, 0.8969)$	$(0.1049, 0.8595, 0.6993)$	$(0.0307, 0.784, 0.4891)$	$(1.0118, -0.1583, -0.0449)$
$(0.2, 0.8, 0.9)$	$(0.0292, 0.5246, 0.9917)$	$(0.2275, 0.3974, 0.8825)$	$(0.1447, -1.3558, 0.9177)$	$(1.0104, 0.2737, -0.084)$
$(0.3, 0.7, 0.6)$	$(0.0505, 2.9182, -7.1678)$	$(0.3412, 1.8571, -4.0133)$	$(0.1669, 0.7341, -19.2625)$	$(1.0102, -0.1482, 1.7664)$
$(0.4, 0.6, 0.7)$	$(0.3135, 0.3515, 0.4656)$	$(0.5936, 0.1812, 0.3119)$	$(0.7777, -0.6088, -0.7666)$	$(1.0027, 0.1229, 0.0703)$

TABLE 5. When c=2, $\eta = 0.01$

λ_N	NL_q	NL_s	NW_q	NW_s
$(0.1, 0.9, 0.8)$	$(0.0003, 0.9967, 0.9731)$	$(0.1024, 0.8758, 0.7587)$	$(0.003, 0.9673, -0.1957)$	$(1.0122, -0.1953, 0.018)$
$(0.2, 0.8, 0.9)$	$(0.0063, 0.856, 0.999)$	$(0.2093, 0.6484, 0.889)$	$(0.0312, 0.2864, 0.9901)$	$(1.0118, -0.0578, -0.0908)$
$(0.3, 0.7, 0.6)$	$(0.0165, 1.9074, -3.0607)$	$(0.3177, 1.2139, -1.7137)$	$(0.0546, 3.9997, -9.0737)$	$(1.0115, -0.8075, 0.8321)$
$(0.4, 0.6, 0.7)$	$(0.1369, -0.2329, 0.7905)$	$(0.4891, -0.1201, 0.5295)$	$(0.3396, -2.0586, 0.3074)$	$(1.0081, 0.4156, -0.0282)$

TABLE 6. When c=3, $\eta = 0.01$

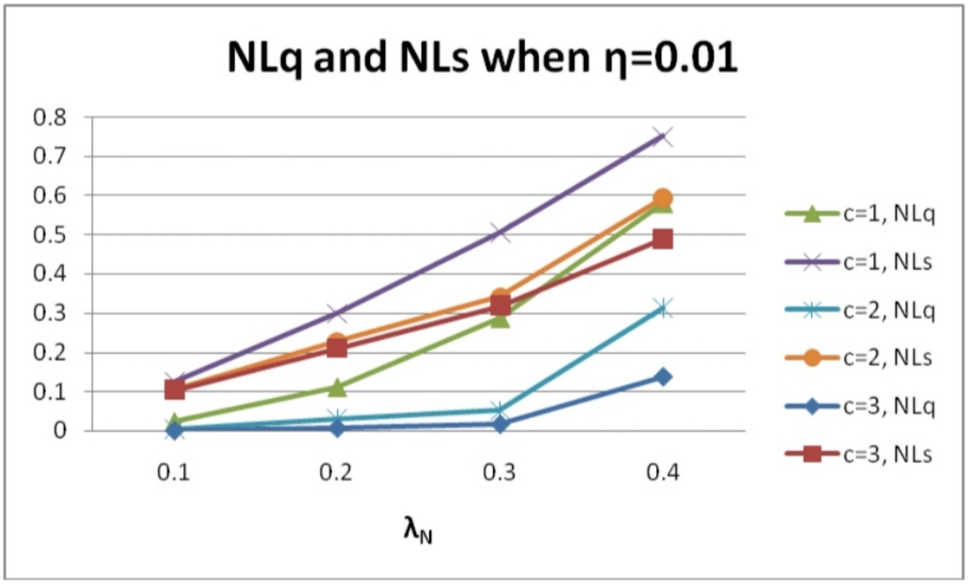

FIGURE 3. NL_q and NL_s when $\eta = 0.01$

In the above line graph, NL_q and NL_s values are plotted with the number of servers to be taken as 1, 2 and 3. Here, we consider a parameter η which is considered as encouraged arrival parameter and it takes the value 0.01 and the heterogeneous service rate is taken as mentioned above. At c=1, NL_q and NL_s gradually increases with increase in λ_N. Also when c=2, NL_q and NL_s increases steadily.

When c=3, both NL_q and NL_s increases slowly with increase in λ_N. To get effective service, a number of 2 servers are enough as the amount of customers waiting in line and system increases with the increasing rate of λ_N with the heterogeneous service rather than increasing the servers as there is slight difference between c=2 and c=3. When the number of servers is increased, it may steer to loss for the service provider.

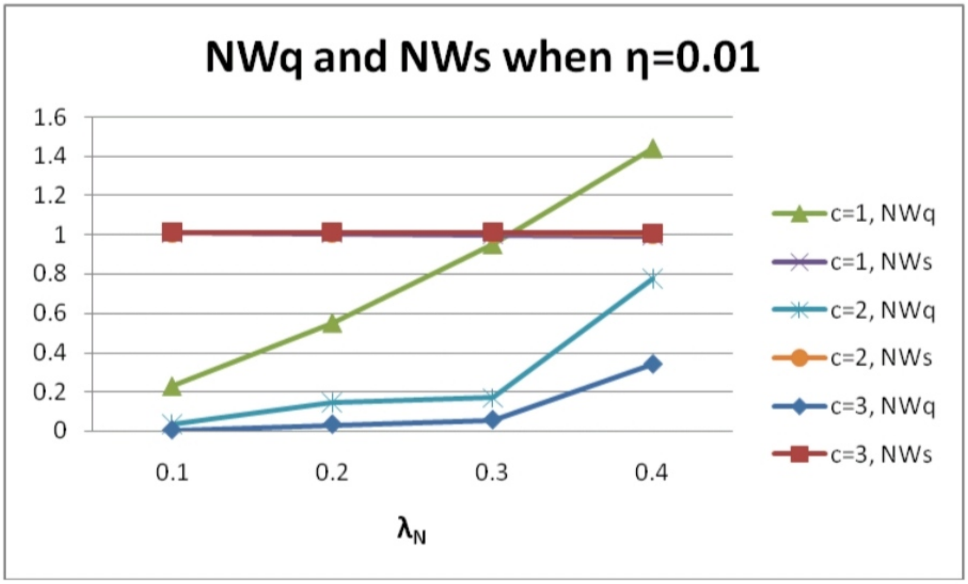

Figure 4. NW_q and NW_s when $\eta = 0.01$

In Fig 4, NW_q and NW_s values are plotted with the number of servers are taken as 1, 2 and 3. Here, we consider a parameter η which is considered as encouraged arrival parameter and it takes the value 0.01. At c=1, 2 and 3, the clients' wait times in the system remains unaltered throughout λ_N. And the waiting time of customers in the queue increases steadily with the increase in λ_N. NW_q at c=2, raises step by step which is effectual to get served besides increasing the server. It may give loss for the service provider.

In the tables above, the values of NW_s and NW_q are not negatives, its their membership values, and it is a single valued neutrosophic off numbers. Here the Neutrosophic M/M/c QM with En. A and heterogeneous service are calculated. When $\eta = 0.005$ and c=1, 2 and 3, the number of customers in the system and queue increases gradually with the increasing λ_N. Also the clients' wait times in the system shows some difference when λ_N increases and waiting time of customers in the queue steadily increases. Also, when $\eta = 0.01$ and c=1, 2 and 3, the number of customers in the system and queue increases steadily with the increasing λ_N. Similarly, the clients' wait times in the system shows slight variation when λ_N increases and the customers in line are waiting longer and longer. As a result of the outcome and line graph, it is easy to suggest that it is sufficient to provide 2 servers for the effective service of customers. Increasing the servers may give loss for the service provider when the arrival rate increases.

4. Result

For single valued neutrosophic M/M/c with encouraged arrival and heterogeneous service rate queuing system, the performance measures with respect to varied arrival and service rates are calculated. Numerical examples of the model are tabulated. In the example, we only know their degrees of membership, so that we assume single valued neutrosophic number to calculate it. Here in the neutrosophic model, both the arrival and service rates depends on Truth, Indeterminacy and False membership functions. They are not considered as valued, but its a membership functions. Also, observed numerical values are examined with suitable line graph. With the obtained result and line graph, it is easy to suggest that it is sufficient to provide **2 servers** for the effective service of customers since they provide effective services and thereby yielding maximum profit. Increasing the servers may give loss for the service provider when the arrival rate increases. At heterogeneous services (i.e) in a varied rate, there may occur a sudden destruction and a customer may balk from the system. From this analysis, we find that even in these conditions, the service provider can maintain the same servers, and there is no need to increase the servers as per our model.

5. Conclusion

M/M/c queuing model with Neutrosophic abstraction with encouraged arrivals and heterogeneous service are depicted here. The description studied here shows that the neutrosophic M/M/c QM with En. A and heterogeneous services can be dealt with uncertain and imprecise cases. It can be further developed with other classical queuing models, also it can be extended to the three types of neutrosophic sets which are under, off, and over respectively. Also the system can be analyzed with different other situations.

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e-Open Maps, e-Closed Maps and e-Homeomorphisms in N-Neutrosophic Crisp Topological Spaces

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Abstract. The concept of $N_{nc}eO$ and $N_{nc}eC$ mappings in $N_{nc}ts$ are introduced and studied some of their related properties in this article. In addition, $N_{nc}eHom$, $N_{nc}eCHom$ and $N_{nc}eT_{\frac{1}{2}}$-space in $N_{nc}ts$ are discussed and establishes some of their related characterizations.

Keywords: $N_{nc}e$-open map, $N_{nc}e$-closed map, $N_{nc}eT_{\frac{1}{2}}$-space, $N_{nc}e$-homeomorphism, $N_{nc}e$-C homeomorphism.

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1. Introduction

Smarandache [14] defined the neutrosophic set on three component neutrosophic sets (T-Truth, F-Falsehood, I-Indeterminacy). Lellis Thivagar et al. [11] was the first given the geometric existence of N topology and in his paper [10] introduced the notion of N_n-open (closed) sets and N_n continuous in N-neutrosophic topological spaces. The concept of N-neutrosophic crisp topological spaces from neutrosophic crisp topological spaces was first explored and investigated by Al-Hamido [1]. As a generalization of closed sets, e-closed sets were introduced and studied by Ekici [7–9]. In 2020, Vadivel and Sundar introduced the concept of N_{nc} γ-open [15], N_{nc} β-open [16] and N_{nc} δ-open sets [18] and their continuous functions [17, 20, 28] and open mappings [19, 21, 22]. The new N_{nc} open sets called N_{nc} e-open sets and its continuous functions are introduced in $N_{nc}ts$ by Vadivel et al. [23–27]. Recently, Das et al. [2–6] introduced b-open sets in different types of neutrosophic topological spaces. In this paper, $N_{nc}e$ open mapping, $N_{nc}e$ closed mapping, $N_{nc}e$ homeomorphism and $N_{nc}e$-C homeomorphism are introduced and some results in $N_{nc}ts$.

2. Preliminaries

Definition 2.1. [13] Let X be a non-empty set. Then F is called a neutrosophic crisp set (in short, ncs) in X if F has the form $F = (F_{01}, F_{02}, F_{03})$, where F_{01}, F_{02}, and F_{03} are subsets of X, then neutrosophic crisp set of types

(i) $F_{01} \cap F_{02} = F_{02} \cap F_{03} = F_{03} \cap F_{01} = \phi$

(ii) $F_{01} \cap F_{02} = F_{02} \cap F_{03} = F_{03} \cap F_{01} = \phi$ and $F_{01} \cup F_{02} \cup F_{03} = X$

(iii) $F_{01} \cap F_{02} \cap F_{03} = \phi$ and $F_{01} \cup F_{02} \cup F_{03} = X$

Definition 2.2. [13] Let $F = (F_{01}, F_{02}, F_{03}), G = (G_{01}, G_{02}, G_{03}) \in ncs(X)$. Then

(i) $\phi_n = (\phi, \phi, X)$,

(ii) $X_n = (X, X, \phi)$,

(iii) $F \subseteq G$, if $F_{01} \subseteq G_{01}, F_{02} \subseteq G_{02}$ and $F_{03} \supseteq G_{03}$.

(iv) $F = G$, if $F \subseteq G$ and $F \subseteq H$

(v) $F^c = (F_{03}, F_{02}^c, F_{01})$

(vi) $F \cap G = (F_{01} \cap G_{01}, F_{02} \cap G_{02}, F_{03} \cup G_{03})$

(vii) $F \cup G = (F_{01} \cup G_{01}, F_{02} \cup G_{02}, F_{03} \cap G_{03})$.

Definition 2.3. [12] A neutrosophic crisp topology (briefly, nct) on a non-empty set X is a family Γ of nc subsets of X satisfying the following axioms

(i) $\phi_n,\ X_n \in \Gamma$.

(ii) $F_1 \cap F_2 \in \Gamma\ \forall\ F_1\ \&\ F_2 \in \Gamma$.

(iii) $\bigcup_b F_b \in \Gamma$, for any $\{F_b : b \in K\} \subseteq \Gamma$.

Then (X, Γ) is a neutrosophic crisp topological space (briefly, $ncts$) in X. The Γ elements are called neutrosophic crisp open sets (briefly, $ncos$) in X and its complement is called neutrosophic crisp closed set (briefly, $nccs$).

Definition 2.4. [1] Let X be a non-empty set. Then ${}_{nc}\Psi_1, {}_{nc}\Psi_2, \cdots, {}_{nc}\Psi_N$ are N-arbitrary crisp topologies defined on X and the collection $N_{nc}\Psi = \{B \subseteq X : B = (\bigcup_{k=1}^{N} F_k) \cup (\bigcap_{k=1}^{N} L_k)$, $F_k, L_k \in {}_{nc}\Psi_k\}$ is called N_{nc}-topology on X if the axioms are satisfied:

(i) $\phi_n,\ X_n \in N_{nc}\Psi$.

(ii) $\bigcup_{k=1}^{\infty} K_k \in N_{nc}\Psi\ \forall\ \{K_k\}_{k=1}^{\infty} \in N_{nc}\Psi$.

(iii) $\bigcap_{k=1}^{n} K_k \in N_{nc}\Psi \ \forall \ \{K_k\}_{k=1}^{n} \in N_{nc}\Psi$.

Then $(X, N_{nc}\Psi)$ is called a N_{nc}-topological space (briefly, $N_{nc}ts$) on X. The $N_{nc}\Psi$ elements are called N_{nc}-open sets ($N_{nc}os$) on X and its complement is called N_{nc}-closed sets ($N_{nc}cs$) on X. The elements of X are known as N_{nc}-sets ($N_{nc}s$) on X.

Definition 2.5. [1,18] Let $(X, N_{nc}\Psi)$ be $N_{nc}ts$ on X and F be a $N_{nc}s$ on X, then the N_{nc} interior of F (briefly, $N_{nc}int(F)$), N_{nc} closure of F (briefly, $N_{nc}cl(F)$), $N_{nc}\delta$ interior of F (briefly, $N_{nc}\delta int(F)$) and $N_{nc}\delta$ closure of F (briefly, $N_{nc}\delta cl(F)$) are defined as

$$N_{nc}int(F) = \cup\{C : C \subseteq F \ \& \ C \text{ is a } N_{nc}os \text{ in } X\}$$

$$N_{nc}cl(F) = \cap\{D : F \subseteq D \ \& \ D \text{ is a } N_{nc}cs \text{ in } X\}$$

$$N_{nc}\delta int(F) = \cup\{C : C \subseteq F \ \& \ C \text{ is a } N_{nc}ros \text{ in } X\}$$

$$N_{nc}\delta cl(F) = \cap\{D : F \subseteq D \ \& \ D \text{ is a } N_{nc}rcs \text{ in } X\}.$$

Definition 2.6. [1,15,18,26,28] Let $(X, N_{nc}\Gamma)$ be any $N_{nc}ts$. Let F be a $N_{nc}s$ in $(X, N_{nc}\Psi)$. Then F is said to be a

(i) N_{nc}-regular (resp. N_{nc}-semi, N_{nc}-pre, N_{nc}-α & N_{nc}-β) open set (briefly, $N_{nc}ros$ (resp. $N_{nc}\mathcal{S}os$, $N_{nc}\mathcal{P}os$, $N_{nc}\alpha os$ & $N_{nc}\beta os$)) if $F = N_{nc}int(N_{nc}cl(F))$ (resp. $F \subseteq N_{nc}cl(N_{nc}int(F))$, $F \subseteq N_{nc}int(N_{nc}cl(F))$, $F \subseteq N_{nc}int(N_{nc}cl(N_{nc}int(F)))$ & $F \subseteq N_{nc}cl(N_{nc}int(N_{nc}cl(F))))$.

(ii) $N_{nc}\delta$ (resp. $N_{nc}\delta$-pre, $N_{nc}\delta$-semi & $N_{nc}e$) open set (briefly, $N_{nc}\delta os$ (resp. $N_{nc}\delta\mathcal{P}os$, $N_{nc}\delta\mathcal{S}os$ & $N_{nc}eos$)) if $F = N_{nc}\delta int(F)$ (resp. $F \subseteq N_{nc}int(N_{nc}\delta cl(F))$, $F \subseteq N_{nc}cl(N_{nc}\delta int(F))$ & $F \subseteq N_{nc}cl(N_{nc}\delta int(F)) \cup N_{nc}int(N_{nc}\delta cl(F)))$.

Definition 2.7. [10,19,21,22,27] Let $(X_1, N_{nc}\Psi)$ and $(X_2, N_{nc}\tau)$ be any two $N_{nc}ts$'s. A map $\zeta : (X_1, N_{nc}\Psi) \to (X_2, N_{nc}\tau)$ is said to be

(i) N_{nc} (resp. $N_{nc}\alpha$, N_{nc} semi, N_{nc} pre, $N_{nc}\gamma$, $N_{nc}\beta$, $N_{nc}\delta$, $N_{nc}\delta$ semi & $N_{nc}\delta$ pre)-open mapping (briefly, $N_{nc}O$ (resp. $N_{nc}\alpha O$, $N_{nc}\mathcal{S}O$, $N_{nc}\mathcal{P}O$, $N_{nc}\gamma O$, $N_{nc}\beta O$, $N_{nc}\delta O$, $N_{nc}\delta\mathcal{S}O$ & $N_{nc}\delta\mathcal{P}O$) if the inverse image of every $N_{nc}os$ in $(X_1, N_{nc}\Psi)$ is a $N_{nc}\alpha os$ (resp. $N_{nc}\mathcal{S}os$, $N_{nc}\mathcal{P}os$, $N_{nc}\gamma os$, $N_{nc}\beta os$, $N_{nc}\delta os$, $N_{nc}\delta\mathcal{S}os$ & $N_{nc}\delta\mathcal{P}os$) in $(X_2, N_{nc}\tau)$.

(ii) N_{nc} (resp. $N_{nc}\alpha$, N_{nc} semi, N_{nc} pre, $N_{nc}\gamma$, $N_{nc}\beta$, $N_{nc}\delta$, $N_{nc}\delta$ semi & $N_{nc}\delta$ pre)-closed mapping (briefly, $N_{nc}C$ (resp. $N_{nc}\alpha C$, $N_{nc}\mathcal{S}C$, $N_{nc}\mathcal{P}C$, $N_{nc}\gamma C$, $N_{nc}\beta C$, $N_{nc}\delta C$, $N_{nc}\delta\mathcal{S}C$ & $N_{nc}\delta\mathcal{P}C$)) if the inverse image of every $N_{nc}cs$ in $(X_1, N_{nc}\Psi)$ is a $N_{nc}\alpha cs$ (resp. $N_{nc}\mathcal{S}cs$, $N_{nc}\mathcal{P}cs$, $N_{nc}\gamma cs$, $N_{nc}\beta cs$, $N_{nc}\delta cs$, $N_{nc}\delta\mathcal{S}cs$ & $N_{nc}\delta\mathcal{P}cs$) in $(X_2, N_{nc}\tau)$.

(iii) N_{nc} (resp. $N_{nc}e$)-continuous (briefly, $N_{nc}Cts$ (resp. $N_{nc}eCts$)) if the inverse image of every $N_{nc}os$ in $(X_2, N_{nc}\tau)$ is a $N_{nc}os$ (resp. $N_{nc}eos$) in $(X_1, N_{nc}\Psi)$.

(iv) N_{nc}-homeomorphism (briefly, $N_{nc}Hom$) if ζ & ζ^{-1} are $N_{nc}Cts$.

Throughout this article, let $(X_1, N_{nc}\Psi)$, $(X_2, N_{nc}\tau)$ and $(X_3, N_{nc}\rho)$ are $N_{nc}ts$'s and $\zeta : (X_1, N_{nc}\Psi) \to (X_2, N_{nc}\tau)$ and $\eta : (X_2, N_{nc}\tau) \to (X_3, N_{nc}\rho)$ are mappings.

3. N-Neutrosophic crisp e-open mapping

Definition 3.1. A mapping ζ is N-neutrosophic crisp e-open (briefly, $N_{nc}eO$) if image of every $N_{nc}eos$ of $(X_1, N_{nc}\Psi)$ is $N_{nc}eos$ in $(X_2, N_{nc}\tau)$.

Theorem 3.2. Let ζ be a function. Then Every

(i) $N_{nc}O$ is a $N_{nc}\alpha O$.
(ii) $N_{nc}\alpha O$ is a $N_{nc}\mathcal{P}O$.
(iii) $N_{nc}\mathcal{P}O$ is a $N_{nc}\gamma O$.
(iv) $N_{nc}\gamma O$ is a $N_{nc}\beta O$.
(v) $N_{nc}\delta O$ is a $N_{nc}O$.
(vi) $N_{nc}\delta O$ is a $N_{nc}\mathcal{S}O$.
(vii) $N_{nc}\delta\mathcal{S}O$ is a $N_{nc}eO$.
(viii) $N_{nc}\mathcal{P}O$ is a $N_{nc}\delta\mathcal{P}O$.
(ix) $N_{nc}\delta\mathcal{P}O$ is a $N_{nc}eO$.
(x) $N_{nc}eO$ is a $N_{nc}\beta O$.

Proof. Proof of (i) to (iii), (iv) and (v) to (vi) are proved in [19], [21] and [22]. We prove only (vii) to (ix).

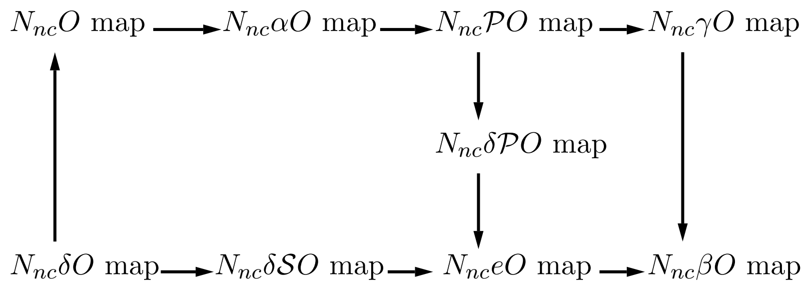
(vii) Let ζ be a $N_{nc}\delta\mathcal{S}O$ mapping and K is a $N_{nc}os$ in X_1. Then $\zeta(K)$ is $N_{nc}\delta\mathcal{S}os$ in X_2. Since every $N_{nc}\delta\mathcal{S}os$ is $N_{nc}eos$ by Proposition 3.1 in [26], $\zeta(K)$ is $N_{nc}eos$ in X_2. Therefore ζ is $N_{nc}eO$ mapping.

(viii) Let ζ be a $N_{nc}\mathcal{P}O$ mapping and K is a $N_{nc}os$ in X_1. Then $\zeta(K)$ is $N_{nc}\mathcal{P}os$ in X_2. Since every $N_{nc}\mathcal{P}os$ is $N_{nc}\delta\mathcal{P}os$ by Proposition 3.1 in [26], $\zeta(K)$ is $N_{nc}\delta\mathcal{P}os$ in X_2. Therefore ζ is $N_{nc}\delta\mathcal{P}O$ mapping.

(ix) Let ζ be a $N_{nc}\delta\mathcal{P}O$ mapping and K is a $N_{nc}os$ in X_1. Then $\zeta(K)$ is $N_{nc}\delta\mathcal{P}os$ in X_2. Since every $N_{nc}\delta\mathcal{P}os$ is $N_{nc}eos$ by Proposition 3.1 in [26], $\zeta(K)$ is $N_{nc}eos$ in X_2. Therefore ζ is $N_{nc}eO$ mapping.

(x) Let ζ be a $N_{nc}eO$ mapping and K is a $N_{nc}os$ in X_1. Then $\zeta(K)$ is $N_{nc}eos$ in X_2. Since every $N_{nc}eos$ is $N_{nc}\beta os$ by Proposition 3.1 in [26], $\zeta(K)$ is $N_{nc}\beta os$ in X_2. Therefore ζ is $N_{nc}\beta O$ mapping. □

Remark 3.3. The following diagram shows $N_{nc}eO$ mapping function in $N_{nc}ts$.



None of these implication is reversible as shown in the following examples.

Example 3.4. Let $X = \{a_o, b_o, c_o, d_o, e_o\} = Y$, ${}_{nc}\Psi_1 = \{\phi_n, X_n, A_o\}$, ${}_{nc}\Psi_2 = \{\phi_n, X_n\}$. $A_o = \langle\{a_o\}, \{\phi\}, \{b_o, c_o, d_o, e_o\}\rangle$, then $2_{nc}\Psi = \{\phi_n, X_n, A_o\}$. Let ${}_{nc}\tau_1 = \{\phi_n, Y_n, B_o, C_o, D_o\}$, ${}_{nc}\tau_2 = \{\phi_n, Y_n\}$. $B_o = \langle\{c_o\}, \{\phi\}, \{a_o, b_o, d_o, e_o\}\rangle$, $C_o = \langle\{a_o, b_o\}, \{\phi\}, \{c_o, d_o, e_o\}\rangle$, $D_o = \langle\{a_o, b_o, c_o\}, \{\phi\}, \{d_o, e_o\}\rangle$, then $2_{nc}\tau = \{\phi_n, Y_n, B_o, C_o, D_o\}$. Define $\zeta : (X, 2_{nc}\Psi) \to (Y, 2_{nc}\tau)$ as identity map, then $2_{nc}eO$ map but not $2_{nc}\delta\mathcal{S}O$ map, then $\zeta(\langle\{a_o\}, \{\phi\}, \{b_o, c_o, d_o, e_o\}\rangle) = \langle\{a_o\}, \{\phi\}, \{b_o, c_o, d_o, e_o\}\rangle$ is a $2_{nc}eos$ but not $2_{nc}\delta\mathcal{S}os$ in Y.

Example 3.5. Let $X = \{a_o, b_o, c_o, d_o, e_o\} = Y$, ${}_{nc}\Psi_1 = \{\phi_n, X_n, A_o\}$, ${}_{nc}\Psi_2 = \{\phi_n, X_n\}$. $A_o = \langle\{c_o, d_o\}, \{\phi\}, \{a_o, b_o, e_o\}\rangle$, then $2_{nc}\Psi = \{\phi_n, X_n, A_o\}$. Let ${}_{nc}\tau_1 = \{\phi_n, Y_n, B_o, C_o, D_o\}$, ${}_{nc}\tau_2 = \{\phi_n, Y_n\}$. $B_o = \langle\{c_o\}, \{\phi\}, \{a_o, b_o, d_o, e_o\}\rangle$, $C_o = \langle\{a_o, b_o\}, \{\phi\}, \{c_o, d_o, e_o\}\rangle$, $D_o = \langle\{a_o, b_o, c_o\}, \{\phi\}, \{d_o, e_o\}\rangle$, then $2_{nc}\tau = \{\phi_n, Y_n, B_o, C_o, D_o\}$. Define $\zeta : (X, 2_{nc}\Psi) \to (Y, 2_{nc}\tau)$ as identity map, then $2_{nc}eO$ map but not $2_{nc}\delta\mathcal{P}O$ map, then $\zeta(\langle\{c_o, d_o\}, \{\phi\}, \{a_o, b_o, e_o\}\rangle) = \langle\{c_o, d_o\}, \{\phi\}, \{a_o, b_o, e_o\}\rangle$ is a $2_{nc}eos$ but not $2_{nc}\delta\mathcal{P}os$ in Y.

Example 3.6. Let $X = \{a_o, b_o, c_o, d_o, e_o\} = Y$, ${}_{nc}\Psi_1 = \{\phi_n, X_n, A_o\}$, ${}_{nc}\Psi_2 = \{\phi_n, X_n\}$. $A_o = \langle\{a_o, d_o\}, \{\phi\}, \{b_o, c_o, e_o\}\rangle$, then $2_{nc}\Psi = \{\phi_n, X_n, A_o\}$. Let ${}_{nc}\tau_1 = \{\phi_n, Y_n, B_o, C_o, D_o\}$, ${}_{nc}\tau_2 = \{\phi_n, Y_n\}$. $B_o = \langle\{c_o\}, \{\phi\}, \{a_o, b_o, d_o, e_o\}\rangle$, $C_o = \langle\{a_o, b_o\}, \{\phi\}, \{c_o, d_o, e_o\}\rangle$, $D_o = \langle\{a_o, b_o, c_o\}, \{\phi\}, \{d_o, e_o\}\rangle$, then $2_{nc}\tau = \{\phi_n, Y_n, B_o, C_o, D_o\}$. Define $\zeta : (X, 2_{nc}\Psi) \to (Y, 2_{nc}\tau)$ as identity map, then $2_{nc}\beta O$ map but not $2_{nc}eO$ map, then $\zeta(\langle\{a_o, d_o\}, \{\phi\}, \{b_o, c_o, e_o\}\rangle) = \langle\{a_o, d_o\}, \{\phi\}, \{b_o, c_o, e_o\}\rangle$ is a $2_{nc}\beta os$ but not $2_{nc}eos$ in Y.

Theorem 3.7. A mapping $\zeta : (X_1, N_{nc}\Psi) \to (X_2, N_{nc}\tau)$ is $N_{nc}eO$ iff for every $N_{nc}s$ φ of $(X_1, N_{nc}\Psi)$, $\zeta(N_{nc}int(\varphi)) \subseteq N_{nc}eint(\zeta(\varphi))$.

Proof. Necessity: Let ζ be a $N_{nc}eO$ & φ be a $N_{nc}os$ in $(X_1, N_{nc}\Psi)$. Now, $N_{nc}int(\varphi) \subseteq \varphi$ implies $\zeta(N_{nc}int(\varphi)) \subseteq \zeta(\varphi)$. Since ζ is a $N_{nc}eO$, $\zeta(N_{nc}int(\varphi))$ is $N_{nc}eos$ in $(X_2, N_{nc}\tau)$ such that $\zeta(N_{nc}int(\varphi)) \subseteq \zeta(\varphi)$ therefore $\zeta(N_{nc}int(\varphi)) \subseteq N_{nc}eint(\zeta(\varphi))$.

Sufficiency: Assume φ is a $N_{nc}os$ of $(X_1, N_{nc}\Psi)$. Then $\zeta(\varphi) = \zeta(N_{nc}int(\varphi)) \subseteq N_{nc}eint(\zeta(\varphi))$. But $N_{nc}eint(\zeta(\varphi)) \subseteq \zeta(\varphi)$. So $\zeta(\varphi) = N_{nc}eint(\varphi)$ which implies $\zeta(\varphi)$ is a $N_{nc}eos$ of $(X_2, N_{nc}\tau)$ and hence ζ is a $N_{nc}eO$. □

Theorem 3.8. If $\zeta : (X_1, N_{nc}\Psi) \to (X_2, N_{nc}\tau)$ is a $N_{nc}eO$ mapping then $N_{nc}int(\zeta^{-1}(\lambda)) \subseteq \zeta^{-1}(N_{nc}eint(\lambda))$ for every $N_{nc}s$ λ of $(X_2, N_{nc}\tau)$.

Proof. Let λ be a $N_{nc}s$ of $(X_2, N_{nc}\tau)$. Then $N_{nc}int(\zeta^{-1}(\lambda))$ is a $N_{nc}os$ in $(X_1, N_{nc}\Psi)$. Since ζ is $N_{nc}eO$, $\zeta(N_{nc}int(\zeta^{-1}(\lambda))$ is $N_{nc}eo$ in $(X_2, N_{nc}\tau)$ and hence $\zeta(N_{nc}int(\zeta^{-1}(\lambda))) \subseteq N_{nc}eint(\zeta(\zeta^{-1}(\lambda))) \subseteq N_{nc}eint(\lambda)$. Thus $N_{nc}int(\zeta^{-1}(\lambda)) \subseteq \zeta^{-1}(N_{nc}eint(\lambda))$. □

Theorem 3.9. A mapping $\zeta : (X_1, N_{nc}\Psi) \to (X_2, N_{nc}\tau)$ is $N_{nc}eO$ iff for each $N_{nc}s$ μ of $(X_2, N_{nc}\tau)$ and for each $N_{nc}cs$ ρ of $(X_1, N_{nc}\Psi)$ containing $\zeta^{-1}(\mu)$ there is a $N_{nc}ecs$ μ of $(X_2, N_{nc}\tau) \ni \mu \subseteq \rho$ & $\zeta^{-1}(\mu) \subseteq \rho$.

Proof. Necessity: Assume ζ is a $N_{nc}eO$. Let μ be the $N_{nc}cs$ of $(X_2, N_{nc}\tau)$ & ρ is a $N_{nc}cs$ of $(X_1, N_{nc}\Psi) \ni \zeta^{-1}(\mu) \subseteq \rho$. Then $\mu = (\zeta^{-1}(\rho^c))^c$ is $N_{nc}ecs$ of $(X_2, N_{nc}\tau) \ni \zeta^{-1}(\mu) \subseteq \rho$.

Sufficiency: Assume ν is a $N_{nc}os$ of $(X_1, N_{nc}\Psi)$. Then $\zeta^{-1}((\zeta(\nu))^c \subseteq \nu^c$ & ν^c is $N_{nc}cs$ in $(X_1, N_{nc}\Psi)$. By hypothesis there is a $N_{nc}ecs$ μ of $(X_2, N_{nc}\tau) \ni (\zeta(\nu))^c \subseteq \mu$ & $\zeta^{-1}(\mu) \subseteq \nu^c$. Therefore $\nu \subseteq (\zeta^{-1}(\mu))^c$. Hence $\mu^c \subseteq \zeta(\nu) \subseteq \zeta((\zeta^{-1}(\mu))^c) \subseteq \mu^c$ which implies $\zeta(\nu) = \mu^c$. Since μ^c is $N_{nc}eos$ of $(X_2, N_{nc}\tau)$. Hence $\zeta(\nu)$ is $N_{nc}eo$ in $(X_2, N_{nc}\tau)$ and thus ζ is $N_{nc}eO$. □

Theorem 3.10. A mapping $\zeta : (X_1, N_{nc}\Psi) \to (X_2, N_{nc}\tau)$ is $N_{nc}eO$ iff $\zeta^{-1}(N_{nc}ecl(\rho)) \subseteq N_{nc}cl(\zeta^{-1}(\rho))$ for every $N_{nc}s$ ρ of $(X_2, N_{nc}\tau)$.

Proof. Necessity: Assume ζ is a $N_{nc}eO$. For any $N_{nc}s$ ρ of $(X_2, N_{nc}\tau)$, $\zeta^{-1}(\rho) \subseteq N_{nc}cl(\zeta^{-1}(\rho))$. Therefore by Theorem 3.9 there exists a $N_{nc}ecs$ μ in $(X_2, N_{nc}\tau) \ni \rho \subseteq \mu$ & $\zeta^{-1}(\mu) \subseteq N_{nc}cl(\zeta^{-1}(\rho))$. Therefore we obtain that $\zeta^{-1}(N_{nc}ecl(\rho)) \subseteq \zeta^{-1}(\mu) \subseteq N_{nc}cl(\zeta^{-1}(\rho))$.

Sufficiency: Assume ρ is a $N_{nc}s$ of $(X_2, N_{nc}\tau)$ & μ is a $N_{nc}cs$ of $(X_1, N_{nc}\Psi)$ containing $\zeta^{-1}(\rho)$. Put $\alpha = N_{nc}cl(\rho)$, then $\rho \subseteq \alpha$ and α is $N_{nc}ec$ & $\zeta^{-1}(\alpha) \subsetneq N_{nc}cl(\zeta^{-1}(\rho)) \subseteq \mu$. Then by Theorem 3.9, ζ is $N_{nc}eO$. □

Theorem 3.11. If ζ & η be two neutrosophic crisp mappings and $\eta \circ \zeta : (X_1, N_{nc}\Psi) \to (X_3, N_{nc}\rho)$ is $N_{nc}eO$. If $\eta : (X_2, N_{nc}\tau) \to (X_3, N_{nc}\rho)$ is $N_{nc}eIrr$ then $\zeta : (X_1, N_{nc}\Psi) \to (X_2, N_{nc}\tau)$ is $N_{nc}eO$ mapping.

Proof. Let μ be a $N_{nc}os$ in $(X_1, N_{nc}\Psi)$. Then $\eta \circ \zeta(\mu)$ is $N_{nc}eos$ of $(X_3, N_{nc}\rho)$ because $\eta \circ \zeta$ is $N_{nc}eO$. Since η is $N_{nc}eIrr$ & $\eta \circ \zeta(\mu)$ is $N_{nc}eos$ of $(X_3, N_{nc}\rho)$ therefore $\eta^{-1}(\eta \circ \zeta(\mu)) = \zeta(\mu)$ is $N_{nc}eos$ in $(X_2, N_{nc}\tau)$. Hence ζ is $N_{nc}eO$. □

Theorem 3.12. If ζ is $N_{nc}O$ and η is $N_{nc}eO$ mappings then $\eta \circ \zeta : (X_1, N_{nc}\Psi) \to (X_3, N_{nc}\rho)$ is $N_{nc}eO$.

Proof. Let μ be a $N_{nc}os$ in $(X_1, N_{nc}\Psi)$. Then $\zeta(\mu)$ is a $N_{nc}os$ of $(X_2, N_{nc}\tau)$ because ζ is a $N_{nc}O$. Since η is $N_{nc}eO$, $\eta(\zeta(\mu)) = (\eta \circ \zeta)(\mu)$ is $N_{nc}eos$ of $(X_3, N_{nc}\rho)$. Hence $\eta \circ \zeta$ is $N_{nc}eO$. □

4. N-Neutrosophic crisp e-closed mapping

Definition 4.1. A mapping $\zeta : (X_1, N_{nc}\Psi) \to (X_2, N_{nc}\tau)$ is N-neutrosophic crisp e-closed (briefly, $N_{nc}eC$) if image of every $N_{nc}cs$ of $(X_1, N_{nc}\Psi)$ is $N_{nc}ecs$ in $(X_2, N_{nc}\tau)$.

Theorem 4.2. Let ζ be a function. Then Every

(i) $N_{nc}C$ is a $N_{nc}\alpha C$.
(ii) $N_{nc}\alpha C$ is a $N_{nc}\mathcal{P}C$.
(iii) $N_{nc}\mathcal{P}C$ is a $N_{nc}\gamma C$.
(iv) $N_{nc}\gamma C$ is a $N_{nc}\beta C$.
(v) $N_{nc}\delta C$ is a $N_{nc}C$.
(vi) $N_{nc}\delta C$ is a $N_{nc}\mathcal{S}C$.
(vii) $N_{nc}\delta\mathcal{S}C$ is a $N_{nc}eC$.
(viii) $N_{nc}\mathcal{P}C$ is a $N_{nc}\delta\mathcal{P}C$.
(ix) $N_{nc}\delta\mathcal{P}C$ is a $N_{nc}eC$.
(x) $N_{nc}eC$ is a $N_{nc}\beta C$.

Proof. Proof of (i) to (iii), (iv) and (v) to (vi) are proved in [19], [21] and [22]. We prove only (vii) to (ix).

(vii) Let ζ be a $N_{nc}\delta\mathcal{S}C$ mapping and K is a $N_{nc}cs$ in X_1. Then $\zeta(K)$ is $N_{nc}\delta\mathcal{S}cs$ in X_2. Since every $N_{nc}\delta\mathcal{S}cs$ is $N_{nc}ecs$ by Proposition 3.1 in [26], $\zeta(K)$ is $N_{nc}ecs$ in X_2. Therefore ζ is $N_{nc}eC$ mapping.

(viii) Let ζ be a $N_{nc}\mathcal{P}C$ mapping and K is a $N_{nc}cs$ in X_1. Then $\zeta(K)$ is $N_{nc}\mathcal{P}cs$ in X_2. Since every $N_{nc}\mathcal{P}cs$ is $N_{nc}\delta\mathcal{P}cs$ by Proposition 3.1 in [26], $\zeta(K)$ is $N_{nc}\delta\mathcal{P}cs$ in X_2. Therefore ζ is $N_{nc}\delta\mathcal{P}C$ mapping.

(ix) Let ζ be a $N_{nc}\delta\mathcal{P}C$ mapping and K is a $N_{nc}cs$ in X_1. Then $\zeta(K)$ is $N_{nc}\delta\mathcal{P}cs$ in X_2. Since every $N_{nc}\delta\mathcal{P}cs$ is $N_{nc}ecs$ by Proposition 3.1 in [26], $\zeta(K)$ is $N_{nc}ecs$ in X_2. Therefore ζ is $N_{nc}eC$ mapping.

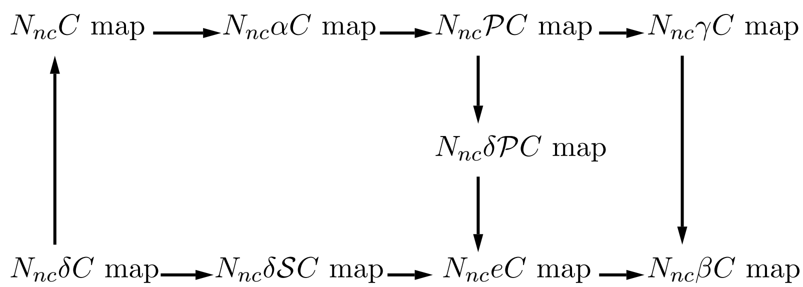
(x) Let ζ be a $N_{nc}eC$ mapping and K is a $N_{nc}cs$ in X_1. Then $\zeta(K)$ is $N_{nc}ecs$ in X_2. Since every $N_{nc}ecs$ is $N_{nc}\beta cs$ by Proposition 3.1 in [26], $\zeta(K)$ is $N_{nc}\beta cs$ in X_2. Therefore ζ is $N_{nc}\beta C$ mapping. □

Example 4.3. In Example 3.4, then $2_{nc}eC$ map but not $2_{nc}\delta\mathcal{S}C$ map, then $\zeta(\langle\{b_o, c_o, d_o, e_o\}, \{\phi\}, \{a_o\}\rangle) = \langle\{b_o, c_o, d_o, e_o\}, \{\phi\}, \{a_o\}\rangle$ is a $2_{nc}ecs$ but not $2_{nc}\delta\mathcal{S}cs$.

Example 4.4. In Example 3.5, then $2_{nc}eC$ map but not $2_{nc}\delta\mathcal{P}C$ map, then $\zeta(\langle\{a_o, b_o, e_o\}, \{\phi\}, \{c_o, d_o\}\rangle) = \langle\{a_o, b_o, e_o\}, \{\phi\}, \{c_o, d_o\}\rangle$ is a $2_{nc}ecs$ but not $2_{nc}\delta\mathcal{P}cs$.

Example 4.5. In Example 3.6, then $2_{nc}\beta C$ map but not $2_{nc}eC$ map, then $\zeta(\langle\{b_o, c_o, e_o\}, \{\phi\}, \{a_o, d_o\}\rangle) = \langle\{b_o, c_o, e_o\}, \{\phi\}, \{a_o, d_o\}\rangle$ is a $2_{nc}\beta cs$ but not $2_{nc}ecs$.

Remark 4.6. The following diagram shows $N_{nc}eC$ mapping function in $N_{nc}ts$.



None of these implication is reversible as shown in the following examples.

Theorem 4.7. A mapping $\zeta : (X_1, N_{nc}\Psi) \to (X_2, N_{nc}\tau)$ is $N_{nc}eC$ iff for each $N_{nc}s$ μ of $(X_2, N_{nc}\tau)$ and for each $N_{nc}os$ λ of $(X_1, N_{nc}\Psi)$ containing $\zeta^{-1}(\mu)$ there is a $N_{nc}eos$ ρ of $(X_2, N_{nc}\tau) \ni \mu \subseteq \rho$ & $\zeta^{-1}(\rho) \subseteq \lambda$.

Proof. Necessity: Assume ζ is a $N_{nc}eC$. Let μ be the $N_{nc}cs$ of $(X_2, N_{nc}\tau)$ & λ is a $N_{nc}os$ of $(X_1, N_{nc}\Psi) \ni \zeta^{-1}(\mu) \subseteq \lambda$. Then $\rho = X_2 - \zeta^{-1}(\lambda^c)$ is $N_{nc}eos$ of $(X_2, N_{nc}\tau) \ni \zeta^{-1}(\rho) \subseteq \lambda$.

Sufficiency: Assume ν is a $N_{nc}cs$ of $(X_1, N_{nc}\Psi)$. Then $(\zeta(\nu))^c$ is a $N_{nc}s$ of $(X_2, N_{nc}\tau)$ & ν^c is $N_{nc}os$ in $(X_1, N_{nc}\Psi) \ni \zeta^{-1}((\zeta(\nu))^c) \subseteq \nu^c$. By hypothesis there is a $N_{nc}eos$ ρ of $(X_2, N_{nc}\tau)$ $\ni (\zeta(\nu))^c \subseteq \rho$ & $\zeta^{-1}(\rho) \subseteq \nu^c$. Therefore $\nu \subseteq (\zeta^{-1}(\rho))^c$. Hence $\rho^c \subseteq \zeta(\rho) \subseteq \zeta((\zeta^{-1}(\rho))^c) \subseteq \rho^c$ which implies $\zeta(\nu) = \rho^c$. Since ρ^c is $N_{nc}ecs$ of $(X_2, N_{nc}\tau)$. Hence $\zeta(\nu)$ is $N_{nc}ec$ in $(X_2, N_{nc}\tau)$ and thus ζ is $N_{nc}eC$. □

Theorem 4.8. If ζ is $N_{nc}C$ & η is $N_{nc}eC$. Then $\eta \circ \zeta : (X_1, N_{nc}\Psi) \to (X_3, N_{nc}\rho)$ is $N_{nc}eC$.

Proof. Let μ be a $N_{nc}cs$ in $(X_1, N_{nc}\Psi)$. Then $\zeta(\mu)$ is $N_{nc}cs$ of $(X_2, N_{nc}\tau)$ because ζ is $N_{nc}C$. Now $(\eta \circ \zeta)(\mu) = \eta(\zeta(\mu))$ is $N_{nc}ecs$ in $(X_3, N_{nc}\rho)$ because η is $N_{nc}eC$. Thus $\eta \circ \zeta$ is $N_{nc}eC$. □

Theorem 4.9. If $\zeta : (X_1, N_{nc}\Psi) \to (X_2, N_{nc}\tau)$ is $N_{nc}eC$ map, then $N_{nc}ecl(\zeta(\rho)) \subsetneq \zeta(N_{nc}cl(\rho))$.

Theorem 4.10. Let ζ & η are $N_{nc}eC$ mappings. If every $N_{nc}ecs$ of $(X_2, N_{nc}\tau)$ is $N_{nc}c$ then, $\eta \circ \zeta : (X_1, N_{nc}\Psi) \to (X_3, N_{nc}\rho)$ is $N_{nc}eC$.

Proof. Let μ be a $N_{nc}cs$ in $(X_1, N_{nc}\Psi)$. Then $\zeta(\mu)$ is $N_{nc}ecs$ of $(X_2, N_{nc}\tau)$ because ζ is $N_{nc}eC$ mapping. By hypothesis $\zeta(\mu)$ is $N_{nc}cs$ of $(X_2, N_{nc}\tau)$. Now $\eta(\zeta(\mu)) = (\eta \circ \zeta)(\mu)$ is $N_{nc}ecs$ in $(X_3, N_{nc}\rho)$ because η is $N_{nc}eC$. Thus $\eta \circ \zeta$ is $N_{nc}eC$. □

Theorem 4.11. The following statements are equivalent for a mapping ζ:

(i) ζ is a $N_{nc}eO$.

(ii) ζ is a $N_{nc}eC$.

(iii) ζ^{-1} is $N_{nc}eCts$.

5. N-Neutrosophic crisp e-homeomorphism

Definition 5.1. A bijection ζ is called a $N_{nc}e$-homeomorphism (briefly $N_{nc}eHom$) if ζ & ζ^{-1} are $N_{nc}eCts$.

Theorem 5.2. Each $N_{nc}Hom$ is a $N_{nc}eHom$.

Proof. Let ζ be $N_{nc}Hom$, then ζ and ζ^{-1} are $N_{nc}Cts$. But every $N_{nc}Cts$ is $N_{nc}eCts$. Hence, ζ and ζ^{-1} is $N_{nc}eCts$. Therefore, ζ is a $N_{nc}eHom$. □

Theorem 5.3. Let ζ be a bijective mapping. The following statements are equivalent, if ζ is $N_{nc}eCts$:

(i) ζ is a $N_{nc}eC$.

(ii) ζ is a $N_{nc}eO$.

(iii) ζ^{-1} is a $N_{nc}eHom$.

Definition 5.4. A $N_{nc}ts$ $(X_1, N_{nc}\Psi)$ is said to be a neutrosophic crisp $eT_{\frac{1}{2}}$ (briefly, $N_{nc}eT_{\frac{1}{2}}$)-space if every $N_{nc}ecs$ is $N_{nc}c$ in $(X_1, N_{nc}\Psi)$.

Theorem 5.5. Let ζ be a $N_{nc}eHom$, then ζ is a $N_{nc}Hom$ if $(X_1, N_{nc}\Psi)$ and $(X_2, N_{nc}\tau)$ are $N_{nc}eT_{\frac{1}{2}}$-space.

Proof. Assume that μ is a $N_{nc}cs$ in $(X_2, N_{nc}\tau)$, then $\zeta^{-1}(\mu)$ is a $N_{nc}ecs$ in $(X_1, N_{nc}\Psi)$. Since $(X_1, N_{nc}\Psi)$ is a $N_{nc}eT_{\frac{1}{2}}$-space, $\zeta^{-1}(\mu)$ is a $N_{nc}cs$ in $(X_1, N_{nc}\Psi)$. Therefore, ζ is $N_{nc}Cts$. By hypothesis, ζ^{-1} is $N_{nc}eCts$. Let ν be a $N_{nc}cs$ in $(X_1, N_{nc}\Psi)$. Then, $(\zeta^{-1})^{-1}(\nu) = \zeta(\nu)$ is a $N_{nc}cs$ in $(X_2, N_{nc}\tau)$, by presumption. Since $(X_2, N_{nc}\tau)$ is a $N_{nc}eT_{\frac{1}{2}}$-space, $\zeta(\nu)$ is a $N_{nc}cs$ in $(X_2, N_{nc}\tau)$. Hence, ζ^{-1} is $N_{nc}Cts$. Hence, ζ is a $N_{nc}Hom$. □

Theorem 5.6. The following statements are equivalent for ζ, if $(X_2, N_{nc}\tau)$ is a $N_{nc}eT_{\frac{1}{2}}$-space:

(i) ζ is $N_{nc}eC$.

(ii) If μ is a $N_{nc}os$ in $(X_1, N_{nc}\Psi)$, then $\zeta(\mu)$ is $N_{nc}eos$ in $(X_2, N_{nc}\tau)$.

(iii) $\zeta(N_{nc}int(\mu)) \subseteq N_{nc}cl(N_{nc}int(\zeta(\mu)))$ for every $N_{nc}s$ μ in $(X_1, N_{nc}\Psi)$.

Proof. (i) $\Rightarrow$ (ii): Obvious.

(ii) $\Rightarrow$ (iii): Let μ be a $N_{nc}s$ in $(X_1, N_{nc}\Psi)$. Then, $N_{nc}int(\mu)$ is a $N_{nc}os$ in $(X_1, N_{nc}\Psi)$. Then, $\zeta(N_{nc}int(\mu))$ is a $N_{nc}eos$ in $(X_2, N_{nc}\tau)$. Since $(X_2, N_{nc}\tau)$ is a $N_{nc}eT_{\frac{1}{2}}$-space, so $\zeta(N_{nc}int(\mu))$ is a $N_{nc}os$ in $(X_2, N_{nc}\tau)$. Therefore, $\zeta(N_{nc}int(\mu)) = N_{nc}int(\zeta(N_{nc}int(\mu))) \subseteq N_{nc}cl(N_{nc}int(\zeta(\mu)))$.

(iii) $\Rightarrow$ (i): Let μ be a $N_{nc}cs$ in $(X_1, N_{nc}\Psi)$. Then, μ^c is a $N_{nc}os$ in $(X_1, N_{nc}\Psi)$. From, $\zeta(N_{nc}int(\mu^c)) \subseteq N_{nc}cl(N_{nc}int(\zeta(\mu^c)))$. Hence, $\zeta(\mu^c) \subseteq N_{nc}cl(N_{nc}int(\zeta(\mu^c)))$. Therefore, $\zeta(\mu^c)$ is $N_{nc}eos$ in $(X_2, N_{nc}\tau)$. Therefore, $\zeta(\mu)$ is a $N_{nc}ecs$ in $(X_1, N_{nc}\Psi)$. Hence, ζ is a $N_{nc}C$. □

Theorem 5.7. Let ζ & η be $N_{nc}eC$, where $(X_1, N_{nc}\Psi)$ and $(X_3, N_{nc}\rho)$ are two $N_{nc}ts$'s and $(X_2, N_{nc}\tau)$ a $N_{nc}eT_{\frac{1}{2}}$-space, then the composition $\eta \circ \zeta$ is $N_{nc}eC$.

Proof. Let μ be a $N_{nc}cs$ in $(X_1, N_{nc}\Psi)$. Since ζ is $N_{nc}ec$ & $\zeta(\mu)$ is a $N_{nc}ecs$ in $(X_2, N_{nc}\tau)$, by assumption, $\zeta(\mu)$ is a $N_{nc}cs$ in $(X_2, N_{nc}\tau)$. Since η is $N_{nc}ec$, then $\eta(\zeta(\mu))$ is $N_{nc}ec$ in $(X_3, N_{nc}\rho)$ & $\eta(\zeta(\mu)) = (\eta \circ \zeta)(\mu)$. Therefore, $\eta \circ \zeta$ is $N_{nc}eC$. □

Theorem 5.8. The following statements are hold for ζ & η:

(i) If $\eta \circ \zeta$ is $N_{nc}eO$ & ζ is $N_{nc}Cts$, then η is $N_{nc}eO$.

(ii) If $\eta \circ \zeta$ is $N_{nc}O$ & η is $N_{nc}eCts$, then ζ is $N_{nc}eO$.

Proof. Obvious. □

6. N-Neutrosophic crisp e-C Homeomorphism

Definition 6.1. A bijection ζ is called a $N_{nc}e$-C homeomorphism (briefly, $N_{nc}eCHom$) if ζ & ζ^{-1} are $N_{nc}eIrr$ mappings.

Theorem 6.2. Each $N_{nc}eCHom$ is a $N_{nc}eHom$.

Proof. Let us assume that μ is a $N_{nc}cs$ in $(X_2, N_{nc}\tau)$. This shows that μ is a $N_{nc}ecs$ in $(X_2, N_{nc}\tau)$. By assumption, $\zeta^{-1}(\mu)$ is a $N_{nc}ecs$ in $(X_1, N_{nc}\Psi)$. Hence, ζ is a $N_{nc}eCts$. Hence, ζ & ζ^{-1} are $N_{nc}eCts$. Hence ζ is a $N_{nc}eHom$. □

Theorem 6.3. If $\zeta : (X_1, N_{nc}\Psi) \to (X_2, N_{nc}\tau)$ is a $N_{nc}eCHom$, then $N_{nc}ecl(\zeta^{-1}(\mu)) \subseteq \zeta^{-1}(N_{nc}cl(\mu))$ for each $N_{nc}ts$ μ in $(X_2, N_{nc}\tau)$.

Proof. Let μ be a $N_{nc}ts$ in $(X_2, N_{nc}\tau)$. Then, $N_{nc}cl(\mu)$ is a $N_{nc}cs$ in $(X_2, N_{nc}\tau)$, and every $N_{nc}cs$ is a $N_{nc}ecs$ in $(X_2, N_{nc}\tau)$. Assume ζ is $N_{nc}eIrr$, $\zeta^{-1}(N_{nc}cl(\lambda))$ is a $N_{nc}ecs$ in $(X_1, N_{nc}\Psi)$, then $N_{nc}cl(\zeta^{-1}(N_{nc}cl(\mu))) = \zeta^{-1}(N_{nc}cl(\mu))$. Here, $N_{nc}ecl\ (\zeta^{-1}(\mu)) \subseteq N_{nc}ecl(\zeta^{-1}(N_{nc}cl(\mu))) = \zeta^{-1}(N_{nc}cl(\mu))$. Therefore, $N_{nc}ecl(\zeta^{-1}(\mu)) \subseteq \zeta^{-1}(N_{nc}cl(\mu))$ for every $N_{nc}s$ μ in $(X_2, N_{nc}\tau)$. □

Theorem 6.4. Let $\zeta : (X_1, N_{nc}\Psi) \to (X_2, N_{nc}\tau)$ be a $N_{nc}eCHom$, then $N_{nc}ecl(\zeta^{-1}(\mu)) = \zeta^{-1}(N_{nc}ecl(\mu))$ for each $N_{nc}s$ μ in $(X_2, N_{nc}\tau)$.

Proof. Since ζ is a $N_{nc}eCHom$, then ζ is a $N_{nc}eIrr$. Let μ be a $N_{nc}s$ in $(X_2, N_{nc}\tau)$. Clearly, $N_{nc}ecl(\mu)$ is a $N_{nc}ecs$ in $(X_1, N_{nc}\Psi)$. Then $N_{nc}ecl(\mu)$ is a $N_{nc}ecs$ in $(X_1, N_{nc}\Psi)$. Since $\zeta^{-1}(\mu) \subseteq \zeta^{-1}(N_{nc}ecl(\mu))$, then $N_{nc}ecl(\zeta^{-1}(\mu)) \subseteq N_{nc}ecl(\zeta^{-1}(N_{nc}ecl(\mu))) = \zeta^{-1}(N_{nc}ecl(\mu))$. Therefore, $N_{nc}ecl(\zeta^{-1}(\mu)) \subseteq \zeta^{-1}(N_{nc}ecl(\mu))$. Let ζ be a $N_{nc}eCHom$. ζ^{-1} is a $N_{nc}eIrr$. Let us consider $N_{nc}s$ $\zeta^{-1}(\mu)$ in $(X_1, N_{nc}\Psi)$, which implies $N_{nc}ecl(\zeta^{-1}(\mu))$ is a $N_{nc}ecs$ in $(X_1, N_{nc}\Psi)$. Hence, $N_{nc}ecl(\zeta^{-1}(\mu))$ is a $N_{nc}ecs$ in $(X_1, N_{nc}\Psi)$. This implies that $(\zeta^{-1})^{-1}(N_{nc}ecl(\zeta^{-1}(\mu))) = \zeta(N_{nc}ecl(\zeta^{-1}(\mu)))$ is a $N_{nc}ecs$ in $(X_2, N_{nc}\tau)$. This proves $\mu = (\zeta^{-1})^{-1}(\zeta^{-1}(\mu)) \subseteq (\zeta^{-1})^{-1}(N_{nc}ecl(\zeta^{-1}(\mu))) = \zeta(N_{nc}ecl(\zeta^{-1}(\mu)))$. Therefore, $N_{nc}ecl(\mu) \subseteq N_{nc}ecl(\zeta(N_{nc}ecl(\zeta^{-1}(\mu)))) = \zeta(N_{nc}ecl(\zeta^{-1}(\mu)))$, since ζ^{-1} is a $N_{nc}eIrr$. Hence, $\zeta^{-1}(N_{nc}ecl(\mu)) \subseteq \zeta^{-1}(\zeta(N_{nc}ecl(\zeta^{-1}(\mu)))) = N_{nc}ecl(\zeta^{-1}(\mu))$. That is, $\zeta^{-1}(N_{nc}ecl(\mu)) \subseteq N_{nc}ecl(\zeta^{-1}(\mu))$. Hence, $N_{nc}ecl(\zeta^{-1}(\mu)) = \zeta^{-1}(N_{nc}\ ecl(\mu))$. □

Theorem 6.5. If ζ & η are $N_{nc}eCHom$'s, then $\eta \circ \zeta$ is a $N_{nc}eCHom$.

Proof. Let ζ and η to be two $N_{nc}eCHom$'s. Assume μ is a $N_{nc}ecs$ in $(X_3, N_{nc}\rho)$. Then, $\eta^{-1}(\mu)$ is a $N_{nc}ecs$ in $(X_2, N_{nc}\tau)$. Then, by hypothesis, $\zeta^{-1}(\eta^{-1}(\mu))$ is a $N_{nc}ecs$ in $(X_1, N_{nc}\Psi)$. Hence, $\eta \circ \zeta$ is a $N_{nc}eIrr$ mapping. Now, let ν be a $N_{nc}ecs$ in $(X_1, N_{nc}\Psi)$. Then, by presumption, $\zeta(\eta)$ is a $N_{nc}ecs$ in $(X_2, N_{nc}\tau)$. Then, by hypothesis, $\eta(\zeta(\nu))$ is a $N_{nc}ecs$ in $(X_3, N_{nc}\rho)$. This implies that $\eta \circ \zeta$ is a $N_{nc}eIrr$. Hence, $\eta \circ \zeta$ is a $N_{nc}eCHom$. □

7. Conclusions

In this paper, the new concept of a $N_{nc}eO$ and $N_{nc}eC$ mappings, $N_{nc}Hom$ and a $N_{nc}eHom$ in $N_{nc}ts$ are studied and discussed their properties. Also, we extended to $N_{nc}eCHom$'s and $N_{nc}eT_{\frac{1}{2}}$-space with some of their properties.

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A New Framework of Interval-valued Neutrosophic in Ẑ -algebra

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Abstract: This article deals about an interval-valued neutrosophic Ẑ-algebra is a mathematical framework which incorporates the concepts of interval-valued neutrosophic sets, Ẑ-algebra and algebraic operations. This innovative algebraic structure addresses the challenges posed by uncertain, imprecise, and indeterminate information in various fields. In this work, we presented the fundamentals of Ẑ-algebra and int_val neutrosophic sets, as well as several of their attributes such as homomorphism and cartesian product.

Keywords: Fuzzy sets, int_val fuzzy sets, neutrosophic set, Ẑ-subalgebra, int_val Ẑ-subalgebra, neutrosophic Ẑ-subalgebra, int_val neutrosophic Ẑ-subalgebra

1. Introduction

The intuitionistic fuzzy set with interval values is the name given to the new concept (IVIFS) which is presented by Attanassov [1] and outlines the fundamentals of IVIFS theory. Chandramouleeswaran [2] proposed Ẑ-algebra, a novel algebraic structure based upon propositional logic, in 2017. In a neutrosophic set, defined a set-theoretic operators, which is known as an interval neutrosophic set (INS), and then several INS properties related to operations and relations over INS [3]. In [4] introduces the phenomenon of int_val fuzzy β- subalgebras and examines a few of their features. This involves some of the information relevant to the theory of an int_val intuitionistic fuzzy subalgebras of β-algebra. Generalized double statistical convergence sequences on ideals in neutrosophic normed spaces were analysed by Jeyaraman et al. [5]. Henceforth, [6] established that each neutrosophic algebra is a direct product of neutrosophic algebras over the neutrosophic field. The ideology of neutrosophic sets in Ẑ-subalgebras is described, also some characteristics of int_val neutrosophic sets in Ẑ-algebras is also discussed. Maissam Jdid et l. [7] formulated Lagrange multipliers and neutrosophic nonlinear programming problems constrained by equality constraints. Manas Karak et al. [8] have introduced an innovative technique aimed at addressing transportation problems using a neutrosophic framework. This novel approach represents a significant stride in effectively handling uncertainty and indeterminacy within transportation scenarios.

Metawee[9] denotes a novel idea of interval_valued neutrosophic in UP-algebra, UP-subalgebras, as well as proved some results and their generalizations. The basic ideology of fuzzy Ẑ-ideal of a Ẑ-algebra under Ẑ-homomorphisms was evaluated, and some of its Cartesian product properties of fuzzy Ẑ-ideals were explored. Every quotient neutrosophic algebra is shown to be quotient algebra, and the concepts of neutrosophic algebra, the ideal, kernel, & neutrosophic quotient algebra are described [10]. The theme of neutrosophic cubic sets is used in β-subalgebra and then $\wp$-union, $\wp$-intersection, $\mathcal{R}$-union, and $\mathcal{R}$-intersection results based on neutrosophic cubic subalgebra is determined. Moreover, the captivating properties of lower and upper-level sets, as well as the homomorphism of neutrosophic cubic β−subalgebras, were explained [11]. The theory of neutrosophic algebra, including its ideal, kernel, and neutrosophic quotient algebra, as well as characterizing some neutrosophic algebra properties and claiming that every quotient neutrosophic algebra is quotient algebra [12]. The authors [13] started exploring an innovative concept for the Fermatean neutrosophic Dombi fuzzy graph. They also discovered a few outcomes of Fermatean neutrosophic Dombi fuzzy graphs' direct, cartesian composition. Shanmugapriya et al. [14] presented a novel concept of neutrosophic fuzzy Sets in Ẑ-algebra.

Samarandache generalises intuitionistic fuzzy sets to neutrosophic set, and many examples are given to distinguish between neutrosophic set as well as intuitionistic fuzzy set [15]. Neutrosophic set is the general framework that was recently proposed. However, from the point of technical view, the neutrosophic set must be specified. An int_val fuzzy set has been used to discuss these various algebraic structures as well as related topics. In [16], the authors have undertaken an insightful exploration into the concept of a fuzzy Ẑ-ideal within the context of a Ẑ-algebra. Furthermore, it was shown that the Cartesian product of fuzzy Ẑ-ideals is a fuzzy Ẑ-ideal. In [17] the authors provided the evidence of Cartesian product of fuzzy Ẑ-subalgebras is always a fuzzy Ẑ-subalgebra. The fundamental principle of a fuzzy Ẑ-subalgebra of Ẑ-algebra and its properties were investigated, and it also discusses how to resolve the inverse image of fuzzy Ẑ-subalgebras and the Ẑ-homomorphism of its image. The author of [18] has made a noteworthy contribution to the field by introducing a novel concept known as MBJ - neutrosophic set within the context of β-algebras. The MBJ - neutrosophic β- subalgebra's homomorphic and inverse images are presented. In MBJ - neutrosophic β- subalgebra, Cartesian product is often examined. A In 1965, Zadeh discovered the fuzzy set, which is very helpful for finding the uncertainties [19]. And again, extended the concept of an int_val fuzzy set as generalization of traditional fuzzy sets, then invented an int_val fuzzy set by using an int_val membership function to represent an interval on the membership scale [20].

This article's main objective is to explain the int_val neutrosophic in Ẑ-algebra. The following are the sections of the paper. The introduction appears in Section 1. Section 2 explained about the necessary definitions and properties of Ẑ-algebra and so on. Section 3 provides a more accurate explanation of neutrosophic in Ẑ-algebra and int_val neutrosophic in Ẑ-algebra. In Section 4, the int_val neutrosophic Ẑ- subalgebra homomorphism is discussed. The cartesian product of two neutrosophic Ẑ-algebras with int_val is defined in Section 5. Section 6 introduces the conclusion of this work.

2. Preliminaries

This section describes the fundamental definitions of fuzzy sets and Ẑ-algebra, as well as their major properties and examples. In the below discussion, the following notations are used such as X denoted by $\mathfrak{W}$, x denoted by ϵ, y denoted as ϱ, and Y is denoted by $\mathfrak{Y}$.

Definition 2.1.[15] Let the fuzzy set from the universal set $\mathfrak{W}$ and it is defining to be ς (ϵ): $\mathfrak{W} \rightarrow [0,1]$ for every element $\epsilon \in \mathfrak{W}$, and ς (ϵ) is known as the membership value of ϵ.

Definition 2.2.[4] The int_val fuzzy set $\mathfrak{W}$ is to be defined on $\bar{\xi}$ ={ ϵ, $\bar{\varsigma}_{\xi}$ (ϵ)/ $\epsilon \in \mathfrak{W}$ }, briefly denoted by, $\bar{\varsigma}_{\xi}(\epsilon)$= $[\varsigma_{\xi}^{L}(\epsilon), \varsigma_{\xi}^{U}(\epsilon)]$, where $\varsigma_{\xi}^{L}(\epsilon)$ & $\varsigma_{\xi}^{U}(\epsilon)$ are the two fuzzy sets in $\mathfrak{W}$ such that

$\varsigma_{\xi}^{L}(\epsilon) \leq \varsigma_{\xi}^{U}(\epsilon)$ for all $\epsilon \in \mathfrak{W}$. Let $\bar{\varsigma}_{\xi}(\epsilon) = [\varsigma_{\xi}^{L}(\epsilon),\ \varsigma_{\xi}^{U}(\epsilon)]\ \forall\ \epsilon \in \mathfrak{W}$.

Let $\mathfrak{D}$[0,1] denote the collection of all closed sub-intervals of [0,1]. If $\varsigma_{\xi}^{L}(\epsilon) = \varsigma_{\xi}^{U}(\epsilon)$ =c, where $0 \leq c \leq 1$, then there exist $\bar{\varsigma}_{\xi}(\epsilon)$ = [c, c] = $\bar{c}$.

For the convenience, ϵ belongs to $\mathfrak{D}$ [0,1] $\forall\ \epsilon \in \mathfrak{W}$,
Thus, the int_val fuzzy set $\bar{\xi}$ is represented as $\bar{\xi}$ ={ ϵ, $\bar{\varsigma}_{\xi}$ (ϵ)/ $\epsilon \in \mathfrak{W}$}, where $\bar{\varsigma}_{\xi}$: $\mathfrak{W} \to \mathfrak{D}$ [0,1].
Now, Define a refined minimum (briefly rmin) of two elements in $\mathfrak{D}$[0,1].
Define the symbols " $\leq$ ", " $\geq$ " & "$=$".
In case, if two elements in $\mathfrak{D}$ [0,1], then it expressed as $\mathfrak{D}_1$:=[a_1, b_1], $\mathfrak{D}_2$:=[a_2, b_2] $\in \mathfrak{D}$ [0,1].
Then, rmin $(\mathfrak{D}_1, \mathfrak{D}_2)$ = [min{ a_1, a_2}, min {b_1, b_2}], $\mathfrak{D}_1 \geq \mathfrak{D}_2$ iff $a_1 \geq a_2$, $b_1 \geq b_2$.
Similarly, there exist $\mathfrak{D}_1 \leq \mathfrak{D}_2$ & $\mathfrak{D}_1 = \mathfrak{D}_2$.

Definition 2.3.[4] Let $\bar{\varsigma}_1$ & $\bar{\varsigma}_2$ are two int_val fuzzy sets on $\mathfrak{W}$, then intersection $\bar{\varsigma}_1 \cap \bar{\varsigma}_2$ of $\bar{\varsigma}_1$ and $\bar{\varsigma}_2$ is referred as

$$(\ \bar{\varsigma}_1 \cap \bar{\varsigma}_2\ )(\epsilon) \geq \text{rmin}\ \{\bar{\varsigma}_1(\epsilon),\ \bar{\varsigma}_2(\epsilon)\}$$

Definition 2.4.[14] Let the neutrosophic fuzzy set ξ ={ϵ : $\varsigma_T(\epsilon), \varsigma_I(\epsilon), \varsigma_F(\epsilon)$ / $\epsilon \in \mathfrak{W}$ }, where $\varsigma_T, \varsigma_I, \varsigma_F$ are fuzzy sets in $\mathfrak{W}$, then it is denoted by $\varsigma_T(\epsilon)$ is true membership function, $\varsigma_I(\epsilon)$ is indeterminate membership function & $\varsigma_F(\epsilon)$ is false membership function.
Definition 2.5. The structure of $\bar{\xi}$ = {(ϵ : $\bar{\varsigma}_T(\epsilon)$, $\bar{\varsigma}_I(\epsilon), \bar{\varsigma}_F(\epsilon)$ / $\epsilon \in \mathfrak{W}$ } is referred to have int_val neutrosophic set in $\mathfrak{W}$, where $\bar{\varsigma}_{T,I,F}$: $\mathfrak{W} \to \mathfrak{D}$ [0,1], $\bar{\varsigma}_T(\epsilon)$ denotes truth int_val membership function, $\bar{\varsigma}_I(\epsilon)$ denotes indeterminate int_val membership function, $\bar{\varsigma}_F(\epsilon)$ denotes false int_val membership function.

Definition 2.6.[2] Suppose $\mathfrak{W}$ be the non-empty subset with the binary operation $*$ & a constant 0, then ($\mathfrak{W}$, $*$, 0) is Ẑ– algebra if,

i) $\epsilon * 0 = 0$
ii) $0 * \epsilon = \epsilon$
iii) $\epsilon * \epsilon = \epsilon$
iv) $\epsilon * \varrho = \varrho * \epsilon$, when $\epsilon \neq 0$ and $\varrho \neq 0\ \forall\ \epsilon, \varrho \in \mathfrak{W}$.
v)

Example 2.7. Let $\mathfrak{W}$= {$0, \omega_1, \omega_2, \omega_3, \omega_4, \omega_5$} be the set defined on $\mathfrak{W}$ with the constant 0 and a binary operations $*$ by introucing cayley's table

$*$	0	ω_1	ω_2	ω_3	ω_4	ω_5
0	0	ω_1	ω_2	ω_3	ω_4	ω_5
ω_1	0	ω_1	ω_5	ω_4	ω_3	ω_2
ω_2	0	ω_5	ω_2	ω_1	ω_5	ω_4
ω_3	0	ω_4	ω_1	ω_3	ω_1	ω_2
ω_4	0	ω_3	ω_5	ω_1	ω_4	ω_3
ω_5	0	ω_2	ω_4	ω_2	ω_3	ω_5

Definition 2.8. If $\mathfrak{W}$ is non-empty subset in neutrosophic Ẑ-algebra, then it's defined by Ẑ-subalgebra of $\mathfrak{W}$,

$$(\epsilon * \varrho) \in \mathfrak{W}, \forall\ \epsilon, \varrho \in \mathfrak{W}.$$

Definition 2.9.[10] Let $(\mathfrak{W}, *, 0)$ be the Ẑ– algebra with the operation $*$ and constant 0 then the neutrosophic set $\xi = \{\epsilon : \varsigma_T, \varsigma_I, \varsigma_F / \epsilon \in \mathfrak{W}\}$, is known to be neutrosophic Ẑ- subalgebra of $\mathfrak{W}$.

i) $\varsigma_T(\epsilon * \varrho) \geq \min\{\varsigma_T(\epsilon), \varsigma_T(\varrho)\}$

ii) $\varsigma_I(\epsilon * \varrho) \geq \min\{\varsigma_I(\epsilon), \varsigma_I(\varrho)\}$

iii) $\varsigma_F(\epsilon * \varrho) \leq \max\{\varsigma_F(\epsilon), \varsigma_F(\varrho)\}$

Definition 2.10.[13] Let $(\mathfrak{W}, *, 0)$ be Ẑ– algebra, then the fuzzy set ς in $\mathfrak{W}$ with membership function ξ_ς it is known as fuzzy Ẑ-subalgebra of a Ẑ-algebra $\mathfrak{W}$, if $\forall \epsilon, \varrho \in \mathfrak{W}$, if the following condition is satisfied

$$\xi_\varsigma(\epsilon * \varrho) \geq \min\{\xi_\varsigma(\epsilon), \xi_\varsigma(\varrho)\}$$

Definition 2.11. Let the Ẑ– algebra $(\mathfrak{W}, *, 0)$ and fuzzy set ς in $\mathfrak{W}$ with a membership function ξ_ς then it is named as Anti-fuzzy Ẑ-subalgebra of a Ẑ-algebra $\mathfrak{W}$, if $\forall\ \epsilon, \varrho \in \mathfrak{W}$, if the following condition is satisfied

$$\xi_\varsigma(\epsilon * \varrho) \leq \max\{\xi_\varsigma(\epsilon), \xi_\varsigma(\varrho)\}$$

Definition 2.12. Let $(\mathfrak{W}, *, 0)$ be a Ẑ– algebra then int_val fuzzy set $\bar{\xi}_\varsigma$ in $\mathfrak{W}$ is referred as an interval_valued fuzzy Ẑ-subalgebra of a Ẑ– algebra $\mathfrak{W}$, if

$$\bar{\xi}_\varsigma(\epsilon * \varrho) \geq \text{rmin}\{\bar{\xi}_\varsigma(\epsilon), \bar{\xi}_\varsigma(\varrho)\}\ \forall\ \epsilon, \varrho \in \mathfrak{W}$$

Definition 2.13.[7] If U be the subset in universe $\mathfrak{W}$, then the rsup property of an int_val fuzzy set $\bar{\varsigma}$ is referred as $\bar{\varsigma}(\epsilon_0) = \underset{\epsilon \in U}{rsup}\,\bar{\varsigma}(\epsilon)$, if $\exists\ \epsilon, \epsilon_0 \in U$.

Definition 2.14. Let $\bar{\varsigma}$ be the int_val neutrosophic fuzzy set in any set of $\mathfrak{W}$ is known as rsup_rsup_rinf property, then the subset U of $\mathfrak{W}$ then $\exists\ \epsilon_0 \in U \ni \bar{\varsigma}_T(\epsilon_0) = \underset{\epsilon \in U}{rsup}(\bar{\varsigma}_T(\epsilon))$,

$\bar{\varsigma}_I(\epsilon_0) = \underset{\epsilon \in U}{rsup}(\bar{\varsigma}_I(\epsilon))$, $\bar{\varsigma}_F(\epsilon_0) = \underset{\epsilon \in U}{rinf}(\bar{\varsigma}_F(\epsilon))$.

Definition 2.15. Let $(\mathfrak{W}, *, 0)$ and $(\mathfrak{W}', *', 0')$ be two Ẑ– algebra, then the mapping from h: $(\mathfrak{W}, *, 0) \rightarrow (\mathfrak{W}', *', 0')$ is known as Ẑ– homomorphism of Ẑ– algebra if

$$\mathfrak{h}(\epsilon * \varrho) = \mathfrak{h}(\epsilon) *' \mathfrak{h}(\varrho)$$

Definition 2.16. Let $\bar{\xi} = \{ \epsilon, \bar{\varsigma}_{T,I,F}(\epsilon) / \epsilon \in \mathfrak{W} \}$ be the neutrosophic set in $\hat{Z}$ and f maps from $\mathfrak{W} \to \mathfrak{Y}$, image of $\bar{\xi}$ under f, f($\bar{\xi}$) represented to be $\{f_{rsup}(\bar{\varsigma}_T), f_{rsup}(\bar{\varsigma}_I), f_{rinf}(\bar{\varsigma}_F), \epsilon \in \mathfrak{Y}\}$

i) $f_{rsup}(\bar{\varsigma}_T)(\varrho) = \begin{cases} \underset{\epsilon \in f^{-1}(\varrho)}{rsup} \bar{\varsigma}_T(\epsilon) & if\ f^{-1}(\varrho) \neq \phi \\ 1 & otherwise \end{cases}$

ii) $f_{rsup}(\bar{\varsigma}_I)(\varrho) = \begin{cases} \underset{\epsilon \in f^{-1}(\varrho)}{rsup} \bar{\varsigma}_I(\epsilon) & if\ f^{-1}(\varrho) \neq \phi \\ 1 & otherwise \end{cases}$

iii) $f_{rinf}(\bar{\varsigma}_F)(\varrho) = \begin{cases} \underset{\epsilon \in f^{-1}(\varrho)}{rinf} \bar{\varsigma}_F(\epsilon) & if\ f^{-1}(\varrho) \neq \phi \\ 0 & otherwise \end{cases}$

Definition 2.17. If $f : \mathfrak{W} \to \mathfrak{Y}$ is a function. Let $\bar{\varsigma}_{T_1,I_1,F_1}$ & $\bar{\varsigma}_{T_2,I_2,F_2}$ be two int_val neutrosophic set in $\mathfrak{W}$ & $\mathfrak{Y}$ respectively, then inverse image of f is represented as $f^{-1}(\bar{\varsigma}_{T_2,I_2,F_2}) = \{ \epsilon, f^{-1}(\bar{\varsigma}_{T_2}(\epsilon)), f^{-1}(\bar{\varsigma}_{I_2}(\epsilon)), f^{-1}(\bar{\varsigma}_{F_2}(\epsilon)), / \epsilon \in \mathfrak{W}\}$ such that $f^{-1}(\bar{\varsigma}_{T_2})\ f(\epsilon) = \bar{\varsigma}_{T_2}(f(\epsilon))$, $f^{-1}(\bar{\varsigma}_{I_2})\ f(\epsilon) = \bar{\varsigma}_{I_2}(f(\epsilon))$, $f^{-1}(\bar{\varsigma}_{F_2})\ f(\epsilon) = \bar{\varsigma}_{F_2}(f(\epsilon))$.

Definition 2.18.[13] Let h be an $\hat{Z}$-endomorphism of int_val neutrosophic $\hat{Z}$-algebras and $\bar{\xi} = \{\epsilon, \bar{\varsigma}_{T,I,F}(\epsilon) / \epsilon \in \mathfrak{W}\}$ be the neutrosophic set in $\mathfrak{W}$, then define a new fuzzy set $\bar{\varsigma}^h$ in $\mathfrak{W}$, as $\bar{\xi}_{\bar{\varsigma}^h}(\epsilon) = \bar{\xi}_{\bar{\varsigma}}(h(\epsilon))\ \forall\ \epsilon \in \mathfrak{W}$.

Definition 2.19.[14] If $\bar{\varsigma}_\xi$ and $\bar{\varsigma}_\zeta$ are the two int_val fuzzy sets of $\mathfrak{W}$, then the cartesian product $\bar{\varsigma}_\xi \times \bar{\varsigma}_\zeta : \mathfrak{W} \times \mathfrak{W} \to \mathcal{D}[0,1]$ is defined as

$$(\bar{\varsigma}_\xi \times \bar{\varsigma}_\zeta)(\epsilon, \varrho) = rmin\{\bar{\varsigma}_\xi(\epsilon), \bar{\varsigma}_\zeta(\varrho)\}\ \forall\ \epsilon \in \mathfrak{W}.$$

3. Interval-valued neutrosophic in Ẑ-algebra

This section describes the definitions on val neutrosophic in Ẑ-algebra in detail.

Definition 3.1. Let ($\mathfrak{W}$, $*$,0) be $\hat{Z}$-algebra. The int_val neutrosophic set $\bar{\xi} = \{ \epsilon: \bar{\varsigma}_T, \bar{\varsigma}_I, \bar{\varsigma}_F(\epsilon)/\epsilon \in \mathfrak{W}\}$ in $\mathfrak{W}$ is known as int_val neutrosophic $\hat{Z}$-algebra of $\mathfrak{W}$, if satisfies the below condition

i) $\bar{\varsigma}_T(\epsilon * \varrho) \geq rmin\{\bar{\varsigma}_T(\epsilon), \bar{\varsigma}_T(\varrho)\}$

ii) $\bar{\varsigma}_I(\epsilon * \varrho) \geq rmin\{\bar{\varsigma}_I(\epsilon), \bar{\varsigma}_I(\varrho)\}$

iii) $\bar{\varsigma}_F(\epsilon * \varrho) \leq rmax\{\bar{\varsigma}_F(\epsilon), \bar{\varsigma}_F(\varrho)\}$

Example 3.2. Consider the example 2.2

$$\bar{\varsigma}_{T,I,F} = \begin{cases} [0.4,0.8] & \epsilon = 0\ when\ (\epsilon = 0, \varrho \neq 0)\ or\ (\epsilon \neq 0, \varrho = 0) \\ [0.3,0.7] & \epsilon = \omega_1, \omega_2 \\ [0.2,0.6] & \epsilon = \omega_4, \omega_5 \end{cases}$$

$$= \{[0.1,0.5] \quad \epsilon = \omega_3$$

Hence, the above example satisfies the condition of int_val neutrosophic of Ẑ-algebra.

Theorem 3.3. Intersection of two int_val neutrosophic Ẑ-algebra of 𝔚 is again an interval-valued neutrosophic Ẑ-algebra of 𝔚.

Proof: Let $\bar{\varsigma}_{T_1,I_1,F_1}$ and $\bar{\varsigma}_{T_2,I_2,F_2}$ are the two neutrosophic int_val neutrosophic Ẑ-algebra of 𝔚. Then,

$$(\bar{\varsigma}_{T_1} \cap \bar{\varsigma}_{T_2})(\epsilon * \varrho) \geq \text{rmin}\{\bar{\varsigma}_{T_1}(\epsilon * \varrho), \bar{\varsigma}_{T_2}(\epsilon * \varrho)\}$$
$$= \{\text{rmin}\{\bar{\varsigma}_{T_1}(\epsilon), \bar{\varsigma}_{T_1}(\varrho)\}, \text{rmin}\{\bar{\varsigma}_{T_2}(\epsilon), \bar{\varsigma}_{T_2}(\varrho)\}\}$$
$$= \{\text{rmin}\{\bar{\varsigma}_{T_1}(\epsilon), \bar{\varsigma}_{T_2}(\epsilon)\}, \text{rmin}\{\bar{\varsigma}_{T_1}(\varrho), \bar{\varsigma}_{T_2}(\varrho)\}\}$$
$$= \text{rmin}\{(\bar{\varsigma}_{T_1 \cap T_2})(\epsilon), (\bar{\varsigma}_{T_1 \cap T_2})(\varrho)\}$$
$$\therefore (\bar{\varsigma}_{T_1} \cap \bar{\varsigma}_{T_2})(\epsilon * \varrho) \geq \text{rmin}\{(\bar{\varsigma}_{T_1 \cap T_2})(\epsilon), (\bar{\varsigma}_{T_1 \cap T_2})(\varrho)\}$$

Similarly, $(\bar{\varsigma}_{I_1} \cap \bar{\varsigma}_{I_2})(\epsilon * \varrho) \geq \text{rmin}\{(\bar{\varsigma}_{I_1 \cap I_2})(\epsilon), (\bar{\varsigma}_{I_1 \cap I_2})(\varrho)\}$

$$(\bar{\varsigma}_{F_1} \cap \bar{\varsigma}_{F_2})(\epsilon * \varrho) \leq \text{rmax}\{\bar{\varsigma}_{F_1}(\epsilon * \varrho), \bar{\varsigma}_{F_2}(\epsilon * \varrho)\}$$
$$= \{\text{rmax}\{\bar{\varsigma}_{F_1}(\epsilon), \bar{\varsigma}_{F_1}(\varrho)\}, \text{rmax}\{\bar{\varsigma}_{F_2}(\epsilon), \bar{\varsigma}_{F_2}(\varrho)\}\}$$
$$= \{\text{rmax}\{\bar{\varsigma}_{F_1}(\epsilon), \bar{\varsigma}_{F_2}(\epsilon)\}, \text{rmax}\{\bar{\varsigma}_{F_1}(\varrho), \bar{\varsigma}_{F_2}(\varrho)\}\}$$
$$= \text{rmax}\{(\bar{\varsigma}_{F_1 \cap F_2})(\epsilon), (\bar{\varsigma}_{F_1 \cap F_2})(\varrho)\}$$
$$\therefore (\bar{\varsigma}_{F_1} \cap \bar{\varsigma}_{F_2})(\epsilon * \varrho) \leq \text{rmax}\{(\bar{\varsigma}_{F_1 \cap F_2})(\epsilon), (\bar{\varsigma}_{F_1 \cap F_2})(\varrho)\}$$

Hence $\bar{\varsigma}_{T_1,I_1,F_1}$ and $\bar{\varsigma}_{T_2,I_2,F_2}$ is an int_val neutrosophic Ẑ-algebra of 𝔚.

Theorem 3.4. Intersection of any set of int_val neutrosophic of Ẑ-algebra of 𝔚 is again an int_val neutrosophic of Ẑ-algebra of 𝔚.

Lemma 3.5. If $\bar{\xi}$ be an int_val neutrosophic Ẑ-subalgebra of 𝔚, then

i) $\bar{\varsigma}_T(0) \geq \bar{\varsigma}_T(\epsilon)$, $\bar{\varsigma}_I(0) \geq \bar{\varsigma}_I(\epsilon)$, & $\bar{\varsigma}_F(0) \leq \bar{\varsigma}_F(\epsilon)$ $\forall\ \epsilon \in$ 𝔚.

ii) $\bar{\varsigma}_T(0) \geq \bar{\varsigma}_T(\epsilon) \geq \bar{\varsigma}_T(\epsilon^*)$, $\bar{\varsigma}_I(0) \geq \bar{\varsigma}_I(\epsilon) \geq \bar{\varsigma}_I(\epsilon^*)$, $\bar{\varsigma}_F(0) \leq \bar{\varsigma}_F(\epsilon) \leq \bar{\varsigma}_F(\epsilon^*)$, where $\epsilon^* = 0 * \epsilon$

Proof: For any $\epsilon \in$ 𝔚,

i) $\bar{\varsigma}_T(0) = [\varsigma_T^L(0), \varsigma_T^U(0)]$

$$\geq [\varsigma_T^L(\epsilon), \varsigma_T^U(\epsilon)]$$
$$= \bar{\varsigma}_T(\epsilon)$$

Likewise, $\bar{\varsigma}_I(0) = [\varsigma_I^L(0), \varsigma_I^U(0)]$

$$\geq [\varsigma_I^L(\epsilon), \varsigma_I^U(\epsilon)]$$
$$= \bar{\varsigma}_I(\epsilon)$$

$\bar{\varsigma}_F(0) = [\varsigma_F^L(0), \varsigma_F^U(0)]$

$$\leq [\varsigma_F^L(\epsilon), \varsigma_F^U(\epsilon)]$$

$= \bar{\varsigma}_F(\epsilon)$

ii) Also, for $\epsilon \in \mathfrak{W}$

$\bar{\varsigma}_T(\epsilon^*) = [\ \varsigma_T^L(\epsilon^*),\ \varsigma_T^U(\epsilon^*)]$

$= [\ \varsigma_T^L(0 * \epsilon),\ \varsigma_T^U(\ 0 * \epsilon)]$

$= [\min(\varsigma_T^L\ (0),\ \varsigma_T^U(\epsilon),\ \min((\varsigma_T^L\ (0),\ \varsigma_T^U(\epsilon))]$

$\bar{\varsigma}_T(\epsilon^*) \geq [\varsigma_T^L\ (\epsilon),\ \varsigma_T^U(\epsilon)]$

$= \bar{\varsigma}_T(\epsilon)$

$\bar{\varsigma}_I(\epsilon^*) = [\ \varsigma_I^L(\epsilon^*),\ \varsigma_I^U(\epsilon^*)]$

$= [\ \varsigma_I^L(0 * \epsilon),\ \varsigma_I^U(\ 0 * \epsilon)]$

$= [\min(\varsigma_I^L\ (0),\ \varsigma_I^U(\epsilon),\ \min((\varsigma_I^L\ (0),\ \varsigma_I^U(\epsilon))]$

$\bar{\varsigma}_I(\epsilon^*) \geq [\varsigma_I^L\ (\epsilon),\ \varsigma_I^U(\epsilon)]$

$= \bar{\varsigma}_I(\epsilon)$

$\bar{\varsigma}_F(\epsilon^*) = [\ \varsigma_F^L(\epsilon^*),\ \varsigma_F^U(\epsilon^*)]$

$= [\ \varsigma_F^L(0 * \epsilon),\ \varsigma_F^U(\ 0 * \epsilon)]$

$= [\max(\varsigma_F^L\ (0),\ \varsigma_F^U(\epsilon),\ \max((\varsigma_F^L\ (0),\ \varsigma_F^U(\epsilon))]$

$\therefore \quad \bar{\varsigma}_F(\epsilon^*) \leq [\varsigma_F^L\ (\epsilon),\ \varsigma_F^U(\epsilon)]$

$= \bar{\varsigma}_F(\epsilon)$

Theorem 3.6. If there is a sequence $\{\epsilon_n\}$ in $\mathfrak{W}$, such that $\lim_{n\to\infty} \bar{\varsigma}_T(\epsilon_n)$ = [1,1], $\lim_{n\to\infty} \bar{\varsigma}_I(\epsilon_n)$ = [1,1], $\lim_{n\to\infty} \bar{\varsigma}_F(\epsilon_n)$ = [0,0]. Let $\bar{\xi}$ be an int_val neutrosophic Ẑ-subalgebra of $\mathfrak{W}$, then $\bar{\varsigma}_T(0)$ = [1,1], $\bar{\varsigma}_I(0)$ = [1,1], $\bar{\varsigma}_F(0)$ = [0,0].

Proof: Let, $\bar{\varsigma}_T(0) \geq \bar{\varsigma}_T(\epsilon)$, for all $\epsilon \in \mathfrak{W}$,

$\bar{\varsigma}_T(0) \geq \bar{\varsigma}_T(\epsilon_n)$

Similarly, $\bar{\varsigma}_I(0) \geq \bar{\varsigma}_I(\epsilon_n)$ & $\bar{\varsigma}_F(0) \leq \bar{\varsigma}_F(\epsilon_n)$ $\forall$ $n \geq 0$

Thus, $[1,1] \geq \bar{\varsigma}_T(0) \geq \lim_{n\to\infty} \bar{\varsigma}_T(\epsilon_n)$ = [1,1]

$\Longrightarrow \bar{\varsigma}_T(0)$ = [1,1]

Similarly, [1,1] $\geq \bar{\varsigma}_I(0) \geq \lim_{n\to\infty} \bar{\varsigma}_I(\epsilon_n)$ = [1,1]

$\Longrightarrow \bar{\varsigma}_I(0)$ = [1,1]

Likewise, [1,1] $\leq \bar{\varsigma}_F(0) \leq \lim_{n\to\infty} \bar{\varsigma}_F(\epsilon_n)$ = [0,0]

$\Longrightarrow \bar{\varsigma}_F(0)$ = [0,0]

Theorem 3.7. Let $\bar{\xi}$= {ϵ: $\bar{\varsigma}_T(\epsilon)$, $\bar{\varsigma}_I(\epsilon)$, $\bar{\varsigma}_F(\epsilon)$ $\forall$ $\epsilon \in \mathfrak{W}$ } such that $[\varsigma_T^L, \varsigma_T^U]$, $[\varsigma_I^L, \varsigma_I^U]$ are fuzzy Ẑ -subalgebra & $[\varsigma_F^L, \varsigma_F^U]$ is anti-fuzzy Ẑ-subalgebra of $\mathfrak{W}$, then $\bar{\xi}$= { ϵ: $\bar{\varsigma}_T(\epsilon)$, $\bar{\varsigma}_I(\epsilon)$, $\bar{\varsigma}_F(\epsilon)\forall$ $\epsilon \in \mathfrak{W}$} is an int_val neutrosophic Ẑ-subalgebra of $\mathfrak{W}$.

Proof: For any ϵ, $\varrho \in \mathfrak{W}$, then

$$\bar{\varsigma}_T(\epsilon * \varrho) = [\varsigma_T^L(\epsilon * \varrho), \varsigma_T^U(\epsilon * \varrho)]$$

$$\geq [\min\{\varsigma_T^L(\epsilon), \varsigma_T^L(\varrho)\}, \min\{\varsigma_T^U(\epsilon), \varsigma_T^U(\varrho)\}]$$

$$= \min\{[\varsigma_T^L(\epsilon), \varsigma_T^U(\epsilon)], [\varsigma_T^L(\varrho), \varsigma_T^U(\varrho)]\}$$

$$= \text{rmin}\{\bar{\varsigma}_T(\epsilon), \bar{\varsigma}_T(\varrho)\}$$

$$\therefore \bar{\varsigma}_T(\epsilon * \varrho) \geq \text{rmin}\{\bar{\varsigma}_T(\epsilon), \bar{\varsigma}_T(\varrho)\}$$

Similarly, $\bar{\varsigma}_I(\epsilon * \varrho) \geq$ rmin $\{\bar{\varsigma}_I(\epsilon), \bar{\varsigma}_I(\varrho)\}$

Hence, $\bar{\varsigma}_T$, $\bar{\varsigma}_I$ are fuzzy Ẑ-subalgebra of $\mathfrak{W}$.

$$\bar{\varsigma}_F(\epsilon * \varrho) = [\varsigma_F^L(\epsilon * \varrho), \varsigma_F^U(\epsilon * \varrho)]$$

$$\leq [\max\{\varsigma_F^L(\epsilon), \varsigma_F^L(\varrho), \max\{\varsigma_F^U(\epsilon), \varsigma_F^U(\varrho)\}]$$

$$= \max\{[\varsigma_F^L(\epsilon), \varsigma_F^U(\epsilon)], [\varsigma_F^L(\varrho), \varsigma_F^U(\varrho)]\}$$

$$= \text{rmax}\{\bar{\varsigma}_F(\epsilon), \bar{\varsigma}_F(\varrho)\}$$

$$\bar{\varsigma}_F(\epsilon * \varrho) \leq \text{rmax}\{\bar{\varsigma}_F(\epsilon), \bar{\varsigma}_F(\varrho)\}$$

Hence, $\bar{\varsigma}_F$ is Anti-fuzzy Ẑ-subalgebra of $\mathfrak{W}$.

$\therefore$ $\bar{\xi}$= {ϵ: $\bar{\varsigma}_T(\epsilon)$, $\bar{\varsigma}_I(\epsilon)$, $\bar{\varsigma}_F(\epsilon)$ $\forall$ $\epsilon \in \mathfrak{W}$ } is an int_val neutrosophic Ẑ-subalgebra of $\mathfrak{W}$.

Theorem 3.8. If $\bar{\xi}$= {ϵ: $\bar{\varsigma}_T(\epsilon)$, $\bar{\varsigma}_I(\epsilon)$, $\bar{\varsigma}_F(\epsilon)$ $\forall$ $\epsilon \in \mathfrak{W}$ } is an int_val neutrosophic Ẑ-subalgebra of $\mathfrak{W}$, then the sets

$$\mathfrak{W}_{\bar{\varsigma}_T} = \{\epsilon \in \mathfrak{W} \,/\, \bar{\varsigma}_T(\epsilon) = \bar{\varsigma}_T(\bar{0})\}$$

$$\mathfrak{W}_{\bar{\varsigma}_I} = \{\epsilon \in \mathfrak{W} \,/\, \bar{\varsigma}_I(\epsilon) = \bar{\varsigma}_I(\bar{0})\}$$

$\mathfrak{W}_{\bar{\varsigma}_F}$= { $\epsilon \in \mathfrak{W}$ / $\bar{\varsigma}_F(\epsilon) = \bar{\varsigma}_F(\bar{0})$} are Ẑ -subalgebra of $\mathfrak{W}$.

Proof: For $\epsilon,\varrho \in \mathfrak{W}_{\bar{\varsigma}_T}$, then

$$\bar{\varsigma}_T(\epsilon) = \bar{\varsigma}_T(\bar{0}) = \bar{\varsigma}_T(\varrho)$$

Now, $\bar{\varsigma}_T(\epsilon * \varrho) \geq$ rmin $\{\bar{\varsigma}_T(\epsilon) * \bar{\varsigma}_T(\varrho)\}$

= rmin $\{\bar{\varsigma}_T(\bar{0}), \bar{\varsigma}_T(\bar{0})\}$

= $\bar{\varsigma}_T(\bar{0})$

$\therefore \bar{\varsigma}_T(\epsilon * \varrho) \geq \bar{\varsigma}_T(\bar{0})$

Similarly, $\bar{\varsigma}_I(\epsilon * \varrho) \geq \bar{\varsigma}_I(\bar{0})$

$\bar{\varsigma}_F(\epsilon * \varrho) \leq$ rmax $\{\bar{\varsigma}_F(\epsilon) * \bar{\varsigma}_F(\varrho)\}$

= rmax $\{\bar{\varsigma}_F(\bar{0}), \bar{\varsigma}_F(\bar{0})\}$

= $\bar{\varsigma}_F(\bar{0})$

$\therefore \bar{\varsigma}_F(\epsilon * \varrho) \leq \bar{\varsigma}_F(\bar{0})$

Hence, $\epsilon * \varrho \in \mathfrak{W}_{\bar{\varsigma}_{T,I,F}}$ is Ẑ -subalgebra of $\mathfrak{W}$.

Theorem 3.9. Given ($\mathfrak{W}$, $*$, 0) & $(\mathfrak{W}', *', 0)$ be the two Ẑ-algebras & f: $\mathfrak{W} \rightarrow \mathfrak{Y}$ is homomorphism of Ẑ - algebras. If $\bar{\xi}$ is an int_val neutrosophic Ẑ-algebra of $\mathfrak{W}$, defined by f $(\bar{\varsigma}_{T,I,F}) = \{\epsilon$, $(\bar{\varsigma}_{T,I,F})(\epsilon) = \bar{\varsigma}_{T,I,F}(\mathrm{f}(\epsilon))\}$ then f $(\bar{\varsigma}_{T,I,F})$ is an int_val neutrosophic Ẑ– subalgebra of $\mathfrak{W}$.

Proof: Given, $\epsilon, \varrho \in \mathfrak{W}$

$(\bar{\varsigma}_{T_f})(\epsilon * \varrho) = \bar{\varsigma}_T(\mathrm{f}(\epsilon * \varrho))$

$= \bar{\varsigma}_T(\mathrm{f}(\epsilon) * \mathrm{f}(\varrho))$

$\geq$ rmin $\{\bar{\varsigma}_T(\mathrm{f}(\epsilon), \bar{\varsigma}_T(\mathrm{f}(\varrho)\}$

$(\bar{\varsigma}_{T_f}) \geq$ rmin $\{(\bar{\varsigma}_{T_f})(\epsilon), (\bar{\varsigma}_{T_f})(\varrho)\}$

Similarly, $(\bar{\varsigma}_{I_f}) \geq$ rmin $\{(\bar{\varsigma}_{I_f})(\epsilon), (\bar{\varsigma}_{I_f})(\varrho)\}$

$(\bar{\varsigma}_{F_f})(\epsilon * \varrho) = \bar{\varsigma}_F(\mathrm{f}(\epsilon * \varrho))$

$= \bar{\varsigma}_F(\mathrm{f}(\epsilon) * \mathrm{f}(\varrho))$

$\leq$ rmax $\{\bar{\varsigma}_F(\mathrm{f}(\epsilon)), \bar{\varsigma}_F(\mathrm{f}(\varrho)\}$

$(\bar{\varsigma}_{F_f}) \leq$ rmax $\{(\bar{\varsigma}_{F_f})(\epsilon), (\bar{\varsigma}_{F_f})(\varrho)\}$

Therefore, f $(\bar{\varsigma}_{T,I,F})$ is an int_val neutrosophic Ẑ– subalgebra of $\mathfrak{W}'$.

4. Homomorphism of Interval-valued (int_val) Neutrosophic Ẑ– Subalgebra

In this section, will look at some methods for investigating results on int_val neutrosophic Ẑ - subalgebra homomorphism.

Theorem. 4.1. If f : $\mathfrak{W} \to \mathfrak{Y}$ is the homomorphism of Ẑ - algebra. If $\bar{\varsigma}_{T,I,F}$ be the int_val neutrosophic Ẑ -subalgebra of $\mathfrak{Y}$, then $f^{-1}(\bar{\varsigma}_{T,I,F}) = \{(f^{-1}(\bar{\varsigma}_T), f^{-1}(\bar{\varsigma}_I), f^{-1}(\bar{\varsigma}_F)/ \epsilon \in \mathfrak{W}\}$ is also the int_val neutrosophic Ẑ - subalgebra of $\mathfrak{Y}$, where $f^{-1}(\bar{\varsigma}_T(\epsilon)) = \bar{\varsigma}_T f(\epsilon)$, $f^{-1}(\bar{\varsigma}_I(\epsilon)) = \bar{\varsigma}_I f(\epsilon)$, $f^{-1}(\bar{\varsigma}_F(\epsilon)) = \bar{\varsigma}_F f(\epsilon)$, for every $\epsilon \in \mathfrak{W}$.

Proof: Given $\bar{\varsigma}_{T,I,F}$ be the int_val neutrosophic Ẑ -subalgebra of $\mathfrak{Y}$,

Let $\epsilon, \varrho \in \mathfrak{W}$

Then, $f^{-1}(\bar{\varsigma}_T(\epsilon * \varrho) = \bar{\varsigma}_T f(\epsilon * \varrho)$

$= \bar{\varsigma}_T(f(\epsilon) * \bar{\varsigma}_T f(\varrho))$

$\geq \text{rmin}\{\bar{\varsigma}_T(f(\epsilon)), (\bar{\varsigma}_T(f(\varrho))\}$

$\geq \text{rmin}\{f(\bar{\varsigma}_T(\epsilon)), f(\bar{\varsigma}_T(\varrho))\}$

$= \text{rmin}\{f^{-1}(\bar{\varsigma}_T(\epsilon)) , f^{-1}(\bar{\varsigma}_T(\varrho))\}$

$f^{-1}(\bar{\varsigma}_T(\epsilon * \varrho) \geq \min\{f^{-1}(\bar{\varsigma}_T(\epsilon)) , f^{-1}(\bar{\varsigma}_T(\varrho))\}$

Similarly, $f^{-1}(\bar{\varsigma}_I)(\epsilon * \varrho) \geq \text{rmin}\{ f^{-1}(\bar{\varsigma}_I(\epsilon)) , f^{-1}(\bar{\varsigma}_I(\varrho))\}$

$f^{-1}(\bar{\varsigma}_F)(\epsilon * \varrho) = \bar{\varsigma}_F(f(\epsilon * \varrho))$

$= \bar{\varsigma}_F(f(\epsilon) * f(\varrho))$

$\leq \text{rmax}\{ \bar{\varsigma}_F(f(\epsilon)), \bar{\varsigma}_F(f(\varrho))\}$

$= \text{rmax}\{f^{-1}(\bar{\varsigma}_F(\epsilon)) , f^{-1}(\bar{\varsigma}_F(\varrho))\}$

$f^{-1}(\bar{\varsigma}_F)(\epsilon * \varrho) \leq \text{rmax}\{f^{-1}(\bar{\varsigma}_F(\epsilon)) , f^{-1}(\bar{\varsigma}_F(\varrho))\}$

$\therefore f^{-1}(\bar{\varsigma}_{T,I,F}) = \{(f^{-1}(\bar{\varsigma}_T), f^{-1}(\bar{\varsigma}_I), f^{-1}(\bar{\varsigma}_F)\}$ is an int_val neutrosophic Ẑ- subalgebra of $\mathfrak{Y}$.

Theorem 4.2. If f : $\mathfrak{W} \to \mathfrak{Y}$ is the homomorphism from Ẑ– algebra $\mathfrak{W}$ *to* $\mathfrak{Y}$. If $\bar{\xi} = (\bar{\varsigma}_{T,I,F})$ be the int_val neutrosophic Ẑ– algebra of $\mathfrak{W}$, then the image of $f(\bar{\xi}) = \{ \epsilon , f_{rsup}(\bar{\varsigma}_T)\, f_{rsup}(\bar{\varsigma}_I), f_{rinf}(\bar{\varsigma}_F)/ \epsilon \in \mathfrak{W}\}$ of $\bar{\xi}$ under f is also the int_val neutrosophic Ẑ– subalgebra of $\mathfrak{Y}$.

Proof: Let $\bar{\xi} = (\bar{\varsigma}_{T,I,F})$ be the int_val neutrosophic Ẑ– subalgebra of $\mathfrak{W}$, let $\varrho_1, \varrho_2 \in \mathfrak{Y}$.

We know that, $\epsilon_1 * \epsilon_2 / \epsilon_1 \in f^{-1}(\varrho_1) \,\&\, \epsilon_2 \in f^{-1}(\varrho_2)\} \subseteq \{ \epsilon \in \mathfrak{W}/\epsilon \in f^{-1}(\varrho_1 * \varrho_2)$

Now,

$f_{rsup}(\bar{\varsigma}_T)((\varrho_1 * \varrho_2) = \text{rsup}\{(\bar{\varsigma}_T)/ \epsilon \in f^{-1}(\varrho_1 * \varrho_2)\}$

$= \text{rsup}\{(\bar{\varsigma}_T)\, \epsilon_1 * \epsilon_2/\epsilon_1 \in f^{-1}(\varrho_1) \,\&\, \epsilon_2 \in f^{-1}(\varrho_2)\}$

$\geq \text{rsup}\{\text{rmin}\{\bar{\varsigma}_T(\epsilon_1), \bar{\varsigma}_T(\epsilon_2)/ \epsilon_1 \in f^{-1}(\varrho_1) \,\&\, \epsilon_2 \in f^{-1}(\varrho_2)\}$

$= \text{rmin}\{\text{rsup}\{\bar{\varsigma}_T(\epsilon_1)/ \epsilon_1 \in f^{-1}(\varrho_1), \bar{\varsigma}_T(\epsilon_2)/ \epsilon_2 \in f^{-1}(\varrho_2)\}\}$

$f_{rsup}(\bar{\varsigma}_T)((\varrho_1 * \varrho_2) \geq \text{rmin}\{f_{\text{rsup}}(\bar{\varsigma}_T(\varrho_1)), f_{\text{rsup}}(\bar{\varsigma}_T(\varrho_2))\}$

Similarly, $f_{rsup}(\bar{\varsigma}_I)\,(\varrho_1 * \varrho_2) \geq \text{rmin}\{f_{\text{rsup}}(\bar{\varsigma}_I(\varrho_1)), f_{\text{rsup}}(\bar{\varsigma}_I(\varrho_2))$

$f_{rinf}(\bar{\varsigma}_F)(\varrho_1 * \varrho_2)$ = rinf $\{\bar{\varsigma}_F(\epsilon) / \epsilon \in f^{-1}(\varrho_1 * \varrho_2)\}$

$\leq$ rinf $\{(\bar{\varsigma}_F)\, \epsilon_1 * \epsilon_2 / \epsilon_1 \in f^{-1}(\varrho_1)\ \&\ \epsilon_2 \in f^{-1}(\varrho_2)\}$

$\leq$ rinf{rmax$\{\bar{\varsigma}_F(\epsilon_1), \varsigma_F(\epsilon_2) / \epsilon_1 \in f^{-1}(\varrho_1)\ \&\ \epsilon_2 \in f^{-1}(\varrho_2)\}$

= rmax {rinf $\{\bar{\varsigma}_F(\epsilon_1) / \epsilon_1 \in f^{-1}(\varrho_1), \bar{\varsigma}_F(\epsilon_2) / \epsilon_2 \in f^{-1}(\varrho_2)\}\}$

=rmax $\{f_{\text{rinf}}(\bar{\varsigma}_F(\varrho_1)), f_{\text{rinf}}(\bar{\varsigma}_F(\varrho_2))\}$

Hence, $f_{\text{rinf}}(\bar{\varsigma}_F)(\varrho_1 * \varrho_2) \leq$ rmax $\{f_{\text{rinf}}(\bar{\varsigma}_F(\varrho_1)), f_{\text{rinf}}(\bar{\varsigma}_F(\varrho_2))\}$

Theorem 4.3. Suppose $f: \mathfrak{W} \to \mathfrak{Y}$ is the homomorphism of Ẑ– algebra. If $\bar{\xi} = \{\epsilon: f(\bar{\varsigma}_{T,I,F})(\epsilon) / \epsilon \in \mathfrak{W}\}$ be an int_val neutrosophic Ẑ– algebra of $\mathfrak{W}$, then its pre-image of $f^{-1}(\bar{\xi}) = \{\epsilon: f^{-1}(\bar{\varsigma}_{T,I,F}) / \epsilon \in \mathfrak{W}\}$ of $\bar{\xi}$ under f is also an int_val neutrosophic Ẑ– subalgebra in $\mathfrak{W}$.

Proof:

$f^{-1}(\bar{\varsigma}_T)(\epsilon * \varrho) = \bar{\varsigma}_T(f(\epsilon * \varrho))$

$= \bar{\varsigma}_T(f(\epsilon) * f(\varrho))$

$\geq$ rmin$\{\bar{\varsigma}_T(f(\epsilon)), \bar{\varsigma}_T(f(\varrho)\}$

=rmin $\{f^{-1}(\bar{\varsigma}_T)(\epsilon), f^{-1}(\bar{\varsigma}_T)(\varrho)\}$

$\therefore f^{-1}(\bar{\varsigma}_T)(\epsilon * \varrho) \geq$ rmin $\{f^{-1}(\bar{\varsigma}_T)(\epsilon), f^{-1}(\bar{\varsigma}_T)(\varrho)\}$

Similarly, $f^{-1}(\bar{\varsigma}_I)(\epsilon * \varrho) \geq$ rmin $\{f^{-1}(\bar{\varsigma}_I)(\epsilon), f^{-1}(\bar{\varsigma}_I)(\varrho)\}$

$f^{-1}(\bar{\varsigma}_F)(\epsilon * \varrho) = \bar{\varsigma}_F(f(\epsilon * \varrho))$

$= \bar{\varsigma}_F(f(\epsilon) * f(\varrho))$

$\leq rmax\{\bar{\varsigma}_F(f(\epsilon)), \bar{\varsigma}_F(f(\varrho))\}$

= rmax $\{f^{-1}(\bar{\varsigma}_F)(\epsilon), f^{-1}(\bar{\varsigma}_F)(\varrho)\}$

$f^{-1}(\bar{\varsigma}_F)(\epsilon * \varrho) \leq$ rmax $\{f^{-1}(\bar{\varsigma}_F)(\epsilon), f^{-1}(\bar{\varsigma}_F)(\varrho)\}$

$\therefore f^{-1}(\bar{\xi}) = \{\epsilon, f^{-1}(\bar{\varsigma}_{T,I,F}) / \epsilon \in \mathfrak{W}\}$ of $\bar{\xi}$ under f is the int_val neutrosophic Ẑ– subalgebra of $\mathfrak{W}$.

Theorem 4.4. If h is a Ẑ-endomorphism of Ẑ –algebra $(\mathfrak{W}, *, 0)$. If $\bar{\xi} = \{\epsilon : \bar{\varsigma}_{T,I,F} / \epsilon \in \mathfrak{W}\}$ be an int_val neutrosophic Ẑ-subalgebra of $\mathfrak{W}$, then $\bar{\xi}^h = \{\epsilon : \bar{\xi}^h_{T,I,F} / \epsilon \in \mathfrak{W}\}$ is also an int_val neutrosophic Ẑ-subalgebra of $\mathfrak{W}$.

Proof: Given, h be an $\hat{Z}$-endomorphism of $\hat{Z}$ −algebra $(\mathfrak{W}, *, 0)$.

Let $\bar{\xi}$ be the int_val neutrosophic $\hat{Z}$-subalgebra of $\mathfrak{W}$,

Let, $\epsilon, \varrho \in \mathfrak{W}$, then

$\bar{\xi}_{T^h}\ (\epsilon * \varrho) = \bar{\xi}_T(\text{h}\ (\epsilon * \varrho)$

$= \bar{\xi}_T(\text{h}\ (\epsilon) * \text{h}\ (\varrho))$

$\geq \text{rmin}\{\ \bar{\xi}_T(\text{h}\ (\epsilon),\ \bar{\xi}_T(\text{h}\ (\varrho)\}$

$\bar{\xi}_{T^h}\ (\epsilon * \varrho) \geq \text{rmin}\{\bar{\xi}_{T^h}\ (\epsilon), \bar{\xi}_{T^h}\ (\varrho)\}$

Similarly, $\bar{\xi}_{I^h}(\ \epsilon * \varrho) \geq \text{rmin}\{\bar{\xi}_{I^h}\ (\epsilon),\ \bar{\xi}_{I^h}\ (\varrho)\}$

$\bar{\xi}_{F^h}\ (\epsilon * \varrho) = \bar{\xi}_F(\text{h}\ (\epsilon * \varrho)$

$= \bar{\xi}_F(\text{h}\ (\epsilon) * \text{h}\ (\varrho))$

$\leq \text{rmax}\{\ \bar{\xi}_F(\text{h}\ (\epsilon),\ \bar{\xi}_F\ (\text{h}\ (\varrho)\}$

$\bar{\xi}_{F^h}\ (\epsilon * \varrho) \leq \text{rmax}\ \{\bar{\xi}_{F^h}\ (\epsilon), \bar{\xi}_{F^h}\ (\varrho)\}$

Hence, $\bar{\xi}^h$ is also an int_val neutrosophic $\hat{Z}$-subalgebra of $\mathfrak{W}$.

Theorem 4.4. Suppose J is the subset of $\mathfrak{W}$. An int_val neutrosophic set $\bar{\xi} = \{\epsilon : \bar{\varsigma}_{T,I,F}\ /\ \epsilon \in \mathfrak{W}\}$ such that $\bar{\varsigma}_{T,I} = \begin{cases} \bar{t} & \epsilon \in J \\ \bar{s} & \epsilon \notin J \end{cases}$, $\bar{\varsigma}_F = \begin{cases} \bar{\alpha} & \epsilon \in J \\ \bar{\beta} & \epsilon \notin J \end{cases}$ where $\bar{t}, \bar{s}, \bar{\alpha}, \bar{\beta} \in \mathfrak{D}\ [0,1]$ with $\bar{t} \geq \bar{s},\ \bar{\alpha} \leq \bar{\beta}$, Then the int_val neutrosophic set $\bar{\xi} = \{\ \epsilon : \bar{\varsigma}_T, \bar{\varsigma}_I, \bar{\varsigma}_F\ /\ \epsilon \in \mathfrak{W}\ \}$ is an int_val neutrosophic of $\hat{Z}$-algebra of $\mathfrak{W}$.

Proof: For $\epsilon, \varrho \in J$

i) $\bar{\varsigma}_T(\epsilon) = \bar{t} = \bar{\varsigma}_T(\varrho)$

$\Longrightarrow\ \bar{\varsigma}_T(\epsilon * \varrho) \geq \text{rmin}\ \{\bar{\varsigma}_T(\epsilon), \bar{\varsigma}_T(\varrho)\}$

$= \text{rmin}\ \{\bar{t}, \bar{t}\}$

$\bar{\varsigma}_T(\epsilon * \varrho) = \bar{t}$

ii) $\bar{\varsigma}_I(\epsilon) = \bar{t} = \bar{\varsigma}_I(\varrho)$

$\Longrightarrow\ \bar{\varsigma}_I(\epsilon * \varrho) \geq \text{rmin}\ \{\bar{\varsigma}_I(\epsilon), \bar{\varsigma}_I(\varrho)\}$

= rmin $\{\bar{t}, \bar{t}\}$

$\bar{\varsigma}_I(\epsilon * \varrho) = \bar{t}$

iii) For, ϵ, ϱ ∈ J

$\bar{\varsigma}_F(\epsilon) = \bar{\alpha} = \bar{\varsigma}_F(\varrho)$

$\Longrightarrow \bar{\varsigma}_F(\epsilon * \varrho) \leq$ rmax $\{\bar{\varsigma}_F(\epsilon), \bar{\varsigma}_F(\varrho)\}$

= rmax $\{\bar{\alpha}, \bar{\alpha}\}$

$\bar{\varsigma}_F(\epsilon * \varrho) = \bar{\alpha}$

Hence, $\bar{\varsigma}_{T,I,F}$ is an int_val neutrosophic Ẑ-algebra of 𝔚.

Theorem 4.5. Let f: 𝔚 → 𝔜 be the homomorphism of Ẑ-algebra. If $\bar{\varsigma}_{T,I,F}$ is the int_val neutrosophic Ẑ-algebra of 𝔚, with the rsup_rsup_rinf property & kerf ⊆ 𝔚$\bar{\varsigma}_{T,I,F}$ then the image of th set $\bar{\xi}$ ={ϵ : $\bar{\varsigma}_{T,I,F}$ / ϵ ∈ 𝔚 } , f($\bar{\varsigma}_{T,I,F}$) is also an int_val neutrosophic Ẑ-algebra of 𝔜.

Proof:

i) Let $f(\epsilon_1) = \varrho_1$, $f(\epsilon_2) = \varrho_2$

$f(\bar{\varsigma}_T)(\varrho_1 * \varrho_2)$ = rsup $\{\bar{\varsigma}_T(\epsilon_1 * \epsilon_2) : \epsilon \in f^{-1}(\varrho_1 * \varrho_2)$

$\geq$ rsup $\{ \bar{\varsigma}_T(\epsilon_1 * \epsilon_2) : \epsilon_1 \in f^{-1}(\varrho_1)$ & $\epsilon_2 \in f^{-1}(\varrho_2)\}$

$\geq$ rsup {rmin$\{ \bar{\varsigma}_T(\epsilon_1), \bar{\varsigma}_T(\epsilon_2)\}$, $\epsilon_1 \in f^{-1}(\varrho_1)$ &$\epsilon_2 \in f^{-1}(\varrho_2)\}$

$\geq$rmin {rsup $\{\bar{\varsigma}_T(\epsilon_1): \epsilon_1 \in f^{-1}(\varrho_1)\}$, rsup $\{\bar{\varsigma}_T(\epsilon_2)$: & $\epsilon_2 \in f^{-1}(\varrho_2)\}\}$

=rmin $\{ \underset{\epsilon_1 \in f^{-1}(\varrho)}{\text{rsup}} \{\bar{\varsigma}_T(\epsilon_1)\}, \underset{\epsilon_2 \in f^{-1}(\varrho)}{\text{rsup}} \{\bar{\varsigma}_T(\epsilon_2)\}\}$

=rmin$\{ f(\bar{\varsigma}_T)(\varrho_1), f(\bar{\varsigma}_T)(\varrho_2)\}$

ii) Similarly, $f(\bar{\varsigma}_I)(\varrho_1 * \varrho_2) \geq$ rmin$\{ f(\bar{\varsigma}_I)(\varrho_1), f(\bar{\varsigma}_I)(\varrho_2)\}$

iii) Let $f(\epsilon_1) = \varrho_1$, $f(\epsilon_2) = \varrho_2$

$f(\bar{\varsigma}_F)(\varrho_1 * \varrho_2)$ = rinf $\{\bar{\varsigma}_F(\epsilon_1 * \epsilon_2) : \epsilon \in f^{-1}(\varrho_1 * \varrho_2)$

$\leq$ rinf $\{ \bar{\varsigma}_F(\epsilon_1 * \epsilon_2) : \epsilon_1 \in f^{-1}(\varrho_1)$ & $\epsilon_2 \in f^{-1}(\varrho_2)\}$

$\leq$ rinf {rmax$\{ \bar{\varsigma}_F(\epsilon_1), \bar{\varsigma}_T(\epsilon_2)\}$, $\epsilon_1 \in f^{-1}(\varrho_1)$ & $\epsilon_2 \in f^{-1}(\varrho_2)\}$

$\leq$rmax {rinf {$\bar{\varsigma}_F(\epsilon_1)$: $\epsilon_1 \in f^{-1}(\varrho_1)$}, rinf {$\bar{\varsigma}_F(\epsilon_2)$: & $\epsilon_2 \in f^{-1}(\varrho_2)$}}

=rmax { $\underset{\epsilon_1 \in f^{-1}(\varrho)}{\text{rinf}}\{\bar{\varsigma}_F(\epsilon_1)\}$, $\underset{\epsilon_2 \in f^{-1}(\varrho)}{\text{rinf}}\{\bar{\varsigma}_F(\epsilon_2)\}$}

=rmax{f($\bar{\varsigma}_F$)(ϱ_1),f($\bar{\varsigma}_F$)(ϱ_2)}

Hence, f ($\bar{\varsigma}_{T,I,F}$) is an int_val neutrosophic Ẑ-algebra of 𝔜.

5. Product of Interval-valued(int_val) neutrosophic Ẑ-algebra

The section that follows the cartesian product of two int_val neutrosophic Ẑ-algebras $\bar{\xi} \times \bar{\zeta}$ of 𝔚 & 𝔜 respectively.

Definition5.1. Let $\bar{\xi}$ = {ϵ, $\bar{\varsigma}_{T,I,F}(\epsilon)$ / $\epsilon \in$ 𝔚} and $\bar{\zeta}$ = {ϱ, $\bar{\varsigma}_{T,I,F}(\varrho)$ / $\varrho \in$ 𝔜} be two int_val neutrosophic sets of 𝔚 & 𝔜 respectively. Then the cartesian product of $\bar{\xi}$ & $\bar{\zeta}$ is referred as $\bar{\xi}$ x $\bar{\zeta}$ then it is defined to be $\bar{\xi}$x $\bar{\zeta}$ = {(ϵ, ϱ), $\bar{\varsigma}_{T_{\bar{\xi}\times\bar{\zeta}}}(\epsilon,\varrho)$, $\bar{\varsigma}_{I_{\bar{\xi}\times\bar{\zeta}}}(\epsilon,\varrho)$, $\bar{\varsigma}_{F_{\bar{\xi}\times\bar{\zeta}}}(\epsilon,\varrho)$/($\epsilon \times \varrho$) $\in \bar{\xi} \times \bar{\zeta}$} where

$\bar{\varsigma}_{T_{\bar{\xi}\times\bar{\zeta}}}: \epsilon \times \varrho \to$ 𝔇 [0,1] ; $\bar{\varsigma}_{I_{\bar{\xi}\times\bar{\zeta}}}: \epsilon \times \varrho \to$ 𝔇 [0,1] and $\bar{\varsigma}_{F_{\bar{\xi}\times\bar{\zeta}}}: \epsilon \times \varrho \to$ 𝔇 [0,1]. $\bar{\varsigma}_{T_{\bar{\xi}\times\bar{\zeta}}}$ = rmin {$\bar{\varsigma}_{T_{\bar{\xi}}}(\epsilon)$, $\bar{\varsigma}_{T_{\bar{\zeta}}}(\varrho)$}; $\bar{\varsigma}_{I_{\bar{\xi}\times\bar{\zeta}}}$ = rmin {$\bar{\varsigma}_{I_{\bar{\xi}}}(\epsilon)$, $\bar{\varsigma}_{I_{\bar{\zeta}}}(\varrho)$}; $\bar{\varsigma}_{F_{\bar{\xi}\times\bar{\zeta}}}$ = rmax {$\bar{\varsigma}_{F_{\bar{\xi}}}(\epsilon)$, $\bar{\varsigma}_{F_{\bar{\zeta}}}(\varrho)$}.

Theorem 5.2. If $\bar{\xi}$ and $\bar{\zeta}$ be two int_val neutrosophic Ẑ-algebra of 𝔚 &𝔜 respectively, then $\bar{\xi} \times \bar{\zeta}$ is an int_val neutrosophic Ẑ-algebra of 𝔚 & 𝔜.

Proof: Let $\bar{\xi}$={ϵ, $\bar{\varsigma}_{T_{\bar{\xi}}}(\epsilon)$, $\bar{\varsigma}_{I_{\bar{\xi}}}(\epsilon)$, $\bar{\varsigma}_{F_{\bar{\xi}}}(\epsilon)$/$\epsilon \in$ 𝔚 } & $\bar{\zeta}$ = { ϱ, $\bar{\varsigma}_{T_{\bar{\zeta}}}(\varrho)$, $\bar{\varsigma}_{I_{\bar{\zeta}}}(\varrho)$, $\bar{\varsigma}_{F_{\bar{\zeta}}}(\varrho)$/$\varrho \in$ 𝔜 } be two int_val neutrosophic sets of 𝔚 &𝔜.

Take $\epsilon = (\epsilon_1, \varrho_1)$ and $\varrho = (\epsilon_2, \varrho_2)$

i) $\bar{\varsigma}_{T_{\bar{\xi}\times\bar{\zeta}}}(\epsilon * \varrho) = \bar{\varsigma}_{T_{\bar{\xi}\times\bar{\zeta}}}((\epsilon_1, \varrho_1) * (\epsilon_2, \varrho_2))$

$= \bar{\varsigma}_{T_{\bar{\xi}\times\bar{\zeta}}}((\epsilon_1 * \varrho_1), (\epsilon_2 * \varrho_2))$

= rmin{$\bar{\varsigma}_{T_{\bar{\xi}}}(\epsilon_1 * \varrho_1)$, $\bar{\varsigma}_{T_{\bar{\zeta}}}(\epsilon_2 * \varrho_2)$}

$\geq$ rmin {rmin {$\bar{\varsigma}_{T_{\bar{\xi}}}(\epsilon_1)$, $\bar{\varsigma}_{T_{\bar{\xi}}}(\epsilon_2)$},rmin{$\bar{\varsigma}_{T_{\bar{\zeta}}}(\varrho_1)$, $\bar{\varsigma}_{T_{\bar{\zeta}}}(\varrho_2)$}}

=rmin{rmin($\bar{\varsigma}_{T_{\bar{\xi}}}(\epsilon_1)$, $\bar{\varsigma}_{T_{\bar{\zeta}}}(\varrho_1)$},rmin{$\bar{\varsigma}_{T_{\bar{\xi}}}(\epsilon_2)$, $\bar{\varsigma}_{T_{\bar{\zeta}}}(\varrho_2)$}}

$=\text{rmin}\{\bar{\varsigma}_{T_{\bar{\xi}\times\bar{\zeta}}}(\epsilon),\ \bar{\varsigma}_{T_{\bar{\xi}\times\bar{\zeta}}}(\varrho)\}$

$\geq\text{rmin}\ \{\bar{\varsigma}_{T_{\bar{\xi}\times\bar{\zeta}}}(\epsilon), \bar{\varsigma}_{T_{\bar{\xi}\times\bar{\zeta}}}(\varrho)\}$

ii) Similarly, $\bar{\varsigma}_{I_{\bar{\xi}\times\bar{\zeta}}}(\epsilon * \varrho) \geq\text{rmin}\ \{\bar{\varsigma}_{I_{\bar{\xi}\times\bar{\zeta}}}(\epsilon), \bar{\varsigma}_{I_{\bar{\xi}\times\bar{\zeta}}}(\varrho)\}$

iii) $\bar{\varsigma}_{F_{\bar{\xi}\times\bar{\zeta}}}(\epsilon * \varrho) = \bar{\varsigma}_{F_{\bar{\xi}\times\bar{\zeta}}}((\epsilon_1, \varrho_1) * (\epsilon_2, \varrho_2))$

$= \bar{\varsigma}_{F_{\bar{\xi}\times\bar{\zeta}}}((\epsilon_1 * \varrho_1), (\epsilon_2 * \varrho_2))$

$= \text{rmax}\{\bar{\varsigma}_{F_{\bar{\xi}}}(\epsilon_1 * \varrho_1),\ \bar{\varsigma}_{F_{\bar{\zeta}}}(\epsilon_2 * \varrho_2)\}$

$\leq\text{rmax}\ \{\text{rmax}\ \{\bar{\varsigma}_{F_{\bar{\xi}}}(\epsilon_1),\ \bar{\varsigma}_{F_{\bar{\xi}}}(\epsilon_2), \text{rmax}\{\bar{\varsigma}_{F_{\bar{\zeta}}}(\varrho_1),\ \bar{\varsigma}_{F_{\bar{\zeta}}}(\varrho_2)\}\}$

$=\text{rmax}\{\text{rmin}(\bar{\varsigma}_{F_{\bar{\xi}}}(\epsilon_1),\ \bar{\varsigma}_{F_{\bar{\zeta}}}(\varrho_1)\}, \text{rmax}\{\bar{\varsigma}_{F_{\bar{\xi}}}(\epsilon_2),\ \bar{\varsigma}_{F_{\bar{\zeta}}}(\varrho_2)\}\}$

$=\text{rmax}\ \{\bar{\varsigma}_{F_{\bar{\xi}\times\bar{\zeta}}}(\epsilon),\ \bar{\varsigma}_{F_{\bar{\xi}\times\bar{\zeta}}}(\varrho)\}$

$\leq\text{rmax}\ \{\bar{\varsigma}_{F_{\bar{\xi}\times\bar{\zeta}}}(\epsilon), \bar{\varsigma}_{F_{\bar{\xi}\times\bar{\zeta}}}(\varrho)\}$

Hence, $\bar{\xi} \times \bar{\zeta}$ is an int_val neutrosophic Ẑ-algebra of $\mathfrak{W}$ & $\mathfrak{Y}$.

Theorem 5.3. If $\bar{\xi}_i = \{\epsilon \in \mathfrak{W}_i / \bar{\varsigma}_{T_{\bar{\xi}_i}}(\epsilon),\ \bar{\varsigma}_{I_{\bar{\xi}_i}}(\epsilon), \bar{\varsigma}_{F_{\bar{\xi}_i}}(\epsilon)\}$ be an int_val neutrosophic Ẑ-algebra of $\mathfrak{W}_i$ respectively, then $\prod_{i=1}^{n} \bar{\xi}_i$ is also an int_val neutrosophic Ẑ-algebra of $\prod_{i=1}^{n} \mathfrak{W}_i$.

Proof:

The Induction process on theorem 5.2.

i) $\prod_{i=1}^{n} \bar{\varsigma}_{T_{\bar{\xi}_i}}(\epsilon_i * \varrho_i) \geq \text{rmin}\ \{\prod_{i=1}^{n} \bar{\varsigma}_{T_{\bar{\xi}_i}}(\epsilon_i), \prod_{i=1}^{n} \bar{\varsigma}_{T_{\bar{\xi}_i}}(\varrho_i)\}$

ii) Similarly, $\prod_{i=1}^{n} \bar{\varsigma}_{I_{\bar{\xi}_i}}(\epsilon_i) \geq\text{rmin}\ \{\prod_{i=1}^{n} \bar{\varsigma}_{I_{\bar{\xi}_i}}(\epsilon_i), \prod_{i=1}^{n} \bar{\varsigma}_{I_{\bar{\xi}_i}}(\varrho_i)\}$

iii) $\prod_{i=1}^{n} \bar{\varsigma}_{F_{\bar{\xi}_i}}(\epsilon_i * \varrho_i) \leq \text{rmax}\ \{\prod_{i=1}^{n} \bar{\varsigma}_{F_{\bar{\xi}_i}}(\epsilon_i), \prod_{i=1}^{n} \bar{\varsigma}_{F_{\bar{\xi}_i}}(\varrho_i)\}$

5. Conclusions

The application of Int val neutrosophic Z-algebra marks a significant advancement in dealing with uncertainty and indeterminate information within various domains. Through its incorporation of interval-valued neutrosophic sets, int val neutrosophic Z-algebra provides a flexible framework for representing and manipulating information that encompasses not only truth and falsity but also the degree of indeterminacy present in real-world scenarios. This work deals about int_val neutrosophic in $\hat{Z}$-algebra using a binary operation $*$ and some of its properties and algebraic structures are also presented. In future, this work may extend to any type of algebra in many ways. This will be used in multiple types of fuzzy sets and their different extensions like int_val intuitionistic neutrosophic $\hat{Z}$ -algebra, cubic neutrosophic in $\hat{Z}$ -algebra.

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Using Neutrosophic Trait Measures to Analyze Impostor Syndrome in College Students after COVID-19 Pandemic with Machine Learning

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Abstract. Impostor syndrome or Impostor phenomenon is a belief that a person thinks their success is due to luck or external factors, not their abilities. This psychological trait is present in certain groups like women. In this paper, we propose a neutrosophic trait measure to represent the psychological concept of the trait-anti trait using refined neutrosophic sets. This study analysed a group of 200 undergraduate students for impostor syndrome, perfectionism, introversion and self-esteem: after the COVID pandemic break in 2021. Data labelling was carried out using these neutrosophic trait measures. Machine learning models like Support Vector Machine(SVM), K-nearest neighbour (K-NN), and random forest were used to model the data; SVM provided the best accuracy of 92.15%.

Keywords: Neutrosophic psychology; Impostor syndrome; Neutrosophic trait measure; SVM, KNN; Random forest

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1. **Introduction**

Impostor Syndrome or Impostor Phenomenon is a person's internal conviction that their success has happened due to pure chance, an external error, or hard work but not their capabilities or intellect. Individuals with impostor syndrome fear that others will someday learn that they are fakes [1]. They feel they do not belong in their working or academic environment despite qualifications, accomplishments, and achievements [2]. Impostor syndrome is appropriate to ethnic, racial, and gender groups [3], especially in creative arts-based careers where success is not readily quantifiable [4]. Impostor syndrome was also thought to be prevalent in women and varies with gender. In 1978, psychologists Pauline Rose Clance and Suzanne Imes presented the concept of an Impostor phenomenon. Their research examined graduate and undergraduate women who were relatively successful but felt "overvalued" by their counterparts or supervisors and further felt like imposters [5]. Furthermore, this research established that the impostor phenomenon happened more in females than males. It is attributed to how different genders synthesize their success. Men believe success comes from within, while women consider that success comes from outside. To complicate matters, it emerges that the more educated and skilled a woman is, the more she doubts her abilities [6]. Due to this, women with IS often work more than others to finally achieve the status they fear they have never earned. [5] coined the term Impostor Phenomenon (IP) to describe a sensation of internal intellectual phoniness that seemed especially widespread among a select group of female high-achievers. They were terrified of being labelled as "impostors" who did not belong "among all these brilliant, intelligent individuals." Rather than skill or competence, many ascribed their success to chance, hard effort, acquaintance with well-connected people, being at the correct place at the perfect time, fates, or individual characteristics such as charisma and the capability to interact effectively.

The Clance IP Scale was created to assess the idea that people are successful by superficial standards but have a false sense of personal ineptitude. The scale evaluates phenomena such as self-doubt and achievement by coincidence [7]. The study assesses the predictive accuracy of several machine learning methods. They proved that approaches, like ensemble learning, are better than simple machine learning algorithms [8]. According to a recent systematic analysis of the literature published in 2020, the prevalence of IS ranges from 9 to 82% [9]. According to [10], the IS prevalence among medical students and trainees was 22 -60%, and 33-44%, respectively. 15% of women dentistry students in the United States reported IS, while 57.8 % of youngsters in Saudi Arabia showed symptoms of Impostor Syndrome [11] [12].

The effect of impostorism on a specific leadership behaviour component of task delegation was analyzed in [13]. The population included 190 managers of various industries, with a prevalence of 74.6%. [3] examined the connection between the impostor phenomenon and racial discrimination in over one hundred and fifty African-American university students aged 18 to 19 through a cross-sectional survey. Respondents with more significant levels of IS also conveyed more survivor guilt feelings. [14] analyzed the prevalence of Impostor syndrome in final-year nursing students in a cross-sectional survey from Australia, the UK and New Zealand. The population selection included over 200 nursing students, of which 45.1% had mild IS, 33.4% were classified as repeatedly having IS feelings, and 8.3% were depicted as frequently experiencing intense IS experiences. A positive weak correlation between IS and preparedness for practice was found. [15] analyzed the connection between IS, perceived prejudice, and mental-health issues among minority trainees. The population sample was 322 College students with a mean age of 21 years (70% women) through a cross-sectional survey.

Perfectionism was described as "insisting of oneself or others a higher quality of performance than is needed by the circumstances". David Burn (1980) [16] produced a perfectionism scale, one of the first instruments to quantify perfectionism. He defined them as one whose ideals are high above reach or reason, who strains unremittingly and compulsively toward unattainable goals. Here, perfectionism is defined as a person's self-description of his or her performance style as perfectionistic, and most psychiatrists would likely agree with this assessment. The multidimensional perfectionism scale [17] is a test used to assess the character trait of perfectionism. Impostor syndrome in the classroom was analyzed in [18] to evaluate the influences of gender, level, grade, GPA, and individual characteristics on impostor syndrome among high school students. The study was conducted on 104 English honours students in grades 9-12, and there were no gender-based differences in impostorism. Impostor Syndrome was analyzed in [19] on 506 college students of mean age 21yrs (79% women) through a cross-sectional survey.

Women were notably more inclined to convey impostor feelings than men. Perfectionism, test anxiety and mental health were mainly related to IS, but low self-esteem was not. In [20] investigated gender disparities in anxiety of success and failure and IS on 104 marketing managers, of which response rate was 92.9% and mean age was 35yrs and 49% were women through a cross-sectional survey. Among male and female managers, significant positive correlations were observed between fear of failure and IS.

In [21], the authors evaluated emotional fatigue and work satisfaction among faculty with IS on 16 and 310 academic faculties for two studies, respectively. Study 1 had 63% women, and Study 2 had 59% women through a cross-sectional survey. Women and men vary in their coping techniques for managing impostor syndrome. [22] analyzed the impostor syndrome among Austrian doctoral students and evaluated gender differences in the impostor syndrome of nearly 631 students. Females had more fear of success and failure and lesser self-esteem than men. Faculties reported higher levels of impostor syndrome and research self-efficacy than non-faculty members. In [23], the authors explored the presence and connection between IS and burnout syndrome in internal medicinal residents.

[2] examined whether IS is a homogeneous construct or whether different types of persons with impostor syndrome can be distinguished based on related characteristics with 242 professionals in administration positions. No association was found between impostor syndrome and gender.

The authors have tried correlating well-being to impostor syndrome and gender role orientation in [24]. The population sample was 379 college students, which was a cross-sectional survey. People with high impostor syndrome scored less in well-being and self-acceptance. Significant differences were found in Impostor Syndrome by gender role orientation. [6] sampled the opinions of five doctors of various disciplines of medicine and their experiences with impostor syndrome. [25] identifies the prevalence of impostor syndrome among computer science students by conducting a cross-sectional survey on 203 college students. Additionally, it validates that the women students had more elevated levels of impostor feelings than the men.

[26] studied IS among first-generation and continuing-generation university students, with a population sample was 388 college students. After researching the relationship between IS level and perfectionism in these populations, only socially stipulated perfectionism was discovered to be mainly related to impostor syndrome among college students.

However, none of these studies thought there could be indeterminacy while making conclusions about these studies. As we see, sometimes, we may be unable to distinguish the presence or absence of impostor syndrome. In those cases, neutrosophic models can be used. Further, all results or conclusions may involve a certain amount of uncertainty in that situation. Neutrosophic will play a significant role.

Neutrosophy was introduced as a generalization to fuzzy theory. It handles the neutralities/indeterminacy present in the real-world scenario [27–31]. Neutrosophy has been recently used in psychology and will be very beneficial in analyzing impostor syndrome problems. Smarandache presented refined neutrosophic set (RNS) in [32] and further evolved into Double Valued NS (DVNS) [33], Triple Refined Indeterminate NS (TRINS) [34], and Multi Refined NS (MRNS) [35]. The indeterminate Likert scaling was defined using TRINS. Neutrosophy and neutrosophic psychology has been used to study several psychological problems [36–38].

This paper proposes the neutrosophic measure for Imposter syndrome based on neutrosophic traits and psychology. This is implemented using data collection using a specifically designed questionnaire after the data preprocessing and applying appropriate machine learning algorithms.

This paper is organized as follows: Section one is an introductory, extensive literature survey regarding impostor syndrome in young adults, and neutrosophy is given. Section two covers basic concepts of neutrosophy, psychology, indeterminate Likert scaling and Impostor syndrome. Neutrosophic measures for Impostor syndrome are introduced in section three.

Section four provides the dataset description along with the methodology and deals with the working of the proposed model, including calculating impostor syndrome scores. Section five deals with data analysis along with the machine learning models used. Results and discussion are given in section six. The last section discusses the limitations and concludes the study.

2. Basic Concepts

This subsection deals with the basic concepts of Neutrosophy and Neutrosophic psychology. The neutrosophic psychological framework is based on Freud's theory of conscious/unconcious memory and preconscious memory, it also includes one more state known as the aconscious state. Based on the neutrosophic theory, the extended neutrosophic psychology is denoted by $(< A >), < NeutA >, < antiA >)$.

Refined Neutrosophic sets are used to capture this, and several applications of neutrosophic psychology can be found in [32].

Definition 2.1. The refined neutrosophic set is described such that the truth T is split into several kinds of truths: $T_1, T_2, \ldots, T_p$, Indeterminate I into different indeterminacies: $I_1, I_2, \ldots, I_r$ and False F into different falsehoods: $F_1, F_2, \ldots, F_s$, where all $0 < p, r, s \leq 1$ are integers, and $p + r + s = n$.

Definition 2.2. [34] Consider X a space of objects with elements in X denoted by x. A TRINS A in X is represented by truth $T_A(x)$, indeterminacy leaning towards truth $IT_A(x)$, indeterminacy $I_A(x)$, indeterminacy leaning towards falsity $IF_A(x)$, and falsity $F_A(x)$ membership functions. For each component $x \in X$, there are

$$0 \leq T_A(x) + IT_A(x) + I_A(x) + IF_A(x) + F_A(x) \leq 5 \tag{1}$$

Thus, a TRINS A can be described by

$$A = \langle x, T_A(x), IT_A(x), INT_A(x), IAT_A(x), AT_A(x) \rangle : x \in X \tag{2}$$

and is characterized by trait $T_A(x)$, indeterminacy trait $IT_A(x)$, neutral trait $INT_A(x)$, indeterminate anti-trait $IAT_A(x)$ and anti trait $AT_A(x)$ membership functions.

Detailed examples and working of TRINS can be obtained from [34, 35, 39].

3. Neutrosophic trait measures

According to neutrosophic psychology, there are several trait and anti-trait pairs. Some of the most common trait and anti-trait pairs related to the Impostor syndrome are

Extraversion-Introversion

Perfectionism-Imperfectionism

Self esteem- Self non-esteem

Honesty - Dishonesty

Let us, for example, consider the extraversion-introversion, trait-anti-trait pair. A person can be an extrovert in some situations, at the same time, he/she can also be an introvert while interacting with some other people. There are also chances that s/he is an ambivert. To accurately capture this kind of personality trait, we have developed a questionnaire. We now proceed to describe the questionnaire.

The questionnaire is framed to cover all these traits and anti-traits pairs. We propose in the following section neutrosophic impostor syndrome measures. Instead of using a complete indeterminate Likert scaling-based questionnaire, we have framed the questionnaire differently since using the indeterminate Likert scaling questionnaire might require more professional support.

We have asked them to answer based on a 5-point Likert scaling to make the evaluation easier. Their answers were combined/aggregated into suitable neutrosophic measures.

We use the concept of TRINS defined by [34]. We extend the concept of TRINS values to psychology and propose a novel architecture for generating neutrosophic values from the questionnaire.

Here, we introduce the concept of neutrosophic trait measures for a trait, which is later used to define neutrosophic impostor syndrome and neutrosophic perfectionist measures.

Definition 3.1. Consider X to be a collection of all trait- anti pair with elements of X denoted by x. A neutrosophic trait S is based on a 5-tuple refined neutrosophic set. It is denoted as a neutrosophic set by

$$S = \langle x, Tr_S(x), ITr_S(x), NTr_S(x), IATr_S(x), ATr_S(x)\rangle : x \in X \tag{3}$$

where $Tr_S(x)$ denotes the degree of presence of trait S, which is based on the truth membership of TRINS, $ITr_S(x)$ denotes the degree of presence of indeterminate trait S, which is based on the truth leaning towards indeterminacy membership of TRINS, $NTr_S(x)$ denotes the degree of presence of neutral trait, which is based on the indeterminate membership of TRINS, $IATr_S(x)$ denotes the degree of presence of indeterminate anti-trait S, which is based on the false leaning towards indeterminacy membership of TRINS, $ATr_S(x)$ denotes the degree of presence of anti-trait S which is based on the false membership of TRINS.

Next, we define the three functions, namely, accuracy function, score function and certainty function.

Definition 3.2. The accuracy function a, defined over the Neutrosophic trait S of x as

$$a(S(x)) = Tr_S(x) - ATr_S(x) \tag{4}$$

Definition 3.3. The score function s, defined over the Neutrosophic trait S of x as

$$s(S(x)) = (Tr_S(x) + ITr_S(x) + (1 - NTr_S(x)) + (1 - IATr_S(x)) + (1 - ATr_S(x)))/5 \tag{5}$$

Definition 3.4. The certainty function c, defined over the Neutrosophic trait S of x as

$$c(S(x)) = Tr_S(x). \tag{6}$$

Neutrosophic Impostor syndrome measure calculations

We propose the Neutrosophic Impostor Syndrome IS as Neutrosophic Trait measure IS in X as given above in 3, it is characterized by the degree of Impostor syndrome using the trait and anti-trait membership values. The Neutrosophic Impostor syndrome is given by

$$IS = \langle x, Tr_{IS}(x), IrT_{IS}(x), NTr_{IS}(x), IATr_{IS}(x), ATr_{IS}(x)\rangle : x \in X \tag{7}$$

It is important to note that the anti-trait of Imposter syndrome is Peter's principle or the Dunning-Kruger effect. Similarly, the neutrosophic perfectionist NP is given by

$$NP = \langle x, Tr_{NP}(x), ITr_{NP}(x), INTr_{NP}(x), IATr_{NP}(x), ATr_{NP}(x)\rangle : x \in X \tag{8}$$

4. System Architecture

Figure 1 provides the proposed framework's architecture. We prepared a separate questionnaire, performed data collection, and pre-processed the data. Neutrosophic trait scores were calculated for each data point, and the data was labelled. Data analysis using exploratory data analysis was done, and then machine learning algorithms were implemented. Discussion and conclusions inferred from EDA and machine learning models are presented in the last module.

4.1. *Questionnaire Design*

We wanted to cover a set of interconnections between impostor syndrome, perfectionism, self-esteem and post-covid confidence levels in students after college reopened. Generally, impostor syndrome questionnaires , are based on the 5-point Likert scale, expect the summation of answers to provide the results.

Our questionnaire was designed innovatively with 29 items to cover impostor characteristics, perfectionism, self-esteem, emotional quotient and introversion-extroversion characteristics. Certain questionnaire sections were framed intentionally, so the respondents had to answer similar concept-based questions at least twice. Some questions were asked in the reversed

[1],https://paulineroseclance.com/pdf/IPTestandscoring.pdf

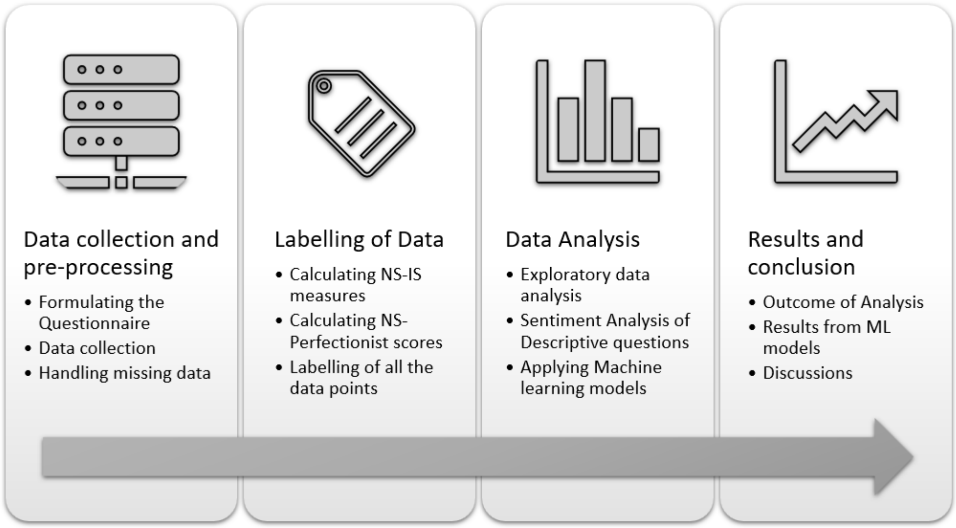


FIGURE 1. The overall system architecture

direction so that a respondent might capture a more genuine picture of themselves in one way or the other.

The survey conducted was anonymous. Questions were on a 5-point Likert scale, from "Strongly Agree" to "Strongly Disagree". The subjects were asked about their age and gender. All the questions were masked and worded positively to hide the direct intentions of the purpose of the questions. The complete questionnaire is provided in Appendix A. Certain salient features are discussed here to highlight its uniqueness.

Impostor Syndrome related: In the questionnaire, 13 items were focused on Impostor Syndrome; they are items 3-5, 8-14, 19-20 and 23 in Appendix 7.

Consider items Q4 and Q8

Q4: Many times, you feel crushed by constructive criticism,
seeing it as evidence of your "ineptness"?
Q8: In rare cases, you feel crushed by constructive criticism,
seeing it as evidence of your "ineptness"?

Item 8 is a reversed question of item 4. If a person has already made up his/her mind that the SA option is the default, or if they are trying to hide things, there are chances that they might exhibit impostor syndrome in the case of reversed questions.

Similarly, consider Q3, Q11 and Q20.

Q3. Do you chalk your success up to fates, luck or error?
Q11. You blame your luck for success rather than hard work.
Q20. Most of your success has been a stroke of luck.

All these questions deal with the luck factor.

Perfectionism related: Generally, impostor syndrome is related to perfection and self-esteem issues. To capture the same, questions related to perfection were asked. Items 6, 7, 15, 17 and 18 are related to perfectionism.

Self-esteem related: Two Likert scale questions were asked regarding their self-image in the house. The second part of the questionnaire consists of descriptive questions on whether the subjects' self-esteem (the question says confidence, though) in their college life had changed after the pandemic and their response to a generalized gender-based self-esteem statement.

IQ-EQ and introversion-extroversion related: Items 21 and 22 are related to IQ and EQ in self-judgment. The introversion-extroversion aspect is dealt with in items 25-27. It also consisted of a quantitative question about how comfortable they can be without being judged in different social settings like home, college, on social media, an online alter-ego and with friends.

4.2. *Methodology*

Data Collection: The dataset has been obtained by conducting a survey consisting of a detailed pen-and-paper questionnaire of 29 items, of which 22 were 5-point Likert scale-based questions and two written questions, covering aspects to measure the participants' levels of experiencing Impostor syndrome, perfectionism and gender. The respondents were 200 university students from Vellore, India.

The survey was conducted in the post-pandemic environment, with in-person classroom attendance. The students who were wiling voluteeraily participated in the study. Their ages ranged from 19 to 22. There was no monetary reward or added incentive for partaking. The study complied with the ethical research regulation of the college from which the respondents were recruited. The collected data was then manually entered into a CSV file by us.

Data pre-processing: Cleaning the manually entered data is an important data pre-processing step:

(1) Many issues were there in data entry; this was cleaned by hardcoding and replacing the anomalies found.
(2) Dropped serial number column for smoother working of machine learning models.
(3) As mentioned earlier, the questionnaire has inverted specific questions. To ensure that each answer's extremes indicated the same phenomenon, the answers (SD $\rightarrow$ SA) have been inverted for some questions (8, 12, 18) to facilitate this.
(4) Filled 'NA' values with 'N' (Neutral) to preserve the item's weight.

Weightage to questions: Of the thirteen items under consideration for Impostor syndrome, items 4, 8, 11-13, 19 & 20 were given a weightage of 2, and others were given a weightage of 1.

4.3. *Labelling of Data*

Neutrosophic Impostor Syndrome measures

The impostor syndrome is calculated from all three functions (accuracy, score and certainty), and a counter variable called result is used. It is initialized to 0.

- Neutrosophic impostor syndrome accuracy cutoff: If $a(IS) \geq 0$, result = result + 1 else result = result.
- Neutrosophic impostor syndrome score cutoff: If $s(IS) \geq 0.53$, result = result + 1 else result = result.
- Neutrosophic impostor syndrome certainty cutoff: If $c(IS) \geq 0.1$, result = result + 1 else result = result.

If the result ≥ 2, the data point is labelled Yes for impostor syndrome.

Similarly, the neutrosophic perfectionist accuracy, score and certainty are calculated, but the labelling is done directly only from item 15. The calculated neutrosophic measures are used for machine learning modelling.

Introversion score calculation: These questions (items 25, 26 and 27) were asked to see how introverted or extroverted the students were. Based on their responses to the questions, an extrovert-introvert score was calculated, which ranged from 0 to 10. If the score was less than or equal to five, the person was labelled an extrovert, while if the score was greater than five, they were labelled introverted.

The first item (25) was asked to show where the students could be themselves without being judged by their surroundings or peers. The next question was used to observe how comfortable the subjects felt at certain places in their daily life, like home, college, friends, workplace, online, and their alter-ego. Students were expected to answer this question in percentages. The following question was targeted to find where they had to pretend to be somebody else they were not. Depending on their responses to all three questions, the score calculation has been divided into six categories: nowhere, home, online and alter-ego, friends, college and workplace. The score variable is altered according to their choice for each question. A detailed score updation is given below: In the first question, if the student chose nowhere, it indicates that they cannot truly be themselves anywhere and are heavily inclined towards being an introvert. The score is then increased by ten. Suppose the student responded that they are comfortable at home. They pretend to be someone they are not at home while being comfortable (home >=50); 1 point is added to the score. If they chose that they felt judged at home but also gave a percentage of being comfortable at home >=70, it could be drawn that they assume they

Table 1. Summary of Dataset after labelling

Title	Overall Value	Female	Male	Undisclosed
Total people	200	41	131	28
number with IS	114	26	68	20
% of people with IS	57%	63.41%	51.91%	71.42%
number of people with perfectionism	57	18	33	6
% of people with perfectionism	28.50%	43.90%	25.19%	21.42%
number of people with IS having perfectionism	36	10	33	3
% of people with IS having perfectionism	31.50%	27.77%	91.66%	8.33%

feel comfortable at a place despite knowing they are judged there. 2 points are then added to the score. If this is not the case, two is removed from the score.

If the student chose online or alter-ego, they pretended to be someone they are not online and answered that they were comfortable(>=60), then; if they were comfortable having an alter-ego (%>=50), then three points are added to the score since it indicates that he/she might have that personality that they think is pretension. However, they cannot express it because they are introverted, and the online community gives a sense of safety to express themselves, else Three is subtracted from the score since they are not comfortable having an online alter-ego, indicating extroverts. Else, if they feel they can be themselves online without being judged and are comfortable in alter-ego(>60%), then the score is increased by 4, else if they were not that comfortable($\leq$ 60%), the score is increased by 1.

If the student chose with friends, they pretended to be someone else in front of their friends and be comfortable around them >60%, and if they feel they could be themselves, the score is increased by two else if they are comfortable being themselves at college >50.

5. Data analysis and Machine learning module

After the labelling of the data, exploratory data analysis was performed on the data; the results are tabulated in Table 1, which is discussed in detail in the next section. Out of the 200 participants, there were 41 females, 131 males and 28 people who chose not to disclose their gender. In total, 114 people had impostor syndrome, which is close to 57%.

Random forest: Random forest is an ML algorithm that uses multiple decision trees to make predictions. It is an extension of the bagging method that uses both bagging and feature randomness to create an uncorrelated forest of decision trees. Random Forest is commonly used for both classification and regression problems. It is a flexible and easy-to-use algorithm that usually produces excellent results, even without hyper-parameter tuning. The dataset was split into two parts: 75% for training and 25% for testing.

Support Vector Machine Support Vector Machine (SVM) is a supervised ML algorithm for classification and regression analysis. It is used to find the best boundary between two data classes by maximizing the margin between them. SVMs are often used in image classification, text classification, and bioinformatics. SVM was implemented for the dataset. The dataset split was a 70:30 ratio. We used a linear kernel to implement the model to avoid overfitting data.

K-Nearest Neighbours (K-NN) It is a simple ML algorithm founded on a supervised learning technique. It uses the likeness between the new case/data and available cases. It classifies a new data point based on the similarity of the dimensional features for each data point. For this model, the dataset was split into an 80:20 ratio.

Sentiment Analysis of descriptive questions

Calculating SVNS values: Expression of sentiment is very complex, but the VADER package is defined to understand online language closely. It has use cases to encompass utf-8 encoded emojis, emoticons (:D, :P, XD), slang words(sux), slang words modified (kinda, friggin), use of all caps(GOOD), use of exclamation points (good!!), and usage of typical negations (not good).

VADER contains inbuilt pre-defined objects like SentimentIntensityAnalyzer() and polarity_scores(). SentimentIntensityAnalyzer() takes input in string format and returns four values: positive (pos), negative(neg), neutral(neu) and compound(comp); where $0 \geq$ pos, neg, neu, comp ≤ 1. The scores denote how much positivity, negativity and neutrality lie in the sentence and the variable "compound" is calculated by normalizing the three scores. The closer the value of the compound is to "1", the higher the positivity of the sentence. If the compound value is ≥ 0.05, it is considered positive; if compound ≤ -0.05, it is negative, or else it is neutral. Since we are focusing on a gender-based study, we asked two questions to the subjects; one based on their views about self-image issues in men and fear of failure in women and the next question describes self-confidence pre and post covid.

6. Results and Discussions

6.1. *Results from EDA*

Of the 200 participants involved in the study, 114 had imposter syndrome. Of the 41 females, 26 had imposter syndrome; out of 131 males, 68 had imposter syndrome. There was a significant difference in the occurrence of Imposter syndrome between females and males, with 63.41% and 51.91%, respectively. It is visible and validated that women are likelier to

Table 2. Results from Machine Learning models

Models	Random forest	SVM	KNN
Train: Test Ratio	75:25	70:30	80:20
Accuracy (%)	83.7	92.15	82.35
Precision(weighted avg)	95.2	90.9	85.7
Recall(weighted avg)	76.9	90.9	85.7

have imposter syndrome. While 28.5% of the total respondents were perfectionists, 43.90% were perfectionists among the females. And only 25.19% of males were perfectionists.

It is interesting to note that 31.5% of people with Imposter syndrome are also perfectionists. However, there is a vast difference between females and males. Only 27.7% of women with imposter syndrome are perfectionists. In contrast, almost 91.66% of men with imposter syndrome are perfectionists. The correlation between perfectionism and imposter syndrome is observed here.

6.2. *Results from Machine Learning Models*

Three machine learning models, namely Random forest, SVM and KNN, were implemented using Python. The results are tabulated in Table 2.

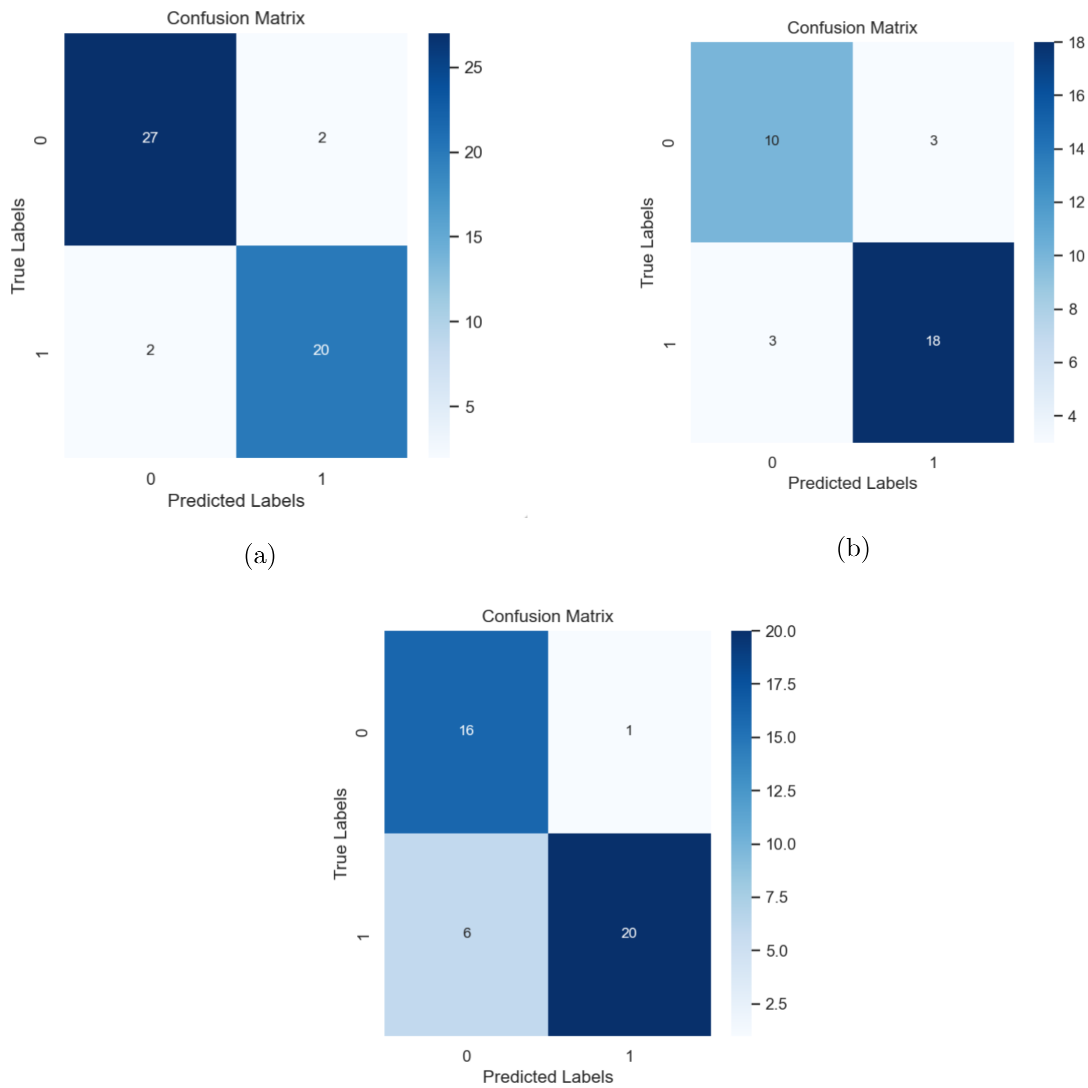
Random forest: The features used in the model include the student's age, their responses to questions, sentiment scores for two descriptive questions, and labels for perfectionist and impostor syndrome. The random forest model had a high accuracy of 83.7%, which means it is suitable for solving problems with many features. Figure 2a gives the confusion matrix for the random forest model.

SVM: SVM was implemented for the dataset. The dataset was split into a 70:30 ratio. We used a linear kernel to implement the model to avoid overfitting data. The best accuracy was obtained with the SVM model; we attained an accuracy of 92.15%. Figure 2b gives the confusion matrix for the SVM model.

KNN: KNN is a supervising machine learning model that stores all the data and classifies a new data point based on the similarity of the dimensional features for each data point. For this model, the dataset was split into an 80:20 ratio. We achieved an accuracy of 80.20%. The confusion matrix for KNN model is given in Figure 2c.

6.3. *Results from sentiment Analysis*

The data set was split based on gender and sentiment analysis was conducted for each long answer question. For the first question, men had a compound score of 0.1491 (positive) and women had 0.2689(positive); meaning women agreed that they face self-image issues more than



(a) (b) (c)

FIGURE 2. Confusion Matrix: (a) Support Vector Model; (b) K-NN ; (c) Random Forest

men. While analysing the second question, men had a compound score of 0.2545 (positive) and women had 0.2580(positive); meaning men and women equally believe that covid has affected their self-confidence.

7. Conclusions

Neutrosophic trait measures were introduced in this paper based on refined neutrosophic sets. A group of 200 students participated in the study; it was conducted and labelled using Neutrosophic trait measures. Impostor syndrome was analysed along with perfectionism, self-esteem issues and introversion. The ratio of students with Impostor syndrome was the same across the genders; no gender-based difference was found. After labelling the data, machine learning models like SVM, KNN and Random forest were implemented. SVM performed the best of the three models.

Limitation and Future Study: The study's primary limitation is based on the fact that the number of female participants is less here since the number of female students in STEM fields is lesser than that of boys in STEM. For future research, gender-oriented studies can be taken up with more participants. With more detailed data collection based on (Q16 and Q25), gender-related studies can be conducted to predict how Impostor syndrome affects the genders in STEM fields. The socio-economic background of the student should have been considered. Similarly, the educational background, like first-generation learners, can be considered in future studies.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Sample Questionnaire

(1) Gender:
(2) Age:
(3) Do you chalk your success up to fates, luck or error?
(4) Many times, you feel crushed by constructive criticism, seeing it as evidence of your "ineptness"?
(5) Do you believe "If I can do it, anybody can"?
(6) Do you agonize over the smallest flaws in your work?
(7) Do you believe that everything you do must be completely perfect?
(8) In rare cases you feel crushed by constructive criticism, seeing it evidence of your "ineptness"?
(9) Do you feel incompetent despite attaining success?
(10) You compare your abilities to people around you and think that others may be more intelligent than you.
(11) You blame your luck for success rather than hard work.
(12) Do you think shortcut to success makes you smarter
(13) You have doubts about your abilities despite people around you trusting you.
(14) Do you often fear not meeting other people's expectations?
(15) I am a perfectionist
(16) Do you feel like a non-valuable member of the family if you don't participate in domestic work?

(17) 'I have to be good at a particular activity to enjoy it' (As in, if you picked up a new hobby like painting, the only way you feel good about doing it is if you are using perfect techniques and doing it the "right" way)

(18) I believe that means is more important than the ends

(19) When people compliment you, you think you are not as accomplished as they think

(20) Most of your success has been a stroke of luck

(21) You have an above-average IQ score

(22) Your emotional quotient is better than your general IQ score

(23) You downplay compliments from others.

(24) Do you feel like an unimportant family member if you don't involve in decision-making process?

(25) Which places do you think you can be yourself without being judged?
Home College With friends Work Online Nowhere

(26) How comfortable are you with being yourself at (please give a percentage)
Home College With friends Work Online Alter Online Egos

(27) What places do you think you pretend to be someone you are not?
Home College With friends Work Online Nowhere

(28) At least 70% of individuals have dissatisfaction in their lives. Women mostly face self-image issues and in men, it is driven by the fear of not being successful or letting people down. Do you agree with this? And in your experience, how have you seen variances to the mentioned scenarios?

(29) As the campus has reopened after conducting classes, examinations, and project reviews online over the past 2 years, do you feel that this might bring up changes in confidence levels, due to the adjustments which may have arisen doubts in self-esteem from the initial change of method?

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Heptagonal Neutrosophic Topology

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Abstract. This article aims at developing the concept of heptagonal neutrosophic topology using heptagonal neutrosophic numbers. The heptagonal neutrosophic union,intersection and complement defined to compare the HNN. The interior, closure, exterior and boundary have introduced to discuss the properties of heptagonal neutrosophic topology and investigated to clarify the new concept and new possibilities in Heptagonal Neutrosophic Topological Space.

Keywords: Neutrosophic set, Heptagonal neutrosophic topology, neutrosophic interior, neutrosophic closure, neutrosophic exterior and neutrosophic boundary

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1. Introduction

Neutrosophic topological spaces have applications in various fields such as decision-making, computer science, and engineering, where the presence of indeterminate, vague, or uncertain information is prevalent. They provide a powerful tool for modeling and analyzing complex systems where classical topological spaces may not be sufficient. Subsequently after Zadeh's [22] introduction of the fuzzy set in the year 1965 with the membership function, the aforesaid fields are developed in various phases with many real life situations. The investigator focused their research in the above fields towards applications in practical problems with the help of intuitionistic fuzzy numbers with membership and non-membership values which was developed by Atanassov.K.T [8] in 1986.

There was a new finding between membership and non-membership values called indeterminacy and combined three values named as neutrosophic numbers which was introduced by Smarandache in 2005 [20]. After the introduction of neutrosophic numbers, investigators employ the concept of neutrosophic numbers and applied in various real life situations exclusively in topological spaces. Consequently, the neutrosophic topological spaces has been introduced by Salama.A.A and Alblowi.S.A in 2012 [4]. Lupia'nez [11–13] applied the neutrosophic concepts in topological spaces and developed a new research dimension in neutrosophic topological spaces.

The neutrosophic numbers from triangular to hexagonal have been published and have been documented their usage in actual life [17, 18]. In recent times (2021) Ali Hamza, Sara Farooq and Muhammad Rafaqat [7] presented Triangular neutrosophic topology. The topologies generated by triangular neutrosophic numbers were introduced by Kungumaraj.E and Narmatha.S [10] in 2022. In this article the extension work of [7] has been done and some of their properties have been investigated. This topological approach will be applied in network analysis, MCDM, image processing and topology optimization process.

This article incorporates five sections. The first section embraces the brief introduction, the second part encircles the preliminary definition and the results which are used in this article, the third section engrosses the main findings of Heptagonal Topological spaces and their properties, the fourth division comprehends the applications of third section which implies the continuous function and their properties of Heptagonal topological spaces. Finally the conclusion part contributes to expound the follow up work of this heptagonal topological space and applications of the same.

2. Preliminaries

Definition 2.1. Let X be a universe of discourse, A_N is a set disclosed in X. An element x from X is noted with respect to neutrosophic set as

$$A_N = \{< x; (\rho(x), \sigma(x), \omega(x)) >: x \in X\}$$

Where $\rho(x)$ is degree of truth membership, $\sigma(x)$ is degree of indeterminacy membership, $\omega(x)$ is degree of falsity membership. And $\rho(x), \sigma(x), \omega(x)$ are real standard or non standard subsets of $]0^-, 1^+[$. That is, There is no restrictions on the sum of $\rho(x), \sigma(x), \omega(x)$.

Definition 2.2. Let S be a space of points (objects), with a generic element in x denoted by S. A single valued neutrosophic set (SVNS) A in S is characterized by truth-membership function T_A, indeterminacy-membership function I_A and falsity-membership function F_A. For each point S in S, $T_A(x), I_A(x), F_A(x) \in [0, 1]$.
When S is continuous, a SVNS A can be written as A $= \int < T(x), I(x), F(x) > /x \in S$.
When S is discrete, a SVNS A can be written as A $=< T(x_i), I(x_i), F(x_i) > /x_i \in S$.

Definition 2.3. A Neutrosophic subset $\tilde{A}^N = (x, \mu_{\tilde{A}^N}(x), \nu_{\tilde{A}^N}(x), \vartheta_{\tilde{A}^N}(x)); x \in X$ of the real line R is called Neutrosophic number if the following conditions holds:
(i) There exist $x \in R$ such that $\mu_{\tilde{A}^N}(x) = 1$ and $\vartheta_{\tilde{A}^N}(x) = 0$

(ii) $\mu_{\tilde{A}^N}(x)$ is continuous function from $R \to [0,1]$ such that $0 \leq \mu_{\tilde{A}^N}(x)+\nu_{\tilde{A}^N}(x)+\vartheta_{\tilde{A}^I}(x) \leq 3$ for all $x \in X$

Definition 2.4. A Triangular Neutrosophic number $\tilde{A}^N$ is an Neutrosophic set in R with the following membership function $\mu_{\tilde{A}^N}(x)$, indetermiancy function$\nu_{\tilde{A}^N}(x)$ and non-membership function $\vartheta_{\tilde{A}^N}(x)$

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, \; if \; a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2}, \; if \; a_2 \leq x \leq a_3 \\ 0, \text{ otherwise} \end{cases} \qquad \nu_{\tilde{A}}(x) = \begin{cases} \frac{a_2-x}{a_2-a_1^{"}}, \; if \; a_1^{'} \leq x \leq a_2 \\ \frac{x-a_2}{a_3^{"}-a_2}, \; if \; a_2 \leq x \leq a_3^{'} \\ 1, \text{ otherwise} \end{cases} \qquad \vartheta_{\tilde{A}}(x) = \begin{cases} \frac{a_2-x}{a_2-a_1^{'}}, \; if \; a_1^{'} \leq x \leq a_2 \\ \frac{x-a_2}{a_3^{'}-a_2}, \; if \; a_2 \leq x \leq a_3^{'} \\ 1, \text{ otherwise} \end{cases}$$

where $a_1^{"} \leq a_1{}' \leq a_1 \leq a_2 \leq a_3 \leq a_3^{'} \leq_3$ " and $\mu_{\tilde{A}^I}(x) + \vartheta_{\tilde{A}^I}(x) \leq 1$, or $\mu_{\tilde{A}^I}(x) = \vartheta_{\tilde{A}^I}(x)$, for all $x \in R$. This TIFN is denoted by $\tilde{A}^I = (a_1, a_2, a_3; a_1{}', a_2, a_3{}')$.

Definition 2.5. Let (X,Y,<, >) be a dual pair, a dual topology on X is a locally convex topology τ so that

$$(X,Y)' \simeq Y$$

Here (X,Y)' denotes the continuous dual of (X,τ) and (X,Y)' $\simeq$ Y means that there is a linear isomorphism.

$$\Psi: Y \longrightarrow (X,Y)'.$$

Definition 2.6. Let $\tau \subseteq N(X)$ then τ is a neutrosophic topology on X if it satisfies the following conditions:

- $X, \phi \in \tau$
- The union and intersection of any number of neutrosophic sets in τ belongs to τ

The pair (X, τ) mentioned as neutrosophic topological space over X.

Definition 2.7. Let $\tau \subseteq N(X)$ be neutrosophic topological space over X then,

- ϕ and X as neutrosophic closed sets over X.
- The union and intersection of any two neutrosophic closed sets is a neutrosophic closed sets over X.

Definition 2.8. A heptagonal neutrosophic number S is defined and described as

$$S =< [(p,q,r,s,t,u,v);\mu], [(p',q',r',s',t',u',v');\gamma], [(p'',q'',r'',s'',t'',u'',v'');\eta] >$$

where $\mu, \gamma, \eta \in [0,1]$.The truth membership function ρ : R $\Rightarrow [0,\mu]$, the indeterminacy membership function σ : R$\Rightarrow [\gamma,1]$, the falsity membership function ω : R $\Rightarrow [\eta,1]$. Using ranking technique of heptagonal neutrosophic number is changed as,

$$\rho(x) = \frac{(p+q+r+s+t+u+v)}{7}$$
$$\sigma(x) = \frac{(p'+q'+r'+s'+t'+u'+v')}{7}$$
$$\omega(x) = \frac{(p''+q''+r''+s''+t''+u''+v'')}{7}$$

Heptagonal Neutrosophic Number Operations

(i) Inclusive: Let X be a non-empty set and A_{HN} and $B_{HN}$are NS of the form A_{HN}=$< x; \rho_{A_{HN}}(x), \sigma_{A_{HN}}(x), \omega_{A_{HN}}(x) >$, B_{HN}=$< x; \rho_{B_{HN}}(x), \sigma_{B_{HN}}(x), \omega_{HN}(x) >$. Then their subsets may be defines as follows,

- $A_{HN} \subseteq B_{HN} \Rightarrow \rho_{A_{HN}}(x) \leq \rho_{B_{HN}}(x); \sigma_{A_{HN}}(x) \geq \sigma_{B_{HN}}(x); \omega_{A_{HN}}(x) \geq \omega_{B_{HN}}(x) \forall x \in$X.
- $B_{HN} \subseteq A_{HN} \Rightarrow \rho_{B_{HN}}(x) \leq \rho_{A_{HN}}(x); \sigma_{B_{HN}}(x) \geq \sigma_{A_{HN}}(x); \omega_{B_{HN}}(x) \geq \omega_{A_{HN}}(x) \forall x \in$X.

(ii)Equality: If $A_{HN} \subseteq B_{HN}$ and $B_{HN} \subseteq A_{HN}$ then A_{HN}=B_{HN} is called Equality of a neutrosophic sets.

(iii)Union and Intersection: Let X be a non empty set and A_{HN} and B_{HN} are in NS of the form A_{HN}=$< x; \rho_{A_{HN}}(x), \sigma_{A_{HN}}(x), \omega_{A_{HN}}(x) >$, B_{HN}=$< x; \rho_{B_{HN}}(x), \sigma_{B_{HN}}(x), \omega_{HN}(x) >$, then $A_{HN} \cup B_{HN}$ and $A_{HN} \cap B_{HN}$ is defined as follows,

- $A_{HN} \cup B_{HN}$=$\{< x; (\rho_{A_{HN}}(x) \vee \rho_{B_{HN}}(x); \sigma_{A_{HN}}(x) \wedge \sigma_{B_{HN}}(x); \omega_{A_{HN}}(x) \wedge \omega_{B_{HN}}(x)) >: x \in X\}$
- $A_{HN} \cap B_{HN}$=$\{< x; (\rho_{A_{HN}}(x) \wedge \rho_{B_{HN}}(x); \sigma_{A_{HN}}(x) \vee \sigma_{B_{HN}}(x); \omega_{A_{HN}}(x) \vee \omega_{B_{HN}}(x)) >: x \in X\}$

(iv)Complement: Let A_{HN}=$< x; \rho_{A_{HN}}(x), \sigma_{A_{HN}}(x), \omega_{A_{HN}}(x) >$ in NS and complement of A^C_{HN} is defined as:

$$A^C_{HN} = \{< x; (\rho(x), \sigma(x), \omega(x)) >: x \in X\}$$

(v)Universal and Empty set: Let A_{HN}=$< x; \rho_{A_{HN}}(x), \sigma_{A_{HN}}(x), \omega_{A_{HN}}(x) >$ in NS and universal setI_A and empty set O_A of A_{HN} is defined as:

- I_A=$\{< x; 1, 0, 0 >: x \in X\}$
- O_A=$\{< x; 0, 1, 1 >: x \in X\}$

Example 2.9. Let A_{HN}, B_{HN} and C_{HN} are HNN and defined as follows,

$$A_{HN} = \{< x; (0{,}72, 0{,}41, 0{,}35, 0{,}81, 0{,}77, 0{,}73, 0{,}77), (0{,}83, 0{,}88, 0{,}93, 0{,}99, 0{,}96, 0{,}90, 0{,}94), (0{,}86, 0{,}99, 0{,}97, 0{,}93, 0{,}94, 0{,}91, 0{,}86) >, < y; (0{,}91, 0{,}32, 0{,}56, 0{,}48, 0{,}81, 0{,}72, 0{,}67), (0{,}78, 0{,}83, 0{,}21, 0{,}38, 0{,}56, 0{,}33, 0{,}98), (0{,}36, 0{,}86, 0{,}96, 0{,}32, 0{,}44, 0{,}56, 0{,}72) >\}$$

$$B_{HN} = \{< x; (0{,}96, 0{,}65, 0{,}73, 0{,}75, 0{,}83, 0{,}56, 0{,}54), (0{,}75, 0{,}95, 0{,}45, 0{,}38, 0{,}79, 0{,}57, 0{,}13), (0{,}59, 0{,}36, 0{,}68, 0{,}47, 0{,}36, 0{,}95, 0{,}44) >, < y; (0{,}38, 0{,}69, 0{,}88, 0{,}98, 0{,}77, 0{,}36, 0{,}98), (0{,}32, 0{,}72, 0{,}42, 0{,}62, 0{,}90, 0{,}22, 0{,}62), (0{,}42, 0{,}52, 0{,}62, 0 = 72, 0{,}36, 0{,}72, 0{,}61) >\}$$

$$C_{HN} = \{< x; (0{,}73, 0{,}74, 0{,}96, 0{,}34, 0{,}85, 0{,}89, 0{,}64), (0{,}46, 0{,}35, 0{,}25, 0{,}96, 0{,}36, 0{,}56, 0{,}16), (0{,}84, 0{,}85, 0{,}37, 0{,}57, 0{,}67, 0{,}22, 0{,}10) >, < y; (0{,}76, 0{,}72, 0{,}78, 0{,}62, 0{,}92, 0{,}56, 0{,}88), (0{,}38, 0{,}98, 0{,}22, 0{,}32, 0{,}54, 0{,}64, 0{,}31), (0{,}86, 0{,}96, 0{,}52, 0{,}22, 0{,}41, 0{,}51, 0{,}32) >\}$$

Using ranking technique by definition 2.4, We get

$$A_{HN} = \{< x; (0{,}65, 0{,}92, 0{,}92) >, < y; (0{,}64, 0{,}58, 0{,}60) >\}$$
$$B_{HN} = \{< x; (0{,}72, 0{,}57, 0{,}55) >, < y; (0{,}72, 0{,}54, 0{,}57) >\}$$
$$C_{HN} = \{< x; (0{,}74, 0{,}44, 0{,}52) >, < y; (0{,}75, 0{,}48, 0{,}53) >\}$$

From definition 2.4 we have,

$$\text{(i)} A_{HN} \subseteq B_{HN};\ B_{HN} \subseteq C_{HN} \Rightarrow A_{HN} \subseteq C_{HN}$$

$$A_{HN} \cup B_{HN} = \{< x; (0{,}65 \vee 0{,}72, 0{,}92 \wedge 0{,}57, 0{,}92 \wedge 0{,}55) >, < y; (0{,}64 \vee 0{,}72, 0{,}58 \wedge 0{,}54, 0{,}60 \wedge 0{,}57) >\}$$

$$A_{HN} \cup B_{HN} = \{< x; (0{,}72, 0{,}57, 0{,}55) >, < y; (0{,}72, 0{,}54, 0{,}57) >\}$$

Similarly,

$$B_{HN} \cup C_{HN} = \{< x; (0{,}74, 0{,}44, 0{,}52) >, < y; (0{,}75, 0{,}48, 0{,}53) >\}$$
$$A_{HN} \cup C_{HN} = \{< x; (0{,}74, 0{,}44, 0{,}52) >, < y; (0{,}75, 0{,}48, 0{,}53) >\}$$

$$(ii) A_{HN} \cap B_{HN} = \{< x; (0{,}65 \wedge 0{,}72, 0{,}92 \vee 0{,}57, 0{,}92 \vee 0{,}55) >, < y; (0{,}64 \wedge 0{,}72, 0{,}58 \vee 0{,}54, 0{,}60 \vee 0{,}57) >\}$$

$$A_{HN} \cap B_{HN} = \{< x; (0{,}65, 0{,}92, 0{,}92) >, < y; (0{,}64, 0{,}58, 0{,}60) >\}$$

Similarly,

$$B_{HN} \cap C_{HN} = \{< x; (0{,}72, 0{,}57, 0{,}55) >, < y; (0{,}72, 0{,}54, 0{,}57) >\}$$
$$A_{HN} \cap C_{HN} = \{< x; (0{,}65, 0{,}92, 0{,}92) >, < y; (0{,}64, 0{,}58, 0{,}60) >\}$$

$$(iii) A^C_{HN} = \{< x; (0{,}65, 1 - 0{,}92, 0{,}92) >, < y; (0{,}64, 1 - 0{,}58, 0{,}60) >\}$$
$$A^C_{HN} = \{< x; (0{,}92, 0{,}08, 0{,}65) >, < y; (0{,}60, 0{,}42, 0{,}64) >\}$$

Similarly

$$B^C_{HN} = \{< x; (0{,}55, 0{,}43, 0{,}72) >, < y; (0{,}57, 0{,}46, 0{,}72) >\}$$
$$C^C_{HN} = \{< x; (0{,}52, 0{,}56, 0{,}74) >, < y; (0{,}53, 0{,}52, 0{,}75) >\}$$

Theorem 2.10. *Let A_{HN},B_{HN} $\in$ N(X), then the following results are true*

1. $A_{HN} \cap A_{HN} = A_{HN}$ *and* $A_{HN} \cup A_{HN} = A_{HN}$
2. $A_{HN} \cap B_{HN} = B_{HN} \cap A_{HN}$ *and* $B_{HN} \cup A_{HN} = A_{HN} \cup B_{HN}$
3. $A_{HN} \cap \phi = \phi$ *and* $A_{HN} \cap X = A_{HN}$
4. $A_{HN} \cup \phi = A_{HN}$ *and* $A_{HN} \cup X = X$
5. $A_{HN} \cap (B_{HN} \cap C_{HN}) = (A_{HN} \cap B_{HN}) \cap C_{HN}$
6. $A_{HN} \cup (B_{HN} \cup C_{HN}) = (A_{HN} \cup B_{HN}) \cup C_{HN}$
7. $A_{HN} \cap (B_{HN} \cup C_{HN}) = (A_{HN} \cap B_{HN}) \cup (A_{HN} \cap C_{HN})$
8. $A_{HN} \cup (B_{HN} \cap C_{HN}) = (A_{HN} \cup B_{HN}) \cap (A_{HN} \cup C_{HN})$
9. $(A^C_{HN})^C = A_{HN}$
10. $A_{HN} \cup A^C_{HN} = X$ *and* $A_{HN} \cap A^C_{HN} = \phi$.

Proof: The results are obvious by the properties of HNN sets.

Theorem 2.11. *Let A_{HN},B_{HN} $\in$ N(X). Then*

1. $(\cup_{i\in I} A_{HN_i})^C = \cap_{i\in I} A^C_{HN_i}$
2. $(\cap_{i\in I} A_{HN_i})^C = \cup_{i\in I} A^C_{HN_i}$

Proof: (i)First verify $(\cup_{i\in I} A_{HN_i})^C \subseteq \cap_{i\in I} A^C_{HN_i}$. Let a $\in (\cup_{i\in I} A_{HN_i})^C$. Thus $a \notin \cup_{i\in I} A_{HN_i}$, so a cannot be in any of the sets A_{HN_i} i.e., for all i $\in$ I, we have $a \notin A_{HN_i}$, hence $a \in A^C_{HN_i}$ for all i $\in$ I. Thus $a \in \cap_{i\in I} A^C_{HN_i}$.Therefore, $(\cup_{i\in I} A_{HN_i})^C \subseteq \cap_{i\in I} A^C_{HN_i}$.
(ii)Now verify $\cap_{i\in I} A^C_{HN_i} \subseteq (\cup_{i\in I} A_{HN_i})^C$. Let $a \in \cap_{i\in I} A^C_{HN_i}$. Thus $a \in A^C_{HN_i}$ for all i$\in$I, hence $a \notin A_{HN_i}$ for all i$\in$I, so $a \notin \cup_{i\in I} A_{HN_i}$, hence $a \in (\cup_{i\in I} A_{HN_i})^C$.Therefore, $\cap_{i\in I} A^C_{HN_i} \subseteq (\cup_{i\in I} A_{HN_i})^C$.

Therefore, $(\cup_{i\in I} A_{HN_i})^C = (\cap_{i\in I} A_{HN_i})^C$.

Theorem 2.12. *Let A_{HN},B_{HN} $\in$ N(X). Then*

1. $B_{HN} \cap (\cup_{i\in I} A_{HN_i})$=$\cup_{i\in I}(B_{HN} \cap A_{HN_i})$
2. $B_{HN} \cup (\cap_{i\in I} A_{HN_i})$=$\cap_{i\in I}(B_{HN} \cup A_{HN_i})$

Proof: (i)Firstly we verify $B_{HN} \cap (\cup_{i\in I} A_{HN_i}) \subseteq \cup_{i\in I}(B_{HN} \cap A_{HN_i})$. If x $\in B_{HN} \cap (\cup_{i\in I} A_{HN_i})$, then x $\in B_{HN}$ and $x \in \cup_{i\in I} A_{HN_i}$. Then x $\in A_{HN_i}$ for some i∈I. Thus, $x \in B_{HN} \cap A_{HN_i}$. Hence, $x \in \cup_{i\in I}(B_{HN} \cap A_{HN_i})$. Therefore, $B_{HN} \cap (\cup_{i\in I} A_{HN_i}) \subseteq \cup_{i\in I}(B_{HN} \cap A_{HN_i})$.
(ii)Now verifying, $\cup_{i\in I}(B_{HN} \cap A_{HN_i}) \subseteq B_{HN} \cap (\cup_{i\in I} A_{HN_i})$. If $x \in \cup_{i\in I}(B_{HN} \cap A_{HN_i})$, then $x \in B_{HN} \cap A_{HN_i}$ for some i∈I. It follows that x $\in B_{HN}$ and x $\in \cup_{i\in I} A_{HN_i}$. Consequently, x $\in B_{HN} \cap (\cup_{i\in I} A_{HN_i})$. Therefore, $\cup_{i\in I}(B_{HN} \cap A_{HN_i}) \subseteq B_{HN} \cap (\cup_{i\in I} A_{HN_i})$. Therefore, $B_{HN} \cap (\cup_{i\in I} A_{HN_i})$=$\cup_{i\in I}(B_{HN} \cap A_{HN_i})$.

3. Heptagonal Neutrosophic topology and its Properties

Definition 3.1. Let X be a set. Let N(x) be a neutrosophic topology, τ be the collection of subsets of N(X) of X, then τ is a heptagonal neutrosophic topology on X, if it satisfy the following conditions;

- N(X) and $\phi \in \tau$
- Union of arbitrarily many elements of τ is an element of τ.
- Intersection of finite elements of τ is an element of τ.

Therefore the pair (X, τ) is a heptagonal neutrosophic topological space over X.
The set in τ are called HN - open set of X. The complement of HN - open set is called HN - closed set.

Example 3.2. Let X=$\{x, y\}$ and $A_{HN} \in$ N(X) then,

$$A_{HN} =\{< x; (0{,}72, 0{,}41, 0{,}35, 0{,}81, 0{,}77, 0{,}73, 0{,}77), (0{,}83, 0{,}88, 0{,}93, 0{,}99, 0{,}96, 0{,}90, 0{,}94), (0{,}86, 0{,}99, 0{,}97, 0{,}93, 0{,}94, 0{,}91, 0{,}86) >, < y; (0{,}91, 0{,}32, 0{,}56, 0{,}48, 0{,}81, 0{,}72, 0{,}67), (0{,}78, 0{,}83, 0{,}21, 0{,}38, 0{,}56, 0{,}33, 0{,}98), (0{,}36, 0{,}86, 0{,}96, 0{,}32, 0{,}44, 0{,}56, 0{,}72) >\}$$

By definition 2.4: We get
A_{HN}=$\{< x; (0{,}65, 0{,}92, 0{,}92) >, < y; (0{,}64, 0{,}58, 0{,}60) >\}$
Hence, τ=$\{\phi, X, A_{HN}\}$ is a heptagonal neutrosophic topology on X.

Example 3.3. Let X={x, y} and B_{HN}, C_{HN} ∈N(X) then,

B_{HN} ={< x; (0,96, 0,65, 0,73, 0,75, 0,83, 0,56, 0,54), (0,75, 0,95, 0,45, 0,38, 0,79, 0,57, 0,13), (0,59, 0,36, 0,68, 0,47, 0,36, 0,95, 0,44) >, < y; (0,38, 0,69, 0,88, 0,98, 0,77, 0,36, 0,98), (0,32, 0,72, 0,42, 0,62, 0,90, 0,22, 0,62), (0,42, 0,52, 0,62, 0 = 72, 0,36, 0,72, 0,61) >}

C_{HN} ={< x; (0,73, 0,74, 0,96, 0,34, 0,85, 0,89, 0,64), (0,46, 0,35, 0,25, 0,96, 0,36, 0,56, 0,16), (0,84, 0,85, 0,37, 0,57, 0,67, 0,22, 0,10) >, < y; (0,76, 0,72, 0,78, 0,62, 0,92, 0,56, 0,88), (0,38, 0,98, 0,22, 0,32, 0,54, 0,64, 0,31), (0,86, 0,96, 0,52, 0,22, 0,41, 0,51, 0,32) >}

By definition 2.4:, We get

$$B_{HN} = \{< x; (0{,}72, 0{,}57, 0{,}55) >, < y; (0{,}72, 0{,}54, 0{,}57) >\}$$

$$C_{HN} = \{< x; (0{,}74, 0{,}44, 0{,}52) >, < y; (0{,}75, 0{,}48, 0{,}53) >\}$$

Let (N(X),τ_1) and (N(X),τ_2) are heptagonal neutrosophic topological space. τ_1={ϕ, B_{HN}, X} and τ_2={ϕ, C_{HN}, X} is a heptagonal neutrosophic topology on X.
$\tau_1 \bigcap \tau_2$={ϕ, X, B_{HN}, C_{HN}} is not a heptagonal neutrosophic topology on X because $B_{HN} \bigcup C_{HN} \notin \tau_1 \bigcap \tau_2$.Whereas,$\tau$={$\phi, X, B_{HN}, C_{HN}, B_{HN} \bigcup C_{HN}, B_{HN} \bigcap C_{HN}$} is a heptagonal neutrosophic topology on X.

Remark: Let (X,τ) be a heptagonal neutrosophic topological space(HNTS). Then (X,τ)^C is the dual topology, whose elements are A^C_{HN} for $A_{HN} \in$ (X,τ). Any open set in τ is known as heptagonal neutrosophic open set(HNOs). Any closed set in τ is known as heptagonal neutrosophic closed set(HNCs) iff it's complement is heptagonal neutrosophic open set.

Definition 3.4. The heptagonal neutrosophic interior and Heptagonal neutrosophic closure are given by,

- HNint(A_{HN})=$\bigcup${O_{HN}/O_{HN} is a HNOs ∈ X where $O_{HN} \subseteq A_{HN}$} and it is the largest HN-open subset of A_{HN}.
- HNcl(B_{HN})=$\bigcap${J_{HN}/J_{HN} is a HNCs ∈ X where $J_{HN} \subseteq B_{HN}$} and it is the smallest HN-closed set containing B_{HN}.

Theorem 3.5. *If X be a set. Let (N(X),τ) is a HN topological space over X and A_{HN}, B_{HN} ∈N(X) then,*

1. *HNint(ϕ)=ϕ and HNint(X)=N(X)*
2. *HNint(A_{HN}) $\subseteq A_{HN}$*
3. *A_{HN} is HN open if and only if A_{HN}=HNint(A_{HN})*

4. *HNint(HNint(A_{HN}))=HNint(A_{HN})*
5. $A_{HN} \subseteq B_{HN} \Rightarrow$ *HNint(A_{HN})⊆HNint(B_{HN})*
6. *HNint(A_{HN})∪HNint(B_{HN})⊆HNint($A_{HN} \cup B_{HN}$)*
7. *HNint(A_{HN})∩HNint(B_{HN})=HNint($A_{HN} \cap B_{HN}$)*

Proof: i) Since ϕ and N(X) are HN-open, then HNint(ϕ)=ϕ and HNint(X)=N(X).
ii) From the definition of heptagonal neutrosophic interior, HNint(A_{HN})⊆ A_{HN}
iii) If A_{HN} is HN-open set over X, then A_{HN} is the largest HN-open set containing A. So, A_{HN}=HNint(A_{HN}).
Conversely, If A_{HN}=HNint(A_{HN}), then A_{HN} is the largest HN-open set containing A_{HN} and hence A_{HN} is HN-open.
iv) As HNint(A_{HN}) is open set, then HNint(HNint(A_{HN}))=HNint(A_{HN}).
v) When $A_{HN} \subseteq B_{HN}$, Also we know that, HNint(A_{HN})⊆ $A_{HN} \subseteq B_{HN}$. As HNint(A_{HN}) is a HN-subset of B_{HN}. So, HNint(A_{HN})⊆HNint(B_{HN}).
vi) It is obvious that, $A_{HN} \subseteq A_{HN} \cup B_{HN}$ and $B_{HN} \subseteq A_{HN} \cup B_{HN}$. From v),
HNint(A_{HN})⊆HNint($A_{HN} \cup B_{HN}$) and HNint(B_{HN})⊆HNint($A_{HN} \cup B_{HN}$)
⇒ HNint(A_{HN})∪HNint(B_{HN})⊆HNint($A_{HN} \cup B_{HN}$).
vii) It is obvious that $A_{HN} \cap B_{HN} \subseteq A_{HN}$ and $A_{HN} \cap B_{HN} \subseteq A_{HN}$. From v) HNint($A_{HN} \cap B_{HN}$)⊆HNint($A_{HN}$) and HNint($A_{HN} \cap B_{HN}$)⊆HNint($A_{HN}$) Also HNint($A_{HN}$)=$A_{HN}$ and HNint(B_{HN})=B_{HN}. Therefore, HNint(A_{HN})∩HNint(B_{HN})⊆ $A_{HN} \cap B_{HN}$
⇒HNint(A_{HN})∩HNint(B_{HN})=HNint($A_{HN} \cap B_{HN}$).

Example 3.6. Let X={x,y} and $A_{HN}, B_{HN}, C_{HN} \in$ N(X) then,

$$A_{HN} = \{< x; (0{,}6, 0{,}6, 0{,}6, 0{,}6, 0{,}6, 0{,}6, 0{,}6), (0{,}6, 0{,}6, 0{,}6, 0{,}6, 0{,}6, 0{,}6, 0{,}6), (0{,}6, 0{,}6, 0{,}6, 0{,}6, 0{,}6, 0{,}6, 0{,}6) >, < y; (0{,}8, 0{,}8, 0{,}8, 0{,}8, 0{,}8, 0{,}8, 0{,}8), (0{,}8, 0{,}8, 0{,}8, 0{,}8, 0{,}8, 0{,}8, 0{,}8), (0{,}8, 0{,}8, 0{,}8, 0{,}8, 0{,}8, 0{,}8, 0{,}8) >\}$$

$$B_{HN} = \{< x; (0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9), (0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9), (0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9) >, < y; (0{,}1, 0{,}1, 0{,}1, 0{,}1, 0{,}1, 0{,}1, 0{,}1), (0{,}1, 0{,}1, 0{,}1, 0{,}1, 0{,}1, 0{,}1, 0{,}1), (0{,}1, 0{,}1, 0{,}1, 0{,}1, 0{,}1, 0{,}1, 0{,}1) >\}$$

$$C_{HN} = \{< x; (0{,}2, 0{,}2, 0{,}2, 0{,}2, 0{,}2, 0{,}2, 0{,}2), (0{,}2, 0{,}2, 0{,}2, 0{,}2, 0{,}2, 0{,}2, 0{,}2), (0{,}2, 0{,}2, 0{,}2, 0{,}2, 0{,}2, 0{,}2, 0{,}2) >, < y; (0{,}4, 0{,}4, 0{,}4, 0{,}4, 0{,}4, 0{,}4, 0{,}4), (0{,}4, 0{,}4, 0{,}4, 0{,}4, 0{,}4, 0{,}4, 0{,}4), (0{,}4, 0{,}4, 0{,}4, 0{,}4, 0{,}4, 0{,}4, 0{,}4) >\}$$

By definition 2.4: we get

$$A_{HN} = \{< x; (0{,}6, 0{,}6, 0{,}6) >, < y; (0{,}8, 0{,}8, 0{,}8) >\}$$
$$B_{HN} = \{< x; (0{,}9, 0{,}9, 0{,}9) >, < y; (0{,}1, 0{,}1, 0{,}1) >\}$$
$$C_{HN} = \{< x; (0{,}2, 0{,}2, 0{,}2) >, < y; (0{,}4, 0{,}4, 0{,}4) >\}$$

HNint(B_{HN})=ϕ and HNint(C_{HN})=ϕ
Since HNint($B_{HN} \cup C_{HN}$)=ϕ therefore, HNint(B_{HN})$\cup C_{HN}$)=ϕ therefore HNint(B_{HN})$\cup$HNint(C_{HN})$\subseteq$HNint($B_{HN} \cup C_{HN}$).

Theorem 3.7. *If X be a set. Let (N(X),τ) is a HN topological space over X and $A_{HN}, B_{HN} \in$N(X) then,*

1. *HNcl(ϕ)=ϕ and HNcl(X)=N(X)*
2. *$A_{HN} \subseteq$HNcl(A_{HN})*
3. *A_{HN} is HN closed if and only if A_{HN}=HNcl(A_{HN})*
4. *HNcl(HNcl(A_{HN}))=HNcl(A_{HN})*
5. *$A_{HN} \subseteq B_{HN} \Rightarrow$ HNcl(A_{HN})$\subseteq$HNcl(B_{HN})*
6. *HNcl($A_{HN} \cup B_{HN}$)=HNcl(A_{HN})$\cup$HNint(B_{HN})*
7. *HNcl($A_{HN} \cap B_{HN}$)$\subseteq$HNcl(A_{HN})$\cup$HNcl(B_{HN})*

Proof: i) If A_{HN} is HN-closed then A_{HN}=HNcl(A_{HN}). Also is ϕ and X are HN-closed, then HNcl(ϕ)=ϕ and HNcl(X)=X.
ii) From the definition of HN-closure. It is obvious from the definition that $A_{HN} \subseteq$HNcl(A_{HN}).
iii) If A_{HN} is HN-closed set over X, then A_{HN} contains A_{HN} and that itself a HN-closed set over X. Then A_{HN} is the smallest HN-closed set containing A_{HN}. So, A_{HN}=HNcl(A_{HN}). Conversely, If A_{HN}=HNcl(A_{HN}), then A_{HN} is the smallest HN-closed set containing A_{HN} and hence A_{HN} is HN-closed.
iv) From above, As A_{HN} is closed, then A_{HN}=HNcl(A_{HN}). As HNcl(A_{HN}) is open set, then HNcl(HNcl(A_{HN}))=HNcl(A_{HN})
v) When $A_{HN} \subseteq B_{HN}$, Since $B_{HN} \subseteq$HNcl(B_{HN}) $\Rightarrow A_{HN} \subseteq$HNcl($B_{HN}$) That is HNcl($B_{HN}$) is a HN-closed set contains A_{HN}. But HNcl(A_{HN}) is the smallest HN-closed set contain A_{HN}. Thus, HNcl(A_{HN})$\subseteq$HNcl(B_{HN}).
vi),vii) is obvious

Example 3.8. Let X={x,y} and A_{HN}, B_{HN} ∈N(X) then,

$$A_{HN} =\{< x; (0{,}4, 0{,}4, 0{,}4, 0{,}4, 0{,}4, 0{,}4, 0{,}4), (0{,}4, 0{,}4, 0{,}4, 0{,}4, 0{,}4, 0{,}4, 0{,}4), (0{,}4, 0{,}4, 0{,}4, 0{,}4, 0{,}4, 0{,}4, 0{,}4) >, < y; (0{,}7, 0{,}7, 0{,}7, 0{,}7, 0{,}7, 0{,}7, 0{,}7), (0{,}7, 0{,}7, 0{,}7, 0{,}7, 0{,}7, 0{,}7, 0{,}7), (0{,}7, 0{,}7, 0{,}7, 0{,}7, 0{,}7, 0{,}7, 0{,}7) >\}$$

$$B_{HN} =\{< x; (0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5), (0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5), (0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5) >, < y; (0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9), (0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9), (0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9) >\}$$

By definition 2.4: we have

$$A_{HN} =\{< x; (0{,}4, 0{,}4, 0{,}4) >, < y; (0{,}7, 0{,}7, 0{,}7) >\}$$
$$B_{HN} =\{< x; (0{,}5, 0{,}5, 0{,}5) >, < y; (0{,}9, 0{,}9, 0{,}9) >\}$$

Then we have,

$$A_{HN} \cup B_{HN} =\{< x; (0{,}5, 0{,}4, 0{,}4) >, < y; (0{,}9, 0{,}7, 0{,}7) >\}$$
$$A_{HN} \cap B_{HN} =\{< x; (0{,}4, 0{,}5, 0{,}5) >, < y; (0{,}7, 0{,}9, 0{,}9) >\}$$

Consider, τ={ϕ,X,$A_{HN}, B_{HN}, A_{HN} \cup B_{HN}, A_{HN} \cap B_{HN}$} is a HN topology. After taking complements,
τ={X,ϕ,A^C_{HN},B^C_{HN},$(A_{HN} \cup B_{HN})^C$,$(A_{HN} \cap B_{HN})^C$}.Where,

$$A^C_{HN} =\{< x; (0{,}4, 0{,}6, 0{,}4) >, < y; (0{,}7, 0{,}3, 0{,}7) >\}$$
$$B^C_{HN} =\{< x; (0{,}5, 0{,}5, 0{,}5) >, < y; (0{,}9, 0{,}1, 0{,}9) >\}$$
$$(A_{HN} \cup B_{HN})^C =\{< x; (0{,}4, 0{,}6, 0{,}5) >, < y; (0{,}7, 0{,}3, 0{,}9) >\}$$
$$(A_{HN} \cap B_{HN})^C =\{< x; (0{,}5, 0{,}5, 0{,}4) >, < y; (0{,}9, 0{,}1, 0{,}7) >\}$$
$$HNcl(A_{HN}) =X$$

$$HNcl(B_{HN}) =X$$
$$A^C_{HN} \cap B^C_{HN} =\{< x; (0{,}4, 0{,}6, 0{,}5) >, < y; (0{,}7, 0{,}3, 0{,}9) >\}$$
$$HNcl(A_{HN} \cap B_{HN}) =(A_{HN} \cup B_{HN})^C$$
$$HNcl(A_{HN} \cap B_{HN}) \subseteq HNcl(A_{HN}) \cap HNcl(B_{HN}).$$

Definition 3.9. Let A_{HN} be a subset of a heptagonal neutrosophic topological space (N(X),τ). A point x∈ A^C_{HN} is said to be an exterior point of A if there exists an open set U containing x such that, U∈ A^C_{HN}. It is denoted by HNext(A_{HN}) and defined as:

HNext(A_{HN})={$\bigcup B; B \subseteq \tau, B \in X - A$}

Theorem 3.10. *If X be a set. Let (N(X),τ) is a HN topological space over X and* A_{HN}, B_{HN} *∈N(X) then*

1. *HNext(ϕ)=X*
2. *HNext(X)=ϕ*
3. *HNext(A_{HN})⊆A^C=X-A_{HN} for any $A_{HN} \subseteq X$*
4. *$A_{HN} \subseteq B_{HN} \Rightarrow$ HNext(B_{HN})⊆HNext(A_{HN})*
5. *HNint(A_{HN})⊆HNext(HNext(A_{HN}))*
6. *HNext($A_{HN} \cup B_{HN}$)=HNext(A_{HN})∩HNext(B_{HN})*
7. *HNext($A_{HN} \cap B_{HN}$)=HNext(A_{HN})∪HNext(B_{HN})*

Proof: i) HNext(ϕ)=HNint(X-ϕ)=X.

ii) HNext(X)=HNint(X-X)=ϕ.

iii) HNext(A_{HN})=int(A^C_{HN})⊆ A^C_{HN}. Since HNint(A_{HN})⊆ A_{HN}.

iv) If $A_{HN} \subseteq B_{HN}$, Then, HNext(B_{HN})=HNint(B^C_{HN}) Also we know that, $A_{HN} \subseteq B_{HN} \Rightarrow B^C_{HN} \subseteq A^C_{HN}$.Also, HNint($B^C_{HN}$)⊆HNint($A^C_{HN}$)

(i) implies, HNint(B_{HN})=HNint(B^C_{HN})⊆HNint(A^C_{HN}) ⊆HNext(A_{HN})

$\Rightarrow$ HNint(B_{HN})⊆HNint(A_{HN})

v) From(iii), HNext(A_{HN})⊆ A^C_{HN}

HNint(A^C_{HN})⊆HNext(HNext(A_{HN}))

HNint(A^C_{HN})⊆HNext(HNext(A_{HN}))

HNint(A_{HN})⊆HNext(HNext(A_{HN}))

vi) HNext($A_{HN} \cup B_{HN}$) =HNint$(A_{HN} \cup B_{HN})^C$

=HNint($A^C_{HN} \cap B^C_{HN}$)

=HNint(A^C_{HN})∩ HNint(B^C_{HN})

HNext($A_{HN} \cup B_{HN}$)=HNext(A_{HN})∩HNext(B_{HN})

vii) HNext($A_{HN} \cap B_{HN}$) =HNint$(A_{HN} \cap B_{HN})^C$

=HNint($A^C_{HN} \cup B^C_{HN}$)

=HNint(A^C_{HN})∪HNint(B^C_{HN})

HNext($A_{HN} \cap B_{HN}$)=HNext(A_{HN})∪HNext(B_{HN})

Definition 3.11. Let A_{HN} be a subset of a heptagonal neutrosophic topological space X and a point x∈X is said to be boundary point of A_{HN} if each open set containing at x intersects both A_{HN} and A^C_{HN}. It is denoted by HNFr(A_{HN}) and defined as:

HNFr(A_{HN})= HNcl(A_{HN})⋂HNcl$(A_{HN})^C$ or

HNFr(A_{HN})= HNcl(A_{HN})-HNint(A_{HN}) or

HNFr(A_{HN})= X-{HNint(A_{HN})⋃HNext(A_{HN})}

Remark: The boundary point is also known as boundary point. the set of all boundary point of a set A_{HN} is called the boundary of A_{HN} or the boundary of A_{HN}, which is denoted by HNfr(A_{HN}). Since by the definition, each boundary point of A_{HN} is also a boundary point of A_{HN}^C ad vice versa, so the boundary of A_{HN} is same as that of A_{HN}^C , i.e. HNfr(A_{HN})=HNfr(A_{HN}^C).

Theorem 3.12. *If A_{HN} is a subset of a HN topological space over X and then the following statements of boundary holds:*

1. *HNcl(X-A_{HN})=X-HNint(A_{HN})*
2. *HNfr(A_{HN})=HNcl(A_{HN})$\cap$ HNint(X-A_{HN})*
3. *HNfr(A_{HN}) is closed*
4. *HNfr(A_{HN})=HNfr(X-A_{HN})*
5. *HNfr(A_{HN})$\cap$ HNint(A_{HN})=ϕ*
6. *HNfr(HNint(A_{HN}))$\subseteq$ HNfr(A_{HN})*
7. *(HNfr(A_{HN}))^C=HNext(A_{HN})$\cup$HNint(A_{HN})*
8. *HNcl(A_{HN})=HNint(A_{HN})$\cup$ HNfr(A_{HN})*

Proof: i) let x$\in$ HNcl(X-A_{HN}) then x is the closure of X-A_{HN}. Then for every U$\in$ τ with x$\in$U, we have that; U$\cap$(X-A_{HN})=ϕ.
So there does not exist a open neighborhood of x that is fully contained in A_{HN}. This x$\notin$ HNint(A_{HN}) i.e., x$\in$(X- HNint(A_{HN})) so, HNcl(X-A_{HN})$\subseteq$X-HNint(A_{HN})
Now, let x$\in$(X- HNint(A_{HN})). Then x$\notin$ HNint(A_{HN}). So for ever open neighborhood U of x, we have that U $\nsubseteq$ A_{HN}. So U$\cap$(X-A_{HN})$\neq$0 for every open neighborhood U of x.Thus x$\in$HNcl(X-A_{HN}) so HNcl(X-A_{HN})$\supseteq$X-HNint(A_{HN})
Therefore, HNcl(X-A_{HN})=X-HNint(A_{HN})
ii) by definition we have HNfr(A_{HN})=HNcl(A_{HN})$\cap$HNint(A_{HN})
Or equivalently, HNfr(A_{HN})=HNcl(A_{HN})$\cap$(X- HNint(A_{HN})
From(i), HNfr(A_{HN})=HNcl(A_{HN})$\cap$HNcl(X-A_{HN})
iii) from2 HNfr(A_{HN}) can be written as as intersection of two closed sets and so HNfr(A_{HN}) is closed.
iv) From(ii), HNfr(A_{HN})=HNcl(A_{HN})$\cap$ HNcl(X-A_{HN}) Since, X-(X-A_{HN})=A_{HN}, also bt the proposition that: HNfr(X-A_{HN})=HNcl(X-A_{HN})$\cap$ HNcl(X-(X-A_{HN})) HNfr(X-A_{HN})= HNcl(X-A_{HN})$\cap$ HNcl(A_{HN})
Comparing, $\Rightarrow$ HNfr(A_{HN})=HNfr(X-A_{HN}).
v) and vi) is obvious
vii) $A_{HN} \in$ N(X).Then,
(HNfr(A_{HN}))^C=(HNcl(A_{HN})$\cap$HNfr(A_{HN}))^C
(HNfr(A_{HN}))^C=(HNcl(A_{HN}))$^C\cup$(HNfr(A_{HN}))^C

$(\text{HNfr}(A_{HN}))^C$=$(\text{HNcl}(A_{HN}))^C \cup (\text{HNint}(A_{HN}))^C$

$(\text{HNfr}(A_{HN}))^C$=$(\text{HNext}(A_{HN})) \cup (\text{HNfr}(A_{HN}))$.

viii) $A_{HN} \in$ N(X).Then, by definition and remark

$\text{HNint}(A_{HN}) \cup \text{HNfr}(A_{HN})$=$\text{HNint}(A_{HN}) \cup (\text{HNcl}(A_{HN}) \cap \text{HNfr}(A_{HN}))$

$\text{HNint}(A_{HN}) \cup \text{HNfr}(A_{HN})$=$\text{HNint}(A_{HN}) \cup (\text{HNcl}(A_{HN}) \cap (\text{HNint}(A_{HN}) \cup \text{HNfr}(A_{HN}))$

$\text{HNint}(A_{HN}) \cup \text{HNfr}(A_{HN})$=$\text{HNcl}(A_{HN}) \cap (\text{HNint}(A_{HN}) \cup \text{HNint}(A_{HN}))^C$

$\text{HNint}(A_{HN}) \cup \text{HNfr}(A_{HN})$=$\text{HNcl}(A_{HN}) \cap$X

$\text{HNint}(A_{HN}) \cup \text{HNfr}(A_{HN})$=$\text{HNcl}(A_{HN})$.

4. Applications of Heptagonal Neutrosophic Topology

Definition 4.1. Let X_{HN} and Y_{HN} are the non-void sets and f: $X_{HN} \longrightarrow Y_{HN}$ be a function, then

1. If A_{HN}=$\{\langle x, [\rho_{A_{HN}}(x), \sigma_{A_{HN}}(x), \omega_{A_{HN}}(x)]\rangle; x \in X_{HN}\}$ is a HN set in X_{HN}, then the image of A_{HN} under $f(A_{HN})$ is denoted by,
   $f(A_{HN})$=$\{\langle y, [f(\rho_{A_{HN}}(y)), f(\sigma_{A_{HN}}(y)), f(\omega_{A_{HN}}(y))]\rangle; y \in Y_{HN}\}$.
2. If B_{HN}=$\{\langle x, [\rho_{A_{HN}}(x), \sigma_{AHN}(x), \omega_{A_{HN}}(x)]\rangle; x \in X_{HN}\}$ is a HN set in X_{HN}, then the inverse-image of B_{HN} under $f^{-1}(B_{HN})$ is denoted by,
   $f^{-1}(B_{HN})$=$\{\langle x, [f^{-1}(\rho_{A_{HN}}(x)), f^{-1}(\sigma_{A_{HN}}(x)), f^{-1}(\omega_{A_{HN}}(x))]\rangle; x \in X_{HN}\}$.

Definition 4.2. A map f: $X_{HN} \longrightarrow Y_{HN}$ is called as heptagonal neutosophic continuous function if the inverse image $f^{-1}(A_{HN})$ of each heptagonal neutosophic open set A_{HN} is the heptagonal neutrosophic open in X_{HN}.

Definition 4.3. A map f: $X_{HN} \longrightarrow Y_{HN}$ is called as heptagonal neutosophic continuous function if the inverse image $f^{-1}(A_{HN})$ of each heptagonal neutrosophic closed set A_{HN} is the heptagonal neutrosophic closed in X_{HN}.

Theorem 4.4. *Let X and Y be a set. Let* A_{HN} $\{A_{HN_i}$*:i*$\in$ *I*$\}$ *be heptagonal neutrosophic set in* X_{HN} *and Let* B_{HN} $\{B_{HN_i}$*:i*$\in$ *I*$\}$ *be heptagonal neutrosophic set in* Y_{HN} *anf f:* $X_{HN} \longrightarrow Y_{HN}$*. Then,*

1. $A_{HN_1} \subseteq A_{HN_2} \Longleftrightarrow f(A_{HN_1}) \subseteq f(A_{HN_2})$
2. $B_{HN_1} \subseteq B_{HN_2} \Longleftrightarrow f^{-1}(B_{HN_1}) \subseteq f^{-1}(B_{HN_2})$
3. $A_{HN} \subseteq f^{-1}(f(A_{HN}))$ *and if f is injective, then* A_{HN}=$f^{-1}(f(A_{HN}))$
4. $f^{-1}(f(B_{HN})) \subseteq B_{HN}$ *and if f is surjective, then* $f^{-1}(f(B_{HN})) = B_{HN}$
5. $f^{-1}(\cup B_{HN_i})$=$\cup f^{-1}(B_{HN_i})$ *and* $f^{-1}(\cap B_{HN_i})$=$\cap f^{-1}(B_{HN_i})$
6. $f^{-1}(\cup A_{HN_i})$=$\cup f^{-1}(A_{HN_i})$ *and* $f^{-1}(\cap A_{HN_i}) \subseteq \cap f^{-1}(A_{HN_i})$ *and if f is injective, then* $f^{-1}(\cap A_{HN_i})$=$\cap f^{-1}(A_{HN_i})$
7. $f^{-1}(1_{HN})$=1_{HN} *and* $f^{-1}(0_{HN})$=0_{HN}

8. $f(1_{HN})$=1_{HN} *and* $f(0_{HN})$=0_{HN} *if f is injective.*

Proof: *The proof is obvious from the basic properties.*

Example 4.5. Let X_{HN}=$\{x, y\}$ and Y_{HN}=$\{x, y\}$and B_{HN}, C_{HN}, D_{HN} ∈N(X) then,

B_{HN} =$\{\langle x; (0{,}96, 0{,}65, 0{,}73, 0{,}75, 0{,}83, 0{,}56, 0{,}54), (0{,}75, 0{,}95, 0{,}45, 0{,}38, 0{,}79, 0{,}57, 0{,}13), (0{,}59, 0{,}36, 0{,}68, 0{,}47, 0{,}36, 0{,}95, 0{,}44)\rangle, \langle y; (0{,}38, 0{,}69, 0{,}88, 0{,}98, 0{,}77, 0{,}36, 0{,}98), (0{,}32, 0{,}72, 0{,}42, 0{,}62, 0{,}90, 0{,}22, 0{,}62), (0{,}42, 0{,}52, 0{,}62, 0{,}72, 0{,}36, 0{,}72, 0{,}61)\rangle\}$

C_{HN} =$\{\langle x; (0{,}73, 0{,}74, 0{,}96, 0{,}34, 0{,}85, 0{,}89, 0{,}64), (0{,}46, 0{,}35, 0{,}25, 0{,}96, 0{,}36, 0{,}56, 0{,}16), (0{,}84, 0{,}85, 0{,}37, 0{,}57, 0{,}67, 0{,}22, 0{,}10) >, \langle y; (0{,}76, 0{,}72, 0{,}78, 0{,}62, 0{,}92, 0{,}56, 0{,}88), (0{,}38, 0{,}98, 0{,}22, 0{,}32, 0{,}54, 0{,}64, 0{,}31), (0{,}86, 0{,}96, 0{,}52, 0{,}22, 0{,}41, 0{,}51, 0{,}32)\rangle\}$

D_{HN} =$\{\langle x; (0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5), (0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5), (0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5)\rangle, \langle y; (0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9), (0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9), (0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9)\rangle\}$

By definition 2.10:, We get

$$B_{HN} = \{\langle(0{,}72, 0{,}57, 0{,}55)\rangle, \langle(0{,}72, 0{,}54, 0{,}57)\rangle\}$$
$$C_{HN} = \{\langle(0{,}74, 0{,}44, 0{,}52)\rangle, \langle(0{,}75, 0{,}48, 0{,}53)\rangle\}$$
$$D_{HN} = \{\langle x; (0{,}5, 0{,}5, 0{,}5)\rangle, \langle y; (0{,}9, 0{,}9, 0{,}9)\rangle\}$$

Then the family E_{HN}=$\{0_{HN}, 1_{HN}, B_{HN}\}$ is a heptagonal neutrosophic topology on X_{HN} and F_{HN}=$\{0_{HN}, 1_{HN}, C_{HN}\}$ is a heptagonal neutrosophic topology on Y_{HN}.
Thus (X_{HN}, B_{HN}) and (Y_{HN}, C_{HN}) are heptagonal neutrosophic toplogical spaces.
Define $f : (X_{HN}, B_{HN}) \longrightarrow (Y_{HN}, C_{HN})$ as f(x)=y, f(y)=x and f(z)=z.

Then, f is heptagonal neutrosophic continuous function.

Theorem 4.6. *Let f:*$X_{HN} \longrightarrow Y_{HN}$ *be a single valued HN function, where*X_{HN} *and* Y_{HN} *are HN topological spaces. Then the following statements are equivalent:*

1. The function f is HN continuous.
2. The inverse image of each HN open set in Y_{HN} is HN open in X_{HN}.

Proof: (i)⟹(ii):

Firstly, assume that f:$X_{HN} \longrightarrow Y_{HN}$ is HN continuous. Let A_{HN} be HN open in Y_{HN}. Then A^C_{HN} is HN closed in Y_{HN}. Since Y_{HN} is HN continuous $f^{-1}(A^C_{HN})$ is HN closed in X_{HN}. But $f^{-1}(A^C_{HN})$=X-$f^{-1}(B_{HN})$. Thus $X_{HN} - f^{-1}(B_{HN})$ is HN closed in X_{HN} and we have that

$f^{-1}(A_{HN})$ is HN open in X. Therefore, (i)$\Longrightarrow$(ii).

(ii)$\Longrightarrow$(i):

Conversely, we assume that the inverse image of each HN open set in Y_{HN} is HN open in X_{HN}. Let B_{HN} be any HN closed set in Y_{HN}. Then B_{HN}^C is HN open in V. By our assumption, $f^{-1}(B_{HN}^C)$ is HN open in X_{HN}. But then, $f^{-1}(B_{HN}^C)$=$X_{HN} - f^{-1}(B_{HN})$. Then $X_{HN} - f^{-1}(B_{HN})$ is HN open in X_{HN} and also $f^{-1}(B_{HN})$ is HN closed in X_{HN}. Therefore f is HN continuous. Hence, (ii)$\Longrightarrow$(i). Therefore (i) and(ii) are equivalent.

Theorem 4.7. *A mapping f:$X_{HN} \longrightarrow Y_{HN}$ is heptagonal neutrosophic continuous iff the inverse image of every heptagonal neutrosophic closed set in Y_{HN} is heptagonal neutrosophic closed in X_{HN}.*

Proof: Firstly we assume that f is a HN continuous. Let A_{HN} be a heptagonal neutrosophic closed set in Y_{HN}. Then A_{HN}^C is open in Y_{HN}. By our assumption, f is HN continuous function, $f^{-1}(A_{HN})$ is HN open in X_{HN}. But then, $f^{-1}(A_{HN}^C)$=$X_{HN} - f^{-1}(A_{HN})$.

Therefore, $f^{-1}(A_{HN})$ is heptagonal neutrosophic closed in X_{HN}.

Conversely, assume the pre image of every heptagonal neutrosophic closed set in Y_{HN} is heptagonal neutrosophic closed in X_{HN}. Let B_{HN} be a HN open set in Y_{HN}, then B_{HN}^C is HN closed in Y_{HN}. By hypothesis that, $f^{-1}(B_{HN}^C)$=$X_{HN} - f^{-1}(B_{HN})$ is HN closed in X_{HN} and so $f^{-1}(B_{HN})$ is HN open in X_{HN}.

Therefore, f is heptagonal neutrosophic continuous.

Theorem 4.8. *A mapping :$X_{HN} \longrightarrow Y_{HN}$ is heptagonal neutrosophic continuous if and only if f(HNcl(A_{HN}))⊂HNcl(f(A_{HN})) for every subset A_{HN} of X_{HN}.* ***Proof:*** Firstly. We assume that f is HN continuous. Let A_{HN} be any subset of X_{HN}. Then HNcl(f(A_{HN})) is a HN closed set in X_{HN}. Since by our assumption f is HN continuous, $f^{-1}(HNcl(f(A_{HN})))$ is HN closed in X_{HN} and it contains A_{HN}. By the definition of HN closure, HNcl(A_{HN}) is the intersection of all HN closed sets containing A_{HN}. Therefore, HNcl(A_{HN})⊆ $f^{-1}(HNcl(f(A_{HN})))$.

Therefore, f(HNcl(A_{HN}))⊂(HNcl(f(A_{HN}))).

Conversely, assume that f(HNcl(A_{HN}))⊂(HNcl(f(A_{HN}))). Let B_{HN} is HN closed in Y_{HN}, f(HNcl($f^{-1}(B_{HN})$))⊆HNcl(B_{HN}). Thus we have, HNcl($f^{-1}(B_{HN})$) $\subseteq f(^{-1}HNcl(B_{HN}))=$ $f^{-1}(B_{HN})$. But we know that $f^{-1}(B_{HN})$ ⊆HNcl($f^{-1}(B_{HN})$). Which then implies that, HNcl($f^{-1}(B_{HN})$)=$f^{-1}(B_{HN})$. Therefore $f^{-1}(B_{HN})$ is HN closed set in X_{HN} for every HN closed set B_{HN} in Y_{HN}.Then by the definition of HN continuity function,

f is heptagonal neutrosophic continuous.

Theorem 4.9. Let (X,τ_X) and (Y,τ_Y) be a heptagonal neutrosopic topological space and let f:$X_{HN} \longrightarrow Y_{HN}$ be the mapping. Then the following statements are equivalent.

1. f is HN continuous map.
2. For each subset $A_{HN} \subseteq X_{HN}$, we have $f(\overline{A}) \subseteq \overline{f(A)}$.
3. For every HN closed subset $B_{HN} \subseteq Y_{HN}$, then the set $f^{-1}(B_{HN})$ is HN closed in X_{HN}.
4. For each x$\in$ X_{HN} and each $B_{HN} \in \tau_Y$ containing f(x), there is some $U_{HN} \in \tau_X$ containing x and such that $f(U_H) \subseteq B_{HN}$.

Proof: We prove the above statements as follows: (i) implies (ii), (ii) implies (iii), (iii) implies (iv) and finally (i) implies (iv).

(i)$\Rightarrow$(ii): Assume that f is a HN continuous mapping. Let $A_{HN} \subseteq X_{HN}$ be a subset. For each x$\in$ $\overline{A_{HN}}$ we have to show that f(x)$\in$ $\overline{f(A_{HN})}$. Fix for such x and letting $B_{HN} \in \tau_Y$ be any HN open subset containing f(x). Since by oue assumption, f is HN continuous, the subset U_{HN}=$f^{-1}(B_{HN})$ is an HN open subsets that contains the element x. Note that $U_{HN} \cap A_{HN} \neq \varnothing$, therefore there exists y$\in$ $A_{HN} \cap u_{HN}$ and f(y)$\in$ $B_{HN} \cap f(A_{HN})$. Since every HN open subset containing f(x) intersects f(A_{HN}) nontrivially,

$$f(\overline{A})HN) \subseteq \overline{f(A_{HN})}.$$

(ii)$\Rightarrow$(iii): Assume that for subset $A_{HN} \subseteq X_{HN}$, we have $f(\overline{A}) \subseteq \overline{f(A)}$. Let $B_{HN} \subseteq Y_{HN}$ be a HN closed subset and let A_{HN}=$f^{-1}(B_{HN})$. We need to show that A_{HN}=$\overline{A_{HN}}$(more specifically that $\overline{A_{HN}} \subseteq A_{HN}$, the opposite containment is always true). So fix that x$\in$ $\overline{A_{HN}}$. Then,

$$\text{f(x)}\in f(\overline{A})HN) \subseteq \overline{f(A_{HN})} \subseteq \overline{B_{HN}}=B_{HN}.$$

That is, f(x)$\in$ B_{HN}. Or in other words x$\in$ $f-1(B_{HN})$=A_{HN} as required.

(iii)$\Rightarrow$(iv): Assume that, for every HN closed subset $B_{HN} \subseteq Y_{HN}$, then the set $f^{-1}(B_{HN})$ is HN closed in X_{HN}. Suppose the pre-images of HN closed sets are HN closed. Fix x$\in$ X_{HN}, and an HN open set $B_{HN} \in \tau_Y$ containing f(x). Then $Y_{HN} - B_{HN}$ is HN closed and hence $f^{-1}(Y_{HN} - B_{HN})$ a HN closed sunset of X_{HN} by our assumption and it does not contains x. But then the complement of this set, $X_{HN} - f^{-1}(Y_{HN} - B_{HN})$, is the HN open and does contains x. So let us fix the HN open set U_{HN} such that,

$$\text{x}\in U_{HN} \subseteq X_{HN} - f^{-1}(Y_{HN} - B_{HN}).$$

Then we have, $f(U_{HN}) \subseteq f(X_{HN} - f^{-1}(Y_{HN} - B_{HN})$=$f(X_{HN}) - (Y_{HN} - B_{HN}) \subseteq B_{HN}$,

$$f(U_{HN}) \subseteq B_{HN} \text{ as required.}$$

(i)$\Rightarrow$(iv):Assuming that, f is HN continuous map. Let x$\in$ X_{HN} and let $B_{HN} \in \tau_Y$ containing f(x). Then the set U_{HN}=$f^{-1}(B_{HN})$ is a HN open subset containing x. Conversely, assume that (iv) holds. Let $B_{HN} \in \tau_Y$ and let x$\in$ $f^{-1}(B_{HN})$. Then f(x)$\in$ B_{HN} and by the hypothesis there exists some $U_{HN_x} \in \tau_Y$ containing x and such that $f(U_{HN_x}) \subseteq B_{HN}$. Thus $U_{HN_x} \subset f^{-1}(B_{HN})$. It follows that $f^{-1}(B_{HN})$=$\bigcup_{x\in f^{-1}(B_{HN})} U_{HN_x}$, which is then the element of τ_X.

Theorem 4.10. A mapping f:$X_{HN} \longrightarrow Y_{HN}$ is heptagonal neutrosophic open function if and only if f(HNint(A_{HN}))⊂HNint(f(A_{HN})) for every subset A_{HN} of X_{HN}. **Proof:** Firstly we assume that, f:$X_{HN} \longrightarrow Y_{HN}$ is heptagonal neutrosophic open function and A_{HN} be a heptagonal neutosophic subset of X_{HN}. Clearly we can see that HNint(A_{HN}) is an HN open set in X_{HN} and HNint(A_{HN}) $\subseteq A_{HN}$. Since by our assumption f is a HN open function, so f(HNint(A_{HN})) is a HN open set in X_{HN}. And f(HNint(A_{HN}))$\subseteq f(A_{HN})$. Since each HN open set is a HN open set and $HNint(f(A_{HN}))$ is the largest HNopen set containing $f(A_{HN})$, so that $HNint(f(A_{HN}))$ is the largest HN open set contained in $f(A_{HN})$. Therefore ,

f(HNint(A_{HN}))⊂HNint(f(A_{HN})) for each HN subset A_{HN} of X_{HN}.

Conversely assume that, f(HNint(A_{HN}))⊂HNint(f(A_{HN})) for every subset A_{HN} of X_{HN}. Let B_{HN} be an HN set in X_{HN}. Therefore, HNint(B_{HN})=B_{HN}. By the hypothesis we have that, f(HNint(B_{HN}))⊂HNint(f(B_{HN})). Which implies that f(B_H)⊆HNint(f(B_H)). Also we have that HNint(f(B_H))⊆ f(B_H). Therefore f(B_H)=HNint(f(B_H)).That is, f(B_{HN}) is the HN open set in X_{HN}. Hence for every HN open set in X_{HN}, f(B_{HN})is the HN open set in X_{HN}. Therefore f is the HN open function.

Example 4.11. Let X_{HN}=$\{x, y\}$ and B_{HN}, C_{HN}, D_{HN} ∈N(X) then,

$$B_{HN} =\{\langle x; (0{,}96, 0{,}65, 0{,}73, 0{,}75, 0{,}83, 0{,}56, 0{,}54), (0{,}75, 0{,}95, 0{,}45, 0{,}38, 0{,}79, 0{,}57, 0{,}13), (0{,}59, 0{,}36, 0{,}68, 0{,}47, 0{,}36, 0{,}95, 0{,}44)\rangle, \langle y; (0{,}38, 0{,}69, 0{,}88, 0{,}98, 0{,}77, 0{,}36, 0{,}98), (0{,}32, 0{,}72, 0{,}42, 0{,}62, 0{,}90, 0{,}22, 0{,}62), (0{,}42, 0{,}52, 0{,}62, 0{,}72, 0{,}36, 0{,}72, 0{,}61)\rangle\}$$

$$C_{HN} =\{\langle x; (0{,}73, 0{,}74, 0{,}96, 0{,}34, 0{,}85, 0{,}89, 0{,}64), (0{,}46, 0{,}35, 0{,}25, 0{,}96, 0{,}36, 0{,}56, 0{,}16), (0{,}84, 0{,}85, 0{,}37, 0{,}57, 0{,}67, 0{,}22, 0{,}10) >, \langle y; (0{,}76, 0{,}72, 0{,}78, 0{,}62, 0{,}92, 0{,}56, 0{,}88), (0{,}38, 0{,}98, 0{,}22, 0{,}32, 0{,}54, 0{,}64, 0{,}31), (0{,}86, 0{,}96, 0{,}52, 0{,}22, 0{,}41, 0{,}51, 0{,}32)\rangle\}$$

$$D_{HN} =\{\langle x; (0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5), (0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5), (0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5, 0{,}5)\rangle, \langle y; (0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9), (0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9), (0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9, 0{,}9)\rangle\}$$

By definition 2.10:, We get

$$B_{HN} =\{\langle(0{,}72, 0{,}57, 0{,}55)\rangle, \langle(0{,}72, 0{,}54, 0{,}57)\rangle\}$$

$$C_{HN} =\{\langle(0{,}74, 0{,}44, 0{,}52)\rangle, \langle(0{,}75, 0{,}48, 0{,}53)\rangle\}$$

$$D_{HN} =\{\langle x; (0{,}5, 0{,}5, 0{,}5)\rangle, \langle y; (0{,}9, 0{,}9, 0{,}9)\rangle\}$$

Then the family E_{HN}={$0_{HN}, 1_{HN}, B_{HN}$} , F_{HN}={$0_{HN}, 1_{HN}, C_{HN}$} and G_{HN}={$0_{HN}, 1_{HN}, D_{HN}$} .Thus (X_{HN}, E_{HN}), (X_{HN}, F_{HN}), (X_{HN}, G_{HN}) are heptagonal neutrosophic topological spaces.
Define $f : (X_{HN}, E_{HN}) \longrightarrow (X_{HN}, F_{HN})$ as f(x)=y, f(y)=x and f(z)=z.
Define $g : (X_{HN}, F_{HN}) \longrightarrow (X_{HN}, G_{HN})$ as g(x)=y, g(y)=z and g(z)=y.
clearly f and g are heptagonal neutrosophic continuous. But gof is not heptagonal neutrosophic continuous. For 1-D is heptagonal neutrosophic closed in (X_{HN}, G_{HN}). $f^{-1}(g^{-1}(1-D))$ is not heptagonal neutrosophic closed in (X_{HN}, E_{HN}). gof is not heptagonal neutrosophic continuous.

Theorem 4.12. A mapping f:$X_{HN} \longrightarrow Y_{HN}$ is heptagonal neutrosophic bijective function. Then the following statements are equivalent:

1. f is HN continuous function.
2. f is HN closed function.
3. f is HN open function.

Proof: (i)$\Longrightarrow$ (ii):
Firslty, assume that, f is HN continuous function, Let A_{HN} be any arbitrary HN closed set in X_{HN}. Then A^C_{HN} is an HN open set in X_{HN}. Since each HN open set is an HN open set, so A^C_{HN} is the Hn open set in X_{HN}. Since f is a bijective function, so that f(A^C_{HN})=f$(A_{HN})^C$ is an HN open set in X_{HN}. Hence f(A_{HN}) is an HN closed set in X_{HN}. Therefore, for each HN closed set in X_{HN}, then f(A_{HN}) is a HN closed set in X_{HN}.
$\Longrightarrow$ f is HN closed function
(ii)$\Longrightarrow$ (iii):
Firstly, assume that, f is HN closed function, Let B_{HN} be any arbitrary HN closed set in X_{HN}. Then B^C_{HN} is an HN closed set in X_{HN}. Since f is a HN closed function, so that f(B^C_{HN})=f$(B_{HN})^C$ is an HN closed set in X_{HN}. Hence f(B_{HN}) is an HN open set in X_{HN}. Therefore, for each HN open set in X_{HN}, then f(A_{HN}) is a HN open set in X_{HN}.
$\Longrightarrow$ f is HN open function.
(iii)$\Longrightarrow$(i):
Firstly, assume that, f is a HN open function. Let C_{HN} be any arbitrary HN open set in Y_{HN}. Then C_{HN} is an HN open set in Y_{HN}. Since each HN open set is an HN open set, so C_{HN} is the HN open set in Y_{HN}. Since f is a bijective function, so that $f^{-1}(C_{HN})$ is an HN open set in Y_{HN}. Again since each HN open set is an HN open set, so $f^{-1}(C_{HN})$ is the HN open set in Y_{HN}. Therefore, for each HN closed set in Y_{HN}, then $f^{-1}((A_{HN}))$ is a HN open set in Y_{HN}.
$\Longrightarrow$ f is HN continuous function.

Example 4.13. Let H_{HN}={x,y}, A_{HN}, B_{HN} and C_{HN} ∈N(X) are defined as follows,

$$A_{HN} = \{\langle x; (0{,}72, 0{,}41, 0{,}35, 0{,}81, 0{,}77, 0{,}73, 0{,}77), (0{,}83, 0{,}88, 0{,}93, 0{,}99, 0{,}96, 0{,}90, 0{,}94), (0{,}86, 0{,}99, 0{,}97, 0{,}93, 0{,}94, 0{,}91, 0{,}86)\rangle, \langle y; (0{,}91, 0{,}32, 0{,}56, 0{,}48, 0{,}81, 0{,}72, 0{,}67), (0{,}78, 0{,}83, 0{,}21, 0{,}38, 0{,}56, 0{,}33, 0{,}98), (0{,}36, 0{,}86, 0{,}96, 0{,}32, 0{,}44, 0{,}56, 0{,}72)\rangle\}$$

$$B_{HN} = \{\langle x; (0{,}96, 0{,}65, 0{,}73, 0{,}75, 0{,}83, 0{,}56, 0{,}54), (0{,}75, 0{,}95, 0{,}45, 0{,}38, 0{,}79, 0{,}57, 0{,}13), (0{,}59, 0{,}36, 0{,}68, 0{,}47, 0{,}36, 0{,}95, 0{,}44)\rangle, \langle y; (0{,}38, 0{,}69, 0{,}88, 0{,}98, 0{,}77, 0{,}36, 0{,}98), (0{,}32, 0{,}72, 0{,}42, 0{,}62, 0{,}90, 0{,}22, 0{,}62), (0{,}42, 0{,}52, 0{,}62, 0 = 72, 0{,}36, 0{,}72, 0{,}61)\rangle\}$$

$$C_{HN} = \{\langle x; (0{,}73, 0{,}74, 0{,}96, 0{,}34, 0{,}85, 0{,}89, 0{,}64), (0{,}46, 0{,}35, 0{,}25, 0{,}96, 0{,}36, 0{,}56, 0{,}16), (0{,}84, 0{,}85, 0{,}37, 0{,}57, 0{,}67, 0{,}22, 0{,}10)\rangle, \langle y; (0{,}76, 0{,}72, 0{,}78, 0{,}62, 0{,}92, 0{,}56, 0{,}88), (0{,}38, 0{,}98, 0{,}22, 0{,}32, 0{,}54, 0{,}64, 0{,}31), (0{,}86, 0{,}96, 0{,}52, 0{,}22, 0{,}41, 0{,}51, 0{,}32)\rangle\}$$

Using De-neutosophication technique: $\frac{(p+q+r+s+t+u+v)}{7}$, We get

$$A_{HN} = \{\langle x; (0{,}65, 0{,}92, 0{,}92)\rangle, \langle y; (0{,}64, 0{,}58, 0{,}60)\rangle\}$$
$$B_{HN} = \{\langle x; (0{,}72, 0{,}57, 0{,}55)\rangle, \langle y; (0{,}72, 0{,}54, 0{,}57)\rangle\}$$
$$C_{HN} = \{\langle x; (0{,}74, 0{,}44, 0{,}52)\rangle, \langle y; (0{,}75, 0{,}48, 0{,}53)\rangle\}$$

Then the family E_{HN}=$\{0_{HN}, 1_{HN}, A_{HN}, B_{HN}\}$ and F_{HN}=$\{0_{HN}, 1_{HN}, C_{HN}\}$ are heptagonal neutrosophic topologies on X_{HN}.
Thus (X_{HN}, E_{HN}) and (X_{HN}, F_{HN}),are heptagonal neutrosophic top0logical spaces.
Define $f : (X_{HN}, E_{HN}) \longrightarrow (X_{HN}, F_{HN})$ as f(x)=y, f(y)=x and f(z)=x.
clearly f is heptagonal neutrosophic continuous. But f is not strongly heptagonal neutrosophic continuous. Since,
D_{HN}=$\{\langle x; (0{,}74, 0{,}44, 0{,}59)\rangle, \langle y; (0{,}5, 0{,}48, 0{,}53)\rangle\}$ is an heptagonal neutrosophic open set in (X_{HN}, F_{HN}), $f^{-1}(D_{HN})$ is not heptagonal neutrosophic open in (X_{HN}, E_{HN}).

5. Conclusions

In this current article, we have introduced heptagonal neutrosophic topology in neutrosophic environments with the help of ranking technique of Heptagonal numbers. Also the Heptagonal neutrosophic set operations are introduced with suitable examples. The Heptagonal neutrosophic interior and closure concepts are also explained to strengthen the HN topology. The theorems and properties of open sets and closed sets of HN topologies are explained with related examples. Further there is a scope to introduce continuous functions, connectedness and compactness based on HN topological spaces. Additionally, Topological Spaces and Bipartite Graph are used in conjunction with the Heptagonal Intuitionistic Fuzzy Number (HIFN) in [16] to solve the Intuitionistic Fuzzy Transportation Problems. Heptagonal Neutrosophic topological spaces can also be used in place of topological spaces and Neutrosophic Heptagonal Numbers can be used as an alternative to HIFN to solve the Neutrosophic Transportation Problems, which is one of the examples of applications of the concepts discussed in this article. We have further planned to expand Multi Criteria Decision Making (MCDM) to discover or select the best answer from the existing with the aid of Neutrosophic soft matrix.

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Neutro Open Set-Based Strong Neutro Metric Space

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Abstract. Memet et al. interposed the concept of neutro metric spaces. They established this concept depend on the neutro axioms from the neutro function one of the ideas of Smarandache. In the concept of neutro metric spaces used the non-negative and negative real numbers. We introduced this notion depend on the non-negative real numbers and redefined the neutro metric space depend on the strong neutro metrics. We investigate the properties of strong neutro metric spaces and present the notion of neutro-open sets and neutro-closed sets. We distinguished between the operations on neutro-open sets in strong neutro metric space and open sets in metric spaces.

Keywords: Strong neutro metric(space), neutro open set, neutro closed set, unified set.

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1. Introduction

Florentin Smarandache [8], considering the facts in the real world, he introduced the theory of neutro algebra. According to him, a system in which everything is right or everything is wrong either does not exist, or if it exists, it is not real. In the theory of neutro algebra, he deals with the issue that some principles may be true and some principles may not be true, and this is closer to the problems in the real world. In 1906 M. Frechet introduced metric spaces [6] as a mathematical tool in the rela world and other researchers continued it in some branches. Recently, Memet et al. introduced the notion of neutro metric spaces [7]. Some researchers have investigated the neutro structures such as [1–6, 9]

We introduce a drawing out of metric spaces, whatever is a distribution of topologic spaces. Our intention in headlining this matter is to design the principles of the matter in kind to contest the matter of metric space theory. We exercise the axioms of metric spaces and illustrate the notion of strong neutro metric spaces and investigate their properties. This paper introduces the notion of open balls in strong metric space with the same as the notion of open balls in metric space and depend on this notion, we present the notion of neutro open balls. We show that the union of any family of unified neutro open sets is a neutro open set and the intersection of any family of chain neutro open sets is a neutro open set, while the intersection of a finite set of an open set is an open set in metric space. Indeed, we distinguished the fundamental structures of metric spaces and fundamental structures of neutro metric spaces.

2. **Preliminaries**

We need materials that have been reviewed before and are effective in our article, so we will address them in this section.

Definition 2.1. [8] For any non-avoided set X, (X, σ) is a neutro algebra, if σ is a neutro operation.

Definition 2.2. [7] Allow $X \neq \emptyset$ and $\sigma : X^2 \to \mathbb{R}$. Then, (X, σ) is titled a *neutro metric space (neutro M. S)* if, there is

$(NM\text{-}1)$ $\big(x, y \in X$ intent, $x\sigma y \geq 0$, $\big(r, s \in X$ intent, $x\sigma y < 0$ or inconclusive $\big)$;

$(NM\text{-}2)$ $\big(x \in X$ intent, $x\sigma x = 0$, $\big(y \in X$ intent, $x\sigma x \neq 0$ or inconclusive $\big)$;

$(NM\text{-}3)$ $\big(x, y, z, w \in X$, intent, $x\sigma z \leq x\sigma y + y\sigma z$, $\big(r, s, t, v \in X$, intent, $x\sigma z > x\sigma y + y\sigma z$ or inconclusive $\big)$;

$(NM\text{-}4)$ $\big(x, y, z, w \in X$, intent, $x\sigma y = y\sigma x$, $\big(r, s \in X$, intent, $r\sigma s \neq s\sigma r$ or inconclusive $\big)$.

3. **Strong neutro open sets**

Definition 3.1. Allow $X \neq \emptyset$ and $\sigma : X^2 \to \mathbb{R}^{\geq 0}$. Then, (X, σ) is titled a *strong neutro M. S* if,

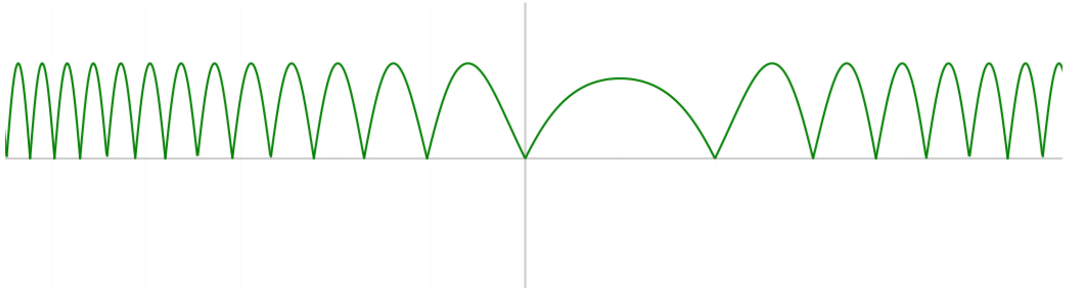
$(NM\text{-}1)$ $\big(x \in X$ intent, $x\sigma x = 0$ and $\big(y \in X$ intent, $y\sigma y \neq 0$ or inconclusive $\big)$;

$(NM\text{-}2)$ $\big(x, y, z, w \in X$, intent, $x\sigma z \leq x\sigma y + y\sigma z$ and $\big(r, s, t, v \in X$, intent, $r\sigma t > r\sigma s + s\sigma t$ or inconclusive $\big)$;

$(NM\text{-}3)$ $\big(x, y, z, w \in X$, intent, $x\sigma y = y\sigma x$ and $\big(r, s \in X$, intent, $r\sigma s \neq s\sigma r$ or inconclusive $\big)$.

Example 3.2. Illustrate $\sigma : \mathbb{R}^2 \to \mathbb{R}^{>0}$ by $x\sigma y = |Sin(xy - x)|$, where $x, y \in \mathbb{R}$. By Figure 1, $x\sigma x = |Sin(x^2 - x)|$ and easy to see that there exists $y \in \mathbb{R}$ in kind $y\sigma y = 0$ and there exists $z \in \mathbb{R}$ in kind $z\sigma z \neq 0$. Accordingly the item $(NM) - 1$ is valid. Inaddition, $x\sigma y = y\sigma x$, infers $|Sin(xy - x)| = |Sin(xy - y)|$. If $x = y$, then $x\sigma y = y\sigma x$ and for $x = 0$ and $y = \frac{\pi}{2}$, we have $x\sigma y \neq y\sigma x$. Accordingly the item $(NM) - 2$ is valid. If $yz = y$ and $y = z$, then $x\sigma z \leq x\sigma y + y\sigma z$ and for $x = \frac{\pi}{2}, y = 1$ and $z = 0$, we get that $x\sigma z > x\sigma y + y\sigma z$ and so the item $(NM) - 3$ is valid.

FIGURE 1. $|Sin(x^2 - x)|$



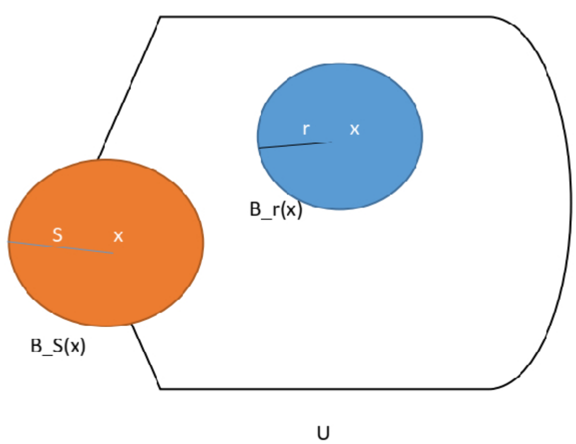
Allow (X, σ) be a strog neutro M. S and $r \in \mathbb{R}^{>0}$. Then $B_r(x) = \{y \in X \mid x\sigma y < r\}$, is the open ball of radius r and center $x \in X$.

Example 3.3. Consider the strog neutro M. S $(\mathbb{R}, \sigma)$ by $x\sigma y = |Sin(xy - x)|$. Computation show that $B_1(x) = \mathbb{R}$ and for all $x \neq 0, B_1(x) = \mathbb{R} \setminus \{\frac{x \pm \frac{\pi}{2}}{x}, \frac{x \pm \frac{3\pi}{2}}{x}, \frac{x \pm \frac{5\pi}{2}}{x}, \ldots\}$. Also for any $r > 1$, we get that $B_r(x) = \mathbb{R}$.

Definition 3.4. Allow (X, σ) be a strog neutro M. S and $U \subseteq X$. Then U is a neutro open set(N. O. S), if $x \in U, r \in \mathbb{R}^{>0}$ in kind $B_r(x) \subseteq U$ and $y \in X, s \in \mathbb{R}^{>0}$ in kind $B_s(x) \nsubseteq U$. From now on, will denote the set of all strog N. O. S of X by $\mathcal{NO}(X)$.

Example 3.5. (i) Allow X be a nonavoid set and $U \subseteq X$. One can see a sample unified set X in Figure 2.

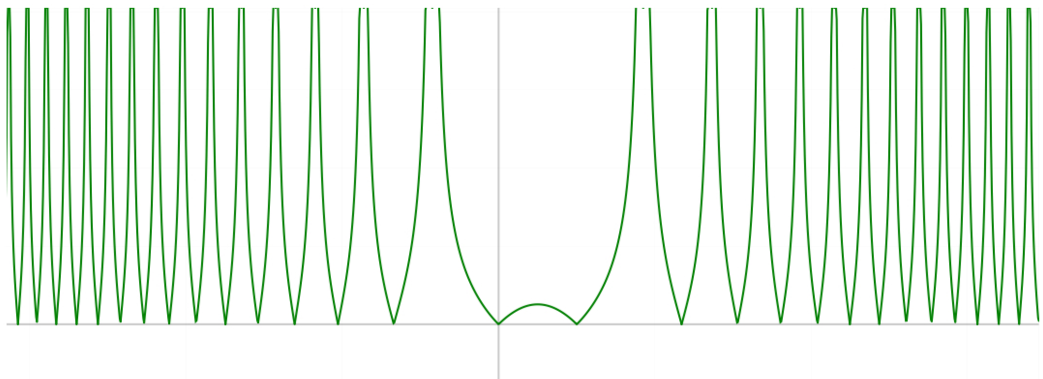
FIGURE 2. Strog N. O. S U



(ii) Illustrate $\sigma : \mathbb{R}^2 \to \mathbb{R}^{>0}$ by $x\sigma y = |tan(xy - x)|$, where $x, y \in \mathbb{R}$. By Figure 3, $x\sigma x = |tan(x^2 - x)|$ and easy to see that there exists $y \in \mathbb{R}$ in kind $y\sigma y = 0$ and there exists $z \in \mathbb{R}$

in kind $z\sigma z \neq 0$. Accordingly the item $(NM)-1$ is valid. As well, $x\sigma y = y\sigma x$, infers $|tan(xy-x)| = |tan(xy-y)|$. If $x=y$, then $x\sigma y = y\sigma x$ and for $x=0$ and $y=\frac{\pi}{4}$, we have $x\sigma y \neq y\sigma x$. Accordingly the item $(NM)-2$ is valid. If $yz=y$ and $y=z$, then $x\sigma z \leq x\sigma y + y\sigma z$ and for $x=\frac{\pi}{4}, y=1$ and $z=0$, we get that $x\sigma z > x\sigma y + y\sigma z$ and so the item $(NM)-3$ is valid.

FIGURE 3. $|tan(x^2-x)|$



Computations show that

$$\begin{aligned}
&B_1(\frac{\pi}{4}) = \{y \in \mathbb{R} \mid y\sigma\frac{\pi}{4} < 1\}\\
&= \{y \in \mathbb{R} \mid |tan(\frac{\pi}{4}y - \frac{\pi}{4})| < 1\}\\
&= \big((-\infty,2)\cap(-\infty,6)\cap(-\infty,10)\cap\dots\big)\cup(4,\infty)\cap(8,\infty)\cap(12,\infty)\cap\dots\\
&= \big(\bigcap_{k\in\mathbb{O}}(-\infty,2k)\big)\cap\big(\bigcap_{k\in\mathbb{N}}(2k,\infty)\big)\\
&= (-\infty,2)\cap\big(\bigcap_{k\in\mathbb{N}}(2k,\infty)\big)\\
&= \emptyset.
\end{aligned}$$

Consider $0 \in U = (-1,\infty)$. Then

$B_1(0) = \{y \in \mathbb{R} \mid y\sigma 0 < 1\} = \{y \in \mathbb{R} \mid |tan(0)| < 1\} = \mathbb{R}$, which $B_1(0) \not\subseteq U$.

Consider $-\frac{\pi}{4} \in U = (-1,\infty)$. Then

$$\begin{aligned}
&B_1(-\frac{\pi}{4}) = \{y \in \mathbb{R} \mid y\sigma - \frac{\pi}{4} < 1\}\\
&= \{y \in \mathbb{R} \mid |tan(-\frac{\pi}{4}y + \frac{\pi}{4})| < 1\}\\
&= \big((0,\infty)\cap(-4,\infty)\cap(-8,\infty)\cap\dots\big)\cap\big((-\infty,-2)\cap(-\infty,-6)\cap(-\infty,-10)\cap\dots\big)\\
&= \bigcap_{k\in\mathbb{W}}(-4k,\infty)\cap\bigcap_{k\in\mathbb{O}}(\infty,-2k) = \emptyset,
\end{aligned}$$

which $B_1(-\frac{\pi}{4}) \subseteq U$. Thence, $(-1, \infty) \in \mathcal{NO}(\mathbb{R})$.

Theorem 3.6. *Allow (X, σ) be a strog neutro M. S. Then $Card(X)geq2$.*

Proof. Allow $Card(X) = 1$ and $x = \{x\}$. By the axiom $NM - 1$, if $x\sigma x = 0$, then must exis $x \neq y \in X$ in kind $y\sigma y \neq 0$, which is a contradication. Accordingly $Card(X) \geq 2$. □

Theorem 3.7. *Any M. S, can be a strong neutro M. S.*

Proof. Allow (X, d) be a M. S and $\lambda \notin X$. Then $(X \cup \{\lambda\}, \sigma)$ is a strong neutro M. S, whichever for $x, y \in X$

$$x\sigma y = \begin{cases} d(x,y) & \text{if } x, y \in X \\ \lambda & \text{if } x = y = \lambda, \end{cases}$$

$\lambda\sigma x = x_0, x\sigma\lambda = y_0$ and $\lambda > \lambda\sigma x + x\sigma\lambda$, which $x_0 \neq y_0, x_0, y_0 \in X$. One can see that the neutro axioms $NM - 1, NM - 2$ and $NM - 3$ are valid. □

Theorem 3.8. *Allow (X, σ) be a strog neutro M. S and $Card(X) = 2$. Then $Card(Range(\sigma)) = 4$.*

Proof. Allow $X = \{a, b\}$. Then illustrate the map $\sigma : X^2 \to \mathbb{R}$ go after:

σ	a	b
a	0	s
b	s'	r

.

We claim that $0 \neq r \neq s \neq s'$. Since $a\sigma a = 0$, $r \in \mathbb{R}$ which that $b\sigma b = r \neq 0$. If for $s \in \mathbb{R}$ consider $a\sigma b = s$, then $s \neq s' \in \mathbb{R}$ in kind $b\sigma a = s'$. Now, we investigate the coming cases:

case 1: if $a\sigma a > a\sigma b + b\sigma b$, then $0 > s+s'$, which is a contradication. Thence, $a\sigma a \leq a\sigma b + b\sigma b$ or $0 \leq s + s'$.

case 2: if $a\sigma b > a\sigma a + a\sigma b$, then $s > s + s'$, which is a contradication. Thence, $a\sigma b \leq a\sigma a + a\sigma b$.

case 3: if $b\sigma a > b\sigma b + b\sigma a$, then $s' > s + s'$, which is a contradication. Thence, $b\sigma a \leq b\sigma b + b\sigma a$.

Since (X, σ) is a strog neutro M. S, we get that $b\sigma b \leq b\sigma a + a\sigma b$. Accordingly $r \leq s + s'$ and so $Card(Range(\sigma)) = 4$. □

Theorem 3.9. *Allow (X, σ) be a strog neutro M. S. Then*

(i) $X \notin \mathcal{NO}(X)$.

(ii) $\emptyset \in \mathcal{NO}(X)$.

Proof. (*i*) Allow $y \in X$ in kind for all $r \in \mathcal{R}^{>0}$, $B_r(y) \nsubseteq X$. If $y\sigma y = 0$, then $\emptyset \neq B_r(y) \subseteq X$, which makes a contradication. If $y\sigma y < 0$, then $\emptyset \neq B_r(y) \subseteq X$, which makes a contradication. If $y\sigma y > 0$, then $\emptyset = B_r(y) \subseteq X$, which makes a contradication. They follow that there for all $y \in X$, $r \in \mathcal{R}^{>0}$, in kind $B_r(y) \subseteq X$ and so $X \notin \mathcal{NO}(X)$.

(*i*) Because there is no point in the empty set, we get that $\emptyset \in \mathcal{NO}(X)$. □

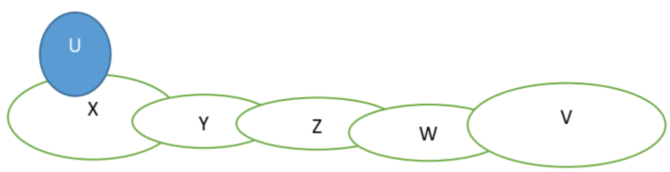
Theorem 3.10. *Allow* (X, σ) *be a strog neutro M. S,* $r \in \mathbb{R}^{>0}$ *and* $x \in X$*. Then* $B_r(x)$ *is not an O. S.*

Proof. Allow $x \in X$ be an arbitrary and for $r \in \mathbb{R}^{>0}$, $B_r(x)$ be an O. S. Thence, $X = \bigcup_{x \in X} B_r(x)$ is an O. S, which it is a contradication by Theorem 3.18. □

Allow X, Y, U be nonavoid sets. Accordingly, X, Y are unified, if $U \nsubseteq X$, then $U \nsubseteq X \cup Y$.

Example 3.11. Allow X, Y, Z, W, U, V be nonavoid sets. Then one can can see the unified sets X, Y, Z, W, U, V in Figure 4.

Figure 4. Unified set X

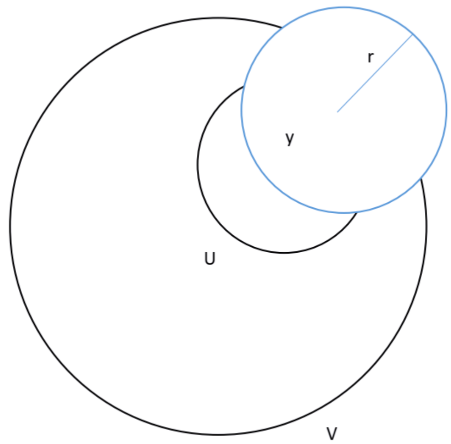


Theorem 3.12. *Allow* (X, σ) *be a strog neutro M. S and* $\{U_i\}_{i \in I} \subseteq \mathcal{NO}(X)$ *be unified. Then* $\bigcup_{i \in I} U_i \in \mathcal{NO}(X)$.

Proof. Since for all $i \in I, U_i \in \mathcal{NO}(X)$, we get $x \in U_i$ and $r \in \mathcal{R}^{>0}$ in kind $B_r(x) \subseteq U_i \subseteq \bigcup_{i \in I} U_i$. As well, $y \in U_i$ and $s \in \mathcal{R}^{>0}$ in kind $B_r(y) \nsubseteq U_i \subseteq$. Since for all $i \in I, U_i$ are unified, we get that $B_r(y) \nsubseteq \bigcup_{i \in I} U_i$. They conclude that $\bigcup_{i \in I} U_i \in \mathcal{NO}(X)$. □

Theorem 3.13. *Allow* (X, σ) *be a strog neutro M. S and* $\{U_i\}_{i \in I} \subseteq \mathcal{NO}(X)$*. If* $\{U_i\}_{i \in I}$ *is a chain, then* $\bigcap_{i \in I} U_i \in \mathcal{NO}(X)$.

FIGURE 5. Unified sets U, V



Proof. If $\bigcap_{i\in I} U_i = \emptyset$, then by Theorem 3.18, $\bigcap_{i\in I} U_i \in \mathcal{NO}(X)$. Allow $\bigcap_{i\in I} U_i \neq \emptyset$. Since $\{U_i\}_{i\in I}$ is a chain, $j \in I$ in kind $\bigcap_{i\in I} U_i = U_j$ and so $\bigcap_{i\in I} U_i \in \mathcal{NO}(X)$. □

Theorem 3.14. *Allow (X, σ) be a strog neutro M. S, $x \in X$ and $r \in \mathcal{R}^{>0}$. Then $B_r(x) \in \mathcal{NO}(X)$.*

Proof. We claim that $x \in X$ in kind $x\sigma x = 0$. If for all $x \in X, x\sigma x \neq 0$, since (X, σ) is a strog neutro M. S, $y \in X$ in kind $x\sigma y > x\sigma x + x\sigma y$, which is contradiaction. Because $x \in X$ in kind $x\sigma x = 0$, we get that $x \in B_r(x)$. Thence, $B_r(x) \neq \emptyset$ and $B_r(x) \subseteq B_r(x)$. As well, we claim that $x \in X$ in kind $x\sigma x \neq 0$. If for all $x \in X, x\sigma x = 0$, since (X, σ) is a strog neutro M. S, $y \in X$ in kind $y\sigma y \leq y\sigma x + x\sigma y$, which is a contradiaction. Allow $x\sigma x = r$. Since (X, σ) is a strog neutro M. S, $z \in X$ in kind $x\sigma z > x\sigma x + x\sigma z > r + x\sigma z$. Thence, $z \notin B_r(x)$ and so $B_r(z) \nsubseteq B_r(x)$. Therefore, Then $B_r(x) \in \mathcal{NO}(X)$. □

Theorem 3.15. *Allow (X, σ) be a strog neutro M. S, $U, V \subseteq X$ and U be unified. If $U \in \mathcal{NO}(X)$ and $U \subseteq V$, then $V \in \mathcal{NO}(X)$.*

Proof. Since $U \in \mathcal{NO}(X)$, we get that there exists $x \in U$ and $s \in \mathbb{R}$, in kind $B_s(x) \subseteq U$. Thence, there exists $x \in V$ and $s \in \mathbb{R}$, in kind $B_s(x) \subseteq V$, because of $U \subseteq V$. As well, there exists $y \in U$ and $r \in \mathbb{R}$, in kind $B_r(y) \nsubseteq U$ as shown in Figure 5, since U, V are unified sets. □

Theorem 3.16. *Allow (X, σ) be a strog neutro M. S. Then*

(i) if a set contains a union of unified open balls, then it is strog N. O. S.
(ii) if a set is strog N. O. S, then it contains a union of unified open balls.

(*iii*) *if a set is strog N. O. S, then it may dose't contain a union of unified open balls.*

Proof. (*i*) Allow $U \subseteq X$ and $U \supseteq \bigcup_{x\in X} B_r(x)$. Then by Theorems 3.12 and 3.14, $\bigcup_{x\in X} B_r(x)$ is a strog N. O. S in X. Now, using Theorem 3.16, U is a N. O. S.

(*ii, iii*) Suppose that U is a strog N. O. S. Thence, $x \in U$ and $r \in \mathbb{R}^{>0}$ in kind $B_r(x) \subseteq U$ and $y \in U$ and $s \in \mathbb{R}^{>0}$ in kind $B_s(y) \nsubseteq U$. Since $\{x\} \subseteq B_r(x) \subseteq U$, we get that $\bigcup_{x\in U} \{x\} \subseteq \bigcup_{\exists\ x\in U} B_r(x) \subseteq U$. Moreover, $\{y\} \subseteq B_s(y) \nsubseteq U$, infers $\bigcup_{\exists\ y\in U} \{y\} \subseteq \bigcup_{\exists\ y\in U} B_s(y) \nsubseteq U$. □

3.1. **Strong neutro closed sets**

Definition 3.17. Allow (X, σ) be a strog neutro M. S and $F \subseteq X$. Then F is a neutro closed set, if $F^c = \{x \in X \mid x \notin F\}$ is a N. O. S From now on, will denote the set of all strog neutro closed set of X by $\mathcal{NC}(X)$.

Theorem 3.18. *Allow*(X, σ) *be a strog neutro M. S. Then*

(*i*) $X \in \mathcal{NC}(X)$.

(*ii*) $\emptyset \notin \mathcal{NC}(X)$.

Proof. (*i*) Since $X^c = \emptyset$, by Theorem 3.18, $\emptyset \in \mathcal{NO}(X)$ and so $X \in \mathcal{NC}(X)$.

(*ii*) Since $\emptyset^c = X$, by Theorem 3.18, $X \notin \mathcal{NO}(X)$ and so $\emptyset \notin \mathcal{NC}(X)$. □

Theorem 3.19. *Allow* (X, σ) *be a strog neutro M. S and* $\{F_i\}_{i\in I} \subseteq \mathcal{NC}(X)$ *be unified. Then* $\bigcap_{i\in I} F_i \in \mathcal{NC}(X)$.

Proof. Since for all $i \in I, F_i \in \mathcal{NC}(X)$, by Theorem 3.12, we get $X \setminus \bigcap_{i\in I} F_i = \bigcup_{i\in I}(X \setminus F_i) \in \mathcal{NO}(X)$. Thence, $\bigcap_{i\in I} F_i \in \mathcal{NC}(X)$. □

Theorem 3.20. *Allow* (X, σ) *be a strog neutro M. S and* $\{F_i\}_{i\in I} \subseteq \mathcal{NC}(X)$. *If* $\{F_i\}_{i\in I}$ *is a chain, then* $\bigcup_{i\in I} F_i \in \mathcal{NC}(X)$.

Proof. Allow $\{F_i\}_{i\in I}$ and $F_i \subseteq F_j \subseteq F_k \subseteq \ldots$ be a chain. Then $F_i^c \supseteq F_j^c \supseteq F_k^c \supseteq \ldots$ and so $\{F_i^c\}_{i\in I}$ is a chain. Since for all $i \in I, F_i \in \mathcal{NC}(X)$, by Theorem 3.13, we get $X \setminus \bigcup_{i\in I} F_i = \bigcap_{i\in I}(X \setminus F_i) \in \mathcal{NO}(X)$. Thence, $\bigcup_{i\in I} F_i \in \mathcal{NC}(X)$. □

4. Conclusions

In this research work, we have been able to deal the connection of M. S and neutro M. S. Indeed, we investigate the diets in M. Ss and add some conditions on the axioms of M. Ss as the neutro metric axioms and so introduce the concepts of open balls, N. O. S, and neutro closed sets. The notion of unified sets recreates an essential role in the basic concepts of strong neutro spaces. We hope to extend these concepts in the real analysis in the next works.

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Optimization of Single-valued Triangular Neutrosophic Fuzzy Travelling Salesman Problem

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Abstract. The travelling salesman problem(TSP) is a classic optimization puzzle, widely studied and celebrated for its significance in operations research, mathematics and computer science. It can also be described as an evolution from a mathematical curiosity to a problem that challenges the computation boundaries, sparks algorithmic innovation, and finds practical applications in various industries. The neutrosophic TSP(NTSP) extends the problem by introducing neutrosophy, handling indeterminacy and inconsistency with distances represented by neutrosophic numbers(NNs). The single-valued triangular fuzzy neutrosophic TSP(SVTFNTSP) goes a step further by incorporating both single-valued triangular fuzzy numbers(SVTFNs) and neutrosophy, representing distances with SVTFNNs. The single-valued triangular fuzzy neutrosophic numbers(SVTFNNs) provide a way to model uncertainty via triangular membership functions, offering a more nuanced representation of uncertain and vague distances. This arises the need to use them and enhances realism in solving complex real-world optimization problems. These extensions adapt the TSP to varying uncertain and vague data, ideal for intricate real-world optimization scenarios. This research article delves into the SVTFNTSP, expressed as a single-valued triangular fuzzy neutrosophic distance matrix(SVTFNDM) with SVTFNNs as its core elements, accounting for both uncertainty and imprecision. The investigation encompasses the formulation and examination of this specialized problem by incorporating a score function to assess defuzzification and optimality, alongside the utilization of a proposed systematic stepwise approach to efficiently ascertain optimal solutions. This approach is practically demonstrated through its application to real-world scenarios, effectively showcasing its feasibility and real-world relevance. Subsequently, through a rigorous comparative analysis with the established methodologies, the superior effectiveness and value of the proposed approach are highlighted, specifically in terms of minimizing total travelling costs. This reaffirms its potential as a robust solution for tackling the SVTFNTSP by underlining its practical utility and enhanced performance.

Keywords : Neutrosophic set, Neutrosophic number, Single-valued triangular fuzzy neutrosophic number, Single-valued triangular fuzzy neutrosophic distance matrix, Travelling salesman problem, Single-valued triangular fuzzy neutrosophic travelling salesman problem, Score function, Range, Optimal solution, Cycle.

1. Introduction

In our daily lives, we frequently encounter a variety of unclear, ambiguous, and inadequate situations. Thus, as an extension of classical sets that enables partial membership(awards a membership grade) for each element - Zadeh [1] developed the idea of fuzzy sets in 1965. The fuzzy set theory has had considerable success in many disciplines because of its capacity to handle uncertainty. Atanassov [2] presented the idea of intuitionistic fuzzy sets in 1983 as an extension of fuzzy sets that not only contains the membership grade but also the non-membership grade of each element due to certain of its constraints. Neutrosophic sets(NS) are an extension of intuitionistic fuzzy sets that incorporate the truth(T), indeterminacy(I), and falsity(F) membership grades for each element. The notion was first described by Smarandache [3] in 1995.

The Travelling Salesman Problem(TSP) is a mathematical challenge that has been studied for centuries, making it difficult to attribute its invention to a single individual. However, the problem gained formal recognition and attention in the 1800s and 1900s as mathematicians explored related concepts. The mathematician and computer scientist George Dantzig is often credited with formulating the TSP in its modern mathematical terms in the 1950s. He introduced the problem as a mathematical challenge and used it to illustrate the concept of linear programming. The problem's historical development involved the contributions of various mathematicians across different time periods. The TSP seeks the shortest route for a salesperson to visit cities once and return to the starting city, minimizing distance or cost. This optimizes route planning and algorithm advancement. The TSP holds profound significance as both a theoretical benchmark and a practical problem-solving tool. As a theoretical challenge, it embodies the complexities of optimization and serves as a yardstick for evaluating algorithmic innovations. In practical realms, the TSP finds applications in diverse fields like logistics, enhancing delivery routes, reducing transportation costs, and improving resource utilization. The problem's versatility underscores its significance in solving complex real-world optimization challenges across industries and domains. Neutrosophic numbers(NNs) hold significant value due to their ability to capture uncertainty, indeterminacy, and inconsistency in a structured manner. They find applications in decision-making, expert systems, and medical diagnoses. They enhance problem-solving by addressing imprecise or incomplete information, enabling more informed choices in diverse domains. Single-valued triangular fuzzy neutrosophic numbers(SVTFNNs) carry substantial importance by seamlessly integrating triangular fuzzy sets and neutrosophy. This fusion enhances the representation of uncertainty and indeterminacy in a comprehensive manner. These numbers find practical applications in decision analysis, risk assessment, and multi-criteria decision-making, where complex and uncertain information is prevalent. The ability of single-valued triangular fuzzy neutrosophic numbers to model both fuzziness and neutrosophy enhances the accuracy of real-world problem-solving, offering a versatile tool to navigate intricate situations and foster well-informed decisions. This ignites interest and fosters a drive to explore and experiment with the single-valued triangular fuzzy neutrosophic traveling salesman problem(SVTFNTSP).

The research landscape surrounding the Traveling Salesman Problem (TSP) and its extensions, such as the Neutrosophic TSP and the Single-Valued Triangular Fuzzy Neutrosophic TSP, has been vibrant and dynamic. Scholars have extensively explored the classic TSP, focusing on developing algorithms, heuristics, and metaheuristics to efficiently find near-optimal solutions for larger instances. The application of the neutrosophic idea is the subject of several recent research publications. Researchers have studied and analysed the issue of completing an assignment in a classical, fuzzy and intuitionistic fuzzy environment [4–8]. In 2019, Prabha and Vimala [9] used the branch and bound method to solve the triangular neutrosophic fuzzy assignment problem, which is demonstrated with an agricultural issue. Using the order relations method, Khalifa Abd El-Wahed et al. [10–12] were able to resolve the neutrosophic fuzzy assignment issue where the matrix elements are interval-valued trapezoidal neutrosophic fuzzy numbers, optimized neutrosophic complex programming using lexicographic order and resolved the interval-type fuzzy linear fractional programming problem in neutrosophic environment using a fuzzy mathematical programming approach respectively. Chakraborty et al [13] proposed a few de-neutrosophication techniques to tackle different forms of triangular fuzzy neutrosophic numbers and also explored on their applications to various fields. A new ranking function of triangular fuzzy neutrosophic numbers put forward by Das et al. [14] and applied to integer programming. Pranab et al. [15] aggregated of triangular fuzzy neutrosophic set information and extended its applications to multi-attribute decision-making. Broumi [16] handled the shortest path problem by using interval valued trapezoidal and triangular fuzzy neutrosophic numbers. The neutrosophic inventory backorder problem was examined by Mullai and Surya [17] and resolved using triangular fuzzy neutrosophic numbers. Smarandache [18] established the Delphi method for evaluating scientific research proposals in a neutrosophic environment. Researchers [19–22] have investigated diverse real-life issues under a neutrosophic environment and resolved the same with the use of single-valued triangular fuzzy neutrosophic matrix games and by developing different score functions for both ranking and turning the neutrosophic data into the appropriate crisp data. Subasri and Selvakumari [23, 24] used the ones assignment method and the branch and bound approach, to solve the travelling salesman problem in a neutrosophic environment utilising triangular and trapezoidal fuzzy distances respectively. S. Dhouib [25] used the Dhouib-Matrix-TSP1 Heuristic to optimise the traveling salesman problem for single-valued triangular fuzzy neutrosophic numbers. Broumi et al. [26] analyzed and answered the shortest path problem under triangular fuzzy neutrosophic environment. Abduallah et al. [27, 28] conducted detailed case studies on leveraging neutrosophic theory in appraisal decision framework and neutrosophic healthcare systems and worked toward sustainable emerging economics based on industry 5.0 and a responsive resilient supply chain based on industry 5.0 respectively. Maissam and Smarandache [29] explored about the use of neutrosophic methods of operation research in the management of corporate work. Uddin et al. [30] introduced a new extension to the intuitionistic fuzzy metric-like spaces. Saleem et al. [31] stablished a unique solution for the integral equations through the intuitionistic extended fuzzy b-metric-like spaces. Ishtiaq, Ahmed et al. [32–35] resolved the non-linear fractional differential equations and guarenteed the existence of some fixed point results in neutrosophic metric, orthogonal neutrosophic metric, generalized neutrosophic metric and neutrosophic metric-like spaces respectively.

Here comes a small discussion about the limitations along with the gaps of the existing algorithms related to this study, which has lead to the need for using the proposed algorithm for solving the single-valued triangular fuzzy neutrosophic traveling salesman problem (SVTFNTSP), its new features and the potential advantages it can offer over existing methods, including the differences between the proposed and the existing methodologies are as follows : Certainly, various algorithms have been proposed for solving the Traveling Salesman Problem (TSP), and each method comes with its own set of advantages and disadvantages. Some of the above mentioned methods involve exploring all possible permutations, leading to exponential time complexity. It becomes impractical for large TSP instances. Some optimization algorithms for TSP, can exhibit exponential growth in the number of nodes explored, which makes it slow for large instances. They can be complex and computationally intensive, particularly when dealing with more intricate TSP instances or constraints. A few methods rely on mass values associated with cities, which may not always be readily available or meaningful for real-world TSP applications. Some methods are based on the assumption that the TSP instance can be represented by a special matrix, which may not hold for all real-world scenarios. It limits its applicability. Many TSP algorithms, including those mentioned, can be sensitive to the initial solution or starting point, potentially leading to suboptimal results if a good initial solution is not found. Some of these methods may struggle with scalability when applied to large TSP instances due to their exponential nature or computational demands. Certain methods may be better suited to specific types of TSP instances and may not perform well on variations or extensions of the problem. Some methods may not guarantee finding the optimal solution but rather provide approximate solutions. For certain applications requiring exact solutions, this can be a limitation. Analyzing the computational complexity and convergence properties of these methods can be challenging, making it difficult to predict their performance in advance. Some TSP algorithms may not be easily parallelizable, limiting their ability to take advantage of modern multi-core processors and distributed computing environments.

The proposed methodology of this research article brings an innovative approach by incorporating the range(a measure of dispersion) to problem-solving. It might have the potential to discover high-quality solutions that outperform existing methods. They employ strategies that are not present in traditional approaches. It is designed to adapt to different problem characteristics and constraints. This adaptability can make it suitable for a wider range of SVTFNTSP instances without extensive customization. It can handle larger and more complex SVTFNTSP instances efficiently. This can be crucial for solving real-world problems of practical significance. It might achieve faster convergence to solutions, potentially reducing the overall computation time for solving SVTFNTSP instances. It has the potential to generalize to other related problems or domains beyond NTSP. This versatility can make them valuable for addressing a broader set of challenges. It is designed to be robust to variations in problem instances or data. They may be less sensitive to changes in input parameters, leading to more reliable performance. In rapidly evolving research fields, using the proposed mathodology can provide a competitive advantage by accessing the latest advancements in optimization and computational intelligence. While existing methods may have established parameter settings and heuristics, the proposed mathodology offers opportunities for customization to better fit the specific requirements and constraints of a given NTSP instance.

This research article explores the single-valued triangular fuzzy neutrosophic traveling salesman problem (SVTFNTSP), which is represented using a single-valued triangular fuzzy neutrosophic distance matrix (SVTFNDM) with SVTFNNs as its fundamental components. This formulation takes into account both uncertainty and imprecision. The study involves the development and investigation of this specialized problem, introducing a score function for defuzzification and optimality assessment. Additionally, it presents a systematic stepwise approach to efficiently determine optimal solutions. The practical applicability of this approach is demonstrated through its implementation in real-world scenarios, highlighting its feasibility and relevance. Furthermore, through a rigorous comparative analysis with established methodologies, the article emphasizes the superior effectiveness and value of the proposed approach, particularly in terms of minimizing total travel costs. This underscores its potential as a robust solution for addressing the SVTFNTSP, emphasizing its practical utility and enhanced performance. The following is how the paper is set up : The abstract and introduction are included in Section 1. We provide some fundamental definitions of a fuzzy set, an intuitionistic fuzzy set, a neutrosophic set and a fuzzy number with their respective examples in the Preliminaries section of Section 2. Neutrosophic number, properties of neutrosophic numbers, single-valued triangular fuzzy neutrosophic number, along with their corresponding examples, travelling salesman problem (TSP), mathematical formulation of TSP and a score function are some of the subjects covered in Section 3. The methodology for solving the single-valued triangular fuzzy neutrosophic travelling salesman issue is presented in Section 4 and comprises defuzzifying the neutrosophic data before using the suggested algorithm step-by-step to get the best answer. The proposed approach to addressing the "Travelling Salesman Problem" in a neutrosophic environment is illustrated in Section 5. Section 6 highlights some significant results and discussions. Section 7 concludes the research article.

2. Preliminaries

Definition 2.1. Let X be a non-empty set. A fuzzy set A in X is characterized by its membership function $\mu_A : X \longrightarrow [0,1]$ and $\mu_A(x)$ is interpreted as the degree of membership of element x in fuzzy set A, for each $x \in X$.

$$A = \{(x, \mu_A(x)) : x \in X\}.$$

Example :

- Variable : Happiness.
- Fuzzy Sets : Unhappy, Neutral, Happy.
- Membership Function : $\mu(Unhappy) = 0.2$; $\mu(Neutral) = 0.5$; $\mu(Happy) = 0.9$.

Definition 2.2. Let X be a non empty set. An Intuitionistic fuzzy set A in X is of the form $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$, where the functions μ_A, $\nu_A : X \longrightarrow [0,1]$ define respectively the degree of membership and the degree of non-membership for every element $x \in X$ to the set A, which is a subset of X.

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

Furthermore, we have $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the intuitionistic fuzzy set index or hesitation margin is the degree of indeterminacy of x in A where $\pi_A(x) \in [0,1]$ i.e; $\pi_A : X \longrightarrow [0,1]$ and $0 \leq \pi_A(x) \leq 1$ for every x∈X.

Example : IFS representing the taste ”Spicy”.

- Element : Dish X.
- Membership degree : $\mu(DishX) = 0.7$(Dish X is ”somewhat” spicy).
- Non-membership degree : $\nu(DishX) = 0.3$(Dish X is ”not very” not spicy).
- Hesitancy degree : $\pi(DishX) = 0.4$(There is moderate uncertainty in the classification).

Definition 2.3. Let X be a non empty set. A Neutrosophic set $A \in X$ is of the form $A = \{(x, T_A(x), I_A(x), F_A(x)) : x \in X\}$, where the functions $T_A, I_A, F_A : X \longrightarrow ^-]0, 1[^+$ define respectively the degree of truth membership, the degree of indeterminacy and the degree of falsity membership for every element $x \in X$ to the set A, which is a subset of X.

$$^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+.$$

Example : Weather Condition. Suppose we want to describe the ”cloudiness” of the sky using a neutrosophic fuzzy set. The set may have the following degrees of membership :

- Truth-membership : 0.7(70 percent cloudy).
- Indeterminacy-membership : 0.2(20 percent uncertain).
- Falsity-membership : 0.1(10 percent not cloudy).

Definition 2.4. The fuzzy set A defined on the set of real numbers is said to be a fuzzy number if A and its membership function $\mu_A(X)$ has the following properties :

(1) A is normal and convex.
(2) A is bounded.
(3) $\mu_A(X)$ is piece - wise continuous.

Example : Fuzzy Number for a Distance - A fuzzy number describing the distance between two cities in kilometers : (300, 350, 400). This indicates that the distance is most likely around 350 km, and there is some degree of membership for distances between 300 km and 400 km.

3. Neutrosophic numbers and its properties

3.1. *Neutrosophic numbers*

Definition 3.1. Let X be a non empty set. A Neutrosophic set $A \in X$ is of the form $A = \{(x, T_A(x), I_A(x), F_A(x)) : x \in X\}$, where the functions $T_A, I_A, F_A : X \longrightarrow [0,1]$ define respectively the degree of truth membership, the degree of indeterminacy membership and the degree of falsity membership for every element $x \in X$ to the set A, which is a subset of X.

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

If in particular, X has only one element, A is called a neutrosophic number, which can be denoted by, $A = (T_A(x), I_A(x), F_A(x))$.

Example : Neutrosophic Number for a Student's Performance Grade - Representing a neutrosophic fuzzy number for a student's performance grade in a subject : (7.2, 7.5, 7.8). This means that the student's grade is most likely around 7.5(truth-membership degree of 7.5), with a small level of indeterminacy(0.3) and a very low level of falsity(0.6).

3.1.1. *Properties of Neutrosophic numbers*

Let $A, B \in X$. Then their operations are defined as,

(1) $(T_A(x), I_A(x), F_A(x)) + (T_B(x), I_B(x), F_B(x)) = (T_A(x) + T_B(x) - T_A(x)T_B(x), I_A(x)I_B(x), F_A(x)F_B(x))$.

(2) $(T_A(x), I_A(x), F_A(x)).(T_B(x), I_B(x), F_B(x)) = (T_A(x)T_B(x), I_A(x) + I_B(x) - I_A(x)I_B(x), F_A(x) + F_B(x) - F_A(x)F_B(x))$.

(3) $k(T_A(x), I_A(x), F_A(x)) = (1 - (1 - T_A(x))k, I_A(x)k, F_A(x)k), (k \in R)$.

(4) $(T_A(x), I_A(x), F_A(x))k = (T_A(x)k, 1 - (1 - I_A(x))k, 1 - (1 - F_A(x))k)(k \in R)$.

3.1.2. *Single-valued triangular fuzzy neutrosophic number*

The single-valued triangular fuzzy neutrosophic number $a = ((a_1, a_2, a_3); \alpha_a, \beta_a, \gamma_a)$, is a neutrosophic set on $\mathbb{R}$, whose truth, indeterminacy and falsehood membership functions are defined as follows, respectively

$$T_a(x) = \left\{ \begin{array}{rcl} \alpha_a(\frac{x-a_1}{a_2-a_1}) & \text{for} & a_1 \leq x \leq a_2 \\ \alpha_a & \text{for} & x = a_2 \\ \alpha_a(\frac{a_3-x}{a_3-a_2}) & \text{for} & a_2 < x \leq a_3 \\ 0 & \text{for} & otherwise \end{array} \right\}.$$

$$I_a(x) = \begin{Bmatrix} \frac{a_2 - x + \beta_a(x - a_1)}{a_2 - a_1} & \text{for} & a_1 \le x \le a_2 \\ \beta_a & \text{for} & x = a_2 \\ \frac{x - a_2 + \beta_a(a_3 - x)}{a_3 - a_2} & \text{for} & a_2 < x \le a_3 \\ 1 & \text{for} & otherwise \end{Bmatrix}.$$

$$F_a(x) = \begin{Bmatrix} \frac{a_2 - x + \gamma_a(x - a_1)}{a_2 - a_1} & \text{for} & a_1 \le x \le a_2 \\ \gamma_a & \text{for} & x = a_2 \\ \frac{x - a_2 + \gamma_a(a_3 - x)}{a_3 - a_2} & \text{for} & a_2 < x \le a_3 \\ 1 & \text{for} & otherwise \end{Bmatrix}.$$

where α_a, β_a, $\gamma_a \in$ [0,1], $a_1, a_2, a_3 \in \mathbb{R}$ and $a_1 \le a_2 \le a_3$.

Example : Single-Valued Triangular Fuzzy Neutrosophic Number for Age - $A = ((6, 7, 8); 1, 0, 0)$. This represents an individual's age, where the truth-membership degree α_a is 1, indicating that the person's age is exactly 7 years (value $a_2 = 7$). The indeterminacy-membership degree β_a and falsity-membership degree γ_a are both 0, indicating that there is no uncertainty or inconsistency associated with this age value.

The Single-valued triangular fuzzy neutrosophic number is expressed using the following Figure 1 :

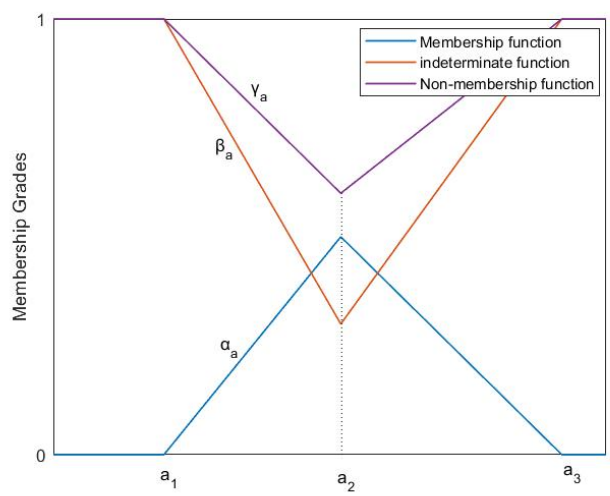

FIGURE 1. Single-valued triangular fuzzy neutrosophic number :

3.2. *Travelling salesman problem(TSP)*

3.2.1. *Description of TSP*

The Traveling salesman problem is a well-known algorithmic problem which consists of a salesman and a set of destinations or points. This problem refers to the challenge of determining the shortest yet effective route for a travelling salesman to visit a list of specific destinations. The salesman has to visit each set of destinations starting from a particular one and returning to the same. His main objective is to find the shortest route from a set of different routes to minimize the total travel cost or the total distance travelled.

3.2.2. *Mathematical formulation of TSP*

The TSP can be formulated mathematically as follows:

$\sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij}\ p_{ij},$

$\sum_{j=1}^{n} p_{ij} = 1,\ i = 1, 2, ...n$

$\sum_{i=1}^{n} p_{ij} = 1,\ j = 1, 2, ...n$

$p_{ij} = 0$ or $1,\ i = 1, 2, ...n$ and $j = 1, 2, ...n,$

where, p_{ij} is a binary variable(if destination i and destination j are not connected, then, $p_{ij} = 0$, else $p_{ij} = 1$) and d_{ij} denotes the distance between destination i and destination j.

3.2.3. *Score function*

The score function as in [26] to be utilized for converting the neutrosophic data of the NTSP into crisp data is as follows :

$$S(a) = \frac{1}{12}((a_1 + 2a_2 + a_3)(2 + \alpha_a - \beta_a - \gamma_a)), \tag{1}$$

where, $a = ((a_1, a_2, a_3); \alpha_a, \beta_a, \gamma_a)$ is a single-valued triangular neutrosophic number, α_a, β_a, $\gamma_a \in$ [0,1], $a_1, a_2, a_3 \in \mathbb{R}$ and $a_1 \leq a_2 \leq a_3$.

4. **Methodology for solving Neutrosohic Travelling salesman problem**

4.1. *Defuzzification of the Neutrosophic data*

The TSP is represented in the form of a matrix called the single-valued triangular fuzzy neutrosophic distance matrix, given by,

$$S = \begin{pmatrix} \infty & s_{12} & . & . & . & s_{1n} \\ s_{21} & \infty & . & . & . & s_{2n} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ s_{n1} & s_{n2} & . & . & . & \infty \end{pmatrix}$$

Here, all its elements are single-valued triangular fuzzy neutrosophic numbers of the form, $a = ((a_1, a_2, a_3); \alpha_a, \beta_a, \gamma_a)$. Using the above score function (1)

$$S(a) = \tfrac{1}{12}((a_1 + 2a_2 + a_3)(2 + \alpha_a - \beta_a - \gamma_a)),$$

each element(single-valued triangular fuzzy neutrosophic number) of the NTSP matrix is defuzzified, hence being converted into their respective crisp numbers.

4.2. *The suggested method for solving the single-valued triangular fuzzy neutrosophic travelling salesman problem :*

The following are the steps involved in the proposed method for solving the single-valued triangular fuzzy neutrosophic travelling salesman problem(SVTFNTSP) :

Step 1 : In the single-valued triangular fuzzy neutrosophic distance matrix(SVTFNDM), the first step is to convert all the single-valued triangular fuzzy neutrosophic data into their corresponding crisp data using the specified score function (1) as mentioned above.

Step 2 : The next step is to calculate the range value of every column of the single-valued triangular fuzzy neutrosophic distance matrix, utilizing the formula, Range = Highest value - Least value and place it below the corresponding columns.

Step 3 : The next task is to find the highest range value from all the range values calculated and select the corresponding column.

Step 4 : Now, choose the least value from the column selected and divide all the remaining entries of the matrix by the selected value. Having performed these few steps, would create certain number of ones in the matrix.

Step 5 : Try choosing exactly one 1 from each row and column. If we are able to do so, the optimal soution(OS) is obtained. If not, draw lines such that all the 1's are covered. Choose the minimum element from the uncovered elements, divide all these elements by the same, multiply the same at the intersection of the drawn lines and the other elements under the drawn line remain the same.

Step 6 : Now, again try selecting exactly one 1 in each row and column. If possible, we can move forward towards the optimal solution. Orelse, repeat step 5 till we are able to choose exactly one 1 in each row and column.

Step 7 : Finally, after being able to select exactly one 1 in each row and column, since it is a TSP, we should check whether the travelling schedule forms a cycle.,i.e., starting from city 1, the schedule should take us through all cities and return back to city 1 itself. If that is the case, we have got the crisp travelling schedule(CTS) along with the total minimal crisp travelling cost(TMCTC) or the total minimal crisp distance travelled(TMCDT) for the given single-valued triangular fuzzy neutrosophic travelling salesman problem, which will themselves serve as the crisp optimal solution(COS) and the crisp optimal travelling cost(COTC) or the crisp optimal distance travelled(CODT) respectively. Orelse, exchange any two rows(cities), one being the city to which the schedule moves from the first city(denoted by city x) and the other row(city), having the highest average among all the other rows, except city 1 and city x(this city is denoted by city y). Hence, exchanging these two cities x and y, would result in the crisp travelling schedule(CTS) along with the total minimal crisp travelling cost(TMCTC) or the total minimal crisp distance travelled(TMCDT) for the given single-valued triangular fuzzy neutrosophic travelling salesman problem, which will themselves serve as the crisp optimal solution(COS) and the crisp optimal travelling cost(COTC) or the crisp optimal distance travelled(CODT) respectively(The main objective of the salesman being to find the shortest route from a set of different routes, thereby minimizing the total travel cost or the total distance travelled).

5. Illustrations for the suggested methodology :

5.1. *Illustration 1 :*

Consider the following symmetric TSP in the form of a single-valued triangular fuzzy neutrosophic distance matrix [25],

$$S = \begin{pmatrix} \infty & s_{12} & s_{13} & s_{14} \\ s_{21} & \infty & s_{23} & s_{24} \\ s_{31} & s_{32} & \infty & s_{34} \\ s_{41} & s_{42} & s_{43} & \infty \end{pmatrix}$$

where its elements(single-valued triangular fuzzy neutrosophic numbers) are as follows :
$s_{12} = s_{21} = \langle(4, 6, 10)\,; 0.8, 0.4, 0.2\rangle$; $s_{13} = s_{31} = \langle(2, 5, 9)\,; 0.7, 0.6, 0.3\rangle$;
$s_{14} = s_{41} = \langle(4, 7, 9)\,; 0.6, 0.6, 0.3\rangle$; $s_{23} = s_{32} = \langle(1, 5, 8)\,; 0.8, 0.5, 0.2\rangle$;
$s_{24} = s_{42} = \langle(2, 7, 9)\,; 0.8, 0.5, 0.4\rangle$; $s_{34} = s_{43} = \langle(1, 5, 10)\,; 0.8, 0.3, 0.1\rangle$.

Step 1 : Using the above specified score function (1),

$$S(a) = \frac{1}{12}((a_1 + 2a_2 + a_3)(2 + \alpha_a - \beta_a - \gamma_a)),$$

converting the given neutrosophic data into their corresponding crisp data, we obtain, $S(s_{12}) = S(s_{21}) = \frac{26\times2.2}{12} = \frac{57.2}{12} = 4.7667$; $S(s_{13}) = S(s_{31}) = \frac{21\times1.8}{12} = \frac{37.8}{12} = 3.15$;

$S(s_{14}) = S(s_{41}) = \frac{27\times1.7}{12} = \frac{45.9}{12} = 3.825$; $S(s_{23}) = S(s_{32}) = \frac{19\times2.1}{12} = \frac{39.9}{12} = 3.325$;

$S(s_{24}) = S(s_{42}) = \frac{25\times1.9}{12} = \frac{47.5}{12} = 3.9583$; $S(s_{34}) = S(s_{43}) = \frac{21\times2.4}{12} = \frac{50.4}{12} = 4.2$.

The finally obtained crisp equivalent TSP matrix is given by,

$$S = \begin{pmatrix} \infty & 4.7667 & 3.15 & 3.825 \\ 4.7667 & \infty & 3.325 & 3.9583 \\ 3.15 & 3.325 & \infty & 4.2 \\ 3.825 & 3.9583 & 4.2 & \infty \\ \end{pmatrix}$$

Step 2 : The next step is to calculate the range value of every column of the single-valued triangular fuzzy neutrosophic distance matrix, utilizing the formula, Range = Highest value - Least value. The required values for the columns 1 - 4 are found to be 1.6167, 1.4417, 1.05 and 0.375 respectively.

Step 3 : Now, the highest range value from all the range values calculated is 1.6167 and the corresponding column is selected, which is found to be column 1.

Step 4 : Next, we choose the least value 3.15 from the column 1 selected as follows :

$$S = \begin{pmatrix} \infty & 4.7667 & 3.15 & 3.825 \\ 4.7667 & \infty & 3.325 & 3.9583 \\ \boxed{3.15} & 3.325 & \infty & 4.2 \\ 3.825 & 3.9583 & 4.2 & \infty \\ \end{pmatrix}$$

Then, we divide all the remaining entries of the matrix by this chosen least value. Having performed these few steps, would create certain number of ones in the matrix, which is shown in the following table :

$$S = \begin{pmatrix} \infty & 1.51 & 1 & 1.22 \\ 1.51 & \infty & 1.06 & 1.26 \\ 1 & 1.06 & \infty & 1.33 \\ 1.22 & 1.26 & 1.33 & \infty \\ \end{pmatrix}$$

The steps having been performed so far has created quite a few number of 1's.

Step 5 : Let us try selecting exactly one 1 in each row and column, as shown below :

$$S = \begin{pmatrix} \infty & 1.51 & \boxed{1} & 1.22 \\ 1.51 & \infty & 1.06 & 1.26 \\ \boxed{1} & 1.06 & \infty & 1.33 \\ 1.22 & 1.26 & 1.33 & \infty \\ \end{pmatrix}$$

We find that it is not possible from the above matrix. Hence in order to reach the optimal solution, we draw lines by covering all the 1's in the matrix, which covers row 1 and column 1. The least element among all the uncovered elements is found to be 1.06. We divide all these uncovered elements by 1.06, multiply the same at the intersection of the drawn lines and the other elements under the drawn line remain the same. The resulting matrix is,

$$S = \begin{pmatrix} \infty & 1.51 & 1 & 1.22 \\ 1.51 & \infty & 1 & 1.19 \\ 1 & 1 & \infty & 1.25 \\ 1.22 & 1.19 & 1.25 & \infty \\ \end{pmatrix}$$

Step 6 : Now, let us again try selecting exactly one 1 in each row and column, which is shown below :

$$S = \begin{pmatrix} \infty & 1.51 & \boxed{1} & 1.22 \\ 1.51 & \infty & 1 & 1.19 \\ \boxed{1} & 1 & \infty & 1.25 \\ 1.22 & 1.19 & 1.25 & \infty \\ \end{pmatrix}$$

Hence, from the above matrix, we find that it is not possible to select exactly one 1 in each row and column. Thus, we cannot move forward towards the optimal solution. So, there is need to repeat step 5. Hence, again we draw lines by covering all the 1's in the matrix, which covers row 3 and column 3. The least element among all the uncovered elements is found to be 1.19. We divide all these uncovered elements by 1.19, multiply the same at the intersection of the drawn lines and the other elements under the drawn line remain the same. The resulting matrix is,

$$S = \begin{pmatrix} \infty & 1.27 & 1 & 1.03 \\ 1.27 & \infty & 1 & 1 \\ 1 & 1 & \infty & 1.25 \\ 1.03 & 1 & 1.25 & \infty \\ \end{pmatrix}$$

At this stage, we are able to select exactly one 1 in each row and column, which is shown below :

$$S = \begin{pmatrix} \infty & 1.27 & \boxed{1} & 1.03 \\ 1.27 & \infty & 1 & \boxed{1} \\ \boxed{1} & 1 & \infty & 1.25 \\ 1.03 & \boxed{1} & 1.25 & \infty \\ \end{pmatrix}$$

Hence, we can move forward towards the optimal solution. So, there is no need to repeat step 5.

Step 7 : The crisp travelling schedule(CTS) here is, $1 \to 3, 2 \to 4, 3 \to 1$ and $4 \to 2$. Thus, we find that this schedule is not a cycle. Since city 1 moves forward to city 3, city x is chosen to be city 3. On calculating the average of all the rows(cities), city 4 is found to have the highest average and hence is considered as city y. On exchanging cities 3 and 4(i.e.,cities x and y), we obtain the cycle(CTS) - $1 \to 3 \to 2 \to 4 \to 1$, which itself serves as the crisp optimal travelling schedule(COTS), where the resulting matrix is,

$$S = \begin{pmatrix} \infty & 1.27 & \boxed{1} & 1.03 \\ 1.27 & \infty & 1 & \boxed{1} \\ 1.03 & \boxed{1} & 1.25 & \infty \\ \boxed{1} & 1 & \infty & 1.25 \end{pmatrix}$$

The total minimal crisp travelling cost(TMCTC) = Rs.(3.15 + 3.325 + 3.9583 + 3.825) = Rs.14.2583, which itself serves as the crisp optimal travelling cost(COTC).

5.2. *Illustration 2 :*

Consider the following symmetric TSP in the form of a single-valued triangular fuzzy neutrosophic distance matrix [25],

$$S = \begin{pmatrix} \infty & s_{12} & s_{13} & s_{14} & s_{15} \\ s_{21} & \infty & s_{23} & s_{24} & s_{25} \\ s_{31} & s_{32} & \infty & s_{34} & s_{35} \\ s_{41} & s_{42} & s_{43} & \infty & s_{45} \\ s_{51} & s_{52} & s_{53} & s_{54} & \infty \end{pmatrix}$$

where its elements(single-valued triangular fuzzy neutrosophic numbers) are as follows :

$s_{12} = s_{21} = \langle (1, 9, 20) ; 0.9, 0.4, 0.1 \rangle$; $s_{13} = s_{31} = \langle (2, 9, 25) ; 0.8, 0.5, 0.1 \rangle$;

$s_{14} = s_{41} = \langle (5, 7, 9) ; 0.9, 0.7, 0.1 \rangle$; $s_{15} = s_{51} = \langle (2, 9, 19) ; 0.4, 0.5, 0.3 \rangle$;

$s_{23} = s_{32} = \langle (3, 9, 14) ; 0.7, 0.3, 0.3 \rangle$; $s_{24} = s_{42} = \langle (5, 8, 13) ; 0.6, 0.2, 0.4 \rangle$;

$s_{25} = s_{52} = \langle (7, 9, 18) ; 0.4, 0.1, 0.1 \rangle$; $s_{34} = s_{43} = \langle (4, 8, 17) ; 0.8, 0.5, 0.2 \rangle$;

$s_{35} = s_{53} = \langle (5, 9, 15) ; 0.9, 0.6, 0.1 \rangle$; $s_{45} = s_{54} = \langle (1, 9, 16) ; 0.7, 0.4, 0.3 \rangle$.

Step 1 : Using the above mentioned score function (1),

$$S(A) = \frac{1}{5}((a_1 + a_2 + a_3) - (\alpha_a + \beta_a + \gamma_a)),$$

converting the given neutrosophic data into their corresponding crisp data, we obtain, $S(s_{12}) = S(s_{21}) = \frac{39 \times 2.4}{12} = \frac{93.6}{12} = 7.8$; $S(s_{13}) = S(s_{31}) = \frac{45 \times 2.2}{12} = \frac{99}{12} = 8.25$;

$S(s_{14}) = S(s_{41}) = \frac{28\times2.1}{12} = \frac{58.8}{12} = 4.9$; $S(s_{15}) = S(s_{51}) = \frac{39\times1.7}{12} = \frac{66.3}{12} = 5.525$;

$S(s_{23}) = S(s_{32}) = \frac{35\times2.1}{12} = \frac{73.5}{12} = 6.125$; $S(s_{24}) = S(s_{42}) = \frac{34\times2}{12} = \frac{68}{12} = 5.6667$;

$S(s_{25}) = S(s_{52}) = \frac{43\times2.2}{12} = \frac{94.6}{12} = 7.8833$; $S(s_{34}) = S(s_{43}) = \frac{37\times2.1}{12} = \frac{77.7}{12} = 6.475$;

$S(s_{35}) = S(s_{53}) = \frac{38\times2.2}{12} = \frac{83.6}{12} = 6.9667$; $S(s_{45}) = S(s_{54}) = \frac{35\times2}{12} = \frac{70}{12} = 5.8333.$

The finally obtained crisp equivalent TSP matrix is given by,

$$S = \begin{pmatrix} \infty & 7.8 & 8.25 & 4.9 & 5.525 \\ 7.8 & \infty & 6.125 & 5.6667 & 7.8833 \\ 8.25 & 6.125 & \infty & 6.475 & 6.9667 \\ 4.9 & 5.6667 & 6.475 & \infty & 5.8333 \\ 5.525 & 7.8833 & 6.9667 & 5.8333 & \infty \end{pmatrix}$$

Step 2 : The next step is to calculate the range value of every column of the single-valued triangular fuzzy neutrosophic distance matrix, utilizing the formula, Range = Highest value - Least value. The required values for the columns 1 - 5 are found to be 3.35, 2.2166, 2.125, 1.575 and 2.3583 respectively.

Step 3 : Now, the highest range value, from all the range values calculated is 3.35 and the corresponding column is selected, which is found to be column 1.

Step 4 : Next, we choose the least value, 4.9 from the column 1 selected, as follows :

$$S = \begin{pmatrix} \infty & 7.8 & 8.25 & 4.9 & 5.525 \\ 7.8 & \infty & 6.125 & 5.6667 & 7.8833 \\ 8.25 & 6.125 & \infty & 6.475 & 6.9667 \\ \boxed{4.9} & 5.6667 & 6.475 & \infty & 5.8333 \\ 5.525 & 7.8833 & 6.9667 & 5.8333 & \infty \end{pmatrix}$$

Then, we divide all the remaining entries of the matrix by this chosen least value. Having performed these few steps, would create certain number of ones in the matrix, which is shown in the following table :

$$S = \begin{pmatrix} \infty & 1.5918 & 1.6837 & 1 & 1.1276 \\ 1.5918 & \infty & 1.25 & 1.1565 & 1.6088 \\ 1.6837 & 1.25 & \infty & 1.3214 & 1.4218 \\ 1 & 1.1565 & 1.3214 & \infty & 1.1905 \\ 1.1276 & 1.6088 & 1.4218 & 1.1905 & \infty \end{pmatrix}$$

The steps having been performed so far has created quite a few number of 1's.

Step 5 : Let us try selecting exactly one 1 in each row and column, as shown below :

$$S = \begin{pmatrix} \infty & 1.5918 & 1.6837 & \boxed{1} & 1.1276 \\ 1.5918 & \infty & 1.25 & 1.1565 & 1.6088 \\ 1.6837 & 1.25 & \infty & 1.3214 & 1.4218 \\ \boxed{1} & 1.1565 & 1.3214 & \infty & 1.1905 \\ 1.1276 & 1.6088 & 1.4218 & 1.1905 & \infty \end{pmatrix}$$

We find that it is not possible from the above matrix. Hence in order to reach the optimal solution, we draw lines by covering all the 1's in the matrix, which covers row 1 and column 1. The least element among all the uncovered elements is found to be 1.1565. We divide all these uncovered elements by 1.1565, multiply the same at the intersection of the drawn lines and the other elements under the drawn line remain the same. The resulting matrix is,

$$S = \begin{pmatrix} \infty & 1.5918 & 1.6837 & 1 & 1.1276 \\ 1.5918 & \infty & 1.0808 & 1 & 1.3911 \\ 1.6837 & 1.0808 & \infty & 1.4259 & 1.2294 \\ 1 & 1 & 1.4259 & \infty & 1.0294 \\ 1.1276 & 1.3911 & 1.2294 & 1.0294 & \infty \end{pmatrix}$$

Step 6 : Now, let us again try selecting exactly one 1 in each row and column, which is shown below :

$$S = \begin{pmatrix} \infty & 1.5918 & 1.6837 & \boxed{1} & 1.1276 \\ 1.5918 & \infty & 1.0808 & 1 & 1.3911 \\ 1.6837 & 1.0808 & \infty & 1.4259 & 1.2294 \\ \boxed{1} & 1 & 1.4259 & \infty & 1.0294 \\ 1.1276 & 1.3911 & 1.2294 & 1.0294 & \infty \end{pmatrix}$$

Hence, from the above matrix, we find that it is not possible to select exactly one 1 in each row and column. Thus, we cannot move forward towards the optimal solution. So, there is need to repeat step 5. Hence, again we draw lines by covering all the 1's in the matrix, which covers row 4 and column 4. The least element among all the uncovered elements is found to be 1.0808. We divide all these uncovered elements by 1.0808, multiply the same at the intersection of the drawn lines and the other elements under the drawn line remain the same. The resulting matrix is,

$$S = \begin{pmatrix} \infty & 1.4728 & 1.5578 & 1 & 1.0433 \\ 1.4728 & \infty & 1 & 1 & 1.2871 \\ 1.5578 & 1 & \infty & 1.4259 & 1.1375 \\ 1 & 1 & 1.4259 & \infty & 1.0294 \\ 1.0433 & 1.2871 & 1.1375 & 1.0294 & \infty \\ \end{pmatrix}$$

At this stage, we are not able to select exactly one 1 in each row and column, which is shown below :

$$S = \begin{pmatrix} \infty & 1.4728 & 1.5578 & \boxed{1} & 1.0433 \\ 1.4728 & \infty & \boxed{1} & 1 & 1.2871 \\ 1.5578 & \boxed{1} & \infty & 1.4259 & 1.1375 \\ \boxed{1} & 1 & 1.4259 & \infty & 1.0294 \\ 1.0433 & 1.2871 & 1.1375 & 1.0294 & \infty \\ \end{pmatrix}$$

Hence, we cannot move forward towards the optimal solution. So, there is need to repeat step 5. Thus, again repeating step 5 for a few number of times, we obtain the resulting matrix, where we are able to select exactly one 1 in each row and column, which is shown below :

$$S = \begin{pmatrix} \infty & 1.5161 & 1.4931 & \boxed{1} & 1 \\ 1.5161 & \infty & \boxed{1} & 1.0433 & 1.2871 \\ 1.4931 & \boxed{1} & \infty & 1.3852 & 1.0592 \\ 1 & 1.0433 & 1.3852 & \infty & \boxed{1} \\ \boxed{1} & 1.2871 & 1.0592 & 1 & \infty \\ \end{pmatrix}$$

Hence, we can move forward towards the optimal solution. So, there is no need to repeat step 5.

Step 7 : The crisp travelling schedule(CTS) here is, $1 \to 4, 2 \to 3, 3 \to 2, 4 \to 5$ and $5 \to 1$. Thus, we find that this schedule is not a cycle. Since city 1 moves forward to city 4, city x is chosen to be city 4. On calculating the average of all the rows(cities), city 3 is found to have the highest average and hence is considered as city y. On exchanging cities 3 and 4(i.e.,cities x and y), we obtain the cycle(CTS) - $1 \to 4 \to 2 \to 3 \to 5 \to 1$, which is the crisp optimal travelling schedule(COTS), where the resulting matrix is,

$$S = \begin{pmatrix} \infty & 1.5161 & 1.4931 & \boxed{1} & 1 \\ 1.5161 & \infty & \boxed{1} & 1.0433 & 1.2871 \\ 1 & 1.0433 & 1.3852 & \infty & \boxed{1} \\ 1.4931 & \boxed{1} & \infty & 1.3852 & 1.0592 \\ \boxed{1} & 1.2871 & 1.0592 & 1 & \infty \end{pmatrix}$$

The total minimal crisp travelling cost(TMCTC) = Rs.(4.9 + 5.6667 + 6.125 + 6.9667 + 5.525) = Rs.29.1834, which itself serves as the crisp optimal travelling cost(COTC).

5.3. *Illustration 3 :*

Consider the following elements(single-valued triangular fuzzy neutrosophic numbers)
$s_{12} = s_{21} = \langle (2, 8, 18) ; 0.8, 0.3, 0.2\rangle$; $s_{13} = s_{31} = \langle (1, 9, 24) ; 0.9, 0.6, 0.2\rangle$;
$s_{14} = s_{41} = \langle (4, 9, 15) ; 0.6, 0.5, 0.2\rangle$; $s_{15} = s_{51} = \langle (3, 6, 13) ; 0.6, 0.3, 0.3\rangle$;
$s_{23} = s_{32} = \langle (6, 9, 19) ; 0.9, 0.4, 0.1\rangle$; $s_{24} = s_{42} = \langle (1, 7, 12) ; 0.9, 0.1, 0.2\rangle$;
$s_{25} = s_{52} = \langle (5, 9, 18) ; 0.9, 0.8, 0.2\rangle$; $s_{34} = s_{43} = \langle (3, 8, 23) ; 0.7, 0.1, 0.1\rangle$;
$s_{35} = s_{53} = \langle (2, 8, 32) ; 0.6, 0.5, 0.4\rangle$; $s_{45} = s_{54} = \langle (2, 5, 11) ; 0.9, 0.3, 0.1\rangle$.

of the symmetric TSP in the form of a single-valued triangular fuzzy neutrosophic distance matrix [25],

$$S = \begin{pmatrix} \infty & s_{12} & s_{13} & s_{14} & s_{15} \\ s_{21} & \infty & s_{23} & s_{24} & s_{25} \\ s_{31} & s_{32} & \infty & s_{34} & s_{35} \\ s_{41} & s_{42} & s_{43} & \infty & s_{45} \\ s_{51} & s_{52} & s_{53} & s_{54} & \infty \end{pmatrix}$$

Step 1 : Using the above specified score function (1),

$$S(a) = \tfrac{1}{12}((a_1 + 2a_2 + a_3)(2 + \alpha_a - \beta_a - \gamma_a)),$$

converting the given neutrosophic data into their corresponding crisp data, we obtain, $S(s_{12}) = S(s_{21}) = \frac{36\times 2.3}{12} = \frac{82.8}{12} = 6.9$; $S(s_{13}) = S(s_{31}) = \frac{43\times 2.1}{12} = \frac{90.3}{12} = 7.525$;

$S(s_{14}) = S(s_{41}) = \frac{37\times 1.9}{12} = \frac{70.3}{12} = 5.8583$; $S(s_{15}) = S(s_{51}) = \frac{28\times 2}{12} = \frac{56}{12} = 4.6667$;

$S(s_{23}) = S(s_{32}) = \frac{43\times 2.4}{12} = \frac{103.2}{12} = 8.6$; $S(s_{24}) = S(s_{42}) = \frac{27\times 2.6}{12} = \frac{70.2}{12} = 5.85$;

$S(s_{25}) = S(s_{52}) = \frac{42\times 2.5}{12} = \frac{105}{12} = 8.75$; $S(s_{34}) = S(s_{43}) = \frac{50\times 1.7}{12} = \frac{85}{12} = 7.0833$;

$S(s_{35}) = S(s_{53}) = \frac{23\times 2.5}{12} = \frac{57.5}{12} = 4.7917$; $S(s_{45}) = S(s_{54}) = \frac{41\times 1.9}{12} = \frac{77.9}{12} = 6.4917$.

The finally obtained crisp equivalent TSP matrix is given by,

$$S = \begin{pmatrix} \infty & 6.9 & 7.525 & 5.8583 & 4.6667 \\ 6.9 & \infty & 8.6 & 5.85 & 6.4917 \\ 7.525 & 8.6 & \infty & 8.75 & 7.0833 \\ 5.8583 & 5.85 & 8.75 & \infty & 4.7917 \\ 4.6667 & 6.4917 & 7.0833 & 4.7917 & \infty \end{pmatrix}$$

Step 2 : The next step is to calculate the range value of every column of the single-valued triangular fuzzy neutrosophic distance matrix, utilizing the formula, Range = Highest value - Least value. The required values for the columns 1 - 5 are found to be 2.8583, 2.75, 1.6667, 3.9583 and 2.4166 respectively.

Step 3 : Now, the highest range value, from all the range values calculated is 3.9583 and the corresponding column is selected, which is found to be column 4.

Step 4 : Next, we choose the least value, 4.7917 from the column 4 selected, as follows :

$$S = \begin{pmatrix} \infty & 6.9 & 7.525 & 5.8583 & 4.6667 \\ 6.9 & \infty & 8.6 & 5.85 & 6.4917 \\ 7.525 & 8.6 & \infty & 8.75 & 7.0833 \\ 5.8583 & 5.85 & 8.75 & \infty & 4.7917 \\ 4.6667 & 6.4917 & 7.0833 & \boxed{4.7917} & \infty \end{pmatrix}$$

Then, we divide all the remaining entries of the matrix by this chosen least value. Having performed these few steps, would create certain number of ones in the matrix, which is shown in the following matrix :

$$S = \begin{pmatrix} \infty & 1.44 & 1.5704 & 1.2226 & 0.9739 \\ 1.44 & \infty & 1.7948 & 1.2209 & 1.3548 \\ 1.5704 & 1.7948 & \infty & 1.8261 & 1.4782 \\ 1.2226 & 1.2209 & 1.8261 & \infty & 1 \\ 0.9739 & 1.3548 & 1.4782 & 1 & \infty \end{pmatrix}$$

The steps having been performed so far has created quite a few number of 1's.

Step 5 : Let us try selecting exactly one 1 in each row and column, as shown below :

$$S = \begin{pmatrix} \infty & 1.44 & 1.5704 & 1.2226 & 0.9739 \\ 1.44 & \infty & 1.7948 & 1.2209 & 1.3548 \\ 1.5704 & 1.7948 & \infty & 1.8261 & 1.4782 \\ 1.2226 & 1.2209 & 1.8261 & \infty & \boxed{1} \\ 0.9739 & 1.3548 & 1.4782 & \boxed{1} & \infty \end{pmatrix}$$

We find that it is not possible from the above matrix. Hence in order to reach the optimal solution, we draw lines by covering all the 1's in the matrix, which covers row 4 and column 4. The least element among all the uncovered elements is found to be 1.2209. We divide all these uncovered elements by 1.2209, multiply the same at the intersection of the drawn lines and the other elements under the drawn line remain the same. The resulting matrix is,

$$S = \begin{pmatrix} \infty & 1.1795 & 1.2863 & 1.0014 & 0.9739 \\ 1.1795 & \infty & 1.4701 & 1 & 1.3548 \\ 1.2863 & 1.4701 & \infty & 1.4957 & 1.4782 \\ 1.0014 & 1 & 1.4957 & \infty & 1 \\ 0.9739 & 1.3548 & 1.4782 & 1 & \infty \end{pmatrix}$$

Step 6 : Now, let us again try selecting exactly one 1 in each row and column, which is shown below :

$$S = \begin{pmatrix} \infty & 1.1795 & 1.2863 & 1.0014 & 0.9739 \\ 1.1795 & \infty & 1.4701 & \boxed{1} & 1.3548 \\ 1.2863 & 1.4701 & \infty & 1.4957 & 1.4782 \\ 1.0014 & \boxed{1} & 1.4957 & \infty & 1 \\ 0.9739 & 1.3548 & 1.4782 & 1 & \infty \end{pmatrix}$$

Hence, from the above matrix, we find that it is not possible to select exactly one 1 in each row and column. Thus, we cannot move forward towards the optimal solution. So, there is need to repeat step 5. Hence, again repeating step 5 for a few number of times, we obtain the resulting matrix, where we are able to select exactly one 1 in each row and column, which is shown below :

$$S = \begin{pmatrix} \infty & 1.2111 & \boxed{1} & 1.1444 & 1 \\ 1.2111 & \infty & 1 & \boxed{1} & 1.2172 \\ 1 & \boxed{1} & \infty & 1.1324 & 1.0054 \\ 1.1444 & 1 & 1.1324 & \infty & \boxed{1} \\ \boxed{1} & 1.2172 & 1.0054 & 1 & \infty \end{pmatrix}$$

Hence, we can move forward towards the optimal solution. So, there is no need to repeat step 5.

Step 7 : The crisp travelling schedule(CTS) here is, $1 \to 3, 2 \to 4, 3 \to 2, 4 \to 5$ and $5 \to 1$. Since we find that this schedule is a cycle(CTS), $1 \to 3 \to 2 \to 4 \to 5 \to 1$, there is no need to exchange the cities and hence this itself serves as the crisp optimal travelling schedule(COTS), where the resulting matrix is the same as that obtained in the previous step which is,

$$S = \begin{pmatrix} \infty & 1.2111 & \boxed{1} & 1.1444 & 1 \\ 1.2111 & \infty & 1 & \boxed{1} & 1.2172 \\ 1 & \boxed{1} & \infty & 1.1324 & 1.0054 \\ 1.1444 & 1 & 1.1324 & \infty & \boxed{1} \\ \boxed{1} & 1.2172 & 1.0054 & 1 & \infty \\ \end{pmatrix}$$

The total minimal crisp travelling cost(TMCTC) = Rs.(7.525 + 8.6 + 5.85 + 4.7917 + 4.6667) = Rs.31.4334, which itself serves as the crisp optimal travelling cost(COTC).

Remark : We observe that the TTC for illustrations 1, 2 and 3 obtained here using the proposed method(PM) differ from those of the corresponding illustrations acquired using the Dhouib-Matrix-TSP1(DM-TSP1) heuristic in [25]. The following section discusses further insights and justifications on the SVTFNTSP that was taken into account in this study.

6. **Results and Discussions :**

The following tables 1, 2 and figures 2, 3, 4 and 5 provide the solutions of the SVTFNTSP obtained using the proposed approach and a comparison of the solutions of the proposed approach here to solve the SVTFNTSP, with an existing method(EM : Dhouib-Matrix-TSP1(DM-TSP1) heuristic) as in [25]. Some of the significant results are shown in these tables and figures.

Table 1. Solutions of the SVTFNTSP obtained using the proposed method :

Illustrations	**CTS**	**TMCTC**
Illustration 1	$1 \to 3 \to 2 \to 4 \to 1$	Rs.14.2583
Illustration 2	$1 \to 4 \to 2 \to 3 \to 5 \to 1$	Rs.29.1834
Illustration 3	$1 \to 3 \to 2 \to 4 \to 5 \to 1$	Rs.31.4334

- The table 1 showcases the crisp travelling schedule(CTS), the total minimal crisp travelling cost(TMCTC) of all the above considered three illustrations, obtained using the proposed method.

TABLE 2. Comparison of the solutions of the SVTFNTSP obtained using the proposed approach, with an existing method as in [25] :

Illustrations	CTS of the EM	CTS of the PM	TMCTC of the EM	TMCTC of the PM
Illustration 1	$1 \to 4 \to 2 \to 3 \to 1$	$1 \to 3 \to 2 \to 4 \to 1$	Rs.14.27	Rs.14.2583
Illustration 2	$1 \to 5 \to 3 \to 2 \to 4 \to 1$	$1 \to 4 \to 2 \to 3 \to 5 \to 1$	Rs.29.20	Rs.29.1834
Illustration 3	$1 \to 3 \to 2 \to 4 \to 5 \to 1$	$1 \to 3 \to 2 \to 4 \to 5 \to 1$	Rs.31.44	Rs.31.4334

- The table 2 compares the CTS and TMCTC calculated using the proposed method(PM) with those of the three illustrations, expressed as real-world problems taken into consideration above, using an existing method(EM : Dhouib-Matrix-TSP1(DM-TSP1) heuristic) as in [25].
- Thus, the above comparison ensures that for the given SVTFNTSPs, the CTS and the TMCTC found by applying the proposed method itself serve as the COTS and the COTC, respectively.
- The same above conclusions can be drawn from the figure 2 which provides an overview of the solutions(COTCs) to the three SVTFNTSPs that were previously taken into consideration and solved using the previously mentioned existing method with that of the same three problems using the proposed method. It does so by demonstrating a sizable amount of variation in the values of the solutions, through the usage of the proposed method.

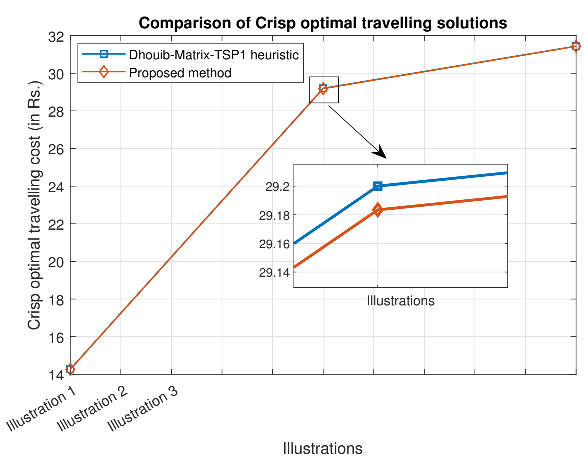

FIGURE 2. An overview of the optimal solutions determined by Dhouib-Matrix-TSP1 heuristic and the proposed approach :

- Figure 2 also shows that the solutions to the aforementioned three SVTFNTSPs obtained using the proposed method(represented by the orange line) are better(in terms of TMCTC/COTC) than those obtained using the other existing method(represented by the blue line) as in [25]. It also presents an even more clear picture of how the solution of the SVTFNTSPs under consideration using the proposed method(represented by the orange line) is better than that of the same SVTFNTSPs using the other existing method(represented by the blue line) as in [25], by providing just a glimpse of an enlarged version of the sample with a considerable amount of variation in the total minimal crisp travelling costs(TMCTC/COTC).

- Knowing that a few classical methods are frequently used to test the optimality of the given TSP and provide better solutions than the other methods, we can infer from the figure 2 that the proposed method also seems to fulfil the same purpose (thereby providing the best possible solution (or) making the total crisp travelling costs as minimal as possible) by giving lower values for the TMCTC of the three SVTFNTSPs taken into consideration in this article, when compared to the Hungarian method thus making it as the COTS along with the COTC.

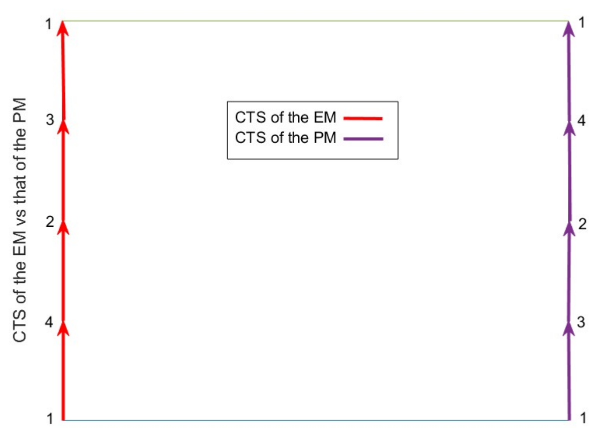

FIGURE 3. A comparison of the CTS(COTS) of illustration 1 using the existing method as in [25] with that of the proposed approach :

- The figure 3 presents a glimpse of the comparison of the (CTS/COTS) of illustration 1 using an existing method($1 \to 4 \to 2 \to 3 \to 1$)(represented by red arrow line) as in [25] and the proposed method($1 \to 3 \to 2 \to 4 \to 1$)(represented by violet arrow line) of the SVTFNTSP taken into consideration respectively.
- The figure 4 presents a glimpse of the comparison of the (CTS/COTS) of illustration 2 using an existing method($1 \to 5 \to 3 \to 2 \to 4 \to 1$)(represented by red arrow line) as in [25] and the proposed method($1 \to 4 \to 2 \to 3 \to 5 \to 1$)(represented by violet arrow line) of the SVTFNTSP taken into consideration respectively.

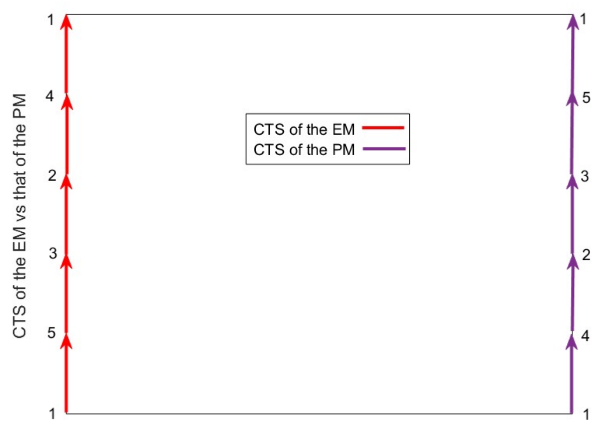

FIGURE 4. A comparison of the CTS(COTS) of illustration 2 using the existing method as in [25] with that of the proposed approach :

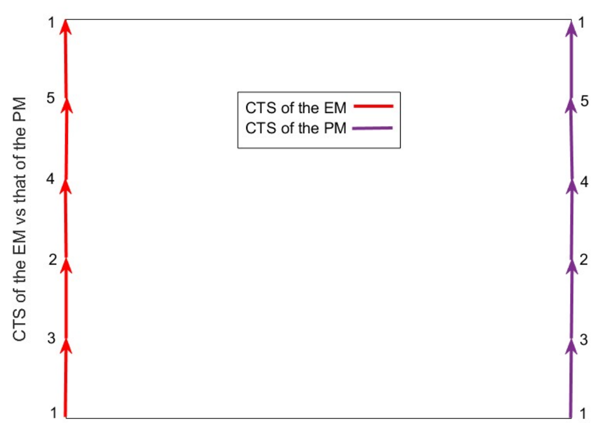

FIGURE 5. A comparison of the CTS(COTS) of illustration 3 using the existing method as in [25] with that of the proposed approach :

- The figure 5 presents a glimpse of the comparison of the (CTS/COTS) of illustration 3 using an existing method($1 \to 3 \to 2 \to 4 \to 5 \to 1$)(represented by red arrow line) as in [25] and the proposed method($1 \to 3 \to 2 \to 4 \to 5 \to 1$)(represented by violet arrow line) of the SVTFNTSP taken into consideration respectively.
- Exploring the single-valued triangular fuzzy neutrosophic travelling salesman problem(SVTFNTSP) comes with several limitations. Firstly, the SVTFNTSP deals with highly uncertain and ambiguous data, making it a complex problem to model and solve accurately. The lack of precise and standardized mathematical representations

for triangular fuzzy neutrosophic data can hinder the development of robust optimization algorithms. Additionally, as SVTFNTSP is an extension of the classic traveling salesman problem(TSP), it inherits its NP-hard complexity, meaning that finding an optimal solution within a reasonable timeframe can be computationally infeasible for large-scale instances. This limitation poses challenges in real-world applications where the problem size can be substantial. Moreover, the availability of real-world data in the form of single-valued triangular fuzzy neutrosophic numbers can be scarce, leading to difficulties in validating and benchmarking proposed algorithms. The lack of well-established benchmarks and standardized datasets makes it challenging to assess the performance of different approaches effectively. Furthermore, the SVTFNTSP introduces additional computational overhead compared to solving the traditional TSP, which can limit its practical applicability. Despite its potential to model uncertainty more accurately, the SVTFNTSP remains an area of ongoing research with open challenges in algorithm development and real-world implementation.

- Having incorporated the range(a measure of dispersion) in the proposed methodology for solving the single-valued triangular fuzzy neutrosophic traveling salesman problem (SVTFNTSP) can offer some potential advantages as follows : The single-valued triangular fuzzy neutrosophic sets in SVTFNTSP are designed to represent indeterminacy and uncertainty in the problem. The range provides a simple way to quantify the spread or variability of neutrosophic values, which can help capture the degree of uncertainty associated with each data point or city in the SVTFNTSP instance. The range is a straightforward measure to calculate and understand. It can be easily computed for neutrosophic values without the need for complex mathematical operations, making it accessible to a wide range of users. The range can be used to create visual representations of neutrosophic data, such as scatter plots or graphs, which can help analysts and decision-makers gain insights into the distribution of data points and associated uncertainty. In some cases, decision-makers may have preferences for solutions with lower or higher ranges. For example, they may prioritize routes with less variability in travel times. In such cases, the range can be used as an additional criterion for evaluating and ranking solutions. The range can be used to compare the dispersion of neutrosophic values across different cities or nodes in the NTSP. This can aid in identifying cities with higher or lower levels of uncertainty, which may influence route planning or decision-making. Analyzing how changes in neutrosophic values affect the range can help assess the sensitivity of the NTSP solution to variations in data, which can be valuable in robust decision-making. The range can complement other measures of dispersion and central tendency, such as standard deviation and mean, providing a more comprehensive view of the data distribution and uncertainty.

7. Conclusion

The SVTFNTSP presents a significant expansion of the traditional TSP, incorporating uncertainty and ambiguity through SVTFNS. This innovative framework empowers decision-makers to more effectively address complex real-world situations by portraying uncertain data more realistically. By taking into account various perspectives, it aids in a better comprehension of the preferences of decision-makers, ultimately enhancing the problem-solving process. Researchers have devised innovative algorithms and methodologies to tackle the intricate nature of single-valued triangular fuzzy neutrosophic data and resolve the same. They have explored a range of optimization techniques, heuristic approaches, and metaheuristics to achieve efficient solutions and enhance computational efficiency. Consequently, this research article earnestly endeavors to assist in this regard by investigating the characteristics, types, and resolutions of SVTFNTSP, introducing a novel method for its resolution. To demonstrate its efficiency and significance, the proposed approach is juxtaposed with specific classical methods. The proposed method consistently demonstrates its effectiveness, advantages, and potential through comparative analyses against alternative methods. It offers reductions in overall travelling costs, optimal solutions, and computational simplification while maintaining solution quality. By effectively leveraging SVTFNS, it adeptly captures uncertainty and ambiguity, yielding assignment solutions that approach optimality. Its ability to strike a balance between accuracy and computational efficiency makes it the preferred choice for real-world problem-solving, particularly in scenarios demanding time and resource optimization. This reduction in computational complexity and overall travelling cost enhances its applicability to larger and more complex SVTFNTSP instances, spanning diverse domains such as supply chain management, transportation, and logistics, where cost-effective solutions hold paramount importance. Notwithstanding its merits, ongoing research endeavors to fine-tune algorithms and address scalability issues, broadening its scope to fully realize its potential. Overall, SVTFNTSP offers a valuable and versatile approach for addressing uncertainty in decision-making, making a substantial contribution to real-world problem-solving across a spectrum of domains while advancing the field of decision theory and optimization techniques.

8. Future work

Future work intends to explore the application of this proposed technique to handle the multi-objective travelling salesman problem and to provide decision-makers in logistics, transportation, and related domains with a robust and effective tool for solving real-world

SVTFNTSP instances, recognizing the considerable ongoing research in this field. The expected outcome of this future work is that, this approach would have the potential to significantly improve route planning and delivery services for sales personnel, resource allocation, portfolio optimization, environmental planning, project scheduling and cost optimization while considering the intricate nature of uncertain data. It will bring forth a range of advantages, a broad scope of applications, and diverse uses. This method offers the advantage of optimizing decision-making by efficiently balancing multiple conflicting objectives, making it ideal for scenarios where objectives include minimizing travel distance, cost, and time while maximizing customer satisfaction or other pertinent criteria. Its scope extends across industries such as supply chain management, tourism, manufacturing, urban planning, logistics and telecommunications, among others. This innovative approach holds the potential to revolutionize decision-making processes, delivering more robust, efficient, and balanced solutions for complex multi-objective scenarios in various domains. It would contribute to advancing the field of decision-making under uncertainty and further demonstrate the versatility and applicability of single-valued triangular fuzzy neutrosophic techniques in solving practical problems. Such an endeavor would augment the findings and benefit the domain of single-valued triangular fuzzy neutrosophic research.

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Interval complex single-valued neutrosophic hypersoft set with Application in Decision Making

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Abstract. The interval complex single-valued neutrosophic hypersoft set (Ξ-set), together with its features and set-theoretic operations, is a new mathematical structure that is discussed in this article. For managing ambiguous and uncertain knowledge, the suggested structure integrates the interval complex single-valued neutrosophic set and hypersoft set. These two elements have already been regarded as trustworthy settings. The first component has the ability to manage information on interval and periodic types, while the second offers a multi-argument domain for concurrent consideration of numerous sub-attributes. The Ξ-set is used to aggregate these sets, allowing for the fusion of various qualities and any related uncertainty. The resultant aggregated sets, which take into account both the attribute values and the associated uncertainty, give a thorough representation of the decision aspect. To assist in decision-making, the method calculates how similar several options are to the optimum option using a distance-based similarity metric. By contrasting the combined sets of several options, the system determines the best option based on the specified selection criteria. Decision-makers can evaluate how changing attribute values may influence their choices using the suggested strategy's endorsement of sensitivity analysis. The efficacy of the recommended decision-support mechanism is demonstrated through a case study with a real-world choice dilemma. The results show how well the framework can handle ambiguity and uncertainty while providing decision-makers with meaningful insights and encouraging rational choices. Finally, the multi-attribute decision-support system based on aggregations of Ξ-set provides a reliable framework for dealing with difficult choice issues that are characterized by ambiguity and vagueness.

Keywords: complex fuzzy set; interval-valued fuzzy set; complex fuzzy soft set; complex intuitionistic fuzzy soft set; complex neutrosophic soft set; hypersoft set; complex fuzzy hypersoft set.

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1. **Introduction**

In the context of an η-set environment with $\mathcal{IV}$ settings, this research article attempts to present the ideas of Ξ-set through the use of theoretical, axiomatic, graphical, and computational approaches. An algorithm is designed for $\mathbb{DSS}$ after conceptualizing the fundamental elementary conceptions of this structure. A real-world application is used to verify the proposed algorithm. The existing pertinent models are explored in detail using the proposed structure, and their generalization is elaborated under specific evaluation aspects. For addressing and modeling uncertainty, ambiguity, and imprecision in $\mathbb{DM}$ processes, $\mathbb{F}$-sets [1] offer a mathematical foundation. They provide more precise and robust analysis by providing more flexible and realistic modeling of real-world occurrences. Artificial intelligence, control systems, pattern recognition, and $\mathbb{DSS}$s are just a few of the areas where fuzzy sets are used. In order to describe complicated and structured uncertainty, a $\mathbb{CF}$-set [2] characterizes a particular feature of the object's uncertainty as a combination of $\mathscr{A}-$term and $\mathscr{P}-$term. Ramot et al. [3, 4] examined the novel idea of $\mathbb{CF}$-sets. The $\mathbb{CF}$-set offers a framework for mathematically expressing $\mathbb{M}_{fn}$ in a set in terms of a complex number. $\mathbb{CF}$-sets have been employed in a number of applications, such as control, pattern recognition, and $\mathbb{DM}$ [5]. $\mathbb{CF}$-sets may be used to simulate intricate connections between input characteristics and output labels in pattern recognition. When designing robust controllers for control, $\mathbb{CF}$-sets can be used to account for noisy and uncertain environments [6]. The $\mathbb{CIF}$-set [7, 8] enables modeling the ambiguous information that incorporates not only the $\mathbb{M}_{fn}$ but also the $\mathbb{N}_{fn}$ which are complex-valued functions. Rani and Garg [9] created $\mathbb{DMR}$ utilizing Hausdorff, Euclidean, and Hamming metrics and studied numerous desirable relations based on these measures [10]. They applied the concept in the $\mathbb{DM}$ process to these $\mathbb{DMR}$, especially in the fields of pattern recognition and medical diagnostics [11]. The complex-valued $\mathbb{M}_{fn}$, $\mathbb{N}_{fn}$ and $\mathbb{I}_{fn}$ are all present in a $\mathcal{CN}$-set. Ali & Smarandache [12] discussed $\mathcal{CN}$-set along with its set theoretic operations and applied in $\mathbb{DM}$ [13]. An extension of the $\mathcal{CN}$-set known as the $\mathbb{IVCN}$-set [14] uses $\mathbb{IVC}$ entries to describe the $\mathbb{M}_{fn}$, $\mathbb{N}_{fn}$ and $\mathbb{I}_{fn}$. Additional uncertainty attributes, such as the degree of vagueness and ambiguity, can be represented using the interval values. The $\mathbb{IVCN}$-sets have been used for a variety of tasks, including diagnosis, image processing, and $\mathbb{DM}$. The $\mathbb{IVCN}$-set has been used to simulate the decision-maker's level of confidence, uncertainty, and ambiguity with reference to various possibilities in the recruitment process [15]. The $\mathbb{IVCN}$-set has been used in image processing to represent the level of uncertainty involved in picture segmentation and recognition [16]. The contributions of scholars [17–19] are significant regarding the handling of uncertainties.

Molodtsov [20] developed $\mathbb{S}$-set theory as a method for handling uncertainty in data analysis and $\mathbb{DM}$. A crisp set that permits the insertion of ambiguous or speculative information is known as a $\mathbb{S}$-set in which each element is connected to a collection of parameters that may be used to symbolize various forms of uncertainty, including haziness, ambiguity, and inconsistent behavior [21,22]. The $\mathbb{S}$-sets have been employed in a wide range of disciplines, including machine learning, image processing, $\mathbb{DM}$, and data mining. The $\mathbb{S}$-sets have been applied to

$\mathbb{DM}$ to simulate human preferences and judgments in cases where the information is lacking or ambiguous. Babitha & Sunil [23] established the idea of $\mathbb{S}$-set relations and studied various related terminologies. Ali et al. [24] presented a number of novel operations and aggregation techniques on $\mathbb{S}$-sets. It has been demonstrated that these new strategies enhance the precision and efficacy of $\mathbb{DM}$ algorithms as as well as the efficiency of pattern recognition and clustering methods [25]. A hybrid notion known as $\mathbb{FS}$-set [26] contains the characteristics of both $\mathbb{F}$-sets and $\mathbb{S}$-sets. Application areas for the $\mathbb{FS}$-set idea include $\mathbb{DM}$ [27] in order to accommodate uncertainty and model inaccurate or incomplete data. The $\mathbb{FS}$-set-based $\mathbb{DM}$ techniques have been proven to be successful in enhancing $\mathbb{DM}$ accuracy and dependability [28, 29]. A hybrid idea known as $\mathbb{IFS}$-set [30] combines the qualities of $\mathbb{S}$-set and $\mathbb{IF}$-set, was presented as a generalisation of $\mathbb{IF}$-set and $\mathbb{S}$-set. By using level $\mathbb{S}$-sets of $\mathbb{IFS}$-sets and providing some illustrated instances, Jiang et al. [31] proposed an adaptable method to $\mathbb{DM}$. They discussed the weighted $\mathbb{IFS}$-sets and their potential use in $\mathbb{DM}$. A hybrid idea known as $\mathcal{NS}$-set contains the characteristics of both $\mathbb{S}$-set and $\mathcal{N}$-set. Maji [32] investigated the idea of a $\mathcal{N}$-set, applied it to $\mathbb{S}$-sets, and developed a $\mathcal{NS}$-set. He defined certain terms, performed some operations, and established some characteristics for the idea of $\mathcal{NS}$-set. In order to construct two $\mathcal{NS}$-sets, Deli & Broumi [33] defined a relation on $\mathcal{NS}$-sets and examined symmetric, transitive, and reflexive $\mathcal{NS}$ relations.

Das & Samanta [34] presented a description of the soft complex set and soft complex number and studied some of its fundamental aspects utilizing $\mathcal{F}$ numbers with the idea of $\mathbb{S}$-set along with the development of distinction and integration of $\mathcal{S}$ functions. The $\mathbb{CFS}$-sets were explored, and the aggregation operation in these sets was examined by Thirunavukarasu et al. [35]. They provided an example of prospective applications that illustrate how aggregation processes may be successfully used in numerous situations with uncertainties and periodicity. The idea of $\mathbb{CIFS}$-set presented by Kumar & Bajaj [36] allowed several parametrization techniques to tackle real-world issues involving $\mathbb{MCDM}$. As a combination of $\mathbb{CF}$-sets, $\mathcal{N}$-sets, and $\mathbb{S}$-sets, Smarandache et al. [37] presented the $\mathcal{CNS}$-set model with some of its fundamental set-theoretic operations. To illustrate the usefulness of this paradigm, a $\mathbb{DM}$ scenario incorporating ambiguous and subjective information was suggested.

1.1. *Research Gap and Motivation*

In the area of $\mathbb{DM}$ under uncertainty, η-set theory [38], a development of $\mathbb{S}$-set theory, has attracted interest. By enabling items to partially belong to distinct sets, it overcomes the drawbacks of conventional set theory. η-sets offer an adaptable framework for simulating ambiguous and uncertain information, enabling more sensible and reliable $\mathbb{DM}$ procedures. η-set applications have been studied in a variety of fields, including healthcare, finance, and environmental management. The literature emphasizes the usefulness and adaptability of η-sets in handling uncertainty, providing interesting directions for further study and real-world applications. Different hybrids with graphical settings [39–41], vague settings [42] and refined settings [43–45] were developed by researchers. However, the contributions of the researchers [46–49] are also worth noting regarding decision-making in hypersoft settings.

Decision-making and uncertainty modeling have both seen a considerable increase in interest in the idea of $\mathbb{IVFHS}$-sets [50]. The $\mathbb{IVFHS}$-sets offer an adaptable framework to deal with ambiguity and uncertainty in $\mathbb{DM}$. Numerous fields, including healthcare [51], banking, supply chain management, environmental assessment, and human resource management [52], have been the subject of research into these applications. According to the research, they are good at capturing and depicting ambiguous and imprecise information, empowering decision-makers to make well-informed decisions. The development of aggregation operations, similarity indices, and $\mathbb{DM}$ techniques based on $\mathbb{IVFHS}$-sets has been the subject of studies. According to the research, $\mathbb{IVFHS}$-sets are useful tools for handling difficult $\mathbb{DM}$ situations with uncertainty and ambiguity. Additional study is required to investigate their applicability in certain fields and to improve their computational efficiency.

The term "interval data" refers to situations in real life when data may be categorized as a set with values ranging from minimum to maximum (lower limits to upper bounds). Data may contain repeating values that correspond to specified parameters. Data repetition can be caused by a variety of sources. This sort of data is classified as periodic. There is currently no adequate model in the literature on fuzzy sets that deals with

(1) sub-attribute values in the form of $\mathbb{DAVS}$,
(2) data of the interval type, and
(3) PN-data, all at once.

The model Ξ-set is being characterized to satisfy the literary requirement. By using the $\mathbb{MAA}$-mapping, which uses the power set of the starting universe (a collection of $\mathbb{IF}$-sets or $\mathcal{N}$-sets) as its domain and maps it to the $\mathbb{CP}$ of the $\mathbb{DAVS}$, case (1) is addressed. Consideration of the lower and upper bounds of reported intervals is used to address scenario (2), whereas case (3) involves the inclusion of the $\mathscr{A}-$term and $\mathscr{P}-$terms into the Argand plane.

1.2. *Paper Layout*

The first section summarizes the literature review and study background of Ξ-set. In Section 2, some elementary notions from literature are discussed to understand the basic knowledge. In Section 3, the novel concept of Ξ-set is initiated along with the aggregation operations of Ξ-set. A $\mathbb{DSS}$ is developed in Section 4 for product selection based on the aggregation

TABLE 1. Abbreviation and notation table.

Full name	Abbreviation	Full name	Abbreviation
Fuzzy set	$\mathbb{F}$-set	Interval-valued $\mathbb{F}$-set	$\mathbb{IVF}$-set
set of all $\mathbb{IVF}$-sets	$\mathscr{C}(\mathbb{U})$	Intuitionistic $\mathbb{F}$-set	$\mathbb{IF}$-set
Complex $\mathbb{IF}$-set	$\mathbb{CIF}$-set	Neutrosophic set	$\mathbb{N}$-set
Interval $\mathbb{N}$-set	$\mathbb{IN}$-set	Complex $\mathbb{F}$-set	$\mathbb{CF}$-set
Interval-valued $\mathbb{CIF}$-set	$\mathbb{IVCIF}$-set	Complex $\mathbb{SVN}$-set	$\mathbb{CSVN}$-set
Complex $\mathbb{SVNS}$-set	$\mathbb{CSVNS}$-set		
Interval $\mathbb{CSVN}$-set	$\mathbb{ICSVN}$-set	Soft set	$\mathbb{S}$-set
Fuzzy $\mathbb{S}$-set	$\mathbb{FS}$-set	Intuitionistic $\mathbb{FS}$-set	$\mathbb{IFS}$-set
Hypersoft set	η-set	Interval $\mathbb{CSVN}$-hypersoft set	Ξ-set
Universal Set	$\mathbb{U}$	Power set of $\mathbb{U}$	$\mathscr{P}(\mathbb{U})$
Single-argument approximate mapping	$\mathbb{SAA}$-mapping	Multi-argument approximate mapping	$\mathbb{MAA}$-mapping
Membership function	$\mathbb{M}_{fn}$	Non-membership function	$\mathbb{N}_{fn}$
Indeterminacy function	$\mathbb{I}_{fn}$	Approximate function	$\mathbb{A}_{fn}$
Amplitude term	$\mathscr{A}-$term	Phase term	$\mathscr{P}-$term
Periodic nature data term	PN-data	Interval valued data	IV-data
Cartesian product	$\mathbb{CP}$	Set of parameters	$\mathbb{SP}$
Disjoint attribute valued set	$\mathbb{DAVS}$	Interval-valued $\mathbb{CFS}$-set	$\mathbb{IVCFS}$-set
Notation	**Description**	**Notation**	**Description**
Unit closed interval	$\mathscr{I}$	ϖ	$[0, 2\pi]$
Collection of all sub-intervals of $\mathscr{I}$	$\mathfrak{I}(\mathscr{I})$		

of Ξ-set aided by the proposed algorithm, and illustrated with the help of a diagram. A comparative analysis of the proposed model with some selected modes has been provided in Section 5 to check its efficiency. Finally, Section 6 concludes the research work.

2. Preliminaries

In this section, Table 1 demonstrates the abbreviations and notations used in this research article.

Definition 2.1. [1] A $\mathbb{F}$-set $\mathbb{A}$ over $\mathbb{U}$ is characterized by a $\mathbb{M}_{fn}$: $\mathbb{A}_m$, where $\mathbb{A}_m : \mathbb{U} \to \mathscr{I}$ is given by $\mathbb{A} = \{(\check{\Upsilon}, \mathbb{A}_m(\check{\Upsilon}))|\check{\Upsilon} \in \mathbb{U}\}$, which assigns a real value within $\mathscr{I}$ to each $\check{\Upsilon} \in \mathbb{U}$ and $\mathbb{A}_m(\check{\Upsilon})$ is $\mathbb{M}_{fn}$ of $\check{\Upsilon} \in \mathbb{U}$.

Definition 2.2. [2] A $\mathbb{CF}$-set $\mathbb{E}$ over $\mathbb{U}$ can be written as $\mathbb{E} = \{(\check{\Upsilon}, \mathbb{E}_m(\check{\Upsilon})) : \check{\Upsilon} \in \mathbb{U}\} = \left\{\left(\check{\Upsilon}, A_m\left(\check{\Upsilon}\right) e^{iP_m(\check{\Upsilon})}\right) : \check{\Upsilon} \in \mathbb{U}\right\}$, where $\mathbb{E}_m$ represents $\mathbb{M}_{fn}$ of $\mathbb{E}$ with $A_m\left(\check{\Upsilon}\right) \in \mathscr{I}$ as $\mathscr{A}-$term and $P_m\left(\check{\Upsilon}\right) \in \varpi$ as $\mathscr{P}-$term and $i = \sqrt{-1}$.

Definition 2.3. [7,8] A $\mathbb{CIF}$-set $\mathbb{F}$ over $\mathbb{U}$ can be written as

$$\mathbb{F} = \{(\check{\Upsilon}, \mathbb{F}_m(\check{\Upsilon}), \mathbb{F}_n(\check{\Upsilon})) : \check{\Upsilon} \in \mathbb{U}\} = \left\{\left(\check{\Upsilon}, A_m(\check{\Upsilon}) e^{iP_m(\check{\Upsilon})}, A_n(\check{\Upsilon}) e^{iP_n(\check{\Upsilon})}\right) : \check{\Upsilon} \in \mathbb{U}\right\}$$

where $\mathbb{F}_m$ and $\mathbb{F}_n$ represents $\mathbb{M}_{fn}$ and $\mathbb{N}_{fn}$ of $\mathbb{F}$ with $A_m(\check{\Upsilon}) \in \mathscr{I}$ as $\mathscr{A}-$term and $P_m(\check{\Upsilon}) \in \varpi$ as $\mathscr{P}-$term of $\mathbb{M}_{fn}$ and $A_n(\check{\Upsilon}) \in \mathscr{I}$ as $\mathscr{A}-$term and $P_n(\check{\Upsilon}) \in \varpi$ as $\mathscr{P}-$term of $\mathbb{N}_{fn}$ such that $0 \le \mathbb{F}_m + \mathbb{F}_n \le 1$ and hesitancy grade $\mathbb{F}_h(\check{\Upsilon}) = 1 - \mathbb{F}_m(\check{\Upsilon}) - \mathbb{F}_n(\check{\Upsilon})$.

Definition 2.4. [12] A $\mathbb{CSVN}$-set $\mathbb{G}$ over $\mathbb{U}$ can be written as

$$\mathbb{G} = \{(\check{\Upsilon}, \mathbb{G}_m(\check{\Upsilon}), \mathbb{G}_n(\check{\Upsilon}), \mathbb{G}_i(\check{\Upsilon})) : \check{\Upsilon} \in \mathbb{U}\} =$$

$$\left\{\left(\check{\Upsilon}, A_m(\check{\Upsilon}) e^{iP_m(\check{\Upsilon})}, A_n(\check{\Upsilon}) e^{iP_n(\check{\Upsilon})}, A_i(\check{\Upsilon}) e^{iP_i(\check{\Upsilon})}\right) : \check{\Upsilon} \in \mathbb{U}\right\}$$

where $\mathbb{G}_m$, $\mathbb{G}_n$ and $\mathbb{G}_i$ represents $\mathbb{M}_{fn}$, $\mathbb{N}_{fn}$ and $\mathbb{I}_{fn}$ of $\mathbb{G}$ with $A_m(\check{\Upsilon}) \in \mathscr{I}$ as $\mathscr{A}-$term, $P_m(\check{\Upsilon}) \in \varpi$ as $\mathscr{P}-$term of $\mathbb{M}_{fn}$, $A_n(\check{\Upsilon}) \in \mathscr{I}$ as $\mathscr{A}-$term, $P_n(\check{\Upsilon}) \in \varpi$ as $\mathscr{P}-$term of $\mathbb{N}_{fn}$ and $A_i(\check{\Upsilon}) \in \mathscr{I}$ as $\mathscr{A}-$term, $P_i(\check{\Upsilon}) \in \varpi$ as $\mathscr{P}-$term of $\mathbb{I}_{fn}$ such that $0 \le \mathbb{G}_m + \mathbb{G}_n + \mathbb{G}_i \le 3$.

Definition 2.5. [20] A $\mathbb{S}$-set $(\mathbb{H}, \Delta)$ over $\mathbb{U}$ is a set of order pairs such that $\mathbb{H} : \Delta \to \mathscr{P}(\mathbb{U})$ is given by

$$(\mathbb{H}, \Delta) = \{(\delta, \mathbb{H}(\check{\Upsilon})) : \delta \in \Delta, \check{\Upsilon} \in \mathbb{U}, \mathbb{H}(\check{\Upsilon}) \in \mathscr{P}(\mathbb{U})\}.$$

Definition 2.6. [37] A set $(\mathbb{N}, \Delta)$ is called $\mathbb{CSVNS}$-set over $\mathbb{U}$ if $\mathbb{N}$ is a parameterized gathering of $\mathbb{CSVN}$-subsets of $\mathbb{U}$ and is given by $\mathbb{N} : \Delta \to \mathscr{P}(\mathbb{U})$ and is defined by

$$(\mathbb{N}, \Delta) = \left\{\left(\delta, \left\{\frac{\mathbb{N}_m(\check{\Upsilon}), \mathbb{N}_n(\check{\Upsilon}), \mathbb{N}_i(\check{\Upsilon})}{\check{\Upsilon}}\right\}\right) : \check{\Upsilon} \in \mathbb{U}, \delta \in \Delta\right\}$$

where $\mathbb{N}_m(\check{\Upsilon}) = A_m(\check{\Upsilon}) e^{iP_m(\check{\Upsilon})}$ represents the $\mathbb{M}_{fn}$ of $\mathbb{N}$ with $A_m(\check{\Upsilon}) \in \mathscr{I}$ as $\mathscr{A}-$term, $P_m(\check{\Upsilon}) \in \varpi$ as $\mathscr{P}-$term, $\mathbb{N}_n(\check{\Upsilon}) = A_n(\check{\Upsilon}) e^{iP_n(\check{\Upsilon})}$ represents the $\mathbb{N}_{fn}$ of $\mathbb{N}$ with $A_n(\check{\Upsilon}) \in \mathscr{I}$ as $\mathscr{A}-$term, $P_n(\check{\Upsilon}) \in \varpi$ as $\mathscr{P}-$term and $\mathbb{N}_i(\check{\Upsilon}) = A_i(\check{\Upsilon}) e^{iP_i(\check{\Upsilon})}$ represents the $\mathbb{I}_{fn}$ of $\mathbb{N}$ with $A_i(\check{\Upsilon}) \in \mathscr{I}$ as $\mathscr{A}-$term, $P_i(\check{\Upsilon}) \in \varpi$ as $\mathscr{P}-$term such that $0 \le \mathbb{N}_m(\check{\Upsilon}) + \mathbb{N}_n(\check{\Upsilon}) + \mathbb{N}_i(\check{\Upsilon}) \le 3$.

Definition 2.7. [38] $(\mathbb{O}, \Delta)$ is called η-set over $\mathbb{U}$ if $\mathbb{O} : \Delta \to \mathscr{P}(\mathbb{U})$ where $\Delta = \prod_{i=1}^{n} \Delta_i$ such that Δ_i are $\mathbb{DAVS}$ of sub-parameters, each set corresponding to a unique parameters $\delta \in \Delta$.

Definition 2.8. If $\mathcal{C}_{\mathcal{SVN}}(\mathbb{U})$ denotes the set containing all $\mathcal{SVN}$-subsets over $\mathbb{U}$ then $\mathcal{SVNHS}$-set $(\mathbb{R}, \Delta)$ is obtained when the mapping $\mathbb{O} : \Delta \to \mathscr{P}(\mathbb{U})$ in Definition 2.7 is replaced by $\mathbb{R} : \Delta \to C_{SVN}(\mathbb{U})$ and all other conditions of Definition 2.7 are remained valid.

Definition 2.9. [56] If $\mathcal{C}_{\mathcal{CSVN}}(\mathbb{U})$ represents the collection of all $\mathbb{CSVN}$-subsets over $\mathbb{U}$ then $\mathcal{CSVNHS}$-set $(\mathbb{V}, \Delta)$ is obtained when the mapping $\mathbb{O} : \Delta \to \mathscr{P}(\mathbb{U})$ in Definition 2.7 is replaced by $\mathbb{V} : \Delta \to \mathcal{C}_{\mathcal{CSVN}}(\mathbb{U})$ and all other conditions of Definition 2.7 are remained valid.

3. **Interval complex single-valued neutrosophic hypersoft set (Ξ-set)**

This section develops the fundamental theory of the Ξ-set.

Definition 3.1. An $\mathbb{ICSVN}$-set $\mathbb{G}_{\mathbb{I}}$ over $\mathbb{U}$ can be written as

$$\mathbb{G}_{\mathbb{I}} = \{(\check{\Upsilon}, \langle \mathbb{G}_{\mathbb{I}m}(\check{\Upsilon}), \mathbb{G}_{\mathbb{I}n}(\check{\Upsilon}), \mathbb{G}_{\mathbb{I}i}(\check{\Upsilon})\rangle) : \check{\Upsilon} \in \mathbb{U}\} =$$
$$\left\{\left(\check{\Upsilon}, A_m(\check{\Upsilon}) e^{iP_m(\check{\Upsilon})}, A_n(\check{\Upsilon}) e^{iP_n(\check{\Upsilon})}, A_i(\check{\Upsilon}) e^{iP_i(\check{\Upsilon})}\right) : \check{\Upsilon} \in \mathbb{U}\right\}.$$

where $\mathbb{G}_{\mathbb{I}m}$ represents $\mathbb{M}_{fn}$ of $\mathbb{G}_{\mathbb{I}}$ with $A_m(\check{\Upsilon}) \in \Im(\mathscr{I})$ as $\mathscr{A}-$term, $P_m(\check{\Upsilon}) \subseteq \varpi$ as $\mathscr{P}-$term, $\mathbb{G}_{\mathbb{I}n}$ represents $\mathbb{N}_{fn}$ with $A_n(\check{\Upsilon}) \in \Im(\mathscr{I})$ as $\mathscr{A}-$term, $P_n(\check{\Upsilon}) \subseteq \varpi$ as $\mathscr{P}-$term and $\mathbb{G}_{\mathbb{I}i}$ represents $\mathbb{I}_{fn}$ with $A_i(\check{\Upsilon}) \in \Im(\mathscr{I})$ as $\mathscr{A}-$term, $P_i(\check{\Upsilon}) \subseteq \varpi$ as $\mathscr{P}-$term and $0 \leq \inf \mathbb{G}_{\mathbb{I}m} + \inf \mathbb{G}_{\mathbb{I}n} + \inf \mathbb{G}_{\mathbb{I}i} \leq \sup \mathbb{G}_{\mathbb{I}m} + \sup \mathbb{G}_{\mathbb{I}n} + \sup \mathbb{G}_{\mathbb{I}i} \leq 3$.

Definition 3.2. Consider two $\mathbb{ICSVN}$-sets

$$\mathbb{G}_{\mathbb{I}}{}^1 = \left\{\left(\check{\Upsilon}, \mathbb{G}_{\mathbb{I}m}^1(\check{\Upsilon}), \mathbb{G}_{\mathbb{I}n}^1(\check{\Upsilon}), \mathbb{G}_{\mathbb{I}i}^1(\check{\Upsilon})\right) : \check{\Upsilon} \in \mathbb{U}\right\}$$

and

$$\mathbb{G}_{\mathbb{I}}{}^2 = \left\{\left(\check{\Upsilon}, \mathbb{G}_{\mathbb{I}m}^2(\check{\Upsilon}), \mathbb{G}_{\mathbb{I}n}^2(\check{\Upsilon}), \mathbb{G}_{\mathbb{I}i}^2(\check{\Upsilon})\right) : \check{\Upsilon} \in \mathbb{U}\right\}$$

having respective $\mathbb{M}_{fn}$: $\mathbb{G}_{\mathbb{I}m}^1(\check{\Upsilon}) = A_m^1(\check{\Upsilon}) e^{iP_m{}^1(\check{\Upsilon})}$, $\mathbb{G}_{\mathbb{I}m}^2(\check{\Upsilon}) = A_m^2(\check{\Upsilon}) e^{iP_m{}^2(\check{\Upsilon})}$, $\mathbb{N}_{fn}$: $\mathbb{G}_{\mathbb{I}n}^1(\check{\Upsilon}) = A_n^1(\check{\Upsilon}) e^{iP_n{}^1(\check{\Upsilon})}$, $\mathbb{G}_{\mathbb{I}n}^2(\check{\Upsilon}) = A_n^2(\check{\Upsilon}) e^{iP_n{}^2(\check{\Upsilon})}$ and $\mathbb{I}_{fn}$: $\mathbb{G}_{\mathbb{I}i}^1(\check{\Upsilon}) = A_i^1(\check{\Upsilon}) e^{iP_i{}^1(\check{\Upsilon})}$, $\mathbb{G}_{\mathbb{I}i}^2(\check{\Upsilon}) = A_i^2(\check{\Upsilon}) e^{iP_i{}^2(\check{\Upsilon})}$.

(1). The *union* of $\mathbb{G}_{\mathbb{I}}{}^1$ and $\mathbb{G}_{\mathbb{I}}{}^2$ is again an $\mathbb{ICSVN}$-set $\mathbb{G}_{\mathbb{I}}{}^3 = \mathbb{G}_{\mathbb{I}}{}^1 \cup \mathbb{G}_{\mathbb{I}}{}^2$, where its $\mathbb{M}_{fn}$, $\mathbb{N}_{fn}$ and $I_{fn} \forall \check{\Upsilon} \in \mathbb{U}$ can be given by

$$\mathbb{G}_{\mathbb{I}m}^3(\check{\Upsilon}) = A_m^3(\check{\Upsilon}) e^{iP_m{}^3(\check{\Upsilon})} = \begin{bmatrix} \max\left(\inf A_m^1(\check{\Upsilon}), \inf A_m^2(\check{\Upsilon})\right), \\ \max\left(\sup A_m^1(\check{\Upsilon}), \sup A_m^2(\check{\Upsilon})\right) \end{bmatrix} e^{i\begin{bmatrix} \max\left(\inf P_m{}^1(\check{\Upsilon}), \inf P_m{}^2(\check{\Upsilon})\right), \\ \max\left(\sup P_m{}^1(\check{\Upsilon}), \sup P_m{}^2(\check{\Upsilon})\right) \end{bmatrix}}.$$

$$\mathbb{G}_{\mathbb{I}n}^3(\check{\Upsilon}) = A_n^3(\check{\Upsilon}) e^{iP_n{}^3(\check{\Upsilon})} = \begin{bmatrix} \min\left(\inf A_n^1(\check{\Upsilon}), \inf A_n^2(\check{\Upsilon})\right), \\ \min\left(\sup A_n^1(\check{\Upsilon}), \sup A_n^2(\check{\Upsilon})\right) \end{bmatrix} e^{i\begin{bmatrix} \min\left(\inf P_n{}^1(\check{\Upsilon}), \inf P_n{}^2(\check{\Upsilon})\right), \\ \min\left(\sup P_n{}^1(\check{\Upsilon}), \sup P_n{}^2(\check{\Upsilon})\right) \end{bmatrix}}$$

$$\mathbb{G}_{\mathbb{I}i}^3(\check{\Upsilon}) = A_i^3(\check{\Upsilon}) e^{iP_i{}^3(\check{\Upsilon})} = \begin{bmatrix} \min\left(\inf A_i^1(\check{\Upsilon}), \inf A_i^2(\check{\Upsilon})\right), \\ \min\left(\sup A_i^1(\check{\Upsilon}), \sup A_i^2(\check{\Upsilon})\right) \end{bmatrix} e^{i\begin{bmatrix} \min\left(\inf P_i{}^1(\check{\Upsilon}), \inf P_i{}^2(\check{\Upsilon})\right), \\ \min\left(\sup P_i{}^1(\check{\Upsilon}), \sup P_i{}^2(\check{\Upsilon})\right) \end{bmatrix}}.$$

(2). The *intersection* of $\mathbb{G}_{\mathbb{I}}{}^1$ and $\mathbb{G}_{\mathbb{I}}{}^2$ is again an $\mathbb{IVCIF}$-set $\mathbb{G}_{\mathbb{I}}{}^4 = \mathbb{G}_{\mathbb{I}}{}^1 \cap \mathbb{G}_{\mathbb{I}}{}^2$, where its $\mathbb{M}_{fn}$, $\mathbb{N}_{fn}$ and $I_{fn} \forall \check{\Upsilon} \in \mathbb{U}$ can be given by

$$\mathbb{G}_{\mathbb{I}m}^4(\check{\Upsilon}) = A_m^4(\check{\Upsilon}) e^{iP_m{}^4(\check{\Upsilon})} = \begin{bmatrix} \min\left(\inf A_m^1(\check{\Upsilon}), \inf A_m^2(\check{\Upsilon})\right), \\ \min\left(\sup A_m^1(\check{\Upsilon}), \sup A_m^2(\check{\Upsilon})\right) \end{bmatrix} e^{i\begin{bmatrix} \min\left(\inf P_m{}^1(\check{\Upsilon}), \inf P_m{}^2(\check{\Upsilon})\right), \\ \min\left(\sup P_m{}^1(\check{\Upsilon}), \sup P_m{}^2(\check{\Upsilon})\right) \end{bmatrix}}.$$

$$\mathbb{G}_{\mathbb{I}n}^4(\check{\Upsilon}) = A_n^4(\check{\Upsilon}) e^{iP_n{}^4(\check{\Upsilon})} = \begin{bmatrix} \max\left(\inf A_n^1(\check{\Upsilon}), \inf A_n^2(\check{\Upsilon})\right), \\ \max\left(\sup A_n^1(\check{\Upsilon}), \sup A_n^2(\check{\Upsilon})\right) \end{bmatrix} e^{i\begin{bmatrix} \max\left(\inf P_n{}^1(\check{\Upsilon}), \inf P_n{}^2(\check{\Upsilon})\right), \\ \max\left(\sup P_n{}^1(\check{\Upsilon}), \sup P_n{}^2(\check{\Upsilon})\right) \end{bmatrix}}.$$

$$\mathbb{G}_{\mathbb{I}_i}^{4}(\check{\Upsilon}) = A_i^4(\check{\Upsilon})\, e^{iP_i^4(\check{\Upsilon})} = \begin{bmatrix} \max\left(\inf A_i^1(\check{\Upsilon}), \inf A_i^2(\check{\Upsilon})\right), \\ \max\left(\sup A_i^1(\check{\Upsilon}), \sup A_i^2(\check{\Upsilon})\right) \end{bmatrix} e^{i\begin{bmatrix} \max\left(\inf P_i^1(\check{\Upsilon}), \inf P_i^2(\check{\Upsilon})\right), \\ \max\left(\sup P_i^1(\check{\Upsilon}), \sup P_i^2(\check{\Upsilon})\right) \end{bmatrix}}.$$

(3). The *complement* of $\mathbb{G}_{\mathbb{I}}{}^{1}$ denoted by

$$(\mathbb{G}_{\mathbb{I}}{}^{1})^{c} = \left\{ \left(\check{\Upsilon}, \mathbb{G}_{\mathbb{I}_m}^{1\,c}(\check{\Upsilon}), \mathbb{G}_{\mathbb{I}_n}^{1\,c}(\check{\Upsilon}) \right) : \check{\Upsilon} \in \mathbb{U} \right\}$$

where
$\mathbb{G}_{\mathbb{I}_m}^{1\,c}(\check{\Upsilon}) = \mathbb{G}_{\mathbb{I}_n}^{1}(\check{\Upsilon})$,
$\mathbb{G}_{\mathbb{I}_n}^{1\,c}(\check{\Upsilon}) = \mathbb{G}_{\mathbb{I}_m}^{1}(\check{\Upsilon})$ and
$\mathbb{G}_{\mathbb{I}_i}^{1\,c} = \left[1 - A_i^{1+}(\check{\Upsilon}), 1 - A_i^{1-}(\check{\Upsilon})\right] e^{i\left[2\pi - P_i^{1+}(\check{\Upsilon}), 2\pi - P_i^{1-}(\check{\Upsilon})\right]}$.

Definition 3.3. A set $(\mathbb{N}_{\mathbb{I}}, \Delta)$ is called $\mathcal{ICSVNS}$-set over $\mathbb{U}$ if $\mathbb{N}_{\mathbb{I}}$ is a parameterized gathering of $\mathcal{IVCSVN}$-subsets of $\mathbb{U}$ and is given by $\mathbb{N}_{\mathbb{I}} : \Delta \to \mathscr{P}(\mathbb{U})$ and is defined by

$$(\mathbb{N}_{\mathbb{I}}, \Delta) = \left\{ \left(\delta, \left\{ \frac{\mathbb{N}_{\mathbb{I}m}(\check{\Upsilon}), \mathbb{N}_{\mathbb{I}n}(\check{\Upsilon}), \mathbb{N}_{\mathbb{I}i}(\check{\Upsilon})}{\check{\Upsilon}} \right\} \right) : \check{\Upsilon} \in \mathbb{U}, \delta \in \Delta \right\}$$

where $\mathbb{N}_{\mathbb{I}m}(\check{\Upsilon}) = A_m(\check{\Upsilon})\, e^{iP_m(\check{\Upsilon})}$ represents the $\mathbb{M}_{fn}$ of $\mathbb{N}_{\mathbb{I}}$ with $A_m(\check{\Upsilon}) \in \mathfrak{I}(\mathscr{I})$ as $\mathscr{A}-$term, $P_m(\check{\Upsilon}) \subseteq \varpi$ as $\mathscr{P}-$term, $\mathbb{N}_{\mathbb{I}n}(\check{\Upsilon}) = A_n(\check{\Upsilon})\, e^{iP_n(\check{\Upsilon})}$ represents the $\mathbb{N}_{fn}$ of $\mathbb{N}_{\mathbb{I}}$ with $A_n(\check{\Upsilon}) \in \mathfrak{I}(\mathscr{I})$ as $\mathscr{A}-$term, $P_n(\check{\Upsilon}) \subseteq \varpi$ as $\mathscr{P}-$term and $\mathbb{N}_{\mathbb{I}i}(\check{\Upsilon}) = A_i(\check{\Upsilon})\, e^{iP_i(\check{\Upsilon})}$ represents the $\mathbb{I}_{fn}$ of $\mathbb{N}_{\mathbb{I}}$ with $A_i(\check{\Upsilon}) \in \mathfrak{I}(\mathscr{I})$ as $\mathscr{A}-$term, $P_i(\check{\Upsilon}) \subseteq \varpi$ as $\mathscr{P}-$term such that $0 \leq \inf \mathbb{N}_{\mathbb{I}m}(\check{\Upsilon}) + \inf \mathbb{N}_{\mathbb{I}n}(\check{\Upsilon}) + \inf \mathbb{N}_{\mathbb{I}i}(\check{\Upsilon}) \leq \sup \mathbb{N}_{\mathbb{I}m}(\check{\Upsilon}) + \sup \mathbb{N}_{\mathbb{I}n}(\check{\Upsilon}) + \sup \mathbb{N}_{\mathbb{I}i}(\check{\Upsilon}) \leq 3$.

Definition 3.4. Consider two $\mathcal{ICSVNS}$-sets $(\mathbb{N}_{\mathbb{I}}{}^{1}, \Delta_1)$ and $(\mathbb{N}_{\mathbb{I}}{}^{2}, \Delta_2)$ having respective $\mathbb{M}_{fn}$: $\mathbb{N}_{\mathbb{I}m}^{1} = A_m^1(\check{\Upsilon})\, e^{iP_m^1(\check{\Upsilon})}$, $\mathbb{N}_{\mathbb{I}m}^{2} = A_m^2(\check{\Upsilon})\, e^{iP_m^2(\check{\Upsilon})}$, $\mathbb{N}_{fn}$: $\mathbb{N}_{\mathbb{I}n}^{1} = A_n^1(\check{\Upsilon})\, e^{iP_n^1(\check{\Upsilon})}$, $\mathbb{N}_{\mathbb{I}n}^{2} = A_n^2(\check{\Upsilon})\, e^{iP_n^2(\check{\Upsilon})}$ and $\mathbb{I}_{fn}$: $\mathbb{N}_{\mathbb{I}i}^{1} = A_i^1(\check{\Upsilon})\, e^{iP_i^1(\check{\Upsilon})}$, $\mathbb{N}_{\mathbb{I}i}^{2} = A_i^2(\check{\Upsilon})\, e^{iP_i^2(\check{\Upsilon})}$

(1) The *union* of $(\mathbb{N}_{\mathbb{I}}{}^{1}, \Delta_1)$ and $(\mathbb{N}_{\mathbb{I}}{}^{2}, \Delta_2)$ is again an $\mathcal{ICSVNS}$-set $(\mathbb{N}_{\mathbb{I}}{}^{3}, \Delta_3) = (\mathbb{N}_{\mathbb{I}}{}^{1}, \Delta_1) \cup (\mathbb{N}_{\mathbb{I}}{}^{2}, \Delta_2)$, where $\Delta_3 = \Delta_1 \cup \Delta_2$, for all $\delta \in \Delta_3$, $\check{\Upsilon} \in \mathbb{U}$, and its $\mathbb{M}_{fn}$, $\mathbb{N}_{fn}$ and $\mathbb{I}_{fn}$ are defined as

$$\mathbb{N}_{\mathbb{I}m}^{3}(\check{\Upsilon}) = \begin{cases} \mathbb{N}_{\mathbb{I}m}^{1}(\check{\Upsilon}) & if\ \delta \in \Delta_1 \backslash \Delta_2 \\ \mathbb{N}_{\mathbb{I}m}^{2}(\check{\Upsilon}) & if\ \delta \in \Delta_2 \backslash \Delta_1 \\ \begin{bmatrix} \max\left(\inf A_m^1(\check{\Upsilon}), \inf A_m^2(\check{\Upsilon})\right), \\ \max\left(\sup A_m^1(\check{\Upsilon}), \sup A_m^2(\check{\Upsilon})\right) \end{bmatrix} e^{i\begin{bmatrix} \max\left(\inf P_m^1(\check{\Upsilon}), \inf P_m^2(\check{\Upsilon})\right), \\ \max\left(\sup P_m^1(\check{\Upsilon}), \sup P_m^2(\check{\Upsilon})\right) \end{bmatrix}} & if\ \delta \in \Delta_1 \cap \Delta_2 \end{cases}$$

$$\mathbb{N}_{\mathbb{I}n}^{3}(\check{\Upsilon}) = \begin{cases} \mathbb{N}_{\mathbb{I}n}^{1}(\check{\Upsilon}) & if\ \delta \in \Delta_1 \backslash \Delta_2 \\ \mathbb{N}_{\mathbb{I}n}^{2}(\check{\Upsilon}) & if\ \delta \in \Delta_2 \backslash \Delta_1 \\ \begin{bmatrix} \min\left(\inf A_n^1(\check{\Upsilon}), \inf A_n^2(\check{\Upsilon})\right), \\ \min\left(\sup A_n^1(\check{\Upsilon}), \sup A_n^2(\check{\Upsilon})\right) \end{bmatrix} e^{i\begin{bmatrix} \min\left(\inf P_n^1(\check{\Upsilon}), \inf P_n^2(\check{\Upsilon})\right), \\ \min\left(\sup P_n^1(\check{\Upsilon}), \sup P_n^2(\check{\Upsilon})\right) \end{bmatrix}} & if\ \delta \in \Delta_1 \cap \Delta_2 \end{cases}$$

$$\mathbb{N}_{\mathbb{I}_i}^{3}(\check{\Upsilon}) = \begin{cases} \mathbb{N}_{\mathbb{I}_i}^{1}(\check{\Upsilon}) & if\ \delta \in \Delta_1 \backslash \Delta_2 \\ \mathbb{N}_{\mathbb{I}_i}^{2}(\check{\Upsilon}) & if\ \delta \in \Delta_2 \backslash \Delta_1 \\ \begin{bmatrix} \min\left(\inf A_i^1(\check{\Upsilon}), \inf A_i^2(\check{\Upsilon})\right), \\ \min\left(\sup A_i^1(\check{\Upsilon}), \sup A_i^2(\check{\Upsilon})\right) \end{bmatrix} e^{i\begin{bmatrix} \min\left(\inf P_i{}^1(\check{\Upsilon}), \inf P_i{}^2(\check{\Upsilon})\right), \\ \min\left(\sup P_i{}^1(\check{\Upsilon}), \sup P_i{}^2(\check{\Upsilon})\right) \end{bmatrix}} & if\ \delta \in \Delta_1 \cap \Delta_2 \end{cases}$$

(2) The *restricted union* of $(\mathbb{N}_{\mathbb{I}}{}^1, \Delta_1)$ and $(\mathbb{N}_{\mathbb{I}}{}^2, \Delta_2)$ denoted by $(\mathbb{N}_{\mathbb{I}}{}^4, \Delta_4) = (\mathbb{N}_{\mathbb{I}}{}^1, \Delta_1) \cup_R (\mathbb{N}_{\mathbb{I}}{}^2, \Delta_2)$, where $\Delta_4 = \Delta_1 \cap \Delta_2$, for all $\delta \in \Delta_4$, $\check{\Upsilon} \in \mathbb{U}$, its $\mathbb{M}_{fn}$, $\mathbb{N}_{fn}$ and $\mathbb{I}_{fn}$ are defined as

$$\mathbb{N}_{\mathbb{I}_m}^{4}(\check{\Upsilon}) = \begin{bmatrix} \max\left(\inf A_m^1(\check{\Upsilon}), \inf A_m^2(\check{\Upsilon})\right), \\ \max\left(\sup A_m^1(\check{\Upsilon}), \sup A_m^2(\check{\Upsilon})\right) \end{bmatrix} e^{i\begin{bmatrix} \max\left(\inf P_m{}^1(\check{\Upsilon}), \inf P_m{}^2(\check{\Upsilon})\right), \\ \max\left(\sup P_m{}^1(\check{\Upsilon}), \sup P_m{}^2(\check{\Upsilon})\right) \end{bmatrix}}$$

$$\mathbb{N}_{\mathbb{I}_n}^{4}(\check{\Upsilon}) = \begin{bmatrix} \min\left(\inf A_n^1(\check{\Upsilon}), \inf A_n^2(\check{\Upsilon})\right), \\ \min\left(\sup A_n^1(\check{\Upsilon}), \sup A_n^2(\check{\Upsilon})\right) \end{bmatrix} e^{i\begin{bmatrix} \min\left(\inf P_n{}^1(\check{\Upsilon}), \inf P_n{}^2(\check{\Upsilon})\right), \\ \min\left(\sup P_n{}^1(\check{\Upsilon}), \sup P_n{}^2(\check{\Upsilon})\right) \end{bmatrix}}$$

$$\mathbb{N}_{\mathbb{I}_i}^{4}(\check{\Upsilon}) = \begin{bmatrix} \min\left(\inf A_i^1(\check{\Upsilon}), \inf A_i^2(\check{\Upsilon})\right), \\ \min\left(\sup A_i^1(\check{\Upsilon}), \sup A_i^2(\check{\Upsilon})\right) \end{bmatrix} e^{i\begin{bmatrix} \min\left(\inf P_i{}^1(\check{\Upsilon}), \inf P_i{}^2(\check{\Upsilon})\right), \\ \min\left(\sup P_i{}^1(\check{\Upsilon}), \sup P_i{}^2(\check{\Upsilon})\right) \end{bmatrix}}$$

(3) The *intersection* of $(\mathbb{N}_{\mathbb{I}}{}^1, \Delta_1)$ and $(\mathbb{N}_{\mathbb{I}}{}^2, \Delta_2)$ denoted by $(\mathbb{N}_{\mathbb{I}}{}^5, \Delta_5) = (\mathbb{N}_{\mathbb{I}}{}^1, \Delta_1) \cap (\mathbb{N}_{\mathbb{I}}{}^2, \Delta_2)$, where $\Delta_5 = \Delta_1 \cap \Delta_2$, for all $\delta \in \Delta_5$, $\check{\Upsilon} \in \mathbb{U}$ its $\mathbb{M}_{fn}$, $\mathbb{N}_{fn}$ and $\mathbb{I}_{fn}$ is defined as

$$\mathbb{N}_{\mathbb{I}_m}^{5}(\check{\Upsilon}) = \begin{bmatrix} \min\left(\inf A_m^1(\check{\Upsilon}), \inf A_m^2(\check{\Upsilon})\right), \\ \min\left(\sup A_m^1(\check{\Upsilon}), \sup A_m^2(\check{\Upsilon})\right) \end{bmatrix} e^{i\begin{bmatrix} \min\left(\inf P_m{}^1(\check{\Upsilon}), \inf P_m{}^2(\check{\Upsilon})\right), \\ \min\left(\sup P_m{}^1(\check{\Upsilon}), \sup P_m{}^2(\check{\Upsilon})\right) \end{bmatrix}}$$

$$\mathbb{N}_{\mathbb{I}_n}^{5}(\check{\Upsilon}) = \begin{bmatrix} \max\left(\inf A_n^1(\check{\Upsilon}), \inf A_n^2(\check{\Upsilon})\right), \\ \max\left(\sup A_n^1(\check{\Upsilon}), \sup A_n^2(\check{\Upsilon})\right) \end{bmatrix} e^{i\begin{bmatrix} \max\left(\inf P_n{}^1(\check{\Upsilon}), \inf P_n{}^2(\check{\Upsilon})\right), \\ \max\left(\sup P_n{}^1(\check{\Upsilon}), \sup P_n{}^2(\check{\Upsilon})\right) \end{bmatrix}}$$

$$\mathbb{N}_{\mathbb{I}_i}^{5}(\check{\Upsilon}) = \begin{bmatrix} \max\left(\inf A_i^1(\check{\Upsilon}), \inf A_i^2(\check{\Upsilon})\right), \\ \max\left(\sup A_i^1(\check{\Upsilon}), \sup A_i^2(\check{\Upsilon})\right) \end{bmatrix} e^{i\begin{bmatrix} \max\left(\inf P_i{}^1(\check{\Upsilon}), \inf P_i{}^2(\check{\Upsilon})\right), \\ \max\left(\sup P_i{}^1(\check{\Upsilon}), \sup P_i{}^2(\check{\Upsilon})\right) \end{bmatrix}}$$

(4) The *extended intersection* of $(\mathbb{N}_{\mathbb{I}}{}^1, \Delta_1)$ and $(\mathbb{N}_{\mathbb{I}}{}^2, \Delta_2)$ denoted by $(\mathbb{N}_{\mathbb{I}}{}^6, \Delta_6) = (\mathbb{N}_{\mathbb{I}}{}^1, \Delta_1) \cap_E (\mathbb{N}_{\mathbb{I}}{}^2, \Delta_2)$, where $\Delta_6 = \Delta_1 \cup \Delta_2$, for all $\delta \in \Delta_6$, $\check{\Upsilon} \in \mathbb{U}$, its $\mathbb{M}_{fn}$, $\mathbb{N}_{fn}$ and $\mathbb{I}_{fn}$ are defined as

$$\mathbb{N}_{\mathbb{I}_m}^{6}(\check{\Upsilon}) = \begin{cases} \mathbb{N}_{\mathbb{I}_m}^{1}(\check{\Upsilon}) & if\ \delta \in \Delta_1 \backslash \Delta_2 \\ \mathbb{N}_{\mathbb{I}_m}^{2}(\check{\Upsilon}) & if\ \delta \in \Delta_2 \backslash \Delta_1 \\ \begin{bmatrix} \min\left(\inf A_m^1(\check{\Upsilon}), \inf A_m^2(\check{\Upsilon})\right), \\ \min\left(\sup A_m^1(\check{\Upsilon}), \sup A_m^2(\check{\Upsilon})\right) \end{bmatrix} e^{i\begin{bmatrix} \min\left(\inf P_m{}^1(\check{\Upsilon}), \inf P_m{}^2(\check{\Upsilon})\right), \\ \min\left(\sup P_m{}^1(\check{\Upsilon}), \sup P_m{}^2(\check{\Upsilon})\right) \end{bmatrix}} & if\ \delta \in \Delta_1 \cap \Delta_2 \end{cases}$$

$$\mathbb{N}_{\mathbb{I}_n}^6(\curlyvee) = \begin{cases} \mathbb{N}_{\mathbb{I}_n}^1(\curlyvee) & if\ \delta \in \Delta_1 \backslash \Delta_2 \\ \mathbb{N}_{\mathbb{I}_n}^2(\curlyvee) & if\ \delta \in \Delta_2 \backslash \Delta_1 \\ \begin{bmatrix} \max\left(\inf A_n^1(\curlyvee), \inf A_n^2(\curlyvee)\right), \\ \max\left(\sup A_n^1(\curlyvee), \sup A_n^2(\curlyvee)\right) \end{bmatrix} e^{i\begin{bmatrix} \max\left(\inf P_n{}^1(\curlyvee), \inf P_n{}^2(\curlyvee)\right), \\ \max\left(\sup P_n{}^1(\curlyvee), \sup P_n{}^2(\curlyvee)\right) \end{bmatrix}} & if\ \delta \in \Delta_1 \cap \Delta_2 \end{cases}$$

$$\mathbb{N}_{\mathbb{I}_i}^6(\curlyvee) = \begin{cases} \mathbb{N}_{\mathbb{I}_i}^1(\curlyvee) & if\ \delta \in \Delta_1 \backslash \Delta_2 \\ \mathbb{N}_{\mathbb{I}_i}^2(\curlyvee) & if\ \delta \in \Delta_2 \backslash \Delta_1 \\ \begin{bmatrix} \max\left(\inf A_i^1(\curlyvee), \inf A_i^2(\curlyvee)\right), \\ \max\left(\sup A_i^1(\curlyvee), \sup A_i^2(\curlyvee)\right) \end{bmatrix} e^{i\begin{bmatrix} \max\left(\inf P_i{}^1(\curlyvee), \inf P_i{}^2(\curlyvee)\right), \\ \max\left(\sup P_i{}^1(\curlyvee), \sup P_i{}^2(\curlyvee)\right) \end{bmatrix}} & if\ \delta \in \Delta_1 \cap \Delta_2 \end{cases}$$

(5) The *complement* of $(\mathbb{N}_{\mathbb{I}}{}^1, \Delta_1)$ denoted by $(\mathbb{N}_{\mathbb{I}}{}^1, \Delta_1)^c = (\mathbb{N}_{\mathbb{I}}{}^{1^c}, \neg\Delta_1)$ such that $\mathbb{N}_{\mathbb{I}}{}^{1^c} : \neg\Delta_1 \to \mathscr{P}(\Bbbk)$ is given by $\mathbb{M}_{fn}$:$\mathbb{N}_{\mathbb{I}_m}^{1\ c}(\curlyvee) = \mathbb{N}_{\mathbb{I}_n}^1(\curlyvee)$, $\mathbb{N}_{fn}$: $\mathbb{N}_{\mathbb{I}_n}^{1^c}(\curlyvee) = \mathbb{N}_{\mathbb{I}_m}^1(\curlyvee)$ and $\mathbb{I}_{fn}$: $\mathbb{N}_{\mathbb{I}_i}^{1^c}(\curlyvee) = \left[1 - \sup \mathbb{N}_{\mathbb{I}_i}^1(\curlyvee), 1 - \inf \mathbb{N}_{\mathbb{I}_i}^1(\curlyvee)\right]$

(6) The *relative complement* of $(\mathbb{N}_{\mathbb{I}}{}^1, \Delta_1)$ denoted by $(\mathbb{N}_{\mathbb{I}}{}^1, \Delta_1)^r$ where $(\mathbb{N}_{\mathbb{I}}{}^1, \Delta_1)^r = (\mathbb{N}_{\mathbb{I}}{}^{1^r}, \Delta_1)$ such that $\mathbb{N}_{\mathbb{I}}{}^{1^r} : \Delta_1 \to \mathscr{P}(\Bbbk)$ is given by $\mathbb{M}_{fn}$:$\mathbb{N}_{\mathbb{I}_m}^{1\ r}(\curlyvee) = \mathbb{N}_{\mathbb{I}_n}^1(\curlyvee)$, $\mathbb{N}_{fn}$: $\mathbb{N}_{\mathbb{I}_n}^{1^r}(\curlyvee) = \mathbb{N}_{\mathbb{I}_m}^1(\curlyvee)$ and $\mathbb{I}_{fn}$: $\mathbb{N}_{\mathbb{I}_i}^{1^r}(\curlyvee) = \left[1 - \sup \mathbb{N}_{\mathbb{I}_i}^1(\curlyvee), 1 - \inf \mathbb{N}_{\mathbb{I}_i}^1(\curlyvee)\right]$

Definition 3.5. Let $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots.., \mathcal{Y}_n$ are $\mathbb{DAVS}$ of n distinct attributes $\zeta_1, \zeta_2, \zeta_3, \ldots.., \zeta_n$ respectively for $n \geq 1, \mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Y}_3 \times \ldots.. \times \mathcal{Y}_n$ and $\Delta(\underline{\delta})$ be a $\mathcal{ICSVNS}$-set defined over $\Bbbk \forall \underline{\delta} = (\beta_1, \beta_2, \beta_3, \ldots.., \beta_n) \in \mathcal{Y}$. Then, the Ξ-set, denoted by $\Omega_{\mathcal{Y}} = (\Delta, \mathcal{Y})$, over $\Bbbk$ is given as

$$\Omega_{\mathcal{Y}} = \{(\underline{\delta}, \Delta(\underline{\delta})) : \underline{\delta} \in \mathcal{Y}, \Delta(\underline{\delta}) \in C_{IV}(\Bbbk)\},$$

where $\Delta : \mathcal{Y} \to C_{IV}(\Bbbk), \Delta(\underline{\delta}) = \emptyset\ if\ \underline{\delta} \notin \mathcal{Y}$ is an $\mathcal{ICSVN}$ $\mathbb{A}_{fn}$ of $\Omega_{\mathcal{Y}}$ and $\Delta(\underline{\delta}) = \langle[\overleftarrow{\Delta}_1(\underline{\delta}), \overrightarrow{\Delta}_1(\underline{\delta})], [\overleftarrow{\Delta}_2(\underline{\delta}), \overrightarrow{\Delta}_2(\underline{\delta})], [\overleftarrow{\Delta}_3(\underline{\delta}), \overrightarrow{\Delta}_3(\underline{\delta})]\rangle$ with lower bounds and upper bounds of $\mathbb{M}_{fn}$, $\mathbb{N}_{fn}$ and $\mathbb{I}_{fn}$ are described as folow

(a) $(\overleftarrow{\Delta}_1(\underline{\delta}) = \overleftarrow{\gamma} e^{i\overleftarrow{\theta}}, \overrightarrow{\Delta}_1(\underline{\delta}) = \overrightarrow{\gamma} e^{i\overrightarrow{\theta}})$ for the $\mathbb{M}_{fn}$ of $\Omega_{\mathcal{Y}}$

(b) $(\overleftarrow{\Delta}_2(\underline{\delta}) = \overleftarrow{\gamma} e^{i\overleftarrow{\theta}}, \overrightarrow{\Delta}_2(\underline{\delta}) = \overrightarrow{\gamma} e^{i\overrightarrow{\theta}})$ for the $\mathbb{N}_{fn}$ of $\Omega_{\mathcal{Y}}$

(c) $(\overleftarrow{\Delta}_3(\underline{\delta}) = \overleftarrow{\gamma} e^{i\overleftarrow{\theta}}, \overrightarrow{\Delta}_3(\underline{\delta}) = \overrightarrow{\gamma} e^{i\overrightarrow{\theta}})$ for the $\mathbb{I}_{fn}$ of $\Omega_{\mathcal{Y}}$ and $\Delta(\underline{\delta})$ is known as $\underline{\delta}$-member of Ξ-set $\forall \underline{\delta} \in \mathcal{Y}$.

Note: $\uplus_{IVCNHS}$ denotes the collection of all Ξ-sets.

Definition 3.6. The *complement* of Ξ-set $(\Delta, \mathcal{Y})$, denoted by $(\Delta, \mathcal{Y})^c$ is stated as

$$(\Delta, \mathcal{Y})^c = \{(\check{\ltimes}, (\Delta(\check{\ltimes}))^c) : \check{\ltimes} \in \mathcal{Y}, (\Delta(\check{\ltimes}))^c \in C_{IV}(\Bbbk)\}$$

where the $\mathscr{A}-$term and $\mathscr{P}-$terms of the $\mathbb{M}_{fn}$ $(\Delta(\check{\ltimes}))^c$ are given by

$(\overleftarrow{\gamma}_{\mathcal{Y}}(\check{\ltimes}))^c = 1 - \overleftarrow{\gamma}_{\mathcal{Y}}(\check{\ltimes})$, $(\overrightarrow{\gamma}_{\mathcal{Y}}(\check{\ltimes}))^c = 1 - \overrightarrow{\gamma}_{\mathcal{Y}}(\check{\ltimes})$ and

$(\overleftarrow{\theta}_{\mathcal{Y}}(\check{\ltimes}))^c = 2\pi - \overleftarrow{\theta}_{\mathcal{Y}}(\check{\ltimes})$, $(\overrightarrow{\theta}_{\mathcal{Y}}(\check{\ltimes}))^c = 2\pi - \overrightarrow{\theta}_{\mathcal{Y}}(\check{\ltimes})$ respectively.

TABLE 2. Tabular Representation of $\S_\Lambda$.

$\S_\Lambda$	$\check{\ltimes}_1$	$\check{\ltimes}_2$	...	$\check{\ltimes}_r$
$\check{\Upsilon}_1$	$\begin{pmatrix} \aleph^1_{\mathbb{X}_\Lambda(\check{\ltimes}_1)}(\check{\Upsilon}_1), \\ \aleph^2_{\mathbb{X}_\Lambda(\check{\ltimes}_1)}(\check{\Upsilon}_1), \\ \aleph^3_{\mathbb{X}_\Lambda(\check{\ltimes}_1)}(\check{\Upsilon}_1) \end{pmatrix}$	$\begin{pmatrix} \aleph^1_{\mathbb{X}_\Lambda(\check{\ltimes}_2)}(\check{\Upsilon}_1), \\ \aleph^2_{\mathbb{X}_\Lambda(\check{\ltimes}_2)}(\check{\Upsilon}_1), \\ \aleph^3_{\mathbb{X}_\Lambda(\check{\ltimes}_2)}(\check{\Upsilon}_1) \end{pmatrix}$	$\cdots$	$\begin{pmatrix} \aleph^1_{\mathbb{X}_\Lambda(\check{\ltimes}_r)}(\check{\Upsilon}_1), \\ \aleph^2_{\mathbb{X}_\Lambda(\check{\ltimes}_r)}(\check{\Upsilon}_1), \\ \aleph^3_{\mathbb{X}_\Lambda(\check{\ltimes}_r)}(\check{\Upsilon}_1) \end{pmatrix}$
$\check{\Upsilon}_2$	$\begin{pmatrix} \aleph^1_{\mathbb{X}_\Lambda(\check{\ltimes}_1)}(\check{\Upsilon}_2), \\ \aleph^2_{\mathbb{X}_\Lambda(\check{\ltimes}_1)}(\check{\Upsilon}_2), \\ \aleph^3_{\mathbb{X}_\Lambda(\check{\ltimes}_1)}(\check{\Upsilon}_2) \end{pmatrix}$	$\begin{pmatrix} \aleph^1_{\mathbb{X}_\Lambda(\check{\ltimes}_2)}(\check{\Upsilon}_2), \\ \aleph^2_{\mathbb{X}_\Lambda(\check{\ltimes}_2)}(\check{\Upsilon}_2), \\ \aleph^3_{\mathbb{X}_\Lambda(\check{\ltimes}_2)}(\check{\Upsilon}_2) \end{pmatrix}$	$\cdots$	$\begin{pmatrix} \aleph^1_{\mathbb{X}_\Lambda(\check{\ltimes}_r)}(\check{\Upsilon}_2), \\ \aleph^2_{\mathbb{X}_\Lambda(\check{\ltimes}_r)}(\check{\Upsilon}_2), \\ \aleph^3_{\mathbb{X}_\Lambda(\check{\ltimes}_r)}(\check{\Upsilon}_2) \end{pmatrix}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\check{\Upsilon}_m$	$\begin{pmatrix} \aleph^1_{\mathbb{X}_\Lambda(\check{\ltimes}_1)}(\check{\Upsilon}_m), \\ \aleph^2_{\mathbb{X}_\Lambda(\check{\ltimes}_1)}(\check{\Upsilon}_m), \\ \aleph^3_{\mathbb{X}_\Lambda(\check{\ltimes}_1)}(\check{\Upsilon}_m) \end{pmatrix}$	$\begin{pmatrix} \aleph^1_{\mathbb{X}_\Lambda(\check{\ltimes}_2)}(\check{\Upsilon}_m), \\ \aleph^2_{\mathbb{X}_\Lambda(\check{\ltimes}_2)}(\check{\Upsilon}_m), \\ \aleph^3_{\mathbb{X}_\Lambda(\check{\ltimes}_2)}(\check{\Upsilon}_m) \end{pmatrix}$	$\cdots$	$\begin{pmatrix} \aleph^1_{\mathbb{X}_\Lambda(\check{\ltimes}_r)}(\check{\Upsilon}_m), \\ \aleph^2_{\mathbb{X}_\Lambda(\check{\ltimes}_r)}(\check{\Upsilon}_m), \\ \aleph^3_{\mathbb{X}_\Lambda(\check{\ltimes}_r)}(\check{\Upsilon}_m) \end{pmatrix}$

Now the aggregation procedures and their conclusive systems for the Ξ-set are established in the form of $\mathcal{CSVNHS}$-set and its cardinal set that results in an aggregate $\mathcal{F}-$set with fuzzy-like features. The terms $\Lambda, \mathfrak{E}, \S_\Lambda$and $\uplus_{ICSVNHS}$are are consistent with definition 3.5. The aggregation operations developed in this research article are modified versions of aggregations discussed in [62].

Definition 3.7. Let $\S_\Lambda \in \uplus_{IVCNHS}$. Assume that $\mathbb{U} = \{\check{\Upsilon}_1, \check{\Upsilon}_2, ..., \check{\Upsilon}_m\}$ and $\mathfrak{E} = \{\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_n\}$ with $\mathcal{L}_1 = \{e_{11}, e_{12}, ..., e_{1n}\}, \mathcal{L}_2 = \{e_{21}, e_{22}, ..., e_{2n}\}, ..., \mathcal{L}_n = \{e_{n1}, e_{n2}, ..., e_{nn}\}$ and $\Lambda = \mathcal{L}_1 \times \mathcal{L}_2 \times ..... \times \mathcal{L}_n = \{\check{\ltimes}_1, \check{\ltimes}_2, ..., \check{\ltimes}_n, ..., \check{\ltimes}_{n^n} = \check{\ltimes}_r\}$, each $\check{\ltimes}_i$ is n-tuple element of Λ and $|\Lambda| = r = n^n$ then $\S_\Lambda$ can be presented in the following tabular notation (see Table 2). Where $\aleph^1_{\mathbb{X}_\Lambda(x)}, \aleph^2_{\mathbb{X}_\Lambda(x)}$ and $\aleph^2_{\mathbb{X}_\Lambda(x)}$ are $\mathbb{M}_{fn}$, $\mathbb{I}_{fn}$ and $\mathbb{N}_{fn}$ of $\mathbb{X}_\Lambda$ respectively with $\mathcal{IVN}$ values. If $\alpha_{ij} = (\aleph^1_{\mathbb{X}_\Lambda(\check{\ltimes}_j)}(\check{\Upsilon}_i), \aleph^2_{\mathbb{X}_\Lambda(\check{\ltimes}_j)}(\check{\Upsilon}_i), \aleph^3_{\mathbb{X}_\Lambda(\check{\ltimes}_j)}(\check{\Upsilon}_i))$, for $i = \mathscr{N}_1^m$ and $j = \mathscr{N}_1^r$ then Ξ-set $\S_\Lambda$ is specifically identified by a matrix,

$$[\alpha_{ij}] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1r} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mr} \end{bmatrix}$$

is called an $m \times r$ Ξ-set matrix..

Definition 3.8. If $\S_\Lambda \in \uplus_{ICSVNHS}$ then cardinal set of $\S_\Lambda$ is defined as

$$\|\S_\Lambda\| = \left\{ (\aleph^1_{\|\S_\Lambda\|}(\check{\ltimes}), \aleph^2_{\|\S_\Lambda\|}(\check{\ltimes}), \aleph^3_{\|\S_\Lambda\|}(\check{\ltimes}))/\check{\ltimes} : \check{\ltimes} \in \Lambda \right\},$$

where $\aleph^1_{\|\S_\Lambda\|}, \aleph^2_{\|\S_\Lambda\|}, \aleph^3_{\|\S_\Lambda\|} : \Lambda \to [0, 1]$ are $\mathbb{M}_{fn}$, $\mathbb{I}_{fn}$ and $\mathbb{N}_{fn}$ of $\|\S_\Lambda\|$ with

$$\aleph^1_{\|\S_\Lambda\|}(\check{\ltimes}), \aleph^2_{\|\S_\Lambda\|}(\check{\ltimes}), \aleph^3_{\|\S_\Lambda\|}(\check{\ltimes}) = \frac{|\mathbb{X}_\Lambda(\check{\ltimes})|}{|\mathbb{U}|}$$

TABLE 3. Tabular Representation of $\|\S_\Lambda\|$.

Λ	$\check{\ltimes}_1$	$\check{\ltimes}_2$	$\cdots$	$\check{\ltimes}_r$
$\aleph_{\|\S_\Lambda\|}$	$\begin{pmatrix} \aleph^1_{\|\S_\Lambda\|}(\check{\ltimes}_1), \\ \aleph^2_{\|\S_\Lambda\|}(\check{\ltimes}_1), \\ \aleph^3_{\|\S_\Lambda\|}(\check{\ltimes}_1) \end{pmatrix}$	$\begin{pmatrix} \aleph^1_{\|\S_\Lambda\|}(\check{\ltimes}_2), \\ \aleph^2_{\|\S_\Lambda\|}(\check{\ltimes}_2), \\ \aleph^3_{\|\S_\Lambda\|}(\check{\ltimes}_2) \end{pmatrix}$	$\cdots$	$\begin{pmatrix} \aleph^1_{\|\S_\Lambda\|}(\check{\ltimes}_r), \\ \aleph^2_{\|\S_\Lambda\|}(\check{\ltimes}_r), \\ \aleph^3_{\|\S_\Lambda\|}(\check{\ltimes}_r) \end{pmatrix}$

respectively. These have $\mathbb{ISVN}$ values.

Note: The collection of all cardinal sets of Ξ-sets is denoted by $\|C_{icsvnhs}(\mathbb{U})\|$ such that $\|C_{icsvnhs}(\mathbb{U})\| \subseteq \mathbb{ISVN}(\Lambda)$.

Definition 3.9. Assume $\S_\Lambda \in C_{icsvnhs}(\mathbb{U})$, $\|\S_\Lambda\| \in \|C_{icsvnhs}(\mathbb{U})\|$ and $\mathfrak{C}$ as in Definition 3.5, then Table 3 represents $\|\S_\Lambda\|$.

If $\alpha_{1j} = (\aleph^1_{\|\S_\Lambda\|}(\check{\ltimes}_j), \aleph^2_{\|\S_\Lambda\|}(\check{\ltimes}_j), \aleph^3_{\|\S_\Lambda\|}(\check{\ltimes}_j))$, for $j = \mathcal{N}_1^r$ then the following matrix represents the cardinal set $\|\S_\Lambda\|$,

$$[\alpha_{ij}]_{1\times r} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1r} \end{bmatrix}$$

and is called *cardinal matrix* of $\|\S_\Lambda\|$.

Definition 3.10. Let $\S_\Lambda \in C_{icsvnhs}(\mathbb{U})$ and $\|\S_\Lambda\| \in \|C_{icsvnhs}(\mathbb{U})\|$. Then Ξ-aggregation operator is defined as

$$\widehat{\S_\Lambda} = A_{icifhs}\left(\|\S_\Lambda\|, \S_\Lambda\right)$$

where

$$A_{icsvnhs} : \|C_{icsvnhs}(\mathbb{U})\| \times C_{icifhs}(\mathbb{U}) \to \mathbb{F}(\mathbb{U}).$$

$\widehat{\S_\Lambda}$ is called the aggregate $\mathcal{F}-$set of Ξ-set $\S_\Lambda$.
Its $\mathbb{M}_{fn}$ is given as

$$\aleph_{\widehat{\S_\Lambda}} : \mathbb{U} \to [0,1]$$

with

$$\aleph_{\widehat{\S_\Lambda}}(\nu) = \frac{1}{|\Lambda|} \sum_{\check{\ltimes} \in \Lambda} \aleph_{Card(\S_\Lambda)}(\check{\ltimes})\aleph_{Card(\mathbb{X}_\Lambda)}(\nu).$$

Definition 3.11. Let $\S_\Lambda \in C_{icsvnhs}(\mathbb{U})$ and $\widehat{\S_\Lambda}$ be its aggregate $\mathcal{F}-$set. Assume $\mathbb{U} = \{\check{\Upsilon}_1, \check{\Upsilon}_2, ..., \check{\Upsilon}_m\}$, then $\widehat{\S_\Lambda}$ can be presented as

$$\begin{bmatrix} \S_{\Lambda} & \vdots & \aleph_{\underbrace{}_{\S_{\Lambda}}} \\ \cdots\cdots\cdots & \vdots & \cdots\cdots\cdots\cdots \\ \check{\Upsilon}_1 & \vdots & \aleph_{\underbrace{}_{\S_{\Lambda}}}(\check{\Upsilon}_1) \\ \check{\Upsilon}_2 & \vdots & \aleph_{\underbrace{}_{\S_{\Lambda}}}(\check{\Upsilon}_2) \\ \vdots & \vdots & \vdots \\ \check{\Upsilon}_m & \vdots & \aleph_{\underbrace{}_{\S_{\Lambda}}}(\check{\Upsilon}_m) \end{bmatrix}$$

If $\alpha_{i1} = \aleph_{\underbrace{}_{\S_{\Lambda}}}(\check{\Upsilon}_i)$ for $i = \mathbb{N}_1^m$ then $\overbrace{\S_{\Lambda}}$ is represented by the matrix,

$$[\alpha_{i1}]_{m\times 1} = \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \vdots \\ \alpha_{m1} \end{bmatrix}$$

which is called *aggregate matrix* of $\overbrace{\S_{\Lambda}}$ over $\mathbb{U}$.

4. Decision support system based on aggregation of Ξ-set

In light of the definitions provided in previous subsection, an algorithm is now described in this section to facilitate the $\mathbb{DSS}$, and the supplied method will be validated with the aid of an example from a real-world scenario.

Algorithm 4.1. The brief description of algorithm 4.1 is displayed in Figure 1.

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Algorithm : $\mathcal{DS}$ Algorithm Based on Aggregations of Ξ-set

==

▷ **Start**

▷ **Input Stage**:

———1. Assume $\mathbb{U}$ as sample space

———2. Assume $\mathfrak{E}$ as $\mathbb{SP}$

———3. Classify $\mathbb{SP}$ into $\mathbb{DAVS}$ $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, ..., \mathcal{L}_n$

▷ **Construction Stage**:

———4. $\Lambda = \mathcal{L}_1 \times \mathcal{L}_2 \times \mathcal{L}_3 \times ... \times \mathcal{L}_n$

———5. Construct Ξ-set $\mathbb{X}_\Lambda$ over $\mathbb{U}$, in compliance with Definition 3.5,

▷ **Computation Stage**:

———6. Determine $\parallel \S_\Lambda \parallel$ for $\mathscr{A}-$term and $\mathscr{P}-$term employing Definition 3.8,

———7. Determine $\overbrace{\S_\Lambda}$ for $\mathscr{A}-$term and $\mathscr{P}-$term employing Definition 3.10,

———8. Determine $\aleph_{\underbrace{}_{\S_{\Lambda}}}(\nu)$ employing Definition 3.10,

▷ **Output Stage**:

———9. Figure out the best alternative by max modulus of $\aleph_{\widehat{\S_\Lambda}}(\nu)$ employing Definition 3.11.

▷**End**

==

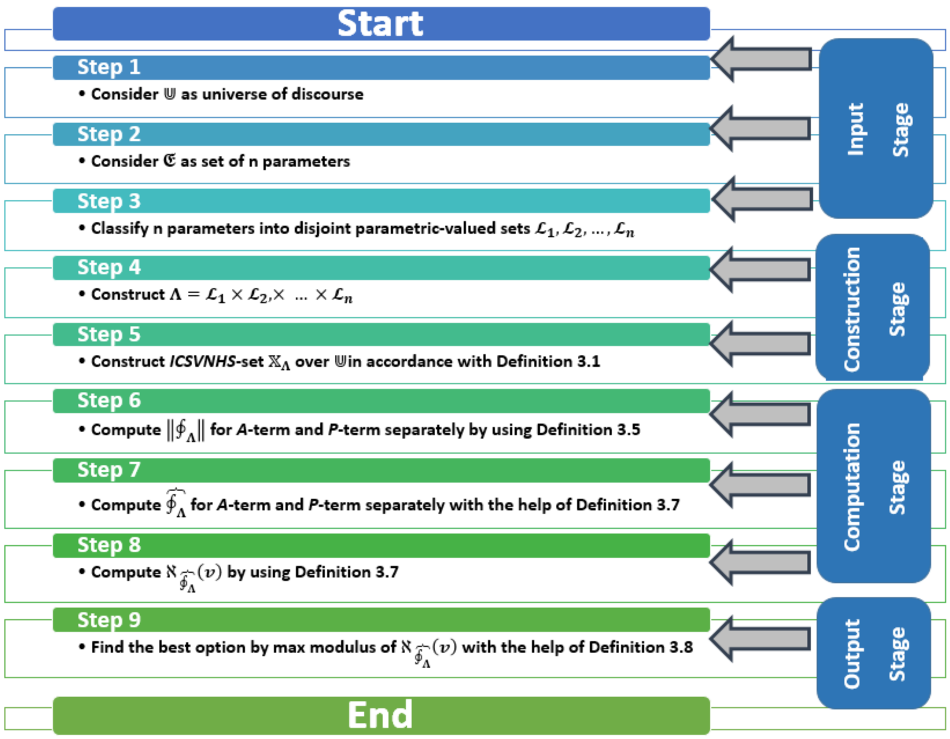

FIGURE 1. $\mathcal{DS}$ Algorithm Based on Aggregations of Ξ-set.

The following real-life example is used to illustrate algorithm:

4.1. *Decision support system based on aggregation of* Ξ*-set*

In this section, a real-world scenario of product selection is discussed based on the aggregation operation of Ξ-set.

Example 4.2. Suppose a person wants to purchase an LED TV from the market. He consults an expert, says Mr. ″*P*″ for the feathers that are necessary to take into consideration while buying a TV. To provide a satisfying viewing experience, numerous elements should be considered when choosing an LED TV's features. Here are some essential characteristics (attributes) that Mr. *P* should take into account:

Screen Size The viewing experience on an LED TV is significantly influenced by the screen size. To choose the right screen size, take into account the room's available area as well as the viewing distance. A more immersive watching experience is often provided by larger displays, but it's essential to make sure the TV is comfortable in the specified space. There are many screen sizes available on the market, but Mr. P preferred 32-inch and 42-inch sizes over others.

Display Technology Although LCD, OLED, and QLED are some of the numerous panel types that are available, LED TVs use LED backlighting technology. Each technology has benefits and disadvantages. While QLED delivers rich colors and high brightness levels, OLED offers great image quality with deep blacks and broad viewing angles. Although less expensive, LCD displays may have contrast and viewing angle restrictions. Mr. P preferred QLED over others.

Resolution The degree of clarity and detail in the material presented on TV depends on its resolution. Full HD (1920x1080 pixels), 4K Ultra HD (3840x2160 pixels), and 8K Ultra HD (7680x4320 pixels) are popular resolutions. In general, higher resolutions provide images that are more realistic and detailed, but the availability of 4K or 8K material should also be taken into account. Due to the unavailability of 4K and 8K, HD is taken into consideration by Mr. P.

Refresh Rate The number of times per second that the TV changes the image on the screen is referred to as the refresh rate. A higher refresh rate, such as 120Hz or 240Hz, helps reduce motion blur in fast-paced situations or sports by allowing for better motion handling. However, a normal refresh rate of 60Hz or 120Hz is generally enough for everyday viewing reasons, so 60Hz and 120Hz refresh rates are preferred.

HDR (High Dynamic Range) The contrast and color accuracy of the presented information are improved with HDR technology. Find TVs that can display HDR content in formats like HDR10, Dolby Vision, or HLG (Hybrid Log-Gamma). Wider color gamuts and more accurate highlights and shadows are possible with HDR-compatible TVs, making for a picture that is more vivid and realistic. Mr. P ignored the HDR attribute.

Smart Features Nowadays, many LED TVs include smart capabilities that provide users access to applications, streaming services, and web surfing. When assessing a TV's smart features, take into account the user interface, the availability of apps, and the simplicity of navigation. Mr. P preferred the LED's having smart features over others.

Connectivity Options Make sure the TV has enough connectors for connecting your gadgets, such as HDMI ports for connecting game consoles, Blu-ray players, or sound systems. For versatility, Mr. P takes into account the LED's accessibility to USB ports, Wi-Fi, Ethernet, and Bluetooth connectivity.

Sound Quality The overall satisfaction is greatly influenced by both the auditory experience and the visual experience, which are both essential. Think about the TV's built-in speakers, or see whether it includes audio-enhancing features like DTS or Dolby Atmos. Mr. P preferred LED's with built-in speakers over others.

Energy Efficiency In general, LED TVs are energy-efficient, but to lower long-term running expenses and environmental effects, it is important to evaluate the energy consumption and energy-saving features of the TV. Mr. P ignored this attribute.

By considering these attributes, one can make an informed decision when buying an LED TV that meets their specific preferences and viewing requirements. There are four types of LED's that are available in market that fulfill the above preferences, so they form the set: $\mathbb{U} = \{\check{\Upsilon}_1, \check{\Upsilon}_2, \check{\Upsilon}_3, \check{\Upsilon}_4\}$. The expert Mr. P considers a $\mathbb{SP}$, $\mathfrak{E} = \{e_1, e_2, ..., \{e_7\}$. For i = 1, 2,..., 7, where the attributes e_i stand for "screen size", "display technology", "resolution", "refresh rate", "smart features", "connectivity options", and "sound quality", respectively Corresponding to each attribute, the $\mathbb{DAVS}$ are: $\mathcal{L}_1 = \{e_{11}, e_{12}\}$; $\mathcal{L}_2 = \{e_{21}\}$; $\mathcal{L}_3 = \{e_{31}\}$; $\mathcal{L}_4 = \{e_{41}, e_{42}\}$; $\mathcal{L}_5 = \{e_{51}\}$; $\mathcal{L}_6 = \{e_{61}\}$ and $\mathcal{L}_7 = \{e_{71}\}$. Then the set $\Lambda = \mathcal{L}_1 \times \mathcal{L}_2 \times ... \times \mathcal{L}_7 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ where each λ_i is a 7-tuple. We construct Ξ-sets $\psi_\Lambda(\lambda_1), \psi_\Lambda(\lambda_2), \psi_\Lambda(\lambda_3), \psi_\Lambda(\lambda_4)$ are defined as,

$$\psi_\Lambda(\lambda_1) = \left\{ \begin{array}{c} \frac{([0.2,0.5],[0.3,0.4],[0.0,0.1])e^{i([0.2,0.8],[0.1,0.3],[0.2,0.3])\pi}}{\check{\Upsilon}_1}, \frac{([0.0,0.2],[0.1,0.3],[0.3,0.5])e^{i([0.2,0.3],[0.1,0.4],[0.1,0.3])\pi}}{\check{\Upsilon}_2}, \\ \frac{([0.0,0.2],[0.2,0.4],[0.0,0.2])e^{i([0.1,0.4],[0.1,0.4],[0.0,0.2])\pi}}{\check{\Upsilon}_3}, \frac{([0.1,0.4],[0.4,0.5],[0.0,0.1])e^{i([0.1,0.3],[0.2,0.4],[0.2,0.3])\pi}}{\check{\Upsilon}_4} \end{array} \right\},$$

$$\psi_\Lambda(\lambda_2) = \left\{ \begin{array}{c} \frac{([0.2,0.3],[0.2,0.5],[0.1,0.2])e^{i([0.0,0.2],[0.1,0.4],[0.1,0.2])\pi}}{\check{\Upsilon}_1}, \frac{([0.1,0.3],[0.2,0.5],[0.0,0.1])e^{i([0.1,0.3],[0.3,0.4],[0.1,0.2])\pi}}{\check{\Upsilon}_2}, \\ \frac{([0.0,0.1],[0.1,0.3],[0.4,0.5])e^{i([0.2,0.4],[0.1,0.3],[0.0,0.2])\pi}}{\check{\Upsilon}_3}, \frac{([0.1,0.3],[0.1,0.4],[0.1,0.3])e^{i([0.1,0.3],[0.2,0.3],[0.2,0.4])\pi}}{\check{\Upsilon}_4} \end{array} \right\},$$

$$\psi_\Lambda(\lambda_3) = \left\{ \begin{array}{c} \frac{([0.2,0.4],[0.1,0.3],[0.1,0.3])e^{i([0.0,0.2],[0.1,0.5],[0.1,0.3])\pi}}{\check{\Upsilon}_1}, \frac{([0.2,0.3],[0.0,0.3],[0.1,0.4])e^{i([0.2,0.4],[0.1,0.4],[0.1,0.2])\pi}}{\check{\Upsilon}_2}, \\ \frac{([0.0,0.2],[0.2,0.3],[0.1,0.5])e^{i([0.1,0.3],[0.2,0.4],[0.1,0.3])\pi}}{\check{\Upsilon}_3}, \frac{([0.0,0.2],[0.1,0.3],[0.1,0.3])e^{i([0.1,0.3],[0.2,0.6],[0.0,0.1])\pi}}{\check{\Upsilon}_4} \end{array} \right\},$$

$$\psi_\Lambda(\lambda_4) = \left\{ \begin{array}{c} \frac{([0.2,0.3],[0.1,0.4],[0.1,0.3])e^{i([0.0,0.2],[0.2,0.5],[0.1,0.2])\pi}}{\check{\Upsilon}_1}, \frac{([0.1,0.5],[0.2,0.3],[0.0,0.1])e^{i([0.1,0.3],[0.3,0.4],[0.1,0.3])\pi}}{\check{\Upsilon}_2}, \\ \frac{([0.1,0.3],[0.1,0.4],[0.1,0.3])e^{i([0.2,0.3],[0.1,0.3],[0.1,0.3])\pi}}{\check{\Upsilon}_3}, \frac{([0.1,0.2],[0.2,0.5],[0.1,0.2])e^{i([0.1,0.3],[0.2,0.3],[0.0,0.3])\pi}}{\check{\Upsilon}_4} \end{array} \right\}.$$

Step 1: Ξ-set $\mathbb{X}_\Lambda$ is written as,

$$\mathbb{X}_\Lambda = \left\{ \begin{array}{c} \left(\lambda_1, \left\{ \begin{array}{c} \frac{([0.2,0.5],[0.3,0.4],[0.0,0.1])e^{i([0.2,0.8],[0.1,0.3],[0.2,0.3])\pi}}{\check{\Upsilon}_1}, \frac{([0.0,0.2],[0.1,0.3],[0.3,0.5])e^{i([0.2,0.3],[0.1,0.4],[0.1,0.3])\pi}}{\check{\Upsilon}_2}, \\ \frac{([0.0,0.2],[0.2,0.4],[0.0,0.2])e^{i([0.1,0.4],[0.1,0.4],[0.0,0.2])\pi}}{\check{\Upsilon}_3}, \frac{([0.1,0.4],[0.4,0.5],[0.0,0.1])e^{i([0.1,0.3],[0.2,0.4],[0.2,0.3])\pi}}{\check{\Upsilon}_4} \end{array} \right\}\right), \\ \left(\lambda_2, \left\{ \begin{array}{c} \frac{([0.2,0.3],[0.2,0.5],[0.1,0.2])e^{i([0.0,0.2],[0.1,0.4],[0.1,0.2])\pi}}{\check{\Upsilon}_1}, \frac{([0.1,0.3],[0.2,0.5],[0.0,0.1])e^{i([0.1,0.3],[0.3,0.4],[0.1,0.2])\pi}}{\check{\Upsilon}_2}, \\ \frac{([0.0,0.1],[0.1,0.3],[0.4,0.5])e^{i([0.2,0.4],[0.1,0.3],[0.0,0.2])\pi}}{\check{\Upsilon}_3}, \frac{([0.1,0.3],[0.1,0.4],[0.1,0.3])e^{i([0.1,0.3],[0.2,0.3],[0.2,0.4])\pi}}{\check{\Upsilon}_4} \end{array} \right\}\right), \\ \left(\lambda_3, \left\{ \begin{array}{c} \frac{([0.2,0.4],[0.1,0.3],[0.1,0.3])e^{i([0.0,0.2],[0.1,0.5],[0.1,0.3])\pi}}{\check{\Upsilon}_1}, \frac{([0.2,0.3],[0.0,0.3],[0.1,0.4])e^{i([0.2,0.4],[0.1,0.4],[0.1,0.2])\pi}}{\check{\Upsilon}_2}, \\ \frac{([0.0,0.2],[0.2,0.3],[0.1,0.5])e^{i([0.1,0.3],[0.2,0.4],[0.1,0.3])\pi}}{\check{\Upsilon}_3}, \frac{([0.0,0.2],[0.1,0.3],[0.1,0.3])e^{i([0.1,0.3],[0.2,0.6],[0.0,0.1])\pi}}{\check{\Upsilon}_4} \end{array} \right\}\right), \\ \left(\lambda_4, \left\{ \begin{array}{c} \frac{([0.2,0.3],[0.1,0.4],[0.1,0.3])e^{i([0.0,0.2],[0.2,0.5],[0.1,0.2])\pi}}{\check{\Upsilon}_1}, \frac{([0.1,0.5],[0.2,0.3],[0.0,0.1])e^{i([0.1,0.3],[0.3,0.4],[0.1,0.3])\pi}}{\check{\Upsilon}_2}, \\ \frac{([0.1,0.3],[0.1,0.4],[0.1,0.3])e^{i([0.2,0.3],[0.1,0.3],[0.1,0.3])\pi}}{\check{\Upsilon}_3}, \frac{([0.1,0.2],[0.2,0.5],[0.1,0.2])e^{i([0.1,0.3],[0.2,0.3],[0.0,0.3])\pi}}{\check{\Upsilon}_4} \end{array} \right\}\right) \end{array} \right\}$$

Step 2: The cardinal is computed as,

$\| \mathbb{X}_\Lambda \| (A - term) =$

$$\left\{ \begin{array}{l} ([0.075, 0.325], [0.250, 0.400], [0.075, 0.225])/\lambda_1, ([0.100, 0.250], [0.150, 0.425], [0.150, 0.275])/\lambda_2, \\ ([0.100, 0.275], [0.100, 0.300], [0.100, 0.375])/\lambda_3, ([0.125, 0.325], [0.150, 0.400], [0.075, 0.225])/\lambda_4 \end{array} \right\}$$

$\| \mathbb{X}_\Lambda \| (P - term) =$

$$\left\{ \begin{array}{l} ([0.150, 0.450], [0.125, 0.375], [0.125, 0.275])/\lambda_1, ([0.100, 0.300], [0.175, 0.350], [0.100, 0.250])/\lambda_2, \\ ([0.100, 0.300], [0.150, 0.475], [0.075, 0.225])/\lambda_3, ([0.100, 0.275], [0.200, 0.375], [0.075, 0.275])/\lambda_4 \end{array} \right\}$$

Step 3: The set $\widehat{\mathbb{X}_\Lambda}$ can be established as,

$\widehat{\mathbb{X}_\Lambda}(A - term) =$

$$\frac{1}{4}\begin{bmatrix} [0.2,0.5],[0.3,0.4],[0.0,0.1] & [0.2,0.3],[0.2,0.5],[0.1,0.2] & [0.2,0.4],[0.1,0.3],[0.1,0.3] & [0.2,0.3],[0.1,0.4],[0.1,0.3] \\ [0.0,0.2],[0.1,0.3],[0.3,0.5] & [0.1,0.3],[0.2,0.5],[0.0,0.1] & [0.2,0.3],[0.0,0.3],[0.1,0.4] & [0.1,0.5],[0.2,0.3],[0.0,0.1] \\ [0.0,0.2],[0.2,0.4],[0.0,0.2] & [0.0,0.1],[0.1,0.3],[0.4,0.5] & [0.0,0.2],[0.2,0.3],[0.1,0.5] & [0.1,0.3],[0.1,0.4],[0.1,0.3] \\ [0.1,0.4],[0.4,0.5],[0.0,0.1] & [0.1,0.3],[0.1,0.4],[0.1,0.3] & [0.0,0.2],[0.1,0.3],[0.1,0.3] & [0.1,0.2],[0.2,0.5],[0.1,0.2] \end{bmatrix}$$

$$\times \begin{bmatrix} [0.075,0.325],[0.250,0.400],[0.075,0.225] \\ [0.100,0.250],[0.150,0.425],[0.150,0.275] \\ [0.100,0.275],[0.100,0.300],[0.100,0.375] \\ [0.125,0.325],[0.150,0.400],[0.075,0.225] \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 0.2 & 0.0 & 0.2 & 0.1 \\ 0.2 & 0.1 & 0.2 & 0.3 \\ 0.0 & 0.4 & 0.1 & 0.1 \\ 0.0 & 0.1 & 0.0 & 0.1 \end{bmatrix} \begin{bmatrix} 0.000 \\ 0.050 \\ 0.075 \\ 0.010 \end{bmatrix} = \begin{bmatrix} 0.004000 \\ 0.005750 \\ 0.007125 \\ 0.001500 \end{bmatrix}$$

$\widehat{\mathbb{X}_\Lambda}(P - term) =$

$$\frac{1}{4}\begin{bmatrix} [0.2,0.8],[0.1,0.3],[0.2,0.3] & [0.0,0.2],[0.1,0.4],[0.1,0.2] & [0.0,0.2],[0.1,0.5],[0.1,0.3] & [0.0,0.2],[0.2,0.5],[0.1,0.2] \\ [0.2,0.3],[0.1,0.4],[0.1,0.3] & [0.1,0.3],[0.3,0.4],[0.1,0.2] & [0.2,0.4],[0.1,0.4],[0.1,0.2] & [0.1,0.3],[0.3,0.4],[0.1,0.3] \\ [0.1,0.4],[0.1,0.4],[0.0,0.2] & [0.2,0.4],[0.1,0.3],[0.0,0.2] & [0.1,0.3],[0.2,0.4],[0.1,0.3] & [0.2,0.3],[0.1,0.3],[0.1,0.3] \\ [0.1,0.3],[0.2,0.4],[0.2,0.3] & [0.1,0.3],[0.2,0.3],[0.2,0.4] & [0.1,0.3],[0.2,0.6],[0.0,0.1] & [0.1,0.3],[0.2,0.3],[0.0,0.3] \end{bmatrix}$$

$$\times \begin{bmatrix} [0.150,0.450],[0.125,0.375],[0.125,0.275] \\ [0.100,0.300],[0.175,0.350],[0.100,0.250] \\ [0.100,0.300],[0.150,0.475],[0.075,0.225] \\ [0.100,0.275],[0.200,0.375],[0.075,0.275] \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 0.5 & 0.0 & 0.0 & 0.1 \\ 0.1 & 0.1 & 0.2 & 0.1 \\ 0.3 & 0.3 & 0.0 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix} \begin{bmatrix} 0.200 \\ 0.025 \\ 0.075 \\ 0.000 \end{bmatrix} = \begin{bmatrix} 0.025000 \\ 0.009375 \\ 0.016875 \\ 0.007500 \end{bmatrix}$$

$$\widehat{\mathbb{X}_\Lambda} = \left\{ 0.004000e^{i0.025000\pi}/\check{\Upsilon}_1, 0.005750e^{i0.009375\pi}/\check{\Upsilon}_2, 0.007125e^{i0.016875\pi}/\check{\Upsilon}_3, 0.001500e^{i0.007500\pi}/\check{\Upsilon}_4 \right\}$$

Assume the modulus value of $\max\left(\aleph_{\overbrace{\mathbb{X}_{\Lambda}}}\right)$
$= \max\{0.004101260482/\check{\Upsilon}_1, 0.005804159727/\check{\Upsilon}_2, 0.007246254583/\check{\Upsilon}_3, 0.001511292293/\check{\Upsilon}_4\}$
$= 0.007246254583/\check{\Upsilon}_3$. This means that the LED $\check{\Upsilon}_3$ may be recommended by Mr. P for purchase.

5. Discussion and comparative analysis

Different $\mathbb{DM}$ algorithmic techniques have already been explored in the literature by [35–37, 55–59, 61] that were based on hybridized complex set architectures with $\mathcal{F}-$set,$\mathcal{IF}-$set, and $\mathcal{SVN}-$set under $\mathcal{S}-$set environment. The lack of several crucial characteristics has a negative impact on the process of $\mathbb{DM}$. For instance, considering "screen size," "screen resolution," "refresh rate," e.t.c., as only attributes in a scenario based on product selection is insufficient because these indicators may have different values (parameters) and sub-values (sub-parameters). It is much more appropriate to further classify these parameters into their $\mathbb{DAVS}$, as we have done in Example 4.2. The aforementioned current $\mathbb{DM}$ models are insufficient for $\mathcal{IV}$ data or $\mathcal{MAA}-$mapping, however, the shortcomings of these models have been solved in the suggested model. By taking into account $\mathcal{MAA}-$mapping, the $\mathbb{DM}$ process will become more dependable and trustworthy. In Table 4, a comparison analysis of proposed model with relevant existing models has been carried out. The Table 4 makes it abundantly clear that our proposed structure, Ξ-set is more flexible and generalized than existing relevant models for the reason that these models [35–37, 55–59, 61] are customized for their particular cases by excluding certain or all features among $\mathbb{M}_{fn}$, $\mathbb{N}_{fn}$, $\mathbb{I}_{fn}$, $\mathbb{SAA}$-mapping, $\mathbb{MAA}$-mapping, $PN-$data and $IV-$data. The visual illustration of this generalization of our suggested structure is shown in Figure 2.

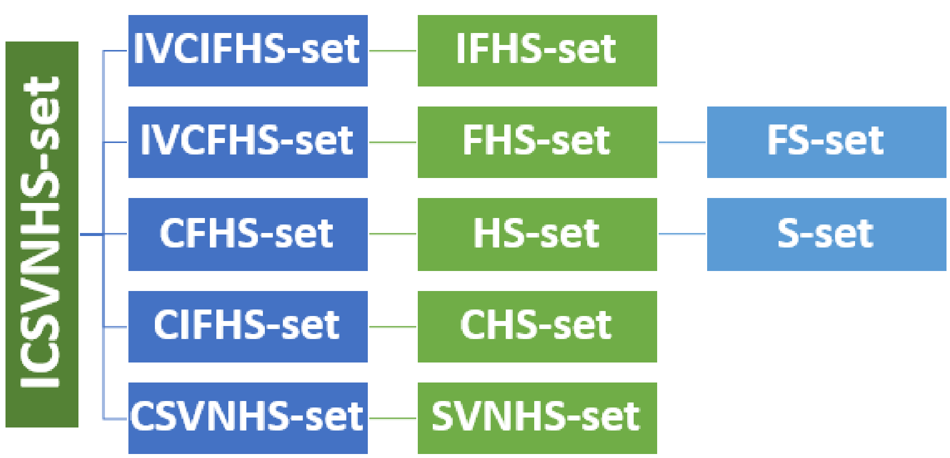

FIGURE 2. Generalization of Proposed Structure

TABLE 4. Comparison analysis of proposed model with some existing relevant models

Authors	Structure	$\mathbb{A}_{fn}$	Remarks
Thirunavukarasu et al. [35]	$\mathbb{CFS}$-set	$\mathbb{SAA}$-mapping	Insufficient for $\mathcal{IV}$ data, $\mathbb{N}_{fn}$, $\mathbb{I}_{fn}$ and partitioning $\mathbb{SP}$ to $\mathbb{DAVS}$.
Fan et al. [58]	$\mathbb{IVCFS}$-set	$\mathbb{SAA}$-mapping	Shows inadequacy for $\mathbb{N}_{fn}$, $\mathbb{I}_{fn}$ and partitioning $\mathbb{SP}$ to $\mathbb{DAVS}$
Selvachandran et al. [61]	$\mathbb{IVCFS}$-set	$\mathbb{SAA}$-mapping	Insufficient for $\mathbb{N}_{fn}$, $\mathbb{I}_{fn}$ and partitioning $\mathbb{SP}$ to $\mathbb{DAVS}$
Kumar et al. [36]	$\mathbb{CIFS}$-set	$\mathbb{SAA}$-mapping	Insufficient for $\mathcal{IV}$ data, $\mathbb{I}_{fn}$ and partitioning $\mathbb{SP}$ to $\mathbb{DAVS}$
Ali et al. [55]	$\mathbb{CIFS}$-set	$\mathbb{SAA}$-mapping	Insufficient for $\mathcal{IV}$ data, $\mathbb{I}_{fn}$ and partitioning $\mathbb{SP}$ to $\mathbb{DAVS}$
Khan et al. [59]	$\mathbb{CIFS}$-set	$\mathbb{SAA}$-mapping	Insufficient for $\mathcal{IV}$ data, $\mathbb{I}_{fn}$ and partitioning $\mathbb{SP}$ to $\mathbb{DAVS}$
Smarandache et al. [37]	$\mathbb{CSVNS}$-set	$\mathbb{SAA}$-mapping	Insufficient for $\mathcal{IV}$ data and partitioning $\mathbb{SP}$ to $\mathbb{DAVS}$
Al-Sharqi et al. [57]	$\mathcal{ICSVNS}$-set	$\mathbb{SAA}$-mapping	Shows inadequacy for partitioning $\mathbb{SP}$ to $\mathbb{DAVS}$
Rahman et al. [56]	$\mathcal{CFHS}$-set	$\mathbb{MAA}$-mapping	Insufficient for $\mathcal{IV}$ data, $\mathbb{N}_{fn}$, $\mathbb{I}_{fn}$.
Rahman et al. [56]	$\mathcal{CIFHS}$-set	$\mathbb{MAA}$-mapping	Insufficient for $\mathcal{IV}$ data and $\mathbb{I}_{fn}$.
Rahman et al. [56]	$\mathcal{CSVNHS}$-set	$\mathbb{MAA}$-mapping	Insufficient for $\mathcal{IV}$ data.
Rahman et al. [60]	$\mathcal{IVCFHS}$-set	$\mathbb{MAA}$-mapping	Insufficient for $\mathbb{N}_{fn}$, $\mathbb{I}_{fn}$.
Proposed Structure	Ξ-set	$\mathbb{MAA}$-mapping	Addresses the restrictions and faults of preceding structures.

5.1. *Merits of proposed Study*

The following are some advantages of the proposed study that are mentioned in this subsection:

(i) The proposed method utilized the Ξ-set concepts to address current $\mathbb{DM}$ difficulties. As a result, this model has enormous potential in the realistic portrayal of computational invasions. The offered approach enables investigators to handle a real-world situation where the periodicity of data in the form of intervals has to be addressed.

(ii) Due to the suggested structure's emphasis on a thorough examination of qualities (sub-attributes) rather than a narrow focus on those traits (attributes), the $\mathbb{DM}$ process is improved, adaptable, and more dependable.

(iii) It discusses the features and qualities of the current relevant structures, i.e., $\mathbb{IVCFHS}$-set, $\mathbb{CFHS}$-set, $\mathbb{CIFHS}$-set, $\mathbb{CSVNHS}$-set, $\mathbb{IVCFS}$-set, $\mathbb{IVCIFS}$-set, $\mathbb{IVCNS}$-set, $\mathbb{CFS}$-set, $\mathbb{CIFS}$-set, $\mathbb{CNS}$-set, etc., so it is not unreasonable to call it the generalized form of all these structures.

TABLE 5. Comparison with existing models under appropriate features

Authors	Structure	$\mathbb{M}_{fn}$	$\mathbb{N}_{fn}$	$\mathbb{I}_{fn}$	$\mathbb{SAA}$-mapping	$\mathbb{MAA}$-mapping	PN- data	IV data
Ali et al. [55]	$\mathbb{CIFS}$-set	✓	✓	×	✓	×	✓	×
Al-Sharqi et al. [57]	$\mathbb{ICSVNS}$-set	✓	✓	✓	✓	×	✓	✓
Fan et al. [58]	$\mathbb{IVCFS}$-set	✓	×	×	✓	×	✓	✓
Khan et al. [59]	$\mathbb{CIFS}$-set	✓	✓	×	✓	×	✓	×
Kumar et al. [36]	$\mathbb{CIFS}$-set	✓	✓	×	✓	×	✓	×
Smarandache et al. [37]	$\mathbb{CSVNS}$-set	✓	✓	✓	✓	×	✓	×
Selvachandran et al. [61]	$\mathbb{IVCFS}$-set	✓	×	×	✓	×	✓	✓
Thirunavukarasu et al. [35]	$\mathbb{CFS}$-set	✓	×	×	✓	×	✓	×
Rahman et al. [56]	$\mathbb{CFHS}$-set	✓	×	×	✓	✓	✓	×
Rahman et al. [56]	$\mathbb{CIFHS}$-set	✓	✓	×	✓	✓	✓	×
Rahman et al. [56]	$\mathbb{CSVNHS}$-set	✓	✓	✓	✓	✓	✓	×
Rahman et al. [60]	$\mathbb{IVCFHS}$-set	✓	×	×	✓	✓	✓	✓
Proposed Structure	Ξ-set	✓	✓	✓	✓	✓	✓	✓

Tables 4 and 5 make it simple to determine the benefit of the proposed study. Table 4 demonstrates the main features of the study. Table 5 demonstrates the dominant features, including $\mathbb{M}_{fn}$, $\mathbb{N}_{fn}$, $\mathbb{I}_{fn}$, $\mathbb{SAA}$-mapping, $\mathbb{MAA}$-mapping, PN data, and IV data of the proposed study.

6. **Conclusion**

This article discusses a novel theoretical framework, the interval complex single-valued neutrosophic hypersoft set (Xi-set), along with its characteristics and set-theoretic operations. The recommended structure blends the interval complex single-valued neutrosophic set and hypersoft set to regulate unclear and unsure knowledge. These two components are already recognized for their dependable settings. While the second provides a multi-argument domain for the concurrent assessment of several sub-attributes, the first component can manage data on intervals and periodic types. The set theoretic operations, including complement, difference, union, and intersection of the Ξ-set, are also described. It has been designed to use aggregate matrices, cardinal sets, aggregate 𝔽-sets, and aggregate matrices as aggregation operators. A 𝔻𝕄 technique that is based on aggregation operators of the Ξ-set has been suggested. To assess the model's flexibility and validity, the suggested structure and its 𝔻𝕊𝕊 in a real-world scenario have been compared with some previously published relevant research. The present work has explored the conceptual basis for a generalized model, that is, Ξ-set, to deal with 𝔻𝕄 real-life situations by using hypothetical data. The authors have pledged to present multiple instance reports based on the Xi-set using actual data. It is feasible to extend hybrid set structures more broadly by including expert sets, prospective fuzzy-set-like models, fuzzy-set-like parameterized families, and algebraic structures.

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A membership based neutrosophic approach for supervised fingerprint image classification

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Abstract. The Neutrosophic Sets (NS) mathematical model is a sophisticated paradigm that effectively addresses uncertainty. This article provides four different methods for the extraction of visual features. The proposal has been investigated with regard to both neutrosophic sets and single-valued NS. The article primarily examines two distinct features: binary and self-intensity approaches. Following that, an attempt was made to classify the images using machine learning techniques. The main objective of this article was exclusively on supervised classification algorithms. The classification of images was performed by using Decision Tree (DT), Random Forest (RF), K Nearest Neighbour (KNN),Naive Bayes (NB), and Logistic Regression (LR) algorithms. Since we have an interest in biometric images, the fingerprint image dataset was chosen for classification. The methods proposed in that research are known to as Membership Based Neutrosophic Binary Image ($MBNI_B$), Membership Based Neutrosophic Self Intensity Image ($MBNI_{SI}$), Membership Based Single Valued Neutrosophic Binary Image ($MBSVNI_B$), and Membership Based Single Valued Neutrosophic Self Intensity Image ($MBSVNI_{SI}$). The proposal possesses a range of improvement accuracy ranging from 5% to 58%.

Keywords: Machine learning; decision tree; random forest; KNN; Naive Bayes; logistic regression; neutrosophic image; fingerprint image

1. Introduction

Zadeh introduced the concept of the fuzzy set as a technique of addressing factors characterized by uncertainty. The fuzzy set employs a membership function that assigns a membership value ranging from 0 to 1 to each component of the set. Atanssov is recognized with the development of the intuitionistic fuzzy set, a mathematical system that assigns both membership and non-membership functions to each element in an existing state. Consequently, it is possible to characterize it in a more precise and definitive way compared to a vague set. Nevertheless, the system's capacity is limited to processing incomplete and uncertain data, rendering it incapable of effectively managing the conflicting data that commonly arises in real-world scenarios. The NS, developed by Smarandache, introduces an innovative structure for addressing uncertainty by assigning memberships to truth, indeterminacy, and falsity. This pioneering invention represents a significant advancement in the field. In contrast to the intuitionistic fuzzy set, it possesses a greater capacity to effectively represent indeterminate or uncertain data. The NS has received significant attention from scholars, and its applications have been incorporated in several domains such as aggregation operators, decision-making, image processing, information measures, graph theory, and algebraic structures. In view of this expansion, we present a comprehensive overview of NS in order to offer a comprehensive understanding of the various concepts, methodologies, and developments related to their extensions. Based on the research findings, it has been observed that several developing countries, like India, China, and Turkey, are actively engaged in the exploration and study of NS. The NS has garnered significant attention from scholars due to its ability to encompass a wide range of descriptive cases. The fuzzy appearance of the neutrosophic scope is more effectively managed by this novel set. Considering the fact that study pertaining to the subject of neutrosophic has been continuing for a span of two decades and is currently garnering the attention of researchers, it is imperative that we adopt a comprehensive perspective in order to identify any prevailing patterns of thought or scientific advancements within the realm of neutrosophic research. In the context of this article, it is important to note that the fundamental definitions of NS, single valued NS and image features [1–3] are regarded as introductory.

The remainder of the paper is organized as follows. Some remarkable related works are indicated in section 2. Then in the section 3 we discussed the some preliminaries and proposed methods. The section 4 explains the implementation of proposed methods in fingerprint datasets. Finally the section 5 concludes the research findings and feature scope of the proposed methods.

List of Symbols and Abbreviations

NS**:** Neutrosophic Sets

DT**:** Decision Tree

RF**:** Random Forest

KNN**:** K Nearest Neighbour

LR**:** Logistic Regression

NB**:** Naive Bayes

SVM**:** Support Vector Machine

CNN: Convolutional Neural Network

$MBNI_B$: Membership Based Neutrosophic Binary Image

$MBNI_{SI}$: Membership Based Neutrosophic Self Intensity Image

$MBSVNI_B$: Membership Based Single Valued Neutrosophic Binary Image

$MBSVNI_{SI}$: Membership Based Single Valued Neutrosophic Self Intensity Image

$\mathcal{L}$: Maximum pixel range 2^8

P_{k_0}: Image padding

g_μ: Mean function

g_σ: Standard deviation function

g_λ: Maximum function

$\mathbb{Z}^+$: Positive integer

Θ_A: Parameter function

ξ_A: Cardinal of Θ_A

2. Related works

The study proposed by the Abdel et al. [4] categorizes 52% of the risks related to real-world data oil, gas, and coal analysis as high risks, 36% as medium risks, and 12% as usual risks throughout every aspect. The paper exhibits the significant efficacy of the neutrosophic technique in the realm of energy analysis. The utilization of neutrosophic statistical concepts is used by Afzal et al. [5] within the domain of LCR meters. The paper provides an in-depth analysis of the correlation between resistance and capacitance through the utilization of NS at particular intervals. The article presents a notable outcome of $30.18 + 40.92IN$, where IN is within the range of [0, 0.26]. Sampling provides the fundamental component of the quick-switching system. Uma et al. [6] utilized a NS-based Poisson distribution and performed a comparative analysis with a fuzzy Poisson distribution using operational characteristic curves. The probability assigned by the suggested model to a set of 1200 samples is 0.45. The automated technique of identifying individuals by evaluating their behavioral and biological characteristics is referred to as biometric recognition. Recognition and verification are two biometric processes utilized to determine an individual's distinct characteristics. A finger knuckle print feature extraction approach, an entropy-based pattern histogram, and a collection of statistical texture features, according to Heidari et al. [7], could be applied to determine how unique a person is. When these approaches were applied to the Poly-U finger knuckle print and finger knuckle print datasets, there was a significant improvement in performance, with increases of 94.91% and 98.5%, respectively. Mohammed et al. [8] applied a watermark recognition technique in their research to secure the confidentiality of patient's healthcare information by implementing a biometric system. To protect the integrity and confidentiality of this data, a cryptographic model has been constructed. Fingerprints, as a biometric characteristic, provide a high degree of specificity for the purpose of biometric identification. Tlhoolebe et al. [9] discovered a 93% classification accuracy rate for machine learning classifiers used to a private dataset of 20 persons ranging in age from 18 to 38 years in their study. The classification challenge demands the use of the KNN, Support Vector Machine (SVM), Kstar, and NB [10] algorithms. A dataset of 400 trials was used to evaluate the suggested approach. The acquired statistics show a 37% false rejection rate and a 27% false acceptance rate. The results shown are quite promising and point to the efficacy of the proposed approach. The findings suggest that the method's efficacy could be improved by combining it with another biometric. Gender classification is one of the fields of biometric authentication. FaceNet feature extraction algorithm was developed by Adhinata et al. [11]. For analyzing the IMDB dataset, the researchers used the KNN, SVM, and DT [12] algorithms. Their research revealed that the KNN method had the highest level of accuracy, with a 95.75% accuracy rate. A database containing a large number of 3408 fingerprint images. Kruti et al. [13] used an SVM classifier on a private dataset to achieve an elevated level of accuracy, reaching 97% in their study. Nguyen et al. [14] attempted to reduce the number of comparisons in automatic fingerprint recognition systems while dealing with large databases in their study. The researchers used a variety of techniques to accomplish this, including RF [15], SVM, Convolutional Neural Network (CNN), and KNN. The FVC 2000, 2002, and 2004 datasets were used to test these techniques. The algorithms were evaluated by analyzing precision, recall, accuracy, and receiver operating characteristic analysis. The results of the research showed the RF algorithm had the highest level of accuracy among the examined algorithms. The value expressed is 96.75%. The SVM method outperforms the competition, with an accuracy rate of 95.5% on two-thirds of the databases. It is proposed that supervised classification approaches such as DT, linear discriminant analysis, medium Gaussian SVM, KNN, and bagged tree ensemble classifiers should be employed to improve the efficiency of fingerprint identification systems.

The authors, Noor et al. [16], applied a methodology that included image enhancement, binarization, and preprocessing techniques in fingerprint analysis. They utilized medium Gaussian SVM classifiers to achieve an impressive accuracy rate of 98.90%. Kumar et al. [17] presented the gravitational search decision forest technique for fingerprint recognition in their research. The suggested approach combines the gravitational search algorithm and the RF method. To discover a suitable solution, the DT method was used to evaluate multiple hypotheses. The method was tested using the NIST SD27 and FVC2004 datasets. The proposal had a 92.56% average recognition rate for the NIST SD27 dataset and a 96.56% recognition rate for the FVC2004 dataset. Table 1 focuses on some of the work achieved so far to recognize fingerprints or classify fingerprints.

Table 1. Related work

Authour(s)	Method	Dataset	Score (%)
Labati et al. [18]	CNN	DB Latent database	89.6
	$CNN+KNN$		46.4
	$CNN+NB$		52.1
	$CNN+SVM$		49.2
Kumar et al. [17]	RF	NIST SD27	92.56
		FVC2004	96.56
Tlhoolebe et al. [9]	KNN, SVM	Private data	93
Jang et al. [19]	DeepPore	High-Resolution-Fingerprint database	93.09
Nguyen et al. [14]	SVM	FVC2000, FVC2002, FVC2004	95.5
Adhinata et al. [11]	KNN	IMDB dataset	95.75
Heidari et al. [7]	Entropy-based pattern	Poly-U finger knuckle print	98.5
Jeon et al. [20]	VGGNet	FVC2000, FVC2002, FVC2004	82.1
	GNet		94.2
	VGGNet + Ensemble		98.3
Saeed et al. [21]	DeepFKTNet Model	FingerPass, FVC2004	98.89
Nahar et al. [22]	Self-Regulating CNN	FVC2004	99.1
Walhazi et al. [23]	Multi-Classifier System	NIST SD27, NIST SD301, FVC2002	100

3. Methods

3.1. *Preliminaries*

Definition 3.1. Let $f(x,y) = \mathcal{I}(i,j)_{m\times n} \in \mathbb{R}^2$ be an image, then the zero padding for neutrosophic image P_{k_0} is defined with respect to h as follows: [24]

$$P_{k_0}(g(x,y)) = \begin{cases} f(x,y) & \text{if } x+h, y+h \leq \max m, \max n \text{ or} \\ & x-h, y-h < \min m, \min n \\ 0 & \text{if } x-h, y-h \geq \min m, \min n \text{ or} \\ & x+h, y+h > \max m, \max n \end{cases} \tag{1}$$

where $k = 2\mathbb{N}+1$, $3 \leq k \leq \min(m,n)$ and $h = k \mod (2)$.

Definition 3.2. Let $f(x,y) = \mathcal{I}(i,j)_{m\times n}$ be an image then the set of arithmetic mean μ values for h of the image is defined as [24]

$$g_\mu(x,y) = \{f_1\mu_1, f_2\mu_2, ...f_c\mu_c\} \tag{2}$$

$$f_c\mu_c = \frac{1}{h^2}\sum_{k=i-\Delta i}^{i+\Delta i}\sum_{l=j-\Delta i}^{j+\Delta i} f_c(k,l) \tag{3}$$

where $c = \{1, 2, ... \min(m, n)\}$ and $\Delta i, \Delta j = \{1, 2, .. \leq h\}$

Definition 3.3. Let $f(x, y) = \mathcal{I}(i, j)_{m\times n}$ be an image then the set of standard deviation σ values for h of the image is defined as [24]

$$\begin{aligned} g_\sigma(x, y) &= \{f_1\sigma_1, f_2\sigma_2, ...f_c\sigma_c\} \\ f_c\sigma_c &= \sqrt{\frac{1}{h^2}\sum_{k=i-\Delta i}^{i+\Delta i}\sum_{l=j-\Delta j}^{j+\Delta j}\left(f_c(k, l) - f_c\mu_c\right)^2} \end{aligned} \tag{4}$$

where $c = \{1, 2, ... \min(m, n)\}$ and $\Delta i, \Delta j = \{1, 2, .. \leq h\}$

Definition 3.4. Let $f(x, y) = \mathcal{I}(i, j)_{m\times n}$ be an image then the set of maximum λ values for h of the image is defined as

$$\begin{aligned} g_\lambda(x, y) &= \{f_1\lambda_1, f_2\lambda_2, ...f_c\lambda_c\} \\ f_c\lambda_c &= \max(f_c(i \pm \Delta i, j \pm \Delta j)) \end{aligned} \tag{5}$$

where $c = \{1, 2, ... \min(m, n)\}$ and $\Delta i, \Delta j = \{1, 2, .. \leq h\}$

3.2. *Proposed Method*

The proposed method involves the utilization of a hypotheses function $H(N_A)$ to determine the values of the neutrosophic membership components. The hypothesis function for the neutrosophic components may vary depending on the approach chosen.

Definition 3.5 $(MBNI_B)$**.** Let $A = \mathcal{I}(i, j)_{m\times n}$ be an image then the neutrosophic components of A is defined as $N_A = \{T_A(i, j), I_A(i, j), F_A(i, j)\}$. The Membership Based Neutrosophic Binary Image $(MBNI_B)$ is formulated as follows:

$$\begin{aligned} \mathcal{NI_B}(A) &= f(A, P_{k_0}, N_A, H(N_A)) \\ \text{where } f(A) &= \mathcal{I}(i, j)_{m\times n} \\ f(P_{k_0}) &= P_{k_0}(A(i, j)) \\ f(N_A) &= \{f(P_{k_0}(T_A)), f(P_{k_0}(I_A)), f(P_{k_0}(F_A))\} \\ f(H(N_A)) &= \{H_1(N_A), H_2(N_A), H_3(N_A), H_4(N_A)\} \end{aligned}$$

$H(N_A)$ refers the four types of hypothesis for the neutrosophic membership functions $f(N_A)$

$H_1(N_{A(i,j)})$**:** $f(P_{k_0}(T_{A(i,j)})) > \max[f(P_{k_0}(I_{A(i,j)})), f(P_{k_0}(F_{A(i,j)}))]$

$H_2(N_{A(i,j)})$**:** $f(P_{k_0}(I_{A(i,j)})) > \max[f(P_{k_0}(T_{A(i,j)})), f(P_{k_0}(F_{A(i,j)}))]$ and $d_1(I_A) > d_2(I_A)$ where
$d_1(I_A) = f(P_{k_0}(I_{A(i,j)})) - f(P_{k_0}(T_{A(i,j)}))$
$d_2(I_A) = f(P_{k_0}(I_{A(i,j)})) - f(P_{k_0}(F_{A(i,j)}))$

$H_3(N_{A(i,j)})$: $f(P_{k_0}(I_{A(i,j)})) > \max[f(P_{k_0}(T_{A(i,j)})), f(P_{k_0}(F_{A(i,j)}))]$ and
$d_1(I_A) < d_2(I_A)$ where
$d_1(I_A) = f(P_{k_0}(I_{A(i,j)})) - f(P_{k_0}(T_{A(i,j)}))$
$d_2(I_A) = f(P_{k_0}(I_{A(i,j)})) - f(P_{k_0}(F_{A(i,j)}))$

$H_4(N_{A(i,j)})$: $f(P_{k_0}(F_{A(i,j)})) > \max[f(P_{k_0}(T_{A(i,j)})), f(P_{k_0}(I_{A(i,j)}))]$

$$\mathcal{NI_B}(A) = \begin{cases} \mathcal{L} - 1 & \text{if } H_1(N_{A(i,j)}) \text{ or } H_2(N_{A(i,j)}) \\ 0 & \text{if } H_3(N_{A(i,j)}) \text{ or } H_4(N_{A(i,j)}) \end{cases} \quad (6)$$

Definition 3.6 ($MBNI_{SI}$)**.** Let $A = \mathcal{I}(i,j)_{m\times n}$ be an image then the neutrosophic components of A is defined as $N_A = \{T_A(i,j), I_A(i,j), F_A(i,j)\}$. The Membership Based Neutrosophic Self Intensity Image ($MBNI_{SI}$) is formulated as follows

$$\begin{aligned} \mathcal{NI_{SI}}(A) &= f(A, P_{k_0}, N_A, H(N_A), \xi(N_A)) \\ \text{where } f(A) &= \mathcal{I}(i,j)_{m\times n} \\ f(P_{k_0}) &= P_{k_0}(A(i,j)) \\ f(N_A) &= \{f(P_{k_0}(T_A)), f(P_{k_0}(I_A)), f(P_{k_0}(F_A))\} \\ f(H(N_A)) &= \{H_1(N_A), H_2(N_A), H_3(N_A), H_4(N_A)\} \\ f(\xi(N_A)) &= \{f_\mu(A), f_\sigma(A), f_\lambda(A)\} \end{aligned}$$

$H(N_A)$ refers the four types of hypothesis for the neutrosophic membership functions $f(N_A)$

$H_1(N_{A(i,j)})$: $f(P_{k_0}(T_{A(i,j)})) > \max[f(P_{k_0}(I_{A(i,j)})), f(P_{k_0}(F_{A(i,j)}))]$

$H_2(N_{A(i,j)})$: $f(P_{k_0}(I_{A(i,j)})) > \max[f(P_{k_0}(T_{A(i,j)})), f(P_{k_0}(F_{A(i,j)}))]$ and
$d_1(I_A) > d_2(I_A)$ where
$d_1(I_A) = f(P_{k_0}(I_{A(i,j)})) - f(P_{k_0}(T_{A(i,j)}))$
$d_2(I_A) = f(P_{k_0}(I_{A(i,j)})) - f(P_{k_0}(F_{A(i,j)}))$

$H_3(N_{A(i,j)})$: $f(P_{k_0}(I_{A(i,j)})) > \max[f(P_{k_0}(T_{A(i,j)})), f(P_{k_0}(F_{A(i,j)}))]$ and
$d_1(I_A) < d_2(I_A)$ where
$d_1(I_A) = f(P_{k_0}(I_{A(i,j)})) - f(P_{k_0}(T_{A(i,j)}))$
$d_2(I_A) = f(P_{k_0}(I_{A(i,j)})) - f(P_{k_0}(F_{A(i,j)}))$

$H_4(N_{A(i,j)})$: $f(P_{k_0}(F_{A(i,j)})) > \max[f(P_{k_0}(T_{A(i,j)})), f(P_{k_0}(I_{A(i,j)}))]$

$$\mathcal{NI_{SI}}(A) = \begin{cases} f(A(i,j)) & \text{if } H_1(N_{A(i,j)}) \\ f_\mu(A(i,j)) & \text{if } H_2(N_{A(i,j)}) \\ \mid f_\mu(A(i,j)) - f_\sigma(A(i,j)) \mid \in \mathbb{Z}^+ & \text{if } H_3(N_{A(i,j)}) \\ 0 & \text{if } H_4(N_{A(i,j)}) \text{ or otherwise} \end{cases} \quad (7)$$

Definition 3.7 ($MBSVNI_B$)**.** Let $A = \mathcal{I}(i,j)_{m\times n}$ be an image then the neutrosophic components of A is defined as $SVN_A = \{T_A(i,j), I_A(i,j), F_A(i,j)\}$. The Membership Based Single

Valued Neutrosophic Binary Image ($MBSVNI_B$) is formulated as follows

$$\begin{aligned}\mathcal{SVNI_B}(A) &= f(A, P_{k_0}, SVN_A, H(SVN_A)) \\ \text{where } f(A) &= \mathcal{I}(i,j)_{m\times n} \\ f(P_{k_0}) &= P_{k_0}(A(i,j)) \\ f(SVN_A) &= \{f(P_{k_0}(T_A)), f(P_{k_0}(I_A)), f(P_{k_0}(F_A))\} \\ f(H(SVN_A)) &= \{H_1(SVN_A), H_2(SVN_A), H_3(SVN_A), H_4(SVN_A)\}\end{aligned}$$

$H(SVN_A)$ refers the four types of hypothesis for the neutrosophic membership functions $f(SVN_A)$

$H_1(SVN_{A(i,j)})$: $f(P_{k_0}(T_{A(i,j)})) > \max[f(P_{k_0}(I_{A(i,j)})), f(P_{k_0}(F_{A(i,j)}))]$

$H_2(SVN_{A(i,j)})$: $f(P_{k_0}(I_{A(i,j)})) > \max[f(P_{k_0}(T_{A(i,j)})), f(P_{k_0}(F_{A(i,j)}))]$ and

$d_1(I_A) > d_2(I_A)$ where

$d_1(I_A) = f(P_{k_0}(I_{A(i,j)})) - f(P_{k_0}(T_{A(i,j)}))$

$d_2(I_A) = f(P_{k_0}(I_{A(i,j)})) - f(P_{k_0}(F_{A(i,j)}))$

$H_3(SVN_{A(i,j)})$: $f(P_{k_0}(I_{A(i,j)})) > \max[f(P_{k_0}(T_{A(i,j)})), f(P_{k_0}(F_{A(i,j)}))]$ and

$d_1(I_A) < d_2(I_A)$ where

$d_1(I_A) = f(P_{k_0}(I_{A(i,j)})) - f(P_{k_0}(T_{A(i,j)}))$

$d_2(I_A) = f(P_{k_0}(I_{A(i,j)})) - f(P_{k_0}(F_{A(i,j)}))$

$H_4(SVN_{A(i,j)})$: $f(P_{k_0}(F_{A(i,j)})) > \max[f(P_{k_0}(T_{A(i,j)})), f(P_{k_0}(I_{A(i,j)}))]$

$$\mathcal{SVNI_B}(A) = \begin{cases} \mathcal{L}-1 & \text{if } H_1(SVN_{A(i,j)}) \text{ or } H_2(SVN_{A(i,j)}) \\ 0 & \text{if } H_3(SVN_{A(i,j)}) \text{ or } H_4(SVN_{A(i,j)}) \end{cases} \tag{8}$$

Definition 3.8 ($MBSVNI_{SI}$)**.** Let $A = \mathcal{I}(i,j)_{m\times n}$ be an image then the neutrosophic components of A is defined as $SVN_A = \{T_A(i,j), I_A(i,j), F_A(i,j)\}$. The Membership Based Single Valued Neutrosophic Self Intensity Image ($MBSVNI_{SI}$) is formulated as follows

$$\begin{aligned}\mathcal{SVNI_{SI}}(A) &= f(A, P_{k_0}, SVN_A, H(SVN_A), \xi(SVN_A)) \\ \text{where } f(A) &= \mathcal{I}(i,j)_{m\times n} \\ f(P_{k_0}) &= P_{k_0}(A(i,j)) \\ f(SVN_A) &= \{f(P_{k_0}(T_A)), f(P_{k_0}(I_A)), f(P_{k_0}(F_A))\} \\ f(H(SVN_A)) &= \{H_1(SVN_A), H_2(SVN_A), H_3(SVN_A), H_4(SVN_A)\} \\ f(\xi(SVN_A)) &= \{f_\mu(A), f_\sigma(A), f_\lambda(A)\}\end{aligned}$$

$H(SVN_A)$ refers the four types of hypothesis for the neutrosophic membership functions $f(SVN_A)$

$H_1(SVN_{A(i,j)})$: $f(P_{k_0}(T_{A(i,j)})) > \max[f(P_{k_0}(I_{A(i,j)})), f(P_{k_0}(F_{A(i,j)}))]$

$H_2(SVN_{A(i,j)})$: $f(P_{k_0}(I_{A(i,j)})) > \max[f(P_{k_0}(T_{A(i,j)})), f(P_{k_0}(F_{A(i,j)}))]$ and
$d_1(I_A) > d_2(I_A)$ where
$d_1(I_A) = f(P_{k_0}(I_{A(i,j)})) - f(P_{k_0}(T_{A(i,j)}))$
$d_2(I_A) = f(P_{k_0}(I_{A(i,j)})) - f(P_{k_0}(F_{A(i,j)}))$

$H_3(SVN_{A(i,j)})$: $f(P_{k_0}(I_{A(i,j)})) > \max[f(P_{k_0}(T_{A(i,j)})), f(P_{k_0}(F_{A(i,j)}))]$ and
$d_1(I_A) < d_2(I_A)$ where
$d_1(I_A) = f(P_{k_0}(I_{A(i,j)})) - f(P_{k_0}(T_{A(i,j)}))$
$d_2(I_A) = f(P_{k_0}(I_{A(i,j)})) - f(P_{k_0}(F_{A(i,j)}))$

$H_4(SVN_{A(i,j)})$: $f(P_{k_0}(F_{A(i,j)})) > \max[f(P_{k_0}(T_{A(i,j)})), f(P_{k_0}(I_{A(i,j)}))]$

$$\mathcal{SVNI}_{\mathcal{SI}}(A) = \begin{cases} f(A(i,j)) & \text{if } H_1(SVN_{A(i,j)}) \\ f_\mu(A(i,j)) & \text{if } H_2(SVN_{A(i,j)}) \\ \mid f_\mu(A(i,j)) - f_\sigma(A(i,j)) \mid \in \mathbb{Z}^+ & \text{if } H_3(SVN_{A(i,j)}) \\ 0 & \text{if } H_4(SVN_{A(i,j)}) \text{ or otherwise} \end{cases} \tag{9}$$

FIGURE 1. Fingerprint image visualization of proposed methods

From the Figure 1 proposal can visualize the patterns of proposed methods fingerprint image.

3.3. *Auto Hyper parameterization*

Definition 3.9. Consider $N\mathcal{I}_A$ be a neutrosophic binary image of the image A then the Bernoulli distribution values for h convolution is calculated as follows:

Let $B(i,j,h)_c = \{N\mathcal{I}_{A(i\pm\Delta i, j\pm\Delta j)^1}, N\mathcal{I}_{A(i\pm\Delta i, j\pm\Delta j)^2}, \cdots N\mathcal{I}_{A(i\pm\Delta i, j\pm\Delta j)^c}\}$ the quantity of successive events $\mathcal{L}-1$ for h and their probability $\bar{\mu}$ are

$$\begin{aligned} \mathcal{B}(N\mathcal{I}_A, e_1^\nu, \bar{\mu}_1^\nu) &= \binom{c}{e_1^\nu} (\bar{\mu}_1^\nu)^{e_1^\nu} (1-\bar{\mu}_1^\nu)^{c-e_1^\nu} \\ \text{where } e_1^\nu &= e_1^\nu(B(i,j,h)_c) = \{n_{\mathcal{L}-1}(e_1), n_{\mathcal{L}-1}(e_2), \cdots n_{\mathcal{L}-1}(e_\nu)\} \\ \bar{\mu}_1^\nu &= \bar{\mu}_1^\nu(B(i,j,h)_c) = \bar{\mu}(B(i,j,h)_1^\nu) \end{aligned} \tag{10}$$

Definition 3.10. Consider $N\mathcal{I}_A$ be a neutrosophic self intensity image of the image A then the Gaussion distribution values for h convolution is calculated as follows:
Let $B(i,j,h)_c = \{N\mathcal{I}_{A(i\pm\Delta i,j\pm\Delta j)^1}, N\mathcal{I}_{A(i\pm\Delta i,j\pm\Delta j)^2}, \cdots N\mathcal{I}_{A(i\pm\Delta i,j\pm\Delta j)^c}\}$ then the mean $\bar{\mu}$ and standard deviation $\bar{\sigma}$ values for h

$$\begin{aligned} \mathcal{G}(N\mathcal{I}_A, \bar{\mu}_{c_1^\nu}, \bar{\sigma}_{c_1^\nu}) &= \frac{1}{\bar{\sigma}_{c_1^\nu}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{N\mathcal{I}_{A(i,j)}-\bar{\mu}_{c_1^\nu}}{\bar{\sigma}_{c_1^\nu}}\right)^2} \qquad (11) \\ \bar{\mu}_{c_1^\nu} &= \bar{\mu}(B(i,j,h)_{c_1^\nu}) \\ \bar{\sigma}_{c_1^\nu} &= \bar{\sigma}(B(i,j,h)_{c_1^\nu}) \end{aligned}$$

Definition 3.11. For the neutrosophic image $N\mathcal{I}_A$ generalized parameters are formulated as follows

$$\begin{aligned} \Theta_A &= \{\theta_1, \theta_2, ...\theta_t\} \qquad (12) \\ \text{where } \theta_t &= \begin{cases} \mathcal{B}(N\mathcal{I}_A, e_1^\nu, \bar{\mu}_1^\nu) & \text{if } N\mathcal{I}_A \text{ is binary image} \\ \mathcal{G}(N\mathcal{I}_A, \bar{\mu}_{c_1^\nu}, \bar{\sigma}_{c_1^\nu}) & \text{if } N\mathcal{I}_A \text{ is self intensity image} \end{cases} \\ \xi_A &= \{\xi_1, \xi_2, ...\xi_t\} \qquad (13) \\ \text{where } \xi_t &= \bar{\bar{\theta}}_t \end{aligned}$$

3.4. *Parameter tuning*

For instance analysis, article suggest default classification data, digits data from the Sklearn dataset [25].

3.4.1. *Decision tree classifier*

DT possesses multiple properties that can be tweaked to improve performance. Maximum depth, minimum sample split, minimum sample leaf, and minimum weighted fraction are among these parameters. This proposal takes into account the first three criteria for classification. The attribute "maximum depth" is in charge of determining the tree's maximum depth. "Tree height" refers to the maximum length of a path from a tree's root to any of its leaves. The minimum split refers to the minimum number of values that must be present in a node before attempting a split operation. To clarify, if a node has only two members and the minimum split criterion is set to 5, the node will enter a terminal state, preventing any further efforts to split it. The minimal sample split denotes the smallest number of entities that can exist in a tree's leaf node. The default value is one-third of the minimum split value

Algorithm 1 Proposed method and hyperparameters tuning algorithm

Require: Input Image database, classification labels

```
for i in image do
    MBNI_B, MBNI_SI
    MBSVNI_B, MBSVNI_SI
    if i is MBNI_B or MBSVNI_B then
        B(NI_A, e_1^ν, μ̄_1^ν) ← Equation 10
    else if i is MBNI_SI or MBSVNI_SI then
        G(NI_A, μ̄_{c_1^ν}, σ̄_{c_1^ν}) ← Equation 11
    end if
    parameters = {Θ_A, ξ_A}
end for
if model = DT then ← Equation 14
    for i = ξ_A do
        h_p(md)
        for j= ξ_A do
            h_p(msl)
            for k = Θ_A do
                h_p(mss)
                model.fit(parameters = i, j, k)
            end for
        end for
    end for
end if
if model = RF then ← Equation 15
    for i = ξ_A do
        h_p(mss)
        for j = ξ_A do
            h_p(msl)
            for k = Θ_A do
                h_p(mw)
                model.fit(parameters = i, j, k)
            end for
        end for
    end for
end if
```

```
if model = LR then ← Equation 16
    for  i = Θ_A do
        h_p(tol)
        model.fit(parameters = i)
    end for
end if
if model = NB then ← Equation 17
    for  i = Θ_A do
        h_p(nb)
        for j =Θ_A  do
            h_p(bin)
            model.fit(parameters = i, j)
        end for
    end for
end if
if model = KNN then ← Equation 18
    for  i = Θ_A do
        h_p(nb)
        for j = Θ_A do
            h_p(ls)
            for k = Θ_A do
                h_p(p)
                model.fit(parameters = i, j, k)
            end for
        end for
    end for
end if
```

specified. The following is the formulation of attribute value selection:

$$\begin{aligned} DT &= f(X, Y, h_p) \quad (14) \\ \text{where } f(h_p) &= \{h_{p(md)}, h_{p(mss)}, h_{p(msl)}\} \\ h_{p(md)} &= \xi_A(2 \times \sqrt{\min(m,n)}) \\ h_{p(mss)} &= \lfloor| \log_e \Theta_A |\rfloor = \lfloor| \log_e \theta_t |\rfloor \\ h_{p(msl)} &= \xi_A \mod (2h) \end{aligned}$$

The utilization of least cost-complexity pruning technique is aimed at mitigating the issue of over fitting in decision tree classification. In the context of cost-complexity pruning, the process is carried out recursively to identify the node that exhibits the lowest level of strength

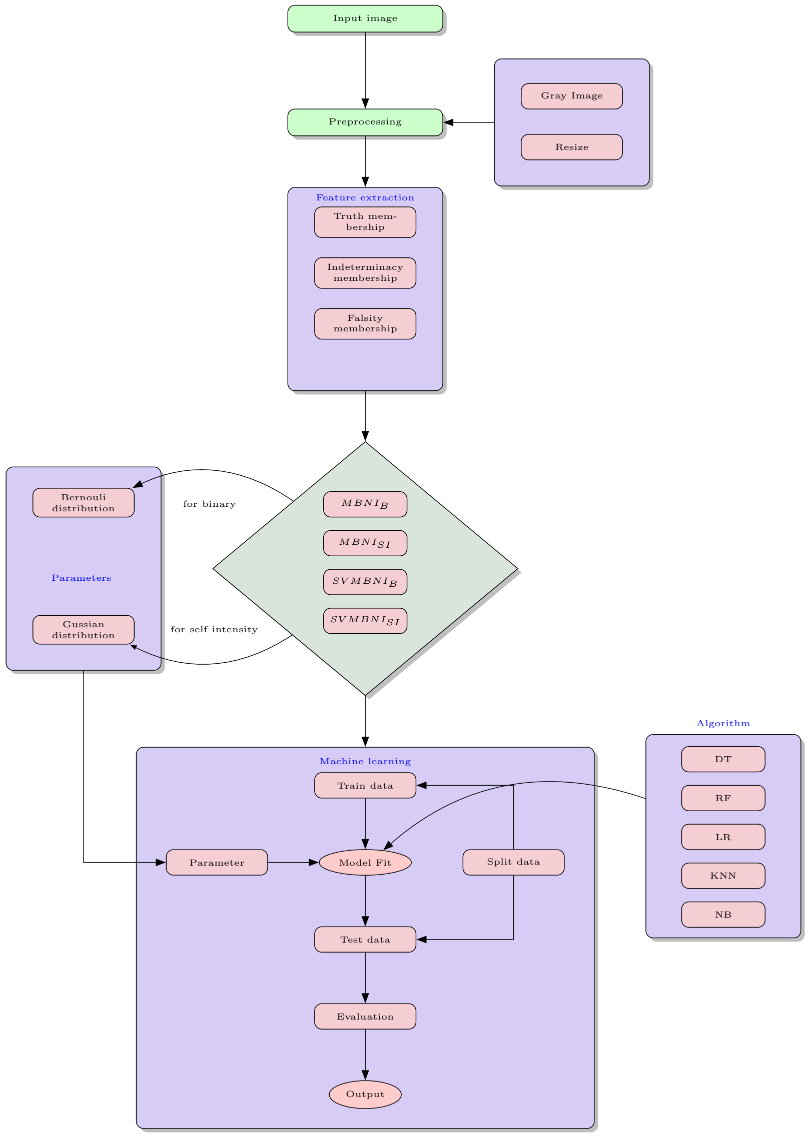


FIGURE 2. Procedure flow chart of the proposed model

or effectiveness. The implementation of an efficient α, which prioritizes the pruning of nodes associated with the intuitive reader interface α, facilitates the identification of the most vulnerable component. The determination of ideal α values in the pruning process involves assessing the effectiveness of α and the corresponding total leaf impurity at each stage. As the value of α increases, a greater portion of the tree will require pruning, resulting in an elevation of the overall impurity of the leaves. The aforementioned equation yields α values of 2 and 4 for the dataset. The association between leaf impurity and α effectiveness for each α value in the training and testing data is shown in Figure 3.

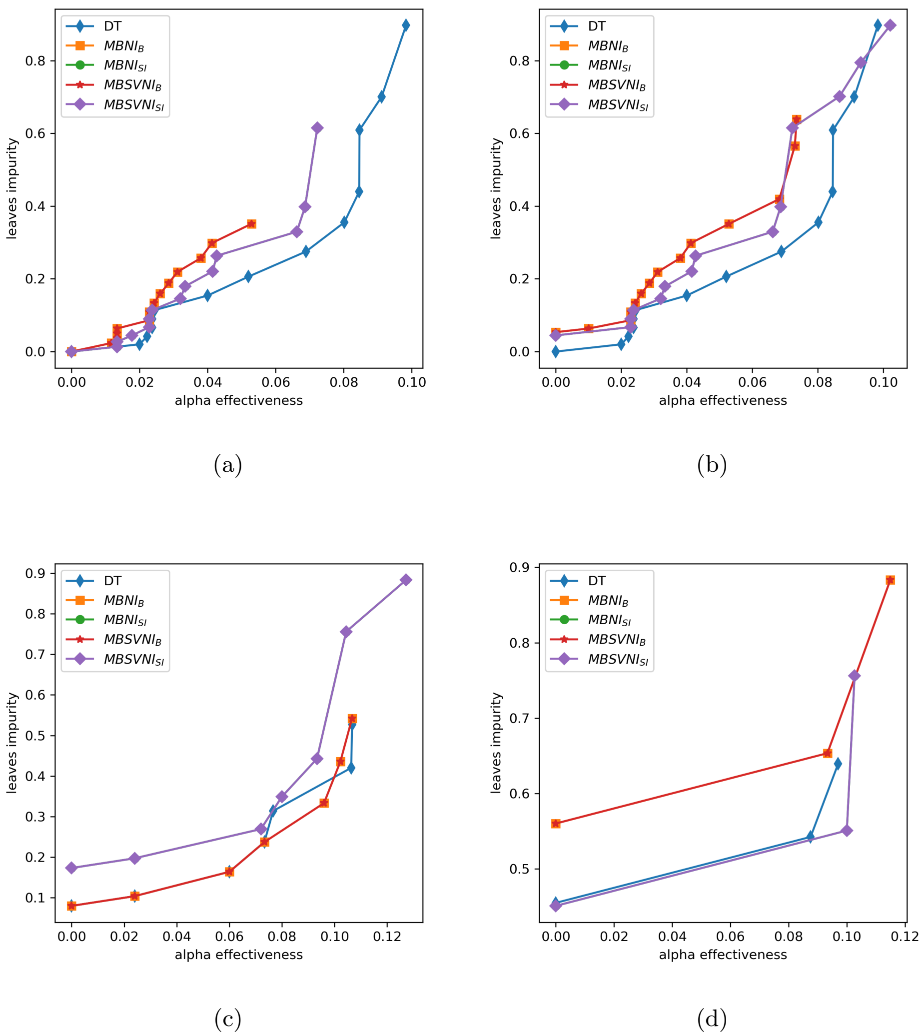


FIGURE 3. Total Impurity vs effective α for training and testing data (a) $\alpha = 2$ training data (b) $\alpha = 4$ training data (c) $\alpha = 2$ testing data (d) $\alpha = 4$ testing data

3.4.2. *Random forest*

The proposal addresses the *RF* algorithm's three hyperparameters: minimum sample split, minimum sample leaf, and minimum weight fraction leaf. *DT* previously covered the minimum sample split and minimum sample leaf. This algorithm will also go through the smallest weighted fraction attribute. This attribute represents the total of the weights required to be at a leaf node. It is comparable to the minimal sample size but uses a fraction of the total number of observations instead. However, the approach of *RF* attribute formulation varies from that of the DT. The RF algorithm's attribute formulation is calculated below.

$$\begin{aligned} RF &= f(X, Y, h_p) \\ \text{where } f(h_p) &= \{h_{p(md)}, h_{p(mss)}, h_{p(msl)}\} \\ h_{p(mss)} &= \xi_A \mod (\sqrt{\min(m,n)}) > 0 \\ h_{p(msl)} &= \xi_A \mod (\sqrt{\min(m,n)}) > 1 \\ h_{p(mw)} &= -\log_2 \Theta_A \times 10^3 \end{aligned} \tag{15}$$

To test the relevance of features, evaluate the mean drop in accuracy of the forest when the features are randomly permuted in out-of-bag samples. This measurement is also known as permutation importance since it shows the former is experimentally biased towards unique predictor variables. This bias stems from an unfair advantage granted by the standard impurity functions to predictors with a large number of values in the case of a single DT. The mean loss in accuracy is significantly biased in this case to exaggerate the impact of associated variables. The comparison plot in Figure 4 revealed the mean decreased accuracy for the consideration data for the traditional RF method and the novel RF method.

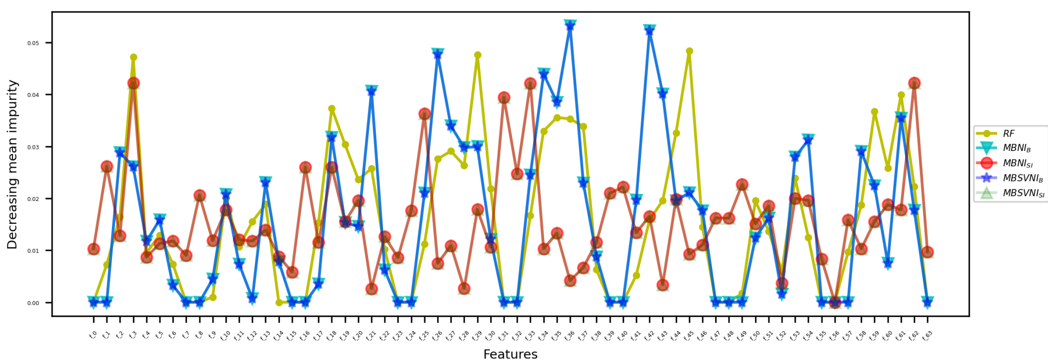
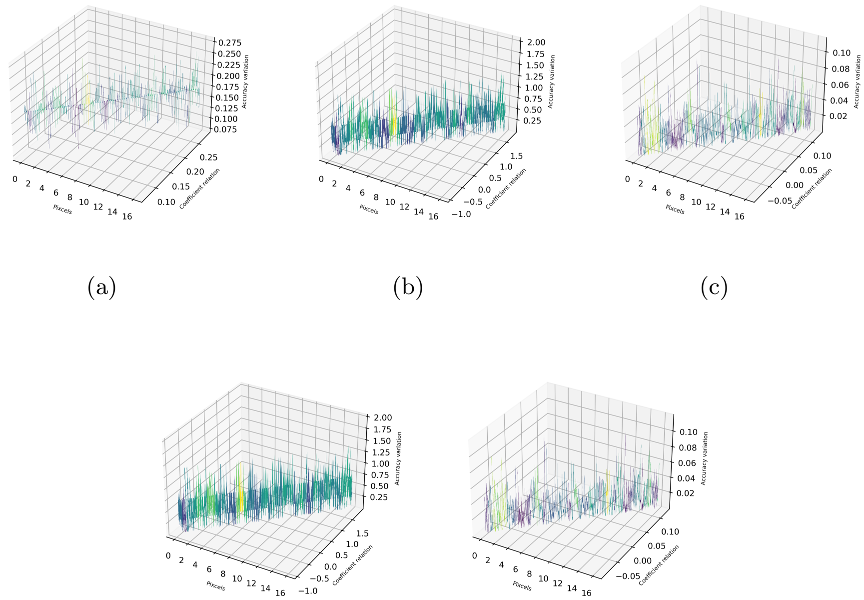


FIGURE 4. Comparison analysis of random forest feature importance

3.4.3. *Logistic regression*

We modify a parameter called tolerance in LR [26]. The size of a tolerance interval is proportional to the size of the population data sample and the population variation. Depending on the data distribution, there are two primary methods for computing tolerance intervals: parametric and non parametric methods. Interval of metric tolerance: To describe coverage and confidence, use knowledge of the population distribution, which is used to refer to a Gaussian distribution. Non parametric tolerance interval: Estimate coverage and confidence using rank statistics, which sometimes results in less precision due to a lack of knowledge about the distribution. The comparison with the pixel relation is accomplished here by varying the



(a) (b) (c)

(d) (e)

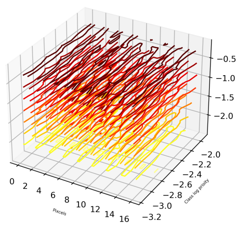
FIGURE 5. Logistic regression analysis (a) Classic method, (b)$MBNI_B$-method, (c) $MBNI_{SI}$-method, (d) $MBSVNI_B$-method, (e) $MBSVNI_{SI}$-method

accuracy of the pixel's correlation coefficients. The relational analysis of the LR technique and NS-based LR approaches is depicted in Figure 5.

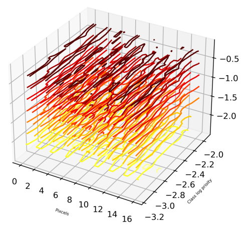
$$\begin{aligned} LR &= f(X, Y, h_p) \qquad (16)\\ \text{where } f(h_p) &= \{h_{p(tol)}\}\\ h_{p(tol)} &= (-\log_2 \Theta_A) \times 10^3 \end{aligned}$$

3.4.4. *Naive Bayes*

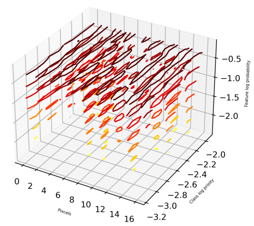
For data with multivariate Bernoulli distributions, Bernoulli NB applies the NB training and classification algorithms. This indicates that several features may exist, but each appears to be a binary-valued variable. The NB method has only two basic parameters: α and



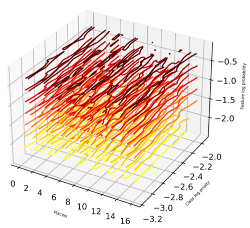
(a)



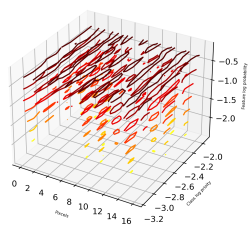
(b)



(c)



(d)



(e)

FIGURE 6. Naive Bayes analysis (a) Classic method, (b)$MBNI_B$-method, (c) $MBNI_{SI}$-method, (d) $MBSVNI_B$-method, (e) $MBSVNI_{SI}$-method

binarization values.

$$\begin{aligned} NB &= f(X, Y, h_p) \quad (17) \\ \text{where } f(h_p) &= \{h_{p(\alpha)}, h_{p(bin)}\} \\ h_{p(\alpha)} &= \mid (\log_{10} \Theta_A \times 10^{-4}) \min(m, n) \mid \\ h_{p(bin)} &= -\log_2 \Theta_A \times 10^{-3} \end{aligned}$$

α is the Laplace smoothing technique used in NB to solve the problem of zero probability with the prior probability and conditional probability. The estimator for a collection of observation counts $X = \{x_1, x_2, ..x_n\}$ from a n-dimensional multinomial distribution with N trials is a smoothed version of the counts as follows:

$$\theta_i = \frac{x_i + \alpha}{N + \alpha n} \quad (i = 1, 2, ..n)$$

Using these attributes, the proposal extracts image properties such as class log priority and feature log likelihood. Figure 6 demonstrates the performance of the analysis with the intensity and the extracted features. Because the feature's probability is more feasible than other ways, the proposed binary method surpasses the classical and self-intensity methods.

3.4.5. *K nearest neighbor*

KNN requires three factors into account: the number of neighborhoods, leaf size, and power parameters. The number of neighbors refers to the number of elements that comprise the classification in a single group. The preceding method, DT, explicitly addresses leaf size. The Minkowski metric is referred to by the power parameter.

$$\begin{aligned} KNN &= f(X, Y, h_p) \quad (18)\\ \text{where } f(h_p) &= \{h_{p(nb)}, h_{p(ls)}, h_{p(p)}\}\\ h_{p(nb)} &= \xi_A \mod (2\sqrt{\min(m,n)})\\ h_{p(ls)} &= \xi_A \mod (\min(m,n)) \times 2\sqrt{\min(m,n)}\\ h_{p(p)} &= \xi_A \mod (\sqrt{\min(m,n)}) \end{aligned}$$

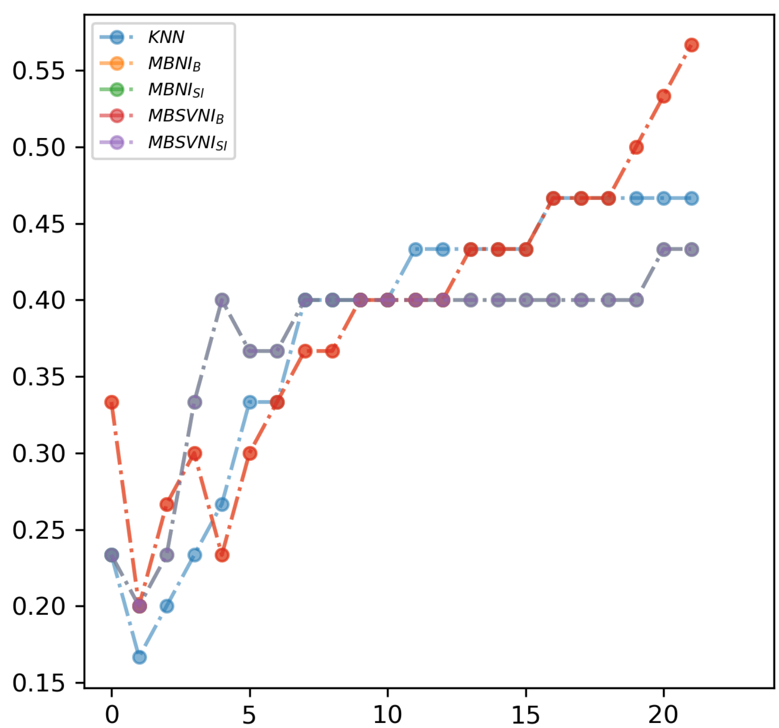


FIGURE 7. K nearest neighbourhood error analysis

The expected error of neighbourhood component analysis can be expressed as

$$err = 1 - \frac{1}{N}\sum_{i,j=1}^{N} P_{ij} y_{ij}$$

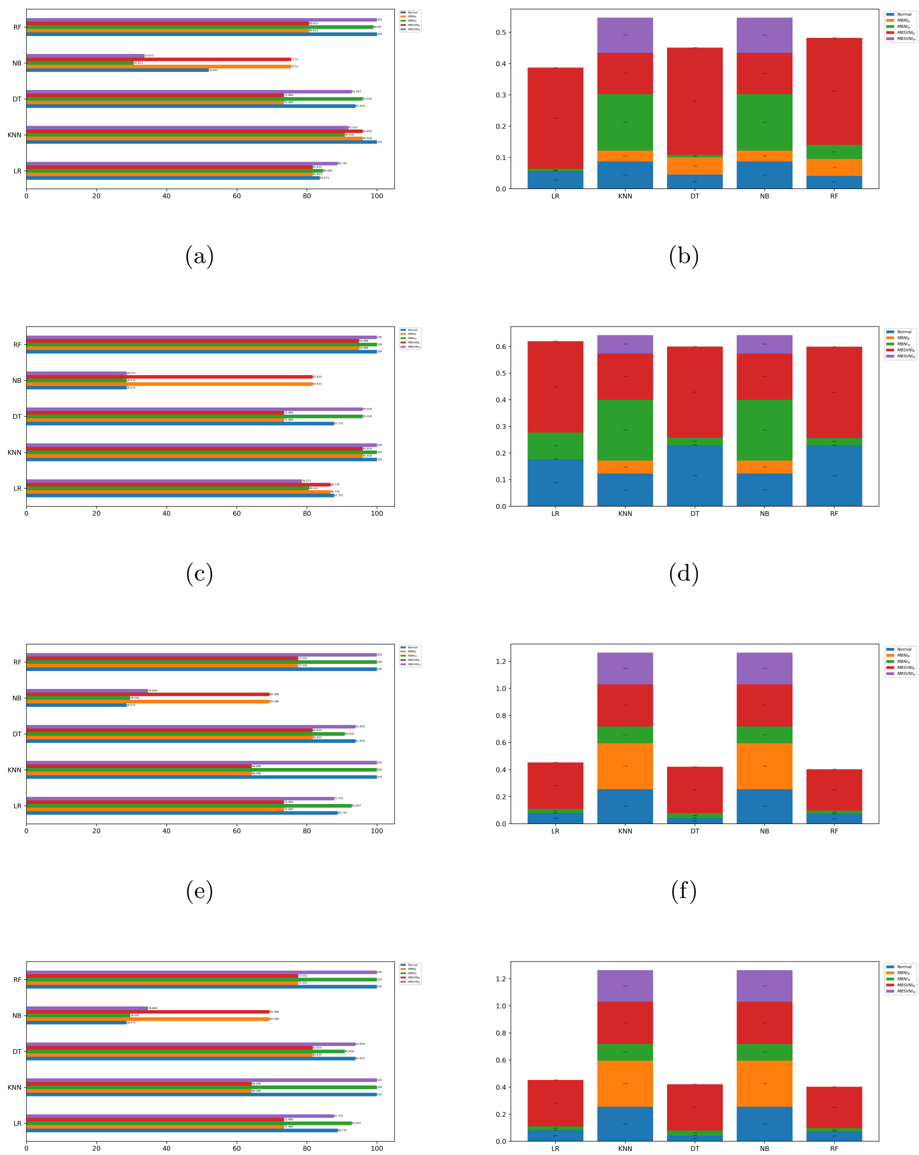
where $y_{ij} = 1$ if $y_i = y_j$ otherwise $y_{ij} = 0$. Figure 7 shows the error comparison analysis of classic KNN method and proposed KNN methods for digit classification dataset. Figure 2 indicates the process manner of the proposal.

4. Results and Discussion

We use hardware that supports the 11th Gen Intel(R) Core(TM) i5-1135G7 @ 2.40GHz 2.42GHz with 16 GB RAM capacity for the analysis. We render the use of Python and Sklearn packages for software [25] support. This article examines the Fingerprint Verification Competition (FVC) databases from 2000 [27] , 2002 [28], and 2004 [29]. Our various databases were collected in FVC2000 using the sensors Secure Desktop Scanner (300×300), TouchChip (256×364), DF-90 (448×478), and synthetic generation based on evolution (240×320), all with 500 dpi. FVC2002 uses three different scanners and the SFinGE synthetic generator to collect fingerprints: Identix TouchView II (388×374), Biometrika FX2000 (296×560), Precise Biometrics 100 SC (300×300), and SFinGE v2.51 (288×384) with a resolution of 500 dpi.

FVC2004 includes the first 100 fingers (800 images) of DB1, DB2, DB3, and DB4. TIF image format, 256 gray-level, uncompressed image resolution (which may vary slightly depending on the database), and 500 dpi. SD 302 [30] is a collection of distributions each containing a logical subset of the images collected for the N2N Fingerprint Challenge. SD 302a for instance only contains friction ridge imagery in Portable Network Graphics (PNG) encoding generated by the Challengers. The data collection was taken from 64.7% female participants, 35.0% male participants, and 0.3% who were not interested in revealing their gender. The labels A-H correspond to the Challenger types IDEMIA, Advanced Optical Systems, Green Bit, Cornell University, Jenetric, Touchless Biometric Systems, Undisclosed, and Clarkson University. The challengers brought their fingerprint capture devices, as well as any computer hardware and software required for fingerprint capture. Challenger wrote or obtained all software used. The challengers brought their fingerprint capture devices, as well as any computer hardware and software required for fingerprint capture. The Challenger obtained all of the software used. Each Challenger was given no more than 5 minutes with a study participant, for a total of 40 minutes of Challenger collection time. Challengers were required to submit a unique image for each finger that could be used with a commercial off-the-shelf (COTS) fingerprint identification system. Challengers could capture more than one finger at a time, but all images must depict only one finger per image. The accuracy of factors is used in this article to evaluate the performance of a fingerprint classification system using machine learning methods (RF, DT, LR, NB, KNN). Table 2 depicts the formulation of metrics formulae.

In general, resize level is an application option to achieve a sustainable result. In this case, the minimum size level = 5 and the maximum size level = 25. Even resize level 64 produces



(a) (b)

(c) (d)

(e) (f)

(g) (h)

FIGURE 8. Resize level = 5, h = 3 comparative analysis; (a) SD302 accuracy (b) SD302 error rate (c) FVC2000 accuracy (d) FVC2000 error rate (e) FVC2002 accuracy (f) FVC2002 error rate (g) FVC2004 accuracy (h) FVC2004 error rate

the same result as size level 25. As a result, with a large size and a more reliable size of 25, the output duration is reduced resize level 64. As a result of the analysis, we chose 5 and 25 as resize factors.

Figure 8 represents the results of the classical method and the proposed neutrosophic basic methods for the obtained hyperparameters. Model selection is based on the consideration of

Table 2. Metrics

Metrics	**Formula**
Root mean square	$\sqrt{\sum_{i=1}^{n} \frac{(\hat{y_i}-y_i)^2}{n}}$
Precision	$\frac{True\ Positive}{True\ Positive+False\ Positive}$
Recall	$\frac{True\ Positive}{True\ Positive+False\ Negative}$
F1-score	$2.\frac{Precision.Recall}{Precision+Recall}$
Accuracy	$\frac{True\ Positive+True\ Negative}{True\ Positive+True\ Negative+False\ Positive+False\ Negative}$

a high level of accuracy while having a low number of errors. $MBNI_{SI}$ and $MBSVNI_{SI}$ perform very well in the SD302 database for the LR algorithm compared to the classical LR algorithm. $MBSVNI_{SI}$, in particular, outperforms others in terms of accuracy and error. The proposed KNN method 5% is less than the classical method for SD302 data, but when considering the error rate, the classical method is extremely large. It indicates that the reliable comfort is lower, but when considering the $MBNI_B$ error rate, it acknowledges that the accuracy level is higher than that of the classical method. The $MBNI_{SI}$ based DT method performs admirably in terms of accuracy and error values. This indicates that $MBNI_{SI}$ is considered for classification while the DT algorithm uses a smaller image size. For the NB algorithm, both the self-intensity methods $MBNI_{SI}$ and $MBSVNI_{SI}$ outperform the classical NB method, but $MBSVNI_{SI}$ has higher error values. While the comparison of these two methods, the $MBNI_{SI}$ method NB algorithm is preferable because of its high accuracy rate and low error rate, and the $MBSVNI_{SI}$ method RF algorithm performs better than other methods.

In the FVC2000 dataset, proposed self-intensity methods ($MBNI_{SI}$, $MBSVNI_{SI}$) perform very well compared to classical NB classification, and the proposed binary methods ($MBNI_B$, $MBSVNI_B$) perform very well compared to classical DT classification with a lower error rate. The proposed binary methods ($MBNI_B$,$MBSVNI_B$) perform equally well in accuracy measures for the RF and KNN algorithms. Based on their error values, the proposed binary method has a shorter error rate than the classical approach. The proposed method LR algorithm outperforms the classical LR algorithm for resizing level 5.

The FVC2002 and FVC2004 dataset performs in the same way as the FVC2000 dataset. Here also proposed $MBNI_{SI}$, $MBSVNI_{SI}$ method NB algorithm perform very well compared to classical NB classification, and the proposed binary methods $MBNI_B$, $MBSVNI_B$ perform very well compared to classical DT classification with a lower error rate and the proposed binary methods ($MBNI_B$, $MBSVNI_B$) outperform the RF and KNN algorithms in terms of accuracy and as well as error rate. The improvement over FVC2000 data is that the

Table 3. Result of the proposed methods

Data	Algorithm	Classic method	$MBNI_B$	$MBNI_{SI}$	$MBSVNI_B$	$MBSVNI_{SI}$
Resize = 5, h = 3						
SD302	LR	83.673±0.056	81.633±0.087	84.694±0.045	81.633±0.087	88.776±0.041
	KNN	100.0±0.0	95.918±0.034	90.816±0.055	95.918±0.034	91.837±0.054
	DT	93.878±0.008	73.469±0.181	95.918±0.008	73.469±0.181	92.857±0.045
	NB	52.041±0.324	75.51±0.132	30.612±0.341	75.51±0.132	33.673±0.342
	RF	100.0±0.0	80.612±0.112	98.98±0.002	80.612±0.112	100.0±0.0
FVC2000	LR	87.755±0.176	86.735±0.123	80.612±0.23	86.735±0.123	78.571±0.229
	KNN	100.0±0.0	95.918±0.049	100.0±0.0	95.918±0.049	100.0±0.0
	DT	87.755±0.1	73.469±0.228	95.918±0.027	73.469±0.228	95.918±0.027
	NB	28.571±0.343	81.633±0.174	28.571±0.343	81.633±0.174	28.571±0.343
	RF	100.0±0.0	94.898±0.069	100.0±0.0	94.898±0.069	100.0±0.0
FVC2002	LR	88.776±0.08	73.469±0.255	92.857±0.042	73.469±0.255	87.755±0.074
	KNN	100.0±0.0	64.286±0.34	100.0±0.0	64.286±0.34	100.0±0.0
	DT	93.878±0.029	81.633±0.123	90.816±0.038	81.633±0.123	93.878±0.023
	NB	28.571±0.343	69.388±0.311	29.592±0.341	69.388±0.311	34.694±0.306
	RF	100.0±0.0	77.551±0.234	100.0±0.0	77.551±0.234	100.0±0.0
FVC2004	LR	88.776±0.08	73.469±0.255	92.857±0.042	73.469±0.255	87.755±0.074
	KNN	100.0±0.0	64.286±0.34	100.0±0.0	64.286±0.34	100.0±0.0
	DT	93.878±0.029	81.633±0.123	90.816±0.038	81.633±0.123	93.878±0.023
	NB	28.571±0.343	69.388±0.311	29.592±0.341	69.388±0.311	34.694±0.306
	RF	100.0±0.0	77.551±0.234	100.0±0.0	77.551±0.234	100.0±0.0
Resize = 25, h=3						
SD302	LR	95.918±0.031	100.0±0.0	100.0±0.0	100.0±0.0	100.0±0.0
	KNN	100.0±0.0	100.0±0.0	100.0±0.0	100.0±0.0	100.0±0.0
	DT	92.857±0.016	90.816±0.064	90.816±0.034	90.816±0.064	90.816±0.034
	NB	53.061±0.322	96.939±0.016	42.857±0.193	96.939±0.016	50.0±0.175
	RF	100.0±0.0	100.0±0.0	100.0±0.0	100.0±0.0	100.0±0.0
FVC2000	LR	100.0±0.0	96.939±0.04	97.959±0.039	96.939±0.04	97.959±0.039
	KNN	100.0±0.0	100.0±0.0	100.0±0.0	100.0±0.0	100.0±0.0
	DT	100.0±0.0	85.714±0.121	100.0±0.0	85.714±0.121	100.0±0.0
	NB	28.571±0.343	84.694±0.154	52.041±0.283	84.694±0.154	53.061±0.281
	RF	100.0±0.0	100.0±0.0	100.0±0.0	100.0±0.0	100.0±0.0
FVC2002	LR	100.0±0.0	100.0±0.0	100.0±0.0	100.0±0.0	98.98±0.005
	KNN	100.0±0.0	100.0±0.0	100.0±0.0	100.0±0.0	100.0±0.0
	DT	98.98±0.005	76.531±0.246	94.898±0.021	76.531±0.246	94.898±0.021
	NB	35.714±0.335	89.796±0.086	65.306±0.419	89.796±0.086	87.755±0.121
	RN	100.0±0.0	100.0±0.0	100.0±0.0	100.0±0.0	100.0±0.0
FVC2004	LR	100.0±0.0	100.0±0.0	100.0±0.0	100.0±0.0	98.98±0.005
	KNN	100.0±0.0	100.0±0.0	100.0±0.0	100.0±0.0	100.0±0.0
	DT	98.98±0.005	76.531±0.246	94.898±0.021	76.531±0.246	94.898±0.021
	NB	35.714±0.335	89.796±0.086	65.306±0.419	89.796±0.086	87.755±0.121
	RF	100.0±0.0	100.0±0.0	100.0±0.0	100.0±0.0	100.0±0.0

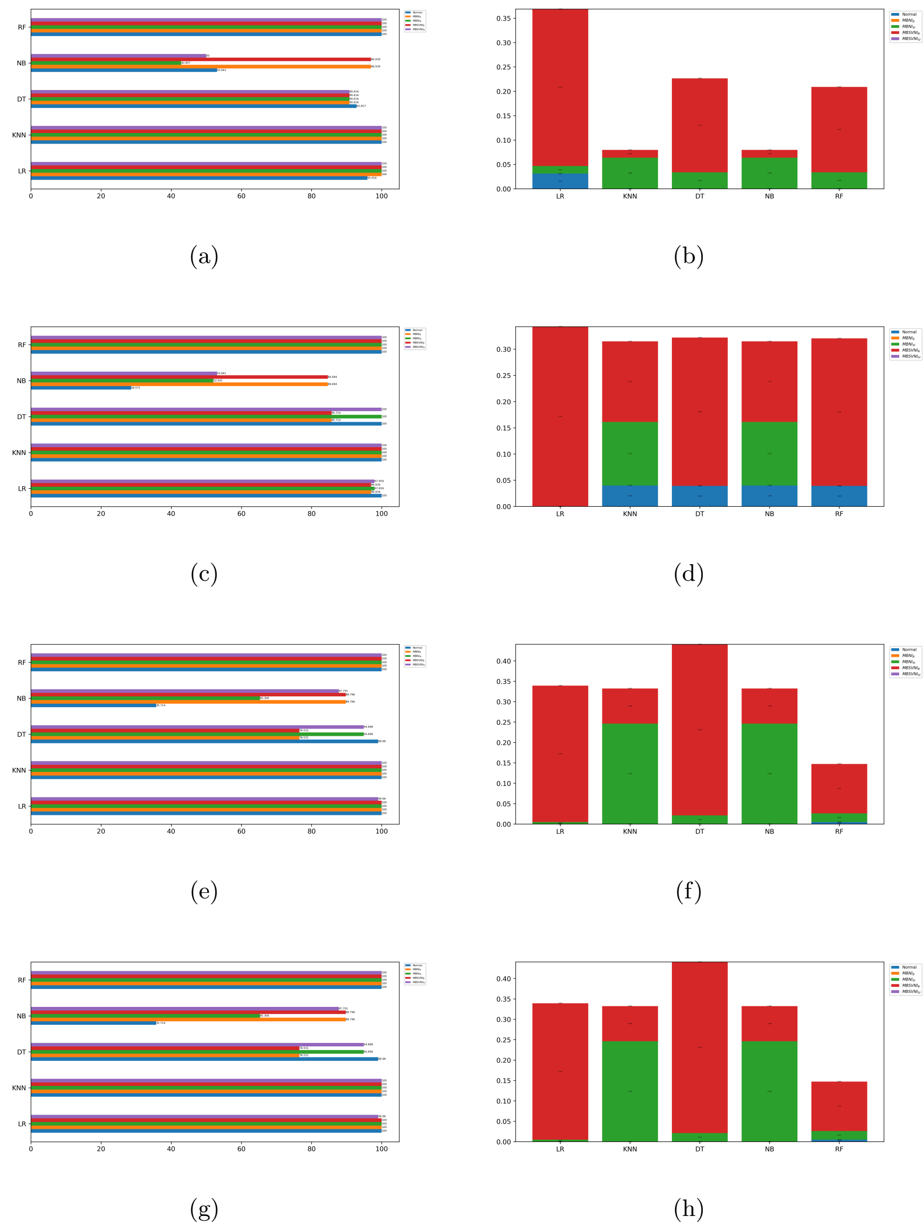


FIGURE 9. Resize level = 25, $h = 3$ comparative analysis; (a) SD302 accuracy (b) SD302 error rate (c) FVC2000 accuracy (d) FVC2000 error rate (e) FVC2002 accuracy (f) FVC2002 error rate (g) FVC2004 accuracy (h) FVC2004 error rate

LR model in FVC2000 is a failure model, whereas it is a successful model here. The sensor type is the underlying cause of these variations in accuracy; moreover, the article suggests that the proposed LR model is considerable if the scanner is an analysis factor.

For resize level 25, most of the proposed method algorithms perform similarly to the classic method algorithm, with the difference being the error rate. From Figure 9 proposal identify classical methods outperform the proposed method algorithm in some scenarios. When the NB algorithm fails on SD302 data, the proposed binary methods $MBNI_B$ and $MBSVNI_B$ achieve a 96% successive model. The traditional LR algorithm achieves successive scores, whereas the proposed method algorithm achieves the maximum point of the score. KNN accomplishes the maximum level of the score for FVC2000 data RF. When using the proposed binary method, NB improves the accuracy level. Except for FVC2000, the other datasets failed the DT algorithm the proposed DT performs well in FVC2000. The above discussion is based on Table 3 observations.

The current research scope of the proposed study is limited to the analysis of fingerprint images. In our study, various data sets were subjected to analysis. The results presented in this part instill a sense of belief within us. The strategies discussed in the related study mostly center around deep learning methodologies, as seen by the collective findings. Our primary objective is to enhance the performance of classical machine learning algorithms through the utilization of NS. The KNN model, as described, has superior performance compared to the other models discussed in the related work section. According to Adhinata [11], the maximum level of the score attainable by the KNN algorithm is 95%. However, the KNN model presented in this study achieved a perfect score of 100% through the utilization of machine learning methodologies. One notable benefit is its compatibility with both binary and self-intensity methodologies. The proposed project effectively implemented the LR and DT algorithms. In the study conducted by Kumar et al. [17], the best performance achieved was reported to be 96%. However, our research endeavors led to an enhancement in performance, resulting in a maximum achievement of 100%. Labati's [18] proposed NB algorithm achieves an accuracy rate of 52%. However, via our enhancements, we were able to significantly improve its performance, resulting in an accuracy rate of 89%. The proposal effectively improves the performance of machine learning algorithms. The classical technique column in Table 3 presents the methodology used by an ordinary machine learning algorithm.

5. Conclusion

This article proposes four new neutrosophic methods to classify fingerprint images. Furthermore, the hyperparameters is determined in order to classify the supervised algorithm. Our primary goal is to achieve the classification of fingerprint images without an individual's assistance. This technique allows researchers to classify fingerprint images for four different datasets without explicitly parameterizing the images. While low-range algorithms demonstrate LR accuracy of 5%, DT accuracy of 8%, and NB accuracy of 58%, high-range algorithms achieve LR accuracy of 5% and NB accuracy of 56%. However, alternative proposed method algorithms achieve higher levels of accuracy with a lower error rate. The proposal makes a significant improvement in the classification performance of the images. This technique will support us in automated, supervised classification in the manner of an AI system. This strategy will be applied in features to unsupervised and other supervised algorithms as well as, if possible, other applications. This article claims that the proposed method's first step will further impact the field of digital images and accomplish desired aims. In order to decide on pixel values and attempt to improve performance in the future, there is also a research gap. As part of our feature work, we will extend this concept to object recognition and other types of image data.

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Exploring Negative-Valued $\mathcal{N}$eutrosophic Structures in the Context of Subalgebras and Ideals in BF-algebras

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Abstract. This scholarly inquiry comprehensively examines Negative-Valued $\mathcal{N}$eutrosophic BF-subalgebras and Negative-Valued $\mathcal{N}$eutrosophic BF-ideals in the context of BF-algebras, aiming to scrutinize their intrinsic characteristics and reveal intricate interrelationships. Employing a systematic and rigorous approach, this study significantly enhances our understanding of these elements within the broader context of algebraic structures, serving as a cornerstone for the advancement of mathematical knowledge in this area and providing a robust framework for future investigations. The findings offer valuable insights, laying the groundwork for further research in this specialized domain and contributing significantly to ongoing academic discourse. By conducting a thorough examination of Negative-Valued $\mathcal{N}$eutrosophic BF-subalgebras and Negative-Valued $\mathcal{N}$eutrosophic BF-ideals, this study facilitates a deeper understanding within the broader landscape of algebraic structures and plays a pivotal role in advancing mathematical knowledge in this specialized field, fostering continued exploration and innovation.

Keywords: BF-algebra; Negative-Valued $\mathcal{N}$eutrosophic Structure; Negative-Valued $\mathcal{N}$eutrosophic BF-Subalgebra; Negative-Valued $\mathcal{N}$eutrosophic BF-ideal.

1. Introduction

A groundbreaking shift in set theory, known as the introduction of fuzzy sets by Zadeh[16] in 1995, marked a significant turning point. In 2002, Neggers and Kim[12] introduced the innovative concept of B-algebra, leading to a multitude of consequential outcomes. Walendziak[15] further extended this framework to formulate BF-algebra, a more general version of B-algebra, and conducted an

extensive investigation into the properties of ideals and normal ideals within BF-algebra.

Atanassov[4] made a significant contribution by introducing the notion of the measure of non-inclusion or falsity (f) and providing an interpretation of intuitionistic fuzzy sets. The term "$\mathcal{N}$eutrosophic", signifying neutrality in thought, was coined by Smarandache, where the primary differentiation is fuzzy/intuitionistic fuzzy logic/sets and $\mathcal{N}$eutrosophic logic/sets lies in the introduction of a third/neutral component. He pioneered the introduction of an autonomous element, representing the level of ambiguity or neutrality, established the $\mathcal{N}$eutrosophic set relies on a triad of constituents, namely (t, i, f), which correspond to authenticity, ambiguity, and falsification. This demonstrates its practical applicability in diverse sectors [1, 2, 3, 8, 14]. Jun et al.[9] introduced a novel mapping characterized by negative-values and developed N-structures. Khan et al.[10] introduced the concept of $\mathcal{N}$eutrosophic N-Structure and employed it within the context of a semi-group. Additionally, Muralikrishna et al. [11] first introduced the concept of Structuere N-ideal within the context of BF-algebra.

Seok-Zun Song et al.[13] Pioneered the idea of $\mathcal{N}$eutrosophic N-ideal in BCK-algebras and conducted an extensive exploration of its various attributes, culminating in the establishment of characterizations for $\mathcal{N}$eutrosophic N-ideal. To set the stage for our discussion, we first provide definitions from [5,6,15] that are essential for the context of this paper.

2. Main contributions to this work

Introducing and extensively examining the concept of Negative-Valued $\mathcal{N}$eutrosophic BF-subalgebras and Negative-Valued $\mathcal{N}$eutrosophic BF-ideals in the context of BF-algebras.

Providing a thorough analysis of the inherent characteristics of Negative-Valued $\mathcal{N}$eutrosophic BF-Subalgebras and Negative-Valued $\mathcal{N}$eutrosophic BF-ideals.

Elucidating the intricate relationships that exist among Negative-Valued $\mathcal{N}$eutrosophic BF-subalgebras and Negative-Valued $\mathcal{N}$eutrosophic BF-ideals.

Conducting a meticulous exploration of the unique properties associated with Negative-Valued $\mathcal{N}$eutrosophic BF-ideals.

Advancing the understanding of BF-algebras and broadening the utility of Negative-Valued $\mathcal{N}$eutrosophic BF-subalgebras and Negative-Valued $\mathcal{N}$ eutrosophic BF-ideals for managing uncertainty in Negative-valued $\mathcal{N}$eutrosophic soft sets.

3. Prerequisites

Notations: Throughout this article, we use the following notations.

TABLE 1

BF-algebra	$\mathcal{BFA}$
Negative-Valued $\mathcal{N}$eutrosophic Structure	$\mathcal{NNS}$
Negative-Valued $\mathcal{N}$eutrosophic BF-ideal	$\mathcal{NNI}$
Negative-Valued $\mathcal{N}$eutrosophic BF-subalgebra	$\mathcal{NNSA}$

Definition 3.1 (15)**.** A $\mathcal{BFA}$ is a structure $\mathrm{S} := (\mathrm{S} \neq \phi, \otimes, 0) \in \mathrm{K}(\tau)$

(I)$\mathrm{t}_1 \otimes \mathrm{t}_1 = 0, - - - - - - - - -(1)$

(II)$\mathrm{t}_1 \otimes 0 = \mathrm{t}_1, - - - - - - -(2)$

(III)$0 \otimes (\mathrm{t}_1 \otimes \mathrm{t}_2) = \mathrm{t}_2 \otimes \mathrm{t}_1, \forall \mathrm{t}_1, \mathrm{t}_2 \in \mathrm{S} - - - - - - -(3)$

Example 3.2 (15)**.** The set $(S = \{0, 1, 2, 3\}, \otimes, 0)$ having the composition table

TABLE 2

$\otimes$	0	1	2	3
0	0	1	2	3
1	1	0	3	0
2	2	3	0	2
3	3	0	2	0

is a $\mathcal{BFA}$.

Example 3.3 (15)**.** Let $S = (R, \otimes, 0)$where $\otimes$ is given by $t_1 \otimes t_2 = \begin{cases} t_1, \mathrm{if} t_2 = 0 \\ t_2, \mathrm{if} t_1 = 0 \\ 0, otherwise \end{cases}$

and set of real numbers(R) is a $\mathcal{BFA}$.

Example 3.4 (6)**.** The set $(S = \{0, 1, 2, 3\}, \otimes, 0)$ having the composition table

Table 3

$\otimes$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

is a $\mathcal{BFA}$.

Example 3.5 (15)**.** Let $S = [0, \infty), \otimes$is defined on S as $t_1 \otimes t_2 = |t_1 - t_2|, \forall t_1, t_2 \in S$ is a $\mathcal{BFA}$.

Note 3.6 (7)**.** *Let* $S = (R, \otimes, 0)$ *where* $\otimes$ *is defined as* $t_1 \otimes t_2 = \begin{cases} t_1, \text{if} t_2 = 0 \\ 0, \text{if} t_1 = 0, t_1 = t_2 \\ t_2 \otimes t_1, otherwise \end{cases}$

is not a $\mathcal{BFA}$.

Definition 3.7 (7, 11)**.** A relation '$\leq$ ' on S is a partial ordering satisfying

$(\forall t_1, t_2 \in S)$, $t_1 \leq t_2 \Leftrightarrow t_1 \otimes t_2 = 0$ $- - - - - - - - - (4)$

Note 3.8 (15)**.** *In any* $\mathcal{BFA}$, $S := (S \neq \phi, \otimes, 0)$, *the following holds:*

$(\forall \mathrm{t}_1 \in \mathrm{S})(0 \otimes (0 \otimes \mathrm{t}_1)) = \mathrm{t}_1 - - - - - - -(5)$

$(\forall \mathrm{t}_1, \mathrm{t}_2 \in \mathrm{S})(0 \otimes \mathrm{t}_1) = (0 \otimes t_2)$ *iff* $t_1 = t_2 - - - - - - -(6)$

$(\forall \mathrm{t}_1, \mathrm{t}_2 \in \mathrm{S})(\mathrm{t}_2 \otimes t_1 = 0)$, *if* $t_1 \otimes t_2 = 0 - - - - - - - (7)$

Definition 3.9 (15)**.** Consider a $\mathcal{BFA}$, $S := (\mathrm{S} \neq \phi, \otimes, 0)$. $M(\neq \phi) \subseteq \mathrm{S}$ is said to be a subalgebra if $\mathrm{t}_1 \otimes \mathrm{t}_2 \in \mathrm{M}, \forall \mathrm{t}_1, \mathrm{t}_2 \in \mathrm{M}$. $- - - - - - - -(8)$

Note 3.10 (15)**.** *It is clear that if M is a subalgebra of S then* $0 \in \mathrm{M}$.

Example 3.11 (15)**.** Consider a $\mathcal{BFA}$, $(\mathrm{S} = \{0, 1, 2, 3\}, \otimes, 0)$ having the composition table

Table 4

$\otimes$	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	1	0	1
3	3	1	1	0

The set $\mathrm{M} = \{0, 1\}$ is a subalgebra of S.

Definition 3.12 (15)**.** Consider a $\mathcal{BFA}$, $S := (S \neq \phi, \otimes, 0)$. $M(\neq \phi) \subseteq S$ is said to be ideal of S if $0 \in M$ $- - - -$ (9)

$(\forall t_1, t_2 \in S)(t_1 \otimes t_2 \in M, t_2 \in M \Rightarrow t_1 \in M)$ - - - (10)

Example 3.13 (15)**.** Consider a $\mathcal{BFA}$, (S $= \{0, 1, 2, 3\}, \otimes$, 0) having the composition table 2

Clearly, $\{0\}$ and S are ideals of S and M $= \{0, 3\} \subseteq$ S is not an ideal of S. ($1 \otimes 3 = 0 \in$ M and $3 \in$ M $\Rightarrow 1 \notin$ M)

4. **Negative-Valued $\mathcal{N}$eutrosophic concept on BF-algebra:**

Represent by $\gamma(\text{S}, [-1, 0])$ be the family of mappings from a set S to $[-1, 0]$ (called, A Negative-Valued mapping on S). A $\mathcal{NNS}$ is denoted by (S, g) of S and g is a Negative-Valued mapping on S. A $\mathcal{NNS}$ over a universe $S \neq \phi$ (see [9]) is

$S_{\mathcal{N}} = \frac{S}{(\aleph_{\mathcal{N}}, I_{\mathcal{N}}, \Psi_{\mathcal{N}})} = \left\{ \frac{t_1}{\aleph_{\mathcal{N}}(t_1), I_{\mathcal{N}}(t_1), \Psi_{\mathcal{N}}(t_1)} / t_1 \in S \right\}$

where $\aleph_{\mathcal{N}}$,$I_{\mathcal{N}}$ and $\Psi_{\mathcal{N}}$ are Negative-Valued mappings on S termed as the "Non-positive truth membership" mapping, the "non-positive indeterminacy membership" mapping and the "non-positive falsity membership" mapping, resp., on S.

A $\mathcal{NNS}$, $S_{\mathcal{N}}$ over S holds:

$(\forall \text{t}_1 \in \text{S})(-3 \leq \aleph_N(\text{t}_1) + \text{I}_N(\text{t}_1) + \Psi_N(\text{t}_1) \leq 0)$

Let us represent $\forall t_1, t_2 \in S$ **,** $\text{t}_1 \text{v} \text{t}_2$ **denotes** $\max\{\text{t}_1, \text{t}_2\}$ **and** $t_1 \wedge t_2$ **denotes** $\min\{t_1, t_2\}$

Definition 4.1. A $\mathcal{NNS}$, $\text{S}_{\mathcal{N}}$ over a $\mathcal{BFA}$, $S := (S \neq \phi, \otimes, 0)$, is a $\mathcal{NNSA}$ if

$i)\aleph_{\mathcal{N}}(t_1 \otimes t_2) \leq \vee \{\aleph_{\mathcal{N}}(t_1), \aleph_{\mathcal{N}}(t_2)\} \, (\forall t_1, t_2 \in S) - - - -(11)$

$ii)I_{\mathcal{N}}(t_1 \otimes t_2) \geq \wedge \{I_{\mathcal{N}}(t_1), I_{\mathcal{N}}(t_2)\} \, (\forall t_1, t_2 \in S) - - - -(12)$

$\text{iii})\Psi_{\mathcal{N}}(t_1 \otimes t_2) \leq \vee \{\Psi_{\mathcal{N}}(t_1), \Psi_{\mathcal{N}}(t_2)\} \, (\forall t_1, t_2 \in S) - - - \, (13)$

Example 4.2. Consider a $\mathcal{BFA}$, (S $= \{0, 1, 2, 3\}$,$\otimes$, 0) having the table 3.

The $\mathcal{NNSA}$ of S is

$S_N = \left\{ \frac{0}{-0.8, -0.1, -0.8}, \frac{1}{-0.8, -0.8, -0.4}, \frac{2}{-0.8, -0.9, -0.4}, \frac{3}{-0.8, -0.9, -0.6} \right\}$

Definition 4.3. A $\mathcal{NNS}$, $S_{\mathcal{N}}$ over a $\mathcal{BFA}$, $S := (S \neq \phi, \otimes, 0)$ is a $\mathcal{NNI}$ of S if

$(i)\aleph_{\mathcal{N}}(0) \leq \aleph_{\mathcal{N}}(t_1) \leq \vee \{\aleph_{\mathcal{N}}(t_1 \otimes t_2), \aleph_{\mathcal{N}}(t_2)\} \, (\forall t_1, t_2 \in S) - - - -(14)$

$(ii)I_{\mathcal{N}}(0) \geq I_{\mathcal{N}}(t_1) \geq \wedge \{I_{\mathcal{N}}(t_1 \otimes t_2), I_{\mathcal{N}}(t_2)\} \, (\forall t_1, t_2 \in S) - - - - - \, (15)$

$(iii)\Psi_{\mathcal{N}}(0) \leq \Psi_{\mathcal{N}}(t_1) \leq \vee \{\Psi_{\mathcal{N}}(t_1 \otimes t_2), \Psi_{\mathcal{N}}(t_2)\} \, (\forall t_1, t_2 \in S) - - - \, (16)$

Example 4.4. Consider a $\mathcal{BFA}$, (S $= \{0, 1, 2, 3\}$,$\otimes$, 0) having the table 3.

The $\mathcal{NNI}$ of S is

$S_{\mathcal{N}} = \left\{ \frac{0}{-0.7, -0.1, -0.8}, \frac{1}{-0.2, -0.8, -0.4}, \frac{2}{-0.6, -0.9, -0.4}, \frac{3}{-0.2, -0.9, -0.6} \right\}$

Proposition 4.5. *If $S_{\mathcal{N}}$ is a $\mathcal{NNI}$ over a $\mathcal{BFA}$, $S := (S \neq \phi, \otimes, 0)$ with $\mathrm{t}_1 \leq \mathrm{t}_2$, $\forall \mathrm{t}_1, \mathrm{t}_2 \in \mathrm{S}$ then*

(i)$\aleph_{\mathcal{N}}(t_1) \leq \aleph_{\mathcal{N}}(t_2) \forall \mathrm{t}_1, \mathrm{t}_2 \in \mathrm{S}$, *i.e* $\aleph_{\mathcal{N}}$ *is order preserving.- - - -(17)*

(ii)$\mathrm{I}_{\mathcal{N}}(t_1) \geq I_{\mathcal{N}}(t_2) \forall \mathrm{t}_1, \mathrm{t}_2 \in \mathrm{S}$, *i.e* $I_{\mathcal{N}}$ *is order reserving.- - - - - (18)*

(iii)$\Psi_{\mathcal{N}}(t_1) \leq \Psi_{\mathcal{N}}(t_2) \forall \mathrm{t}_1, \mathrm{t}_2 \in \mathrm{S}$, *i.e* $\Psi_{\mathcal{N}}$ *is order preserving.- - - -(19)*

Proof.

Given $S_{\mathcal{N}}$ is a $\mathcal{NNI}$ over a $\mathcal{BFA}$, $S := (S \neq \phi, \otimes, 0)$ with $\mathrm{t}_1 \leq \mathrm{t}_2$, $\forall \mathrm{t}_1, \mathrm{t}_2 \in \mathrm{S}$

$\Rightarrow \mathrm{Since} \mathrm{t}_1 \leq \mathrm{t}_2 \Rightarrow \mathrm{t}_1 \otimes \mathrm{t}_2 = 0 (\mathrm{by}(4))$

To prove i) : $S_{\mathcal{N}}$ is a $\mathcal{NNI}$

$\Rightarrow \aleph_{\mathcal{N}}(t_1) \leq \vee\{\aleph_{\mathcal{N}}(t_1 \otimes t_2), \aleph_{\mathcal{N}}(t_2)\}$ *(by(14))*

$\Rightarrow \aleph_{\mathcal{N}}(t_1) \leq \vee\{\aleph_{\mathcal{N}}(0), \aleph_{\mathcal{N}}(t_2)\}$

$\Rightarrow \aleph_{\mathcal{N}}(t_1) \leq \aleph_{\mathcal{N}}(t_2)$ *(by(14))*

$\Rightarrow \aleph_{\mathcal{N}}$ *is order preserving.*

To prove ii) : $S_{\mathcal{N}}$ is a $\mathcal{NNI}$

$\Rightarrow \mathrm{I}_{\mathcal{N}}(t_1) \geq \wedge\{\mathrm{I}_{\mathcal{N}}(t_1 \otimes t_2), \mathrm{I}_{\mathcal{N}}(t_2)\}$ *(by(15))*

$\Rightarrow \mathrm{I}_{\mathcal{N}}(t_1) \geq \wedge\{\mathrm{I}_{\mathcal{N}}(0), \mathrm{I}_{\mathcal{N}}(t_2)\}$

$\Rightarrow \mathrm{I}_{\mathcal{N}}(t_1) \geq \mathrm{I}_{\mathcal{N}}(t_2)$ *(by(15))*

$\Rightarrow \mathrm{I}_{\mathcal{N}}$ *is order reserving.*

To prove iii) : $S_{\mathcal{N}}$ is a $\mathcal{NNI}$

$\Rightarrow \Psi_{\mathcal{N}}(t_1) \leq \vee\{\Psi_{\mathcal{N}}(t_1 \otimes t_2), \Psi_{\mathcal{N}}(t_2)\}$ *(by(16))*

$\Rightarrow \Psi_{\mathcal{N}}(t_1) \leq \vee\{\Psi_{\mathcal{N}}(0), \Psi_{\mathcal{N}}(t_2)\}$

$\Rightarrow \Psi_{\mathcal{N}}(t_1) \leq \Psi_{\mathcal{N}}(t_2)$ *(by(16))*

$\Rightarrow \Psi_{\mathcal{N}}$ *is order preserving.*

Theorem 4.6. *If $S_{\mathcal{N}}$ is a $\mathcal{NNI}$ over a $\mathcal{BFA}$, $S := (S \neq \phi, \otimes, 0)$ then $S_{\mathcal{N}}$ is a $\mathcal{NNSA}$ of S.*

Proof.

Let $S_{\mathcal{N}}$ be a $\mathcal{NNI}$ of S, $\forall \mathrm{t}_1, \mathrm{t}_2 \in \mathrm{S}$

$\Rightarrow \aleph_{\mathcal{N}}(0) \leq \aleph_{\mathcal{N}}(t_1) \leq \vee\{\aleph_{\mathcal{N}}(t_1 \otimes t_2), \aleph_{\mathcal{N}}(t_2)\}$ **(by (14))**

$\Rightarrow \mathrm{I}_{\mathcal{N}}(0) \geq \mathrm{I}_{\mathcal{N}}(t_1) \geq \wedge\{\mathrm{I}_{\mathcal{N}}(t_1 \otimes t_2), \mathrm{I}_{\mathcal{N}}(t_2)\}$**(by (15))**

$\Rightarrow \Psi_{\mathcal{N}}(0) \leq \Psi_{\mathcal{N}}(t_1) \leq \vee\{\Psi_{\mathcal{N}}(t_1 \otimes t_2), \Psi_{\mathcal{N}}(t_2)\}$**(by (16))**

Put $t_1 = t_1 \otimes t_2$ in (14)

$\Rightarrow \aleph_{\mathcal{N}}(t_1 \otimes t_2) \leq \vee\{\aleph_{\mathcal{N}}(t_1 \otimes t_2 \otimes t_2), \aleph_{\mathcal{N}}(t_2)\}$

$\Rightarrow \aleph_{\mathcal{N}}(t_1 \otimes t_2) \leq \vee\{\aleph_{\mathcal{N}}(t_1), \aleph_{\mathcal{N}}(t_2)\} (by(1)\&(2))$

Similarly we can prove for $I_{\mathcal{N}}$ and $\Psi_{\mathcal{N}}$ also Hence, $S_{\mathcal{N}}$ is a $\mathcal{NNSA}$ of S

Note 4.7. *The Converse of the above theorem need not be true.*

Example 4.8. Suppose we have a $\mathcal{BFA}$[5], $(\mathrm{S} = \{0,1,2\}, \otimes, 0)$ having the Composition table

TABLE 5

$\otimes$	0	1	2
0	0	1	2
1	1	0	0
2	2	0	0

The $\mathcal{NNS}$ of S is,

$S_{\mathcal{N}} = \left\{ \frac{0}{-0.5,0,-0.9}, \frac{1}{-0.5,0,0}, \frac{2}{0,0,-0.5} \right\}$

is not a $\mathcal{NNI}$ but $\mathcal{NNSA}$.

Since $\aleph_{\mathcal{N}}(t_1) = \aleph_{\mathcal{N}}(2) = 0 \not\leq \vee\{\aleph_{\mathcal{N}}(2 \otimes 1) = -0.5, \aleph_{\mathcal{N}}(1) = -0.5\}$

The following theorem is an adequate condition for $\mathcal{NNSA}$ to be $\mathcal{NNI}$.

Theorem 4.9. *If S_N be a $\mathcal{NNSA}$ over a $\mathcal{BFA}$ $S := (S \neq \phi, \otimes, 0)$ with $t_1 \otimes t_2 \leq t_3, \forall \mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3 \in$ S and*

$\aleph_{\mathcal{N}}(t_1) \leq \vee\{\aleph_{\mathcal{N}}(t_2), \aleph_{\mathcal{N}}(t_3)\}$ $(\forall \mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3 \in \mathrm{S})$

$I_{\mathcal{N}}(t_1) \geq \wedge\{I_{\mathcal{N}}(t_2), I_{\mathcal{N}}(t_3)\}$ $(\forall \mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3 \in \mathrm{S})$

$\Psi_{\mathcal{N}}(t_1) \leq \vee\{\Psi_{\mathcal{N}}(t_2), \Psi_{\mathcal{N}}(t_3)\}$ $(\forall \mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3 \in \mathrm{S})$

then $S_{\mathcal{N}}$ is a $\mathcal{NNI}$ of S

Proof. *Let $S_{\mathcal{N}}$ be a $\mathcal{NNSA}$ of S with $t_1 \otimes t_2 \leq t_3, \forall t_1, t_2, t_3 \in S$*

$\Rightarrow \aleph_{\mathcal{N}}(t_1 \otimes t_2) \leq \vee\{\aleph_{\mathcal{N}}(t_1), \aleph_{\mathcal{N}}(t_2)\}$ *(by (11))*

Put $\mathrm{t}_1 = \mathrm{t}_2$

$\Rightarrow \aleph_{\mathcal{N}}(t_1 \otimes t_1) \leq \vee\{\aleph_{\mathcal{N}}(t_1), \aleph_{\mathcal{N}}(t_1)\}$

$\Rightarrow \aleph_{\mathcal{N}}(0) \leq \aleph_{\mathcal{N}}(t_1)$ *(by(1))*

and $\aleph_{\mathcal{N}}(t_1) \leq \vee\{\aleph_{\mathcal{N}}(t_1 \otimes t_2), \aleph_{\mathcal{N}}(t_2)\} \Leftrightarrow \aleph_{\mathcal{N}}(t_1) \leq \vee\{\aleph_{\mathcal{N}}(t_3), \aleph_{\mathcal{N}}(t_2)\}$ *(by(17))*

and $\mathrm{I}_{\mathcal{N}}(t_1 \otimes t_2) \geq \wedge\{\mathrm{I}_{\mathcal{N}}(t_1), I_{\mathcal{N}}(t_2)\}$ *(by(12))*

Put $\mathrm{t}_1 = \mathrm{t}_2$

$\Rightarrow \mathrm{I}_{\mathcal{N}}(t_1 \otimes t_1) \leq \wedge\{\mathrm{I}_{\mathcal{N}}(t_1), I_{\mathcal{N}}(t_1)\}$

$\Rightarrow \mathrm{I}_{\mathcal{N}}(0) \leq I_{\mathcal{N}}(t_1)$ *(by(1))*

and $\mathrm{I}_{\mathcal{N}}(t_1) \geq \wedge\{\mathrm{I}_{\mathcal{N}}(t_1 \otimes t_2), I_{\mathcal{N}}(t_2)\} \Leftrightarrow I_{\mathcal{N}}(t_1) \geq \wedge\{I_{\mathcal{N}}(t_3), I_{\mathcal{N}}(t_2)\}$ *(by(18))*

Similarly, we can prove for $\Psi_{\mathcal{N}}$ also.

Hence $S_{\mathcal{N}}$is a $\mathcal{NNI}$ of S.

Theorem 4.10. *If $S_{\mathcal{N}}$ is a $\mathcal{NNI}$ over a $\mathcal{BFA}$, $S := (S \neq \phi, \otimes, 0)$ with $t_1 \otimes t_2 \leq t_3, \forall t_1, t_2, t_3 \in S$ then i)$\aleph_{\mathcal{N}}(t_1) \leq \vee\{\aleph_{\mathcal{N}}(t_2), \aleph_{\mathcal{N}}(t_3)\}$*

ii)$I_{\mathcal{N}}(t_1) \geq \wedge\{I_{\mathcal{N}}(t_2), I_{\mathcal{N}}(t_3)\}$

$iii)\Psi_{\mathcal{N}}(t_1) \leq \vee\{\Psi_{\mathcal{N}}(t_2), \Psi_{\mathcal{N}}(t_3)\}$

Proof. *Given* $\mathrm{t}_1 \otimes \mathrm{t}_2 \leq \mathrm{t}_3, \forall \mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3 \in S$

To prove (i)*:*$S_{\mathcal{N}}$ *is* $\mathcal{NNI}$

$\Rightarrow \aleph_{\mathcal{N}}(0) \leq \aleph_{\mathcal{N}}(t_1) \leq \vee\{\aleph_{\mathcal{N}}(t_1 \otimes t_2), \aleph_{\mathcal{N}}(t_2)\}$ *(by(14))*

$\Rightarrow \aleph_{\mathcal{N}}(t_1) \leq \vee\{\aleph_{\mathcal{N}}(t_3), \aleph_{\mathcal{N}}(t_2)\}$*(by Proposition 4.5)*

To prove (ii)*:*$S_{\mathcal{N}}$ *is* $\mathcal{NNI}$

$\Rightarrow I_{\mathcal{N}}(0) \geq I_{\mathcal{N}}(t_1) \geq \wedge\{I_{\mathcal{N}}(t_1 \otimes t_2), I_{\mathcal{N}}(t_2)\}$ *(by(15))*

$\Rightarrow I_{\mathcal{N}}(t_1) \geq \wedge\{I_{\mathcal{N}}(t_3), I_{\mathcal{N}}(t_2)\}$ *(by Proposition 4.5)*

To prove (iii)*:*$S_{\mathcal{N}}$ *is* $\mathcal{NNI}$

$\Rightarrow \Psi_{\mathcal{N}}(0) \leq \Psi_{\mathcal{N}}(t_1) \leq \vee\{\Psi_{\mathcal{N}}(t_1 \otimes t_2), \Psi_{\mathcal{N}}(t_2)\}$ *(by(16))*

$\Rightarrow \Psi_{\mathcal{N}}(t_1) \leq \vee\{\Psi_{\mathcal{N}}(t_3), \Psi_{\mathcal{N}}(t_2)\}$ *(by Proposition 4.5)*

Note 4.11. *Applying induction on n and from the Theorem 4.10, we have*

Theorem 4.12. *If* $S_{\mathcal{N}}$ *is a* $\mathcal{NNI}$ *over a* $\mathcal{BFA}$*,* $S := (S \neq \phi, \otimes, 0)$ *then for any* $\mathrm{p}, \mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3, ..., \mathrm{a}_\mathrm{n} \in \mathrm{S}$ *and*

$(....((\mathrm{p} \otimes \mathrm{a}_1) \otimes \mathrm{a}_2) \otimes ....)) \otimes \mathrm{a}_\mathrm{n} = 0$ *implies*

$i)\aleph_{\mathcal{N}}(p) \leq \vee\{\aleph_{\mathcal{N}}(a_1), \aleph_{\mathcal{N}}(a_2), ..., \aleph_{\mathcal{N}}(a_n)\}$

$ii)I_{\mathcal{N}}(p) \geq \wedge\{I_{\mathcal{N}}(a_1), I_{\mathcal{N}}(a_2), ..., I_{\mathcal{N}}(a_n)\}$

$iii)\Psi_{\mathcal{N}}(p) \leq \vee\{\Psi_{\mathcal{N}}(a_1), \Psi_{\mathcal{N}}(a_2), ..., \Psi_{\mathcal{N}}(a_n)\}$

5. Conclusions:

The investigation of $\mathcal{NNSA}$ and $\mathcal{NNI}$ within the context of $\mathcal{BFA}$ has led to several conclusions.

Firstly, the study has provided a thorough analysis of the inherent characteristics of $\mathcal{NNSA}$ and $\mathcal{NNI}$. This analysis has helped in understanding the properties and behaviors of these structures within BF-algebras.

Secondly, the investigation has revealed the intricate relationships that exist between $\mathcal{NNSA}$ and $\mathcal{NNI}$. By exploring these relationships, researchers have gained insights into how these structures interact and influence each other within the broader context of algebraic structures.

Furthermore, the study has delved into the unique properties associated with $\mathcal{NNI}$. By examining these properties, researchers have enhanced their understanding of $\mathcal{NNI}$ and its potential applications in managing uncertainty in Negative-Valued $\mathcal{N}$eutrosophic soft sets.

Overall, the investigation of $\mathcal{NNSA}$ and $\mathcal{NNI}$ within the context of $\mathcal{BFA}$ has contributed significantly to the field. It has expanded our comprehension of these

structures and their relationships, paving the way for further research and advancements in this specialized domain of mathematics

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A Neutrosophic Bézier Curve Model using Control Points

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Abstract. Uncertain and ambiguous data is usually presented as a data point. When dealing with uncertain data, it is challenging to deal with incompleteness, imprecision, and incertitude; therefore, various mathematical models were developed to resolve issues concerning uncertainty points. To overcome this difficulty, a mathematical model with redefined control points characterised by three important components of the neutrosophic set was produced. The neutrosophic set can then be translated by creating models based on the neutrosophic theory and its relationships to produce control points. Hence, this paper represents a curve generated by combining neutrosophic control points with the Bézier basic functions using their relation as a Neutrosophic Bézier Curve. With an illustration example, we show how to visualise neutrosophic data sets into Neutrosophic Bézier Curve and their relationship.

Keywords: Neutrosophic; Spline curve; Bézier curve; control point, data visualization

1. Introduction

Computer Aided Graphic Design (CAGD) is to create three-dimensional curve models and visualisations. The design process is about identifying and solving environmental problems to realise the need to improve lifestyle in the age of technology. Work on this topic has been going on for several years [35]. Data points are also collected from physical objects or environments. Once data is collected using various specialised tools and procedures, such as echo sounding and data error, some information loss or inaccuracy occurs. Spline modelling is one of the simplest and most powerful three-dimensional object creation methods. Spline modelling also enables users to create designs faster than conventional modelling techniques. The basis of this new method for designing the Bézier Curve has been introduced by [2] This method allows the curve to fit the control polygon by moving a point of the Bézier Curve. The framework of this new method for designing the Bézier Curve has been constructed.

Defining and interpreting accurate data from actual events and scenarios dealing with ambiguous data is challenging. [3] discovered a fuzzy set to handle uncertainty data in 1965 as an extension of the classical notion set. These are well studied and documented in the literature [3–9]. Then, [10, 11] introduced an intuitionistic fuzzy set in 1983. On the other hand, the data point obtained is difficult to understand as it is affected by noise and certainty. Several studies and reviews have been undertaken to explore the uncertainty problem by considering data modelling and data reduction problems, as stated in [13–15]. To represent real data points, curves and the surface is necessary, as mentioned in [12]. Hence, using the concept of fuzzy set theory, fuzzy number, fuzzy point and fuzzy relation, a new data point defined as fuzzy point relation has represented uncertainty data. Therefore, considerable research has been conducted to explore the Fuzzy Spline Model to visualise a fuzzy data set geometrically [16]. [17] describe Type-2 Fuzzy Bézier Curve Modeling and [18] implement the Interval Type-2 Fuzzy Logic System Model in Measuring the Index Value of the Underground Economy in Malaysia from 2001 to 2010. Later, [19] introduced a new concept of intuitionistic fuzzy sets with geometric modelling called the Intuitionistic Fuzzy Bézier Curve model using intuitionistic sets. They develop a spline model for data problems that involve an intuitionistic set. The intuitionistic data problem set was converted into a point relationship and blended with a spline to be visualised geometrically by curve and surface.

However, previous studies on Fuzzy Bézier Curve blended with spline functions only involve fuzzy and intuitionistic sets, while a problem involving a neutrosophic set has not yet been extensively developed. Therefore, this paper discusses and introduces a new model of the Neutrosophic Bézier Curve to represent data visualised with spline functions in geometric modelling. Neutrosophic sets (NSs) proposed by [20–25] which is a generalisation of fuzzy sets and intuitionistic fuzzy set, is a powerful tool to deal with incomplete, indeterminate and inconsistent information which exist in the real world. Neutrosophic set theory is a fabulous mathematical technique that can be applied to various fields. Uncertain data was analysed and visualised using a new type of geometric modelling more on neutrosophic. In addition, [26] conducted research on neutrosophic data problems to produce the Bézier curve and Bézier surface. However, [26] only used Neutrosophic Data with Basic Spline to generate three distinct curves. They did not clearly explain the relationship between neutrosophic point relation and basic spline. Furthermore, they did not introduce or identify the CAGD characteristics. As a result, this research did not meet the properties of CAGD, which are data prediction and accessible design.

Such a result, a Neutrosophic Bézier Curve model will be introduced in this paper. The Neutrosophic set can then be translated by creating models based on the neutrosophic theory and its relationships. The curve is generated by combining Neutrosophic control points with the Bézier basic functions and using their relation to represent the curve. Lastly, this new model is visualised using numerical examples of neutrosophic data sets with randomly selected membership values. Based on the visualization's findings, it is anticipated that the evaluation and analysis process will be simpler to carry out and have significant advantages in several areas, particularly the issue of uncertainty in the representation of real problems.

2. **Model Construction Method**

2.1. *Bézier Curve*

Pierre Bézier has derived the mathematical basis of curves and surface techniques from geometrical considerations as in [27, 28]. Later, around the 1970s, Forrest (1972) and Gordon and Reisenfeld (1974) found the connection between the work of Bézier and the classical Bernstein polynomials. They discovered that the Bernstein polynomials are the basis functions for Bézier curves and surfaces. The curve is necessary and inevitable for representing data points [29]. However, the nature of the data point obtained is difficult to understand, process and describe as it is affected by noise and uncertainty. Usually, data with uncertainty characteristics will be ignored or removed from a data set, disregarding its effect on the resulting curve and surface. Hence, the evaluation and analysis process will be incomplete. If there exists an element of uncertainty, the data set should be filtered so that it can be used to generate a curve of a model that wants to be investigated. Therefore an appropriate approach is needed to visualise and overcome this problem.

2.2. *Neutrosophic Set*

This section will begin with a summary of laws in neutrosophic sets as defined in [30]. Neutrosophic Set as an expansion of Intuitionistic Fuzzy Set where in Intuitionistic Fuzzy Set, the com- ponents T known as membership, I known as inconsistency and F known as non-membership are restricted either $t+i+f=1$ or $t^2+f^2 \leq 1$, if T,I,F are all reduced to the points t,i,f respectively, or $sup\,T + sup\,I + sup\,F = 1$ if T,I,F are subsets of $[0,1]$. But in Neutrosophic Set, there is no restriction on T(truth-membership), I(indeterminacy-membership), F(false-membership) other than they are subsets of $]^-0,1^+[$ thus, $^-0 \leq inf\,T + inf\,I + inf\,F \leq sup\,T + sup\,I + sup\,F \leq 3^+$ [30] .

Definition 1. *[30]Let E be a universe of discourse, and W a set included in E. An element x from E is noted with respect to the set W as $x(T,I,F)$. x belongs to W and define as follows: true value in the set denoted as t, indeterminate value in the set as i and false value in the set as f, where t varies in T_W, i varies in I_W, f varies in F_W. T_W, I_W, F_W are functions depending on many known or unknown* *parameters.* T_W, I_W, F_W *are real standard or non-standard subset of* $]^-0, 1^+[$. *That is*

$$T_W : X[0,1]$$

$$I_W : X[0,1]$$

$$F_W : X[0,1]$$

$$where\ ^-0 \leq T_W + I_W + F_W \leq 3^+$$

Definition 2. *Let a crisp set M is fixed and let* $B^* \subset M$. *A Neutrosophic set* B^* *in M is an object of the following*

$$B^* = \{(x, \mu_B(x), \gamma_B(x), \pi_B(x)) | x \in X\} \tag{1}$$

where functions $\mu_B : X \to [0,1], \gamma_B : X \to [0,1], \pi_B : X \to [0,1]$ *define the degree of membership, the degree of non membership and the degree of indeterminate of the element* $x \in M$ *to the set B*, respectively and for every* $x \in M$.

$$0 \leq \mu_B(x) + \gamma_B(x) + \pi_B(x) \leq 3$$

where $\mu_B(x) + \gamma_B(x) + \pi_B(x)$ *are independent membership of element* $x \in M$ *to set B*.*

2.3. *Neutrosophic Number and Neutrosophic Point Relation*

Prior research in Fuzzy systems (FSs) and Intuitionistic Fuzzy systems (IFSs) discussed the result in uncertainty. Still, these methods cannot be successfully solved when decently, unacceptable, and decision-maker declaration is uncertain. Therefore, some theories are mandatory for solving the problem with uncertainty. Hence, the Neutrosophic Sets (NSs) reflect three membership which is truth membership, indeterminacy membership and falsity membership will introduce named as Neutrosophic Curve. Furthermore, Neutrosophic Set is more practical and can solve the data than FSs and IFSs, which are involved with inconsistent, incomplete and uncertain data.

The concept of Neutrosophic Set is used to develop Neutrosophic Point Relation. Generally, neutrosophic point relations are data sets defined on a universal set which are Cartesian products of $X \times Y$ that are mapping from $X \to Y$. It represents the strength of the association between elements of the two sets. Neutrosophic Point Relation is defined and used as a converter from the definition of Neutrosophic data points to introduce Neutrosophic Control Point.

Definition 3. *Let R and S be a space points with non-empty sets and* $r, s \subseteq \mathrm{R} \times \mathrm{S}$, *then Neutrosophic point relation is defined as*

$$\mathbb{M}^* = ((r_i, s_i), \mu_{R\times S}(r_i, s_i), \gamma_{R\times S}(r_i, s_i), \pi_{R\times S}(r_i, s_i) | (\mu_{R\times S}(r_i, s_i), \gamma_{R\times S}(r_i, s_i), \pi_{R\times S}(r_i, s_i)) \in \mathrm{R} \times \mathrm{S} \tag{2}$$

where (r_i, s_i) *is a point relation and* M *is a neutrosophic point relation space on* $\mathrm{R} \times \mathrm{S}$ *and functions* $\mu_B : X \to [0,1], \gamma_B : X \to [0,1], \pi_B : X \to [0,1]$ *define truth membership, indeterminacy membership and falsity membership respectively.*

$$0 \leq \mu_B(x) + \gamma_B(x) + \pi_B(x) \leq 3$$

Definition 4. *Let* $r, s \subseteq \mathrm{R} \times \mathrm{S}$ *with*

$$\tilde{M} = \{(r_i, y_i) \,|y_i \in (0,1)\} \text{ and } \tilde{N} = \{(s_i, y_i) \,|y_i \in (0,1)\} \tag{3}$$

represent two neutrosophic points. Then

$$\widetilde{S} = \{((r_i, s_i), \mu_{R\times S}(r_i, s_i), \gamma_{R\times S}(r_i, s_i), \pi_{R\times S}(r_i, s_i)) \,|0 \leq \mu_{R\times S}(x) + \gamma_{R\times S}(x) + \pi_{R\times S}(x) \leq 3\} \tag{4}$$

is a neutrosophic point relation on $\widetilde{M}$ *and* $\widetilde{N}$ *if*

$$\mu_s(r_i, s_i) \leq \mu_M(r_i), \forall (r_i, s_i) \in R \times S,$$

$$\gamma_s(r_i, s_i) \leq \gamma_M(r_i), \forall (r_i, s_i) \in R \times S$$

$$\pi_s(r_i, s_i) \leq \pi_M(r_i), \forall (r_i, s_i) \in R \times S$$

and

$$\mu_s(r_i, s_i) \leq \mu_N(r_i) \forall (r_i, s_i) \in R \times S$$

$$\gamma_s(r_i, s_i) \leq \gamma_N(r_i), \forall (r_i, s_i) \in R \times S$$

$$\pi_s(r_i, s_i) \leq \pi_N(r_i), \forall (r_i, s_i) \in R \times S$$

Neutrosophic point relation is a subset of the Cartesian product of a set that can be used to represent the data with a connection between variables, attributes or quantities. It can also visualize into the spline the dependencies and correlations of variables.

2.4. *Neutrosophic Control Point Relations*

Neutrosophic spline model in the context of geometric modeling results when each coefficient geometry spline model redefined through neutrosophic fuzzy approach until produced a form of control points. A Bézier curve is a curve that is determined by its control polygon. Bézier curve is a parametric curve used in computer graphics and related fields. The Bézier curve is a parametric curve $B(t)$ that is a polynomial function of the parameter, t. The polynomial degree depends on the number of points used to define the curve. This paper employs neutrosophic control point relation using the neutrosophic point relation we introduced in the previous section and produces an approximating curve. The approximating curve does not pass through the interior points but is attracted to them. This section discussed blending Neutrosophic Control Point Relation with Bézier function to produce Neutrosophic Bézier Curves. Next, the curve is generated with the blending and recursive processes. The Neutrosophic Control Points are defined as follows:

Definition 5. *Neutrosophic control point relation can be defined as a set of* $n+1$ *points that shows positions and coordinates of a location and is used to describe a curve which is denoted by*

$$\tilde{C}_{PR_i} = \left\{\tilde{C}_{PR_0}, \tilde{C}_{PR_1}, \ldots, \tilde{C}_{PR_s}\right\} \tag{5}$$

and can be written as

$$\left\{\left(\left(p_i, q_i\right), \mu_{p\times q}\left(p_i, q_i\right)\right)_1, \left(\left(p_i, q_i\right), \mu_{p\times q}\left(p_i, q_i\right)\right)_2, \ldots, \left(\left(p_i, q_i\right), \mu_{p\times q}\left(p_i, q_i\right)\right)_n\right\}$$

where the neutrosophic control point relation is also control the shape of a curve.

2.5. *Neutrosophic Bézier Model*

In a previous study, [31] had come out with the design and tuning of fuzzy control surfaces with Bézier functions. Hence, [32] and [33] use fuzzy set theory, uncertainty data and technique of interpolation to build rational Bézier curve and followed by [34] whose used Bézier curve modeling to interpret intuitionistic data problem. The idea of constructing Neutrosophic Bézier Model starts with the new Neutrosophic Control Point. let $\widetilde{C}_{PR}$ be a Neutrosophic Control Point Relations defined by Neutrosophic Point Relation and $B(t)$ be a Bézier curve with parameter, t, hence by blending it, Neutrosophic Bézier Curve is defined as follow.

$$\widetilde{B}(t) = \sum_{i=0}^{n} \widetilde{C}_{PR} B_{n,i}(t),\ \ 0 \le t \le 1 \tag{6}$$

$$\text{with}$$

$$\widetilde{B^{\mu}}(t) = \sum_{i=0}^{n} \widetilde{C}_{PR} B_{n,i}(t),\ \ 0 \le t \le 1$$

$$\widetilde{B^{\lambda}}(t) = \sum_{i=0}^{n} \widetilde{C}_{PR} B_{n,i}(t),\ \ 0 \le t \le 1$$

$$\widetilde{B^{\pi}}(t) = \sum_{i=0}^{n} \widetilde{C}_{PR} B_{n,i}(t),\ \ 0 \le t \le 1$$

with Bernstein polynomials or blending function,

$B_{n,i}(t) = \begin{pmatrix} n \\ i \end{pmatrix}^{i} t^i (1-t)^{n-i}$ where $\begin{pmatrix} n \\ i \end{pmatrix} = \frac{n!}{i!(n-i)!}$ are the binomial coefficients.

For degree of n Neutrosophic Bézier also can be written as

$$\widetilde{B}(t) = \widetilde{C}_{PR_0} B_{n,0} + \widetilde{C}_{PR_1} B_{n,1} + \ldots + \widetilde{C}_{PR_n} B_{n,d} \tag{7}$$

3. **Results**

This paper uses cubic Bézier curve approximation to show how Neutrosophic Bézier curve will represent in the graph and illustrate Neutrosophic Bézier Model.

3.1. *Example 1*

Let $C_0^* = (1,3), C_1^* = (3,6), C_2^* = (6,2)$ and $C_3^* = (9,5)$ be an Neutrosophic Control Point Relation. Hence, truth-membership, indeterminacy-membership, false-membership is summarized as follows:

TABLE 1. Examples of some NCP and its respective degrees.

NCP	truth-membership, μ_C (C_i^*)	indeterminacy-membership, ν_C (C_i^*)	false-membership, π_C (C_i^*)
C_0^*	0.3	0.6	0.1
C_1^*	0.8	0.1	0.1
C_2^*	0.7	0.1	0.2
C_3^*	0.2	0.4	0.4

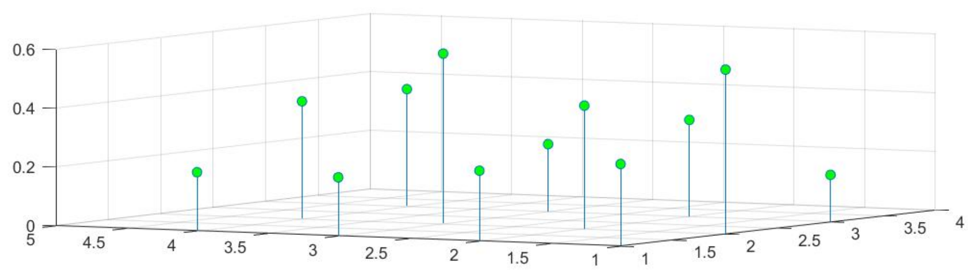

FIGURE 1. Neutrosophic Control Points

A Bézier curve is a curve that is determined by its control polygon. The Bézier curve is a parametric curve $B(t)$ that is a polynomial function of the parameter, t. Here, we will employ Neutrosophic Control Point relation using neutrosophic point relation. Figure 1 shows Neutrosophic Control Point that results from Neutrosophic Data Point in Table 1. Next, by blending it with Bézier curve, the following graphs of Neutrosophic Bézier curve are sketched for truth-membership, indeterminacy-membership, false-membership and all membership respectively.

TABLE 2. Neutrosophic Bézier curve for truth-membership

NCP	truth-membership μ_C
C_0	$\langle(1,3);0.3\rangle$
C_1	$\langle(3,6);0.8\rangle$
C_2	$\langle(6,2);0.7\rangle$
C_3	$\langle(9,5);0.2\rangle$

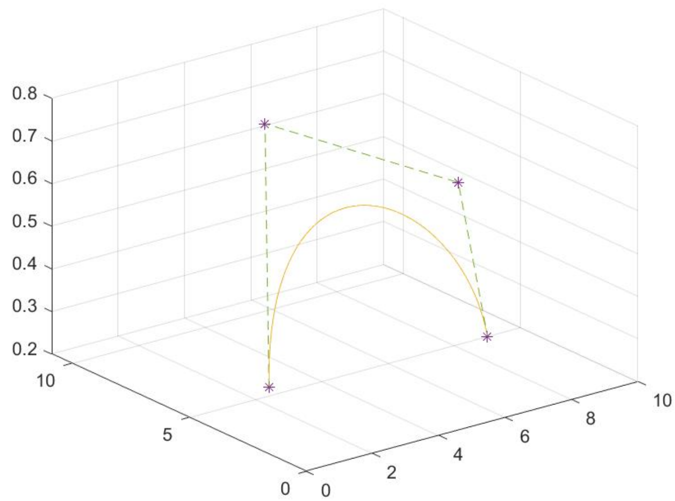

FIGURE 2. Neutrosophic Bézier curve for truth-membership

Figure 2 above is Neutrosophic Bézier curve produced from Neutrosophic Control Point with truth membership, μ_C in Table 2.

TABLE 3. Neutrosophic Bézier curve for indeterminacy-membership

NCP	indeterminacy-membership v_C
C_0	$\langle(1,3);0.6\rangle$
C_1	$\langle(3,6);0.1\rangle$
C_2	$\langle(6,2);0.1\rangle$
C_3	$\langle(9,5);0.1\rangle$

Figure 3 shows Neutrosophic Bézier curve produced from Neutrosophic Control Point with indeterminacy membership, v_C in Table 3.

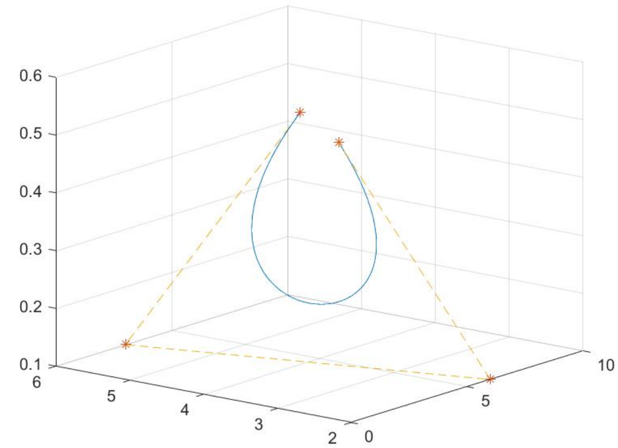

FIGURE 3. Neutrosophic Bézier curve for indeterminacy-membership

TABLE 4. Neutrosophic Bézier curve for false-membership

NCP	false-membership π_C
C_0	$\langle(1,3);0.1\rangle$
C_1	$\langle(3,6);0.1\rangle$
C_2	$\langle(6,2);0.2\rangle$
C_3	$\langle(9,5);0.4\rangle$

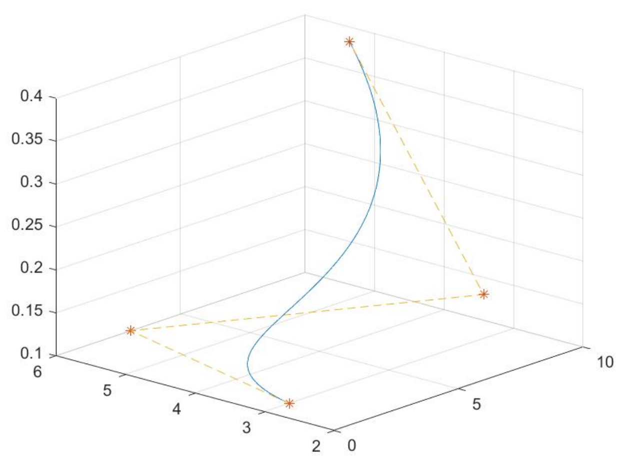

FIGURE 4. Neutrosophic Bézier curve for false-membership

Figure 4 represents Neutrosophic Bézier curve produced from Neutrosophic Control Point with false membership, π_C in Table 4.

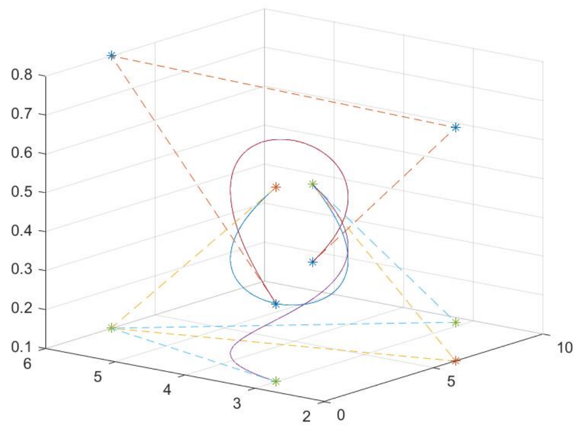

FIGURE 5. Neutrosophic Bézier curve for all membership degree

Figure 5 is the combination of all membership degrees to produce Neutrosophic Bézier Curve where

$$\widetilde{B}(t) = \sum_{i=0}^{3} \widetilde{C}_{PR} B_{3,i}(t), \quad 0 \leq t \leq 1 \tag{8}$$

with

$$\widetilde{B^{\mu}}(t) = \sum_{i=0}^{3} \widetilde{C}_{PR} B_{3,i}(t), \ \ 0 \leq t \leq 1$$

$$\widetilde{B^{\lambda}}(t) = \sum_{i=0}^{3} \widetilde{C}_{PR} B_{3,i}(t), \ \ 0 \leq t \leq 1$$

$$\widetilde{B^{\pi}}(t) = \sum_{i=0}^{3} \widetilde{C}_{PR} B_{3,i}(t), \ \ 0 \leq t \leq 1$$

4. **Conclusions**

A new model is proposed to represent a visualisation of neutrosophic data set which called as Neutrosophic Bézier Curve model. Neutrosophic Bézier Curve model approximation is an optimal method for modeling data with uncertainty data since it is defined by truth-membership T, indeterminacy-membership I, false-membership F. Based on this definition blended with the control point, an approximation Neutrosophic Bézier Curve has been developed. All three curves representing the data will solve complex uncertainty data in graphic design and visualisation problems.

The proposed model is described in basic terms and illustrates the final results. As a result, additional deep research with new definitions and ideas is required to depict the processes in greater detail. This generalised model must be applied to real data to achieve the intended visualisation and analysis. Organisation can use the data to increase productivity and employee satisfaction by showing the importance of various employee satisfaction in logistic services as stated in [35]. The Neutrosophic data combined with visualisation with Bézier will provide complete knowledge of the study and explain the problems studied with its reasoning. The system and the resulting model will contribute to the Neutrosophic Modeling techniques area.

This paper could also be extended to other spline models such as B-Spline and NURBS (Non-Uniform Rational B-Spline), and also in future works, especially in the development of management decision-making field, stochastic processes, stock market, remote sensing, data mining, real-time tracking, routing and wireless sensor networks.

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Some Results for Compatible maps on Neutrosophic Metric Spaces

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Abstract. The neutrosophic theory has been effectively used to address uncertainty and ambiguity. Neutrosophic Metric Space (NMS) was introduced by Krisci and Simsek in 2020. Following that, several kinds of compatible maps and their characteristics were investigated in the context of Intuitionistic fuzzy metric spaces and fuzzy metric spaces. In this paper, the author introduce the notion of compatible maps of type (α) and type (β) in neutrosophic metric space. For this purpose, four non-comparable mappings are used to prove the basic results. Furthermore, we prove several common fixed points results for compatible maps of type (α) and type (β) in neutrosophic metric space and provide a non-trivial examples.

Keywords: Fixed point; Neutrosophic metric Space; Compatible maps.

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1. Introduction

The concept of metric spaces and the Banach contraction principle serve as the foundation of fixed point theory. The openness of metric space attracts a huge number of academics to the axiomatic interpretation. Following Zadeh's [29] introduction of the idea of fuzzy sets (FSs), many academics offered a variety of generalisations for classical structures. The idea of Fuzzy Metric Space (FMS) was first put forth in 1975 by Kramosil and Michalek [14]. Later, George and Veeramani [6] redefined the concept of FMS. Following then, several researchers looked at the FMS characteristics and produced numerous fixed point results. Intuitionistic Fuzzy Sets(IFSs) was introduced by Atanassov [1] with the concept of non - membership to

FSs. Park [22] defined Intuitionistic Fuzzy Metric Space (IFMS) from the concept of IFSs and given some fixed point results. In FMS and IFMS various fixed point theorems has been proved by Alaca et al [2]. Grabiec [20] gave fuzzy interpretation of Banach and Edelstein fixed point theorems in the sense of Kramosil and Michalek. Weakly commuting maps in metric spaces were first proposed by Sessa [24], who started the trend of enhancing commutativity in fixed point theorems. Jungck [24] soon enlarged this concept to compatible maps. Smarandache [25,26] established the new idea called Neutrosophic logic and Neutrosophic Set (NS) in 1998. In general, the ideas of FS and IFS deal with degrees of membership and non-membership, respectively. By incorporating a degree of indeterminacy, the neutrosophic set generalises fuzzy and intuitionistic fuzzy sets. Hence several researchers have made studies on the concept of neutrosophic set. Parimala Mani et al. [8,9]obtained decision making applications form Neutrosophic Support Soft Topological Spaces. Sahin et al. [23]studied adequacy of online education using Hausdorff Measures based on neutrosophic quadruple sets. Recently, Sahin and Kargin [19] obtained neutrosophic triplet metric spaces and neutrosophic triplet normed spaces. Kirisci and Simsek [15] established the concept of neutrosophic metric spaces (NMSs) that deals with membership, non-membership and naturalness functions and derived various fixed point theorems for neutrosophic metric space. Sowndrarajan and Jeyaraman et al. [12,27] studied Banach and Edelstein contraction fixed point results for neutrosophic metric space.
In this manuscript, we introduce the notion of compatible maps of type (α) and type (β) in neutrosophic metric space. We also establish fixed point results by using four mappings and obtain a non trivial example

2. **Preliminaries**

Definition 2.1 [26] Let Σ be a non-empty fixed set. A Neutrosophic Set N in Σ is a collection of elements in the form $N = \{\langle a, \xi_N(a), \varrho_N(a), \nu_N(a)\rangle : a \in \Sigma\}$ where the functions $\xi_N(a), \varrho_N(a)$ and $\nu_N(a)$ represent the degree of membership, degree of indeterminacy and the degree of non-membership respectively of each element a $\in$ N to the set Σ.

Definition 2.2 [10] A continuous t - norm (CTN) is a function $\star : [0,1] \times [0,1] \to [0,1]$ that satisfies the following conditions;
For all $\varrho_1, \varrho_2, \varrho_3, \varrho_4 \in$ [0, 1]
(i) $\varrho_1 \star 1 = \varrho_1$;
(ii) If $\varrho_1 \leq \varrho_3$ and $\varrho_2 \leq \varrho_4$ then $\varrho_1 \star \varrho_2 \leq \varrho_3 \star \varrho_4$;
(iii) $\star$ is continuous;
(iv) $\star$ is commutative and associative.

Definition 2.3 [10] A continuous t - co norm (CTC) is a function $\diamond : [0,1] \times [0,1] \to [0,1]$ that satisfies the following conditions;

For all $\varrho_1, \varrho_2, \varrho_3, \varrho_4 \in [0,\ 1]$

(i) $\varrho_1 \diamond 0 = \varrho_1$;

(ii) If $\varrho_1 \leq \varrho_3$ and $\varrho_2 \leq \varrho_4$ then $\varrho_1 \diamond \varrho_2 \leq \varrho_3 \diamond \varrho_4$;

(iii) $\diamond$ is continuous;

(iv) $\diamond$ is commutative and associative.

3. Neutrosophic Metric Spaces

In this section, we define basic concepts of neutrosophic metric space and prove various properties of the space with suitable examples.

Definition 3.1 [27] A 6 - tuple $(\Sigma, \Lambda, \aleph, \beth, \star, \diamond)$ is called Neutrosophic Metric Space(NMS), if Σ is an arbitrary non empty set, $\star$ is a neutrosophic CTN, $\diamond$ is a neutrosophic CTC and $\Lambda, \aleph, \beth$ are neutrosophic sets on $\Sigma^2 \times \mathbb{R}^+$ satisfying the following conditions:

For all $\varrho, \varsigma, \omega \in \Sigma, \vartheta \in \mathbb{R}^+$

(i) $0 \leq \Lambda(\varrho, \varsigma, \vartheta) \leq 1;\ 0 \leq \aleph(\varrho, \varsigma, \vartheta) \leq 1;\ 0 \leq \beth(\varrho, \varsigma, \vartheta) \leq 1$;

(ii) $\Lambda(\varrho, \varsigma, \vartheta) + \aleph(\varrho, \varsigma, \vartheta) + \beth(\varrho, \varsigma, \vartheta) \leq 3$;

(iii) $\Lambda(\varrho, \varsigma, \vartheta) = 1$ if and only if $\varrho = \varsigma$;

(iv) $\Lambda(\varrho, \varsigma, \vartheta) = \Lambda(\varsigma, \varrho, \vartheta)$ for $\vartheta > 0$;

(v) $\Lambda(\varrho, \varsigma, \vartheta) \star \Lambda(\varsigma, \varrho, \mu) \leq \Lambda(\varrho, \omega, \vartheta + \mu)$, for all $\vartheta, \mu > 0$;

(vi) $\Lambda(\varrho, \varsigma, .) : [0, \infty) \to [0,1]$ is neutrosophic continuous ;

(vii) $lim_{\vartheta \to \infty} \Lambda(\varrho, \varsigma, \vartheta) = 1$ for all $\vartheta > 0$;

(viii) $\aleph(\varrho, \varsigma, \vartheta) = 0$ if and only if $\varrho = \varsigma$;

(ix) $\aleph(\varrho, \varsigma, \vartheta) = \aleph(\varsigma, \varrho, \vartheta)$ for $\vartheta > 0$;

(x) $\aleph(\varrho, \varsigma, \vartheta) \diamond \aleph(\varrho, \omega, \mu) \geq \aleph(\varrho, \omega, \vartheta + \mu)$, for all $\vartheta, \mu > 0$;

(xi) $\aleph(\varrho, \varsigma, .) : [0, \infty) \to [0,1]$ is neutrosophic continuous ;

(xii) $lim_{\vartheta \to \infty} \aleph(\varrho, \varsigma, \vartheta) = 0$ for all $\vartheta > 0$;

(xiii) $\beth(\varrho, \varsigma, \vartheta) = 0$ if and only if $\varrho = \varsigma$;

(xiv) $\beth(\varrho, \varsigma, \vartheta) = \beth(\varsigma, \varrho, \vartheta)$ for $\vartheta > 0$;

(xv) $\beth(\varrho, \varsigma, \vartheta) \diamond \beth(\varrho, \omega, \mu) \geq \beth(\varrho, \omega, \vartheta + \mu)$, for all $\vartheta, \mu > 0$;

(xvi) $\beth(\varrho, \varsigma, .) : [0, \infty) \to [0,1]$ is neutrosophic continuous ;

(xvii) $lim_{\vartheta \to \infty} \beth(\varrho, \varsigma, \vartheta) = 0$ for all $\vartheta > 0$;

(xviii) If $\vartheta > 0$ then $\Lambda(\varrho, \varsigma, \vartheta) = 0, \aleph(\varrho, \varsigma, \vartheta) = 1, \beth(\varrho, \varsigma, \vartheta) = 1$.

Then $(\Lambda, \aleph, \beth)$ is called neutrosophic metric on Σ. The functons $\Lambda, \aleph$ and $\beth$ denote degree of closedness, neturalness and non - closedness between ϱ and ς with respect to ϑ respectively.

Example 3.2 [27] Let (Σ, d) be a metric space. $\Lambda, \aleph, \beth : \Sigma^2 \times \mathbb{R}^+ \to [0,1]$ defined by

$$\Lambda(\varrho,\varsigma,\vartheta) = \frac{\vartheta}{\vartheta + d(\varrho,\varsigma)}; \quad \aleph(\varrho,\varsigma,\vartheta) = \frac{d(\varrho,\varsigma)}{\vartheta + d(\varrho,\varsigma)}; \quad \beth(\varrho,\varsigma,\vartheta) = \frac{d(\varrho,\varsigma)}{\vartheta}$$

for all $\varrho, \varsigma \in \Sigma$ and $\vartheta > 0$. where $\varrho \star \varsigma = min\{\varrho,\varsigma\}$ and $\varrho \diamond \varsigma = max\{\varrho,\varsigma\}$. Then $(\Sigma, \Lambda, \aleph, \beth, \star, \diamond)$ is called NMS induced by a standard neutrosophic metric.

Definition 3.3 Let $(\Sigma, \Lambda, \aleph, \beth, \star, \diamond)$ be neutrosophic metric space. Then

(a) $\{\varrho_n\}$ in Σ is converging to a point $\varrho \in \Sigma$ if for each $\vartheta > 0$

$$lim_{n\to\infty}\Lambda(\varrho_n,\varrho,\vartheta) = 1; \quad lim_{n\to\infty}\aleph(\varrho_n,\varrho,\vartheta) = 0; \quad lim_{n\to\infty}\beth(\varrho_n,\varrho,\vartheta) = 0.$$

(b) $\{\varrho_n\}$ in Σ is called a Cauchy if for each $\epsilon > 0$ and $\vartheta > 0$ there exist $n \in \mathbb{N}$ such that $\Lambda(\varrho_{n+p},\varrho_n,\vartheta) = 1$; $\aleph(\varrho_{n+p},\varrho_n,\vartheta) = 0$; $\beth(\varrho_{n+p},\varrho_n,\vartheta) = 0$.

(c) $(\Sigma, \Lambda, \aleph, \beth, \star, \diamond)$ is said to be complete NMS if every Cauchy sequence is convergence in it.

Lemma 3.4 Let $\{\varrho_n\}$ be a sequence in a NMS $(\Sigma, \Lambda, \aleph, \beth, \star, \diamond)$. If there exist a number $k \in (0, 1)$such that for all $\varrho, \varsigma \in \Lambda$ and $\vartheta > 0$

$$\begin{aligned} \Lambda(\varrho_{n+2},\varrho_{n+1},k\vartheta) &\geq \Lambda(\varrho_{n+1},\varrho_n,k\vartheta), \\ \aleph(\varrho_{n+2},\varrho_{n+1},k\vartheta) &\leq \aleph(\varrho_{n+1},\varrho_n,k\vartheta), \\ \beth(\varrho_{n+2},\varrho_{n+1},k\vartheta) &\leq \beth(\varrho_{n+1},\varrho_n,k\vartheta) \end{aligned} \tag{1}$$

for all $\vartheta > 0$ and $n = 1, 2, 3\cdots$, then $\{\varrho_n\}$ is a Cauchy sequence in Λ

Proof. By Mathematical induction, we have

$$\begin{aligned} \Lambda(\varrho_{n+2},\varrho_{n+1},\vartheta) &\geq \Lambda(\varrho_2,\varrho_1,\frac{\vartheta}{k^n}), \\ \aleph(\varrho_{n+2},\varrho_{n+1},\vartheta) &\leq \aleph(\varrho_2,\varrho_1,\frac{\vartheta}{k^n}), \\ \beth(\varrho_{n+2},\varrho_{n+1},\vartheta) &\leq \beth(\varrho_2,\varrho_1,\frac{\vartheta}{k^n}) \end{aligned} \tag{2}$$

for all $\vartheta > 0$ and $n = 1, 2, \ldots$.

$$\begin{aligned} \Lambda(\varrho_n,\varrho_{n+p},\vartheta) &\geq \Lambda(\varrho_1,\varrho_2,\frac{\vartheta}{pk^{n-1}}) \star \cdots \star \Lambda(\varrho_1,\varrho_2,\frac{\vartheta}{pk^{n+p-2}}), \\ \aleph(\varrho_n,\varrho_{n+p},\vartheta) &\leq \aleph(\varrho_1,\varrho_2,\frac{\vartheta}{pk^{n-1}}) \diamond \cdots \diamond \aleph(\varrho_1,\varrho_2,\frac{\vartheta}{pk^{n+p-2}}), \\ \beth(\varrho_n,\varrho_{n+p},\vartheta) &\leq \beth(\varrho_1,\varrho_2,\frac{\vartheta}{pk^{n-1}}) \diamond \cdots \diamond \beth(\varrho_1,\varrho_2,\frac{\vartheta}{pk^{n+p-2}}). \end{aligned} \tag{3}$$

Therefore, from equation(1),we have

$$\lim_{n\to\infty} \Lambda(\varrho_1, \varrho_{n+p}, \vartheta) \geq 1 \star 1 \star \cdots \star 1 \geq 1,$$
$$\lim_{n\to\infty} \aleph(\varrho_1, \varrho_{n+p}, \vartheta) \leq 0 \star 0 \diamond \cdots \diamond 0 \leq 0 \tag{4}$$
$$\lim_{n\to\infty} \beth(\varrho_1, \varrho_{n+p}, \vartheta) \leq 0 \star 0 \diamond \cdots \diamond 0 \leq 0$$

which implies that $\{\varrho_n\}$ is a Cauchy sequence in Λ. □

Definition 3.5 Let Φ and Ψ be two mappings from neutrosophic metric space Σ into itself. The mappings are said to be compatible if

$$\lim_{n\to\infty} \Lambda(\Phi\Psi(\varrho_n), \Psi\Phi(\varrho_n), \vartheta) = 1, \ \lim_{n\to\infty} \aleph(\Phi\Psi(\varrho_n), \Psi\Phi(\varrho_n), \vartheta) = 0, \ \lim_{n\to\infty} \beth(\Phi\Psi(\varrho_n), \Psi\Phi(\varrho_n), \vartheta) = 0. \tag{5}$$

for all $\vartheta > 0$ whenever $\{\varrho_n\} \subset \Lambda$ such that $lim_{n\to\infty}\Phi(\varrho_n) = lim_{n\to\infty}\Psi(\varrho_n) = \varrho$ for some $\varrho \in \Sigma$.

Definition 3.6 Let Φ and Ψ be two mappings from NMS Σ into itself. The mappings are said to be compatible maps of type(α) if

$$\lim_{n\to\infty} \Lambda(\Phi\Psi(\varrho_n), \Psi\Psi(\varrho_n), \vartheta) = 1 \quad and \quad \lim_{n\to\infty} \Lambda(\Psi\Phi(\varrho_n), \Phi\Phi(\varrho_n), \vartheta) = 1,$$
$$\lim_{n\to\infty} \aleph(\Phi\Psi(\varrho_n), \Psi\Psi(\varrho_n), \vartheta) = 0 \quad and \quad \lim_{n\to\infty} \aleph(\Psi\Phi(\varrho_n), \Phi\Phi(\varrho_n), \vartheta) = 0,$$
$$\lim_{n\to\infty} \beth(\Phi\Psi(\varrho_n), \Psi\Psi(\varrho_n), \vartheta) = 0 \quad and \quad \lim_{n\to\infty} \beth(\Psi\Phi(\varrho_n), \Phi\Phi(\varrho_n), \vartheta) = 0.$$

for all $\vartheta > 0$ whenever $\{\varrho_n\} \subset \Lambda$ such that $lim_{n\to\infty}\Phi(\varrho_n) = lim_{n\to\infty}\Psi(\varrho_n) = \varrho$ for some $\varrho \in \Sigma$.

Definition 3.7 Let Φ and Ψ be two mappings from NMS Σ into itself. The mappings are said to be compatible maps of type(β) if for all $\vartheta > 0$

$$\lim_{n\to\infty} \Lambda(\Phi\Phi(\varrho_n), \Psi\Psi(\varrho_n), \vartheta) = 1, \ \lim_{n\to\infty} \aleph(\Phi\Phi(\varrho_n), \Psi\Psi(\varrho_n), \vartheta) = 0, \ \lim_{n\to\infty} \beth(\Phi\Phi(\varrho_n), \Psi\Psi(\varrho_n), \vartheta) = 0.$$

for all $\vartheta > 0$ whenever $\{\varrho_n\} \subset \Lambda$ such that $lim_{n\to\infty}\Phi(\varrho_n) = lim_{n\to\infty}\Psi(\varrho_n) = \varrho$ for some $\varrho \in \Sigma$.

Proposition 3.8 Let Σ be a NMS and Φ, Ψ be continuous mapping from Σ into itself. Then Φ and Ψ be compatible if and only if they are compatible of type(α).

Proof: Let $\{\varrho_n\} \subset \Lambda$ such that $lim_{n\to\infty}\Phi(\varrho_n) = lim_{n\to\infty}\Psi(\varrho_n) = \varrho$ for some $\varrho \in \Lambda$. Since Φ is continuous, we have $lim_{n\to\infty}\Phi\Phi(\varrho_n) = lim_{n\to\infty}\Phi\Psi(\varrho_n) = \Phi\Psi$. Also, since Φ, Ψ are compatible,

$$\lim_{n\to\infty} \Lambda(\Phi\Psi(\varrho_n), \Psi\Phi(\varrho_n), \vartheta) = 1, \ \lim_{n\to\infty} \aleph(\Phi\Psi(\varrho_n), \Psi\Phi(\varrho_n), \vartheta) = 0, \ \lim_{n\to\infty} \beth(\Phi\Psi(\varrho_n), \Psi\Phi(\varrho_n), \vartheta) = 0.$$

for all $\vartheta > 0$. From the inequality,

$$\Lambda(\Phi\Phi(\varrho_n), \Psi\Phi(\varrho_n), \vartheta) \geq \Lambda(\Phi\Phi(\varrho_n), \Phi\Psi(\varrho_n), \frac{\vartheta}{2}) \star \Lambda(\Phi\Psi(\varrho_n), \Psi\Phi(\varrho_n), \frac{\vartheta}{2}),$$
$$\aleph(\Phi\Phi(\varrho_n), \Psi\Phi(\varrho_n), \vartheta) \leq \aleph(\Phi\Phi(\varrho_n), \Phi\Psi(\varrho_n), \frac{\vartheta}{2}) \diamond \aleph(\Phi\Psi(\varrho_n), \Psi\Phi(\varrho_n), \frac{\vartheta}{2}),$$
$$\beth(\Phi\Phi(\varrho_n), \Psi\Phi(\varrho_n), \vartheta) \leq \beth(\Phi\Phi(\varrho_n), \Phi\Psi(\varrho_n), \frac{\vartheta}{2}) \diamond \beth(\Phi\Psi(\varrho_n), \Psi\Phi(\varrho_n), \frac{\vartheta}{2}).$$

Therefore,

$$\lim_{n\to\infty} \Lambda(\Phi\Phi(\varrho_n), \Psi\Phi(\varrho_n), \vartheta) = 1, \ \lim_{n\to\infty} \aleph(\Phi\Phi(\varrho_n), \Psi\Phi(\varrho_n), \vartheta) = 0, \ \lim_{n\to\infty} \beth(\Phi\Phi(\varrho_n), \Psi\Phi(\varrho_n), \vartheta) = 0.$$

Also we get,

$$\lim_{n\to\infty} \Lambda(\Psi\Psi(\varrho_n), \Phi\Psi(\varrho_n), \vartheta) = 1, \ \lim_{n\to\infty} \aleph(\Psi\Psi(\varrho_n), \Phi\Psi(\varrho_n), \vartheta) = 0, \ \lim_{n\to\infty} \beth(\Psi\Psi(\varrho_n), \Phi\Psi(\varrho_n), \vartheta) = 0.$$

Hence Φ and Ψ are compatible of type α.

Conversely, Let $\{\varrho_n\} \subset \Lambda$ such that $\lim_{n\to\infty} \Phi(\varrho_n) = \lim_{n\to\infty} \Phi(\varrho_n) = \varrho$ for some $\varrho \in \Lambda$.

Since Ψ is also continuous, we have

$$\lim_{n\to\infty} \Psi\Phi(\varrho_n) = \lim_{n\to\infty} \Psi\Psi(\varrho_n) = \Psi\varrho$$

Since Φ and Ψ are compatible of type (α), we get

$$\Lambda(\Phi\Psi(\varrho_n), \Psi\Psi(\varrho_n), \frac{\vartheta}{2}) = \Lambda(\Psi\Phi(\varrho_n), \Phi\Phi(\varrho_n), \frac{\vartheta}{2}) = 1,$$
$$\aleph(\Phi\Psi(\varrho_n), \Psi\Psi(\varrho_n), \frac{\vartheta}{2}) = \aleph(\Psi\Phi(\varrho_n), \Phi\Phi(\varrho_n), \frac{\vartheta}{2}) = 0,$$
$$\beth(\Phi\Psi(\varrho_n), \Psi\Psi(\varrho_n), \frac{\vartheta}{2}) = \beth(\Psi\Phi(\varrho_n), \Phi\Phi(\varrho_n), \frac{\vartheta}{2}) = 0$$

for all $\vartheta > 0$. Thus from the inequality,

$$\Lambda(\Phi\Psi(\varrho_n), \Psi\Phi(\varrho_n), \vartheta) \geq \Lambda(\Phi\Psi(\varrho_n), \Psi\Psi(\varrho_n), \frac{\vartheta}{2}) \star \Lambda(\Psi\Psi(\varrho_n), \Psi\Phi(\varrho_n), \frac{\vartheta}{2}),$$
$$\aleph(\Phi\Psi(\varrho_n), \Psi\Phi(\varrho_n), \vartheta) \leq \aleph(\Phi\Psi(\varrho_n), \Psi\Psi(\varrho_n), \frac{\vartheta}{2}) \diamond \aleph(\Psi\Psi(\varrho_n), \Psi\Phi(\varrho_n), \frac{\vartheta}{2}),$$
$$\beth(\Phi\Phi(\varrho_n), \Psi\Phi(\varrho_n), \vartheta) \leq \beth(\Phi\Phi(\varrho_n), \Phi\Psi(\varrho_n), \frac{\vartheta}{2}) \diamond \beth(\Phi\Psi(\varrho_n), \Psi\Phi(\varrho_n), \frac{\vartheta}{2})$$

Therefore

$$\lim_{n\to\infty} \Lambda(\Phi\Phi(\varrho_n), \Psi\Phi(\varrho_n), \vartheta) = 1, \ \lim_{n\to\infty} \aleph(\Phi\Phi(\varrho_n), \Psi\Phi(\varrho_n), \vartheta) = 0, \ \lim_{n\to\infty} \beth(\Phi\Phi(\varrho_n), \Psi\Phi(\varrho_n), \vartheta) = 0.$$

Hence Φ and Ψ are compatible maps. □

Proposition 3.9 Let $(\Sigma, \Lambda, \aleph, \beth, \star, \diamond)$ be a NMS and Φ, Ψ be self mappings from Σ into itself. If Φ, Ψ are compatible maps of type (α) and $\Phi(\varrho) = \Psi(\varrho)$ for some $\varrho \in \Sigma$, then $\Phi\Psi(\varrho) = \Psi\Psi(\varrho) = \Psi\Phi(\varrho) = \Phi\Phi(\varrho)$

*Proof:*Let $\{\varrho_n\} \subset \Sigma$ defined by $\lim_{n\to\infty} \varrho_n = \varrho$ for some $\varrho \in \Sigma$ and $n = 1, 2, \ldots$ and $\Phi(\varrho) = \Psi(\varrho)$. Then we have

$$\lim_{n\to\infty} \Phi(\varrho_n) = \lim_{n\to\infty} \Psi(\varrho_n) = \Phi(\varrho) = \Psi(\varrho).$$

Since, Φ, Ψ are compatible of type(α), we get

$$\Lambda(\Phi\Psi(\varrho), \Psi\Psi(\varrho), \vartheta) = \lim_{n\to\infty} \Lambda(\Phi\Psi(\varrho_n), \Psi\Psi(\varrho_n), \vartheta) = 1,$$
$$\aleph(\Phi\Psi(\varrho), \Psi\Psi(\varrho), \vartheta) = \lim_{n\to\infty} \aleph(\Phi\Psi(\varrho_n), \Psi\Psi(\varrho_n), \vartheta) = 0,$$
$$\beth(\Phi\Psi(\varrho), \Psi\Psi(\varrho), \vartheta) = \lim_{n\to\infty} \beth(\Phi\Psi(\varrho_n), \Psi\Psi(\varrho_n), \vartheta) = 0.$$

Therefore $\Phi\Psi(\varrho) = \Psi\Psi(\varrho)$. Also, we have $\Psi\Phi(\varrho) = \Phi\Phi(\varrho)$.
Since $\Phi(\varrho) = \Psi(\varrho)$, $\Psi\Psi(\varrho) = \Phi\Psi(\varrho)$. Hence $\Phi\Psi(\varrho) = \Psi\Psi(\varrho) = \Psi\Phi(\varrho) = \Phi\Phi(\varrho)$. □

Proposition 3.10 Let $(\Sigma, \Lambda, \aleph, \beth, \star, \diamond)$ be a NMS and Φ, Ψ be two self maps from Σ into itself. If Φ, Ψ are compatible maps of type $(\alpha)$and $\{\varrho_n\} \subset \Sigma$ such that
$\lim_{n\to\infty} \Phi(\varrho_n) = \lim_{n\to\infty} \Psi(\varrho_n) = \varrho$ for some $\varrho \in \Sigma$, then

(i) $\lim_{n\to\infty} \Psi\Phi(\varrho_n) = \Phi\varrho$ if Φ is continuous at $\varrho \in \Sigma$,
(ii) $\Phi\Psi(\varrho) = \Psi\Phi(\varrho)$ and $\Phi(\varrho) = \Psi(\varrho)$ if Φ, Ψ are continuous at $\varrho \in \Sigma$.

Proof:(i) Since Φ is continuous at ϱ and $\lim_{n\to\infty} \Phi(\varrho_n) = \varrho$, $\lim_{n\to\infty} \Phi\Phi(\varrho_n) = \Phi\varrho$. Also we have Φ, Ψ are compatible maps of ype (α), Then

$$\lim_{n\to\infty} \Lambda(\Psi\Phi(\varrho_n), \Phi\Phi(\varrho_n), \vartheta) = 1, \lim_{n\to\infty} \aleph(\Psi\Phi(\varrho_n), \Phi\Phi(\varrho_n), \vartheta) = 0, \lim_{n\to\infty} \beth(\Psi\Phi(\varrho_n), \Phi\Phi(\varrho_n), \vartheta) = 0.$$

for all $\vartheta > 0$. From the definition (3.1),

$$\lim_{n\to\infty} \Lambda(\Psi\Phi(\varrho_n), \Phi(\varrho), \vartheta) \geq \lim_{n\to\infty} \Lambda(\Psi\Phi(\varrho_n), \Phi\Phi(\varrho_n), \frac{\vartheta}{2}) \star \Lambda(\Phi\Phi(\varrho_n), \Phi(\varrho), \frac{\vartheta}{2}) \geq 1,$$
$$\lim_{n\to\infty} \aleph(\Psi\Phi(\varrho_n), \Phi(\varrho), \vartheta) \leq \lim_{n\to\infty} \aleph(\Psi\Phi(\varrho_n), \Phi\Phi(\varrho_n), \frac{\vartheta}{2}) \diamond \aleph(\Phi\Phi(\varrho_n), \Phi(\varrho), \frac{\vartheta}{2}) \leq 0,$$
$$\lim_{n\to\infty} \beth(\Psi\Phi(\varrho_n), \Phi(\varrho), \vartheta) \leq \lim_{n\to\infty} \beth(\Psi\Phi(\varrho_n), \Phi\Phi(\varrho_n), \frac{\vartheta}{2}) \diamond \beth(\Phi\Phi(\varrho_n), \Phi(\varrho), \frac{\vartheta}{2}) \leq 0.$$

Hence $\lim_{n\to\infty} \Psi\Phi(\varrho_n) = \Phi(\varrho)$.
(ii) we have $\lim_{n\to\infty} \Phi(\varrho_n) = \lim_{n\to\infty} \Psi(\varrho_n) = \varrho$. and Φ, Ψ are continuous at $\varrho \in \Sigma$. From the result(i) we have, $\lim_{n\to\infty} \Phi\Psi(\varrho_n) = \Phi(\varrho)$ and $\lim_{n\to\infty} \Psi\Phi(\varrho_n) = \Psi(\varrho)$. Since the limit is always unique, so we obtain $\Phi(\varrho) = \Psi(\varrho)$. By Proposition(3.9), Hence, we prove that $\Phi\Psi(\varrho) = \Psi\Phi(\varrho)$. □

Example 3.11 Let $\Sigma = [0, \infty)$ be a metric d which is defined by $d(\varrho, \varsigma) = |\varrho - \varsigma|$, where $\star$ and $\diamond$ defined by $a \star b = min\{a, b\}, \quad a \diamond b = max\{a, b\}$. we define $(\Lambda, \aleph, \beth)$ by

$$\Lambda(\varrho, \varsigma, \vartheta) = \left(exp\left(\frac{d(\varrho, \varsigma)}{\vartheta}\right)\right)^{-1},$$

$$\aleph(\varrho, \varsigma, \vartheta) = \frac{exp\left(\frac{d(\varrho,\varsigma)}{\vartheta}\right) - 1}{exp\left(\frac{d(\varrho,\varsigma)}{\vartheta}\right)},$$

$$\beth(\varrho, \varsigma, \vartheta) = exp\left(\frac{d(\varrho, \varsigma)}{\vartheta}\right).$$

for all $\varrho, \varsigma \in \Sigma$ and $\vartheta > 0$. Then $(\Sigma, \Lambda, \aleph, \beth, \star, \diamond)$ is a NMS. Let Φ, Ψ be defined by

$$\Phi(\varrho) = \begin{cases} 1, & \text{if for all } \ \varrho \in [0, 1] \\ 1 + \varrho, & \text{if for all } \ \varrho \in (1, \infty) \end{cases}$$

$$\Psi(\varrho) = \begin{cases} 1, & \text{if for all } \ \varrho \in [0, 1] \\ 1 + \varrho, & \text{if for all } \ \varrho \in [0, 1) \end{cases}$$

Let $\{\varrho_n\}$ be a sequence in Σ such that $\lim_{n\to\infty} \Phi\varrho_n = \lim_{n\to\infty} \Psi\varrho_n = \omega$. From the definition of Φ, Ψ, ϱ and $\lim_{n\to\infty} \varrho_n = 0$. Since Φ, Ψ are discontinuous at $\varrho = 1$, Therefore (Φ, Ψ) are compatible maps of type (β).

4. **Main Results**

In this section, we present some interesting concepts such as compatible maps of of type (α) and type (β) in neutrosophic metric space with suitable examples. Also we prove some fixed point theorems using compatible mapping of type (α).

Theorem 4.1 Let $(\Sigma, \Lambda, \aleph, \beth, \star, \diamond)$ be a complete neutrosophic metric space with $\vartheta \star \vartheta \geq \vartheta$, $\vartheta \diamond \vartheta \leq \vartheta$ for all $\vartheta \in [0, 1]$ and satisfy the condition (1). Let $\Phi, \Psi, \Omega, \Lambda$ and Γ be mappings from Σ into itself such that

(i) $\Gamma(\Sigma) \subset \Phi\Psi(\Sigma)$, $\Gamma(\Sigma) \subset \Omega\Lambda(\Sigma)$;

(ii) There exists $k \in (0, 1)$ such that for all $\varrho, \varsigma \in \Sigma$, $\beta \in (0, 2)$ and $\vartheta > 0$

$$\begin{aligned}
\Lambda(\Gamma\varrho, \Gamma\varsigma, k\vartheta) &\geq \Lambda(\Phi\Psi\varrho, \Gamma\varrho, \vartheta) \star \Lambda(\Omega\Gamma\varsigma, \Gamma\varsigma, \vartheta) \star \Lambda(\Omega\Gamma\varsigma, \Gamma\varrho, \beta\vartheta) \\
&\star \Lambda(\Phi\Psi\varrho, \Gamma\varsigma, (2-\beta)\vartheta) \star \Lambda(\Phi\Psi\varrho, \Omega\Gamma\varsigma, \vartheta), \\
\aleph(\Gamma\varrho, \Gamma\varsigma, k\vartheta) &\leq \aleph(\Phi\Psi\varrho, \Gamma\varrho, \vartheta) \diamond \aleph(\Omega\Gamma\varsigma, \Gamma\varsigma, \vartheta) \diamond \aleph(\Omega\Gamma\varsigma, \Gamma\varrho, \beta\vartheta) \\
&\diamond \aleph(\Phi\Psi\varrho, \Gamma\varsigma, (2-\beta)\vartheta) \diamond \aleph(\Phi\Psi\varrho, \Omega\Gamma\varsigma, \vartheta), \\
\beth(\Gamma\varrho, \Gamma\varsigma, k\vartheta) &\leq \beth(\Phi\Psi\varrho, \Gamma\varrho, \vartheta) \diamond \beth(\Omega\Gamma\varsigma, \Gamma\varsigma, \vartheta) \diamond \beth(\Omega\Gamma\varsigma, \Gamma\varrho, \beta\vartheta) \\
&\diamond \beth(\Phi\Psi\varrho, \Gamma\varsigma, (2-\beta)\vartheta) \diamond \beth(\Phi\Psi\varrho, \Omega\Gamma\varsigma, \vartheta).
\end{aligned}$$

(iii) $\Gamma\Psi = \Psi\Gamma$, $\Gamma\Lambda = \Lambda\Gamma$, $\Phi\Psi = \Psi\Phi$ and $\Omega\Lambda = \Lambda\Omega$,

(iv) Φ and Ψ are continuous,

(v) Γ and $\Phi\Psi$ are compatible of type (α),

(vi) $\Lambda(\varrho, \Omega\Gamma\varrho, \vartheta) \geq \Lambda(\varrho, \Phi\Psi\varrho, \vartheta)$, $\aleph(\varrho, \Omega\Gamma\varrho, \vartheta) \leq \aleph(\varrho, \Phi\Psi\varrho, \vartheta)$,
$\beth(\varrho, \Omega\Gamma\varrho, \vartheta) \leq \beth(\varrho, \Phi\Psi\varrho, \vartheta)$ for all $\varrho \in \Sigma$ and $\vartheta > 0$.

Then $\Phi, \Psi, \Omega, \Lambda$ and Γ have a common fixed point in Σ.

Proof: Since $\Gamma(\Sigma) \subset \Phi\Psi(\Sigma)$ for fixed $\varrho_0 \in \Lambda$, we choose a point $\varrho_1 \in \Lambda$ such that $\Gamma\varrho_0 = \Phi\Psi\varrho_1$. Since $\Gamma(\Sigma) \subset \Omega\Lambda(\Lambda)$, we take $\varrho_2 \in \Lambda$ for this point ϱ_1 such that $\Phi\varrho_1 = \Omega\Lambda\varrho_2$. Consider a sequence $\{\varsigma_n\} \subset \Lambda$, bu mathematical induction,

$$\varsigma_{2n} = \Gamma\varrho_{2n} = \Phi\Psi\varrho_{2n+1},\ \varsigma_{2n+1} = \Gamma\varrho_{2n+1} = \Phi\Psi\varrho_{2n+2}$$

for $n = 1, 2, \ldots$. From (ii)we have

$$\begin{aligned}
\Lambda(\varsigma_{2n+1}, \varsigma_{2n+2}, k\vartheta) = \Lambda(\Gamma\varrho_{2n+1}, \Gamma\varrho_{2n+2}, k\vartheta) &\geq \Lambda(\varsigma_{2n}, \varsigma_{2n+1}, \vartheta) \star \Lambda(\varsigma_{2n+1}, \varsigma_{2n+2}, \vartheta) \\
&\star \Lambda(\varsigma_{2n+1}, \varsigma_{2n+1}, \vartheta) \star \Lambda(\varsigma_{2n}, \varsigma_{2n+2}, (1+q)\vartheta) \\
&\star \Lambda(\varsigma_{2n}, \varsigma_{2n+1}, \vartheta), \\
\aleph(\varsigma_{2n+1}, \varsigma_{2n+2}, k\vartheta) = \aleph(\Gamma\varrho_{2n+1}, \Gamma\varrho_{2n+2}, k\vartheta) &\leq \aleph(\varsigma_{2n}, \varsigma_{2n+1}, \vartheta) \diamond \aleph(\varsigma_{2n+1}, \varsigma_{2n+2}, \vartheta) \\
&\diamond \aleph(\varsigma_{2n+1}, \varsigma_{2n+1}, \vartheta) \diamond \aleph(\varsigma_{2n}, \varsigma_{2n+2}, (1+q)\vartheta) \qquad (6) \\
&\diamond \aleph(\varsigma_{2n}, \varsigma_{2n+1}, \vartheta), \\
\beth(\varsigma_{2n+1}, \varsigma_{2n+2}, k\vartheta) = \beth(\Gamma\varrho_{2n+1}, \Gamma\varrho_{2n+2}, k\vartheta) &\leq \beth(\varsigma_{2n}, \varsigma_{2n+1}, \vartheta) \diamond \beth(\varsigma_{2n+1}, \varsigma_{2n+2}, \vartheta) \\
&\diamond \beth(\varsigma_{2n+1}, \varsigma_{2n+1}, \vartheta) \diamond \beth(\varsigma_{2n}, \varsigma_{2n+2}, (1+q)\vartheta) \\
&\diamond \beth(\varsigma_{2n}, \varsigma_{2n+1}, \vartheta)
\end{aligned}$$

for all $\vartheta > 0$ and $\beta = 1$ - q with q $\in (0, 1)$.

Since $\star, \diamond$ are continuous also $\Lambda(\varrho, \varsigma, .), \aleph(\varrho, \varsigma, .)$ and $\beth(\varrho, \varsigma, .)$ are continuous, let $q \to 1$ in the above equation, we get

$$\begin{aligned}
\Lambda(\varsigma_{2n+1}, \varsigma_{2n+2}, k\vartheta) &\geq \Lambda(\varsigma_{2n}, \varsigma_{2n+1}, \vartheta) \star \Lambda(\varsigma_{2n+1}, \varsigma_{2n+2}, \vartheta), \\
\aleph(\varsigma_{2n+1}, \varsigma_{2n+2}, k\vartheta) &\leq \aleph(\varsigma_{2n}, \varsigma_{2n+1}, \vartheta) \diamond \aleph(\varsigma_{2n+1}, \varsigma_{2n+2}, \vartheta), \qquad (7) \\
\beth(\varsigma_{2n+1}, \varsigma_{2n+2}, k\vartheta) &\leq \beth(\varsigma_{2n}, \varsigma_{2n+1}, \vartheta) \diamond \beth(\varsigma_{2n+1}, \varsigma_{2n+2}, \vartheta)
\end{aligned}$$

Also we have

$$\begin{aligned}
\Lambda(\varsigma_{2n+2}, \varsigma_{2n+3}, k\vartheta) &\geq \Lambda(\varsigma_{2n+1}, \varsigma_{2n+2}, \vartheta) \star \Lambda(\varsigma_{2n+2}, \varsigma_{2n+3}, \vartheta), \\
\aleph(\varsigma_{2n+2}, \varsigma_{2n+3}, k\vartheta) &\leq \aleph(\varsigma_{2n+1}, \varsigma_{2n+2}, \vartheta) \diamond \aleph(\varsigma_{2n+2}, \varsigma_{2n+3}, \vartheta), \qquad (8) \\
\beth(\varsigma_{2n+2}, \varsigma_{2n+3}, k\vartheta) &\leq \beth(\varsigma_{2n+1}, \varsigma_{2n+2}, \vartheta) \diamond \beth(\varsigma_{2n+2}, \varsigma_{2n+3}, \vartheta).
\end{aligned}$$

From equation (7) and (8)

$$\Lambda(\varsigma_{2n+1},\varsigma_{2n+2},k\vartheta) \geq \Lambda(\varsigma_n,\varsigma_{n+1},\vartheta)\star\Lambda(\varsigma_{n+1},\varsigma_{n+2},\vartheta),$$
$$\aleph(\varsigma_{2n+1},\varsigma_{2n+2},k\vartheta) \leq \aleph(\varsigma_n,\varsigma_{n+1},\vartheta)\diamond\aleph(\varsigma_{n+1},\varsigma_{n+2},\vartheta),$$
$$\beth(\varsigma_{2n+1},\varsigma_{2n+2},k\vartheta) \leq \beth(\varsigma_n,\varsigma_{n+1},\vartheta)\diamond\beth(\varsigma_{n+1},\varsigma_{n+2},\vartheta).$$

for $n = 1, 2, \ldots$. Then for positive integers n and p,

$$\Lambda(\varsigma_{2n+1},\varsigma_{2n+2},k\vartheta) \geq \Lambda(\varsigma_n,\varsigma_{n+1},\vartheta)\star\Lambda(\varsigma_{n+1},\varsigma_{n+2},\frac{\vartheta}{k^p}),$$
$$\aleph(\varsigma_{2n+1},\varsigma_{2n+2},k\vartheta) \leq \aleph(\varsigma_n,\varsigma_{n+1},\vartheta)\diamond\aleph(\varsigma_{n+1},\varsigma_{n+2},\frac{\vartheta}{k^p}),$$
$$\beth(\varsigma_{2n+1},\varsigma_{2n+2},k\vartheta) \leq \beth(\varsigma_n,\varsigma_{n+1},\vartheta)\diamond\beth(\varsigma_{n+1},\varsigma_{n+2},\frac{\vartheta}{k^p}).$$

Since

$$lim_{n\to\infty}\Lambda(\varsigma_{n+1},\varsigma_{n+2},k\vartheta) = 1,\ lim_{n\to\infty}\aleph(\varsigma_{n+1},\varsigma_{n+2},k\vartheta) = 0, lim_{n\to\infty}\beth(\varsigma_{n+1},\varsigma_{n+2},k\vartheta) = 0,$$

we have

$$\Lambda(\varsigma_{n+1},\varsigma_{n+2},k\vartheta) \geq \Lambda(\varsigma_n,\varsigma_{n+1},\vartheta),$$
$$\aleph(\varsigma_{n+1},\varsigma_{n+2},k\vartheta) \leq \aleph(\varsigma_n,\varsigma_{n+1},\vartheta),$$
$$\beth(\varsigma_{n+1},\varsigma_{n+2},k\vartheta) \leq \beth(\varsigma_n,\varsigma_{n+1},\vartheta).$$

By lemma(3.4), Since Σ is complete, so $\{\varsigma_n\}$ is a Cauchy sequence which is converges to a point $\varrho \in \Sigma$. Also $\{\Gamma\varrho_n\}, \{\Phi\Psi\varrho_{2n+1}\}, \{\Omega\Lambda\varrho_{2n+2}\}$ are subsequences of $\{\varsigma_n\}$, $\lim_{n\to\infty}\Gamma\varrho_n = \varrho = \lim_{n\to\infty}\Phi\Psi\varrho_{2n+1} = \lim_{n\to\infty}\Omega\Lambda\varrho_{2n+2}$. Also, since Φ, Ψ are continuous and $\Gamma\Phi\Psi$ are compatible of type (α), by proposition (3.9), we have $\lim_{n\to\infty}\Gamma\Phi\Psi(\varrho_{2n+1}) = \Phi\Psi\varrho$ and $\lim_{n\to\infty}(\Phi\Psi)^2\varrho_{2n+1} = \Phi\Psi\varrho$. By(ii) with $\beta = 1$, we obtain

$$\begin{aligned}\Lambda(\Gamma\Phi\Psi\varrho_{2n+1},\Gamma\varrho_{2n+2},k\vartheta) \geq\ & \Lambda((\Phi\Psi)^2\varrho_{2n+1},\Gamma\Phi\Psi\varrho_{2n+1},\vartheta)\star\Lambda(\Omega\Lambda\varrho_{2n+2},\Gamma\varrho_{2n+2},\vartheta)\\ &\star\Lambda(\Omega\Gamma\varrho_{2n+2},\Gamma\Phi\Psi\varrho,\vartheta)\star\Lambda((\Phi\Psi)^2\varrho_{2n+1},\Gamma\Phi\varrho_{2n+2},\vartheta)\\ &\star\Lambda((\Phi\Psi)^2\varrho_{2n+1},\Omega\Lambda\varrho_{2n+2},\vartheta),\\ \aleph(\Gamma\Phi\Psi\varrho_{2n+1},\Gamma\varrho_{2n+2},k\vartheta) \leq\ & \aleph((\Phi\Psi)^2\varrho_{2n+1},\Gamma\Phi\Psi\varrho_{2n+1},\vartheta)\diamond\aleph(\Omega\Lambda\varrho_{2n+2},\Gamma\varrho_{2n+2},\vartheta)\\ &\diamond\aleph(\Omega\Gamma\varrho_{2n+2},\Gamma\Phi\Psi\varrho,\vartheta)\diamond\aleph((\Phi\Psi)^2\varrho_{2n+1},\Gamma\Phi\varrho_{2n+2},\vartheta)\\ &\diamond\aleph((\Phi\Psi)^2\varrho_{2n+1},\Omega\Lambda\varrho_{2n+2},\vartheta),\\ \beth(\Gamma\Phi\Psi\varrho_{2n+1},\Gamma\varrho_{2n+2},k\vartheta) \leq\ & \beth((\Phi\Psi)^2\varrho_{2n+1},\Gamma\Phi\Psi\varrho_{2n+1},\vartheta)\diamond\beth(\Omega\Lambda\varrho_{2n+2},\Gamma\varrho_{2n+2},\vartheta)\\ &\diamond\beth(\Omega\Gamma\varrho_{2n+2},\Gamma\Phi\Psi\varrho,\vartheta)\diamond\beth((\Phi\Psi)^2\varrho_{2n+1},\Gamma\Phi\varrho_{2n+2},\vartheta)\\ &\diamond\beth((\Phi\Psi)^2\varrho_{2n+1},\Omega\Lambda\varrho_{2n+2},\vartheta),\end{aligned}$$

which implies that

$$\begin{aligned}\Lambda(\Phi\Psi\varrho,\varrho,k\vartheta) &= lim_{n\to\infty}\Lambda(\Gamma\Phi\Psi\varrho_{2n+1},\Gamma\varrho_{2n+2},k\vartheta)\\ &\geq 1\star 1\star\Lambda(\varrho,\Phi\Psi\varrho,\vartheta)\star\Lambda(\Phi\Psi\varrho,\varrho,\vartheta)\star\Lambda(\Phi\Psi\varrho,\varrho,\vartheta),\\ \aleph(\Phi\Psi\varrho,\varrho,k\vartheta) &= lim_{n\to\infty}\aleph(\Gamma\Phi\Psi\varrho_{2n+1},\Gamma\varrho_{2n+2},k\vartheta)\\ &\leq 0\diamond 0\diamond\aleph(\varrho,\Phi\Psi\varrho,\vartheta)\diamond\aleph(\Phi\Psi\varrho,\varrho,\vartheta)\diamond\aleph(\Phi\Psi\varrho,\varrho,\vartheta),\\ \beth(\Phi\Psi\varrho,\varrho,k\vartheta) &= lim_{n\to\infty}\beth(\Gamma\Phi\Psi\varrho_{2n+1},\Gamma\varrho_{2n+2},k\vartheta)\\ &\leq 0\diamond 0\diamond\beth(\varrho,\Phi\Psi\varrho,\vartheta)\diamond\beth(\Phi\Psi\varrho,\varrho,\vartheta)\diamond\beth(\Phi\Psi\varrho,\varrho,\vartheta.\end{aligned}$$

Hence, by lemma (3.4), $\Phi\Psi\varrho=\varrho$. Also, by(vi), since $\Lambda(\varrho,\Omega\Gamma\varrho,\vartheta)\geq\Lambda(\varrho,\Phi\Psi\varrho,\vartheta)=1$ and $\aleph(\varrho,\Omega\Gamma\varrho,\vartheta)\leq\aleph(\varrho,\Phi\Psi\varrho,\vartheta)=0$ and $\beth(\varrho,\Omega\Gamma\varrho,\vartheta)\leq\beth(\varrho,\Phi\Psi\varrho,\vartheta)=0$ for all $\vartheta>0$, we get $\Omega\Lambda\varrho=\varrho$. By(ii) with $\beta=1$, we have

$$\begin{aligned}\Lambda(\Gamma\Phi\Psi\varrho,\Gamma\varrho,k\vartheta) &\geq \Lambda((\Phi\Psi)^2\varrho_{2n+1},\Gamma\Phi\Psi\varrho_{2n+1},\vartheta)\star\Lambda(\Omega\Lambda\varrho,\Gamma\varrho,\vartheta)\\ &\quad\star\Lambda(\Omega\Gamma\varrho,\Gamma\Phi\Psi\varrho_{2n+1},\vartheta)\star\Lambda((\Phi\Psi)^2\varrho_{2n+1},\Gamma\varrho,\vartheta)\\ &\quad\star\Lambda((\Phi\Psi)^2\varrho_{2n+1},\Omega\Lambda\varrho,\vartheta),\\ \aleph(\Gamma\Phi\Psi\varrho,\Gamma\varrho,k\vartheta) &\leq \aleph((\Phi\Psi)^2\varrho_{2n+1},\Gamma\Phi\Psi\varrho_{2n+1},\vartheta)\diamond\aleph(\Omega\Lambda\varrho,\Gamma\varrho,\vartheta)\\ &\quad\diamond\Lambda(\Omega\Gamma\varrho,\Gamma\Phi\Psi\varrho_{2n+1},\vartheta)\diamond\aleph((\Phi\Psi)^2\varrho_{2n+1},\Gamma\varrho,\vartheta)\\ &\quad\diamond\aleph((\Phi\Psi)^2\varrho_{2n+1},\Omega\Lambda\varrho,\vartheta),\\ \beth(\Gamma\Phi\Psi\varrho,\Gamma\varrho,k\vartheta) &\leq \beth((\Phi\Psi)^2\varrho_{2n+1},\Gamma\Phi\Psi\varrho_{2n+1},\vartheta)\diamond\beth(\Omega\Lambda\varrho,\Gamma\varrho,\vartheta)\\ &\quad\diamond\beth(\Omega\Gamma\varrho,\Gamma\Phi\Psi\varrho_{2n+1},\vartheta)\diamond\beth((\Phi\Psi)^2\varrho_{2n+1},\Gamma\varrho,\vartheta)\\ &\quad\diamond\beth((\Phi\Psi)^2\varrho_{2n+1},\Omega\Lambda\varrho,\vartheta).\end{aligned}$$

Thus

$$\begin{aligned}\Lambda(\Phi\Psi\varrho,\Gamma\varrho,k\vartheta) &= lim_{n\to\infty}\Lambda(\Gamma\Phi\Psi\varrho_{2n+1},\Gamma\varrho,k\vartheta)\\ &\geq 1\star 1\star 1\star\Lambda(\Phi\Psi\varrho,\Gamma\varrho,\vartheta)\star 1\\ &\geq \Lambda(\Phi\Psi\varrho,\Gamma\varrho,k\vartheta),\\ \aleph(\Phi\Psi\varrho,\Gamma\varrho,k\vartheta) &= lim_{n\to\infty}\aleph(\Gamma\Phi\Psi\varrho_{2n+1},\Gamma\varrho,k\vartheta)\\ &\leq 0\diamond 0\diamond 0\diamond\aleph(\Phi\Psi\varrho,\Gamma\varrho,\vartheta)\diamond 0\\ &\leq \aleph(\Phi\Psi\varrho,\Gamma\varrho,k\vartheta),\\ \beth(\Phi\Psi\varrho,\Gamma\varrho,k\vartheta) &= lim_{n\to\infty}\beth(\Gamma\Phi\Psi\varrho_{2n+1},\Gamma\varrho,k\vartheta)\\ &\leq 0\diamond 0\diamond 0\diamond\beth(\Phi\Psi\varrho,\Gamma\varrho,\vartheta)\diamond 0\\ &\leq \beth(\Phi\Psi\varrho,\Gamma\varrho,k\vartheta).\end{aligned}$$

By using the by Lemma (3.4), we get $\Phi\Psi\varrho = \Gamma\varrho = \varrho$. Now we will prove that $\Psi\varrho = \varrho$. By(ii) with $\beta = 1$ and (iii), we obtain

$$\Lambda(\Psi\varrho, \varrho, k\vartheta) = \Lambda(\Psi\Gamma\varrho, \Gamma\varrho, \vartheta) = \Lambda(\Gamma\Psi\varrho, \Gamma, k\vartheta) \geq \Lambda(\Phi\Phi\varrho, \Gamma\Psi, \vartheta) \star \Lambda(\Omega\Lambda\varrho, \Gamma\varrho, \vartheta) \star \Lambda(\Omega\Gamma\varrho, \Gamma\Psi\varrho, \vartheta) \star \Lambda(\Phi\Phi\varrho, \Gamma\varrho, \vartheta) \star \Lambda(\Phi\Phi\varrho, \Omega\Gamma\varrho, \vartheta),$$

$$\aleph(\Psi\varrho, \varrho, k\vartheta) = \Lambda(\Psi\Gamma\varrho, \Gamma\varrho, \vartheta) = \aleph(\Gamma\Psi\varrho, \Gamma, k\vartheta) \leq \aleph(\Phi\Phi\varrho, \Gamma\Psi, \vartheta) \diamond \aleph(\Omega\Lambda\varrho, \Gamma\varrho, \vartheta) \diamond \aleph(\Omega\Gamma\varrho, \Gamma\Psi\varrho, \vartheta) \diamond \aleph(\Phi\Phi\varrho, \Gamma\varrho, \vartheta) \diamond \aleph(\Phi\Phi\varrho, \Omega\Gamma\varrho, \vartheta),$$

$$\beth(\Psi\varrho, \varrho, k\vartheta) = \beth(\Psi\Gamma\varrho, \Gamma\varrho, \vartheta) = \beth(\Gamma\Psi\varrho, \Gamma, k\vartheta) \leq \beth(\Phi\Phi\varrho, \Gamma\Psi, \vartheta) \diamond \beth(\Omega\Lambda\varrho, \Gamma\varrho, \vartheta) \diamond \beth(\Omega\Gamma\varrho, \Gamma\Psi\varrho, \vartheta) \diamond \beth(\Phi\Phi\varrho, \Gamma\varrho, \vartheta) \diamond \beth(\Phi\Phi\varrho, \Omega\Gamma\varrho, \vartheta).$$

Therefore, we get $\Psi\varrho = \varrho$. Since $\Phi\Psi\varrho = \varrho$, hence $\Phi\varrho = \varrho$. Next we show that $\Lambda\varrho = \varrho$. By(ii) with $\beta = 1$ and (iii), we get

$$\Lambda(\Lambda\varrho, \varrho, k\vartheta) = \Lambda(\Lambda\Gamma\varrho, \Gamma\varrho, k\vartheta) = \Lambda(\Gamma\varrho, \Lambda\Gamma\varrho, k\vartheta) = 1 \star 1 \star \Lambda(\Lambda\varrho, \varrho, \vartheta) \star \Lambda(\varrho, \Gamma\varrho, \vartheta) \star \Lambda(\varrho, \Gamma\varrho, \vartheta),$$

$$\aleph(\Lambda\varrho, \varrho, k\vartheta) = \aleph(\Lambda\Gamma\varrho, \Gamma\varrho, k\vartheta) = \aleph(\Gamma\varrho, \Lambda\Gamma\varrho, k\vartheta) = 0 \diamond 0 \diamond \aleph(\Lambda\varrho, \varrho, \vartheta) \diamond \aleph(\varrho, \Gamma\varrho, \vartheta) \diamond \aleph(\varrho, \Gamma\varrho, \vartheta),$$

$$\beth(\Lambda\varrho, \varrho, k\vartheta) = \beth(\Lambda\Gamma\varrho, \Gamma\varrho, k\vartheta) = \beth(\Gamma\varrho, \Lambda\Gamma\varrho, k\vartheta) = 0 \diamond 0 \diamond \beth(\Lambda\varrho, \varrho, \vartheta) \diamond \beth(\varrho, \Gamma\varrho, \vartheta) \diamond \beth(\varrho, \Gamma\varrho, \vartheta).$$

which implies that $\vartheta\varrho = \varrho$. Since $\Omega\Lambda\varrho = \varrho$, we have $\Omega\varrho = \Omega\Lambda\varrho = \varrho$. Hence, we get $\Phi\varrho = \Psi\varrho = \Omega\varrho = \Lambda\varrho = \Omega\varrho = \varrho$, that is ϱ is a common fixed point of $\Phi, \Psi, \Omega, \Lambda$ and Γ.
Uniqueness of the fixed point ϱ follows from (ii). Hence ϱ is unique common fixed point of the five mappings $\Phi, \Psi, \Omega, \Lambda$ and Γ. □

Corollary 4.2 Let Σ be a complete neutrosophic metric space with $\vartheta \star \vartheta \geq \vartheta$, $\vartheta \diamond \vartheta \leq \vartheta$ for all $\vartheta \in [0, 1]$. Let Φ, Ψ and Γ be mappings from Σ into itself such that

(i) $\Gamma(\Sigma) \subset \Phi(\Sigma)$, $\Gamma(\Sigma) \subset \Omega(\Sigma)$;

(ii) There exists $k \in (0, 1)$ such that for all $\varrho, \varsigma \in \Sigma$, $\beta \in (0, 2)$ and $\vartheta > 0$

$$\Lambda(\Gamma\varrho, \Gamma\varsigma, k\vartheta) \geq \Lambda(\Phi\varrho, \Gamma\varrho, \vartheta) \star \Lambda(\Omega\varsigma, \Gamma\varsigma, \vartheta) \star \Lambda(\Phi\varrho, \Omega\varsigma, \beta\vartheta) \star \Lambda(\Phi\varrho, \Gamma\varsigma, (2-\beta)\vartheta) \star \Lambda(\Omega\varsigma, \Gamma\varrho, \vartheta),$$

$$\aleph(\Gamma\varrho, \Gamma\varsigma, k\vartheta) \leq \aleph(\Phi\varrho, \Gamma\varrho, \vartheta) \diamond \aleph(\Omega\varsigma, \Gamma\varsigma, \vartheta) \diamond \aleph(\Phi\varrho, \Omega\varsigma, \beta\vartheta) \diamond \aleph(\Phi\varrho, \Gamma\varsigma, (2-\beta)\vartheta) \diamond \aleph(\Omega\varsigma, \Gamma\varrho, \vartheta),$$

$$\beth(\Gamma\varrho, \Gamma\varsigma, k\vartheta) \leq \beth(\Phi\varrho, \Gamma\varrho, \vartheta) \diamond \beth(\Omega\varsigma, \Gamma\varsigma, \vartheta) \diamond \beth(\Phi\varrho, \Omega\varsigma, \beta\vartheta) \diamond \beth(\Phi\varrho, \Gamma\varsigma, (2-\beta)\vartheta) \diamond \beth(\Omega\varsigma, \Gamma\varrho, \vartheta).$$

(iii) Φ is continuous,

(iv) Γ and Φ are compatible of type (α),

(vi) $\Lambda(\varrho, \Omega\varrho, \vartheta) \geq \Lambda(\varrho, \Phi\varrho, \vartheta)$, $\aleph(\varrho, \Omega\varrho, \vartheta) \leq \aleph(\varrho, \Phi\varrho, \vartheta)$, $\beth(\varrho, \Omega\varrho, \vartheta) \leq \beth(\varrho, \Phi\varrho, \vartheta)$. for all $\varrho \in \Sigma$ and $\vartheta > 0$.

Then Φ, Ω and Γ have a common fixed point in Σ.

Proof. Suppose I_X be the identity mapping on Σ. We prove this corollary by using theorem (4.1) with $\Psi = \Gamma = I_X$. □

Example 4.3 Let $\Sigma = \{\frac{1}{n}; n \in \mathbb{N}\} \cup \{0\}$ be a metric defined by $d(\varrho, \varsigma) = |\varrho - \varsigma|$. For all $\varrho, \varsigma \in \Sigma$ and $\vartheta \in (0, \infty)$, define

$$\Lambda(\varrho, \varsigma, \vartheta) = \frac{\vartheta}{\vartheta + |\varrho - \varsigma|}; \quad \aleph(\varrho, \varsigma, \vartheta) = \frac{|\varrho - \varsigma|}{\vartheta + |\varrho - \varsigma|}; \quad \beth(\varrho, \varsigma, \vartheta) = \frac{|\varrho - \varsigma|}{\vartheta}$$

Clearly $(\Sigma, \Lambda, \aleph, \beth, \star, \diamond)$ is a complete neutrosophic metric space on Σ. Here $\star$ is defined by $\varrho \star \varsigma = min\{\varrho, \varsigma\}$ and $\diamond$ is defined as $\varrho \diamond \varsigma = max\{\varrho, \varsigma\}$ respectively.

Let $\Phi, \Psi, \Omega, \Lambda$ and Γ is defined by

$$\Phi(\varrho) = \frac{\varrho}{4}, \ \Psi(\varrho) = \frac{\varrho}{6}, \ \Omega(\varrho) = \frac{\varrho}{2}, \ \Lambda(\varrho) = \frac{\varrho}{3}, \ \Gamma(\varrho) = \frac{\varrho}{36}.$$

Then we have $\Gamma(\Sigma) \subset \Phi(\Sigma)$, $\Gamma(\Sigma) \subset \Omega(\Sigma)$; It is evident that $\Phi, \Psi, \Omega, \Lambda$ and Γ are continuous. Also the all conditions of Theorem(4.1) has been satisfied. Hence 0 is a unique fixed point of $\Phi, \Psi, \Omega, \Lambda$ and Γ .

5. Conclusion:

In this paper, we establish a novel concept termed Neutrosophic Metric Space (NMS) and investigate its many features. In the context of NMS, compatible maps of type (α) and type (β) definitions are defined, and various fixed point results are proven for five mappings. In addition, we provided several instances to support our findings. Additionally, neutrosophic normed space, neutrosophic triplet b-metric space, and neutrosophic triplet bipolar metric spaces can all be included in the concept of compatible mappings.

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Single valued neutrosophic ordered subalgebras of ordered BCI-algebras

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Abstract: In order to apply neutrosophic set theory to ordered BCI-algebra, the notion of single valued neutrosophic ordered subalgebra is introduced and several properties are investigated. The conditions under which single valued neutrosophic level subsets become ordered subalgebras are explored. It is investigated when the T-neutrosophic q-set, I-neutrosophic q-set and F-neutrosophic q-set can be ordered subalgebras. A special set $\mathcal{X}_0^1$ is created and conditions are established in which it becomes an ordered subalgebra.

Keywords: Ordered BCI-algebra, single valued neutrosophic ordered subalgebra, ordered subalgebra, single valued neutrosophic level subset, T- (resp., F- and I-) neutrosophic q-set, T- (resp., F- and I-) neutrosophic $\in\vee q$-set.

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1 Introduction

The neutrosophic set was introduced in the 1990s by Florentin Smarandache as an extension of fuzzy sets and intuitionistic fuzzy sets. This set is a useful device for handling uncertainty, ambiguity and incomplete information as a mathematical framework that extends the concept of classical sets. There is no room for uncertainty in classical set theory. This is because elements only belong to or do not belong to a set in classical set theory. However, in many real-world situations, we encounter imprecise or uncertain information. Neutrosophic sets provide a way to represent and reason with such information. One can use neutrosophic sets in various fields and applications, for example, decision-making, image processing, pattern recognition, expert systems, artificial intelligence, etc. Neutrosophic sets can be applied to algebraic structures to handle uncertainty and indeterminacy in mathematical operations, for example, neutrosophic field, neutrosophic ring, neutrosophic linear algebra, neutrosophic group, etc. The neutrosophic set is also applied to logical algebras (see [1], [2], [5], [6], [7], [8],[9], [10],[11], [12]). These applications show how neutrosophic sets can be incorporated into algebraic structures to accommodate uncertainty and indeterminacy in mathematical operations. By extending classical algebraic structures to neutrosophic sets, it becomes possible to perform algebraic calculations and analyses in scenarios lacking accurate or complete information. K. Iséki [3] first introduced the BCI-algebra. This algebra is a generalized version of the BCK-algebra introduced by K. Iséki and S. Tanaka [4] so as to generalize the set difference in set theory. Recently, Yang et al. [13] attempted to generalize BCI-algebra, and introduced the ordered BCI-algebra.

To apply neutrosophic set theory to the ordered BCI-algebra is the aim of this paper. The notion of single valued neutrosophic ordered subalgebras in ordered BCI-algebras is introduced, and several properties are investigated. We explore The conditions under which single valued neutrosophic level subsets become ordered subalgebras are explored. When the T-neutrosophic q-set, I-neutrosophic q-set and F-neutrosophic q-set can be ordered subalgebras is looked at. A special set $\mathcal{Q}_0^1$ is made and the conditions that it becomes an ordered subalgebra are found.

2 Preliminaries

Definition 2.1 ([13])**.** Suppose that Q is a set with a constant "ϵ", a binary operation " $\rightarrow$ " and a binary relation " $\leq_Q$ ". $\mathbf{Q} := (Q, \rightarrow, \epsilon, \leq_Q)$ is said to be an *ordered BCI-algebra* (for simplicity, OBCI-algebra) if $\mathbf{Q}$ satisfies:

$$(\forall \mathfrak{w}, \mathfrak{z}, \mathfrak{u} \in Q)(\epsilon \leq_Q (\mathfrak{w} \rightarrow \mathfrak{z}) \rightarrow ((\mathfrak{z} \rightarrow \mathfrak{u}) \rightarrow (\mathfrak{w} \rightarrow \mathfrak{u}))), \tag{2.1}$$
$$(\forall \mathfrak{w}, \mathfrak{z} \in Q)(\epsilon \leq_Q \mathfrak{w} \rightarrow ((\mathfrak{w} \rightarrow \mathfrak{z}) \rightarrow \mathfrak{z})), \tag{2.2}$$
$$(\forall \mathfrak{w} \in Q)(\epsilon \leq_Q \mathfrak{w} \rightarrow \mathfrak{w}), \tag{2.3}$$
$$(\forall \mathfrak{w}, \mathfrak{z} \in Q)(\epsilon \leq_Q \mathfrak{w} \rightarrow \mathfrak{z},\ \epsilon \leq_Q \mathfrak{z} \rightarrow \mathfrak{w} \ \Rightarrow\ \mathfrak{w} = \mathfrak{z}), \tag{2.4}$$
$$(\forall \mathfrak{w}, \mathfrak{z} \in Q)(\mathfrak{w} \leq_Q \mathfrak{z} \ \Leftrightarrow\ \epsilon \leq_Q \mathfrak{w} \rightarrow \mathfrak{z}), \tag{2.5}$$
$$(\forall \mathfrak{w}, \mathfrak{z} \in Q)(\epsilon \leq_Q \mathfrak{w},\ \mathfrak{w} \leq_Q \mathfrak{z} \ \Rightarrow\ \epsilon \leq_Q \mathfrak{z}). \tag{2.6}$$

Obviously $\mathbf{Q} := (Q, \rightarrow, \epsilon, \leq_Q)$ with $Q = \{\epsilon\}$ is an OBCI-algebra, which is said to be the *trivial OBCI-algebra*.

Proposition 2.2 ([13])**.** *Let $\mathbf{Q} := (Q, \rightarrow, \epsilon, \leq_Q)$ be an OBCI-algebra. The following hold in $\mathbf{Q}$:*

$$(\forall \mathfrak{w} \in Q)(\epsilon \rightarrow \mathfrak{w} = \mathfrak{w}). \tag{2.7}$$
$$(\forall \mathfrak{w}, \mathfrak{z}, \mathfrak{u} \in Q)(\mathfrak{u} \rightarrow (\mathfrak{z} \rightarrow \mathfrak{w}) = \mathfrak{z} \rightarrow (\mathfrak{u} \rightarrow \mathfrak{w})). \tag{2.8}$$
$$(\forall \mathfrak{w}, \mathfrak{z}, \mathfrak{u} \in Q)(\epsilon \leq_Q \mathfrak{w} \rightarrow \mathfrak{z} \ \Rightarrow\ \epsilon \leq_Q (\mathfrak{z} \rightarrow \mathfrak{u}) \rightarrow (\mathfrak{w} \rightarrow \mathfrak{u})). \tag{2.9}$$
$$(\forall \mathfrak{w}, \mathfrak{z}, \mathfrak{u} \in Q)(\epsilon \leq_Q \mathfrak{w} \rightarrow \mathfrak{z},\ \epsilon \leq_Q \mathfrak{z} \rightarrow \mathfrak{u} \ \Rightarrow\ \epsilon \leq_Q \mathfrak{w} \rightarrow \mathfrak{u}). \tag{2.10}$$
$$(\forall \mathfrak{w}, \mathfrak{z}, \mathfrak{u} \in Q)(\epsilon \leq_Q (\mathfrak{u} \rightarrow (\mathfrak{z} \rightarrow \mathfrak{w})) \rightarrow (\mathfrak{z} \rightarrow (\mathfrak{u} \rightarrow \mathfrak{w}))). \tag{2.11}$$
$$(\forall \mathfrak{w}, \mathfrak{z}, \mathfrak{u} \in Q)(\epsilon \leq_Q \mathfrak{u} \rightarrow (\mathfrak{z} \rightarrow \mathfrak{w}) \ \Rightarrow\ \epsilon \leq_Q \mathfrak{z} \rightarrow (\mathfrak{u} \rightarrow \mathfrak{w})). \tag{2.12}$$
$$(\forall \mathfrak{w}, \mathfrak{z} \in Q)(((\mathfrak{w} \rightarrow \mathfrak{z}) \rightarrow \mathfrak{z}) \rightarrow \mathfrak{z} = \mathfrak{w} \rightarrow \mathfrak{z}). \tag{2.13}$$
$$(\forall \mathfrak{w} \in Q)((\mathfrak{w} \rightarrow \mathfrak{w}) \rightarrow \mathfrak{w} = \mathfrak{w}). \tag{2.14}$$
$$(\forall \mathfrak{w}, \mathfrak{z}, \mathfrak{u} \in Q)(\epsilon \leq_Q (\mathfrak{z} \rightarrow \mathfrak{u}) \rightarrow ((\mathfrak{w} \rightarrow \mathfrak{z}) \rightarrow (\mathfrak{w} \rightarrow \mathfrak{u}))). \tag{2.15}$$
$$(\forall \mathfrak{w}, \mathfrak{z}, \mathfrak{u} \in Q)(\epsilon \leq_Q \mathfrak{w} \rightarrow \mathfrak{z} \ \Rightarrow\ \epsilon \leq_Q (\mathfrak{u} \rightarrow \mathfrak{w}) \rightarrow (\mathfrak{u} \rightarrow \mathfrak{z})). \tag{2.16}$$

Definition 2.3 ([13])**.** Let A be a subset of Q. A is said to be

- a *subalgebra* of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$ if it satisfies:

$$(\forall \mathfrak{w}, \mathfrak{z} \in Q)(\mathfrak{w}, \mathfrak{z} \in A \Rightarrow \mathfrak{w} \to \mathfrak{z} \in A). \tag{2.17}$$

- an *ordered subalgebra* of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$ if it satisfies:

$$(\forall \mathfrak{w}, \mathfrak{z} \in Q)(\mathfrak{w}, \mathfrak{z} \in A, \epsilon \leq_Q \mathfrak{w}, \epsilon \leq_Q \mathfrak{z} \Rightarrow \mathfrak{w} \to \mathfrak{z} \in A). \tag{2.18}$$

We recall that all subalgebras are ordered subalgebras, whereas the converse is not necessarily true (see [13]).

Let Q be a non-empty set. A *single valued neutrosophic set* in Q is a structure of the form:

$$\mathcal{C}_\sim := \{\langle \mathfrak{y}; \widetilde{\mathcal{C}}_T(\mathfrak{y}), \widetilde{\mathcal{C}}_I(\mathfrak{y}), \widetilde{\mathcal{C}}_F(\mathfrak{y})\rangle \mid \mathfrak{y} \in Q\}$$

where $\widetilde{\mathcal{C}}_F : Q \to [0,1]$ is a false membership function, $\widetilde{\mathcal{C}}_I : Q \to [0,1]$ is an indeterminate membership function, and $\widetilde{\mathcal{C}}_T : Q \to [0,1]$ is a truth membership function. For brevity, the symbol $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ is used for the single valued neutrosophic set

$$\mathcal{C}_\sim := \{\langle \mathfrak{y}; \widetilde{\mathcal{C}}_T(\mathfrak{y}), \widetilde{\mathcal{C}}_I(\mathfrak{y}), \widetilde{\mathcal{C}}_F(\mathfrak{y})\rangle \mid \mathfrak{y} \in Q\}.$$

Given a single valued neutrosophic set $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ in Q, we consider the following sets.

$$\begin{aligned}
\mathcal{Q}(\widetilde{\mathcal{C}}_T; \varrho) &:= \{\mathfrak{y} \in Q \mid \widetilde{\mathcal{C}}_T(\mathfrak{y}) \geq \varrho\}, \\
\mathcal{Q}(\widetilde{\mathcal{C}}_I; \sigma) &:= \{\mathfrak{y} \in Q \mid \widetilde{\mathcal{C}}_I(\mathfrak{y}) \geq \sigma\}, \\
\mathcal{Q}(\widetilde{\mathcal{C}}_F; \delta) &:= \{\mathfrak{y} \in Q \mid \widetilde{\mathcal{C}}_F(\mathfrak{y}) \leq \delta\},
\end{aligned}$$

which are called *single valued neutrosophic level subsets* of Q where $\varrho, \sigma, \delta \in [0,1]$.

3 Single valued neutrosophic ordered subalgebras

Here the notion of single valued neutrosophic (ordered) subalgebras is introduced and several properties are investigated. Unless otherwise specified, we henceforth denote an OBCI-algebra by $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$.

Definition 3.1. Let $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ be a single valued neutrosophic set in Q. $\mathcal{C}_\sim$ is said to be a *single valued neutrosophic subalgebra* of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$ if $\mathcal{C}_\sim$ satisfies:

$$(\forall \mathfrak{y}, \mathfrak{k} \in Q) \begin{pmatrix} \widetilde{\mathcal{C}}_T(\mathfrak{y} \to \mathfrak{k}) \geq \min\{\widetilde{\mathcal{C}}_T(\mathfrak{y}), \widetilde{\mathcal{C}}_T(\mathfrak{k})\} \\ \widetilde{\mathcal{C}}_I(\mathfrak{y} \to \mathfrak{k}) \geq \min\{\widetilde{\mathcal{C}}_I(\mathfrak{y}), \widetilde{\mathcal{C}}_I(\mathfrak{k})\} \\ \widetilde{\mathcal{C}}_F(\mathfrak{y} \to \mathfrak{k}) \leq \max\{\widetilde{\mathcal{C}}_F(\mathfrak{y}), \widetilde{\mathcal{C}}_F(\mathfrak{k})\} \end{pmatrix}. \tag{3.1}$$

Definition 3.2. Let $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ be a single valued neutrosophic set in Q. $\mathcal{C}_\sim$ is said to be a *single* *valued neutrosophic ordered subalgebra* of $\mathbf{Q} := (Q, \rightarrow, \epsilon, \leq_Q)$ if $\mathcal{C}_\sim$ satisfies:

$$(\forall \mathfrak{y}, \mathfrak{k} \in Q)\left(\epsilon \leq_Q \mathfrak{y},\ \epsilon \leq_Q \mathfrak{k} \ \Rightarrow \begin{cases} \widetilde{\mathcal{C}}_T(\mathfrak{y} \rightarrow \mathfrak{k}) \geq \min\{\widetilde{\mathcal{C}}_T(\mathfrak{y}), \widetilde{\mathcal{C}}_T(\mathfrak{k})\} \\ \widetilde{\mathcal{C}}_I(\mathfrak{y} \rightarrow \mathfrak{k}) \geq \min\{\widetilde{\mathcal{C}}_I(\mathfrak{y}), \widetilde{\mathcal{C}}_I(\mathfrak{k})\} \\ \widetilde{\mathcal{C}}_F(\mathfrak{y} \rightarrow \mathfrak{k}) \leq \max\{\widetilde{\mathcal{C}}_F(\mathfrak{y}), \widetilde{\mathcal{C}}_F(\mathfrak{k})\} \end{cases} \right). \tag{3.2}$$

Example 3.3. Let $Q = \{0, \epsilon, j, 1\}$ be a set, where 0 and 1 are the least element and the greatest element of Q, respectively. A binary operation " $\rightarrow$ " on Q is provided by the table below:

$\rightarrow$	1	ϵ	j	0
1	1	0	0	0
ϵ	1	ϵ	j	0
j	1	j	ϵ	0
0	1	1	1	1

Let $\leq_Q := \{(1, 1), (j, 1)\}, (\epsilon, 1), (j, j), (0, j), (\epsilon, \epsilon), (0, \epsilon), (0, 0)\}$. Then $\mathbf{Q} := (Q, \rightarrow, \epsilon, \leq_Q)$ is an OBCI-algebra (see [13]).

(i) Let $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ be a single valued neutrosophic set in Q provided by the table below:

Q	$\widetilde{\mathcal{C}}_T(\mathfrak{y})$	$\widetilde{\mathcal{C}}_I(\mathfrak{y})$	$\widetilde{\mathcal{C}}_F(\mathfrak{y})$
1	0.68	0.73	0.31
ϵ	0.68	0.73	0.31
j	0.24	0.49	0.59
0	0.68	0.73	0.31

Clearly $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ is a single valued neutrosophic subalgebra of $\mathbf{Q} := (Q, \rightarrow, \epsilon, \leq_Q)$.

(ii) Let $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ be a single valued neutrosophic set in Q provided by the table below:

Q	$\widetilde{\mathcal{C}}_T(\mathfrak{y})$	$\widetilde{\mathcal{C}}_I(\mathfrak{y})$	$\widetilde{\mathcal{C}}_F(\mathfrak{y})$
1	0.38	0.25	0.63
ϵ	0.64	0.76	0.29
j	0.38	0.25	0.63
0	0.64	0.76	0.29

Clearly $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ is a single valued neutrosophic ordered subalgebra of $\mathbf{Q} := (Q, \rightarrow, \epsilon, \leq_Q)$.

It is certain that every single valued neutrosophic subalgebra is a single valued neutrosophic ordered subalgebra. However, as the following example shows, the converse is not necessarily true. From this point of view, we can say that the single valued neutrosophic ordered subalgebra is a generalization of the single valued neutrosophic subalgebra.

$\rightarrow$	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
1	1	0	0	0	0
$\frac{3}{4}$	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
$\frac{1}{2}$	1	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	0
$\frac{1}{4}$	1	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	0
0	1	1	1	1	1

Example 3.4. Suppose that $Q = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ is a set with a binary operation " $\rightarrow$ " provided by the table: and that $\leq_Q$ is the natural order in Q. Certainly $\mathbf{Q} := (Q, \rightarrow, \frac{3}{4}, \leq_Q)$ is an OBCI-algebra (see [13]). Suppose that $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ is a single valued neutrosophic set in Q provided by the table:

Q	$\widetilde{\mathcal{C}}_T(\mathfrak{y})$	$\widetilde{\mathcal{C}}_I(\mathfrak{y})$	$\widetilde{\mathcal{C}}_F(\mathfrak{y})$
1	0.37	0.25	0.63
$\frac{3}{4}$	0.68	0.76	0.29
$\frac{1}{2}$	0.37	0.25	0.63
$\frac{1}{4}$	0.37	0.25	0.63
0	0.68	0.76	0.29

It is routine to check that $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ is a single valued neutrosophic ordered subalgebra of $\mathbf{Q} := (Q, \rightarrow, \frac{3}{4}, \leq_Q)$. However, $\mathcal{C}_\sim$ is not a single valued neutrosophic subalgebra of $\mathbf{Q} := (Q, \rightarrow, \frac{3}{4}, \leq_Q)$ since

$$\widetilde{\mathcal{C}}_T(0 \rightarrow \tfrac{3}{4}) = \widetilde{\mathcal{C}}_T(1) = 0.37 \not\geq 0.68 = \min\{\widetilde{\mathcal{C}}_T(0), \widetilde{\mathcal{C}}_T(\tfrac{3}{4})\}$$

and/or

$$\widetilde{\mathcal{C}}_F(0 \rightarrow \tfrac{3}{4}) = \widetilde{\mathcal{C}}_F(1) = 0.63 \not\leq 0.29 = \max\{\widetilde{\mathcal{C}}_F(0), \widetilde{\mathcal{C}}_F(\tfrac{3}{4})\}.$$

Theorem 3.5. *A single valued neutrosophic set $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ in Q is a single valued neutrosophic ordered subalgebra of $\mathbf{Q} := (Q, \rightarrow, \epsilon, \leq_Q)$ if and only if $\mathcal{C}_\sim$ satisfies:*

$$(\forall \mathfrak{y}, \mathfrak{k} \in Q)\left(\begin{array}{c} \epsilon \leq_Q \mathfrak{y},\, \epsilon \leq_Q \mathfrak{k},\, \mathfrak{y} \in \mathcal{Q}(\widetilde{\mathcal{C}}_T; \varrho_\mathfrak{y}),\, \mathfrak{k} \in \mathcal{Q}(\widetilde{\mathcal{C}}_T; \varrho_\mathfrak{k}) \\ \Rightarrow \mathfrak{y} \rightarrow \mathfrak{k} \in \mathcal{Q}(\widetilde{\mathcal{C}}_T; \min\{\varrho_\mathfrak{y}, \varrho_\mathfrak{k}\}) \end{array}\right), \tag{3.3}$$

$$(\forall \mathfrak{w}, \mathfrak{z} \in Q)\left(\begin{array}{c} \epsilon \leq_Q \mathfrak{w},\, \epsilon \leq_Q \mathfrak{z},\, \mathfrak{w} \in \mathcal{Q}(\widetilde{\mathcal{C}}_I; \sigma_\mathfrak{w}),\, \mathfrak{z} \in \mathcal{Q}(\widetilde{\mathcal{C}}_I; \sigma_\mathfrak{z}) \\ \Rightarrow \mathfrak{w} \rightarrow \mathfrak{z} \in \mathcal{Q}(\widetilde{\mathcal{C}}_I; \min\{\sigma_\mathfrak{w}, \sigma_\mathfrak{z}\}) \end{array}\right), \tag{3.4}$$

$$(\forall \mathfrak{y}, \mathfrak{z} \in Q)\left(\begin{array}{c} \epsilon \leq_Q \mathfrak{y},\, \epsilon \leq_Q \mathfrak{z},\, \mathfrak{y} \in \mathcal{Q}(\widetilde{\mathcal{C}}_F; \delta_\mathfrak{y}),\, \mathfrak{z} \in \mathcal{Q}(\widetilde{\mathcal{C}}_F; \delta_\mathfrak{z}) \\ \Rightarrow \mathfrak{y} \rightarrow \mathfrak{z} \in \mathcal{Q}(\widetilde{\mathcal{C}}_F; \max\{\delta_\mathfrak{y}, \delta_\mathfrak{z}\}) \end{array}\right). \tag{3.5}$$

Proof. Assume that $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ is a single valued neutrosophic ordered subalgebra of $\mathbf{Q} := (Q, \rightarrow, \epsilon, \leq_Q)$. Let $\mathfrak{y}, \mathfrak{k} \in Q$ be such that $\epsilon \leq_Q \mathfrak{y}$, $\epsilon \leq_Q \mathfrak{k}$, $\mathfrak{y} \in \mathcal{Q}(\widetilde{\mathcal{C}}_T; \varrho_\mathfrak{y})$ and $\mathfrak{k} \in \mathcal{Q}(\widetilde{\mathcal{C}}_T; \varrho_\mathfrak{k})$. Then $\widetilde{\mathcal{C}}_T(\mathfrak{y}) \geq \varrho_\mathfrak{y}$ and $\widetilde{\mathcal{C}}_T(\mathfrak{k}) \geq \varrho_\mathfrak{k}$, which imply that $\widetilde{\mathcal{C}}_T(\mathfrak{y} \rightarrow \mathfrak{k}) \geq \min\{\widetilde{\mathcal{C}}_T(\mathfrak{y}), \widetilde{\mathcal{C}}_T(\mathfrak{k})\} \geq \min\{\varrho_\mathfrak{y}, \varrho_\mathfrak{k}\}$. Hence $\mathfrak{y} \rightarrow \mathfrak{k} \in \mathcal{Q}(\widetilde{\mathcal{C}}_T; \min\{\varrho_\mathfrak{y}, \varrho_\mathfrak{k}\})$. Similarly, one is capable of verifying that $\mathfrak{w} \rightarrow \mathfrak{z} \in \mathcal{Q}(\widetilde{\mathcal{C}}_I; \min\{\sigma_\mathfrak{w}, \sigma_\mathfrak{z}\})$ for all $\mathfrak{w} \in$

$\mathcal{Q}(\widetilde{\mathcal{C}}_I;\sigma_{\mathfrak{w}})$ and $\mathfrak{z} \in \mathcal{Q}(\widetilde{\mathcal{C}}_I;\sigma_{\mathfrak{z}})$ with $\epsilon \leq_Q \mathfrak{w}$ and $\epsilon \leq_Q \mathfrak{z}$. Now, let $\mathfrak{y}, \mathfrak{z} \in Q$ be such that $\epsilon \leq_Q \mathfrak{y}$, $\epsilon \leq_Q \mathfrak{z}$, $\mathfrak{y} \in \mathcal{Q}(\widetilde{\mathcal{C}}_F;\delta_{\mathfrak{y}})$ and $\mathfrak{z} \in \mathcal{Q}(\widetilde{\mathcal{C}}_F;\delta_{\mathfrak{z}})$. Then $\widetilde{\mathcal{C}}_F(\mathfrak{y}) \leq \delta_{\mathfrak{y}}$ and $\widetilde{\mathcal{C}}_F(\mathfrak{z}) \leq \delta_{\mathfrak{z}}$. Thus

$$\widetilde{\mathcal{C}}_F(\mathfrak{y} \to \mathfrak{z}) \leq \max\{\widetilde{\mathcal{C}}_F(\mathfrak{y}), \widetilde{\mathcal{C}}_F(\mathfrak{z})\} \leq \max\{\delta_{\mathfrak{y}}, \delta_{\mathfrak{z}}\},$$

and so $\mathfrak{y} \to \mathfrak{z} \in \mathcal{Q}(\widetilde{\mathcal{C}}_F; \max\{\delta_{\mathfrak{y}}, \delta_{\mathfrak{z}}\})$.

Conversely, suppose that $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ is a single valued neutrosophic set in Q that satisfies the conditions (3.3), (3.4) and (3.5). If $\widetilde{\mathcal{C}}_T(\mathfrak{w} \to \mathfrak{z}) < \min\{\widetilde{\mathcal{C}}_T(\mathfrak{w}), \widetilde{\mathcal{C}}_T(\mathfrak{z})\}$ for some $\mathfrak{w}, \mathfrak{z} \in Q$ with $\epsilon \leq_Q \mathfrak{w}$ and $\epsilon \leq_Q \mathfrak{z}$, then

$$\widetilde{\mathcal{C}}_T(\mathfrak{w} \to \mathfrak{z}) < \varrho_{\mathfrak{o}} \leq \min\{\widetilde{\mathcal{C}}_T(\mathfrak{w}), \widetilde{\mathcal{C}}_T(\mathfrak{z})\}$$

for some $\varrho_{\mathfrak{o}} \in (0, 1]$. Hence $\mathfrak{w}, \mathfrak{z} \in \mathcal{Q}(\widetilde{\mathcal{C}}_T; \varrho_{\mathfrak{o}})$ and $\mathfrak{w} \to \mathfrak{z} \notin \mathcal{Q}(\widetilde{\mathcal{C}}_T; \varrho_{\mathfrak{o}})$, a contradiction, and thus

$$\widetilde{\mathcal{C}}_T(\mathfrak{y} \to \mathfrak{k}) \geq \min\{\widetilde{\mathcal{C}}_T(\mathfrak{y}), \widetilde{\mathcal{C}}_T(\mathfrak{k})\}$$

for all $\mathfrak{y}, \mathfrak{k} \in Q$ with $\epsilon \leq_Q \mathfrak{y}$ and $\epsilon \leq_Q \mathfrak{k}$. Similarly, we can obtain $\widetilde{\mathcal{C}}_I(\mathfrak{y} \to \mathfrak{k}) \geq \min\{\widetilde{\mathcal{C}}_I(\mathfrak{y}), \widetilde{\mathcal{C}}_I(\mathfrak{k})\}$ for all $\mathfrak{y}, \mathfrak{k} \in Q$ with $\epsilon \leq_Q \mathfrak{y}$ and $\epsilon \leq_Q \mathfrak{k}$. Let there be $\mathfrak{z}, \mathfrak{u} \in Q$ so that $\epsilon \leq_Q \mathfrak{z}$, $\epsilon \leq_Q \mathfrak{u}$ and $\widetilde{\mathcal{C}}_F(\mathfrak{z} \to \mathfrak{u}) > \max\{\widetilde{\mathcal{C}}_F(\mathfrak{z}), \widetilde{\mathcal{C}}_F(\mathfrak{u})\}$. If we take $\delta := \max\{\widetilde{\mathcal{C}}_F(\mathfrak{z}), \widetilde{\mathcal{C}}_F(\mathfrak{u})\}$, then $\mathfrak{z}, \mathfrak{u} \in \mathcal{Q}(\widetilde{\mathcal{C}}_F;\delta)$ and $\mathfrak{z} \to \mathfrak{u} \notin \mathcal{Q}(\widetilde{\mathcal{C}}_F;\delta)$, a contradiction. Hence $\widetilde{\mathcal{C}}_F(\mathfrak{k} \to \mathfrak{o}) \leq \max\{\widetilde{\mathcal{C}}_F(\mathfrak{k}), \widetilde{\mathcal{C}}_F(\mathfrak{o})\}$ for all $\mathfrak{k}, \mathfrak{o} \in Q$ with $\epsilon \leq_Q \mathfrak{k}$ and $\epsilon \leq_Q \mathfrak{o}$. Consequently, $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ is a single valued neutrosophic ordered subalgebra of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$. □

Theorem 3.6. *Given a single valued neutrosophic set $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ in Q, its nonempty single valued neutrosophic level subsets $\mathcal{Q}(\widetilde{\mathcal{C}}_T;\varrho)$, $\mathcal{Q}(\widetilde{\mathcal{C}}_I;\sigma)$ and $\mathcal{Q}(\widetilde{\mathcal{C}}_F;\delta)$ are ordered subalgebras of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$ for all $\varrho, \sigma \in (0.5, 1]$ and $\delta \in [0, 0.5)$ if and only if the following fact is established.*

$$(\forall \mathfrak{y}, \mathfrak{k} \in Q)\left(\epsilon \leq_Q \mathfrak{y},\ \epsilon \leq_Q \mathfrak{k} \ \Rightarrow \begin{cases} \max\{\widetilde{\mathcal{C}}_T(\mathfrak{y} \to \mathfrak{k}), 0.5\} \geq \min\{\widetilde{\mathcal{C}}_T(\mathfrak{y}), \widetilde{\mathcal{C}}_T(\mathfrak{k})\} \\ \max\{\widetilde{\mathcal{C}}_I(\mathfrak{y} \to \mathfrak{k}), 0.5\} \geq \min\{\widetilde{\mathcal{C}}_I(\mathfrak{y}), \widetilde{\mathcal{C}}_I(\mathfrak{k})\} \\ \min\{\widetilde{\mathcal{C}}_F(\mathfrak{y} \to \mathfrak{k}), 0.5\} \leq \max\{\widetilde{\mathcal{C}}_F(\mathfrak{y}), \widetilde{\mathcal{C}}_F(\mathfrak{k})\} \end{cases} \right). \tag{3.6}$$

Proof. Assume that the nonempty single valued neutrosophic level subsets $\mathcal{Q}(\widetilde{\mathcal{C}}_T;\varrho)$, $\mathcal{Q}(\widetilde{\mathcal{C}}_I;\sigma)$ and $\mathcal{Q}(\widetilde{\mathcal{C}}_F;\delta)$ are ordered subalgebras of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$ for all $\varrho, \sigma \in (0.5, 1]$ and $\delta \in [0, 0.5)$. Let there be $\mathfrak{w}, \mathfrak{z} \in Q$ that satisfies $\epsilon \leq_Q \mathfrak{w}$, $\epsilon \leq_Q \mathfrak{z}$ and $\max\{\widetilde{\mathcal{C}}_I(\mathfrak{w} \to \mathfrak{z}), 0.5\} < \min\{\widetilde{\mathcal{C}}_I(\mathfrak{w}), \widetilde{\mathcal{C}}_I(\mathfrak{z})\} := \sigma$. We get $\sigma \in (0.5, 1]$ and $\mathfrak{w}, \mathfrak{z} \in \mathcal{Q}(\widetilde{\mathcal{C}}_I;\sigma)$. Since $\mathcal{Q}(\widetilde{\mathcal{C}}_I;\sigma)$ is an ordered subalgebra of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$, it follows that $\mathfrak{w} \to \mathfrak{z} \in \mathcal{Q}(\widetilde{\mathcal{C}}_I;\sigma)$. Hence $\widetilde{\mathcal{C}}_I(\mathfrak{w} \to \mathfrak{z}) \geq \sigma = \min\{\widetilde{\mathcal{C}}_I(\mathfrak{w}), \widetilde{\mathcal{C}}_I(\mathfrak{z})\}$, a contradiction. Thus

$$\max\{\widetilde{\mathcal{C}}_I(\mathfrak{y} \to \mathfrak{k}), 0.5\} \geq \min\{\widetilde{\mathcal{C}}_I(\mathfrak{y}), \widetilde{\mathcal{C}}_I(\mathfrak{k})\}$$

for all $\mathfrak{y}, \mathfrak{k} \in Q$ with $\epsilon \leq_Q \mathfrak{y}$ and $\epsilon \leq_Q \mathfrak{k}$. In a similar way, we have

$$\max\{\widetilde{\mathcal{C}}_T(\mathfrak{y} \to \mathfrak{k}), 0.5\} \geq \min\{\widetilde{\mathcal{C}}_T(\mathfrak{y}), \widetilde{\mathcal{C}}_T(\mathfrak{k})\}$$

for all $\mathfrak{y}, \mathfrak{k} \in Q$ with $\epsilon \leq_Q \mathfrak{y}$ and $\epsilon \leq_Q \mathfrak{k}$. If we say

$$\min\{\widetilde{\mathcal{C}}_F(\mathfrak{w} \to \mathfrak{z}), 0.5\} > \max\{\widetilde{\mathcal{C}}_F(\mathfrak{w}), \widetilde{\mathcal{C}}_F(\mathfrak{z})\} := \delta$$

for some $\mathfrak{w}, \mathfrak{z} \in Q$ satisfying $\epsilon \leq_Q \mathfrak{w}$ and $\epsilon \leq_Q \mathfrak{z}$, then $\delta \in [0, 0.5)$ and $\mathfrak{w}, \mathfrak{z} \in \mathcal{Q}(\widetilde{\mathcal{C}}_F; \delta)$. But $\mathfrak{w} \to \mathfrak{z} \notin \mathcal{Q}(\widetilde{\mathcal{C}}_F; \delta)$, a contradiction. Thus $\min\{\widetilde{\mathcal{C}}_F(\mathfrak{y} \to \mathfrak{k}), 0.5\} \leq \max\{\widetilde{\mathcal{C}}_F(\mathfrak{y}), \widetilde{\mathcal{C}}_F(\mathfrak{k})\}$ for all $\mathfrak{y}, \mathfrak{k} \in Q$ with $\epsilon \leq_Q \mathfrak{y}$ and $\epsilon \leq_Q \mathfrak{k}$.

Conversely, a single valued neutrosophic set $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ in Q satisfies the condition (3.6). Assume that $\mathfrak{y}, \mathfrak{k} \in Q$ are such that $\epsilon \leq_Q \mathfrak{y}$ and $\epsilon \leq_Q \mathfrak{k}$. If $\mathfrak{y}, \mathfrak{k} \in \mathcal{Q}(\widetilde{\mathcal{C}}_T; \varrho) \cap \mathcal{Q}(\widetilde{\mathcal{C}}_I; \sigma)$ for all $\varrho, \sigma \in (0.5, 1]$, then $\max\{\widetilde{\mathcal{C}}_T(\mathfrak{y} \to \mathfrak{k}), 0.5\} \geq \min\{\widetilde{\mathcal{C}}_T(\mathfrak{y}), \widetilde{\mathcal{C}}_T(\mathfrak{k})\} \geq \varrho > 0.5$ and

$$\max\{\widetilde{\mathcal{C}}_I(\mathfrak{y} \to \mathfrak{k}), 0.5\} \geq \min\{\widetilde{\mathcal{C}}_I(\mathfrak{y}), \widetilde{\mathcal{C}}_I(\mathfrak{k})\} \geq \sigma > 0.5.$$

Hence $\widetilde{\mathcal{C}}_T(\mathfrak{y} \to \mathfrak{k}) \geq \varrho$ and $\widetilde{\mathcal{C}}_I(\mathfrak{y} \to \mathfrak{k}) \geq \sigma$, that is, $\mathfrak{y} \to \mathfrak{k} \in \mathcal{Q}(\widetilde{\mathcal{C}}_T; \varrho) \cap \mathcal{Q}(\widetilde{\mathcal{C}}_I; \sigma)$. Now, if $\mathfrak{y}, \mathfrak{k} \in \mathcal{Q}(\widetilde{\mathcal{C}}_F; \delta)$ for all $\delta \in [0, 0.5)$, then $\min\{\widetilde{\mathcal{C}}_F(\mathfrak{y} \to \mathfrak{k}), 0.5\} \leq \max\{\widetilde{\mathcal{C}}_F(\mathfrak{y}), \widetilde{\mathcal{C}}_F(\mathfrak{k})\} \leq \delta < 0.5$ and so $\widetilde{\mathcal{C}}_F(\mathfrak{y} \to \mathfrak{k}) \leq \delta$, i.e., $\mathfrak{y} \to \mathfrak{k} \in \mathcal{Q}(\widetilde{\mathcal{C}}_F; \delta)$. Therefore $\mathcal{Q}(\widetilde{\mathcal{C}}_T; \varrho)$, $\mathcal{Q}(\widetilde{\mathcal{C}}_I; \sigma)$ and $\mathcal{Q}(\widetilde{\mathcal{C}}_F; \delta)$ are ordered subalgebras of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$ for all $\varrho, \sigma \in (0.5, 1]$ and $\delta \in [0, 0.5)$. □

Given a single valued neutrosophic set $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ in Q and $\varrho, \sigma, \delta \in [0, 1]$, we consider the sets:

$$\begin{aligned} T_q(\mathcal{C}_\sim, \varrho) &:= \{\mathfrak{y} \in Q \mid \widetilde{\mathcal{C}}_T(\mathfrak{y}) + \varrho > 1\}, \\ I_q(\mathcal{C}_\sim, \sigma) &:= \{\mathfrak{y} \in Q \mid \widetilde{\mathcal{C}}_I(\mathfrak{y}) + \sigma > 1\}, \\ F_q(\mathcal{C}_\sim, \delta) &:= \{\mathfrak{y} \in Q \mid \widetilde{\mathcal{C}}_F(\mathfrak{y}) + \delta < 1\}, \end{aligned}$$

which are called the T-*neutrosophic* q-*set*, I-*neutrosophic* q-*set* and F-*neutrosophic* q-*set*, respectively, of $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$. Also, we consider the sets:

$$\begin{aligned} T_{\in\vee q}(\mathcal{C}_\sim, \varrho) &= \mathcal{Q}(\widetilde{\mathcal{C}}_T; \varrho) \cup T_q(\mathcal{C}_\sim, \varrho), \\ I_{\in\vee q}(\mathcal{C}_\sim, \sigma) &= \mathcal{Q}(\widetilde{\mathcal{C}}_I; \sigma) \cup I_q(\mathcal{C}_\sim, \sigma), \\ F_{\in\vee q}(\mathcal{C}_\sim, \delta) &= \mathcal{Q}(\widetilde{\mathcal{C}}_F; \delta) \cup F_q(\mathcal{C}_\sim, \delta), \end{aligned}$$

which are called the T-*neutrosophic* $\in\vee q$-*set*, I-*neutrosophic* $\in\vee q$-*set* and F-*neutrosophic* $\in\vee q$-*set*, respectively, of $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$.

Theorem 3.7. *If $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ is a single valued neutrosophic ordered subalgebra of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$, then its nonempty T- (resp., I- and F-) neutrosophic q-set $T_q(\mathcal{C}_\sim, \varrho)$ (resp., $I_q(\mathcal{C}_\sim, \sigma)$ and $F_q(\mathcal{C}_\sim, \delta)$) is an ordered subalgebra of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$ for all $\varrho \in (0, 1]$ (resp., $\sigma \in (0, 1]$ and $\delta \in [0, 1)$).*

Proof. Suppose that $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ is a single valued neutrosophic ordered subalgebra of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$, and that $\mathfrak{y}, \mathfrak{k} \in Q$ are so that $\epsilon \leq_Q \mathfrak{y}$, $\epsilon \leq_Q \mathfrak{k}$ and $\mathfrak{y}, \mathfrak{k} \in T_q(\mathcal{C}_\sim, \varrho)$ for $\varrho \in (0, 1]$. Then $\widetilde{\mathcal{C}}_T(\mathfrak{y}) + \varrho > 1$ and $\widetilde{\mathcal{C}}_T(\mathfrak{k}) + \varrho > 1$. It follows that

$$\widetilde{\mathcal{C}}_T(\mathfrak{y} \to \mathfrak{k}) + \varrho \geq \min\{\widetilde{\mathcal{C}}_T(\mathfrak{y}), \widetilde{\mathcal{C}}_T(\mathfrak{k})\} + \varrho = \min\{\widetilde{\mathcal{C}}_T(\mathfrak{y}) + \varrho, \widetilde{\mathcal{C}}_T(\mathfrak{k}) + \varrho\} > 1.$$

Hence $\mathfrak{y} \to \mathfrak{k} \in T_q(\mathcal{C}_\sim, \varrho)$, and therefore $T_q(\mathcal{C}_\sim, \varrho)$ is an ordered subalgebra of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$ for all $\varrho \in (0, 1]$. Similarly, we can verify that $I_q(\mathcal{C}_\sim, \sigma)$ is an ordered subalgebra of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$ for all $\sigma \in (0, 1]$. Now, let $\mathfrak{y}, \mathfrak{k} \in F_q(\mathcal{C}_\sim, \delta)$ for all $\delta \in [0, 1)$ and $\mathfrak{y}, \mathfrak{k} \in Q$ with $\epsilon \leq_Q \mathfrak{y}$ and $\epsilon \leq_Q \mathfrak{k}$. Then $\widetilde{\mathcal{C}}_F(\mathfrak{y}) + \delta < 1$

and $\widetilde{\mathcal{C}}_F(\mathfrak{k}) + \delta < 1$, which imply that

$$\widetilde{\mathcal{C}}_F(\mathfrak{y} \to \mathfrak{k}) + \delta \leq \max\{\widetilde{\mathcal{C}}_F(\mathfrak{y}), \widetilde{\mathcal{C}}_F(\mathfrak{y})\} + \delta = \max\{\widetilde{\mathcal{C}}_F(\mathfrak{y}) + \delta, \widetilde{\mathcal{C}}_F(\mathfrak{y}) + \delta\} < 1.$$

Hence $\mathfrak{y} \to \mathfrak{k} \in F_q(\mathcal{C}_\sim, \delta)$ for all $\delta \in [0, 1)$ and $\mathfrak{y}, \mathfrak{k} \in Q$ with $\epsilon \leq_Q \mathfrak{y}$ and $\epsilon \leq_Q \mathfrak{k}$, and therefore $F_q(\mathcal{C}_\sim, \delta)$ is an ordered subalgebra of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$ for all $\delta \in [0, 1)$. □

Theorem 3.8. *Suppose that a single valued neutrosophic set $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ in Q satisfies:*

$$(\forall \mathfrak{y}, \mathfrak{k} \in Q) \left(\begin{array}{c} \epsilon \leq_Q \mathfrak{y},\ \epsilon \leq_Q \mathfrak{k},\ \mathfrak{y} \in T_q(\mathcal{C}_\sim, \varrho_{\mathfrak{y}}),\ \mathfrak{k} \in T_q(\mathcal{C}_\sim, \varrho_{\mathfrak{k}}) \\ \Rightarrow \mathfrak{y} \to \mathfrak{k} \in T_{\in\vee q}(\mathcal{C}_\sim, \min\{\varrho_{\mathfrak{y}}, \varrho_{\mathfrak{k}}\}) \end{array} \right), \tag{3.7}$$

$$(\forall \mathfrak{w}, \mathfrak{z} \in Q) \left(\begin{array}{c} \epsilon \leq_Q \mathfrak{w},\ \epsilon \leq_Q \mathfrak{z},\ \mathfrak{w} \in I_q(\mathcal{C}_\sim, \sigma_{\mathfrak{w}}),\ \mathfrak{k} \in I_q(\mathcal{C}_\sim, \sigma_{\mathfrak{z}}) \\ \Rightarrow \mathfrak{w} \to \mathfrak{z} \in I_{\in\vee q}(\mathcal{C}_\sim, \min\{\sigma_{\mathfrak{w}}, \sigma_{\mathfrak{z}}\}) \end{array} \right), \tag{3.8}$$

$$(\forall \mathfrak{y}, \mathfrak{z} \in Q) \left(\begin{array}{c} \epsilon \leq_Q \mathfrak{y},\ \epsilon \leq_Q \mathfrak{z},\ \mathfrak{y} \in F_q(\mathcal{C}_\sim, \delta_{\mathfrak{y}}),\ \mathfrak{z} \in F_q(\mathcal{C}_\sim, \delta_{\mathfrak{z}}) \\ \Rightarrow \mathfrak{y} \to \mathfrak{z} \in F_{\in\vee q}(\mathcal{C}_\sim, \max\{\delta_{\mathfrak{y}}, \delta_{\mathfrak{z}}\}) \end{array} \right), \tag{3.9}$$

The nonempty T-neutrosophic q-set $T_q(\mathcal{C}_\sim, \varrho)$, I-neutrosophic q-set $I_q(\mathcal{C}_\sim, \sigma)$ and F-neutrosophic q-set $F_q(\mathcal{C}_\sim, \delta)$ are ordered subalgebras of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$ for all $\varrho, \sigma \in (0.5, 1]$ and $\delta \in [0, 0.5)$).

Proof. Suppose that a single valued neutrosophic set $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ satisfies (3.7), (3.8) and (3.9). Let $\mathfrak{y}, \mathfrak{k} \in Q$ be such that $\epsilon \leq_Q \mathfrak{y}$ and $\epsilon \leq_Q \mathfrak{k}$. If $\mathfrak{y}, \mathfrak{k} \in T_q(\mathcal{C}_\sim, \varrho) \cap I_q(\mathcal{C}_\sim, \sigma)$ for all $\varrho, \sigma \in (0.5, 1]$, then $\mathfrak{y} \to \mathfrak{k} \in T_{\in\vee q}(\mathcal{C}_\sim, \varrho) \cap I_{\in\vee q}(\mathcal{C}_\sim, \sigma)$ by (3.7) and (3.8). Hence $\mathfrak{y} \to \mathfrak{k} \in \mathcal{Q}(\widetilde{\mathcal{C}}_T; \varrho)$ or $\mathfrak{y} \to \mathfrak{k} \in T_q(\mathcal{C}_\sim, \varrho)$; and $\mathfrak{y} \to \mathfrak{k} \in \mathcal{Q}(\widetilde{\mathcal{C}}_I; \sigma)$ or $\mathfrak{y} \to \mathfrak{k} \in I_q(\mathcal{C}_\sim, \sigma)$. If $\mathfrak{y} \to \mathfrak{k} \in \mathcal{Q}(\widetilde{\mathcal{C}}_T; \varrho) \cap \mathcal{Q}(\widetilde{\mathcal{C}}_I; \sigma)$, then $\widetilde{\mathcal{C}}_T(\mathfrak{y} \to \mathfrak{k}) \geq \varrho > 1 - \varrho$ and $\widetilde{\mathcal{C}}_I(\mathfrak{y} \to \mathfrak{k}) \geq \sigma > 1 - \sigma$ since $\varrho, \sigma \in (0.5, 1]$. Hence $\mathfrak{y} \to \mathfrak{k} \in T_q(\mathcal{C}_\sim, \varrho) \cap I_q(\mathcal{C}_\sim, \sigma)$, and so $T_q(\mathcal{C}_\sim, \varrho)$ and $I_q(\mathcal{C}_\sim, \sigma)$ are ordered subalgebras of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$. If $\mathfrak{y}, \mathfrak{k} \in F_q(\mathcal{C}_\sim, \delta)$, then $\mathfrak{y} \to \mathfrak{k} \in F_{\in\vee q}(\mathcal{C}_\sim, \delta)$ by (3.9). Hence $\mathfrak{y} \to \mathfrak{k} \in \mathcal{Q}(\widetilde{\mathcal{C}}_F; \delta)$ or $\mathfrak{y} \to \mathfrak{k} \in F_q(\mathcal{C}_\sim, \delta)$ If $\mathfrak{y} \to \mathfrak{k} \in \mathcal{Q}(\widetilde{\mathcal{C}}_F; \delta)$, then $\widetilde{\mathcal{C}}_F(\mathfrak{y} \to \mathfrak{k}) \leq \delta < 1 - \delta$ since $\delta < 0.5$. Thus $\mathfrak{y} \to \mathfrak{k} \in F_q(\mathcal{C}_\sim, \delta)$, and therefore $F_q(\mathcal{C}_\sim, \delta)$ is an ordered subalgebra of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$. □

Proposition 3.9. *Given a single valued neutrosophic set $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ in Q, if the nonempty T-neutrosophic q-set $T_q(\mathcal{C}_\sim, \varrho)$, I-neutrosophic q-set $I_q(\mathcal{C}_\sim, \sigma)$ and F-neutrosophic q-set $F_q(\mathcal{C}_\sim, \delta)$ are ordered subalgebras of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$ for all $\varrho, \sigma \in (0, 0.5]$ and $\delta \in [0.5, 1)$, then the following assertion is valid.*

$$(\forall \mathfrak{y}, \mathfrak{k} \in Q)(\forall \varrho_{\mathfrak{y}}, \varrho_{\mathfrak{k}} \in (0, 0.5]) \left(\begin{array}{c} \epsilon \leq_Q \mathfrak{y},\ \epsilon \leq_Q \mathfrak{k},\ \mathfrak{y} \in T_q(\mathcal{C}_\sim, \varrho_{\mathfrak{y}}),\ \mathfrak{k} \in T_q(\mathcal{C}_\sim, \varrho_{\mathfrak{k}}) \\ \Rightarrow \mathfrak{y} \to \mathfrak{k} \in \mathcal{Q}(\widetilde{\mathcal{C}}_T; \max\{\varrho_{\mathfrak{y}}, \varrho_{\mathfrak{k}}\}) \end{array} \right), \tag{3.10}$$

$$(\forall \mathfrak{w}, \mathfrak{z} \in Q)(\forall \sigma_{\mathfrak{w}}, \sigma_{\mathfrak{z}} \in (0, 0.5]) \left(\begin{array}{c} \epsilon \leq_Q \mathfrak{w},\ \epsilon \leq_Q \mathfrak{z},\ \mathfrak{w} \in I_q(\mathcal{C}_\sim, \sigma_{\mathfrak{w}}),\ \mathfrak{z} \in I_q(\mathcal{C}_\sim, \sigma_{\mathfrak{z}}) \\ \Rightarrow \mathfrak{w} \to \mathfrak{z} \in \mathcal{Q}(\widetilde{\mathcal{C}}_I; \max\{\sigma_{\mathfrak{w}}, \sigma_{\mathfrak{z}}\}) \end{array} \right), \tag{3.11}$$

$$(\forall \mathfrak{y}, \mathfrak{z} \in Q)(\forall \delta_{\mathfrak{y}}, \delta_{\mathfrak{z}} \in [0.5, 1)) \left(\begin{array}{c} \epsilon \leq_Q \mathfrak{y},\ \epsilon \leq_Q \mathfrak{z},\ \mathfrak{y} \in F_q(\mathcal{C}_\sim, \delta_{\mathfrak{y}}),\ \mathfrak{z} \in F_q(\mathcal{C}_\sim, \delta_{\mathfrak{z}}) \\ \Rightarrow \mathfrak{y} \to \mathfrak{z} \in \mathcal{Q}(\widetilde{\mathcal{C}}_F; \min\{\delta_{\mathfrak{y}}, \delta_{\mathfrak{z}}\}) \end{array} \right), \tag{3.12}$$

Proof. Suppose that $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ is a single valued neutrosophic set in Q and that the nonempty T-neutrosophic q-set $T_q(\mathcal{C}_\sim, \varrho)$, I-neutrosophic q-set $I_q(\mathcal{C}_\sim, \sigma)$ and F-neutrosophic q-set $F_q(\mathcal{C}_\sim, \delta)$ are ordered subalgebras of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$ for all $\varrho, \sigma \in (0, 0.5]$ and $\delta \in [0.5, 1)$. For every $\mathfrak{y}, \mathfrak{k} \in Q$ with $\epsilon \leq_Q \mathfrak{y}$ and $\epsilon \leq_Q \mathfrak{k}$, let $\varrho_{\mathfrak{y}}, \varrho_{\mathfrak{k}} \in (0, 0.5]$ be such that $\mathfrak{y} \in T_q(\mathcal{C}_\sim, \varrho_{\mathfrak{y}})$ and $\mathfrak{k} \in T_q(\mathcal{C}_\sim, \varrho_{\mathfrak{k}})$. Then

$\mathfrak{y}, \mathfrak{k} \in T_q(\mathcal{C}_\sim, \max\{\varrho_\mathfrak{y}, \varrho_\mathfrak{k}\})$ and $\max\{\varrho_\mathfrak{y}, \varrho_\mathfrak{k}\} \in (0, 0.5]$, and so $\mathfrak{y} \to \mathfrak{k} \in T_q(\mathcal{C}_\sim, \max\{\varrho_\mathfrak{y}, \varrho_\mathfrak{k}\})$. It follows that $\widetilde{\mathcal{C}}_T(\mathfrak{y} \to \mathfrak{k}) > 1 - \max\{\varrho_\mathfrak{y}, \varrho_\mathfrak{k}\} \geq \max\{\varrho_\mathfrak{y}, \varrho_\mathfrak{k}\}$. Hence $\mathfrak{y} \to \mathfrak{k} \in \mathcal{Q}(\widetilde{\mathcal{C}}_T; \max\{\varrho_\mathfrak{y}, \varrho_\mathfrak{k}\})$, i.e., (3.10) is valid. By the similar way, we can get the result (3.11). For every $\mathfrak{y}, \mathfrak{z} \in Q$ with $\epsilon \leq_Q \mathfrak{y}$ and $\epsilon \leq_Q \mathfrak{z}$, let $\mathfrak{y} \in F_q(\mathcal{C}_\sim, \delta_\mathfrak{y})$ and $\mathfrak{z} \in F_q(\mathcal{C}_\sim, \delta_\mathfrak{z})$ for all $\delta_\mathfrak{y}, \delta_\mathfrak{z} \in [0.5, 1)$. Then $\mathfrak{y}, \mathfrak{z} \in F_q(\mathcal{C}_\sim, \min\{\delta_\mathfrak{y}, \delta_\mathfrak{z}\})$ and $\min\{\delta_\mathfrak{y}, \delta_\mathfrak{z}\} \in [0.5, 1)$. Hence $\mathfrak{y} \to \mathfrak{z} \in F_q(\mathcal{C}_\sim, \min\{\delta_\mathfrak{y}, \delta_\mathfrak{z}\})$, which implies that $\widetilde{\mathcal{C}}_F(\mathfrak{y} \to \mathfrak{z}) + \min\{\delta_\mathfrak{y}, \delta_\mathfrak{z}\} < 1$. It follows that $\widetilde{\mathcal{C}}_F(\mathfrak{y} \to \mathfrak{z}) < 1 - \min\{\delta_\mathfrak{y}, \delta_\mathfrak{z}\} \leq \min\{\delta_\mathfrak{y}, \delta_\mathfrak{z}\}$. Therefore $\mathfrak{y} \to \mathfrak{z} \in \mathcal{Q}(\widetilde{\mathcal{C}}_F; \min\{\delta_\mathfrak{y}, \delta_\mathfrak{z}\})$, which proves (3.12). □

Proposition 3.10. *Given a single valued neutrosophic set $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ in Q, if the nonempty T-neutrosophic q-set $T_q(\mathcal{C}_\sim, \varrho)$, I-neutrosophic q-set $I_q(\mathcal{C}_\sim, \sigma)$ and F-neutrosophic q-set $F_q(\mathcal{C}_\sim, \delta)$ are ordered subalgebras of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$ for all $\varrho, \sigma \in (0.5, 1]$ and $\delta \in [0, 0.5)$, then the following hold true.*

$$(\forall \mathfrak{y}, \mathfrak{k} \in Q)(\forall \varrho_\mathfrak{y}, \varrho_\mathfrak{k} \in (0.5, 1]) \left(\begin{array}{c} \epsilon \leq_Q \mathfrak{y},\ \epsilon \leq_Q \mathfrak{k},\ \mathfrak{y} \in \mathcal{Q}(\widetilde{\mathcal{C}}_T; \varrho_\mathfrak{y}),\ \mathfrak{k} \in \mathcal{Q}(\widetilde{\mathcal{C}}_T; \varrho_\mathfrak{k}) \\ \Rightarrow \mathfrak{y} \to \mathfrak{k} \in T_q(\mathcal{C}_\sim, \max\{\varrho_\mathfrak{y}, \varrho_\mathfrak{k}\}) \end{array} \right), \tag{3.13}$$

$$(\forall \mathfrak{w}, \mathfrak{z} \in Q)(\forall \sigma_\mathfrak{w}, \sigma_\mathfrak{z} \in (0.5, 1]) \left(\begin{array}{c} \epsilon \leq_Q \mathfrak{w},\ \epsilon \leq_Q \mathfrak{z},\ \mathfrak{w} \in \mathcal{Q}(\widetilde{\mathcal{C}}_I; \sigma_\mathfrak{w}),\ \mathfrak{z} \in \mathcal{Q}(\widetilde{\mathcal{C}}_I; \sigma_\mathfrak{z}) \\ \Rightarrow \mathfrak{w} \to \mathfrak{z} \in I_q(\mathcal{C}_\sim, \max\{\sigma_\mathfrak{w}, \sigma_\mathfrak{z}\}) \end{array} \right), \tag{3.14}$$

$$(\forall \mathfrak{y}, \mathfrak{z} \in Q)(\forall \delta_\mathfrak{y}, \delta_\mathfrak{z} \in [0, 0.5)) \left(\begin{array}{c} \epsilon \leq_Q \mathfrak{y},\ \epsilon \leq_Q \mathfrak{z},\ \mathfrak{y} \in \mathcal{Q}(\widetilde{\mathcal{C}}_F; \delta_\mathfrak{y}),\ \mathfrak{z} \in \mathcal{Q}(\widetilde{\mathcal{C}}_F; \delta_\mathfrak{z}) \\ \Rightarrow \mathfrak{y} \to \mathfrak{z} \in F_q(\mathcal{C}_\sim, \min\{\delta_\mathfrak{y}, \delta_\mathfrak{z}\}) \end{array} \right), \tag{3.15}$$

Proof. Suppose that $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ is a single valued neutrosophic set in Q and that the nonempty T-neutrosophic q-set $T_q(\mathcal{C}_\sim, \varrho)$, I-neutrosophic q-set $I_q(\mathcal{C}_\sim, \sigma)$ and F-neutrosophic q-set $F_q(\mathcal{C}_\sim, \delta)$ are ordered subalgebras of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$ for all $\varrho, \sigma \in (0.5, 1]$ and $\delta \in [0, 0.5)$. For every $\mathfrak{y}, \mathfrak{k} \in Q$ with $\epsilon \leq_Q \mathfrak{y}$ and $\epsilon \leq_Q \mathfrak{k}$, let $\sigma_\mathfrak{w}, \sigma_\mathfrak{z} \in (0.5, 1]$ be such that $\mathfrak{w} \in \mathcal{Q}(\widetilde{\mathcal{C}}_I; \sigma_\mathfrak{w})$ and $\mathfrak{z} \in \mathcal{Q}(\widetilde{\mathcal{C}}_I; \sigma_\mathfrak{z})$. Then $\widetilde{\mathcal{C}}_I(\mathfrak{w}) \geq \sigma_\mathfrak{w} > 1 - \sigma_\mathfrak{w} \geq 1 - \max\{\sigma_\mathfrak{w}, \sigma_\mathfrak{z}\}$ and $\widetilde{\mathcal{C}}_I(\mathfrak{z}) \geq \sigma_\mathfrak{z} > 1 - \sigma_\mathfrak{z} \geq 1 - \max\{\sigma_\mathfrak{w}, \sigma_\mathfrak{z}\}$, which induce $\mathfrak{w}, \mathfrak{z} \in I_q(\mathcal{C}_\sim, \max\{\sigma_\mathfrak{w}, \sigma_\mathfrak{z}\})$ and $\max\{\sigma_\mathfrak{w}, \sigma_\mathfrak{z}\} \in (0.5, 1]$. Since $I_q(\mathcal{C}_\sim, \max\{\sigma_\mathfrak{w}, \sigma_\mathfrak{z}\})$ is an ordered subalgebra of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$, we have $\mathfrak{w} \to \mathfrak{z} \in I_q(\mathcal{C}_\sim, \max\{\sigma_\mathfrak{w}, \sigma_\mathfrak{z}\})$. In a similar way, one obtains get $\mathfrak{y} \to \mathfrak{k} \in T_q(\mathcal{C}_\sim, \max\{\varrho_\mathfrak{y}, \varrho_\mathfrak{k}\})$ for all $\varrho_\mathfrak{y}, \varrho_\mathfrak{k} \in (0.5, 1]$ and $\mathfrak{y}, \mathfrak{k} \in Q$ with $\epsilon \leq_Q \mathfrak{y}$ and $\epsilon \leq_Q \mathfrak{k}$. For every $\mathfrak{y}, \mathfrak{z} \in Q$ with $\epsilon \leq_Q \mathfrak{y}$ and $\epsilon \leq_Q \mathfrak{z}$, let $\delta_\mathfrak{y}, \delta_\mathfrak{z} \in [0, 0.5)$ be such that $\mathfrak{y} \in \mathcal{Q}(\widetilde{\mathcal{C}}_F; \delta_\mathfrak{y})$ and $\mathfrak{z} \in \mathcal{Q}(\widetilde{\mathcal{C}}_F; \delta_\mathfrak{z})$. Then $\widetilde{\mathcal{C}}_F(\mathfrak{y}) \leq \delta_\mathfrak{y} < 1 - \delta_\mathfrak{y} \leq 1 - \min\{\delta_\mathfrak{y}, \delta_\mathfrak{z}\}$ and $\widetilde{\mathcal{C}}_F(\mathfrak{z}) \leq \delta_\mathfrak{z} < 1 - \delta_\mathfrak{z} \leq 1 - \min\{\delta_\mathfrak{y}, \delta_\mathfrak{z}\}$. Hence $\mathfrak{y}, \mathfrak{z} \in F_q(\mathcal{C}_\sim, \min\{\delta_\mathfrak{y}, \delta_\mathfrak{z}\})$, and so $\mathfrak{y} \to \mathfrak{z} \in F_q(\mathcal{C}_\sim, \min\{\delta_\mathfrak{y}, \delta_\mathfrak{z}\})$ since $\min\{\delta_\mathfrak{y}, \delta_\mathfrak{z}\} \in [0, 0.5)$ and $F_q(\mathcal{C}_\sim, \min\{\delta_\mathfrak{y}, \delta_\mathfrak{z}\})$ is an ordered subalgebra of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$. □

Proposition 3.11. *Given a single valued neutrosophic set $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ in Q, let the nonempty T-neutrosophic $\in\vee q$-set $T_{\in\vee q}(\mathcal{C}_\sim, \varrho)$, I-neutrosophic $\in\vee q$-set $I_{\in\vee q}(\mathcal{C}_\sim, \sigma)$ and F-neutrosophic $\in\vee q$-set $F_{\in\vee q}(\mathcal{C}_\sim, \delta)$ be ordered subalgebras of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$ for all $\varrho, \sigma \in (0, 1]$ and $\delta \in [0, 1)$. The following assertions are established.*

$$(\forall \mathfrak{y}, \mathfrak{k} \in Q)(\forall \varrho_\mathfrak{y}, \varrho_\mathfrak{k} \in (0, 1]) \left(\begin{array}{c} \epsilon \leq_Q \mathfrak{y},\ \epsilon \leq_Q \mathfrak{k},\ \mathfrak{y} \in T_q(\mathcal{C}_\sim, \varrho_\mathfrak{y}),\ \mathfrak{k} \in T_q(\mathcal{C}_\sim, \varrho_\mathfrak{k}) \\ \Rightarrow \mathfrak{y} \to \mathfrak{k} \in T_{\in\vee q}(\mathcal{C}_\sim, \max\{\varrho_\mathfrak{y}, \varrho_\mathfrak{k}\}) \end{array} \right), \tag{3.16}$$

$$(\forall \mathfrak{w}, \mathfrak{z} \in Q)(\forall \sigma_\mathfrak{w}, \sigma_\mathfrak{z} \in (0, 1]) \left(\begin{array}{c} \epsilon \leq_Q \mathfrak{w},\ \epsilon \leq_Q \mathfrak{z},\ \mathfrak{w} \in I_q(\mathcal{C}_\sim, \sigma_\mathfrak{w}),\ \mathfrak{k} \in I_q(\mathcal{C}_\sim, \sigma_\mathfrak{z}) \\ \Rightarrow \mathfrak{w} \to \mathfrak{z} \in I_{\in\vee q}(\mathcal{C}_\sim, \max\{\sigma_\mathfrak{w}, \sigma_\mathfrak{z}\}) \end{array} \right), \tag{3.17}$$

$$(\forall \mathfrak{y}, \mathfrak{z} \in Q)(\forall \delta_\mathfrak{y}, \delta_\mathfrak{z} \in [0, 1)) \left(\begin{array}{c} \epsilon \leq_Q \mathfrak{y},\ \epsilon \leq_Q \mathfrak{z},\ \mathfrak{y} \in F_q(\mathcal{C}_\sim, \delta_\mathfrak{y}),\ \mathfrak{z} \in F_q(\mathcal{C}_\sim, \delta_\mathfrak{z}) \\ \Rightarrow \mathfrak{y} \to \mathfrak{z} \in F_{\in\vee q}(\mathcal{C}_\sim, \min\{\delta_\mathfrak{y}, \delta_\mathfrak{z}\}) \end{array} \right). \tag{3.18}$$

Proof. Given a single valued neutrosophic set $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ in Q, suppose that the nonempty T-neutrosophic $\in\vee q$-set $T_{\in\vee q}(\mathcal{C}_\sim, \varrho)$, I-neutrosophic $\in\vee q$-set $I_{\in\vee q}(\mathcal{C}_\sim, \sigma)$ and F-neutrosophic $\in\vee q$-set $F_{\in\vee q}(\mathcal{C}_\sim, \delta)$

are ordered subalgebras of $\mathbf{Q} := (Q, \rightarrow, \epsilon, \leq_Q)$ for all $\varrho, \sigma \in (0,1]$ and $\delta \in [0,1)$. Let $\mathfrak{y}, \mathfrak{k} \in Q$ be so that $\epsilon \leq_Q \mathfrak{y}$, $\epsilon \leq_Q \mathfrak{k}$, $\mathfrak{y} \in T_q(\mathcal{C}_\sim, \varrho_\mathfrak{y})$ and $\mathfrak{k} \in T_q(\mathcal{C}_\sim, \varrho_\mathfrak{k})$ for all $\varrho_\mathfrak{y}, \varrho_\mathfrak{k} \in (0,1]$. Then $\mathfrak{y} \in T_{\in\vee q}(\mathcal{C}_\sim, \varrho_\mathfrak{y}) \subseteq T_{\in\vee q}(\mathcal{C}_\sim, \max\{\varrho_\mathfrak{y}, \varrho_\mathfrak{k}\})$ and $\mathfrak{k} \in T_{\in\vee q}(\mathcal{C}_\sim, \varrho_\mathfrak{k}) \subseteq T_{\in\vee q}(\mathcal{C}_\sim, \max\{\varrho_\mathfrak{y}, \varrho_\mathfrak{k}\})$, which imply from the hypothesis that $\mathfrak{y} \rightarrow \mathfrak{k} \in T_{\in\vee q}(\mathcal{C}_\sim, \max\{\varrho_\mathfrak{y}, \varrho_\mathfrak{k}\})$. By the similarly way, $\mathfrak{w} \rightarrow \mathfrak{z} \in I_{\in\vee q}(\mathcal{C}_\sim, \max\{\sigma_\mathfrak{w}, \sigma_\mathfrak{z}\})$ is established for all $\sigma_\mathfrak{w}, \sigma_\mathfrak{z} \in (0,1]$ and $\mathfrak{w}, \mathfrak{z} \in Q$ satisfying $\epsilon \leq_Q \mathfrak{w}$, $\epsilon \leq_Q \mathfrak{z}$, $\mathfrak{w} \in I_q(\mathcal{C}_\sim, \sigma_\mathfrak{w})$ and $\mathfrak{z} \in I_q(\mathcal{C}_\sim, \sigma_\mathfrak{z})$. For every $\mathfrak{y}, \mathfrak{z} \in Q$ satisfying $\epsilon \leq_Q \mathfrak{y}$ and $\epsilon \leq_Q \mathfrak{z}$, let $\mathfrak{y} \in F_q(\mathcal{C}_\sim, \delta_\mathfrak{y})$ and $\mathfrak{z} \in F_q(\mathcal{C}_\sim, \delta_\mathfrak{z})$ for all $\delta_\mathfrak{y}, \delta_\mathfrak{z} \in [0,1)$. Then $\mathfrak{y} \in F_q(\mathcal{C}_\sim, \delta_\mathfrak{y}) \subseteq F_q(\mathcal{C}_\sim, \min\{\delta_\mathfrak{y}, \delta_\mathfrak{z}\})$ and $\mathfrak{z} \in F_q(\mathcal{C}_\sim, \delta_\mathfrak{z}) \subseteq F_q(\mathcal{C}_\sim, \min\{\delta_\mathfrak{y}, \delta_\mathfrak{z}\})$. Since $F_q(\mathcal{C}_\sim, \min\{\delta_\mathfrak{y}, \delta_\mathfrak{z}\})$ is an ordered subalgebra of $\mathbf{Q} := (Q, \rightarrow, \epsilon, \leq_Q)$, we obtain $\mathfrak{y} \rightarrow \mathfrak{z} \in F_{\in\vee q}(\mathcal{C}_\sim, \min\{\delta_\mathfrak{y}, \delta_\mathfrak{z}\})$, as required. □

Given a single valued neutrosophic set $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ in Q, consider the set:

$$\mathcal{Q}_0^1 := \{\mathfrak{y} \in Q \mid \widetilde{\mathcal{C}}_T(\mathfrak{y}) > 0,\ \widetilde{\mathcal{C}}_I(\mathfrak{y}) > 0,\ \widetilde{\mathcal{C}}_F(\mathfrak{y}) < 1\}. \tag{3.19}$$

We find conditions for the set $\mathcal{Q}_0^1$ in (3.19) to be an ordered subalgebra of $\mathbf{Q} := (Q, \rightarrow, \epsilon, \leq_Q)$.

Theorem 3.12. *If $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ is a single valued neutrosophic ordered subalgebra of $\mathbf{Q} := (Q, \rightarrow, \epsilon, \leq_Q)$, then the set $\mathcal{Q}_0^1$ in (3.19) is an ordered subalgebra of $\mathbf{Q} := (Q, \rightarrow, \epsilon, \leq_Q)$.*

Proof. Suppose that $\mathfrak{y}, \mathfrak{k} \in Q$ are such that $\epsilon \leq_Q \mathfrak{y}$ and $\epsilon \leq_Q \mathfrak{k}$ and $\mathfrak{y}, \mathfrak{k} \in \mathcal{Q}_0^1$. Then $\widetilde{\mathcal{C}}_T(\mathfrak{y}) > 0$, $\widetilde{\mathcal{C}}_I(\mathfrak{y}) > 0$, $\widetilde{\mathcal{C}}_F(\mathfrak{y}) < 1$, $\widetilde{\mathcal{C}}_T(\mathfrak{k}) > 0$, $\widetilde{\mathcal{C}}_I(\mathfrak{k}) > 0$, and $\widetilde{\mathcal{C}}_F(\mathfrak{k}) < 1$. Suppose also that $\widetilde{\mathcal{C}}_T(\mathfrak{y} \rightarrow \mathfrak{k}) = 0$, $\widetilde{\mathcal{C}}_I(\mathfrak{y} \rightarrow \mathfrak{k}) = 0$ and $\widetilde{\mathcal{C}}_F(\mathfrak{y} \rightarrow \mathfrak{k}) = 1$, respectively. Using (3.2) derives to $0 = \widetilde{\mathcal{C}}_T(\mathfrak{y} \rightarrow \mathfrak{k}) \geq \min\{\widetilde{\mathcal{C}}_T(\mathfrak{y}), \widetilde{\mathcal{C}}_T(\mathfrak{k})\} > 0$,

$$0 = \widetilde{\mathcal{C}}_I(\mathfrak{y} \rightarrow \mathfrak{k}) \geq \min\{\widetilde{\mathcal{C}}_I(\mathfrak{y}), \widetilde{\mathcal{C}}_I(\mathfrak{k})\} > 0, \text{ and } 1 = \widetilde{\mathcal{C}}_F(\mathfrak{y} \rightarrow \mathfrak{k}) \leq \min\{\widetilde{\mathcal{C}}_F(\mathfrak{y}), \widetilde{\mathcal{C}}_F(\mathfrak{k})\} < 1,$$

respectively. This is a contradiction, and thus $\widetilde{\mathcal{C}}_T(\mathfrak{y} \rightarrow \mathfrak{k}) > 0$, $\widetilde{\mathcal{C}}_I(\mathfrak{y} \rightarrow \mathfrak{k}) > 0$ and $\widetilde{\mathcal{C}}_F(\mathfrak{y} \rightarrow \mathfrak{k}) < 1$, respectively. Hence $\mathfrak{y} \rightarrow \mathfrak{k} \in \mathcal{Q}_0^1$, and therefore $\mathcal{Q}_0^1$ is an ordered subalgebra of $\mathbf{Q} := (Q, \rightarrow, \epsilon, \leq_Q)$. □

Theorem 3.13. *If a single valued neutrosophic set $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ in Q satisfies:*

$$(\forall \mathfrak{y}, \mathfrak{k} \in Q)(\forall \varrho_\mathfrak{y}, \varrho_\mathfrak{k} \in (0,1]) \left(\begin{array}{c} \epsilon \leq_Q \mathfrak{y},\ \epsilon \leq_Q \mathfrak{k},\ \mathfrak{y} \in \mathcal{Q}(\widetilde{\mathcal{C}}_T; \varrho_\mathfrak{y}),\ \mathfrak{k} \in \mathcal{Q}(\widetilde{\mathcal{C}}_T; \varrho_\mathfrak{k}) \\ \Rightarrow \mathfrak{y} \rightarrow \mathfrak{k} \in T_q(\mathcal{C}_\sim, \min\{\varrho_\mathfrak{y}, \varrho_\mathfrak{k}\}) \end{array} \right), \tag{3.20}$$

$$(\forall \mathfrak{w}, \mathfrak{z} \in Q)(\forall \sigma_\mathfrak{w}, \sigma_\mathfrak{z} \in (0,1]) \left(\begin{array}{c} \epsilon \leq_Q \mathfrak{w},\ \epsilon \leq_Q \mathfrak{z},\ \mathfrak{w} \in \mathcal{Q}(\widetilde{\mathcal{C}}_I; \sigma_\mathfrak{w}),\ \mathfrak{z} \in \mathcal{Q}(\widetilde{\mathcal{C}}_I; \sigma_\mathfrak{z}) \\ \Rightarrow \mathfrak{w} \rightarrow \mathfrak{z} \in I_q(\mathcal{C}_\sim, \min\{\sigma_\mathfrak{w}, \sigma_\mathfrak{z}\}) \end{array} \right), \tag{3.21}$$

$$(\forall \mathfrak{y}, \mathfrak{z} \in Q)(\forall \delta_\mathfrak{y}, \delta_\mathfrak{z} \in [0,1)) \left(\begin{array}{c} \epsilon \leq_Q \mathfrak{y},\ \epsilon \leq_Q \mathfrak{z},\ \mathfrak{y} \in \mathcal{Q}(\widetilde{\mathcal{C}}_F; \delta_\mathfrak{y}),\ \mathfrak{z} \in \mathcal{Q}(\widetilde{\mathcal{C}}_F; \delta_\mathfrak{z}) \\ \Rightarrow \mathfrak{y} \rightarrow \mathfrak{z} \in F_q(\mathcal{C}_\sim, \max\{\delta_\mathfrak{y}, \delta_\mathfrak{z}\}) \end{array} \right), \tag{3.22}$$

then the set $\mathcal{Q}_0^1$ in (3.19) is an ordered subalgebra of $\mathbf{Q} := (Q, \rightarrow, \epsilon, \leq_Q)$.

Proof. Suppose that $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ is a single valued neutrosophic set in Q that satisfies the conditions (3.20), (3.21) and (3.22). For every $\mathfrak{y}, \mathfrak{k} \in Q$ with $\epsilon \leq_Q \mathfrak{y}$ and $\epsilon \leq_Q \mathfrak{k}$, let $\mathfrak{y}, \mathfrak{k} \in \mathcal{Q}_0^1$. Then $\widetilde{\mathcal{C}}_T(\mathfrak{y}) > 0$, $\widetilde{\mathcal{C}}_I(\mathfrak{y}) > 0$, $\widetilde{\mathcal{C}}_F(\mathfrak{y}) < 1$, $\widetilde{\mathcal{C}}_T(\mathfrak{k}) > 0$, $\widetilde{\mathcal{C}}_I(\mathfrak{k}) > 0$, and $\widetilde{\mathcal{C}}_F(\mathfrak{k}) < 1$. It is clear that $\mathfrak{y} \in \mathcal{Q}(\widetilde{\mathcal{C}}_T; \widetilde{\mathcal{C}}_T(\mathfrak{y})) \cap \mathcal{Q}(\widetilde{\mathcal{C}}_I; \widetilde{\mathcal{C}}_I(\mathfrak{y})) \cap \mathcal{Q}(\widetilde{\mathcal{C}}_F; \widetilde{\mathcal{C}}_F(\mathfrak{y}))$ and $\mathfrak{k} \in \mathcal{Q}(\widetilde{\mathcal{C}}_T; \widetilde{\mathcal{C}}_T(\mathfrak{k})) \cap \mathcal{Q}(\widetilde{\mathcal{C}}_I; \widetilde{\mathcal{C}}_I(\mathfrak{k})) \cap \mathcal{Q}(\widetilde{\mathcal{C}}_F; \widetilde{\mathcal{C}}_F(\mathfrak{k}))$. Suppose

$$\widetilde{\mathcal{C}}_T(\mathfrak{y} \rightarrow \mathfrak{k}) = 0, \widetilde{\mathcal{C}}_I(\mathfrak{y} \rightarrow \mathfrak{k}) = 0 \text{ and } \widetilde{\mathcal{C}}_F(\mathfrak{y} \rightarrow \mathfrak{k}) = 1,$$

respectively. We get $\widetilde{\mathcal{C}}_T(\mathfrak{y} \to \mathfrak{k}) + \min\{\widetilde{\mathcal{C}}_T(\mathfrak{y}), \widetilde{\mathcal{C}}_T(\mathfrak{k})\} = \min\{\widetilde{\mathcal{C}}_T(\mathfrak{y}), \widetilde{\mathcal{C}}_T(\mathfrak{k})\} \leq 1$,

$$\widetilde{\mathcal{C}}_I(\mathfrak{y} \to \mathfrak{k}) + \min\{\widetilde{\mathcal{C}}_I(\mathfrak{y}), \widetilde{\mathcal{C}}_I(\mathfrak{k})\} = \min\{\widetilde{\mathcal{C}}_I(\mathfrak{y}), \widetilde{\mathcal{C}}_I(\mathfrak{k})\} \leq 1,$$

and $\widetilde{\mathcal{C}}_F(\mathfrak{y} \to \mathfrak{k}) + \max\{\widetilde{\mathcal{C}}_F(\mathfrak{y}), \widetilde{\mathcal{C}}_F(\mathfrak{k})\} \geq 1$, respectively. Then $\mathfrak{y} \to \mathfrak{k} \notin T_q(\mathcal{C}_\sim, \min\{\widetilde{\mathcal{C}}_T(\mathfrak{y}), \widetilde{\mathcal{C}}_T(\mathfrak{k})\})$,

$$\mathfrak{y} \to \mathfrak{k} \notin I_q(\mathcal{C}_\sim, \min\{\widetilde{\mathcal{C}}_I(\mathfrak{y}), \widetilde{\mathcal{C}}_I(\mathfrak{k})\}),$$

and $\mathfrak{y} \to \mathfrak{k} \notin F_q(\mathcal{C}_\sim, \max\{\widetilde{\mathcal{C}}_F(\mathfrak{y}), \widetilde{\mathcal{C}}_F(\mathfrak{k})\})$, respectively. It contradicts conditions (3.20), (3.21) and (3.22), respectively. Hence $\widetilde{\mathcal{C}}_T(\mathfrak{y} \to \mathfrak{k}) > 0$, $\widetilde{\mathcal{C}}_I(\mathfrak{y} \to \mathfrak{k}) > 0$ and $\widetilde{\mathcal{C}}_F(\mathfrak{y} \to \mathfrak{k}) < 1$, that is, $\mathfrak{y} \to \mathfrak{k} \in \mathcal{Q}_0^1$. Therefore $\mathcal{Q}_0^1$ is an ordered subalgebra of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$. □

Theorem 3.14. *Suppose that a single valued neutrosophic set $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ in Q satisfies:*

$$(\forall \mathfrak{y}, \mathfrak{k} \in Q)(\forall \varrho_{\mathfrak{y}}, \varrho_{\mathfrak{k}} \in (0,1]) \left(\begin{array}{c} \epsilon \leq_Q \mathfrak{y},\ \epsilon \leq_Q \mathfrak{k},\ \mathfrak{y} \in T_q(\mathcal{C}_\sim, \varrho_{\mathfrak{y}}),\ \mathfrak{k} \in T_q(\mathcal{C}_\sim, \varrho_{\mathfrak{k}}) \\ \Rightarrow \mathfrak{y} \to \mathfrak{k} \in \mathcal{Q}(\widetilde{\mathcal{C}}_T; \min\{\varrho_{\mathfrak{y}}, \varrho_{\mathfrak{k}}\}) \end{array} \right), \tag{3.23}$$

$$(\forall \mathfrak{w}, \mathfrak{z} \in Q)(\forall \sigma_{\mathfrak{w}}, \sigma_{\mathfrak{z}} \in (0,1]) \left(\begin{array}{c} \epsilon \leq_Q \mathfrak{w},\ \epsilon \leq_Q \mathfrak{z},\ \mathfrak{w} \in I_q(\mathcal{C}_\sim, \sigma_{\mathfrak{w}}),\ \mathfrak{z} \in I_q(\mathcal{C}_\sim, \sigma_{\mathfrak{z}}) \\ \Rightarrow \mathfrak{w} \to \mathfrak{z} \in \mathcal{Q}(\widetilde{\mathcal{C}}_I; \min\{\sigma_{\mathfrak{w}}, \sigma_{\mathfrak{z}}\}) \end{array} \right), \tag{3.24}$$

$$(\forall \mathfrak{y}, \mathfrak{z} \in Q)(\forall \delta_{\mathfrak{y}}, \delta_{\mathfrak{z}} \in [0,1)) \left(\begin{array}{c} \epsilon \leq_Q \mathfrak{y},\ \epsilon \leq_Q \mathfrak{z},\ \mathfrak{y} \in F_q(\mathcal{C}_\sim, \delta_{\mathfrak{y}}),\ \mathfrak{z} \in F_q(\mathcal{C}_\sim, \delta_{\mathfrak{z}}) \\ \Rightarrow \mathfrak{y} \to \mathfrak{z} \in \mathcal{Q}(\widetilde{\mathcal{C}}_F; \max\{\delta_{\mathfrak{y}}, \delta_{\mathfrak{z}}\}) \end{array} \right). \tag{3.25}$$

The set $\mathcal{Q}_0^1$ in (3.19) is an ordered subalgebra of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$.

Proof. Suppose that $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ is a single valued neutrosophic set in Q that satisfies the conditions (3.23), (3.24) and (3.25). For every $\mathfrak{y}, \mathfrak{k} \in Q$ with $\epsilon \leq_Q \mathfrak{y}$ and $\epsilon \leq_Q \mathfrak{k}$, let $\mathfrak{y}, \mathfrak{k} \in \mathcal{Q}_0^1$. We have $\widetilde{\mathcal{C}}_T(\mathfrak{y}) > 0$, $\widetilde{\mathcal{C}}_I(\mathfrak{y}) > 0$, $\widetilde{\mathcal{C}}_F(\mathfrak{y}) < 1$, $\widetilde{\mathcal{C}}_T(\mathfrak{k}) > 0$, $\widetilde{\mathcal{C}}_I(\mathfrak{k}) > 0$, and $\widetilde{\mathcal{C}}_F(\mathfrak{k}) < 1$. Then $\widetilde{\mathcal{C}}_T(\mathfrak{y}) + 1 > 1$, $\widetilde{\mathcal{C}}_I(\mathfrak{y}) + 1 > 1$, $\widetilde{\mathcal{C}}_F(\mathfrak{y}) + 0 < 1$, $\widetilde{\mathcal{C}}_T(\mathfrak{k}) + 1 > 1$, $\widetilde{\mathcal{C}}_I(\mathfrak{k}) + 1 > 1$, and $\widetilde{\mathcal{C}}_F(\mathfrak{k}) + 0 < 1$. Hence $\mathfrak{y}, \mathfrak{k} \in T_q(\mathcal{C}_\sim, 1) \cap I_q(\mathcal{C}_\sim, 1) \cap F_q(\mathcal{C}_\sim, 0)$. If $\widetilde{\mathcal{C}}_T(\mathfrak{y} \to \mathfrak{k}) = 0$, $\widetilde{\mathcal{C}}_I(\mathfrak{y} \to \mathfrak{k}) = 0$ and $\widetilde{\mathcal{C}}_F(\mathfrak{y} \to \mathfrak{k}) = 1$, respectively, then $\widetilde{\mathcal{C}}_T(\mathfrak{y} \to \mathfrak{k}) < 1 = \min\{1,1\}$, $\widetilde{\mathcal{C}}_I(\mathfrak{y} \to \mathfrak{k}) < 1 = \min\{1,1\}$, and $\widetilde{\mathcal{C}}_F(\mathfrak{y} \to \mathfrak{k}) > 0 = \max\{0,0\}$, respectively. Thus $\mathfrak{y} \to \mathfrak{k} \notin \mathcal{Q}(\widetilde{\mathcal{C}}_T; \min\{1,1\}) \cap \mathcal{Q}(\widetilde{\mathcal{C}}_I; \min\{1,1\}) \cap \mathcal{Q}(\widetilde{\mathcal{C}}_F; \max\{0,0\})$, a contradiction. Hence $\widetilde{\mathcal{C}}_T(\mathfrak{y} \to \mathfrak{k}) > 0$, $\widetilde{\mathcal{C}}_I(\mathfrak{y} \to \mathfrak{k}) > 0$ and $\widetilde{\mathcal{C}}_F(\mathfrak{y} \to \mathfrak{k}) < 1$, that is, $\mathfrak{y} \to \mathfrak{k} \in \mathcal{Q}_0^1$. Therefore $\mathcal{Q}_0^1$ is an ordered subalgebra of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$. □

Theorem 3.15. *If a single valued neutrosophic set $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ in Q satisfies:*

$$(\forall \mathfrak{y}, \mathfrak{k} \in Q)(\forall \varrho_{\mathfrak{y}}, \varrho_{\mathfrak{k}} \in (0,1]) \left(\begin{array}{c} \epsilon \leq_Q \mathfrak{y},\ \epsilon \leq_Q \mathfrak{k},\ \mathfrak{y} \in T_q(\mathcal{C}_\sim, \varrho_{\mathfrak{y}}),\ \mathfrak{k} \in T_q(\mathcal{C}_\sim, \varrho_{\mathfrak{k}}) \\ \Rightarrow \mathfrak{y} \to \mathfrak{k} \in T_q(\mathcal{C}_\sim, \min\{\varrho_{\mathfrak{y}}, \varrho_{\mathfrak{k}}\}) \end{array} \right), \tag{3.26}$$

$$(\forall \mathfrak{w}, \mathfrak{z} \in Q)(\forall \sigma_{\mathfrak{w}}, \sigma_{\mathfrak{z}} \in (0,1]) \left(\begin{array}{c} \epsilon \leq_Q \mathfrak{w},\ \epsilon \leq_Q \mathfrak{z},\ \mathfrak{w} \in I_q(\mathcal{C}_\sim, \sigma_{\mathfrak{w}}),\ \mathfrak{z} \in I_q(\mathcal{C}_\sim, \sigma_{\mathfrak{z}}) \\ \Rightarrow \mathfrak{w} \to \mathfrak{z} \in I_q(\mathcal{C}_\sim, \min\{\sigma_{\mathfrak{w}}, \sigma_{\mathfrak{z}}\}) \end{array} \right), \tag{3.27}$$

$$(\forall \mathfrak{y}, \mathfrak{z} \in Q)(\forall \delta_{\mathfrak{y}}, \delta_{\mathfrak{z}} \in [0,1)) \left(\begin{array}{c} \epsilon \leq_Q \mathfrak{y},\ \epsilon \leq_Q \mathfrak{z},\ \mathfrak{y} \in F_q(\mathcal{C}_\sim, \delta_{\mathfrak{y}}),\ \mathfrak{z} \in F_q(\mathcal{C}_\sim, \delta_{\mathfrak{z}}) \\ \Rightarrow \mathfrak{y} \to \mathfrak{z} \in F_q(\mathcal{C}_\sim, \max\{\delta_{\mathfrak{y}}, \delta_{\mathfrak{z}}\}) \end{array} \right), \tag{3.28}$$

then the set $\mathcal{Q}_0^1$ in (3.19) is an ordered subalgebra of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$.

Proof. Suppose that $\mathcal{C}_\sim := (\widetilde{\mathcal{C}}_T, \widetilde{\mathcal{C}}_I, \widetilde{\mathcal{C}}_F)$ is a single valued neutrosophic set in Q satisfying the conditions (3.26), (3.27) and (3.28), and $\mathfrak{y}, \mathfrak{k} \in Q$ be such that $\epsilon \leq_Q \mathfrak{y}$ and $\epsilon \leq_Q \mathfrak{k}$. If $\mathfrak{y}, \mathfrak{k} \in \mathcal{Q}_0^1$, then $\widetilde{\mathcal{C}}_T(\mathfrak{y}) > 0$, $\widetilde{\mathcal{C}}_I(\mathfrak{y}) > 0$,

$\widetilde{\mathcal{C}}_F(\mathfrak{y}) < 1$, $\widetilde{\mathcal{C}}_T(\mathfrak{k}) > 0$, $\widetilde{\mathcal{C}}_I(\mathfrak{k}) > 0$, and $\widetilde{\mathcal{C}}_F(\mathfrak{k}) < 1$. Hence $\widetilde{\mathcal{C}}_T(\mathfrak{y}) + 1 > 1$, $\widetilde{\mathcal{C}}_I(\mathfrak{y}) + 1 > 1$, $\widetilde{\mathcal{C}}_F(\mathfrak{y}) + 0 < 1$, $\widetilde{\mathcal{C}}_T(\mathfrak{k})+1 > 1$, $\widetilde{\mathcal{C}}_I(\mathfrak{k})+1 > 1$, and $\widetilde{\mathcal{C}}_F(\mathfrak{k})+0 < 1$. Thus $\mathfrak{y}, \mathfrak{k} \in T_q(\mathcal{C}_\sim, 1) \cap I_q(\mathcal{C}_\sim, 1) \cap F_q(\mathcal{C}_\sim, 0)$. If $\widetilde{\mathcal{C}}_T(\mathfrak{y} \to \mathfrak{k}) = 0$, $\widetilde{\mathcal{C}}_I(\mathfrak{y} \to \mathfrak{k}) = 0$ and $\widetilde{\mathcal{C}}_F(\mathfrak{y} \to \mathfrak{k}) = 1$, respectively, then $\widetilde{\mathcal{C}}_T(\mathfrak{y} \to \mathfrak{k}) + 1 = 1$, $\widetilde{\mathcal{C}}_I(\mathfrak{y} \to \mathfrak{k}) + 1 = 1$ and $\widetilde{\mathcal{C}}_F(\mathfrak{y} \to \mathfrak{k}) + 0 = 1$, respectively. It follows that $\mathfrak{y} \to \mathfrak{k} \notin T_q(\mathcal{C}_\sim, \min\{1, 1\}) \cap I_q(\mathcal{C}_\sim, \min\{1, 1\}) \cap F_q(\mathcal{C}_\sim, \max\{0, 0\})$, a contradiction. Hence $\widetilde{\mathcal{C}}_T(\mathfrak{y} \to \mathfrak{k}) > 0$, $\widetilde{\mathcal{C}}_I(\mathfrak{y} \to \mathfrak{k}) > 0$ and $\widetilde{\mathcal{C}}_F(\mathfrak{y} \to \mathfrak{k}) < 1$, i.e., $\mathfrak{y} \to \mathfrak{k} \in \mathcal{Q}_0^1$. Consequently, $\mathcal{Q}_0^1$ is an ordered subalgebra of $\mathbf{Q} := (Q, \to, \epsilon, \leq_Q)$. □

4 Conclusion

Smarandache proposed a single-valued neutrosophic set as a part of neutrosophic theory. This set is an extension of classical set theory that allows for the representation of inconsistent, indeterminate and uncertain information in a more comprehensive manner. Single-valued neutrosophic sets have been applied in various fields, for example, image processing, decision making, medical diagnosis, natural language processing, etc., due to their ability to handle uncertainty and imprecision. Of course, it is well known that research on single-valued neutrosophic sets applied to algebraic structures is also actively underway. To apply the single-valued neutrosophic set to ordered BCI-algebras is the aim of this paper. We introduced the notion of single valued neutrosophic ordered subalgebras in ordered BCI-algebras, and investigated several related properties. We explored the conditions under which single valued neutrosophic level subsets become ordered subalgebras, and when the T-neutrosophic q-set, I-neutrosophic q-set and F-neutrosophic q-set could become ordered subalgebras. We created a special set $\mathcal{Q}_0^1$ and found the conditions that it becomes an ordered subalgebra. Based on the ideas and results of this paper, in the future we will investigate a neutrosophic set version for several types of filters in ordered BCI-algebras.

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Properties of neutrosophic $\varkappa$-ideals in subtraction semigroups

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Abstract. Our aim is to explore the idea of neutrosophic $\mathfrak{N}-$ideals in near-subtraction semigroups in this article and obtain some outcomes that are equivalent to them. We also illustrate the notion of a neutrosophic $\varkappa-$ intersection. Additionally, in a near-subtraction semigroup, we examine the term homomorphism of a neutrosophic $\varkappa-$ structure and establish some conclusions based on a homomorphic neutrosophic $\varkappa-$ structure preimage of a neutrosophic $\varkappa-$ left (respectively, right) ideal.
Keywords: Semigroups; Subtraction semigroups; neutrosophic $\varkappa-$structures, neutrosophic $\varkappa-$ ideals, homomorphism.

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1. **Introduction**

In [26], Schein investigated the systems of the type $(\Sigma, \circ, \setminus)$, where Σ is a family of functions closed under the composition $\circ$ of functions (and therefore $(\Sigma, \circ)$ is a function semigroup) and the set theoretic subtraction $\setminus$ (and therefore $(\Sigma, \setminus)$ is a subtraction algebra). In [29], Zelinka examined Schein's suggestion for the multiplication structure and discovered a method for resolving a challenge in a kind of subtraction algebra, namely atomic subtraction algebras. In subtraction algebras [11], Jun et al. proposed the idea of ideals by examining the characterisation of ideals. In [10], Jun et al. explored the ideals produced by a set and its associated outcomes. Dheena et al. [1], formed the ideas of near-subtraction semigroups as well as strongly regular near-subtraction semigroups. They found an equivalent assertion for a near-subtraction semigroup to be strongly regular.

Zadeh [30] developed the idea that a fuzzy subset φ of a set K is a map from K into $[0, 1]$. Since then, this concept has been effectively used in a range of applications, including image processing, control systems, engineering, robotics, industrial automation, and optimisation.

In subtraction algebras, Lee et al. [14] established the term fuzzy ideal and made some assertions that a fuzzy set is to be a fuzzy ideal. Prince Williams [28] coined the terms fuzzy ideals and fuzzy intersection in near-subtraction semigroups and homomorphic fuzzy images and preimages of a near-subtraction semigroup.

In [16], Molodtsov introduced a concept, namely the soft set $(F, \Im)$, which is a mapping from $\Im$ into the power set of $\mathbb{U}$ given a base universe set $\mathbb{U}$ and the gathering of attributes $\Im$. Jun et al. [12] extended Molodtsov's concept to hybrid structures, a concept that is similar to the theories of soft and fuzzy sets, and proved a number of hybrid structure attributes for a gathering of parameter values over a base universe set. The authors further explored the ideas of hybrid subalgebras, and hybrid fields based on this approach. Several authors produced hybrid concepts in a variety of algebraic structures (See [2–5, 15, 17, 18, 20–23]).

Smarandache came up with neutrophophic sets as a way to deal with the constant unpredictability. It makes intuitionistic fuzzy sets as well as fuzzy sets more broad. Neutrosophic sets can be described by these three things: their membership functions for indeterminacy (I), falsity (F), and truth (T). These sets can be used in a lot of different ways to deal with the problems that come from unclear information. A neutrosophic set can tell the difference between membership functions that are absolute and those that are relative. Smarandache used these collections for non-standard analyses like sports choices (losing, tying, and winning), control theory, decision-making theory, and so on. This area has been studied by several authors(See [8, 9, 27]).

Khan et al. examined ϵ-neutrosophic $\varkappa$-subsemigroup and a semigroup in [13]. Elavarasan et al. [6] examined the idea of neutrosophic $\varkappa$-ideals in semigroups. Elavarasan et al. presented neutrosophic filters and bi-filters in a semigroup and examined their properties in [7]. Muhiuddin et al. provided the definitions and characteristics of neutrosophic $\varkappa$-interior ideals as well as neutrosophic $\varkappa$- ideals in ordered semigroups in [19].

Porselvi et al. proposed neutrosophic $\varkappa$-interior ideal structure as well as neutrosophic $\varkappa$-simple in semigroups in [25], and they obtained comparable statements for the two types of structures. Porselvi et al. [24] described numerous characteristics of a neutrosophic $\varkappa$-bi-ideal structure in a semigroup and showed that when a semigroup is regular left duo, both a neutrosophic $\varkappa$-right ideal and a neutrosophic $\varkappa$-bi-ideal are identical. They discussed analogous claims for the regular semigroup with regard to the neutrosophic $\varkappa$-product.

This article explores the idea of neutrosophic $\varkappa-$ideal in near-subtraction semigroups and its associated characteristics. Additionally, we provide examples of a neutrosophic $\varkappa$-left ideal that is not a neutrosophic $\varkappa$-right ideal and vice versa. Moreover, we examine and discuss the neutrosophic $\varkappa$-image, neutrosophic $\varkappa$-intersection, and neutrosophic $\varkappa$-preimage of a near-subtraction semigroup using homomorphism.

2. Preliminaries of subtraction semigroups

We compile some basic definitions for near-subtraction semigroups in this portion, which we will use in the next section.

Definition 2.1. [26] A set $\Im(\neq \emptyset)$ with the binary operation " $-$ " that fulfils the below assertions is referred to as a subtraction algebra. $\forall q_0, l_0, i_0 \in \Im$,

(i) $q_0 - (l_0 - q_0) = q_0$.

(ii) $q_0 - (q_0 - l_0) = l_0 - (l_0 - q_0)$.

(iii) $(q_0 - l_0) - i_0 = (q_0 - i_0) - l_0$.

The following are some characteristics of a subtraction algebra:

(i) $q_0 - 0 = q_0$ and $0 - q_0 = 0$.

(ii) $(q_0 - l_0) - q_0 = 0$.

(iii) $(q_0 - l_0) - l_0 = q_0 - l_0$.

(iv) $(q_0 - l_0) - (l_0 - q_0) = q_0 - l_0$, where $0 = q_0 - q_0$ is an element that is independent on the choice of $q_0 \in \Im$.

Definition 2.2. [29] A set $\Im(\neq \emptyset)$ with the binary operations " $-$ " and "." that satisfies the following requirements is referred to as a subtraction semigroup:

(i) $(\Im, -)$ and $(\Im, .)$ are a subtraction algebra and a semigroup, respectively.

(ii) $l_0(l_1 - l_2) = l_0 l_1 - l_0 l_2$ and $(l_0 - l_1)l_2 = l_0 l_2 - l_1 l_2$ $\forall l_0, l_1, l_2 \in \Im$.

Definition 2.3. [29] A set $\Im(\neq \emptyset)$ with the binary operations " $-$ " and "." that satisfy the following requirements is referred to as a near-subtraction semigroup (NSS for short):

(i) $(\Im, -)$ and $(\Im, .)$ are a subtraction algebra and a semigroup, respectively.

(ii) $(l_0 - l_1)l_2 = l_0 l_2 - l_1 l_2$ $\forall l_0, l_1, l_2 \in \Im$.

Clearly $0 l_0 = 0$ $\forall l_0 \in \Im$.

Hereafter, $\Im$ represents the near-subtraction semigroup.

Definition 2.4. If $l_0 - l_1 \in L$ whenever $l_0, l_1 \in L$, then a subset $L(\neq \emptyset)$ of $\Im$ is said to be a subalgebra of $\Im$.

Definition 2.5. Let $(\Im, -, .)$ be a NSS. A subset $\Re(\neq \emptyset)$ of $\Im$ is referred as

(i) a right ideal whenever $\Re$ is a subalgebra of $(\Im, -)$ and $\Re\Im \subseteq \Re$.

(ii) a left ideal whenever $\Re$ is a subalgebra of $(\Im, -)$ and $p_1 c_1 - p_1(w_1 - c_1) \in \Re$ $\forall p_1, w_1 \in \Im; c_1 \in \Re$.

(iii) an ideal whenever $\Re$ is both a right and a left ideal.

3. Preliminaries of Neutrosophic $\varkappa$- structures

This portions outlines the basic ideas of neutrosophic $\varkappa$-structures of $\Im$, which are essential for the sequel.

For a set $Q(\neq \emptyset)$, $\mathcal{F}(Q,\mathbb{I}^-)$ is the family of functions with negative-values from a set Q to $\mathbb{I}^-$, where $\mathbb{I}^- = [-1,0]$. An element $k_1 \in \mathcal{F}(Q,\mathbb{I}^-)$ is known as a $\varkappa$-function on Q and $\varkappa$-structure denotes (Q,k_1) of X.

Definition 3.1. [12] For a set $Q(\neq \emptyset)$, a *neutrosophic $\varkappa$- structure* of Q is described as below:

$$Q_M := \frac{Q}{(T_M,I_M,F_M)} = \left\{\frac{v_0}{(T_M(v_0),I_M(v_0),F_M(v_0))} \ : \ v_0 \in Q\right\},$$

where T_M on Q means the negative truth membership function, I_M on Q means the negative indeterminacy membership function and F_M on Q means the negative false membership function.

Note 3.2. *Q_M satisfies the requirement: $-3 \leq T_M(b_1) + I_M(b_1) + F_M(b_1) \leq 0$ $\forall b_1 \in Q$.*

Definition 3.3. [13] For a set $Q(\neq \emptyset)$, let $Q_J := \frac{Q}{(T_J,I_J,F_J)}$ and $Q_V := \frac{Q}{(T_V,I_V,F_V)}$,

(i) Q_J is defined as a *neutrosophic $\varkappa$-substructure* of Q_V, represented by $Q_J \subseteq Q_V$, if it fulfils the below criteria: for any $z_0 \in Q$,

$$T_J(z_0) \geq T_V(z_0), I_J(z_0) \leq I_V(z_0), F_J(z_0) \geq F_V(z_0).$$

If $Q_J \subseteq Q_V$ and $Q_V \subseteq Q_J$, then $Q_J = Q_V$.

(ii) The intersection of Q_J and Q_V is a neutrosophic $\varkappa$-structure over Q and is defined as follows: $Q_J \cap Q_V = Q_{J\cap V} = (Q; T_{J\cap V}, I_{J\cap V}, F_{J\cap V})$, where

$$(T_J \cap T_V)(h_0) = T_{J\cap V}(h_0) = T_J(h_0) \vee T_V(h_0),$$
$$(I_J \cap I_V)(h_0) = I_{J\cap V}(h_0) = I_J(h_0) \wedge I_V(h_0),$$
$$(F_J \cap F_V)(h_0) = F_{J\cap V}(h_0) = F_J(h_0) \vee F_V(h_0) \ for\ any\ h_0 \in Q.$$

Definition 3.4. For $V_0 \subseteq Q \neq \emptyset$, consider the neutrosophic $\varkappa$-structure

$$\chi_{V_0}(Q_D) = \frac{Q}{(\chi_V(T)_D, \chi_V(I)_D, \chi_V(F)_D)},$$

where

$$\chi_{V_0}(T)_D : Q \to \mathbb{I}^-,\ j_1 \to \begin{cases} -1 & if\ j_1 \in V_0 \\ 0 & if\ j_1 \notin V_0, \end{cases}$$
$$\chi_{V_0}(I)_D : Q \to \mathbb{I}^-,\ j_1 \to \begin{cases} 0 & if\ j_1 \in V_0 \\ -1 & if\ j_1 \notin V_0, \end{cases}$$
$$\chi_{V_0}(F)_D : Q \to \mathbb{I}^-,\ j_1 \to \begin{cases} -1 & if\ j_1 \in V_0 \\ 0 & if\ j_1 \notin V_0, \end{cases}$$

which is described as the *characteristic neutrosophic $\varkappa$-structure* of V_0 over Q.

Definition 3.5. [12] For a nonempty set Q, let $Q_N = \frac{Q}{(T_N, I_N, F_N)}$ and $\eth, \varphi, \Theta \in \mathbb{I}^-$ with $-3 \leq \eth + \varphi + \Theta \leq 0$. Consider the following sets:

$T_N^{\eth} = \{c_1 \in Q \mid T_N(c_1) \leq \eth\}, I_N^{\varphi} = \{c_1 \in Q \mid I_N(c_1) \geq \varphi\}, F_N^{\Theta} = \{c_1 \in Q | F_N(c_1) \leq \Theta\}$. Then the set $Q_N(\eth, \varphi, \Theta) = \{c_1 \in Q | T_N(c_1) \leq \eth, I_N(c_1) \geq \varphi, F_N(c_1) \leq \Theta\}$ is referred as a $(\eth, \varphi, \Theta)$-*level set* of Q_N. Note that $Q_N(\eth, \varphi, \Theta) = T_N^{\eth} \cap I_N^{\varphi} \cap F_N^{\Theta}$.

4. Neutrosophic $\varkappa$-ideals in subtraction semigroups

The idea of neutrosophic $\varkappa-$ ideals in near-subtraction is defined in this portion. We also develop a case where a neutrosophic $\varkappa-$ right ideal is not a neutrosophic $\varkappa-$ left ideal, and vice versa, and we describe certain properties of a neutrosophic $\varkappa-$ structure's homomorphism in a near-subtraction semigroup.

Definition 4.1. A neutrosophic $\varkappa$-structure $\Im_B = \frac{\Im}{(T_B, I_B, F_B)}$ of $\Im$ is defined as a *neutrosophic $\varkappa$-ideal* of $\Im$ if it meets the below axioms:

(i) $(\forall g_0, l_0 \in \Im) \begin{pmatrix} T_B(g_0 - l_0) \leq T_B(g_0) \vee T_B(l_0) \\ I_B(g_0 - l_0) \geq I_B(g_0) \wedge I_B(l_0) \\ F_B(g_0 - l_0) \leq F_B(g_0) \vee F_B(l_0) \end{pmatrix}$.

(ii) $(\forall s_0, j_0, l_0 \in \Im) \begin{pmatrix} T_B(s_0 l_0 - s_0(j_0 - l_0)) \leq T_B(l_0) \\ I_B(s_0 l_0 - s_0(j_0 - l_0)) \geq I_B(l_0) \\ F_B(s_0 l_0 - s_0(j_0 - l_0)) \leq F_B(l_0) \end{pmatrix}$.

(iii) $(\forall l_0, q_0 \in \Im) \begin{pmatrix} T_B(l_0 q_0) \leq T_B(l_0) \\ I_B(l_0 q_0) \geq I_B(l_0) \\ F_B(l_0 q_0) \leq F_B(l_0) \end{pmatrix}$.

Note that $\Im_B$ of $\Im$ is a *neutrosophic $\varkappa-$left ideal* when (i) and (ii) are hold, and $\Im_B$ of $\Im$ is a *neutrosophic $\varkappa-$right ideal* when (i) and (iii) are hold.

Notation 1. *Let $\Im$ be a NSS. Then we use the below notations:*

(i) $\mathscr{N}_{\mathfrak{I}}(\Im)$ is the gathering of all neutrosophic $\varkappa-$ ideals of $\Im$.

(ii) $\mathscr{N}_{\mathfrak{R}}(\Im)$ is the gathering of all neutrosophic $\varkappa-$ right ideals of $\Im$.

(iii) $\mathscr{N}_{\mathfrak{L}}(\Im)$ is the gathering of all neutrosophic $\varkappa-$ left ideals of $\Im$.

Here are a few examples of neutrosophic $\varkappa$-ideals.

Example 4.2. Let $\Im = \{0, i_0, p_0\}$ be a set with two operations " $-$ " and "." that are represented by the below tables:

-	0	i_0	p_0
0	0	0	0
i_0	i_0	0	i_0
p_0	p_0	p_0	0

.	0	i_0	p_0
0	0	0	0
i_0	0	i_0	0
p_0	i_0	0	p_0

Then $(\Im, -, .)$ is a NSS. Define a neutrosophic $\varkappa$-structure $\Im_N := \{\frac{0}{(w,l,w_1)}, \frac{i_0}{(r,k,r_1)}, \frac{p_0}{(y,v,y_1)}\}$ of $\Im$ for $v, k, l, w, w_1, r, r_1, y, y_1 \in [-1, 0]$.

(i) If $y > r = w; v < k = l$ and $y_1 > r_1 = w_1$, then $\Im_N \in \mathscr{N}_{\mathfrak{I}}(\Im)$.

(ii) If $y = r > w; k = v < l$ and $y_1 = r_1 > w_1$, then $\Im_N \in \mathscr{N}_{\mathfrak{R}}(\Im)$, but $\Im_N \notin \mathscr{N}_{\mathfrak{L}}(\Im)$ as $T_N(p_0.0-p_0(p_0-0)) = T_N(i_0) = r \nleq w = T_N(0); I_N(p_0.0-p_0(p_0-0)) = I_N(i_0) = k \ngeq l = I_N(0)$ and $F_N(p_0.0 - p_0(p_0 - 0)) = F_N(i_0) = r_1 \nleq w_1 = F_N(0)$.

(iii) If $r > y > w; k < v < l$ and $r_1 > y_1 > w_1$, then $\Im_N$ is neither in $\mathscr{N}_{\mathfrak{R}}(\Im)$ nor in $\mathscr{N}_{\mathfrak{L}}(\Im)$ as $T_N(p_0.0 - p_0(i_0 - 0)) = T_N(i_0) = r \nleq w = T_N(0)$, $I_N(p_0.0 - p_0(i_0 - 0)) = I_N(i_0) = k \ngeq l = I_N(0)$, $F_N(p_0.0 - p_0(i_0 - 0)) = F_N(i_0) = r_1 \nleq w_1 = F_N(0)$ and $T_N(p_0.0) = T_N(i_0) = r \nleq y = T_N(p_0)$, $I_N(p_0.0) = I_N(i_0) = k \ngeq v = I_N(p_0)$, $F_N(p_0.0) = F_N(i_0) = r_1 \nleq y_1 = F_N(p_0)$. But it fulfils the assertion (i) of Definition 4.1.

Example 4.3. Let $\Im = \{0, r, l, k\}$ be a set with two operations " $-$ " and "." are given by

-	0	r	l	k
0	0	0	0	0
r	r	0	k	l
l	l	0	0	l
k	k	0	k	0

.	0	r	l	k
0	0	0	0	0
r	0	r	l	k
l	0	0	0	0
k	0	r	l	k

Then $(\Im, -, .)$ is a NSS. For $p, w, n, m, m_1, y, y_1, s, s_1 \in [-1, 0]$, define a neutrosophic $\varkappa$-structure $\Im_N := \{\frac{0}{(m,p,m_1)}, \frac{r}{(y,w,y_1)}, \frac{l}{(s,n,s_1)}, \frac{k}{(s,n,s_1)}\}$ of $\Im$. If $s > y > m, n < w < p$ and $s_1 > y_1 > m_1$, then $\Im_N \in \mathscr{N}_{\mathfrak{L}}(\Im)$, but $\Im_N \notin \mathscr{N}_{\mathfrak{R}}(\Im)$ as $T_N(r.l) = T_N(l) = s \nleq y = T_N(r)$, $I_N(r.l) = I_N(l) = n \ngeq w = I_N(r)$ and $F_N(r.l) = F_N(l) = s_1 \nleq y_1 = F_N(r)$.

Theorem 4.4. *For $\Im_N = \frac{\Im}{(T_N, I_N, F_N)}$, the listed assertions are equivalent:*

(i) For any $\varrho, \lambda, \nu \in \mathbb{I}^-$, $\Im_N(\varrho, \lambda, \nu)(\neq \phi)$ of $\Im$ is a left(right) ideal,

(ii) $\Im_N \in \mathscr{N}_{\mathfrak{L}}(\Im)$ $(\mathscr{N}_{\mathfrak{R}}(\Im))$.

Proof: $(i) \Rightarrow (ii)$ Let $c, z \in \Im$. Then $T_N(c) = q_1; F_N(c) = r_1; I_N(c) = t_1$ and $T_N(z) = q_2; F_N(z) = r_2; I_N(z) = t_2$, for some $q_1, q_2, t_1, t_2, r_1, r_2 \in \mathbb{I}^-$.

If $q = max\{q_1, q_2\}; t = min\{t_1, t_2\}$ and $r = max\{r_1, r_2\}$, then $T_N(c) \leq q; I_N(c) \geq t; F_N(c) \leq r$ and $T_N(z) \leq q; I_N(z) \geq t; F_N(z) \leq r$, so $c, z \in \Im_N(q, t, r)$. By assumption, we get $c - z \in \Im_N(q, t, r)$ which implies $T_N(c-z) \leq q = T_N(c) \vee T_N(z)$; $I_N(c-z) \geq t = I_N(c) \wedge I_N(z)$; $F_N(c-z) \leq r = F_N(c) \vee F_N(z)$.

For any $n_0, v \in \Im$, we have $n_0c-n_0(v-c) \in \Im_N(q_1, t_1, r_1)$ which implies $T_N(n_0c-n_0(v-c)) \leq q_1 = T_n(c), I_N(n_0c-n_0(v-c)) \geq t_1 = I_N(c), F_N(n_0c-n_0(v-c)) \leq r_1 = F_N(c)$. So $\Im_N \in \mathscr{N}_{\mathfrak{L}}(\Im)$.

Also, for $r \in \Im$, we have $cr \in \Im_N(q_1, t_1, r_1)$ which implies $T_N(cr) \leq q_1 = T_N(c); I_N(cr) \geq t_1 = I_N(c); F_N(cr) \leq r_1 = F_N(c)$. So $\Im_N \in \mathscr{N}_{\mathfrak{R}}(\Im)$.

$(ii) \Rightarrow (i)$ Let $q, z \in \Im_N(\varrho, \lambda, \nu)$. Then $T_N(q - z) \leq T_N(q) \vee T_N(z) \leq \varrho$; $I_N(q - z) \geq I_N(q) \wedge I_N(z) \geq \lambda$ and $F_N(q - z) \leq F_N(q) \vee F_N(z) \leq \nu$ which imply $q - z \in \Im_N(\varrho, \lambda, \nu)$.

Also, $T_N(qz) \leq T_N(q) \leq \varrho$; $I_N(qz) \geq I_N(q) \geq \lambda$ and $F_N(qz) \leq F_N(q) \leq \nu$ imply that $qz \in \Im_N(\varrho, \lambda, \nu)$. So $\Im_N(\varrho, \lambda, \nu)$ of $\Im$ is a right ideal.

For $l \in \Im_N(\varrho, \lambda, \nu)$ and $s, j \in \Im$, we have $T_N(sl - s(j-l)) \leq T_N(l) = \varrho; I_N(sl - s(j-l)) \geq I_N(l) = \lambda$ and $F_N(sl - s(j-l)) \leq F_n(l) = \nu$ which imply $sl - s(j-l) \in \Im_N(\varrho, \lambda, \nu)$.

So, $\Im_N(\varrho, \lambda, \nu)$ of $\Im$ is a left ideal.

We have the succeeding corollary as a outcome of the Theorem 4.4.

Corollary 4.5. *For $\emptyset \neq D \subseteq \Im$, a neutrosophic $\varkappa$- structure $\Im_N = \frac{\Im}{(T_N, I_N, F_N)}$ of $\Im$ is characterized as below: For $g_1, l_1, \omega_1, t_1, s_1, v_1 \in [-1, 0]$,*

$$T_N(y_0) := \begin{cases} g_1 & if\ y_0 \in D \\ l_1 & otherwise \end{cases}; \quad I_N(y_0) := \begin{cases} \omega_1 & if\ y_0 \in D \\ t_1 & otherwise, \end{cases}; \quad F_N(y_0) := \begin{cases} s_1 & if\ y_0 \in D \\ v_1 & otherwise, \end{cases}$$

where $g_1 < l_1; \omega_1 > t_1$ and $s_1 < v_1$ in $[-1, 0]$, the mentioned below statements are equivalent:

(i) $\Im_N \in \mathscr{N}_{\mathfrak{L}}(\Im)(\mathscr{N}_{\mathfrak{R}}(\Im))$,

(ii) D of $\Im$ is a left(right) ideal.

Corollary 4.6. *For $\emptyset \neq L \subseteq \Im$ and $\Im_N = \frac{\Im}{(T_N, I_N, F_N)}$, the listed below statements are equivalent:*

(i) $\chi_L(\Im_N) \in \mathscr{N}_{\mathfrak{L}}(\Im)(\mathscr{N}_{\mathfrak{R}}(\Im))$,

(ii) L of $\Im$ is a left(right) ideal.

Theorem 4.7. *Let $\Im_N = \frac{\Im}{(T_N, I_N, F_N)} \in \mathscr{N}_{\mathfrak{L}}(\Im)(\mathscr{N}_{\mathfrak{R}}(\Im))$. Then the sets $T_N^0 = \{c_1 \in Q \mid T_N(c_1) = T_N(0)\}, I_N^0 = \{c_1 \in Q \mid I_N(c_1) = I_N(0)\}, F_N^0 = \{c_1 \in Q | F_N(c_1) = F_N(0)\}$ of $\Im$ are left (right) ideals.*

Proof: For $l_0, w_0 \in T_N^0 \cap I_N^0 \cap F_N^0$, we have $T_N(l_0 - w_0) \leq T_N(l_0) \vee T_N(w_0) = T_N(0)$, $I_N(l_0 - w_0) \geq I_N(l_0) \wedge I_N(w_0) = I_N(0)$ and $F_N(l_0 - w_0) \leq F_N(l_0) \vee F_N(w_0) = F_N(0)$. So $l_0 - w_0 \in T_N^0 \cap I_N^0 \cap F_N^0$.

For $s \in \Im$, we have $T_N(sl_0 - s(w_0 - l_0)) \leq T_N(l_0) = T_N(0)$, $I_N(sl_0 - s(w_0 - l_0)) \geq I_N(l_0) = I_N(0)$ and $F_N(sl_0 - s(w_0 - l_0)) \leq F_N(l_0) = F_N(0)$. So $sl_0 - s(w_0 - l_0) \in T_N^0 \cap I_N^0 \cap F_N^0$.

Therefore T_N^0, I_N^0 and F_N^0 are left ideals.

Theorem 4.8. *Let $\Im_J := \frac{\Im}{(T_J, I_J, F_J)}$ and $\Im_W := \frac{\Im}{(T_W, I_W, F_W)}$ be the neutrosophic $\varkappa$-structures in $\Im$. If $\Im_J, \Im_W \in \mathscr{N}_{\mathfrak{L}}(\Im)(\mathscr{N}_{\mathfrak{R}}(\Im))$, then $\Im_J \cap \Im_W \in \mathscr{N}_{\mathfrak{L}}(\Im)(\mathscr{N}_{\mathfrak{R}}(\Im))$.*

Proof: Let $w_1, f_1 \in \Im$. Then

$$\begin{aligned} T_{J \cap W}(f_1 - w_1) &= (T_J \cap T_W)(f_1 - w_1) \\ &= T_J(f_1 - w_1) \vee T_W(f_1 - w_1) \\ &\leq \{T_J(f_1) \vee T_J(w_1)\} \vee \{T_W(f_1) \vee T_W(w_1)\} \\ &= (T_J \cap T_W)(f_1) \vee (T_J \cap T_W)(w_1) = T_{J \cap W}(f_1) \vee T_{J \cap W}(w_1), \end{aligned}$$

$$\begin{aligned} I_{J\cap W}(f_1-w_1) &= (I_J\cap I_W)(f_1-w_1) \\ &= I_J(f_1-w_1)\wedge I_W(f_1-w_1) \\ &\geq \{I_J(f_1)\wedge I_J(w_1)\}\wedge\{I_W(f_1)\wedge I_W(w_1)\} \\ &= (I_J\cap I_W)(f_1)\wedge(I_J\cap I_W)(w_1) = I_{J\cap W}(f_1)\wedge I_{J\cap W}(w_1), \\ F_{J\cap W}(f_1-w_1) &= (F_J\cap F_W)(f_1-w_1) \\ &= F_J(f_1-w_1)\vee F_W(f_1-w_1) \\ &\leq \{F_J(f_1)\vee F_J(w_1)\}\vee\{F_W(f_1)\vee F_W(w_1)\} \\ &= (F_J\cap F_W)(f_1)\vee(F_J\cap F_W)(w_1) = F_{J\cap W}(f_1)\vee F_{J\cap W}(w_1). \end{aligned}$$

For $s_1 \in \Im$, we have

$$\begin{aligned} T_{J\cap W}(s_1w_1-s_1(f_1-w_1)) &= (T_J\cap T_W)(s_1w_1-s_1(f_1-w_1)) \\ &= T_J(s_1w_1-s_1(f_1-w_1))\vee T_W(s_1w_1-s_1(f_1-w_1)) \\ &\leq T_J(w_1)\vee T_W(w_1) = (T_J\cap T_W)(w_1), \\ I_{J\cap W}(s_1w_1-s_1(f_1-w_1)) &= (I_J\cap I_W)(s_1w_1-s_1(f_1-w_1)) \\ &= I_J(s_1w_1-s_1(f_1-w_1))\wedge I_W(s_1w_1-s_1(f_1-w_1)) \\ &\geq I_J(w_1)\wedge I_W(w_1) = (I_J\cap I_W)(w_1), \\ F_{J\cap W}(s_1w_1-s_1(f_1-w_1)) &= (F_J\cap F_W)(s_1w_1-s_1(f_1-w_1)) \\ &= F_J(s_1w_1-s_1(f_1-w_1))\vee F_W(s_1w_1-s_1(f_1-w_1)) \\ &\leq F_J(w_1)\vee F_W(w_1) = (F_J\cap F_W)(w_1). \end{aligned}$$

So, $\Im_J \cap \Im_W \in \mathscr{N}_{\mathfrak{L}}(\Im)$.

Hereafter, the symbols $\Im$ and $\Im'$ denote the near-subtraction semigroups.

Definition 4.9. A homomorphism ξ of $\Im$ into $\Im'$ such that $\xi(w_1 - a_1) = \xi(w_1) - \xi(a_1)$ and $\xi(w_1a_1) = \xi(w_1)\xi(a_1)\ \forall w_1, a_1 \in \Im$ is defined.

Definition 4.10. Consider a mapping $\Omega : \mathbb{N} \to \mathbb{M}$, where $\mathbb{N}, \mathbb{M} \neq \{\phi\}$. Suppose $\mathbb{M}_S := \frac{\mathbb{M}}{(T_S,I_S,F_S)}$ over $\mathbb{M}$ is a neutrosophic $\varkappa$-structure. Then, under Ω, the preimage of $\mathbb{M}_S$ is described as a neutrosophic $\varkappa$-structure $\Omega^{-1}(\mathbb{M}_S) = \frac{\mathbb{N}}{(\Omega^{-1}(T_S),\Omega^{-1}(I_S),\Omega^{-1}(F_S))}$ over $\mathbb{N}$, where $\Omega^{-1}(T_S)(l_0) = T_S(\Omega(l_0))$, $\Omega^{-1}(I_S)(l_0) = I_S(\Omega(l_0))$ and $\Omega^{-1}(F_S)(l_0) = F_S(\Omega(l_0))$ for all $l_0 \in \mathbb{N}$.

Theorem 4.11. *Let $\Omega : \Im \to \Im'$ be a homomorphism of NSS. If $\Im'_S \in \mathscr{N}_{\mathfrak{J}}(\Im')$, where $\Im'_S := \frac{\Im'}{(T_S,I_S,F_S)}$, then $\Omega^{-1}(\Im'_S) \in \mathscr{N}_{\mathfrak{J}}(\Im)$.*

Proof: Let $k_0, g_0 \in \Im$. Then

$$\begin{aligned} \Omega^{-1}(T_S)(k_0-g_0) = T_S(\Omega(k_0-g_0)) &= T_S(\Omega(k_0)-\Omega(g_0)) \\ &\leq T_S(\Omega(k_0))\vee T_S(\Omega(g_0)) = \Omega^{-1}(T_S)(k_0)\vee\Omega^{-1}(T_S)(g_0), \end{aligned}$$

$$\begin{aligned}\Omega^{-1}(I_S)(k_0-g_0) = I_S(\Omega(k_0-g_0)) &= I_S(\Omega(k_0)-\Omega(g_0))\\ &\geq I_S(\Omega(k_0)) \wedge I_S(\Omega(g_0)) = \Omega^{-1}(I_S)(k_0) \wedge \Omega^{-1}(I_S)(g_0),\\ \Omega^{-1}(F_S)(k_0-g_0) = F_S(\Omega(k_0-g_0)) &= F_S(\Omega(k_0)-\Omega(g_0))\\ &\leq F_S(\Omega(k_0)) \vee F_S(\Omega(g_0)) = \Omega^{-1}(F_S)(k_0) \vee \Omega^{-1}(F_S)(g_0).\end{aligned}$$

Let $q_0 \in \Im$. Then

$$\begin{aligned}\Omega^{-1}(T_S)(q_0k_0 - q_0(g_0-k_0)) &= T_S(\Omega(q_0k_0 - q_0(g_0-k_0)))\\ &= T_S(\Omega(q_0k_0) - \Omega(q_0(g_0-k_0)))\\ &= T_S(\Omega(q_0)\Omega(k_0) - \Omega(q_0)(\Omega(g_0)-\Omega(k_0)))\\ &\leq T_S(\Omega(k_0)) = \Omega^{-1}(T_S)(k_0),\\ \Omega^{-1}(I_S)(q_0k_0 - q_0(g_0-k_0)) &= I_S(\Omega(q_0k_0 - q_0(g_0-k_0)))\\ &= I_S(\Omega(q_0k_0) - \Omega(q_0(g_0-k_0)))\\ &= I_S(\Omega(q_0)\Omega(k_0) - \Omega(q_0)(\Omega(g_0)-\Omega(k_0)))\\ &\geq I_S(\Omega(k_0)) = \Omega^{-1}(I_S)(k_0),\\ \Omega^{-1}(F_S)(q_0k_0 - q_0(g_0-k_0)) &= F_S(\Omega(q_0k_0 - q_0(g_0-k_0)))\\ &= F_S(\Omega(q_0k_0) - \Omega(q_0(g_0-k_0)))\\ &= F_S(\Omega(q_0)\Omega(k_0) - \Omega(q_0)(\Omega(g_0)-\Omega(k_0)))\\ &\leq F_S(\Omega(k_0)) = \Omega^{-1}(F_S)(k_0).\end{aligned}$$

Also,

$$\Omega^{-1}(T_S)(k_0g_0) = T_S(\Omega(k_0g_0) = T_S(\Omega(k_0)\Omega(g_0)) \leq T_S(\Omega(k_0)) = \Omega^{-1}(T_S)(k_0),$$
$$\Omega^{-1}(I_S)(k_0g_0) = I_S(\Omega(k_0g_0) = I_S(\Omega(k_0)\Omega(g_0)) \geq I_S(\Omega(k_0)) = \Omega^{-1}(I_S)(k_0),$$
$$\Omega^{-1}(F_S)(k_0g_0) = F_S(\Omega(k_0g_0) = F_S(\Omega(k_0)\Omega(g_0)) \leq F_S(\Omega(k_0)) = \Omega^{-1}(F_S)(k_0).$$

So, $\Omega^{-1}(\Im'_S) \in \mathscr{N}_{\mathfrak{I}}(\Im)$.

Definition 4.12. Consider a onto map $\Omega : \mathbb{N} \to \mathbb{M}$, where $\mathbb{N}, \mathbb{M} \neq \{\phi\}$. Suppose $\mathbb{N}_{\mathscr{B}} := \frac{\mathbb{N}}{(T_{\mathscr{B}}, I_{\mathscr{B}}, F_{\mathscr{B}})}$ over $\mathbb{N}$ is a neutrosophic $\varkappa$-structure. Then, under Ω, the image of $\mathbb{N}_{\mathscr{B}}$ is described as a neutrosophic $\varkappa$-structure

$$\Omega(\mathbb{N}_{\mathscr{B}}) = \frac{\mathbb{M}}{(\Omega(T_{\mathscr{B}}), \Omega(I_{\mathscr{B}}), \Omega(F_{\mathscr{B}}))}$$

over $\mathbb{M}$, where, for all $y_2 \in \mathbb{M}$,

$$\Omega(T_{\mathscr{B}})(y_2) = \bigwedge_{y_1 \in \Omega^{-1}(y_2)} T_{\mathscr{B}}(y_1),$$
$$\Omega(I_{\mathscr{B}})(y_2) = \bigvee_{y_1 \in \Omega^{-1}(y_2)} I_{\mathscr{B}}(y_1),$$
$$\Omega(F_{\mathscr{B}})(y_2) = \bigwedge_{y_1 \in \Omega^{-1}(y_2)} F_{\mathscr{B}}(y_1).$$

Theorem 4.13. *Let* $\xi : \Im \to \Im'$ *be an onto homomorphism of NSS and* $\Im'_{\mathscr{Z}} := \frac{\Im'}{(T_{\mathscr{Z}}, I_{\mathscr{Z}}, F_{\mathscr{Z}})}$ *is a neutrosophic* $\varkappa$*-structure of* $\Im'$. *If* $\xi^{-1}(\Im'_{\mathscr{Z}}) \in \mathscr{N}_{\mathfrak{I}}(\Im)$, *then* $\Im'_{\mathscr{Z}} \in \mathscr{N}_{\mathfrak{I}}(\Im')$.

Proof: Let $v_0', r_0' \in \Im'$. Then $\exists\ v_0, r_0 \in \Im$ such that $\xi(v_0) = v_0'$ and $\xi(r_0) = r_0'$. Now,

$$\begin{aligned}
T_{\mathscr{Z}}(v_0' - r_0') = T_{\mathscr{Z}}(\xi(v_0) - \xi(r_0)) = T_{\mathscr{Z}}(\xi(v_0 - r_0)) &= \xi^{-1}(T_{\mathscr{Z}})(v_0 - r_0) \\
&\leq \xi^{-1}(T_{\mathscr{Z}})(v_0) \vee \xi^{-1}(T_{\mathscr{Z}})(r_0) \\
&= T_{\mathscr{Z}}(\xi(v_0)) \vee T_{\mathscr{Z}}(\xi(r_0)) \\
&= T_{\mathscr{Z}}(v_0') \vee T_{\mathscr{Z}}(r_0'), \\
I_{\mathscr{Z}}(v_0' - r_0') = I_{\mathscr{Z}}(\xi(v_0) - \xi(r_0)) = I_{\mathscr{Z}}(\xi(v_0 - r_0)) &= \xi^{-1}(I_{\mathscr{Z}})(v_0 - r_0) \\
&\geq \xi^{-1}(I_{\mathscr{Z}})(v_0) \wedge \xi^{-1}(I_{\mathscr{Z}})(r_0) \\
&= I_{\mathscr{Z}}(\xi(v_0)) \wedge I_{\mathscr{Z}}(\xi(r_0)) \\
&= I_{\mathscr{Z}}(v_0') \wedge I_{\mathscr{Z}}(r_0'), \\
F_{\mathscr{Z}}(v_0' - r_0') = F_{\mathscr{Z}}(\xi(v_0) - \xi(r_0)) = F_{\mathscr{Z}}(\xi(v_0 - r_0)) &= \xi^{-1}(F_{\mathscr{Z}})(v_0 - r_0) \\
&\leq \xi^{-1}(F_{\mathscr{Z}})(v_0) \vee \xi^{-1}(F_{\mathscr{Z}})(r_0) \\
&= F_{\mathscr{Z}}(\xi(v_0)) \vee F_{\mathscr{Z}}(\xi(r_0)) \\
&= F_{\mathscr{Z}}(v_0') \vee F_{\mathscr{Z}}(r_0').
\end{aligned}$$

Let $s_0' \in \Im'$. Then $\exists s \in \Im$ such that $\xi(s) = s_0'$. Now

$$\begin{aligned}
T_{\mathscr{Z}}(s_0'v_0' - s_0'(r_0' - v_0')) &= T_{\mathscr{Z}}(\xi(s)\xi(v_0) - \xi(s)(\xi(r_0) - \xi(v_0))) \\
&= T_{\mathscr{Z}}(\xi(sv_0) - \xi(s)\xi(r_0 - v_0)) \\
&= T_{\mathscr{Z}}(\xi(sv_0) - \xi(s(r_0 - v_0))) \\
&= T_{\mathscr{Z}}(\xi(sv_0 - s(r_0 - v_0))) \\
&= \xi^{-1}(T_{\mathscr{Z}})(sv_0 - s(r_0 - v_0)) \leq \xi^{-1}(T_{\mathscr{Z}})(v_0) = T_{\mathscr{Z}}(\xi(v_0)) = T_{\mathscr{Z}}(v_0'), \\
I_{\mathscr{Z}}(s_0'v_0' - s_0'(r_0' - v_0')) &= I_{\mathscr{Z}}(\xi(s)\xi(v_0) - \xi(s)(\xi(r_0) - \xi(v_0))) \\
&= I_{\mathscr{Z}}(\xi(sv_0) - \xi(s)\xi(r_0 - v_0)) \\
&= I_{\mathscr{Z}}(\xi(sv_0) - \xi(s(r_0 - v_0))) \\
&= I_{\mathscr{Z}}(\xi(sv_0 - s(r_0 - v_0))) \\
&= \xi^{-1}(I_{\mathscr{Z}})(sv_0 - s(r_0 - v_0)) \geq \xi^{-1}(I_{\mathscr{Z}})(v_0) = I_{\mathscr{Z}}(\xi(v_0)) = I_{\mathscr{Z}}(v_0'), \\
F_{\mathscr{Z}}(s_0'v_0' - s_0'(r_0' - v_0')) &= F_{\mathscr{Z}}(\xi(s)\xi(v_0) - \xi(s)(\xi(r_0) - \xi(v_0))) \\
&= F_{\mathscr{Z}}(\xi(sv_0) - \xi(s)\xi(r_0 - v_0)) \\
&= F_{\mathscr{Z}}(\xi(sv_0) - \xi(s(r_0 - v_0))) \\
&= F_{\mathscr{Z}}(\xi(sv_0 - s(r_0 - v_0))) \\
&= \xi^{-1}(F_{\mathscr{Z}})(sv_0 - s(r_0 - v_0)) \leq \xi^{-1}(F_{\mathscr{Z}})(v_0) = F_{\mathscr{Z}}(\xi(v_0)) = F_{\mathscr{Z}}(v_0').
\end{aligned}$$

Also,

$$T_{\mathscr{Z}}(v_0'r_0') = T_{\mathscr{Z}}(\xi(v_0r_0)) = \xi^{-1}(T_{\mathscr{Z}})(v_0r_0) \leq \xi^{-1}(T_{\mathscr{Z}})(v_0) = T_{\mathscr{Z}}(\xi(v_0)) = T_{\mathscr{Z}}(v_0'),$$
$$I_{\mathscr{Z}}(v_0'r_0') = I_{\mathscr{Z}}(\xi(v_0r_0)) = \xi^{-1}(I_{\mathscr{Z}})(v_0r_0) \geq \xi^{-1}(I_{\mathscr{Z}})(v_0) = I_{\mathscr{Z}}(\xi(v_0)) = I_{\mathscr{Z}}(v_0'),$$
$$F_{\mathscr{Z}}(v_0'r_0') = F_{\mathscr{Z}}(\xi(v_0r_0)) = \xi^{-1}(F_{\mathscr{Z}})(v_0r_0) \leq \xi^{-1}(F_{\mathscr{Z}})(v_0) = F_{\mathscr{Z}}(\xi(v_0)) = F_{\mathscr{Z}}(v_0').$$

So, $\Im'_{\mathscr{Z}} \in \mathscr{N}_{\mathfrak{I}}(\Im')$.

Definition 4.14. A neutrosophic $\varkappa$- structure $\Im_{\mathscr{B}} := \frac{\Im}{(T_{\mathscr{B}}, I_{\mathscr{B}}, F_{\mathscr{B}})}$ is defined to fulfils the sup property in $\Im$ if $\forall\, S \subseteq \Im$, $\exists\, l_0 \in S : T_{\mathscr{B}}(l_0) = \bigwedge_{l\in S} T_{\mathscr{B}}(l);\ I_{\mathscr{B}}(l_0) = \bigvee_{l\in S} I_{\mathscr{B}}(l);\ F_{\mathscr{B}}(l_0) = \bigwedge_{l\in S} F_{\mathscr{B}}(l)$.

Proposition 4.15. *A homomorphic image of a neutrosophic $\varkappa$-ideal having sup property is a neutrosophic $\varkappa$-ideal.*

Proof: Let $\varrho : \Im \to \Im'$ be a homomorphism of NSS and let $\Im_{\mathscr{Z}} := \frac{\Im}{(T_{\mathscr{Z}}, I_{\mathscr{Z}}, F_{\mathscr{Z}})}$ of $\Im$ be a neutrosophic $\varkappa$-ideal having sup property.

Suppose $\varrho(b), \varrho(w) \in \Im'$ and let $b_0 \in \varrho^{-1}(\varrho(b))$ and $w_0 \in \varrho^{-1}(\varrho(w))$ be such that

$$T_{\mathscr{Z}}(b_0) = \bigwedge_{k_0\in\varrho^{-1}(\varrho(b))} T_{\mathscr{Z}}(k_0),\quad I_{\mathscr{Z}}(b_0) = \bigvee_{k_0\in\varrho^{-1}(\varrho(b))} I_{\mathscr{Z}}(k_0),\quad F_{\mathscr{Z}}(b_0) = \bigwedge_{k_0\in\varrho^{-1}(\varrho(b))} F_{\mathscr{Z}}(k_0),$$
$$T_{\mathscr{Z}}(w_0) = \bigwedge_{k_0\in\varrho^{-1}(\varrho(b))} T_{\mathscr{Z}}(k_0),\quad I_{\mathscr{Z}}(w_0) = \bigvee_{k_0\in\varrho^{-1}(\varrho(b))} I_{\mathscr{Z}}(k_0),\quad F_{\mathscr{Z}}(w_0) = \bigwedge_{k_0\in\varrho^{-1}(\varrho(b))} F_{\mathscr{Z}}(k_0).$$

Then

$$\begin{aligned}
\varrho(T_{\mathscr{Z}})(\varrho(b) - \varrho(w)) &= \bigwedge_{z\in\varrho^{-1}(\varrho(b)-\varrho(w))} T_{\mathscr{Z}}(z) \leq T_{\mathscr{Z}}(b_0) \vee T_{\mathscr{Z}}(w_0)\\
&= \left(\bigwedge_{k_0\in\varrho^{-1}(\varrho(b))} T_{\mathscr{Z}}(k_0)\right) \vee \left(\bigwedge_{k_0\in\varrho^{-1}(\varrho(w))} T_{\mathscr{Z}}(k_0)\right)\\
&= \varrho(T_{\mathscr{Z}})(\varrho(b)) \vee \varrho(T_{\mathscr{Z}})(\varrho(w)),\\
\varrho(I_{\mathscr{Z}})(\varrho(b) - \varrho(w)) &= \bigvee_{z\in\varrho^{-1}(\varrho(b)-\varrho(w))} I_{\mathscr{Z}}(z) \geq I_{\mathscr{Z}}(b_0) \wedge I_{\mathscr{Z}}(w_0)\\
&= \left(\bigvee_{k_0\in\varrho^{-1}(\varrho(b))} I_{\mathscr{Z}}(k_0)\right) \wedge \left(\bigvee_{k_0\in\varrho^{-1}(\varrho(w))} I_{\mathscr{Z}}(k_0)\right)\\
&= \varrho(I_{\mathscr{Z}})(\varrho(b)) \wedge \varrho(I_{\mathscr{Z}})(\varrho(w)),\\
\varrho(F_{\mathscr{Z}})(\varrho(b) - \varrho(w)) &= \bigwedge_{z\in\varrho^{-1}(\varrho(b)-\varrho(w))} F_{\mathscr{Z}}(z) \leq F_{\mathscr{Z}}(b_0) \vee F_{\mathscr{Z}}(w_0)\\
&= \left(\bigwedge_{k_0\in\varrho^{-1}(\varrho(b))} F_{\mathscr{Z}}(k_0)\right) \vee \left(\bigwedge_{k_0\in\varrho^{-1}(\varrho(w))} F_{\mathscr{Z}}(k_0)\right)\\
&= \varrho(F_{\mathscr{Z}})(\varrho(b)) \vee \varrho(F_{\mathscr{Z}})(\varrho(w)).
\end{aligned}$$

Given $\varrho(s) \in \Im'$ and let $s_0 \in \varrho^{-1}(\varrho(s))$. Then

$$\begin{aligned}
\varrho(T_{\mathscr{X}})(\varrho(s)\varrho(b) - \varrho(s)(\varrho(w) - \varrho(b))) &= \bigwedge_{z \in \varrho^{-1}(\varrho(s)\varrho(b) - \varrho(s)(\varrho(w) - \varrho(b)))} T_{\mathscr{X}}(z) \\
&\leq T_{\mathscr{X}}(b_0) = \bigwedge_{k_0 \in \varrho^{-1}(\varrho(b))} T_{\mathscr{X}}(k_0) = \varrho(T_{\mathscr{X}})(\varrho(b)), \\
\varrho(I_{\mathscr{X}})(\varrho(s)\varrho(b) - \varrho(s)(\varrho(w) - \varrho(b))) &= \bigvee_{z \in \varrho^{-1}(\varrho(s)\varrho(b) - \varrho(s)(\varrho(w) - \varrho(b)))} I_{\mathscr{X}}(z) \\
&\geq I_{\mathscr{X}}(b_0) = \bigvee_{k_0 \in \varrho^{-1}(\varrho(b))} I_{\mathscr{X}}(k_0) = \varrho(I_{\mathscr{X}})(\varrho(b)), \\
\varrho(F_{\mathscr{X}})(\varrho(s)\varrho(b) - \varrho(s)(\varrho(w) - \varrho(b))) &= \bigwedge_{z \in \varrho^{-1}(\varrho(s)\varrho(b) - \varrho(s)(\varrho(w) - \varrho(b)))} F_{\mathscr{X}}(z) \\
&\leq F_{\mathscr{X}}(b_0) = \bigwedge_{k_0 \in \varrho^{-1}(\varrho(b))} F_{\mathscr{X}}(k_0) = \varrho(F_{\mathscr{X}})(\varrho(b)).
\end{aligned}$$

Also,

$$\begin{aligned}
\varrho(T_{\mathscr{X}})(\varrho(b)\varrho(w)) &= \bigwedge_{z \in \varrho^{-1}(\varrho(b)\varrho(w))} T_{\mathscr{X}}(z) \leq T_{\mathscr{X}}(b_0) = \bigwedge_{k_0 \in \varrho^{-1}(\varrho(b))} T_{\mathscr{X}}(k_0) = \varrho(T_{\mathscr{X}})(\varrho(b)), \\
\varrho(I_{\mathscr{X}})(\varrho(b)\varrho(w)) &= \bigvee_{z \in \varrho^{-1}(\varrho(b)\varrho(w))} I_{\mathscr{X}}(z) \geq I_{\mathscr{X}}(b_0) = \bigvee_{k_0 \in \varrho^{-1}(\varrho(b))} I_{\mathscr{X}}(k_0) = \varrho(I_{\mathscr{X}})(\varrho(b)), \\
\varrho(F_{\mathscr{X}})(\varrho(b)\varrho(w)) &= \bigwedge_{z \in \varrho^{-1}(\varrho(b)\varrho(w))} F_{\mathscr{X}}(z) \leq F_{\mathscr{X}}(b_0) = \bigwedge_{k_0 \in \varrho^{-1}(\varrho(b))} F_{\mathscr{X}}(k_0) = \varrho(F_{\mathscr{X}})(\varrho(b)).
\end{aligned}$$

Hence $\varrho(\Im_{\mathscr{X}})$ is a neutrosophic $\varkappa$-ideal of $\varrho(\Im)$.

5. Conclusion

We defined and examined neutrosophic $\varkappa-$ ideals in near-subtraction semigroups in this article. We formed ideals for a neutrosophic $\varkappa-$ ideal in a near-subtraction semigroup, and we also obtained various aspects of the neutrosophic $\varkappa-$ image as well as the neutrosophic $\varkappa-$ preimage of a near-subtraction semigroup using homomorphism mapping. In our future research work, we will explore the notion of a neutrosophic $\varkappa-$ prime ideal and its related properties in near-subtraction semigroups using the ideas and findings presented in this paper.

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Enhancing Failure Mode and Effect Analysis with Neutrosophic Inverse Soft Expert Sets

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Abstract. In this study, we introduce a novel concept, the Neutrosophic Inverse Soft Expert Set (NISES), and apply it to the Failure Mode and Effect Analysis (FMEA) framework. Developed by NASA, FMEA is a robust tool for addressing industrial challenges. Our approach leverages the Evaluation based on Distance from Average Solution (EDAS) algorithm to solve FMEA problems. We implement this methodology in a real-world scenario involving a steam valve with eight distinct failure modes. Through rigorous analysis, we employ the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) to rank the identified failure modes. Comparing our FMEA model, which integrates rough set theory and TOPSIS, with the conventional method, we demonstrate the superior efficiency of our approach. Additionally, we extend the application of Neutrosophic Inverse Soft Expert Sets using the Additive Ratio Assessment-Simplified Version (ARAS-SV) method. This innovative method facilitates a quantitative assessment of alternative options based on multiple attributes, allowing for a precise determination of the optimal choice.

Keywords: Soft set, inverse soft set, neutrosophic set, neutrosophic inverse soft set, Failure Mode and Effect Analysis, Additive Ratio Assessment-Simplified Version method

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1. Introduction

The Failure Mode and Effect Analysis (FMEA) process constitutes a pivotal cornerstone in contemporary engineering and industrial practices. It stands as an indispensable methodology not only for identifying potential failures within a given model but also for effecting requisite measures to rectify them, ultimately ensuring the seamless operation of machinery and systems. This approach finds extensive application in a diverse array of industries, including aviation, automotive, and automation, where its effectiveness in real-life scenarios is unequivocally acknowledged.

Central to the FMEA process are three critical risk factors: Severity (S), Occurrence (O), and Detection (D). These elements collectively contribute to the calculation of the Risk Priority Number (RPN), which, in turn, serves as the guiding metric for prioritizing and executing the FMEA process for a specific model. Notably, the relative weightings assigned to Severity, Occurrence, and Detection may vary, depending on the specific FMEA methodology employed, reflecting the nuanced nature of risk assessment.

FMEA stands as an efficient and indispensable tool for mitigating uncertainties that invariably arise in practical, real-world situations. Its application transcends mere fault detection; it encompasses a systematic approach to preemptively predict the potential order of failure in a given model, significantly enhancing the proactive management of operational risks. The versatility of FMEA is further underscored by its adaptability to distinct scenarios, where the weights attributed to Severity, Occurrence, and Detection may either be uniformly distributed or differentially ranked, contingent upon the specific FMEA technique in use.

In light of the existing body of research on FMEA techniques, we propose the hypothesis that the integration of Neutrosophic Inverse Soft Expert Sets (NISES) in conjunction with the Evaluation Based on Distance from Average Solution (EDAS) method will yield a more efficient and accurate assessment of risk factors in complex systems. This hypothesis is grounded in the potential of NISES to capture uncertainties in expert judgments and the robustness of the EDAS method in evaluating the performance of alternatives. Through rigorous testing and comparative analysis, we aim to substantiate this hypothesis and contribute to the advancement of risk assessment methodologies.

This introduction sets the stage for a comprehensive exploration of the nuanced methodologies and applications associated with FMEA. In the ensuing sections, we delve into a rich tapestry of literature, encompassing a spectrum of innovative approaches and models that have significantly advanced the field. The motivation for the present study arises from the endeavor to incorporate the groundbreaking concept of neutrosophic set theory into the FMEA framework, opening up new vistas for enhanced risk assessment and decision-making. As we proceed, we embark on a journey through fundamental concepts, detailed methodologies, and in-depth comparative analyses, collectively contributing to a deeper understanding of FMEA's evolving landscape.

Our research represents a groundbreaking exploration at the intersection of risk assessment methodologies and decision-making processes. In this study, we introduce a novel framework by incorporating Neutrosophic Inverse Soft Expert Sets into the well-established domain of Failure Mode and Effect Analysis. This innovative approach stems from the pioneering work of Smarandache, who introduced the concept of Neutrosophic Sets as a unified framework for handling uncertainty. We extend this idea to address critical issues in FMEA, particularly in situations where conventional risk assessment models may fall short in capturing the intricate nuances of complex systems.

Unlike traditional software-dependent approaches, our research adopts a manual, hands-on methodology, which allows for meticulous scrutiny and customization of the assessment process. Through a detailed literature review, we have identified gaps and limitations in existing FMEA models, which our study seeks to address. Our approach offers a systematic means of evaluating risk factors by considering the expertise of individuals (experts) in a neutrosophic form, allowing for the expression of uncertain and indeterminate information.

To validate our approach, we conducted extensive empirical testing, drawing inspiration from influential studies in the field. Our results reveal not only the feasibility but also the potential superiority of NISES-based FMEA in capturing uncertainties and providing more accurate risk assessments. The empirical outcomes of our research affirm the innovative nature of our approach and its capacity to enhance risk management practices in various domains, from engineering to healthcare.

By introducing this novel framework and eschewing reliance on software tools, we underscore the importance of human expertise in risk evaluation. Our research contributes not only to the field of risk assessment but also to the broader discourse on decision-making under uncertainty. It opens new horizons for further exploration, encouraging scholars and practitioners to embrace the versatility and effectiveness of Neutrosophic Inverse Soft Expert Sets as a valuable tool for managing and mitigating risks in an ever-evolving world of complexity and ambiguity.

1.1. *Literature review*

The landscape of Failure Mode and Effect Analysis (FMEA) has been enriched by a wealth of research contributions. Song et al. [20] addressed a specific case involving a steam valve system, employing the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) method in conjunction with a rough set approach. This study demonstrated the efficacy of integrating advanced decision-making techniques with FMEA to enhance system reliability.

Zadeh's pioneering work [23] on fuzzy sets introduced a transformative approach to address shortcomings in RPN for FMEA models. Chang et al. [6] further extended this concept by integrating grey theory with fuzzy sets, augmenting risk assessment methodologies. Chin et al. [5] introduced Data Envelopment Analysis (DEA) into FMEA, presenting an alternative perspective for risk evaluation.

Gilchrist's innovative model [8] for FMEA opened new avenues for analysis, while Liu et al. [15] combined grey theory with fuzzy evidential reasoning, enriching risk assessment strategies. Pillay et al. [18] introduced a modified FMEA model with approximate reasoning, contributing a unique viewpoint.

Xu et al.'s work [22] on fuzzy assessment techniques in FMEA advanced risk evaluation methods. Zavadskas et al. [9] made a significant contribution with the introduction of the Evaluation based on Distance from Average Solution (EDAS) method, further expanding the FMEA toolkit.

Molodtsov's introduction of soft set theory [17] marked a revolutionary shift in uncertainty management. Feng's hybrid models [7], combining soft set theory with other structures, further elevated risk assessment strategies.

Hu-Chen Liu et al.'s integration of risk evaluation concepts with fuzzy digraph and matrix theory [15] provided a fresh perspective on FMEA. Akram et al.'s introduction of TOPSIS and ELECTRE I method using pythagorean fuzzy information [3] added diversity to the repertoire of approaches available in FMEA.

Our approach, integrating Neutrosophic Inverse Soft Expert Sets (NISES) into Failure Mode and Effect Analysis (FMEA), introduces a novel framework for risk assessment. To provide a clear overview of the literature landscape and how our approach stands out, we present a comprehensive table summarizing studies based on their assumptions, methods, and results.

Our integration of NISES in FMEA stands out as a novel contribution, streamlining risk assessment and offering adaptability to uncertainties. This comprehensive table highlights the unique perspective our approach brings to Failure Mode and Effect Analysis, distinguishing it from prior methodologies based on their underlying assumptions, methods, and results.

Author	Assumptions	Methods Employed	Results and Contributions
Song et al.	Standard FMEA assumptions, TOPSIS, Rough Set	Established foundational FMEA techniques	Introduced a robust framework for failure mode assessment
Zadeh	Utilizes Fuzzy Sets	Introduced a transformative approach to FMEA	Revolutionized risk assessment through fuzzy logic
Chang et al.	Embraces Grey Theory in Fuzzy Sets	Expanded risk assessment methodologies in FMEA	Provided a comprehensive framework for handling uncertainties
Chin et al.	Applies Data Envelopment Analysis (DEA) assumptions	Integrated DEA for alternative perspectives	Enhanced decision-making through DEA in FMEA
Gilchrist	Innovates in FMEA Modeling	Pioneered a unique model for failure mode assessment	Introduced a novel approach for comprehensive risk evaluation
Liu et al.	Utilizes Grey Theory, Fuzzy Evidential Reasoning assumptions	Advanced risk assessment strategies in FMEA	Enhanced risk assessment by combining multiple uncertainty sources
Pillay et al.	Incorporates Modified FMEA assumptions with Approximate Reasoning	Introduced a novel approach for FMEA	Enhanced risk assessment through tailored approximate reasoning
Xu et al.	Applies Fuzzy Assessment of FMEA assumptions	Elevated risk evaluation techniques in FMEA	Provided a more nuanced approach to risk assessment using fuzzy logic
Zavadskas et al.	Leverages Evaluation based on Distance from Average Solution (EDAS) assumptions	Significant contribution to FMEA methodology	Improved risk assessment through a novel evaluation approach
Molodtsov	Utilizes Soft Set Theory assumptions	Revolutionized uncertainty management in FMEA	Provided a comprehensive framework for handling uncertainties using soft sets
Feng	Applies Hybrid Models combining Soft Set Theory assumptions	Elevated risk assessment strategies in FMEA	Enhanced risk assessment by integrating multiple methodologies
Hu-Chen Liu et al.	Utilizes Risk Evaluation with Fuzzy Digraph and Matrix Theory assumptions	Provided a fresh perspective on FMEA risk evaluation	Enhanced risk assessment by combining fuzzy digraphs and matrix theory
Akram et al.	Incorporates TOPSIS, ELECTRE I with Pythagorean Fuzzy Information assumptions	Enhanced diversity of approaches in FMEA	Provided a versatile approach to risk assessment using multiple methodologies
Smarandache	Applies Neutrosophic Sets assumptions	Unified uncertainty structures under neutrosophic sets	Introduced a novel framework for handling uncertainties using neutrosophic sets

The empirical results of our study, which integrates Neutrosophic Inverse Soft Expert Sets (NISES) into Failure Mode and Effect Analysis (FMEA), have unveiled promising advancements in risk assessment methodologies. Building upon the foundational research of Zadeh [23] , Chang [6] , Chin [5], Gilchrist [8], Liu [15] , Pillay [18] , Xu [22] , Zavadskas [9], Molodtsov [17] , Feng [7] , Hu-Chen Liu [15] , and Akram [3] , our innovative approach offers a fresh perspective on addressing uncertainties in complex systems. Through rigorous empirical testing, we have demonstrated the effectiveness of NISES in enhancing the accuracy of risk evaluation. The integration of NISES with FMEA has not only showcased its potential to provide more nuanced insights but has also yielded practical implications for risk mitigation strategies. Our findings contribute to the ever-evolving landscape of risk assessment and underscore the value of incorporating Neutrosophic Inverse Soft Expert Sets in decision-making processes within a variety of domains.

2. Preliminaries

Throughout this paper, let U denote universe set, Υ represent parameter set, $P(U)$ denotes power set of U, $P(\Upsilon)$ denotes the power set of Υ, $\mathbb{H}$ being set of experts, Θ represents a set of opinions and N_S^U denotes the collection of all neutrosophic subsets of U.

Definition 2.1. [17] For a given universe set U with parameter Υ, a soft set is mapping from S to $P(U)$, where $S \subseteq \Upsilon$.

Definition 2.2. [10] Let $P(\Upsilon)$ be the set of all subsets of parameter set Υ. A pair (F, U) is called an inverse soft set over Υ, where F is a mapping given by

$$F : U \to P(\Upsilon).$$

Definition 2.3. [4] The mapping from set $\mathfrak{A}$ to the power set of U constitutes a soft expert set, where $\mathfrak{A} \subseteq Z$, $Z = \Upsilon \times \mathbb{H} \times \Theta$, Υ is a set of parameters, $\mathbb{H}$ is a set of experts and Θ is the set of opinions.

Definition 2.4. [21] Consider a mapping $\Xi_\Upsilon : U \to P(\mathfrak{A})$, where U denotes the universe set and Υ denotes the set of parameters. Then the pair $\mathfrak{P} = (\Xi_\Upsilon, U)$ is defined as inverse soft expert sets, where $\mathfrak{A} \subseteq Z$, $Z = \Upsilon \times \mathbb{H} \times \Theta$, Υ is a set of parameters, $\mathbb{H}$ is a set of experts and Θ is the set of opinions.

Definition 2.5. [19] A neutrosophic set (N-sets) is defined by

$$A = \{< u, T_A(u), I_A(u), F_A(u) >; u \in U, T_A(u), I_A(u), F_A(u) \in [0, 1]\},$$

where u being the generic element of U, T_A being truth-membership function, I_A being indeterminacy-membership function and F_A represents falsity-membership function.

3. Neutrosophic inverse soft expert sets

Definition 3.1. Consider a mapping,

$$F : N_S^U \to P(Z)$$

where N_S^U denotes the collection of all neutrosophic subsets of U , then the pair (F, N_S^U) is called as neutrosophic inverse soft expert set (NISES).

Example 3.2. Let $U = \{\vartheta_1, \vartheta_2, \vartheta_3\}$ be a universe set , $\Upsilon = \{\beth_1, \beth_2\}$ be a set of parameters and $\mathbb{H} = \{\varrho_1, \varrho_2\}$ be a set of experts. Suppose that $F : N_S^U \to P(Z)$ is a function defined as follows.

TABLE 1. Neutrosophic inverse soft expert set

(F, N_S^U)	$(\gimel_1, \varrho_1, 1)$	$(\gimel_1, \varrho_1, 0)$	$(\gimel_1, \varrho_2, 1)$	$(\gimel_1, \varrho_2, 0)$	$(\gimel_2, \varrho_1, 1)$	$(\gimel_2, \varrho_1, 0)$	$(\gimel_2, \varrho_2, 1)$	$(\gimel_2, \varrho_2, 0)$
ϑ_1	(0.3,0.4,0.7)	(0.7,0.5,0.2)	(0.8,0.7,0.3)	(0.2,0.3,0.7)	(0.4,0.6,0.4)	(0.7,0.3,0.6)	(0.9,0.3,0.3)	(0.4,0.6,0.1)
ϑ_2	(0.5,0.2,0.9)	(0.3,0.5,0.6)	(0.3,0.1,0.5)	(0.4,0.7,0.9)	(0.9,0.3,0.5)	(0.1,0.7,0.3)	(0.1,0.4,0.7)	(0.2,0.1,0.5)
ϑ_3	(0.2,0.5,0.8)	(0.4,0.1,0.6)	(0.4,0.9,0.4)	(0.1,0.4,0.6)	(0.6,0.2,0.9)	(0.1,0.5,0.5)	(0.6,0.3,0.2)	(0.3,0.6,0.7)

Thus, we can view the neutrosophic inverse soft expert set (F, N_S^U) as a collection of approximations as follows.

$$(F, N_S^U) = \Bigg[\Bigg\{(F, \vartheta_1) = \Big\{\frac{(\gimel_1, \varrho_1, 1)}{(0.3, 0.4, 0.7)}, \frac{(\gimel_1, \varrho_1, 0)}{(0.7, 0.5, 0.2)}, \frac{(\gimel_1, \varrho_2, 1)}{(0.8, 0.7, 0.3)}, \frac{(\gimel_1, \varrho_2, 0)}{(0.2, 0.3, 0.7)}, \frac{(\gimel_2, \varrho_1, 1)}{(0.4, 0.6, 0.4)}, \frac{(\gimel_2, \varrho_1, 0)}{(0.7, 0.3, 0.6)}, \frac{(\gimel_2, \varrho_2, 1)}{(0.9, 0.3, 0.3)}, \frac{(\gimel_2, \varrho_2, 0)}{(0.4, 0.6, 0.1)}\Big\}\Bigg\},$$
$$\Bigg\{(F, \vartheta_2) = \Big\{\frac{(\gimel_1, \varrho_1, 1)}{(0.5, 0.2, 0.9)}, \frac{(\gimel_1, \varrho_1, 0)}{(0.3, 0.5, 0.6)}, \frac{(\gimel_1, \varrho_2, 1)}{(0.3, 0.1, 0.5)}, \frac{(\gimel_1, \varrho_2, 0)}{(0.4, 0.7, 0.9)}, \frac{(\gimel_2, \varrho_1, 1)}{(0.9, 0.3, 0.5)}, \frac{(\gimel_2, \varrho_1, 0)}{(0.1, 0.7, 0.3)}, \frac{(\gimel_2, \varrho_2, 1)}{(0.1, 0.4, 0.7)}, \frac{(\gimel_2, \varrho_2, 0)}{(0.2, 0.1, 0.5)}\Big\}\Bigg\},$$
$$\Bigg\{(F, \vartheta_3) = \Big\{\frac{(\gimel_1, \varrho_1, 1)}{(0.2, 0.5, 0.8)}, \frac{(\gimel_1, \varrho_1, 0)}{(0.4, 0.1, 0.6)}, \frac{(\gimel_1, \varrho_2, 1)}{(0.4, 0.9, 0.4)}, \frac{(\gimel_1, \varrho_2, 0)}{(0.1, 0.4, 0.6)}, \frac{(\gimel_2, \varrho_1, 1)}{(0.6, 0.2, 0.9)}, \frac{(\gimel_2, \varrho_1, 0)}{(0.1, 0.5, 0.5)}, \frac{(\gimel_2, \varrho_2, 1)}{(0.6, 0.3, 0.2)}, \frac{(\gimel_2, \varrho_2, 0)}{(0.3, 0.6, 0.7)}\Big\}\Bigg\}\Bigg].$$

Then (F, N_S^U) is a neutrosophic inverse soft expert set over (N_S^U, Z).

Definition 3.3. Let $(F, N_S^U)_A$ be a neutrosophic inverse soft expert set over (N_S^U, Z). An agree-neutrosophic inverse soft expert set is denoted as $(F, N_S^U)_A^1$ defined as,

$$(F, N_S^U)_A^1 = \{F(\psi); \psi \in \Upsilon \times \mathbb{H} \times \{1\}\}.$$

Definition 3.4. Let $(F, N_S^U)_A$ be a neutrosophic inverse soft expert set over (N_S^U, Z). A disagree-neutrosophic inverse soft expert set is denoted as $(F, N_S^U)_A^0$ defined as,

$$(F, N_S^U)_A^0 = \{F(\psi); \psi \in \Upsilon \times \mathbb{H} \times \{0\}\}.$$

Example 3.5. Consider example 3.2.Then the agree-neutrosophic inverse soft expert set $(F, N_S^U)_A^1$ is

$$(F, N_S^U)_A^1 = \Bigg[\Bigg\{(F, \vartheta_1) = \Big\{\frac{(\gimel_1, \varrho_1, 1)}{(0.3, 0.4, 0.7)}, \frac{(\gimel_1, \varrho_2, 1)}{(0.8, 0.7, 0.3)}, \frac{(\gimel_2, \varrho_1, 1)}{(0.4, 0.6, 0.4)}, \frac{(\gimel_2, \varrho_2, 1)}{(0.9, 0.3, 0.3)}\Big\}\Bigg\},$$
$$\Bigg\{(F, \vartheta_2) = \Big\{\frac{(\gimel_1, \varrho_1, 1)}{(0.5, 0.2, 0.9)}, \frac{(\gimel_1, \varrho_2, 1)}{(0.3, 0.1, 0.5)}, \frac{(\gimel_2, \varrho_1, 1)}{(0.9, 0.3, 0.5)}, \frac{(\gimel_2, \varrho_2, 1)}{(0.1, 0.4, 0.7)}\Big\}\Bigg\},$$
$$\Bigg\{(F, \vartheta_3) = \Big\{\frac{(\gimel_1, \varrho_1, 1)}{(0.2, 0.5, 0.8)}, \frac{(\gimel_1, \varrho_2, 1)}{(0.4, 0.9, 0.4)}, \frac{(\gimel_2, \varrho_1, 1)}{(0.6, 0.2, 0.9)}, \frac{(\gimel_2, \varrho_2, 1)}{(0.6, 0.3, 0.2)},\Big\}\Bigg\}\Bigg].$$

and the disagree-neutrosophic inverse soft expert set $(F, N_S^U)_A^0$ is

$$(F, N_S^U)_A^0 = \left[\left\{(F, \vartheta_1) = \left\{\frac{(\gimel_1, \varrho_1, 0)}{(0.7, 0.5, 0.2)}, \frac{(\gimel_1, \varrho_2, 0)}{(0.2, 0.3, 0.7)}, \frac{(\gimel_2, \varrho_1, 0)}{(0.7, 0.3, 0.6)}, \frac{(\gimel_2, \varrho_2, 0)}{(0.4, 0.6, 0.1)}\right\}\right\},\right.$$
$$\left\{(F, \vartheta_2) = \left\{\frac{(\gimel_1, \varrho_1, 0)}{(0.3, 0.5, 0.6)}, \frac{(\gimel_1, \varrho_2, 0)}{(0.4, 0.7, 0.9)}, \frac{(\gimel_2, \varrho_1, 0)}{(0.1, 0.7, 0.3)}, \frac{(\gimel_2, \varrho_2, 0)}{(0.2, 0.1, 0.5)}\right\}\right\},$$
$$\left.\left\{(F, \vartheta_3) = \left\{\frac{(\gimel_1, \varrho_1, 0)}{(0.4, 0.1, 0.6)}, \frac{(\gimel_1, \varrho_2, 0)}{(0.1, 0.4, 0.6)}, \frac{(\gimel_2, \varrho_1, 0)}{(0.1, 0.5, 0.5)}, \frac{(\gimel_2, \varrho_2, 0)}{(0.3, 0.6, 0.7)}\right\}\right\}\right].$$

Definition 3.6. Let $(F, N_S^U)_A$ be a neutrosophic inverse soft expert set over (N_S^U, Z). Then the complement of $(F, N_S^U)_A$ denoted by $(F, N_S^U)_A^C$ is defined as,

$$(F, N_S^U)_A^C = \widetilde{C}(F(\psi)); \forall \psi \in U$$

where $\widetilde{c}$ is neutrosophic inverse soft expert complement.

Example 3.7. Consider $(F, N_S^U)_A$ over (N_S^U, Z) as given in Example 3.2. By using the complement for $(F, N_S^U)_A$, we obtain $(F, N_S^U)_A^C$ which is defined as,

$$(F, N_S^U)_A^c = \left[\left\{(F, \vartheta_1) = \left\{\frac{(\gimel_1, \varrho_1, 1)}{(0.7, 0.4, 0.3)}, \frac{(\gimel_1, \varrho_1, 0)}{(0.2, 0.5, 0.7)}, \frac{(\gimel_1, \varrho_2, 1)}{(0.3, 0.7, 0.8)}, \frac{(\gimel_1, \varrho_2, 0)}{(0.7, 0.3, 0.2)}, \frac{(\gimel_2, \varrho_1, 1)}{(0.4, 0.6, 0.4)},\right.\right.\right.$$
$$\left.\left.\frac{(\gimel_2, \varrho_1, 0)}{(0.6, 0.3, 0.7)}, \frac{(\gimel_2, \varrho_2, 1)}{(0.3, 0.3, 0.9)}, \frac{(\gimel_2, \varrho_2, 0)}{(0.1, 0.6, 0.4)}\right\}\right\},$$
$$\left\{(F, \vartheta_2) = \left\{\frac{(\gimel_1, \varrho_1, 1)}{(0.9, 0.2, 0.5)}, \frac{(\gimel_1, \varrho_1, 0)}{(0.6, 0.5, 0.3)}, \frac{(\gimel_1, \varrho_2, 1)}{(0.5, 0.1, 0.3)}, \frac{(\gimel_1, \varrho_2, 0)}{(0.9, 0.7, 0.4)}, \frac{(\gimel_2, \varrho_1, 1)}{(0.5, 0.3, 0.9)},\right.\right.$$
$$\left.\left.\frac{(\gimel_2, \varrho_1, 0)}{(0.3, 0.7, 0.1)}, \frac{(\gimel_2, \varrho_2, 1)}{(0.7, 0.4, 0.1)}, \frac{(\gimel_2, \varrho_2, 0)}{(0.5, 0.1, 0.2)}\right\}\right\},$$
$$\left\{(F, \vartheta_3) = \left\{\frac{(\gimel_1, \varrho_1, 1)}{(0.8, 0.5, 0.2)}, \frac{(\gimel_1, \varrho_1, 0)}{(0.6, 0.1, 0.4)}, \frac{(\gimel_1, \varrho_2, 1)}{(0.4, 0.9, 0.4)}, \frac{(\gimel_1, \varrho_2, 0)}{(0.6, 0.4, 0.1)}, \frac{(\gimel_2, \varrho_1, 1)}{(0.9, 0.2, 0.6)},\right.\right.$$
$$\left.\left.\left.\frac{(\gimel_2, \varrho_1, 0)}{(0.5, 0.5, 0.1)}, \frac{(\gimel_2, \varrho_2, 1)}{(0.2, 0.3, 0.6)}, \frac{(\gimel_2, \varrho_2, 0)}{(0.7, 0.6, 0.3)}\right\}\right\}\right].$$

4. FMEA with Neutrosophic Inverse Soft Expert Sets and EDAS

Problem statement

Let's revisit the problem addressed by Song et al. [20]. They tackled an issue with a steam valve system in a power plant, which exhibited eight distinct failure modes. Their approach involved employing FMEA based on rough group preference by similarity to ideal solution. They began by computing rough interval weights for the risk factors and then constructed a crisp evaluation matrix for the failure modes. Each failure mode (indexed as i = 1,2,...,m) was evaluated against criteria (indexed as j = S,O,D) using conventional scores. To incorporate uncertainties, they transformed crisp elements in the group decision matrix into rough number forms, resulting in a rough group decision-making matrix. Furthermore, they computed rough sequences and average rough intervals along with their respective intervals. By determining the weighted normalized decision matrix in rough number form, they obtained a comprehensive evaluation. Additionally, they defined positive and negative ideal solutions and calculated the separation of each failure mode from these benchmarks. Finally, they compared their approach with fuzzy FMEA, conventional FMEA, and rough FMEA, ultimately concluding the steam valve problem based on their ranking values.

The motivation for our present study stems from the preceding work. We have taken up the same steam valve system in a power plant featuring eight distinct failure modes as the focal point. Utilizing the FMEA approach, we've adopted the EDAS method, incorporating the neutrosophic inverse soft expert set (NISES) as a key tool in solving the problem. The subsequent section elucidates the failure modes and their respective solutions in a clear and accessible manner. In contrast to rough interval weights, we've opted for attribute weights. We then proceed to construct a decision matrix (DM) employing NISES, accounting for *i* failure modes (i = 1,2,...,m) against the three criteria (j = S,O,D). This process involves the computation of positive distance average (PDA) and negative distance average (NDA) matrices, weighted normalized positive distance averages ($WNPDA_i$) and weighted normalized negative distance averages ($WNNDA_i$), as well as assessment scores (AS_i). Finally, we conclude the evaluation with a final ranking based on (AS_i).

The algorithm is presented below and the comparative analysis of our new approach with existing Song et al. [20] approach is presented in the next section.

4.1. *Algorithm*

We now present the algorithm on failure mode and effect analysis approach using evaluation based on distance from average solution method with neutrosophic inverse soft expert set.

Input:NISES.

Output: Ranking the alternatives.

Step 1. Choose the criteria that reveals about failure data.

Step 2. The decision making matrix (D) using NISES is constructed.

$$\widetilde{DM} = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ m \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ \vdots & \vdots & \vdots \\ r_{m1} & r_{m2} & r_{m3} \end{pmatrix} \end{array} \tag{1}$$

Step 3. Define average solution as

$$AV_j = \frac{\sum_{i=1}^{m} r_{ij}}{m}. \tag{2}$$

Step 4. Calculate positive distance average (PDA) and negative distance average (NDA) matrices as follows.

$$PDA = [PDA_{ij}]_{m\times 3} \tag{3}$$

$$NDA = [NDA_{ij}]_{m\times 3} \tag{4}$$

where,

$$PDA_{ij} = \frac{max(0, (AV_j - r_{ij}))}{AV_j}; i = 1, 2..., m, j = 1, 2, 3 \tag{5}$$

$$NDA_{ij} = \frac{max(0, (r_{ij} - AV_j)}{AV_j}; i = 1, 2..., m, j = 1, 2, 3 \tag{6}$$

Step 5. Determine weighted sum of positive distance average (WSPDA) and weighted sum of negative distance average (WSNDA) .

$$WSPDA_i = \sum_{j=1}^{3} PDA_{ij} \times w_j; i = 1, 2..., m \tag{7}$$

$$WSNDA_i = \sum_{j=1}^{3} NDA_{ij} \times w_j; i = 1, 2..., m \tag{8}$$

Step 6. Calculate weighted normalized positive distance average (WNPDA) and weighted normalized negative distance average (WNNDA)

$$WNPDA_i = \frac{WPDA_i}{max_i(WPDA_i)}; i = 1, 2..., m \tag{9}$$

$$WNNDA_i = \frac{WNDA_i}{max_i(WNDA_i)}; i = 1, 2..., m \tag{10}$$

Step 7. The assessment score (AS_i) for each alternatives is calculated as follows.

$$AS_i = \frac{1}{2}(WNPDA_i + WNNDA_i) \tag{11}$$

Step 8. Perform final ranking by arranging the assessment score of alternatives in descending order .

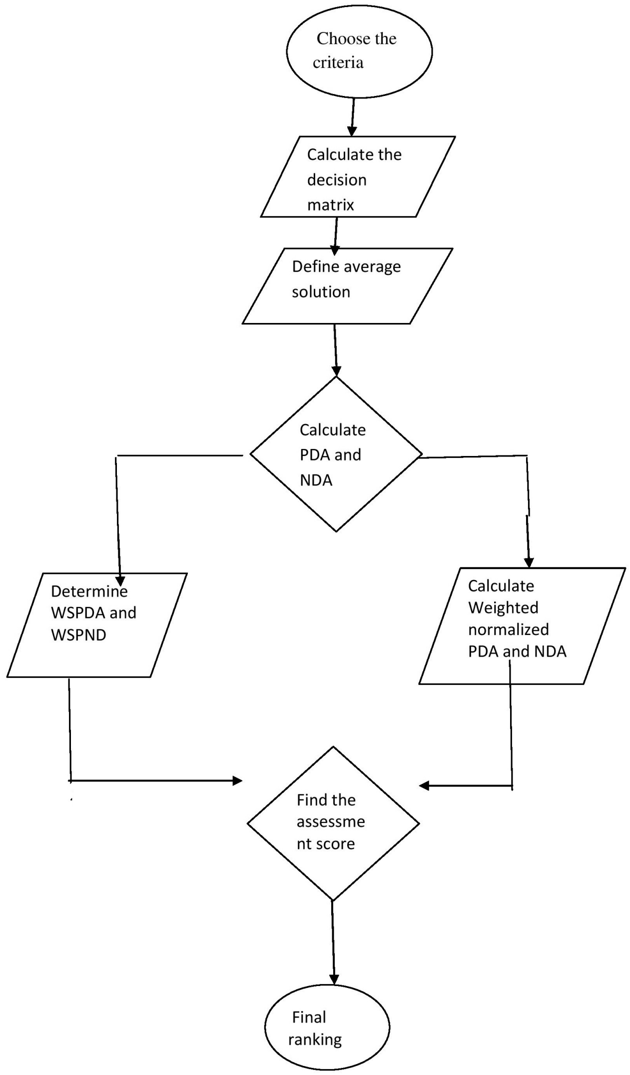

Figure 1. Algorithm on FMEA approach using EDAS

5. **Comparative Analysis**

In this section, we focus on the steam valve system within a power plant, where failures may manifest under various circumstances. These failure modes encompass instances such as prolonged shutting time (Mode 1), improper sealing causing leakage (Mode 2), steam leakage from the valve shaft (Mode 3), valve replacements (Mode 4), valve obstruction during operation (Mode 5), fractures in the valve shaft (Mode 6), failure of the valve shaft bolster bearing (Mode 7), and excessive noise in the system (Mode 8), particularly while a steam valve is in operation within the plant. A prior study [20] addressed this specific scenario using the TOPSIS method within the framework of FMEA for the steam valve system. Notably, they employed rough set theory as a pivotal tool to substantiate their findings and

arguments
We present the table of steam valve system using FMEA model as presented by Song et al., (2013) in table 2.

Table 2. Tabular representation of the steam valve system

S.No	Failure Mode	Causes	Effects of failure	Detection measures
1	Prolong of shutting time	Counter-intuitive spring decision	Over boost of steam turbine rotor and parts mishap	Valve seal test
2	Not being firmly closed	Little bushing leeway, shaft twisting	Cutting edge erosion of steam turbine	Valve break test
3	Steam spill around valve shaft	Compaction power of firing filler isn't sufficient	Misuse of substance water and warm misfortune	Assessment in the wake of pressing evacuation
4	Valve changes	Water driven chamber spills	Problem in regular opening	closing of valve with hazardous activity
5	Valve jam in activity	Due to procedure and material imperfections	Valve can't open and close	Valve activity test
6	Crack of valve shaft	Weariness break under rotating pressure	Stumbling of turbine	Metallographic tests on the crack hole
7	Breakdown of valve shaft bolster bearing	Low quality of bearing material and long haul milage	Anomalous activity of valve framework	Dismantle examination
8	Over the top commotion framework	Framework vibration because of outlandish parts	Make the client feel awkward	Change working condition, recurrence estimation of valve

In our current investigation, we have retained the focus on the eight potential failure modes occurring within the plant. To validate the robustness of our findings, we have employed the Evaluation Based on Distance from Average Solution method, leveraging Neutrosophic Inverse Soft Expert Set as a crucial tool. This rigorous evaluation serves to establish the superiority of our approach in comparison to the existing work [20]. It is noteworthy that we have diligently assigned weights to the factors

Severity (S), Occurrence (O), and Detection (D), and subsequently validated the outcomes, ensuring a comprehensive and reliable assessment.

5.1. *NISES Group Decision Making Procedure*

Step 1. The failure mode criteria are 1, 2, 3, 4, 5, 6, 7, 8. The problem of steam valve system discussed in [20] is considered with the same eight failure modes.

Step 2. Create the decision making matrix.

TABLE 3. Tabular representation of rating for failure modes with RPN in ANISES

No.	Failure mode	Severity				Occurrence				Detection			
		$\sqsupset_1$	$\sqsupset_2$	$\sqsupset_3$	$\sqsupset_4$	$\sqsupset_1$	$\sqsupset_2$	$\sqsupset_3$	$\sqsupset_4$	$\sqsupset_1$	$\sqsupset_2$	$\sqsupset_3$	$\sqsupset_4$
1	Long shutting time of valve	(0.8,0.6,0.3)	(0.4,0.5,0.8)	(0.5,0.3,0.1)	(0.4,0.4,0.5)	(0.5,0.3,0.4)	(0.9,0.1,0.5)	(0.5,0.4,0.1)	(0.5,0.2,0.5)	(0.4,0.9,0.4)	(0.6,0.2,0.9)	(0.6,0.3,0.2)	(0.8,0.4,0.2)
2	Not being firmly closed	(0.5,0.2,0.9)	(0.3,0.1,0.5)	(0.9,0.3,0.5)	(0.1,0.4,0.7)	(0.1,0.5,0.6)	(0.2,0.4,0.6)	(0.1,0.5,0.2)	(0.3,0.1,0.4)	(0.5,0.3,0.4)	(0.9,0.1,0.5)	(0.5,0.4,0.1)	(0.5,0.2,0.5)
3	Steam spill around valve shaft	(0.2,0.5,0.8)	(0.4,0.9,0.4)	(0.6,0.2,0.9)	(0.6,0.3,0.2)	(0.8,0.4,0.2)	(0.5,0.3,0.2)	(0.9,0.8,0.4)	(0.4,0.2,0.6)	(0.9,0.3,0.3)	(0.3,0.2,0.9)	(0.8,0.4,0.1)	(0.4,0.6,0.2)
4	Valve changes	(0.4,0.5,0.6)	(0.7,0.2,0.3)	(0.1,0.5,0.9)	(0.3,0.5,0.8)	(0.4,0.8,0.2)	(0.5,0.6,0.8)	(0.1,0.3,0.8)	(0.3,0.5,0.1)	(0.3,0.4,0.7)	(0.8,0.2,0.4)	(0.4,0.3,0.2)	(0.1,0.3,0.5)
5	Valve jam in activity	(0.3,0.4,0.7)	(0.8,0.2,0.4)	(0.4,0.3,0.2)	(0.1,0.3,0.5)	(05,0.3,0.9)	(0.7,0.5,0.3)	(0.5,0.7,0.2)	(0.9,0.5,0.2)	(0.9,0.3,0.3)	(0.3,0.2,0.9)	(0.8,0.4,0.1)	(0.4,0.6,0.2)
6	Crack of valve shaft	(0.3,0.4,0.7)	(0.8,0.7,0.3)	(0.4,0.6,0.4)	(0.9,0.3,0.3)	(0.3,0.2,0.9)	(0.8,0.4,0.1)	(0.4,0.6,0.2)	(0.1,0.9,0.7)	(0.9,0.2,0.3)	(0.4,0.6,0.3)	(0.1,0.7,0.3)	(0.3,0.9,0.1)
7	Breakdown of valve shaft bolster bearing	(0.9,0.2,0.3)	(0.4,0.6,0.3)	(0.1,0.7,0.3)	(0.3,0.9,0.1)	(0.2,0.3,0.1)	(0.4,0.5,0.2)	(0.9,0.4,0.9)	(0.4,0.2,0.5)	(0.4,0.8,0.2)	(0.5,0.6,0.8)	(0.1,0.3,0.8)	(0.3,0.5,0.1)
8	Over the top commotion framework	(0.3,0.4,0.8)	(0.7,0.9,0.1)	(0.2,0.4,0.3)	(0.3,0.8,0.2)	(0.9,0.4,0.9)	(0.8,0.7,0.3)	(0.6,0.2,0.9)	(0.1,0.5,0.9)	(0.1,0.3,0.5)	(05,0.3,0.9)	(0.5,0.4,0.1)	(0.5,0.2,0.5)

TABLE 4. Tabular representation of rating for failure modes with RPN in DNISES

No.	Failure mode	Severity				Occurrence				Detection			
		$\sqsupset_1$	$\sqsupset_2$	$\sqsupset_3$	$\sqsupset_4$	$\sqsupset_1$	$\sqsupset_2$	$\sqsupset_3$	$\sqsupset_4$	$\sqsupset_1$	$\sqsupset_2$	$\sqsupset_3$	$\sqsupset_4$
1	Long shutting time of valve	(0.1,0.7,0.9)	(0.4,0.6,0.5)	(0.8,0.3,0.5)	(0.5,0.1,0.7)	(0.8,0.3,0.3)	(0.4,0.4,0.8)	(0.4,0.2,0.6)	(0.8,0.1,0.7)	(0.3,0.6,0.7)	(0.6,0.2,0.9)	(0.4,0.6,0.1)	(0.4,0.4,0.7)
2	Not being firmly closed	(0.3,0.5,0.6)	(0.4,0.7,0.9)	(0.1,0.7,0.3)	(0.2,0.1,0.5)	(0.9,0.2,0.9)	(0.3,0.1,0.7)	(0.8,0.5,0.2)	(0.8,0.7,0.1)	(0.4,0.6,0.1)	(0.4,0.3,0.7)	(0.8,0.2,0.5)	(0.7,0.3,0.8)
3	Steam spill around valve shaft	(0.4,0.1,0.6)	(0.1,0.4,0.6)	(0.1,0.5,0.5)	(0.3,0.6,0.7)	(0.6,0.2,0.9)	(0.6,0.3,0.4)	(0.3,0.2,0.6)	(0.1,0.8,0.9)	(0.1,0.7,0.6)	(0.6,0.4,0.5)	(0.9,0.2,0.3)	(0.4,0.6,0.3)
4	Valve changes	(0.2,0.5,0.8)	(0.4,0.7,0.9)	(0.3,0.5,0.4)	(0.8,0.9,0.3)	(0.4,0.1,0.9)	(0.7,0.2,0.9)	(0.3,0.6,0.8)	(0.5,0.2,0.4)	(0.6,0.4,0.5)	(0.9,0.2,0.3)	(0.4,0.6,0.3)	(0.4,0.2,0.9)
5	Valve jam in activity	(0.4,0.7,0.3)	(0.3,0.5,0.2)	(0.4,0.9,0.2)	(0.4,0.6,0.1)	(0.4,0.3,0.7)	(0.8,0.2,0.5)	(0.7,0.3,0.8)	(0.9,0.3,0.8)	(0.7,0.5,0.2)	(0.2,0.3,0.7)	(0.7,0.3,0.7)	(0.8,0.9,0.5)
6	Crack of valve shaft	(0.7,0.5,0.2)	(0.2,0.3,0.7)	(0.7,0.3,0.7)	(0.4,0.6,0.1)	(0.4,0.4,0.7)	(0.9,0.3,0.3)	(0.4,0.2,0.9)	(0.4,0.8,0.1)	(0.2,0.5,0.8)	(0.4,0.7,0.9)	(0.3,0.5,0.4)	(0.8,0.9,0.3)
7	Breakdown of valve shaft bolster bearing	(0.4,0.8,0.3)	(0.2,0.4,0.7)	(0.1,0.7,0.6)	(0.6,0.4,0.5)	(0.9,0.2,0.3)	(0.4,0.6,0.3)	(0.5,0.3,0.6)	(0.2,0.9,0.8)	(0.8,0.2,0.5)	(0.7,0.3,0.8)	(0.1,0.8,0.9)	(0.4,0.6,0.1)
8	Over the top commotion framework	(0.3,0.7,0.9)	(0.4,0.7,0.8)	(0.1,0.2,0.9)	(0.3,0.4,0.2)	(0.3,0.4,0.8)	(0.7,0.8,0.2)	(0.1,0.4,0.3)	(0.9,0.2,0.5)	(0.1,0.7,0.6)	(0.6,0.4,0.5)	(0.4,0.6,0.5)	(0.6,0.2,0.1)

Remark 5.1. (i) Now we find the Agree - NISES as follows,
(max of degree of membership $\{\sqsupset_1, \sqsupset_2, \sqsupset_3, \sqsupset_4\}$, min of degree of non- membership $\{\sqsupset_1, \sqsupset_2, \sqsupset_3, \sqsupset_4\}$, min of degree of indeterminacy $\{\sqsupset_1, \sqsupset_2, \sqsupset_3, \sqsupset_4\}$).
(ii) Now we find the Disagree-NISES as follows,

(min of degree of membership $\{\beth_1, \beth_2, \beth_3, \beth_4\}$, max of degree of non- membership $\{\beth_1, \beth_2, \beth_3, \beth_4\}$, min of degree of indeterminacy $\{\beth_1, \beth_2, \beth_3, \beth_4\}$).

Table 5. Tabular representation of RPN in Agree - NISES

Failure Mode	Severity	Occurrence	Detection
1	(0.8,0.3,0.8)	(0.9,0.1,0.5)	(0.8,0.2,0.9)
2	(0.9,0.1,0.9)	(0.3,0.1,0.6)	(0.9,0.1,0.5)
3	(0.6,0.2,0.9)	(0.9,0.2,0.6)	(0.9,0.2,0.9)
4	(0.7,0.2,0.9)	(0.5,0.3,0.8)	(0.8,0.2,0.7)
5	(0.8,0.2,0.7)	(0.9,0.3,0.9)	(0.9,0.2,0.9)
6	(0.9,0.3,0.7)	(0.8,0.2,0.9)	(0.9,0.2,0.3)
7	(0.9,0.2,0.3)	(0.9,0.2,0.9)	(0.5,0.3,0.8)
8	(0.7,0.4,0.8)	(0.9,0.2,0.9)	(0.5,0.2,0.9)

Table 6. Tabular representation of RPN in Disagree - NISES

Failure Mode	Severity	Occurrence	Detection
1	(0.1,0.7,0.5)	(0.4,0.4,0.3)	(0.3,0.6,0.1)
2	(0.1,0.7,0.3)	(0.3,0.7,0.1)	(0.4,0.6,0.1)
3	(0.1,0.6,0.5)	(0.1,0.8,0.4)	(0.1,0.7,0.3)
4	(0.2,0.9,0.3)	(0.3,0.6,0.4)	(0.4,0.6,0.3)
5	(0.3,0.9,0.1)	(0.4,0.3,0.5)	(0.2,0.9,0.2)
6	(0.2,0.6,0.1)	(0.4,0.8,0.1)	(0.2,0.9,0.3)
7	(0.1,0.8,0.3)	(0.2,0.9,0.3)	(0.1,0.8,0.1)
8	(0.1,0.7,0.2)	(0.1,0.8,0.2)	(0.1,0.7,0.1)

Remark 5.2. Now we can find the NISES by using the following way,
(max of degree of membership $\{\beth_1, \beth_2, \beth_3, \beth_4\}$, min of degree of indeterminacy $\{\beth_1, \beth_2, \beth_3, \beth_4\}$, min of degree of non- membership $\{\beth_1, \beth_2, \beth_3, \beth_4\}$).

TABLE 7. Tabular representation of RPN in NISES

Failure Mode	Severity	Occurrence	Detection
1	(0.1,0.3,0.5)	(0.4,0.1,0.3)	(0.3,0.2,0.1)
2	(0.1,0.1,0.3)	(0.3,0.1,0.1)	(0.4,0.1,0.1)
3	(0.1,0.2,0.5)	(0.1,0.2,0.4)	(0.1,0.2,0.3)
4	(0.2,0.2,0.3)	(0.3,0.3,0.4)	(0.4,0.2,0.3)
5	(0.3,0.2,0.1)	(0.4,0.3,0.5)	(0.2,0.2,0.2)
6	(0.2,0.3,0.1)	(0.4,0.2,0.1)	(0.2,0.2,0.3)
7	(0.1,0.2,0.3)	(0.2,0.2,0.3)	(0.1,0.3,0.1)
8	(0.1,0.4,0.2)	(0.1,0.2,0.2)	(0.1,0.2,0.1)

Remark 5.3. $\underline{lim(NISES)}$ or $\underline{lim} = \dfrac{\text{degree of membership + degree of indeterminacy}}{2}$

$$\overline{lim(NISES)} \text{ or } \overline{lim} = \frac{\text{degree of indeterminacy + degree of non-membership}}{2}.$$

TABLE 8. NISES failure modes assessment matrix

Failure Mode	Severity	Occurrence	Detection
1	[0.2,0.4]	[0.25,0.2]	[0.25,0.15]
2	[0.1,0.2]	[0.2,0.1]	[0.25,0.1]
3	[0.15,0.35]	[0.15,0.3]	[0.15,0.25]
4	[0.2,0.25]	[0.3,0.35]	[0.3,0.25]
5	[0.25,0.15]	[0.35,0.4]	[0.2,0.2]
6	[0.25,0.2]	[0.3,0.15]	[0.2,0.25]
7	[0.15,0.25]	[0.2,0.25]	[0.2,0.2]
8	[0.25,0.3]	[0.15,0.2]	[0.15,0.15]

Calculate the decision matrix for failure mode, using the formula $|\underline{lim(NISES)} - \overline{lim(NISES)}|$.

$$\widetilde{DM} = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} \begin{bmatrix} 0.2 & 0.05 & 0.1 \\ 0.1 & 0.1 & 0.15 \\ 0.2 & 0.15 & 0.1 \\ 0.05 & 0.05 & 0.05 \\ 0.1 & 0.05 & 0 \\ 0.05 & 0.15 & 0.05 \\ 0.1 & 0.05 & 0 \\ 0.05 & 0.05 & 0 \end{bmatrix}$$

Step 3. Find AV of all attributes as follows.

$$AV_1 = \frac{0.05 + 0.1 + 0.2 + 0.05 + 0.1 + 0.2 + 0.1 + 0.05}{8} = 0.09$$

$$AV_2 = \frac{0.15 + 0.1 + 0.15 + 0.05 + 0.05 + 0.05 + 0.05 + 0.05}{8} = 0.08$$

$$AV_3 = \frac{0.05 + 0.15 + 0.1 + 0.05 + 0 + 0.1 + 0 + 0}{8} = 0.05$$

Step 4. The values of PDA solution for first attribute 'S' are given below

$$PDA_{11} = \frac{max(0, (0.09 - 0.2))}{0.09} = 0$$

$$PDA_{21} = \frac{max(0, (0.09 - 0.1))}{0.09} = 0$$

$$PDA_{31} = \frac{max(0, (0.09 - 0.2))}{0.09} = 0$$

$$PDA_{41} = \frac{max(0, (0.09 - 0.05))}{0.09} = 0.444$$

$$PDA_{51} = \frac{max(0, (0.09 - 0.1))}{0.09} = 0$$

$$PDA_{61} = \frac{max(0, (0.09 - 0.05))}{0.09} = 0.444$$

$$PDA_{71} = \frac{max(0, (0.09 - 0.1))}{0.09} = 0$$

$$PDA_{81} = \frac{max(0, (0.09 - 0.05))}{0.09} = 0.444$$

Other values of the PDA solution is provided in Table 9 .

Table 9. Values of PDA solution

FM	S	O	D
1	0	0.375	0
2	0	0	0
3	0	0	0
4	0.444	0.375	0
5	0	0.375	0
6	0.444	0	0
7	0	0.375	0
8	0.444	0.375	0

'S'- NDA solution is given below.

$$NDA_{21} = \frac{max(0, (0.2 - 0.09))}{0.09} = 1.222$$

$$NDA_{22} = \frac{max(0, (0.1 - 0.09))}{0.09} = 0.111$$

$$NDA_{23} = \frac{max(0, (0.2 - 0.09))}{0.09} = 1.222$$

$$NDA_{24} = \frac{max(0, (0.05 - 0.09))}{0.09} = 0$$

$$NDA_{25} = \frac{max(0, (0.1 - 0.09))}{0.09} = 0.111$$

$$NDA_{26} = \frac{max(0, (0.05 - 0.09))}{0.09} = 0$$

$$NDA_{27} = \frac{max(0, (0.1 - 0.09))}{0.09} = 0.111$$

$$NDA_{28} = \frac{max(0, (0.05 - 0.09))}{0.09} =0$$

Table10 indicates the other values of the NDA solution namely 'O' and 'D'.

Table 10. Values of NDA solution

FM	S	O	D
1	1.222	0	1
2	0.111	0.250	2
3	1.222	0.875	1
4	0	0	0
5	0.111	0	0
6	0	0.875	0
7	0.111	0	0
8	0	0	0

Step 5. Determine WSPDA and WSNDA for all alternatives, using attribute weights. By assigning equal weights to all the criteria we have the following table.

Table 11. Weight attributes

Attribute	S	O	D
ω_j	1/3	1/3	1/3

TABLE 12. Values of the weighted positive distances

FM	S	O	D	Sum
1	0	0.124	0	0.124
2	0	0	0	0
3	0	0	0	0
4	0.147	0.124	0	0.271
5	0	0.124	0	0.124
6	0.147	0	0	0.147
7	0	0.124	0	0.124
8	0.147	0.124	0	0.271

TABLE 13. Values of the weighted negative distances

FM	S	O	D	Sum
1	0.403	0	0.33	0.733
2	0.037	0.083	0.66	0.779
3	0.403	0.289	0.33	1.022
4	0	0	0	0
5	0.037	0	0	0.037
6	0	0.289	0	0.289
7	0.037	0	0	0.037
8	0	0	0	0

Step 6. Determine the weighted normalized PDA of each failure mode from Equation (9)

$$WNPDA_1 = \frac{0.124}{0.271} = 0.458$$

$$WNPDA_2 = \frac{0}{0.271} = 0$$

$$WNPDA_3 = \frac{0}{0.271} = 0$$

$$WNPDA_4 = \frac{0.271}{0.271} = 1$$

$$WNPDA_5 = \frac{0.124}{0.271} = 0.458$$

$$WNPDA_6 = \frac{0.147}{0.271} = 0.542$$

$$WNPDA_7 = \frac{0.124}{0.271} = 0.458$$

$$WNPDA_8 = \frac{0.271}{0.271} = 1$$

Next we determine the weighted normalized NDA of each failure mode from Equation (10)

$$WNNDA_1 = \frac{0.733}{1.022} = 0.717$$

$$WNNDA_2 = \frac{0.779}{1.022} = 0.782$$

$$WNNDA_3 = \frac{1.022}{1.022} = 1$$

$$WNNDA_4 = \frac{0}{1.022} = 0$$

$$WNNDA_5 = \frac{0.037}{1.022} = 0.036$$

$$WNNDA_6 = \frac{0.289}{1.022} = 0.283$$

$$WNNDA_7 = \frac{0.037}{1.022} = 0.036$$

$$WNNDA_8 = \frac{0}{1.022} = 0$$

Step 7. Determine the assessment score using the Equation (11)

$$AS_1 = \frac{1}{2}(0.458 + 0.717) = 0.588$$

$$AS_2 = \frac{1}{2}(0 + 0.782) = 0.391$$

$$AS_3 = \frac{1}{2}(0 + 1) = 0.5$$

$$AS_4 = \frac{1}{2}(1 + 0) = 0.5$$

$$AS_5 = \frac{1}{2}(0.458 + 0.036) = 0.247$$

$$AS_6 = \frac{1}{2}(0.542 + 0.283) = 0.413$$

$$AS_7 = \frac{1}{2}(0.458 + 0.036) = 0.247$$

$$AS_8 = \frac{1}{2}(1 + 0) = 0.5$$

Step 8. Ranking the failure mode

$$AS_1 \succ AS_3 \approx AS_4 \approx AS_8 \approx AS_6 \succ AS_2 \succ AS_5 \approx AS_7.$$

5.2. *Comparison of Song et al. [20] approach and our approach*

A comparison of Song et al. [20] approach and our approach is provided in Table 14 below.

TABLE 14. Comparison of the two approaches

Ranking	Alternative (s)	Best Alternative
Existing	$1 > 7 > 5 > 6 > 8 > 4 > 3 > 2$	1
Our approach	$1 > 3 \approx 4 \approx 8 \approx 6 > 2 > 5 \approx 7$	1

Both, our approach and the method proposed by Song et al. [20] yield equivalent results. However, when juxtaposed with Song et al.'s method, our approach boasts a streamlined process and straightforward calculations that are more intuitive and easier to comprehend. This comparative analysis is also visually represented through a graph, as illustrated below.

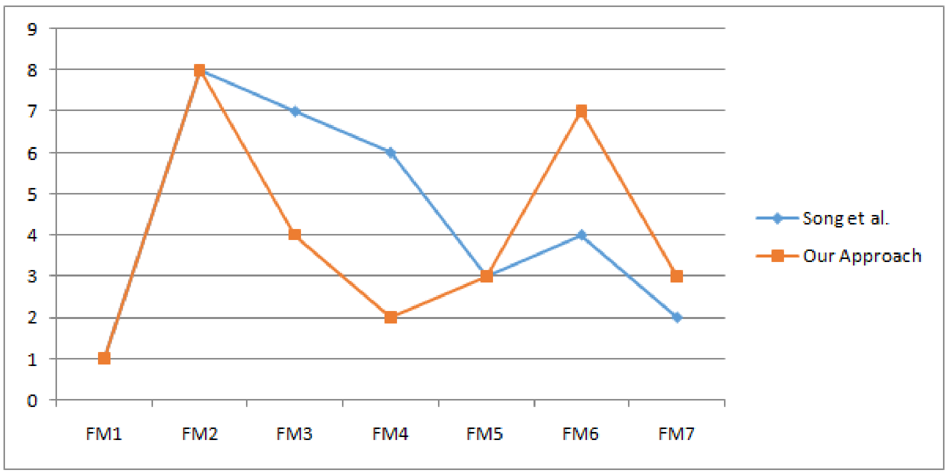

FIGURE 2. Comparative analysis of our approach with [20]

In the subsequent section, we delve into another application of the Neutrosophic Inverse Soft Expert Set, namely the Additive Ratio Assessment Simplified Version method. What sets this approach apart is its novel computation of optimal score values, which relies on the lower and upper limits of neutrosophic inverse soft expert sets. This innovation represents a significant advancement compared to the methodology employed in the Additive Ratio Assessment method by Zavadskas et al. [24]

We proceed by presenting an algorithm for the Additive Ratio Assessment Simplified Version method utilizing neutrosopic inverse soft expert sets. The algorithm consists of eight key steps. Central to this process is the construction of an $m \times n$ decision matrix (r_{ij}) where m signifies the cardinality of the universal set $|U|$, and n represents the cardinality of set $|\gimel|$. This decision matrix is then evaluated based on input from the decision makers. Subsequently, a Weighted Normalized Decision Matrix (WNDM) is derived, and an optimal score value is computed using the optimality function (OF). Following this, the Utility Degree (UD) is calculated, and the conclusion is determined based on the value of the utility degree.

6. **Additive Ratio Assessment-Simplified Version Method in neutrosophic inverse soft expert set**

Zavadskas et al. [24] pioneered the concept of the Additive Ratio Assessment (ARAS) method. The novelty of this method lies in its ability to facilitate the selection of the optimal alternative, taking into account the number of attributes. The final ranking of alternatives is accomplished by assessing the utility degree of each alternative. In the following section, we introduce the algorithm for the Additive Ratio Assessment - Simplified Version (ARAS-SV) method as outlined below.

6.1. *Algorithm on additive ratio assessment-simplified version Method using neutrosophic inverse soft expert set*

Step 1. Construct the decision matrix based on the information received from the decision maker using NISES and remark (5.3), namely $X = \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ \vdots & \vdots & \dots & \vdots \\ r_{m1} & r_{m2} & \dots & r_{mn} \end{bmatrix}$ (or) $X = (r_{ij})_{m \times n}$.

Step 2. Normalized Decision Matrix (NDM_{ij}) is defined as follows.

$$NDM_{ij} = \frac{r_{ij}}{\sum\limits_{i=1}^{m} r_{ij}}; j = 1, 2..., n \tag{12}$$

Step 3. Choose the weight of attributes w_j from the decision maker.

Step 4. Form the weighted normalized decision matrix (WNDM) as follows.

$$WNDM_{ij} = r^*_{ij} \cdot w_j; i = 1, 2, ..., m, j = 1, 2..., n \tag{13}$$

Step 5. Construct the optimality function (OF) as follows.

$$OF_i = \sum_{j=1}^{n} WNDM; i = 1, 2, ..., m \tag{14}$$

Step 6. Calculate optimality score value using optimality function defined in remark (5.3) as follows.

$$S_i = \frac{\underline{lim} + \overline{lim}}{2} \tag{15}$$

Step 7. Calculate the utility degree (UD) using this formula

$$UD_i = \frac{S_i}{V_0}, i = 1, 2, ..., m, \tag{16}$$

where V_0 is the maximum value of S_i.

Step 8. UD_i values are arranged in descending order in order to find the final ranking.

6.2. *Illustrative Example*

Problem statement

Imagine a scenario where a patient needs to make a crucial decision about selecting the most suitable doctor among four experts, each specializing in different fields of medical treatment. The challenge at hand is to make an informed choice based on various parameters. We denote the four doctors as $U = \{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$ and define a set of parameters $\Upsilon = \{\daleth_1, \daleth_2, \daleth_3, \daleth_4, \daleth_5, \daleth_6\}$. These parameters encompass factors such as hospital expenditure ($\daleth_1$), the efficiency of diagnosis by doctors ($\daleth_2$), doctor availability ($\daleth_3$), hospital provisions ($\daleth_4$), doctors' experience in treating the specific disease ($\daleth_5$), and the distance of the hospital from the patient's residence ($\daleth_6$). To navigate this decision-making process systematically, we employ the ARAS-SV method, breaking it down step by step as follows.

The problem is to choose a best doctor by a patient based on the parameters, listed.

Let us apply ARAS-SV method in the above situation step by step below.

Construct NISES as follows.

Table 15. Neutrosophic inverse soft expert sets

N_S^U	$(\daleth_1, \varrho_1, 1)$	$(\daleth_1, \varrho_1, 0)$	$(\daleth_1, \varrho_2, 1)$	$(\daleth_1, \varrho_2, 0)$	$(\daleth_2, \varrho_1, 1)$	$(\daleth_2, \varrho_1, 0)$	$(\daleth_2, \varrho_2, 1)$	$(\daleth_2, \varrho_2, 0)$	$(\daleth_3, \varrho_1, 1)$	$(\daleth_3, \varrho_1, 0)$	$(\daleth_3, \varrho_2, 1)$	$(\daleth_3, \varrho_2, 0)$
ϑ_1	(0.2,0.4,0.9)	(0.5,0.1,0.7)	(0.9,0.7,0.3)	(0.4,0.8,0.1)	(0.1,0.2,0.3)	(0.8,0.2,0.4)	(0.9,0.4,0.2)	(0.4,0.7,0.3)	(0.3,0.4,05)	(0.8,0.7,03)	(0.6,0.3,0.8)	(0.6,0.9,0.1)
ϑ_2	(0.3,0.8,0.1)	(0.5,0.3,0.1)	(0.8,0.5,0.6)	(0.4,0.2,0.8)	(0.9,0.8,0.6)	(0.3,0.5,0.7)	(0.5,0.3,0.2)	(0.8,0.2,0.4)	(0.4,0.2,0.6)	(0.6,0.2,0.5)	(0.4,0.5,0.6)	(0.9,0.4,0.5)
ϑ_3	(0.3,0.6,0.1)	(0.4,0.5,0.1)	(0.9,0.2,0.5)	(0.1,0.9,0.2)	(0.4,0.2,0.2)	(0.6,0.3,0.7)	(0.3,0.6,0.7)	(0,0.3,0.8)	(0.1,0.7,0.9)	(0.3,0.7,0.1)	(0.6,0.2,0.9)	(0.2,1,0.8)
ϑ_4	(0.2,0.4,0.7)	(0.1,0.4,0.2)	(0.7,0.4,0.9)	(1,0.8,0.3)	(0.4,0.8,0.1)	(0.7,0.6,0.3)	(0.5,0.5,0.5)	(0.3,0.8,0)	(0.4,0.1,0.3)	(0.5,0.9,0.2)	(0.8,0.5,0.3)	(0.6,0.4,0.1)

.

Table 16. Tabular representation of Agree - NISES

N_S^U	$(\daleth_1, \varrho_1, 1)$	$(\daleth_1, \varrho_2, 1)$	$(\daleth_2, \varrho_1, 1)$	$(\daleth_2, \varrho_2, 1)$	$(\daleth_3, \varrho_1, 1)$	$(\daleth_3, \varrho_2, 1)$
ϑ_1	(0.2,0.4,0.9)	(0.9,0.7,0.3)	(0.1,0.2,0.3)	(0.9,0.4,0.2)	(0.3,0.4,0.5)	(0.6,0.3,0.8)
ϑ_2	(0.3,0.8,0.1)	(0.8,0.5,0.6)	(0.9,0.8,0.6)	(0.5,0.3,0.2)	(0.4,0.2,0.6)	(0.4,0.5,0.6)
ϑ_3	(0.3,0.6,0.1)	(0.9,0.2,0.5)	(0.4,0.2,0.2)	(0.3,0.6,0.7)	(0.1,0.7,0.9)	(0.6,0.2,0.9)
ϑ_4	(0.2,0.4,0.7)	(0.7,0.4,0.9)	(0.4,0.8,0.1)	(0.5,0.5,0.5)	(0.4,0.1,0.3)	(0.8,0.5,0.3)

.

Table 17. Tabular representation of Disagree - NISES

N_S^U	$(\gimel_1, \varrho_1, 0)$	$(\gimel_1, \varrho_2, 0)$	$(\gimel_2, \varrho_1, 0)$	$(\gimel_2, \varrho_2, 0)$	$(\gimel_3, \varrho_1, 0)$	$(\gimel_3, \varrho_2, 0)$
ϑ_1	(0.5,0.1,0.7)	(0.4,0.8,0.1)	(0.8,0.2,0.4)	(0.4,0.7,0.3)	(0.8,0.7,0.3)	(0.6,0.9,0.1)
ϑ_2	(0.5,0.3,0.1)	(0.4,0.2,0.8)	(0.3,0.5,0.7)	(0.8,0.2,0.4)	(0.6,0.2,0.5)	(0.9,0.4,0.5)
ϑ_3	(0.4,0.5,0.1)	(0.1,0.9,0.2)	(0.6,0.3,0.7)	(0,0.3,0.8)	(0.3,0.7,0.1)	(0.2,1,0.8)
ϑ_4	(0.1,0.4,0.2)	(1,0.8,0.3)	(0.7,0.6,0.3)	(0.3,0.8,0)	(0.5,0.9,0.2)	(0.6,0.4,0.1)

Following the procedure adopted in Remark 5.2, we calculate NISES as follows,

Table 18. Tabular representation of NISES

N_S^U	$\gimel_1$	$\gimel_2$	$\gimel_3$	$\gimel_4$	$\gimel_5$	$\gimel_6$
ϑ_1	(0.2,0.1,0.7)	(0.4,0.7,0.1)	(0.1,0.2,0.3)	(0.4,0.4,0.1)	(0.3,0.4,0.3)	(0.6,0.3,0.1)
ϑ_2	(0.3,0.3,0.1)	(0.4,0.2,0.6)	(0.3,0.5,0.6)	(0.5,0.2,0.2)	(0.4,0.2,0.5)	(0.4,0.4,0.5)
ϑ_3	(0.3,0.5,0.1)	(0.1,0.2,0.2)	(0.4,0.2,0.2)	(0,0.3,0.7)	(0.1,0.7,0.1)	(0.2,0.2,0.8)
ϑ_4	(0.1,0.4,0.2)	(0.1,0.4,0.3)	(0.4,0.6,0.1)	(0.3,0.5,0)	(0.4,0.1,0.2)	(0.6,0.4,0.1)

Step 1. Define the decision matrix X using the decision makers information as namely from Table 18 and remark (5.3) as follows.

$$X = \begin{array}{c} \\ \vartheta_1 \\ \vartheta_2 \\ \vartheta_3 \\ \vartheta_4 \end{array} \begin{array}{cccccc} \gimel_1 & \gimel_2 & \gimel_3 & \gimel_4 & \gimel_5 & \gimel_6 \\ \end{array} \begin{pmatrix} (.15,.4) & (.65,.4) & (.15,.25) & (.4,.25) & (.35,.35) & (.45,.2) \\ (.4,.2) & (.3,.4) & (.4,.55) & (.35,.2) & (.3,.35) & (.4,.45) \\ (.4,.3) & (.15,.2) & (.3,.2) & (.15,.5) & (.4,.4) & (.2,.5) \\ (.25,.3) & (.25,.35) & (.5,.35) & (.4,.25) & (.25,.15) & (.5,.25) \end{pmatrix}$$

Step 2. Calculate the NDM using the equation (12).

$$\begin{array}{c} \\ \vartheta_1 \\ \vartheta_2 \\ \vartheta_3 \\ \vartheta_4 \end{array} \begin{array}{cccccc} \gimel_1 & \gimel_2 & \gimel_3 & \gimel_4 & \gimel_5 & \gimel_6 \end{array} \begin{pmatrix} (.125,.333) & (.481,.296) & (.111,.185) & (.308,.208) & (.269,.280) & (.290,.143) \\ (.333,.166) & (.222,.296) & (.296,.407) & (.269,.166) & (.231,.280) & (.258,.321) \\ (.333,.250) & (.111,.146) & (.222,.148) & (.115,.417) & (.308,.320) & (.129,.357) \\ (.208,.250) & (.185,.259) & (.370,.259) & (.308,.208) & (.192,.120) & (.321,.179) \end{pmatrix}$$

Step 3. Form the weight of attributes w_j from the decision maker namely patient as follows.

$\gimel_1$ = cost of hospital expenditure = 0.1

$\gimel_2$ = diagnosing efficiency of doctors = 0.2

$\gimel_3$ = availability of doctors = 0.2

$\gimel_4$ = hospital provisions = 0.2

$\gimel_5$ = doctors experience in curing the disease = 0.2

$\gimel_6$ = the hospital distance from the patient house = 0.1

Attribute	$\gimel_1$	$\gimel_2$	$\gimel_3$	$\gimel_4$	$\gimel_5$	$\gimel_6$
w_j	0.1	0.2	0.2	0.2	0.2	0.1

Step 4. Construct the weighted normalized decision matrix using the equation (13)

$$\begin{array}{c} \\ \vartheta_1 \\ \vartheta_2 \\ \vartheta_3 \\ \vartheta_4 \end{array} \begin{array}{c} \begin{array}{cccccc} \gimel_1 & \gimel_2 & \gimel_3 & \gimel_4 & \gimel_5 & \gimel_6 \end{array} \\ \begin{pmatrix} (.013, .033) & (.096, .059) & (.022, .037) & (.062, .042) & (.054, .029) & (.029, .014) \\ (.033, .025) & (.022, .029) & (.044, .030) & (.023, .083) & (.026, .071) & (.013, .036) \\ (.033, .017) & (.044, .059) & (.059, .081) & (.054, .033) & (.052, .064) & (.026, .032) \\ (.021, .025) & (.037, .052) & (.074, .052) & (.062, .044) & (.064, .036) & (.032, .018) \end{pmatrix} \end{array}$$

Step 5. Calculate the optimality function using the equation (14)

$OF_1 = (0.013, 0.033)+(0.096, 0.059)+(0.022, 0.037)+(0.062, 0.042)+(0.054, 0.029)+(0.029, 0.014) = (0.276, 0.214).$

$OF_2 = (0.033, 0.025)+(0.022, 0.029)+(0.044, 0.030)+(0.023, 0.083)+(0.026, 0.071)+(0.013, 0.036) = (0.161, 0.274).$

$OF_3 = (0.033, 0.017)+(0.044, 0.059)+(0.059, 0.081)+(0.054, 0.033)+(0.052, 0.064)+(0.026, 0.032) = (0.268, 0.286).$

$OF_4 = (0.021, 0.025)+(0.037, 0.052)+(0.074, 0.052)+(0.062, 0.044)+(0.064, 0.036)+(0.032, 0.018) = (0.290, 0.227).$

Step 6. Construct optimal score value using optimality function as follows.

$$S_i = \frac{\underline{lim} + \overline{lim}}{2}$$

$$S_1 = \frac{0.276 + 0.214}{2} = 0.245$$

$$S_2 = \frac{0.161 + 0.274}{2} = 0.218$$

$$S_3 = \frac{0.268 + 0.286}{2} = 0.277$$

$$S_4 = \frac{0.290 + 0.227}{2} = 0.209$$

Step 7. Construct the utility degree using the equation (16)

$$UD_1 = \frac{0.245}{0.277} = 0.884$$

$$UD_2 = \frac{0.218}{0.277} = 0.787$$

$$UD_3 = \frac{0.277}{0.277} = 1$$

$$UD_4 = \frac{0.209}{0.277} = 0.755$$

Step 8. The final ranking of alternatives and conclusion.

Finally, the third doctor ϑ_3 is the best choice to patient for treatment as per the final ranking.

$\vartheta_3 > \vartheta_1 > \vartheta_2 > \vartheta_4$.

7. **Result and discussion**

The integration of the Neutrosophic Inverse Soft Expert Sets technique into our Failure Mode and Effect Analysis approach has yielded a host of insightful outcomes. Through a meticulous comparative analysis with the methodology proposed by Song et al., several distinct advantages of our approach have come to light.

One prominent finding is the enhanced efficiency in the assessment of Risk Priority Numbers. By harnessing the power of NISES, we have devised a streamlined and transparent system for allocating weights to Severity (S), Occurrence (O), and Detection (D). This enhancement not only expedites the computation process but also enables a more intuitive evaluation of risk factors. In practical terms, this translates to swifter and more precise decision-making, a crucial attribute in industries where rapid response to potential failures is imperative.

Furthermore, our approach showcases commendable resilience in scenarios characterized by uncertainties and imprecise information. The inherent adaptability of neutrosophic sets allows us to effectively navigate the complexities of real-world situations. This adaptability proves invaluable in industries subject to dynamic and swiftly changing environments, providing a robust framework for risk assessment. Additionally, the NISES technique exhibits noteworthy versatility in accommodating a wide spectrum of expert judgments and assessments. Its adaptability to varying levels of expertise within a team ensures that insights from experts of different domains can be seamlessly integrated into the analysis. This inclusive approach not only fortifies the reliability of the results but also fosters a collaborative decision-making environment, a critical aspect in complex industrial settings.

In conclusion, the integration of NISES into FMEA constitutes a significant leap forward in the realm of risk assessment methodologies. Its impact is evidenced not only in the streamlined computation process but also in its adeptness at handling uncertainties and its inclusivity in expert assessments.

As industries continue to evolve, the NISES technique is poised to be a formidable and indispensable tool in navigating the intricate landscape of risk assessment and decision-making.

Our results exhibit superiority through a streamlined computation process facilitated by the integration of Neutrosophic Inverse Soft Expert Sets. This simplification not only accelerates the assessment of Risk Priority Numbers but also enhances the transparency and intuitiveness of the evaluation process. The assignment of weights to Severity (S), Occurrence (O), and Detection (D) factors is executed with greater efficacy, eliminating potential complexities and uncertainties in the weighting process. This, in turn, leads to a more accurate and reliable risk assessment. The adaptability of our approach to uncertainties and imprecise information, owing to the NISES technique, ensures its effectiveness in dynamic and rapidly changing environments. Additionally, our approach excels in inclusivity, accommodating a diverse range of expert judgments and assessments. This feature enables insights from experts with varying levels of expertise to be seamlessly integrated into the analysis, resulting in a more comprehensive and reliable evaluation. Ultimately, our approach yields equivalent optimal alternatives while offering potential for rapid decision-making, positioning it as a valuable tool in industries where timely and precise decision-making is critical.

8. **Limitations**

While the Neutrosophic Inverse Soft Expert Sets technique presents promising advancements in Failure Mode and Effect Analysis, it is essential to acknowledge its limitations.

1. Dependence on Expert Judgments: Like any expert-based approach, the effectiveness of NISES relies heavily on the quality and reliability of expert assessments. Inaccurate or biased judgments can introduce errors into the analysis, potentially leading to suboptimal decisions.

2. Sensitivity to Parameter Selection: The choice of parameters, such as the thresholds for Risk Priority Numbers or the weighting factors, can significantly influence the results. Selecting inappropriate values may lead to skewed assessments and potentially incorrect prioritization of failure modes.

3. Complexity of Implementation: Implementing the NISES technique may require a certain level of familiarity with neutrosophic theory and soft computing concepts. This complexity could pose a challenge for practitioners without a strong background in these areas.

4. Limited Historical Data: In situations where there is a scarcity of historical data or prior instances of similar failure modes, the accuracy and reliability of the NISES technique may be compromised. This is especially pertinent in novel or highly specialized industries.

5. Difficulty in Quantifying Soft Expert Opinions: Soft expert opinions, inherent to the NISES technique, can be challenging to quantify objectively. This subjectivity introduces an additional layer of uncertainty, potentially impacting the precision of the results.

6. Computational Overhead:Depending on the scale and complexity of the FMEA, the computational requirements for implementing NISES may be higher compared to more conventional approaches. This could lead to longer processing times, particularly for large-scale analyses.

7. Lack of Standardization: As a relatively new methodology, NISES may not yet have established standardized procedures or widely-accepted best practices. This can lead to variability in its application across different industries and contexts.

8. Potential for Overfitting: In situations where the NISES technique is applied to a limited dataset, there is a risk of overfitting, where the model may perform exceptionally well on the available data but struggle to generalize to new, unseen scenarios.

It's important to recognize these limitations and consider them in the context of specific applications. Addressing these challenges through ongoing research and refinement of the methodology will be crucial in realizing the full potential of NISES in FMEA.

9. **Conclusion and Future Work**

In conclusion, the integration of the NISES technique into FMEA approach presents a significant advancement in risk assessment methodologies. The simplified computation of RPN weights enhances the practicality and accessibility of the method, making it a valuable tool for industries facing complex decision-making scenarios.

Looking ahead, our research aims to explore the potential extensions of this approach into the realms of soft-rough fuzzy set and soft fuzzy rough set methodologies within the context of FMEA. This expansion holds promise for further refinement and enhancement of risk assessment techniques, catering to a broader spectrum of industries and applications.

Additionally, we plan to delve deeper into the application of neutrosophic sets within our approach. This presents an exciting avenue for research, with the potential to revolutionize risk analysis methodologies by incorporating a broader spectrum of uncertainties and complexities. By leveraging the power of neutrosophic sets, we anticipate even greater strides in the field of risk assessment and decision-making.

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Plithogenic Statistical Study of Environmental Audit and Corporate Social Responsibility in the Junín Region, Peru

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Abstract. Caring for the environment is a transversal task that concerns all professions, including accounting, with an emphasis on auditing; on the other hand, corporate social responsibility seeks efficiency between environmental, social, and economic aspects. This research aims to determine the relationship between environmental auditing and corporate social responsibility in the Junín Region, Peru. To meet this objective, surveys were applied to a randomly selected sample (121 Chartered Public Accountants) and interviews (12 Auditors attached to the Audit Chapter) as research instruments, the sample was obtained from the members of the College of Public Accountants of Junín, which made it possible through the Concurrent Triangulation Design to apply a holistic vision that allowed to compensate and strengthen the credibility of the investigation. To combine all these statistical results, the Plithogenic Statistic was used as a tool for processing the collected data. One of the advantages offered by the plithogenic theory is the possibility of combining knowledge from different sources, which allows us to capture the holistic and dynamic nature of the phenomena. In this case, there is a phenomenon that responds to different branches of knowledge that overlap in a complex way, such as the environmental and ecological aspect, with the economic-financial aspect, as well as the social and educational aspect, which present contradictory components among themselves.

Keywords: Environmental audit, corporate social responsibility, environmental care, Concurrent Triangulation Design, plithogenic statistics, plithogenic refined statistics, plithogenic neutrosophic statistics, neutrosophic statistics.

1 Introduction

This research stems from the need to care for the environment, which has become a very well-written green speech. Within the economic aspects where the social and environmental spheres revolve, the desire for control emerges, which is systematized through the environmental audit. Faced with this concern, the objective of the investigation is formulated, which is to determine the relationship that exists between environmental auditing and corporate social responsibility in the Junín Region, Peru.

The main theme of the study is based on the analysis of the variables Environmental Audit and Corporate Social Responsibility, both of which are important and consistent, hence the reason for evaluating the relationship between them. Over the years, the deterioration of nature has been observed, whose main depletion factor is directed by human beings, and companies pollute and destroy the environment. As part of the common welfare is the protection of future generations, for this inevitably control of the use of natural resources must be exercised, through the environmental audit. If substantial changes are not made in the way of developing the Environmental Audit, as well as the requirement of Corporate Social Responsibility, the disappearance of most of life on planet Earth will soon be observed.

An *a priori* analysis of the studied variables allows us to realize that both variables are contradictory to each other, at least partially. Especially if we practice the most widespread and successful economic models in the purely economic sense, those that do not take into account the damage to the environment that is inflicted during the production and service process. Until today, the polluting model, destructive of ecosystems, has predominated. However, more and more policies are included in private companies and States on the protection of the environment, since the old model will fail in the long term, also economically speaking.

This paper shows the results of the surveys carried out on 121 public accountants, and also 12 auditors were interviewed. This information was corroborated with each other by Concurrent Triangulation Design, confirming and compensating for the results [1, 2]. The qualitative results were converted to neutrosophic scales to process the indeterminacy that is typical of any decision-making process.

We carry out Plithogenic Statistics [3-7] instead of Classical Statistics because it involves processing variables of a different nature with a certain contradictory relationship with each other in a phenomenon that is multivariate, where there are indeterminate aspects, due to the lack of knowledge of how to dynamically run an economically profitable company, which fulfills its corporate purpose but also respects the laws and the environment. To combine so many benefits, we understand that the right measure must be found, where a 100% acceptable result will never be obtained in all these aspects.

The Plithogenic Statistics was introduced by Professor F. Smarandache, which according to his own words is defined as: “Plithogenic Statistics (PS) encompasses the analysis and observations of the events studied by the Plithogenic Probability. Plithogenic Statistics is a generalization of classical Multivariate Statistics, and it is a simultaneous analysis of many outcome neutrosophic /indeterminate variables, and it as well as a multi-indeterminate statistic.” ([7]).

Also, “The Plithogenic Probability of an event to occur is composed of the chances that the event occurs concerning all random variables (parameters) that determine it. The Plithogenic Probability, based on Plithogenic Variate Analysis, is a multi-dimensional probability (“plitho” means “many”, synonymous with “multi”). We may say that it is a probability of sub-probabilities, where each sub-probability describes the behavior of one variable. We assume that the event we study is produced by one or more variables. Each variable is represented by a Probability Distribution (Density) Function (PDF).” [7].

This paper consists of a Materials and Methods section, where the basics of Plithogeny are explained as well as some statistical and other tools used in this work. Section 3 contains the results obtained from the study we carried out. The article ends with the conclusions.

2 Materials and Methods

This section is dedicated to summarizing the basic principles of the theories that will be applied in solving the problem. The first one of them is the notion of Plithogenic Sets and Plithogenic Statistics.

2.1 Basic Notions on Plithogeny

According to F. Smarandache, “Plithogeny is the genesis or origination, creation, formation, development, and evolution of new entities from dynamics and organic fusions of contradictory and/or neutrals and/or non-contradictory multiple old entities. Plithogeny pleads for the connections and unification of theories and ideas in any field. As “entities” in this study, we take the ‘knowledge’ in various fields, such as soft sciences, hard sciences, arts, and letters theories, etc.” [3, 8, 9]

A *Plithogenic Set* is a non-empty set P whose elements within the domain of discourse U ($P \subseteq U$) are characterized by one or more attributes $A_1, A_2, \cdots, A_m, m \geq 1$, where each attribute can have a set of possible values within the spectrum S of values (states), such that S can be a finite, infinite, discrete, continuous, open, or closed set.

Each element $x \in P$ is characterized by all the possible values of the attributes that are inside the set $V = \{v_1, v_2, \cdots, v_n\}$. The value of an attribute has a *degree of appurtenance* $d(x, v)$ of an element x in the set P, about a certain given criterion. The degree of appurtenance can be either fuzzy, intuitionistic fuzzy, or neutrosophic, among others.

That means,

$$\forall x \in P, d: P \times V \rightarrow \mathcal{P}([0,1]^z) \quad (1)$$

Where $d(x, v) \subseteq [0,1]^z$ and $\mathcal{P}([0,1]^z)$ is the power set of $[0,1]^z$. $z = 1$ (the *fuzzy degree of appurtenance*), $z = 2$ (the *intuitionistic fuzzy degree of appurtenance*), or $z = 3$ (the *neutrosophic degree of appurtenance*).

Whether the cardinality of V is greater than or equal to 1, $c: V \times V \rightarrow [0,1]$ is called an *attribute value contradiction degree function* between any pair of attributes v_a, v_b, which satisfies the following axioms:

- $c(v_a, v_a) = 0$,
- $c(v_a, v_b) = c(v_b, v_a)$.

c defined as above, is denoted by c_F to indicate that this is a function called *fuzzy attributes value contradiction degree function*. It is generally defined like $c_{IF}: V \times V \rightarrow [0,1]^2$ as an *intuitionistic attributes value contradiction function* and like $c_N: V \times V \rightarrow [0,1]^3$ to indicate a *neutrosophic attributes value contradiction function*.

Thus, the Plithogenic Set is characterized by (P, a, V, d, c), which is constituted by the set P, the set a of attributes, the set V of values, the appurtenance function d and the c called *value contradiction degree function*.

The contradiction function in practice is applied to compare the contradiction of all attributes concerning a dominant attribute in case it exists, which is the most important one compared to the others.

On the other hand (U, a, V, d, c) is called *Plithogenic Probability*, where U is the event space E. A Plithogenic Probability is the probability that an event occurs in all the random variables that determine it. Where each random variable can be classical, (T,I,F)-neutrosophic, I-neutrosophic, (T,F)-intuitionistic fuzzy, (T,N,F)-picture fuzzy, (T,N,F)-spherical fuzzy, or (other fuzzy extensions) distribution function. In this way, the Plithogenic Probability generalizes classical multivariate Probability.

For its part, *Plithogenic Statistics* includes the analysis and observations obtained through the methods of the Plithogenic Probability [3, 6-7]. Plithogenic Statistics generalizes classical multivariate Statistics.

The *Refined Probabilities* are decomposed into more than one element of truth, or into more than one element of indeterminacy, or into more than one element of falsehood [3, 10]. That is, they are of the form; $(T_1, T_2, \cdots, T_p, I_1, I_2, \cdots, I_q, F_1, F_2, \cdots, F_r)$, where at least one of the indices p, q, or r is strictly bigger than 1.

2.2 Other tools used in the research

In this sub-section, we describe other tools that are used to solve the problem.

Cronbach's alpha makes it possible to quantify the level of reliability of a measurement scale for the unobservable magnitude constructed from the n observed variables. Cronbach's Alpha is calculated using the variances with Equation 2 ([11]):

$$\alpha = \left[\frac{k}{k-1}\right]\left[1 - \frac{\sum_{i=1}^{k} S_i^2}{S_t^2}\right] \quad (2)$$

Where:
S_i^2 is the variance of the ith item,
S_t^2 is the variance of all the observed values,
k is the number of questions or items.

Based on the correlation among items, the *Standard Cronbach's Alpha* is defined as follows in Equation 3:

$$\alpha_{stand} = \frac{kp}{1+p(k-1)} \quad (3)$$

Where:
k is the number of questions or items.
p is the mean of the linear correlations among the items.
Alphas bigger than 0.7 or 0.8 are enough to consider the scale reliable.

Spearman's Rho Correlation Coefficient results in a measure of the correlation between two variables. It is a non-parametric test, therefore it does not need to be verified that the sample satisfies a given distribution.

In the analyzed sample, the results were compared using Spearman's Rho Correlation Coefficient, which is calculated by Equation 4 ([12]):

$$\rho = 1 - \frac{6\sum_{i=1}^{N} D_i^2}{N(N^2-1)} \quad (4)$$

Where D is the difference between the corresponding x-y order statistics. N is the number of data pairs. $\rho \in [-1, 1]$, where 0 means no correlation, 1 means maximum positive correlation, and -1 means maximum negative correlation.

Finally, we address the *Concurrent Triangulation Design* ([1, 2]). This model is probably the most popular and is used when the researcher intends to confirm or corroborate results and perform cross-validation between quantitative and qualitative data, as well as take advantage of each method and minimize its weaknesses. It may happen that confirmation or corroboration is not presented. Quantitative and qualitative data on the research problem are simultaneously collected and analyzed at approximately the same time. During interpretation and discussion, the two kinds of results are fully explained, and comparisons of the databases are generally made. These are discussed "side by side", that is, the statistical results of each variable or quantitative hypothesis are included, followed by qualitative categories and segments, as well as the grounded theory that confirms or not the quantitative findings.

3 The Plithogenic statistical studies

This study focuses on the following two variables V1 and V2:

V1- **Environmental Audit**: This implies a systematic and independent examination or evaluation process to determine if the audited party complies with the audit objectives. This is the result of the application of an important methodology for the audited party to continuously improve its operations and guarantee a better framework of environmental protection. Using the audit, steps are taken to eliminate identified deficiencies. It is based on four main dimensions:

D11: Systemic evaluation,
D12: Business efficiency,
D13: Strategic Foresight,
D 14: Protective role.

V2- **Corporate Social Responsibility**: This is a sensible and coherent pact or commitment that fully respects the mission of the company, protecting its strengths, and considering the economic, social, and environmental expectations of all the parties involved. It consists of the following four dimensions:

D21 = Economic,
D22 = Legal,
D23 = Ethics,
D24 = Voluntary.

The objective of this research is to find a relationship between both variables and their dimensions. As well as the quantification of the probabilities of the behavior of both of them.

For the study, there were 121 unionized accountants for the quantitative study, they were randomly selected with simple random sampling. For the qualitative study, 12 ordinary and independent auditors were involved. Table 1 summarizes the research methods applied.

TECHNIQUE	INSTRUMENTS	CHARACTERISTICS
Reference analysis	Records	Bibliographic material, scientific articles, current reports, and others.
Interview	Interview guide	It consists of 8 open questions (4 of each variable)
Survey	Questionnaire	It consists of 24 questions with responses on a Likert scale (12 questions for each variable).

Table 1: Research Techniques and Instruments.

The questionnaire was evaluated by judges to determine its reliability. The validation of the expert judges determines that the research instrument regarding Category 1: Environmental audit is found in 13 items at a high qualification level and 3 items at a moderate qualification level, in Category 2: Corporate Social Responsibility it is found in 14 items at a high qualification level and 2 items at a moderate qualification level, determining that the instrument has a favorable evaluation.

The internal consistency of the quantitative instrument analyzed by Cronbach's Alpha determines a value of 0.99; therefore, the survey has a high level of reliability.

The survey was then applied where the 121 public accountants had to express their opinion on the situation of the two variables and their dimensions. To capture the multidimensionality of the problem, it was decided to apply the theory of Plithogenic Refined Probabilities. This was applied in two phases, which are mentioned below:

Phase 1. The experts were asked their opinions on each of the dimensions of the variables, based on a Likert scale with the components ([13-14]): Strongly Disagree, Disagree, Undecided, Agree, Strongly Agree.

1.1 For these results, we calculated their relative frequency in percent.

1.2. These frequency values were converted into Plithogenic Refined Probabilities to express the behavior of these dimensions in a general way in the region. Percentages are converted to Plithogenic Neutrosophic Probabilities

Phase 2: Data are processed by using techniques of Plithogenic Neutrosophic Probabilities.

The results of the survey in Phase 1 are shown in the following Tables 2-9:

	Absolute frequency	Percentage
Strongly disagree	28	23.1
Disagree	29	24.0
Undecided	23	19.0
Agree	41	33.9
Total	121	100.0

Table 2: Frequency table of Dimension 1 Systemic Evaluation of Variable 1 Environmental Audit in the Junín Region.

	Absolute frequency	Percentage
Strongly disagree	26	21.5
Disagree	45	37.2
Undecided	19	15.7
Agree	22	18.2
Strongly agree	9	7.4
Total	121	100.0

Table 3: Frequency table of Dimension 2 Business Efficiency of Variable 1 Environmental Audit in the Junín Region.

	Absolute frequency	Percentage
Strongly disagree	25	20.7
Disagree	27	22.3
Undecided	23	19.0
Agree	31	25.6
Strongly agree	15	12.4
Total	121	100.0

Table 4: Frequency table of Dimension 3 Strategic Prospective of Variable 1 Environmental Audit in the Junín Region

	Absolute frequency	Percentage
Disagree	32	26.4
Undecided	41	33.9
Agree	3. 4	28.1
Strongly agree	14	11.6
Total	121	100.0

Table 5: Frequency table of Dimension 4 Protective Role of Variable 1 Environmental Audit in the Junín Region.

	Absolute frequency	Percentage
Disagree	39	32.2
Undecided	37	30.6
Agree	37	30.6
Strongly agree	8	6.6
Total	121	100.0

Table 6: Frequency table of Dimension 1 Economic of Variable 2 Corporate social responsibility in the Junín Region.

	Absolute frequency	Percentage
Disagree	49	40.5
Undecided	40	33.1
Agree	28	23.1
Strongly agree	4	3.3
Total	121	100.0

Table 7: Frequency table of Dimension 2 Legal of Variable 2 Corporate social responsibility in the Junín Region.

	Absolute frequency	Percentage
Strongly disagree	30	24.8
Disagree	32	26.4
Undecided	26	21.5
Agree	19	15.7
Strongly agree	14	11.6
Total	121	100.0

Table 8: Frequency table of Dimension 3 Ethics of Variable 2 Corporate social responsibility in the Junín Region

	Absolute frequency	Percentage
Disagree	61	50.4
Undecided	13	10.7
Agree	33	27.3
Strongly agree	14	11.6
Total	121	100.0

Table 9: Frequency table of Dimension 4 Voluntary of Variable 2 Corporate social responsibility in the Junín Region

These results are plotted in Figure 1.

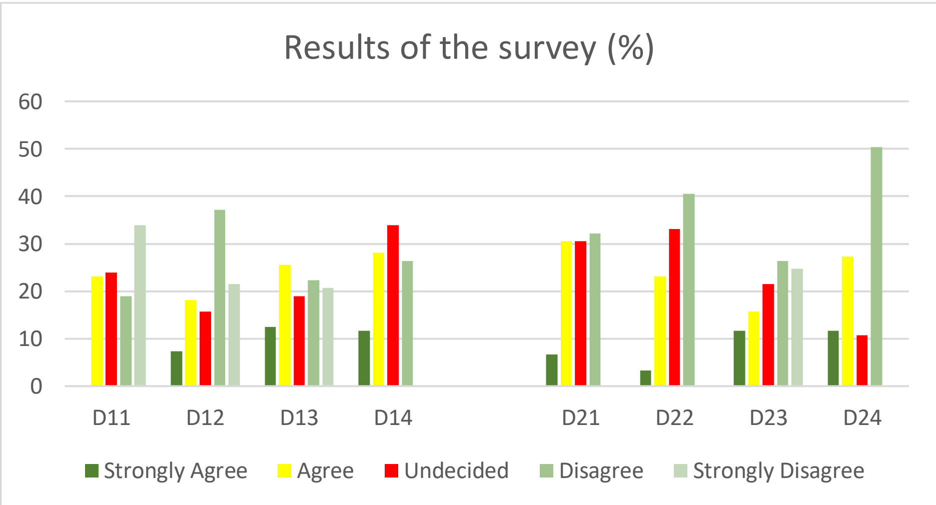

Figure 1: Results of the percentages for the 8 dimensions according to Tables 2-9.

To obtain the probabilities of each variable and its dimensions, they were represented by values of the type $(p_1, p_2, pI, np_2, np_1)$ that mean:

p_1- "Strongly sure" probability that the variable (dimension) is occurring properly,

p_2- "Sure" probability that the variable (dimension) is occurring properly,

pI- "Unsure" probability that the variable (dimension) is occurring properly,

np_2- "Sure" probability that the variable (dimension) is not occurring properly,

np_1- "Totally sure" probability that the variable (dimension) is not occurring properly,

In this case, p_1 is matched with the percentage corresponding to "Strongly disagree" from Tables 2-9; $p_2$is matched to "Agree"; pI with "Undecided"; $np_2$with "Disagree"; and $np_1$with "Strongly disagree".

For example, the probabilities of D11 are $(0,33.9,19.0,24.0,23.1)$, based on the results in Table 2. For clarity these probabilities can be converted to Plithogenic Neutrosophic Probabilities such that: $(p_1 + p_2, pI, np_2 + np_1)$, that is, continuing with the example we have that the Plithogenic Neutrosophic Probability of D11 is (33.9,19.0,47.1).

Table 10 contains the results of each of the Plithogenic Refined Probabilities (PRP) and the Plithogenic Neutrosophic Probabilities (PNP) of all dimensions.

Dimension	PRP	PNP
D11	(0,33.9,19.0,24.0,23.1)	(33.9,19.0,47.1)
D12	(7.4,18.2,15.7,37.2,21.5)	(25.6,15.7,58.7)
D13	(12.4,25.6,19.0,22.3,20.7)	(38.0,19.0,43.0)
D14	(11.6,28.1,33.9,26.4,0)	(39.7,33.9,26.4)
D21	(6.6,30.6,30.6,32.2,0)	(37.2,30.6,32.2)
D22	(3.3,23.1,33.1,40.5,0)	(26.4,33.1,40.5)
D23	(11.6,15.7,21.5,26.4,24.8)	(27.3,21.5,51.2)
D24	(11.6,27.3,10.7,50.4,0)	(38.9,10.7,50.4)

Table 10: Plithogenic Refined Probabilities and Plithogenic Neutrosophic Probabilities calculated for the 8 dimensions.

From the values of the PNPs in Table 10, it can be seen that the probabilities of good results are less than or equal to 39.7% at the most, which can be assessed as "less than acceptable".

If we take $\wedge_p$ as the plitogenic conjunction between probabilities of the PNP type, where $(p_A, I_A, np_A) \wedge_p (p_B, I_B, np_B) = (p_A \wedge p_B, I_A \vee I_B, np_A \vee np_B)$, such that $\wedge$ is the t-norm minimum of fuzzy logic and $\vee$ is the t-conorm maximum. Then, calculating $PNP(V1) = {\wedge_p}_{i=1}^{4} \text{PNP}(D1i)$ and $PNP(V2) = {\wedge_p}_{j=1}^{4} \text{PNP}(D2j)$ we have:

$PNP(V1) = (25.6,33.9,58.7)$, while $PNP(V2) = (26.4,33.1,51.2)$. This shows that both variables have a low probability of being considered to have good behavior in the Junín region. These results are plotted in Figure 2.

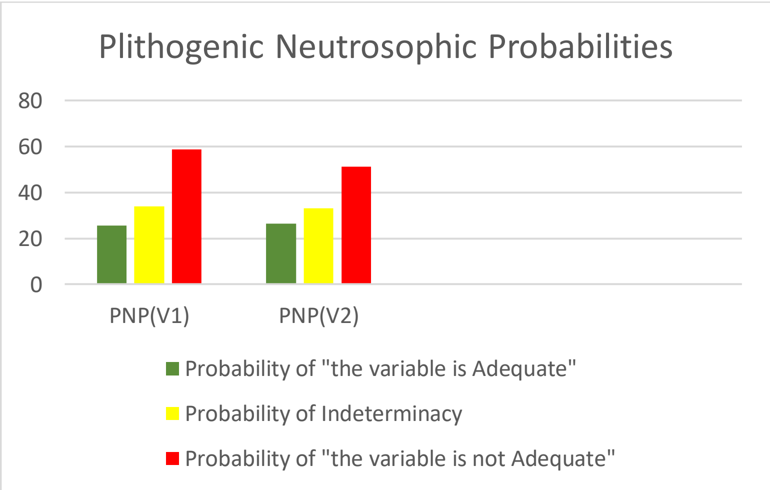

Figure 2: Graphical representation of $PNP(V1)$(on the left) and $PNP(V2)$(on the right) with chart graphs. Veracity appears in green, indeterminacy in yellow, and falsehood in red.

Next, we consider the relationship between both variables. To do this, we make the following conversion for each of the respondents, as can be seen in Table 11:

Value on the Likert Scale	Conversion to the form $a + bI$
Strongly disagree	0
Disagree	0.3
Undecided	I
Agree	0.6
Strongly agree	1

Table 11: Likert scale conversion rules into numbers of the form $a + bI$ ([15, 16]), with a and b constants.

For each Likert-type response from the 121 respondents, the data was processed by converting them into numbers $a + bI$ as it was indicated in Table 11. Then, the opinion of the ith expert is added as follows:

- The opinions on the 8 dimensions of a given expert are converted into the form $a + bI$ according to the rules in Table 11.
- The expert's opinions for D11, D12, D13, and D14 are added together and this is considered his/her opinion about V1. In the same way, his/her opinions on D21, D22, D23, and D24 are added; this is considered his/her opinion on V2.

Let us recall that the sum between $a_1 + b_1 I$ and $a_2 + b_2 I$ is defined as $(a_1 + a_2) + (b_1 + b_2)I$ ([15, 16]).

For example, if expert X evaluates D11 as "Undecided", D12 as "Disagree", D13 as "Agree" and D14 as "Strongly Agree", then the conversion would be according to Table 11 as follows:

I for D11, 0.3 for D12, 0.6 for D13, and 1 for D14. To calculate the value of the X's opinion on the variable V1 we have that it is equal to $I + 0.3 + 0.6 + 1 = 1.9 + I$.

Additionally, in this article, we define an order relationship between numbers of the form $a + bI$, as follows:

$a_1 + b_1 I \preccurlyeq a_2 + b_2 I$ ($a_1 + b_1 I$ is less than or equal to $a_2 + b_2 I$) if and only if $a_1 < a_2$, or if $a_1 = a_2$, then $b_1 > b_2$.

Spearman's Rho coefficient can be calculated, which only requires the ordinal number of the data to be calculated. This result can be seen in Table 12.

			Environmental audit	Corporate social responsibility
Spearman's Rho	Environmental audit	Correlation coefficient	1.000	598 **
		Next (bilateral)	.	.000
		No.	121	121
	Corporate social responsibility	Correlation coefficient	.598 **	1.000
		Next (bilateral)	.000	.
		No.	121	121

**. The correlation is significant at the 0.01 level (bilateral).

Table 12: Correlation between variable 1 (Environmental audit) and variable 2 (Corporate social responsibility)

According to Spearman's Rho bivariate correlation analysis, a moderate positive correlation (0.598) is found between the Environmental audit variable and the Corporate social responsibility variable in the Junín region, the p-value has been 0.000 <0.05, therefore, the null hypothesis H_0 is rejected. This means that the present problem of environmental auditing in the Junín Region is associated with the problem of corporate social responsibility; in other words, if environmental auditing practices are carried out, there will be awareness of corporate social responsibility, which can be applied to business environments of mining, industry, and commerce in the Junín Region.

Table 13 contains the interpretation in linguistic form of the correlation values:

Ratio range	Relationship
"-1"	"Great and perfect negative relationship"
"(-0.9 to -0.99)"	"Very high negative ratio"
"(-0.7 to -0.89)"	"High Negative Ratio"
"(-0.4 to -0.69)"	"Moderate Negative Ratio"
"(-0.2 to -0.39)"	"Low Negative Ratio"
"(-0.01 to -0.19)"	"Very low negative ratio"
"0"	"Nil"
"(0.0 to 0.19)"	"Very low positive ratio"

“(0.2 to 0.39)”	“Low positive ratio”
“(0.4 to 0.69)”	**“Moderate Positive Ratio”**
“(0.7 to 0.89)”	“High positive ratio”
“(0.9 to 0.99)”	“Very high positive ratio”
"1"	“Great and perfect positive relationship”

Table 13: Linguistic scale of interpretation of Spearman's rho coefficient between two variables.

Note that between V1 and V2 there is a moderate positive relationship.

In the other part of the investigation, in parallel, interviews were conducted with the 12 auditors attached to the Audit Chapter. They were individually asked the following question during the interview: How could an environmental audit be developed that allows corporate social responsibility and vice versa?

In summary, they responded that to develop an environmental audit it is necessary to generate environmental policies, where the government intervenes directly through laws and regulations in the legislature, execution organized by sectors through the Ministry of Economy and Finance (MEF), the National Superintendence of Tax Administration (SUNAT in Spanish), etc., and compliance with these through the judiciary.

They consider that it is important to apply the win-win policy, invest to face the negative threats of the environment, maximize its results, and a factual study that guides sustainability, inevitably with long-term observable results, in practice it implies moderating the use of inputs such as plastics, paper, water, acquisition of non-polluting equipment; as well as the reduction of waste, garbage, etc. good practices that involve the economy, are fair wages, reasonable prices, payment of taxes. Carrying out environmental controls, periodic audits, and obtaining “green” certifications, implies budget allocation for process improvement, which is a clear investment that allows compliance with environmental regulations. It focuses on the generation and distribution of added value among collaborators and shareholders.

In the triangulation, the general hypothesis is confirmed, through quantitative data because there is a moderate positive relationship between environmental auditing and corporate social responsibility in the Junín Region, Peru. In the qualitative aspect, it is confirmed that environmental audits are required to achieve corporate social responsibility and that variable 1 is effectively related to variable 2.

Conclusion

This paper studied the relationship between environmental auditing and corporate social responsibility in the Peruvian region of Junín. As well as the state of these two variables in the region. According to the results obtained, it is concluded that there is a direct relationship of moderate positive scale ($r = 0.598$) between environmental auditing and corporate social responsibility in the Junín region, the p-value has been $0.000<0.05$, therefore the hypothesis that there is a positive relationship between these two variables is accepted. The presence of difficulties in the performance of both variables was observed, since the probability values obtained gave results of less than 40%, while the probability that the results are not adequate gave values above 50%.

Beyond the obtained result to carry out this particular study, the relevance of using Plithogenic Statistics is shown for solving real-life problems. With the support of this tool, we were able to combine statistical results between variables of a different nature.

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Computation of Neutrosophic Soft Topology using Python

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Abstract. While other programming languages are either losing ground or stagnant, Python's popularity is growing. Soft sets lack the ability to eliminate the uncertainties present in conventional approaches, whereas neutrosophic sets are capable of handling confusing and contradictory data. In order to reduce the number of human computations performed to compute union, intersection and complement of neutrosophic soft sets, programs are developed in Python. Additionally, Python is used to compute Neutrosophic soft topological operators like interior and closure. The developed python programs are documented in this paper.

Keywords: neutrosophic soft; topological operators; python; topology; open set

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1. **Introduction**

Programming languages play a vital role in solving mathematical problems. Over the last decade, constructing codes using a programming language for mathematical problems has undergone unexpected growth. In earlier times, rigid computer programs could only solve limited-size models. But today, they consist of highly developed and adaptable information processing systems with modules that manage incoming data, provide powerful computational algorithms and present the model's results in a manner that is acceptable to an application-oriented user. Even basic mathematical problems need an unreasonably high amount of computational effort if solved manually. Computational programs make these calculations easier by providing the results in no time. Python is currently one of the most popular programming languages among developers and tech companies. Python is highly preferable for its simplicity.

Zadeh [29] developed the fuzzy set ($\mathcal{FS}$) theory in 1965. It has developed into a very important tool for addressing problems with uncertainity. Molodtsov [18] presented the soft set theory in 1991, which addresses uncertainty. In his work, he established the core ideas of this novel theory and successfully applied it to a number of areas, including optimisation, algebraic structures, operations research, clustering, game theory, medical diagnosis, lattice, topology, data analysis, and decision-making under uncertainty. The notion of intuitionistic fuzzy set was developed by Atanassov [1] by generalizing the fuzzy set theory. Intuitionistic fuzzy set contains both belongingness and non-belongingness values. But it is incapable of handling the ambiguity and contradictory data present in any system.

Smarandache [25,26] presented the novel idea of a neutrosophic set ($\mathcal{NS}$) as a generalisation of crisp sets, fuzzy set theory, and intuitionistic fuzzy set theory. The novel branch of philosophy known as neurosophy generalises fuzzy logic, intuitionistic fuzzy logic, and paraconsistent logic. Neutrosophic logic serves as a mathematical toolbox for issues involving inconsistent, incomplete and ambiguous knowledge. Neutrosophic logic and sets are used in many different areas, including financial dataset detection, investigation of the rise and collapse of the new economy,relational database systems, semantic web services, information systems, etc.

A theoretical framework for soft set theory was constructed by Maji et al. [16] in 2003. They also constructed operations like union (OR) and intersection (AND) of two soft sets as well as put forth certain hypotheses regarding these operations. Maji et al. initiated the study of $\mathcal{F}_{\mathcal{S}}$ sets in [13]. The idea of IF_{S} sets was presented by Maji et al. [14], who also defined new operations on it and examined some of their characteristics. By combining the ideas of soft set and neutrosophic set, Maji [17] initiated the study on neutrosophic soft sets ($\mathcal{N}_{\mathcal{S}}\mathcal{S}$s) and presented a solution for a decision-making problem utilizing the neutrosophic soft set, which was later modified by Deli and Broumi [9].

C.L. Chang [5] was the first to put forth the theory of fuzzy topological spaces ($\mathcal{FTS}$) in 1968, along with some additional definitions of basic topological ideas including open set, closed set, continuity, and compactness. A thorough analysis of the construction of $\mathcal{FTS}$ was conducted by Lowen [12]. Coker [6] proposed intuitionistic fuzzy topological space ($\mathcal{IFTS}$) in 1995. Several operations on $\mathcal{IFTS}$ were described by Coker et al. [7,8].

Soft topological spaces ($\mathcal{STS}s$) was developed by Shabir and Naz [22]. Tanay and Kandemir [27] defined $\mathcal{F}_{\mathcal{S}}$ interior, $\mathcal{F}_{\mathcal{S}}$ basis, $\mathcal{F}_{\mathcal{S}}$ neighbourhood, and $\mathcal{F}_{\mathcal{S}}$ subspace topology and developed fuzzy soft topology. The theory of $\mathcal{IF}_{\mathcal{S}}$ topological space was documented in [2, 19]. A topological structure on neutrosophic soft sets was built by Bera and Mahapatra [3,4] and they examined its structural characterizations as well as the theory related to topological space, including neighbourhood, boundary interior, closure, base, subspace, continuous mappings, separation axioms, compactness and connectedness.

Zahariev [30] constructed a Matlab software package for solving fuzzy linear systems of equations and inequalities in fuzzy algebras. In 2014, Salama et al., [21] provided an Excel package for calculating neutrosophic data. Salama et al., [20] devised and implemented a neutrosophic data operation by utilizing c # programming language, Microsoft Visual Studio and NET Framework in 2014. In [11], Karunambigai and Kalaivani constructed a Matlab program for computing the power of an intuitionistic fuzzy matrix, the index matrix and the strength of connectedness of an intuitionistic fuzzy graph. Topal et al., [28] developed hard-coded Python programs for determining the score, accuracy, certainity matrix of a bipolar neutrosophic matrix ($\mathcal{BNM}$) and for computing $\mathcal{BNM}$ intersection , $\mathcal{BNM}$ union, $\mathcal{BNM}$ complement, $\mathcal{BNM}$ addition, $\mathcal{BNM}$ product, $\mathcal{BNM}$ transpose and $\mathcal{BNM}$ composition. El-Ghareeb [10] created an Open Source Python Neutrosophic package for single-valued neutrosophic numbers and sets, interval-valued neutrosophic numbers and sets.

Sleem et al., [24] documented interval valued neutrosophic sets ($\mathcal{IVNS}$) software work that operates on $\mathcal{IVNS}$ and normalization for $\mathcal{IVNS}$ matrices. Sidiropoulos et.al., [23] presented the Python library and algorithms for fuzzy set measures. Till now researchers have developed algorithms for determining operations on neutrosophic numbers, but none of these programs can be dealt with neutrosophic soft sets or neutrosophic soft topological operators. In this context, soft code python programs have been developed for the operators in neutrosophic soft topological space in this paper.

2. Background of Neutrosophic Soft Sets

We first present the basic definitions of Neutrosophic soft sets ($\mathcal{N_S S}$s).

Definition 2.1. [25] Let $\mathfrak{U}$ represent the initial universe and $\mathfrak{T};\ \mathfrak{I};\ \mathfrak{F}\ :\ \mathfrak{U} \to]^{-}0, 1^{+}[$ and $^{-}0 \leq \mathfrak{T}_{\mathfrak{L}}(\xi) + \mathfrak{L}(\xi) + \mathfrak{F}_{\mathfrak{L}}(\xi) \leq\ 3^{+}$, a $\mathcal{NS}$ is written as:

$$\mathfrak{L} = \{<\xi,\ \mathfrak{T}_{\mathfrak{L}}(\xi),\ \mathfrak{I}_{\mathfrak{L}}(\xi), \mathfrak{F}_{\mathfrak{L}}(\xi) >: \xi \in \mathfrak{U}\}$$

Example 2.2. Assume that $\mathfrak{U} = \{\xi_1, \xi_2, \xi_3\}$, where ξ_1 represents the battery life, ξ_2 represents the price, ξ_3 represents the performance of an electronic scooter. The e-scooter company gives its assurance of its product. Based on the opinion of some experts regarding the company's assurance on the battery life, price and performance, the neutrosophic set can be defined as

$$(\mathfrak{N}, \Phi) = <\left\{\frac{\xi_1}{0.7, 0.4, 0.5}, \frac{\xi_2}{0.4, 0.5, 0.5}, \frac{\xi_3}{0.3, 0.3, 0.4}\right\}>$$

Definition 2.3. [9] Let the universal set be represented by $\mathcal{U}$ and $\mathfrak{T}_{f_{\mathcal{P}(\varsigma)}}(\xi),\ \mathfrak{I}_{f_{\mathcal{P}(\varsigma)}}(\xi), \mathfrak{F}_{f_{\mathcal{P}(\varsigma)}}(\xi) \in [0,1]$ represent the "truth", "indeterminacy", "falsity" functions of $f_{\mathcal{P}(\varsigma)}$ where $f_{\mathcal{P}} : \Phi \to \mathcal{P}(\mathcal{U})$ and Φ-set of attributes . Then the $\mathcal{N_S S}$ is written as $\mathcal{P} = \left\{\left(\varsigma, \left\{<\xi,\ \mathfrak{T}_{f_{\mathcal{P}(\varsigma)}}(\xi),\ \mathfrak{I}_{f_{\mathcal{P}(\varsigma)}}(\xi), \mathfrak{F}_{f_{\mathcal{P}(\varsigma)}}(\xi) >: \xi \in \mathcal{U}\right\}\right) : \varsigma \in \Phi\right\}$

Example 2.4. Let $\mathfrak{U}$ be the set of laptops of different companies under consideration and Φ be the set of parameters. $\Phi = \{$battery life, compact, less weight, build quality, expensive, maximum storage, RAM size, good display quality$\}$. suppose that, there are three laptops in the universe $\mathfrak{U}$ given by $\{\xi_1, \xi_2, \xi_3\}$ and the set of parameters $\{\varsigma_1, \varsigma_2\}$, where ς_1 represents the parameter 'battery life', ς_2 represents the parameter 'expensive'.

Then the neutrosophic soft sets is written as,

$$(\mathfrak{M}, \Phi) = \left\{ \begin{array}{c} < \varsigma_1, \left\{ \frac{\xi_1}{0.9,0.4,0.3}, \frac{\xi_2}{0.5,0.3,0.5}, \frac{\xi_3}{0.4,0.1,0.3} \right\} >, \\ < \varsigma_2, \left\{ \frac{\xi_1}{0.7,0.1,0.4}, \frac{\xi_2}{0.6,0.3,0.2} \frac{\xi_3}{0.6,0.1,0.5} \right\} > \end{array} \right\}$$

Definition 2.5. [9] Let $\mathfrak{U}$ represent the initial universe and ($\mathfrak{H}$, Φ) be a $\mathcal{N}_{\mathcal{S}}\mathcal{S}$ over $\mathfrak{U}$

(1) $\mathfrak{H}$ is called an absolute $\mathcal{N}_{\mathcal{S}}\mathcal{S}$ if $\mathfrak{T}_{f_{\mathcal{H}(\varsigma)}}(\xi) = 1$, $\mathfrak{I}_{f_{\mathcal{H}(\varsigma)}}(\xi) = 0$, $\mathfrak{F}_{f_{\mathcal{H}(\varsigma)}}(\xi) = 0 \ \forall \ \xi \in \mathcal{U}$ and $\varsigma \in \Phi$ (written symbolically as 1_u).

(2) $\mathfrak{H}$ is called a null $\mathcal{N}_{\mathcal{S}}\mathcal{S}$ if $\mathfrak{T}_{f_{\mathcal{H}(\varsigma)}}(\xi) = 0$, $\mathfrak{I}_{f_{\mathcal{H}(\varsigma)}}(\xi) = 1$, $\mathfrak{F}_{f_{\mathcal{H}(\varsigma)}}(\xi) = 1 \ \forall \ \xi \in \mathcal{U}$ and $\varsigma \in \Phi$ (written symbolically as ϕ_u).

3. Implementing Python for Computations in Neutrosophic Soft Environment

3.1. *Union and Intersection of Neutrosophic Soft Sets*

Definition 3.1. [9] For any two $\mathcal{N}_{\mathcal{S}}\mathcal{S}s$ ($\mathfrak{H}$, Φ) and ($\mathfrak{G}$, Φ) over $\mathfrak{U}$, union and intersection are given by,

$$\mathfrak{H} \cup \mathfrak{G} = \mathfrak{P} = \{\left(\varsigma, \left\{< \xi,\ \mathfrak{T}_{f_{\mathfrak{P}(\varsigma)}}(\xi),\ \mathfrak{I}_{f_{\mathfrak{P}(\varsigma)}}(\xi), \mathfrak{F}_{f_{\mathfrak{P}(\varsigma)}}(\xi) >: \xi \in \mathfrak{U}\right\}\right) : \varsigma \in \Phi\}$$

where

$$\mathfrak{T}_{f_{\mathfrak{P}(\varsigma)}}(\xi) = \max\left(\mathfrak{T}_{f_{\mathfrak{H}(\varsigma)}}(\xi), \mathfrak{T}_{f_{\mathfrak{G}(\varsigma)}}(\xi)\right),$$

$$\mathfrak{I}_{f_{\mathfrak{P}(\varsigma)}}(\xi) = \min\left(\mathfrak{I}_{f_{\mathfrak{H}(\varsigma)}}(\xi), \mathfrak{I}_{f_{\mathfrak{G}(\varsigma)}}(\xi)\right),$$

$$\mathfrak{F}_{f_{\mathfrak{P}(\varsigma)}}(\xi) = \min\left(\mathfrak{F}_{f_{\mathfrak{H}(\varsigma)}}(\xi), \mathfrak{F}_{f_{\mathfrak{G}(\varsigma)}}(\xi)\right),$$

$$\mathfrak{H} \cap \mathfrak{G} = \mathfrak{Q} = \{\left(\varsigma, \left\{< \xi,\ \mathfrak{T}_{f_{\mathfrak{Q}(\varsigma)}}(\xi),\ \mathfrak{I}_{f_{\mathfrak{Q}(\varsigma)}}(\xi), \mathfrak{F}_{f_{\mathfrak{Q}(\varsigma)}}(\xi) >: \xi \in \mathfrak{U}\right\}\right) : \varsigma \in \Phi\}$$

where

$$\mathfrak{T}_{f_{\mathfrak{Q}(\varsigma)}}(\xi) = \min\left(\mathfrak{T}_{f_{\mathfrak{H}(\varsigma)}}(\xi), \mathfrak{T}_{f_{\mathfrak{G}(\varsigma)}}(\xi)\right),$$

$$\mathfrak{I}_{f_{\mathfrak{Q}(\varsigma)}}(\xi) = \max\left(\mathfrak{I}_{f_{\mathfrak{H}(\varsigma)}}(\xi), \mathfrak{I}_{f_{\mathfrak{G}(\varsigma)}}(\xi)\right),$$

$$\mathfrak{F}_{f_{\mathfrak{Q}(\varsigma)}}(\xi) = \max\left(\mathfrak{F}_{f_{\mathfrak{H}(\varsigma)}}(\xi), \mathfrak{F}_{f_{\mathfrak{G}(\varsigma)}}(\xi)\right)$$

Example 3.2. Consider $\mathfrak{U} = \{\xi_1, \xi_2\}$, the attributes $\Phi = \{\varsigma_1, \varsigma_2\}$. Let the two $\mathcal{N_S S}$s $(\mathfrak{H}, \Phi)$ and $(\mathfrak{G}, \Phi)$ be given by

$$(\mathfrak{H}, \Phi) = \left\{ \begin{array}{c} <\varsigma_1, \left\{\frac{\xi_1}{0.4,0.5,0.2}, \frac{\xi_2}{0.6,0.2,0.1}\right\}>, \\ <\varsigma_2, \left\{\frac{\xi_1}{0.9,0.4,0.3}, \frac{\xi_2}{0.5,0.6,0.3}\right\}> \end{array} \right\}$$

$$(\mathfrak{G}, \Phi) = \left\{ \begin{array}{c} <\varsigma_1, \left\{\frac{\xi_1}{0.8,0.3,0.4}, \frac{\xi_2}{0.2,0.4,0.6}\right\}>, \\ <\varsigma_2, \left\{\frac{\xi_1}{0.8,0.5,0.7}, \frac{\xi_2}{0.2,0.5,0.8}\right\}> \end{array} \right\}$$

Then

$$\mathfrak{H} \cup \mathfrak{G} = \left\{ \begin{array}{c} <\varsigma_1, \left\{\frac{\xi_1}{0.8,0.3,0.2}, \frac{\xi_2}{0.6,0.2,0.1}\right\}>, \\ <\varsigma_2, \left\{\frac{\xi_1}{0.9,0.4,0.3}, \frac{\xi_2}{0.5,0.5,0.3}\right\}> \end{array} \right\}$$

$$\mathfrak{H} \cap \mathfrak{G} = \left\{ \begin{array}{c} <\varsigma_1, \left\{\frac{\xi_1}{0.4,0.5,0.4}, \frac{\xi_2}{0.2,0.4,0.6}\right\}>, \\ <\varsigma_2, \left\{\frac{\xi_1}{0.8,0.5,0.7}, \frac{\xi_2}{0.2,0.6,0.8}\right\}> \end{array} \right\}$$

3.1.1. *Function definition*

The following functions define the union and intersection of any two $\mathcal{N_S S}s$

```
def nss_union(l1, l2):
    """Return tuple comprised of the maximum, minimum,
    minimum value in each tuple argument."""
    result = [[((max(l1[g][d][0], l2[g][d][0])),
               (min(l1[g][d][1], l2[g][d][1])),
               (min(l1[g][d][2], l2[g][d][2])))
        for d in range(elements)] for g in range(attributes)]
    return (result)
```

```
def nss_intersection(l1, l2):
    """Return tuple comprised of the minimum, maximum,
    maximum value in each tuple argument."""
    result = [[((min(l1[g][d][0], l2[g][d][0])),
               (max(l1[g][d][1], l2[g][d][1])),
               (max(l1[g][d][2], l2[g][d][2])))
        for d in range(elements)] for g in range(attributes)]
    return (result)
```

3.1.2. *Program and intuitive description*

```
from pprint import pprint

attributes = int(input("Enter the number of attributes : "))
elements = int(input("Enter the number of elements in the universal
    ↪ set: "))

print("Enter the elements of N s set:A")
l1 = [[tuple(map(float , input().split(" "))) for x in range(
    ↪ elements)] for g in range(attributes)]

print("Enter the elements of N s set:B")
l2= [[tuple(map(float , input().split(" "))) for x in range(elements
    ↪ )] for g in range(attributes)]

def nss_union(l1, l2):
    """Return tuple comprised of the maximum, minimum, minimum
        ↪ value in each tuple argument."""
    result = [[((max(l1[g][d][0], l2[g][d][0])),
                (min(l1[g][d][1], l2[g][d][1])),
                (min(l1[g][d][2], l2[g][d][2]))) for d in range(
                    ↪ elements)] for g in range(attributes)]
    return (result)

print("Union of Neutrosophic soft sets is")
pprint(nss_union(l1, l2))

def nss_intersection(l1, l2):
    """Return tuple comprised of the minimum, maximum, maximum
        ↪ value in each tuple argument."""
    result = [[((min(l1[g][d][0], l2[g][d][0])),
                (max(l1[g][d][1], l2[g][d][1])),
                (max(l1[g][d][2], l2[g][d][2]))) for d in range(
                    ↪ elements)] for g in range(attributes)]
    return (result)
```

```
print("Intersection of Neutrosophic soft sets is")
pprint(nss_intersection(l1, l2))
```

The program initially gets the value of no. of. attributes and no. of. elements in the universal set and stores in the variable namely attributes and elements respectively. Gets the elements of neutrosophic soft sets. Compute the union and intersection of the $\mathcal{N}_{\mathcal{S}}\mathcal{S}$s using the defined functions.

```
Enter the number of attributes : 2
Enter the number of elements in the universal set: 2
Enter the elements of N_s set:A
.4 .5 .2
.6 .2 .1
.9 .4 .3
.5 .6 .3
Enter the elements of N_s set:B
.8 .3 .4
.2 .4 .6
.8 .5 .7
.2 .5 .8
Union of Neutrosophic soft sets is
[[(0.8, 0.3, 0.2), (0.6, 0.2, 0.1)], [(0.9, 0.4, 0.3), (0.5, 0.5, 0.3)]]
Intersection of Neutrosophic soft sets is
[[(0.4, 0.5, 0.4), (0.2, 0.4, 0.6)], [(0.8, 0.5, 0.7), (0.2, 0.6, 0.8)]]
```

FIGURE 1. Output of the union and intersection of neutrosophic soft sets program

3.2. *Neutrosophic Soft Subset and Superset*

Definition 3.3. [9] For any two $\mathcal{N}_{\mathcal{S}}\mathcal{S}s$ $(\mathfrak{H}, \Phi)$ and $(\mathfrak{G}, \Phi)$ over $\mathfrak{U}$,

(1) $(\mathfrak{H}, \Phi) \subseteq (\mathfrak{G}, \Phi)$ if $\mathfrak{T}_{\mathfrak{H}(\varsigma)}(\xi) \leq \mathfrak{T}_{\mathfrak{G}(\varsigma)}(\xi)$, $\mathfrak{I}_{\mathfrak{H}(\varsigma)}(\xi) \geq \mathfrak{I}_{\mathfrak{G}(\varsigma)}(\xi)$, $\mathfrak{F}_{\mathfrak{H}(\varsigma)}(\xi) \geq \mathfrak{F}_{\mathfrak{G}(\varsigma)}(\xi)$, $\forall \varsigma \in \Phi,\ \xi \in \mathfrak{U}$.

(2) $(\mathfrak{H}, \Phi) \supseteq (\mathfrak{G}, \Phi)$ if $\mathfrak{T}_{\mathfrak{H}(\varsigma)}(\xi) \geq \mathfrak{T}_{\mathfrak{G}(\varsigma)}(\xi)$, $\mathfrak{I}_{\mathfrak{H}(\varsigma)}(\xi) \leq \mathfrak{I}_{\mathfrak{G}(\varsigma)}(\xi)$, $\mathfrak{F}_{\mathfrak{H}(\varsigma)}(\xi) \leq \mathfrak{F}_{\mathfrak{G}(\varsigma)}(\xi)$, $\forall \varsigma \in \Phi,\ \xi \in \mathfrak{U}$.

3.2.1. *Function definition*

The following defined functions identify whether the given $\mathcal{N}_{\mathcal{S}}\mathcal{S}$ is a neutrosophic soft subset or superset of another $\mathcal{N}_{\mathcal{S}}\mathcal{S}$.

```
def nss_sub(l1, l2):
    for g in range(attributes):
        for d in range(elements):
            if (((l1[g][d][0] <= l2[g][d][0]) and
            (l1[g][d][1] >= l2[g][d][1]) and
            (l1[g][d][2] >= l2[g][d][2]))):
                return 1
            return 0
```

The function returns the value 1, if the condition is true, else it returns the value 0.

```
def nss_sup(l1, l2):
    for g in range(attributes):
        for d in range(elements):
            if (((l1[g][d][0] >= l2[g][d][0]) and
            (l1[g][d][1] <= l2[g][d][1]) and
            (l1[g][d][2] <= l2[g][d][2]))):
                return 1
            return 0
```

The function returns the value 1, if the condition is true, else it returns the value 0.

3.3. *Complement of a Neutrosophic Soft Set*

Definition 3.4. [9] For any $\mathcal{N}_{\mathcal{S}}\mathcal{S}$ ($\mathfrak{H}$, Φ) over $\mathfrak{U}$, the complement is given by,

$$\mathfrak{H}^c = \left\{ \left(\varsigma, \left\{ < \xi,\ \mathfrak{F}_{f_{\mathfrak{H}(\varsigma)}}(\xi), 1 - \ \mathfrak{I}_{f_{\mathfrak{H}(\varsigma)}}(\xi), \mathfrak{T}_{f_{\mathfrak{H}(\varsigma)}}(\xi) >: \xi \in \mathfrak{U} \right\} \right) : \varsigma \in \Phi \right\}$$

3.3.1. *Function definition*

The following defined function gives the complement of a $\mathcal{N}_{\mathcal{S}}\mathcal{S}$

```
def nss_compl(m):
    """Return tuple comprised of the complement value
    in each tuple argument."""
    result = [[((m[g][d][2]),
              (round((1 - m[g][d][1]), 1)),
              (m[g][d][0])) for d in range(elements)]
              for g in range(attributes)]
    return (result)
```

3.3.2. *Program and intuitive description*

```
from pprint import pprint

attributes = int(input("Enter the number of attributes : "))
elements = int(input("Enter the number of elements in the universal
  ↪ set: "))

print("Enter the elements of N s set:")
A = [[tuple(map(float, input().split(" "))) for d in range(elements
  ↪ )] for g in range(attributes)]

def nss_compl(m):
    """Return tuple comprised of the complement value in each tuple
      ↪ argument."""
    result = [[((m[g][d][2]), (round((1 - m[g][d][1]), 1)), (m[g][d
      ↪ ][0])) for d in range(elements)] for g in range(attributes
      ↪ )]
    return (result)

print("The complement of the Neutrosophic soft set is:")
pprint(nss_compl(A))
```

The program initially gets the value of the number of attributes and number of elements in the universal set and stores it in the variable namely attributes and elements respectively. Gets the elements of $\mathcal{N_S S}$s. Compute the complement of the $\mathcal{N_S S}$s using the defined functions.

```
Enter the number of attributes : 2
Enter the number of elements in the universal set: 2
Enter the elements of N_s set:
.4 .5 .2
.6 .2 .1
.9 .4 .3
.5 .6 .3
The complement of the Neutrosophic soft set is:
[[(0.2, 0.5, 0.4), (0.1, 0.8, 0.6)], [(0.3, 0.6, 0.9), (0.3, 0.4, 0.5)]]
```

FIGURE 2. Output of the complement of a neutrosophic soft set program

3.4. *Neutrosophic Soft Topology*

Definition 3.5. [3] Let the family of all $\mathcal{N}_\mathcal{S}\mathcal{S}$s over $\mathfrak{U}$ via parameters in Φ be represented as $\mathcal{N}_\mathcal{S}\mathcal{S}(\mathfrak{U}, \Phi)$ and $\tau_u \subset \mathcal{N}_\mathcal{S}\mathcal{S}(\mathfrak{U}, \Phi)$. Then τ_u is known as a $\mathcal{N}_\mathcal{S}$ topology on $(\mathfrak{U},\ \Phi)$ if the following conditions are true.

(1) $\phi_u,\ 1_u \in \tau_u$
(2) $\forall\ \mathfrak{D}_1, \mathfrak{D}_2 \in \tau_u \Rightarrow \mathfrak{D}_1 \cap \mathfrak{D}_2 \in \tau_u$
(3) for $\cup_{i\in J}\mathfrak{D}_i \in \tau_u,\ \forall\ \{\mathfrak{D}_i : i \in J\} \subseteq \tau_u$

Then $(\mathfrak{U},\ \Phi,\ \tau_u)$ is called as a neutrosophic soft topological space $(\mathcal{N}_\mathcal{S}\mathcal{T}\mathcal{S})$. Every member of τ_u is called as neutrosophic soft open set $(\mathcal{N}_\mathcal{S}\mathcal{O}\mathcal{S})$ and its complement is termed as neutrosophic soft closed set $(\mathcal{N}_\mathcal{S}\mathcal{C}\mathcal{S})$.

Example 3.6. Let $\mathfrak{U} = \{\xi_1, \xi_2, \xi_3\}$ and the attributes $\Phi = \{\varsigma_1, \varsigma_2\}$. Consider the $\mathcal{N}_\mathcal{S}\mathcal{S}$s

$$(\mathfrak{N}, \Phi) = \left\{ \begin{array}{l} < \varsigma_1, \left\{ \frac{\xi_1}{0.7,0.4,0.5}, \frac{\xi_2}{0.4,0.5,0.5}, \frac{\xi_3}{0.3,0.3,0.4} \right\} >, \\ < \varsigma_2, \left\{ \frac{\xi_1}{0.6,0.2,0.4}, \frac{\xi_2}{0.5,0.4,0.3}, \frac{\xi_3}{0.4,0.6,0.5} \right\} > \end{array} \right\}$$

$$(\mathfrak{P}, \Phi) = \left\{ \begin{array}{l} < \varsigma_1, \left\{ \frac{\xi_1}{0.8,0.3,0.4}, \frac{\xi_2}{0,5,0.4,0.3}, \frac{\xi_3}{0.7,0.1,0.2} \right\} >, \\ < \varsigma_2, \left\{ \frac{\xi_1}{0.7,0.1,0.3}, \frac{\xi_2}{0.6,0.2,0.1}, \frac{\xi_3}{0.7,0.4,0.3} \right\} > \end{array} \right\}$$

Then, $\tau_u = \{\ \phi_u,\ 1_u,\ \ (\mathfrak{N}, \Phi), (\mathfrak{P}, \Phi)\}$ forms a $\mathcal{N}_\mathcal{S}$ topology.

3.4.1. *Program and intuitive description*

```
from pprint import pprint
import numpy as np

print("Enter the collection of Neutrosophic Soft Sets along with 
    ↪ null and absolute Neutrosophic Soft Sets")
m = int(input("Number of Neutrosophic Soft Sets: "))
attributes = int(input("Enter the Number of attributes : "))
```

```
elements = int(input("Enter the Number of elements in the Universe
    ↪ Set: "))

PNS_Sets = []
a = [[0] * elements for i in range(attributes)]
for i in range(m):
    print("Enter the Primary Neutrosophic soft set:", i + 1)
    ns_set = [[tuple(map(float, input().split(" "))) for d in range
        ↪ (elements)]
              for g in range(attributes)]
    PNS_Sets.append(ns_set)

def nss_compl(m):
    """Return tuple comprised of the complement values in each tuple
        ↪ argument."""
    result = [[((m[g][d][2]), (round((1 - m[g][d][1]), 1)), (m[g][d
        ↪ ][0])) for d in range(elements)] for g in range(attributes
        ↪ )]
    return (result)

def ns_union(m1, m2):
    """Return tuple comprised of the max, min, min value in each
        ↪ tuple argument."""
    result = [[((max(m1[g][d][0], m2[g][d][0])), (min(m1[g][d][1],
        ↪ m2[g][d][1])),
              (min(m1[g][d][2], m2[g][d][2]))) for d in range(
                  ↪ elements)] for g in range(attributes)]
    return (result)

def ns_int(m1, m2):
    """Return tuple comprised of the min, max, max value in each
        ↪ tuple argument."""
    result = [[((min(m1[g][d][0], m2[g][d][0])), (max(m1[g][d][1],
        ↪ m2[g][d][1])),
              (max(m1[g][d][2], m2[g][d][2]))) for d in range(
                  ↪ elements)] for g in range(attributes)]
```

```
    return (result)

Union = []
for m1 in range(len(PNS_Sets)):
    for m2 in range(len(PNS_Sets)):
        x1 = ns_union(PNS_Sets[m1], PNS_Sets[m2])
        Union.append(x1)

res = [Union[i] for i in range(len(Union)) if i == Union.index(
    ↪ Union[i]) ]

Intersection = []
for m1 in range(len(PNS_Sets)):
    for m2 in range(len(PNS_Sets)):
        x2 = ns_int(PNS_Sets[m1], PNS_Sets[m2])
        Intersection.append(x2)

res1 = [Intersection[i] for i in range(len(Intersection)) if i ==
    ↪ Intersection.index(Intersection[i]) ]

Topology = []
for m1 in res:
    Topology.append(m1)
for m2 in res1:
    Topology.append(m2)

topology = []
for m1 in range(len(Topology)):
    for m2 in range(len(Topology)):
        y1 = ns_union(Topology[m1], Topology[m2])
        y2 = ns_int(Topology[m1], Topology[m2])
        topology.append(y1)
        topology.append(y2)

print("The Neutrosophic Soft Topology is:")
```

```
res3 = [topology[i] for i in range(len(topology)) if i == topology.
    ↪ index(topology[i]) ]
pprint(res3)
```

The above program is developed for framing $\mathcal{N}_{\mathcal{S}}$ topology using the numpy package. Program initially gets the values of the number of $\mathcal{N}_{\mathcal{S}}\mathcal{S}$s, number of attributes and number of elements. As the second step, $\mathcal{N}_{\mathcal{S}}\mathcal{O}\mathcal{S}$s are collected as input and stored in the list called PNS_Sets. The $\mathcal{N}_{\mathcal{S}}$-complement, $\mathcal{N}_{\mathcal{S}}$-union function and $\mathcal{N}_{\mathcal{S}}$-intersection function are defined for computing complement, union and intersection of $\mathcal{N}_{\mathcal{S}}\mathcal{S}$s respectively. In the third step, the program creates a list of union and intersection of the $\mathcal{N}_{\mathcal{S}}\mathcal{S}$s in the list PNS_Sets and are appended to the list "topology". The result is finally printed using pprint module.

```
Enter the collection of Neutrosophic Soft Sets along with null and absolute Neutrosophic Soft Sets
Number of Neutrosophic Soft Sets: 6
Enter the Number of attributes : 2
Enter the Number of elements in the Universe Set: 2
Enter the Primary Neutrosophic soft set: 1
1 0 0
1 0 0
1 0 0
1 0 0
Enter the Primary Neutrosophic soft set: 2
0 1 1
0 1 1
0 1 1
0 1 1
Enter the Primary Neutrosophic soft set: 3
.8 .3 .4
.5 .4 .3
.7 .1 .2
.7 .1 .3
Enter the Primary Neutrosophic soft set: 4
.3 .3 .4
.6 .2 .4
.5 .4 .3
.4 .6 .5
Enter the Primary Neutrosophic soft set: 5
.1 .9 .8
.2 .9 .8
.1 .8 .7
.3 .7 .8
Enter the Primary Neutrosophic soft set: 6
.3 .3 .4
.6 .2 .4
.5 .4 .3
.4 .6 .5
The Neutrosophic Soft Topology is:
[[[(1.0, 0.0, 0.0), (1.0, 0.0, 0.0)], [(1.0, 0.0, 0.0), (1.0, 0.0, 0.0)]],
 [[(0.0, 1.0, 1.0), (0.0, 1.0, 1.0)], [(0.0, 1.0, 1.0), (0.0, 1.0, 1.0)]],
 [[(0.8, 0.3, 0.4), (0.5, 0.4, 0.3)], [(0.7, 0.1, 0.2), (0.7, 0.1, 0.3)]],
 [[(0.3, 0.3, 0.4), (0.6, 0.2, 0.4)], [(0.5, 0.4, 0.3), (0.4, 0.6, 0.5)]],
 [[(0.1, 0.9, 0.8), (0.2, 0.9, 0.8)], [(0.1, 0.8, 0.7), (0.3, 0.7, 0.8)]],
 [[(0.8, 0.3, 0.4), (0.6, 0.2, 0.3)], [(0.7, 0.1, 0.2), (0.7, 0.1, 0.3)]],
 [[(0.3, 0.3, 0.4), (0.5, 0.4, 0.4)], [(0.5, 0.4, 0.3), (0.4, 0.6, 0.5)]]]
```

FIGURE 3. Output of the Neutrosophic Soft Topology framing program

3.4.2. *Program and intuitive description*

```
from pprint import pprint
import numpy as np

print("Enter the collection of Neutrosophic Soft Sets")
m = int(input("Number of Neutrosophic Soft Sets: "))
attributes = int(input("Enter the Number of attributes : "))
elements = int(input("Enter the Number of elements in the Universe
    ↪ Set: "))

NS_Sets = []
a = [[0] * elements for i in range(attributes)]
for i in range(m):
    print("Enter the Neutrosophic soft set:", i + 1)
    ns_set = [[tuple(map(float, input().split(" "))) for d in range
        ↪ (elements)] for g in range(attributes)]
    NS_Sets.append(ns_set)

def ns_union(n1, n2):
    """Return tuple comprised of the max, min, min value in each
        ↪ tuple argument."""
    result = [[((max(n1[g][d][0], n2[g][d][0])),
                (min(n1[g][d][1], n2[g][d][1])),
                (min(n1[g][d][2], n2[g][d][2]))) for d in range(
                    ↪ elements)] for g in range(attributes)]
    return (result)

def ns_int(n1, n2):
    """Return tuple comprised of the min, max, max value in each
        ↪ tuple argument."""
    result = [[((min(n1[g][d][0], n2[g][d][0])),
                (max(n1[g][d][1], n2[g][d][1])),
                (max(n1[g][d][2], n2[g][d][2]))) for d in range(
                    ↪ elements)] for g in range(attributes)]
    return (result)
```

```
Topology = []
for n1 in NS_Sets:
    for n2 in NS_Sets:
        if (np.array_equal(n1, n2) == False):
            x1 = ns_union(n1, n2)
            x2 = ns_int(n1, n2)
            res1 = any(np.array_equal(x1, m) for m in NS_Sets)
            res2 = any(np.array_equal(x2, m) for m in NS_Sets)
            Topology.append(res1)
            Topology.append(res2)

if all(Topology) == True:
    print("The Collection forms a Neutrosophic Soft Topology")
else:
    print("Does not form a Neutrosophic Soft Topology")
```

The above program is developed for verifying whether a forms a $\mathcal{N}_{\mathcal{S}}$ topology or not. The program initially gets the values of the number of $\mathcal{N}_{\mathcal{S}}\mathcal{OS}$s, number of attributes and number of elements. In the second step, $\mathcal{N}_{\mathcal{S}}\mathcal{OS}$s are collected as input and stored in the list called NS_Sets . The $\mathcal{N}_{\mathcal{S}}$-union function and $\mathcal{N}_{\mathcal{S}}$-intersection function are defined for computing union and intersection of $\mathcal{N}_{\mathcal{S}}\mathcal{OS}$s respectively.

In the third step, the program creates a list of Boolean values. The list is created by taking two $\mathcal{N}_{\mathcal{S}}\mathcal{OS}$s from the list of $\mathcal{N}_{\mathcal{S}}\mathcal{OS}$s and checking the condition if the two $\mathcal{N}_{\mathcal{S}}\mathcal{OS}$s are equal, if they are equal then exists the loop, if not then the program computes the values of $\mathcal{N}_{\mathcal{S}}$ union and $\mathcal{N}_{\mathcal{S}}$ intersection of the two $\mathcal{N}_{\mathcal{S}}\mathcal{OS}$s using pre-defined $\mathcal{N}_{\mathcal{S}}$-union and $\mathcal{N}_{\mathcal{S}}$-intersection function. In the next step, the program checks whether the computed values of $\mathcal{N}_{\mathcal{S}}$ intersection and $\mathcal{N}_{\mathcal{S}}$ union are in the list of $\mathcal{N}_{\mathcal{S}}\mathcal{OS}$s utilizing the built-in 'any' function which returns only Boolean values as result. The results are appended to the list 'Topology'

In the final step, the program checks if all the values in the list 'Topology' are true utilizing the built-in 'all' function, if the result is true then the collection of input values forms a $\mathcal{N}_{\mathcal{S}}\mathcal{TS}$ or else it does not.

```
Enter the collection of Neutrosophic Soft Sets
Number of Neutrosophic Soft Sets: 7
Enter the Number of attributes : 2
Enter the Number of elements in the Universe Set: 2
Enter the Neutrosophic soft set: 1
1 0 0
1 0 0
1 0 0
1 0 0
Enter the Neutrosophic soft set: 2
0 1 1
0 1 1
0 1 1
0 1 1
Enter the Neutrosophic soft set: 3
.8 .3 .4
.5 .4 .3
.7 .1 .2
.7 .1 .3
Enter the Neutrosophic soft set: 4
.3 .3 .4
.6 .2 .4
.5 .4 .3
.4 .6 .5
Enter the Neutrosophic soft set: 5
.1 .9 .8
.2 .9 .8
.1 .8 .7
.3 .7 .8
Enter the Neutrosophic soft set: 6
.8 .3 .4
.6 .2 .3
.7 .1 .2
.7 .1 .3
Enter the Neutrosophic soft set: 7
.3 .3 .4
.5 .4 .4
.5 .4 .3
.4 .6 .5
The Collection forms a Neutrosophic Soft Topology
```

FIGURE 4. Output of the Neutrosophic Soft Topology program

3.5. *Operators of Neutrosophic Soft Topological Space*

Definition 3.7. [3] Let $\mathcal{N_S T S}$ be represented as $(\mathfrak{U}, \Phi, \tau_u)$ and $(\mathfrak{L}, \Phi)$ be any arbitrary $\mathcal{N_S S}$. Then the closure of a $\mathcal{N_S S}$ $(\mathfrak{L}, \Phi)$ is the intersection of all $\mathcal{N_S CS}s$ containing $(\mathfrak{L}, \Phi)$.

i.e., $\mathcal{N_S}cl(\mathfrak{L}, \Phi) = \cap\{(\mathfrak{F}, \Phi) : (\mathfrak{L}, \Phi) \subseteq (\mathfrak{F}, \Phi), (\mathfrak{F}, \Phi)$ is a $\mathcal{N_S CS}$ in $\mathfrak{U}.\}$

Definition 3.8. [3] Let $\mathcal{N_S T S}$ be represented as $(\mathfrak{U}, \Phi, \tau_u)$ and $(\mathfrak{L}, \Phi)$ be any arbitrary $\mathcal{N_S S}$. Then the interior of a $\mathcal{N_S S}$ $(\mathfrak{L}, \Phi)$ is the union of all $\mathcal{N_S OS}s$ contained in $(\mathfrak{L}, \Phi)$.

i.e., $\mathcal{N_S}int(\mathfrak{L}, \Phi) = \cup\{(\mathfrak{G}, \Phi) : (\mathfrak{G}, \Phi) \subseteq (\mathfrak{L}, \Phi)$, $(\mathfrak{G}, \Phi)$ is a $\mathcal{N_S OS}$ in $\mathfrak{U}$ $\}$.

Example 3.9. Consider the Example 3.6,

$$(\mathfrak{N},\Phi)^c = \left\{ \begin{array}{l} <\varsigma_1, \left\{ \frac{\xi_1}{0.5,0.6,0.7}, \frac{\xi_2}{0.5,0.5,0.4}, \frac{\xi_3}{0.4,0.7,0.3} \right\}>, \\ <\varsigma_2, \left\{ \frac{\xi_1}{0.4,0.8,0.6}, \frac{\xi_2}{0.3,0.6,0.5}, \frac{\xi_3}{0.5,0.4,0.4} \right\}> \end{array} \right\}$$

$$(\mathfrak{P},\Phi)^c = \left\{ \begin{array}{l} <\varsigma_1, \left\{ \frac{\xi_1}{0.4,0.7,0.8}, \frac{\xi_2}{0.3,0.6,0.5}, \frac{\xi_3}{0.2,0.9,0.7} \right\}>, \\ <\varsigma_2, \left\{ \frac{\xi_1}{0.3,0.9,0.7}, \frac{\xi_2}{0.1,0.8,0.6}, \frac{\xi_3}{0.3,0.6,0.7} \right\}> \end{array} \right\}$$

Then, ${\tau_u}^c = \{\ \phi_u,\ 1_u,\ (\mathfrak{N},\Phi)^c, (\mathfrak{P},\Phi)^c\}$ is the collection of $\mathcal{N_S CS}$s.

Let $(\mathfrak{S},\Phi)$ be an arbitrary $\mathcal{N_S S}$ defined as

$$(\mathfrak{S},\Phi) = \left\{ \begin{array}{l} <\varsigma_1, \left\{ \frac{\xi_1}{0.2,0.8,0.9}, \frac{\xi_2}{0.3,0.7,0.7}, \frac{\xi_3}{0.1,0.9,0.8} \right\}>, \\ <\varsigma_2, \left\{ \frac{\xi_1}{0.2,0.9,0.8}, \frac{\xi_2}{0.1,0.8,0.7}, \frac{\xi_3}{0.3,0.7,0.8} \right\}> \end{array} \right\}$$

Here $\mathcal{N_S}cl(\mathfrak{S},\Phi) = (\mathfrak{P},\Phi)^c$ and $\mathcal{N_S}int(\mathfrak{S},\Phi) = \phi_u$

3.5.1. *Program and intuitive description*

```
from pprint import pprint

m = int(input("Number of Neutrosophic Soft Open Sets: "))
attributes = int(input("Enter the Number of attributes : "))
elements = int(input("Enter the Number of elements in the Universe
    ↪ Set: "))

NS_Sets = []
p = [[0] * elements for g in range(attributes)]
for i in range(m):
    print("Enter the Neutrosophic soft set:", i + 1)
    ns_set = [[tuple(map(float, input().split(" "))) for e in range
        ↪ (elements)]
              for a in range(attributes)]
    NS_Sets.append(ns_set)


def nss_compl(m):
    """Return tuple comprised of the complement value in each tuple
        ↪ argument."""
    result = [[((m[a][e][2]), (round((1 - m[a][e][1]), 1)), (m[a][e
        ↪ ][0])) for e in range(elements)] for a in range(attributes
        ↪ )]
    return (result)
```

```
complement = []
for m in NS_Sets:
    x = nss_compl(m)
    complement.append(x)

mA = int(input("Number of Arbitrarty Neutrosophic Soft Sets: "))

Arbitrary_Sets = []
p = [[0] * elements for i in range(attributes)]
for i in range(mA):
    print("Enter the Artbitrary Neutrosophic soft set:", i + 1)
    arbi_set = [[tuple(map(float, input().split(" "))) for e in
        ↪ range(elements)]
              for a in range(attributes)]
    Arbitrary_Sets.append(arbi_set)

def superset(A, m):
    return (((m[g][d][0] >= A[g][d][0]) and (m[g][d][1] <= A[g][d
        ↪ ][1]) and (m[g][d][2] <= A[g][d][2])))

def subset(A, m):
    return (((m[g][d][0] <= A[g][d][0]) and (m[g][d][1] >= A[g][d
        ↪ ][1]) and (m[g][d][2] >= A[g][d][2])))

def nss_union(m1, m2):
    """Return tuple comprised of the max, min, min value in each
        ↪ tuple argument."""
    result = [[((max(m1[g][d][0], m2[g][d][0])), (min(m1[g][d][1],
        ↪ m2[g][d][1])),
              (min(m1[g][d][2], m2[g][d][2]))) for d in range(
                  ↪ elements)] for g in range(attributes)]
    return (result)

def nss_intersection(m1, m2):
```

```
    """Return tuple comprised of the min, max, max value in each
       ↪ tuple argument."""
    result = [[((min(m1[g][d][0], m2[g][d][0])), (max(m1[g][d][1],
       ↪ m2[g][d][1])),
                (max(m1[g][d][2], m2[g][d][2]))) for d in range(
                   ↪ elements)] for g in range(attributes)]
    return (result)

for m1 in Arbitrary_Sets:
    Superset = []
    for m in complement:
        call = []
        for g in range(attributes):
            for d in range(elements):
                v = superset(m1, m)
                call.append(v)
        if all(call):
            Superset.append(m)
    closure = Superset[0]
    for i in range(len(Superset)):
        closure = nss_intersection(closure, Superset[i])
    print("Closure:")
    pprint(closure)

for m1 in Arbitrary_Sets:
    Subset = []
    for m in NS_Sets:
        cal2 = []
        for g in range(attributes):
            for d in range(elements):
                l = subset(m1, m)
                cal2.append(l)
        if all(cal2):
            Subset.append(m)
    interior = Subset[0]
    for i in range(len(Subset)):
```

```
        closure = nss_union(interior, Subset[i])
    print("Interior:")
    pprint(interior)
```

The $\mathcal{N}_\mathcal{S}$-complement, $\mathcal{N}_\mathcal{S}$-superset, $\mathcal{N}_\mathcal{S}$-subset $\mathcal{N}_\mathcal{S}$-union and $\mathcal{N}_\mathcal{S}$-intersection functions are defined in this program. Program initially gets the values of the number of $\mathcal{N}_\mathcal{S}\mathcal{O}\mathcal{S}$s, number of attributes and number of elements. In the second step, $\mathcal{N}_\mathcal{S}\mathcal{O}\mathcal{S}$s are collected as input and stored in the list called NS_Sets. Then it creates the list of complement using the $\mathcal{N}_\mathcal{S}$-Complement function. In the third step, the program gets the arbitrary $\mathcal{N}_\mathcal{S}\mathcal{S}s$ and creates a list of arbitrary $\mathcal{N}_\mathcal{S}\mathcal{S}s$.

For each element in the list of arbitrary $\mathcal{N}_\mathcal{S}\mathcal{S}$, the program compiles a list of supersets (subsets) by taking $\mathcal{N}_\mathcal{S}\mathcal{C}\mathcal{S}$s ($\mathcal{N}_\mathcal{S}\mathcal{O}\mathcal{S}$s) from the list of complement ($\mathcal{N}_\mathcal{S}\mathcal{O}\mathcal{S}s$) using the pre-defined superset (subset) function. Then the program computes the closure (interior) of the arbitrary set by taking $\mathcal{N}_\mathcal{S}\mathcal{S}$ from the list of supersets (subsets) and using the $\mathcal{N}_\mathcal{S}$-intersection ($\mathcal{N}_\mathcal{S}$-union) function, prints the value of the closure(interior) of the arbitrary set.

```
Number of Neutrosophic Soft Open Sets: 4
Enter the Number of attributes : 2
Enter the Number of elements in the Universe Set: 3
Enter the Neutrosophic soft set: 1
.7 .4 .5
.4 .5 .5
.3 .3 .4
.6 .2 .4
.5 .4 .3
.4 .6 .5
Enter the Neutrosophic soft set: 2
.8 .3 .4
.5 .4 .3
.7 .1 .2
.7 .1 .3
.6 .2 .1
.7 .4 .3
Enter the Neutrosophic soft set: 3
1 0 0
1 0 0
1 0 0
1 0 0
1 0 0
1 0 0
Enter the Neutrosophic soft set: 4
0 1 1
0 1 1
0 1 1
0 1 1
0 1 1
0 1 1
Number of Arbitrarty Neutrosophic Soft Sets: 1
Enter the Artbitrary Neutrosophic soft set: 1
.2 .8 .9
.3 .7 .7
.1 .9 .8
.2 .9 .8
.1 .8 .7
.3 .7 .8
```

```
Number of Arbitrarty Neutrosophic Soft Sets: 1
Enter the Artbitrary Neutrosophic soft set: 1
.2 .8 .9
.3 .7 .7
.1 .9 .8
.2 .9 .8
.1 .8 .7
.3 .7 .8
Closure:
[[(0.4, 0.7, 0.8), (0.3, 0.6, 0.5), (0.2, 0.9, 0.7)],
 [(0.3, 0.9, 0.7), (0.1, 0.8, 0.6), (0.3, 0.6, 0.7)]]
Interior:
[[(0.0, 1.0, 1.0), (0.0, 1.0, 1.0), (0.0, 1.0, 1.0)],
 [(0.0, 1.0, 1.0), (0.0, 1.0, 1.0), (0.0, 1.0, 1.0)]]
```

Figure 5. Output of closure and interior of a neutrosophic soft set program

4. **Conclusion**

In this paper, python functions for union, intersection, superset, subset and complement for working in $\mathcal{N}_{\mathcal{S}}$ environment were defned and python program for union, intersection and complement in $\mathcal{N}_{\mathcal{S}}$ environment were furnished. Further python program for computing closure and interior of a $\mathcal{N}_{\mathcal{S}}\mathcal{S}$ in $\mathcal{N}_{\mathcal{S}}$ environment was provided. Moreover, Python program was constructed to verify whether a particular collection of $\mathcal{N}_{\mathcal{S}}\mathcal{S}$s forms a $\mathcal{N}_{\mathcal{S}}$-topology or not. User HTML interface can be developed using Pyscript in near future. In due course, the python programme can be built to find the optimal solution to any decision-making problem with complex data.

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Decarbonization Transportation: Evaluating Role of Cyber Security in Transportation sector based on Neutrosophic Techniques in a Climate of Uncertainty

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Abstract

Recently, transport systems that produce low levels of carbon emissions have emerged as an essential component of several nations' goals for achieving sustainable development. These systems also play a very significant role in the process of developing low-carbon cities. On the other hand, the safety of low-carbon modes of transport has been put in jeopardy in several different ways. For instance, assaults that result in a denial of service present a significant danger to the networks that connect electric cars to the grid. Several different strategies for defending against these dangers have been developed to lessen their impact. However, these strategies are only applicable to certain kinds of situations or assaults. Therefore, the purpose of this paper is to examine the security element from a holistic approach, present an overview of the obstacles and future directions of cyber security technologies in low-carbon transportation, and overcome such challenges. To begin, the low-carbon transport services are positioned in this article based on the idea of low-carbon transport and the significance of low-carbon transport. After that, using the network's design and the manner of communication as a lens, this article defines the common threats posed by attacks on the network. An additional consideration is given to the associated defensive technologies as well as the pertinent security recommendations, this time from the point of view of data security, network management security, and network application security. To improve this notion of safeguarding the gride and communication, it is necessary to evaluate ability of network against sniffing and spoofing or from any various methods of attacks. This motivates the study to employ Multi- Criteria Decision Making (MCDM) entailed in entropy for weighting the benchmarks which employed in Combined Compromise Solution (CoCoSo) toward obtaining optimal intelligent transportation systems (ITrSs) through ranking various ITrSs. In this study the utilized techniques are powered by uncertainty theory is Triangular Neutrosophic Sets (TriNSs). These utilized techniques contributed to constructing robust hybridization model. This model implemented in real case study to ensure its validity on decision making.

Keywords: Low-Carbon Transportation; Sustainable Environment; Cyber security; Holistic Approach; Multi- Criteria Decision Making (MCDM); Triangular Neutrosophic Sets (TriNSs).

1. Introduction

Both the energy and transport industries are responsible for the majority of the world's total carbon emissions. The transformation of energy and transport networks into low-carbon models is unavoidable if the strategic aim of reaching "carbon peaking and neutralization" is to be accomplished [1].

Our everyday lives, including the industries of energy and transportation, are being revolutionized by the spread of digital technology. Digitalization is a significant development that gives choices for reducing energy demand and carbon emissions [2]; nevertheless, it has been questioned as to whether the local energy savings from networked digital devices might compensate for the increased energy usage of the devices. Digitalization is an important trend that provides alternatives for reducing energy demand and carbon emissions. Additionally, digital technologies have the potential to contribute to the intelligent and environmentally friendly design of the future generation of transportation systems.

Both the academic world and the business world are interested in how decarbonization might be facilitated in a timely and cost-effective manner by digitalization [3]. This is an important development. Digitization comprises sensing, transmission, and computing, i.e., data production, data transfer, data storage and transformation, and data application (data value generation). When seen through the lens of the data value stream, the production of data of a high standard is strongly dependent on the development of infrastructure, which may include sensors for energy and transportation. In the meanwhile, the development of technologies known as 5G and 6G helps to speed up the transmission of data, which is necessary to meet the requirements of the big data age for the timely delivery of large quantities of data. Data has the potential to generate values in a wide variety of application situations, and digitalization may be able to contribute to the industry's overall growth and success if it is supported by more complex procedures and algorithms.

The dependability and stability of low-carbon transportation systems are significantly improved when cyber security measures are included [4]. To begin, ensuring the confidentiality, enforceability, and non-repudiation of one's data is the primary responsibility of a company's data security team [5]. Second, the implementation of network management security, which should include trust management, misbehavior detection, an intrusion detection system, and a firewall, is required in order to guarantee the dependability of the data that is sent inside a network system, as well as its integrity and accuracy. In conclusion, personalized network applications result in new security needs, and it is important to pay attention to both edge computing security and software-defined security in this context.

In order to accomplish this goal, the focus of this article is on the analysis and research of the functioning of a low-carbon ecosystem, with the primary emphasis being placed on the cyber security of low-carbon transportation. The purpose of this piece is to shed light on the significance and effect of cyber security in low-carbon transportation, as well as to encourage the coordinated development of electric cars, transportation, energy, information, and cyber security. Specifically, the article will focus on revealing the relevance and impact of cyber security in low-carbon transportation. The following is the most important contributions that this article makes:

- The low-carbon transport service is positioned in this assessment based on the idea of low-carbon transport and the significance of low-carbon transport.
- A discussion of the problems with cyber security and the solutions to those problems.
- Challenges and potential avenues of study have been highlighted in light of the recent trend toward the development of low-carbon modes of transportation.
- Construct evaluation model responsible for evaluating intelligent transportation systems.
- Applying the constructed evaluation model in real enterprises of transportation.

2. Theoretical Background

The development of low-carbon transportation might be helped forward by the integration of transportation, energy, and information networks. On the other hand, several application scenarios and information interactions in low-carbon transportation would expose its vulnerabilities to attack. A security defense system must be implemented in order to repel assaults in application situations or communication modalities. In the first part of this section, we will go through several common application situations and low-carbon transportation component types. In the second place, it examines common information and communication technologies, such as E-Mobility, smart grids, in-vehicle connectivity, and communication between vehicles and other things. The latter part of this section focuses on the possible cyber dangers and common assaults on low-carbon transportation.

2.1 Carbon-neutral transportation

Charging stations and battery swapping stations are potential sources of electricity for electric cars [6]. Charging stations, battery swap stations, and other similar facilities may all get their power from the smart grid. The demand for energy from electric cars is processed through charging stations and battery swap stations, which function as an intermediate.

The commercialization of electric cars is making steady progress due to the many positive impacts these vehicles may have on their surrounding environments. Traditional automobile manufacturers, such as Mercedes-Benz and BMW, as well as the electric vehicle manufacturer Tesla, have been preparing their transition towards the electrification of vehicles in response to the current worldwide trend.

Compatibility is one of the fundamental ideas behind the smart grid, which encompasses both centralized and decentralized power-producing infrastructures as well as access to a wide variety of energy storage solutions. The term "distributed power" refers to the production and storage of electricity by a wide variety of low-capacity devices that are linked to either the smart grid or the distribution system. These distributed energy resources are also known as "distributed energy resources." The distributed energy resources system is a decentralized, modular, and more adaptable technology that utilizes renewable energy such as solar energy and wind energy on a local scale. The system also makes use of modular solar panels and wind turbines. If not, centralized electricity would be provided by conventional power stations, which include coal, gas, and nuclear power plants, as well as hydropower dams and large-scale solar power stations. The smart grid that is enabled by 5G and artificial intelligence will contribute to an improvement in the efficiency of energy transit and utilization.

The low battery capacity of electric cars combined with the length of journeys taken inside metropolitan areas results in the need for regular charging of electric vehicles. Charging stations are often installed in areas that have a large concentration of electric vehicles, such as shopping malls and parking lots. In general, this means that charging stations are located in high-density areas. In the event that the charging station is unable to detect the arrival of electric cars that have a need for charging, however, there is a possibility that a charging service congestion may arise. Because of advances in communication technology between vehicles and other objects, it is now feasible to exchange information of this kind in order to supervise the charging process in the most effective manner.

2.2 Information and communication technologies applied to low-carbon modes of transportation

In spite of the fact that significant attention has been dedicated to the development of electric cars from both academic and industrial perspectives, the growth of the electric vehicle sector is being hampered by problems such as insufficient charging facilities, a lack of standardization, and inconsistent norms. Electric cars and charging stations are seeing tremendous expansion as a direct result of the concerted efforts of companies and governments all over the globe.

The charging pile's communication method may primarily be broken down into two categories: wired communication and wireless communication modalities. The most common forms of the wired communication method include industrial serial bus and wired Ethernet, among others. When it comes to data transfer, industrial serial bus systems are more dependable; nevertheless, they come with a number of drawbacks, including high complexity, low communication capacity, poor flexibility, high building costs, and limited scalability. Despite its complicated wiring, limited flexibility, high construction cost, and limited expansibility, the wired Ethernet network has a larger capacity and provides data transfer that is dependable.

2.3 Disputes over safety in low-carbon modes of transportation

In the recent past, the idea of a smart grid has emerged as a result of advancements in information and communications technologies. The beneficial information is constantly being transmitted and monitored inside the smart grid in order to initiate decision-making on the management of the power system. However, despite the fact that the access network makes the operation of the system more efficient, this improvement will put the system's security at risk since it is dependent on the flow of information. Because communication networks are susceptible to cyber assaults and hostile infiltration, the smart grid is not immune to the attacks that hackers have been carrying out in increasing numbers over the last several years, regardless of whether they are motivated by profit or politics.

"GPS spoofing" is a method that may be used to attack an autonomous vehicle's multi-sensor fusion positioning technique, which ultimately results in the car losing control of itself. This was discovered via research. This safety concern has sounded the alarm for manufacturers, who in recent years have increased their efforts to commercialize autonomous driving.

2.4 Attacks that are typical in low-carbon transportation

One of the most significant difficulties that nations all over the globe are now confronting is that of maintaining adequate levels of cyber security. The advancement and promotion of low-carbon modes of transport are impossible to do without the accompaniment of cyber security [7]. The foundation of successful low-carbon city development is a foolproof and comprehensive cyber security protection and prevention system. Nevertheless, low-carbon transport is being subjected to a large number of assaults and safety hazards. Jamming, spoofing, data dimension, denial of service, botnet, and sybil assaults are the most common types of low-carbon transportation threats [8]. Other types of attacks include dos attacks and data dimension attacks. Be aware that, in addition to traditional forms of cyber assault, there is also the possibility of a physical attack, which involves the use of forceful methods to target electric cars, charging piles, and other forms of firmware, among other things. In this context, it is necessary to perform routine safeguarding and maintenance on such important facilities.

3. Safety Measures for Network Management

The dependability, integrity, and accuracy of data that is shared inside a network system are ensured by the security of network management, which includes trust management, identification of inappropriate behavior, intrusion detection systems, and firewalls.

3.1 Confidence administration

Standard technologies that need much greater processing power, such as intrusion detection, password encryption, and decryption technology, are not suitable because of the restricted capacity of cars. Alternately, the function of the trust mechanism is to assess the amount of confidence that may be placed in vehicles by using the interaction history of those vehicles. By using this as assessment advice, it is possible to give up on hostile cars and promote trustworthy vehicles for data exchange.

The majority of the literature that is based on the entity-based trust model assesses the trustworthiness of vehicles. In this instance, direct trust and indirect suggestion trust are used together in order to identify automobiles that are not to be trusted or that are harmful.

Model of trust that is based on data, the model of trust that is based on data seeks to determine how reliable the data level is. In this case, the trust model calls for the collection of data from a wide range of sources, such as the cars themselves, their immediate neighbors, and roadside units.

The mixed trust model is one that, by default, incorporates the positive aspects of both traditional and alternative models of trust. Not only does it determine the degree to which cars can be trusted, but it also computes the accuracy of the data. The trustworthiness of vehicles has an effect on the dependability of data as a result of the influence of contact behavior, and the trustworthiness of data, in turn, reflects the dependability of vehicles as a result of the forwarding route that a data will be traveled. This is the purpose of the combined trust model, and it is inherent in its design.

3.2 Detection of inappropriate behavior

It is also possible to identify malicious vehicles based on the actions taken by network participants. These actions include forwarding, altering, discarding, and selective discarding of data. The benefit of using techniques that are based on network behavior is that they are wholly unaffected by the information that is being sent.

It is challenging to assess the fabricated data when there are just a few cars collecting enough messages for detection. Another way of saying this is that it is challenging to determine the forged data based only on the data content level.

3.3 Intrusion detection system

Because of power outages and natural calamities, the smart grid is very susceptible to denial of service assaults. Denial-of-service attacks are made with the intention of delaying, blocking, or otherwise disrupting communications, which may have a significant negative impact on the functioning of a network. An intrusion detection system that is based on deep learning has the potential to improve the detection and mitigation of denial-of-service attacks [9]. For the purpose of analyzing the features of information flow, it is put at the smart grid's edge. The typical pattern of behavior shown by information flow is created on the basis of spatiotemporal characteristics and context relations. In addition to that, the real-time schedulability analysis will extract the timing requirements as well as the model parameters of the information flow. After that, the intrusion detection system will develop a real-time model that precisely characterizes the temporal characteristics. This model will then be used to reflect the typical behavior of smart grid systems. Finally, with the help of packet attribute analysis and a data consistency model that is based on deep learning, it is possible to determine the usual behavior pattern of a physical information system as well as the origin of a cyber-assault.

Because of its usefulness in rapidly detecting assaults, the intrusion detection system garners a lot of attention. The network or host intrusion detection system will, as a general rule, identify any irregularity in the system and sound an alert if it exists. The intrusion detection system may be broken down into two groups according to the technological basis of intrusion detection:

- Signature-based intrusion detection system: This intrusion detection system analyzes known attacks to extract their distinguishing features and patterns, which is called the signature [10]. The signature-based intrusion detection system has the advantage of a high detection rate for known attacks, but the disadvantage is that it is not able to detect unknown or new attacks.
- Using supervised or unsupervised learning approaches to construct models based on characteristics, anomaly-based intrusion detection systems are another form of intrusion detection systems that are also known as behavior-based intrusion detection systems [11]. The model is able to distinguish between regular and aberrant patterns of network traffic, and it also has the power to detect undiscovered and novel forms of assault. Statistical and machine learning methods are used in the execution of this approach.

3.4 Firewall

The divide between public and private networks may be represented by a firewall. It is able to identify assaults and filter the flow of harmful traffic via the network [12]. The next-generation firewall has the capability to integrate denial-of-service attack detection technologies with network protocol identification capabilities. The former is used to filter attack flow and lower the danger of attack, while the latter is used to deal with injection and spoofing. Both of these functions are important for preventing attacks.

It is possible to identify the internet protocol address of the host computer or server that is connected to the network thanks to the smart grid system. Therefore, the function of the firewall known as packet filtering may be accomplished using a whitelist. To begin, the firewall analyses the flow of network traffic based on the kind of protocol, the number of ports, and the internet protocol address of the destination. This allows it to determine whether or not the traffic conforms to the standards, restrictions, and whitelists that have been set. After that, the position of the firewall may be adjusted so that it is in accordance with the structure of the smart grid network.

In order to defend against a wide range of assaults, firewalls for electric vehicles are also now in development. The network configuration information for an in-vehicle network is generally unchangeable, which is one of the properties of this kind of network. In most cases, the original equipment manufacturer is in possession of a communication matrix or database that lays out the guidelines for how electronics inside the vehicle are to communicate with one another. As a direct consequence of this, the "whitelist" filtering rules are often used as the foundation for the firewall technology used in an in-vehicle network.

4. Motivations of Study

Two significant trends in the upcoming decades are expected to be decarbonization and digitalization [13]. Massive amounts of data will be generated as the already pervasive process of digitization proceeds, particularly in the energy and transportation networks. How this data may support and promote data security has become a critical concern.

Hence. This study focuses on embracing cybersecurity technologies in intelligent transportation to guarantee confidentiality, unforgeability, and non-repudiation for smart grid. Accordingly, it is important to evaluate the enterprises of transportation which are deploying and embracing cybersecurity toward digitalization and decarbonization dimensions. We are conducting a survey for transportation enterprises for contributing to the evaluation process.

Accordingly, we constructed an evaluation model which is responsible for evaluating the enterprises which obtained from the conducted survey. This model relies on mathematical techniques and uncertainty theory to improve decisions in uncertainty environments [18-22].

5. Methodology of Evaluation

5.1 Recognition of principal ingredients

- One of the important procedures in study's evaluation methodology is recognizing principal and influential aspects.
- These aspects entailed in set of benchmarks {Bs}= {b_1, b_2,b_3,…b_n} also, set of nominees of Transportation enterprises systems as {TESs}={TES_1,TES_2,…TES_n}.
- Choice and communicate with decision makers (DecMs) who related to our interested study's scope.

5.2 Express the most and least significant benchmark through benchmarks' weights

- We are leveraging TriNSs scale in [14],[15] to contribute to evaluation process where DecMs are rating criteria based on the determined scale.
- Entropy is employed as a technique of MCDM techniques with the support of TriNSs s as branch of neutrosophic theory for generating criteria's weights. Hence, Neutrosophic decision matrices are constructed based on DecMs'preferences.
- Score function in Eq. (1) turns the constructed matrices to deneutrosophic matrices as mentioned in Ref [16].

$$s(b_{ij}) = \frac{(C_{ij}+D_{ij}+F_{ij})}{9} * (2 + \alpha - \beta - \theta) \quad (1)$$

where C_{ij}, D_{ij}, F_{ij} pointed to lower, middle, upper. Also, α, β, θ are truth, indeterminacy, and falsity.

- Eq.(2) aggregates deneutrosophic matrices into single decision matrix to construct an aggregated matrix.

$$Z_{ij} = \frac{(\sum_{j=1}^{N} b_{ij})}{S} \quad (2)$$

Where b_{ij} refers to value of criterion in matrix, S refers to number of decision makers.

- An aggregated matrix is normalizing through Eq. (3).

$$\mho_{ij} = \frac{z_{ij}}{\sum_{j=1}^{m} z_{ij}} \quad (3)$$

Where $\sum_{j=1}^{m} z_{ij}$ represents sum of each criterion in aggregated single decision matrix per column

- Entropy computes easily via utilizing Eq. (4).

$$e_{j=-h\sum_{i=1}^{m}\mho_{ij}}\ \ln \mho_{ij} \tag{4}$$

$$\text{Where} h=\frac{1}{\ln(Q)} \tag{5}$$

Q refers to number of alternatives

- Weight vectors generated through leveraging Eq.(6).

$$\text{w\_vector}_{j=}\frac{1-e_j}{\sum_{j=1}^{n}(1-e_j)} \tag{6}$$

5.3 Find out the best and the worst intelligence transportation system accordance to cyber security through ranking

Herein we are exploiting CoCoSo according to [17] as MCDM ranker technique for ranking nominees of ITrSs through improving the utilized ranker technique by TriNSs through implementing several steps:

- an aggregated decision matrix generated from previous steps of obtaining benchmarks' weights has been leveraged. Furthermore, Eq.s (7) and (8) are responsible for normalizing single matrix.

$$\text{Nor}_{ij}=\frac{Z_{ij-\min(Z_{ij})}}{\max(Z_{ij})-\min(Z_{ij})}\ \text{, for beneficial criteria} \tag{7}$$

$$\text{Nor}_{ij}=\frac{\max(Z_{ij})-Z_{ij}}{\max(Z_{ij})-\min(Z_{ij})}\ \text{, for non-beneficial criteria} \tag{8}$$

- Sum of weighted matrix is generated based on Eq. (9).

$$\text{Sum\_weighted}_i=\sum_{j=1}^{n}\ \text{w\_vector}_j*\text{Nor}_{ij} \tag{9}$$

- Eq. (10) is deploying for calculating power of weighted matrix.

$$\text{Power}_j=\sum_{j=1}^{n}\left(\text{Nor}_{ij}\right)^{\text{Sum\_weighted}_{ij}} \tag{10}$$

- Three different appraisal score for ITrSs candidates are calculating through following Eq.s.

$$\text{Score}_{ia}==\frac{S_i+P_i}{\sum_{i=1}^{m}S_i+P_i} \tag{11}$$

$$\text{Score}_{ib}=\frac{S_i}{\min_i S_i}+\frac{P_i}{\min_i P_i} \tag{12}$$

$$\text{Score}_{ic}=\frac{\lambda S_i+(1-\lambda)P_i}{\lambda \max_i S_i+(1-\lambda)\max_i P_i}\ ,0\le\lambda\le 1 \tag{13}$$

Where S_i indicates to sum of each raw in sum of raw in weighted matrix whilst P_i refers to sum of raw in power of weighted matrix.

- The final rank is obtaining via Eq. (14).

$$K_{i=}(K_{ia}*K_{ib}*K_{ic})^{1/3}+\frac{1}{3}(K_{ia}+K_{ib}+K_{ic}) \tag{14}$$

6. Numerical case study

The validation process is considered important in this study. Hence, we applied our constructed model of evaluation on real transportation enterprises which embracing study's notion. The evaluation for systems of these enterprises is performed based on set of benchmarks.

The validation process is conducting through following dimensions:

6.1 First dimension: benchmarks and intelligent transportation systems(alternatives).

- The ITrSs which embracing study's notion are identifying where four alternatives are contributed to validation process. Accordingly, the benchmarks are determined as in Figure 1.
- Therefore, five DecMs are contributed to rate alternatives of ITrSs based determined benchmarks through utilizing triangular Neutrosophic scale in [15].

6.2 Second dimension: Determining benchmark's weights.

- An aggregated matrix is constructed through calculating average for five Neutrosophic decision matrices as listed in Table 1.
- Whilst Table 2 represents normalization of an aggregated matrix.
- Entropy matrix is constructed based on Eqs. (4),(5) and Table 3 is generated.
- Moreover, vector of weights is generated through implementing Eq.(6) and Figure 2 represents benchmarks' criteria. According to this Figure B3 considered the optimal criterion where its value of weight is the highest compared with others otherwise B4 where its value of weights is the least one.

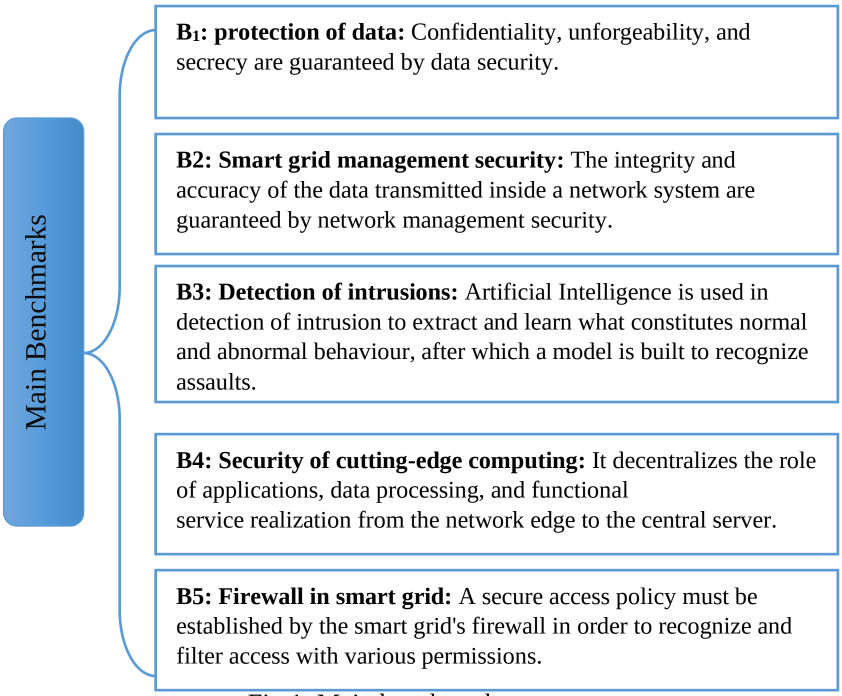

Fig.1. Main benchmarks

Table1. An aggregated decision matrix

	B_1	**B_2**	**B_3**	**B_4**	**B_5**
$ITrSs_1$	4.656666667	4.946666667	5.456666667	4.283333333	5.253333333
$ITrSs_2$	5.583333333	6.103333333	4.726666667	5.593333333	6.706666667
$ITrSs_3$	4.683333333	4.653333333	6.123333333	4.64	4.81
$ITrSs_4$	6.65	4.596666667	7.233333333	5.623333333	5.996666667

Table2. Normalized an aggregated decision matrix.

	B_1	**B_2**	**B_3**	**B_4**	**B_5**

ITrSs1	0.215852905	0.2436782	0.23180402	0.212677921	0.230746706
ITrSs2	0.258807169	0.3006568	0.20079298	0.277722608	0.294582723
ITrSs3	0.217088999	0.2292282	0.26012461	0.230387289	0.211273792
ITrSs4	0.308250927	0.2264368	0.30727839	0.279212181	0.263396779

Table3. Entropy Matrix

	B1	**B2**	**B3**	**B4**	**B5**
ITrSs1	-0.330936629	-0.344050886	-0.338865722	-0.329220395	-0.338374972
ITrSs2	-0.349822408	-0.361325095	-0.322369282	-0.355799453	-0.360037656
ITrSs3	-0.33159213	-0.337661701	-0.350282368	-0.338207048	-0.32844632
ITrSs4	-0.362762369	-0.33632417	-0.362588849	-0.356214232	-0.351395988
$-h\sum_{i=1}^{m} \mho_{ij}\, ln\mho_{ij}$	-1.375113537	-1.379361852	-1.374106221	-1.379441128	-1.378254936

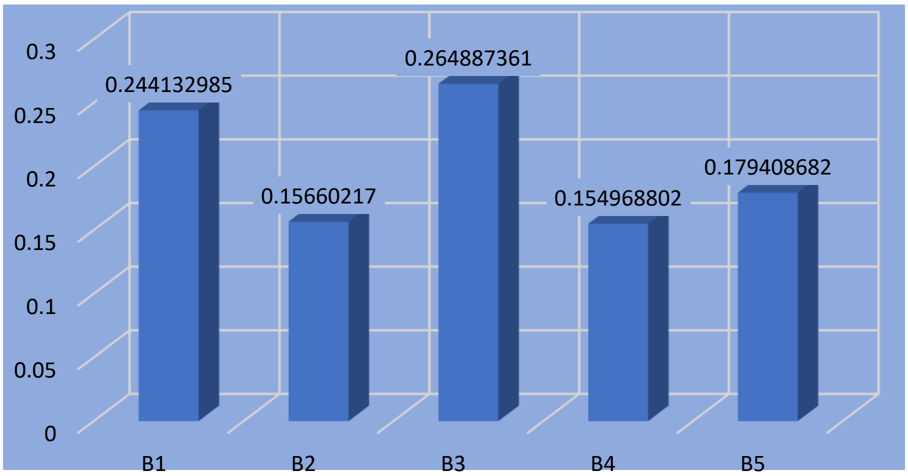

Fig.2. benchmarks Weights

6.3 Third dimension: Ranking alternatives of ITrSs

- In this dimension, an aggregated matrix which generated from second dimension is normalized by Eq.(7) where benchmarks are considering beneficial and Table 4 has been generated.
- Weighted decision matrix has been produced through multiplying normalized matrix with weights of benchmarks which generated from entropy-TriNSs. The result of this process is illustrated in Table 5.
- The power of sum weighted matrix has been computed via Eq.(10) and results are showcased in Table 6.
- Candidates' appraisal scores have been calculated through Eq.(11),(12),(13). And its scores are appeard in Table 7. According to this Table, we concluded that ITrS4 is the best otherwise ITrS1 is the least one.
- Finally, the final rank for candidates ITrSs. Eq.(14) is utilized for obtaining the candidates' final rank which showcased in Figure 3 where the results in this Figure agreed with ranking in Table7and emphasized that ITrS4 is the best otherwise ITrS1 is the least one

Table 4. Normalized an aggregated matrix based on CoCoSo-TriNSs

	B_1	B_2	B_3	B_4	B_5
$ITrS_1$	0	0.232300885	0.291223404	0	0.233743409
$ITrS_2$	0.464882943	1	0	0.97761194	1
$ITrS_3$	0.013377926	0.037610619	0.557180851	0.266169154	0
$ITrS_4$	1	0	1	1	0.625659051

Table 5. Weighted decision matrix based on CoCoSo-TriNSs

	B_1	B_2	B_3	B_4	B_5
$ITrS_1$	0	0.036378823	0.077141399	0	0.041935597
$ITrS_2$	0.113493261	0.15660217	0	0.151499351	0.179408682
$ITrS_3$	0.003265993	0.005889905	0.147590165	0.041247915	0
$ITrS_4$	0.244132985	0	0.264887361	0.154968802	0.112248666

Table 6. Powe of Weighted decision matrix based on CoCoSo-TriNSs

	B_1	B_2	B_3	B_4	B_5
$ITrS_1$	0	0.795650225	0.721240842	0	0.770453377
$ITrS_2$	0.829445018	1	0	0.996497271	1
$ITrS_3$	0.348810625	0.598260777	0.856480736	0.814549312	0
$ITrS_4$	1	0	1	1	0.919308384

Table 7. Various scores of candidates

	K_{ia}	K_{ib}	K_{ic}	Rank
$ITrS_1$	0.169858453	1.672656822	1.605882339	4
$ITrS_2$	0.307824659	1.14460306	2.910247763	2
$ITrS_3$	0.195815279	1.713475377	1.851284357	3
$ITrS_4$	0.326501609	1.672656822	3.086824095	1

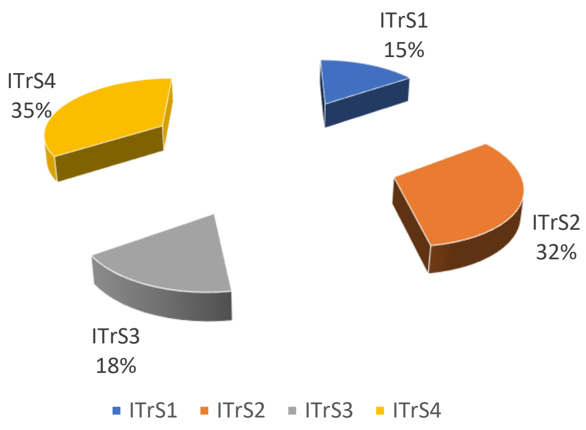

Fig.3. Final Score for various ITrSs candidates

7. Future directions

In point of fact, there are still questions that need answers from communities. The fundamental objective is to advance the intelligence level of cars within a low-carbon transportation system in order to make use of more potent digitalized services and cyber security technologies. This will be accomplished via the promotion of this direction. In accordance with the integrated control for low-carbon transportation, it is desirable to investigate other new approaches such as detachable security technology and zero-knowledge proof. These hypothetical trajectories are presented in the following order:

- The information and communication systems of automobiles, artificial intelligence, software, the Internet, and other technologies are deeply incorporated into the intelligent linked vehicles.
- Combining several technologies in order to perform certain tasks has become more common in recent years, thanks both to the ongoing development of existing technologies and the birth of brand-new technologies in fields ranging from hardware design to software algorithms. However, these technologies are not coupled to a particular degree, and they also need a balance to be struck between their functions and their performances.
- People will have a greater propensity to purchase electric cars when the policies that promote them and when environmental preservation are taken into consideration. The proliferation of electric cars has brought to light the pressing need to improve internet connectivity inside vehicles and their ability to communicate with one another. Then, when a significant number of electric cars are linked to the smart grid, the influence on the system of the smart grid will become more significant. In light of this, one of the potential areas of focus for future study is the integrated control issue of smart charging for electric vehicles and collaborative technology for smart grids.
- Large amounts of disparate data coming from a variety of sources are required to ensure the safety of intelligent transportation systems. Zero-knowledge proof is progressively being included in intelligent transportation systems in order to guarantee the system's computational security. The goal of zero-knowledge proof is to convince the verifier that the prover is in possession of the proof without revealing any confidential information.
- In light of the fact that low-carbon transport is still in its infancy, it is still required to devise high-level policy and speed up technological innovation.

8. Conclusion

Most recent studies emphasized that energy and transportation systems are becoming more intelligent, sustainable, and efficient courtesy to digital technologies like sensors, 5G, IoT, and data trading.

Hence, this article begins by providing an introduction to the idea of low-carbon transport as well as its historical context. This is done in light of the growing focus on attaining low-carbon transport and the requirement of securing the system via the use of Cybersecurity technology ∘n the era of intelligent transportation. After that, this article classifies and reviews emerging defense technologies from the aspects of data security, network management security, and network application security, covering up-to-date technical advances that have been contributing to communities. The classification and review process based on identifying typical attacks within the ecosystem of a low-carbon transportation system, and it covers recent technical advancements that have been helping communities. Also, evaluating the transportation enterprises which embracing our notion of deploying technologies that support cybersecurity toward safeguarding smart grid and information against any attack. Through the survey that was conducted for the enterprises, we communicated with four ITrSs which contribute to the evaluation process.

Hence, we constructed robust hybrid model which relied on mathematical techniques of MCDM techniques. These techniques are hybridized with uncertainty theory of neutrosophic which has main role of supporting MCDM techniques in situations characterized with ambiguity and inability to preferences and making decisions. Moreover, we employed entropy based on TriNSs to obtain weights for five benchmarks. The obtained benchmarks' weights are contributed to the process of ranking four ITrSs through multiplying the weights by normalized matrix. This matrix generated from utilizing CoCoSo based on TriNSs which responsible for ranking ITrSs after that recommending the best and worst ITrS.

In addition to the existing contribution, this article highlighted several future directions that cover the cyber-secure low-carbon transportation system from the evolution of vehicles, compatibility of defense technologies integration, and potential impact on unlocking the cyber security and system reliability. These future directions cover all these topics and more.

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Examples of NeutroHyperstructures on Biological Inheritance

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Abstract: : In 1934, Marty introduced the concept of hyperstructures, which serves as a generalization of algebraic structures. Hyperstructures have applications in various fields, including biology, where they prove useful for analyzing the different types of hyperstructures in inheritance. On the other hand, NeutroHyperstructures combine Neutrosophic sets with hyperstructures, offering a promising avenue to handle uncertainty in inheritance analysis. Inspired by the intriguing variety of hyperstructures observed in inheritance phenomena, this paper takes on the purpose of thoroughly examining the types of NeutroHyperstructures present in multiple biological inheritance examples. The study focuses on analyzing inheritance patterns in Mirabilis Jalapa flowers, Shorthorn Cattle coat color, and blood types (ABO, ABO with rhesus, MN, MN with rhesus, and the Kidd system) through the lens of NeutroHyperstructures. Through this meticulous analysis, the research aims to contribute significant insights into the genetic inheritance processes, unveiling the role of NeutroHyperstructures in governing diverse biological traits. The findings offer valuable implications for the field of mathematical biology, presenting novel perspectives on inheritance modeling and establishing the potential of NeutroHyperstructures to effectively address uncertainty in genetics and inheritance studies. This study fosters a deeper understanding of complex biological inheritance and opens new avenues for practical applications in the realm of genetics and related disciplines.

Keywords: Biological Inheritance; Hyperstructures; Hypergroup; NeutroHyperstructures; NeutroHypergroup

1. Introduction

In 1995, Smarandache introduced the notion of Neutrosophy as a new branch in philosophy. Initially, ideas were seen as either "True" or "False". However, in neutrosophic concepts, ideas can be viewed as "True", "False" or "Indeterminate". One of the research related to neutrosophic sets is the neutrosophic quadruple set on algebraic structures [1-2]. Neutrosophic sets find many applications in various fields, including on Economics [3], supply chain [4], and operations research [5]. As neutrosophic research develops, these concepts can be applied to abstract structures. One such research development is NeutroAlgebra, introduced by Smarandache in 2019 [6 - 7]. In the NeutroAlgebra concept, operations are partially well-defined, partially false, and partially indeterminate, while axioms is partially true, partially false, and partially indeterminate.

On the other hand, in 1934 Marty introduced the concept of hyperstructures, a generalization of algebraic structures [9]. By applying the concept of neutrosophy to hyperstructures, a new concept called NeutroHyperstructures was defined [10-11]. on of the research developments in

NeutroHyperstructures is the definition of LA-Hyperstructures by Mirvakili, et al., namely Neutro-LA-Semihypergroup and Neutro-H_v-Semihypergroup [12].

Furthermore, hyperstructures have numerous applications in different fields including Physics [13], Chemistry [14], and Biology [15 – 16]. Inspired by NeutroHyperstructures and the applications of hyperstructures in Chemistry, the authors studied the applications of NeutroHypestructures in Chemical reactions [17].

This paper aims to analyze the types of NeutroHyperstructures in several examples of biological inheritance. The examples examined in this paper include Mirabilis Jalapa flowers, coat color of Shorthorn Cattle, ABO blood type, ABO with rhesus, MN, MN with rhesus, and kidd system. This paper is organized as follows: After the Introduction, Section 2 presents the basic theories used in this study. In Section 3 we analyze the NeutroHyperstructures contained in Biological Inheritance, focusing on examples like Mirabilis Jalapa flowers, coat color of Shorthorn Cattle, ABO blood type, ABO with rhesus, MN, MN with rhesus, and Kidd system. Finally, in Section 4 we provide the conclusion based on the results of the research conducted.

2. Preliminaries

In this section, we recall some concepts of NeutroHyperstructures taken from Ibrahim and Agboola [10] and Al-Tahan et al. [11].

Definition 2.1 [11] Let G be a nonempty set and "$\boxminus$" be a hyperoperation in G. Then the operation "$\otimes$" is called a NeutroHyperoperation in G if some (or all) of the following conditions are satisfied with $(T, I, F) \notin \{(1,0,0), (0,0,1)\}$.

1. There exist $p, q \in G$ with $p \otimes q \subseteq G$ (degree of truth "T")
2. There exist $p, q \in G$ with $p \otimes q \nsubseteq G$ (degree of falsity "F")
3. There exist $p, q \in G$ with $p \otimes q$ is indeterminate in G (degree of indeterminacy "I")

Example 2.2 Let $G = \{u, v, w\}$ and define a hyperoperation "$\otimes$" as follows.

Table 1. $(G,\otimes)$

$\otimes$	u	v	w
u	u	?	$\{u, w\}$
v	?	v	$\{v, w\}$
w	$\{u, w\}$	$\{v, w\}$	v

Then, $(G,\otimes)$ is a NeutroHyperoperation because there exist $u, v \in G$ such that $u \otimes v$ is indeterminate.

Definition 2.3 [11] Let G be a nonempty set and "$\otimes$" be a hyperoperation in G. Then "$\otimes$" is called *AntiHyperoperation* in G if for every $p, q \in G, p \otimes q \nsubseteq G$.

Example 2.4 Based on Example 2.2, $(\{u, w\},\otimes)$ is an AntiHyperoperation since $w \otimes w = v \nsubseteq \{u, w\}$.

Definition 2.5 [11] Let G be a nonempty set and "$\otimes$" be a hyperoperation on G. Then "$\otimes$" is called NeutroAssociative on G if there exist $p, q, r, x, y, z, a, b, c \in G$ that satisfiy some (or all) of the following conditions with $(T, I, F) \notin \{(1,0,0), (0,0,1)\}$.

1. $p \otimes (q \otimes r) = (p \otimes q) \otimes r$ (degree of truth "T")
2. $x \otimes (y \otimes z) \neq (x \otimes y) \otimes z$ (degree of falsity "F")
3. $a \otimes (b \otimes c)$ is indeterminate or $(a \otimes b) \otimes c$ is indeterminate (degree of indeterminacy "I")

Example 2.6 Based on Example 2.2, $(G,\otimes)$ is NeutroAssociative since there exists $u, v, w \in G$ such that $u \otimes (v \otimes w)$ is indeterminate.

Definition 2.7 [11] Let G be a nonempty set and "$\otimes$" be a hyperoperation in G. Then "$\otimes$" is called NeutroWeakAssociative in G if there exists $a, b, c, x, y, z, p, q, r \in G$ that satisfy some (or all) of the following conditions with $(T, I, F) \notin \{(1,0,0), (0,0,1)\}$.

1. $[a \otimes (b \otimes c)] \cap [(a \otimes b) \otimes c] \neq \emptyset$ (degree of truth "T")
2. $[x \otimes (y \otimes z)] \cap [(x \otimes y) \otimes z] = \emptyset$ (degree of falsity "F")
3. $p \otimes (q \otimes r)$ is indeterminate or $(p \otimes q) \otimes r$ is indeterminate (degree of indeterminacy "I")

Example 2.8 Based on Example 2.2, $(G,\otimes)$ is NeutroWeakAssociative since there exists $u, v, w \in G$ such that $u \otimes (v \otimes w)$ is indeterminate.

Definition 2.9 [11] Let G be a nonempty set and "$\otimes$" be a hyperoperation in G. Then $(G,\otimes)$ is called a NeutroHypergroupoid if "$\otimes$" is a NeutroHyperoperation, a NeutroSemihypergroup if "$\otimes$" is NeutroAssociative but not an AntiHyperoperation, and Neutro H_v-semigroup if "$\otimes$" is NeutroWeakAssociative but not an AntiHyperoperation.

Definition 2.10 [10] A NeutroHypergroupoid $(G,\otimes)$ is called a NeutroHypergroup if G is a NeutroSemihypergroup and there exist $e, f, g \in G$ that satisfy some (or all) of the following conditions with $(\mathrm{T}, \mathrm{I}, \mathrm{F}) \notin \{(1,0,0), (0,0,1)\}$. (This following condition is called a NeutroReproduction axiom).

1. $e \otimes G = G \otimes e = G$ (degree of truth "T")
2. $f \otimes G \neq G \otimes f \neq G$ (degree of falsity "F")
3. $g \otimes G$ or $G \otimes g$ indeterminate (degree of indeteminacy "I")

Example 2.11 Let $H = \{f, y, m\}$. Define a hyperoperation "$\otimes$" as follows.

Table 2. $(H,\otimes)$

$\otimes$	f	y	m
f	f	$\{f, y\}$	m
y	f	y	y
m	$\{f, m\}$	y	m

One can easily see that $(H,\otimes)$ is a NeutroHypergroup.

3. Main Results

In this section, the results of the research obtained are presented. In this section, "$\boxplus$" is defined as the result of mating.

Based on [16], in the case of flower color inheritance in the four o'clock plant (*Mirabilis Jalapa*), suppose R, P, and W respectively represent the color of the flowers of *Mirabilis Jalapa* namely red, pink, and white. Let $G = \{R, P, W\}$, the result of $(G, ⊞)$ is given as follows.

Table 3. $(G, ⊞)$

$⊞$	R	P	W
R	R	$\{R, P\}$	P
P	$\{R, P\}$	G	$\{R, P\}$
W	P	$\{R, P\}$	W

Theorem 3.1 $(G, ⊞)$ is a NeutroHypergroup.
Proof. It is clear that $(G, ⊞)$ is not an Antihyperoperation. Next, $P ⊞ (P ⊞ P) = (P ⊞ P) ⊞ P = G$, an $R ⊞ (W ⊞ W) = P \neq \{R, P\} = (R ⊞ W) ⊞ W$. Then, $(G, ⊞)$ is a NeutroSemihypergroup. Furthermore, for every $P \in G, P ⊞ G = G ⊞ P = G$, $R ⊞ G = G ⊞ R \neq G$. Thus, $(G, ⊞)$ is a NeutroHypergroup.

Remark 3.2 Based on [16], $(G, ⊞)$ is not a NeutroH_v-semigroup because G is a H_v-semigroup. Clearly, there exists no element in G that satisfies falsify or the indeterminacy for NeutroH_v-semigroup.

Furthermore, based on [16], in the case of coat color inheritance of Shorthorn Cattle, suppose R, G, and W represent the colors of the Shorthorn Cattle coat, which respectively state red, reddish gray, and white. Let $K = \{R, G, W\}$, the result of $(K, ⊞)$ is given as follows.

Table 4. $(K, ⊞)$

$⊞$	R	G	W
R	R	$\{R, G\}$	G
G	$\{R, G\}$	K	$\{G, W\}$
W	G	$\{G, W\}$	W

Theorem 3.3 $(K, ⊞)$ is a NeutroHypergroup.
Proof. It is clear that $(K, ⊞)$ is not an Antihyperoperation. Next, $G ⊞ (G ⊞ G) = (G ⊞ G) ⊞ G = K$ and $R ⊞ (W ⊞ W) = G \neq \{G, W\} = (R ⊞ W) ⊞ W$. Then, $(K, ⊞)$ is a NeutroSemihypergroup. Furthermore, $R ⊞ K = K ⊞ R \neq K$ and $G ⊞ K = K ⊞ G = K$. Thus, $(G, ⊞)$ is a NeutroHypergroup.
Remark 3.4 $(K, ⊞)$ is not a NeutroH_v-semigroup. The reason is same as in Remark 3.2.

Next, we want to analyze NeutroHyperstructures in the inheritance of traits from blood groups including the ABO, MN, ABO with Rhesus, MN with Rhesus systems, and Kidd System.

The ABO blood group system was introduced by Karl Landsteiner in 1900 [18]. Based on [16], suppose $M = \{O, A, B, AB\}$ represents the set of ABO system blood groups. The result $(M, ⊞)$ is given as follows.

Table 5. $(M, ⊞)$

$⊞$	O	A	B	AB
O	O	$\{O, A\}$	$\{O, B\}$	$\{A, B\}$
A	$\{O, A\}$	$\{AB, O, A\}$	M	$\{AB, A, B\}$
B	$\{O, B\}$	M	$\{O, B\}$	$\{AB, A, B\}$
AB	$\{A, B\}$	$\{AB, A, B\}$	$\{AB, A, B\}$	$\{AB, A, B\}$

Theorem 3.5 $(M, ⊞)$ is a NeutroHypergroup.
Proof. It is clear that $(M, ⊞)$ is not an Antihyperoperation. Next, $O ⊞ (O ⊞ O) = (O ⊞ O) ⊞ O$ and, $O ⊞ (A ⊞ AB) = \{O, A, B\} \neq \{A, B, AB\} = (O ⊞ A) ⊞ AB$. Then, $(M, ⊞)$ is a NeutroSemihypergroup. Furthermore, $A ⊞ M = M ⊞ A = M$ and $O ⊞ M = M ⊞ O \neq M$. Thus, $(M, ⊞)$ is a NeutroHypergroup.

Theorem 3.6 Let $M' = \{O, A, B\}$ and $M" = \{O, B, AB\}$. Then $(M', ⊞)$ is a NeutroSubhypergroup and $(M", ⊞)$ is a NeutroSemihypergroup.

Proof. First, we want to show that $(M',\boxplus)$ is a NeutroSemihypergroup. It is clear that $(M',\boxplus)$ is not an Antihyperstructure. Next, from Theorem 3.5, we can deduce that $(M',\boxplus)$ is a NeutroSemihypergroup. Next, $O \boxplus M' = M' \boxplus O = M'$ and $B \boxplus M' = M' \boxplus B \neq M$. Thus, $(M',\boxplus)$ is a NeutroSubhypergroup. Next, we want to show that $(M",\boxplus)$ is a NeutroSemihypergroup. Next, $B \boxplus (B \boxplus B) = (B \boxplus B) \boxplus B$ and for every $O \boxplus (O \boxplus AB) = \{O, A, B\} \neq \{A, B\} = (O \boxplus O) \boxplus AB$.Thus, $(M",\boxplus)$ is a NeutroSemihypergroup.

Remark 3.7 $(M",\boxplus)$ is not a NeutroSubhypergroup since it does not satisfy the NeutroReproduction axiom.

Theorem 3.8 Let $M_3 = \{A, B, AB\}$. Then, $(M_3,\boxplus)$ is a NeutroSubhypergroup
Proof. The proof is similar to that of Theorem 3.5.

Furthermore, we want to include the rhesus factor in the ABO blood group system. The rhesus (Rh) blood group system was discovered by Karl Landsteiner and Alexander S. Wiener in 1940 [19]. Let $M = \{O, A, B, AB\}$ represent the set of ABO blood group system, and $R = \{Rh^+, Rh^-\}$ is represent the rhesus set. We obtain the ABO blood group set with rhesus

$$N = M \times R = \{O^-, O^+, A^-, A^+, B^-, B^+, AB^+, AB^-\}.$$

$(N,\boxplus)$ is presented by Table 8. Based on Table 8, we have the following result.

Theorem 3.9 $(N,\boxplus)$ is a NeutroHypergroup.
Proof. It is clear that $(N,\boxplus)$ is not an AntiHyperoperation. Next, $O^- \boxplus (O^- \boxplus O^-) = (O^- \boxplus O^-) \boxplus O^-$ and $O^+ \boxplus (AB^- \boxplus AB^-) = \{A^+, A^-B^+, B^-O^+, O^-\} \neq \{AB^+, AB^-, A^+, A^-, B^+, B^-\} = (O^+ \boxplus AB^-) \boxplus AB^-$. Then, $(N,\boxplus)$ is a NeutroSemihypergroup. Now $A^+ \boxplus N = N \boxplus A^+ = N$ and $B^- \boxplus N = N \boxplus B^- = N$. Thus, $(N,\boxplus)$ is a NeutroHypergroup.

Theorem 3.10 Let $N_1 = \{O^+, O^-, A^-\}$. Then, $(N_1,\boxplus)$ is a NeutroHypergroup.

Table 6. $(N_1,\boxplus)$

$\boxplus$	O^+	O^-	A^-
O^+	O^+ O^-	O^+ O^-	A^+ A^- O^+ O^-
O^-	O^+ O^-	O^-	A^- O^-
A^-	A^+ A^- O^+ O^-	A^- O^-	A^- O^-

Proof. $O^+ \boxplus (O^+ \boxplus A^-) = O^+ \boxplus \{A^+, A^-, O^+, O^-\} =$ undefined since $O^+ \boxplus A^+$ is undefined. So, $(N_1,\boxplus)$ satisfies the degree of indeterminacy axiom for NeutroAssociative. Therefore, $(N_1,\boxplus)$ is a NeutroSemihypergroup. To prove the NeutroReproduction Axiom, it is similar to Theorem 3.9. Thus, $(N_1,\boxplus)$ is a NeutroHypergroup.

Remark 3.11 Based on Table 8, it is obvious that $(\{O^+, O^-, B^-\},\boxplus)$ is a NeutroHypergroup.

Theorem 3.12 Let $N_2 = \{O^+, O^-, A^+, A^-, B^-\}$. Then, $(N_2,\boxplus)$ is a NeutroHypergroup.
Proof. The proof is similar to that of Theorem 3.10.

Theorem 3.13 Let $N_3 = \{O^+, O^-, A^+, A^-, B^+, B^-\}$. Then, $(N_3,\boxplus)$ is a NeutroHypergroup.
Proof. The proof is similar to that of Theorem 3.10.

Next, we want to investigate the NeutroHyperstructures related to the MN blood group. This blood group system was discovered by Karl Landsteiner and P. Levine in 1927 [15]. Suppose X is the set of possible blood types possessed by the marriage of two individuals, namely $X = \{M, N, MN\}$. $(X,\boxplus)$ is presented in Table 9.

Theorem 3.14 $(X,\boxplus)$ is a NeutroHypergroup.
Proof. The proof is similar to that of Theorem 3.10.

Furthermore, we want to include the rhesus factor in the MN blood group. Let $X = \{M, N, MN\}$ and $R = \{Rh^+, Rh^-\}$. We get $P = X \times R = \{M^+, M^-, N^+, N^-, MN^+, MN^-\}$. $(P,\boxplus)$ is presented in Table 10.

Theorem 3.15 $(P,\boxplus)$ is a NeutroHypergroup.
Proof. The proof is similar to that of Theorem 3.10.

Next, we want to analyze the NeutroHyperstructures contained in the Kidd blood group. Kidd blood group was discovered in 1951 in a patient named Mrs. Kidd [20]. The phenotypes of the Kidd blood group are as follows.

Table 7. Phenotypes of Kidd Blood Groups [20]

Phenotypes	**Frequency**
$Jk^{(a+b+)}$	**50% Caucasians**, 41% Blacks, 49% Asians
$Jk^{(a+b-)}$	26% Caucasians, **51% Blacks**, 23% Asians
$Jk^{(a-b+)}$	23% Caucasians, 8% Blacks, **27% Asians**
$Jk^{(a-b-)}$	**0.9% Polynesians**

Table 8. $(N,\boxplus)$

$\boxplus$	O^+	O^-	A^+	A^-	B^+	B^-	AB^+	AB^-
O^+	O^+ O^-	O^+ O^-	A^+ A^- O^+ O^-	A^+ A^- O^+ O^-	B^+ B^- O^+ O^-	B^+ B^- O^+ O^-	A^+ A^- B^+ B^-	A^+ A^- B^+ B^-
O^-	O^+ O^-	O^-	A^+ A^- O^+ O^-	A^- O^-	B^+ B^- O^+ O^-	B^- O^-	A^+ A^- B^+ B^-	A^- B^-
A^+	A^+ A^- O^+ O^-	A^+ A^- O^+ O^-	A^+ A^- O^+ O^-	A^+ A^- O^+ O^-	N	N	AB^+ AB^- A^+ A^- B^+ B^-	AB^+ AB^- A^+ A^- B^+ B^-
A^-	A^+ A^- O^+ O^-	A^- O^-	A^+ A^- O^+ O^-	A^- O^-	N	AB^- A^- B^- O^-	AB^+ AB^- A^+ A^- B^+ B^-	AB^- A^- B^-
B^+	B^+ B^- O^+ O^-	B^+ B^- O^+ O^-	N	N	B^+ B^- O^+ O^-	B^+ B^- O^+ O^-	AB^+ AB^- A^+ A^- B^+ B^-	AB^+ AB^- A^+ A^- B^+ B^-
B^-	B^+ B^- O^+ O^-	B^- O^-	N	AB^- A^- B^- O^-	B^+ B^- O^+ O^-	B^- O^-	AB^+ AB^- A^+ A^- B^+ B^-	AB^- A^- B^-
AB^+	A^+ A^- O^+ O^-	A^+ A^- O^+ O^-	AB^+ AB^- A^+ A^- B^+ B^-	AB^+ AB^- A^+ A^- B^+ B^-	AB^+ AB^- A^+ A^- B^+ B^-	AB^+ AB^- A^+ A^- B^+ B^-	AB^+ AB^- A^+ A^- B^+ B^-	AB^+ AB^- A^+ A^- B^+ B^-
AB^-	A^+ A^- O^+ O^-	A^- B^-	AB^+ AB^- A^+ A^- B^+ B^-	AB^- A^- B^-	AB^+ AB^- A^+ A^- B^+ B^-	AB^- A^- B^-	AB^+ AB^- A^+ A^- B^+ B^-	AB^- A^- B^-

Table 9. $(X,\boxplus)$

$\boxplus$	M	N	MN
M	M	M, MN	MN
N	M, MN	X	N, MN
MN	MN	N, MN	N

Table 10. $(P,\boxplus)$

$\boxplus$	M^+	M^-	N^+	N^-	MN^+	MN^-
M^+	M^+ M^-	M^+ M^-	MN^+ MN^-	MN^+ MN^-	M^+ M^- MN^+ MN^-	M^+ M^- MN^+ MN^-
M^-	M^+ M^-	M^-	MN^+ MN^-	MN^-	M^+ M^- MN^+ MN^-	M^- MN^-
N^+	MN^+ MN^-	MN^+ MN^-	N^+ N^-	N^+ N^-	N^+ N^- MN^+ MN^-	N^+ N^- MN^+ MN^-
N^-	MN^+ MN^-	MN^-	N^+ N^-	N^-	N^+ N^- MN^+ MN^-	N^- MN^-
MN^+	M^+ M^- MN^+ MN^-	M^+ M^- MN^+ MN^-	N^+ N^- MN^+ MN^-	N^+ N^- MN^+ MN^-	P	P
MN^-	M^+ M^- MN^+ MN^-	M^- MN^-	N^+ N^- MN^+ MN^-	N^- MN^-	P	M^- N^- MN^-

Furthermore, let $Y = \{Jk^{(a+b+)}, Jk^{(a+b-)}, Jk^{(a-b+)}, JK^{(a-b-)}\}$. The result of $(Y,\boxplus)$ is in Table 11. (Note : Here, $Jk^{(a+b+)} \boxplus Jk^{(a+b+)} = \{Jk^{(a+a+)}, Jk^{(a+b+)}\}$. We ignore $Jk^{(a+a+)}$ because it is not in the phenotypes. Here, $Jk^{(a+b+)} \boxplus Jk^{(a+b+)} = Jk^{(a+b+)}$).

Table 11. $(Y,\boxplus)$

$\boxplus$	$Jk^{(a+b+)}$	$Jk^{(a+b-)}$	$Jk^{(a-b+)}$	$Jk^{(a-b-)}$
$Jk^{(a+b+)}$	$Jk^{(a+b+)}$	$Jk^{(a+b+)}$ $Jk^{(a+b-)}$	$Jk^{(a+b+)}$ $Jk^{(a-b+)}$	$Jk^{(a+b-)}$ $Jk^{(a-b+)}$
$Jk^{(a+b-)}$	$Jk^{(a+b+)}$ $Jk^{(a+b-)}$	$Jk^{(a+b-)}$	$Jk^{(a+b+)}$ $Jk^{(a-b-)}$	$Jk^{(a+b-)}$ $Jk^{(a-b-)}$
$Jk^{(a-b+)}$	$Jk^{(a+b+)}$ $Jk^{(a-b+)}$	$Jk^{(a+b+)}$ $Jk^{(a-b-)}$	$Jk^{(a-b+)}$	$Jk^{(a-b-)}$ $Jk^{(a-b+)}$
$JK^{(a-b-)}$	$Jk^{(a+b-)}$ $Jk^{(a-b+)}$	$Jk^{(a+b-)}$ $Jk^{(a-b-)}$	$Jk^{(a-b-)}$ $Jk^{(a-b+)}$	$Jk^{(a-b-)}$

Theorem 3.16 $(Y,\boxplus)$ is a NeutroSemihypergroup.
Proof. It is clear that $(Y,\boxplus)$ is not an AntiHyperstructures. Next, $Jk^{(a+b+)} \boxplus \left(Jk^{(a+b+)} \boxplus Jk^{(a+b+)}\right) = \left(Jk^{(a+b+)} \boxplus Jk^{(a+b+)}\right) \boxplus Jk^{(a+b+)}$ and $Jk^{(a+b-)} \boxplus \left(Jk^{(a+b-)} \boxplus Jk^{(a-b+)}\right) = \left\{Jk^{(a+b+)}, Jk^{(a+b-)}, Jk^{(a-b-)}\right\} \neq \left\{Jk^{(a+b+)}, Jk^{(a-b-)}\right\} = \left(Jk^{(a+b-)} \boxplus Jk^{(a+b-)}\right) \boxplus Jk^{(a-b+)}$.
Thus, $(Y,\boxplus)$ is a NeutroSemihypergroup.

Remark 3.17 $(Y,\boxplus)$ is not a NeutroHypergroup since $(Y,\boxplus)$ doesn't satisfy the NeutroReproduction Axiom.

Theorem 3.18 Let $Y_1 = \{Jk^{(a+b+)}, Jk^{(a+b-)}, Jk^{(a-b+)}\}$. Then, $(Y_1,\boxplus)$ is a NeutroHypergroup
Proof. The proof is similar to that of Theorem 3.10.

Remark 3.19 It is clear that $(Y_2 = \{Jk^{(a+b+)}, Jk^{(a+b-)}, Jk^{(a-b-)}\}, \boxplus)$ and $(Y_3 = \{Jk^{(a+b+)}, Jk^{(a+b-)}, Jk^{(a-b-)}\}, \boxplus)$ are NeutroHypergroups.

4. Conclusions

Based on the previous explanations, we have investigated NeutroHyperstructures related to color inheritance in Mirabilis Jalapa and coat color, as well as the inheritance of blood types ABO, ABO with rhesus, MN, MN with rhesus, and the Kidd system. The types of NeutroHyperstructures obtained include NeutroHypergroup for Mirabilis Jalapa, coat color, ABO blood groups, ABO with rhesus, MN blood groups, and MN with rhesus and NeutroSemihypergroup for Kidd Blood Groups. For future research, we can investigate the types of NeutroHyperstructures in other fields.

5. Future Work

For future research, we can investigate the types of NeutroHyperstructures in other fields.

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A Novel Approach to the Algebraic Structure of Neutrosophic SuperHyper Algebra

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Abstract. Hyperalgebras and BCI algebras extend classical algebraic structures, and these specific structures offer tools and frameworks for studying various operations and logic in algebra. Many authors continued their research in hyperstructures to explore different logical algebras. SuperHyper Algebra is one of the significant advancements in algebra which has been developed recently. In this article, we propose a generalized concept, namely, SuperHyper BCI-Algebra, and investigate some of its properties. We also define SuperHyper Subalgebra, and some of its characteristics are examined. Finally, we extend our vision to Neutrosophic SuperHyper BCI-Algebra.

Keywords: Hyper BCI-Algebra, SuperHyper operation, SuperHyper Groupoid, SuperHyper BCI-Algebra, SuperHyper Subalgebra and Neutrosophic SuperHyper BCI-Algebra.

1. Introduction

The conceptual frameworks of fuzzy logic and fuzzy sets have been widely applied in several situations that involve uncertainty. This idea was first proposed by L. Zadeh [1]. As a result of an element's ambiguity or partial belongingness to the set, fuzzy sets are effective at dealing with uncertainty. Fuzzy set theory does not provide for hesitation or ambiguity in membership degrees. Atanassov [2] initiated the idea of intuitionistic fuzzy sets to include uncertainty in membership degrees. Still, some scenarios cannot address issues with incomplete data. Smarandache [3] developed neutrosophic set theory, a key factor dealing with indeterminacy. Sets containing elements that have independent degrees of truth, indeterminate and false memberships over the unit interval -]0, 1[+ are called Neutrosophic sets. Many applications of Neutrosophic logic have been developed, especially in Decision-Making difficulties. The following articles highlight the theoretical developments of Neutrosophic logic [4, 5], and its applications [6–9].

The investigation of BCK-algebras was initiated by K. Iseki in 1996 [10, 11], which extended the concepts of set-theoretic difference and propositional calculus. The algebraic representations of the set difference and its properties from set theory and the implicational functor from logical systems comprise the class of logical algebras known as BCI-algebras. They are highly related to numerous logical algebras and partially ordered commutative monoids. The combinators B, C, K, and I in combinatory logic are the origins of their names [12]. F. Marty [13] proposed the hyperstructure theory (also known as multialgebras), and numerous researchers have studied hyper BCK-algebras. The hyperstructure theory has now been applied to a wide range of mathematical structures, with applications in both pure and applied mathematics. Then, various researchers developed this new field. Many significant results emerged throughout the following decades, but one of the most blooming hyperstructures has been described since the 1970s. In [14], hyperstructures were applied to BCI-algebras and defined as hyper BCI-algebra as a generalization of a BCK-algebra and various related characteristics were examined. Hyper BCK-algebras is a natural evolution from classical BCK-algebras. In a standard BCK-algebra, combining two elements yields another element, whereas in a Hyper BCK-algebra, combining two elements results in a set. Also, many results have been studied on hyper BCK-algebras [15–17]. Recently, the area of hyperstructure theory has received a lot of attention. Hyper BCK/BCI-algebras were applied to various mathematical fields, such as topology, functional analysis, coding theory, group theory, etc. Since then, a significant research on the theory of BCK-algebras has been published in the literature. [18–21].

In [22], the authors extended the concept of fuzzy hyper BCK-subalgebras by introducing the concept of fuzzy BCK-subalgebras and formulated the notion of extendable fuzzy BCK-subalgebras. Also, the fuzzification of the implicative hyper BCK-ideals and their attributes are explored in [23]. In [24], the notion of Intuitionistic fuzzy hyper BCK-Ideals of Hyper BCK-Algebra was introduced. A study on intuitionistic fuzzy Lie sub-superalgebras and intuitionistic fuzzy ideals of Lie superalgebras was published in [25]. Many researchers have also investigated the intuitionistic fuzzification of ideals and subalgebras in BCK/BCI-algebras [26–30].

One of the developing fields during the last few decades is the study of the algebraic properties of neutrosophic logic. In [31], Neutrosophic BCI/BCK algebra was introduced. Neutrosophic subalgebra and ideals in BCK/BCI algebra were extensively discussed in [32–36]. Theoretical aspects concerned with introducing the concept of NeutroHyperGroups and presenting their basic properties and examples were discussed in [37]. The conceptions of the BMBJ-Neutrosophic Hyper-BCK-Ideals, MBJ-neutrosophic hyper BCK-ideal and MBJ neutrosophic strong and weak hyper BCK-ideal were investigated in [38, 39]. In [40], Smarandache initiated the most generalized algebra called SuperHyper Algebras, and in [41], Smarandache defined a SuperHyperGraph (SHG) and added the SuperVertices in classical HyperGraph. The most generalized algebra, SuperHyper Algebra, and numerous variations of Hyper structures were the inspiration for this study. In this paper,

- A generalized concept SuperHyper Algebra was extended to SuperHyper groupoids.
- A hybridization of BCI algebra and SuperHyper Algebra is explored as SuperHyper BCI-Algebra over a SuperHyper groupoids.
- A framework including SuperHyper subalgebras, and Neutrosophic SuperHyper BCI-algebras are discussed

The structure of the paper is organized as follows: In the preliminaries section, we present the fundamental concepts that are pertinent to this study. In the next section, we define SuperHyper groupoid and SuperHyper BCI-Algebra and investigate some of its characteristics. In section 4, SuperHyper subalgebra and its properties were discussed. Further, we generalize our perspective to Neutrosophic SuperHyper BCI-Algebra. Finally, we have summarized our key findings and suggested avenues for future research.

2. **Preliminaries**

Definition 2.1. [3] For any subset U of S, a neutrosophic set U on S is of the form $U = \{(\alpha, T_U(\alpha), I_U(\alpha), F_U(\alpha))|\alpha \in S\}$, where $T_U, I_U, F_U : S \to [0,1]$ represents, truth, indeterminacy and falsity membership functions respectively and $0 \leq T_U(\alpha) + I_U(\alpha) + F_U(\alpha) \leq 3$.

Definition 2.2. [14] Consider a not empty set B with the binary operation '$\circ$' and the constant 0. If the ensuing axioms are true, then $(B, \circ, 0)$ is known as a BCI-algebra,

(1) $((\upsilon_1 \circ \upsilon_2) \circ (\upsilon_1 \circ \upsilon_3)) \circ (\upsilon_3 \circ \upsilon_2) = 0$
(2) $(\upsilon_1 \circ (\upsilon_1 \circ \upsilon_2)) \circ \upsilon_2 = 0$
(3) $\upsilon_1 \circ \upsilon_1 = 0$
(4) $\upsilon_1 \circ \upsilon_2 = 0$ and $\upsilon_2 \circ \upsilon_1 = 0 \implies \upsilon_1 = \upsilon_2$

$\forall\, \upsilon_1, \upsilon_2, \upsilon_3 \in B$

Definition 2.3. [14] Consider a non-empty set B and $\Diamond : B \times B \to P^*(B)$ where $P^*(B)$ represents the power set of $B \setminus \{0\}$. Let P, $Q \subseteq B$, then the notation $P\Diamond Q$ is the collection $\bigcup\limits_{p \in P, q \in Q} p\Diamond q$. Then $(B, \Diamond)$ is said to be a hyper groupoid and $\Diamond$ is called a Hyperoperation on B. Also $r \ll s$ denotes $0 \in r\Diamond s$ and for any two subsets P, Q of B, $P \ll Q$ means that $\forall\, p \in P$, $\exists\, q \in Q$ such that $p \ll q$.

Definition 2.4. [14] A hyper groupoid $(B, \Diamond)$ with a constant element 0 is called a hyper BCI-algebra if it satisfies the following conditions:

(H1) $(\zeta\Diamond\eta)\Diamond(\eta\Diamond\theta) \ll \zeta\Diamond\eta$
(H2) $(\zeta\Diamond\eta)\Diamond\theta = (\zeta\Diamond\theta)\Diamond\eta$
(H3) $\zeta \ll \zeta$
(H4) $\zeta \ll \eta$ and $\eta \ll \zeta \implies \zeta = \eta$
(H5) $0\Diamond(0\Diamond\zeta) \ll \zeta$

$\forall\, \zeta, \eta, r \in B$

Notation 1. *[40] $P^n_*(B)$ denotes the k^{th} powerset of the set B and none of $P^i(B), i = 1, 2, ..., k$ contain the empty set Φ.*

Definition 2.5. [40] A classical - type Binary SuperHyper Operation $\circ^*_{(2,k)}$ is defined as follows: $\circ^*_{(2,k} : B^2 \rightarrow P^k_*(B)$, where $P^k_*(B)$ is the kth - power set of the set B, with no empty-set Φ.

3. **SuperHyper BCI-Algebra**

In this section, we define SuperHyper BCI-Algebra and investigate some of its characteristics. The following notation generalizes the notation $x \in B$ to the power sets. This notation will help to define algebraic structure in SuperHyper theory.

Notation 2. *For a set B and an element τ , $\tau \prec B$ denotes $\tau \in B$. By induction, for a collection $\mathcal{C} \in P^n_*(B)$ and an element τ, $\tau \prec \mathcal{C}$ denotes $\tau \prec \mathcal{X}$ for some $\mathcal{X} \in \mathcal{C}$.*

The following example provides an overview of the above notation.

Example 3.1. Let $B = \{0, 1, 2, 3\}$ and $\mathcal{C} = \{\{\{0\}, \{1, 3\}\}, \{\{1\}, \{4, 3\}\}, \{\{1, 3\}\}\} \in P^3_*(B)$ then $0 \prec \mathcal{C}$, since $0 \in \{0\}$, $0 \prec \{\{0\}\} \in \mathcal{C}$. Similarly $1 \prec \mathcal{C}$, $3 \prec \mathcal{C}$ and $4 \prec \mathcal{C}$ but $2 \not\prec \mathcal{C}$ since 2 is not in any of the collection of $\mathcal{C}$.

Smarandache [40] has introduced the algebraic operations in SuperHyper theory. Here, an algebraic operation in SuperHyper groupoid was investigated.

Definition 3.2. Let S denote a non-empty set and $\oslash$ be a SuperHyper operation on S defined as a function $\oslash : S \times S \rightarrow P^n_*(S)$. Here $P^n_*(S)$ denotes the nth powerset of the set $S \backslash \Phi$. For any two subsets X and Y of S, $X \oslash Y$ is denotes the collection $\bigcup\limits_{\mathfrak{x} \in X, \mathfrak{y} \in Y} \mathfrak{x} \oslash \mathfrak{y}$ and for any two collection $\mathcal{C}$ and $\mathcal{D}$ of $P^i_*(S)$, $\mathcal{C} \oslash \mathcal{D}$ denotes the collection $\bigcup\limits_{\mathcal{X} \in \mathcal{C}, \mathcal{Y} \in \mathcal{D}} \mathcal{X} \oslash \mathcal{Y}$, $\forall i = 1, 2, ..., n$.

The set S with a SuperHyper operation $\oslash$ and all those above said notations is called as SuperHyper groupoid and denoted by $(S, \oslash)$.

Notation 3. *Furthermore, we say $\tau \ll \rho$ if $0 \prec \tau \oslash \rho \in P^{n-1}_*(S)$. For all $\mathcal{X}$, $\mathcal{Y} \subseteq S$, $\mathcal{X} \ll \mathcal{Y}$ represents that $\forall\, \mathfrak{x} \in \mathcal{X}$, $\exists\, \mathfrak{y} \in \mathcal{Y}$ such that $\mathfrak{x} \ll \mathfrak{y}$ and for every $\mathcal{C}$ and $\mathcal{D} \in P^i_*(S)$, $\mathcal{C} \ll \mathcal{D}$ means that for every $\mathcal{X} \in \mathcal{C}$, $\exists\, \mathcal{Y} \in \mathcal{D}$ such that $\mathcal{X} \ll \mathcal{Y}$, $\forall i = 1, 2, ..., n$.*

Definition 3.3. A SuperHyper groupoid $(S, \oslash)$ that contains a constant 0 is described as a SuperHyper BCI-algebra under the following conditions:

(SH1) $(\kappa \oslash \mu) \oslash (\lambda \oslash \mu) \ll \kappa \oslash \lambda$

(SH2) $(\kappa \oslash \lambda) \oslash \mu = (\kappa \oslash \mu) \oslash \lambda$

(SH3) $\kappa \ll \kappa$

(SH4) $\kappa \ll \lambda$ and $\lambda \ll \kappa \implies \kappa = \lambda$

(SH5) $0 \oslash (0 \oslash \kappa) \ll \kappa$

for every $\kappa, \lambda, \mu \in S$

Example 3.4. Let $S = \{0, 1\}$ and $\oslash : S \times S \to P_*^2(S)$. Consider the table below:

$\oslash$	0	1
0	{{0}, {0,1}}	{{0},{1}}
1	{{1}}	{{0},{1},{0,1}}

Then $(S, \oslash)$ is a SuperHyper BCI-algebra.

The examples below illustrate the existence of SuperHyper BCI-algebra.

Example 3.5. Suppose we have a hyper BCI-Algebra $(S, \Diamond, 0)$. A SuperHyper operation $\oslash$ on S is defined as $\tau \oslash \rho = P_*^{n-1}(\tau \Diamond \rho)$ for all $\tau, \rho \in S$. Here $(S, \oslash)$ forms a SuperHyper BCI-algebra.

Example 3.6. Let $\oslash : S \times S \to P_*^n(S)$ be a SuperHyper operation on $S = [0, \infty)$. Then we define $(\tau \oslash \rho)$ as

$$(\tau \oslash \rho) = \begin{cases} P_*^{n-1}[0, \tau], & \text{if } \tau \leq \rho \\ P_*^{n-1}(0, \rho], & \text{if } \tau > \rho \neq 0 \\ P_*^{n-1}\{\tau\} = \{\tau\}, & \text{if } \rho = 0 \end{cases}$$

for all $\tau, \rho \in S$. Then $(S, \oslash)$ is a SuperHyper BCI-algebra.

The following theorem discusses some characteristics of SuperHyper BCI-algebra.

Proposition 3.7. *In any SuperHyper BCI-algebra, the following holds.*

(i) $\mu \ll 0 \implies \mu = 0$

(ii) $0 \prec \mu \oslash (\mu \oslash 0)$

(iii) $\mu \ll \mu \oslash 0$

(iv) $0 \oslash (\mu \oslash \lambda) \ll \lambda \oslash \mu$

(v) $\mathcal{X} \ll \mathcal{X}$

(vi) $\mathcal{X} \subseteq \mathcal{Y} \implies \mathcal{X} \ll \mathcal{Y}$

(vii)$\mathcal{X} \ll P_*^i(\{0\}) \implies \mathcal{X} = P_*^i(\{0\}) \, \forall \mathcal{X} \subseteq P_*^{i-1}(S)$

(viii) $\mu \oslash 0 \ll P_*^n(\{\lambda\}) \implies \mu \ll \lambda$

(ix) $\mu \oslash \lambda = P_*^n(\{0\}) \implies (\mu \oslash \kappa) \oslash (\lambda \oslash \kappa) = P_*^n(\{0\})$ *and* $\mu \oslash \kappa \ll \lambda \oslash \kappa$

(x) $\mathcal{X} \oslash P_*^i(\{0\}) = P_*^n(\{0\}) \implies \mathcal{X} = P_*^i(\{0\}) \forall \mathcal{X} \subseteq P_*^{i-1}(\{0\})$

(xi) $(\mathcal{X} \oslash \mathcal{Y}) \oslash \mathcal{C} = (\mathcal{X} \oslash \mathcal{C}) \oslash \mathcal{Y}$

for all $\mu, \lambda, \kappa \in S$ *and for all non-empty subsets* $\mathcal{X}, \mathcal{Y}$ *and* $\mathcal{C}$ *of* $P_*^i(S), i = 1, 2, ...n$

Proof :

(i) Let $\mu \ll 0$, then $0 \prec \mu \oslash 0$. By (SH3), $0 \ll 0$, so that $0 \prec 0 \oslash (\mu \oslash 0)$. Also, $(0 \oslash 0) \oslash (\mu \oslash 0) \ll 0 \oslash \mu$, which implies $0 \ll 0 \oslash \mu$. Hence, $0 \prec 0 \oslash (0 \oslash \mu)$. Now, by (SH5), $0 \prec 0 \oslash (0 \oslash \mu) \ll \mu$.Then $0 \ll x$. Therefore by (SH4), $\mu = 0$

(ii) Since $\mu \ll \mu$, $0 \prec (\mu \oslash 0) \oslash (\mu \oslash 0) = (\mu \oslash (\mu \oslash 0)) \oslash 0$. Then $\exists$, a λ from $\mu \oslash (\mu \oslash 0)$ such that $\lambda \ll 0$. By (i) $\lambda = 0$. Hence $0 \prec \mu \oslash (\mu \oslash 0)$.

(iii) By (ii) it follows.

(iv) Since $\lambda \ll \lambda \implies 0 \oslash (\mu \oslash \lambda) \subseteq (\lambda \oslash \lambda) \oslash (\mu \oslash \lambda) \ll \lambda \oslash \mu$

(v) Since $\mu \ll \mu \implies \mathcal{X} \ll \mathcal{X}\ \forall, \mathcal{X} \in P_*^1(S)$. By induction it follwos that $\mathcal{X} \ll \mathcal{X}, \forall \mathcal{X} \in P_*^i(S), i = 2, ...n$

(vi) Trivial.

(vii) Let $\mathcal{X} \ll P_*^i\{0\}$ and let $\mu \in \mathcal{X}$. Then $\mu \ll 0 \implies \mu = 0$. Therefore $\mathcal{X} = P_*^i\{0\}$.

(viii) We know that $0 \prec (\mu \oslash 0) \oslash \lambda$= $(\mu \oslash \lambda) \oslash 0$, so there exists $\mu \prec \mu \oslash \lambda$ such that $0 \prec \mu \oslash 0$, ie, $\mu \ll 0$, which implies that $\mu = 0 \prec \mu \oslash \lambda$ by(i). Hence $\mu \ll \lambda$.

(ix) Let $\lambda \ll \kappa$ .Then $(\mu \oslash \kappa) \oslash 0 \subseteq (\mu \oslash \kappa) \oslash (\mu \oslash \lambda) \ll \mu \oslash \lambda$ (by SH1).Hence, $(\mu \oslash \kappa) \oslash 0 \ll \mu \oslash \lambda$. This means that for each $\mathfrak{a} \prec \mu \oslash \kappa$, $\exists\ \mathfrak{b} \prec (\mu \oslash \lambda)$ such that $\mathfrak{a} \oslash 0 \ll \{\mathfrak{b}\}$. Hence by (vii), $\mathfrak{a} \oslash \mathfrak{b}$.Hence $\mu \oslash \kappa \ll \mu \oslash \lambda$.

(x) It follows from (i).

(xi) It follows from (SH2)

4. SuperHyper Subalgebra

In this segment, we define SuperHyper subalgebra and examine a few of its characteristics.

Definition 4.1. Let $(S, \oslash)$ denotes the SuperHyper BCI-Algebra and $S' \subset S$ such that $0 \in S'$. If S' is a SuperHyper BCI-Algebra corresponding to the SuperHyper operation $\oslash$ on S, then S' is called as SuperHyper subalgebra of S.

Theorem 4.2. *Let S' be the subset of a SuperHyper BCI-algebra $(S, \oslash)$ such that $S' \neq \emptyset$. Then S' is a SuperHyper subalgebra of S iff the restricted map $\oslash|_{S'} : S' \times S' \to P_*^n(S')$ is a binary SuperHyper operation.*

Proof: $(\Rightarrow)$ Obvious.
$(\Leftarrow)$ It is easy to verify (SH1), (SH2),(SH3),(SH4) & (SH5). Hence it is need to show that $0 \in S'$. Since $\oslash|S'$ is a binary SuperHyper operation, $\tau \oslash \rho \subseteq P_*^{n-1}(S')\ \forall \tau, \rho \in S'$. Then $\tau \ll \tau\ \forall \tau \in S'$, we have $0 \prec \tau \oslash \tau$. Hence $0 \prec P_*^{n-1}(S')$ ie., $0 \in S'$.

Example 4.3. Let $(S, \oslash)$ be a SuperHyper BCI-algebra as in example 3.6 and let $S' = [0, \mathfrak{a}]$ for every $\mathfrak{a} \in [0, \infty)$. Then $(S', \oslash)$ is a SuperHyper subalgebra.

Proof: For any $\tau, \rho \in S'$,

$$(\tau \oslash \rho) = \begin{cases} P_*^{n-1}[0,\tau], & \text{if } \tau \leq \rho \\ P_*^{n-1}(0,\rho], & \text{if } \tau > \rho \neq 0 \\ P_*^{n-1}\{\tau\} = \{\tau\}, & \text{if } \rho = 0 \end{cases}$$

Clearly, $P_*^{n-1}[0,\tau]$, $P_*^{n-1}(0,\rho]$ and $P_*^{n-1}\{\tau\}$ are subsets of $P_*^n([0,\infty))$. Hence $\oslash|_{S'}$ is a binary SuperHyper operation from $S' \times S' \to P_*^n(S')$.

Theorem 4.4. *Consider a SuperHyper BCI-algebra $(S, \oslash)$. Then the set*

$$K(S) = \{\tau \oslash S | 0 \oslash \tau = P_*^n(\{0\})\}$$

is a SuperHyper subalgebra of S whensoever $K(S) \neq \Phi$.

Proof: Let $\tau, \rho \in K(S)$ and $\mathfrak{a} \prec \tau \oslash \rho$. Then $0 \oslash (\tau \oslash \rho) = (0 \oslash \rho) \oslash (\tau \oslash \rho) \ll 0 \oslash \tau = 0$. Therefore by 3.7 (vii) $0 \oslash (\tau \oslash \rho) = \{0\}$. Hence $\tau \oslash \rho \subseteq K(S)$. Hence by theorem 4.2 $K(S)$ is a non-empty SuperHyper subalgebra.

Theorem 4.5. *Suppose there is a SuperHyper BCI-algebra defined by $(S, \oslash)$. Then $S'_1 = \{\tau \in S | \tau \oslash (\tau \oslash 0) = 0\}$ is a SuperHyper subalgebra of S whenever $S'_1 \neq \Phi$.*

Proof: Proof follows from the proposition 3.7 and theorem 4.2.

Theorem 4.6. *Let $(S, \oslash)$ be a SuperHyper BCI-algebra. If $S'_2 = \{\tau \in S | 0 \oslash x = P_{n-1}^*(\{0\})\}$ is non-empty then S'_2 is SuperHyper subalgebra.*

Proof: Let $\tau, \rho \in S'_2$. Then $0 \oslash \tau = P_*^{n-1}(\{0\})$ and $0 \oslash \rho = P_*^{n-1}(\{0\})$. Now, $0 \oslash (\tau \oslash \rho) = (0 \oslash \rho) \oslash (\tau \oslash \rho) \ll 0 \oslash \tau = P_*^{n-1}(\{0\})$. Hence by proposition 3.7 (viii) $0 \oslash (\tau \oslash \rho) = P_*^{n-1}(\{0\})$. Therefore, for any $\mathfrak{a} \prec \tau \oslash \rho$, $0 \oslash \mathfrak{a} = P_*^{n-1}(\{0\})$. ie., $\mathfrak{a} \in S'_2$ which implies $\tau \oslash \rho \subseteq P_*^{n-1}(S'_2)$.

5. Neutrosophic SuperHyper BCI-Algebra

In this section, the concept of Neutrosophic SuperHyper BCI-Algebra is introduced.

Definition 5.1. For any neutrosophic set $A =< T_A, I_A, F_A >$ in S we define the following notations.
(i) Let B be a subset of S. Then

$$T_B = inf\{T(\tau)|\tau \in B\}, I_B = inf\{I(\tau)|\tau \in B\}, F_B = inf\{F(\tau)|\tau \in B\}.$$

(ii) Let $\mathcal{Y}$ be an element in $P_i^*(S)$. Then

$$T_{\mathcal{Y}} = inf\{T(\mathcal{X})|\mathcal{X} \in \mathcal{Y}\}, I_{\mathcal{Y}} = inf\{I(\mathcal{X})|\mathcal{X} \in \mathcal{Y}\}, F_{\mathcal{Y}} = inf\{F(\mathcal{X})|\mathcal{X} \in \mathcal{Y}\}.$$

The following defines neutrosophic SuperHyper BCI-algebra.

Definition 5.2. In the realm of SuperHyper BCI-algebras, let S be the designated algebraic structure, and consider a neutrosophic set $U =< T_U, I_U, F_U >$ within S. We define U as a neutrosophic SuperHyper BCI-algebra of S when it adheres to the ensuing conditions for all elements τ and ρ in S.
$T(\tau \oslash \rho) \geq min(T(\tau), T(\rho)), I(\tau \oslash \rho) \geq min(I(\tau), I(\rho))$ & $F(\tau \oslash \rho) \leq max(F(\tau), F(\rho))$.

Example 5.3. Let $S = \{0, 1\}$ and $\oslash : S \times S \to P_*^2(S)$. Consider the table below:

$\oslash$	0	1
0	{{0}}	{{0}}
1	{{1}}	{{0},{1},{0,1}}

Then $(S, \oslash)$ is a SuperHyper BCI-algebra.We characterize a neutrosophic set A on S by

S	$T_A(\tau)$	$I_A(\tau)$	$F_A(\tau)$
0	0.71	0.63	0.18
1	0.53	0.42	0.67

Here $T(0 \oslash 1) = T(\{\{0\}\}) = 0.71 \geq min(T(0), T(0)) = 0.53$, Similarly we can verify for other values. Therefore A is a neutrosophic SuperHyper BCI-algebra.

Example 5.4. Consider the SuperHyper BCI-algebra as in example 3.4 and a neutrosophic set defined in the example 5.3. Here A is not Neutrosophic SuperHyper BCI-algebra because $T(0 \oslash 0) = T(\{\{0\}, \{0, 1\}\}) = 0.53 \ngeq min(T(0), T(0)) = 0.71$.

Proposition 5.5. *Let A be Neutrosophic SuperHyper BCI-algebra of S then the following holds. If $A =< T_A, I_A, F_A >$ then there exist τ, ρ and $\gamma \in S$ such that (i) $T_A(0) \geq T_A(\tau)$, (ii)$I_A(0) \geq I_A(\rho)$ and (iii)$F_A(0) \leq F_A(\gamma)$*

Proof:(i) By proposition 3.7 $\tau \ll 0$, then $\tau = 0$, so that $0 \not\prec \tau \oslash 0$, $\forall \tau \neq 0$. By definition $T(\tau \oslash 0) \geq min(T(\tau), T(0))$.
Case:1 Suppose $T(\tau) \leq T(0)$, then it is proved.
Case:2 If $T(\tau) > T(0), T(\tau \oslash 0) \geq T(0)$, then $\exists\ \kappa \prec \tau \oslash 0$ such that $T(\kappa) \leq T(0)$.
Similarly, we can prove for Indeterminacy and Falsity membership.

6. Conclusion

We have introduced the SuperHyper Groupoid, SuperHyper BCI algebra, SuperHyper Subalgebra, and Neutrosophic SuperHyper BCI-Algebra, which are the most extensive forms of algebras. With appropriate examples, we have discussed the characterizations of SuperHyper BCI algebra and SuperHyper subalgebras. Finally, we expanded our notion to Neutrosophic SuperHyper BCI-Algebra. These ideas will pave the way for additional theoretical research on SuperHyper theory. In [42] the extensions soft set to the HyperSoft Set, IndetermSoft Set, IndetermHyperSoft Set, and TreeSoft Set, and their practical applications are highlighted. The generalized notion of SuperHyper BCI algebra can be used to explore more hyper algebraic structures in the future, and it can be extended to rough sets, soft sets and extensions of soft sets.

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The symbolic plithogenic differentials calculus

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Abstract: In this paper, we were keen to present the concept of the symbolic plithogenic differentials calculus, Where the symbolic plithogenic differentiable is defined. In addition, properties of the symbolic plithogenic differentiation are introduced. Also, we prove the derivation rules of the symbolic plithogenic functions.

Keywords: symbolic plithogenic differentials; symbolic plithogenic functions; derivative symbolic plithogenic functions.

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1. Introduction and Preliminaries

To The genesis, origination, formation, development, and evolution of new entities through dynamics of contradictory and/or neutral and/or noncontradictory multiple old entities is known as plithogenic. Plithogeny advocates for the integration of theories from several fields. We use numerous "knowledges" from domains like soft sciences, hard sciences, arts and literature theories, etc. as "entities" in this study, this is what Smarandache introduced, as he presented a study on plithogeny, plithogenic set, logic, probability, and statistics [2], in addition to presenting introduction to the symbolic plithogenic algebraic structures (revisited), through which he discussed several ideas, including mathematical operations on plithogenic numbers [1]. Also, an overview of plithogenic set and symbolic plithogenic algebraic structures was discussed by him [3]. It is thought that the symbolic n-plithogenic sets are a good place to start when developing algebraic extensions for other classical structures including rings, vector spaces, modules, and equations [4-5-6-7].

Alhasan also presented several papers on calculus, in which he discussed neutrosophic definite and indefinite integrals. He also presented the most important applications of definite integrals in neutrosophic logic [8-9].

Integration is important in human life, and one of its most important applications is the calculation of area, size and arc length. In our reality we find things that cannot be precisely defined, and that contain an indeterminacy part. This is the reason for studying neutrosophic integration and methods of its integration in this paper.

Smarandache presented the division operation in the plithogenic field as follows [1]:

Division of Symbolic Plithogenic Components

$$\frac{P_i}{P_j} = \begin{cases} x_0 + x_1P_1 + x_2P_2 + \cdots + x_jP_j + P_i & x_0 + x_1 + x_2 + \cdots + x_j = 0 \quad i > j \\ x_0 + x_1P_1 + x_2P_2 + \cdots + x_iP_i & x_0 + x_1 + x_2 + \cdots + x_i = 1 \quad i = j \\ \emptyset & i < j \end{cases}$$

where all coefficients $x_0, x_1, x_2, \dots, x_i, \dots \in$ SPS.

Division of Symbolic Plithogenic Numbers

Let consider two symbolic plithogenic numbers as below:

$$PN_r = a_0 + a_1P_1 + a_2P_2 + \cdots + a_rP_r$$

$$PN_s = b_0 + b_1P_1 + b_2P_2 + \cdots + b_sP_s$$

$$\frac{PN_r}{PN_s} = \begin{cases} none, one \ \ many & r \geq s \\ \emptyset & r < s \end{cases}$$

This study covered a number of subjects; in the first, which included an introduction and information of plithogenic filed. We presented the symbolic plithogenic differentials calculusin the main discussion section. The paper's conclusion is provided in the final.

Main Discussion

The symbolic plithogenic differentials

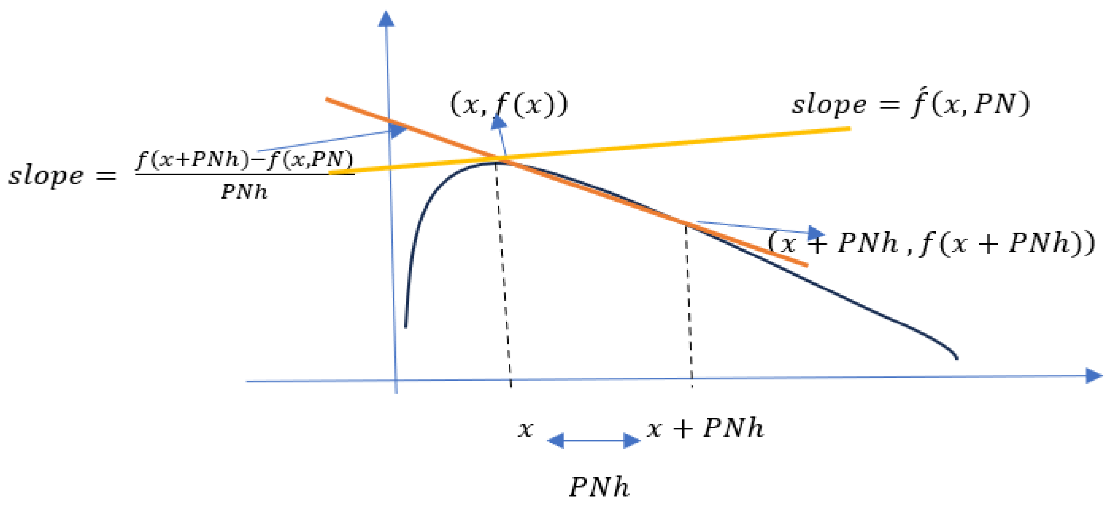
Definition 1

Let $f: SPS \to SPS$, if:

$$\lim_{PNh \to 0} \frac{f(x + PNh) - f(x, PN)}{PNh}$$

exist, then we say that the function $f(x, PN)$ is differentiable with respect to x and it is given by the formula:

$$\acute{f}(x, PN) = \lim_{PNh \to 0} \frac{f(x + PNh) - f(x, PN)}{PNh}$$



where $PNh = h_0 + h_1P_1 + h_2P_2 + \cdots + h_nP_n \in SPS$ is amount of small change in x.

then, $PNh \to 0$ is equivalent to: $h_0 \to 0$, $h_1 \to 0$, $h_2 \to 0$, ….. , and $h_n \to 0$

Notes:

1) The tangent slop to $f(x, PN)$ at $x_0 \in SPS$, where $x_0 = a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n$, is:

$$m_{PN} = \acute{f}(a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n).$$

2) The equation of the tangent to $f(x, PN)$ at $x_0 = a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n$ is:

$$y - f(a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n) = \acute{f}(a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n)(x - a_0 - a_1P_1 - a_2P_2 - \cdots - a_nP_n)$$

Example 1

Differentiate $f(x, PN) = (3P_2 + 2)x^2$ with respect to x using definition, and find an equation of the tangent line to the curve at $x_0 = P_1 + 1$

solution:

$$\acute{f}(x, PN) = \lim_{PNh \to 0} \frac{f(x + PNh) - f(x, PN)}{PNh}$$

$$\acute{f}(x, PN) = \lim_{PNh \to 0} \frac{(3P_2 + 2)(x + PNh)^2 - (3P_2 + 2)x^2}{PNh}$$

$$\acute{f}(x, PN) = \lim_{PNh \to 0} \frac{(3P_2 + 2)(x^2 + 2(PNh)x + (PNh)^2) - (3P_2 + 2)x^2}{PNh}$$

$$= \lim_{PNh \to 0} \frac{(3P_2 + 2)x^2 + (3P_2 + 2)(2(PNh)x + (PNh)^2) - (3P_2 + 2)x^2}{PNh}$$

$$= \lim_{PNh \to 0} \frac{PNh(3P_2 + 2)(2x + PNh)}{PNh}$$

$$= \lim_{PNh \to 0} (3P_2 + 2)(2x + PNh)$$

$$= (3P_2 + 2)(2x + 0)$$

$$\Rightarrow \qquad \acute{f}(x, PN) = (6P_2 + 4)x$$

Let's find the tangent equation:

$$m_{PN} = \acute{f}(P_1 + 1) = (6P_2 + 4)(P_1 + 1) = 12P_2 + 4P_1 + 4$$

$$f(P_1 + 1) = (P_1 + 1)^2 = 3P_1 + 1$$

then:

$$y - f(P_1 + 1) = \acute{f}(P_1 + 1)(x - P_1 - 1)$$

$$y - 3P_1 - 1 = (12P_2 + 4P_1 + 4)(x - P_1 - 1)$$

$$y - 3P_1 - 1 = (12P_2 + 4P_1 + 4)x + (12P_2 + 4P_1 + 4)(-P_1 - 1)$$

$$y - 3P_1 - 1 = (12P_2 + 4P_1 + 4)x - 12P_2 - 4P_1 - 4P_1 - 12P_2 - 4P_1 - 4$$

$$y - 3P_1 - 1 = (12P_2 + 4P_1 + 4)x - 24P_2 - 12P_1 - 4$$

$$y = (12P_2 + 4P_1 + 4)x - 24P_2 - 9P_1 - 4$$

Example 2

Differentiate $f(x, PN) = sin\big((P_4 - 5P_1 + 7)x + 4P_1\big)$ with respect to x using definition.

solution:

$$\acute{f}(x, PN) = \lim_{PNh \to 0} \frac{f(x + PNh) - f(x, PN)}{PNh}$$

$$\acute{f}(x, I) = \lim_{PNh \to 0} \frac{sin\big((P_4 - 5P_1 + 7)(x + PNh) + 4P_1\big) \ - sin\big((P_4 - 5P_1 + 7)x + 4P_1\big)}{PNh}$$

$$= \lim_{PNh \to 0} \frac{sin\big((P_4 - 5P_1 + 7)x + 4P_1 + (P_4 - 5P_1 + 7)(x + PNh)\big) \ - sin\big((P_4 - 5P_1 + 7)x + 4P_1\big)}{PNh}$$

$$= \lim_{PNh \to 0} \frac{cos\left((P_4 - 5P_1 + 7)x + 4P_1 + \frac{(P_4 - 5P_1 + 7)}{2} PNh\right) sin\left(\frac{(P_4 - 5P_1 + 7)}{2} PNh\right)}{\frac{PNh}{2}}$$

$$= \lim_{PNh \to 0} cos\left((P_4 - 5P_1 + 7)x + 4P_1 + \frac{(P_4 - 5P_1 + 7)}{2} PNh\right) \lim_{PNh \to 0} \frac{sin\left(\frac{(P_4 - 5P_1 + 7)}{2} PNh\right)}{\frac{PNh}{2}}$$

$$= \lim_{PNh \to 0} cos\left((P_4 - 5P_1 + 7)x + 4P_1 + \frac{(P_4 - 5P_1 + 7)}{2} PNh\right) \lim_{PNh \to 0} \frac{(P_4 - 5P_1 + 7) sin\left(\frac{(P_4 - 5P_1 + 7)}{2} PNh\right)}{\frac{(P_4 - 5P_1 + 7)PNh}{2}}$$

$$= cos\big((P_4 - 5P_1 + 7)x + 4P_1\big)(P_4 - 5P_1 + 7) \quad (1)$$

$$\Rightarrow \qquad \acute{f}(x, PN) = (P_4 - 5P_1 + 7)\, cos\big((P_4 - 5P_1 + 7)x + 4P_1\big)$$

Example 3

Differentiate $f(x, I) = \sqrt{7P_5 x - 3P_1 + 1}$ with respect to x using definition.

solution:

$$\acute{f}(x, PN) = \lim_{PNh \to 0} \frac{f(x + PNh) - f(x, PN)}{PNh}$$

$$\acute{f}(x, PN) = \lim_{PNh \to 0} \frac{\sqrt{7P_5(x + PNh) - 3P_1 + 1} - \sqrt{7P_5 x - 3P_1 + 1}}{PNh}$$

$$= \lim_{PNh\to 0} \frac{\sqrt{7P_5(x+PNh)-3P_1+1}-\sqrt{7P_5x-3P_1+1}}{PNh} \frac{\sqrt{7P_5(x+PNh)-3P_1+1}+\sqrt{7P_5x-3P_1+1}}{\sqrt{7P_5(x+PNh)-3P_1+1}+\sqrt{7P_5x-3P_1+1}}$$

$$= \lim_{PNh\to 0} \frac{7P_5(x+PNh)-3P_1+1-7P_5x+3P_1-1}{PNh\left(\sqrt{7P_5(x+PNh)-3P_1+1}+\sqrt{7P_5x-3P_1+1}\right)}$$

$$= \lim_{PNh\to 0} \frac{7P_5x+(7P_5)PNh-3P_1+1-7P_5x+3P_1-1}{PNh\left(\sqrt{7P_5(x+PNh)-3P_1+1}+\sqrt{7P_5x-3P_1+1}\right)}$$

$$= \lim_{PNh\to 0} \frac{(7P_5)PNh}{PNh\left(\sqrt{7P_5(x+PNh)-3P_1+1}+\sqrt{7P_5x-3P_1+1}\right)}$$

$$= \lim_{PNh\to 0} \frac{7P_5}{\sqrt{7P_5(x+PNh)-3P_1+1}+\sqrt{7P_5x-3P_1+1}}$$

$$\Rightarrow \qquad \acute{f}(x,PN) = \frac{7P_5}{2\sqrt{7P_5x-3P_1+1}}$$

The rules of the symbolic plithogenic derivatives

Let $PN_s = a_0 + a_1P_1 + a_2P_2 + \cdots + a_rP_r$, $PN_r = b_0 + b_1P_1 + b_2P_2 + \cdots + b_sP_s \in SPS$, then we can prove each of the following, using the definition 1:

1) $\frac{d}{dx}[PC] = 0$; where $PC = c_0 + c_1P_1 + c_2P_2 + \cdots + c_rP_r$ is symbolic plithogenic constant.

2) $\frac{d}{dx}[PN_sx + PN_r] = PN_s$

3) $\frac{d}{dx}[PN_sx^n] = nPN_sx^{n-1}; n \ is$ real number.

4) $\frac{d}{dx}[e^{PN_sx+PN_r}] = PN_se^{PN_sx+PN_r}$

5) $\frac{d}{dx}(PN_r)^x = (PN_r)^x \ ln(PN_r)$; where $PN_r > 0$

6) $\frac{d}{dx}\left[PN_r \, log_{PN_s} x\right] = \frac{PN_r}{x \ ln(PN_s)}$; where $PN_s > 0$, and $\frac{PN_r}{ln(PN_s)}$ is divisible.

7) $\frac{d}{dx}[ln(PN_sx + PN_r)] = \frac{PN_s}{PN_sx + PN_r}$

8) $\frac{d}{dx}\left[\sqrt{PN_sx + PC}\right] = \frac{PN_s}{2\sqrt{PN_sx + PC}}$

9) $\frac{d}{dx}[sin(PN_sx + PN_r)] = PN_scos(PN_sx + PN_r)$

10) $\frac{d}{dx}[cos(PN_sx + PN_r)] = -PN_ssin(PN_sx + PN_r)$

11) $\frac{d}{dx}[tan(PN_s x + PN_r)] = PN_s sec^2(PN_s x + PN_r)$

12) $\frac{d}{dx}[cot(PN_s x + PN_r)] = -PN_s csc^2(PN_s x + PN_r)$

13) $\frac{d}{dx}[sec(PN_s x + PN_r)] = PN_s sec(PN_s x + PN_r) tan(PN_s x + PN_r)$

14) $\frac{d}{dx}[csc(PN_s x + PN_r)] = -PN_s csc(PN_s x + PN_r) cot(PN_s x + PN_r)$

Proof (3)

$$\frac{d}{dx}[PN_s x^n] = \lim_{PNh \to 0} \frac{f(x + PNh) - f(x, PN)}{PNh}$$

$$= \lim_{PNh \to 0} \frac{PN_s (x + PNh)^n - PN_s x^n}{PNh}$$

$$= \lim_{PNh \to 0} \frac{\left[PN_s x^n + nPN_s x^{n-1}(PNh) + \frac{n(n-1)}{2!} PN_s x^{n-2}(PNh)^2 + \cdots + nPN_s x(PNh)^{n-1} + (PNh)^n\right] - PN_s x^n}{PNh}$$

$$= \lim_{PNh \to 0} \left[\frac{nPN_s x^{n-1}(PNh) + \frac{n(n-1)}{2!} PN_s x^{n-2}(PNh)^2 + \cdots + nPN_s x(PNh)^{n-1} + (PNh)^n}{PNh}\right]$$

$$= \lim_{PNh \to 0} \left[nPN_s x^{n-1} + \frac{n(n-1)}{2!} PN_s x^{n-2}(PNh) + \cdots + nPN_s x(PNh)^{n-2} + (PNh)^{n-1}\right]$$

$$= nPN_s x^{n-1} + 0 + \cdots\cdots + 0 + 0$$

$$\Rightarrow \quad \frac{d}{dx}[PN_s x^n] = nPN_s x^{n-1}$$

Example 4

1) $\frac{d}{dx}(P_7 - 8P_4 + 1) = 0$

2) $\frac{d}{dx}[(-4P_3 - 3P_1)x - 7P_5 - 3P_1 + 5] = -4P_3 - 3P_1$

3) $\frac{d}{dx}[(3P_1 + 5)x^5] = (15P_1 + 25)x^4$

4) $\frac{d}{dx}\left[e^{(P_6 + 53P_3)x + 73P_2 + 4}\right] = (P_6 + 53P_3)e^{(P_6 + 53P_3)x + 73P_2 + 4}$

5) $\frac{d}{dx}(1 + P_1 + 2P_2 + P_3)^x = (1 + 2P_2 + P_1)^x \;\; ln(2P_2 + P_1)$

$$= (1 + 2P_2 + P_1 + P_3)^x \quad [ln1 + (ln3 - ln1)P_1 + (ln4 - ln3)P_2 + (ln5 - ln4)P_3]$$

$$= (1 + 2P_2 + P_1 + P_3)^x \quad \left[(ln3)P_1 + \left(ln\frac{4}{3}\right)P_2 + \left(ln\frac{5}{4}\right)P_2\right]$$

6) $$\frac{d}{dx}\left[P_4 \, log_{(1+2P_1)} \, x\right] = \frac{P_4}{x \, ln(1 + 2P_1)} = \left(\frac{P_4}{ln(1 + 2P_1)}\right)\frac{1}{x}$$

$$= \left(\frac{P_4}{(ln3)P_1}\right)\frac{1}{x} = \frac{1}{(ln3)}(x_0 + x_1P_1 + P_4)\frac{1}{x}$$

$$= \left(\frac{x_0 + x_1P_1 + P_4}{ln3}\right)\frac{1}{x}$$

where:

$$\frac{P_4}{P_1} = x_0 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4 \qquad \Longrightarrow \qquad P_4 = x_0P_1 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4$$

$$\Longrightarrow \qquad P_2 = (x_0 + x_1)P_1 + x_2P_2 + x_3P_3 + x_4P_4$$, then:

$$x_0 + x_1 = 0 \;,\; x_2 = 0 \;,\; x_3 = 0 \text{ and } x_4 = 1$$

hence: $\frac{P_4}{P_1} = x_0 + x_1P_1 + P_4$, where: $x_0 + x_1 = 0$

7) $$\frac{d}{dx}\left[ln\big((7 + 5P_2 + P_3)x + 6 + 7P_1 + P_4\big)\right] = \frac{7 + 5P_2 + P_3}{(7 + 5P_2 + P_3)x + 6 + 7P_1 + P_4}$$

8) $$\frac{d}{dx}\left[\sqrt{(4 + 8P_7 + P_4)x + 2 + P_3}\right] = \frac{4 + 8P_7 + P_4}{2\sqrt{(4 + 8P_7 + P_4)x + 2 + P_3}}$$

9) $$\frac{d}{dx}\left[sin\big((9 - P_7)x + P_4\big)\right] = (9 - P_7)cos\big((9 - P_7)x + P_4\big)$$

10) $$\frac{d}{dx}\left[cos\big((5P_2 + P_1 - 4)x + P_8 + 2\big)\right] = (-5P_2 - P_1 + 4)sin\big((5P_2 + P_1 - 4)x + P_8 + 2\big)$$

11) $$\frac{d}{dx}\left[tan\big((P_7 + 4P_5 + 6)x + 6\big)\right] = (P_7 + 4P_5 + 6)sec^2\big((P_7 + 4P_5 + 6)x + 6\big)$$

12) $$\frac{d}{dx}\left[csc\big((8P_6 + 6)x + 7P_5 + 3\big)\right]$$
$$= (-8P_6 - 6)csc\big((8P_6 + 6)x + 7P_5 + 3\big)cot\big((8P_6 + 6)x + 7P_5 + 3\big)$$

Properties of the symbolic plithogenic differentiation:

I. Derivative of sum or difference of the symbolic plithogenic functions.

Suppose that $f(x, PN)$ and $g(x, PN)$ are any two differentiable symbolic plithogenic functions, then:

$$\frac{d}{dx}[f(x, PN) \pm g(x, PN)] = \frac{d}{dx}[f(x, PN)] \pm \frac{d}{dx}[g(x, PN)]$$

Proof:

$$\frac{d}{dx}[f(x,PN)+g(x,PN)]=$$

$$=\lim_{PNh\to 0}\frac{f(x+PNh)\pm g(x+PNh)-[f(x,PN)+g(x,PN)]}{PNh}$$

$$=\lim_{PNh\to 0}\frac{[f(x+PNh)-f(x,PN)]\pm[g(x+PNh)-g(x,PN)]}{PNh}$$

$$=\lim_{PNh\to 0}\left[\frac{f(x+PNh)-f(x,PN)}{PNh}\pm\frac{f(x+PNh)-f(x,PN)}{PNh}\right]$$

$$=\lim_{PNh\to 0}\frac{f(x+PNh)-f(x,PN)}{PNh}\pm\lim_{PNh\to 0}\frac{f(x+PNh)-f(x,PN)}{PNh}$$

$$=\frac{d}{dx}[f(x,PN)\ \ ]\pm\frac{d}{dx}[g(x,PN)]$$

Example 5

1) $\frac{d}{dx}\left[5P_2x^4+cot\big((7+P_2+P_3)x\big)\right]=20P_2x^3-(7+P_2+P_3)csc^2\big((7+P_2+P_3)x\big)$

2) $\frac{d}{dx}[(7+P_5)x+ln(P_2x)]=7+P_5+\frac{P_2}{P_2x}=7+P_5+(x_0+x_1P_1+x_2P_2)\frac{1}{x}$

where:

$$\frac{P_2}{P_2}=x_0+x_1P_1+x_2P_2 \qquad \Rightarrow \qquad P_2=x_0P_2+x_1P_2+x_2P_2$$

$$\Rightarrow \qquad P_2=(x_0+x_1+x_2)P_2$$, then:

$$x_0+x_1+x_2=1$$

hence: $\frac{P_2}{P_2}=x_0+x_1P_1+x_2P_2$, where: $x_0+x_1+x_2=1$

II. Derivative of product of a symbolic plithogenic constant and the symbolic plithogenic function

$$\frac{d}{dx}[PC.f(x,PN)]=PC.\frac{d}{dx}\ \ [f(x,PN)]$$

Proof:

$$\frac{d}{dx}[PC.f(x,PN)]=\lim_{PNh\to 0}\frac{PC.f(x+PNh)-PC.f(x,PN)}{PNh}$$

$$=\lim_{PNh\to 0}PC\left[\frac{f(x+PNh)-f(x,PN)}{PNh}\right]$$

$$=PC\lim_{PNh\to 0}\left[\frac{f(x+PNh)-f(x,PN)}{PNh}\right]$$

$$= PC\frac{d}{dx}\ \ [f(x,PN)]$$

III. Derivative of product of two the symbolic plithogenic functions

$$\frac{d}{dx}[f(x,PN).\,g(x,PN)] = f(x,PN)\frac{d}{dx}[g(x,PN)] + g(x,PN)\frac{d}{dx}[f(x,PN)]$$

Proof:

$$\frac{d}{dx}[f(x,PN).\,g(x,PN)] =$$

$$= \lim_{PNh\to 0}\frac{f(x+PNh).\,g(x+PNh) - f(x,PN).\,g(x,PN)}{PNh}$$

$$= \lim_{PNh\to 0}\frac{f(x+PNh).\,g(x+PNh) - f(x+PNh)g(x,PN) + f(x+PNh)g(x,PN) - f(x,PN).\,g(x,PN)}{PNh}$$

$$= \lim_{PNh\to 0}\left[f(x+PNh)\frac{g(x+PNh) - g(x,PN)}{PNh} + g(x,PN)\frac{f(x+PNh) - f(x,PN)}{PNh}\right]$$

$$= \lim_{PNh\to 0} f(x+PNh)\lim_{PNh\to 0}\frac{g(x+PNh) - g(x,PN)}{PNh} + \lim_{PNh\to 0} g(x,PN)\lim_{PNh\to 0}\frac{f(x+PNh) - f(x,PN)}{PNh}$$

$$= f(x,PN)\frac{d}{dx}[g(x,PN)] + g(x,PN)\frac{d}{dx}[f(x,PN)]$$

Example 6

1) $\frac{d}{dx}\left[4P_4x^2 sin\big((P_2+P_3)x\big)\ \ \right] = 8x.\,sin\big((P_2+P_3)x\big) + (4P_2+4P_3)cos\big((P_2+P_3)x\big)$

2) $\frac{d}{dx}\left[x\sqrt{(P_7-3)x+P_5}\right] = \sqrt{(P_7-3)x+P_5} + \frac{P_7-3}{2\sqrt{(P_7-3)x+P_5}}$

V. Derivative of quotient of two the symbolic plithogenic functions

$$\frac{d}{dx}\left[\frac{f(x,PN)}{g(x,PN)}\right] = \frac{g(x,PN)\frac{d}{dx}[f(x,PN)] - f(x,PN)\frac{d}{dx}[g(x,PN)]}{\big(g(x,PN)\big)^2}$$

Proof:

$$\frac{d}{dx}\left[\frac{f(x,PN)}{g(x,PN)}\right] = \lim_{PNh\to 0}\frac{\frac{f(x+PNh)}{g(x+PNh)} - \frac{f(x,PN)}{g(x,PN)}}{PNh}$$

$$= \lim_{PNh\to 0}\frac{f(x+PNh).\,g(x,PN) - f(x,PN).\,g(x,PN) - f(x,PN).\,g(x+PNh) + f(x,PN).\,(x,PN)}{PNh.\,g(x,PN).\,g(x+PNh)}$$

$$= \lim_{PNh\to 0}\left[\frac{g(x,PN)\frac{f(x+PNh) - f(x,PN)}{PNh} - f(x,PN)\frac{g(x+PNh) - g(x,PN)}{PNh}}{g(x,PN).\,g(x+PNh)}\right]$$

$$=\frac{\lim_{PNh\to 0}\left[g(x,PN)\frac{f(x+PNh)-f(x,PN)}{PNh}\right]-\lim_{PNh\to 0}\left[f(x,PN)\frac{g(x+PNh)-g(x,PN)}{PNh}\right]}{\lim_{PNh\to 0}[g(x,PN).g(x+PNh)]}$$

$$=\frac{\lim_{PNh\to 0}g(x,PN)\lim_{PNh\to 0}\frac{f(x+PNh)-f(x,PN)}{PNh}-\lim_{PNh\to 0}f(x,PN)\lim_{PNh\to 0}\frac{g(x+PNh)-g(x,I)}{PNh}}{\lim_{PNh\to 0}g(x,PN).\lim_{PNh\to 0}g(x+PNh)}$$

$$=\frac{g(x,PN)\frac{d}{dx}[f(x,PN)]-f(x,PN)\frac{d}{dx}[g(x,PN)]}{g(x,PN).g(x,PN)}$$

$$=\frac{d}{dx}\left[\frac{f(x,PN)}{g(x,PN)}\right]=\frac{g(x,PN)\frac{d}{dx}[f(x,PN)]-f(x,PN)\frac{d}{dx}[g(x,PN)]}{\left(g(x,PN)\right)^2}$$

Example 7

1) $$\frac{d}{dx}\left[\frac{e^{P_3x+2P_7-3}}{P_2x}\right]=\frac{P_3xe^{P_3x+2P_7-3}-P_2e^{P_3x+2P_7-3}}{P_2x^2}$$

$$=\frac{(3+I)xe^{(3+I)x+5I}-e^{(3+I)x+5I}}{(3+4I)x^2}$$

$$=(\frac{1}{3}-\frac{4}{21}I)\left[\frac{(3+I)xe^{(3+I)x+5I}-e^{(3+I)x+5I}}{x^2}\right]$$

2) $$\frac{d}{dx}\left[\frac{P_3}{P_2x}\right]=\frac{-P_3P_2}{P_2x^2}=\frac{-P_3}{P_2x^2}=(x_0+x_1P_1+x_2P_2-P_3)\frac{1}{x^2}$$

where:

$$\frac{-P_3}{P_2}=x_0+x_1P_1+x_2P_2+x_3P_3 \quad\Longrightarrow\quad -P_3=x_0P_2+x_1P_2+x_2P_2+x_3P_3$$

$$\Longrightarrow\quad -P_3=(x_0+x_1+x_2)P_2+x_3P_3$$, then:

$$x_0+x_1+x_2=0\ ,\ x_3=-1$$

hence: $\frac{-P_3}{P_2}=x_0+x_1P_1+x_2P_2-P_3$, where: $x_0+x_1=0$

5. Conclusions

In our daily lives, derivatives are crucial for tasks like figuring out how to calculate acceleration, displacement, and velocity as a function of time in rectilinear motion, other. In this article, we discussed the concept of The symbolic plithogenic differentials calculus, where we presented the rules of The symbolic plithogenic differentials, taking into account the mathematical operations on them.

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An Introduction to some Methods for Solving A Large System Linear Neutrosophic Equations

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Abstract: This paper aims to extend the methods of steepest descent and conjugate directions to the neutrosophic field R(I). the generalizations were built similarly to the classic algorithms, starting by generalizing the quadratic forms to R(I). Geometric isometry (AH-Isometry) S was used as the tool, and many examples are presented in the main paragraphs. The simple extension method can be generalized to other linear and non-linear methods.

Keywords: Quadratic forms; neutrosophic field; Steepest Descent; Conjugate Directions; Neutrosophic Matrix.

1. Introduction

Methods for the steepest descent [1], conjugate directions [2], and conjugate gradients [3] were constructed and derived from quadratic forms.

It is not hard to see the advantages of The Method of Steepest Descent; it is simple, easy, and popular. It uses a zig-zag path from an arbitrary point until it converges to the solution.

Conjugate direction methods have been developed to speed up the slow convergence of the steepest descent method; more details can be found in [4].

The nature of the consideration of how these two methods work in the neutrosophic field will not change.

Despite the great importance of optimization in modern mathematics, the importance of neutrosophic in prediction, and the existence of many studies that have developed many optimization concepts in the field of neutrosophic, as described in [5-13]. However, no study has established a method for solving large linear system neutrosophic equations.

This paper lays a foundation for specifically addressing this problem by extending the steepest descent and conjugate directions methods to the R(I).

To ensure more effective and general results, the matrix A_N was treated such that its real section differs from its neutrosophic section.

AH-isometry S was used to speed up the results because it is a simple and effective tool that saves the properties of the classic study in R(I), which is defined as followed:

$$S : R(I) \to R \times R$$
$$S(a + bI) = (a, a + b)$$

where $R(I)=\{a+bI\ ;a,b\in R\}$.

And its invert is

$$S^{-1}:R\times R\rightarrow R(I)$$

With some basic properties

$$S\left[(a+bI)+(c+dI)\right]=S(a+bI)+S(c+dI)$$

$$S\left[(a+bI).(c+dI)\right]=S(a+bI).S(c+dI).$$

Other details and applications of this tool can be found in [14-18]. Starting from derivate more generalized quadratic form

$$f(x_N)=\frac{1}{2}x_N^T A_N x_N - b_N^T x_N + c_N$$

The methods for steepest descent and conjugate directions were generalized to the neutrosophic field. It can be seen that the classical definitions used in this paper were simply extended to R(I).

2. Preliminaries

Steepest descent and conjugate directions are the most popular iterative methods for solving large systems of linear equations. Each method is effective for systems of the form

$$A\xi=b. \tag{1}$$

where ξ is an unknown vector, b is a known vector, and A is a known, square, positive-definite (or positive-indefinite) matrix.

A matrix A is positive-definite if, for every nonzero vector ξ,

$$\xi^T A\xi>0. \tag{2}$$

Where ξ^T is the Transpose of ξ.

The Quadratic Form

A quadratic form is a scalar, quadratic function of a vector with the form

$$f(\xi)=\frac{1}{2}\xi^T A\xi-b^T\xi+c, \tag{3}$$

Where A is a matrix, ξ and b are vectors, and c is a scalar constant.

However, condition (2) is not a very intuitive idea, as it affects the shape of quadratic forms.

The gradient of a quadratic form is defined to be

$$f'(\xi)=\begin{bmatrix}\frac{\partial}{\partial\xi_1}f(\xi)\\ \frac{\partial}{\partial\xi_2}f(\xi)\\ \vdots\\ \frac{\partial}{\partial\xi_n}f(\xi)\end{bmatrix}. \tag{4}$$

One can apply Equation (4) to Equation (3), and derive, then, we obtain

$$f'(\xi)=\frac{1}{2}A^T\xi+\frac{1}{2}A\xi-b \tag{5}$$

If A is symmetric, this equation reduces to

$$f'(\xi)=A\xi-b \tag{6}$$

Setting the gradient to zero, we obtain Equation (1).

3. The Neutrosophic Quadratic Form

Definition (1): The neutrosophic form of a matrix $A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$ could be written as following:

$$A_N=\begin{bmatrix}a_{11}^N & a_{12}^N\\ a_{12}^N & a_{22}^N\end{bmatrix}$$

where $a_{ij}^N=a_{ij}+Ia_{ij}$, $i,j=1,2$.

In fact, $A_N:=A+IA=\begin{bmatrix}a_{11}^N & a_{12}^N\\ a_{21}^N & a_{22}^N\end{bmatrix}+I\begin{bmatrix}a_{11}^N & a_{12}^N\\ a_{21}^N & a_{22}^N\end{bmatrix}=\begin{bmatrix}a_{11}^N & a_{12}^N\\ a_{21}^N & a_{22}^N\end{bmatrix}+\begin{bmatrix}a_{11}^N I & a_{12}^N I\\ a_{21}^N I & a_{22}^N I\end{bmatrix}=\begin{bmatrix}a_{11}^N & a_{12}^N\\ a_{21}^N & a_{22}^N\end{bmatrix}$.

When A_N is a symmetric matrix, then their componentes are symmetrics. One can considered a more generalized form of A_N by taking non-coincide components.

Transpose of A_N is defined as: $A_N^T:=[A+IA]^T=A^T+IA^T=\begin{bmatrix}a_{11} & a_{21}\\ a_{12} & a_{22}\end{bmatrix}+I\begin{bmatrix}a_{11} & a_{21}\\ a_{12} & a_{22}\end{bmatrix}=\begin{bmatrix}a_{11}^N & a_{21}^N\\ a_{12}^N & a_{22}^N\end{bmatrix}$.

For $m=2, n=1$, we have $A_N^T=\begin{bmatrix}a_{11}^N & a_{21}^N\end{bmatrix}$., and $A_N=\begin{bmatrix}a_{11}^N\\ a_{21}^N\end{bmatrix}$

By looking to the equation $A\xi=b$, where $A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$, $\xi=\begin{bmatrix}\xi_{11}\\ \xi_{21}\end{bmatrix}$, $b=\begin{bmatrix}b_{11}\\ b_{21}\end{bmatrix}$,

and despite it is easy to find the neutrosophic form of it, we will consider more generalized form:

$$A_N x_N=b_N,$$

where

$$A_N=A+I\Lambda=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}+I\begin{bmatrix}\lambda_{11} & \lambda_{12}\\ \lambda_{21} & \lambda_{22}\end{bmatrix},\ x_N=\xi+I\eta=\begin{bmatrix}\xi_{11}\\ \xi_{21}\end{bmatrix}+I\begin{bmatrix}\eta_{11}\\ \eta_{21}\end{bmatrix},\ b_N=b+IB=\begin{bmatrix}b_{11}\\ b_{21}\end{bmatrix}+I\begin{bmatrix}B_{11}\\ B_{21}\end{bmatrix},$$

that we can rewrite it as

$$A\xi+I\left(A\eta+\Lambda\xi+\Lambda\eta\right)=b+IB. \tag{7}$$

Note that, if we make the neutrosophic part in (7) is equal to zero, then we will find the classical equation $A\xi=b$.

The Neutrosophic Quadratic Form

To find The Neutrosophic Quadratic Form, we begin Form the well-known relation

$$f(\xi)=\frac{1}{2}\xi^T A\xi-b^T\xi+c.$$

That which, according to the neutrosophic form, simply, we have

$$f(x_N)=\frac{1}{2}x_N^T A_N x_N-b_N^T x_N+c_N,\ c_N=c+Id. \tag{8}$$

By taking S for the both sides of (8):

$$\begin{aligned}
S[f(x_N)] &= S(1/2)S(x_N^T)S(A_N)S(x_N)-S(b_N^T)S(x_N)+S(c_N)\\
&=(1/2,1/2)(\xi,\xi+\eta)^T(A,A+\Lambda)(\xi,\xi+\eta)-(b,b+B)^T(\xi,\xi+\eta)+(c,c+d)\\
&=(1/2,1/2)(\xi^T,\xi^T+\eta^T)(A,A+\Lambda)(\xi,\xi+\eta)-(b^T,b^T+B^T)(\xi,\xi+\eta)+(c,c+d)\\
&=\left[\frac{1}{2}\xi^T A\xi,\frac{1}{2}(\xi^T+\eta^T)(A+\Lambda)(\xi+\eta)\right]-\left[b^T\xi,(b^T+B^T)(\xi+\eta)\right]+(c,c+d)\\
&=\left[\frac{1}{2}\xi^T A\xi-b^T\xi+c,\frac{1}{2}(\xi^T+\eta^T)(A+\Lambda)(\xi+\eta)-(b^T+B^T)(\xi+\eta)+(c+d)\right].
\end{aligned}$$

Taking S^{-1}, then we obtain

$$\begin{aligned}
f(x_N)&=\frac{1}{2}\xi^T A\xi-b^T\xi+c+\\
&+I\left\{\frac{1}{2}(\xi^T+\eta^T)(A+\Lambda)(\xi+\eta)-(b^T+B^T)(\xi+\eta)+(c+d)-\left[\frac{1}{2}\xi^T A\xi-b^T\xi+c\right]\right\}.
\end{aligned}$$

Derivate the last function with respect to ξ,η respectively:

$$\begin{aligned}
f'^N_\xi&=\frac{\partial f(x_N)}{\partial\xi}=\frac{1}{2}(A+A^T)\xi-b+I\left[\frac{1}{2}(A+\Lambda+A^T+\Lambda^T)(\xi+\eta)-(b+B)-\left(\frac{1}{2}(A+A^T)\xi-b\right)\right]\\
&=\frac{1}{2}(A+A^T)\xi-b+I\left[\frac{1}{2}(\Lambda+\Lambda^T)(\xi+\eta)+\frac{1}{2}(A+A^T)\eta-B\right].\\
f'^N_\eta&=\frac{\partial f(x_N)}{\partial\eta}=I\left[\frac{1}{2}(A+\Lambda+A^T+\Lambda^T)(\xi+\eta)-(b+B)\right].
\end{aligned}$$

By making $f'(x_N)=0$, then it leads to solve the system:

$$(*)\quad\begin{cases}
\frac{1}{2}(A+A^T)\xi-b=0 & (9)\\
\frac{1}{2}(\Lambda+\Lambda^T)(\xi+\eta)+\frac{1}{2}(A+A^T)\eta-B=0 & (10)\\
\frac{1}{2}(A+\Lambda+A^T+\Lambda^T)(\xi+\eta)-(b+B)=0 & (11)
\end{cases}$$

If A_N is symmetric, then, $A=A^T,\Lambda=\Lambda^T$, and the system becomes:

$$(**)\quad\begin{cases}
A\xi-b=0 & (12)\\
\Lambda(\xi+\eta)+A\eta-B=0 & (13)\\
(A+\Lambda)(\xi+\eta)-(b+B)=0 & (14)
\end{cases}$$

In both systems, we note that the third equation is a summation of the first and the second equation.

Examples:

1. Let $A_N = \begin{bmatrix} 1+I & 4+3I \\ -1+2I & -2-I \end{bmatrix}$, then $A = \begin{bmatrix} 1 & 4 \\ -1 & -2 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$. $x_N = \begin{bmatrix} \xi_1 + I\eta_1 \\ \xi_2 + I\eta_2 \end{bmatrix}$, then $\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$. $b_N = \begin{bmatrix} 2+I \\ -3-2I \end{bmatrix}$, then $b = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

By returning to $(*)$ relations, we have for (9):

$$A^T = \begin{bmatrix} 1 & -1 \\ 4 & -2 \end{bmatrix}, \text{ and } (1/2)\left(A + A^T\right) = \begin{bmatrix} 1 & 3/2 \\ 3/2 & -2 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 3/2 \\ 3/2 & -2 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \xrightarrow{\text{yields}} \xi = \begin{bmatrix} -2/17 \\ 24/17 \end{bmatrix}.$$

And for (10), we have

$$\frac{1}{2}\left(\Lambda + \Lambda^T\right)(\xi + \eta) = \begin{bmatrix} 1 & 5/2 \\ 5/2 & -1 \end{bmatrix}\begin{bmatrix} -2/17 + \eta_1 \\ 24/17 + \eta_2 \end{bmatrix} = \begin{bmatrix} 58/17 + \eta_1 + (5/2)\eta_2 \\ -29/17 + (5/2)\eta_1 - \eta_2 \end{bmatrix},$$

$$\frac{1}{2}\left(A + A^T\right)\eta = \begin{bmatrix} 1 & 3/2 \\ 3/2 & -2 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \eta_1 + (3/2)\eta_2 \\ (3/2)\eta_1 - 2\eta_2 \end{bmatrix},$$

and

$$\begin{bmatrix} 58/17 + \eta_1 + (5/2)\eta_2 \\ -29/17 + (5/2)\eta_1 - \eta_2 \end{bmatrix} + \begin{bmatrix} \eta_1 + (3/2)\eta_2 \\ (3/2)\eta_1 - 2\eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

By solving the system

$$\begin{cases} 58/17 + 2\eta_1 + 4\eta_2 = 1, \\ -29/17 + 4\eta_1 - 3\eta_2 = -2. \end{cases}$$

We obtain

$$\eta = \begin{bmatrix} -13/34 \\ -7/17 \end{bmatrix}, \text{ and } x_N = \begin{bmatrix} (-2/17) + I(-13/34) \\ (24/17) + I(-7/17) \end{bmatrix}.$$

This solution satisfied (11), since

$$\frac{1}{2}\left(A + \Lambda + A^T + \Lambda^T\right)(\xi + \eta) = \begin{bmatrix} 2 & 4 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix} = b + B.$$

2. Let A_N be a symmetric matrix, i.e. $A_N = \begin{bmatrix} 2-I & 5+2I \\ 5+2I & -1+3I \end{bmatrix}$, $x_N = \begin{bmatrix} \xi_1 + I\eta_1 \\ \xi_2 + I\eta_2 \end{bmatrix}$, and $b_N = \begin{bmatrix} 1+2I \\ 2+3I \end{bmatrix}$.

By returning to the $(**)$, we have for (12):

$$\begin{bmatrix} 2 & 5 \\ 5 & -1 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

$$\begin{bmatrix} 2\xi_1 + 5\xi_2 \\ 5\xi_1 - \xi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

then

$$\xi = \begin{bmatrix} 11/27 \\ 1/27 \end{bmatrix}.$$

and for (13):

$$\begin{bmatrix}-1 & 2\\ 2 & 3\end{bmatrix}\begin{bmatrix}(11/27)+\eta_1\\ (1/27)+\eta_2\end{bmatrix}+\begin{bmatrix}2 & 5\\ 5 & -1\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}=\begin{bmatrix}2\\ 3\end{bmatrix},$$

then we find that

$$\eta=\begin{bmatrix}266/1269\\ 385/1269\end{bmatrix},$$

and

$$x_N=\begin{bmatrix}11/27\\ 1/27\end{bmatrix}+I\begin{bmatrix}266/1269\\ 385/1269\end{bmatrix}.$$

The solution satisfies (14):

$$(A+\Lambda)(\xi+\eta)-(b+B)=\begin{bmatrix}1 & 7\\ 7 & 2\end{bmatrix}\begin{bmatrix}29/47\\ 16/47\end{bmatrix}-\begin{bmatrix}3\\ 5\end{bmatrix}=0.$$

A special case

If $A=\Lambda,\ \xi=\eta,\ b=B$ then the neutrosophic Quadratic Form is:

$$f(x_N)=\frac{1}{2}\xi^T A\xi-b^T\xi+c+I\left(\frac{3}{2}\xi^T A\xi-3b^T\xi+c\right), \tag{15}$$

And

$$f'(x_N)=\frac{1}{2}A\xi+\frac{1}{2}A^T\xi-b+I\left(\frac{3}{2}A\xi+\frac{3}{2}A^T\xi-3b\right),$$

by making $f'(x_N)=0$, that leads to

$$\left(A+A^T\right)\xi=2b. \tag{16}$$

This means that the neutrosophic quadratic form of the classical form can be found directly by solving the well-known equation (16), which turns out to be (1) when A is symmetric, which we will see later in an example.

4. The Method of neutrosophic Steepest Descent

Derivate the function (8) by using the relation (4), we find

$$f'(x_N)=\frac{\partial f(x_N)}{\partial\xi}+\frac{\partial f(x_N)}{\partial\eta}=$$
$$=\frac{1}{2}\left(A+A^T\right)\xi-b+I\left[\left(A+\Lambda+A^T+\Lambda^T\right)(\xi+\eta)-(2b+2B)-\left(\frac{1}{2}\left(A+A^T\right)\xi-b\right)\right]$$

Let us denote $x_{(i)}^N=\xi_{(i)}+\eta_{(i)}I$ to $x_N=\xi+\eta I$ in step (i), then

$$-f'\left(x_{(i)}^N\right)=b-\frac{1}{2}\left(A+A^T\right)\xi_{(i)}+I\left[(2b+2B)-\left(A+\Lambda+A^T+\Lambda^T\right)\left(\xi_{(i)}+\eta_{(i)}\right)-\left(b-\frac{1}{2}\left(A+A^T\right)\xi_{(i)}\right)\right]$$
$$=b+(b+2B)I-\left\{\frac{1}{2}\left(A+A^T\right)+\left[\frac{1}{2}\left(A+A^T\right)+\left(\Lambda+\Lambda^T\right)\right]I\right\}\xi_{(i)}-$$
$$-\left\{\frac{1}{2}\left(A+A^T\right)+\left[\frac{1}{2}\left(A+A^T\right)+\left(\Lambda+\Lambda^T\right)\right]I\right\}\eta_{(i)}I.$$

Putting $k_N=b+(b+2B)I$, $t_N=\frac{1}{2}\left(A+A^T\right)+\left[\frac{1}{2}\left(A+A^T\right)+\left(\Lambda+\Lambda^T\right)\right]I$, and suppose that

$$r_{(i)}^{N} = -f'\left(x_{(i)}^{N}\right),$$

then

$$r_{(i)}^{N} = k_N - t_N x_{(i)}^{N}, \quad (17)$$

Let us define the neutrosophic error as

$$e_{(i)}^{N} = x_{(i)}^{N} - x_N = \left(\xi_{(i)} + \eta_{(i)} I\right) - \left(\xi + \eta I\right) = (\xi_{(i)} - \xi) + \left(\eta_{(i)} - \eta\right) I \ .$$

Note that $r_{(i)}^{N} = -t_N e_{(i)}^{N}$.

Let us start from a point $x_{(0)}^{N}$, and we will choose a point $x_{(1)}^{N}$ such that $x_{(1)}^{N} = x_{(0)}^{N} + \alpha_N . r_{(0)}^{N}$

where α_N is a neutrosophic number, which is minimizes f when $\frac{df\left(x_{(1)}^{N}\right)}{d\alpha_N}$ is equal to zero.

To determine α_N, we have $f'\left(x_{(i)}^{N}\right) = -r_{(i)}^{N}$ and

$$\frac{df\left(x_{(1)}^{N}\right)}{d\alpha_N} = \left[f'\left(x_{(1)}^{N}\right)\right]^T \frac{d\left(x_{(1)}^{N}\right)}{d\alpha_N} = \left[f'\left(x_{(1)}^{N}\right)\right]^T r_{(0)}^{N}$$

Setting the last expression to zero, then $\left[r_{(1)}^{N}\right]^T r_{(0)}^{N} = 0$, by using (11), we have

$$\left[k_N - t_N x_{(1)}^{N}\right]^T r_{(0)}^{N} = 0$$

$$\left[k_N - t_N \left(x_{(0)}^{N} + \alpha_N . r_{(0)}^{N}\right)\right]^T r_{(0)}^{N} = 0$$

$$\left[k_N - t_N x_{(0)}^{N}\right]^T r_{(0)}^{N} - \alpha_N \left[t_N r_{(0)}^{N})\right]^T r_{(0)}^{N} = 0$$

$$\left[k_N - t_N x_{(0)}^{N}\right]^T r_{(0)}^{N} = \alpha_N \left[t_N r_{(0)}^{N})\right]^T r_{(0)}^{N}$$

$$\left[r_{(0)}^{N}\right]^T r_{(0)}^{N} = \alpha_N \left[r_{(0)}^{N})\right]^T \left[t_N\right]^T r_{(0)}^{N}$$

$$\alpha_N = \frac{\left[r_{(0)}^{N}\right]^T r_{(0)}^{N}}{\left[r_{(0)}^{N}\right]^T \left[t_N\right]^T r_{(0)}^{N}}.$$

This formula can be rewritten with more specifically way as followed:
By returning to (5), it is easy to see that

$$-f'(\xi) = b - \frac{1}{2}\left(A + A^T\right)\xi ,$$

and

$$\left[2f'^{N}_{\eta} - f'(\xi)\right] I = \left[\left(A + \Lambda + A^T + \Lambda^T\right)(\xi + \eta) - (2b + 2B) - \left(\frac{1}{2}\left(A + A^T\right)\xi - b\right)\right] I .$$

Then

$$r_{(0)}^{N} = -f'\left(x_{(0)}^{N}\right) = -f'\left(\xi_{(0)}\right) + I\left[-2f'^{N}_{\eta_{(0)}} + f'\left(\xi_{(0)}\right)\right],$$

and one can write

$$\left[r_{(0)}^{N}\right]^{T}=\left[-f'\left(x_{(0)}^{N}\right)\right]^{T}=\left[-f'\left(\xi_{(0)}\right)\right]^{T}+I\left[\left[-2f'^{N}_{\eta_{(0)}}\right]^{T}+\left[f'\left(\xi_{(0)}\right)\right]^{T}\right],$$

$$S\left\{\left[r_{(0)}^{N}\right]^{T}r_{(0)}^{N}\right\}=S\left\{\left[r_{(0)}^{N}\right]^{T}\right\}S\left\{r_{(0)}^{N}\right\}=\left(\left[-f'\left(\xi_{(0)}\right)\right]^{T},\left[-2f'^{N}_{\eta_{(0)}}\right]^{T}\right)\left(-f'\left(\xi_{(0)}\right),-2f'^{N}_{\eta_{(0)}}\right)=$$

$$=\left(\left[-f'\left(\xi_{(0)}\right)\right]^{T}\left(-f'\left(\xi_{(0)}\right)\right),\left[-2f'^{N}_{\eta_{(0)}}\right]^{T}\left(-2f'^{N}_{\eta_{(0)}}\right)\right).$$

and

$$S^{-1}\left(S\left\{\left[r_{(0)}^{N}\right]^{T}r_{(0)}^{N}\right\}\right)=\left[-f'\left(\xi_{(0)}\right)\right]^{T}\left(-f'\left(\xi_{(0)}\right)\right)+I\left(\left[-2f'^{N}_{\eta_{(0)}}\right]^{T}\left(-2f'^{N}_{\eta_{(0)}}\right)-\left[-f'\left(\xi_{(0)}\right)\right]^{T}\left(-f'\left(\xi_{(0)}\right)\right)\right).$$

Suppose that

$$\varphi=\left[-f'\left(\xi_{(0)}\right)\right]^{T}\left(-f'\left(\xi_{(0)}\right)\right),\ \psi=\left[-2f'^{N}_{\eta_{(0)}}\right]^{T}\left(-2f'^{N}_{\eta_{(0)}}\right)-\left[-f'\left(\xi_{(0)}\right)\right]^{T}\left(-f'\left(\xi_{(0)}\right)\right).$$

Then

$$\left[r_{(0)}^{N}\right]^{T}r_{(0)}^{N}=\varphi+I\psi .$$

And by using the same tool S, we find

$$S^{-1}\left(S\left\{\left[r_{(0)}^{N}\right]^{T}\left[t_{N}\right]^{T}r_{(0)}^{N}\right\}\right)=\left[r_{(0)}^{N}\right]^{T}\left[t_{N}\right]^{T}r_{(0)}^{N}=$$

$$=\left[-f'\left(\xi_{(0)}\right)\right]^{T}\left[\frac{1}{2}\left(A+A^{T}\right)\right]^{T}\left(-f'\left(\xi_{(0)}\right)\right)+I\left[\left(\left[-2f'^{N}_{\eta_{(0)}}\right]^{T}\right)\left[\left(A+A^{T}\right)^{T}+\left(\Lambda+\Lambda^{T}\right)^{T}\right]\left(-2f'^{N}_{\eta_{(0)}}\right)-\right.$$

$$\left.-\left[-f'\left(\xi_{(0)}\right)\right]^{T}\left[\frac{1}{2}\left(A+A^{T}\right)\right]^{T}\left(-f'\left(\xi_{(0)}\right)\right)\right].$$

Putting

$$\Phi=\left[-f'\left(\xi_{(0)}\right)\right]^{T}\left(\frac{1}{2}\left(A^{T}+A\right)\right)\left(-f'\left(\xi_{(0)}\right)\right),$$

$$\Psi=\left[-2f'^{N}_{\eta_{(0)}}\right]^{T}\left[(A^{T}+A)+\left(\left(\Lambda^{T}+\Lambda\right)\right)\right]\left(-2f'^{N}_{\eta_{(0)}}\right)-\left[-f'\left(\xi_{(0)}\right)\right]^{T}\left(\frac{1}{2}\left(A^{T}+A\right)\right)\left(-f'\left(\xi_{(0)}\right)\right).$$

Then $\left[r_{(0)}^{N}\right]^{T}\left[t_{N}\right]^{T}r_{(0)}^{N}=\Phi+I\Psi$ and α_N takes the form

$$\alpha_{N}=\frac{\varphi+I\psi}{\Phi+I\Psi}.$$

$$S\left(\alpha_{N}\right)=S\left(\frac{\varphi+I\psi}{\Phi+I\Psi}\right)=\frac{S\left(\varphi+I\psi\right)}{S\left(\Phi+I\Psi\right)}=\frac{\left(\varphi,\varphi+\psi\right)}{\left(\Phi,\Phi+\Psi\right)}=\left(\frac{\varphi}{\Phi},\frac{\varphi+\psi}{\Phi+\Psi}\right)$$

$$S^{-1}\left(S\left(\alpha_{N}\right)\right)=\alpha_{N}=\frac{\varphi}{\Phi}+I\left[\frac{\varphi+\psi}{\Phi+\Psi}-\frac{\varphi}{\Phi}\right],$$

or

$$\alpha_{N}=\frac{\left[-f'\left(\xi_{(0)}\right)\right]^{T}\left(-f'\left(\xi_{(0)}\right)\right)}{\left[-f'\left(\xi_{(0)}\right)\right]^{T}\left(\frac{1}{2}\left(A^{T}+A\right)\right)\left(-f'\left(\xi_{(0)}\right)\right)}+$$

$$+I\left[\frac{\left[-2f'^{N}_{\eta(0)}\right]^{T}\left(-2f'^{N}_{\eta(0)}\right)}{\left[-2f'^{N}_{\eta(0)}\right]^{T}\left[(A^{T}+A)+\left(\left(\Lambda^{T}+\Lambda\right)\right)\right]\left(-2f'^{N}_{\eta(0)}\right)}-\frac{\left[-f'\left(\xi_{(0)}\right)\right]^{T}\left(-f'\left(\xi_{(0)}\right)\right)}{\left[-f'\left(\xi_{(0)}\right)\right]^{T}\left(\frac{1}{2}\left(A^{T}+A\right)\right)\left(-f'\left(\xi_{(0)}\right)\right)}\right].$$

Notice that the real part of α_N is the well-known classic form of α .
As a result, The Method of neutrosophic Steepest Descent is:

$$r^{N}_{(i)} = k_{N} - t_{N}x^{N}_{(i)},$$

$$\alpha^{N}_{(i)} = \frac{\left[r^{N}_{(i)}\right]^{T} r^{N}_{(i)}}{\left[r^{N}_{(i)}\right]^{T}\left[t_{N}\right]^{T} r^{N}_{(i)}},$$

$$x^{N}_{(i+1)} = x^{N}_{(i)} + \alpha^{N}_{(i)}.r^{N}_{(i)}.$$

Under conditions of multiplication of matrix, we can premultiplying both sides of the last equation by $-t_N$ and adding k_N , then we have

$$r^{N}_{(i+1)} = r^{N}_{(i)} - \alpha^{N}_{(i)}t_{N}.r^{N}_{(i)} \tag{18}$$

Example 1:

Let us start with $A_N = \begin{bmatrix} 4-4I & 2+2I \\ 2+2I & 2-2I \end{bmatrix}$, $b_N = \begin{bmatrix} -1+I \\ 1-I \end{bmatrix}$, and $x^{N}_{(i)} = \begin{bmatrix} -\frac{1}{4}+\frac{1}{4}I \\ 0 \end{bmatrix}$.

$$k_N = b + (b + 2B)I = \begin{bmatrix} -1+I \\ 1-I \end{bmatrix},$$

$$t_N = \frac{1}{2}\left(A + A^T\right) + \left[\frac{1}{2}\left(A + A^T\right) + \left(\Lambda + \Lambda^T\right)\right]I = \begin{bmatrix} 4-4I & 2+6I \\ 2+6I & 2-2I \end{bmatrix},$$

$$r^{N}_{(i)} = k_N - t_N x^{N}_{(i)} = \begin{bmatrix} -1+I \\ 1-I \end{bmatrix} - \begin{bmatrix} 4-4I & 2+6I \\ 2+6I & 2-2I \end{bmatrix}\begin{bmatrix} -\frac{1}{4}+\frac{1}{4}I \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{2}-\frac{3}{2}I \end{bmatrix}.$$

$$\left[r^{N}_{(i)}\right]^{T} r^{N}_{(i)} = \frac{9}{4} - \frac{9}{4}I.$$

$$\left[r^{N}_{(i)}\right]^{T}\left[t_N\right]^{T} r^{N}_{(i)} = \frac{9}{2} - \frac{9}{2}I.$$

$$\alpha^{N}_{(i)} = \frac{\frac{9}{4}(1-I)}{\frac{9}{2}(1-I)} = \frac{1}{2}.$$

By using (18), we have

$$r^{N}_{(i+1)} = \begin{bmatrix} 0 \\ \frac{3}{2}-\frac{3}{2}I \end{bmatrix} - \left(\frac{1}{2}\right)\begin{bmatrix} 4-4I & 2+6I \\ 2+6I & 2-2I \end{bmatrix}\begin{bmatrix} 0 \\ \frac{3}{2}-\frac{3}{2}I \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}+\frac{3}{2}I \\ 0 \end{bmatrix}.$$

$$\left[r^{N}_{(i+1)}\right]^{T} r^{N}_{(i+1)} = \begin{bmatrix} -\frac{3}{2}+\frac{3}{2}I & 0 \end{bmatrix}\begin{bmatrix} -\frac{3}{2}+\frac{3}{2}I \\ 0 \end{bmatrix} = \frac{9}{4} - \frac{9}{4}I.$$

$$\left[r^{N}_{(i+1)}\right]^{T}\left[t_N\right]^{T} r^{N}_{(i+1)} = \begin{bmatrix} -\frac{3}{2}+\frac{3}{2}I & 0 \end{bmatrix}\begin{bmatrix} 4-4I & 2+6I \\ 2+6I & 2-2I \end{bmatrix}\begin{bmatrix} -\frac{3}{2}+\frac{3}{2}I \\ 0 \end{bmatrix} = 9 - 9I.$$

Hence

$$\alpha_{(i+1)}^{N}=\frac{\frac{9}{4}(1-I)}{9(1-I)}=\frac{1}{4}.$$

Example 2:

Let $A_N=\begin{bmatrix}4-\frac{1}{2}I & 2-\frac{1}{2}I\\ -2+\frac{1}{2}I & -1\end{bmatrix}$, $b_N=\begin{bmatrix}1-I\\ -1+2I\end{bmatrix}$, and $x_{(i)}^{N}=\begin{bmatrix}\frac{1}{4}-\frac{1}{4}I\\ 0\end{bmatrix}$.

Then

$$k_N=b+(b+2B)I=\begin{bmatrix}1-I\\ -1+3I\end{bmatrix},$$

$$t_N=\frac{1}{2}(A+A^T)+\left[\frac{1}{2}(A+A^T)+(\Lambda+\Lambda^T)\right]I=\begin{bmatrix}4+3I & 2I\\ 2I & -1+2I\end{bmatrix},$$

$$r_{(i)}^{N}=k_N-t_N x_{(i)}^{N}=\begin{bmatrix}1-I\\ -1+3I\end{bmatrix}-\begin{bmatrix}4+3I & 2I\\ 2I & -1+2I\end{bmatrix}\begin{bmatrix}\frac{1}{4}-\frac{1}{4}I\\ 0\end{bmatrix}=\begin{bmatrix}0\\ -1+3I\end{bmatrix}.$$

$$\left[r_{(i)}^{N}\right]^T r_{(i)}^{N}=1+3I.$$

$$\left[r_{(i)}^{N}\right]^T [t_N]^T r_{(i)}^{N}=-1+5I.$$

$$\alpha_{(i)}^{N}=\frac{1+3I}{-1+5I}=-1+2I.$$

By using (18), we have

$$r_{(i+1)}^{N}=\begin{bmatrix}0\\ -1+3I\end{bmatrix}-(-1+2I)\begin{bmatrix}4+3I & 2I\\ 2I & -1+2I\end{bmatrix}\begin{bmatrix}0\\ -1+3I\end{bmatrix}=\begin{bmatrix}-4I\\ 0\end{bmatrix}.$$

$$\left[r_{(i+1)}^{N}\right]^T r_{(i+1)}^{N}=\begin{bmatrix}-4I & 0\end{bmatrix}\begin{bmatrix}-4I\\ 0\end{bmatrix}=16I.$$

$$\left[r_{(i+1)}^{N}\right]^T [t_N]^T r_{(i+1)}^{N}=\begin{bmatrix}-4I & 0\end{bmatrix}\begin{bmatrix}4+3I & 2I\\ 2I & -1+2I\end{bmatrix}\begin{bmatrix}-4I\\ 0\end{bmatrix}=112I.$$

Hence

$$\alpha_{(i+1)}^{N}=\frac{16I}{112I}=\frac{1}{7}I.$$

5. The method of Neutrosophic Conjugate Directions

As in the classic case, we use coordinate axes as search directions. Every step consists of two paths, the first leads to the correct $x_{(1)}^{N}$ – coordinate, and the second hit the desired point, after n steps, it will be done.

Definition (2): Let A_N be a symmetric matrix, two nonzero vectors $d_{(i)}^{N}, d_{(j)}^{N}$ are said to be A_N - orthogonal, if

$$\left[d_{(i)}^{N}\right]^T .A_N .d_{(j)}^{N}=0_N$$

where $i\neq j$.

In general, we take

$$x_{(i+1)}^{N}=x_{(i)}^{N}+\alpha_{(i)}^{N}d_{(i)}^{N} \tag{19}$$

By taking into consideration the fact that $e_{(i+1)}^{N}$ should be orthogonal to $d_{(i)}^{N}$, we can find the value of $\alpha_{(i)}^{N}$, which eliminates the need to step in the direction $d_{(i)}^{N}$ again. We have

$$\left[d_{(i)}^N\right]^T .e_{(i+1)}^N = 0_N$$

$$\left[d_{(i)}^N\right]^T .\left(e_{(i)}^N + \alpha_{(i)}^N d_{(i)}^N\right) = 0_N$$

Which leads to

$$\alpha_{(i)}^N = \frac{-\left[d_{(i)}^N\right]^T e_{(i)}^N}{\left[d_{(i)}^N)\right]^T d_{(i)}^N}. \tag{20}$$

Knowing $e_{(i)}^N$ guarantees computing $\alpha_{(i)}^N$, As it has known in classical case, the solution is to be the two vectors $d_{(i)}^N$ and $d_{(j)}^N$ are A_N -orthogonal. To make $e_{(i+1)}^N$ be A_N -orthogonal to $d_{(i)}^N$, it is sufficient to find the minimum point along the direction $d_{(i)}^N$:

$$\frac{d}{d\alpha^N} f\left(x_{(i+1)}^N\right) = 0_N$$

$$\left[f'\left(x_{(i+1)}^N\right)\right]^T . \frac{d}{d\alpha^N} x_{(i+1)}^N = 0_N$$

$$-\left[r_{(i+1)}^N\right]^T d_{(i)}^N = 0_N$$

$$\left[d_{(i)}^N\right]^T .A_N .e_{(i+1)}^N = 0_N$$

Now, the equation (20) becomes with A_N -orthogonal search directions, as following:

$$\alpha_{(i)}^N = \frac{-\left[d_{(i)}^N\right]^T .A_N .e_{(i)}^N}{\left[d_{(i)}^N)\right]^T .A_N .d_{(i)}^N}$$

$$\alpha_{(i)}^N = \frac{\left[d_{(i)}^N\right]^T .r_{(i)}^N}{\left[d_{(i)}^N)\right]^T .A_N .d_{(i)}^N}.$$

Which could be to calculate.

Let us write $\alpha_{(i)}^N$ as $\alpha_{(i)}^N = X + IY$. Suppose that $d_{(i)}^N = d_{(i)} + ID_{(i)}$, then $\left[d_{(i)}^N)\right]^T = d_{(i)}^T + ID_{(i)}^T$.

$$S\left\{\left[d_{(i)}^N\right]^T .r_{(i)}^N\right\} = S\left(\left[d_{(i)}^N\right]^T\right) S\left(r_{(i)}^N\right) = \left(d_{(i)}{}^T , d_{(i)}{}^T + D_{(i)}{}^T\right)\left(-f'\left(\xi_{(i)}\right), -2f'{}^N_{\eta_{(i)}}\right)$$

$$= \left(d_{(i)}{}^T\left(-f'\left(\xi_{(i)}\right)\right), \left(d_{(i)}{}^T + D_{(i)}{}^T\right)\left[-2f'{}^N_{\eta_{(i)}}\right]\right).$$

Then

$$\left[d_{(i)}^N\right]^T .r_{(i)}^N = d_{(i)}{}^T\left(-f'\left(\xi_{(i)}\right)\right) + I\left[\left(d_{(i)}{}^T + D_{(i)}{}^T\right)\left[-2f'{}^N_{\eta_{(i)}}\right] - d_{(i)}{}^T\left(-f'\left(\xi_{(i)}\right)\right)\right].$$

$$S\left\{\left[d_{(i)}^N)\right]^T .A_N .d_{(i)}^N\right\} = \left(\left(d_{(i)}{}^T , d_{(i)}{}^T + D_{(i)}{}^T\right)\right)\left(A, A + \Lambda\right)\left(d_{(i)}, d_{(i)} + D_{(i)}\right)$$

$$= \left(d_{(i)}{}^T A.d_{(i)}, \left(d_{(i)}{}^T + D_{(i)}{}^T\right)\left(A + \Lambda\right)\left(d_{(i)} + D_{(i)}\right)\right).$$

Hence

$$\left[d_{(i)}^N)\right]^T .A_N .d_{(i)}^N = d_{(i)}{}^T A.d_{(i)} + I\left[\left(d_{(i)}{}^T + D_{(i)}{}^T\right)\left(A + \Lambda\right)\left(d_{(i)} + D_{(i)}\right) - d_{(i)}{}^T A.d_{(i)}\right].$$

Then we can write

$$S\left(\alpha_{(i)}^{N}\right)=S\left(\frac{\left[d_{(i)}^{N}\right]^{T}.r_{(i)}^{N}}{\left[d_{(i)}^{N})\right]^{T}.A_{N}.d_{(i)}^{N}}\right)=\frac{S\left(\left[d_{(i)}^{N}\right]^{T}.r_{(i)}^{N}\right)}{S\left(\left[d_{(i)}^{N})\right]^{T}.A_{N}.d_{(i)}^{N}\right)}$$

$$=\frac{\left(d_{(i)}^{\ T}\left(-f'\left(\xi_{(i)}\right)\right),\left(d_{(i)}^{\ T}+D_{(i)}^{\ T}\right)\left[-2f'^{N}_{\eta(i)}\right]\right)}{\left(d_{(i)}^{\ T}A.d_{(i)},\left(d_{(i)}^{\ T}+D_{(i)}^{\ T}\right)(A+\Lambda)\left(d_{(i)}+D_{(i)}\right)\right)}$$

$$=\left(\frac{d_{(i)}^{\ T}\left(-f'\left(\xi_{(i)}\right)\right)}{d_{(i)}^{\ T}A.d_{(i)}},\frac{\left(d_{(i)}^{\ T}+D_{(i)}^{\ T}\right)\left[-2f'^{N}_{\eta(i)}\right]}{\left(d_{(i)}^{\ T}+D_{(i)}^{\ T}\right)(A+\Lambda)\left(d_{(i)}+D_{(i)}\right)}\right)$$

Finally

$$\alpha_{(i)}^{N}=\frac{d_{(i)}^{\ T}\left(-f'\left(\xi_{(i)}\right)\right)}{d_{(i)}^{\ T}A.d_{(i)}}+I\left[\frac{\left(d_{(i)}^{\ T}+D_{(i)}^{\ T}\right)\left[-2f'^{N}_{\eta(i)}\right]}{\left(d_{(i)}^{\ T}+D_{(i)}^{\ T}\right)(A+\Lambda)\left(d_{(i)}+D_{(i)}\right)}-\frac{d_{(i)}^{\ T}\left(-f'\left(\xi_{(i)}\right)\right)}{d_{(i)}^{\ T}A.d_{(i)}}\right].$$

We can see that the classical $\alpha_{(i)}$ is the real part of $\alpha_{(i)}^{N}$.

Example:

To find the minimizer of

$$f\left(x_{N}\right)=\frac{1}{2}x_{N}^{T}\begin{bmatrix}4-4I & 2+2I\\ 2+2I & 2-2I\end{bmatrix}x_{N}-\begin{bmatrix}-1+I\\ 1-I\end{bmatrix}^{T}x_{N}.$$

Let $x_{(0)}^{N}=\begin{bmatrix}0\\0\end{bmatrix}$, and A_{N} -conjugate directions $d_{(0)}^{N}=\begin{bmatrix}1-I\\0\end{bmatrix}$ and $d_{(1)}^{N}=\begin{bmatrix}-3/8-(3/8)I\\ 3/4+(3/4)I\end{bmatrix}$.

$$t_{N}=\begin{bmatrix}4-4I & 2+6I\\ 2+6I & 2-2I\end{bmatrix}.$$

$$r_{(0)}^{N}=k_{N}-t_{N}x_{(0)}^{N}=k_{N}=\begin{bmatrix}-1+I\\ 1-I\end{bmatrix}.$$

$$\alpha_{(0)}^{N}=\frac{\begin{bmatrix}1-I & 0\end{bmatrix}\begin{bmatrix}-1+I\\ 1-I\end{bmatrix}}{\begin{bmatrix}1-I & 0\end{bmatrix}\begin{bmatrix}4-4I & 2+2I\\ 2+2I & 2-2I\end{bmatrix}\begin{bmatrix}1-I\\ 0\end{bmatrix}}=\frac{-1+I}{4-4I}=\frac{-1}{4}.$$

Thus

$$x_{(1)}^{N}=x_{(0)}^{N}+\alpha_{(0)}^{N}d_{(0)}^{N}=\begin{bmatrix}-1/4+(1/4)I\\ 0\end{bmatrix}.$$

And

$$r_{(1)}^{N}=k_{N}-t_{N}x_{(1)}^{N}=\begin{bmatrix}0\\ \frac{3}{2}-\frac{3}{2}I\end{bmatrix}.$$

$$\alpha_{(1)}^{N}=\frac{\begin{bmatrix}-\frac{3}{8}-\frac{3}{8}I & \frac{3}{4}+\frac{3}{4}I\end{bmatrix}\begin{bmatrix}0\\ \frac{3}{2}-\frac{3}{2}I\end{bmatrix}}{\begin{bmatrix}-\frac{3}{8}-\frac{3}{8}I & \frac{3}{4}+\frac{3}{4}I\end{bmatrix}\begin{bmatrix}4-4I & 2+2I\\ 2+2I & 2-2I\end{bmatrix}\begin{bmatrix}-\frac{3}{8}-\frac{3}{8}I\\ \frac{3}{4}+\frac{3}{4}I\end{bmatrix}}=\frac{\frac{9}{8}-\frac{9}{8}I}{\frac{9}{16}-\frac{153}{16}I}=2-2I.$$

And

$$x_{(2)}^{N} = x_{(1)}^{N} + \alpha_{(1)}^{N} d_{(1)}^{N} = \begin{bmatrix} -1+I \\ \frac{3}{2} - \frac{3}{2} I \end{bmatrix}.$$

Let us prove that the algorithm of Neutrosophic Conjugate Directions can compute x in n steps: Using the error term as a linear neutrosophic combination of the neutrosophic search directions

$$e_{(0)}^{N} = \sum_{j=1}^{n-1} \delta_j^N d_{(j)}^{N}$$

where δ_j^N are neutrosophic numbers one can compute simply.

Since that the search directions are A_N -orthogonal, then we have

$$\left[d_{(k)}^{N}\right]^T .A_N .e_{(0)}^{N} = \sum_j \delta_{(j)}^N \left[d_{(k)}^{N}\right]^T .A_N .d_{(j)}^{N}$$
$$= \delta_{(k)}^N \left[d_{(k)}^{N}\right]^T .A_N .d_{(k)}^{N}$$

hence

$$\delta_{(k)}^N = \frac{\left[d_{(k)}^{N}\right]^T .A_N .e_{(0)}^{N}}{\left[d_{(k)}^{N}\right]^T .A_N .d_{(k)}^{N}} = \frac{\left[d_{(k)}^{N}\right]^T .A_N .\left(e_{(0)}^{N} + \sum_{i=1}^{k-1} \alpha_{(i)}^N d_{(i)}^N\right)}{\left[d_{(k)}^{N}\right]^T .A_N .d_{(k)}^{N}} = \frac{\left[d_{(k)}^{N}\right]^T .A_N .e_{(k)}^{N}}{\left[d_{(k)}^{N}\right]^T .A_N .d_{(k)}^{N}}$$

We can see that $\alpha_{(k)}^N = -\delta_{(k)}^N$, hence we can write

$$e_{(i)}^{N} = e_{(0)}^{N} + \sum_{j=0}^{i-1} \alpha_{(j)}^N d_{(j)}^N = \sum_{j=0}^{n-1} \delta_{(j)}^N d_{(j)}^N - \sum_{j=0}^{i-1} \delta_{(j)}^N d_{(j)}^N$$
$$= \sum_{j=0}^{n-1} \delta_{(j)}^N d_{(j)}^N - \sum_{j=0}^{i-1} \delta_{(j)}^N d_{(j)}^N = \sum_{j=i}^{n-1} \delta_{(j)}^N d_{(j)}^N.$$

6. Conclusions

In this study, the neutrosophic quadratic form, method of neutrosophic steepest descent, and method of neutrosophic conjugate directions were introduced. Many examples have been discussed. The author did not address the topic of convergence study, so it did not adhere to the examples concerned with the fulfillment of the condition $x_N^T A_N x_N > 0$.

It is possible to work in many research directions, such as Markov chains. It is also possible to introduce a disturbance operator on the behavior of the moving point and then work on processing and correction to reach the optimal solution. One can also return to [14] and examine the application of the (AH) in generalizing stable distributions.

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